Qualitative Properties of Hybrid Singular Systems

by

Humeyra Kiyak

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Examining Committee Membership

The following served on the Examining Committee for this thesis. The decision of the Examining Committee is by majority vote.

External Examiner: Pei Yu

Professor, Dept. of Applied Mathematics, University of Western Ontario

Supervisor: Xinzhi Liu

Professor, Dept. of Applied Mathematics, University of Waterloo

Internal Member: Edward R. Vrscay

Professor, Dept. of Applied Mathematics, University of Waterloo

Internal Member: Jun Liu

Assoc. Professor, Dept. of Applied Mathematics, University of Waterloo

Internal-External Member: Sherman Shen

Professor, Dept. of Electrical and Computer Engineering, University of Waterloo

Author's Declaration

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

A singular system model is mathematically formulated as a set of coupled differential and algebraic equations. Singular systems, also referred to as descriptor or differential algebraic systems, have extensive applications in power, economic, and biological systems. The main purpose of this thesis is to address the problems of stability and stabilization for singular hybrid systems with or without time delay.

First, some sufficient conditions on the exponential stability property of both continuous and discrete impulsive switched singular systems with time delay (ISSSD) are proposed. We address this problem for the continuous system in three cases: all subsystems are stable, the system consists of both stable and unstable subsystems,and all subsystems are unstable. For the discrete system, we focus on when all subsystems are stable, and the system consists of both stable and unstable subsystems. The stability results for both the continuous and the discrete system are investigated by first using the multiple Lyapunov functions along with the average-dwell time (ADT) switching signal to organize the jumps among the system modes and then resorting the Halanay Lemma.

Second, an optimal feedback control only for continuous ISSSD is designed to guarantee the exponential stability of the closed-loop system. Moreover, a Luenberger-type observer is designed to estimate the system states such that the corresponding closed-loop error system is exponentially stable. Similarly, we have used the multiple Lyapunov functions approach with the ADT switching signal and the Halanay Lemma.

Third, the problem of designing a sliding mode control (SMC) for singular systems subject to impulsive effects is addressed in continuous and discrete contexts. The main objective is to design an SMC law such that the closed-loop system achieves stability. Designing a sliding surface, analyzing a reaching condition and designing an SMC law are

investigated throughly. In addition, the discrete SMC law is slightly modified to eliminate chattering.

Last, mean square admissibility for singular switched systems with stochastic noise in continuous and discrete cases is investigated. Sufficient conditions that guarantee mean square admissibility are developed by using linear matrix inequalities (LMIs).

Acknowledgements

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My deepest gratitude and love go to my family for their support and prayers including my wonderful siblings, Erdem, Selim and Ayse. I continue to be thankful to my beloved husband, Mehmet Yilmaz, for his emotional support, understanding and love.

Finally, I am grateful to my God, Allah (c.c). Without Allah (c.c), I could not have achieved all the things that I have today in my life.

Dedication

This is dedicated to my dearest parents, and beloved husband.

Table of Contents

List of Figures

List of Notations

$\mathbb{N}$	the set of natural numbers.
$\mathbb{Z}$	the set of all integers.
$\mathbb{Z}^+$	the set of all nonnegative integers.
$\mathbb{R}$	real number set.
$\mathbb{R}^+$	the set of all nonnegative real numbers.
$\mathbb{R}^n$	n-dimensional Euclidean space.
$\mathbb{R}^{n\times m}$	the space of $n \times m$ real matrices.
$\mathcal{C}(A, B)$	the set of all continuous functions mapping A into B.
$\mathcal{C}^1(A, B)$	the set of all continuously differentiable functions mapping A into B.
$\mathcal{C}_d$	set of all continuous functions defined from $[-d, 0]$ to $\mathbb{R}^n$
x^T	transpose of a vector x.
$\|x\|$	norm of a vector of x.

sup	supremum, the least upper bound.
inf	infimum, the greatest lower bound.
$\|A\|$	norm of a matrix A.
A^T	transpose of a matrix A.
A^{-1}	inverse of a matrix A.
$\lambda_{\max}(A)$	maximum eigenvalue of a matrix A.
$\lambda_{\min}(A)$	minimum eigenvalue of a matrix A.
$A > 0$	A is a symmetric positive definite matrix.
$A \geq 0$	A is a symmetric positive semi definite matrix.
$A < 0$	A is a symmetric negative definite matrix.
$A \leq 0$	A is a symmetric positive semi definite matrix.
$\mathbb{E}$	mathematical expectation or mean
$(\Omega, \mathcal{F}, \mathbb{P})$	a complete probability space.
$W(t)$	a standard Wiener process defined on $(\Omega, \mathcal{F}, \mathbb{P})$.
$\mathcal{L}_{ad}(\Omega; L^p[a, b])$	the family of $\mathbb{R}^n$-valued $\mathcal{F}_t$-adapted process $f(t)$ such that $\int_a^b \|f(t)\|^p dt < \infty$ for all $t \in [a, b]$
$\mathcal{C}^{1,2}(\mathbb{R}^+ \times \mathbb{R}^n; \mathbb{R}^+)$	the set of all functions from $\mathbb{R}^+ \times \mathbb{R}^n$ to $\mathbb{R}^+$ that are continuously differentiable in the first variable and twice continuously differentiable in the second variable.

Chapter 1

Introduction

The thesis is concerned with singular systems which are encountered in many applications. For instance, a singular system has the form

$$E\dot{x}(t) = Ax(t) + g(t, x(t)) \tag{1.1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $E \in \mathbb{R}^{n \times n}$ is a singular matrix with rank $(E) = r < n$, and $g(t, x(t)) : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^n$ is the vector-valued differentiable function.

In the remainder of this chapter, we shall present some motivations for the study of singular systems, and provide the scope of this thesis.

1.1 Motivation

Singular systems, also referred to as descriptor systems or differential algebraic systems, consist of differential equations, which describe dynamics of subsystems, and the algebraic

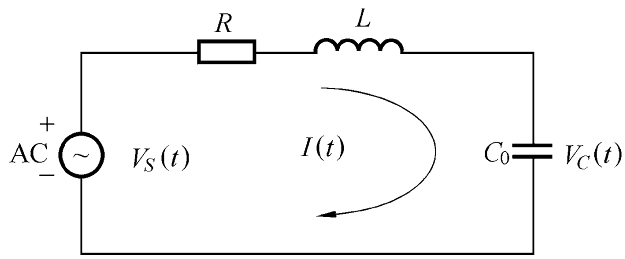

Figure 1.1: A single-loop circuit network

equations, which characterize coupling by constraints such as joints ([3],[4]). Many different real world or man-made systems such as economic systems, power systems, control systems, chemical processes, etc., can be modeled by singular systems ([5]-[6],[7]). See the following motivating examples for illustrations.

Example 1.1.1. ***(Electrical Networks)*** *[8]-[9] Electrical networks can be composed by subsystems of network elements (such as resistor, capacitors, inductors) and by coupling due to Kirchoff's laws. Let consider a simple circuit network as shown in Figure 1.1. Using basic circuit theory and Kirchoff's law, the following equations, which describe the system, are obtained as*

$$\begin{aligned} L\dot{I}(t) &= V_L(t), \\ \dot{V}_C(t) &= \frac{1}{C_0} I(t), \\ RI(t) &= V_R(t), \\ V_L(t) + V_C(t) + V_R(t) &= V_S(t) \end{aligned}$$

where $R, L,$ *and* C_0 *stand for resistor, inductor, and capacitor, respectively,* $V_R(t)$*,* $V_L(t)$*, and* $V_C(t)$ *are their voltages, respectively.* $V_S(t)$ *is the voltage source which is taken as the* *control input. If we choose the state $x(t)$, control input $u(t)$ and output $y(t)$ as*

$$x(t) = \begin{bmatrix} I(t) \\ V_L(t) \\ V_C(t) \\ V_R(t) \end{bmatrix}, \quad u(t) = V_S(t), \quad y(t) = V_C(t)$$

the above equations can be written in the following singular linear system form:

$$E\dot{x} = Ax + Bu,$$
$$y = Cx$$

with

$$E = \begin{bmatrix} L & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{1}{C_0} & 0 & 0 & 0 \\ -R & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}.$$

Example 1.1.2. ***(Chemical Processes)*** *[6] Models of chemical processes typically consist of differential equations describing the dynamic balances of mass and energy while additionally algebraic equations account for thermodynamic equilibrium relations, steady-state assumptions, empirical correlations, and so on.*

Three main difficulties are observed in singular systems, compared with that of nonsingular systems:

1. Satisfying the existence and uniqueness of solutions

   Existence and uniqueness of solutions are determined by regularity in both linear and nonlinear singular systems ([10]). Moreover, constraints on nonlinear perturbation term are required for solutions to nonlinear singular systems.

2. Calculating of derivatives of the Lyapunov functions

   Lyapunov direct method has been the most popular and efficient approach for stability analysis of singular systems. However, for singular systems, because of the mixed differential-algebraic nature, the selection of Lyapunov function and calculation of its derivative along the motions of the systems are more difficult ([11]).

3. Occurring impulses and jumps in the state of the system

   Inconsistent initial condition of singular systems may lead to finite instantaneous jump ([12]). When such jumps reach a certain extent, they could be physically destructive to the system. In addition to inconsistent initial condition, index of a given singular system causes impulses in the state.

Many practical systems involve a mixture continuous dynamics which consist of continuous differential equations representing the evolution of the system, and discrete dynamics which include difference equations representing jumps or impulsive actions in the system states. Systems in which these two kinds of dynamics coexist and interact are usually called *hybrid systems*. For example, impulsive systems can be naturally viewed as a class of hybrid systems whereas switching (switched) systems is considered as a class of hybrid systems. Moreover, impulsive switched systems in which impulses arise as a result of switching is another class of hybrid system. In this theses, we focused on these three types of hybrid systems.

The dynamics of an impulsive system are usually given by a pair of equations. That is, an impulsive system consists of a differential equation that describes a continuous evolutionary process, and a difference equation that governs the discrete impulsive actions ([13], [14],[15]). The main characteristic of an impulsive system is that the system is subjected to abrupt changes at certain moments between the intervals of the continuous evolutions. The durations of these changes sufficiently small when compared to the total duration of the process. These changes can be reasonably well-approximated by instantaneous changes of the state. Impulsive phenomena are ubiquitous in the real world or man-made systems, such as mass-spring systems, biological systems, epidemic disease models, and so forth. Theoretically, impulsive systems have richer properties than the corresponding non-impulsive ones. However, stability is the most important property of impulsive systems. The reasons are twofold ([15]). Firstly, when stable system undergo impulses, the system may not conserve its stability due to up or down jump discontinuities. Secondly, the impulses may play as a stabilizing factor in an unstable system if they are well-timed in the sense that they are logically formalized.

A switched system is composed of a family of continuous-time subsystems and a rule that controls the switching between them ([1], [16]). Therefore, the dynamics of the system is determined by both the subsystems and the switching signal (law). It is clear that while the continuous dynamics of the switched systems consist of continuous subsystems, the discrete dynamics include the switching signal which illustrates as abrupt change of mode of the continuous dynamics. For instance, in applications, the discrete dynamics can be the abrupt change of climates or environment; a thermostat turning the heat on and off.

There has been reasonable progress in the study of stability of switched systems. Most of the work has been made on designing an appropriate switching rule to stabilize the system ([1],[16],[17]). In some studies, it was shown that if the switched system has stable subsystems, then the entire system is stable provided that the dwell time, the time between any two consecutive switchings, is sufficiently large. Later, it was showed that a similar result holds when the average interval between the consecutive switchings is no less than dwell time, leading to the average dwell time concept. This approach was also used to prove the stability of switched systems consist of stable and unstable subsystems.

The stability of switched systems has been also studied using either common Lyapunov function method or multiple Lyapunov function method. Since different subsystems of a switched system may have very different structures, the differential inequalities imposed by a common Lyapunov function for all subsystems can become a very restrictive. Therefore, the method of multiple Lyapunov functions provides a more powerful tool for stability of switched systems ([18]).

Another class of hybrid systems are impulsive switched systems which consist of switches of states and abrupt changes at the switching instants ([19],[20],[21]). From control point of view, switching and impulsive control is an effective method in the sense that it allows stabilization of a complex system by using only small control impulses in different modes.

When the singular and impulsive switching phenomena are simultaneously encountered, the impulsive switched singular systems are naturally arisen. That is, when subsystems are singular systems, the impulsive switched system becomes a class of impulsive switched singular systems ([22],[23],[24]). Because of switches among multiple singular subsystems, it is inevitably difficult to analyze such systems. Besides regularity and impulse nature, the problem of consistent initial condition is also important to analyze these systems. It is well recognized that the singular system may exhibit finite instantaneous jumps due to inconsistent initial conditions. In impulsive switched singular systems, it cannot be guaranteed that the states at switching points satisfy the consistent initial condition of the next activated subsystem under arbitrary switching. On the other hand, instantaneously state jumps are unavoidable in impulsive switched singular systems even if all subsystems are regular and impulse-free. It should be noted that the accumulation of jumps can be destructive even though every jump strength is small, especially when the switching is very fast. Thus, in a switched singular systems the following questions arise as ([25],[26],[27]):

Q1. How to describe the state jumps in impulsive switched singular systems?

Q2. How state jump affect the stability of the entire system?

Q3. How to choose a switching law such that the entire system is stable?

To best of our knowledge, there are limited results dealing with the above questions in the literature. Thus, it is crucially needed to study the impulsive switched singular systems.

Time delay is unavoidable in many physical systems whose future state depends not only on the present state but also on the past state. Therefore, physical systems can be modeled more realistically by including some of the historical values of the system states; this leads to systems with delay differential equations. However, the study of systems with time delay is usually more challenging than that of ordinary systems since the delay may cause oscillations or chaotic behaviors in the system. Moreover, time delays may be the main cause of instability of dynamical systems. Thus, many important and interesting results have been reported on the stability analysis and control problems of delayed dynamical singular and nonsingular systems.

In many mathematical models, systems are driven by some inherent noise having a stochastic structure. Therefore, this stochasticity in the design of these systems leads to stochastic systems (SSs). The theory of SSs is more sophisticated compared to the deterministic systems. As a result, many tools utilized in analyzing deterministic problems cannot be carried over to handle the corresponding stochastic problems. For instance, stochastic integrals may not be in the sense of the classical Leibniz-Newton calculus, but in the sense of Itô calculus. Also, while solution of a deterministic system is represented by a single sample path, in a stochastic system solution consists of an infinite sequence of the sample paths because of the randomness. Thus, the qualitative notions are redefined in the SSs. Stability, among these notions, has received incredible attention ([28]-[29]).

Considering random noise in a switched system leads to a stochastic switched system. People have mostly worked SSs with Markovian switching in which the switching is a Markovian chain ([30]-[31]). Singular systems with Markovian switching have also gotten attention ([31]-[32]). On the other hand, singular subsystems of a switched system perturbed by a Wiener process are also considered as a stochastic switched singular system. Stability analysis of this kind of system have been obtained in terms of linear matrix inequalities (LMIs).

1.2 Thesis Organization

This thesis is organized as follows: In Chapter 2, the required mathematical background including some definitions and theorems are given. Chapter 3 addresses the exponential stability problem for continuous and discrete impulsive switched singular systems with time-delay, respectively. In Chapter 4, an optimal feedback control for impulsive switched singular systems with time-delay has been designed to guarantee the exponential stability of the closed-loop system. In addition, a Luenberger-type observer is designed to estimate the system states such that the corresponding closed-loop error system is exponentially stable. Firstly, some sufficient conditions on the exponential stability property of the continuous impulsive switched systems have been proposed in Chapter 5. To obtain this objective, a sliding surface is designed on which the sliding motion of the system state happens, then a sliding mode control law is designed to force the system state to reach, stay and slide on the sliding surface. Secondly, a similar sliding surface and control design are adopted to discrete impulsive switched singular systems to obtain some sufficient conditions for the exponential stability of the full order system. Chapter 6 provides sufficient conditions for mean square admissibility in terms of LMIs for continuous and discrete switched singular systems perturbed by a Wiener process, respectively.

Chapter 2

Mathematical Background

This chapter provides necessary background about impulsive systems, switched systems, delay systems, singular systems and stochastic systems.

2.1 Switched Systems

2.1.1 Preliminaries

A switched system, which is a type of hybrid system, consists of mode-dependent continuous/discrete dynamics and a logic-based switching rule which triggers abrupt transitions between modes. Switched systems arise in two contexts: (i) a system which exhibits sudden changes in its dynamics; (ii) when switching control is used to stabilize a continuous system.

A switched system with switching signal σ can be described by

$$\dot{x} = f_{\sigma(t,x(t))}(t, x(t)), \tag{2.1}$$

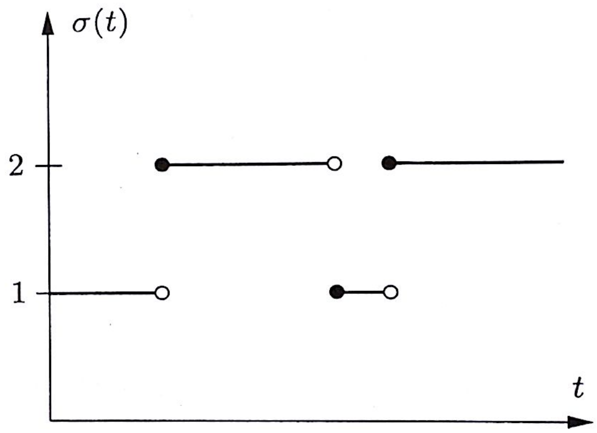

Figure 2.1: Time-dependent switching [1]

where $x \in \mathbb{R}$ is the system state, the switching signal (switching law or switching rule) $\sigma(t,x) : \mathbb{R}^+ \times \mathbb{R}^n \rightarrow \Xi$ is a function taking values in an index set $\Xi = \{1, 2, \ldots, N\}$ for some $N \in \mathbb{N}$. $f_{\sigma(t,x(t))}$ is a family of functions defined on $\mathbb{R}^+ \times \mathbb{R}^n$ and indexed by Ξ as $\{f_i, i \in \Xi\}$.

There are two specific types of switching signal:

- **time-dependent switching:** switching signal σ is a function of time t. That is, each instant t specifies a subsystem $\dot{x} = f_{\sigma(t)}(t, x)$. For an illustration of a simple time-dependent switching signal, see Figure 2.1.

- **state-dependent switching:** σ is a function of the state x. The space is divided into finite (or infinite) number of regions by a family of switching surfaces. A single subsystem is activated in each of these regions, and the system changes mode when crossing a switching surface.

  In Figure 2.2, the thick curves denote the switching surfaces, the thin curves with ar-

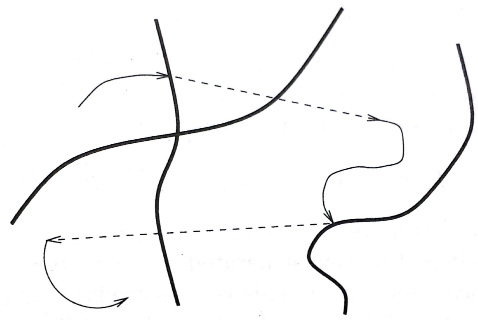

Figure 2.2: State-dependent switching [1]

rows denote the continuous portions of the trajectory, and the dashed lines symbolize the jumps.

Note that it is difficult to make a distinction between state-dependent and time dependent switching because every possible trajectory of system with state-dependent switching is also a solution of the system with time-dependent switching for suitably defined switching signal. In view of this, a nonlinear switched system is given by

$$\begin{cases} \dot{x} = f_{\sigma(t)}(t, x), \quad t \geq t_0, \\ x(t_0) = x_0, \end{cases} \tag{2.2}$$

where the switching signal $\sigma(t) : [t_0, \infty) \to \Xi$ is assumed to be right-continuous constant function taking values in a finite compact set $\Xi = \{1, 2, \ldots, N\}$ for some $N \in \mathbb{N}$.

The existence and uniqueness of solution to system (2.2) guarantees by continuity in t and Lipschitz in x for each $f_i(t, x),\ i \in \Xi$. The solution of (2.2) is parameterized by both the initial condition and the switching signal σ. To show this dependency, solution denotes as $x(t) = x(t; x_0, \sigma)$.

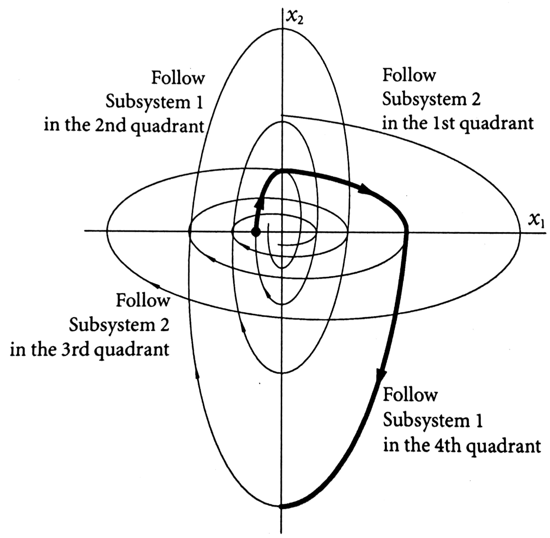

Figure 2.3: Switching between stable subsystems [2]

2.1.2 Stability of Switched Systems

The investigation for sufficient conditions guaranteeing stability is an important problems in switching systems. Because of the switching signal, a switched system can be unstable even if all subsystems are stable ([33]). This case is well illustrated in Figure 2.3, where a system switching between two stable subsystems, i.e. $\Xi = 1, 2$. Similarly, the switched system can be stable depending on a switching signal even though all subsystems are unstable. Figure 2.4 illustrates the case when two subsystems are unstable. Based on these cases, the main problems in stability of switched systems are divided into the following three categories:

1. Find sufficient conditions that guarantee stability of a switched system under arbitrary switching ([17]-[34]).

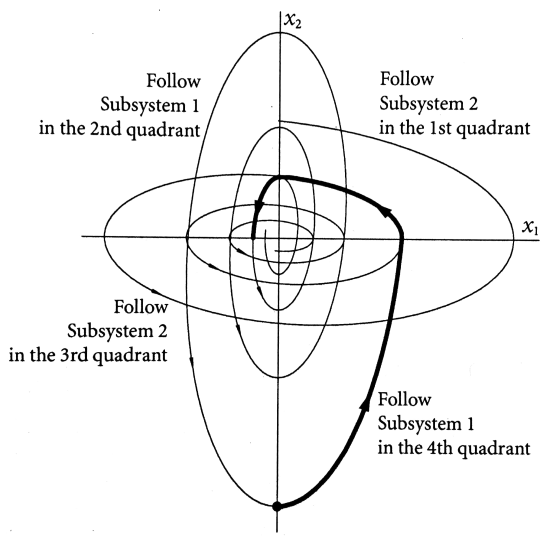

Figure 2.4: Switching between unstable subsystems [2]

2. Identify the switching signals such that a switched system is stable ([35],[36]).

3. Design a particular switching signal to stabilize a switched system ([18],[37]).

Some details about each of these categories are given below.

First Category: Stability under arbitrary switching

A challenge to analyze the stability of switched systems is that even if the all subsystems are stable, the switched system might be unstable, illustrated in the following example.

Example 2.1.1. *Consider the following switched system*

$$\dot{x} = A_{\sigma(t)}x(t) \tag{2.3}$$

with $\Xi = \{1, 2\}$ *and*

$$A_1 = \begin{bmatrix} -0.1 & 1 \\ -10 & -0.1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -0.1 & 10 \\ -1 & -0.1 \end{bmatrix}.$$

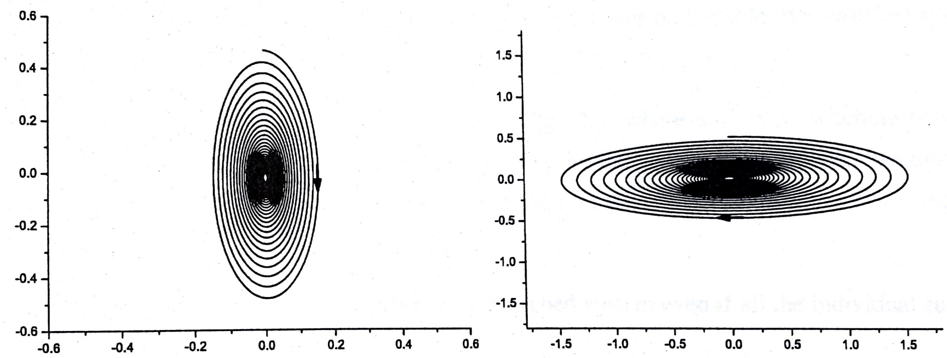

Figure 2.5: Phase portraits for subsystems in Example 2.1.1 [2]

Both A_1 and A_2 are Hurwitz matrices since they have eigenvalues with negative real parts. Therefore, both subsystems are stable (see Figure 2.5 for an illustration). Consider the following state-dependent switching rule when $x_1x_2 < 0$, then subsystem 1 is active, otherwise subsystem 2 is active. Under this switching rule, the switched system is unstable, as shown in Figure 2.6.

For stability under arbitrary switching to be possible, a necessary condition is that subsystems must be stable. However, it is not a sufficient condition for stability.

Theorem 2.1.1. *[1] Let $V \in C^1(\mathbb{R}^n, \mathbb{R}^+)$ be a positive definite and radially unbounded function and $W \in C(\mathbb{R}^n, \mathbb{R}^+)$ be positive definite function. If*

$$\nabla V(x) \cdot f_i(x) \leq -W(x) \tag{2.4}$$

for all $x \in \mathbb{R}^n$ and for all $i \in \Xi$ then the switched system

$$\begin{cases} \dot{x} = f_{\sigma(t)}(x), \quad t \geq t_0, \\ x(t_0) = x_0, \end{cases} \tag{2.5}$$

is globally asymptotically stable for arbitrary switching.

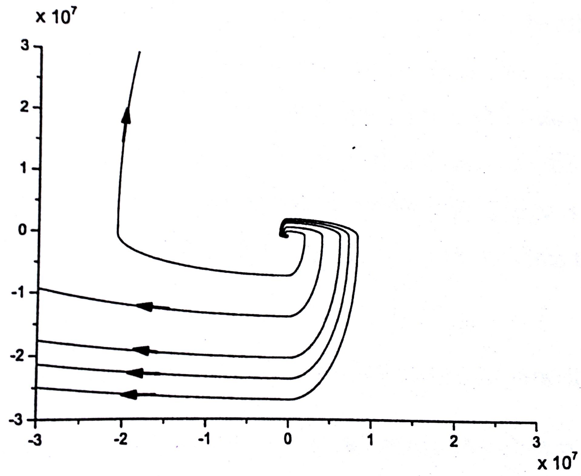

Figure 2.6: Phase-portrait for the switched system in Example 2.1.1 [2]

Second Category: Stability under slow switching

Again consider a switched system composed entirely of stable subsystems. The switched system might be unstable. The remedy to this situation is that the switched system dwells in each of its stable subsystems as long as possible. That is, a switching signal is designed such that the running time of each single mode is sufficiently large. Thus, the switching effect diminishes and this ensures that the switched system maintains the same stability property. This type of switching is called *slow switching*. To characterize the slow switching suppose that the switched system (2.2) switches at fixed time t_k for $k \in \mathbb{N}$. Then, the running time between any two switching moments is called ***dwell time*** and is denoted by T_D. This type of switching signals can be represented by

$$\Xi_{\inf}(T_D) = \{T_D \mid \inf\{t_k - t_{k-1}\} \geq T_D,\ \forall k \in \mathbb{N}\} \tag{2.6}$$

for some $T_D > 0$.

In brief, if all subsystems are stable and dwell time T_D sufficiently large, then stability of the switched system is preserved.

Determining a dwell time may be too restrictive since the (exponential) stability is an asymptotic property. Instead, ***the average dwell time***, T_a can be taken sufficiently large as the concept in [37]. This type of switching signals satisfies

$$N_\sigma(t, t_0) \leq N_0 + \frac{t - t_0}{T_a}, \ \forall t \geq t_0 \tag{2.7}$$

where $N_\sigma(t, t_0)$ is the number of switches between t_0 and t, and N_0 is the chatter bound.

Theorem 2.1.2. *[37] Consider system (2.2). Assume there exist Lyapunov functions $V_i \in C^1(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R})$, $i \in \Xi$, positive constants $c_1,\ c_2,\ \lambda$ and p such that*

$$c_1\|x\|^p \leq V_i(t, x) \leq c_2\|x\|^p,$$

and

$$\dot{V}_i(t, x) \leq -\lambda\|x\|^p, \quad V_i(t, x) \leq \mu V_j(t, x)$$

for all $i,\ j \in \Xi$. Then system (2.2) is exponentially stable for every switching signal with average dwell time T_a satisfying

$$T_a > \frac{c_1 \ln \mu}{\lambda}.$$

Thirth Category: Stability under designed switching

This category can be viewed as a control problem where switching control is used to stabilize an unstable continuous system. In [35], Wicks *et al.* first constructed a stabilizing time-dependent switching rule for a linear switched system which is in the form

$$\dot{x} = A_{\sigma(t)}(x(t)) \tag{2.8}$$

with $\sigma(t) \in \Xi = \{1, 2\}$ and $A_{\sigma(t)}$ for all $\sigma(t) \in \Xi$ is not Hurwitz. If there exists a scalar $0 < \alpha < 1$ such that the convex combination

$$\alpha A_1 + (1 - \alpha)A_2$$

is Hurwitz, then there is a stabilizing state-dependent switching rule for the system (2.8).

The state-dependent switching rule is expendable to a linear switched system with m subsystems if there exist constants $\alpha_i > 0$ satisfying $\sum_{i=1}^{m} \alpha_i = 1$ such that the convex combination matrix $\sum_{i=1}^{m} \alpha_i A_i$ is Hurwitz.

2.2 Impulsive Systems

2.2.1 Preliminaries

To construct a system of impulsive differential equations, the Dirac delta function is used ([38],[15]). The Dirac delta function is often defined, in a nonrigorous way, as follows,

$$\delta(t) = \lim_{\varepsilon \to 0} I_\varepsilon(t)$$

where

$$I_\varepsilon(t) = \begin{cases} \dfrac{1}{\varepsilon}, & 0 \le t \le \varepsilon, \\ 0, & t > \varepsilon. \end{cases}$$

The Dirac delta function is also defined more rigorously in terms of integration as follows,

$$\int_{-\infty}^{\infty} f(t)\delta(t-a)dt = f(a),$$

where f is a continuous function. Now, consider the following control system

$$\begin{aligned} \dot{x}(t) &= f(t, x(t)) + u(t), \qquad (2.9) \\ x(t_0) &= x_0, \end{aligned}$$

where $u \in \mathbb{R}^n$ is the system input of the form

$$u(t) = \sum_{k=1}^{\infty} I_k(x(t))\delta(t - t_k), \qquad (2.10)$$

with I_k being control gain matrix with appropriate dimensions and $\delta(\cdot)$ being the Dirac delta function, and t_k being a strictly increasing sequence $\{t_k\}_{k=1}^{\infty}$ with $\lim_{k\to\infty} t_k = \infty$.

Integrating (2.9) over $[t_k, t_k + h]$ gives

$$x(t_k + h) - x(t_k) = \int_{t_k}^{t_k+h} \Big[f(s, x(s)) + \sum_{k=1}^{\infty} I_k(x(s))\delta(s - t_k)\Big] ds,$$

where h is sufficiently small. As $h \to 0^+$, we obtain

$$\Delta x(t_k) = x(t_k^+) - x(t_k) = I_k(x(t_k)),$$

where $x(t_k^+) = \lim_{h\to 0^+} x(t_k + h)$, and $x(t_k) = x(t_k^-)$. Apparently, the control acts as an impulsive force at time instant t_k. As a result, system (2.9) can be rewritten as:

$$\begin{cases} \dot{x}(t) = f(t, x(t)), & t \neq t_k \\ \Delta x(t) = I_k(x(t)), & t = t_k \\ x(t_0^+) = x_0, \end{cases} \tag{2.11}$$

where $x(t) \in \mathbb{R}^n$ is system state, $f(t, x) : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^n$, $I_k(x(t)) \in \mathbb{R}^n$ is the impulse amount at time t_k, $\{t_k\}_{k=1}^{\infty}$ are the fixed impulsive times that form an increasing sequence satisfying $t_{k-1} < t_k$ and $\lim_{k\to\infty} t_k = \infty$. $\Delta x = x(t^+) - x(t^-)$ where $x(t^-)$ (and $x(t^+)$) is the state just before (and just after) the impulsive action with $x(t^+) = \lim_{s\to t^+} x(s)$. The solution x is assumed to be left-continuous, i.e., $x(t_k^-) = x(t_k)$. This system is called an impulsive system.

Next, existence and uniqueness of system (2.11) is established by the following theorem:

Theorem 2.2.1. ***(Existence and Uniqueness)*** *[14] Assume $f \in C^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$ and $x + I_k(x) \in \mathbb{R}^n$ for each $k \in \mathbb{N}$ and $x \in \mathbb{R}^n$. Then, for each $(t_0, x_0) \in \mathbb{R} \times \mathbb{R}^n$ there exists a unique solution of (2.11).*

If $\phi(t) = \phi(t, t_0, x_0)$ on the interval (α, β) containing t_0 is a solution of system (2.11), then it satisfies the following [14]:

(i) $(t, \phi(t)) \in \mathbb{R}^+ \times \mathbb{R}^n$ for $t \in (\alpha, \beta)$ and $\phi(t_0^+) = x_0$ where $x_0 \in \mathbb{R}^n$.

(ii) For $t \in (\alpha, \beta)$, $t \neq t_k$, $\dot{\phi}(t) = f(t, \phi(t))$.

(iii) $\phi(t)$ is continuous from the left in (α, β), and if $t = t_k$, then $\phi(t_k^+) = \phi(t_k) + I_k(\phi(t_k))$.

The solution of system (2.11) evolves as follows: the system states starts when $t \neq t_0$. Then, whenever $t \neq t_k$, the system process in governed by the ordinary differential equation $\dot{x}(t) = f(t, x(t))$ until $t = t_1$. At this moment, the process is subject to an impulse and instantly changes by some amount $I_k x(t)$, given by the difference equation in (2.11), causing a jump discontinuity in the system state. For $t > t_1$, if $t \neq t_2$ holds, the process continues according to the differential equation $\dot{x}(t) = f(t, x(t))$ until an impulsive action occurs again. This continues in the same manner as long as the solution exists.

2.2.2 Stability of Impulsive Systems

In applications, the most significant property of impulsive systems is stability. When a stable system is subject to impulses, the system may lose or maintain its stability due to impulses. On the other hand, under some conditions impulses stabilize some system even when the underlying systems are unstable ([39], [40], [41]).

Definition 2.2.1. *Let $x(t) = x(t, t_0, x_0)$ be a given solution of (2.11) existing for $t \geq t_0$. Then, the trivial solution of (2.11) is said to be stable if for any $\varepsilon > 0$ and $t_0 \in \mathbb{R}^+$, there exists a $\delta = \delta(t_0, \varepsilon)$ such that*

$$\|x_0\| < \delta \text{ implies } \|x(t)\| < \varepsilon, \quad t \geq t_0.$$

The following theorem represents the sufficient conditions that guarantee exponential stability of a simple impulsive system given by

$$\begin{cases} \dot{x}(t) = Ax(t), & t \neq t_k \\ \Delta x(t) = B_k x(t), & t = t_k \end{cases} \tag{2.12}$$

where $x \in \mathbb{R}^n$ is the system state variable, and $A \in \mathbb{R}^{nxn}$ is a system coefficient matrix, and $B_k \in \mathbb{R}^{nxn}$ is constant matrices.

Theorem 2.2.2. *[39] Assume that the eigenvalues of A have negative real parts. Then the trivial solution of system (2.12) is globally exponentially stable if the following inequality holds:*

$$\ln \alpha_k - \nu(t_k - t_{k-1}) \leq 0, \quad k = 1, 2, \ldots \tag{2.13}$$

where $\alpha_k = \frac{\lambda_{\max}\left([I+B_k]^T P[I+B_k]\right)}{\lambda_{\min}(P)}$ with P being a positive definite matrix satisfying

$$A^T P + PA = -Q$$

for any positive definite matrix Q, $0 < \nu < \xi$, and $\xi = \frac{\lambda_{\min}(Q)}{\lambda_{\min}(P)}$.

2.3 Delay Systems

2.3.1 Preliminaries

The main difference between ordinary and delay differential equations is initial data. In ordinary differential equations, the initial condition is given at a specific time while a continuous function defined on a finite interval is the initial data for delay differential conditions.

We use the following notations: let $\mathcal{C}_d$ with $d > 0$ being a time delay be the set of all continuous functions defined from $[-d, 0]$ to $\mathbb{R}^n$. Then, function $x_t : [-d, 0] \to \mathbb{R}^n$ is defined by $x_t(s) = x(t+s)$ for all $s \in [-d, 0]$. Moreover, d-norm of this function is defined by $\|x_t\|_d = \sup_{t-d\leq\theta\leq t} \|x(\theta)\|$ where $\|.\|$ is the Euclidean norm on $\mathbb{R}^n$.

The initial value problem of a delay system is described by

$$\begin{cases} \dot{x}(t) = f(t, x_t) \\ x_{t_0}(s) = \phi(s), \quad s \in [-d, 0] \end{cases} \tag{2.14}$$

where $f : \mathbb{R}^+ \times \Omega \to \mathbb{R}^n$ for $\Omega \subset C_d$.

Theorem 2.3.1. ***(Existence)*** *If $f : \mathbb{R}^+ \times \Omega \to \mathbb{R}^n$ is continuous where $\Omega \subset C_d$ is an open set, then for any $(t_0, \phi) \in \mathbb{R}^+ \times \Omega$ there exists a solution of system (2.14).*

Definition 2.3.1. *A function $f(t, \psi)$ defined on $\mathbb{R}^+ \times \Omega$ is said to be Lipschitz in ψ if there exists a constant $L > 0$ such that for any $(t, \psi_1), (t, \psi_2) \in \mathbb{R}^+ \times \Omega$,*

$$\|f(t, \psi_1) - f(t, \psi_2)\| \leq L \|\psi_1 - \psi_2\|.$$

Theorem 2.3.2. ***(Uniqueness)*** *If $f(t, x)$ is continuous in t and Lipschitz in x, then for any $(t_0, \phi) \in \mathbb{R}^+ \times \Omega$, there exists a unique solution of (2.14).*

2.3.2 Stability of Delay Systems

We require the following definition in order to state the stability notions of delay systems.

Definition 2.3.2. *Suppose that $f(t, 0) = 0$ for all $t \in \mathbb{R}^+$, which guarantees that system (2.14) possesses a trivial solution $x(t) = 0$. The trivial solution of system (2.14) is said to be*

(i) ***stable*** *if for any* $\varepsilon > 0$ *and* $t_0 \in \mathbb{R}^+$, *there exists a* $\delta = \delta(t_0, \varepsilon) > 0$ *such that*

$$\|\phi\|_d < \delta \text{ implies } \|x(t)\| < \varepsilon \text{ for any } t \geq t_0 - d;$$

(ii) ***unstable*** *if (i) does not hold;*

(iii) ***uniformly stable*** *if (i) holds for* $\delta = \delta(\varepsilon)$;

(iv) ***asymptotically stable*** *if it is stable and there exists a positive constant* $\delta = \delta(t_0)$ *such that*

$$\|\phi\|_d < \delta \text{ implies } \lim_{t\to\infty} x(t) = 0;$$

(v) ***exponentially stable*** *if there exist positive constants* k, α, *and* λ *such that*

$$\|x(t)\| \leq \alpha \|\phi\|_d e^{-\lambda(t-t_0)} \text{ for all } \|\phi\|_d < k \text{ and } t \geq t_0.$$

The Lyapunov method is an efficient method for stability of delay systems. There are two Lyapunov methods commonly used for delay systems: the Lyapunov-Krasovskii functional method and the Razumikhin technique. To guarantee asymptotic stability of a time-delay system by the Lyapunov-Krasovskii method a positive definite functional with a negative definite time derivative along the trajectory of the system has to be found. On the other hand, a positive definite function whose time-derivative is negative definite under the Razumikhin condition guarantees the asymptotic stability of the system by the Razumikhin technique. By utilizing the Razumikhin technique, the following lemma was established by Halanay [42].

Lemma 2.3.1. *[42] Assume that* u *is a continuous nonnegative function defined on* $[t_0 - d, b)$ *and satisfies*

$$\dot{u}(t) \leq -\alpha u(t) + \beta \sup_{\sigma\in[t-d,t]} u(\sigma), \qquad t \in [t_0, b)$$

where α and β are positive constants satisfying $\alpha > \beta > 0$. Then, there exists a positive constant ξ such that

$$u(t) \leq \sup_{\sigma \in [t_0 - d, t_0]} u(\sigma) e^{-\xi(t-t_0)}, \qquad t \in [t_0, b)$$

where ξ is a unique positive solution of

$$\xi = \alpha - \beta e^{\xi d}.$$

In the following theorem, using the Halanay lemma sufficient conditions are obtained for exponential stability of linear delay system

$$\dot{x}(t) = Ax(t) + Bx(t-d), \tag{2.15}$$

where A, B are constant matrices with appropriate dimensions.

Theorem 2.3.3. *[43] The origin of system (2.15) is exponentially stable if matrix A is Hurwitz and the following inequality is satisfied*

$$-\frac{\lambda_{\min}(Q) - \beta^*}{\lambda_{\max}(P)} + \frac{\beta^*}{\lambda_{\min}(P)} < 0$$

where P and Q are positive definite matrices satisfying the Lyapunov equation

$$A^T P + P^T A = -Q$$

and $\beta^* = \|PB\|$.

2.4 Stochastic Systems

In this section, we firstly introduce some notations and definitions.

2.4.1 Preliminaries

Probability theory is used to analyze random experiments which are called elementary events and usually denoted by w. These elementary events are grouped in a sample space Ω. $\mathcal{F}$ denotes the family of all interesting events of Ω. For further purposes, $\mathcal{F}$ is required to be σ-algebra as defined below.

Definition 2.4.1. *$\mathcal{F}$ is said to be a σ-algebra on Ω if the following conditions hold:*

1. *the empty set $\emptyset \in \mathcal{F}$;*
2. *if $A \in \mathcal{F}$, then $A^C \in \mathcal{F}$ where A^C is the complement of A;*
3. *if $\{A_i\}_{i\geq 1} \in \mathcal{F}$, $\cup_{i\geq 1} A_i \in \mathcal{F}$.*

Definition 2.4.2. *A real valued function $X : \Omega \to \mathbb{R}$ is said to be a random variable or $\mathcal{F}$-measurable if $\{w|\ X(w) \leq x\} \in \mathcal{F}$ for all $x \in \mathbb{R}$.*

If $(\Omega, \mathcal{F})$ is a measure space, then the elements of $\mathcal{F}$ are called $\mathcal{F}$-measurable sets.

Definition 2.4.3. *A function $\mathbb{P} : \mathcal{F} \to [0, 1]$ is said to be a probability measure on the measurable space $(\Omega, \mathcal{F})$ if*

1. *$\mathbb{P}(\emptyset) = 0$ and $\mathbb{P}(\Omega) = 1$;*
2. *for $\{A_i\}_{i\geq 1} \subset \mathcal{F}$ where $A_i \cap A_j = \emptyset$ for all $i \neq j$,*

$$\mathbb{P}\left(\cup_{i\geq 1} A_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(A_i).$$

Furthermore, the triplet $(\Omega, \mathcal{F}, \mathbb{P})$ is called a probability space. If the σ-algebra is complete; that is, $\mathcal{F} = \mathcal{F}^C$ where $\mathcal{F}^C$ is the complement of $\mathcal{F}$, then the probability space is said to be complete.

Let $X_1(w), X_2(w), \ldots,$ a sequence of random variables, and $X(w)$ be defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Definition 2.4.4. *The sequence $\{X_k(w)\}_{k\geq 1}$ is said to converge to $X(w)$ with probability one (w.p.1) or almost surely (a.s.) if*

$$\mathbb{P}\left\{w|\ \lim_{k\to\infty} X_k(w) = X(w)\right\} = 1.$$

Definition 2.4.5. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space. A sequence of increasing sub-σ-algebra $\{\mathcal{F}_t\}_{t\geq 0}$ of $\mathcal{F}$ is said to be a filtration.*

Definition 2.4.6. *A stochastic process $X(t)$ is a family of random variables*

$$\left\{X_t(w)|\ t \in I \subset \mathbb{R}^+,\ w \in \Omega\right\}.$$

Moreover, $X(t)$ is a continuous stochastic process if for all $w \in \Omega$, $X_t(w)$ is continuous for all $t \in \mathbb{R}$.

Definition 2.4.7. *The probabilistic behavior of a random variable is described by its distribution function $F(x)$ defined by*

$$F(x) = \mathbb{P}\{w|\ X(w) \leq x\} \text{ for all } x \in \mathbb{R}.$$

Assume that X is a continuous random variable, then there exists a non-negative and integrable function $f(x)$ such that

$$F(x) = \int_{-\infty}^{x} f(s)ds,$$

which implies that $f(x) = \dfrac{dF(x)}{dx}$ which is called *the probability density function (p.d.f.)* of X.

Definition 2.4.8. *The mathematical expectation (or mean or the first moment) of a continuous stochastic process $X(t)$ is defined by*

$$m(t) = \mathbb{E}\left[X(t)\right] = \int_{-\infty}^{\infty} x f(x,t) dx,$$

where $f(x,t)$ is the p.d.f. of $x = X(t)$;
the mean square (or the second moment) of the stochastic process $X(t)$ is defined by

$$m_2(t) = \mathbb{E}\left[X^2(t)\right] = \int_{-\infty}^{\infty} x^2 f(x,t) dx;$$

the variance of the stochastic process $X(t)$ is

$$Var\left[X(t)\right] = \mathbb{E}\left[(X(t) - m(t))^2\right] = m_2(t) - m^2(t).$$

Definition 2.4.9. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$. An continuous (a.s.) stochastic process $W(t)$ for all $t \in \mathbb{R}^+$ is said to be Wiener (or Brownian motion) process if*

1. $\mathbb{P}\{w | \; W(0) = 0\} = 1$;

2. *for any $0 \leq s < t < \infty$, the increment $W(t) - W(s)$ is independent of $\mathcal{F}_s$;*

3. *for any $t \in \mathbb{R}^+$ and $h > 0$, the increment $W(t+h) - W(t)$ is Gaussian (or normally) distributed with*

$$\mathbb{E}\left[W(t+h) - W(t)\right] = \mu h;$$

$$\mathbb{E}\left[(W(t+h) - W(t))^2\right] = \sigma^2 h,$$

where the mean $\mu \in \mathbb{R}$ and the variance σ^2 is a positive constant. The stochastic process $W(t)$ defines a standard Wiener process if the mean $\mu = 0$ and the variance $\sigma^2 = 1$.

A typical n-dimensional nonlinear stochastic system is defined by

$$dx(t) = f(t, x(t))dt + g(t, x(t))dW(t), \qquad t \in [t_0, T], \tag{2.16}$$
$$x(t_0) = x_0,$$

for any $t_0, T \in \mathbb{R}^+$ with $T > t_0$, where $x(t)$ is $\mathbb{R}^n$-valued stochastic process, $W(t) = (W_1(t), W_2(t), \cdots, W_m(t))^T$ is an m-dimensional Wiener process, $f : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^n$ is the deterministic drift coefficient of the process x, $g : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^{n\times m}$ is the diffusion coefficient of the process x, and x_0 is an $\mathcal{F}_{t_0}$-measurable $\mathbb{R}^n$-valued random variable such that $\mathbb{E}\left[\|x_0\|^2\right] < \infty$.

The stochastic integral equation corresponding to initial value problem (2.16) is

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))ds + \int_{t_0}^{t} g(s, x(s))dW(s), \tag{2.17}$$

where the first integral is a Riemann integral (a.s.) and the second one is an Itô integral satisfying the following properties:

1. $\mathbb{E}\left[\int_{t_0}^{t} g(s, x(s))dW(s)\right] = 0;$

2. $\mathbb{E}\left\|\int_{t_0}^{t} g(s, x(s))dW(s)\right\|^2 = \int_{t_0}^{t} \mathbb{E}\left\|g(s, x(s))\right\|^2 dW(s).$

In the following part, we will present the solution of (2.16).

Definition 2.4.10. *Let* $(\Omega, \mathcal{F}, \mathcal{F}_{t\geq 0}, \mathbb{P})$ *be a complete probability space. For any* $w \in \Omega$, $a, b \in \mathbb{R}^+$ *with* $a < b$, *and* $p \geq 1$, *a random process* $f(t, w)$ *is belongs to class* $\mathcal{L}_{ad}(\Omega, L^p[a, b])$ *if it is* $\mathcal{F}_t$*-adapted and almost all its sample paths are pth integrable in the Riemann sense.*

Definition 2.4.11. *For any* $t_0, T \in \mathbb{R}^+$, *the* $\mathbb{R}^n$*-valued stochastic process* $x(t) = x(t, t_0, x_0)$ *is said to be a solution of the initial value problem in (2.16) if the following properties hold:*

1. *$x(t)$ is continuous and $\mathcal{F}_t$-adapted;*

2. *the $\mathbb{R}^n$-valued $f \in \mathcal{L}_{ad}(\Omega; L^1[a,b])$ and the $\mathbb{R}^{n\times m}$-valued $g \in \mathcal{L}_{ad}(\Omega; L^2[a,b])$;*

3. *for any $t \in [t_0, T]$, $x(t)$ satisfies the stochastic differential equation in (2.16) w.p.1;*

4. *at $t = t_0$, x satisfies the initial condition in (2.16) w.p.1.*

Moreover, a solution $x(t)$ is said to be unique if

$$\mathbb{P}\left\{x(t) = y(t),\ \forall t \in [t_0, T]\right\} = 1,$$

where $y(t)$ is any other solution.

In stochastic differential equations, we use the stochastic version of the chain rule, which is called Itô formula.

Definition 2.4.12. ***(Itô Formula)*** *For any $t_0 \in \mathbb{R}^+$ and $t \geq t_0$, let $x(t)$ be an $\mathbb{R}^n$-dimensional Itô process, i.e., $\mathbb{R}^n$-valued continuous adapted process satisfying*

$$dx(t) = f(t, x(t))dt + g(t, x(t))dW(t), \quad (a.s.)$$

where $f \in \mathcal{L}_{ad}(\Omega; L^1[a,b])$ and $g \in \mathcal{L}_{ad}(\Omega; L^2[a,b])$. Let $V \in \mathcal{C}^{1,2}(\mathbb{R}^+ \times \mathbb{R}^n; \mathbb{R})$. Then, for any $t \geq t_0$, V is a real-valued Itô process satisfying

$$dV(t,x) = \mathcal{L}V(t,x)dt + V_x(t,x)g(t,x)dW(t), \quad (a.s.)$$

where

$$\mathcal{L}V(t,x) = V_t(t,x) + V_x(t,x)f(t,x) + \frac{1}{2}tr\left[g^T(t,x)V_{xx}(t,x)g(t,x)\right].$$

If $V \in \mathcal{C}^{1,2}(\mathbb{R}^+ \times \mathbb{R}^n; \mathbb{R})$, *then*

$$V_t(t,x) = \frac{\partial V}{\partial t}, \quad V_x(t,x) = \left[\frac{\partial V}{\partial x_1}, \frac{\partial V}{\partial x_2}, \dots, \frac{\partial V}{\partial x_n}\right],$$

$$V_{xx}(t,x) = \left[\frac{\partial^2 V}{\partial x \partial x_j}\right]_{n\times n} = \begin{bmatrix} \frac{\partial^2 V}{\partial x_1 \partial x_1} & \frac{\partial^2 V}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 V}{\partial x_1 \partial x_n} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial^2 V}{\partial x_n \partial x_1} & \frac{\partial^2 V}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 V}{\partial x_n \partial x_n} \end{bmatrix}.$$

The operator $\mathcal{L}$ *is called the averaged derivative or infinitesimal diffusion operator at a point* (t,x) *and can be generally defined as*

$$\mathcal{L}V(t,x) = \lim_{h\to 0^+} \frac{1}{h}\Big[\mathbb{E}\big[V(t+h, x(t+h))\big] - V(t,x)\Big].$$

2.4.2 Stability of Stochastic Systems

Now we will present the definition of some stability properties of the trivial solution of system (2.16).

Definition 2.4.13. *For any* $t \geq t_0$ *with* $t_0 \in \mathbb{R}^+$, *let* $x(t) = x(t, t_0, x_0)$ *be a solution of system (2.16). The trivial solution of (2.16) is said to be*

- ***almost surely stable*** *(or* ***stable w.p.1****) if for any* ε, $\varepsilon^* > 0$, *and* $t_0 \in \mathbb{R}^+$, *there exists* $\delta = \delta(\varepsilon, \varepsilon^*, t_0)$ *such that*

$$\|x_0\| < \delta \quad \textit{implies} \quad \mathbb{P}\{w | \sup_{t\geq t_0} \|x(t)\| > \varepsilon^*\} < \varepsilon;$$

- ***pth moment stable*** *if for any* $\varepsilon > 0$ *and* $t_0 \in \mathbb{R}^+$, *there exists* $\delta = \delta(\varepsilon, t_0)$ *such that for* $p > 0$

$$\|x_0\|^p < \delta \quad \textit{implies} \quad \mathbb{E}[\sup_{t\geq t_0} \|x(t)\|^p] < \varepsilon;$$

- ***asymptotically stable*** *if for any* $\varepsilon \in (0,1)$, *there exists* $\delta = \delta(\varepsilon, t_0)$ *such that*

$$\|x_0\| < \delta \quad \textit{implies} \quad \mathbb{P}\{w| \lim_{t\to\infty} \sup \|x(t)\| = 0\} < 1 - \varepsilon;$$

- ***almost surely asymptotically stable*** *if it is almost surely stable and*

$$\mathbb{P}\{w| \lim_{t\to\infty} \sup \|x(t)\| = 0\} = 1;$$

- ***pth moment asymptotically stable*** *if if it is stable in the pth moment and*

$$\lim_{t\to\infty} \mathbb{E}[\sup_{t\geq t_0} \|x(t)\|^p = 0;$$

- ***pth moment exponentially stable*** *if there exist positive constants* p, α, *and* β *such that, for any* $t_0 \in \mathbb{R}^+$,

$$\|x_0\|^p < \delta \quad \textit{implies} \quad \mathbb{E}[\|x(t)\|^p] \leq \alpha \, \|x_0\|^p \, e^{-\beta(t-t_0)}.$$

2.5 Singular Systems

This section is devoted to singular systems in both the continuous and discrete cases, respectively. Some basic definitions for singular systems are given. Then, we focus on stability of singular systems in both cases.

2.5.1 Continuous Singular Systems

State space variable method is used to obtain state space models of singular systems using physical variables such as speed, weight, or temperature, which are sufficient to characterize the system. A set of equations can be established by the physical relationships among the variables. This set of equations can be arranged in the following form

$$E\dot{x}(t) = f(t, x(t)) \tag{2.18}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $E \in \mathbb{R}^{n\times n}$ is a singular matrix with rank $(E) = r < n$, and $f(t, x(t)) : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^n$ is the vector-valued differentiable function. The system (2.18) is called *a general nonlinear singular system.*

When $f(t, x(t))$ is equal to $Ax(t) + g(t, x(t))$ where $A \in \mathbb{R}^{n\times n}$ is the coefficient matrix, $g(t, x(t)) : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^n$ is the vector-valued nonlinear perturbation, the general nonlinear singular system (2.18) reduces to the following form:

$$E\dot{x}(t) = Ax(t) + g(t, x(t)). \tag{2.19}$$

Equation (2.19) describes *a nonlinear singular system.*

Definition 2.5.1. *System (2.19) (or matrix pair (E, A)) is regular if there exists a constant scalar $\gamma \in \mathbb{C}$ such that*

$$\det(\gamma E - A) \neq 0.$$

Moreover, the matrix pair (E, A) is said to be impulse free if $\deg(\det(\gamma E - A)) = rank(E)$.

Theorem 2.5.1. *[44] System (2.19) is regular if and only if there exist two nonsingular matrices $\tilde{Q}, \tilde{P}$ such that $\tilde{Q} = \begin{bmatrix} \tilde{Q}^1 \\ \tilde{Q}^2 \end{bmatrix}$ and $\tilde{P} = \begin{bmatrix} \tilde{P}^1 & \tilde{P}^2 \end{bmatrix}$ where $\tilde{Q}^1 \in \mathbb{R}^{n_1\times n}$, $\tilde{Q}^2 \in \mathbb{R}^{n_2\times n}$, $\tilde{P}^1 \in \mathbb{R}^{n\times n_1}$, $\tilde{P}^2 \in \mathbb{R}^{n\times n_2}$ and the following standard decomposition holds:*

$$\tilde{Q}E\tilde{P} = diag\,(I_{n_1}, N)\,, \quad \tilde{Q}A\tilde{P} = diag\,(A_1, I_{n_2})\,, \quad \tilde{Q}g(t,x) = \begin{bmatrix} g_1(t, x_1, x_2) \\ g_2(t, x_1, x_2) \end{bmatrix}, \tag{2.20}$$

where $n_1 + n_2 = n$, $A_1 \in \mathbb{R}^{n_1\times n_1}$, $N \in \mathbb{R}^{n_2\times n_2}$ is a nilpotent matrix with nilpotent index h, $g_1 \in \mathbb{R}^{n_1}$ and $g_2 \in \mathbb{R}^{n_2}$.

Corollary 2.5.1. *[44] Let (E, A) be regular and n_1 be the dimension of the decomposition (2.20), then*

$$\deg(\det(\gamma E - A)) = n_1.$$

Corollary 2.5.2. *[44] If the matrix pair (E, A) is impulse free, then*

$$\deg(\det(\gamma E - A)) = r$$

where $r = rank(E)$.

Let

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = P^{-1}x, \quad x_1 \in \mathbb{R}^{n_1}, \quad x_2 \in \mathbb{R}^{n_2},$$

and system (2.19) be regular. By the above theorem, system (2.19) is a restricted system equivalent to

$$\dot{x_1} = A_1 x_1 + g_1(t, x_1, x_2), \tag{2.21}$$

$$N\dot{x_2} = x_2 + g_2(t, x_1, x_2). \tag{2.22}$$

Subsystems (2.21) and (2.22) are called *the slow subsystem* and *the fast subsystem*, respectively. x_1 and x_2 are called *the slow substate* and *fast substate*, respectively ([10]).

The nilpotent matrix N causes impulse terms in the solution of subsystem (2.22). This reason implies the following result.

Corollary 2.5.3. *[8] System (2.19) is impulse free if and only if $N = 0$.*

In the literature, there are two kinds of solution to the singular systems: distributional and classical solutions ([10], [45],[46]). We consider only classical solutions expressed by classical functions, which are continuously differentiable. A classical solution to a singular system requires consistent initial values ([10]). Necessary conditions for consistent initial conditions have been proposed by [12] and [47]. The characterization of the set of consistent initial conditions $\mathcal{H}_0$ is given as follows.

Corollary 2.5.4. *Let the function $g_2(t, x_1, x_2)$ be $(h-1)$ times continuously differentiable when given system (2.19) is regular. Then, the set of consistent initial conditions is given by*

$$\mathcal{H}_0 = \left\{ \eta \mid [0 \ I_{n_1}]P^{-1}\eta = -\sum_{j=0}^{h-1} N^j g_2^{(j)}(t_0) \right\}$$

where $I_{n_1} \in \mathbb{R}^{n_1 \times n_1}$ is an identity matrix, and P is defined in decomposition form.

The following theorem is the existence and uniqueness theorem for solutions of (2.19).

Theorem 2.5.2. *[47],[48] Given the regular nonlinear singular system (2.19) is of index h and a consistent initial condition x_0 for the system at t_0. If the following conditions (a) and (b) are satisfied in the closed domain $D : \|y - y_0\| \leq K, \ t_0 \leq t \leq t_0 + R$ where $y = P^{-1}x$ and $y_0 = P^{-1}x_0$:*

(a) the function $g_2(x_1, x_2, t)$ is h times differentiable with respect to x_1, x_2, and t in the domain $G \supset D$;

(b) the function $g_1(x_1, x_2, t)$ and $\sum_{i=0}^{h-1} N^i g_2^{(i+1)}(x_1, x_2, t)$ are continuous and satisfy the Lipschitz condition in D.

Then, the regular nonlinear singular system has a unique solution satisfying $x(t_0) = x_0$ and defined on the interval $[t_0, t_0 + \alpha]$, where $\alpha = \min(R, K/M)$ and

$$M = \max \left\{ \sup_D \|A_1 x_1 + g_1(x_1, x_2, t)\|, \sup_D \left\| \sum_{i=0}^{h-1} N^i g_2^{(i+1)}(x_1, x_2, t) \right\| \right\}.$$

Before finishing this subsection, it is necessary to stress the importance of the condition underlying the existence and uniqueness of solution to nonlinear singular systems. For a nonlinear singular system (2.19) to have a unique solution, the function $g_2(x_1, x_2, t)$ appearing in the decomposition of the nonlinear perturbation $g(t, x(t))$ needs to be h times piecewise continuously differentiable.

2.5.2 Stability Definitions and Theorems For Continuous Case

Consider the following singular system

$$E\dot{x}(t) = f(t, x(t)) \tag{2.23}$$
$$x(t_0) = x_0$$

where $f(t, x(t)) : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is piecewise continuous function which guarantee existence and uniqueness of solution to (2.23) and assume that $f(t, 0) \equiv 0$ for all t so that the system has a trivial solution.

Definition 2.5.2. *The trivial solution of system (2.23) is said to be stable if for every $\varepsilon > 0$ and any $t_0 \in \mathbb{R}^+$ there exists a $\delta = \delta(t_0, \varepsilon) > 0$ such that $\|x_0\| < \delta$ implies $\|x(t; t_0, x_0)\| < \varepsilon$ for all $t \geq t_0$.*

Definition 2.5.3. *The trivial solution of system (2.23) is said to be uniformly stable if for every $\varepsilon > 0$ and any $t_0 \in \mathbb{R}^+$ there exists a $\delta = \delta(\varepsilon) > 0$ such that $\|x_0\| < \delta$ implies $\|x(t; t_0, x_0)\| < \varepsilon$ for all $t \geq t_0$.*

Definition 2.5.4. *The trivial solution of system (2.23) is said to be asymptotically stable if it is stable and there exists a $\eta = \eta(t_0) > 0$ such that*

$$\lim_{t \to \infty} x(t; t_0, x_0) = 0$$

whenever $\|x_0\| < \eta$.

Definition 2.5.5. *The trivial solution of system (2.23) is said to be uniformly asymptotically stable if it is uniformly stable and for every* $\varepsilon > 0$ *there exists a* $\delta > 0$ *and* $T = T(\varepsilon) > 0$ *such that for all* $t_0 \in \mathbb{R}^+$, $\|x_0\| < \delta$ *implies* $\|x(t; t_0, x_0)\| < \varepsilon$ *for all* $t \geq t_0 + T$.

Now consider system (2.23) in a special form described by

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + g(t, x(t)) \\ x(t_0) &= x_0 \end{aligned} \tag{2.24}$$

with index h and x_0 being a consistent initial condition. Assume that system (2.24) is regular, and satisfies the conditions in Theorem 2.5.2. Also, to ensure that system has a trivial solution, assume that $g(t, 0) \equiv 0$.

Definition 2.5.6. *Regular system (2.24) is said to be exponentially stable if there exist* $\alpha,\ \beta > 0$ *such that its state* $x(t)$ *satisfies*

$$\|x(t)\| \leq \alpha \|Ex_0\| e^{-\beta(t-t_0)}, \qquad t \geq t_0.$$

Definition 2.5.7. *System (2.24) is said to be E-exponentially stable if there exist constants* $\alpha,\ \beta > 0$ *such that*

$$\|Ex(t)\| \leq \alpha \|Ex_0\| e^{-\beta(t-t_0)}, \qquad t \geq t_0.$$

E-exponential stability means stability of the slow variable x_1.

The following lemma characterizes the relationship between the exponential stability and the E-exponential stability for system (2.24).

Lemma 2.5.1. *[44],[49] For system (2.24) with $g(t, x) = 0$, the E-exponential stability is equivalent to the exponential stability.*

The following theorem gives a criteria for the stability of regular nonlinear singular systems.

Theorem 2.5.3. *[10] The singular system (2.24) is stable if and only if*

$$\sigma(E, A) \subset \mathbb{C}^- = \{s \in \mathbb{C}|\ \mathrm{Re}(s) < 0\};$$

where $\sigma(E, A) = \{s \in \mathbb{C}|\ \det(sE - A) = 0\}$ and all perturbations satisfy

$$\|g(t, x)\| \leq \gamma \|x\|$$

with sufficiently small γ.

The above theorem indicates that the stability of the singular system is determined by eigenvalues of the matrix pair and the perturbation which is bounded by a linear growth bound.

Definition 2.5.8. *System (2.24) is called admissible if it is stable and impulse-free.*

Theorem 2.5.4. *[50],[51],[52] If system (2.24) with $g(t, x) = 0$ is admissible, then for each $Y > 0$ there exists $X > 0$ satisfying*

$$E^T XA + A^T XE = -E^T Y E.$$

2.5.3 Discrete Singular Systems

Consider discrete time singular system

$$Ex(n + 1) = Ax(n) + g(n, x(n)), \tag{2.25}$$

where $x \in \mathbb{R}^N$, $E, A \in \mathbb{R}^{N\times N}$ are system coefficient matrices where E being singular with $\text{rank}(E) = r < N$, and $g(n, x(n)) : \mathbb{Z}^+ \times \mathbb{R}^N \to \mathbb{R}^N$.

Definition 2.5.9. *System (2.25) is regular if there exists a constant scalar $\gamma \in \mathbb{C}$ such that*

$$\det(\gamma E - A) \neq 0.$$

Moreover, the matrix pair (E, A) is said to be casual if $\deg(\det(\gamma E - A)) = rank(E)$.

Remark 2.5.1. *Impulse freeness in continuous time singular systems is called as casual in discrete time singular systems.*

Theorem 2.5.1, Corollary 2.5.1, Corollary 2.5.2, and Corollary 2.5.3 in continuous singular systems are valid for the discrete singular system in (2.25).

2.5.4 Stability Definitions and Theorems For Discrete Case

Consider the following singular system

$$Ex(n+1) = f(n, x(n)) \tag{2.26}$$
$$x(n_0) = x_0$$

where $f(n, x(n)) : \mathbb{Z}^+ \times \mathbb{R}^N \to \mathbb{R}^N$ is piecewise continuous function and assume that there exists a unique solution to (2.26).

Definition 2.5.10. *The trivial solution of system (2.26) is said to be stable if for every $\varepsilon > 0$ and any $n_0 \in \mathbb{Z}^+$ there exists a $\delta = \delta(n_0, \varepsilon) > 0$ such that $\|x_0\| < \delta$ implies $\|x(n; n_0, x_0)\| < \varepsilon$ for all $n \geq n_0$. It is uniformly stable if δ may be chosen independent of n_0. If it is not stable, it is said to be unstable.*

Definition 2.5.11. *The solution of system (2.26) is said to be exponentially stable if for the initial condition* $x(n_0) = x_0$ *there exist constants* $0 < \varepsilon < 1$ *and* $M \geq 1$ *such that* $\|x(n)\| \leq M\|Ex_0\|\varepsilon^{n-n_0}$ *for any* $n \geq n_0$.

Definition 2.5.12. *System (2.26) is said to be E-exponentially stable if there exist constants* $0 < \varepsilon < 1$ *and* $M \geq 1$ *such that* $\|Ex(n)\| \leq M\|Ex_0\|\varepsilon^{n-n_0}$ *for any* $n \geq n_0$.

Let us consider $f(n, x(n))$ in (2.26) in a special form described by $f(n, x(n)) = Ax(n) + g(n, x(n))$. In this case, the system in (2.26) is defined as

$$\begin{aligned} Ex(n+1) &= Ax(n) + g(n, x(n)) \\ x(n_0) &= x_0 \end{aligned} \tag{2.27}$$

Definition 2.5.13. *System (2.27) is said to be stable if system (2.26) is regular and all eigenvalues of the system are within the unit circle centered at the origin.*

Theorem 2.5.5. *[53],[54] Suppose that* $g(n, x(n)) = 0$ *and matrix* A *is invertible, then linear singular system (2.27) is asymptotically stable if and only if there exists* $X > 0$ *satisfying*

$$A^T XA - E^T XE = -Y,$$

where $Y > 0$.

Chapter 3

Stability of Impulsive Switched Singular Systems with Time-Delay: Continuous and Discrete

In this chapter, exponential stability of impulsive switched singular systems with time-delay (ISSSD) is studied. The stability results for both continuous and discrete ISSSD have been investigated by using the multiple Lyapunov functions along with the average-dwell time (ADT) switching signal to organize the jumps among the system modes and by resorting the Halanay lemma. Numerical examples with simulations are also represented to further clarify the proposed methodology.

3.1 Impulsive Switched Singular Systems with Time-Delay: Continuous

Consider the following impulsive switched singular systems with time-delay

$$\begin{aligned} E_{\sigma(t)}\dot{x}(t) &= A_{\sigma(t)}x(t) + f_{\sigma(t)}(t, x(t-d)), \qquad && t \neq t_k \\ \triangle x(t) &= B_k x(t^-), && t = t_k \\ x_{t_0^-}(s) &= \phi(s), \qquad s \in [-d, 0], \end{aligned} \tag{3.1}$$

where $x \in \mathbb{R}^n$ is the system state variable, and $A_{\sigma(t)}, B_k, E_{\sigma(t)} \in \mathbb{R}^{n\times n}$ are system coefficient matrices where $E_{\sigma(t)}$ being singular with $\text{rank}(E_{\sigma(t)}) = r < n$, the matrix pairs $(E_{\sigma(t)}, A_{\sigma(t)})$ being regular, and B_k being constant matrices. The switching signal $\sigma(t) : [t_0, \infty) \to \Xi$ is a piecewise constant function taking values in a finite compact set $\Xi = \{1, 2, \dots, N\}$ for some $N \in \mathbb{N}$. $\{t_k\}_{k=1}^{\infty}$ are the impulsive times that form an increasing sequence satisfying $t_{k-1} < t_k$ and $\lim_{k\to\infty} t_k = \infty$. $\Delta x = x(t^+) - x(t^-)$ where $x(t^-)$ (and $x(t^+)$) is the state just before (and just after) the impulsive action with $x(t^+) = \lim_{s\to t^+} x(s)$. The solution x is assumed to be right-continuous, i.e., $x(t_k^+) = x(t_k)$. For all $t \geq t_0$, the delayed state vector is defined by $x_t = x(t+s)$, where $s \in [-d, 0]$ with d being a positive constant representing the time delay. $f_{\sigma(t)}(t, x(t-d)) : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^n$ are piecewise continuous vector-valued functions ensuring the existence and uniqueness of solutions for system (3.1) with $f_{\sigma(t)}(t, 0) \equiv 0,\ t \in \mathbb{R}^+$ and and there exist constant matrices $F_{\sigma(t)}$ such that

$$\|f_{\sigma(t)}(t, x) - f_{\sigma(t)}(t, x^*)\| \leq \|F_{\sigma(t)}E_{\sigma(t)}(x - x^*)\| \tag{3.2}$$

for all (t, x), $(t, x^*) \in D = \{(t, x) :\ t \in \mathbb{R}^+,\ \|E_{\sigma(t)}x\| < \rho\}$.

3.1.1 Systems with Stable Subsystems

Consider the system in (3.1) with stable subsystems. Before stating sufficient conditions that guarantee exponential stability of this system, we first present the following lemmas that are needed in the proof of main theorems.

Lemma 3.1.1 ([42]). *Assume that u is a continuous nonnegative function defined on $[t_0 - d, b)$ and satisfies*

$$\dot{u}(t) \leq -\alpha u(t) + \beta \sup_{\sigma \in [t-d,t]} u(\sigma), \qquad t \in [t_0, b)$$

where α and β are positive constants satisfying $\alpha > \beta > 0$. Then, there exists a positive constant ξ such that

$$u(t) \leq \sup_{\sigma \in [t_0-d,t_0]} u(\sigma) e^{-\xi(t-t_0)}, \qquad t \in [t_0, b)$$

where ξ is a unique positive solution of

$$\xi = \alpha - \beta e^{\xi d}.$$

Lemma 3.1.2. *For any vectors $x, y \in \mathbb{R}^n$ and a scalar $\varepsilon > 0$, the following inequality holds:*

$$2x^T y \leq \frac{1}{\varepsilon} x^T x + \varepsilon y^T y. \tag{3.3}$$

In the following theorem, some sufficient conditions are established to guarantee the exponential stability of system (3.1).

Theorem 3.1.1. *For any $i \in \Xi$, assume that each subsystem of (3.1) is admissible. Then, the trivial solution of (3.1) is exponentially stable if the following assumptions hold:*

(i) For any $i, j \in \Xi$ there exists $\gamma_k > 1$ such that

$$(I + B_k)^T E_j^T X_j E_j (I + B_k) \leq \gamma_k E_i^T X_i E_i, \tag{3.4}$$

where X_i and X_j are defined in the proof.

(ii) For any t_0, the switching law satisfies

$$N(t, t_0) \leq N_0 + \frac{t - t_0}{T_a}$$

where $N(t, t_0)$ represents the number of switchings in (t, t_0), and N_0 and T_a are the chatter bound and average dwell time to be defined, respectively.

Proof. Let $x(t) = x(t, t_0, x_0)$ be the solution of the system (3.1). For $t \in [t_{k-1}, t_k)$, define

$$\upsilon_i(t) = V_i(x(t)) = x^T(t) E_i^T X_i E_i x(t), \qquad t \neq t_k, \quad i = \sigma(t)$$

as a Lyapunov function candidate for i^{th} subsystem. Then, derivative of υ_i along the trajectory of (3.1) is given by

$$\begin{aligned} \dot{\upsilon}_i(t) &= \dot{x}^T(t) E_i^T X_i E_i x(t) + x^T(t) E_i^T X_i E_i \dot{x}(t) \\ &= x^T(t) \left(A_i^T X_i E_i + E_i^T X_i A_i \right) x(t) + f_i^T(t, x(t-d)) X_i E_i x(t) + x^T(t) E_i^T X_i f_i(t, x(t-d)) \\ &= -x^T(t) E_i^T Y_i E_i x(t) + 2 f_i^T(t, x(t-d)) X_i E_i x(t) \end{aligned} \tag{3.5}$$

where $-E_i^T Y_i E_i = A_i^T X_i E_i + E_i^T X_i A_i$ for any $Y_i > 0$.

Using Lemma 3.1.2 and inequality (3.2), we can obtain that

$$\begin{aligned} 2 f_i^T(t, x(t-d)) X_i E_i x(t) &\leq \frac{1}{\varepsilon_i} f_i^T(t, x(t-d)) f_i(t, x(t-d)) + \varepsilon_i x^T E_i^T(t) X_i^T X_i E_i x(t) \\ &\leq \frac{1}{\varepsilon_i} \| f_i(t, x(t-d)) \|^2 + \varepsilon_i \| E_i x(t) \|^2 \lambda_{\max}(X_i^2) \\ &\leq \frac{1}{\varepsilon_i} \| F_i E_i x(t-d) \|^2 + \varepsilon \| E_i x(t) \|^2 \lambda_{\max}(X_i^2) \end{aligned} \tag{3.6}$$

Substituting (3.6) into (3.5), we obtain

$$
\begin{aligned}
\dot{\upsilon}_i(t) &\leq -x^T(t)E_i^T Y_i E_i x(t) + \frac{1}{\varepsilon_i}\|F_i E_i x(t-d)\|^2 + \varepsilon_i \|E_i x(t)\|^2 \lambda_{\max}(X_i^2) \\
&\leq -\lambda_{\min}(Y_i)\|E_i x(t)\|^2 + \frac{1}{\varepsilon_i}\|F_i\|^2\|E_i x(t-d)\|^2 + \varepsilon_i\|E_i x(t)\|^2\lambda_{\max}(X_i^2) \\
&= \Big(-\lambda_{\min}(Y_i) + \varepsilon_i\lambda_{\max}(X_i^2)\Big)\|E_i x(t)\|^2 + \frac{1}{\varepsilon_i}\|F_i\|^2\|E_i x(t-d)\|^2 \\
&\leq -\alpha_i \upsilon_i(t) + \beta_i \sup_{\sigma\in[t-d,t]} \upsilon_i(\sigma) \qquad (3.7)
\end{aligned}
$$

where $\alpha_i = \dfrac{\lambda_{\min}(Y_i) - \varepsilon_i\lambda_{\max}(X_i^2)}{\lambda_{\max}(X_i)}$, $\beta_i = \dfrac{\|F_i\|^2}{\varepsilon_i\lambda_{\min}(X_i)}$.

Applying Lemma 3.1.1 to (3.7), we obtain the solution of differential inequality (3.7) for $t \in [t_{k-1}, t_k)$

$$
\upsilon_i(t) \leq \sup_{\sigma\in[t_{k-1}-d,t_{k-1}]} \upsilon_i(\sigma)e^{-\xi_i(t-t_{k-1})}, \qquad (3.8)
$$

where ξ_i is a unique positive solution of

$$
\xi_i = \alpha_i - \beta_i e^{\xi_i d}.
$$

On the other hand, for $t = t_k$, $k = 1, 2, 3, \ldots$, suppose $\sigma(t_k) = j$, it follows from (3.1) and (3.4) that

$$
\begin{aligned}
\upsilon_j(t_k) &= x^T(t_k)E_j^T X_j E_j x(t_k) \\
&= x^T(t_k^-)(I+B_k)^T E_j^T X_j E_j (I+B_k)x(t_k^-) \\
&\leq \gamma_k x^T(t_k^-)E_i^T X_i E_i x(t_k^-) \\
&= \gamma_k \upsilon_i(t_k^-). \qquad (3.9)
\end{aligned}
$$

Using (3.8) and (3.9) successively on each subinterval leads to the following results. For $t \in [t_0, t_1)$, we have

$$
\upsilon_{i_1}(t) \leq \sup_{\sigma\in[t_0-d,t_0]} \upsilon_{i_1}(\sigma)e^{-\xi_{i_1}(t-t_0)}, \qquad (3.10)
$$

For $t \in [t_1, t_2)$, we have

$$\upsilon_{i_2}(t) \leq \sup_{\sigma \in [t_1-d, t_1]} \upsilon_{i_2}(\sigma) e^{-\xi_{i_2}(t-t_1)}. \tag{3.11}$$

From (3.9), we can result in the following inequality

$$\begin{aligned} \sup_{\sigma \in [t_1-d, t_1]} \upsilon_{i_2}(\sigma) &\leq \gamma_1 \sup_{\sigma \in [t_1-d, t_1]} \upsilon_{i_1}(\sigma^-) \\ \sup_{\sigma \in [t_1-d, t_1]} \upsilon_{i_2}(\sigma) e^{-\xi_{i_2}(t-t_1)} &\leq \gamma_1 \sup_{\sigma \in [t_1-d, t_1]} \upsilon_{i_1}(\sigma^-) e^{-\xi_{i_2}(t-t_1)}. \end{aligned} \tag{3.12}$$

By (3.10) and (3.12), we have

$$\upsilon_{i_2}(t) \leq \gamma_1 \sup_{\sigma \in [t_1-d, t_1]} \upsilon_{i_1}(\sigma^-) e^{-\xi_{i_2}(t-t_1)}. \tag{3.13}$$

Suppose that $t_k - d \in [t_{k-1}, t_k)$, by (3.10), we can find $\sup_{\sigma \in [t_1-d, t_1]} \upsilon_{i_1}(\sigma^-)$ as

$$\sup_{\sigma \in [t_1-d, t_1]} \upsilon_{i_1}(\sigma^-) = \sup_{\sigma \in [t_0-d, t_0]} \upsilon_{i_1}(\sigma) e^{-\xi_{i_1}(t_1-d-t_0)}. \tag{3.14}$$

Thus, if we rewrite (3.13), we obtain for $t \in [t_1, t_2)$

$$\upsilon_{i_2}(t) \leq \gamma_1 \sup_{\sigma \in [t_0-d, t_0]} \upsilon_{i_1}(\sigma) e^{-\xi_{i_1}(t_1-d-t_0)} e^{-\xi_{i_2}(t-t_1)}. \tag{3.15}$$

Similarly, for $t \in [t_2, t_3)$ we can write the following inequality using (3.8)

$$\upsilon_{i_3}(t) \leq \sup_{\sigma \in [t_2-d, t_2]} \upsilon_{i_3}(\sigma) e^{-\xi_{i_3}(t-t_2)}. \tag{3.16}$$

Also, from (3.9) we have

$$\begin{aligned} \sup_{\sigma \in [t_2-d, t_2]} \upsilon_{i_3}(\sigma) &\leq \gamma_2 \sup_{\sigma \in [t_2-d, t_2]} \upsilon_{i_2}(\sigma^-) \\ \sup_{\sigma \in [t_2-d, t_2]} \upsilon_{i_3}(\sigma) e^{-\xi_{i_3}(t-t_2)} &\leq \gamma_2 \sup_{\sigma \in [t_2-d, t_2]} \upsilon_{i_2}(\sigma^-) e^{-\xi_{i_3}(t-t_2)}. \end{aligned} \tag{3.17}$$

By (3.16) and (3.17), we have

$$\upsilon_{i_3}(t) \leq \gamma_2 \sup_{\sigma\in[t_2-d,t_2]} \upsilon_{i_2}(\sigma^-)e^{-\xi_{i_3}(t-t_2)}. \tag{3.18}$$

Using (3.14), we can find $\sup_{\sigma\in[t_2-d,t_2]} \upsilon_{i_2}(\sigma^-)$ in (3.18) as

$$\sup_{\sigma\in[t_2-d,t_2]} \upsilon_{i_2}(\sigma^-) = \gamma_1 \sup_{\sigma\in[t_0-d,t_0]} \upsilon_{i_1}(\sigma)e^{-\xi_{i_1}(t_1-d-t_0)}e^{-\xi_{i_2}(t_2-d-t_1)} \tag{3.19}$$

Therefore, we obtain for $t \in [t_2, t_3)$

$$\upsilon_{i_3}(t) \leq \gamma_1\gamma_2 \sup_{\sigma\in[t_0-d,t_0]} \upsilon_{i_1}(\sigma)e^{-\xi_{i_1}(t_1-d-t_0)}e^{-\xi_{i_2}(t_2-d-t_1)}e^{-\xi_{i_3}(t-t_2)}. \tag{3.20}$$

In general, for $t \in [t_{k-1}, t_k)$

$$\upsilon_{i_k}(t) \leq \gamma_1\gamma_2\dots\gamma_{k-1} \sup_{\sigma\in[t_0-d,t_0]} \upsilon_{i_1}(\sigma)e^{-\xi_{i_1}(t_1-d-t_0)}e^{-\xi_{i_2}(t_2-d-t_1)}\dots e^{-\xi_{i_k}(t-t_{k-1})}. \tag{3.21}$$

Let $\xi = \min\{\xi_{i_j};\ i \in \Xi \text{ and } j = 1, 2, \dots, k\}$, so inequality (3.21) becomes

$$\upsilon_{i_k}(t) \leq \gamma_1 e^{\xi_{i_1}d}\gamma_2 e^{\xi_{i_2}d}\dots\gamma_{k-1}e^{\xi_{i_{k-1}}d} \sup_{\sigma\in[t_0-d,t_0]} \upsilon_{i_1}(\sigma)e^{-\xi(t-t_0)}. \tag{3.22}$$

Let $\gamma = \max\{\gamma_l;\ \ l = 1, 2, \dots, k-1\}$ and $\xi^* = \max\{\xi_{i_l};\ \ i \in \Xi,\ l = 1, 2, \dots, k-1\}$. Then, inequality (3.22) is written as

$$\upsilon_{i_k}(t) \leq \gamma^{k-1} \sup_{\sigma\in[t_0-d,t_0]} \upsilon_{i_1}(\sigma)e^{(i-1)\xi^* d}e^{-\xi(t-t_0)}$$

Applying assumption (ii) with $N_0 = \frac{\eta}{\ln\gamma+\xi^* d}$, where η is an arbitrary constant, and $T_a = \frac{\ln\gamma+\xi^* d}{\xi-\xi^{**}}$, $(\xi > \xi^{**})$ leads to

$$\upsilon_{i_k}(t) \leq \sup_{\sigma\in[t_0-d,t_0]} \upsilon_{i_1}(\sigma)e^{\eta-\xi^{**}(t-t_0)},$$

which implies that

$$\|E_i x(t)\| \ \ \leq \ \ \mu\|E_i x(t_0-d)\|e^{(\eta-\xi^{**}(t-t_0))/2}, \qquad t \geq t_0, \tag{3.23}$$

where $\mu = \sqrt{\lambda_M/\lambda_m}$ defined for $\lambda_M = \max\{\lambda_{\max}(X_i);\ i \in \Xi\}$ and $\lambda_m = \min\{\lambda_{\min}(X_i);\ i \in \Xi\}$. Thus, the trivial solution of the system (3.1) is E-exponentially stable.

Let

$$\tilde{P}_i^{-1} x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \tag{3.24}$$

then it follows from the standard decomposition form that system (3.1) is equivalent to

$$\dot{x}_1(t) = A_{1i} x_1(t) + \tilde{Q}_i^1 f_i(t, x(t-d)) \tag{3.25}$$

$$0 = x_2(t) + \tilde{Q}_i^2 f_i(t, x_t) \tag{3.26}$$

where $i = 1, 2, \ldots, N,\ x_1 \in \mathbb{R}^r,\ x_2 \in \mathbb{R}^{n-r},\ \tilde{Q}_i = \begin{bmatrix} \tilde{Q}_i^1 \\ \tilde{Q}_i^2 \end{bmatrix},\ \tilde{Q}_i^1 \in \mathbb{R}^{r \times n},\ \tilde{Q}_i^2 \in \mathbb{R}^{(n-r)\times n}$, $\tilde{P}_i = \begin{bmatrix} \tilde{P}_i^1 & \tilde{P}_i^2 \end{bmatrix}$, $\tilde{P}_i^1 \in \mathbb{R}^{n \times r}$, and $\tilde{P}_i^2 \in \mathbb{R}^{n \times (n-r)}$.

By (3.24), we can write

$$x(t) = \tilde{P}_i \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

then

$$\begin{aligned} \tilde{Q}_i E_i x(t) &= \tilde{Q}_i E_i \tilde{P}_i \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \\ &= \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \\ &= \begin{bmatrix} x_1(t) \\ 0 \end{bmatrix}. \end{aligned} \tag{3.27}$$

From (3.27) and (3.23), we have

$$\begin{aligned}\sup_{t-d\le\theta\le t}\|x_1(\theta)\| &= \sup_{t-d\le\theta\le t}\|\tilde{Q}_iE_ix(\theta)\|\\ &\le \|\tilde{Q}_i\|\sup_{t-d\le\theta\le t}\|E_ix(\theta)\|\\ &= \mu\|\tilde{Q}_i\|\|E_ix(t_0-d)\|e^{(\eta-\xi^{**}(t-d-t_0))/2}\end{aligned}\tag{3.28}$$

which implies that x_1 is exponentially stable.

We need to show that x_2 is also exponentially stable. It follows from (3.26) and (3.2) that

$$\begin{aligned}\sup_{t-d\le\theta\le t}\|x_2(\theta)\| &\le \|\tilde{Q}_i^2\|\sup_{t-d\le\theta\le t}\|F_iE_ix(\theta)\|\\ &= \|\tilde{Q}_i^2\|\sup_{t-d\le\theta\le t}\|F_iE_i\tilde{P}_i^1x_1(\theta)+F_iE_i\tilde{P}_i^2x_2(\theta)\|\\ &\le \|\tilde{Q}_i^2\|\ \|F_iE_i\tilde{P}_i^1\|\sup_{t-d\le\theta\le t}\|x_1(\theta)\|+\|\tilde{Q}_i^2\|\|F_iE_i\tilde{P}_i^2\|\sup_{t-d\le\theta\le t}\|x_2(\theta)\|\\ \left(1-\|\tilde{Q}_i^2\|\|F_iE_i\tilde{P}_i^2\|\right)\sup_{t-d\le\theta\le t}\|x_2(\theta)\| &\le \|\tilde{Q}_i^2\|\|F_iE_i\tilde{P}_i^1\|\sup_{t-d\le\theta\le t}\|x_1(\theta)\|\\ \sup_{t-d\le\theta\le t}\|x_2(\theta)\| &\le \frac{\|\tilde{Q}_i^2\|\|F_iE_i\tilde{P}_i^1\|}{1-\|\tilde{Q}_i^2\|\|F_iE_i\tilde{P}_i^2\|}\sup_{t-d\le\theta\le t}\|x_1(\theta)\|\\ &\le \frac{\|\tilde{Q}_i^2\|\|F_iE_i\tilde{P}_i^1\|}{1-\|\tilde{Q}_i^2\|\|F_iE_i\tilde{P}_i^2\|}\mu\|\tilde{Q}_i\|\|E_ix(t_0-d)\|e^{(\eta-\xi^{**}(t-d-t_0))/2}\end{aligned}\tag{3.29}$$

where $1>\|\tilde{Q}_i^2\|\|F_iE_i\tilde{P}_i^2\|$. This shows that x_2 is exponentially stable. Thus, the trivial solution of system (3.1) is exponentially stable. □

Example 3.1.1. *Consider the impulsive switched singular system with time-delay given by (3.1) where* $x=\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix}$, $\sigma(t)\in\Xi=\{1,2\}$, $E_1=E_2=\begin{bmatrix}4&0\\ 2&0\end{bmatrix}$ *with* $rank(E_i)=1$,

$B_k = 0.2I$, $d = 0.3339$ *and*

$$A_1 = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}, \quad f_1(t, x(t-d)) = \begin{bmatrix} \frac{1}{15}\tanh(x_1(t-d)) & \frac{1}{15}\tanh(x_2(t-d)) \end{bmatrix}^T,$$

$$A_2 = \begin{bmatrix} -4 & 1 \\ 1 & -4 \end{bmatrix}, \quad f_2(t, x(t-d)) = \begin{bmatrix} \frac{1}{2}\tanh(x_1(t-d)) & \frac{1}{2}\tanh(x_2(t-d)) \end{bmatrix}^T.$$

Also, initial function is $\phi(t) = [2 \quad 1.6]^T$. *The Lipschitz conditions (3.2) are satisfied with* $F_1 = \begin{bmatrix} 0.0333 & -0.0333 \\ 0 & 0 \end{bmatrix}$ *and* $F_2 = \begin{bmatrix} 0.25 & -0.25 \\ 0 & 0 \end{bmatrix}$. *From the Jordan canonical form of* $(\gamma E - A)^{-1}E$ *for* $\gamma \in \mathbb{C}$, *we find that*

$$Q_1 = \begin{bmatrix} \frac{1}{5} & \frac{1}{10} \\ -\frac{2}{5} & \frac{4}{5} \end{bmatrix}, \quad P_1 = \begin{bmatrix} 1 & 0 \\ \frac{4}{5} & -\frac{1}{2} \end{bmatrix}, \quad Q_2 = \begin{bmatrix} \frac{4}{9} & \frac{1}{9} \\ -\frac{1}{12} & \frac{1}{6} \end{bmatrix}, \quad P_2 = \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{1}{3} & -\frac{4}{3} \end{bmatrix},$$

such that $Q_1E_1P_1 = Q_2E_2P_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad Q_1A_1P_1 = \begin{bmatrix} -\frac{3}{10} & 0 \\ 0 & 1 \end{bmatrix}, \quad Q_2A_2P_2 = \begin{bmatrix} -\frac{5}{6} & 0 \\ 0 & 1 \end{bmatrix}$.
From the decomposition form it is clear that the systems are impulse free. Moreover, the eigenvalues of (E_1, A_1) *and* (E_2, A_2) *are negative.*

$X_1 = \begin{bmatrix} 1.6667 & 0 \\ 0 & 1.6667 \end{bmatrix} > 0$ *satisfies* $A_1^T X_1 E_1 + E_1^T X_1 A_1 = -E_1^T Y_1 E_1$ *for any* $Y_1 = I > 0$.
Similarly, $X_2 = \begin{bmatrix} 0.6667 & 0 \\ 0 & 0.3333 \end{bmatrix} > 0$ *satisfying* $A_2^T X_2 E_2 + E_2^T X_2 A_2 = -E_2^T Y_2 E_2$ *for any* $Y_2 = I > 0$. *Hence,* $\alpha_1 = 0.0167$, $\beta_1 = 0.0038$, $\alpha_2 = 0.5001$, *and* $\beta_2 = 0.25$. *The mode decay rates are* $\xi_1 = 0.01$ *and* $\xi_2 = 0.0014$. *Thus,* $\xi = \min\{\xi_1, \xi_2\} = 0.0014$. $\gamma_k = 1.44$ *so that the inequality* $(I + B_k)^T E_j^T X_j E_j (I + B_k) \leq \gamma_k E_i^T X_i E_i$ *is satisfied. Thus,* $\gamma = \max\{\gamma_k\} = 1.44$. *Then, the average dwell time* $T_a = 0.4452$. *Figure 3.1 shows that the solution of the singular system vanishes exponentially after running mode 1 and 2 on the first and second interval, respectively.*

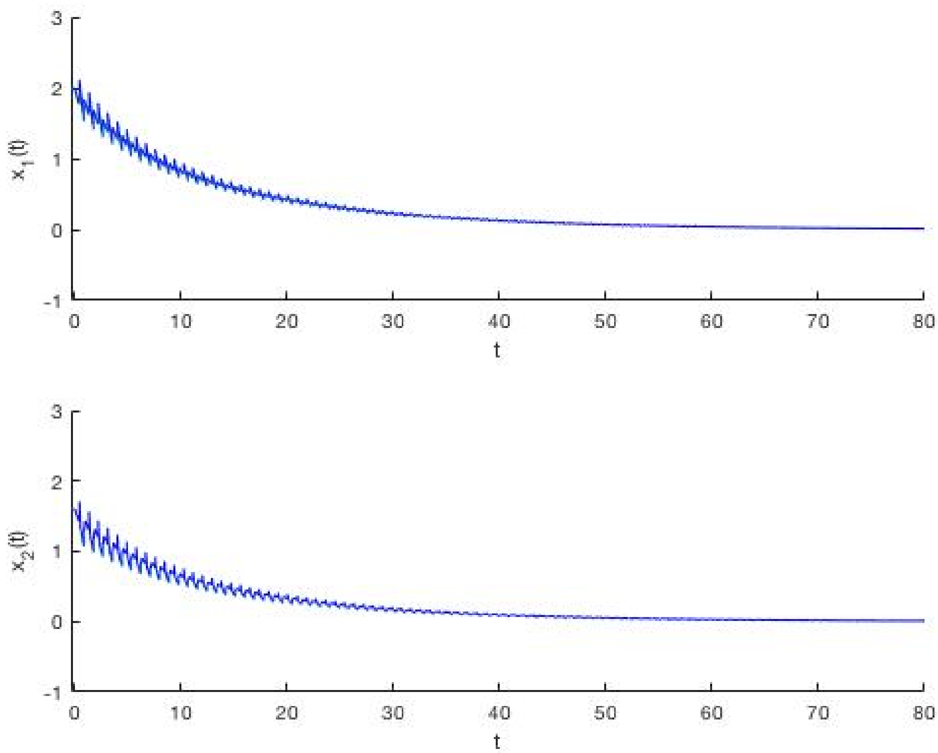

Figure 3.1: ISSSD with stable subsystems

3.1.2 Systems with Stable and Unstable Subsystems

Consider again the system in (3.1) with $\Xi = \Xi_u \cup \Xi_s$ where Ξ_u and Ξ_s represent the index sets of unstable and stable subsystems, respectively. Lemma 3.1.1 and the following lemmas are important in proving the main theorem.

Lemma 3.1.3. *[43] For $a \in \mathbb{R}$, with $a > 0$, and $t_0 \in \mathbb{R}^+$, let $u : [t_0, t_0 + a) \to \mathbb{R}^+$ satisfy the following delay differential inequality*

$$\dot{u}(t) \leq \alpha u(t) + \beta \sup_{\sigma \in [t-d,t]} u(\sigma), \qquad t \in [t_0, t_0 + a).$$

Assume that $\alpha + \beta > 0$. Then, there exist positive constants ξ and k such that

$$u(t) \leq k e^{\xi(t-t_0)}, \qquad t \in [t_0, t_0 + a)$$

where $\xi = \alpha + \beta$ and $k = \sup_{\sigma \in [t_0-d,t_0]} u(\sigma)$.

Theorem 3.1.2. *For any $i \in \Xi$, assume that each subsystem of (3.1) is impulse free. Then, the trivial solution of (3.1) is exponentially stable if the following assumptions hold:*

(A1) For any $i, j \in \Xi$ there exists $\gamma_k < 1$ such that

$$(I + B_k)^T E_j^T X_j E_j (I + B_k) \leq \gamma_k E_i^T X_i E_i. \tag{3.30}$$

(A2) for any t_0, the switching law guarantees that

$$\inf_{t \geq t_0} \frac{T^-(t_0, t)}{T^+(t_0, t)} \geq \frac{\lambda^+ + \lambda^*}{\lambda^- - \lambda^*} \tag{3.31}$$

where $\lambda^* \in (0, \lambda^-)$; furthermore, there exists $0 < \nu < \lambda^*$ such that

(i) for $i \in \Xi_u$

$$\ln \gamma_k - \nu(t_k - t_{k-1}) \leq 0, \quad k = 1, 2, \ldots, l \tag{3.32}$$

(ii) for $i \in \Xi_s$

$$\ln \gamma_k - \nu(t_k - t_{k-1}) + \xi_{m_k} d \leq 0, \quad k = l+1, l+2, \ldots, m-1. \tag{3.33}$$

where ξ_{m_k} is a unique positive solution of

$$\xi_{m_k} = \alpha_{m_k} - \beta_{m_k} e^{\xi_{m_k} d},$$

where $\alpha_{m_k} = \dfrac{\lambda_{\min}(Y_m) - \varepsilon_m \|X_m\|^2}{\lambda_{\max}(X_m)}$ *and* $\beta_{m_k} = \dfrac{\|F_m\|^2}{\varepsilon_m \lambda_{\min}(X_m)}$.

Proof. Let $x(t) = x(t, t_0, x_0)$ be the solution of the system (3.1). For any $i \in \Xi$ and $t \in [t_{k-1}, t_k)$, define

$$\upsilon_i(t) = V_i(x(t)) = x^T(t) E_i^T X_i E_i x(t), \qquad t \neq t_k, \quad i = \sigma(t)$$

as a Lyapunov function candidate for i^{th} subsystem. Then, derivative of υ_i along the trajectory of (3.1) is given by

$$\begin{aligned} \dot{\upsilon}_i(t) &= \dot{x}^T(t) E_i^T X_i E_i x(t) + x^T(t) E_i^T X_i E_i \dot{x}(t) \\ &= x^T(t) \left(A_i^T X_i E_i + E_i^T X_i A_i \right) x(t) + 2 f_i^T(t, x_t) X_i E_i x(t). \end{aligned} \tag{3.34}$$

For $i \in \Xi_s$, we have

$$\dot{\upsilon}_i(t) = -x^T(t) E_i^T Y_i E_i x(t) + 2 f_i^T(t, x_t) X_i E_i x(t) \tag{3.35}$$

where $-E_i^T Y_i E_i = A_i^T X_i E_i + E_i^T X_i A_i$ for any $Y_i > 0$. Thus, using the Lipschitz condition (3.2) we obtain

$$\dot{\upsilon}_i(t) \leq -\alpha_i \upsilon_i(t) + \beta_i \sup_{\sigma \in [t-d, t]} \upsilon_i(\sigma) \tag{3.36}$$

where $\alpha_i = \dfrac{\lambda_{\min}(Y_i) - \varepsilon_i \lambda_{\max}(X_i^2)}{\lambda_{\max}(X_i)}$, and $\beta_i = \dfrac{\|F_i\|^2}{\varepsilon_i \lambda_{\min}(X_i)}$.

By Lemma (3.1.1), we obtain the solution of differential inequality (3.36) for $t \in [t_{k-1}, t_k)$

$$\upsilon_i(t) \le \sup_{\sigma \in [t_{k-1}-d, t_{k-1}]} \upsilon_i(\sigma) e^{-\xi_i(t-t_{k-1})}, \tag{3.37}$$

where ξ_i is a unique positive solution of

$$\xi_i = \alpha_i - \beta_i e^{\xi_i d}.$$

Let δ_i $(i \in \Xi_u)$ be a positive constant such that the matrix pairs $(E_i, A_i - \delta_i E_i)$ has eigenvalues with negative real parts. Then, for each $Y_i > 0$ there exists $X_i > 0$ satisfying

$$(A_i - \delta_i E_i)^T X_i E_i + E_i^T X_i (A_i - \delta_i E_i) = -E_i^T Y_i E_i. \tag{3.38}$$

Thus, we have

$$\dot{\upsilon}_i(t) = x^T(t)(-E_i^T Y_i E_i + 2\delta_i E_i^T X_i E_i) x(t) + 2 f_i^T(t, x_t) X_i E_i x(t). \tag{3.39}$$

By Lipschitz condition (3.2), we obtain

$$\dot{\upsilon}_i(t) \le \alpha_i^* \upsilon_i(t) + \beta_i^* \sup_{\sigma \in [t-d, t]} \upsilon_i(\sigma) \tag{3.40}$$

where $\alpha_i^* = \dfrac{-\lambda_{\min}(Y_i) + 2\delta_i \lambda_{\max}(X_i) + \varepsilon_i \lambda_{\max}(X_i^2)}{\lambda_{\min}(X_i)}$ and $\beta_i^* = \dfrac{\|F_i\|^2}{\varepsilon_i \lambda_{\min}(X_i)}$.

By Lemma 3.1.3, the solution of (3.40) is obtained for $t \in [t_{k-1}, t_k)$

$$\upsilon_i(t) \le \sup_{\sigma \in [t_{k-1}-d, t_{k-1}]} \upsilon_i(\sigma) e^{\zeta_i(t-t_{k-1})}, \tag{3.41}$$

where $\zeta_i = \alpha_i^* + \beta_i^* > 0$.

On the other hand, for $t = t_k$, $k = 1, 2, 3, \ldots$, suppose $\sigma(t_k) = j$, it follows from (3.1) and (3.30) that

$$\begin{aligned} \upsilon_j(t_k) &= x^T(t_k) E_j^T X_j E_j x(t_k) \\ &= x^T(t_k^-)(I + B_k)^T E_j^T X_j E_j (I + B_k) x(t_k^-) \\ &\le \gamma_k x^T(t_k^-) E_i^T X_i E_i x(t_k^-) \\ &= \gamma_k \upsilon_i(t_k^-). \end{aligned} \tag{3.42}$$

Using (3.37) and (3.42) successively on each subinterval leads to the following results. For $t \in [t_0, t_1)$, we have

$$\upsilon_{i_1}(t) \leq \sup_{\sigma \in [t_0-d, t_0]} \upsilon_{i_1}(\sigma) e^{-\xi_{i_1}(t-t_0)}, \tag{3.43}$$

For $t \in [t_1, t_2)$, we have

$$\upsilon_{i_2}(t) \leq \sup_{\sigma \in [t_1-d, t_1]} \upsilon_{i_2}(\sigma) e^{-\xi_{i_2}(t-t_1)}. \tag{3.44}$$

From (3.42), we can result in the following inequality

$$\begin{aligned} \sup_{\sigma \in [t_1-d, t_1]} \upsilon_{i_2}(\sigma) &\leq \gamma_1 \sup_{\sigma \in [t_1-d, t_1]} \upsilon_{i_1}(\sigma^-) \\ \sup_{\sigma \in [t_1-d, t_1]} \upsilon_{i_2}(\sigma) e^{-\xi_{i_2}(t-t_1)} &\leq \gamma_1 \sup_{\sigma \in [t_1-d, t_1]} \upsilon_{i_1}(\sigma^-) e^{-\xi_{i_2}(t-t_1)}. \end{aligned} \tag{3.45}$$

By (3.44) and (3.45), we have

$$\upsilon_{i_2}(t) \leq \gamma_1 \sup_{\sigma \in [t_1-d, t_1]} \upsilon_{i_1}(\sigma^-) e^{-\xi_{i_2}(t-t_1)}. \tag{3.46}$$

Suppose that $t_k - d \in [t_{k-1}, t_k)$, by (3.43), we can find $\sup_{\sigma \in [t_1-d, t_1]} \upsilon_{i_1}(\sigma^-)$ as

$$\sup_{\sigma \in [t_1-d, t_1]} \upsilon_{i_1}(\sigma^-) = \sup_{\sigma \in [t_0-d, t_0]} \upsilon_{i_1}(\sigma) e^{-\xi_{i_1}(t_1-d-t_0)}. \tag{3.47}$$

Thus, if we rewrite (3.46), we obtain for $t \in [t_1, t_2)$

$$\upsilon_{i_2}(t) \leq \gamma_1 \sup_{\sigma \in [t_0-d, t_0]} \upsilon_{i_1}(\sigma) e^{-\xi_{i_1}(t_1-d-t_0)} e^{-\xi_{i_2}(t-t_1)}. \tag{3.48}$$

Similarly, for $t \in [t_2, t_3)$ we obtain

$$\upsilon_{i_3}(t) \leq \gamma_1 \gamma_2 \sup_{\sigma \in [t_0-d, t_0]} \upsilon_{i_1}(\sigma) e^{-\xi_{i_1}(t_1-d-t_0)} e^{-\xi_{i_2}(t_2-d-t_1)} e^{-\xi_{i_3}(t-t_2)}. \tag{3.49}$$

In general, for $t \in [t_{k-1}, t_k)$

$$\upsilon_{i_k}(t) \leq \gamma_1 \gamma_2 \dots \gamma_{k-1} \sup_{\sigma \in [t_0-d, t_0]} \upsilon_{i_1}(\sigma) e^{-\xi_{i_1}(t_1-d-t_0)} e^{-\xi_{i_2}(t_2-d-t_1)} \dots e^{-\xi_{i_k}(t-t_{k-1})}. \tag{3.50}$$

Now, use (3.41) and (3.42) successively on each subinterval. For $t \in [t_0, t_1)$, we have

$$\upsilon_{i_1}(t) \leq \sup_{\sigma \in [t_0 - d, t_0]} \upsilon_{i_1}(\sigma) e^{\zeta_{i_1}(t - t_0)}, \tag{3.51}$$

For $t \in [t_1, t_2)$, we have

$$\upsilon_{i_2}(t) \leq \sup_{\sigma \in [t_1 - d, t_1]} \upsilon_{i_2}(\sigma) e^{\zeta_{i_2}(t - t_1)}. \tag{3.52}$$

From (3.42), we get the following inequality

$$\begin{aligned} \sup_{\sigma \in [t_1 - d, t_1]} \upsilon_{i_2}(\sigma) &\leq \gamma_1 \sup_{\sigma \in [t_1 - d, t_1]} \upsilon_{i_1}(\sigma^-) \\ \sup_{\sigma \in [t_1 - d, t_1]} \upsilon_{i_2}(\sigma) e^{\zeta_{i_2}(t - t_1)} &\leq \gamma_1 \sup_{\sigma \in [t_1 - d, t_1]} \upsilon_{i_1}(\sigma^-) e^{\zeta_{i_2}(t - t_1)}. \end{aligned} \tag{3.53}$$

By (3.52) and (3.53), we have

$$\upsilon_{i_2}(t) \leq \gamma_1 \sup_{\sigma \in [t_1 - d, t_1]} \upsilon_{i_1}(\sigma^-) e^{\zeta_{i_2}(t - t_1)}. \tag{3.54}$$

By (3.51), we can find $\sup_{\sigma \in [t_1 - d, t_1]} \upsilon_{i_1}(\sigma^-)$ as

$$\sup_{\sigma \in [t_1 - d, t_1]} \upsilon_{i_1}(\sigma^-) = \sup_{\sigma \in [t_0 - d, t_0]} \upsilon_{i_1}(\sigma) e^{\zeta_{i_1}(t_1 - t_0)}. \tag{3.55}$$

Thus, if we substitute (3.55) into (3.54), we obtain for $t \in [t_1, t_2)$

$$\upsilon_{i_2}(t) \leq \gamma_1 \sup_{\sigma \in [t_0 - d, t_0]} \upsilon_{i_1}(\sigma) e^{\zeta_{i_1}(t_1 - t_0)} e^{\zeta_{i_2}(t - t_1)}. \tag{3.56}$$

Similarly, for $t \in [t_2, t_3)$ we can obtain

$$\upsilon_{i_3}(t) \leq \gamma_1 \gamma_2 \sup_{\sigma \in [t_0 - d, t_0]} \upsilon_{i_1}(\sigma) e^{\zeta_{i_1}(t_1 - t_0)} e^{\zeta_{i_2}(t_2 - t_1)} e^{\zeta_{i_3}(t - t_2)}. \tag{3.57}$$

In general, for $t \in [t_{k-1}, t_k)$

$$\upsilon_{i_k}(t) \leq \gamma_1 \gamma_2 \dots \gamma_{k-1} \sup_{\sigma \in [t_0 - d, t_0]} \upsilon_{i_1}(\sigma) e^{\zeta_{i_1}(t_1 - t_0)} e^{\zeta_{i_2}(t_2 - t_1)} \dots e^{\zeta_{i_k}(t - t_{k-1})}. \tag{3.58}$$

To obtain a general estimate, let us run l unstable modes and switch l times from an unstable mode, and run $m-l$ stable modes and switch $m-l-1$ times from a stable mode. Then, for $t \in [t_{k-1}, t_k)$

$$\upsilon_{m_k}(t) \le \prod_{j=1}^{l} \gamma_j e^{\zeta_{m_j}(t_j - t_{j-1})} \times \prod_{s=l+1}^{m-1} \gamma_s e^{\xi_{m_s} d} e^{-\xi_{m_s}(t_s - t_{s-1})} \times \sup_{\sigma \in [t_0 - d, t_0]} \upsilon_{m_1}(\sigma) e^{-\xi_{m_k}(t - t_{m-1})}. \tag{3.59}$$

Let

$$\lambda^+ = \max\{\zeta_{m_j} : \ j = 1, 2, \dots, l\}$$
$$\lambda^- = \min\{\xi_{m_s} : \ s = l+1, l+2, \dots, k\}$$

and denote by $T^+(t_0, t)$ and $T^-(t_0, t)$ the total activation time of unstable and stable modes, respectively. Then, for $t \in [t_{k-1}, t_k)$, we have

$$\upsilon_{m_k}(t) \le \prod_{j=1}^{l} \gamma_j e^{\lambda^+ T^+} \times \prod_{s=l+1}^{m-1} \gamma_s e^{\xi_{m_s} d} e^{-\lambda^- T^-} \times \sup_{\sigma \in [t_0 - d, t_0]} \upsilon_{m_1}(\sigma). \tag{3.60}$$

Choose $\lambda^* \in (0, \lambda^-)$, and assume that the switching law satisfies (3.31) where this condition implies that for any $t \ge t_0$

$$\begin{aligned} (\lambda^+ + \lambda^*) T^+ &\le (\lambda^- - \lambda^*) T^- \\ -\lambda^- T^- + \lambda^+ T^+ &\le -\lambda^* T^- - \lambda^* T^+ \\ &= -\lambda^* (T^- + T^+) \\ &= -\lambda^* (t - t_0) \end{aligned}$$

Thus, by condition (3.31), we obtain

$$\upsilon_{m_k}(t) \le \prod_{j=1}^{l} \gamma_j \times \prod_{s=l+1}^{m-1} \gamma_s e^{\xi_{m_s} d} \times \sup_{\sigma \in [t_0 - d, t_0]} \upsilon_{m_1}(\sigma) e^{-\lambda^* (t - t_0)}. \tag{3.61}$$

Then, for $\nu < \lambda^*$ positive scalar, we have the following inequality

$$\begin{aligned}
\upsilon_{m_k}(t) &\leq \prod_{j=1}^{l} \gamma_j \times \prod_{s=l+1}^{m-1} \gamma_s e^{\xi_{m_s} d} \times \sup_{\sigma \in [t_0-d, t_0]} \upsilon_{m_1}(\sigma) e^{-\nu(t-t_0)} e^{-(\lambda^*-\nu)(t-t_0)} \\
&= \sup_{\sigma \in [t_0-d, t_0]} \upsilon_{m_1}(\sigma) \gamma_1 e^{-\nu(t_1-t_0)} \gamma_2 e^{-\nu(t_2-t_1)} \ldots \gamma_l e^{-\nu(t_l-t_{l+1})} \\
&\times \gamma_{l+1} e^{-\nu(t_{l+1}-t_l)+\xi_{m_{(l+1)}} d} \gamma_{l+2} e^{-\nu(t_{l+2}-t_{l+1})+\xi_{m_{(l+2)}} d} \\
&\ldots \gamma_{m-1} e^{-\nu(t_{m-1}-t_{m-2})+\xi_{m_{(m-1)}} d} e^{-\nu(t-t_{m-1})} e^{-(\lambda^*-\nu)(t-t_0)}.
\end{aligned}$$

By (3.32) and (3.33), we obtain

$$\upsilon_{m_k}(t) \leq \sup_{\sigma \in [t_0-d, t_0]} \upsilon_{m_1}(\sigma) e^{-(\lambda^*-\nu)(t-t_0)} \tag{3.62}$$

which implies that

$$\|E_m x(t)\| \leq \mu \|E_m x_{t_0}\|_d e^{-(\lambda^*-\nu)(t-t_0)/2}, \quad t \geq t_0 \tag{3.63}$$

where $\mu = \sqrt{\lambda_M/\lambda_m}$ defined for $\lambda_M = \max\{\lambda_{\max}(X_i);\ i \in \Xi\}$ and $\lambda_m = \min\{\lambda_{\min}(X_i);\ i \in \Xi\}$. Thus, the trivial solution of the system (3.1) is E-exponentially stable.

Let

$$\tilde{P}_i^{-1} x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \tag{3.64}$$

then it follows from the standard decomposition form that system (3.1) is equivalent to

$$\dot{x}_1(t) = A_{1i} x_1(t) + \tilde{Q}_i^1 f_i(t, x(t-d)) \tag{3.65}$$

$$0 = x_2(t) + \tilde{Q}_i^2 f_i(t, x(t-d)) \tag{3.66}$$

where $i = 1, 2, \ldots, N$, $x_1 \in \mathbb{R}^r$, $x_2 \in \mathbb{R}^{n-r}$, $\tilde{Q}_i = \begin{bmatrix} \tilde{Q}_i^1 \\ \tilde{Q}_i^2 \end{bmatrix}$, $\tilde{Q}_i^1 \in \mathbb{R}^{r \times n}$, $\tilde{Q}_i^2 \in \mathbb{R}^{(n-r) \times n}$, $\tilde{P}_i = \begin{bmatrix} \tilde{P}_i^1 & \tilde{P}_i^2 \end{bmatrix}$, $\tilde{P}_i^1 \in \mathbb{R}^{n \times r}$, and $\tilde{P}_i^2 \in \mathbb{R}^{n \times (n-r)}$.

By (3.64), we can write

$$x(t) = \tilde{P}_i \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

then

$$\begin{aligned} \tilde{Q}_i E_i x(t) &= \tilde{Q}_i E_i \tilde{P}_i \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \\ &= \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \\ &= \begin{bmatrix} x_1(t) \\ 0 \end{bmatrix}. \end{aligned} \tag{3.67}$$

From (3.63) and (3.67), we have

$$\begin{aligned} \|x_{1_t}\|_d &= \|\tilde{Q}_i E_i x_t\|_d \\ &\leq \|\tilde{Q}_i\| \|E_i x_t\|_d \\ &= \|\tilde{Q}_i\| \sup_{t-d \leq \theta \leq t} \|E_i x(\theta)\| \\ &= \mu \|\tilde{Q}_i\| \|E_i x_{t_0}\|_d e^{-(\lambda^* - \nu)(t-d-t_0)/2} \end{aligned} \tag{3.68}$$

which implies that x_1 is exponentially stable.

We need to show that x_2 is also exponentially stable. It follows from (3.66) and (3.2) that

$$\begin{aligned} \|x_{2_t}\|_d &\leq \|\tilde{Q}_i^2\| \|F_i E_i x_t\|_d \\ &= \|\tilde{Q}_i^2\| \|F_i E_i \tilde{P}_i^1 x_{1_t} + F_i E_i \tilde{P}_i^2 x_{2_t}\|_d \\ &\leq \|\tilde{Q}_i^2\| \|F_i E_i \tilde{P}_i^1\| \|x_{1_t}\|_d + \|Q^2\| \|F_i E_i \tilde{P}_i^2\| \|x_{2_t}\|_d \end{aligned}$$

$$\big(1 - \|\tilde{Q}_i^2\|\|F_iE_i\tilde{P}_i^2\|\big)\|x_{2_t}\|_d \leq \|\tilde{Q}_i^2\|\|F_iE_i\tilde{P}_i^1\|\|x_{1_t}\|_d$$

$$\|x_{2_t}\|_d \leq \frac{\|\tilde{Q}_i^2\|\|F_iE_i\tilde{P}_i^1\|}{1 - \|\tilde{Q}_i^2\|\|F_iE_i\tilde{P}_i^2\|}\|x_{1_t}\|_d$$

$$\leq \frac{\|\tilde{Q}_i^2\|\|F_iE_i\tilde{P}_i^1\|}{1 - \|\tilde{Q}_i^2\|\|F_iE_i\tilde{P}_i^2\|}\mu\|\tilde{Q}_i\|\|E_ix_{t_0}\|_d e^{-(\lambda^*-\nu)(t-d-t_0)/2} \tag{3.69}$$

where $1 > \|\tilde{Q}_i^2\|\|F_iE_i\tilde{P}_i^2\|$. This shows that x_2 is exponentially stable. Thus, the trivial solution of system (3.1) is exponentially stable.

In fact one can use the **average dwell time** to achieve a similar result. To do so, from (3.61) we have

$$\begin{aligned}
\upsilon_{m_k}(t) &\leq \prod_{j=1}^{m-1}\gamma_j e^{\xi_{m_j}d} \times \sup_{\sigma\in[t_0-d,t_0]}\upsilon_{m_1}(\sigma)e^{-\lambda^*(t-t_0)} \\
&\leq \prod_{j=1}^{m-1}\gamma e^{\xi^* d} \times \sup_{\sigma\in[t_0-d,t_0]}\upsilon_{m_1}(\sigma)e^{-\lambda^*(t-t_0)} \\
&= \sup_{\sigma\in[t_0-d,t_0]}\upsilon_{m_1}(\sigma)e^{(m-1)\ln\varrho-\lambda^*(t-t_0)}
\end{aligned}$$

where $\gamma = \max\{\gamma_j;\ j = 1,2,\dots,m-1\}$, $\xi^* = \max\{\xi_{m_j};\ j = 1,2,\dots,m-1\}$, and $\varrho = \gamma e^{\xi^* d}$. Applying average dwell time with $N_0 = \frac{\eta}{\ln\varrho}$, where η is an arbitrary constant, and $T_a = \frac{\ln\varrho}{\lambda^*-\lambda}$, leads to

$$\upsilon_{m_k}(t) \leq \sup_{\sigma\in[t_0-d,t_0]}\upsilon_{m_1}(\sigma)e^{\eta-\lambda(t-t_0)}$$

which implies that

$$\|E_m x(t)\| \leq \mu\|E_m x_{t_0}\|_d e^{(\eta-\lambda(t-t_0))/2}, \qquad t \geq t_0$$

where $\mu = \sqrt{\lambda_M/\lambda_m}$ defined for $\lambda_M = \max\{\lambda_{\max}(X_i);\ i\in\Xi\}$ and $\lambda_m = \min\{\lambda_{\min}(X_i);\ i\in\Xi\}$. Thus, the trivial solution of the system (3.1) is E-exponentially stable.

By the same manner used in (A2); (i) and (ii), we obtain that

$$\|x_{1_t}\|_d \le \mu\|\tilde{Q}_i\|\|E_i x_{t_0}\|_d e^{(\eta-\lambda(t-d-t_0))/2},$$
$$\|x_{2_t}\|_d \le \frac{\|\tilde{Q}_i^2\|\|F_i E_i \tilde{P}_i^1\|}{1-\|\tilde{Q}_i^2\|\|F_i E_i \tilde{P}_i^2\|}\mu\|\tilde{Q}_i\|\|E_i x_{t_0}\|_d e^{(\eta-\lambda(t-d-t_0))/2}.$$

Thus, the trivial solution of (3.1) is exponentially stable. □

Example 3.1.2. *Consider the impulsive switched singular system with time-delay given by (3.1) where* $x = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$, $\sigma(t) \in \Xi = \{1, 2\}$, $E_1 = E_2 = \begin{bmatrix} 4 & 0 \\ 2 & 0 \end{bmatrix}$ *with* $rank(E_i) = 1$ *for* $i = 1, 2$, $B_k = -0.2I$, $d = 0.2202$ *and*

$$A_1 = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}, \quad f_1(t, x(t-d)) = \begin{bmatrix} \frac{1}{15}\tanh(x_1(t-d)) & \frac{1}{15}\tanh(x_2(t-d)) \end{bmatrix}^T,$$

$$A_2 = \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}, \quad f_2(t, x(t-d)) = \begin{bmatrix} \frac{1}{15}\tanh(x_1(t-d)) & \frac{1}{15}\tanh(x_2(t-d)) \end{bmatrix}^T.$$

Also, initial function is $\phi(t) = [0.5 - t^2 \quad 0.4 + t]^T$. *The Lipschitz conditions are satisfied with* $F_1 = F_2 = \begin{bmatrix} 0.0333 & -0.0333 \\ 0 & 0 \end{bmatrix}$. *From the Jordan canonical form of* $(\gamma E - A)^{-1}E$ *for* $\gamma \in \mathbb{C}$, *we find that*

$$\tilde{Q}_1 = \begin{bmatrix} 0.2 & 0.1 \\ -0.4 & 0.8 \end{bmatrix}, \quad P_1 = \begin{bmatrix} 1 & 0 \\ 0.8 & -0.5 \end{bmatrix}, \quad Q_2 = \begin{bmatrix} 0.0816 & -0.0204 \\ -0.1429 & 0.2857 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 3.5 & 0 \\ 1 & 1 \end{bmatrix},$$

such that $Q_1E_1P_1 = Q_2E_2P_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad Q_1A_1P_1 = \begin{bmatrix} -0.3 & 0 \\ 0 & 1 \end{bmatrix}, \quad Q_2A_2P_2 = \begin{bmatrix} 1.0714 & 0 \\ 0 & 1 \end{bmatrix}.$
From the decomposition form it is clear that the systems are impulse free. Moreover, while the eigenvalues of (E_1, A_1) *are negative, the eigenvalues of* (E_2, A_2) *are positive.*

$X_1 = \begin{bmatrix} 1.6667 & 0 \\ 0 & 1.6667 \end{bmatrix} > 0$ *satisfies* $A_1^T X_1 E_1 + E_1^T X_1 A_1 = -E_1^T Y_1 E_1$ *for* $Y_1 = I > 0$. *Similarly,* $X_2 = \begin{bmatrix} 1.1154 & -0.6923 \\ -0.6923 & 1 \end{bmatrix} > 0$ *satisfying* $(A_2 - \delta_2 E_2)^T X_2 E_2 + E_2^T X_2 (A_2 - \delta_2 E_2) = -E_2^T Y_2 E_2$ *for* $Y_2 = I > 0$ *and* $\delta_2 = 2$. *Hence,* $\alpha_{1_s} = 0.0167$, $\beta_{1_s} = 0.0038$, $\alpha^*_{2_j} = 2.7818$, *and* $\beta^*_{2_j} = 0.0069$ *where* $s = 1, 3, 5, \ldots$ *and* $j = 2, 4, 6, \ldots$. *The mode decay rate is* $\xi_{1_s} = 0.01$ *and the growth rate is* $\zeta_{2_j} = 2.7887$. *Thus,* $\xi^* = \max\{\xi_{1_s}\} = 0.01$. $\gamma_k = 0.64$ *so that the inequality* $(I + B_k)^T E_j^T X_j E_j (I + B_k) \leq \gamma_k E_i^T X_i E_i$ *is satisfied. Thus,* $\gamma = \max\{\gamma_k\} = 0.64$. $\lambda^* \in (0, \lambda^-)$ *where* $\lambda^- = \min\{\xi_{1_s}\}$. *Then, from the average dwell time assumption* $T_a = 0.4401$. *Figure 3.2 shows that the solution of the singular system vanishes exponentially after running mode 1 and 2 on the first and second interval, respectively.*

3.1.3 Systems with Unstable Subsystems

This section is an extension of the study in [55] to the singular system in (3.1) with $\Xi = \Xi_u$ where Ξ_u represents the index sets of unstable subsystems. Also, we consider the system in (3.1) in case the fact that singular matrices $E_{\sigma(t)}$ are the same for each subsystem. First of all, we describe the problem, and give the necessary definitions and lemmas.

Definition 3.1.1. *The ith* ($i \in \Xi = \{1, 2, \ldots, N\}$) *mode of (3.1) is the subsystem running during the time interval* $[t_{k-1}, t_k]$ *described by*

$$\begin{aligned} E\dot{x}(t) &= A_i x(t) + f_i(t, x(t-d)), && t \in [t_{k-1}, t_k), \\ \triangle x(t) &= B_k x(t), && t = t_k. \end{aligned}$$

Assume that there exists a linear convex combination which is either stable or unstable.

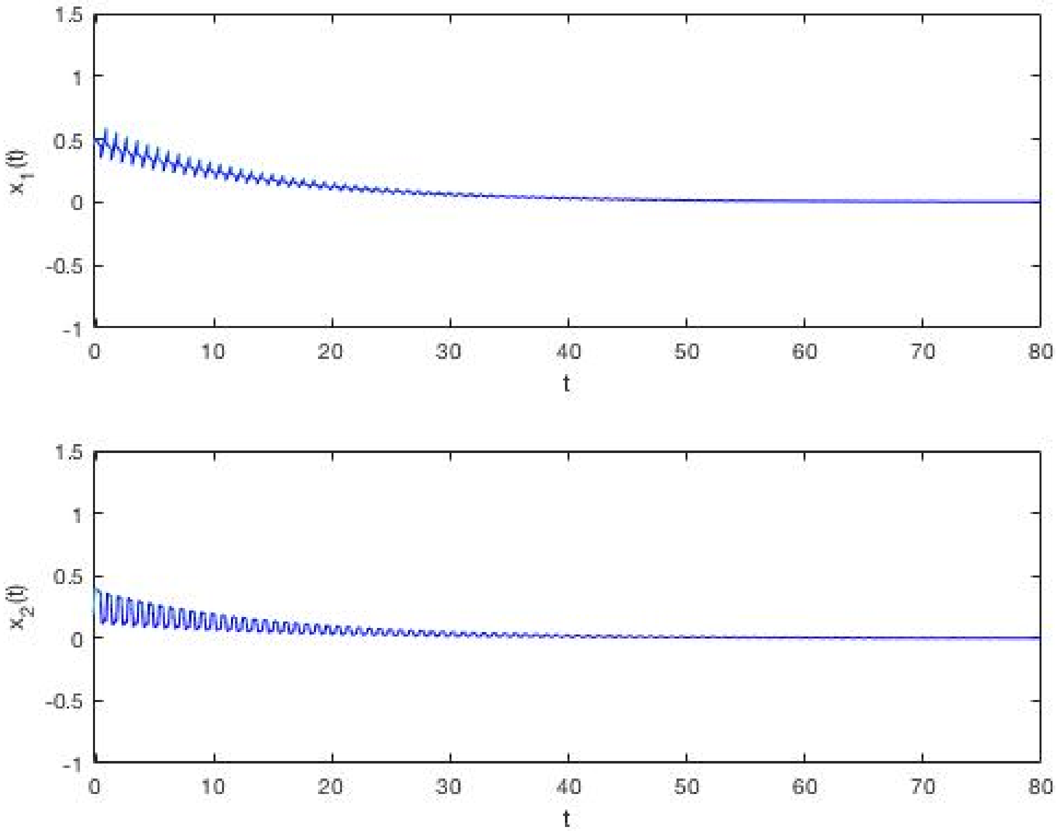

Figure 3.2: ISSSD with stable and unstable subsystems

Next, we will design switching rules and corresponding switching regions for the both cases.

Case 1: There exists an unstable convex combination of subsystems. That is, there exists a linear convex combination

$$\sum_{i=1}^{N} \alpha_i A_i,$$

where $\alpha_i \in (0,1)$ with $\sum_{i=1}^{N} \alpha_i = 1$ such that all the eigenvalues of matrix pair $(E, \sum_{i=1}^{N} \alpha_i A_i)$ have positive real parts. Therefore, for any symmetric positive definite matrix $Y \in \mathbb{R}^{n\times n}$, there exists a positive definite matrix $X \in \mathbb{R}^{n\times n}$ such that

$$\begin{aligned}\left(\sum_{i=1}^{N} \alpha_i A_i\right)^T XE + E^T X \left(\sum_{i=1}^{N} \alpha_i A_i\right) &= E^T Y E \\ \Rightarrow \sum_{i=1}^{N} \alpha_i \left(A_i^T XE + E^T XA_i\right) &= E^T Y E \end{aligned} \tag{3.70}$$

Moreover, the switching region Ω_i is constructed as

$$\Omega_i = \left\{x \in \mathbb{R}^n |\ x^T \left(A_i^T XE + E^T XA_i\right) x \le \xi x^T E^T Y E x\right\}, \qquad i = 1, 2, \ldots, N, \tag{3.71}$$

where $\xi \ge 1$ is some constant which can be adjusted in order to allow the switching regions overlapping each other.

Case 2: There exists a stable convex combination of subsystems. Suppose that there exists a linear convex combination

$$\sum_{i=1}^{N} \alpha_i A_i,$$

where $\alpha_i \in (0,1)$ with $\sum_{i=1}^{N} \alpha_i = 1$ such that all the eigenvalues of matrix pair $(E, \sum_{i=1}^{N} \alpha_i A_i)$ have negative real parts. Therefore, for any symmetric positive definite matrix $Y \in \mathbb{R}^{n\times n}$, there exists a positive definite matrix $X \in \mathbb{R}^{n\times n}$ such that

$$\left(\sum_{i=1}^{N} \alpha_i A_i\right)^T XE + E^T X \left(\sum_{i=1}^{N} \alpha_i A_i\right) = -E^T Y E$$

$$\Rightarrow \sum_{i=1}^{N} \alpha_i \left(A_i^T XE + E^T X A_i\right) = -E^T Y E \tag{3.72}$$

Similarly, the switching region Ω_i^* is constructed as

$$\Omega_i^* = \left\{x \in \mathbb{R}^n |\ x^T \left(A_i^T XE + E^T X A_i\right) x \leq -\frac{1}{\xi^*} x^T E^T Y E x\right\}, \qquad i = 1, 2, \ldots, N, \tag{3.73}$$

where $\xi^* \geq 1$ is some constant. ξ^* can be adjusted, so the switching regions overlap each other.

Definition 3.1.2 (Minimum rule)**.** *At each switching we decide the next mode by the minimum rule defined as*

$$i(x) = \arg\min \left[x^T \left(A_i^T XE + E^T X A_i\right) x\right].$$

To obtain the stability of the system in (3.1) we proposed the **switching rule (SR)** given by

(1) Choose the initial mode by minimum rule applied to $x(t_0)$.

(2) Stay in the ith mode as long as the state in the switching region Ω_i (Case 1) or Ω_i^* (Case 2).

(3) If the state hits the boundary of Ω_i or Ω_i^*, determine the jth mode using the minimum rule and switch to the jth mode.

In both cases we can obtain the stability results using common Lyapunov function method. Thus,for any $t \geq t_0$ the common Lyapunov function is defined by

$$V(x(t)) = x^T(t)E^T XEx(t), \tag{3.74}$$

where X is defined in (3.70) and (3.72).

Definition 3.1.3. *Let us define the following class of function for later use:*

$$K_1 = \{g \in C(\mathbb{R}^+, \mathbb{R}^+)|\ g(0) = 0 \ \textit{and}\ g(s) > 0\},$$

$$K_3 = \{g \in C(\mathbb{R}^+, \mathbb{R}^+)|\ g(0) = 0,\ g(s) > 0,\ \textit{and}\ g\ \textit{is nondcreasing in}\ s\}.$$

$$S(\rho) = \{x \in \mathbb{R}^n|\ \ \|x\| \leq \rho\ \textit{for}\ \rho \geq 0\}.$$

Let $J \subset \mathbb{R}^+$ be an interval of the form $[a, b)$ where $0 \leq a < b < \infty$ and $D \subset \mathbb{R}^n$ be an open set. Now, consider the impulsive system given by

$$\begin{aligned} \dot{x} &= f(t, x_t), \quad t \neq t_k, \\ \Delta x(t) &= I(t, x_{t^-}), \quad t = t_k, \end{aligned} \tag{3.75}$$

where $x \in PC(\mathbb{R}^+, \mathbb{R}^n)$, functionals $f, I : J \times PC([-d, 0], D) \to \mathbb{R}^n$, and $\Delta x = x(t) - x(t^-)$. $\{t_k\}_{k=1}^{\infty}$ are the impulsive times that form an increasing sequence satisfying $t_{k-1} < t_k$ and $\lim_{k\to\infty} t_k = \infty$.

Definition 3.1.4. *Given a function $V : J \times D \to \mathbb{R}^+$, the upper right-hand derivative of V with respect to system (3.75) is defined by*

$$D^+V_{(3.75)}(t, \psi(0)) = \limsup_{h\to 0^+} \frac{1}{h} \left[V(t + h, \psi(0)) + hf(t, \psi) - V(t, \psi(0))\right], \tag{3.76}$$

for $(t, \psi) \in J \times PC([-d, 0], D)$.

Lemma 3.1.4. *[56] Let be* $P \in \mathbb{R}^{n \times n}$ *a positive definite matrix, and* $Q \in \mathbb{R}^{n \times n}$ *a symmetric matrix. Then, the following inequality is valid for all* $x \in \mathbb{R}^n$

$$\lambda_{\min}(P^{-1}Q)x^T P x \leq x^T Q x \leq \lambda_{\max}(P^{-1}Q)x^T P x.$$

Lemma 3.1.5. *[57] Assume that there exist functions* $a, b, c \in K_1$, $p \in PC(\mathbb{R}^+, \mathbb{R}^+)$, $g \in K_3$, *and* $V : [-d, \infty) \times S(\rho) \to \mathbb{R}^+$ *where* V *is continuous on* $[-d, \infty) \times S(\rho)$ *and on* $[t_{k-1}, t_k) \times S(\rho)$ *for* $k = 1, 2, \dots,$ $\lim_{(t,y)\to(t_k^-, x)} V(t, y) = V(t_k^-, x)$ *exists. Moreover,* V, *restricted to* $\mathbb{R}^+ \times S(\rho)$, *is locally Lipschitz in* x *and the following conditions are satisfied:*

(i) $b(\|x\|) \leq V(t, x) \leq a(\|x\|)$ *for all* $(t, x) \in [-d, \infty) \times S(\rho)$;

(ii) $D^+V_{(3.75)}(t, \psi(0)) \leq p(t)c(V(t, \psi(0)))$ *for all* $t \neq t_k$ *in* $\mathbb{R}^+$ *and* $\psi \in PC([-d, 0], S(\rho))$ *whenever* $V(t, \psi(0)) \geq g(V(t + s, \psi(s)))$ *for* $s \in [-d, 0]$;

(iii) $V(t_k, \psi(0) + I(t_k, \psi)) \leq g(V(t_k^-, \psi(0))$ *for all* $(t_k, \psi) \in \mathbb{R}^+ \times PC([-d, 0], S(\rho_1))$ *for which* $\psi(0^-) = \psi(0)$;

(iv) $\tau = \sup_{k\in\mathbb{N}} \{t_k - t_{k-1}\} < \infty$, $M_1 = \sup_{t\geq t_0} \int_t^{t+\tau} p(s)ds < \infty$, *and* $M_2 = \inf_{q>0} \int_{g(q)}^{q} \dfrac{ds}{c(s)} > M_1$.

Then, the trivial solution of (3.75) is uniformly asymptotically stable.

Lemma 3.1.6. *[57] Assume that there exist functions* $a, b, c \in K_1$, $p \in PC(\mathbb{R}^+, \mathbb{R}^+)$, g, $\hat{g} \in K_3$ *where* $s \leq \hat{g}(s) < g(s)$ *for* $s > 0$, *and* $V : [-d, \infty) \times S(\rho) \to \mathbb{R}^+$ *where* V *is continuous on* $[-d, \infty) \times S(\rho)$ *and on* $[t_{k-1}, t_k) \times S(\rho)$ *for* $k = 1, 2, \dots,$ $\lim_{(t,y)\to(t_k^-, x)} V(t, y) = V(t_k^-, x)$ *exists. Moreover,* V, *restricted to* $\mathbb{R}^+ \times S(\rho)$, *is locally Lipschitz in* x *and the following conditions are satisfied:*

(i) $b(\|x\|) \leq V(t,x) \leq a(\|x\|)$ *for all* $(t,x) \in [-d,\infty) \times S(\rho)$;

(ii) $D^+V_{(3.75)}(t,\psi(0)) \leq -p(t)c(V(t,\psi(0)))$ *for all* $t \neq t_k$ *in* $\mathbb{R}^+$ *and* $\psi \in PC([-d,0],S(\rho))$ *whenever* $g(V(t+s,\psi(0))) \geq V(t,\psi(s))$ *for* $s \in [-d,0]$;

(iii) $V(t_k,\psi(0)+I(t_k,\psi)) \leq \hat{g}(V(t_k^-,\psi(0))$ *for all* $(t_k,\psi) \in \mathbb{R}^+ \times PC([-d,0],S(\rho_1))$ *for which* $\psi(0^-) = \psi(0)$;

(iv) $\mu = \inf_{k\in\mathbb{N}} \{t_k - t_{k-1}\} < \infty$, $M_1 = \inf_{t\geq t_0} \int_t^{t+\mu} p(s)ds > M_2$ *where* $M_2 = \sup_{q>0} \int_q^{g(q)} \dfrac{ds}{c(s)}$.

Then, the trivial solution of (3.75) is uniformly asymptotically stable.

Now, we state and prove the main theorems in the two cases described above. We obtain the first three theorems for Case 1, and the last three ones are about Case 2.

Theorem 3.1.3. *Assume that the following assumptions hold:*

(i) *there exists a linear convex combination*

$$\sum_{i=1}^{N} \alpha_i A_i,$$

where $\alpha_i \in (0,1)$ *with* $\sum_{i=1}^N \alpha_i = 1$ *such that all the eigenvalues of matrix pair* $(E, \sum_{i=1}^N \alpha_i A_i)$ *have positive real parts;*

(ii) *the equation in (3.70) holds;*

(iii) *there exist functions* $a_1, a_2 \in C([t_0,\infty],\mathbb{R}^+)$ *such that*

$$\max_{i\in\Xi} \left\{2x^T(t)E^T X f_i(t,x(t-d))\right\} \leq a_1(t)\left\|Ex(t)\right\|^2 + a_2(t)\left\|Ex(t-d)\right\|^2;$$

(iv) there exists some constant $0 < \alpha < 1$ *with*

$$\alpha^2 \geq \max_{k\in\mathbb{N}} \left\{\lambda_{\max}(X^{-1}C_k^T X C_k)\right\} \tag{3.77}$$

where $C_k = I + B_k$*,* C_k *and singular matrix* E *are commute, i.e.* $C_k E = E C_k$ *such that*

$$\frac{1}{\lambda_{\min}(X)} \left\{\tau\xi\lambda_{\max}(Y) + \sup_{t\geq t_0} \int_t^{t+\tau} \left[a_1(s) + \frac{a_2(s)}{\alpha^2}\right] ds\right\} + 2\ln\alpha < 0, \tag{3.78}$$

where $\tau = \sup_{k\in\mathbb{N}}\{t_k - t_{k-1}\} < \infty$ *and* $\xi_i \geq 1$ *defined in (3.71).*

Then, the trivial solution of system (3.1) with switching rule SR is uniformly asymptotically stable.

Proof. Define the common Lyapunov function for the switched singular system in (3.1)

$$V(x(t)) = x^T(t)E^T X E x(t),$$

then

$$\lambda_{\min}(X)\left\|Ex\right\|^2 \leq V(x(t)) \leq \lambda_{\max}(X)\left\|Ex\right\|^2.$$

Now, assume that the ith mode is active on $[t_{k-1}, t_k)$. Then, $x(t) \in \Omega_i$ for $t \in [t_{k-1}, t_k)$ under switching rule SR, so $x^T\left(A_i^T X E + E^T X A_i\right)x \leq \xi x^T E^T Y E x$ where $\xi \geq 1$ by (3.71) and by condition (iii) the time derivative of the Lyapunov function is obtained as

$$\begin{aligned}
\dot{V}(x(t)) &= \dot{x}^T(t)E^T X E x(t) + x^T(t)E^T X E\dot{x}(t)\\
&= \left[A_i x(t) + f_i(t, x(t))\right]^T X E x(t)\\
&\quad + x^T(t)E^T X\left[A_i x(t) + f_i(t, x(t-d))\right]\\
&= x^T(t)\left[A_i^T X E + E^T X A_i\right]x(t) + 2x^T(t)E^T X f_i(t, x(t-d))\\
&\leq \xi x^T(t)E^T Y E x(t) + 2x^T(t)E^T X f_i(t, x(t-d))
\end{aligned}$$

$$\leq \xi\lambda_{\max}(Y)\left\|Ex(t)\right\|^2 + 2\left|x^T(t)E^TXf_i(t,x(t-d))\right|$$

$$\leq \frac{1}{\lambda_{\min}(X)}\left\{\left[\xi\lambda_{\max}(Y) + a_1(t)\right]V(x(t)) + a_2(t)\sup_{\sigma\in[-d,0]}V(x(t+\sigma))\right\}. \tag{3.79}$$

Choose a function $g(s) = \alpha^2 s$ where $0 < \alpha < 1$ by using Lemma 3.1.5. Whenever $V(x(t)) \geq g\left(V(x(t+s))\right)$ for $s \in [-d, 0]$, then from (3.79) we obtain

$$\dot{V}(x(t)) \leq \frac{1}{\lambda_{\min}(X)}\left\{\xi\lambda_{\max}(Y) + a_1(t) + \frac{a_2(t)}{\alpha^2}\right\}V(x(t)),$$
$$= p(t)c(V(x(t))),$$

where $p(t) = \frac{1}{\lambda_{\min}(X)}\left\{\xi\lambda_{\max}(Y) + a_1(t) + \frac{a_2(t)}{\alpha^2}\right\}$ and $c(V(x(t))) = V(x(t))$.

On the other hand, by the system (3.1) at $t = t_k$ we obtain

$$\begin{aligned} V(x(t_k)) &= x^T(t_k)E^TXEx(t_k) \\ &= x^T(t_k^-)(I+B_k)^TE^TXE(I+B_k)x(t_k^-) \\ &= x^T(t_k^-)C_k^TE^TXEC_kx(t_k^-), \end{aligned} \tag{3.80}$$

where $I + B_k = C_k$. Therefore, by Lemma 3.1.4 and assumption (iv), $V(x(t_k))$ in (3.80) becomes

$$\begin{aligned} V(x(t_k)) &= x^T(t_k^-)E^TC_k^TXC_kEx(t_k^-) \\ &\leq \lambda_{\max}(X^{-1}C_k^TXC_k)x^T(t_k^-)E^TXEx(t_k^-) \\ &\leq \max_{i\in\Xi}\left\{\lambda_{\max}(X^{-1}C_k^TXC_k)\right\}V(x(t_k^-)) \\ &\leq \alpha^2V(x(t_k^-)). \end{aligned}$$

Then, we have

$$M_1 = \sup_{t\geq t_0}\int_t^{t+\tau}p(s)ds = \frac{1}{\lambda_{\min}(X)}\left\{\tau\xi\lambda_{\max}(Y) + \sup_{t\geq t_0}\int_t^{t+\tau}\left[a_1(s) + \frac{a_2(s)}{\alpha^2}\right]ds\right\},$$

and

$$M_2 = \inf_{q>0} \int_{g(q)}^{q} \frac{ds}{c(s)} = \inf_{q>0} \int_{\alpha^2 q}^{q} \frac{ds}{s} = -2 \ln \alpha.$$

By (3.78) in assumption (iv) $M_2 > M_1$. Therefore, all conditions in Lemma 3.1.5 are satisfied. That means the trivial solution of the system in (3.1) with switching rule SR is uniformly asymptotically stable. □

The inequality in (3.77) is obtained by using Lemma 3.1.4. Instead of that lemma, using different approach we can obtain the following theorem.

Theorem 3.1.4. *Assume that all conditions in Theorem 3.1.3 hold except that the inequality in (3.77) is replaced by*

$$\alpha^2 \geq \frac{\lambda_{\max}(X)}{\lambda_{\min}(X)} \max_{k \in \Xi}(\|C_k\|^2).$$

Then the trivial solution of the system in (3.1) with switching rule SR is uniformly asymptotically stable.

Proof. Similarly we obtain inequality (3.79) in the previous proof. Then, at $t = t_k$ we have

$$\begin{aligned} V(x(t_k)) &= x^T(t_k) E^T X E x(t_k) \\ &= x^T(t_k^-) E^T C_k^T X_i C_k E x(t_k^-) \\ &\leq \lambda_{\max}(X) \left\| E x(t_k^-) \right\|^2 \|C_k\|^2 \\ &\leq \frac{\lambda_{\max}(X)}{\lambda_{\min}(X)} V(x(t_k^-)) \max_{k \in \Xi}(\|C_k\|^2) \\ &\leq \alpha^2 V(x(t_k^-)), \end{aligned}$$

where $\alpha^2 \geq \dfrac{\lambda_{\max}(X)}{\lambda_{\min}(X)} \max_{k \in \Xi}(\|C_k\|^2)$. □

In the next theorem, we use Lemma 3.1.4 to evaluate the derivative of common Lyapunov function in Theorem 3.1.3.

Theorem 3.1.5. *Assume that all conditions in Theorem 3.1.3 hold except that the inequality in (3.78) is replaced by*

$$\tau\xi\lambda_{\max}(X^{-1}Y) + \frac{1}{\lambda_{\min}(X)}\left\{\sup_{t\geq t_0}\int_t^{t+\tau}\left[a_1(s) + \frac{a_2(s)}{\alpha^2}\right]ds\right\} + 2\ln\alpha < 0. \tag{3.81}$$

Then the trivial solution of the system in (3.1) with switching rule SR is uniformly asymptotically stable.

Proof. We will use Lemma 3.1.4 to obtain

$$\dot{V}(x(t)) \leq p(t)c(V(x(t))),$$

with $p(t)$ replaced by

$$p(t) = \xi\lambda_{\max}(X^{-1}Y) + \frac{1}{\lambda_{\min}(X)}\left\{a_1(t) + \frac{a_2(t)}{\alpha^2}\right\} \quad \text{for any } t \geq t_0.$$

□

Example 3.1.3. *Consider the ISSSD given by (3.1) with $\sigma(t) = \{1, 2\}$, the same singular matrix*

$$E = \begin{bmatrix} 4 & 0 \\ 2 & 0 \end{bmatrix}$$

in each subsystem,

$$A_1 = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}, \quad C_1 = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}, \qquad f_1(t, x(t-1)) = \begin{bmatrix} \frac{1}{15}\tanh(x_1(t-1)) & \frac{1}{15}\tanh(x_2(t-1)) \end{bmatrix}^T,$$

$$A_2 = \begin{bmatrix} 1 & 2 \\ 1 & -2 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 1.01 & 0 \\ 0 & 1.01 \end{bmatrix}, \qquad f_2(t, x(t-1)) = \begin{bmatrix} \frac{1}{15}\tanh(x_1(t-1)) & \frac{1}{15}\tanh(x_2(t-1)) \end{bmatrix}^T,$$

and the initial function is $\phi(t) = \begin{bmatrix} 0.25 + t & 1 - t \end{bmatrix}^T$. *The Lipschitz conditions are satisfied with* $F_1 = F_2 = \begin{bmatrix} 0.0333 & -0.0333 \\ 0 & 0 \end{bmatrix}$.

Both singular subsystems are unstable since the matrix pairs (E, A_i) *for* $i = 1, 2$ *have eigenvalues with positive real parts. Let the convex combination*

$$\sum_{i=1}^{2} \alpha_i A_i = 0.5A_1 + 0.5A_2,$$

where $\alpha_i = 0.5$ *for* $i = 1, 2$*, then the matrix pair* $(E, 0.5A_1 + 0.5A_2)$ *has an eigenvalue with positive real part. Also, we have*

$$\max_{i \in \Xi} \left\{ 2x^T(t) E^T X f_i(t, x(t-d)) \right\} \leq a_1(t) \left\| Ex(t) \right\|^2 + a_2(t) \left\| Ex(t-d) \right\|^2;$$

where $a_1(t) = 1$, $a_2(t) = 0.0244$. *Inequality (3.77) is satisfied by choosing* $\alpha = 0.2$*, and the inequality in (3.78) for* $\tau = 1$ *implies*

$$\frac{1}{\lambda_{\min}(X)} \left\{ \tau \xi \lambda_{\max}(Y) + \sup_{t \geq t_0} \int_t^{t+\tau} \left[a_1(s) + \frac{a_2(s)}{\alpha^2} \right] ds \right\} + 2 \ln \alpha = \{\xi + 1.1220\} + 2 \ln(0.2) < 0,$$

which yields $1 \leq \xi < 2.09693$. *Therefore, the trivial solution of system (3.1) with switching rule SR is uniformly asymptotically stable when* $\xi \in [1, 2.09693)$ *as shown in the Figure 3.3.*

In these three theorems a Hurwitz linear convex combination of unstable subsystems do not required. However, in the following results it is assumed that there exists a Hurwitz linear convex combination of unstable subsystems.

Theorem 3.1.6. *Assume that the following assumptions hold:*

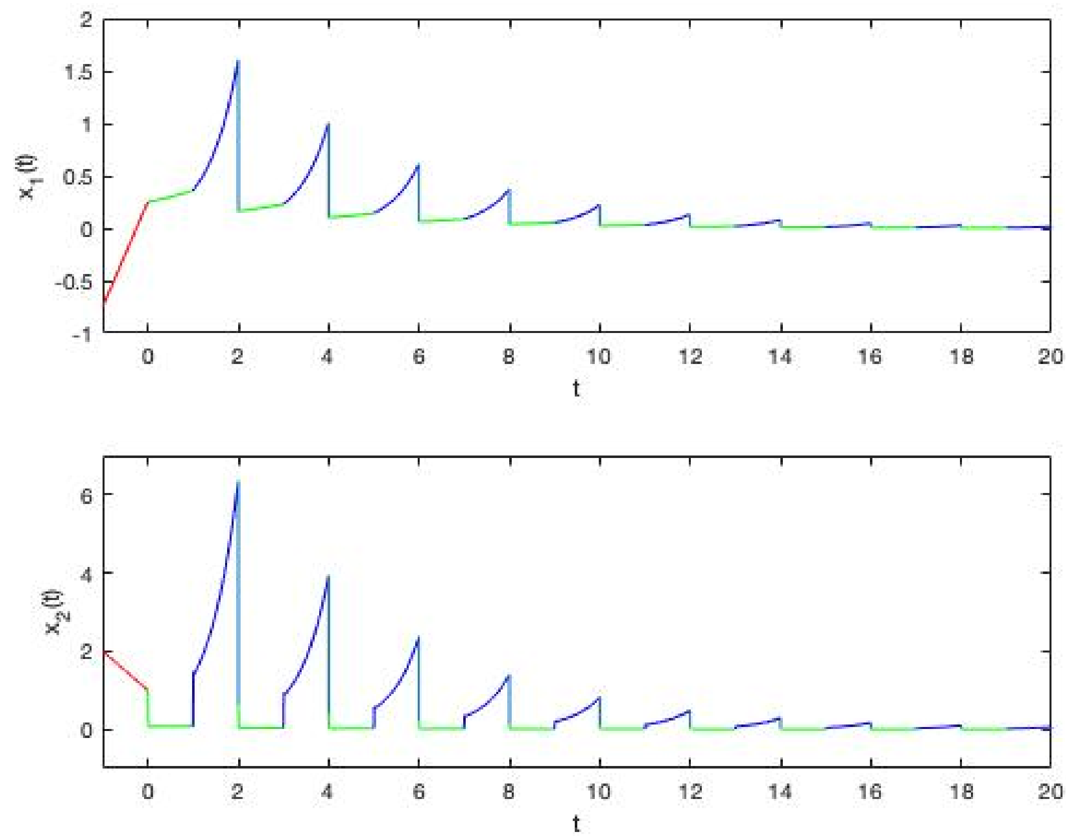

Figure 3.3: ISSSD with unstable subsystems in unstable convex combination case

(i) there exists a linear convex combination

$$\sum_{i=1}^{N} \alpha_i A_i,$$

where $\alpha_i \in (0,1)$ *with* $\sum_{i=1}^{N} \alpha_i = 1$ *such that all the eigenvalues of matrix pair* $(E, \sum_{i=1}^{N} \alpha_i A_i)$ *have negative real parts;*

(ii) the equation in (3.72) holds;

(iii) there exist functions $a_1, a_2 \in C([t_0, \infty], \mathbb{R}^+)$ *such that*

$$\max_{i \in \Xi} \left\{ 2x^T(t) E^T X f_i(t, x(t-d)) \right\} \leq a_1(t) \left\| Ex(t) \right\|^2 + a_2 \left\| Ex(t-d) \right\|^2;$$

(iv) there exists some constant $\alpha \geq 1$ *with*

$$\alpha^2 \geq \max_{k \in \mathbb{N}} \left\{ \lambda_{\max}(X^{-1} C_k^T X C_k) \right\} \tag{3.82}$$

where $C_k = I + B_k$*,* C_k *and singular matrix* E *are commute, i.e.* $C_k E = E C_k$ *such that*

$$\frac{\lambda_{\min}(Y)}{\xi^* \lambda_{\max}(X)} \geq \frac{1}{\lambda_{\min}(X)} \left\{ a_1(t) + a_2(t)\beta^2 \right\}, \tag{3.83}$$

and

$$2 \ln \beta - \frac{\mu \lambda_{\min}(Y)}{\xi^* \lambda_{\max}(X)} + \sup_{t \geq t_0} \int_t^{t+\mu} \frac{1}{\lambda_{\min}(X)} \left\{ a_1(s) + a_2(s)\beta^2 \right\} ds < 0, \tag{3.84}$$

where $\beta > \alpha$*,* $\mu = \inf_{k \in \mathbb{N}} \{t_k - t_{k-1}\} > 0$ *and* $\xi^* \geq 1$ *defined in (3.73).*

Then the trivial solution of the system in (3.1) with switching rule SR is uniformly asymptotically stable.

Proof. Let the common Lyapunov function for the switched singular system in (3.1)

$$V(x(t)) = x^T(t)E_i^T X_i E_i x(t),$$

then

$$\lambda_{\min}(X)\left\|Ex\right\|^2 \leq V(x(t)) \leq \lambda_{\max}(X)\left\|Ex\right\|^2.$$

Now, assume that the ith mode is active on $[t_{k-1}, t_k)$. Then, $x(t) \in \Omega_i^*$ for $t \in [t_{k-1}, t_k)$ by switching rule SR, so $x^T\left(A_i^T XE + E^T XA_i\right)x \leq -\frac{1}{\xi^*}x^T E^T Y Ex$ where $\xi^* \geq 1$ by (3.73) and by condition (iii) the time derivative of the Lyapunov function is obtained as

$$\begin{aligned}
\dot{V}(x(t)) &= \dot{x}^T(t)E^T XEx(t) + x^T(t)E^T XE\dot{x}(t) \\
&= \left[A_i x(t) + f_i(t, x(t))\right]^T XEx(t) \\
&\quad + x^T(t)E^T X\left[A_i x(t) + f_i(t, x(t-d))\right] \\
&= x^T(t)\left[A_i^T XE + E^T XA_i\right]x(t) + 2x^T(t)E^T Xf_i(t, x(t-d)) \\
&\leq -\frac{1}{\xi_i^*}x^T(t)E^T Y Ex(t) + 2\left|x^T(t)E^T Xf_i(t, x(t-d))\right| \\
&\leq -\frac{1}{\xi^*}\lambda_{\min}(Y)\left\|Ex(t)\right\|^2 + a_1(t)\left\|Ex(t)\right\|^2 + a_2(t)\left\|Ex(t-d)\right\|^2 \\
&\leq -\frac{\lambda_{\min}(Y)}{\xi^*\lambda_{\max}(X)}V(x(t)) + \frac{1}{\lambda_{\min}(X)}\left\{a_1(t)V(x(t)) + a_2(t)\sup_{\sigma\in[-d,0]}V(x(t+\sigma))\right\}.
\end{aligned} \tag{3.85}$$

Choose a function $\hat{g}(s) = \alpha^2 s$, $g(s) = \beta^2 s$ where $\beta > \alpha$ and α is defined in (3.82). Whenever $g\left(V(x(t))\right) \geq V(x(t+s))$ for $s \in [-d, 0]$, then from (3.85) we can obtain that

$$\begin{aligned}
\dot{V}(x(t)) &\leq -\frac{\lambda_{\min}(Y)}{\xi_i^*\lambda_{\max}(X)}V(x(t)) + \frac{1}{\lambda_{\min}(X)}\left\{a_1(t) + a_2(t)\beta^2\right\}V(x(t)), \\
&= -p(t)c(V(x(t))),
\end{aligned}$$

where $p(t) = \dfrac{\lambda_{\min}(Y)}{\xi^* \lambda_{\max}(X)} - \dfrac{1}{\lambda_{\min}(X)} \{a_1(t) + a_2(t)\beta^2\} \geq 0$ and $c(V(x(t))) = V(x(t))$.

On the other hand, by the system (3.1) at $t = t_k$ we have

$$\begin{aligned} V(x(t_k)) &= x^T(t_k) E^T X E x(t_k) \\ &= x^T(t_k^-)(I + B_k)^T E^T X E (I + B_k) x(t_k^-) \\ &= x^T(t_k^-) C_k^T E^T X E C_k x(t_k^-), \end{aligned} \tag{3.86}$$

where $I + B_k = C_k$. Therefore, by Lemma 3.1.4 and assumption (iv), $V(x(t_k))$ in (3.86) becomes

$$\begin{aligned} V(x(t_k)) &= x^T(t_k^-) E^T C_k^T X C_k E x(t_k^-) \\ &\leq \lambda_{\max}(X^{-1} C_k^T X C_k) x^T(t_k^-) E^T X E x(t_k^-) \\ &\leq \max_{k \in \mathbb{N}} \left\{ \lambda_{\max}(X^{-1} C_k^T X C_k) \right\} V(x(t_k^-)) \\ &\leq \alpha^2 V(x(t_k^-)). \end{aligned}$$

Then, we have

$$M_1 = \inf_{t \geq t_0} \int_t^{t+\mu} p(s) ds = \frac{\mu \lambda_{\min}(Y)}{\xi^* \lambda_{\max}(X)} - \sup_{t \geq t_0} \int_t^{t+\mu} \frac{1}{\lambda_{\min}(X)} \left\{ a_1(s) + a_2(s)\beta^2 \right\} ds,$$

and

$$M_2 = \sup_{q>0} \int_q^{g(q)} \frac{ds}{c(s)} = \sup_{q>0} \int_q^{\beta^2 q} \frac{ds}{s} = 2 \ln \beta.$$

The inequality in (3.84) in assumption (iv) implies $M_1 > M_2$. Therefore, all conditions in Lemma 3.1.6 are satisfied. That means the trivial solution of the system in (3.1) with switching rule SR is uniformly asymptotically stable. □

The following two theorems are similarly obtained as Theorem 3.1.4 and Theorem 3.1.5. Thus, we omitted the proofs of these theorems.

Theorem 3.1.7. *Assume that conditions in Theorem 3.1.6 hold except that the inequality in (3.82) is replaced by*

$$\alpha^2 \geq \frac{\lambda_{\max}(X)}{\lambda_{\min}(X)} \max_{k\in\Xi}(\|C_k\|^2).$$

Then the trivial solution of the system in (3.1) with switching rule SR is uniformly asymptotically stable.

Theorem 3.1.8. *Assume that all conditions in Theorem 3.1.6 hold except that the inequalities in (3.83) and (3.84) are replaced by*

$$\frac{\lambda_{\min}(X^{-1}Y)}{\xi^*} \geq \frac{1}{\lambda_{\min}(X)} \left\{a_1(t) + a_2(t)\beta^2\right\},$$

and

$$2\ln\beta - \frac{\mu\lambda_{\min}(X^{-1}Y)}{\xi^*} + \sup_{t\geq t_0}\int_t^{t+\mu} \frac{1}{\lambda_{\min}(X)} \left\{a_1(s) + a_2(s)\beta^2\right\} ds < 0,$$

where $\beta > \alpha$, $\mu = \inf_{k\in\mathbb{N}}\{t_k - t_{k-1}\} > 0$ and $\xi^* \geq 1$ defined in (3.73), respectively. Then the trivial solution of the system in (3.1) with switching rule SR is uniformly asymptotically stable.

Example 3.1.4. *Consider the ISSSD given by (3.1) with $\sigma(t) = \{1, 2\}$, the same singular matrix*

$$E = \begin{bmatrix} 4 & 0 \\ 2 & 0 \end{bmatrix}$$

in each subsystem,

$$A_1 = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}, \quad C_1 = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}, \qquad f_1(t, x(t-1)) = \begin{bmatrix} \frac{1}{15}\tanh(x_1(t-1)) & \frac{1}{15}\tanh(x_2(t-1)) \end{bmatrix}^T,$$

$$A_2 = \begin{bmatrix} 1 & 2 \\ 1 & -2 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}, \qquad f_2(t, x(t-1)) = \begin{bmatrix} \frac{1}{15}\tanh(x_1(t-1)) & \frac{1}{15}\tanh(x_2(t-1)) \end{bmatrix}^T,$$

and the initial function is $\phi(t) = \begin{bmatrix} 0.25+t & 1-t \end{bmatrix}^T$. *The Lipschitz conditions are satisfied with* $F_1 = F_2 = \begin{bmatrix} 0.0333 & -0.0333 \\ 0 & 0 \end{bmatrix}$.

Both singular subsystems are unstable since the matrix pairs (E, A_i) *for* $i = 1, 2$ *have eigenvalues with positive real parts. Let the convex combination*

$$\sum_{i=1}^{2} \alpha_i A_i = 0.2A_1 + 0.1A_2,$$

where $\alpha_1 = 0.2$ *and* $\alpha_2 = 0.1$, *then the matrix pair* $(E, 0.2A_1 + 0.1A_2)$ *has an eigenvalue with positive real part. Also, we have*

$$\max_{i\in\Xi} \left\{2x^T(t)E^T X f_i(t, x(t-d))\right\} \leq a_1(t) \left\|Ex(t)\right\|^2 + a_2(t) \left\|Ex(t-d)\right\|^2;$$

where $a_1(t) = 1$, $a_2(t) = 5.5$. *Inequality (3.82) is satisfied by choosing* $\alpha = 3.6$, *and the inequality in (3.84) for* $\mu = 1$ *implies*

$$2\ln\beta - \frac{\mu\lambda_{\min}(Y)}{\xi^*\lambda_{\max}(X)} + \sup_{t\geq t_0}\int_t^{t+\mu} \frac{1}{\lambda_{\min}(X)} \left\{a_1(s) + a_2(s)\beta^2\right\} ds = 0.2994 - \frac{1}{\xi^*} < 0,$$

which yields $1 \leq \xi^* < 3.34$. *Therefore, the trivial solution of system (3.1) with switching rule SR is uniformly asymptotically stable when* $\xi^* \in [1, 3.34)$ *as shown in the Figure 3.4.*

3.2 Impulsive Switched Singular Systems with Time-Delay: Discrete

First, we will introduce some notations that we used. Let $\mathbb{R}$ be the real numbers, $\mathbb{R}^+$ the positive real numbers, $\mathbb{Z}$ the integers, $\mathbb{Z}^+$ the positive integers, $\mathbb{N}$ the natural numbers, i.e.,

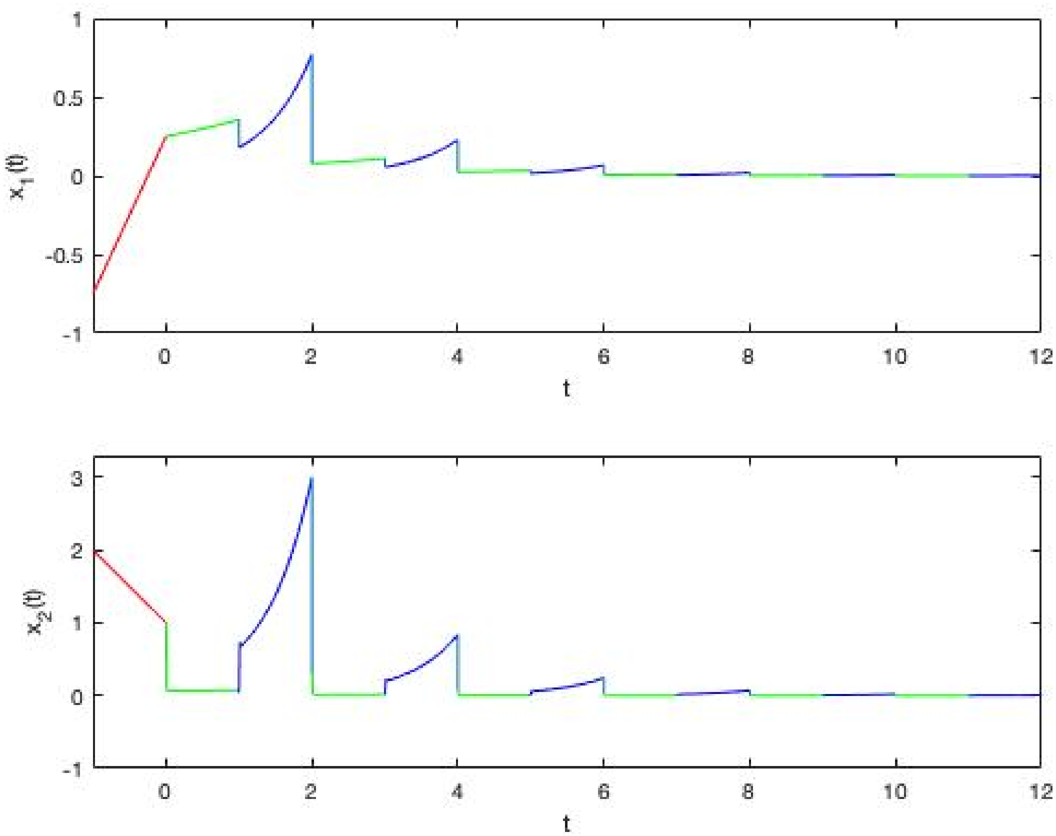

Figure 3.4: ISSSD with unstable subsystems in stable convex combination case

$\mathbb{N} = \{1, 2, \dots\}$, and for some positive integer d, let $\mathbb{N}_{-d} = \{-d, -d+1, \dots, -1, 0\}$. Let $\mathbb{R}^N$ denotes the N-dimensional real space. For a given integer d, let $C = \{\phi : \mathbb{N}_{-d} \to \mathbb{R}^N\}$. For any $\phi \in C$, we define $\|\phi\|_d = \max_{\theta \in \mathbb{N}_{-d}}\{\phi(\theta)\}$.

Consider the impulsive switching singular discrete system with time-delay:

$$\begin{aligned} E_{\sigma(n)}x(n+1) &= A_{\sigma(n)}x(n) + f_{\sigma(n)}(n, x(n-d)), && n_{k-1}^+ \le n < n_k \\ \triangle x(n) &= B_k x(n), && n = n_k, \quad k \in \mathbb{N} \\ x_{n_0} &= \phi, \end{aligned} \tag{3.87}$$

where $x \in \mathbb{R}^N$, $n_0 \in \mathbb{Z}^+$, $f_{\sigma(n)}(n, x(n-d)) : \mathbb{Z}^+ \times \mathbb{R}^N \to \mathbb{R}^N$, $\phi \in C$ and $x_{n_0} \in C$ is defined by $x_{n_0}(s) = x(n_0 + s)$ for any $s \in \mathbb{N}_{-d}$ with $d \in \mathbb{N}$ representing the delay in system (3.87). $A_{\sigma(n)}, B_k, E_{\sigma(n)} \in \mathbb{R}^{N\times N}$ are system coefficient matrices where $E_{\sigma(n)}$ being singular with $\text{rank}(E_{\sigma(n)}) = r < N$, $A_{\sigma(n)}$ being invertible, the matrix pairs $(E_{\sigma(n)}, A_{\sigma(n)})$ being regular, and B_k being constant matrices. $\sigma(n) : \mathbb{N} \to \Xi$ is a switching rule taking values $\sigma(n) = i$ in a finite compact set $\Xi = \{1, 2, \dots, M\}$ for some $M \in \mathbb{N}$. $\{n_k\}_{k=0}^{\infty}$ are the impulsive times that form an increasing sequence satisfying $n_{k-1} < n_k$ and $\lim_{k\to\infty} n_k = \infty$. $\Delta x(n_k) = x(n_k^+) - x(n_k)$ where $x(n_k^+)$ is the state just after the impulsive action and $x_{n_k^+}(s) = x(n_k^+ + s)$ for $s \in \mathbb{N}_{-d}$, $k \in \mathbb{N}$. We assume $f_{\sigma(n)}(n, 0) \equiv 0$ and for all $(n, x(n-d))$, $(n, x^*(n-d)) \in \mathbb{Z}^+ \times \mathbb{R}^N$

$$\|f_{\sigma(n)}(n, x(n-d)) - f_{\sigma(n)}(n, x^*(n-d))\| \le \|F_{\sigma(n)}E_{\sigma(n)}(x(n-d) - x^*(n-d))\| \tag{3.88}$$

where $F_{\sigma(n)}$ are constant matrices with appropriate dimension so that system (3.87) admits the unique solution. The solution of system (3.87) is denoted by $x(n) = x(n; n_0, \phi)$ for any given initial condition $n_0 \in \mathbb{Z}^+$ and $\phi \in C$.

3.2.1 Systems with Stable Subsystems

Before stating the conditions to ensure exponential stability of ISSSD (3.13) with stable subsystems, we will introduce the following lemma that we will use in the proof of main theorem.

Lemma 3.2.1. *[58] Let $d > 0$ be a natural number, and $\{x_n\}_{n\geq -d}$ be a sequence of real numbers satisfying the inequality*

$$\triangle x_n \leq -ax_n + b\max\{x_n, x_{n-1}, \ldots, x_{n-d}\}, \quad n \geq 0, \tag{3.89}$$

where $\triangle x_n = x_{n+1} - x_n$.
If $0 < b < a \leq 1$, then there exists a constant $\lambda_0 \in (0, 1)$ such that

$$x_n \leq \max\{0, x_0, x_{-1}, \ldots, x_{-d}\}\lambda_0^n, \quad n \geq 0. \tag{3.90}$$

Moreover, λ_0 can be chosen as the smallest root in the interval $(0, 1)$ of the equation

$$\lambda^{d+1} + (a-1)\lambda^d - b = 0. \tag{3.91}$$

Theorem 3.2.1. *For any $i \in \Xi$, assume that each subsystem of (3.87) is admissible. Then, the trivial solution of (3.87) is exponentially stable if the following assumptions hold:*

(i) For any $i, j \in \Xi$ there exists $\gamma_k > 1$ such that

$$(I + B_k)^T E_j^T X_j E_j (I + B_k) \leq \gamma_k E_i^T X_i E_i, \tag{3.92}$$

where X_i is positive definite matrix satisfying the Lyapunov equation

$$A_i^T X_i A_i - E_i^T X_i E_i = -Y_i$$

for any $Y_i > 0$.

(ii) For any $n \geq n_0$, the switching law satisfies the ADT condition

$$N(n_0, n) \leq N_0 + \frac{n - n_0}{T_a},$$

where N denotes the number of switchings in (n_0, n), T_a is the average dwell time and N_0 is the chatter bound.

Proof. Let $x(n) = x(n; n_0, \phi)$ be the solution of the system (3.87). For $n \in [n_{k-1}, n_k)$, define

$$V_i(x(n)) = x^T(n) E_i^T X_i E_i x(n), \quad i = \sigma(n) \tag{3.93}$$

as a Lyapunov function candidate for i^{th} subsystem. The variation of V_i is defined as

$$\triangle V_i(x(n)) = V_i(x(n+1)) - V_i(x(n)).$$

Thus, the variation of V_i relative to system (3.87) is

$$\begin{aligned}
\triangle V_i(x(n)) &= x^T(n+1) E_i^T X_i E_i x(n+1) - x^T(n) E_i^T X_i E_i x(n) \\
&= \left[x^T(n) A_i^T + f_i^T(n, x(n-d))\right] X_i \left[A_i x(n) + f_i(n, x(n-d))\right] - x^T(n) E_i^T X_i E_i x(n) \\
&= x^T(n) A_i^T X_i A_i x(n) + x^T(n) A_i^T X_i f_i(n, x(n-d)) + f_i^T(n, x(n-d)) X_i A_i x(n) \\
&\quad + f_i^T(n, x(n-d)) X_i f_i(n, x(n-d)) - x^T(n) E_i^T X_i E_i x(n) \\
&= x^T(n) \left[A_i^T X_i A_i - E_i^T X_i E_i\right] x(n) + 2 f_i^T(n, x(n-d)) X_i A_i x(n) \\
&\quad + f_i^T(n, x(n-d)) X_i f_i(n, x(n-d)) \\
&= -x^T(n) Y_i x(n) + 2 f_i^T(n, x(n-d)) X_i A_i x(n) + f_i^T(n, x(n-d)) X_i f_i(n, x(n-d))
\end{aligned}$$

where $A_i^T X_i A_i - E_i^T X_i E_i = -Y_i$ for any $Y_i > 0$.

Using the Lipschitz condition (3.88), we obtain that

$$
\begin{aligned}
2f_i^T(n,x(n-d))X_iA_ix(n) &\le \frac{1}{\varepsilon_i}f_i^T(n,x(n-d))f_i(n,x(n-d)) + \varepsilon_i x^T(n)A_i^TX_i^2A_ix(n) \\
&\le \frac{1}{\varepsilon_i}\|f_i(n,x(n-d))\|^2 + \varepsilon_i\|A_ix(n)\|^2\lambda_{\max}(X_i^2) \\
&\le \frac{1}{\varepsilon_i}\|F_iE_ix(n-d)\|^2 + \varepsilon_i\|A_ix(n)\|^2\lambda_{\max}(X_i^2) \\
&\le \frac{1}{\varepsilon_i}\|F_i\|^2\|E_ix(n-d)\|^2 + \varepsilon_i\|A_i\|^2\|x(n)\|^2\lambda_{\max}(X_i^2)
\end{aligned}
$$

and

$$
\begin{aligned}
f_i^T(n,x(n-d))X_if_i(n,x(n-d)) &\le \lambda_{\max}(X_i)\|f_i(n,x(n-d))\|^2 \\
&\le \lambda_{\max}(X_i)\|F_iE_ix(n-d)\|^2 \\
&\le \lambda_{\max}(X_i)\|F_i\|^2\|E_ix(n-d)\|^2.
\end{aligned}
$$

Thus, we obtain

$$
\begin{aligned}
\triangle V_i(x(n)) &\le -x^T(n)Y_ix(n) + \Big[\frac{1}{\varepsilon_i} + \lambda_{\max}(X_i)\Big]\|F_i\|^2\|E_ix(n-d)\|^2 + \varepsilon_i\|A_i\|^2\|x(n)\|^2\lambda_{\max}(X_i^2) \\
&\le \Big[-\lambda_{\min}(Y_i) + \varepsilon_i\|A_i\|^2\lambda_{\max}(X_i^2)\Big]\|x(n)\|^2 + \Big[\frac{1}{\varepsilon_i} + \lambda_{\max}(X_i)\Big]\|F_i\|^2\|E_ix_n\|_d^2 \\
&\le -\alpha_iV_i(x(n)) + \beta_i\max_{s\in\mathbb{N}_{-d}}V_i(x(n+s)) \qquad (3.94)
\end{aligned}
$$

where $\alpha_i = \frac{\lambda_{\min}(Y_i)-\varepsilon_i\|A_i\|^2\lambda_{\max}(X_i^2)}{\lambda_{\max}(E_i^TX_iE_i)} > 0$, and $\beta_i = \frac{[1+\varepsilon_i\lambda_{\max}(X_i)]\|F_i\|^2}{\varepsilon_i\lambda_{\min}(X_i)} > 0$.
By Lemma (3.2.1) we obtain the solution of (3.94) for $n \in [n_{k-1}^+, n_k)$ as

$$
V_i(x(n)) \le \max_{\theta\in\mathbb{N}_{-d}}\{V_i(x(n_{k-1}^+ + \theta))\}\lambda_{0_i}^{(n-n_{k-1})} \qquad (3.95)
$$

where λ_{0_i} is the smallest root in the interval $(0,1)$ of the equation

$$
\lambda^{d+1} + (\alpha_i - 1)\lambda^d - \beta_i = 0.
$$

On the other hand, for $n = n_k$, $k = 1, 2, 3, \ldots$, suppose $\sigma(n_k) = j$, it follows from (3.87) and (3.92) that

$$\begin{aligned} V_j(x(n_k^+)) &= x^T(n_k^+)E_j^T X_j E_j x(n_k^+) \\ &= x^T(n_k)(I + B_k)^T E_j^T X_j E_j (I + B_k) x(n_k) \\ &\le \gamma_k x^T(n_k) E_i^T X_i E_i x(n_k) \\ &= \gamma_k V_i(x(n_k)). \end{aligned} \tag{3.96}$$

Using (3.95) and (3.96) successively on each subinterval leads to the following results. For instance, for $n \in [n_0^+, n_1)$, we have

$$V_{i_1}(x(n)) \le \max_{\theta \in \mathbb{N}_{-d}} \{V_{i_1}(x(n_0^+ + \theta))\} \lambda_{0_{i_1}}^{(n-n_0)} \tag{3.97}$$

For $n \in [n_1^+, n_2)$, we have

$$V_{i_2}(x(n)) \le \max_{\theta \in \mathbb{N}_{-d}} \{V_{i_2}(x(n_1^+ + \theta))\} \lambda_{0_{i_2}}^{(n-n_1)} \tag{3.98}$$

From (3.96), we obtain

$$\begin{aligned} &\max_{\theta \in \mathbb{N}_{-d}} \{V_{i_2}(x(n_1^+ + \theta))\} \le \gamma_1 \max_{\theta \in \mathbb{N}_{-d}} \{V_{i_1}(x(n_1 + \theta))\} \\ \Rightarrow &\max_{\theta \in \mathbb{N}_{-d}} \{V_{i_2}(x(n_1^+ + \theta))\} \lambda_{0_{i_2}}^{(n-n_1)} \le \gamma_1 \max_{\theta \in \mathbb{N}_{-d}} \{V_{i_1}(x(n_1 + \theta))\} \lambda_{0_{i_2}}^{(n-n_1)} \end{aligned} \tag{3.99}$$

Thus, by (3.98) and (3.99), we obtain

$$\begin{aligned} V_{i_2}(x(n)) \le \max_{\theta \in \mathbb{N}_{-d}} \{V_{i_2}(x(n_1^+ + \theta))\} \lambda_{0_{i_2}}^{(n-n_1)} &\le \gamma_1 \max_{\theta \in \mathbb{N}_{-d}} \{V_{i_1}(x(n_1 + \theta))\} \lambda_{0_{i_2}}^{(n-n_1)} \\ \Rightarrow V_{i_2}(x(n)) &\le \gamma_1 \max_{\theta \in \mathbb{N}_{-d}} \{V_{i_1}(x(n_1 + \theta))\} \lambda_{0_{i_2}}^{(n-n_1)} \end{aligned} \tag{3.100}$$

We suppose that $n_k - d \in [n_{k-1}, n_k)$. Then, we can find $\max_{\theta \in \mathbb{N}_{-d}} \{V_{i_1}(x(n_1 + \theta))\}$ by (3.97) as

$$\max_{\theta \in \mathbb{N}_{-d}} \{V_{i_1}(x(n_1 + \theta))\} = \max_{\theta \in \mathbb{N}_{-d}} \{V_{i_1}(x(n_0^+ + \theta))\} \lambda_{0_{i_1}}^{(n_1-d-n_0)},$$

so inequality (3.100) becomes

$$V_{i_2}(x(n)) \leq \gamma_1 \max_{\theta\in\mathbb{N}_{-d}}\{V_{i_1}(x(n_0^+ + \theta))\}\lambda_{0_{i_1}}^{(n_1-d-n_0)}\lambda_{0_{i_2}}^{(n-n_1)} \qquad n \in [n_1^+, n_2). \tag{3.101}$$

Similarly, for $n \in [n_2^+, n_3)$ we can obtain the following inequality using (3.95)

$$V_{i_3}(x(n)) \leq \max_{\theta\in\mathbb{N}_{-d}}\{V_{i_3}(x(n_2^+ + \theta))\}\lambda_{0_{i_3}}^{(n-n_2)}. \tag{3.102}$$

Also, from (3.96) we have

$$\begin{aligned} &\max_{\theta\in\mathbb{N}_{-d}}\{V_{i_3}(x(n_2^+ + \theta))\} \leq \gamma_2 \max_{\theta\in\mathbb{N}_{-d}}\{V_{i_2}(x(n_2 + \theta))\} \\ \Rightarrow &\max_{\theta\in\mathbb{N}_{-d}}\{V_{i_3}(x(n_2^+ + \theta))\}\lambda_{0_{i_3}}^{(n-n_2)} \leq \gamma_2 \max_{\theta\in\mathbb{N}_{-d}}\{V_{i_2}(x(n_2 + \theta))\}\lambda_{0_{i_3}}^{(n-n_2)}. \end{aligned} \tag{3.103}$$

Thus, by (3.102) and (3.103) we obtain that

$$V_{i_3}(x(n)) \leq \gamma_2 \max_{\theta\in\mathbb{N}_{-d}}\{V_{i_2}(x(n_2 + \theta))\}\lambda_{0_{i_3}}^{(n-n_2)}.$$

Inequality (3.101) gives

$$\max_{\theta\in\mathbb{N}_{-d}}\{V_{i_2}(x(n_2 + \theta))\} = \gamma_1 \max_{\theta\in\mathbb{N}_{-d}}\{V_{i_1}(x(n_0^+ + \theta))\}\lambda_{0_{i_1}}^{(n_1-d-n_0)}\lambda_{0_{i_2}}^{(n_2-d-n_1)}.$$

Thus, we obtain the following inequality for $n \in [n_2^+, n_3)$

$$\Rightarrow V_{i_3}(x(n)) \leq \gamma_1\gamma_2 \max_{\theta\in\mathbb{N}_{-d}}\{V_{i_1}(x(n_0^+ + \theta))\}\lambda_{0_{i_1}}^{(n_1-d-n_0)}\lambda_{0_{i_2}}^{(n_2-d-n_1)}\lambda_{0_{i_3}}^{(n-n_2)}.$$

In general, for $n \in [n_{k-1}^+, n_k)$

$$V_{i_k}(x(n)) \leq \gamma_1\gamma_2\dots\gamma_{k-1} \max_{\theta\in\mathbb{N}_{-d}}\{V_{i_1}(x(n_0^+ + \theta))\}\lambda_{0_{i_1}}^{(n_1-d-n_0)}\lambda_{0_{i_2}}^{(n_2-d-n_1)}\lambda_{0_{i_3}}^{(n_3-d-n_2)}\dots\lambda_{0_{i_k}}^{(n-n_{k-1})}. \tag{3.104}$$

Let $\lambda = \max\{\lambda_{0_{i_j}},\ i \in \mathbb{N},\ j = 1, 2, \dots, k\}$, so inequality (3.104) becomes

$$V_{i_k}(x(n)) \leq \gamma_1\lambda_{0_{i_1}}^{-d}\gamma_2\lambda_{0_{i_2}}^{-d}\dots\gamma_{k-1}\lambda_{0_{i_{k-1}}}^{-d} \max_{\theta\in\mathbb{N}_{-d}}\{V_{i_1}(x(n_0^+ + \theta))\}\lambda^{(n-n_0)}. \tag{3.105}$$

Let $\gamma = \max\{\gamma_i,\ i = 1, 2, \ldots, k-1\}$ and $\tilde{\lambda} = \min\{\lambda_{0_{i_j}},\ i \in \mathbb{N},\ j = 1, 2, \ldots, k-1\}$, so by (3.105) we obtain

$$\begin{aligned} V_{i_k}(x(n)) &\le \gamma^{(k-1)}(\tilde{\lambda}^{-d})^{(k-1)} \max_{\theta \in \mathbb{N}_{-d}}\{V_{i_1}(x(n_0^+ + \theta))\}\lambda^{(n-n_0)} \\ &= \left(\gamma\tilde{\lambda}^{-d}\right)^{(k-1)} \max_{\theta \in \mathbb{N}_{-d}}\{V_{i_1}(x(n_0^+ + \theta))\}\lambda^{(n-n_0)} \\ &= \mu^{(k-1)} \max_{\theta \in \mathbb{N}_{-d}}\{V_{i_1}(x(n_0^+ + \theta))\}\lambda^{(n-n_0)} \end{aligned} \tag{3.106}$$

where $\mu = \gamma\tilde{\lambda}^{-d}$. Since $\gamma > 1$ and $\tilde{\lambda}^{-d} > 1$, $\mu > 1$. Thus, we have

$$\begin{aligned} V_{i_k}(x(n)) &\le \mu^{(k-1)}\lambda^{(n-n_0)} \max_{\theta \in \mathbb{N}_{-d}}\{V_{i_1}(x(n_0^+ + \theta))\} \\ &= \lambda^{(k-1)\frac{\ln\mu}{\ln\lambda}}\lambda^{(n-n_0)} \max_{\theta \in \mathbb{N}_{-d}}\{V_{i_1}(x(n_0^+ + \theta))\} \\ &= \lambda^{(n-n_0)\frac{(k-1)\ln\mu}{(n-n_0)\ln\lambda}}\lambda^{(n-n_0)} \max_{\theta \in \mathbb{N}_{-d}}\{V_{i_1}(x(n_0^+ + \theta))\} \\ &= \lambda^{(n-n_0)\left[\frac{(k-1)\ln\mu}{(n-n_0)\ln\lambda}+1\right]} \max_{\theta \in \mathbb{N}_{-d}}\{V_{i_1}(x(n_0^+ + \theta))\} \end{aligned} \tag{3.107}$$

For simplicity, choose $N_0 = 0$ in assumption (ii). In this case, from ADT condition we obtain

$$N(n_0, n) \le \frac{n - n_0}{T_a} \Rightarrow \frac{N(n_0, n)}{n - n_0} \le \frac{1}{T_a}$$

where $N(n_0, n) = k - 1$.

Thus, the inequality in (3.107) becomes

$$\begin{aligned} V_{i_k}(x(n)) &\le \lambda^{(n-n_0)\left[\frac{N(n_0,n)\ln\mu}{(n-n_0)\ln\lambda}+1\right]} \max_{\theta \in \mathbb{N}_{-d}}\{V_{i_1}(x(n_0^+ + \theta))\} \\ &\le \left(\lambda^{\frac{\ln\mu}{T_a\ln\lambda}+1}\right)^{(n-n_0)} \max_{\theta \in \mathbb{N}_{-d}}\{V_{i_1}(x(n_0^+ + \theta))\} \\ &= \lambda^{\rho(n-n_0)} \max_{\theta \in \mathbb{N}_{-d}}\{V_{i_1}(x(n_0^+ + \theta))\} \end{aligned} \tag{3.108}$$

where $\rho = \frac{\ln\mu}{T_a \ln\lambda} + 1$, λ^ρ is a decay rate and $\frac{\ln\mu}{T_a \ln\lambda} + 1 > 0$ which implies $T_a > -\frac{\ln\mu}{\ln\lambda}$.

Let

$$\tilde{P}_i^{-1} x(n) = \begin{bmatrix} x_1(n) \\ x_2(n) \end{bmatrix}, \text{ and } \tilde{Q}_i^{-T} X_i \tilde{Q}_i^{-1} = \begin{bmatrix} X_{1i} & X_{2i} \\ X_{2i}^T & X_{3i} \end{bmatrix} \tag{3.109}$$

where $x_1(n)$ and $x_2(n)$ are called slow and fast sub-state of system (3.87), respectively. Then, it follows from the standard decomposition form that system (3.87) is equivalent to

$$x_1(n+1) = A_{1_i} x_1(n) + \tilde{Q}_i^1 f_i(n, x(n-d)) \tag{3.110}$$

$$0 = x_2(n) + \tilde{Q}_i^2 f_i(n, x(n-d)) \tag{3.111}$$

where $i = 1, 2, \ldots, M$, $x_1 \in \mathbb{R}^r$, $x_2 \in \mathbb{R}^{N-r}$, $\tilde{Q}_i = \begin{bmatrix} \tilde{Q}_i^1 \\ \tilde{Q}_i^2 \end{bmatrix}$, $\tilde{Q}_i^1 \in \mathbb{R}^{r \times N}$, $\tilde{Q}_i^2 \in \mathbb{R}^{(N-r) \times N}$, $\tilde{P}_i = \begin{bmatrix} \tilde{P}_i^1 & \tilde{P}_i^2 \end{bmatrix}$, $\tilde{P}_i^1 \in \mathbb{R}^{N \times r}$, and $\tilde{P}_i^2 \in \mathbb{R}^{N \times (N-r)}$.

Using the relationship (3.109), Lyapunov function candidate for i^{th} subsystem (3.93) can be rewritten as

$$\begin{aligned} V_i(x(n)) &= x^T(n) E_i^T X_i E_i x(n) \\ &= x^T(n) \tilde{P}_i^{-T} \tilde{P}_i^T E_i^T \tilde{Q}_i^T \tilde{Q}_i^{-T} X_i \tilde{Q}_i^{-1} \tilde{Q}_i E_i \tilde{P}_i \tilde{P}_i^{-1} x(n) \\ &= x_1^T(n) X_{1i} x_1(n) > 0, \quad \forall x_1(n) \neq 0. \end{aligned} \tag{3.112}$$

Then one can obtain from (3.108) by using (3.112) that

$$\begin{aligned} \lambda_{\min}(X_{1i_k}) \|x_1(n)\|^2 &\leq \lambda_{\max}(X_{i_1}) \max_{\theta \in \mathbb{N}_{-d}} \{\|E_i x(n_0^+ + \theta)\|^2\} \lambda^{\rho(n-n_0)} \\ \Rightarrow \|x_1(n)\| &\leq \sqrt{\frac{\lambda_{\max}(X_{i_1})}{\lambda_{\min}(X_{1i_k})}} \max_{\theta \in \mathbb{N}_{-d}} \{\|E_i x(n_0^+ + \theta)\|\} \lambda^{\rho(n-n_0)/2}, \end{aligned} \tag{3.113}$$

which implies that x_1 is exponentially stable.

We need to show that x_2 is also exponentially stable. It follows from the Lipschitz condition in (3.88) and (3.111) that

$$
\begin{aligned}
\|x_2(n)\| &\le \|\tilde{Q}_i^2\|\|F_iE_ix(n-d)\| \\
&= \|\tilde{Q}_i^2\|\|F_iE_i\tilde{P}_i^1x_1(n-d) + F_iE_i\tilde{P}_i^2x_2(n-d)\| \\
&\le \|\tilde{Q}_i^2\|\|F_iE_i\tilde{P}_i^1\|\|x_1(n-d)\| + \|\tilde{Q}_i^2\|\|F_iE_i\tilde{P}_i^2\|\|x_2(n-d)\| \\
&\le \|\tilde{Q}_i^2\|\|F_iE_i\tilde{P}_i^1\| \max_{\theta\in\mathbb{N}_{-d}}\{\|x_1(n+\theta)\|\} + \|\tilde{Q}_i^2\|\|F_iE_i\tilde{P}_i^2\| \max_{\theta\in\mathbb{N}_{-d}}\{\|x_2(n+\theta)\|\}.
\end{aligned}
$$

Then taking maximum of both sides of above inequality gives

$$
\begin{aligned}
\max_{\theta\in\mathbb{N}_{-d}}\{\|x_2(n+\theta)\|\} &\le \|\tilde{Q}_i^2\| \left(\|F_iE_i\tilde{P}_i^1\| \max_{\theta\in\mathbb{N}_{-d}}\{\|x_1(n+\theta)\|\} + \|F_iE_i\tilde{P}_i^2\| \max_{\theta\in\mathbb{N}_{-d}}\{\|x_2(n+\theta)\|\}\right) \\
\max_{\theta\in\mathbb{N}_{-d}}\{\|x_2(n+\theta)\|\} &\le \frac{\|\tilde{Q}_i^2\|\|F_iE_i\tilde{P}_i^1\|}{1-\|\tilde{Q}_i^2\|\|F_iE_i\tilde{P}_i^2\|} \max_{\theta\in\mathbb{N}_{-d}}\{\|x_1(n+\theta)\|\} \\
&\le \frac{\|\tilde{Q}_i^2\|\|F_iE_i\tilde{P}_i^1\|}{1-\|\tilde{Q}_i^2\|\|F_iE_i\tilde{P}_i^2\|} \sqrt{\frac{\lambda_{\max}(X_{i_1})}{\lambda_{\min}(X_{1i_k})}} \max_{\theta\in\mathbb{N}_{-d}}\{\|Ex(n_0^+ + \theta)\|\}\lambda^{\rho(n-d-n_0)/2}
\end{aligned}
\tag{3.114}
$$

where $1 > \|\tilde{Q}_i^2\|\|F_iE_i\tilde{P}_i^2\|$. This shows that x_2 is exponentially stable. Thus, the trivial solution of system (3.87) is exponentially stable. □

For efficiency of the above theoretical result, please look at the following numerical example.

Example 3.2.1. *Consider the discrete ISSSD given by (3.87) where* $x = \begin{bmatrix} x_1(n) & x_2(n) & x_3(n) \end{bmatrix}^T$,

$$
\sigma(n) \in \Xi = \{1,2\},\ E_1 = E_2 = \begin{bmatrix} -2 & -5 & 3 \\ 1 & 1 & 0 \\ 0 & 1 & -1 \end{bmatrix},\ B_k = 0.5I,
$$

$$
A_1 = \begin{bmatrix} -1.6 & -0.5 & -2.1 \\ 0 & -0.1 & 0.1 \\ 0.2 & -0.1 & 0.3 \end{bmatrix}, \quad f_1(n, x(n-1)) = \frac{1}{10}\begin{bmatrix} \tan^{-1}(x_1(n-1)) \\ \tan^{-1}(x_2(n-1)) \\ \tan^{-1}(x_3(n-1)) \end{bmatrix},
$$

$$A_2 = \begin{bmatrix} 4.2 & 5 & 0.2 \\ -1 & -0.8 & -0.2 \\ -0.4 & -0.8 & 0.4 \end{bmatrix} \quad f_2(n, x(n-1)) = \frac{1}{15} \begin{bmatrix} \tanh(x_1(n-1)) \\ \tanh(x_2(n-1)) \\ \tanh(x_3(n-1)) \end{bmatrix}.$$

The initial function is given by $\phi(n) = [1-n \quad -2+n \quad 1+n]^T$. *The Lipschitz condition in (3.88) is satisfied with* $F_1 = \begin{bmatrix} 0.0333 & 0.1667 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ *in stable subsystem 1 and* $F_2 = \begin{bmatrix} 0.0222 & 0.1111 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ *in stable subsystem 2. We calculated that* $\alpha_1 = 0.5270$, $\beta_1 = 0.0018$, $\alpha_2 = 0.6110$ *and* $\beta_2 = 0.0057$. *By Lemma 3.2.1,* $\lambda_{0_1} = 0.4770$ *and* $\lambda_{0_2} = 0.4031$. *Also,* $\gamma_k = 2.25$ *so that the inequality* $(I + B_k)^T E_j^T X_j E_j (I + B_k) \leq \gamma_k E_i^T X_i E_i$ *is satisfied. Thus, the system is exponentially stable under ADT switching with* $T_a > 2.3229$ *seconds. The simulation is shown in Figure 3.5.*

3.2.2 Systems with Stable and Unstable Subsystems

Consider the system in (3.87) with $\Xi = \Xi_u \cup \Xi_s$ where Ξ_u and Ξ_s represent the index sets of unstable and stable subsystems, respectively. Following lemma and Lemma 3.2.1 will be used in the main theorem of this subsection.

Lemma 3.2.2. *Let* $d > 0$ *be a natural number, and* $\{x_n\}_{n \geq -d}$ *be a sequence of positive real numbers satisfying the inequality*

$$\triangle x(n) \leq a x(n) + b \max_{s \in \mathbb{N}_{-d}} \{x(n+s)\}, \quad n \geq n_0, \tag{3.115}$$

where $\triangle x(n) = x(n+1) - x(n)$ *and* $\mathbb{N}_{-d} = \{-d, \ldots, -2, -1, 0\}$.

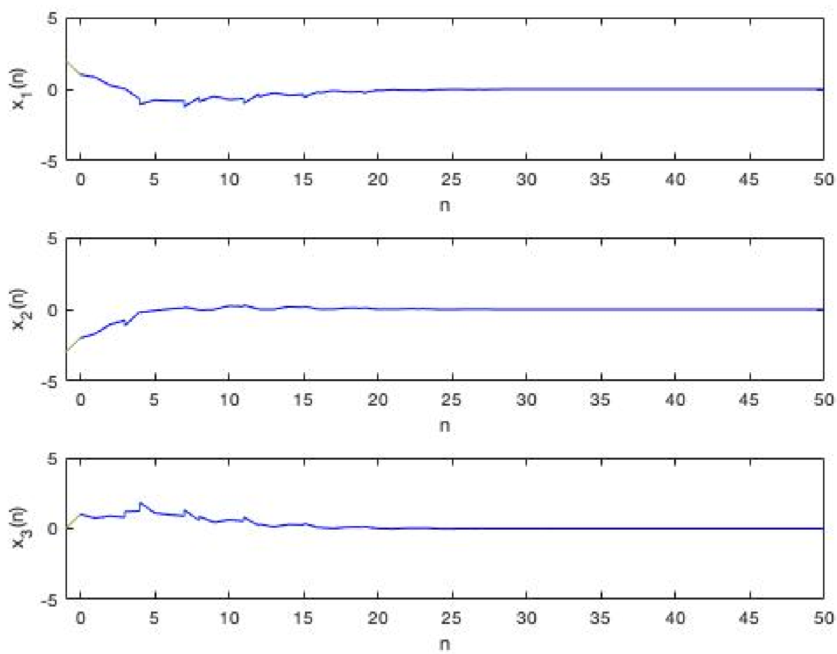

Figure 3.5: State responses of the discrete ISSSD

Assume that $a > 0$ and $b > 0$, then the solution of difference inequality (3.115) is given by

$$x(n) \leq (a+b+1)^{n-n_0} \max_{s\in\mathbb{N}_{-d}}\{x(n_0+s)\}, \quad n \geq n_0. \tag{3.116}$$

Proof. Consider

$$\triangle y(n) = ay(n) + b \max_{s\in\mathbb{N}_{-d}}\{y(n+s)\}, \quad n \geq n_0 \tag{3.117}$$

with the initial condition

$$y(n) = x(n), \quad n \in \{n_0-d, \ldots, n_0-2, n_0-1, n_0\}. \tag{3.118}$$

Since $a > 0$ and $b > 0$ in difference delay equation (3.117), $y(n)$ is increasing. Thus, $\max_{s\in\mathbb{N}_{-d}}\{y(n+s)\} = y(n)$ for all $n \geq n_0$. As a result, equation (3.117) becomes

$$\triangle y(n) = (a+b)y(n), \quad n \geq n_0$$

where $\triangle y(n) = y(n+1) - y(n)$.
Thus, the solution of above delay difference equation is

$$y(n) = (a+b+1)^{n-n_0} \max_{s\in\mathbb{N}_{-d}}\{y(n_0+s)\}, \quad n \geq n_0$$

Claim that $x(n) \leq y(n)$ for all $n \geq n_0$. If it was not true, there would exist n^* such that $x(n^*+1) > y(n^*+1)$ and $x(n) \leq y(n)$ for all $n_0 \leq n \leq n^*$. Thus,

$$\begin{aligned} \triangle y(n^*) &= y(n^*+1) - y(n^*) \\ &< x(n^*+1) - x(n^*) \\ &= \triangle x(n^*) \\ &\leq ax(n^*) + b \max_{s\in\mathbb{N}_{-d}}\{x(n^*+s)\} \\ &\leq ay(n^*) + b \max_{s\in\mathbb{N}_{-d}}\{y(n^*+s)\}. \end{aligned}$$

That is,

$$\triangle y(n^*) < ay(n^*) + b \max_{s\in\mathbb{N}_{-d}}\{y(n^*+s)\}$$

which contradicts with equation (3.117). That means the claim $x(n) \leq y(n)$ for all $n \geq n_0$ is correct. In other words, we obtain that

$$x(n) \leq (a+b+1)^{n-n_0} \max_{s\in\mathbb{N}_{-d}}\{y(n_0+s)\}, \quad n \geq n_0.$$

By using the initial condition (3.118), we conclude the solution of delay difference equation (3.115) as

$$\begin{aligned} x(n) &\leq (a+b+1)^{n-n_0} \max_{s\in\mathbb{N}_{-d}}\{y(n_0+s)\} \\ &= (a+b+1)^{n-n_0} \max_{s\in\mathbb{N}_{-d}}\{x(n_0+s)\} \quad \text{ for all } n \geq n_0. \end{aligned}$$

□

Theorem 3.2.2. *For any $i \in \Xi = \Xi_u \cup \Xi_s$, assume that each subsystem of (3.87) is impulse free. Then, the trivial solution of (3.87) is exponentially stable if the following assumptions hold:*

(A1) For any $i, j \in \Xi$ there exists $\gamma_k > 1$ such that

$$(I+B_k)^T E_j^T X_j E_j (I+B_k) \leq \gamma_k E_i^T X_i E_i, \tag{3.119}$$

where X_i is positive definite matrix satisfying the Lyapunov equation

$$A_i^T X_i A_i - E_i^T X_i E_i = -Y_i$$

for any $Y_i > 0$.

(A2) Let

$$\lambda_+ = \max\{\lambda^*_{0_{i_j}} : \ j = 1, 2, \ldots, l\}$$

$$\lambda_- = \max\{\lambda_{0_{i_p}} : \ p = l+1, l+2, \ldots, k\}$$

where $\lambda^*_{0_{i_j}} = \alpha^*_i + \beta^*_i + 1$ *with* α^*_i, β^*_i *are positive numbers defined later in the proof, and* $\lambda_{0_{i_p}}$ *is the smallest root in the interval* $(0, 1)$ *of the equation*

$$\lambda^{d+1} + (\alpha_i - 1)\lambda^d - \beta_i = 0.$$

with α_i, β_i *are positive numbers defined later in the proof as well. Let also* $T^+(n_0, n)$ *be the total activation time of unstable modes,* $T^-(n_0, n)$ *be the total activation time of stable modes, and for any* n_0*, assume that the switching law guarantees that*

$$\frac{T^-(n_0, n)}{T^+(n_0, n)} > \frac{\ln \lambda_+ - \ln \lambda_*}{\ln \lambda_* - \ln \lambda_-} \tag{3.120}$$

where $0 < \lambda_- < \lambda_* < 1$*. Furthermore, for any* $n \geq n_0$*, the switching law holds the ADT condition, which is*

$$N(n_0, n) \leq N_0 + \frac{n - n_0}{T_a}$$

where N *denotes the number of switchings in* (n_0, n)*,* T_a *is the average dwell time and* N_0 *is the chatter bound.*

Proof. Let $x\,(n) = x(n; n_0, \phi)$ be the solution of the system (3.87). For any $i \in \Xi$ and $n \in [n_{k-1}, n_k)$, define

$$V_i(x(n)) = x^T(n) E_i^T X_i E_i x(n), \quad i = \sigma(n) \tag{3.121}$$

as a Lyapunov function candidate for i^{th} subsystem. The variation of V_i is defined as

$$\triangle V_i(x(n)) = V_i(x(n+1)) - V_i(x(n)).$$

Thus, the variation of V_i relative to system (3.87) is

$$\begin{aligned}
\triangle V_i(x(n)) &= x^T(n+1)E_i^T X_i E_i x(n+1) - x^T(n)E_i^T X_i E_i x(n) \\
&= \big[x^T(n)A_i^T + f_i^T(n, x(n-d))\big] X_i \big[A_i x(n) + f_i(n, x(n-d))\big] - x^T(n)E_i^T X_i E_i x(n) \\
&= x^T(n)A_i^T X_i A_i x(n) + x^T(n)A_i^T X_i f_i(n, x(n-d)) + f_i^T(n, x(n-d)) X_i A_i x(n) \\
&\quad + f_i^T(n, x(n-d)) X_i f_i(n, x(n-d)) - x^T(n)E_i^T X_i E_i x(n) \\
&= x^T(n)\big[A_i^T X_i A_i - E_i^T X_i E_i\big] x(n) + 2 f_i^T(n, x(n-d)) X_i A_i x(n) \\
&\quad + f_i^T(n, x(n-d)) X_i f_i(n, x(n-d)) \qquad (3.122)
\end{aligned}$$

For $i \in \Xi_S$, we obtain

$$\triangle V_i(x(n)) \leq -x^T(n)Y_i x(n) + 2 f_i^T(n, x(n-d)) X_i A_i x(n) + f_i^T(n, x(n-d)) X_i f_i(n, x(n-d))$$

where $A_i^T X_i A_i - E_i^T X_i E_i = -Y_i$ for any $Y_i > 0$.

Using the Lipschitz condition in (3.88) and Lemma 3.1.2, for any $\varepsilon_i > 0$ we obtain that

$$\begin{aligned}
2 f_i^T(n, x(n-d)) X_i A_i x(n) &\leq \frac{1}{\varepsilon_i} f_i^T(n, x(n-d)) f_i(n, x(n-d)) + \varepsilon_i x^T(n) A_i^T X_i^2 A_i x(n) \\
&\leq \frac{1}{\varepsilon_i}\|f_i(n, x(n-d))\|^2 + \varepsilon_i \|A_i x(n)\|^2 \lambda_{\max}(X_i^2) \\
&\leq \frac{1}{\varepsilon_i}\|F_i E_i x(n-d)\|^2 + \varepsilon_i \|A_i x(n)\|^2 \lambda_{\max}(X_i^2) \\
&\leq \frac{1}{\varepsilon_i}\|F_i\|^2 \|E_i x(n-d)\|^2 + \varepsilon_i \|A_i\|^2 \|x(n)\|^2 \lambda_{\max}(X_i^2) \quad (3.123)
\end{aligned}$$

and

$$\begin{aligned}
f_i^T(n, x(n-d)) X_i f_i(n, x(n-d)) &\leq \lambda_{\max}(X_i)\|f_i(n, x(n-d))\|^2 \\
&\leq \lambda_{\max}(X_i)\|F_i E_i x(n-d)\|^2 \\
&\leq \lambda_{\max}(X_i)\|F_i\|^2 \|E_i x(n-d)\|^2. \qquad (3.124)
\end{aligned}$$

Thus, we obtain

$$\begin{aligned}
\triangle V_i(x(n)) &\le -x^T(n)Y_ix(n) + \left[\frac{1}{\varepsilon_i} + \lambda_{\max}(X_i)\right]\|F_i\|^2\|E_ix(n-d)\|^2 + \varepsilon_i\|A_i\|^2\|x(n)\|^2\lambda_{\max}(X_i^2) \\
&\le -\lambda_{\min}(Y_i)\|x(n)\|^2 + \varepsilon_i\|A_i\|^2\|x(n)\|^2\lambda_{\max}(X_i^2) + \left[\frac{1}{\varepsilon_i} + \lambda_{\max}(X_i)\right]\|F_i\|^2\|E_ix(n-d)\|^2 \\
&= \Big[-\lambda_{\min}(Y_i) + \varepsilon_i\|A_i\|^2\lambda_{\max}(X_i^2)\Big]\|x(n)\|^2 + \left[\frac{1}{\varepsilon_i} + \lambda_{\max}(X_i)\right]\|F_i\|^2\|E_ix_n\|_d^2 \\
&\le -\alpha_iV_i(x(n)) + \beta_i \max_{s\in\mathbb{N}_{-d}} V_i(x(n+s)) \qquad (3.125)
\end{aligned}$$

where $\alpha_i = \frac{\lambda_{\min}(Y_i)-\varepsilon_i\|A_i\|^2\lambda_{\max}(X_i^2)}{\lambda_{\max}(E_i^TX_iE_i)} > 0$ with $P>0$, and $\beta_i = \frac{[1+\varepsilon_i\lambda_{\max}(X_i)]\|F_i\|^2}{\varepsilon_i\lambda_{\min}(X_i)} > 0$.
By Lemma 3.2.1 we obtain the solution of (3.125) for $n \in [n_{k-1}^+, n_k)$ as

$$V_i(x(n)) \le \max_{s\in\mathbb{N}_{-d}}\{V_i(x(n_{k-1}^+ + s))\}\lambda_{0_i}^{(n-n_{k-1})}, \qquad (3.126)$$

where λ_{0_i} is the smallest root in the interval $(0,1)$ of the equation

$$\lambda^{d+1} + (\alpha_i - 1)\lambda^d - \beta_i = 0.$$

Let δ_i $(i \in \Xi_u)$ be a positive constant such that all eigenvalues of the matrix pairs $(E_i + \delta_iE_i, A_i)$ are located in the unit circle. Then, for any $Y_i > 0$, there exists $X_i > 0$ satisfying

$$A_i^TX_iA_i - (E_i+\delta_iE_i)^TX_i(E_i+\delta_iE_i) = -Y_i. \qquad (3.127)$$

Plugging (3.127) into (3.122) we obtain

$$\begin{aligned}
\triangle V_i(x(n)) \le\ & x^T(n)\left[-Y_i + \delta_i^2E_i^TX_iE_i + 2\delta_iE_i^TX_iE_i\right]x(n) \\
& + 2f_i^T(n,x(n-d))X_iA_ix(n) + f_i^T(n,x(n-d))X_if_i(n,x(n-d)). \qquad (3.128)
\end{aligned}$$

By (3.123)-(3.124), inequality (3.128) becomes

$$
\begin{aligned}
\triangle V_i(x(n)) \leq\; & x^T(n)\left[-Y_i + \delta_i^2 E_i^T X_i E_i + 2\delta_i E_i^T X_i E_i\right] x(n) \\
& + \frac{1}{\varepsilon_i}\|F_i\|^2\|E_i x(n-d)\|^2 + \varepsilon_i\|A_i\|^2\|x(n)\|^2\lambda_{\max}(X_i^2) + \lambda_{\max}(X_i)\|F_i\|^2\|E_i x(n-d)\|^2 \\
\leq\; & \Big[-\lambda_{\min}(Y_i) + \varepsilon_i\|A_i\|^2\lambda_{\max}(X_i^2)\Big]\|x(n)\|^2 + (\delta_i^2 + 2\delta_i)\lambda_{\max}(X_i)\|E_i x(n)\|^2 \\
& + \left[\frac{1}{\varepsilon_i} + \lambda_{\max}(X_i)\right]\|F_i\|^2\|E_i x_n\|_d^2 \\
\leq\; & \left[-\frac{\lambda_{\min}(Y_i)}{\|E_i\|^2} + \frac{\varepsilon_i\|A_i\|^2\lambda_{\max}(X_i^2)}{\|E_i\|^2} + (\delta_i^2 + 2\delta_i)\lambda_{\max}(X_i)\right]\|E_i x(n)\|^2 \\
& + \left[\frac{1}{\varepsilon_i} + \lambda_{\max}(X_i)\right]\|F_i\|^2\|E_i x_n\|_d^2 \\
\leq\; & \alpha_i^* V_i(x(n)) + \beta_i^* \max_{s\in\mathbb{N}_{-d}} V_i(x(n+s))
\end{aligned} \tag{3.129}
$$

where $\alpha_i^* = \frac{-\lambda_{\min}(Y_i)+\varepsilon_i\|A_i\|^2\lambda_{\max}(X_i^2)+(\delta_i^2+2\delta_i)\lambda_{\max}(X_i)\|E_i\|^2}{\|E_i\|^2\lambda_{\min}(X_i)} > 0$, $\beta_i^* = \frac{[1+\varepsilon_i\lambda_{\max}(X_i)]\|F_i\|^2}{\varepsilon_i\lambda_{\min}(X_i)} > 0$, and $\varepsilon_i > 0$ such that $-\lambda_{\min}(Y_i) + \varepsilon_i\|A_i\|^2\lambda_{\max}(X_i^2) < 0$.

By Lemma 3.2.2 the solution of (3.129) is obtained for $n \in [n_{k-1}^+, n_k)$ as

$$
V_i(x(n)) \leq \max_{s\in\mathbb{N}_{-d}}\{V_i(x(n_{k-1}^+ + s))\}\lambda_{0_i}^{*(n-n_{k-1})}, \tag{3.130}
$$

where $\lambda_{0_i}^* = \alpha_i^* + \beta_i^* + 1$.

On the other hand, for $n = n_k$, $k = 1, 2, 3, \ldots$, suppose $\sigma(n_k) = j$, it follows from (3.87) and (3.119) that

$$
V_j(x(n_k^+)) \leq \gamma_k V_i(x(n_k)). \tag{3.131}
$$

Using (3.126) and (3.131) successively on each subinterval leads to the following results. For instance, for $n \in [n_0^+, n_1)$, we have

$$
V_{i_1}(x(n)) \leq \max_{s\in\mathbb{N}_{-d}}\{V_{i_1}(x(n_0^+ + s))\}\lambda_{0_{i_1}}^{(n-n_0)} \tag{3.132}
$$

For $n \in [n_1^+, n_2)$, we have

$$V_{i_2}(x(n)) \leq \max_{s\in\mathbb{N}_{-d}}\{V_{i_2}(x(n_1^+ + s))\}\lambda_{0_{i_2}}^{(n-n_1)} \tag{3.133}$$

From (3.131), we obtain

$$\begin{aligned} &\max_{s\in\mathbb{N}_{-d}}\{V_{i_2}(x(n_1^+ + s))\} \leq \gamma_1 \max_{s\in\mathbb{N}_{-d}}\{V_{i_1}(x(n_1 + s))\} \\ \Rightarrow &\max_{s\in\mathbb{N}_{-d}}\{V_{i_2}(x(n_1^+ + s))\}\lambda_{0_{i_2}}^{(n-n_1)} \leq \gamma_1 \max_{s\in\mathbb{N}_{-d}}\{V_{i_1}(x(n_1 + s))\}\lambda_{0_{i_2}}^{(n-n_1)} \end{aligned} \tag{3.134}$$

Thus, by (3.133) and (3.134), we obtain

$$\begin{aligned} V_{i_2}(x(n)) \leq \max_{s\in\mathbb{N}_{-d}}\{V_{i_2}(x(n_1^+ + s))\}\lambda_{0_{i_2}}^{(n-n_1)} &\leq \gamma_1 \max_{s\in\mathbb{N}_{-d}}\{V_{i_1}(x(n_1 + s))\}\lambda_{0_{i_2}}^{(n-n_1)} \\ \Rightarrow V_{i_2}(x(n)) &\leq \gamma_1 \max_{s\in\mathbb{N}_{-d}}\{V_{i_1}(x(n_1 + s))\}\lambda_{0_{i_2}}^{(n-n_1)} \end{aligned} \tag{3.135}$$

We suppose that $n_k - d \in [n_{k-1}, n_k)$. Then, we can find $\max_{s\in\mathbb{N}_{-d}}\{V_{i_1}(x(n_1 + s))\}$ by (3.131) as

$$\max_{s\in\mathbb{N}_{-d}}\{V_{i_1}(x(n_1 + s))\} = \max_{s\in\mathbb{N}_{-d}}\{V_{i_1}(x(n_0^+ + s))\}\lambda_{0_{i_1}}^{(n_1-d-n_0)},$$

so inequality (3.135) becomes

$$V_{i_2}(x(n)) \leq \gamma_1 \max_{s\in\mathbb{N}_{-d}}\{V_{i_1}(x(n_0^+ + s))\}\lambda_{0_{i_1}}^{(n_1-d-n_0)}\lambda_{0_{i_2}}^{(n-n_1)} \qquad n \in [n_1^+, n_2). \tag{3.136}$$

Similarly, for $n \in [n_2^+, n_3)$ we can obtain the following inequality

$$V_{i_3}(x(n)) \leq \gamma_1\gamma_2 \max_{\theta\in\mathbb{N}_{-d}}\{V_{i_1}(x(n_0^+ + \theta))\}\lambda_{0_{i_1}}^{(n_1-d-n_0)}\lambda_{0_{i_2}}^{(n_2-d-n_1)}\lambda_{0_{i_3}}^{(n-n_2)}. \tag{3.137}$$

In general, for $n \in [n_{k-1}^+, n_k)$

$$V_{i_k}(x(n)) \leq \gamma_1\gamma_2\ldots\gamma_{k-1} \max_{\theta\in\mathbb{N}_{-d}}\{V_{i_1}(x(n_0^+ + \theta))\}\lambda_{0_{i_1}}^{(n_1-d-n_0)}\lambda_{0_{i_2}}^{(n_2-d-n_1)}\lambda_{0_{i_3}}^{(n_3-d-n_2)}\ldots\lambda_{0_{i_k}}^{(n-n_{k-1})}. \tag{3.138}$$

Now, use (3.130) and (3.131) successively on each subinterval. For $n \in [n_0^+, n_1)$, we have

$$V_{i_1}(x(n)) \leq \max_{s \in \mathbb{N}_{-d}} \{V_{i_1}(x(n_0^+ + s))\} \lambda_{0_{i_1}}^{*(n-n_0)} \tag{3.139}$$

For $n \in [n_1^+, n_2)$, we have

$$V_{i_2}(x(n)) \leq \max_{s \in \mathbb{N}_{-d}} \{V_{i_2}(x(n_1^+ + s))\} \lambda_{0_{i_2}}^{*(n-n_1)} \tag{3.140}$$

From (3.131), we obtain

$$\max_{s \in \mathbb{N}_{-d}} \{V_{i_2}(x(n_1^+ + s))\} \leq \gamma_1 \max_{s \in \mathbb{N}_{-d}} \{V_{i_1}(x(n_1 + s))\}$$

$$\Rightarrow \max_{s \in \mathbb{N}_{-d}} \{V_{i_2}(x(n_1^+ + s))\} \lambda_{0_{i_2}}^{*(n-n_1)} \leq \gamma_1 \max_{s \in \mathbb{N}_{-d}} \{V_{i_1}(x(n_1 + s))\} \lambda_{0_{i_2}}^{*(n-n_1)} \tag{3.141}$$

Thus, by (3.140) and (3.141), we obtain

$$V_{i_2}(x(n)) \leq \max_{s \in \mathbb{N}_{-d}} \{V_{i_2}(x(n_1^+ + s))\} \lambda_{0_{i_2}}^{*(n-n_1)} \leq \gamma_1 \max_{s \in \mathbb{N}_{-d}} \{V_{i_1}(x(n_1 + s))\} \lambda_{0_{i_2}}^{*(n-n_1)}$$

$$\Rightarrow V_{i_2}(x(n)) \leq \gamma_1 \max_{s \in \mathbb{N}_{-d}} \{V_{i_1}(x(n_1 + s))\} \lambda_{0_{i_2}}^{*(n-n_1)} \tag{3.142}$$

We can find $\max_{s \in \mathbb{N}_{-d}} \{V_{i_1}(x(n_1 + s))\}$ by (3.139) as

$$\max_{s \in \mathbb{N}_{-d}} \{V_{i_1}(x(n_1 + s))\} = \max_{s \in \mathbb{N}_{-d}} \{V_{i_1}(x(n_0^+ + s))\} \lambda_{0_{i_1}}^{*(n_1-n_0)},$$

so inequality (3.142) becomes

$$V_{i_2}(x(n)) \leq \gamma_1 \max_{s \in \mathbb{N}_{-d}} \{V_{i_1}(x(n_0^+ + s))\} \lambda_{0_{i_1}}^{*(n_1-n_0)} \lambda_{0_{i_2}}^{*(n-n_1)} \qquad n \in [n_1^+, n_2). \tag{3.143}$$

Similarly, for $n \in [n_2^+, n_3)$ we can obtain the following inequality

$$V_{i_3}(x(n)) \leq \gamma_1 \gamma_2 \max_{s \in \mathbb{N}_{-d}} \{V_{i_1}(x(n_0^+ + s))\} \lambda_{0_{i_1}}^{*(n_1-n_0)} \lambda_{0_{i_2}}^{*(n_2-n_1)} \lambda_{0_{i_3}}^{*(n-n_2)}. \tag{3.144}$$

In general, for $n \in [n_{k-1}^+, n_k)$

$$V_{i_k}(x(n)) \leq \gamma_1 \gamma_2 \dots \gamma_{k-1} \max_{s \in \mathbb{N}_{-d}} \{V_{i_1}(x(n_0^+ + s))\} \lambda_{0_{i_1}}^{*(n_1-n_0)} \lambda_{0_{i_2}}^{*(n_2-n_1)} \lambda_{0_{i_3}}^{*(n_3-n_2)} \dots \lambda_{0_{i_k}}^{*(n-n_{k-1})}. \tag{3.145}$$

To obtain a general estimate, let us run l unstable modes and switch l times from an unstable mode, and run $m-l$ stable modes and switch $m-l-1$ times from a stable mode. Then, for $n \in [n_{k-1}, n_k)$

$$V_{m_k}(n) \leq \prod_{j=1}^{l} \gamma_j \lambda_{0_{m_j}}^{*(n_j-n_{j-1})} \times \prod_{p=l+1}^{m-1} \gamma_p \lambda_{0_{m_p}}^{-d} \lambda_{0_{m_p}}^{(n_p-n_{p-1})} \times \max_{s\in\mathbb{N}_{-d}} \{V_{m_1}(x(n_0^+ + s))\} \lambda_{0_{m_k}}^{(n-n_{m-1})}. \tag{3.146}$$

Let

$$\lambda_+ = \max\{\lambda_{0_{i_j}}^* \ : \ j = 1, 2, \dots, l\}$$
$$\lambda_- = \max\{\lambda_{0_{i_p}} \ : \ p = l+1, l+2, \dots, k\}$$

and denote by $T^+(n_0, n)$ and $T^-(n_0, n)$ the total activation time of unstable and stable modes, respectively. Then, for $n \in [n_{k-1}, n_k)$, we have

$$V_{m_k}(n) \leq \prod_{j=1}^{l} \gamma_j \lambda_+^{T^+} \times \prod_{p=l+1}^{m-1} \gamma_p \lambda_{0_{m_p}}^{-d} \lambda_-^{T^-} \times \max_{s\in\mathbb{N}_{-d}} \{V_{m_1}(x(n_0^+ + s))\}. \tag{3.147}$$

Choose λ_* such that $0 < \lambda_- < \lambda_* < 1$, and assume that the switching law satisfies (3.120) where this condition implies that for any $n \geq n_0$

$$T^+ (\ln \lambda_+ - \ln \lambda_*) < T^- (\ln \lambda_* - \ln \lambda_-)$$
$$T^+ \ln \left(\frac{\lambda_+}{\lambda_*}\right) < T^- \ln \left(\frac{\lambda_*}{\lambda_-}\right)$$
$$\ln \left(\frac{\lambda_+}{\lambda_*}\right)^{T^+} < \ln \left(\frac{\lambda_*}{\lambda_-}\right)^{T^-}$$
$$\ln \left(\frac{\lambda_+}{\lambda_*}\right)^{T^+} - \ln \left(\frac{\lambda_*}{\lambda_-}\right)^{T^-} < 0$$

$$\ln\left(\frac{\lambda_+^{T^+}\lambda_-^{T^-}}{\lambda_*^{(T^++T^-)}}\right) < 0$$
$$\frac{\lambda_+^{T^+}\lambda_-^{T^-}}{\lambda_*^{(T^++T^-)}} < 1$$
$$\lambda_+^{T^+}\lambda_-^{T^-} < \lambda_*^{(n-n_0)}.$$

Thus, we obtain

$$V_{m_k}(n) \leq \prod_{j=1}^{l}\gamma_j \times \prod_{p=l+1}^{m-1}\gamma_p\lambda_{0_{m_p}}^{-d} \times \max_{s\in\mathbb{N}_{-d}}\{V_{m_1}(x(n_0^+ + s))\}\lambda_*^{(n-n_0)}. \tag{3.148}$$

From (3.148) one can obtain

$$\begin{aligned} V_{m_k}(n) &\leq \prod_{p=1}^{m-1}\gamma_p\lambda_{0_{m_p}}^{-d} \times \max_{s\in\mathbb{N}_{-d}}\{V_{m_1}(x(n_0^+ + s))\}\lambda_*^{(n-n_0)} \\ &\leq \prod_{p=1}^{m-1}\gamma\tilde{\lambda}^{-d} \times \max_{s\in\mathbb{N}_{-d}}\{V_{m_1}(x(n_0^+ + s))\}\lambda_*^{(n-n_0)} \end{aligned} \tag{3.149}$$

where $\gamma = \max\{\gamma_p,\ p = 1, 2, \ldots, m-1\}$ and $\tilde{\lambda} = \min\{\lambda_{0_{m_p}},\ m \in \mathbb{N},\ p = 1, 2, \ldots, m-1\}$, so we obtain

$$\begin{aligned} V_{m_k}(x(n)) &\leq \gamma^{(m-1)}(\tilde{\lambda}^{-d})^{(m-1)} \max_{\theta\in\mathbb{N}_{-d}}\{V_{m_1}(x(n_0^+ + \theta))\}\lambda_*^{(n-n_0)} \\ &= \left(\gamma\tilde{\lambda}^{-d}\right)^{(m-1)} \max_{\theta\in\mathbb{N}_{-d}}\{V_{m_1}(x(n_0^+ + \theta))\}\lambda_*^{(n-n_0)} \\ &= \mu^{(m-1)} \max_{\theta\in\mathbb{N}_{-d}}\{V_{m_1}(x(n_0^+ + \theta))\}\lambda_*^{(n-n_0)} \end{aligned} \tag{3.150}$$

where $\mu = \gamma\tilde{\lambda}^{-d}$. Thus, we have

$$\begin{aligned} V_{m_k}(x(n)) &\leq \mu^{(m-1)}\lambda_*^{(n-n_0)} \max_{\theta\in\mathbb{N}_{-d}}\{V_{m_1}(x(n_0^+ + \theta))\} \\ &= \lambda_*^{(m-1)\frac{\ln\mu}{\ln\lambda_*}}\lambda_*^{(n-n_0)} \max_{\theta\in\mathbb{N}_{-d}}\{V_{m_1}(x(n_0^+ + \theta))\} \\ &= \lambda_*^{(n-n_0)\frac{(m-1)\ln\mu}{(n-n_0)\ln\lambda_*}}\lambda_*^{(n-n_0)} \max_{\theta\in\mathbb{N}_{-d}}\{V_{m_1}(x(n_0^+ + \theta))\} \\ &= \lambda_*^{(n-n_0)\left[\frac{(m-1)\ln\mu}{(n-n_0)\ln\lambda_*}+1\right]} \max_{\theta\in\mathbb{N}_{-d}}\{V_{m_1}(x(n_0^+ + \theta))\}. \end{aligned} \tag{3.151}$$

For simplicity, choose the chatter bound $N_0 = 0$. In this case, we obtain

$$N(n_0, n) \leq \frac{n-n_0}{T_a} \Rightarrow \frac{N(n_0,n)}{n-n_0} \leq \frac{1}{T_a}$$

where $N(n_0, n) = m - 1$. Therefore, we can write down inequality (3.151) as

$$\begin{aligned} V_{m_k}(x(n)) &\leq \lambda_*^{(n-n_0)\left[\frac{N(n_0,n)\ln\mu}{(n-n_0)\ln\lambda_*}+1\right]} \max_{\theta\in\mathbb{N}_{-d}}\{V_{m_1}(x(n_0^+ + \theta))\} \\ &\leq \left(\lambda_*^{\frac{\ln\mu}{T_a\ln\lambda_*}+1}\right)^{(n-n_0)} \max_{\theta\in\mathbb{N}_{-d}}\{V_{m_1}(x(n_0^+ + \theta))\} \\ &= \lambda_*^{\rho(n-n_0)} \max_{\theta\in\mathbb{N}_{-d}}\{V_{m_1}(x(n_0^+ + \theta))\} \end{aligned} \tag{3.152}$$

where $\rho = \frac{\ln\mu}{T_a\ln\lambda_*} + 1$, λ^ρ is a decay rate and $T_a > -\frac{\ln\mu}{\ln\lambda_*}$.

By using decomposition of the system (3.87) similar to Theorem 3.2.1, we can similarly obtain following inequalities which show the sub-states x_1 and x_2 are exponentially stable

$$\|x_1(n)\| \leq \sqrt{\frac{\lambda_{\max}(X_{m_1})}{\lambda_{\min}(X_{1m_k})}} \max_{\theta\in\mathbb{N}_{-d}}\{\|E_m x(n_0^+ + \theta)\|\}\lambda_*^{\rho(n-n_0)/2}, \tag{3.153}$$

$$\|x_2(n)\| \leq \frac{\|\tilde{Q}_m^2\|\|F_m\tilde{P}_m^1\|}{1-\|\tilde{Q}_m^2\|\|F_m\tilde{P}_m^2\|}\sqrt{\frac{\lambda_{\max}(X_{m_1})}{\lambda_{\min}(X_{1m_k})}} \max_{\theta\in\mathbb{N}_{-d}}\{\|E_m x(n_0^+ + \theta)\|\}\lambda_*^{\rho(n-d-n_0)/2}. \tag{3.154}$$

where $1 > \|\tilde{Q}_m^2\|\|F_m\tilde{P}_m^2\|$. This implies that the entire system is exponentially stable. □

Example 3.2.2. *Consider the discrete ISSSD given by (3.87) where* $x = \begin{bmatrix} x_1(n) & x_2(n) & x_3(n) \end{bmatrix}^T$, $\sigma(n) \in \Xi = \{1, 2\}$, $E_1 = E_2 = \begin{bmatrix} -2 & -5 & 3 \\ 1 & 1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$, $B_k = 0.9I$,

$$A_1 = \begin{bmatrix} -1.6 & -0.5 & -2.1 \\ 0 & -0.1 & 0.1 \\ 0.2 & -0.1 & 0.3 \end{bmatrix}, \quad f_1(n, x(n-1)) = \frac{1}{10} \begin{bmatrix} \tan^{-1}(x_1(n-1)) \\ \tan^{-1}(x_2(n-1)) \\ \tan^{-1}(x_3(n-1)) \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 4.2 & 5 & 1.2 \\ -1 & -1.8 & -0.2 \\ -0.4 & -1.5 & 1.4 \end{bmatrix} \quad f_2(n, x(n-1)) = \frac{1}{15} \begin{bmatrix} \tanh(x_1(n-1)) \\ \tanh(x_2(n-1)) \\ \tanh(x_3(n-1)) \end{bmatrix}.$$

The initial function is given by $\phi(n) = [1-n \quad -2+n \quad 1+n]^T$. *The Lipschitz condition in (3.88) is satisfied with* $F_1 = \begin{bmatrix} 0.0333 & 0.1667 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ *in stable subsystem 1 and* $F_2 = \begin{bmatrix} 0.0222 & 0.1111 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ *in stable subsystem 2. We calculated that* $\alpha_1 = 0.5270$, $\beta_1 = 0.0018$, $\alpha_2^* = 0.2199$ *and* $\beta_2^* = 2.1190$. *By Lemma 3.2.1 and Lemma 3.2.2,* $\lambda_{0_1} = 0.4770$ *and* $\lambda_{0_2}^* = 3.3389$, *respectively. Also,* $\gamma_k = 3.61$ *so that the inequality* $(I + B_k)^T E_j^T X_j E_j (I + B_k) \leq \gamma_k E_i^T X_i E_i$ *is satisfied. Thus, the system is exponentially stable under ADT switching with* $T_a > 2.7245$ *seconds. The simulation of the state responses is shown in Figure 3.6.*

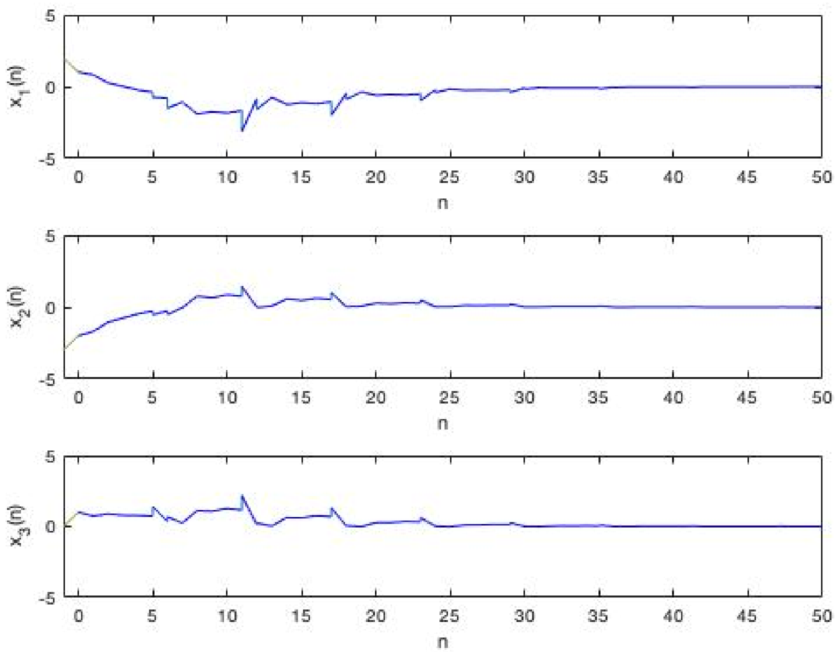

Figure 3.6: State responses of the discrete ISSSD

Chapter 4

Optimal Control and State Estimation for ISSSD

In this chapter, some sufficient conditions on the exponential stability property of the optimal closed-loop system have been firstly proposed. Secondly, an optimal feedback control for the system has been designed to guarantee the exponential stability of the closed-loop system. Moreover, a Luenberger-type observer is designed to estimate the system states such that the corresponding closed-loop error system is exponentially stable. The stability results have been investigated by using the multiple Lyapunov functions along with the average-dwell time (ADT) switching signal to organize the jumps among the system modes. Also, the Halanay lemma that is already introduced in Lemma 3.1.1 is used as a methodology. All theoretical results are illustrated with numerical results.

4.1 Preliminaries on Controllability and Observability

Consider the regular singular system

$$E\dot{x}(t) = Ax(t) + Bu(t) \tag{4.1}$$
$$y(t) = Cx(t)$$

where $x \in \mathbb{R}^n,\ u \in \mathbb{R}^p,\ y \in \mathbb{R}^q$ are the system state, control input and output, respectively; and $A, E \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times p}$ and $C \in \mathbb{R}^{q\times n}$ are system coefficient matrices where E being singular with $\text{rank}(E) = r < n$.

As pointed out in the Chapter 2, two nonsingular matrices $\tilde{Q}$ and $\tilde{P}$ exist such that the system in (4.1) is equivalent to

$$\dot{x}_1(t) = A_1x_1(t) + B_1u(t) \tag{4.2a}$$
$$y_1 = C_1x_1(t)$$
$$N\dot{x}_2(t) = x_2(t) + B_2u(t) \tag{4.2b}$$
$$y_2 = C_2x_2(t)$$

and the original system output is given by

$$y(t) = C_1x_1(t) + C_2x_2(t)$$

where $x_1 \in \mathbb{R}^{n_1}$ and $x_2 \in \mathbb{R}^{n_2}$ are slow and fast sub-state, respectively; $N \in \mathbb{R}^{n_2\times n_2}$ is nilpotent matrix with the index h.

Following two theorems state the definition of controllability and observability in singular system (4.1), respectively.

Theorem 4.1.1. *[8],[59]* ***(Controllability)***

(1) *Slow subsystem (4.2a) is controllable if and only if*

$$rank[sE - A,\ B] = n, \quad \forall s \in \mathbb{C},\ s \text{ finite}$$

(2) *The following statements are equivalent.*

(a) *Fast subsystem (4.2b) is controllable.*

(b) $rank[B_2,\ NB_2,\ N^2B_2,\ \ldots,\ N^{h-1}B_2] = n_2$.

(c) $rank[N \quad B_2] = n_2$.

(d) $rank[E \quad B] = n$.

(3) *The following statements are equivalent.*

(a) *System (4.1) is controllable.*

(b) *Both its slow and fast subsystems are controllable.*

(c) $rank[B_1,\ A_1B_1,\ A_1^2B_1,\ \ldots,\ A_1^{r-1}B_1] = n_1$ *and* $rank[B_2,\ NB_2,\ N^2B_2,\ \ldots,\ N^{h-1}B_2] = n_2$.

(d) $rank[sE - A,\ B] = n, \quad \forall s \in \mathbb{C},\ s$ *finite, and* $rank[E \quad B] = n$.

Theorem 4.1.2. *[8],[59]* ***(Observability)***

(1) *Slow subsystem (4.2a) is observable if and only if*

$$rank \begin{bmatrix} sE - A \\ C \end{bmatrix} = n, \quad \forall s \in \mathbb{C},\ s \text{ finite}$$

(2) *The following statements are equivalent.*

(a) *Fast subsystem (4.2b) is observable.*

(b) *rank* $\begin{bmatrix} C_2 \\ C_2N \\ C_2N^2 \\ \vdots \\ C_2N^{h-1} \end{bmatrix} = n_2.$

(c) *rank* $\begin{bmatrix} N \\ C_2 \end{bmatrix} = n_2.$

(d) *rank* $\begin{bmatrix} E \\ C \end{bmatrix} = n.$

(3) *The following statements are equivalent.*

(a) *System (4.1) is observable.*

(b) *Both its slow and fast subsystems are observable.*

(c) *rank* $\begin{bmatrix} sE - A \\ C \end{bmatrix} = n, \quad \forall s \in \mathbb{C},$ *s finite, and rank* $\begin{bmatrix} E \\ C \end{bmatrix} = n.$

4.2 Problem Formulation

Consider an ISSSD defined in the following form

$$\begin{aligned} E_{\sigma(t)}\dot{x}(t) &= A_{\sigma(t)}x(t) + B_{\sigma(t)}u + f_{\sigma(t)}(t, x(t-d)), && t \neq t_k \\ \triangle x(t) &= I_k x(t^-), && t = t_k \\ y(t) &= C_{\sigma(t)}x(t) \\ x_{t_0^-}(s) &= \phi(s), \qquad s \in [-d, 0], \end{aligned} \tag{4.3}$$

where $x \in \mathbb{R}^n$ is the system state variable, $A_{\sigma(t)}, E_{\sigma(t)}, I_k \in \mathbb{R}^{n\times n}$, $B_{\sigma(t)} \in \mathbb{R}^{n\times p}$ and $C_{\sigma(t)} \in \mathbb{R}^{q\times n}$ are system coefficient matrices where $E_{\sigma(t)}$ being singular with $\text{rank}(E_{\sigma(t)}) = r < n$, I_k being constant matrices, $u \in \mathbb{R}^p$ and $y \in \mathbb{R}^q$ are a system input and output, respectively. $K_{\sigma(t)} \in \mathbb{R}^{p\times n}$ is control gain matrix of the form $u(t) = K_{\sigma(t)}x(t)$. The switching signal $\sigma(t) : [t_0, \infty) \to \Xi$ is a piecewise constant function taking values in a finite compact set $\Xi = \{1, 2, \dots, N\}$ for some $N \in \mathbb{N}$. $\{t_k\}_{k=1}^{\infty}$ are the impulsive times that form an increasing sequence satisfying $t_{k-1} < t_k$ and $\lim_{k\to\infty} t_k = \infty$. $\Delta x = x(t^+) - x(t^-)$ where $x(t^-)$ (and $x(t^+)$) is the state just before (and just after) the impulsive action with $x(t^+) = \lim_{s\to t^+} x(s)$. The solution x is assumed to be right-continuous, i.e., $x(t_k^+) = x(t_k)$. x_{t_0} is defined by $x_{t_0}(s) = x(t_0 + s)$ for $s \in [-d, 0]$ with d being a positive constant representing the time delay. $f_{\sigma(t)}(t, x(t-d)) : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^n$ are piecewise continuous vector-valued functions with $f_{\sigma(t)}(t, 0) \equiv 0$ and there exist constant matrices $F_{\sigma(t)}$ such that

$$\|f_{\sigma(t)}(t, x) - f_{\sigma(t)}(t, x^*)\| \le \|F_{\sigma(t)}E_{\sigma(t)}(x - x^*)\| \tag{4.4}$$

for all (t, x), $(t, x^*) \in D = \{(t, x) : \ t \in \mathbb{R}^+, \ \|E_{\sigma(t)}x\| < \rho\}$. Assume that each subsystem in (4.3) is regular, controllable and observable.

4.3 Optimal Control Design

Stability of the optimal closed-loop system for any given control matrix is analyzed in this section. Moreover, the problem of designing an optimal controller is handled.

Assume that a linear state feedback optimal controller of the form $u(t) = K_i x(t)$ exists such that for any $i \in \Xi$, each subsystem in the optimal closed-loop system of system (4.3)

$$\begin{aligned} E_i\dot{x}(t) &= (A_i + B_iK_i)x(t) + f_i(t, x(t-d)), && t \neq t_k \\ \triangle x(t) &= I_k x(t^-), && t = t_k \\ x_{t_0^-}(s) &= \phi(s), \qquad s \in [-d, 0]. \end{aligned} \tag{4.5}$$

is asymptotically stable. This assumption implies that for any $i \in \Xi$ there is a Lyapunov function $V_i(x(t)) = x^T(t)E_i^TX_iE_ix(t)$; that means for $X_i > 0$ the time derivative of $V_i(x(t))$ along the trajectory of (4.5) is negative definite.

Theorem 4.3.1. ***(Stability)*** *For any $i \in \Xi$, assume that each subsystem of (4.5) is admissible. Then, the trivial solution of (4.5) is exponentially stable if the following conditions hold:*

(i) For any symmetric positive definite matrix Y_i and symmetric positive definite matrix R_i there exists a positive definite matrix X_i satisfying the algebraic Riccati-like equation

$$E_i^TX_iA_i + A_i^TX_iE_i + E_i^TX_iB_iK_i + K_i^TB_i^TX_iE_i + \varepsilon_iE_i^TX_i^2E_i + K_i^TR_iK_i = -E_i^TY_iE_i, \tag{4.6}$$

where ε_i is positive constant.

(ii) For any $i, j \in \Xi$ there exists $\gamma_k > 1$ such that

$$(I + I_k)^TE_j^TX_jE_j(I + I_k) \leq \gamma_kE_i^TX_iE_i. \tag{4.7}$$

(iii) For any t_0, the switching law satisfies

$$N(t_0, t) \leq N_0 + \frac{t - t_0}{T_a} \tag{4.8}$$

where $N(t_0, t)$ represents the number of switchings in (t_0, t), and N_0 and T_a are the chatter bound and average dwell time, respectively.

Proof. Given $\varepsilon_i \in (0, \rho)$ choose $\delta < \varepsilon_i$ and $\phi \in \mathbb{R}^n$ such that $\|E_i\phi\|_d < \delta$. For $t_0 \in \mathbb{R}^+$, let $x(t) = x(t, t_0, \phi)$ be the solution of system (4.5) such that $x(t_0+s) = \phi(s)$ with $s \in [-d, 0]$. We claim $\|E_i x(t)\| < \varepsilon_i$ for all $t \geq t_0$. Suppose the claim is not true, then there exists a $t^* > t_0$ such that

$$\|E_i x(t^*)\| = \varepsilon_i \text{ and } \|E_i x(t)\| < \varepsilon_i \text{ for all } t \in [t_0, t^*). \tag{4.9}$$

For $t \in [t_{k-1}, t_k) \subseteq [t_0, t^*]$, define

$$\upsilon_i(t) = V_i(x(t)) = x^T(t)E_i^T X_i E_i x(t), \qquad i = \sigma(t)$$

as a Lyapunov function candidate for i^{th} subsystem. Since υ_i is positive semi definite, υ_i satisfies the following inequality

$$\lambda_{\min}(X_i)\|E_i x(t)\|^2 \leq \upsilon_i \leq \lambda_{\max}(X_i)\|E_i x(t)\|^2, \tag{4.10}$$

where $\lambda_{\min}(X_i)$ and $\lambda_{\max}(X_i)$ are minimum and maximum eigenvalue of matrix X_i, respectively.

Derivative of υ_i along the trajectory of (4.5) is given by

$$\begin{aligned}\dot{\upsilon}_i(t) &= x^T(t)E_i^T X_i(A_i + B_iK_i)x(t) + x^T(t)(A_i + B_iK_i)^T X_i E_i x(t) \\ &\quad + 2f_i^T(t, x(t-d))X_i E_i x(t)\end{aligned} \tag{4.11}$$

Using inequality (4.4), we obtain that

$$2f_i^T(t, x(t-d))X_i E_i x(t) \leq \frac{1}{\varepsilon_i}\|F_i\|^2\|E_i x(t-d)\|^2 + \varepsilon_i x^T(t)E_i^T(t)X_i^2 E_i x(t). \tag{4.12}$$

Substituting (4.12) into (4.11) gives the following inequality

$$\begin{aligned}\dot{\upsilon}_i(t) &\leq x^T(t)E_i^T X_i A_i x(t) + x^T(t)E_i^T X_i B_i K_i x(t) + x^T(t)A_i^T X_i E_i x(t) + x^T(t)K_i^T B_i^T X_i E_i x(t) \\ &\quad + \frac{1}{\varepsilon_i}\|F_i\|^2\|E_i x(t-d)\|^2 + \varepsilon_i x^T(t)E_i^T X_i^2 E_i x(t)\end{aligned} \tag{4.13}$$

By algebraic Riccati-like equation (4.6), we obtain

$$\dot{\upsilon}_i(t) \leq -x^T(t)E_i^T W_i E_i x(t) + \frac{1}{\varepsilon_i}\|F_i\|^2\|E_i x(t-d)\|^2$$

where $E_i^T W_i E_i = K_i^T R_i K_i + E_i^T Y_i E_i$. Thus, we have

$$\dot{\upsilon}_i(t) \leq -\alpha_i \upsilon_i(t) + \beta_i \sup_{\sigma\in[t-d,t]} \upsilon_i(\sigma) \qquad \text{for all } t \in [t_{k-1}, t_k) \tag{4.14}$$

where $\alpha_i = \dfrac{\lambda_{\min}(W_i)}{\lambda_{\max}(X_i)}$, and $\beta_i = \dfrac{\|F_i\|^2}{\varepsilon_i \lambda_{\min}(X_i)}$.
Applying the Halanay Lemma to (4.14), we obtain the solution of differential inequality (4.14) for $t \in [t_{k-1}, t_k)$ as

$$\upsilon_i(t) \leq \sup_{\sigma\in[t_{k-1}-d,t_{k-1}]} \upsilon_i(\sigma)e^{-\xi_i(t-t_{k-1})}, \tag{4.15}$$

where ξ_i is a unique positive solution of

$$\xi_i = \alpha_i - \beta_i e^{\xi_i d}.$$

On the other hand, let us suppose $\sigma(t_k) = j$, $k = 1, 2, 3, \cdots$. It follows from (4.5) and (4.7) that

$$\upsilon_j(t_k) = x^T(t_k^-)(I + I_k)^T E_j^T X_j E_j (I + I_k) x(t_k^-) \leq \gamma_k \upsilon_i(t_k^-). \tag{4.16}$$

Using (4.15) and (4.16) successively on each subinterval leads to in general, for $i \in \Xi$ and $t \in [t_{k-1}, t_k)$

$$\upsilon_{i_k}(t) \leq \gamma_1\gamma_2\ldots\gamma_{k-1} \sup_{\sigma\in[t_0-d,t_0]} \upsilon_{i_1}(\sigma)e^{-\xi_{i_1}(t_1-d-t_0)}e^{-\xi_{i_2}(t_2-d-t_1)}\ldots e^{-\xi_{i_k}(t-t_{k-1})}. \tag{4.17}$$

Let $\xi = \min\{\xi_{i_j};\ i \in \Xi \text{ and } j = 1, 2, \ldots, k\}$, $\gamma = \max\{\gamma_l;\quad l = 1, 2, \ldots, k-1\}$ and $\xi^* = \max\{\xi_{i_l};\quad i \in \Xi,\ l = 1, 2, \ldots, k-1\}$. Then, inequality (4.17) becomes

$$\upsilon_{i_k}(t) \leq \gamma^{k-1} \sup_{\sigma\in[t_0-d,t_0]} \upsilon_{i_1}(\sigma)e^{(k-1)\xi^* d}e^{-\xi(t-t_0)}.$$

By ADT condition (4.8) with $N_0 = \frac{\eta}{\ln\gamma+\xi^* d}$, where η is an arbitrary constant, and $T_a = \frac{\ln\gamma+\xi^* d}{\xi-\xi^{**}}$, $(\xi > \xi^{**})$, we obtain

$$\upsilon_{i_k}(t) \le \sup_{\sigma\in[t_0-d,t_0]} \upsilon_{i_1}(\sigma)e^{\eta-\xi^{**}(t-t_0)} \text{ for all } t \ge t_0 \tag{4.18}$$

which implies the following inequality

$$\lambda_{\min}(X_{i_k})\|E_i x(t)\|^2 \le \lambda_{\max}(X_{i_1})\|E_i\phi(t)\|_d^2 e^{\eta-\xi^{**}(t-t_0)}, \quad t \ge t_0. \tag{4.19}$$

Since $e^{-\xi^{**}(t-t_0)} \le 1$ and $\|E_i\phi(t)\|_d < \delta$, we have

$$\lambda_{\min}(X_{i_k})\|E_i x(t)\|^2 \le \lambda_{\max}(X_{i_1})\|E_i\phi(t)\|_d^2 e^{\eta-\xi^{**}(t-t_0)} < e^{\eta}\lambda_{\max}(X_{i_1})\delta^2. \tag{4.20}$$

Choose $\delta > 0$ such that $e^{\eta}\lambda_{\max}(X_{i_1})\delta^2 < \lambda_{\min}(X_{i_k})\varepsilon_i^2$. Thus, by inequality (4.20) we obtain

$$\lambda_{\min}(X_{i_k})\|E_i x(t)\|^2 < \lambda_{\max}(X_{i_1})\delta^2 < \lambda_{\min}(X_{i_k})\varepsilon_i^2$$
$$\Rightarrow \|E_i x(t)\| < \varepsilon_i \text{ for all } t \in [t_0, t^*]$$

which leads a contradiction at $t = t^*$. This shows $\|E_i x(t)\| < \varepsilon_i$ for all $t \ge t_0$.

By inequality (4.19), we obtain

$$\|E_i x(t)\| \le \mu\|E_i\phi(t)\|_d e^{(\eta-\xi^{**}(t-t_0))/2}, \quad t \ge t_0, \tag{4.21}$$

where $\mu = \sqrt{\lambda_M/\lambda_m}$ with $\lambda_M = \max\{\lambda_{\max}(X_i);\ i \in \Xi\}$ and $\lambda_m = \min\{\lambda_{\min}(X_i);\ i \in \Xi\}$. Thus, the trivial solution of system (4.5) is E-exponentially stable.

By using decomposition form of system (4.5), one may obtain

$$\|x_1(t)\| \le \mu\|\tilde{Q}_i\|\|E_i\phi(t)\|_d e^{(\eta-\xi^{**}(t-d-t_0))/2}, \tag{4.22}$$

which implies that slow sub-state x_1 is exponentially stable, and

$$\|x_2(t)\| \le \frac{\|\tilde{Q}_i^2\|\|F_iE_i\tilde{P}_i^1\|}{1-\|\tilde{Q}_i^2\|\|F_iE_i\tilde{P}_i^2\|}\mu\|\tilde{Q}_i\|\|E_i\phi(t)\|_d e^{(\eta-\xi^{**}(t-d-t_0))/2}. \tag{4.23}$$

where $\|\tilde{Q}_i^2\|\|F_iE_iP^2\| < 1$. This shows that fast sub-state x_2 is exponentially stable. Thus, by (4.22) and (4.23) the trivial solution of system (4.5) is exponentially stable. □

Remark 4.3.1. *The algebraic Riccati-like equation given in (4.6) guarantees that the Lyapunov function is decreasing along the trajectory of system (4.5); that is, the continuous system is stabilized by the feedback controller. On the other hand, impulse manners as a destabilizing perturbation by condition (ii). The role of ADT condition in condition (iii) is to organize the switching among the system modes which eventually guarantees the exponential stability.*

Having established the stability result, we can now design an optimal controller stabilizing the closed-loop system (4.5).

Theorem 4.3.2. ***(Optimal control design)*** *For any $i \in \Xi$, assume that each subsystem of (4.5) is admissible. Then, system (4.5) under the feedback controllers $u(t) = K_i x(t)$ with $K_i = -R_i^{-1} B_i^T X_i E_i$ is exponentially stable if the following assumptions hold:*

(i) For any symmetric positive-semi definite matrix Q_i and symmetric positive definite matrix R_i there exists a positive definite matrix X_i satisfying the algebraic algebraic Riccati-like inequality

$$E_i^T X_i A_i + A_i^T X_i E_i - E_i^T X_i B_i R_i^{-1} B_i^T X_i E_i + \varepsilon_i E_i^T X_i^2 E_i \leq -Q_i, \tag{4.24}$$

where ε_i is positive constant.

(ii) For any $i, j \in \Xi$ there exists $\gamma_k > 1$ such that

$$(I + I_k)^T E_j^T X_j E_j (I + I_k) \leq \gamma_k E_i^T X_i E_i. \tag{4.25}$$

(iii) For any t_0, the switching law satisfies

$$N(t_0, t) \leq N_0 + \frac{t - t_0}{T_a} \tag{4.26}$$

where $N(t_0, t)$ represents the number of switchings in (t_0, t), and N_0 and T_a are the chatter bound and average dwell time, respectively.

Proof. For any $i \in \Xi$, let the cost function (or the performance index) be defined as follows

$$J_i(u) = \int_{t_{k-1}}^{t_k} (x^T Q_i x + u^T R_i u) dt$$

where $Q_i = Q_i^T \geq 0$ and $R_i = R_i^T > 0$. Our goal is to construct a stabilizing state feedback controller of the form $u = K_i x$ that minimizes the cost function. Such a linear controller is denoted by u^*. By the optimal control law, we need to find u^* which minimizes the following function

$$g_i(x, u) = \frac{d\upsilon_i}{dt} + x^T Q_i x + u^T R_i u$$

where

$$\upsilon_i(t) = V_i(x(t)) = x^T(t) E_i^T X_i E_i x(t), \qquad t \in [t_{k-1}, t_k), \quad i = \sigma(t)$$

as a Lyapunov function candidate for i^{th} subsystem with $X_i = X_i^T > 0$. Thus, we have

$$g_i(x, u) = 2x^T E_i^T X_i A_i x + 2x^T E_i^T X_i B_i u + 2x^T E_i^T X_i f_i(t, x(t-d)) + x^T Q_i x + u^T R_i u$$

and

$$\frac{\partial g_i}{\partial u} = 2x^T E_i^T X_i B_i + 2u^T R_i.$$

Thus, for $u = u^* = -R_i^{-1} B_i^T X_i E_i x$, the function $g_i(x, u)$ has minimum value since the second derivative of $g_i(x, u)$ with respect to u has positive value for $u^* = -R_i^{-1} B_i^T X_i E_i x$. By plugging u^* into system (4.5) we obtain

$$E_i \dot{x} = (A_i - B_i R_i^{-1} B_i^T X_i E_i) x + f_i(t, x(t-d)). \tag{4.27}$$

Our optimal controller satisfies the equation

$$\frac{d\upsilon_i}{dt}|_{u=u^*} + x^T Q_i x + u^{*T} R_i u^* = 0$$

$$2x^T E_i^T X_i E_i \dot{x}|_{u=u^*} + x^T Q_i x + x^T E_i^T X_i B_i R_i^{-1} R_i R_i^{-1} B_i^T X_i E_i x = 0$$

$$x^T E_i^T X_i A_i x + x^T A_i^T X_i E_i x - x^T E_i^T X_i B_i R_i^{-1} B_i^T X_i E_i x + 2 f_i^T(t, x(t-d)) X_i E_i x(t) + x^T Q_i x = 0 \tag{4.28}$$

Since

$$2f_i^T(t,x(t-d))X_iE_ix(t) \leq \frac{L_i^2}{\varepsilon_i}\|x(t-d)\|^2 + \varepsilon_i x^T(t)E_i^TX_i^2E_ix(t) \tag{4.29}$$

where ε_i is a positive constant and $L_i = \|F_iE_i\|$, we obtain

$$x^T\Big[E_i^TX_iA_i + A_i^TX_iE_i - E_i^TX_iB_iR_i^{-1}B_i^TX_iE_i + \varepsilon_iE_i^TX_i^2E_i + Q_i\Big]x + \frac{L_i^2}{\varepsilon_i}\|x(t-d)\|^2 \geq 0.$$

Thus, we obtain the algebraic Riccati-like inequality as

$$E_i^TX_iA_i + A_i^TX_iE_i + \varepsilon_iE_i^TX_i^2E_i - E_i^TX_iB_iR_i^{-1}B_i^TX_iE_i \leq -Q_i.$$

The rest is done as proof of the previous Theorem 4.3.1 . □

Remark 4.3.2. *Optimal controller $u = -R_i^{-1}B_i^TX_iE_ix$ minimizing the cost function*

$$J_i(u) = \int_{t_{k-1}}^{t_k}(x^TQ_ix + u^TR_iu)dt$$

subject to system (4.3) requires solving the algebraic Riccati-like equation given by (4.24) which guarantees the existence of the positive definite matrix X_i for all $i \in \Xi$. On the other word, an appropriate Lyapunov function which is decreasing along the trajectory of the closed-loop system is found by solving the algebraic Riccati-like equation given in (4.24).

Example 4.3.1. *Consider the ISSSD given by (4.3) where $x = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$, $\sigma(t) \in \Xi = \{1,2\}$, $E_1 = E_2 = \begin{bmatrix} 4 & 0 \\ 2 & 0 \end{bmatrix}$ with $rank(E_i) = 1$, $I_k = 0.8I$, $d = 0.5$ and*

$$A_1 = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad B_1 = B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

$$f_1(t,x(t-d)) = f_2(t,x(t-d)) = \Big[\tfrac{1}{75}\tanh(x_1(t-d)) \quad \tfrac{1}{75}\tanh(x_2(t-d))\Big]^T.$$

Initial function is $\phi(t) = [0.5 - t \quad -5.72 + t]^T$. *Also,* $L_1 = 0.0133$ *in subsystem 1 and* $L_2 = 0.0133$ *in subsystem 2 are satisfied inequality (4.29).*

The control gain matrices are computed as

$$K_1 = \begin{bmatrix} 5.72 & 0 \end{bmatrix} \text{ and } K_2 = \begin{bmatrix} 5.8 & 0 \end{bmatrix}$$

where $X_1 = \begin{bmatrix} 3 & -3.14 \\ -3.14 & 3.42 \end{bmatrix} > 0$, $X_2 = \begin{bmatrix} 3 & -3.1 \\ -3.1 & 3.3 \end{bmatrix} > 0$, *and* $R_1 = R_2 = 1$. *Hence, for any* $Q_1 = Q_2 = \begin{bmatrix} 20 & 0 \\ 0 & 0 \end{bmatrix} \geq 0$, *we obtain* $\alpha_1 = 0.1573$, $\beta_1 = 0.1447$, $\alpha_2 = 0.1599$, *and* $\beta_2 = 0.1264$. *By Halanay inequality, the mode decay rates are computed as* $\xi_1 = 0.0110$ *and* $\xi_2 = 0.0297$. *Also,* $\gamma_k = 3.24$ *so that the inequality* $(I + B_k)^T E_j^T X_j E_j (I + B_k) \leq \gamma_k E_i^T X_i E_i$ *is satisfied. The system is simulated for 5 seconds and ADT is 1.15 seconds. Figure 4.1 depicts the slow and fast sub-states.*

4.4 State Estimation

State feedback control is designed under the assumption that all state variables are accessible for measurement. In practical applications, this assumption may not hold since it is either impossible or too expensive to measure all the state variables. In such cases, a state observer or state estimator is needed in order to apply state feedback. Motivated by this consideration, the state estimation technique is used to estimate the unmeasured state variables in this section.

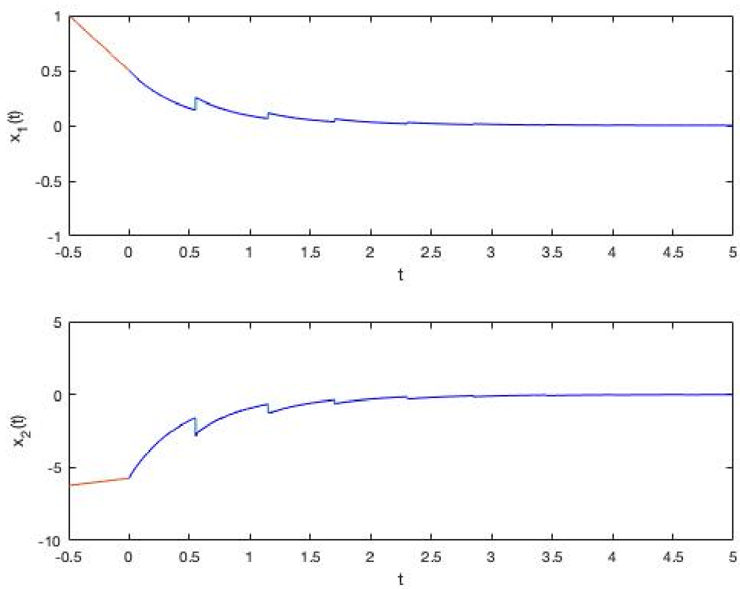

Figure 4.1: State response of the corresponding closed-loop system

For any $i \in \Xi$, the observer dynamic is considered as

$$\begin{aligned} E_i\dot{\hat{x}} &= A_i\hat{x} + B_iu + f_i(t, \hat{x}(t-d)) + L(y - C_i\hat{x}), && t \neq t_k \\ \triangle\hat{x}(t) &= I_k\hat{x}(t^-), && t = t_k \\ \hat{y}(t) &= C_i\hat{x}(t) \\ \hat{x}_{t_0^-}(s) &= \hat{\phi}(s), \qquad s \in [-d, 0], \end{aligned} \tag{4.30}$$

where $L \in \mathbb{R}^{n\times q}$ is estimator or observer gain matrix.

Let us define the estimation error for system state as $e(t) = x(t) - \hat{x}(t)$ and the state estimation error system is

$$\begin{aligned} E_i\dot{e}(t) &= (A_i - LC_i)e(t) + f_i(t, x(t-d)) - f_i(t, \hat{x}(t-d)), && t \neq t_k \\ e(t) &= (I + I_k)e(t^-), && t = t_k \\ e_{t_0^-}(s) &= \varphi(s), \qquad s \in [-d, 0]. \end{aligned} \tag{4.31}$$

Assumption 4.4.1. *Suppose that the matrix pairs $(E_i, A_i - LC_i)$ are regular.*

Theorem 4.4.1. ***(State estimation)*** *For any $i \in \Xi$, assume that each subsystem of (4.31) is admissible. Then, the trivial solution of the error system (4.31) is exponentially stable if the following assumptions hold:*

(i) For any $i, j \in \Xi$ there exists $\gamma_k > 1$ such that

$$(I + I_k)^T E_j^T X_j E_j (I + I_k) \leq \gamma_k E_i^T X_i E_i \tag{4.32}$$

where X_i is a positive definite solution of the Lyapunov equation

$$(A_i - LC_i)^T X_i E_i + E_i^T X_i (A_i - LC_i) = -E_i^T Y_i E_i \quad \text{for any } Y_i > 0. \tag{4.33}$$

(ii) For any t_0, the switching law satisfies

$$N(t_0, t) \leq N_0 + \frac{t - t_0}{T_a} \tag{4.34}$$

where $N(t_0, t)$ represents the number of switchings in (t_0, t), and N_0 and T_a are the chatter bound and average dwell time, respectively.

Proof. Given $\varepsilon_i \in (0, \rho)$ choose $\delta < \varepsilon_i$ and $\varphi \in \mathbb{R}^n$ such that $\|E_i\varphi\|_d < \delta$. For $t_0 \in \mathbb{R}^+$, let $e(t) = e(t, t_0, \varphi)$ be the solution of system (4.31) such that $e(t_0+s) = \varphi(s)$ with $s \in [-d, 0]$. We claim $\|E_i e(t)\| < \varepsilon_i$ for all $t \geq t_0$. Suppose the claim is not true, then there exists a $t^* > t_0$ such that

$$\|E_i e(t^*)\| = \varepsilon_i \text{ and } \|E_i e(t)\| < \varepsilon_i \text{ for all } t \in [t_0, t^*). \tag{4.35}$$

For $t \in [t_{k-1}, t_k)$, define

$$\upsilon_i(t) = V_i(e(t)) = e^T(t) E_i^T X_i E_i e(t), \quad i = \sigma(t)$$

as a Lyapunov function candidate for i^{th} subsystem. Then, derivative of υ_i along the trajectory of (4.31) is given by

$$\dot{\upsilon}_i(t) = -e^T(t) E_i^T Y_i E_i e(t) + 2\big[f_i(t, x(t-d)) - f_i(t, \hat{x}(t-d))\big]^T X_i E_i e(t). \tag{4.36}$$

By the Lipschitz condition in (4.4), we have

$$2\big[f_i(t, x(t-d)) - f_i(t, \hat{x}(t-d))\big]^T X_i E_i e(t) \leq \frac{L_i^2}{\varepsilon_i}\|E_i e(t-d)\|^2 + \varepsilon_i \|E_i e(t)\|^2 \|X_i\|^2 \tag{4.37}$$

where $F_{\sigma(t)} \in \mathbb{R}^{n\times n}$ are constant matrices and ε_i are positive constants.

Substituting (4.37) into (4.36), we obtain

$$\begin{aligned} \dot{\upsilon}_i(t) &\leq -e^T(t) E_i^T Y_i E_i e(t) + \frac{L_i^2}{\varepsilon_i}\|E_i(x(t-d) - \hat{x}(t-d))\|^2 + \varepsilon_i \|E_i e(t)\|^2 \|X_i\|^2 \\ &\leq -\alpha_i \upsilon_i(t) + \beta_i \sup_{\sigma\in[t-d,t]} \upsilon_i(\sigma) \end{aligned} \tag{4.38}$$

where $\alpha_i = \dfrac{\lambda_{\min}(Y_i) - \varepsilon_i \|X_i\|^2}{\lambda_{\max}(X_i)}$ and $\beta_i = \dfrac{L_i^2}{\varepsilon_i \lambda_{\min}(X_i)}$.
By Halanay Inequality, the solution of differential inequality (4.38) for $t \in [t_{k-1}, t_k)$ is obtained as

$$\upsilon_i(t) \leq \sup_{\sigma \in [t_{k-1}-d, t_{k-1}]} \upsilon_i(\sigma) e^{-\xi_i(t-t_{k-1})}, \tag{4.39}$$

where ξ_i is a unique positive solution of

$$\xi_i = \alpha_i - \beta_i e^{\xi_i d}.$$

On the other hand, let us suppose $\sigma(t_k) = j,\ k = 1, 2, 3, \ldots$. It follows from (4.31) and (4.32) that

$$\upsilon_j(t_k) = e^T(t_k^-)(I + I_k)^T E_j^T X_j E_j (I + I_k) e(t_k^-) \leq \gamma_k \upsilon_i(t_k^-). \tag{4.40}$$

By using (4.39) and (4.40) on each subinterval, we generally obtain

$$\upsilon_{i_k}(t) \leq \gamma^{k-1} \sup_{\sigma \in [t_0-d, t_0]} \upsilon_{i_1}(\sigma) e^{(k-1)\xi^* d} e^{-\xi(t-t_0)}, \qquad t \in [t_{k-1}, t_k) \tag{4.41}$$

where $\gamma = \max\{\gamma_l;\quad l = 1, 2, \ldots, k-1\}$, $\xi^* = \max\{\xi_{i_l};\quad i \in \Xi,\ l = 1, 2, \ldots, k-1\}$, and $\xi = \min\{\xi_{i_j};\ i \in \Xi \text{ and } j = 1, 2, \ldots, k\}$.

Applying ADT condition (4.34) with $N_0 = \frac{\eta}{\ln \gamma + \xi^* d}$, where η is an arbitrary constant, and $T_a = \frac{\ln \gamma + \xi^* d}{\xi - \xi^{**}}$, $(\xi > \xi^{**})$ to inequality (4.41) results in

$$\upsilon_{i_k}(t) \leq \sup_{\sigma \in [t_0-d, t_0]} \upsilon_{i_1}(\sigma) e^{\eta - \xi^{**}(t-t_0)} \text{ for all } t \geq t_0. \tag{4.42}$$

Thus, similar to Theorem 4.3.1 we can obtain a contradiction using inequality (4.42). However, we skip this part for this theorem.

Because of property of positive semi definite Lyapunov function υ_{i_k}, we obtain the following inequality

$$\|E_i e(t)\| \leq \mu \|E_i \varphi(t)\|_d e^{(\eta - \xi^{**}(t-t_0))/2}, \qquad t \geq t_0, \tag{4.43}$$

where $\mu = \sqrt{\lambda_M/\lambda_m}$ with $\lambda_M = \max\{\lambda_{\max}(X_i);\ i \in \Xi\}$ and $\lambda_m = \min\{\lambda_{\min}(X_i);\ i \in \Xi\}$. This means that trivial solution of the system (4.31) is E-exponentially stable.

Using decomposition form of system (4.31) as done in previous theorems we obtain that

$$\|e_1(t)\| \leq \mu\|\tilde{Q}_i\|\|E_i\varphi(t)\|_d e^{(\eta-\xi^{**}(t-d-t_0))/2}, \tag{4.44}$$

and

$$\|e_2(t)\| \leq \frac{L_i\|\tilde{Q}_i^2\|\|E_i\tilde{P}_i^1\|}{1-L_i\|\tilde{Q}_i^2\|\|E_i\tilde{P}_i^2\|}\mu\|\tilde{Q}_i\|\|E_i\varphi(t)\|_d e^{(\eta-\xi^{**}(t-d-t_0))/2} \tag{4.45}$$

where $i = 1, 2, \ldots, N$, $e_1 \in \mathbb{R}^r$, $e_2 \in \mathbb{R}^{n-r}$, $\tilde{Q}_i = \begin{bmatrix} \tilde{Q}_i^1 \\ \tilde{Q}_i^2 \end{bmatrix}$, $\tilde{Q}_i^1 \in \mathbb{R}^{r\times n}$, $\tilde{Q}_i^2 \in \mathbb{R}^{(n-r)\times n}$, $\tilde{P}_i = \begin{bmatrix} \tilde{P}_i^1 & \tilde{P}_i^2 \end{bmatrix}$, $\tilde{P}_i^1 \in \mathbb{R}^{n\times r}$, $\tilde{P}_i^2 \in \mathbb{R}^{n\times(n-r)}$, and $L\|\tilde{Q}_i^2\|\|E_i\tilde{P}_i^2\| < 1$. Inequality (4.44) and (4.45) show that e_1 and e_2 are exponentially stable. Thus, the trivial solution of system (4.31) is exponentially stable. That means $\hat{x}(t) \to x(t)$ as $t \to \infty$. $\square$

Remark 4.4.1. *Theorem 4.4.1 provides sufficient conditions to ensure exponential stability of the state estimation error system given by (4.31). The Lyapunov equation given in (4.33) guarantees the existence of the positive definite matrix X_i for all $i \in \Xi$. Switching among the subsystems is organized by the ADT condition.*

Example 4.4.1. *Consider the ISSSD given by (4.3) where $x = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$, $\sigma(t) \in \Xi = \{1, 2\}$, $E_1 = E_2 = \begin{bmatrix} 4 & 0 \\ 2 & 0 \end{bmatrix}$ with $rank(E_i) = 1$ for $i \in \Xi$, $I_k = 0.3I$, $d = 0.5$ and*

$$A_1 = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}, \quad B_1 = B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

$$f_1(t, x(t-d)) = f_2(t, x(t-d)) = \begin{bmatrix} \frac{1}{150}\tanh(x_1(t-d)) & \frac{1}{150}\tanh(x_2(t-d)) \end{bmatrix}^T,$$

$$C_1 = C_2 = \begin{bmatrix} 1 & 1 \end{bmatrix}.$$

Also, initial function is $\phi(t) = [1 \quad -1.4286]^T$.

The error-state system with time-delay is

$$\begin{aligned} E_i\dot{e}(t) &= (A_i - LC_i)e(t) + f_i(t, x_t) - f_i(t, \hat{x}_t), & t \neq t_k \\ e(t) &= (I + I_k)e(t^-), & t = t_k \\ \varphi(s) &= \begin{bmatrix} 1 - s & -1.4286 + s \end{bmatrix}^T, \qquad s \in [-0.5, 0] \end{aligned}$$

where the observer gain matrix $L = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$. $X_1 = \begin{bmatrix} 3 & 4 \\ 4 & 7 \end{bmatrix} > 0$ *satisfies* $(A_1 - LC_1)^T X_1 E_1 + E_1^T X_1 (A_1 - LC_1) = -E_1^T Y_1 E_1$ *for any* $Y_1 = I > 0$. *Similarly,* $X_2 = \begin{bmatrix} 0.3333 & -0.6666 \\ -0.6666 & 2.9999 \end{bmatrix} > 0$ *satisfying* $(A_2 - LC_2)^T X_2 E_2 + E_2^T X_2 (A_2 - LC_2) = -E_2^T Y_2 E_2$ *for any* $Y_2 = I > 0$. *Hence,* $\alpha_1 = 0.0554$, $\beta_1 = 0.0084$, $\alpha_2 = 0.3110$, *and* $\beta_2 = 0.1404$. *The mode decay rates are* $\xi_1 = 0.0466$ *and* $\xi_2 = 0.1482$. *Thus,* $\xi = \min\{\xi_1, \xi_2\} = 0.0466$ *and* $\xi^* = \max\{\xi_1, \xi_2\} = 0.1482$. *Moreover,* $\gamma_k = 1.69$ *so that the inequality* $(I + B_k)^T E_j^T X_j E_j (I + B_k) \leq \gamma_k E_i^T X_i E_i$ *is satisfied. The error system is simulated for 12 seconds, and ADT is computed as 1 second. Figure 4.2 depicts estimation errors of the slow and fast sub-states. All errors converges to zero which shows the effectiveness of the proposed observer.*

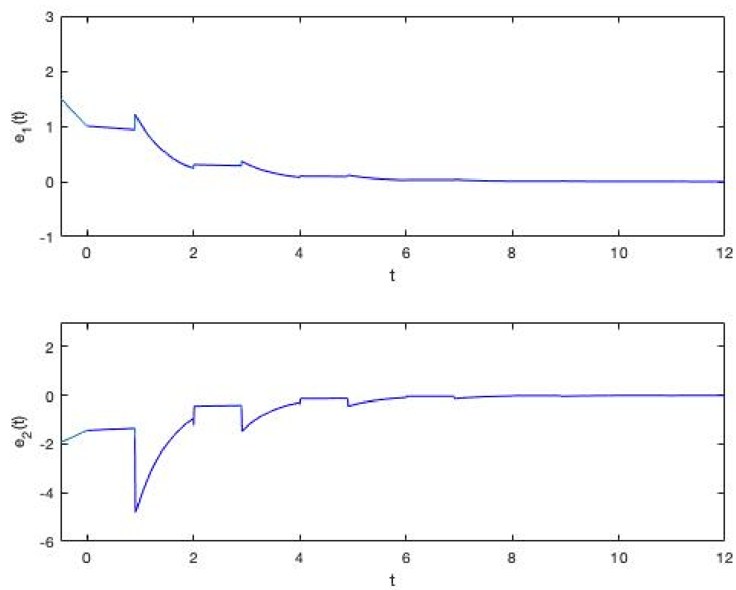

Figure 4.2: Upper: estimation error of slow sub-state. Lower: estimation error of fast sub-state.

Chapter 5

Sliding Mode Control for Impulsive Singular Systems: Continuous and Discrete

The main purpose of this chapter is to address the problem of stabilization for continuous and discrete impulsive singular systems via a sliding mode control (SMC). Firstly, some sufficient conditions on the exponential stability property of the continuous impulsive systems have been proposed. To obtain this objective, a sliding surface is designed on which the sliding motion of the system state happens, then a sliding mode control law is designed to force the system state to reach, stay and slide on the sliding surface. Secondly, a similar sliding surface and control design are adopted to discrete impulsive switched singular systems to obtain some sufficient conditions for the exponential stability of the full order system. Numerical examples with simulations are represented to further clarify the theoretical results.

5.1 Problem Formulation

The sliding mode design approach comprises two steps. The first one is to design of a sliding surface on which the sliding motion will take place. The second step is to design of a control law which forces the system state trajectories to reach and slide on the surface. In SMC, the reachability condition is a necessary condition to guarantee existence of the sliding mode. By the reachability condition, the system reaches the sliding surface from any initial conditions in a finite time and remain on this surface after reach it. Details of the design procedures are given in the main results.

5.2 Continuous Sliding Mode Control

Consider the impulsive singular system with nonlinear perturbation of the form

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + Bu(t) + f(t, x(t)), && t \neq t_k \\ \triangle x(t) &= I_k x(t), && t = t_k \\ x(t_0) &= x_0, \end{aligned} \tag{5.1}$$

where $x(t) \in \mathbb{R}^n$ for all $t \geq t_0$ with $t_0 \in \mathbb{R}^+$ is the system state variable, and $A, I_k, E \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times p}$ are system coefficient matrices where E being singular with rank$(E) = r < n$, $u(t) \in \mathbb{R}^p$ is the input vector, $f(x(t), t) \in \mathbb{R}^n$ is the perturbation term and $I_k \in \mathbb{R}^{n\times n}$ is an $n \times n$ constant matrix for each $k \in \mathbb{N}$. $\{t_k\}_{k=1}^{\infty}$ are the impulsive times that form an increasing sequence satisfying $t_{k-1} < t_k$ and $\lim_{k\to\infty} t_k = \infty$. $\Delta x = x(t^+) - x(t^-)$ where $x(t^-)$ (and $x(t^+)$) is the state just before (and just after) the impulsive action with $x(t^+) = \lim_{s\to t^+} x(s)$. The solution x is assumed to be left-continuous, i.e., $x(t_k^-) = x(t_k)$. $f(t, x(t)) : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^n$ are piecewise continuous vector-valued functions with $f(t, 0) \equiv 0$. Assume that the following assumptions are valid:

(A1) System (5.1) is regular and controllable.

(A2) B has full column rank; that is, $\text{rank}(B) = p$ where $1 \leq p < n$.

(A3) The perturbation term $f(t, x(t))$ is Lipschitz in x for all $t \geq t_0$ and satisfies the following matching condition:

$$f(t, x(t)) = B\bar{f}(t, x(t))$$

where $\bar{f}(t, x(t)) \in \mathbb{R}^p$ bounded by

$$\|\bar{f}(t, x(t))\| \leq \varepsilon \|x(t)\|$$

with $\varepsilon > 0$ being positive constant.

Remark 5.2.1. *The stability of singular systems requires regularity of the system expressed by assumption (A1). On the other hand, the sliding motion is insensitive to the nonlinear perturbation in the system by (A3). This assumption makes SMC an attractive one for designing robust controllers.*

Remark 5.2.2. *The matching condition given in assumption (A3) means that $f(x(t), t)$ lies in the range space of B, i.e. $\mathcal{R}(f(x(t), t)) \subset \mathcal{R}(B)$, so it is possible to write $f(t, x(t)) = B\bar{f}(t, x(, t)$ for some $\bar{f}(t, x(t)) \in \mathbb{R}^p$.*

5.2.1 Sliding Mode Control Design

Consider system (5.1) with p inputs. Then, the p-dimensional vector of sliding mode hyper-surface is defined by

$$\mathscr{S}(x) = \{x(t) \mid \quad S(x) = 0\},$$

where switching function is

$$S(x) = \begin{bmatrix} s_1(x) \\ s_2(x) \\ \vdots \\ s_p(x) \end{bmatrix}_{p\times 1} = GEx,$$

where s_i is a scalar-valued function for $i = 1, 2, \ldots, p$, $G \in \mathbb{R}^{p\times n}$ has full row rank, i.e. $\text{rank}(G) = p$ such that $GB \in \mathbb{R}^{p\times p}$ is an invertible matrix, which is guaranteed because $\text{rank}(B) = p$ (as stated in Assumption (A2)). Then, the time derivative of $S(x)$ along the trajectories of (5.1) is given by

$$\dot{S}(x) = GE\dot{x} = GAx + GBu + GB\bar{f}(t, x(t)).$$

Also, $\dot{S}(x) = 0$ when system (5.1) remains on the sliding surface, which leads to the p-dimensional equivalent control input

$$u^{eq}(t) = -(GB)^{-1}GAx(t) - \bar{f}(x(t), t) =: Kx(t) - \bar{f}(t, x(t)). \tag{5.2}$$

Thus, the closed-loop equivalent reduced system during the sliding motion is given by

$$\begin{aligned} E\dot{x}(t) &= \big(I - B(GB)^{-1}G\big)Ax(t) = (A + BK)x(t) =: A^{eq}x(t), && t \neq t_k \\ \triangle x(t) &= I_k x(t), && t = t_k \\ x(t_0) &= x_0, \end{aligned} \tag{5.3}$$

where the matrix pairs $\big(E, A^{eq}\big)$ is regular and stable.

5.2.2 Reaching Condition

Define the Lyapunov function of the state $S(x)$ by

$$V(S(x)) = \frac{1}{2}S^T(x)S(x) \tag{5.4}$$

and require that the time derivative along the trajectories of the closed-loop full system with multiple inputs

$$\dot{V}(S(x)) = \frac{\partial V}{\partial S(x)}\frac{dS(x)}{dt} = S^T(x)\dot{S}(x) = S^T(x)(GAx + GBu + GB\bar{f}(t, x(t)) < 0$$

which is guaranteed if the control input u takes the form

$$u(t) = u^{eq}(t) - (GB)^{-1}\operatorname{diag}(\eta)\operatorname{Sgn}(S(x(t))) \tag{5.5}$$

where $\operatorname{diag}(\eta)$ is an $p \times p$ diagonal matrix with positive entries η and Sgn is the p-dimensional signum vector function defined as follows:

$$\operatorname{Sgn}(S(x(t))) = \begin{bmatrix} \operatorname{sgn}(s_1(x(t))) \\ \operatorname{sgn}(s_2(x(t))) \\ \vdots \\ \operatorname{sgn}(s_p(x(t))) \end{bmatrix}_{p\times 1}, \text{ where } \operatorname{sgn}(s_i(x(t))) = \begin{cases} 1, & \text{if } s_i(x(t)) > 0, \\ 0, & \text{if } s_i(x(t)) = 0, \\ -1, & \text{if } s_i(x(t)) < 0, \end{cases} \tag{5.6}$$

for $i = 1, 2, \ldots, p$.

Therefore, the full order closed-loop system with multiple inputs outside the sliding surface is given by

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + Bu(t) + f(t, x(t)), && t \neq t_k \\ \triangle x(t) &= I_k x(t), && t = t_k \\ x(t_0) &= x_0, \end{aligned} \tag{5.7}$$

where the control law has the form in (5.5).

On the sliding surface, $S(x(t)) = 0$, we have $\operatorname{Sgn}(S(x(t))) = 0$ which leads $u(t) = u^{eq}(t)$.

In this case, the closed-loop system with multiple inputs on the sliding surface is given by

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + Bu^{eq}(t) + f(t, x(t)) = (I - B(GB)^{-1}G)Ax(t), && t \neq t_k \\ \triangle x(t) &= I_k x(t), && t = t_k \\ x(t_0) &= x_0. \end{aligned} \tag{5.8}$$

where $u^{eq}(t) = -(GB)^{-1}GAx(t) - \bar{f}(t, x(t))$.

Now, by the following theorem one may check the control input in (5.5) satisfies reachability condition. Moreover, an upper bound for reaching time can be computed by the following theorem.

Theorem 5.2.1. *Suppose that sliding surface is given by*

$$\mathscr{S}(x) = \{x(t) \mid \quad S(x) = 0\}, \tag{5.9}$$

where switching function is $S(x) = GEx$. Then the trajectory of system (5.1) can be driven on the sliding surface in a finite time, and it subsequently remains on it if the sliding mode controller is designed as

$$u(t) = u^{eq}(t) - (GB)^{-1}\operatorname{diag}(\eta)\operatorname{Sgn}(S(x(t))) \tag{5.10}$$

where r-dimensional equivalent control input

$$u^{eq}(t) = -(GB)^{-1}GAx(t) - \bar{f}(t, x(t)) \tag{5.11}$$

such that $GB \in \mathbb{R}^{p\times p}$ is an invertible matrix, diag(η) *is an $p \times p$ diagonal matrix with positive diagonal elements, and* Sgn *is the p-dimensional signum vector function.*

Proof. Define the Lyapunov function candidate by

$$V(S(x)) = \frac{1}{2}S^T(x)S(x).$$

Then, the time derivative along the trajectory of (5.1)

$$\begin{aligned}\dot{V}(S(x)) &= S^T(x)\dot{S}(x) \\ &= S^T(x)\left(GE\dot{x}\right) \\ &= S^T(x)\left(GAx + GBu + GB\bar{f}(t,x(t))\right).\end{aligned}$$

Substituting the control law in (5.10) into the above equation, we have

$$\begin{aligned}\dot{V}(S(x)) &= S^T(x)GAx \\ &\quad + S^T(x)\left(GB\right)\left(-(GB)^{-1}GAx(t) - \bar{f}(t,x(t)) - (GB)^{-1}\operatorname{diag}(\eta)\operatorname{Sgn}(S(x(t)))\right) \\ &\quad + S^T(x)GB\bar{f}(t,x(t)) \\ &= S^T(x)GAx - S^T(x)\left(GB\right)\left(GB\right)^{-1}GAx(t) - S^T(x)GB\bar{f}(t,x(t)) \\ &\quad - S^T(x)\left(GB\right)\left(GB\right)^{-1}\operatorname{diag}(\eta)\operatorname{Sgn}(S(x(t))) + S^T(x)GB\bar{f}(t,x(t)) \\ &= -S^T(x)\operatorname{diag}(\eta)\operatorname{Sgn}(S(x(t))).\end{aligned}$$

Plugging in definition of each factor into the above equations gives

$$\begin{aligned}\dot{V}(S(x)) &= -\begin{bmatrix} s_1(x) \\ s_2(x) \\ \vdots \\ s_p(x)\end{bmatrix}^T \begin{bmatrix} \eta_1 & 0 & \cdots & 0 \\ 0 & \eta_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \eta_p \end{bmatrix} \begin{bmatrix} \operatorname{sgn}(s_1(x)) \\ \operatorname{sgn}(s_2(x)) \\ \vdots \\ \operatorname{sgn}(s_p(x))\end{bmatrix} \\ &= -\left[s_1(x)\eta_1\operatorname{sgn}(s_1(x) + s_2(x)\eta_2\operatorname{sgn}(s_2(x) + \cdots + s_p(x)\eta_p\operatorname{sgn}(s_p(x)\right] \\ &= -\left[\eta_1|s_1(x)| + \eta_2|s_2(x)| + \cdots + \eta_p|s_p(x)|\right] \\ &\leq -\min_{i=1,2,\ldots,p}\{\eta_i\} - \left[|s_1(x)| + |s_2(x)| + \cdots + |s_p(x)|\right] \\ &= -\min_{i=1,2,\ldots,p}\{\eta_i\}\,\|S(x)\|_1\end{aligned}$$

$$\le -\min_{i=1,2,\dots,p}\{\eta_i\}\sqrt{p}\,\|S(x)\|$$
$$\le -\min_{i=1,2,\dots,p}\{\eta_i\}\sqrt{2p}V^{\frac{1}{2}}$$

which implies

$$\dot{V}(S(x)) = S^T(x)\dot{S}(x) < 0.$$

Therefore, the state of system (5.1) will reach the sliding surface in finite time and subsequently remain on it.

We can also calculate the finite time as follows: by integrating the last inequality from t_k to t and t_{k-1} to t_k for $k = 1, 2, \dots,$ we have

$$\begin{aligned}
V^{\frac{1}{2}}(S(x(t))) - V^{\frac{1}{2}}(S(x(t_k))) &\le -\frac{\sqrt{2p}}{2}\min_{i=1,2,\dots,p}\{\eta_i\}(t - t_k),\\
V^{\frac{1}{2}}(S(x(t_k))) - V^{\frac{1}{2}}(S(x(t_{k-1}))) &\le -\frac{\sqrt{2p}}{2}\min_{i=1,2,\dots,p}\{\eta_i\}(t_k - t_{k-1}),\\
V^{\frac{1}{2}}(S(x(t_{k-1}))) - V^{\frac{1}{2}}(S(x(t_{k-2}))) &\le -\frac{\sqrt{2p}}{2}\min_{i=1,2,\dots,p}\{\eta_i\}(t_{k-1} - t_{k-2}),\\
&\vdots\\
V^{\frac{1}{2}}(S(x(t_1))) - V^{\frac{1}{2}}(S(x(t_0))) &\le -\frac{\sqrt{2p}}{2}\min_{i=1,2,\dots,p}\{\eta_i\}(t_1 - t_0).
\end{aligned} \tag{5.12}$$

Summing both sides of (5.12) gives

$$V^{\frac{1}{2}}(S(x(t))) - V^{\frac{1}{2}}(S(x(t_0))) \le -\frac{\sqrt{2p}}{2}\min_{i=1,2,\dots,p}\{\eta_i\}(t - t_0). \tag{5.13}$$

It can be seen from (5.13) that there exists a time t^* such that $V(S(x(t^*))) = 0$ and consequently $S = 0$ for $t \ge t^*$ which means that the system trajectories can reach to the sliding surface in a finite time t^*. Thus, this finite time t^* is calculated

$$
\begin{aligned}
V^{\frac{1}{2}}(S(x(t^*))) - V^{\frac{1}{2}}(S(x(t_0))) &\leq -\frac{\sqrt{2p}}{2} \min_{i=1,2,\ldots,p} \left\{\eta_{i_j}\right\}(t^* - t_0), \\
\Rightarrow -V^{\frac{1}{2}}(S(x(t_0))) &\leq -\frac{\sqrt{2p}}{2} \min_{i=1,2,\ldots,p} \left\{\eta_i\right\}(t^* - t_0) \\
\Rightarrow t^* &\leq \frac{\sqrt{2}}{\sqrt{p}\min_{i=1,2,\ldots,p}\left\{\eta_i\right\}} V^{\frac{1}{2}}(S(x(t_0))) + t_0
\end{aligned}
$$

□

Designing of the control input on the sliding surface guarantees stability of closed loop system (5.8). However, impulse affects the stability of the entire closed loop system. Therefore, the following theorem gives sufficient conditions to ensure exponential stability of entire system on the sliding surface.

Theorem 5.2.2. ***(Stability result)*** *Assume that the following assumptions hold:*

(i) *System (5.1) is admissible during the sliding motion so that for any $Y > 0$, there exist $X > 0$, the solution of*

$$E^T X(A + BK) + (A + BK)^T XE = -E^T Y E \tag{5.14}$$

where the control gain matrix $K = -(GB)^{-1}GA$.

(ii) *Singular matrix E and $I + I_k$ are commute.*

(iii) *The following inequality holds*

$$\ln \gamma_k - \nu(t_k - t_{k-1}) \leq 0, \qquad k = 1, 2, \ldots, \tag{5.15}$$

where $\gamma_k = \frac{\lambda_{\max}\left[(I+I_k)^T X(I+I_k)\right]}{\lambda_{\min}(X)}$, $0 < \nu < \alpha$ and $\alpha = \frac{\lambda_{\min}(Y)}{\lambda_{\max}(X)}$.

Then, the control law (5.5) guarantees that the system (5.1) be exponentially stable.

Proof. For all $t \geq t_0$ with $t_0 \in \mathbb{R}^+$, let $x(t) = x(t, t_0, x_0)$ be the solution of the system (5.1). For $t \in (t_{k-1}, t_k]$, define

$$\upsilon(t) = V(x(t)) = x^T(t)E^T XEx(t), \qquad t \neq t_k,$$

as the Lyapunov function candidate. Then, the time derivative of υ along the trajectory of (5.1) during the sliding motion is given by

$$\begin{aligned}
\dot{\upsilon}(t) &= 2\dot{x}^T(t)E^T XEx(t) \\
&= \dot{x}^T(t)E^T XEx(t) + x^T(t)E^T XE\dot{x}(t) \\
&= x^T(t)\big(A + BK\big)^T XEx(t) + x^T(t)E^T X\big(A + BK\big)x(t) \\
&= -x^T(t)E^T YEx(t) \\
&\leq -\lambda_{\min}(Y)\|Ex(t)\|^2 \\
&\leq -\alpha\upsilon(t)
\end{aligned}$$

where $\alpha = \frac{\lambda_{\min}(Y)}{\lambda_{\max}(X)}$. Therefore, we have

$$\upsilon(t) \leq \upsilon(t_{k-1}^+)e^{-\alpha(t-t_{k-1})}, \qquad t \in (t_{k-1}, t_k]. \tag{5.16}$$

On the other hand, at $t = t_k^+$, by assumption (ii) we have

$$\begin{aligned}
\upsilon(t_k^+) &= x^T(t_k^+)E^T XEx(t_k^+) \\
&= x^T(t_k)(I + I_k)^T E^T XE(I + I_k)x(t_k) \\
&\leq \gamma_k \upsilon(t_k),
\end{aligned} \tag{5.17}$$

where $\gamma_k = \frac{\lambda_{\max}\big[(I+I_k)^T X(I+I_k)\big]}{\lambda_{\min}(X)}$. Now, using (5.16) and (5.17) successively on each subinterval leads to the following results: for $t \in (t_0, t_1]$, we have

$$\upsilon(t) \leq \upsilon(t_0^+)e^{-\alpha(t-t_0)}. \tag{5.18}$$

For $t \in (t_1, t_2]$, we have

$$\begin{aligned} \upsilon(t) &\le \upsilon(t_1^+)e^{-\alpha(t-t_1)} \\ &\le \gamma_1 \upsilon(t_1)e^{-\alpha(t-t_1)} \\ &= \gamma_1 \upsilon(t_0^+)e^{-\alpha(t_1-t_0)}e^{-\alpha(t-t_1)} \\ &= \gamma_1 \upsilon(t_0^+)e^{-\alpha(t-t_0)}. \end{aligned} \tag{5.19}$$

In general, for $t \in (t_{k-1}, t_k]$, we obtain

$$\begin{aligned} \upsilon(t) &\le \upsilon(t_0^+)\gamma_1\gamma_2 \dots \gamma_{k-1}e^{-\alpha(t-t_0)} \\ &= \upsilon(t_0^+)\gamma_1\gamma_2 \dots \gamma_{k-1}e^{-\nu(t-t_0)}e^{-(\alpha-\nu)(t-t_0)} \\ &= \upsilon(t_0^+)\gamma_1 e^{-\nu(t_1-t_0)}\gamma_2 e^{-\nu(t_2-t_1)} \dots \gamma_{k-1}e^{-\nu(t_{k-1}-t_{k-2})}e^{-\nu(t-t_{k-1})}e^{-(\alpha-\nu)(t-t_0)}. \end{aligned} \tag{5.20}$$

By assumption (iii), we have

$$\upsilon(t) \le \upsilon(t_0^+)e^{-(\alpha-\nu)(t-t_0)}, \qquad t \ge t_0 \tag{5.21}$$

which implies that

$$\|Ex(t)\| \le \mu \|Ex(t_0^+)\| e^{-(\alpha-\nu(t-t_0))/2}, \qquad t \ge t_0 \tag{5.22}$$

where $\mu = \sqrt{\frac{\lambda_{\max}(X)}{\lambda_{\min}(X)}}$. Thus, the trivial solution of system (5.1) during the sliding motion is E-exponentially stable which is equivalent to its exponential stability. This completes the proof of stability of full order closed-loop system (5.7). □

Remark 5.2.3. *The algebraic Riccati equation given in (5.14) guarantees that the Lyapunov function be decreasing along the trajectory of system (5.1); that is, the continuous system is stable due to the control law u in (5.5).*

Example 5.2.1. *Consider the ISS given by (5.1) where* $x = \begin{bmatrix} x_1(t) & x_2(t) & x_3(t) \end{bmatrix}^T$,

$$E = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \ A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \ B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$\bar{f}(t, x(t)) = \begin{bmatrix} x_1^2(t) + x_2^2(t) \\ 0 \end{bmatrix}, \ \text{and } I_k = 0.1I,$$

where $I \in \mathbb{R}^{n \times n}$ *identity matrix and* $k = 1, 2, \cdots$. *The initial condition is given by* $x_0 = \begin{bmatrix} 1 & -2 & 3 \end{bmatrix}^T$. *Choosing*

$$G = \begin{bmatrix} 1 & 1 & -1 \\ -1 & 2 & -1 \end{bmatrix},$$

the equivalent control is

$$u^{eq} = Kx(t) - \bar{f}$$

where

$$K = \begin{bmatrix} 2 & -1 & 0 \\ 3 & 0 & -1 \end{bmatrix},$$

the corresponding equivalent system is

$$E\dot{x}(t) = A^{eq}x(t)$$

where

$$A^{eq} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 0 \\ 3 & 0 & 0 \end{bmatrix},$$

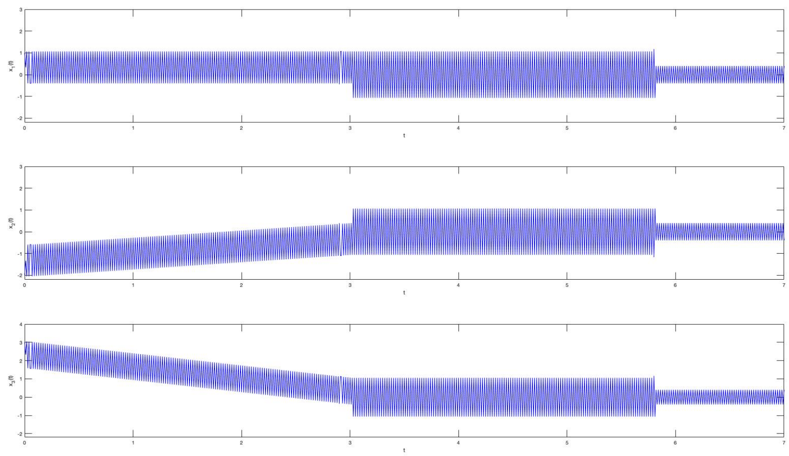

Figure 5.1: Impulsive switched singular system sub-states.

which the matrix pair (E, A^{eq}) *is stable, and the feedback control law is given by*

$$u(t) = u^{eq}(t) - (GB)^{-1}\,\mathrm{diag}(\eta)\,\mathrm{Sgn}(S(x(t)))$$

where

$$(GB)^{-1} = \begin{bmatrix} -1 & 1 \\ -2 & 1 \end{bmatrix}, \quad \eta = [1.1 \quad 1], \quad S(x(t)) = \begin{bmatrix} 1 & 2 & 1 \\ -1 & 1 & 2 \end{bmatrix} x(t).$$

$\alpha = 0.6745$ *and* $\gamma_k = 7.0522$ *are computed. Therefore,* $t_k - t_{k-1} \geq 2.8959$. *Exponential stability of the system by the designed continuous sliding mode control, the control input and sliding surface are simulated in Figure 5.1, Figure 5.2, and Figure 5.3 respectively.*

5.3 Discrete Sliding Mode Control

The finite sampling frequency in discrete sliding mode control (DSMC) makes control input invariant between two sampling instants. This means that when system dynamics cross

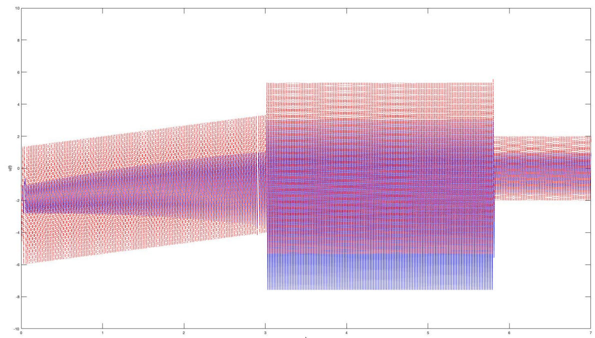

Figure 5.2: Control input $u = [u_1 \quad u_2]^T$.

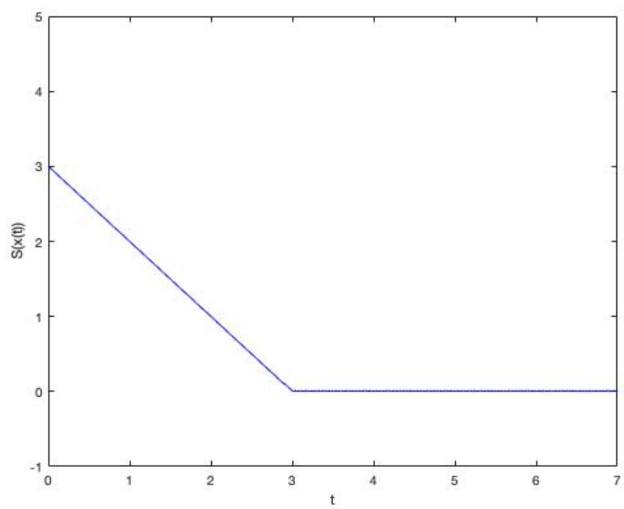

Figure 5.3: Sliding function $S(x(t))$.

the sliding surface between sampling instants, the control input cannot immediately take measures to make the system remain on the sliding surface. Therefore, in DSMC, system states cannot remain on the sliding surface, but can remain in a neighborhood of the sliding surface in which switching function is equal to zero. This sliding-like motion in discrete time is called a quasi-sliding mode or pseudo-sliding mode which is stated in the following definition.

Consider the single input discrete time system given by:

$$x(n+1) = Ax(n) + Bu(n), \qquad n = 0, 1, 2, \ldots \tag{5.23}$$

where $x \in \mathbb{R}^{N\times 1}$, $u \in \mathbb{R}^1$, $A \in \mathbb{R}^{N\times N}$ and $B \in \mathbb{R}^{N\times 1}$ are matrices of appropriate dimensions.

Definition 5.3.1. *System (5.23) is said to exhibit a pseudo-sliding mode if there exists an integer K such that, for all $n > K$, whenever $x(n)$ is contained in a neighborhood $\mathcal{B}_s$ of $\{x(n):\ s(n) = 0\}$ where $s(n)$ is the switching function, then $x(n+1)$ is also contained in the $\mathcal{B}_s$.*

In general, $\mathcal{B}_s$ can be defined in two ways [60]:

$$\mathcal{B}_s = \{x(n):\ |s(n)| \leq \varepsilon\}, \tag{5.24}$$

$$\mathcal{B}_s = \{x(n):\ |s(n)| \leq \varepsilon \|x(n)\|_1\} \text{ where } \|x(n)\|_1 = \sum_{i=1}^{N} |x(n)|. \tag{5.25}$$

The set defined in (5.24) is called a boundary layer type while the set defined in (5.25) is called a sliding or switching region type. These two sets are illustrated in Figure 5.4.

Remark 5.3.1. *For clarity, the quasi-sliding mode definition, the boundary layer in (5.24) and the switching region in (5.25)are given for systems with single input.*

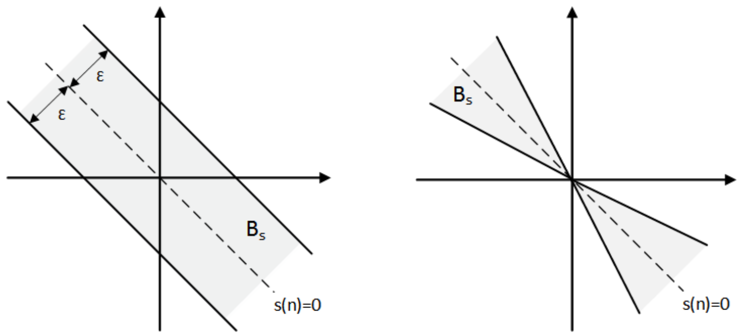

Figure 5.4: Boundary layer (**left**) and switching region (**right**)

The state trajectory of a DSMC system has the following properties [61]:

(P1) The trajectory moves monotonically toward the sliding surface by starting from any initial state, and cross the sliding surface in finite time.

(P2) Once the trajectory has crossed the switching plane the first time, it cross the surface again in every successive sampling period. Therefore, a zigzag motion about the sliding surface occurs.

(P3) The size of each successive zigzagging step is non-increasing and the trajectory stays within B_s.

Consider a discrete impulsive singular system with perturbation of the form

$$\begin{aligned} Ex(n+1) &= Ax(n) + Bu(n) + f(n, x(n)), && n_k^+ < n \leq n_{k+1} \\ \triangle x(n) &= I_k x(n), && n = n_k, \quad k \in \mathbb{N} \\ x(n_0) &= x_0, \end{aligned} \tag{5.26}$$

where $x(n) \in \mathbb{R}^N$ is the system state at the sampling instant n for all $n \geq n_0$ with $n_0 \in \mathbb{N}$, and $A, E, I_k \in \mathbb{R}^{N\times N}$ and $B \in \mathbb{R}^{N\times p}$are system coefficient matrices where E being singular with $\operatorname{rank}(E) = r < N$, $u(n) \in \mathbb{R}^p$ is the input vector and $I_k \in \mathbb{R}^{N\times N}$ being constant matrices. $\{n_k\}_{k=0}^{\infty}$ are the impulsive times that form an increasing sequence satisfying $n_{k-1} < n_k$ and $\lim_{k\to\infty} n_k = \infty$. $\Delta x(n_k) = x(n_k^+) - x(n_k)$ where $x(n_k^+)$ is the value of x at n_k with impulse, and $x(n_k)$ is the value of x at n_k without impulse. $f(n, x(n)) : \mathbb{Z}^+ \times \mathbb{R}^N \to \mathbb{R}^N$ are piecewise continuous vector-valued functions with $f(n, 0) \equiv 0$.

Addition to assumption (A2), the system in (5.26) holds the following assumptions:

(A4) The system in (5.26) is regular and controllable.

(A5) System (5.26) satisfies the following matching condition:

$$f(n, x(n)) = B\bar{f}(n, x(n))$$

with $\bar{f}(n, x(n)) \in \mathbb{R}^p$ bounded by

$$\|\bar{f}(n, x(n))\| \leq \varepsilon \|x(n)\|$$

where $\varepsilon > 0$ is positive constants.

5.3.1 Quasi-Sliding Mode Control Design

Consider system (5.26) with p inputs. Then, the p-dimensional vector of quasi-sliding mode hyper-surface is defined by

$$\mathscr{S}(n) = \{x(n) \mid \quad S(n) = 0\},$$

where switching function is

$$S(n) = \begin{bmatrix} s_1(n) \\ s_2(n) \\ \vdots \\ s_p(n) \end{bmatrix}_{p\times 1} = GEx(n), \tag{5.27}$$

where $s_i(n)$ is a scalar-valued function $(i = 1, 2, \ldots, p)$, and $G \in \mathbb{R}^{p\times N}$ has full row rank, i.e. rank$G = p$ such that $GB \in \mathbb{R}^{p\times p}$ is an invertible matrix. The ideal quasi-sliding mode satisfies

$$S(n+1) = S(n) = 0, \qquad n = n_0, n_1, n_2, \ldots \tag{5.28}$$

From (5.28) and (5.26), one can get

$$GAx(n) + GBu(n) + GB\bar{f}(n, x(n)) = S(n) = 0.$$

Solving this equation for u gives the equivalent control input as

$$u^{eq}(n) = -(GB^{-1}GAx(n) - \bar{f}(n, x(n)) =: Kx(n) - \bar{f}(n, x(n)), \tag{5.29}$$

where $GB \in \mathbb{R}^{p\times p}$ is an invertible matrix. Thus, the closed-loop equivalent system during the ideal quasi-sliding motion is given by

$$\begin{aligned} Ex(n+1) &= \big(I - B(GB)^{-1}G\big)Ax(n) = (A + BK)x(n) =: A^{eq}x(n), && n_{k-1}^{+} < n \leq n_k \\ \triangle x(n) &= I_k x(n), && n = n_k, \quad k \in \mathbb{N} \\ x(n_0) &= x_0 \end{aligned} \tag{5.30}$$

where (E, A^{eq}) is regular and stable.

5.3.2 Reaching Condition

The necessary reaching condition for the quasi-sliding mode is expressed as

$$\triangle S(n)S(n) < 0$$

where $\triangle S(n) = S(n+1) - S(n)$. On the other hand, the sufficient reaching condition is given by

$$|s_i(n+1)| < |s_i(n)|, \quad i = 1, 2, \ldots, p, \tag{5.31}$$

which has the following equivalent form:

$$s_i^2(n+1) < s_i^2(n).$$

Remark 5.3.2. *From condition (5.31), it can be seen that DSMC is designed such that the switching function is decreased at every sampling index n.*

Condition (5.31) is satisfied by the following reaching law,

$$S(n+1) = \operatorname{diag}(q)S(n) - \operatorname{diag}(\eta)\operatorname{Sgn}(S(n)), \tag{5.32}$$

where $\operatorname{diag}(q) = \operatorname{diag}\{q_1, q_2, \ldots, q_p\} \in \mathbb{R}^{p\times p}$, $\operatorname{diag}(\eta) = \operatorname{diag}\{\eta_1, \eta_2, \ldots, \eta_p\} \in \mathbb{R}^{p\times p}$, $0 < q_i < 1$, $\eta_i > 0$ and Sgn is the p-dimensional signum vector function defined as follows:

$$\operatorname{Sgn}(S(n)) = \begin{bmatrix} \operatorname{sgn}(s_1(n)) \\ \operatorname{sgn}(s_2(n)) \\ \vdots \\ \operatorname{sgn}(s_p(n)) \end{bmatrix}_{p\times 1}, \text{ where } \operatorname{sgn}(s_i(n)) = \begin{cases} 1, & \text{if } s_i(n) > 0, \\ 0, & \text{if } s_i(n) = 0, \\ -1, & \text{if } s_i(n) < 0, \end{cases} \tag{5.33}$$

for $i = 1, 2, \ldots, p$.

From (5.32), one may get

$$s_i(n+1) = q_i s_i(n) - \eta_i \operatorname{sgn}(s_i(n)).$$

From this expression, the following is obtained

$$s_i^2(n+1) - s_i^2(n) = -(1-q_i^2)\left(|s_i(n)| + \frac{q_i\eta_i}{1-q_i^2}\right)^2 + \frac{\eta_i^2}{1-q_i^2},$$

where $|s_i(n)| = s_i(n)\,\mathrm{sgn}(s_i(n))$. Therefore, $s_i^2(n+1) < s_i^2(n)$ if $|s_i(n)| > \dfrac{\eta_i}{1+q_i}$. In other word, the reaching law (5.32) satisfies the reaching condition (5.31) only if $|s_i(n)| > \dfrac{\eta_i}{1+q_i}$.

By (5.27), the incremental change of $S(n)$ is

$$\begin{aligned} S(n+1) - S(n) &= GEx(n+1) - GEx(n) \\ &= GAx(n) + GBu(n) + GB\bar{f}(n, x(n)) - GEx(n). \end{aligned} \tag{5.34}$$

On the other hand, from reaching law (5.32) we have

$$\begin{aligned} S(n+1) - S(n) &= (\mathrm{diag}(q) - I)S(n) - \mathrm{diag}(\eta)\,\mathrm{Sgn}(S(n)), \\ &= (\mathrm{diag}(q) - I)GEx(n) - \mathrm{diag}(\eta)\,\mathrm{Sgn}(GEx(n)) \end{aligned} \tag{5.35}$$

where $I \in \mathbb{R}^{p\times p}$ is identity matrix. Comparing equations (5.34) and (5.35) gives

$$GAx(n) + GBu(n) + GB\bar{f}(n, x(n)) - GEx(n) = (\mathrm{diag}(q) - I)GEx(n) - \mathrm{diag}(\eta)\,\mathrm{Sgn}(GEx(n)).$$

Solving for $u(n)$ gives the control law

$$u(n) = u^{eq}(n) + (GB)^{-1}\big[\,\mathrm{diag}(q)S(n) - \mathrm{diag}(\eta)\,\mathrm{Sgn}(S(n))\big] \tag{5.36}$$

where $u^{eq}(n) = -(GB)^{-1}GAx(n) - \bar{f}(n, x(n))$ and $S(n) = GEx(n)$. Therefore, the closed-loop full system with multiple inputs is given by

$$\begin{aligned} Ex(n+1) &= Ax(n) + Bu(n) + f(n, x(n)), & n_k^+ < n \le n_{k+1} \\ \triangle x(n) &= I_k x(n), & n = n_k, \quad k \in \mathbb{N} \\ x(n_0) &= x_0, \end{aligned} \tag{5.37}$$

where the control input is $u(n) = u^{eq}(n) + (GB)^{-1}\big[\operatorname{diag}(q)S(n) - \operatorname{diag}(\eta)\operatorname{Sgn}(S(n))\big]$ and $u^{eq}(n) = -(GB)^{-1}GAx(n) - \bar{f}(n, x(n))$.

5.3.3 Boundary Layer Type $\mathcal{B}_s$

The neighborhood $\mathcal{B}_s$ expressed in the definition of a pseudo-sliding mode (see Definition 5.3.1) is defined as either boundary layer type by (5.24) or switching region type by (5.25). In boundary layer type, switching occurs on the sliding surface, but control input is affected by the distance from the current system states to the sliding surface. On the contrary, the input $u(n)$ is equal to the equivalent control input $u^{eq}(n)$ when the switching function is in the switching region ([60]). However, we only consider the boundary layer type one in this thesis.

First of all, we should obtain the boundary layer type neighborhoods for the system in (5.26) with multiple inputs, represented by $\mathcal{B}_S$, by using (5.24). According to (5.24), we should also determine ε in the boundary layer type $\mathcal{B}_s = \{x(n) : \ |s(n)| \leq \varepsilon\}$.
By the reaching law in (5.32), we have

$$s_i(n+1) = q_i s_i(n) - \eta_i \operatorname{sgn}(s_i(n)), \quad i = 1, 2, \ldots, p. \tag{5.38}$$

The sign of the first term of the right hand side in (5.38) is the same as $s_i(n)$ and the second term of the right hand side in (5.38) is opposite to that of $s_i(n)$. By the definition of quasi-sliding mode, the sign of $s_i(n+1)$ must be opposite to that of $s_i(n)$. For instance, if $s_i(n) > 0$, then by (5.38) we have

$$s_i(n+1) = q_i s_i(n) - \eta_i < 0$$

which implies

$$s_i(n) < \frac{\eta_i}{q_i}. \tag{5.39}$$

If $s_i(n) < 0$, then by (5.38) we have

$$s_i(n+1) = q_i s_i(n) + \eta_i > 0$$

which implies

$$s_i(n) > -\frac{\eta_i}{q_i}. \tag{5.40}$$

Therefore, the boundary layer type $\mathcal{B}_s$ is obtained and represented as

$$\mathcal{B}_S = \{x(n) : \ |s_i(n)| < \varepsilon_i\},$$

where $\varepsilon_i = \dfrac{\eta_i}{q_i}$.

Remark 5.3.3. *In DSMC, the state trajectory reach and stay within above specified boundary layer which is a neighborhood of the sliding surface.*

5.3.4 Chattering Elimination

The control law is already obtained as

$$u(n) = u^{eq}(n) + (GB)^{-1}\big[\operatorname{diag}(q)S(n) - \operatorname{diag}(\eta)\operatorname{Sgn}(S(n))\big]$$

where $u^{eq}(n) = -(GB)^{-1}GAx(n) - \bar{f}(x(n), n)$. The discontinuous signum function in the control law causes chattering so that to reduce or eliminate chattering, the signum function in reaching law (5.32) is replaced by a high-slope saturation function; that is, the reaching law is taken as

$$S(n+1) = \operatorname{diag}(q)S(n) - \operatorname{diag}(\eta)\operatorname{Sat}\left(\frac{S(n)}{\phi}\right), \tag{5.41}$$

where the norm of $\phi = \begin{bmatrix} \phi_1 & \phi_2 & \dots & \phi_p \end{bmatrix}^T$ is the boundary layer thickness and Sat is the p-dimensional saturation vector function defined by

$$\text{Sat}\left(\frac{S(n)}{\phi}\right) = \begin{bmatrix} \text{sat}\left(\frac{s_1(n)}{\phi_1}\right) \\ \text{sat}\left(\frac{s_2(n)}{\phi_2}\right) \\ \vdots \\ \text{sat}\left(\frac{s_p(n)}{\phi_p}\right) \end{bmatrix}_{p\times 1}, \text{ where } \text{sat}\left(\frac{s_i(n)}{\phi_i}\right) = \begin{cases} \text{sgn}\left(\frac{s_i(n)}{\phi_i}\right), & \text{if } |s_i(n)| > \phi_i, \\ \frac{s_i(n)}{\phi_i}, & \text{if } |s_i(n)| \leq \phi_i, \end{cases} \tag{5.42}$$

for $i = 1, 2, \dots, p$.

From (5.41), one may get

$$s_i(n+1) = q_i s_i(n) - \eta_i \,\text{sat}\left(\frac{s_i(n)}{\phi_i}\right) = \begin{cases} q_i s_i(n) - \eta_i \,\text{sgn}\left(\frac{s_i(n)}{\phi_i}\right), & \text{if } |s_i(n)| > \phi_i, \\ q_i s_i(n) - \eta_i \frac{s_i(n)}{\phi_i}, & \text{if } |s_i(n)| \leq \phi_i. \end{cases}$$

In order to find a condition on boundary layer thickness ϕ_i, we consider case $|s_i(n)| \leq \phi_i$. Then, we have

$$s_i^2(n+1) - s_i^2(n) = -(1-q_i^2)s_i^2(n) - 2q_i\eta_i \frac{s_i^2(n)}{\phi_i} + \eta_i^2 \frac{s_i^2(n)}{\phi_i^2}.$$

Therefore, $s_i^2(n+1) < s_i^2(n)$, which is an equivalent form of the reaching condition in (5.31) if $\left| q_i - \frac{\eta_i}{\phi_i} \right| < 1$ or equivalently $\phi_i > \frac{\eta_i}{q_i + 1}$.

Theorem 5.3.1. ***(Stability result)*** *Assume that the following assumptions hold:*

(i) *System (5.26) is admissible during the quasi-sliding motion, so for any $Y > 0$, there exists $X > 0$ satisfying the Riccati equation*

$$(A + BK)^T X (A + BK) - E^T X E = -Y, \tag{5.43}$$

where the control gain matrix $K = -(GB)^{-1}GA$.

(ii) *Singular matrix E and $I + I_k$ are commute.*

(iii) *The following inequality holds*

$$\ln \gamma_k + \nu(t_k - t_{k-1}) \ln(1 - \alpha) \leq 0, \qquad k = 1, 2, \ldots, \tag{5.44}$$

where $\gamma_k = \frac{\lambda_{\max}\left[(I+I_k)^T X (I+I_k)\right]}{\lambda_{\min}(X)}$, $0 < \nu < 1$ *and* $\alpha = \frac{\lambda_{\min}(Y)}{\lambda_{\max}(X)}$.

Then, the control law in (5.36) guarantees that system (5.26) is exponentially stable.

Proof. For any $n \geq n_0$ with $n_0 \in \mathbb{N}$, let $x(n) = x(n; n_0, x_0)$ be the solution of system (5.26). For $n \in (n_{k-1}, n_k]$, define

$$V(x(n)) = x^T(n) E^T X E x(n), \tag{5.45}$$

as a Lyapunov function candidate. Then, the variation of V relative to system (5.26) during the quasi-sliding motion (i.e. $S(n)$ is p-dimensional 0 vector) is

$$\begin{aligned}
\triangle V(x(n)) &= x^T(n+1) E^T X E x(n+1) - x^T(n) E^T X E x(n) \\
&= (Ax(n) + BKx(n))^T X (Ax(n) + BKx(n)) \\
&\quad - x^T(n) E^T X E x(n) \\
&= x(n)^T \Big((A + BK)^T X (A + BK) - E^T X E \Big) x(n) \\
&= -x(n)^T Y x(n) \\
&\leq -\lambda_{\min}(Y) \|x(n)\|^2 \\
&\leq -\alpha V(x(n))
\end{aligned} \tag{5.46}$$

where $\alpha = \frac{\lambda_{\min}(Y)}{\lambda_{\max}(E^T X E)}$. Thus, we have

$$V(x(n)) \leq V(x(n_{k-1}^+))(1 - \alpha)^{(n - n_{k-1})}, \qquad n \in (n_{k-1}, n_k]. \tag{5.47}$$

On the other hand, for $n = n_k^+$, $k = 1, 2, 3, \dots$, by assumption (ii) we have

$$\begin{aligned} V(x(n_k^+)) &= x^T(n_k^+)E^T X E x(n_k^+) \\ &= x^T(n_k)(I + I_k)^T E^T X E(I + I_k)x(n_k) \\ &\le \gamma_k V(x(n_k)) \end{aligned} \tag{5.48}$$

where $\gamma_k = \frac{\lambda_{\max}\left[(I+I_k)^T X(I+I_k)\right]}{\lambda_{\min}(X)}$.

Using (5.47) and (5.48) successively on each subinterval leads to the following results: for $n \in (n_0, n_1]$, we have

$$V(x(n)) \le V(x(n_0^+))(1-\alpha)^{(n-n_0)}. \tag{5.49}$$

For $t \in (n_1, n_2]$, we have

$$\begin{aligned} V(x(n)) &\le V_(n_1^+)(1-\alpha)^{(n-n_1)} \\ &\le \gamma_1 V(x(n_1))(1-\alpha)^{(n-n_1)} \\ &= \gamma_1 V(x(n_0^+))(1-\alpha)^{(n_1-n_0)}(1-\alpha)^{(n-n_1)} \\ &= \gamma_1 V(x(n_0^+))(1-\alpha)^{(n-n_0)}. \end{aligned} \tag{5.50}$$

In general, for $n \in (n_0, n_k]$, we obtain

$$V(x(n)) \le V(x(n_0^+))\gamma_1\gamma_2 \dots \gamma_{k-1}(1-\alpha)^{(n-n_0)}. \tag{5.51}$$

Then, for $0 < \nu < 1$, one may obtain

$$\begin{aligned} V(x(n)) &\le \gamma_1\gamma_2 \dots \gamma_{k-1}(1-\alpha)^{\nu(n-n_0)}(1-\alpha)^{(1-\nu)(n-n_0)}V(x(n_0^+)) \\ &= V(x(n_0^+))\gamma_1(1-\alpha)^{\nu(n_1-n_0)}\gamma_2(1-\alpha)^{\nu(n_2-n_1)} \dots \\ &\quad \gamma_{k-1}(1-\alpha)^{\nu(n_{k-1}-n_{k-2})}(1-\alpha)^{\nu(n-n_{k-1})}(1-\alpha)^{(1-\nu)(n-n_0)}. \end{aligned} \tag{5.52}$$

By assumption (iii), (5.52) becomes

$$V(x(n)) \leq V(x(n_0^+))(1-\alpha)^{(1-\nu)(n-n_0)}. \tag{5.53}$$

By definition of Lyapunov candidate function, (5.53) implies

$$\|Ex(n)\| \leq \mu \|Ex(n_0^+)\|(1-\alpha)^{\rho(n-n_0)/2}, \qquad n \geq n_0,$$

where $\mu = \sqrt{\frac{\lambda_{\max}(X)}{\lambda_{\min}(X)}}$. Thus, the trivial solution of system (5.26) during the sliding motion is E-exponentially stable. For system (5.26) on the sliding surface, its E-exponential stability is equivalent to its exponential stability. Thus, the trivial solution of system (5.26) is exponentially stable. □

Example 5.3.1. *Consider the discrete ISS given by (5.26) where* $x(n) = \begin{bmatrix} x_1(n) & x_2(n) & x_3(n) \end{bmatrix}^T$,

$$E = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}, A = \begin{bmatrix} 2 & -1 & 2 \\ 0 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix}, \; B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$\bar{f}(n, x(n)) = \begin{bmatrix} 0 \\ x_2(n) \end{bmatrix}, \; \textit{and } I_k = 0.1I,$$

where $I \in \mathbb{R}^{n \times n}$ *identity matrix and* $k = 1, 2, \cdots$. *The initial condition is given by* $x_0 = \begin{bmatrix} 1.0565 & 1.4435 & 0 \end{bmatrix}^T$. *Choosing*

$$G = \begin{bmatrix} -1 & 1 & 0 \\ -1 & -1 & -2 \end{bmatrix},$$

the equivalent control is

$$u^{eq} = Kx(n) - \bar{f}$$

where

$$K = \begin{bmatrix} 2 & -2 & 2 \\ -2 & 0 & -4 \end{bmatrix},$$

the corresponding equivalent system is

$$E\dot{x}(n) = A^{eq}x(n)$$

where

$$A^{eq} = \begin{bmatrix} 2 & -1 & 2 \\ 2 & -1 & 2 \\ -2 & 1 & -2 \end{bmatrix},$$

which matrix pair (E, A^{eq}) *is stable and the feedback control law is given by*

$$u(n) = u^{eq}(n) + (GB)^{-1}\left[\,\mathrm{diag}(q)S(n) - \mathrm{diag}(\eta)\,\mathrm{Sgn}(S(n))\right]$$

where

$$(GB)^{-1} = \begin{bmatrix} 1 & 0 \\ -0.5 & -0.5 \end{bmatrix},\ q = [0.1 \quad 0.2],\ \eta = [1.04 \quad 1.14],\ S(x(n)) = \begin{bmatrix} -1 & 0 & 1 \\ -1 & -2 & -1 \end{bmatrix} x(n).$$

$\alpha = 0.9303$ *and* $\gamma_k = 6.7067$ *are obtained. Thus,* $t_k - t_{k-1} \geq 2.3816$ *for* $\nu = 0.3$. *In Figure 5.5, exponential stability of the system by the designed discrete sliding mode control in equation (5.36) is illustrated. In this case, the system exhibits chattering. However, it is clear that the control law with saturation function (5.41) successfully remove chattering as shown in Figure 5.6. Moreover, the control input designed using signum function and saturation function are shown in Figure 5.7 and Figure 5.8, respectively.*

Remark 5.3.4. *Figure 5.6 and Figure 5.8 show that the modified control law with saturation function instead of sign gives better performance. The system in Example 5.3.1 exhibits*

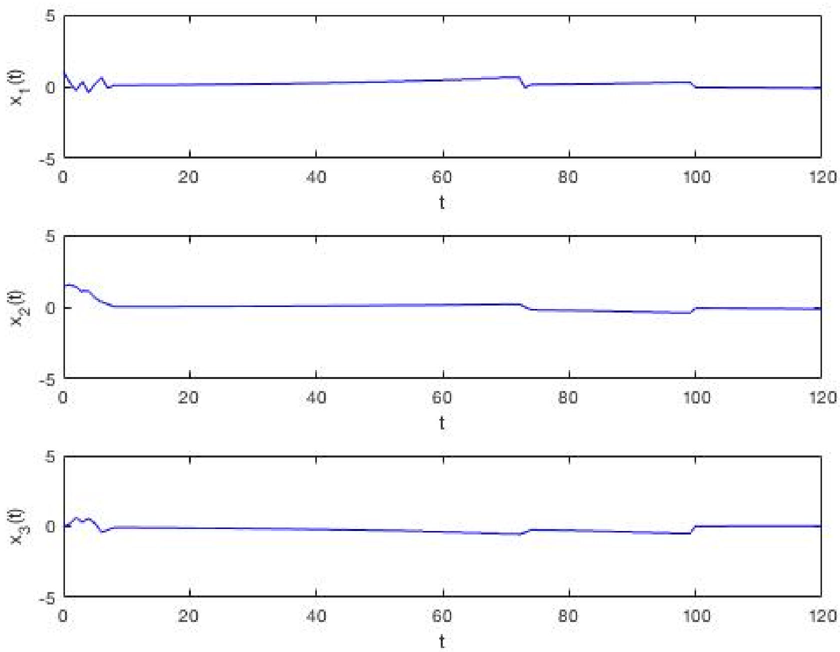

Figure 5.5: System sub-states in control law with sign function

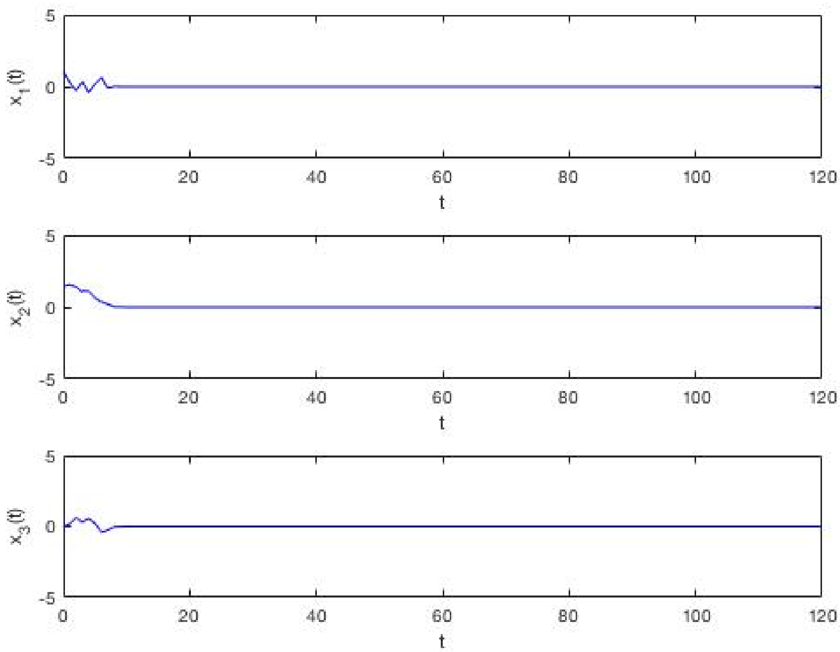

Figure 5.6: System sub-states in control law with saturation function

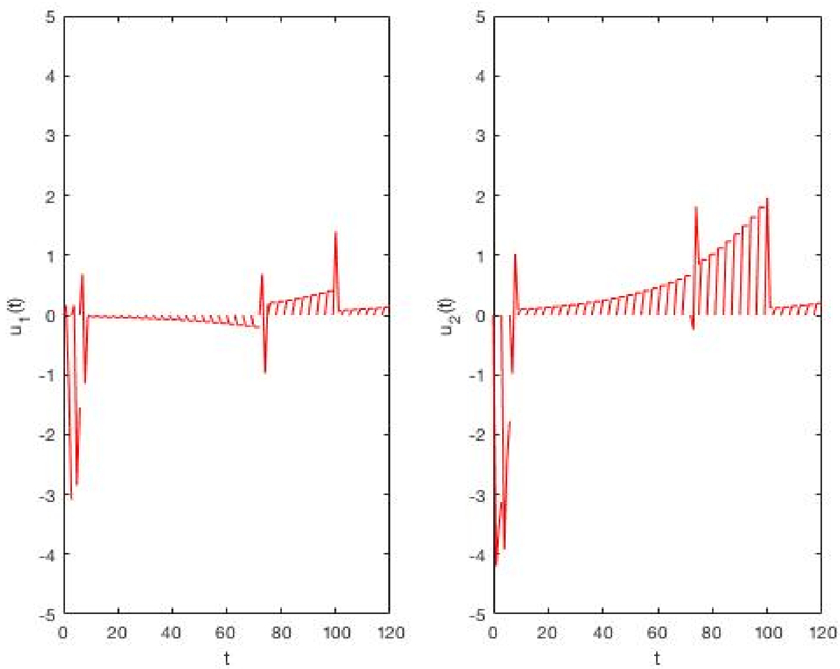

Figure 5.7: Control input $u = [u_1 \quad u_2]^T$ with sign function

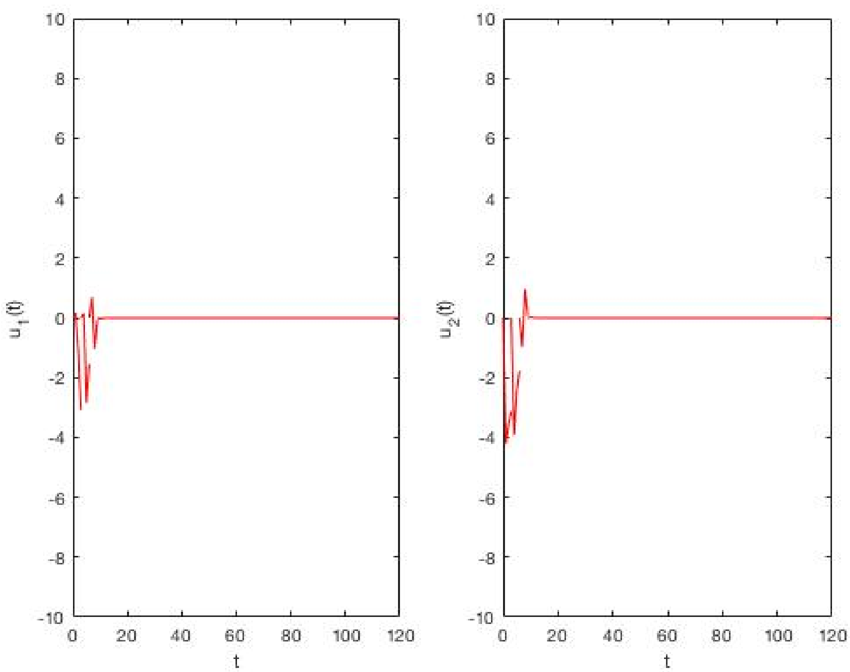

Figure 5.8: Control input $u = [u_1 \quad u_2]^T$ with saturation function

considerable chattering in Figure 5.5 and Figure 5.7. On the contrast, it is clear from Figure 5.6 and Figure 5.8 that the modified control law with saturation function successfully removed chattering. In addition to chattering reduction, settling time is also reduced in control law with saturation function.

Chapter 6

Stochastic Switched Singular Systems: Continuous and Discrete

This chapter establishes the mean square admissibility for continuous and discrete stochastic switched singular systems. Linear matrix inequalities (LMIs) and average dwell time (ADT) are used to develop sufficient conditions that guarantee the admissibility. The theoretical results are illustrated by numerical simulations.

6.1 Stochastic Switched Singular Systems: Continuous

Consider the following stochastic linear singular system

$$
\begin{aligned}
E_{\sigma(t)}dx(t) &= A_{\sigma(t)}x(t)dt + C_{\sigma(t)}x(t)dW(t), \qquad (6.1)\\
x(t_0) &= x_0
\end{aligned}
$$

where $x(t) \in \mathbb{R}^n$ is the system state, and $E_{\sigma(t)}, A_{\sigma(t)} \in \mathbb{R}^{n\times n}$, and $C_{\sigma(t)} \in \mathbb{R}^{n\times n}$ are system coefficient matrices where $E_{\sigma(t)}$ being singular with $\text{rank}(E_{\sigma(t)}) = r < n$, the switching signal $\sigma(t) : [t_0, \infty) \to \Xi$ is a piecewise constant function taking values in a compact set $\Xi = \{1, 2, \ldots, N\}$ for some $N \in \mathbb{N}$, $W(t) \in \mathbb{R}$ is a standard Wiener process defined on the complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

6.1.1 Preliminaries

In this subsection, we recall some basic definitions and lemmas which are used in the following subsections.

Definition 6.1.1. *System (6.1) is said to be exponentially stable in the mean square if there exist positive constant M and λ such that for any initial condition $x(t_0) = x_0$;*

$$\mathbb{E}\left[\|x(t)\|^2\right] \le M\|x_0\|^2 e^{-\lambda(t-t_0)}, \qquad \forall t \ge t_0 \tag{6.2}$$

Definition 6.1.2. *The system in (6.1) is said to be mean square admissible if it is regular, impulse-free and stable in the mean square.*

When the regularity of system $E\dot{x} = Ax$ is not known, it is always possible to choose two nonsingular matrices Q and P such that the following dynamic decomposition form holds:

$$QEP = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}, \quad QAP = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}.$$

Lemma 6.1.1. *[62] System $E\dot{x} = Ax$ is impulse-free if and only if A_4 is nonsingular.*

Lemma 6.1.2. ***(Schur Complement)*** *[62] Given any real matrices P_1, P_2 and P_3 with $P_1 = P_1^T$ and $P_3 > 0$. Then, we have*

$$P_1 + P_2 P_3^{-1} P_2^T < 0,$$

if and only if

$$\begin{bmatrix} P_1 & P_2 \\ P_2^T & -P_3 \end{bmatrix} < 0.$$

Lemma 6.1.3. *[62] Let*

$$\mathcal{N} = \begin{bmatrix} \mathcal{P} & \mathcal{X} \\ \mathcal{Y} & \mathcal{Z} \end{bmatrix},$$

where $\mathcal{P}$, $\mathcal{X}$,$\mathcal{Y}$ and $\mathcal{Z}$ are any real given matrices with appropriate dimensions such that

$$\mathcal{N} + \mathcal{N}^T < 0.$$

Then, $\mathcal{Z}$ is nonsingular and

$$\mathcal{P} + \mathcal{P}^T - \mathcal{X}\mathcal{Z}^{-1}\mathcal{Y} - \mathcal{Y}^T\mathcal{Z}^{-T}\mathcal{X}^T < 0.$$

Definition 6.1.3. *For any $t_0 \in \mathbb{R}^+$ and $t \geq t_0$, let $x(t)$ be an $\mathbb{R}^n$-dimensional Itô process, i.e., $\mathbb{R}^n$-valued continuous adapted process satisfying*

$$dx(t) = f(t, x(t))dt + g(t, x(t))dW(t), \quad (a.s.)$$

where $f \in \mathcal{L}_{ad}(\Omega; L^1[a, b])$ and $g \in \mathcal{L}_{ad}(\Omega; L^2[a, b])$. Let $V \in \mathcal{C}^{1,2}(\mathbb{R}^+ \times \mathbb{R}^n; \mathbb{R})$. Then, for any $t \geq t_0$, V is a real-valued Itô process satisfying

$$dV(t, x) = \mathcal{L}V(t, x)dt + V_x(t, x)g(t, x)dW(t), \quad (a.s.)$$

where

$$\mathcal{L}V(t, x) = V_t(t, x) + V_x(t, x)f(t, x) + \frac{1}{2}tr\left[g^T(t, x)V_{xx}(t, x)g(t, x)\right].$$

If $V \in \mathcal{C}^{1,2}(\mathbb{R}^+ \times \mathbb{R}^n; \mathbb{R})$, *then*

$$V_t(t,x) = \frac{\partial V}{\partial t}, \quad V_x(t,x) = \left[\frac{\partial V}{\partial x_1}, \frac{\partial V}{\partial x_2}, \dots, \frac{\partial V}{\partial x_n}\right],$$

$$V_{xx}(t,x) = \left[\frac{\partial^2 V}{\partial x \partial x_j}\right]_{n\times n} = \begin{bmatrix} \frac{\partial^2 V}{\partial x_1 \partial x_1} & \frac{\partial^2 V}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 V}{\partial x_1 \partial x_n} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial^2 V}{\partial x_n \partial x_1} & \frac{\partial^2 V}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 V}{\partial x_n \partial x_n} \end{bmatrix}.$$

The operator $\mathcal{L}$ *is called the averaged derivative or infinitesimal diffusion operator at a point* (t,x) *and can be generally defined as*

$$\mathcal{L}V(t,x) = \lim_{h\to 0^+} \frac{1}{h}\Big[\mathbb{E}\big[V(t+h, x(t+h))\big] - V(t,x)\Big].$$

6.1.2 Stability Result

In this subsection, we shall establish sufficient conditions for the mean square admissibility for system (6.1).

Theorem 6.1.1. *The stochastic singular system in (6.1) is mean square admissible if*

(i) there exist matrices $X_i > 0$ *and* R_i *such that*

$$E_i^T X_i E_i \geq 0, \tag{6.3}$$

$$A_i^T X_i E_i + E_i^T X_i A_i + A_i^T S_i R_i + R_i^T S_i^T A_i + C_i^T X_i C_i < 0 \tag{6.4}$$

where $S_i \in \mathbb{R}^{n\times(n-r)}$ *is any matrix with full column rank and satisfies* $E_i^T S_i = 0$,

(ii) condition $rank(E_i, C_i) = rank(E_i)$ *holds,*

(iii) for any t_0, the switching law satisfies the ADT condition

$$N(t, t_0) \leq N_0 + \frac{t - t_0}{T_a},$$

where the chatter bound $N_0 = \dfrac{\eta}{\ln \mu}$ such that η is an arbitrary constant and $\mu > 1$, and average dwell time $T_a = \dfrac{\ln \mu}{\lambda - \lambda^*}$ such that $\lambda > \lambda^*$.

Proof. Suppose that there exist matrices $X_i > 0$ and R_i such that (6.3) and (6.4) hold and $rank(E_i, C_i) = rank(E_i)$. We first show that the regularity and impulse-freeness of system (6.1). It is always possible to choose two nonsingular matrices $\tilde{Q}_i$ and $\tilde{P}_i$ such that

$$\tilde{Q}_i E_i \tilde{P}_i = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \quad \tilde{Q}_i A_i \tilde{P}_i = \begin{bmatrix} A_{1i} & A_{2i} \\ A_{3i} & A_{4i} \end{bmatrix}. \tag{6.5}$$

Under the condition $rank(E_i, C_i) = rank(E_i)$,

$$\tilde{Q}_i C_i \tilde{P}_i = \begin{bmatrix} C_{1i} & C_{2i} \\ 0 & 0 \end{bmatrix} \tag{6.6}$$

where the partitions of $\tilde{Q}_i A_i \tilde{P}_i$ and $\tilde{Q}_i C_i \tilde{P}_i$ are compatible with that of $\tilde{Q}_i E_i \tilde{P}_i$.
In view of $X_i > 0$ and the condition $rank(E_i, C_i) = rank(E_i)$, $C_i^T X_i C_i \geq 0$. Therefore, (6.4) becomes

$$A_i^T X_i E_i + E_i^T X_i A_i + A_i^T S_i R_i + R_i^T S_i^T A_i < 0. \tag{6.7}$$

Let $\bar{X}_i = X_i E_i + S_i R_i$, so (6.3) and (6.7) become

$$E_i^T \bar{X}_i = \bar{X}_i^T E_i \geq 0, \tag{6.8}$$

$$A_i^T \bar{X}_i + \bar{X}_i^T A_i < 0. \tag{6.9}$$

Let

$$\tilde{Q}_i^{-T}\bar{X}_i\tilde{P}_i = \begin{bmatrix} \bar{X}_{1i} & \bar{X}_{2i} \\ \bar{X}_{3i} & \bar{X}_{4i} \end{bmatrix}. \tag{6.10}$$

Pre- and post-multiplying (6.8) by $\tilde{P}_i^T$ and $\tilde{P}_i$, respectively, gives

$$\begin{bmatrix} \bar{X}_{1i} & \bar{X}_{2i} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \bar{X}_{1i}^T & 0 \\ \bar{X}_{2i}^T & 0 \end{bmatrix} \geq 0,$$

which implies that $\bar{X}_{2i} = 0$. Moreover, it implies $\bar{X}_{1i} = \bar{X}_{1i}^T \geq 0$ by Schur complement. Noting (6.9) and

$$\alpha(A_i^T\bar{X}_i) \leq \frac{1}{2}\alpha(A_i^T\bar{X}_i + \bar{X}_i^TA_i) < 0$$

where $\alpha(A_i^T\bar{X}_i) = \max_{\{\lambda| \ \det(\lambda I - A_i^T\bar{X}_i)=0\}} Re(\lambda)$. That is, $A_i^T\bar{X}_i$ is nonsingular, so is $\bar{X}_i$. Since $\bar{X}_{1i} = \bar{X}_{1i}^T \geq 0$, this also implies that $\bar{X}_{1i} > 0$.

Now, pre- and post-multiplying (6.9) by $\tilde{P}_i^T$ and $\tilde{P}_i$, respectively, one can obtain

$$\begin{bmatrix} A_{1i}^T\bar{X}_{1i} + \bar{X}_{1i}A_{1i} + A_{3i}^T\bar{X}_{3i} + \bar{X}_{3i}^TA_{3i} & A_{3i}^T\bar{X}_{4i} + \bar{X}_{1i}A_{2i} + \bar{X}_{3i}^TA_{4i} \\ \bar{X}_{4i}^TA_{3i} + A_{2i}^T\bar{X}_{1i} + A_{4i}^T\bar{X}_{3i} & A_{4i}^T\bar{X}_{4i} + \bar{X}_{4i}^TA_{4i} \end{bmatrix} < 0. \tag{6.11}$$

By Schur complement, above matrix inequality is negative definite if and only if

$$A_{4i}^T\bar{X}_{4i} + \bar{X}_{4i}^TA_{4i} < 0.$$

This together with

$$\alpha(\bar{X}_{4i}^TA_{4i}) \leq \frac{1}{2}\alpha(A_{4i}^T\bar{X}_{4i} + \bar{X}_{4i}^TA_{4i}) < 0$$

implies $\bar{X}_{4i}^TA_{4i}$ is nonsingular, so A_{4i} is nonsingular, too. Therefore, each subsystem in (6.1) is impulse-free by Lemma 6.1.1.

To show regularity of each subsystem in (6.1), we need to show determinant of $(sE_i - A_i)$ is not identically zero. Mathematically,

$$
\begin{aligned}
\det(sE_i - A_i) &= \det(s\tilde{Q}_i E_i \tilde{P}_i - \tilde{Q}_i A_i \tilde{P}_i) \\
&= \det\left(\begin{bmatrix} sI - A_{1i} & -A_{2i} \\ -A_{3i} & -A_{4i} \end{bmatrix}\right) \\
&= \det(-A_{4i})\det(sI - (A_{1i} - A_{2i}A_{4i}^{-1}A_{3i})).
\end{aligned}
$$

Since A_{4i} is nonsingular, $\det(-A_{4i}) \neq 0$. We also can find an $s \in \mathbb{C}$ such that $\det(sI - (A_{1i} - A_{2i}A_{4i}^{-1}A_{3i})) \neq 0$. Therefore, we proved the regularity of system (6.1).

Next, we will show that system (6.1) is stochastically stable. Since A_{4i} is nonsingular, we can set

$$
\hat{Q}_i = \begin{bmatrix} I & -A_{2i}A_{4i}^{-1} \\ 0 & I \end{bmatrix} \tilde{Q}_i, \quad \hat{P}_i = \tilde{P}_i \begin{bmatrix} I & 0 \\ -A_{4i}^{-1}A_{3i} & I \end{bmatrix}. \tag{6.12}
$$

By (6.5), (6.6) and (6.10), one can obtain that

$$
\hat{E}_i = \hat{Q}_i E_i \hat{P}_i = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \tag{6.13}
$$

$$
\hat{A}_i = \hat{Q}_i A_i \hat{P}_i = \begin{bmatrix} A_{1i} - A_{2i}A_{4i}^{-1}A_{3i} & 0 \\ 0 & A_{4i} \end{bmatrix}, \tag{6.14}
$$

$$
\hat{C}_i = \hat{Q}_i C_i \hat{P}_i = \begin{bmatrix} C_{1i} - C_{2i}A_{4i}^{-1}A_{3i} & C_{2i} \\ 0 & 0 \end{bmatrix}, \tag{6.15}
$$

$$
\hat{Q}_i^{-T}\bar{X}_i\hat{P}_i = \begin{bmatrix} \bar{X}_{1i} & 0 \\ A_{4i}^{-T}A_{2i}^{T}\bar{X}_{1i} + \bar{X}_{3i} - \bar{X}_{4i}A_{4i}^{-1}A_{3i} & \bar{X}_{4i} \end{bmatrix}. \tag{6.16}
$$

Then, system (6.1) has the following restricted equivalent form

$$dx_1(t) = \tilde{A}_{1i}x_1(t)dt + \tilde{C}_{1i}x_1(t)dW, \tag{6.17}$$

$$x_2(t) = 0, \tag{6.18}$$

where

$$\hat{P}_i^{-1}x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \quad \tilde{A}_{1i} = A_{1i} - A_{2i}A_{4i}^{-1}A_{3i}, \quad \tilde{C}_{1i} = C_{1i} - C_{2i}A_{4i}^{-1}A_{3i}.$$

It follows from (6.8) and (6.9) that $\bar{X}_{1i} > 0$, so we can define the Lyapunov candidate function for i^th subsystem

$$V_i(x_1(t)) = x_1^T(t)\bar{X}_{1i}x_1(t). \tag{6.19}$$

The infinitesimal operator $\mathcal{L}$ of the Markov process acting on the Lyapunov candidate function $V_i(x_1(t))$ is given by

$$\mathcal{L}V_i(x_1(t)) = V_{i_t}(x_1(t)) + V_{i_{x_1}}(x_1(t))\tilde{A}_{1i}x_1(t) + \frac{1}{2}tr\big[x_1^T(t)\tilde{C}_{1i}^TV_{i_{x_1x_1}}(x_1(t))\tilde{C}_{1i}x_1(t)\big].$$

Notice that $V_{i_t}(x_1(t)) = 0$, $V_{i_{x_1}}(x_1(t)) = 2x_1^T(t)\bar{X}_{1i}$ and $V_{i_{x_1x_1}}(x_1(t)) = 2\bar{X}_{1i}$, which implies

$$\begin{aligned}\mathcal{L}V_i(x_1(t)) &= 2x_1^T(t)\bar{X}_{1i}\tilde{A}_{1i}x_1(t) + x_1^T(t)\tilde{C}_{1i}^T\bar{X}_{1i}\tilde{C}_{1i}x_1(t),\\ &= x_1^T(t)\bar{X}_{1i}\tilde{A}_{1i}x_1(t) + x_1^T(t)\tilde{A}_{1i}^T\bar{X}_{1i}x_1(t) + x_1^T(t)\tilde{C}_{1i}^T\bar{X}_{1i}\tilde{C}_{1i}x_1(t)\\ &= x_1^T(t)\left[\bar{X}_{1i}\tilde{A}_{1i} + \tilde{A}_{1i}^T\bar{X}_{1i} + \tilde{C}_{1i}^T\bar{X}_{1i}\tilde{C}_{1i}\right]x_1(t).\end{aligned} \tag{6.20}$$

Now, we can express the inequality in (6.4)

$$A_i^T\bar{X}_i + \bar{X}_i^TA_i + C_i^TXC_i < 0. \tag{6.21}$$

Let

$$\hat{Q}_i^{-T}X_i\hat{Q}_i^{-1} = \begin{bmatrix} X_{1i} & X_{2i} \\ X_{2i}^T & X_{4i} \end{bmatrix}.$$

Noting that $E_i^T \bar{X}_i = E_i^T X_i E_i$ since $E_i^T S_i = 0$. Pre- and post-multiplying this by $\hat{P}_i^T$ and $\hat{P}_i$, respectively, one can get

$$\left(\hat{P}_i^T E_i^T \hat{Q}_i^T\right)\left(\hat{Q}_i^{-T} \bar{X}_i \hat{P}_i\right) = \left(\hat{P}_i^T E_i^T \hat{Q}_i^T\right)\left(\hat{Q}_i^{-T} X_i \hat{Q}_i^{-1}\right)\left(\hat{Q}_i E_i \hat{P}_i\right),$$

which implies

$$\bar{X}_{1i} = X_{1i}.$$

Therefore,

$$\hat{Q}_i^{-T} X_i \hat{Q}_i^{-1} = \begin{bmatrix} \bar{X}_{1i} & X_{2i} \\ X_{2i}^T & X_{4i} \end{bmatrix}. \tag{6.22}$$

Pre- and post-multiplying (6.21) by $\hat{P}_i^T$ and $\hat{P}_i$, respectively, we obtain

$$\begin{aligned} &\left(\hat{P}_i^T A_i^T \hat{Q}_i^T\right)\left(\hat{Q}_i^{-T} \bar{X}_i \hat{P}_i\right) + \left(\hat{P}_i^T \bar{X}_i^T \hat{Q}_i^{-1}\right)\left(\hat{Q}_i A_i \hat{P}_i\right) \\ &\quad + \left(\hat{P}_i^T C_i^T \hat{Q}_i^T\right)\left(\hat{Q}_i^{-T} X_i \hat{Q}_i^{-1}\right)\left(\hat{Q}_i C_i \hat{P}_i\right) < 0. \end{aligned} \tag{6.23}$$

Substituting the expressions in (6.13)-(6.16) and (6.22) to (6.23) gives

$$\begin{bmatrix} \bar{X}_{1i}\tilde{A}_{1i} + \tilde{A}_{1i}^T \bar{X}_{1i} + \tilde{C}_{1i}^T \bar{X}_{1i} \tilde{C}_{1i} & * \\ * & * \end{bmatrix} < 0,$$

where $*$ represents a matrix is not used in the following. Above linear matrix inequality implies that

$$\bar{X}_{1i}\tilde{A}_{1i} + \tilde{A}_{1i}^T \bar{X}_{1i} + \tilde{C}_{1i}^T \bar{X}_{1i} \tilde{C}_{1i} < 0.$$

Thus, we can say $\bar{X}_{1i}\tilde{A}_{1i} + \tilde{A}_{1i}^T \bar{X}_{1i} + \tilde{C}_{1i}^T \bar{X}_{1i} \tilde{C}_{1i} = -Y_i$ where $Y_i > 0$. This together with (6.20) gives

$$\begin{aligned} \mathcal{L}V_i(x_1(t)) &= -x_1(t)^T Y_i x_1(t) \\ &\le -\lambda_{\min}(Y_i)\|x_1(t)\|^2, \quad \text{for all } t \in [t_{k-1}, t_k). \end{aligned}$$

Combining this with general definition of the average derivative $\mathcal{L}$ yields

$$\frac{d}{dt}\mathbb{E}\big[V_i(x_1(t))\big] = \mathbb{E}\big[\mathcal{L}V_i(x_1(t))\big] \leq -\lambda_{\min}(Y_i)\mathbb{E}\big[\|x_1(t)\|^2\big] \leq -\frac{\lambda_{\min}(Y_i)}{\lambda_{\max}(\bar{X}_{1i})}\mathbb{E}\big[V_i(x_1(t))\big].$$

From the last one, we obtain

$$\mathbb{E}\big[V_i(x_1(t))\big] \leq \mathbb{E}\big[V_i(x_1(t_{k-1}))\big]e^{-\lambda_i(t-t_{k-1})}, \qquad t \in [t_{k-1}, t_k) \tag{6.24}$$

where $\lambda_i = \dfrac{\lambda_{\min}(Y_i)}{\lambda_{\max}(\bar{X}_{1i})}$.
From (6.24), for $t \in [t_0, t_1)$, one can obtain

$$\mathbb{E}\big[V_{i_1}(x_1(t))\big] \leq \mathbb{E}\big[V_{i_1}(x_1(t_0))\big]e^{-\lambda_{i_1}(t-t_0)}, \tag{6.25}$$

From (6.19), we have for any $i, j \in \Xi$

$$V_j(x_1(t)) \leq \mu V_i(x_1(t)), \tag{6.26}$$

where $\mu = \dfrac{\max\{\lambda_{\max}(\bar{X}_{1j})\}}{\min\{\lambda_{\min}(\bar{X}_{1j})\}}$.
For $t \in [t_1, t_2)$, from (6.24) and (6.25) we have

$$\begin{aligned}\mathbb{E}\big[V_{i_2}(x_1(t))\big] &\leq \mathbb{E}\big[V_{i_2}(x_1(t_1))\big]e^{-\lambda_{i_2}(t-t_1)},\\ &\leq \mu\mathbb{E}\big[V_{i_1}(x_1(t_1))\big]e^{-\lambda_{i_2}(t-t_1)},\\ &\leq \mu\mathbb{E}\big[V_{i_1}(x_1(t_0))\big]e^{-\lambda_{i_1}(t_1-t_0)}e^{-\lambda_{i_2}(t-t_1)}.\end{aligned}$$

Namely, we have

$$\mathbb{E}\big[V_{i_2}(x_1(t))\big] \leq \mu\mathbb{E}\big[V_{i_1}(x_1(t_0))\big]e^{-\lambda_{i_1}(t_1-t_0)}e^{-\lambda_{i_2}(t-t_1)}. \tag{6.27}$$

Similarly, for $t \in [t_2, t_3)$,

$$\begin{aligned}\mathbb{E}\big[V_{i_3}(x_1(t))\big] &\leq \mathbb{E}\big[V_{i_3}(x_1(t_2))\big]e^{-\lambda_{i_3}(t-t_2)},\\ &\leq \mu^2\mathbb{E}\big[V_{i_1}(x_1(t_0))\big]^{-\lambda_{i_1}(t_1-t_0)}e^{-\lambda_{i_2}(t_2-t_1)}e^{-\lambda_{i_3}(t-t_2)}.\end{aligned} \tag{6.28}$$

Therefore, generally, we have for $t \in [t_{k-1}, t_k)$

$$\mathbb{E}\big[V_{i_k}(x_1(t))\big] \leq \mu^{k-1}\mathbb{E}\big[V_{i_1}(x_1(t_0))\big]e^{-\lambda_{i_1}(t_1-t_0)}e^{-\lambda_{i_2}(t_2-t_1)}\dots e^{-\lambda_{i_k}(t-t_{k-1})}. \tag{6.29}$$

Let $\lambda = \min_{j=1,2,\dots,k}\{\lambda_{i_j}\}$, so inequality (6.29) is

$$\mathbb{E}\big[V_{i_k}(x_1(t))\big] \leq e^{(k-1)\ln\mu}e^{-\lambda(t-t_0)}\mathbb{E}\big[V_{i_1}(x_1(t_0))\big]. \tag{6.30}$$

By ADT condition with $N_0 = \dfrac{\eta}{\ln\mu}$, where η is an arbitrary constant and $\mu > 1$, and $T_a = \dfrac{\ln\mu}{\lambda - \lambda^*}$, where $\lambda > \lambda^*$, one may obtain

$$\mathbb{E}\big[V_{i_k}(x_1(t))\big] \leq e^{\eta-\lambda^*(t-t_0)}\mathbb{E}\big[V_{i_1}(x_1(t_0))\big], \tag{6.31}$$

which implies

$$\mathbb{E}\big[\|x_1(t)\|^2\big] \leq \alpha e^{\eta-\lambda^*(t-t_0)}\|x_1(t_0)\|^2, \tag{6.32}$$

where $\alpha = \dfrac{\max\{\lambda_{\max}(\bar{X}_{1i})\}}{\min\{\lambda_{\min}(\bar{X}_{1i})\}}$ for all $i \in \Xi$.

Therefore, (6.32) together with (6.18) and $\hat{P}_i^{-1}x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$ implies exponential stability in the mean square of the system in (6.1). Since it has already been shown that system (6.1) is regular and impulse-free, the system in (6.1) is mean square admissible. Thus, the proof is completed. □

Example 6.1.1. *Consider system (6.1) where* $x = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$, $\sigma(t) \in \Xi = \{1, 2\}$,

$$E_1 = E_2 = \begin{bmatrix} 1 & 1 & 0.5 \\ -0.5 & 1.5 & 1.75 \\ 1 & 1 & 0.5 \end{bmatrix}, \quad A_1 = \begin{bmatrix} -10 & 5 & 6.5 \\ 2 & -5.5 & -1.25 \\ -9 & 4 & 8.5 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} -5 & 8 & 3 \\ -2 & 2 & -1 \\ -8 & -5 & 4 \end{bmatrix}, C_1 = C_2 = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix},$$

and initial condition $x(0) = [1 \quad -1 \quad -1]^T$.

Choose

$$S_i = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

which has full column rank and satisfies $E_i^T S_i = 0$ *where* $i = 1, 2$.

By solving the linear matrix inequalities in (6.3)-(6.4), we obtain the solution as

$$X_1 = \begin{bmatrix} 0.5403 & 0.0066 & -0.5025 \\ 0.0066 & 0.1429 & 0.0066 \\ -0.5025 & 0.0066 & 0.5403 \end{bmatrix}, \; R_1 = \begin{bmatrix} -0.0955 & 0.1108 & 0.5073 \end{bmatrix}$$

$$X_2 = \begin{bmatrix} 0.7903 & -0.1535 & -0.5524 \\ -0.1535 & 0.8490 & -0.1535 \\ -0.5524 & -0.1535 & 0.7903 \end{bmatrix}, \; R_2 = \begin{bmatrix} 0.2817 & -0.2298 & -0.2122 \end{bmatrix}$$

After necessary calculations for ADT, we obtained that ADT is 1.5. Therefore, the system is illustrated in Figure 6.1.

6.2 Stochastic Switched Singular Systems: Discrete

This section studies admissibility in the mean square of discrete stochastic switched singular systems.

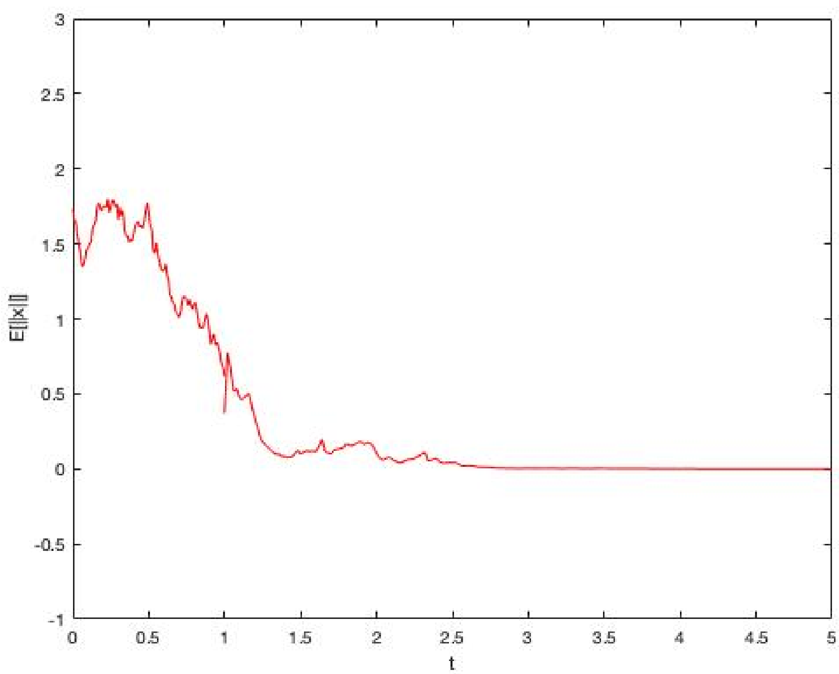

Figure 6.1: Mean of $\|x(t)\|$

Consider the following discrete-time stochastic linear singular system

$$E_{\sigma(t)}x(n+1) = A_{\sigma(t)}x(n) + C_{\sigma(t)}x(n)W(n), \qquad (6.33)$$
$$x(n_0) = x_0$$

where $x(n) \in \mathbb{R}^N$ is the system state, and $E_{\sigma(t)}, A_{\sigma(t)} \in \mathbb{R}^{N\times N}$, and $C_{\sigma(t)} \in \mathbb{R}^{N\times N}$ are system coefficient matrices where E being singular with $\text{rank}(E_{\sigma(t)}) = r < N$, $\sigma(n) : \mathbb{N} \to \Xi$ is a switching rule taking values $\sigma(n) = i$ in a compact set $\Xi = \{1, 2, \dots, M\}$ for some $M \in \mathbb{N}$, $W(n) \in \mathbb{R}$ is a standard Wiener process defined on the complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

6.2.1 Preliminaries

The following definitions and lemmas will be used in the proof of the main results.

Definition 6.2.1. *For system*

$$Ex(n+1) = Ax(n),$$

its generalized spectral radius is defined as

$$\rho(E, A) = \max_{\lambda\in\{\det(sE-A)=0\}} |\lambda|.$$

Lemma 6.2.1. *[63] If there exist nonsingular matrices $\tilde{Q}_i$, $\tilde{P}_i \in \mathbb{R}^{N\times N}$ such that one of the following conditions is satisfied, then system (6.33) has a unique solution.*

(i)

$$\tilde{Q}_i E_i \tilde{P}_i = \begin{bmatrix} I_r & 0 \\ 0 & N \end{bmatrix}, \quad \tilde{Q}_i A_i \tilde{P}_i = \begin{bmatrix} A_{1i} & 0 \\ 0 & I_{N-r} \end{bmatrix}, \quad \tilde{Q}_i C_i \tilde{P}_i = \begin{bmatrix} C_{1i} & C_{2i} \\ 0 & 0 \end{bmatrix}, \qquad (6.34)$$

where $N \in \mathbb{R}^{(N-r)\times(N-r)}$ is a nilpotent matrix, $I_r \in \mathbb{R}^{r\times r}$ and $I_{N-r} \in \mathbb{R}^{(N-r)\times(N-r)}$ identity matrices, $A_{1i}, C_{1i} \in \mathbb{R}^{r\times r}$, and $C_{2i} \in \mathbb{R}^{r\times(N-r)}$.

(ii)

$$\tilde{Q}_iE_i\tilde{P}_i = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}, \quad \tilde{Q}_iA_i\tilde{P}_i = \begin{bmatrix} A_{1i} & 0 \\ 0 & I_{N-r} \end{bmatrix}, \quad \tilde{Q}_iC_i\tilde{P}_i = \begin{bmatrix} C_{1i} & C_{2i} \\ 0 & C_{3i} \end{bmatrix}, \tag{6.35}$$

where $I_r \in \mathbb{R}^{r\times r}$ and $I_{N-r} \in \mathbb{R}^{(N-r)\times(N-r)}$ identity matrices, $A_{1i}, C_{1i} \in \mathbb{R}^{r\times r}$, $C_{2i} \in \mathbb{R}^{r\times(N-r)}$, and $C_{3i} \in \mathbb{R}^{(N-r)\times(N-r)}$.

When the regularity of system $Ex(n+1) = Ax(n)$ is not known, it is always possible to choose two nonsingular matrices $\tilde{Q}$ and $\tilde{P}$ such that

$$\tilde{Q}E\tilde{P} = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \quad \tilde{Q}A\tilde{P} = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}.$$

Lemma 6.2.2. *[62] The pair (E, A) is casual if and only if A_4 is nonsingular.*

Remark 6.2.1. *In discrete singular systems, impulse freeness is called as casual.*

6.2.2 Stability Result

This subsection deals with establishing the mean square admissibility of (6.33) using LMIs together with ADT approach.

Theorem 6.2.1. *The stochastic singular system in (6.33) is mean square admissible if*

(i) there exists a matrix $X_i = X_i^T$ such that

$$E_i^T X_i E_i \geq 0, \tag{6.36}$$

$$A_i^T X_i A_i - E_i^T X_i E_i + C_i^T X_i C_i < 0, \tag{6.37}$$

(ii) condition $rank(E_i, C_i) = rank(E_i)$ holds,

(iii) for any n_0, the switching law satisfies the ADT condition

$$N(t,t_0) \leq N_0 + \frac{t-t_0}{T_a},$$

where the chatter bound $N_0 = 0$, and average dwell time $T_a > -\frac{\ln \mu}{\ln \alpha}$.

Proof. Suppose that there exists a matrix $X_i = X_i^T$ such that (6.36) and (6.37) hold. We first show that system (6.33) is regular and causal. To do this end, we choose two nonsingular matrices $\tilde{Q}_i$ and $\tilde{P}_i$ such that

$$\tilde{Q}_i E_i \tilde{P}_i = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \quad \tilde{Q}_i A_i \tilde{P}_i = \begin{bmatrix} A_{1i} & A_{2i} \\ A_{3i} & A_{4i} \end{bmatrix}. \tag{6.38}$$

Under the condition $rank(E_i, C_i) = rank(E_i)$,

$$\tilde{Q}_i C_i \tilde{P}_i = \begin{bmatrix} C_{1i} & C_{2i} \\ 0 & 0 \end{bmatrix} \tag{6.39}$$

where the partitions of $\tilde{Q}_i A_i \tilde{P}_i$ and $\tilde{Q}_i C_i \tilde{P}_i$ are compatible with that of $\tilde{Q}_i E_i \tilde{P}_i$.

Let

$$\tilde{Q}_i^{-T} X_i \tilde{Q}_i^{-1} = \begin{bmatrix} X_{1i} & X_{2i} \\ X_{2i}^T & X_{3i} \end{bmatrix}. \tag{6.40}$$

Pre- and post-multiplying (6.36) by $\tilde{P}_i^T$ and $\tilde{P}_i$, respectively, gives

$$\begin{bmatrix} X_{1i} & 0 \\ 0 & 0 \end{bmatrix} \geq 0, \tag{6.41}$$

which implies $X_{1i} \geq 0$ by Schur complement.

Similarly, pre- and post-multiplying (6.37) by $\tilde{P}_i^T$ and $\tilde{P}_i$, respectively, we derive

$$\begin{bmatrix} W_{1i} & W_{2i} \\ W_{2i}^T & W_{3i} \end{bmatrix} < 0, \tag{6.42}$$

where

$$W_{1i} = A_{1i}^T X_{1i} A_{1i} + A_{3i}^T X_{2i}^T A_{1i} + A_{1i}^T X_{2i} A_{3i} + A_{3i}^T X_{3i} A_{3i} - X_{1i} + C_{1i}^T X_{1i} C_{1i},$$
$$W_{2i} = A_{1i}^T X_{1i} A_{2i} + A_{3i}^T X_{2i}^T A_{2i} + A_{1i}^T X_{2i} A_{4i} + A_{3i}^T X_{3i} A_{4i} + C_{1i}^T X_{1i} C_{2i},$$
$$W_{3i} = A_{2i}^T X_{1i} A_{2i} + A_{4i}^T X_{2i}^T A_{2i} + A_{2i}^T X_{2i} A_{4i} + A_{4i}^T X_{3i} A_{4i} + C_{2i}^T X_{1i} C_{2i}.$$

The inequality (6.42) implies $W_{3i} < 0$. Since $X_{1i} \geq 0$, we have

$$A_{2i}^T X_{1i} A_{2i} + C_{2i}^T X_{1i} C_{2i} \geq 0.$$

Therefore, from W_{3i} we have

$$A_{4i}^T X_{2i}^T A_{2i} + A_{2i}^T X_{2i} A_{4i} + A_{4i}^T X_{3i} A_{4i} < 0. \tag{6.43}$$

From (6.43), it follows that A_{4i} is nonsingular. Therefore, subsystems in (6.33) are casual by Lemma 6.2.2.

On the other hand, to show regularity of each subsystem in (6.33), we need to show $\det(sE_i - A_i) \neq 0$, so

$$\begin{aligned}
\det(sE_i - A_i) &= \det(s\tilde{Q}_i E_i \tilde{P}_i - \tilde{Q}_i A_i \tilde{P}_i) \\
&= \det\left(\begin{bmatrix} sI - A_{1i} & -A_{2i} \\ -A_{3i} & -A_{4i} \end{bmatrix}\right) \\
&= \det(-A_{4i}) \det(sI - (A_{1i} - A_{2i} A_{4i}^{-1} A_{3i}).
\end{aligned}$$

Since A_{4i} is nonsingular, $\det(-A_{4i}) \neq 0$. We also can find an $s \in \mathbb{C}$ such that $\det(sI - (A_{1i} - A_{2i} A_{4i}^{-1} A_{3i}) \neq 0$. Therefore, we proved the regularity of system (6.33).

Let

$$\hat{P}_i = \tilde{P}_i \begin{bmatrix} I & 0 \\ -A_{4i}^{-1} A_{3i} & A_{4i}^{-1} \end{bmatrix}, \tag{6.44}$$

so we derive

$$\tilde{Q}_i E_i \hat{P}_i = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \tag{6.45}$$

$$\tilde{Q}_i A_i \hat{P}_i = \begin{bmatrix} A_{1i} - A_{2i}A_{4i}^{-1}A_{3i} & A_{2i}A_{4i}^{-1} \\ 0 & I \end{bmatrix}, \tag{6.46}$$

$$\tilde{Q}_i C_i \hat{P}_i = \begin{bmatrix} C_{1i} - C_{2i}A_{4i}^{-1}A_{3i} & C_{2i}A_{4i}^{-1} \\ 0 & 0 \end{bmatrix}. \tag{6.47}$$

It is shown that (6.45)-(6.47) satisfy Lemma 6.2.1-(ii) under assumption $rank(E_i, C_i) = rank(E_i)$, so system (6.33) has a unique solution under the consistent initial condition and is casual.

Now, we need to prove that system (6.33) is mean square asymptotically stable. Let

$$\hat{P}_i^{-1} x(n) = \begin{bmatrix} x_1(n) \\ x_2(n) \end{bmatrix},$$

where $x_1(n) \in \mathbb{R}^r$, $x_2 \in \mathbb{R}^{n-r}$, then system (6.33) is equivalent to

$$x_1(n+1) = \tilde{A}_{1i} x_1(n) + \tilde{C}_{1i} x_1(n) W(n), \tag{6.48}$$

$$x_2(n) = 0, \tag{6.49}$$

with $\tilde{A}_{1i} = A_{1i} - A_{2i}A_{4i}^{-1}A_{3i}$, and $\tilde{C}_{1i} = C_{1i} - C_{2i}A_{4i}^{-1}A_{3i}$.

Substituting (6.45)-(6.47) and (6.40) into (6.37), we obtain

$$\tilde{A}_{1i}^T X_{1i} \tilde{A}_{1i} - X_{1i} + \tilde{C}_{1i}^T X_{1i} \tilde{C}_{1i} < 0, \tag{6.50}$$

which implies together with (6.41) $X_{1i} > 0$.

Define Lyapunov candidate function

$$V_i(x(n)) = x^T(n)E_i^T X_i E_i x(n) = x_1^T(n)X_{1i}x_1(n) > 0. \tag{6.51}$$

Therefore, one can obtain

$$\mathbb{E}\left\{V_i(x_1(n+1))\right\} - V_i(x_1(n)) = x_1^T(n)\left[\tilde{A}_{1i}^T X_{1i}\tilde{A}_{1i} - X_{1i}\right]x_1(n).$$

Since $X_{1i} > 0$, by (6.50)

$$\mathbb{E}\left\{V_i(x_1(n+1))\right\} - V_i(x_1(n)) = x_1^T(n)\left[\tilde{A}_{1i}^T X_{1i}\tilde{A}_{1i} - X_{1i}\right]x_1(n) < 0.$$

Thus, there exists a positive constant δ_i such that

$$\mathbb{E}\left\{V_i(x_1(n+1))\right\} - V_i(x_1(n)) < -\delta_i \left\|x_1(n)\right\|^2. \tag{6.52}$$

Also, by Lyapunov candidate function we have

$$\lambda_{\min}\left(X_{1i}\right)\left\|x_1(n)\right\|^2 \leq V_i(x_1(n)) \leq \lambda_{\max}\left(X_{1i}\right)\left\|x_1(n)\right\|^2$$

Using this and (6.52) gives

$$\mathbb{E}\left\{V_i(x_1(n+1))\right\} - V_i(x_1(n)) < -\delta_i \left\|x_1(n)\right\|^2 \leq -\frac{\delta_i}{\lambda_{\max}\left(X_{1i}\right)}V_i(x_1(n))$$

$$\Rightarrow \quad \mathbb{E}\left\{V_i(x_1(n+1))\right\} \leq \alpha_i V_i(x_1(n))$$

where $\alpha_i = \left(1 - \dfrac{\delta_i}{\lambda_{\max}\left(X_{1i}\right)}\right)$ and $0 < \alpha_i < 1$.

For $n \in [n_0, n_1)$, by iteration, one may obtain

$$\mathbb{E}\left\{V_{i_1}(x_1(n))\right\} \leq \alpha_{i_1}^{n-n_0}V_{i_1}(x_1(n_0)). \tag{6.53}$$

Similarly, for $n \in [n_1, n_2)$

$$\mathbb{E}\left\{V_{i_2}(x_1(n))\right\} \leq \alpha_{i_2}^{n-n_1} V_{i_2}(x_1(n_1)). \tag{6.54}$$

On the other hand, from (6.51), we have for any $i, j \in \Xi$

$$V_j(x_1(t)) \leq \mu V_i(x_1(t)), \tag{6.55}$$

where $\mu = \dfrac{\max\{\lambda_{\max}(X_{1j})\}}{\min\{\lambda_{\min}(X_{1j})\}}$.

Therefore, inequality (6.54) becomes

$$\mathbb{E}\left\{V_{i_2}(x_1(n))\right\} \leq \mu \alpha_{i_2}^{n-n_1} \alpha_{i_1}^{n_1-n_0} V_{i_1}(x_1(n_0)). \tag{6.56}$$

Generally, one may obtain for $n \in [n_{k-1}, n_k)$

$$\mathbb{E}\left\{V_{i_k}(x_1(n))\right\} \leq \mu^{(k-1)} \alpha_{i_k}^{n-n_{k-1}} \alpha_{i_{k-1}}^{n_{k-1}-n_{k-2}} \dots \alpha_{i_1}^{n_1-n_0} V_{i_1}(x_1(n_0)). \tag{6.57}$$

Let $\alpha = \max_{j=1,2,\dots,k}\{\alpha_{i_j}\}$, so

$$\begin{aligned}
\mathbb{E}\left\{V_{i_k}(x_1(n))\right\} &\leq \mu^{(k-1)} \alpha^{n-n_0} V_{i_1}(x_1(n_0)) \\
&= \alpha^{(k-1)\frac{\ln\mu}{\ln\alpha}} \alpha^{(n-n_0)} V_{i_1}(x_1(n_0)) \\
&= \alpha^{(n-n_0)\left[\frac{(k-1)\ln\mu}{(n-n_0)\ln\alpha}+1\right]} V_{i_1}(x_1(n_0)).
\end{aligned} \tag{6.58}$$

For simplicity, choose the chatter bound $N_0 = 0$ in ADT condition. Therefore, from inequality (6.58), we obtain

$$\begin{aligned}
\mathbb{E}\left\{V_{i_k}(x_1(n))\right\} &\leq \left(\alpha^{\frac{\ln\mu}{T_a \ln\alpha}+1}\right)^{(n-n_0)} V_{i_1}(x_1(n_0)) \\
&= \alpha^{\rho(n-n_0)} V_{i_1}(x_1(n_0))
\end{aligned} \tag{6.59}$$

where $\rho = \frac{\ln \mu}{T_a \ln \alpha} + 1$, and $T_a > -\frac{\ln \mu}{\ln \alpha}$.

Thus,

$$\mathbb{E}\left\{\|x_1(n)\|^2\right\} \leq \mu\alpha^{\rho(n-n_0)} \|x_1(n_0)\|^2 \tag{6.60}$$

Taking the limit on (6.60) as $n \to \infty$, we have

$$\lim_{n\to\infty} \mathbb{E}\left\{\|x_1(n)\|^2\right\} = 0.$$

This together with (6.49) implies system (6.33) is asymptotically stable in the mean square. We already proved the regularity and causality of the system (6.33). Therefore, system (6.33) is mean square admissible. □

Example 6.2.1. *Consider system (6.33) with where* $x = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$, $\sigma(t) \in \Xi = \{1, 2\}$,

$$E_1 = E_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad A_1 = \begin{bmatrix} -0.2 & 0.1 \\ 0.3 & 0.2 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -0.5 & 0.4 \\ 0.6 & 0.5 \end{bmatrix},$$

$$C_1 = C_2 = \begin{bmatrix} 0.3 & 0.2 \\ 0 & 0 \end{bmatrix},$$

and initial condition $x(0) = [-1 \quad 1.5]^T$.

The solution of LMIs (6.36)-(6.37) is obtained as follows:

$$X_1 = \begin{bmatrix} 1 & -2 \\ -2 & -1 \end{bmatrix}$$

$$X_2 = \begin{bmatrix} 67.7778 & -55.0113 \\ -55.0113 & 29.7959 \end{bmatrix}$$

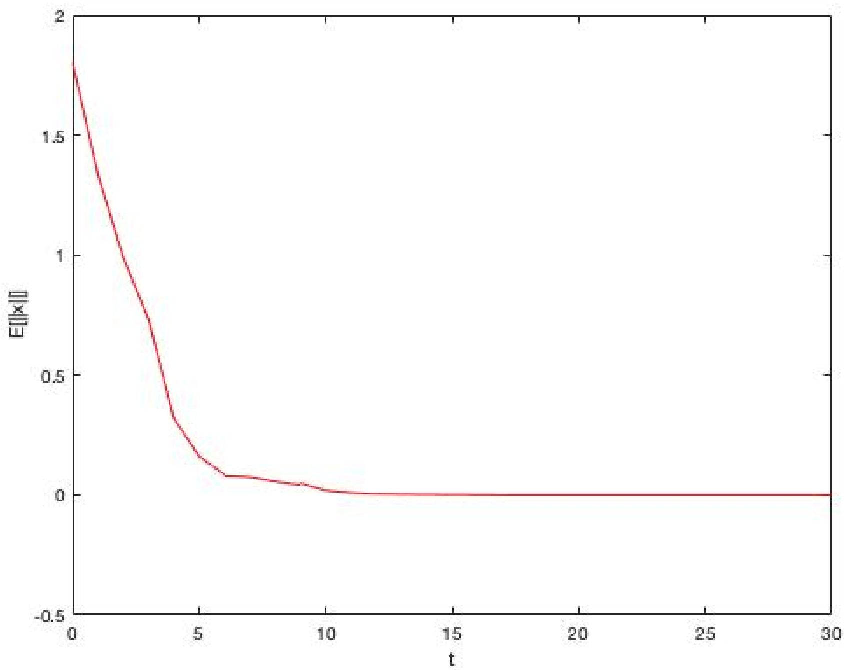

Figure 6.2: Mean of $\|x(t)\|$

which are symmetric matrices. Therefore, there exist positive constants $\delta_1 = 3$ *and* $\delta_2 = 236$ *such that inequality (6.52) holds, so we calculate* $\alpha_1 = 0.5702$, $\alpha_2 = 0.4503$, $\mu = 61.5094$ *and* $\alpha = \max\{\alpha_1,\ \alpha_2\} = 0.5702$. *As a result, ADT is* $T_a > 7.3321$. *The system is illustrated in Figure 6.2.*

Chapter 7

Conclusions and Future Research

In this chapter, the possible future research directions are discussed.

7.1 Lyapunov-Razumikhin Technique

In Chapter 3, the stability problem for continuous and discrete impulsive switched singular systems with time delay has been addressed. The stability conditions have been investigated by Halanay inequalities. However, the Lyapunov functional method and the Lyapunov-Razumikhin technique are two commonly used approaches to establish sufficient conditions for stability of delay systems. For many delay systems, Razumikhin technique appears to be easier to establish sufficient conditions for stability than to construct appropriate Lyapunov functionals. The Lyapunov functionals are quite complicated. Especially for large-scale systems the construction of suitable Lyapunov functionals are challenging. Lyapunov-Razumikhin technique is based on Lyapunov functions whereas suitable Lyapunov functional candidate, which is divided into two parts: a function part and a functional part, are constructed in Lyapunov functional method. When dealing with impulsive stabilization, the value of the function part can be effectively brought down by the impulse while the impulse cannot change value of the functional part at each impulsive instant. Thus, Razumikhin technique is worthwhile to investigate the stability of delay systems. Future work can be done to study stability theory of impulse switched singular systems with time-delay by using Razumikhin technique.

7.2 Invariance Principle

In 1960's, the classical invariance principle was constructed to be a powerful tool for the stability analysis of autonomous ordinary differential equations by LaSalle. Moreover, numerous extensions of the original invariance principle were derived for various differential systems such as invariance principles for impulsive systems, invariance principles for impulsive switched systems, and invariance principles for switched delay systems.

Invariance principles were developed for singular systems with jumps and switched singular systems under arbitrary and dwell-time switching signals in [64] and [65], respectively. Although the invariance principle is already studied for impulsive singular systems in [66] and switched singular systems in [67], the idea of invariance principles for impulsive switched singular with and without time-delay systems has not yet been addressed and explored in the literature. Thus, future work can be done to fill this gap by extending La'Salle's invariance principles to such systems.

7.3 Comparison Principle

In Chapter 6, we have studied the stability analysis for stochastic switched singular systems which is one of the most important research topic in stochastic systems, as well. In our investigation, we have used LMIs to obtain the sufficient conditions for mean square admissibility of the system. However, different from the traditional Lyapunov stability theory, people use mostly the comparison principle to obtain some stability criteria of a stochastic system.

In classic stability analysis, we compute an upper bound on the norm of the solution $x(t)$ of the equation $\dot{x} = f(t, x)$ without computing the solution. The comparison principle is one of the tools to obtain that boundary. The comparison principle compares the solution of the differential inequality $\dot{v} \leq f(t, v(t))$ with the solution of the differential equation $\dot{u} = f(t, u)$. The comparison lemma is given as follows in [68]:

Lemma 7.3.1. *(**Comparison Lemma**) Consider the scalar differential equation*

$$\dot{u} = f(t, u), \qquad u(t_0) = u_0,$$

where $f(t, u)$ is continuous in t and locally Lipschitz in u for all $t \geq 0$ and all $u \in J \subset \mathbb{R}$. Let $[t_0, T)$ be a maximal interval of existence of the solution $u(t)$, and suppose that $u(t) \in J$ for all $t \in [t_0, T)$. Let $v(t)$ be a continuous and differentiable in t whose derivative $\dot{v}$ satisfies the differential inequality

$$\dot{v} \leq f(t, v(t)), \qquad v(t_0) \leq u_0$$

with $v(t) \in J$ for all $t \in [t_0, T)$. Then, $v(t) \leq u(t)$ for all $t \in [t_0, T)$.

Because of comparison principles, the stability properties of a stochastic system can be derived by the corresponding stability properties of a deterministic system. For instance, some stability results for switched singular systems in the literature can be used to investigate the stability properties of stochastic switched singular systems.

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