Packing Directed Joins

by

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AUTHOR'S DECLARATION FOR ELECTRONIC SUBMISSION OF A THESIS

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

Edmonds and Giles [4] conjectured that the maximum number of directed joins in a packing is equal to the minimum weight of a directed cut, for any weighted directed graph. This is dual to the Lucchesi-Younger Theorem [10], [9] which proves that the maximum number of directed cuts in a packing is equal to the minimum weight of a directed join, for any weighted directed graph. Schrijver [13], Feofiloff and Younger [5], [6] proved that the conjecture does hold for directed graphs with directed paths from every source to every sink. Schrijver [13] noted that this implies versions of Menger's Theorem, Gupta's Theorem, and Edmonds's Branching Theorem [13].

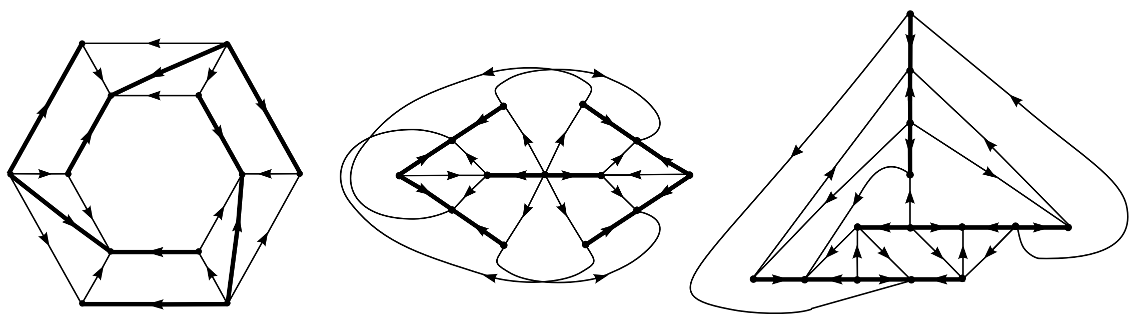

Figure 1: The counterexamples due to Schrijver, and Cornuéjols and Guenin.

Surprisingly, the unweighted version of the Edmonds-Giles Conjecture is open [16], and the general conjecture is not true. The first counterexample is due to Schrijver [12], and Younger [17] showed the author that it is the smallest member in an infinite family. Cornuéjols and Guenin [7] discovered two additional counterexamples. Despite its importance, there is very little known about counterexamples to the Edmonds-Giles Conjecture. In this thesis, we provide a general framework which explains the known counterexamples, describe new counterexamples that are variations of existing ones, show that there is a common structure in all minimal counterexamples, extend the infinite family of Younger to include one of the counterexamples of Cornuéjols and Guenin, and show that the known counterexamples are the "smallest" possible.

Acknowledgements

I would like to extend a warm thank you to D.H. Younger for introducing this topic to me, and for providing several key insights that proved helpful along the way. Bertrand Guenin deserves much credit for being an excellent, and available supervisor, and for allowing me to research my chosen topic. Thank you Bertrand! I would also like to thank the Combinatorics & Optimization Department for providing a friendly, and productive, environment. Finally, thank you for buying me the Snoopy calculator Mom!

Dedicated to Christopher Lee Rios (1971-2000) and Wesley Willis (1963-2003):

"Could have gone to college and been a mathematician.
Bad decisions kept me out the game."

"Rock over London, Rock on Chicago!"

Contents

Chapter 1

Introduction

Strong connection is a concept of fundamental importance in the study of directed graphs. Directed graph $D = (N, A)$ is *strongly connected* if there exists a directed path from u to w for all $u, w \in N$.

A *cut* induced by $\emptyset \subsetneq X \subsetneq N$ is the set of arcs $\delta(X) = \delta_D(X) = \{(u, w) \in A : u \in X \text{ and } w \notin X\}$. A cut d is a *directed cut* if $d = \delta(X)$ and $\delta(N \backslash X) = \emptyset$ for some $\emptyset \subsetneq X \subsetneq N$, and d is inclusion-wise minimal with this property. It is well known that a directed graph is strongly connected if and only if it does not contain a directed cut.

A *directed join* is a minimal subset of A whose contraction makes the directed graph strongly connected. Alternatively, a directed join is a minimal set of arcs that has a non-empty intersection with every directed cut.

Many classical results in directed graph theory can be phrased as problems involving directed cuts and directed joins. In particular, one central theme is the pursuit of largest possible collections of disjoint directed cuts, or disjoint directed joins. The former pursuit, discussed in the next section, is well understood, thanks to the Lucchesi-Younger Theorem [10] (see also [9]). Progress on the latter has been slower; the goal of this thesis is to re-energize this pursuit through new results and new understanding.

1.1 Disjoint Directed Cuts

A collection of directed cuts is *pairwise disjoint* if no two of the directed cuts have an arc in common. The *size* of a directed join is the number of arcs contained in it.

Remark 1.1. *The maximum number of pairwise disjoint directed cuts is less than or* *equal to the size of the smallest directed join, for any directed graph.*

This remark follows from the fact that, in any directed graph, every directed cut has a non-empty intersection with every directed join. Hence, in a collection of pairwise disjoint directed cuts, each directed cut uses at least one of the arcs in any directed join of smallest size. The remarkable fact is that equality can always be reached.

Theorem 1.2 (Lucchesi-Younger). *The maximum number of pairwise disjoint directed cuts is equal to the size of the smallest directed join, for all directed graphs.*

The question of equality can be extended to weighted directed graphs. A *weighted directed graph* (D, ω) is a directed graph D with with arc set A and node set N, together with non-negative integer arc weights $\omega \in \mathbb{Z}_+^A$. A collection of directed cuts is a *packing* if each arc a is present in at most $\omega(a)$ of the directed cuts. The *weight* of a directed join is the sum of $\omega(a)$ over all arcs a in the directed join.

Remark 1.3. *The maximum number of directed cuts in a packing is less than or equal to the smallest weight of a directed join, for all weighted directed graphs.*

The Lucchesi-Younger Theorem also proves that equality can always be reached for weighted directed graphs.

Theorem 1.4 (Lucchesi-Younger). *The maximum number of directed cuts in a packing is equal to the smallest weight of a directed join, for all weighted directed graphs.*

It is interesting to note that the unweighted and weighted versions of the Lucchesi-Younger Theorem are equivalent. Proving the unweighted version from the weighted version is trivial. For the other direction, if $\omega(a) = k$ for $k > 0$, then replace a by a directed path of length k in the unweighted directed graph. This is also known as replacing a by k arcs in *series* (Figure (1.1)).

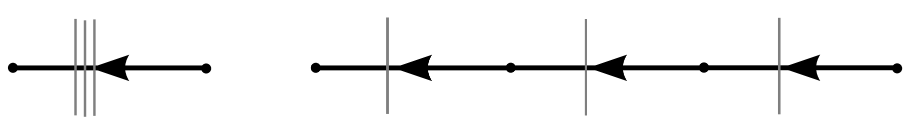

Figure 1.1: Directed cuts "spread out" while simulating weighted arcs with series arcs.

Furthermore, arcs with $\omega(a) = 0$ can also be simulated in an unweighted directed graph. Such simulation requires two properties. Firstly, directed cuts containing a must not be present in packings of largest size. Secondly, directed joins can effectively include a without increasing their size. These properties are met exactly by contracting a (Remark 1.5).

Remark 1.5. *The directed cuts in D/a are exactly those in D that do not include a. That is, $\delta_{D/a}(X)$ is a directed cut if and only if $\delta_D(X)$ is a directed cut and $a \notin \delta_D(X)$.*

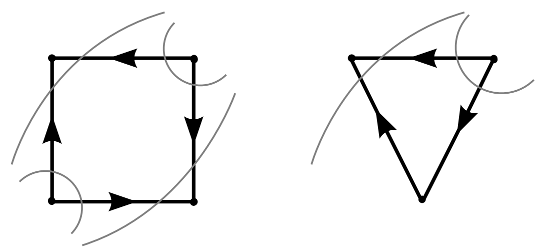

Figure 1.2: Contracting an arc removes all directed cuts containing it.

1.2 Disjoint Directed Joins

In this section, directed joins and directed cuts trade places. Although analogous inequalities can be stated, the analogous theorems are either unproven or untrue.

A collection of directed joins is *pairwise disjoint* if no two of the directed joins have an arc in common. The *size* of a directed cut is the number of arcs contained in it.

Remark 1.6. *The maximum number of pairwise disjoint directed joins is less than or equal to,the size of the smallest directed cut, for any directed graph.*

Again, this remark follows from the fact that in any directed graph, every directed join has a non-empty intersection with every directed cut. Hence, in a collection of pairwise disjoint directed joins, each directed join uses at least one of the arcs in any directed cut of smallest size.

Conjecture 1.7 (Woodall). *The maximum number of pairwise disjoint directed joins is equal to the size of the smallest directed cut, for all directed graphs.*

Woodall's Conjecture [16] has been open for over 20 years. Again, the question of equality can be extended to weighted directed graphs. A collection of directed joins is a *packing* if each arc a is present in at most $\omega(a)$ of the directed joins. The *weight* of a directed cut is the sum of $\omega(a)$ over all arcs a in the directed cut.

Remark 1.8. *The maximum number of directed joins in a packing is less than or equal to the smallest weight of a directed cut, for all weighted directed graphs.*

Edmonds and Giles [4] conjectured the weighted version of Woodall's Conjecture.

Conjecture 1.9 (Edmonds-Giles). *The maximum number of directed joins in a packing is equal to the smallest weight of a directed cut, for all weighted directed graphs.*

However, there is a fundamental difference between generalizing Woodall's Conjecture and generalizing the unweighted Lucchesi-Younger Theorem. In particular, there is no known way of showing that the conjectures of Woodall and Edmonds-Giles are equivalent.

On one hand, if the Edmonds-Giles Conjecture was true, then it would trivially imply Woodall's Conjecture. For the other direction, if $\omega(a) = k$ for $k > 0$, then replace a by k copies of the arc in the unweighted directed graph. This is also known as replacing a, by k *parallel* arcs (Figure (1.3)).

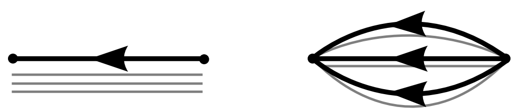

Figure 1.3: Directed joins "spread out" while simulating weighted arcs with parallel arcs.

However, simulating arcs with $\omega(a) = 0$ in an unweighted directed graph cannot be done in the same manner. Any potential simulation requires two properties. Firstly, directed joins containing a must not be present in packings of largest size. Secondly, directed cuts can effectively include a without increasing their size. Although these properties are met by deleting a, an unwanted side-effect of introducing new directed cuts invalidates this approach (Remark (1.10)).

Remark 1.10. *The directed cuts in $D \backslash a$ are exactly those in D together with the cuts in D that are prevented from being directed only by a. That is, $\delta_{D \backslash a}(X)$ is a directed cut if and only if $\delta_D(X)$ is a directed cut* or $\delta_D(N \backslash X) = \{a\}$.

Therefore, when simulating an arc of weight 0 by deleting it, there is no guarantee that this deletion will maintain the size of the smallest directed cut. Hence, there is no guarantee that this approach will reveal a sufficiently large packing of directed joins.

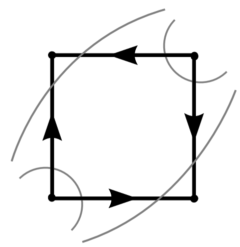

Figure 1.4: Deleting an arc can introduce new directed cuts.

Incidentally it is not difficult to describe when the directed cuts of D and $D\backslash a$ are in one-to-one correspondence. A *transitive* arc in directed graph D is an arc $a = (u, w)$ where D contains a directed path from u to w that does not use arc a.

Remark 1.11. *The directed cuts in $D\backslash a$ are exactly those in D if and only if a is a transitive arc in D. That is, if a is a transitive arc in D then $\delta_{D\backslash a}(X)$ is a directed cut if and only if $\delta_D(X)$ is a directed cut.*

1.2.1 Schrijver's (D_1, ω_1) and Minimality

In fact, the Edmonds-Giles Conjecture is not true. Schrijver [12] first demonstrated this by constructing the weighted directed graph (D_1, ω_1) (Figure (1.5)). In this weighted directed graph, every arc is either weight 0 or weight 1. By convention, arcs with $\omega(a) = 0$ are indicated by thin lines, and arcs with $\omega(a) = 1$ are indicated by thick lines.

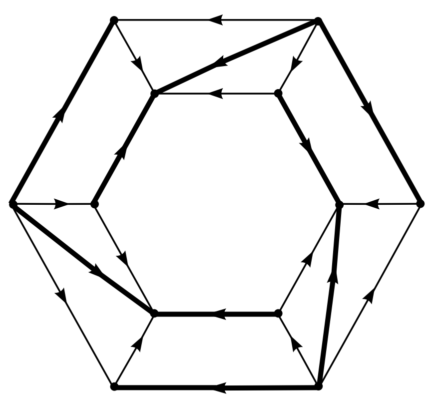

Figure 1.5: Schrijver's (D_1, ω_1).

In order to analyze this graph we need to introduce several ideas. A *source* is a node that is not the destination of any arc. A *sink* is a node that is not the origin of any arc. A *trivial cut* is a directed cut of the form $\delta(\{n\})$ or $\delta(N\backslash\{n\})$ for some node $n \in N$. Relative to a path P, every arc in P is in one of two directions. We differentiate between these two directions by arbitrarily calling one direction the *forward* direction and the other the *backward* direction. A path P is *alternating* if every pair of adjacent arcs in P has the property that one of the arcs is forward and one is backward. We will let $arcs(P)$ and $nodes(P)$ denote the arcs and nodes included in P, respectively.

Remark 1.12. *If P is an alternating path and d is a directed cut, such that $|arcs(P) \cap d| \geq 2$, then $arcs(P) \cap d$ contains at least one forward arc of P and one backward arc of P.*

Figure (1.5) shows that the arcs of weight 1 can be partitioned into three disjoint alternating paths P_1, P_2, and P_3. Furthermore, every node that is internal to one of the paths is either a source or a sink inducing a trivial cut of weight 2.

We now show that (D_1, ω_1) fails the Edmonds-Giles Conjecture. Let $\tau = \tau(D, \omega_1)$ denote the smallest weight of a directed cut in (D, ω) and $\nu = \nu(D, \omega)$ denote the size of the largest packing of directed joins in (D, ω). Notice that $\tau = 2$ and this is obtained by every trivial cut, and no other directed cut. Suppose that $\nu = 2$. Therefore, the arcs of weight 1 can be partitioned into two directed joins J_1 and J_2. Due to the trivial cuts of weight 2, J_1 and J_2 neither include nor exclude adjacent arcs along P_1, P_2, or P_2. In particular, since each path is alternating, J_1 and J_2 must divide the forward and backward arcs of each path between them. Up to exchanging J_1 and J_2 there are exactly four such partitions. Therefore, by Remark (1.12), J_1 and J_2 both have a non-empty intersection with each directed cut that intersects P_1, P_2, or P_3 more than once. However, for each of the four possible partitions, there exists a directed cut that intersects each path at most once, and is disjoint from either J_1 or J_2. Since each of the four special cuts does not intersect P_1, P_2, or P_3 more than once, we call these cuts *crossing cuts* (Figure (1.6)).

The above discussion shows that it is essentially the trivial cuts and the crossing cuts that make it impossible for J_1 and J_2 to both be directed joins. This reasoning can also be used to understand the other known counterexamples. Interestingly, it is possible to find J_1 and J_2 which intersect each of the trivial cuts and crossing cuts except any single specified trivial cut or crossing cut.

Additional counterexamples to the Edmonds-Giles Conjecture can be made by modifying (D_1, ω_1). The simplest modification is to add transitive arcs of weight 0. From

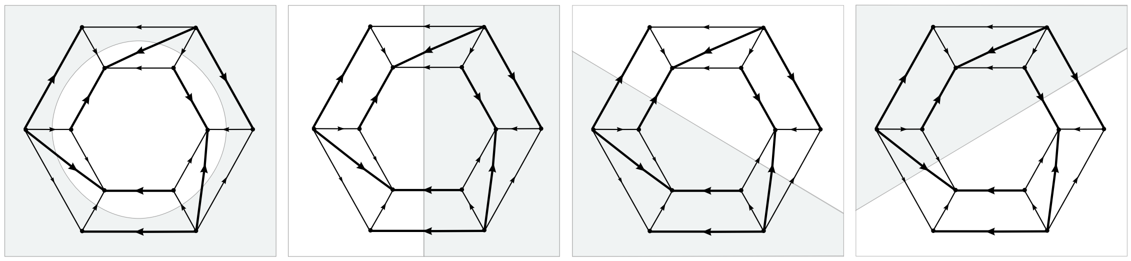

Figure 1.6: The four crossing cuts of (D_1, ω_1). Left to right d_1, d_2, d_3, d_4.

Remark (1.11) this addition does not change the sets of nodes which induce directed cuts, nor does it change the weight of any directed cut. The transitive arcs on the left of Figure (1.7) are the only arcs that can be added to (D_1, ω_1) without eliminating a trivial cut or a crossing cut.

Although it is not possible to simply add non-transitive arcs to (D_1, ω_1), there are other modifications that produce additional counterexamples. For example, if (D', ω') is a modification of (D_1, ω_1) where $\tau(D', \omega') = \tau(D_1, \omega_1) = 2$, and the trivial and crossing cuts of (D_1, ω_1) exist in (D', ω') with the same non-zero weight arcs, then one would expect $\nu(D', \omega') = \nu(D_1, \omega_1) = 1$. Figure (1.7) shows one such counterexample.

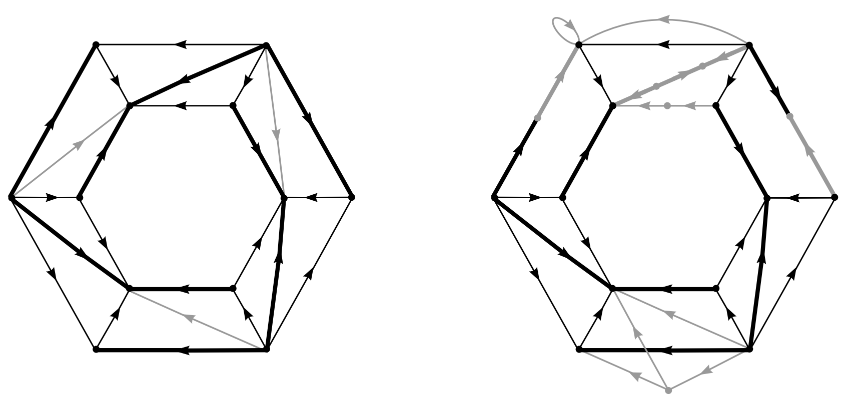

Figure 1.7: Modifications of (D_1, ω_1) failing the Edmonds-Giles Conjecture.

Younger showed the author [17] that (D_1, ω_1) is the smallest member of an infinite family of counterexamples, found by generalizing Schrijver's example to any odd number of paths (Figure (1.8)). Directed graphs in this D_1 *family* can be reduced to D_1 by arc contractions.

Given the abundance of possible modifications, it becomes natural to ask for the *essen-*

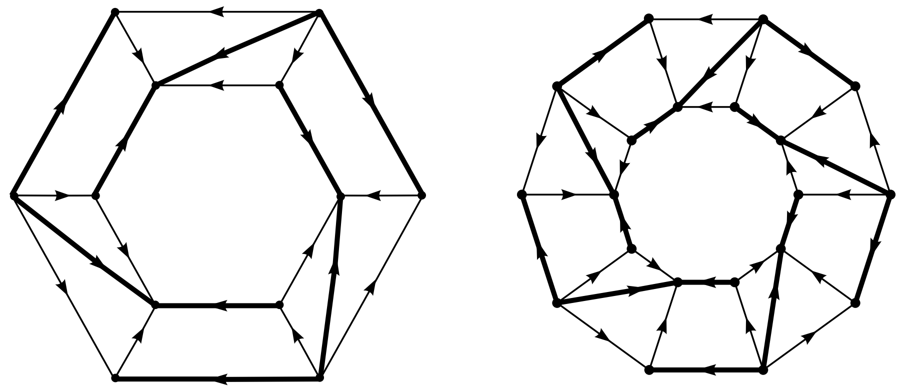

Figure 1.8: Schrijver's three path counterexample and its five path generalization.

tial counterexamples. We say that a directed graph D fails the Edmonds-Giles Conjecture if there exists an arc weight ω such that (D, ω) fails the Edmonds-Giles Conjecture. If no such ω exists, then the directed graph satisfies the Edmonds-Giles Conjecture. The task of finding the essential directed graphs that fail the Edmonds-Giles Conjecture is simplified by the following well-known sequence of ideas.

Let $\mathbb{O}$ be a set of operations where each operation maps directed graphs to directed graphs. Say that D is an $\mathbb{O}$-*minor* of D', or simply a *minor* of D', if it is possible to obtain D by applying operations in $\mathbb{O}$ to D'. Specifically, D is a *strict minor* of D' if D is a minor of D' and $D \neq D'$.

Suppose that $\mathbb{O}$ produces minors that have the following properties:

- If a directed graph satisfies the Edmonds-Giles Conjecture then every one of its minors also satisfies it. Satisfying the conjecture is said to be *closed* under $\mathbb{O}$.

- No infinite sequence of directed graphs has the property that every directed graph is a strict minor of the previous one. In other words, repeatedly taking strict minors always results in a directed graph that has no strict minors.

- If a directed graph has no strict minors, then it satisfies the Edmonds-Giles Conjecture.

Thus, every directed graph D' that fails the Edmonds-Giles Conjecture has a minor D that fails the conjecture, and furthermore, every strict minor of D satisfies the conjecture. We call such a minor $\mathbb{O}$-*minimal*, or simply *minimal*, with respect to failing the conjecture.

An excluded minor characterization of the directed graphs that satisfy the Edmonds-Giles Conjecture would be a set of operations together with a corresponding complete list of the minimal directed graphs that fail the conjecture. One shortcoming of our current understanding of the Edmonds-Giles Conjecture is that it is not known whether maximum sized collections of disjoint directed joins can be found in polynomial time [15]. Another shortcoming is the sizeable gap between the existing counterexamples and the handful of directed graph classes for which the conjecture is known to hold [13], [5], [6], [11]. Discovering an excluded minor characterization would not only provide a beautiful new result, but it could also contribute towards overcoming these important shortcomings.

1.2.2 Cornuéjols and Guenin's (D_2, ω_2) and (D_3, ω_3)

Let $D' = (N', A')$ and $D = (N, A)$ be directed graphs. D' is *contractible* to D if $D'/C = D$ for some $C \subseteq A'$. D' is a *transitive extension* of D if $N' = N$ and $A' \backslash A$ consists only of transitive arcs of D.

Remark 1.13. *Any directed graph that is contractible to a transitive extension of D_1, fails the Edmonds-Giles Conjecture.*

Cornuéjols and Guenin [7] proved this remark with a simple argument. Suppose $D = (N, A)$ is contractible to a transitive extension of D_1. In particular, suppose $D/C\backslash T = D_1$ for disjoint $C, T \subseteq A$ with T consisting only of transitive arcs in D_1. Then (D, ω) fails the Edmonds-Giles Conjecture for $\omega \in \mathbb{Z}_+^A$ defined as follows:

$$\omega(a) \begin{cases} 0 & \text{if } a \in T \\ \tau(D_1, \omega_1) = 2 & \text{if } a \in C \\ \omega_1(a) & \text{otherwise.} \end{cases}$$

Together with this observation, Cornuéjols and Guenin presented two additional counterexamples, (D_2, ω_2) and (D_3, ω_3), to the Edmonds-Giles Conjecture (Figure (1.9)). In particular, D_1, D_2, and D_3 are not contractible to transitive extensions of one another.

These new counterexamples share much in common with (D_1, ω_1). For example, the trivial cuts together with the existence of exactly four crossing cuts prevent packings of two directed joins (Figures (1.11), (1.12)). Also, only transitive arcs can be added to D_2 and D_3 without making the result satisfy the Edmonds-Giles Conjecture. The relationship between D_1 and D_3 is dramatically illustrated by a new embedding of D_3 in

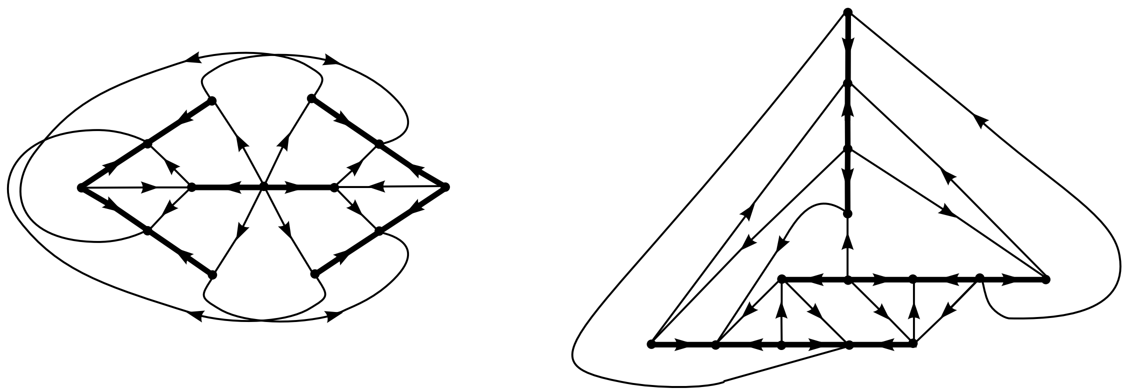

Figure 1.9: On the left is (D_2, ω_2), on the right is (D_3, ω_3).

Figure (1.10). From this embedding it is easy to find the four crossing cuts of (D_3, ω_3). It is also interesting to note that this embedding shows that a transitive extension of D_3 is contractible to D_1.

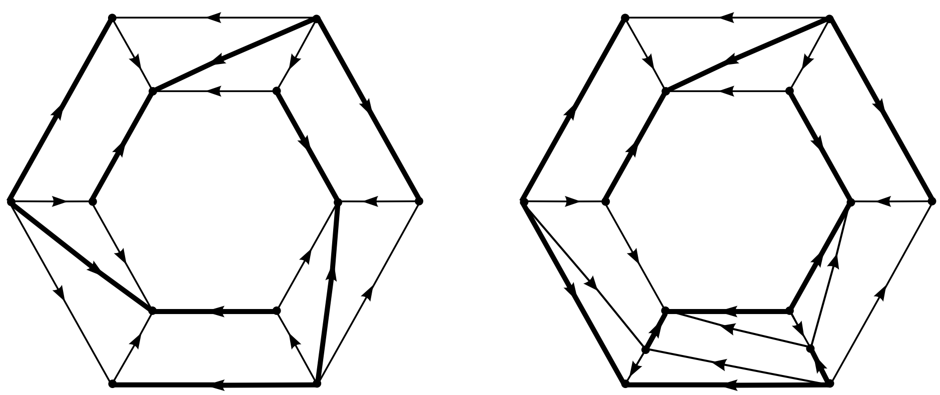

Figure 1.10: (D_1, ω_1) and a new embedding of (D_3, ω_3).

Remark 1.14. *Any directed graph that is contractible to a transitive extension of D_1, D_2, or D_3, fails the Edmonds-Giles Conjecture.*

Remark (1.14) follows in the same way as Remark (1.13). Hoping that they had found an excluded minor characterization of the directed graphs that satisfy the Edmonds-Giles Conjecture, Cornuéjols and Guenin asked the following question, which is the converse of Remark (1.14). In the next section, we show that the answer to this question is no.

Question 1.15 (Cornuéjols-Guenin). *If D fails the Edmonds-Giles Conjecture then*

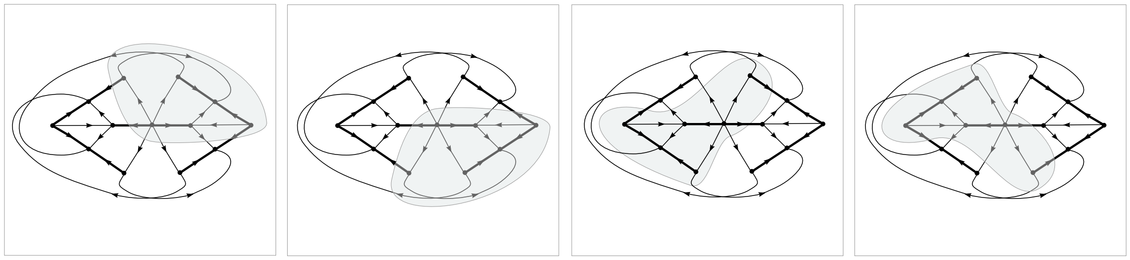

Figure 1.11: The four crossing cuts of (D_2, ω_2).

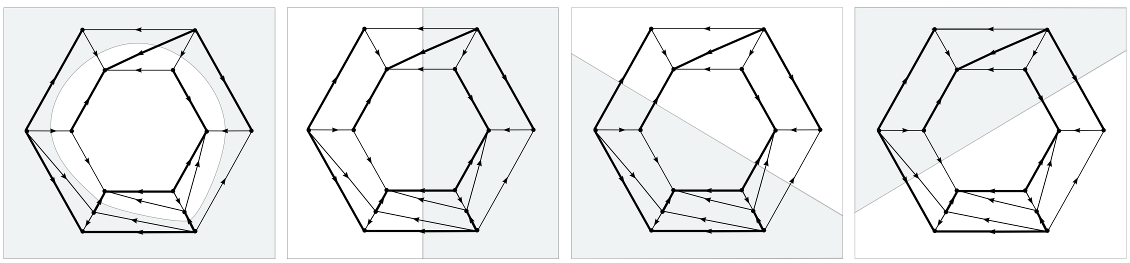

Figure 1.12: The four crossing cuts of (D_3, ω_3).

is it true that D is contractible to a transitive extension of D_1, D_2, or D_3?

1.2.3 New Counterexamples

One key step towards an excluded minor characterization is to understand how existing counterexamples can be modified into new counterexamples. Cornuéjols and Guenin have already pointed out the importance of contraction and transitive arcs in this regard. In this section, we present several new modifications that produce counterexamples that are not contractible to transitive extensions of D_1, D_2, or D_3. The goal is not just to expand the list of minimal counterexamples with respect to contraction and transitive arcs, but also to motivate the minimality operations that we will be using in subsequent chapters.

Modification One

The first modification involves the deletion of certain non-transitive arcs of weight 0 in (D_2, ω_2) and (D_3, ω_3). The following remark is a consequence of Remark (1.10).

Remark 1.16. *If (D, ω) fails the Edmonds-Giles Conjecture and $\omega(a) = 0$ and $\tau(D, \omega) = \tau(D \backslash a, \omega)$ then $(D \backslash a, \omega)$ also fails the Edmonds-Giles Conjecture.*

Remark (1.16) states that new counterexamples can be found from old counterexamples by deleting arcs of weight 0, so long as the deletion does not introduce directed cuts of weight smaller than τ. Applying this idea to (D_2, ω_2) and (D_3, ω_3) uncovers arcs that can be deleted (Figure (1.13)), while such deletions are not possible in (D_1, ω_1).

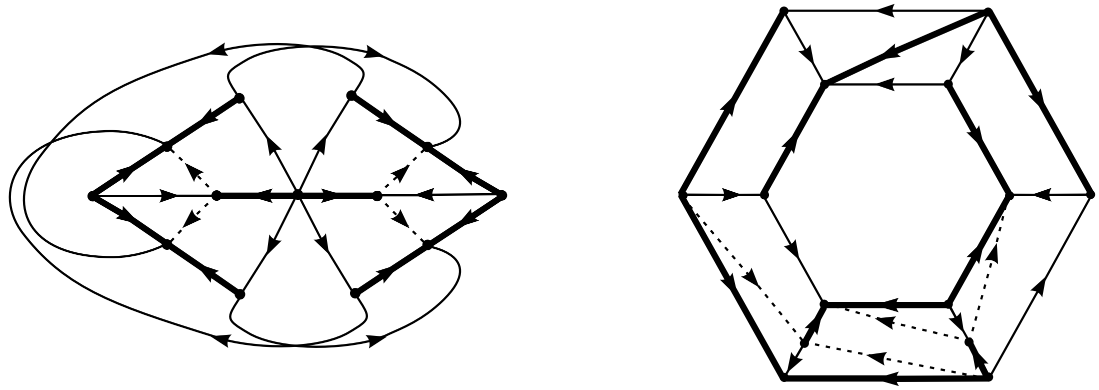

Figure 1.13: New counterexamples by deleting at most one dashed arc per node.

Notice that, in this figure, deleting any two dashed arcs adjacent to the same node results in a directed cut of weight 1. However, independently deleting at most one dashed arc adjacent to each node does not decrease the value of τ. Therefore, from Remark (1.16) the resulting directed graphs fail the Edmonds-Giles Conjecture.

Further counterexamples can be uncovered by noticing that an arc b that would be transitive in D may not be transitive in $D \backslash a$. Therefore, $(D \backslash a) \cup b$ will be another new counterexample (Figure (1.14)).

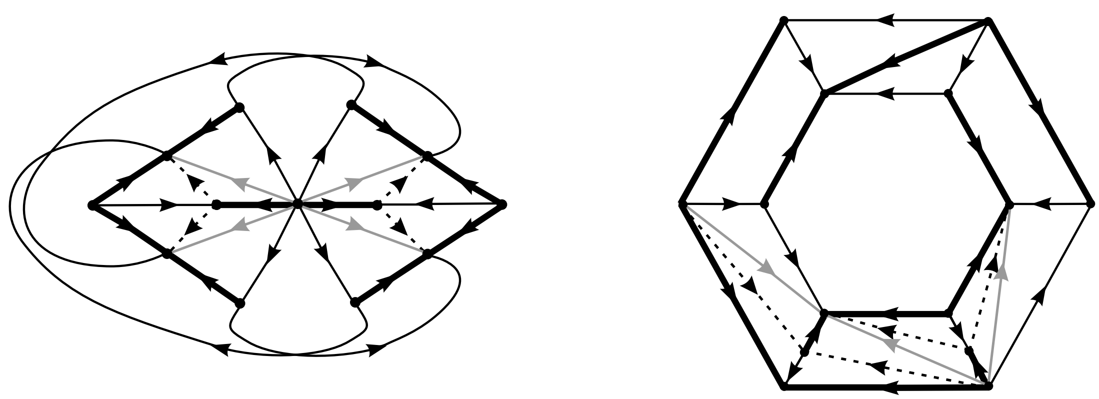

Figure 1.14: Additional counterexamples by adding non-transitive weight 0 arcs.

Deletion is not a closed operation for the property of satisfying the Edmonds-Giles Conjecture, so we will let $(D_2, \omega_2)^+$ and $(D_3, \omega_3)^+$ denote the small families of counterexamples found by deleting arcs of weight 0 from (D_2, ω_2) and (D_3, ω_3).

Modification Two

The second modification involves the addition of a node and arcs of weight 0 incident to this node. One must be careful to ensure that the result fails the Edmonds-Giles Conjecture and is not contractible to the original. In order to illustrate the important aspects of the construction, we show three failed attempts to modify (D_1, ω_1) in Figure (1.15).

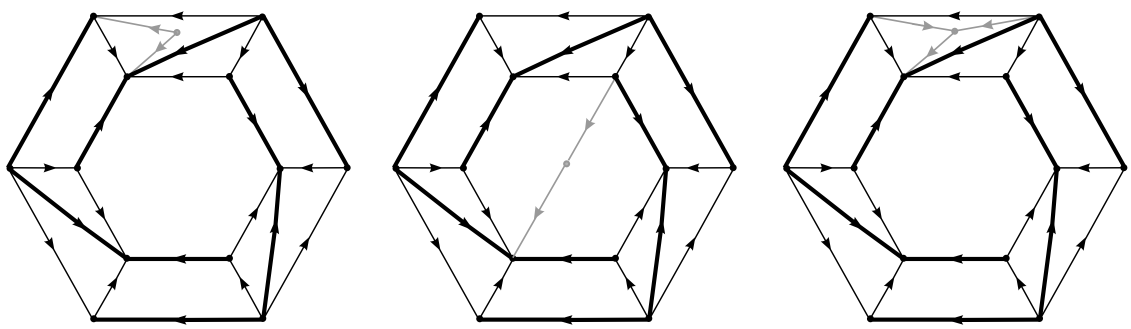

Figure 1.15: Modifications where $\tau = \nu$, or the directed graph contracts to D_1.

The first attempt has introduced the directed cut of weight 0. The second attempt has $\tau = 2$, but it has eliminated one of the crossing cuts that ensured $\nu < 2$. The third attempt has $\nu = 1 < 2 = \tau$ but it is contractible to D_1.

These failures suggest the following approach. Suppose (D, ω) fails the Edmonds-Giles Conjecture where

- r_1, r_2 are distinct sources in D
- s_1, s_2 are distinct sinks in D
- D contains a directed path from r_i to s_j for $i, j \in \{1, 2\}$

Form (D', ω') from (D, ω) by adding a node n and arcs of weight 0, (r_1, n), (r_2, n), (n, s_1), and (n, s_2). This construction ensures that if $\delta_D(X)$ is a directed cut then $\delta_{D'}(X)$ or $\delta_{D'}(X \cup \{n\})$ is a directed cut containing the same non-zero weight arcs as $\delta_D(X)$.

Hence, (D', ω') also fails the Edmonds-Giles Conjecture. Furthermore, D' is not contractible to D since contracting an arc that is adjacent to a node such as n will result in an arc incident to two sources or to two sinks. When applying this idea to (D_2, ω_2) and (D_3, ω_3) we find that each can support this construction in two different ways (Figure(1.16)). No such addition is possible in (D_1, ω_1).

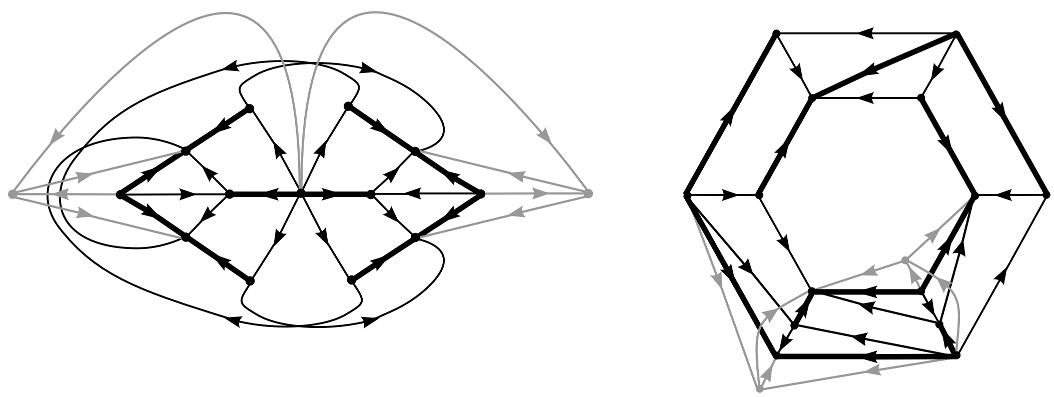

Figure 1.16: New counterexamples with added nodes and arcs.

Further counterexamples can be uncovered by repeated application of the idea, with complex complex counterexamples being formed by using added nodes from one addition during a subsequent addition (Figure (1.17)). In Chapter 4, a clique substitution will be used to eliminate all of these modifications.

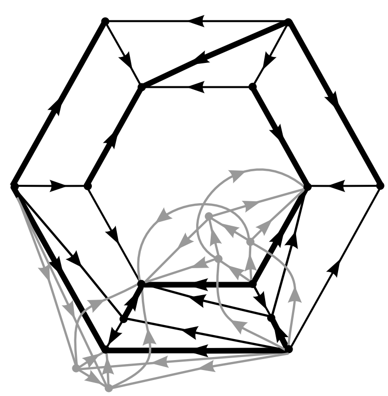

Figure 1.17: Additional counterexamples can be formed by repeatedly applying this idea.

Modification Three

The third type of modification involves the manipulation of crossing cuts. Let us recall the four crossing cuts of (D_1, ω_1) that are given in Figure (1.6). Notice that two of the crossing cuts, d_1 and d_2, intersect in arc a, where $\omega_1(a) = 1$ and a is the middle arc on one of the alternating paths. Our goal is to spread out d_1 and d_2 so that they no longer share this arc of weight 1. Towards this goal, let us consider what happens to d_1 and d_2 when the modifications in Figure (1.18) are made.

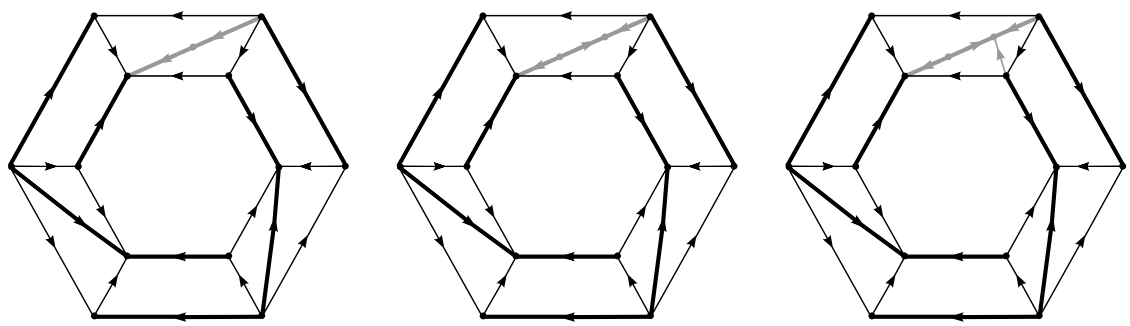

Figure 1.18: Modifications where $\tau = \nu$ or the directed graph contracts to D_1.

In the first case, the arc a has been replaced by two arcs of weight 1 in series, a_1, a_2. This modification has the effect of duplicating the crossing cuts, since $d_1 \backslash \{a\} \cup \{a_1\}$, $d_1 \backslash \{a\} \cup \{a_2\}$, $d_2 \backslash \{a\} \cup \{a_1\}$, and $d_2 \backslash \{a\} \cup \{a_2\}$ are all crossing cuts. However, this modification is not very useful since the weighted directed graph satisfies the Edmonds-Giles Conjecture, and it can be contracted to D_1.

In the second case, the arc a has been replaced by an alternating path of length three where each arc is weight 1. Again, this duplicates the crossing cuts d_1 and d_2. Also, the modified directed graph can be contracted to D_1 in two different ways. The one difference is that this weighted directed graph fails the Edmonds-Giles Conjecture.

The third case is similar to the second, except that an additional arc of weight 0 has been added. This important arc has the effect of eliminating the duplication of d_1. Although the directed graph can still be contracted to D_1, the additional arc has made it so that this can be done in only one way. The weighted directed graph fails the Edmonds-Giles Conjecture.

These observations lead to the modifications of (D_1, ω_1) found in Figure (1.19). Effectively these modifications have separated d_1 and d_2 by forcing d_1 and d_2 to use different arcs of weight 1. The final result are weighted directed graphs that fail the Edmonds-

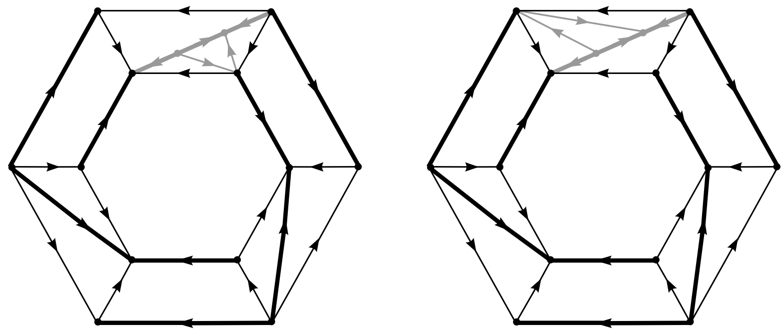

Figure 1.19: New counterexamples by modifying (D_1, ω_1).

Giles Conjecture, are minimal with respect to contraction in this regard, and the directed graphs are not contractible to transitive extensions of D_1, D_2, or D_3. It is also interesting to note that one of the new arcs of weight 1 is not included in any crossing cut. This is a property that is not present in (D_1, ω_1), (D_2, ω_2), or (D_3, ω_3).

Figure (1.20) shows that additional counterexamples can be uncovered by manipulating the arcs of weight 0, as was done with the first modification. Also, it should be noted that this modification can be independently done on any of the alternating paths in (D_1, ω_1) and on the alternating path of length three in (D_3, ω_3). In Chapter 4, a folding operation will be used to eliminate all of these modifications.

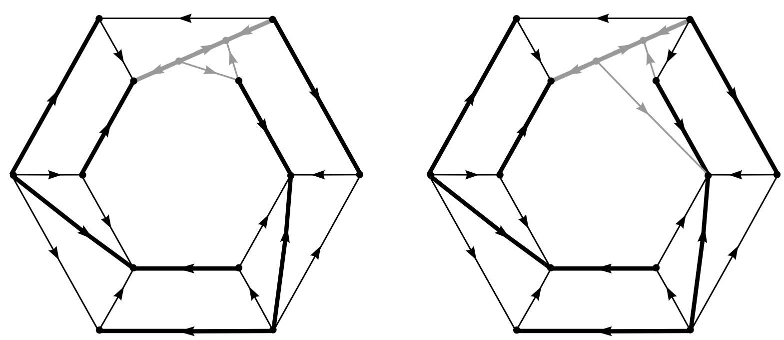

Figure 1.20: Additional counterexamples made by altering the weight 0 arcs.

1.3 Subsequent Chapters

Chapter 2 introduces the concept of *clutters*. Several results show that clutters are of particular interest to this topic, and that this topic is of particular interest to clutters.

In fact, Cornuéjols and Guenin discovered (D_2, ω_2) and (D_3, ω_3), while writing a paper with Margot [2], by running a computer search designed to find special types of clutters.

Chapter 3 describes source-sink connected graphs and outlines three existing proofs, each showing that the Edmonds-Giles Conjecture holds for any weighted graph in this class. The class is generalized to graphs containing a super-source and a super-sink, and it is conjectured that the Edmonds-Giles Conjecture holds for weighted graphs in this class.

Chapter 4 begins with a new definition for minimality. This definition leads to a common tree structure for all arcs of non-zero weight in any weighted directed graph that minimally fails the Edmonds-Giles Conjecture.

Chapter 5 further investigates the case when $\tau = 2$. The results from the previous chapter show that the path structure of (D_1, ω_1), (D_2, ω_2), and (D_3, ω_3) is not coincidental. In fact, we find that at least three paths are required for any minimal counterexample. Accompanying this path structure is a new global argument, called the *trace*. The trace leads to an expansion of the infinite D_1 family [17] found in Figure (1.8) to the infinite $D_{1,3}$ family. The smallest, and second smallest, members of this new family are found in Figures (1.10) and (1.21), respectively.

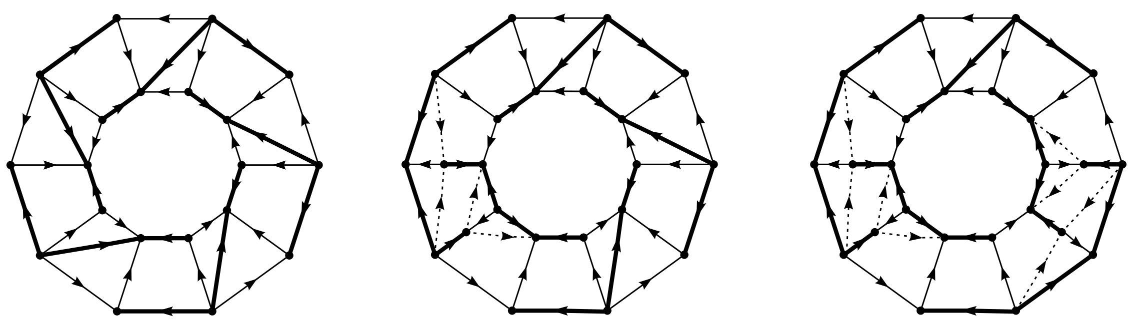

Figure 1.21: $D_{1,3}$ family with five paths

Chapters 6 and 7 are the remuneration for the two previous chapters, raising hope that new results can be found in this area. In particular, Chapter 6 shows that (D_1, ω_1), $(D_2, \omega_2)^+$, and $(D_3, \omega_3)^+$, are the only minimal counterexamples with three paths. Hence, the known counterexamples (D_1, ω_1), $(D_2, \omega_2)^+$, and $(D_3, \omega_3)^+$ can now be well understood as the only minimal counterexamples with additional restrictions of having $\tau = 2$ and three paths. Chapter 7 continues the investigation by showing that there are no minimal counterexamples with four paths.

Chapter 2

Clutters

This chapter defines clutters, blockers of clutters, minors of clutters, as well as several standard properties and results. It also introduces the term partitionable, and states a related conjecture due to Cornuéjols, Guenin, and Margot. Finally, these terms are applied to directed cuts and directed joins. For an excellent treatment of clutters, see Cornuéjols [1].

2.1 Basic Definitions

A *clutter* H is a finite set of vertices, $V(H)$, together with a finite set of edges, $E(H)$, such that each edge is a subset of the vertices, and no edge is a subset of another edge.

Let H be a clutter, ω a weight function mapping vertices to non-negative integers, and k a positive integer. We consider the following parameters:

$$\nu_k(H,\omega) = \frac{1}{k}\max\{r : \text{ there exists a list of } r \text{ edges in } E(H) \text{ with repetition allowed, such that no vertex } v \in V(H) \text{ is contained in more than } k\omega(v) \text{ members of this list.}\}$$

$$\tau_k(H,\omega) = \frac{1}{k}\min\{r : \text{ there exists a list of vertices in } V(H) \text{ with repetition allowed, whose total element-wise } \omega \text{ sum is } r\text{, such that no edge in } E(H) \text{ contains fewer than } k \text{ members of this list.}\}$$

Let $\nu(H,\omega)$ represent $\nu_1(H,\omega)$, and $\tau(H,\omega)$ represent $\tau_1(H,\omega)$. Since any list giving of vertices or edges can be replicated k times

$$\nu(H,\omega) \leq \nu_k(H,\omega) \tag{2.1}$$
$$\tau(H,\omega) \geq \tau_k(H,\omega) \tag{2.2}$$

for any positive integer k.

2.2 Linear Programming

In order to further understand ν and τ, consider the following linear program formulation. Let $M = M(H)$ be the $(0,1)$ incidence matrix for clutter H, where the columns index $V(H)$, and each row is the characteristic vector of vertices, for each edge in $E(H)$.

$$\nu^*(H,\omega) = max\{y1 : y \geq 0, yM \leq \omega\} \tag{2.3}$$
$$\tau^*(H,\omega) = min\{\omega x : x \geq 0, Mx \geq 1\} \tag{2.4}$$

The linear programs form a primal-dual pair. From linear programming duality,

$$\nu^*(H,\omega) = \tau^*(H,\omega) \tag{2.5}$$

Furthermore, the maximum value obtained by (2.3), where y/k is integer, is $\nu_k(H,\omega)$. Likewise, the minimum value obtained by (2.4), where x/k is integer, is $\tau_k(H,\omega)$. Thus, $\nu^*(H,\omega) \geq \nu_k(H,\omega)$ and $\tau^*(H,\omega) \leq \tau_k(H,\omega)$, for any positive integer k. Also from weak duality, for any positive integer k,

$$\nu_k(H,\omega) \leq \tau_k(H,\omega). \tag{2.6}$$

Therefore, from (2.1), (2.5), (2.6), we have the following chain of inequalities:

$$\nu(H,\omega) \leq \nu_k(H,\omega) \leq \nu^*(H,\omega) = \tau^*(H,\omega) \leq \tau_k(H,\omega) \leq \tau(H,\omega). \tag{2.7}$$

2.3 Properties

The chain of inequalities in (2.7) can be used to define several well known properties of clutters. Let H be a clutter, and k be a positive integer.

- if $\tau^*(H,\omega) = \tau(H,\omega)$, for all ω, then H is *ideal*.
- if $\nu_k(H,\omega) = \tau_k(H,\omega)$, for all ω, then H is *1/k-Mengerian*.
- if $\nu(H,\omega) = \tau(H,\omega)$, for all ω, then H is *Mengerian*.
- if $\nu(H,1) = \tau(H,1)$, then H *packs*.

Remark 2.1. *If H is Mengerian, then H is ideal.*

The converse of the Remark (2.1), is not true. Let Q_6 be the following clutter:

$$M(Q_6) \;=\; \begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}$$

Remark 2.2. *Q_6 is ideal, but does not pack, so Q_6 is not Mengerian.*

Classes of clutters, that arise from combinatorial objects, are often studied. For a historical example, consider directed graph $D = (N, A)$, and specified nodes $s, t \in N$. Let $H = (A, P)$ be the clutter of st-paths, where P is the set of arc minimal directed paths, from s to t.

In this case, $\nu(H,\omega)$ is the size of the largest packing of st-paths, and $\tau(H,\omega)$ is the smallest weight of an st-cut, for any ω.

From Menger's Theorem, $\nu(H,\omega) = \tau(H,\omega)$, so for any directed graph, the clutter of st-cuts, is Mengerian.

2.4 Blocker

The *blocker* of clutter $H = (V, E)$, is denoted $b(H)$, and is the unique clutter (V, E'), where E' is the set of all minimal subsets $A' \subseteq V$, such that $A' \cap A \neq \emptyset$, for all $A \in E$.

Proposition 2.3. *[1] H is ideal, if and only if, $b(H)$ is ideal.*

Edmonds and Giles showed that clutters come in pairs, because the blocker of the blocker of a clutter, is the original clutter.

Proposition 2.4. *[1] $b(b(H)) = H$.*

Example 2.5. *For any graph, the clutter of st-cuts and the clutter of st-paths form a blocking pair.*

Example 2.6. *For any graph, the clutter of T-joins and the clutter of T-cuts form a blocking pair.*

In particular, we will see that, for any graph the clutter of directed cuts and the clutter of directed joins form a blocking pair.

2.5 Contraction, Deletion, Minors

For clutter $H = (V, E)$, define $H \backslash v = (V - \{v\}, \{A \in E : v \notin A\})$, and $H/v = b(b(H \backslash v))$. Both $H \backslash v$ and H/v are clutters. These two operations are called *deletion* and *contraction*, respectively. It is not hard to show that the two operations commute with each other.

With regards to the properties listed in section (2.3), contraction and deletion are equivalent to ensuring an appropriate value of $\omega(v)$, for $\omega \in \mathbb{Z}_+^V$.

Remark 2.7. *Contracting v is equivalent to setting $\omega(v) = \infty$.*

Remark 2.8. *Deleting v is equivalent to setting $\omega(v) = 0$.*

If clutter G can be obtained from clutter H, by a sequence of contractions and deletions, then G is called a *minor* of H. If G is a minor of H, and $G \neq H$, then G is a *strict minor*, of H.

The properties of Mengerian, $1/k$-Mengerian, and ideal, are closed under taking minors. The following property is also closed under taking minors.

2.6 Partitionable

For clutter H, we introduce the following terminology: H is *partitionable*, if $\tau(H, \omega) \geq 2$ implies $\nu(H, \omega) \geq 2$, for all weight functions ω.

Conjecture 2.9. *[Cornuéjols-Guenin-Margot] [2] If H is ideal, and non-Mengerian, and every minor of H is Mengerian, then for some weight function ω*

$$2 = \nu(H, \omega) > \tau(H, \omega) = 1.$$

Remark 2.10. *Conjecture (2.9) would imply that an ideal clutter is Mengerian if and only if it is partitionable.*

Cornuéjols, Guenin, and Margot [2] further defined the Q_6*-property*, and demonstrated its importance, with Theorem (2.12).

Definition 2.11. *Clutter H has the Q_6-property, if H does not pack, and $V(H)$ can be partitioned into non-empty sets $I_1, \ldots, I_6$, such that there are $S_1, \ldots, S_4 \in E(H)$, where*

$$S_1 = I_1 \cup I_3 \cup I_5, \qquad S_2 = I_1 \cup I_4 \cup I_6,$$
$$S_3 = I_2 \cup I_4 \cup I_5, \qquad S_4 = I_2 \cup I_3 \cup I_6.$$

Theorem 2.12. *If H is ideal, and does not pack, but every strict minor of H does pack, and $\tau(H, 1) = 2$, then H has the Q_6-property.*

2.7 Directed Cuts and Directed Joins

For directed graph, $D = (N, A)$, let $C_D = (A, C)$ be the clutter of directed cuts, where C is the set of all arc minimal directed cuts, in D. Let $J_D = (A, J)$ be the clutter of directed joins, where J is the set of all arc minimal directed joins, in D.

In this section, C_D and J_D are shown to form a blocking pair, the effect of contracting an arc in D is translated to C_D and J_D, and J_D is shown to be ideal.

Proposition 2.13. $b(C_D) = J_D$

Proof. Suppose $b \in b(C_D)$. Therefore b is a minimal set of arcs that intersects every directed cut. Since the contraction of $a \in A$, eliminates every directed cut containing a, and creates no new directed cuts, then D/b contains no directed cuts. Hence, D/b is strongly connected. Since b is a minimal such set of arcs, $b \in J_D$.

Suppose that $j \in J_D$. Therefore, j is a minimal set of arcs, such that D/j is strongly connected. Therefore, D/j contains no directed cuts. Since the contraction of $a \in A$, eliminates every directed cut containing a, and creates no new directed cuts, then j must only intersect each directed cut. Since j is a minimal such set of arcs, $j \in b(C_D)$. ☐

Proposition 2.14. *$C_{D/a} = C_D \backslash a$ and $J_{D/a} = C_D/a$.*

Proof. The first statement follows from Remark (1.5). The second statement follows from the first statement, Remark (2.13), and from the definition of contraction and deletion:

$$\begin{aligned} J_{D/a} &= b(C_{D/a}) \\ &= b(C_D \backslash a) \\ &= b(C_D)/a \\ &= J_D/a. \end{aligned}$$

□

Proposition 2.15. *J_D is ideal for all directed graphs D.*

Proof. From the Lucchesi-Younger Theorem, C_D is Mengerian for all directed graphs. Therefore, by Remark 2.1, C_D is ideal. By Proposition 2.15, $b(C_D) = J_D$. Therefore, by Proposition 2.3, J_D is ideal. □

Remark 2.16. *$\tau^*(J_D, \omega)$ is the smallest weight of a directed cut, for all directed graphs D, and all weight functions ω.*

If the Cornuéjols-Guenin-Margot Conjecture was true, then Remark (2.16) would imply that if directed graph D failed the Edmonds-Giles Conjecture, then there would exist an ω such that

$$1 = \nu(D, \omega) < \tau(D, \omega) = 2.$$

Therefore, there is special significance to studying weighted directed graphs with $\tau = 2$.

Chapter 3

Source-Sink Connected

A directed graph is, *source-sink connected*, if it contains directed paths, connecting every source, to every sink (Figure (3.1)).

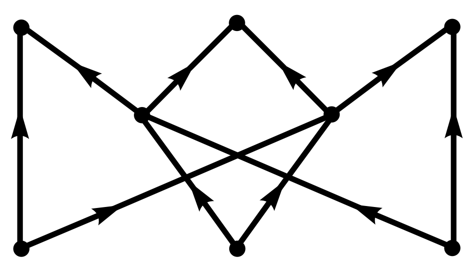

Figure 3.1: A source-sink connected directed graph.

Schrijver [13], Feofiloff and Younger [6], and Feofiloff [5] have shown that every source-sink connected directed graph satisfies the Edmonds-Giles Conjecture, for all possible weightings.

Theorem 3.1. *The maximum number of directed joins in a packing is equal to the smallest weight of a directed cut, for all weighted source-sink connected graphs.*

This chapter outlines three known proofs to this result, each of which varies considerably, from the others. The chapter concludes by generalizing the class of source-sink connected graphs to super-source super-sink graphs and offers the conjecture that the Edmonds-Giles Conjecture holds for this new class.

3.1 Proof by Schrijver

This section outlines the proof of Theorem (3.1), presented in Schrijver [15], and first given by Schrijver [13].

The proof has its foundations in a rooted arborescences theorem proven by Edmonds [3]. An *r-arborescence* is a set of arcs forming a directed spanning tree, rooted at node r, where every arc in the tree is directed away from r. An *r-cut* is the set of arcs with origin in R, and destination in $\overline{R}$, for any node set R with $r \in R$.

Theorem 3.2 (Disjoint Arborescences). *The maximum number of pairwise disjoint r-arborescences, is equal to, the size of the smallest r-cut, for any directed graph, and any specified node r.*

This theorem is used to prove a theorem by Schrijver [13] (see also [14]) on bibranchings. For node partition R, S, an *R-S bibranching* is a set of arcs that includes a directed path from r to S for all $r \in R$, and a directed path from R to s for all $s \in S$. An *R-S bicut* is the set of arcs with origin in U and destination in $\overline{U}$ if $U \subseteq R$, and is the set of arcs with origin in $\overline{U}$ and destination in U if $U \subseteq S$.

Theorem 3.3 (Disjoint Bibranchings). *The maximum number of pairwise disjoint $R-S$ bibranchings is equal to the size of the smallest $R-S$ bicut, for any directed graph, and any node partition R, S.*

Schrijver's proof of Theorem (3.1) also uses two elementary facts about directed cuts. Let W be a path, or a cycle, and $\delta(X)$ be a directed cut. If W is not directed, then it has arcs in two distinct directions called *forwards* and *backwards*. Say that arcs a and b are *consecutive* in $W \cap \delta(X)$ if it is possible to travel along W so that there is no other arc in $W \cap \delta(X)$ occurring between the a and b.

Remark 3.4. *In the intersection of a directed cut and a path, consecutive arcs are in opposite directions.*

Remark 3.5. *The intersection of a directed cut and a cycle, contains an equal number of forwards and backwards arcs.*

Remark (3.5) conveniently shows why directed cuts and directed cycles do not intersect.

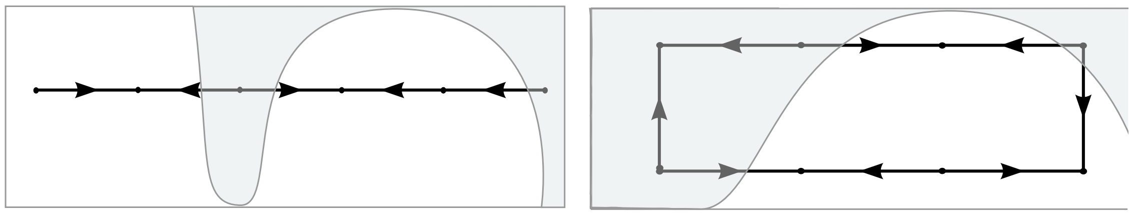

Figure 3.2: A directed cut intersecting a path, and a cycle.

Consider a potential counterexample to Theorem (3.1), $(D = (N, A), \omega)$, which minimizes the value

$$|N| + \sum_{a \in A} \omega(a). \tag{3.1}$$

Since $|A|$ is independent of (3.1), transitive arcs of weight 0 may be added to (D, ω), without consequence. This is because such arcs do not change the directed cuts, the weight of any directed cut, or the value of (3.1). Hence, we may assume the following:

Remark 3.6. *If there is a directed path from u to v, then $(u, v) \in A$.*

Furthermore, by minimizing over (3.1), it is possible to show the following:

Remark 3.7. *If $\omega(a) > 0$, then a is in a directed cut of weight $\tau(D, \omega)$.*

Remark 3.8. *If $\omega(\delta(X)) = \tau(D, \omega)$, then $\delta(X)$ is a trivial directed cut.*

Remark 3.9. *If $\omega(a) > 0$, then a is adjacent to a source, or a sink.*

Remarks (3.7), (3.8), are also valid for the definition of minimality, discussed in Chapter 4, and appear as Proposition (4.20), and Proposition (4.13), respectively. The next step is to prove the following lemma.

Lemma 3.10. *If (u, v) and (u', v') are arcs, with non-zero ω weight, and there is a directed path from u' to v, then either u' is a source or v is a sink.*

Proof. Otherwise, from Remark (3.9), it must be that u is a source and v' is a sink. Since D is source-sink connected, there is a directed path from u to v'. From Remark (3.6), $(u, v') \in A$ (Figure (3.3)).

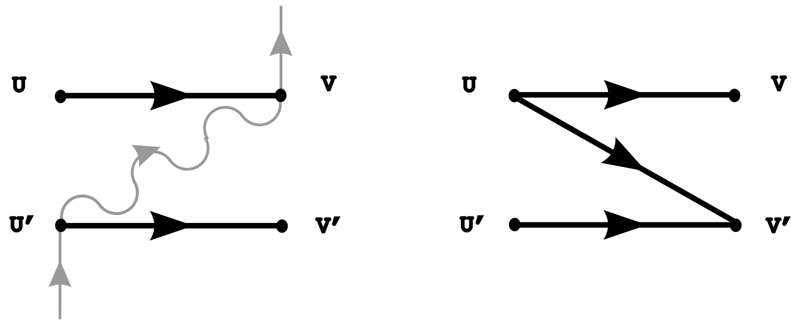

Figure 3.3: If u' non-source and v non-sink, then arc (u, v') exists.

Consider a new weight, ω', where

$$\omega'(a) = \begin{cases} \omega(a) + 1 & \text{if } a = (u, v') \\ \omega(a) - 1 & \text{if } a \in \{(u, v), (u', v')\} \\ \omega(a) & \text{otherwise.} \end{cases}$$

First of all, we show that replacing ω with ω' does not change the weight of the smallest directed cut. Notice that the weight of every trivial cut is equal in (D, ω) and (D, ω'). Furthermore, from Remark 3.4 and the path containing arcs

$$(u, v), (u, v'), (u', v'),$$

if a directed cut intersects both (u, v) and (u', v'), then it also intersects (u, v'). Hence, the weight of every non-trivial directed cut in (D, ω') is at most one less than in (D, ω). Therefore, from Remark (3.8), $\tau(D, \omega) = \tau(D, \omega') = \tau$.

Since (D, ω') also lowers the value of (3.1) set by (D, ω), there exists a packing $J_1, \ldots, J_\tau$ of directed joins in (D, ω'). Suppose that J_1 includes (u, v'). Consider J_1', where

$$J_1' \quad = \quad J_1 - \{(u, v')\} \cup \{(u, v), (u', v')\}).$$

We show that J_1' is also a directed join. From the hypothesis there is a directed path from u' to v. Therefore, there is a cycle containing nodes u, v, u', v' and the arcs

$$(u, v), (u, v'), (u', v').$$

From this cycle, and Remark (3.5), if a directed cut intersects (u, v') then it must also intersect (u, v), or (u', v'). Therefore, J_1' is a directed join and $J_1', J_2, \ldots, J_\tau$ contradicts the choice of (D, ω). □

From this lemma, it is possible to partition N into R and S where every source is in R, every sink is in S, $\delta(R)$ is a directed cut, and every arc $a \in \delta(R)$ with $\omega(a) > 0$ has a source as its origin and a sink as its destination.

To prove this, consider the set of arcs A', where

$$A' = \{(v, u) : (u, v) \in A,\ \omega((u, v)) > 0, \text{ and } u \text{ is not a source, or } v \text{ is not a sink.}\}$$

Let R be a set of nodes where $n \in R$ if and only if there is a directed path from n to a source in $(N, A \cup A')$.

From this definition of R every source is included in R, $\delta(R)$ is a directed cut, and every $a \in \delta(R)$ with $\omega(a) > 0$ has a source as its origin and a sink as its destination. Furthermore, R does not contain a sink, otherwise there would be a directed path from a sink to a source in $(N, A \cup A')$. Choosing the shortest such path can be used to contradict Lemma (3.10).

The next step is to construct the graph

$$D' = (N, A_\omega \cup A_0^{-1})$$

where A_ω contains $\omega(a)$ copies of every arc $a \in A$, and A_0^{-1} contains τ copies of (v, u) for every arc (u, v), where the combined weight of arcs from u to v is 0.

Now, for any $U \subseteq N$, if $\delta(U)$ is a directed cut in D, then A_ω has at least τ arcs with origin in U and destination in $\overline{U}$. Furthermore, if $\delta(U)$ is not a directed cut in D, then A_0^{-1} has at least τ arcs with origin in U and destination in $\overline{U}$.

Hence, for any $U \subseteq N$, D' has at least τ arcs exiting U. Therefore, by Theorem (3.3), there are τ disjoint R-S bibranchings in D'. Schrijver finishes the proof by showing that each of these bibranchings, when intersected with A_ω, results in a directed join.

3.2 Proof by Feofiloff-Younger

A *source-side cut* is a directed cut, $\delta(X)$, where X contains no sinks. A *sink-side cut* is a directed cut, where X contains every source. A *side cut* is either a source-side cut or a sink-side cut. On the other hand, a *cross cut* is a directed cut, $\delta(X)$, where X contains a sink and does not contain every source (Figure (3.4)).

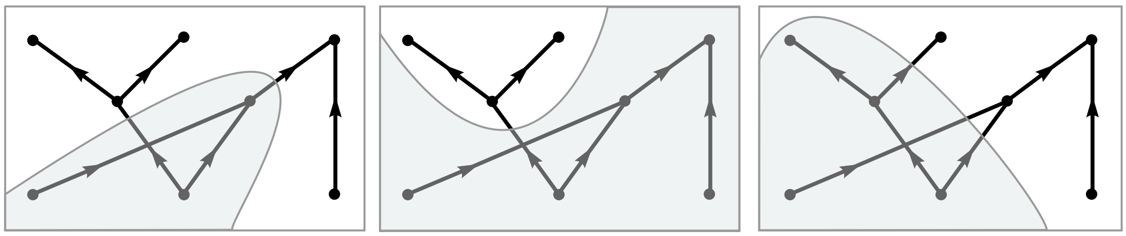

Figure 3.4: From left to right: a source-side cut, a sink-side cut, and a cross cut.

A *side join* is a set of arc, that intersects every side cut. Feofiloff-Younger reformulated the source-sink connected theorem as a statement on side joins.

Theorem 3.11 (Feofiloff-Younger). *The maximum number of side joins in a packing is equal to the smallest weight of a side cut, for all weighted directed graphs.*

Here we show that Theorem (3.1) and Theorem (3.11) are equivalent. In a source-sink connected graph suppose $\delta(X)$ is a directed cut. If X contains a sink, then X must contain every source; otherwise, there is a directed path from a source outside of X to a sink inside of X contradicting that $\delta(X)$ is a directed cut. Therefore, in a source-sink connected graph every directed cut is a side cut, and Theorem (3.11) implies Theorem (3.1).

On the other hand, if (D, ω) is a weighted directed graph then let (D', ω') be the weighted directed graph obtained by adding a weight 0 arc from every source to every sink. The smallest weight of any directed cut in (D', ω'), denoted $\tau(D', \omega')$, is the smallest weight of any side cut in (D, ω). From Theorem (3.1), it is possible to find a packing of $\tau(D', \omega')$ directed joins in (D', ω'). In particular, this packing is a packing of side joins in (D, ω), and is equal in size, to the smallest weight of any side cut in (D, ω). Therefore, Theorem (3.1) implies Theorem (3.11).

The first step of the Feofiloff-Younger proof is to reduce the problem, to one of bi-side joins. A *bi-side join* j is a set of arcs where the arcs in j that are adjacent to a source intersect every source-side cut, and the arcs in j that are adjacent to a sink intersect every sink-side cut. Every bi-side join is a side join, but the converse is not true.

In order to complement the definition of a bi-side join, the notion of bi-weight needs to be introduced. The *bi-weight* of a source-side cut (respectively, sink-side cut) is the total weight of every arc in the directed cut that is adjacent to a source (respectively, sink).

Theorem 3.12 (Feofiloff-Younger). *The maximum number of bi-side joins in a packing is equal to the smallest bi-weight of a side cut, for all weighted directed graphs.*

Feofiloff and Younger show that Theorem (3.12) implies Theorem (3.11). The bi-side join theorem is proved algorithmically. In particular, bi-side joins are built one arc at a time, always ensuring that enough room is left over for the remaining bi-side joins to be built.

3.3 Proof by Feofiloff

Feofiloff [5] shows that any node n in (D, ω) that is neither a source nor a sink can be *split* into two nodes: one source n_+, and one sink n_- (Figure (3.5)). As long as a total weight of at least $\tau(D, \omega)$ is placed on arcs from n_+ to n_-, the resulting weighted directed graph will satisfy the Edmonds-Giles Conjecture if and only if (D, ω) satisfies the Edmonds-Giles Conjecture.

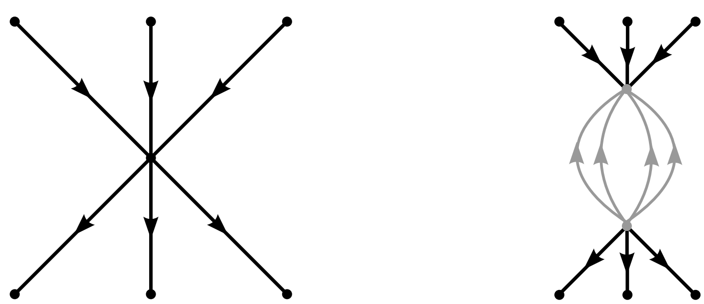

Figure 3.5: Splitting a node into a source and a sink.

By splitting every node that is neither a source nor a sink, the result is a directed bipartite graph with sources forming one side of the node partition, and sinks forming the other side of the node partition. The idea for this transformation was provided by Lucchesi.

In this form, one may find a packing of τ source-side joins. A *source-side join* is a set of arcs that intersects every source-side cut. This packing may or may not be a packing of side joins. The key to the proof is that any source-side join can be modified to intersect strictly more sink-side cuts. Moreover, the method used to improve the source-side join ensures that each element of the initial packing continues to be a source-side join. By repeated application of the method, each source-side join is converted into a side join.

Therefore, the procedure can be used to prove the desired theorem, and furthermore, the procedure is completed in polynomial time.

3.4 Super-Source Super-Sink

A node is a *super-source* if it is a source, and there exist directed paths from the node to every sink. A node is a *super-sink* if it is a sink, and there exist directed paths from every source to the node. A directed graph is *super-source super-sink* if it contains both a super-source and a super-sink.

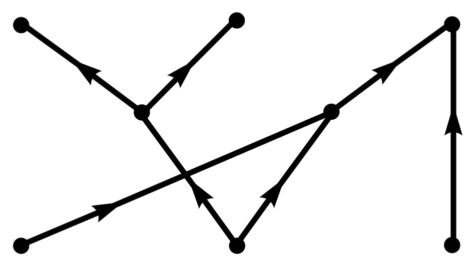

Figure 3.6: A super-source super-sink directed graph.

Conjecture 3.13. *The maximum number of directed joins in a packing is equal to the smallest weight of a directed cut, for all weighted super-source super-sink graphs.*

Notice that in a source-sink connected graph every source is a super-source, and every sink is a super-sink. Therefore, Conjecture (3.13) would generalize Theorem (3.1) from a condition on all sources and sinks to a condition on a single source and a single sink.

For example, in (D_1, ω_1) there are directed paths joining each of the three sources to two of the sinks, and directed paths joining two of the sources to each of the three sinks. Figure (3.7) shows that attempting to create a super-source or a super-sink in (D_1, ω_1) by adding the necessary directed path will violate one of its special directed cuts.

It is also worth noting that Conjecture (3.13) is the weakest condition that could be placed on the sources and sinks, since D_2 contains a super-source (Figure (3.8)).

In order to further support the conjecture, let us show that the property of having a super-source and a super-sink is closed under several operations that are used in the upcoming chapters. Hence, if there is a weighted super-source super-sink graph that

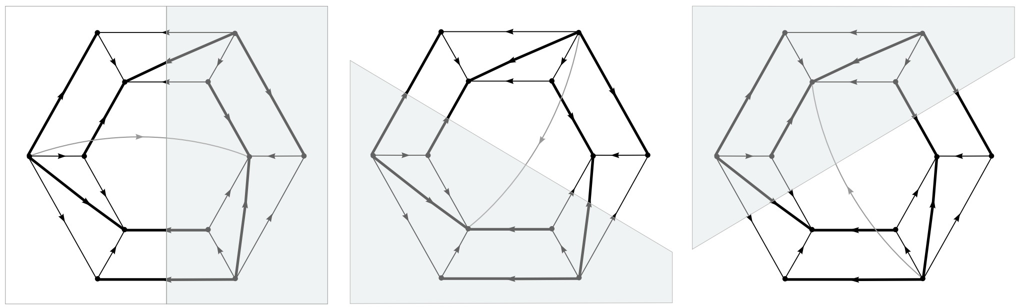

Figure 3.7: Modifying D_1 to have a super-source or super-sink violates one special cut.

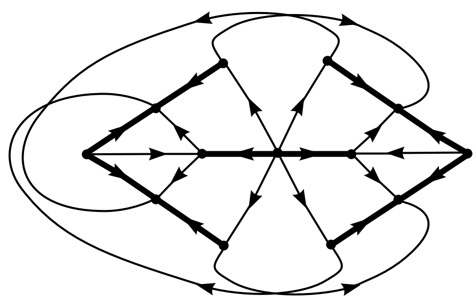

Figure 3.8: The center node in this embedding of D_2 is a super-source.

fails the Edmonds-Giles Conjecture then there is also such a graph that is minimal with respect to these operations.

Chapter 4 gives a formal definition of the following operations: loop deletion, parallel arc deletion, arc contraction (Figure (3.9)), and bi-clique substitution (Figure (3.10)).

Closure for loop deletion and parallel arc deletion follow easily. For the case of arc contraction, difficulty only arises if a super-source or a super-sink is the origin or destination of the contracted arc. Let D be a directed graph with super-sink n and arc $a = (u, n)$. Let n' be the node in D/a that results from contracting arc a. Since n is a super-sink in D there exists a directed path from every source in D to n. Since every source in D, except possibly u, is also a source in D', then these directed paths show that in D/a there exists a directed path from every source in D/a to n. If n' is not a super-sink in D/a, then it must be that n' is not a sink. Therefore, there exists a directed path in D/a from n' to some sink n''. By composing directed paths it is easy to see that n'' is a super-sink in D/a. The analogous result holds for super-sources, and therefore, super-source super-sink graphs are closed under arc contraction.

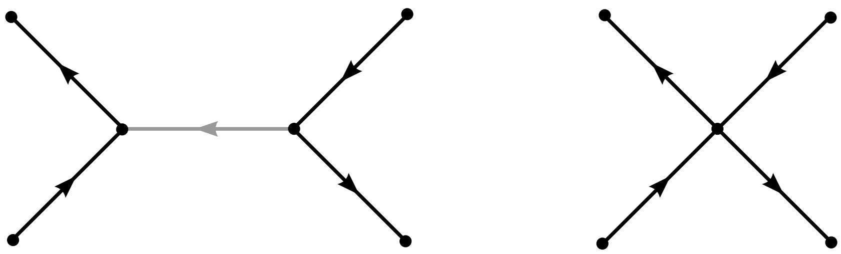

Figure 3.9: Example of arc contraction.

Next, we show closure for bi-clique substitutions. Let n be a node where there are at least two arcs with n as the origin, and at least two arcs with n as the destination. A *bi-clique substitution* on n is an operation that deletes n, and adds the arc (u, w) for every pair of nodes u, w where (u, n) and (n, w) are arcs.

Suppose that D' is the result of performing a bi-clique substitution on node n in directed graph D. Consider any directed path from a source to a sink in D. If the path does not intersect n then it exists in D'. If the path does intersect n then a replacement path, using one of the newly introduced arcs can be found in D'.

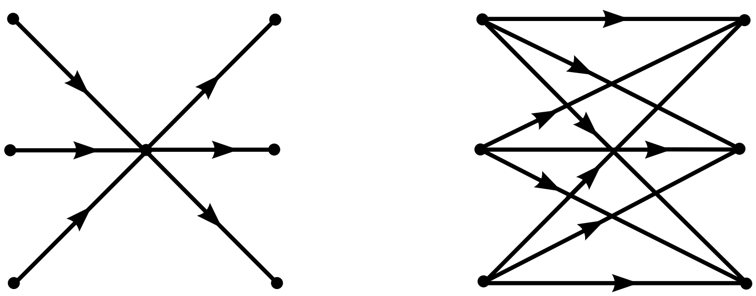

Figure 3.10: Example of bi-clique substitution.

A final note on Conjecture (3.13) is given in Section (6.4). If the conjecture turns out to be false, it will be of interest to investigate how far along the spectrum from source-sink connected to super-source super-sink the Edmonds-Giles Conjecture holds.

Chapter 4

Minimally non-Knitted

knitted: [adj] *made by intertwining threads in a series of connected loops.*

In this chapter, the term *knitted* is introduced to represent weighted directed graphs that satisfy the Edmonds-Giles Conjecture. The name is chosen because contracting a directed join results in a strongly connected graph. Since every arc in a strongly connected graph is in a directed cycle, it is as if the graph is a "series of connected loops".

The ultimate goal of this chapter is to study the properties of minimally non-knitted weighted directed graphs. Towards this goal, the concept of an *augmented directed graph* is introduced. Minimally non-knitted augmented directed graphs are studied, and then extended to minimally non-knitted weighted directed graphs by adding the smallest possible set of weights that maintain the property of being non-knitted.

The main result is that the non-zero weight arcs in any minimally non-knitted weighted directed graph, can be uniquely partitioned into *s-trees*. This structure is interesting in itself, and is also the starting point for the next chapter.

4.1 Augmented Directed Graphs

An augmented directed graph $[D, Z]$, is a directed graph $D = (N, A)$, and a specified arc subset, $Z \subseteq A$, where every arc in Z is forced to have weight 0 or ∞. Hence, arcs in Z are forced to be contracted or deleted in the clutter of directed joins (Remark 2.7 and 2.8). If an arc weight vector for D satisfies the restrictions imposed by Z then we say the weight is *valid* for $[D, Z]$, or simply *valid*. Effectively, the weight ∞ represents any value as large as, τ, the smallest weight of a directed cut.

Definition 4.1. *An augmented directed graph* $[D, Z]$ *is* knitted *if the maximum number of directed joins in a packing is equal to the smallest weight of a directed cut, for all valid weights* Z*.*

Augmented directed graphs are studied here because they give stronger minimality results than directed graphs. For example, a special type of biclique substitution allows us to avoid one of the problems with the Cornuéjols and Guenin question (Figures (1.16) and (4.1)).

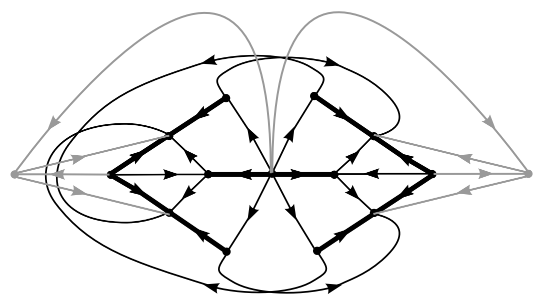

Figure 4.1: Minimality using augmented directed graphs avoids the above problem.

A *minor* of an augmented directed graph, is any augmented directed graph that can be obtained from it by a sequence of the following operations: loop deletion, parallel arc deletion, arc contraction, Z-transfer, transitive-Z deletion, Z-biclique substitution, and folding.

Next we describe each operation, and show that the property of being knitted is closed under the operation. In each case, let $[D, Z]$ be a knitted augmented directed graph, for $D = (N, A)$, and $[D', Z']$ be the result of applying the particular operation.

Loop or parallel arc deletion

$[D', Z'] = [D \backslash a', Z \backslash a']$, for loop or parallel arc a'. Closure can easily be seen by considering the clutter. Alternatively, a packing of directed joins for $[D', Z']$ with arbitrary weight ω' can be found from a packing of directed joins for $[D, Z]$ with weight ω such that

$$\omega(a) = \begin{cases} \omega'(a) & \text{if } a \neq a' \\ 0 & \text{if } a = a'. \end{cases} .$$

Arc contraction

$[D', Z'] = [D/a', Z\backslash a']$, for arc a'. From the definition of Z, any arc in Z may have its weight changed to ∞. Thus, any arc in D may have its weight changed to ∞. By using Remark (2.14), a packing of directed joins for $[D', Z']$ with arbitrary valid weight ω', can be found from a packing of directed joins for $[D, Z]$ with weight ω such that

$$\omega(a) = \begin{cases} \omega'(a) & \text{if } a \neq a' \\ \infty & \text{if } a = a'. \end{cases}$$

Z-transfer

$[D', Z'] = [D, Z \cup \{a'\}]$. If $a' \in A\backslash Z$, then we can perform a *Z-transfer* on a'. Notice that every weight that is valid for $[D', Z']$ is also valid for $[D, Z]$. Therefore, directed joins for $[D', Z']$ with arbitrary valid weight ω', can be found from a packing of directed joins for $[D, Z]$ with weight ω'.

Transitive-Z deletion

$[D', Z'] = [D\backslash a', Z\backslash a']$. If $a' \in Z$, and a' is a transitive arc, then a *transitive-Z deletion* can be performed on a'. This operation does not change the directed cuts (Remark (1.10)), nor does it change the weight of any directed cut. Therefore, directed joins for $[D', Z']$ with arbitrary valid weight ω', can be found from a packing of directed joins for $[D, Z]$ with weight ω

$$\omega(a) = \begin{cases} \omega'(a) & \text{if } a \neq a' \\ 0 & \text{if } a = a'. \end{cases}$$

Z-biclique substitution

$[D', Z'] = [D\backslash A_r \cup A_{uw}, Z\backslash A_r \cup A_{uw}]$. Let r be a node in $D = (N, A)$, and let $A_r \subseteq A$ be the arcs that have r as the origin or destination. Furthermore, let A_{uw} contain the arc (u, w) for every pair of nodes u, w where $(u, r) \in A$ and $(r, w) \in A$. Suppose that $A_r \subseteq Z$, there are at least two arcs in A_r where r is the origin, and there are at least two arcs in A_r where r is the destination. Given this situation, a *Z-biclique substitution* can be performed on r (Figure (4.2)).

Now we show that the property of being knitted is closed under this operation. In particular, we must show how the arcs in A_{uw} can be simulated by the arcs in A_r, given

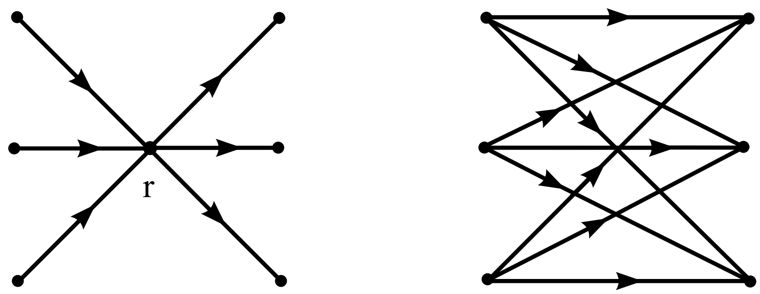

Figure 4.2: The biclique operation on node r. Every arc in the figure is in Z.

that both sets of arcs are limited to weights of 0 and ∞. At first it may seem that there will be an information problem, since there can be many more arcs in A_{uw} than in D_r. However, suppose that any pair of arcs (u_1, w_1) and (u_2, w_2) in A_{uw} are given a weight of ∞. This is equivalent to contracting both (u_1, w_1) and (u_2, w_2) in D'. However, from the definition of A_{uw}, the arcs (u_1, w_2) and (u_2, w_1) are also present in D'. Therefore, from Remark (3.4), if (u_1, w_1) and (u_2, w_2) are contracted, then the directed cycle formed in D' by (u_1, w_2) and (u_2, w_1) can also be contracted. In particular, the nodes u_1, u_2, w_1, w_2 can be identified in D'. In general, every node in D' that is either the origin or destination of an arc in A_{uw} with weight ∞, can be contracted to a single node. This simplified behaviour can be simulated in A_r.

Specifically, a packing of directed joins for $[D', Z']$ with arbitrary valid weight ω', can be found from a packing of directed joins for $[D, Z]$ with weight ω

$$\omega(a) = \begin{cases} \omega'(a) & \text{if } a \notin A_{uw} \\ \infty & \text{if } a = (u', r) \text{ or } a = (r, w'), \text{ and } \exists (u', w') \in A_{uw} \text{ with } \omega(u', w') = \infty \\ 0 & \text{otherwise.} \end{cases}$$

Folding

$[D', Z'] = [D \circ (w_1, w_2) \circ (u_1, u_2), Z \circ (w_1, w_2) \circ (u_1, u_2)]$. The notation $D \circ (x, y)$ represents the directed graph D with the nodes x and y identified. Likewise, $Z \circ (x, y)$ is the set of arcs Z with x and y identified. If $D = (N, A)$ contains an alternating path of length three,

$$P = (w_2, (w_2, u_2), u_2, (w_1, u_2), w_1, (w_1, u_1), u_1) = (w_2, f_2, u_2, b, w_1, f_1, u_1)$$

with the following properties:

- (F1) $f_1, b, f_2 \in A \backslash Z$
- (F2) w_1 is a source, u_2 is a sink
- (F3) $\delta(\{w_1\}) \cap (A\backslash Z) = \{f_1, b\}$ and $\delta(\{N\backslash u_2\}) \cap (A\backslash Z) = \{b, f_2\}$
- (F4) there is a directed path from w_2 to u_1 (and not vice versa)
- (F5) if $(x, u_2) \in Z$ and $(w_1, y) \in Z$ where $x \neq w_1, w_2$ and $y \neq u_1, u_2$ then $x = y$ or there is a directed path from x to y.

Then a *fold* can be performed on $[D, Z]$ resulting in $[D', Z']$ where D' and Z' result from D and Z respectively by identifying w_1 and w_2 into w and identifying u_1 and u_2 into u. In particular, let $D = (N, A)$ and $D' = (N', A')$ where

$$\begin{aligned} N &= \{n_1, n_2, \ldots, n_g, u_1, w_1, u_2, w_2\} \\ N' &= \{n_1, n_2, \ldots, n_g, u, w\} \\ A &= \{a_1, a_2, \ldots, a_h, f_1, b, f_2\} \\ A' &= \{a'_1, a'_2, \ldots, a'_h, f = (u, w)\} \end{aligned}$$

and $a'_i = a_i$ for $i \in \{1, \ldots, h\}$ except that if a_i is incident to u_1 or u_2 then a'_i is incident to u, and if a_i is incident to w_1 or w_2 then a'_i is incident to w. Finally, $Z' = \{a'_i | a_i \in Z, i \in \{1, 2, \ldots, h\}\}$.

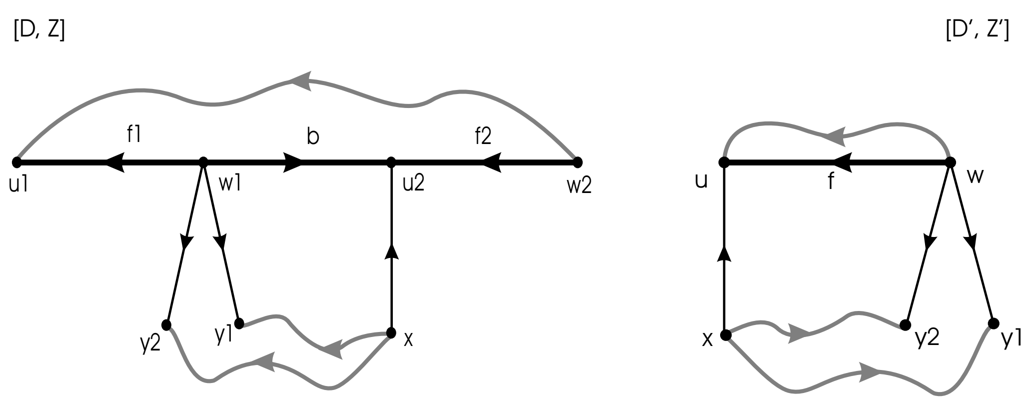

Figure 4.3: The folding operation on $[D, Z]$ resulting in $[D', Z']$.

In order to understand why knitted is closed under folding, it is necessary to understand how the directed cuts of $[D, Z]$ relate to the directed cuts of $[D', Z']$.

Let $d' = \delta_{D'}(X')$, where $u \notin X'$ and $w \in X'$, be a cut that is not necessarily directed. Given the following definitions for d_N, d_L and d_R, the next three remarks follow from (F2) and (F3). The fourth remark follows from (F5). See Figure (4.4).

- $d_N = \delta_D(X_N)$ where $X_N = X' \backslash \{w\} \cup \{w_1, w_2\}$
- $d_L = \delta_D(X_L)$ where $X_L = X' \backslash \{w\} \cup \{w_1, u_2, w_2\}$
- $d_R = \delta_D(X_R)$ where $X_R = X' \backslash \{w\} \cup \{w_2\}$

Remark 4.2. *d_N is a directed cut in D $\iff$ d' is a directed cut in D'. Moreover, $\forall i$ $a'_i \in d' \cap (A' \backslash Z') \iff a_i \in d_N \cap (A \backslash Z)$. Also, $f \in d' \cap (A' \backslash Z')$ and $f_1, b, f_2 \in d_N \cap (A \backslash Z)$.*

Remark 4.3. *If d_L is a directed cut in D then d' is a directed cut in D'. Moreover, $\forall i$ $a'_i \in d' \cap (A' \backslash Z') \iff a_i \in d_L \cap (A \backslash Z)$. Also, $f \in d' \cap (A' \backslash Z')$ and $f_1 \in d_L \cap (A \backslash Z)$.*

Remark 4.4. *If d_R is a directed cut in D then d' is a directed cut in D'. Moreover, $\forall i$ $a'_i \in d' \cap (A' \backslash Z') \iff a_i \in d_R \cap (A \backslash Z)$. Also, $f \in d' \cap (A' \backslash Z')$ and $f_2 \in d_R \cap (A \backslash Z)$.*

Remark 4.5. *If d' is a directed cut in D' then d_L is a directed cut in D or d_R is a directed cut in D.*

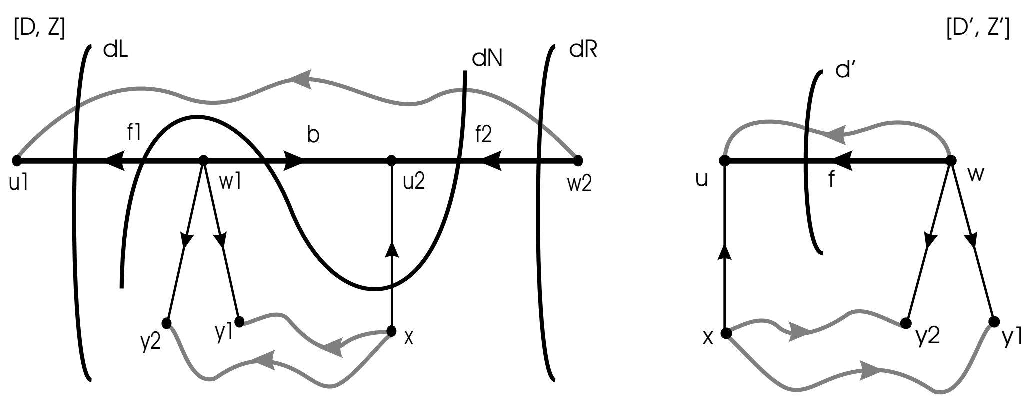

Figure 4.4: The relationships between d', d_N, d_L, and d_R.

Let $d'_{all} = \delta_{D'}(X')$ where $u, w \in X'$ and $d'_{none} = \delta_{D'}(X')$ where $u, w \notin X'$ be cuts that are not necessarily directed. Given the following definitions for d_{all} and d_{none}, the next two remarks follow from (F3):

- $d_{all} = \delta_D(X)$ where $X = X'\backslash\{u, w\} \cup \{u_1, w_1, u_2, w_2\}$
- $d_{none} = \delta_D(X)$ where $X = X'$.

Remark 4.6. *d'_{all} is a directed cut in D' $\iff$ d_{all} is a directed cut in D. Moreover, $\forall i$ $a'_i \in d'_{all} \cap (A'\backslash Z')$ $\iff$ $a_i \in d_{all} \cap (A\backslash Z)$. Also, $f \notin d' \cap (A'\backslash')$ and $f_1, b, f_2 \notin d \cap (A\backslash Z)$.*

Remark 4.7. *d'_{none} is a directed cut in D' $\iff$ d_{none} is a directed cut in D. Moreover, $\forall i$ $a'_i \in d'_{none} \cap (A'\backslash Z')$ $\iff$ $a_i \in d_{none} \cap (A\backslash Z)$. Also, $f \notin d' \cap (A'\backslash')$ and $f_1, b, f_2 \notin d \cap (A\backslash Z)$.*

Now we are ready to show that knitted is closed under folding. Suppose that $[D, Z]$ is knitted. Let $D' = (N', A')$ and consider an arbitrary $\omega' \in \mathbb{Z}_+^{A'}$ that is valid for $[D', Z']$. We will show that $\nu(D', \omega') = \tau(D', \omega')$. Let $\tau = \tau(D', \omega')$ and $k = \omega(f)$. Define $\omega \in \mathbb{Z}_+^A$ as follows

$$\omega(a) = \begin{cases} k & \text{if } a = f_1 \text{ or } a = f_2 \\ \tau - k & \text{if } a = b \\ \omega'(a'_i) & \text{if } a = a_i. \end{cases}$$

Notice that ω is valid for $[D, Z]$ since $f_1, b, f_2 \in (A\backslash Z)$ by (F1). Now we show that $\tau = \tau(D, \omega) = \tau(D', \omega')$. From (F2) and (F3) we know that $\tau(D, \omega) \leq \tau$ since

$$\omega(\delta(\{w_1\})) = \omega(f_1) + \omega(b) = k + (\tau - k) = \tau.$$

In order to prove that $\tau(D, \omega) \geq \tau$ consider a directed cut $d = \delta_D(X)$ chosen arbitrarily in D. In each of the following cases we use the fact ω is defined so that $\forall i$ $\omega(a_i) = \omega'(a'_i)$.

Case One: $f_1, b, f_2 \notin d$. Thus $u_1, w_1, u_2, w_2 \in X$ or $u_1, w_1, u_2, w_2 \notin X$. Therefore, by Remark (4.6) or (4.7) there is a directed cut d' in D' where $\omega(d) = \omega'(d') \geq \tau$.

Case Two: $b \in d$. From (F4) there is a cycle with f_1, b, f_2 and the directed path from w_2 to u_1. Since f_1 and f_2 are the only arcs in one direction in this cycle, then by Proposition (3.5) if $b \in d$ then $f_1 \in d$ or $f_2 \in d$. Therefore, since $\omega(f_1) = \omega(f_2)$

$$\omega(d) \geq \omega(f_1) + \omega(b) = k + (\tau - k) = \tau.$$

Case Three: $f_1 \in d$ and $b \notin d$. From Remark (3.4) this implies $f_2 \notin d$. From Remark (4.3) there is a directed cut d in D where $\omega(d) - \omega(f_1) = \omega'(d') - \omega(f)$. Hence,

$\omega(d) = \omega'(d') \geq \tau$.

Case Four: $f_2 \in d$ and $b \notin d$. From Remark (3.4) this implies $f_1 \notin d$. From Remark (4.4) there is a directed cut d in D where $\omega(d) - \omega(f_2) = \omega'(d') - \omega(f)$. Hence, $\omega(d) = \omega'(d') \geq \tau$.

Therefore, we have shown that $\tau = \tau(D, \omega) = \tau(D', \omega')$. Since $[D, Z]$ is knitted then (D, ω) has a packing of τ directed joins, $\mathbb{J} = J_1, J_2, \ldots, J_\tau$. The next remark is true since w_1 and u_2 are sources/sinks that induce directed cuts of weight τ in (D, ω).

Remark 4.8. *For each $J_i \in \mathbb{J}$ either $f_1, f_2 \in J_i$ and $b \notin J_i$, or $f_1, f_2 \notin J_i$ and $b \in J_i$.*

Our goal is to show that we can translate $\mathbb{J}$ into a packing of τ directed joins for (D', ω'). For $i = 1, 2, \ldots, \tau$ define

$$J'_i = \begin{cases} J_i \backslash \{f_1, f_2\} \cup \{f\} & \text{if } f_1, f_2 \in J_i \\ J_i \backslash \{b\} & \text{if } f_1, f_2 \notin J_i \end{cases}$$

First let us show that $\mathbb{J}' = J'_1, J'_2, \ldots, J'_\tau$ is a packing in (D', ω'). This follows from the fact that $\mathbb{J}$ is a packing for (D, ω) and

- $a'_j \in J'_i \iff a_j \in J_i$ and $\omega'(a'_j) = \omega(a_j)$
- $f \in J'_i \iff f_1, f_2 \in J_i$ and $\omega'(f) = \omega(f_1) = \omega(f_2)$

Next let us show that J'_i is a directed join of D' for $i = 1, 2, \ldots, \tau$. Let $d' = \delta_{D'}(X')$ be any directed cut in D', and J'_i be any of the directed joins in $\mathbb{J}'$. We will show that $d' \cap J'_i \neq \emptyset$. In each of the following cases we use that $J_i \subseteq (A \backslash Z)$ and $J'_i \subseteq (A' \backslash' Z)$. This assumption is justified since $\mathbb{J}$ and $\mathbb{J}'$ are packings, and each directed join need only contain a single copy of each arc.

Case One: $f \notin d'$. Therefore, $u, w \in X'$ or $u, w \notin X'$. Therefore, by Remark (4.6) or (4.7) there is a directed cut d in D where

$$(d' \cap J'_i \neq \emptyset) \iff (d \cap J_i \neq \emptyset)$$

Since J_i is a directed join of D we have that $d \cap J_i \neq \emptyset$. Hence $d' \cap J'_i \neq \emptyset$.

Case Two: $f \in d'$. We may assume that $f \notin J'_i$ or else $f \in d' \cap J'_i$. Since $f \notin J'_i$ then $f_1, f_2 \notin J_i$. By Remark (4.5) there is a directed cut d in D where

$$(d' \cap J'_i \neq \emptyset) \iff (d \cap J_i \neq \emptyset)$$

Since J_i is a directed join of D we have that $d \cap J_i \neq \emptyset$. Hence $d' \cap J_i' \neq \emptyset$.

Therefore, a packing of size τ has been found in (D', ω'). Hence, if $[D, Z]$ is knitted then $[D', Z']$ is knitted. Thus, knitted is closed under folding.

Properties from Minimality

Every augmented directed graph, has a finite number of minors. This is due to the following chain of observations. No operation increases the number of nodes. Every operation that does not decrease the number of nodes, does not increase the number of arcs. Every operation that does not decrease the number of arcs, does transfer an arc into Z.

Furthermore, the empty augmented graph, containing no arcs, is knitted, and is always eventually obtained by repeated applications of the minor operations. Therefore, we are justified in the following definition:

Definition 4.9. *An augmented directed graph is* minimally non-knitted *if it is not knitted and every one of its minors is knitted.*

There are four properties of minimally non-knitted augmented directed graphs that will be presented. In the first, contraction is used to show that in minimally non-knitted augmented directed graphs, the directed cuts of minimum weight, τ, are always trivial directed cuts. The first step is to prove a lemma about directed joins. This proof considers the *union* and *intersection* of two directed cuts.

Remark 4.10. *If $\delta(X_1)$ and $\delta(X_2)$ are directed cuts where $X_1 \cup X_2 \neq N$, then $\delta(X_1 \cup X_2)$ is a directed cut.*

Remark 4.11. *If $\delta(X_1)$ and $\delta(X_2)$ are directed cuts where $X_1 \cap X_2 \neq \emptyset$, then $\delta(X_1 \cap X_2)$ is a directed cut.*

For notational purposes, if $D = (N, A)$ and $K \subseteq N$, where $D[K]$ is connected, then D/K represents the sequence of arc contractions that results in K being contracted to a single node. For $K_1, K_2 \subseteq N$, $\delta(K_1, K2)$ represents the set of arcs from any node in K_1 to any node in K_2. Also, for $a \in A$ and $\omega \in \mathbb{Z}_+^A$ we will let ω/a and $\omega \backslash a$ represent weight vectors for D/a and $D \backslash a$, and extend this to allow ω/K for $K \subseteq N$ as above.

The next lemma establishes that, under the right condition, directed joins can be glued together across directed cuts.

Lemma 4.12. *If $\delta(K)$ is a directed cut, J_1 is a directed join of D/K, J_2 is a directed join of $D/\overline{K}$, $(J_1 \cup J_2) \cap \delta(K) = \{a\}$, then $J_1 \cup J_2$ is a directed join of D.*

Proof. Consider any directed cut $\delta(I)$ in D. The nodes of D can be partitioned into, $K \cap I$, $K \cap \overline{I}$, $\overline{K} \cap I$, and $\overline{K} \cap \overline{I}$ (Figure (4.5)).

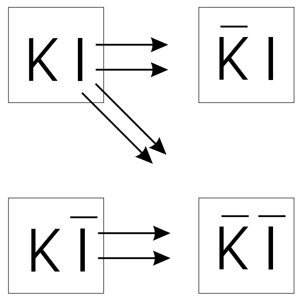

Figure 4.5: Arc a must be in $\delta(KI, \overline{KI})$, $\delta(KI, \overline{K}I)$ or $\delta(K\overline{I}, \overline{KI})$.

If the destination of a is not in $\overline{K} \cap \overline{I}$, then $(J_1 \cup J_2) \cap \delta(K \cup I) = \emptyset$, contradicting that J_1 is a directed join of D/K.

Likewise, if the origin of a is not in $K \cap I$, then $(J_1 \cup J_2) \cap \delta(K \cap I) = \emptyset$, contradicting that J_2 is a directed join of $D/\overline{K}$.

Therefore, $a \in (J_1 \cup J_2) \cap \delta(K \cap I, \overline{K} \cap \overline{I})$. Therefore, $a \in (J_1 \cup J_2) \cap \delta(I)$, and so $J_1 \cup J_2$ is a directed join of D. □

Proposition 4.13. *If augmented directed graph $[D, Z]$ is minimally non-knitted, and ω is a valid weight preventing $[D, Z]$ from being knitted, then every directed cut of weight $\tau(D, \omega)$ is trivial.*

Proof. For sake of contradiction, assume the hypothesis and the existence of a directed cut $\delta(K)$, where $\omega(\delta(K)) = \tau(D, \omega)$, and $\delta(K)$ is not trivial.

Since $\delta(K)$ is a minimum weight cut, both D/K and $D/\overline{K}$ are connected. Since $\delta(K)$ is not trivial, $[D/K, Z/K]$ and $[D/\overline{K}, Z/\overline{K}]$ are minors of $[D, Z]$. Since $[D, Z]$ is minimally non-knitted, both $(D/K, \omega/K)$ and $(D/\overline{K}, \omega/\overline{K})$ have directed join packings of size at least $\tau(D, \omega)$. Since $\delta(K)$ is a directed cut in both D/K and $D/\overline{K}$, these directed join packings are of size exactly $\tau(D, \omega)$.

For each $a \in \delta(K)$, a directed join J_1 of $(D/K, \omega/K)$, and a directed join J_2 of $(D/\overline{K}, \omega/K)$ exist, where

$$J_1 \cap \delta(K) = J_2 \cap \delta(K) = \{a\}$$

Therefore, from Lemma (4.12), $J_1 \cup J_2$ is a directed join of D. Since $\omega(\delta(K)) = \tau(D, \omega)$, then $\tau(D, \omega)$ such directed joins exist. This collection of directed joins is a packing in D with ω, contradicting the fact that $[D, Z]$ is non-knitted. Therefore, no such $\delta(K)$ exists. □

The three additional properties of minimally non-knitted augmented directed graphs are, more or less, direct consequences of the minor operations.

Remark 4.14. *If augmented directed graph $[D, Z]$ is minimally non-knitted, and ω is a valid weight preventing $[D, Z]$ from being knitted, then an arc has ω weight 0, if and only if, the arc is in Z.*

Proof. For the first direction, if $a \in Z$ then $\omega(a) = 0$ or $\omega(a) = \infty$. If $\omega(a) = \infty$, then the minor corresponding to contracting a is also non-knitted. For the other direction, if $a \notin Z$ and $\omega(a) = 0$, then the minor corresponding to Z-transfer on a is also non-knitted. Both directions contradict the fact that $[D, Z]$ is minimally non-knitted. □

Remark 4.15. *If augmented directed graph $[D, Z]$ is minimally non-knitted, then every node r is the origin or destination of at least one arc outside of Z.*

Proof. Proof by contradiction. Consider an ω that is a valid weight preventing $[D, Z]$ from being knitted. Apply Remark (4.14) so every arc in Z has ω weight 0. If r is a source or a sink, then it induces a directed cut of weight 0.

Now suppose r is the origin and destination of at least two arcs. In this case, a packing of $\tau(D, \omega)$ directed joins exists from the minor obtained by applying the Z-biclique substitution to r. This packing is also a packing in $[D, Z]$ with ω.

In the remaining case, without loss of generality, r is the destination of exactly one arc, say a (Figure (4.6)). In this case, a packing of $\tau(D, \omega)$ directed joins exists from the minor obtained by contracting a. This packing is also a packing in $[D, Z]$ with ω. □

Proposition 4.16. *If augmented directed graph $[D, Z]$ is minimally non-knitted, and ω is a valid weight preventing $[D, Z]$ from being knitted, then every transitive arc of D is not in Z.*

Proof. This is a direct consequence of the transitive-Z deletion operation. □

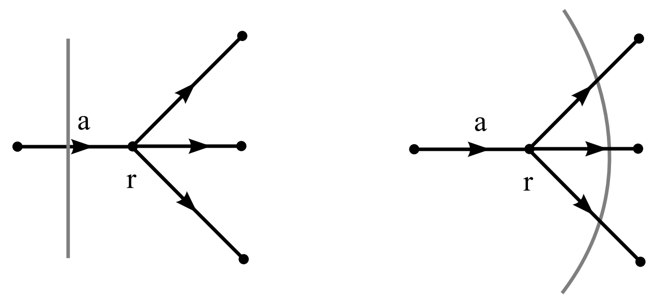

Figure 4.6: If $a \in \delta(X)$ then $\delta(X \cup \{r\})$ is also a directed cut.

4.2 Minimum Weight

In this section, the notion of knitted is extended from augmented directed graphs to weighted directed graphs (Definition (4.17)). This is done by adding non-zero weights to every arc that does not have its weight fixed by the augmented directed graph's set Z. Every minimally non-knitted augmented directed graph has at least one weight assignment that results in a non-knitted weighted directed graph. The resulting weighted directed graph is *minimally non-knitted* if such a weight assignment is chosen to be *minimum.* This section also discusses two important properties of minimally non-knitted weighted directed graphs.

Definition 4.17. *A weighted directed graph is* knitted *if the maximum number of directed joins in a packing is equal to the smallest weight of a directed cut.*

The process of extending augmented directed graphs to weighted directed graphs involves setting weight values in two steps. Therefore, it is useful to be able to"separate" the resulting weights vectors $\omega \in \mathbb{Z}_+^A$ into their zero and non-zero values. From Remark (4.14) the zero values can be represented by the set $Z \subseteq A$. The non-zero values can be represented by a vector in $\mathbb{N}^{A \backslash Z}$, where $\mathbb{N}$ is the set of positive integers. Formally, for $\omega \in \mathbb{Z}_+^A$, $\omega_\mathbb{N} \in \mathbb{N}^{A \backslash Z}$, $Z \subseteq A$, we say that $\omega = \omega_\mathbb{N} \cup Z$ if

$$\omega(a) = \begin{cases} \omega_\mathbb{N}(a) & \text{if } a \in A \backslash Z \\ 0 & \text{if } a \in Z. \end{cases}$$

The next step in defining minimality for weighted directed graphs is to define a partial order for vectors in $\mathbb{N}^{A \backslash Z}$. This is done in general, where $\omega_1, \omega_2 \in \mathbb{S}^E$ for any totally-ordered set $\mathbb{S}$ and arbitrary set E. Informally, $\omega_1 < \omega_2$, if ω_1 has more small elements of $\mathbb{S}$

than ω_2. Often this is referred to as a lexicographical ordering. Formally, for $\omega_1, \omega_2 \in \mathbb{S}^E$, say that $\omega_1 < \omega_2$, if $\exists k \in \mathbb{S}$ such that the following two conditions hold:

$$|\{e \in E : \omega_1(e) = i\}| = |\{e \in E : \omega_2(e) = i\}|, \text{ for all } i \in S \text{ where } i < k \quad (4.1)$$

$$|\{e \in E : \omega_1(e) = k\}| > |\{e \in E : \omega_2(e) = k\}| \quad (4.2)$$

Definition 4.18. *A weighted directed graph* $(D, \omega_{\mathbb{N}} \cup Z)$ *is* minimally non-knitted *if the augmented directed graph* $[D, Z]$ *is minimally non-knitted, and* $(D, \omega'_{\mathbb{N}} \cup Z)$ *is knitted, for every* $\omega'_{\mathbb{N}} < \omega_{\mathbb{N}}$.

The task of working with the two different notions of minimal weights, is simplified by the following lemma. The lemma allows us to work with ω, instead of its components $\omega_{\mathbb{N}} \cup Z$.

Lemma 4.19. *If* (D, ω) *is minimally non-knitted, and* $\omega' < \omega$ *and* $Z \subseteq Z'$, *then* (D, ω') *is knitted.*

Proof. In the first case, suppose that $\exists a \in Z' \setminus Z$. Therefore, since $[D, Z]$ is a minimally non-knitted augmented directed graph, $[D, Z \cup \{a\}]$ is knitted. In particular, it is knitted for ω'. Hence (D, ω') is knitted.

In the second case, $Z = Z'$. Since $\omega' < \omega$, it follows that $\omega'_{\mathbb{N}} < \omega_{\mathbb{N}}$. Therefore, since (D, ω) is a minimally non-knitted weighted directed graph, (D, ω') is knitted. □

Notice that this lemma, together with Remark (4.15), essentially allows us to ignore the special set Z. Instead, we can think of minimality on ω as consisting of two tiers; one for weight 0 and the second for the remaining weights. Such an approach will be taken in subsequent chapters when τ is restricted to two. An immediate consequence of Definition (4.18) is the following:

Proposition 4.20. *If* (D, ω) *is minimally non-knitted, then every arc* a *with* $\omega(a) > 0$, *is in a directed cut of weight* $\tau(D, \omega)$.

Proof. For contradiction, suppose there is such an arc a'. Consider a new weight, $\omega' \in \mathbb{Z}_+^A$ where

$$\omega'(a) = \begin{cases} \omega(a) & \text{if } a \neq a' \\ \omega(a) - 1 & \text{if } a = a'. \end{cases}$$

Notice that $\tau(D,\omega) = \tau(D,\omega')$ since a' is not in a directed cut of minimum weight. Furthermore, $Z \subseteq Z'$. Therefore, by Lemma (4.19), (D,ω') is knitted. Therefore, there is a packing of τ directed joins in (D,ω'). This packing is also a packing in (D,ω), contradicting that (D,ω) is non-knitted. □

A second consequence of Definition (4.18), the main result of this section, requires more machinery to be built up before it can be proven. This machinery includes a property, *accommodating*, common to many packing and covering problems, and an operation, *pushing*, that will be used to reveal directed joins with this property.

A directed join J is *accommodating* in (D,ω) if $\omega(a) > 0$ for all $a \in J$, and

$$\omega(\delta(X)) - |J \cap \delta(X)| \geq \tau(D,\omega) - 1$$

for all directed cuts $\delta(X)$ in D. In other words, a directed join is accommodating if its removal leaves enough room for a potential packing of $\tau(D,\omega) - 1$ directed joins.

Remark 4.21. *If (D,ω) is minimally non-knitted, then it does not contain an accommodating directed join.*

Proof. Suppose that (D,ω) does contain an accommodating directed join J. Consider (D,ω'), where

$$\omega'(a) = \begin{cases} \omega(a) & \text{if } a \notin J \\ \omega(a) - 1 & \text{if } a \in J \end{cases}$$

Since $\omega' < \omega$, and $Z \subseteq Z'$, then by Lemma (4.19), (D,ω') is knitted. Furthermore, since J is accommodating

$$\tau(D,\omega') = \tau(D,\omega) - 1.$$

Therefore, there is a packing of $\tau(D,\omega) - 1$ directed joins in (D,ω'). However, this packing together with J, is a packing of $\tau(D,\omega)$ directed joins in (D,ω). This contradicts the fact that (D,ω) is non-knitted. □

Next, we introduce an operation that acts on the non-zero weights $\omega_N^{A\setminus Z}$.

As discussed in the Chapter 1, every path that is not directed, has arcs in two distinct directions. More formally, if

$$P = (n_1, a_1, n_2, a_2, \ldots, n_k)$$

is a path, then a_i is a *forward* arc in P if $a_i = (n_i, n_{i+1})$, otherwise $a_i = (n_{i+1}, n_i)$ is a *backward* arc in P.

Given weight vector ω, and arc-simple walk P, define ω' to be the result of *pushing* P, where

$$\omega'(a) = \begin{cases} \omega(a) + 1 & \text{if } a \text{ is a forward arc in } P \\ \omega(a) - 1 & \text{if } a \text{ is a backward arc in } P \\ \omega(a) & \text{otherwise.} \end{cases}$$

The pushing operation may also be applied to cycles, and doing so forms the basis of the next proposition.

Proposition 4.22. *If (D, ω) is minimally non-knitted, then every cycle has an arc a with $\omega(a) = 0$.*

Proof. Otherwise, suppose C is a cycle in (D, ω), where every arc of C has non-zero weight. Without loss of generality, suppose that C contains a backward arc b of weight k, where

$$k = min\{\omega(a) : a \text{ is an arc of } C\}$$

Let ω' be the result of pushing cycle C. First of all, $\omega' \in \mathbb{Z}_+^A$ because C had no arc a with $\omega(a) = 0$. For the same reasoning, $Z \subseteq Z'$.

From Remark (3.5), every directed cut intersects the same number of forward arcs and backward arcs of C. Therefore,

$$\omega(\delta(X)) \quad = \quad \omega'(\delta(X)), \tag{4.3}$$

for every directed cut $\delta(X)$ in D. Therefore, $\tau(D, \omega') = \tau(D, \omega)$.

Furthermore, since C has a backward arc b with $\omega(b) = k$, then $\omega'(b) = k - 1$. Therefore, $\omega' < \omega$.

Therefore, all of the conditions for Lemma (4.19) are satisfied. Therefore, (D, ω') is knitted. Therefore, there is a packing of τ directed joins in (D, ω'). Let J be one of the directed joins in such a packing.

Since J is part of a packing of τ directed joins in (D, ω'), then for all directed cuts $\delta(X)$ in D,

$$\omega'(\delta(X)) - |J \cap \delta(X)| \geq \tau - 1. \tag{4.4}$$

However, (4.3) and (4.4), together give

$$\omega(\delta(X)) - |J \cap \delta(X)| \geq \tau - 1.$$

for all directed cuts $\delta(X)$ in D. Therefore, J is accommodating in (D, ω). This contradicts that (D, ω) is minimally non-knitted by Remark (4.21). □

The proof presented for Proposition (4.22) gives a slightly stronger result than stated.

4.3 S-Tree Structure

A subgraph T of (D, ω) is a *source-sink-tree*, or simply an *s-tree*, with weight k cuts, if T is a tree of non-zero ω weight arcs, where every internal (non-leaf) node of T is a source or sink that induces a directed cut of weight k in (D, ω). An s-tree is maximal in (D, ω) if no other s-tree in (D, ω) contains it. All of the results in this chapter lead to the following Proposition.

Proposition 4.23. *If (D, ω) is minimally non-knitted then its non-zero weight arcs can be uniquely partitioned into maximal s-trees of (D, ω) with weight $\tau(D, \omega)$ cuts. Furthermore, each s-tree has at least two arcs, and every node of D is included in at least one of these s-trees.*

Proof. Let (D, ω) be a minimally non-knitted weighted directed graph. First verify that the relation of being in a common s-tree, forms an equivalence relation on the non-zero weight arcs.

The relation is easily reflexive and symmetric. Next we show that the relation is transitive. Suppose arcs a and b are in an s-tree T_1, and arcs b and c are in an s-tree T_2. Let $T = T_1 \cup T_2$. Certainly T contains both a and c. It remains to show that T is an s-tree. We begin by showing that T is a tree. Since b is in both T_1 and T_2, T is connected. From Proposition (4.22), T is acyclic. Hence, T is a tree. Next, we show that T is in fact an s-tree. As a preliminary step, let us examine arc b closely. From Proposition (4.20), arc b is in a directed cut of minimum weight. From Proposition (4.13), this minimum weight directed cut is a trivial cut. Let s be a source or sink that induces a minimum weight directed cut and is the origin or destination of b. Now we prove that every node in T that is not a source or sink that induces a minimum weight directed cut, must in fact be a leaf of T. This would complete the proof that T is an s-tree. Consider any node n in

T that is not a source or sink that induces a minimum weight directed cut. Since T_1 and T_2 are s-trees, node n has degree at most one in both T_1 and T_2. Therefore, if n is not a leaf in T then it has degree one in T_1, degree one in T_2, and degree two in T. Let m_1 be the node adjacent to n in T_1, and let m_2 be the node adjacent to n in T_2. Since s is a source or sink that does induce a minimum weight directed cut, $s \neq n$. Furthermore, s is a node in both T_1 and T_2 because b is an arc in both T_1 and T_2. Therefore, there is a path from s to n in T_1, and a path from s to n in T_1 (Figure (4.7)). From Proposition (4.22), the union of these two paths must not contain any cycles. Therefore, it must be that, $m_1 = m_2$. Therefore, n has degree one in T so n is a leaf of T. Hence, only sources and sinks that induce minimum weight directed cuts are internal nodes of T. This completes the proof that T is an s-tree. Therefore, the relation of two non-zero weight arcs being in a common s-tree is reflexive, symmetric, and transitive, so the relation is an equivalence relation.

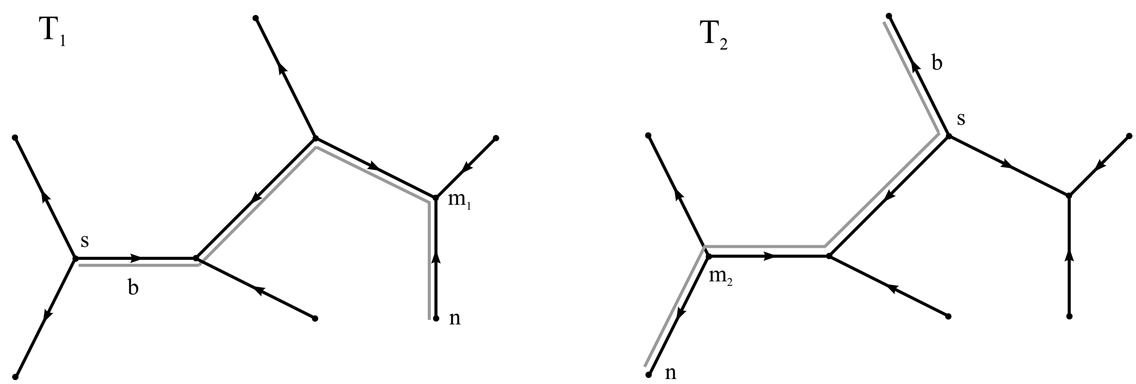

Figure 4.7: Deducing that n is a leaf of $T = T_1 \cup T_2$.

Therefore, the arcs of non-zero weight may be partitioned into equivalence classes. For each equivalence class, consider the union U of every s-tree that contains an arc of that class. In particular, from Proposition (4.22), U must be acyclic. Furthermore, if an internal node n of U is not a source or sink that induces a directed cut of minimum weight, then there are two arcs a_1, a_2 adjacent to n. In particular, since a_1, a_2 are in the same equivalence class, they must be in an s-tree together. However, this is not possible since the s-tree would have n as an internal node. Therefore, the union U is also an s-tree. Furthermore, since the union contains all of the possible arcs in the any of the s-trees in the equivalence class, the union is a maximal s-tree. Therefore, the non-zero weight arcs in a minimally non-knitted weighted directed graph, can be partitioned uniquely into maximal s-trees.

It remains to show that each maximal s-tree has at least two arcs. Every s-tree contains a source or sink that induces a minimum weight directed cut. From Proposition (2.7), every trivial directed cut of minimum weight must contain at least two non-zero weight arcs. Since these two arcs form an s-tree, every maximal s-tree contains at least two arcs.

Finally it remains to show that every node is contained in a maximal s-tree. This reduces to showing that every node is the origin or destination of a non-zero weight arc, and this has been shown in Remark (4.15). □

Chapter 5

Minimally non-2-Knitted

In this chapter, we specialize the previous chapter by studying minimally non-knitted weighted directed graphs with $\tau = 2$. These *minimally non-2-knitted* weighted directed graphs have special significance due to Remark (2.10). The arguments found here provide a framework for the remaining chapters, and begin by translating results from previous chapters. The following remark is a consequence of Propositions (2.7) and (4.13).

Remark 5.1. *Minimally non-2-knitted weighted directed graphs only have arcs of weight* 0 *and* 1*, and every directed cut of weight* 2 *is a trivial cut.*

An *s-path* of (D, ω) is an s-tree of (D, ω) with cuts of weight 2, that is a path. A weighted directed graph is an *s-path graph* with p paths, if its non-zero weight arcs are weight 1, these arcs uniquely partition into p maximal s-paths with cuts of weight 2, each of these s-paths has length at least two, and every node in the graph is in an s-path.

Proposition 5.2. *Minimally non-2-knitted weighted directed graph are s-path graphs.*

Proof. By Proposition (4.23) we can partition the non-zero weight arcs into s-trees with weight $\tau = 2$ cuts. Let T be one of these s-trees. By Remark (5.1), T contains only arcs of weight 1. Let n be an internal node of T. Since n is not a leaf in T, it must have degree at least two in T. However, n cannot have degree greater than two in T, otherwise it could not be a source or sink inducing a directed cut of weight 2. Hence, every internal node of T has degree two in T, so T is an s-path. □

For the remainder of the thesis we will be assuming that (D, ω) is an s-path graph with s-paths $P_1, \dots P_p$, where $D = (N, A)$ and $A = A_0 \cup A_1$ with

$$A_0 = \{a \in A | \omega(a) = 0\}$$
$$A_1 = \{a \in A | \omega(a) = 1\}$$

5.1 Crossing Sets

A *crossing set* J is a subset of non-zero arcs such that, for each s-path P, J exclusively contains the forward arcs or the backward arcs of P. For every crossing set, there is exactly one crossing set that is disjoint from it. Such pairs have special significance.

Remark 5.3. *If J_1, J_2 is a packing of two directed joins in an s-path graph then J_1 and J_2 are disjoint crossing sets.*

Remark (5.3) follows from the fact that J_1 and J_2 must intersect every trivial directed cut on every s-path. Therefore, neither J_1 nor J_2 can include or exclude adjacent arcs on any s-path. Therefore, J_1 and J_2 must either contain the forward or backward arcs of each s-path, matching the definition of crossing sets.

Since crossing sets contain either the forward or backward arcs of each s-path, it is possible to represent each crossing set as a binary number. Define function b, for *binary*, to map crossing sets to p-digit binary numbers, where p is equal to the number of s-paths in the given s-path graph. For crossing set $J \subseteq A_1$, let $b(J) \in \{0, 1\}^p$ be the following:

$$b(J)[i] = \begin{cases} 0 & \text{if } J \text{ contains the backward arcs of } P_i \\ 1 & \text{if } J \text{ contains the forward arcs of } P_i. \end{cases}$$

Notice that if J_1 and J_2 are disjoint crossing sets then $b(J_1) = \overline{b(J_2)}$ where $\overline{x}$ is the bit-wise complement of x.

5.2 Crossing Cuts

A *crossing cut* is a directed cut that intersects each s-path at most once. Crossing cuts should not be confused with the concept of cross cuts (Chapter 3).

Let $\delta(X)$ be a crossing cut and P be an s-path. Either $\delta(X)$ intersects a forward or backward arc of P, or $\delta(X)$ *skips* P by having X include or exclude every node in

P. Define a function q mapping crossing cuts to *quaternary* numbers (Figure (5.1)). If $d = \delta(X)$ is a crossing cut, then let $q(d) \in \{0, 1, +, -\}^p$ be defined as follows:

$$q(d)[i] = \begin{cases} 0 & \text{if } \delta(X) \cap arcs(P_i) \text{ is a backward arc of } P_i \\ 1 & \text{if } \delta(X) \cap arcs(P_i) \text{ is a forward arc of } P_i \\ + & \text{if } nodes(P_i) \subseteq X \\ - & \text{if } nodes(P_i) \cap X = \emptyset. \end{cases}$$

Relation to Crossing Sets

From Remark (3.4) every directed cut that is not a crossing cut must intersect an s-path in both a forward arc and a backward arc. Therefore, every crossing set intersects every non-crossing cut, and we have justified the next two remarks.

Remark 5.4. *Crossing set J is a directed join if and only if there does not exist a crossing cut d where $d \cap J = \emptyset$.*

Remark 5.5. *An s-path graph with $\tau = 2$ has $\nu = 1$ if and only if for every pair of disjoint crossing sets, J_1 and J_2, there is a crossing cut d where $d \cap J_1 = \emptyset$ (equivalently, $d \cap A_1 \subseteq J_2$), or $d \cap J_2 = \emptyset$ (equivalently, $d \cap A_1 \subseteq J_1$).*

The above remark quickly gives us a nice result.

Proposition 5.6. *There do not exist minimally non-2-knitted s-path graphs with one or two paths.*

Proof. Let (D, ω) be an s-path graph with one or two paths. Notice that (D, ω) has at least one pair of disjoint crossing sets J_1, J_2. If $\tau(D, \omega) = 0$ or $\tau(D, \omega) = 1$ then (D, ω) is knitted. Otherwise, if $\tau(D, \omega) = 2$ and $\nu(D, \omega) = 1$ then by Remark (5.5), there must be a crossing cut d where $d \cap J_1 = \emptyset$ or $d \cap J_2 = \emptyset$ or else $\nu(D, \omega) = 2$. However, since a crossing cut d is not trivial, then by Remark (5.1) $\omega(d) \geq 3$, and so d must intersect at least three s-paths. This is not possible; so (D, ω) is knitted. □

In order to compare crossing cuts and crossing sets, we introduce the function s, for *set*, mapping the symbolic representation of a crossing cut to a set of binary numbers representing the crossing sets that contain the non-zero weight arcs of the crossing cut. For the sake of future arguments, the domain of the map is generalized from $\{0, 1, +, -\}^p$

to $\{0, 1, +, -, \pm\}^p$ so that $\pm$ can indicate when the distinction between $+$ and $-$ is not made. If $u \in \{0, 1, +, -, \pm\}^p$ let $s(u)$ be defined to contain every $w \in \{0, 1\}^p$ where

$$w[i] = \begin{cases} 0 & \text{if } u[i] = 0 \\ 1 & \text{if } u[i] = 1 \end{cases}$$

For example, $s(0{+}{-}) = s(0{-}{\pm}) = \{000, 001, 010, 011\}$. Crossing cut d is *minimal* if there does not exist crossing cut d' with $s(q(d)) \subsetneq s(q(d))$.

Remark 5.7. *Let J be a crossing set and d a crossing cut. The following are true*

- $b(J) \in s(q(d)) \iff d \cap A_1 \subseteq J$
- $\overline{b(J)} \in s(q(d)) \iff d \cap J = \emptyset$.

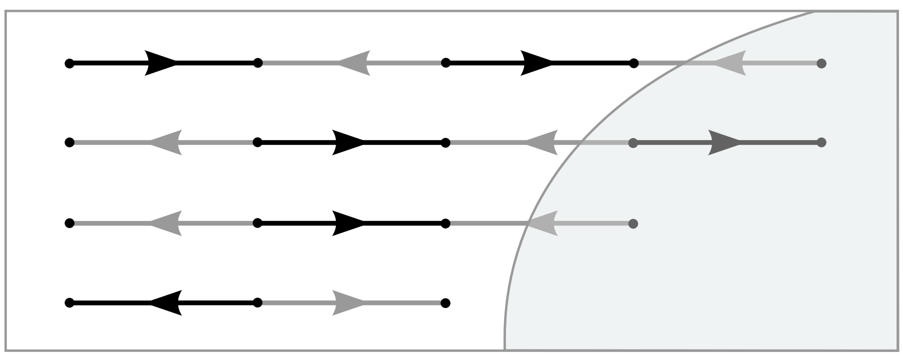

Figure 5.1: From top to bottom, s-paths P_1, P_2, P_3, and P_4, where the forward arcs of each s-path are directed from left to right. Disjoint crossing sets J_1, J_2 are represented by the grey and black arcs, where $b(J_1) = 0001$ and $b(J_2) = 1110$. Crossing cut d skips P_4 with $q(d) = 000-$ and $s(q(d)) = \{0000, 0001\}$. Notice $b(J_1) = 0001 \in s(q(d))$ so the non-zero weight arcs of d are a subset of J_1.

Pushing S-Paths

By pushing weight along s-paths, the existence of small collections of crossing cuts can be determined. In the next chapter we will see that this idea relates the known counterexamples of the Edmonds-Giles Conjecture to the Q_6-property.

Recall that if W is an arc-simple walk, then pushing on W changes the weight vector

ω into ω' where

$$\omega'(a) = \begin{cases} \omega(a) + 1 & \text{if } a \text{ is forwards in } W \\ \omega(a) - 1 & \text{if } a \text{ is backwards in } W \\ \omega(a) & \text{otherwise.} \end{cases}$$

In particular, if W traverses s-path P, from its beginning to its end, then the weight of every forward arc in P is increased by one, and the weight of every backward arc in P is decreased by one. This will be referred to as pushing the s-path in its *forward direction*. Pushing P in its *backward direction* increases the weight of every backward arc in P by one, and decreases the weight of every forward arc in P by one.

Lemma 5.8. *If (D, ω) is minimally non-2-knitted then $\tau(D, \omega) = \tau(D, \omega')$ where ω' is the result of pushing weight (in either direction) along any single s-path.*

Proof. Without loss of generality, suppose that ω' is the result of pushing s-path P in the forward direction. The only directed cuts that could have $\omega'(d) < \omega(d)$ are those that contain a backward arc of P. If d is a directed cut of weight $\tau(D, \omega)$ in (D, ω) that contains a backward arc of P then by Proposition (5.2), d must be a trivial cut containing exactly one forward and one backward arc of P. Hence,

$$\omega'(d) = \omega(d) + 1 - 1 = \omega(d) \geq \tau(D, \omega).$$

Otherwise, if d is any other directed cut containing a backward arc of P then $\omega(d) \geq \tau(D, \omega) + 1$. Let f be the number of times d intersects a forward arc of P and b be the number of times d intersects a backward arc of P. By Remark (3.4), $f \geq b - 1$. Hence,

$$\omega'(d) = \omega(d) + f - b \geq \omega(d) + b - 1 - b = \omega(d) - 1 \geq \tau(D, \omega).$$

□

Therefore, pushing an s-path does not change the smallest weight of a directed cut. Furthermore, in terms of minimum weight vectors, $\omega' < \omega$, because pushing lowers the weight of at least one arc to zero. Hence, pushing any s-path on a minimally non-2-knitted, results in a knitted weighted directed graph with a packing of two directed joins. Examining this packing gives information on the crossing cuts of the original.

Proposition 5.9. *For any s-path P, there are two crossing cuts that intersect the forward arcs of P, and do not intersect any other s-path in the same direction. Furthermore, there* *are two crossing cuts that intersect the backward arcs of P, and do not intersect any other s-path in the same direction.*

Proof. In (D, ω) push s-path P in its forward direction, resulting in knitted (D, ω') with a packing J_1, J_2 of directed joins. By Remark (4.21) both J_1 and J_2 are directed joins in (D, ω) that are not accommodating. If J_1 is not accommodating in (D, ω), then there is a crossing cut d_1 where $d_1 \cap A_1 \subseteq J_1$. However, J_1 is accommodating in (D, ω'), and the only arcs that have greater weight in ω' than ω are the forward arcs in P. Thus, it must be that d_1 intersects P in its forward direction. The same argument for J_2 can be made, resulting in crossing cut d_2. In particular, d_1 and d_2 must not share an arc of the same direction on any s-path other than P, otherwise J_1 and J_2 would share an arc on an s-path other than P. The same argument can be applied to the backward direction of P, giving the two crossing cuts that intersect backward arcs of P and do not intersect any other s-path in the same direction. □

Union and Intersection

By taking unions and intersections, the existence of certain crossing cuts imply the existence of additional crossing cuts. Let $\delta(X)$ and $\delta(X')$ be two crossing cuts that intersect s-path P in opposite directions. Say that $\delta(X)$ and $\delta(X')$ are *together* on P if $nodes(P) \cap (X \cap X') = \emptyset$. Otherwise, it must be that $nodes(P) \subseteq (X \cup X')$, and we say that $\delta(X)$ and $\delta(X')$ are *apart* on P (Figure (5.2)).

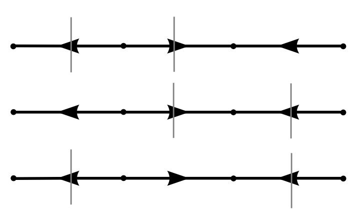

Figure 5.2: Crossing cuts that are together and apart.

Remark 5.10. *If $d = \delta(X)$ and $d' = \delta(X')$ are crossing cuts, then $d \cap d'$ is a crossing cut $\iff$ d and d' are not apart on any s-path and $X \cap X' \neq \emptyset$. On the other hand, $d \cup d'$ is a crossing cut $\iff$ d and d' are not together on any s-path and $X \cup X' \neq N$.*

A set of crossing cuts is *closed* if the result of any union and/or intersection of crossing cuts from the set is either not a crossing cut, or is already in the set. For example, $\{d_1, d_2\}$ would not be a closed set of crossing cuts if $q(d_1) = 000$ and $q(d_2) = 001$.

5.3 Trace

Remark (5.5) shows the importance of understanding what types of crossing cuts are present in an s-path graph. For this reason we introduce a global view of the crossing cuts, called the *trace*. The trace eliminates much of the information on individual crossing cuts, but as we will see, there are several ways to infer this lost information from the trace. Before illustrating the utility of the trace, we first provide a handful of definitions.

Hypercubes, Terms, Maxterms, and Reversibility

A *trace*, $\mathbb{T}$, is any $\mathbb{T} \in 2^{\{0,1\}^p}$. Visually, a trace can be represented on a p-dimensional hypercube, by colouring the nodes whose co-ordinates are included in the trace. The *trace* of an s-path graph (D, ω) is $\mathbb{T} = \mathbb{T}(D, \omega)$, where the following union is done over every crossing cut d of (D, ω)

$$\mathbb{T} = \bigcup_d s(q(d)).$$

To make the distinction between traces that arise from s-path graphs, and those that are just subsets of p-digits binary numbers, we say that a trace $\mathbb{T}$ is *valid* for (D, ω), or simply *valid*, if there exists an s-path graph (D, ω) such that $\mathbb{T} = \mathbb{T}(D, \omega)$, otherwise the trace is *invalid*. Invalid traces do exist, and they result from the fact that crossing cuts of any weighted directed graph have the restriction that they are intersection and union closed. Figure (5.3) shows an invalid trace, where the x denotes nodes that are not included in the trace.

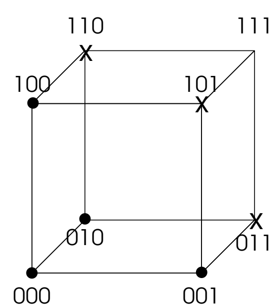

Figure 5.3: This invalid trace is not the trace of any s-path graph with three paths.

Define $t \subseteq \mathbb{T}$ to be a *term* of $\mathbb{T}$ if there exists $q \in \{0, 1, +, -, \pm\}^p$ such that $t = s(q)$. If $\mathbb{T}$ is valid for (D, ω) then every crossing cut of (D, ω) corresponds to a term in $\mathbb{T}$. However, the correspondence does not necessarily reverse. If $\mathbb{T}$ is a valid trace, then a term t is *reversible* in $\mathbb{T}$ if every s-path graph (D, ω) with $\mathbb{T} = \mathbb{T}(D, \omega)$ has a crossing cut d where $t = s(q(d))$.

A term t is a *maxterm* if there does not exist any other term t' such that $t \subsetneq t'$. In the visual representation, a term is any subcube whose nodes are entirely coloured, while a maxterm is any maximal subcube with this property. Those familiar with boolean function simplification and Karnaugh Maps may recognize these concepts [8]. An entire trace is *reversible* if every maxterm in the trace is reversible. Figure (5.4) shows a valid trace that is not reversible.

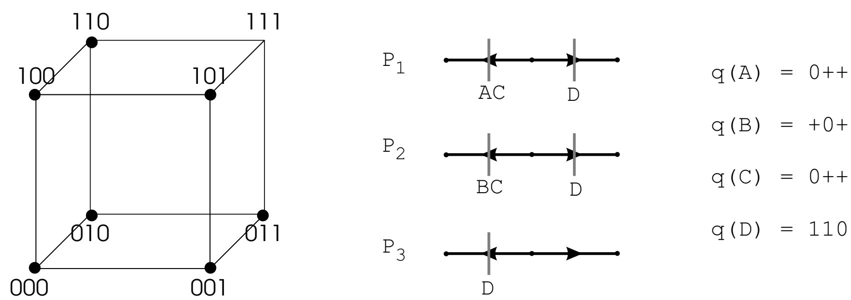

Figure 5.4: This trace is not reversible since the maxterm $t = s(\pm\pm 0)$ is not reversible in the above closed set of crossing cuts.

Opposite Pairs and Half-Traces

An alternative representation of the trace is possible by Remark (5.7). In the following expression, the outer union is done over the crossing cuts d of (D, ω), and the inner union is done over the crossing sets J which satisfy $d \cap A_1 \subseteq J$

$$\mathbb{T} = \bigcup_{d} \bigcup_{d \cap A_1 \subseteq J} b(J).$$

From Remark (5.5) this alternate representation leads to the following proposition, which we will refer to as the result on opposite pairs

Proposition 5.11. *If $\mathbb{T}$ is the trace of an s-path graph with p paths and $\tau = 2$ then $\nu = 1$, if and only if, for every $x \in \{0, 1\}^p$ either $x \in \mathbb{T}$ or $\overline{x} \in \mathbb{T}$.*

When studying traces, this proposition allows us to limit our scope to those traces satisfying $x \in \mathbb{T}$ or $\overline{x} \in \mathbb{T}$. A second notion of opposite pairs is possible to obtain from

Proposition (5.9). Define function

$$r_b : x \in \{0,1\}^p \rightarrow \{0,1\}^{p-1}$$

for $b \in \{1, \ldots, p\}$ as follows. Let $r_b(x) = x'$ where for $j = 1, 2, \ldots, p-1$

$$x'[j] = \begin{cases} x[j] & \text{if } j < b \\ x[j+1] & \text{if } j > b \end{cases}$$

Intuitively, r_b has the effect of removing the b^{th} bit. Define $\mathbb{S} \in 2^{\{0,1\}^{p-1}}$ to be a *half-trace* of $\mathbb{T} \in 2^{\{0,1\}^p}$, on the a^{th} side of b for $a \in \{0,1\}$ and $b \in \{1, \ldots, p\}$ if

$$\mathbb{S} \quad = \quad \{r_b(x) | x \in \mathbb{T} \text{ and } x[b] = a\}$$

Intuitively, a half-trace is any hypercube of one smaller dimension, with "normalized" co-ordinates. By Proposition (5.9), it is now possible to limit our scope even further

Proposition 5.12. *If $\mathbb{S}$ is a half-trace of $\mathbb{T}(D,\omega)$ for minimally non-2-knitted (D,ω) then there exists $x \in \mathbb{S}$ where $\overline{x} \in \mathbb{S}$ as well.*

In other words, Proposition (5.12) ensures that in every half-trace there is an opposite pair of nodes included in that half-trace. Figure (5.5) shows a trace that satisfies the criterion of Proposition (5.11) but not the criterion of Proposition (5.12).

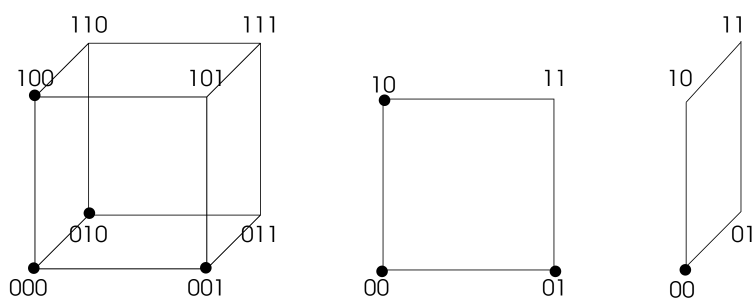

Figure 5.5: The "front" half-trace does, but the "right" does not, include opposite nodes.

Crossing Cuts and Non-Zero Weight Arcs

Proposition (5.11) allows us to prove two precise lemmas regarding crossing cuts and the role of individual non-zero weight arcs. Essentially, these lemmas show that certain non-zero weight arcs must be included in every crossing cut of a particular type. The desire to extend these lemmas to make more general statements is discussed after the lemmas are proven.

Lemma 5.13. *If (D, ω) is minimally non-2-knitted and a is an arc that is at one end of a maximal s-path in (D, ω), then there exists $x \in \mathbb{T}(D, \omega)$ such that if d is a crossing cut where $x \in s(q(d))$ then $a \in d$.*

Proof. If $\mathbb{T}(D, \omega) \subseteq \mathbb{T}(D/a, \omega/a)$ then by Proposition (5.11) we have that, $(D/a, \omega/a)$ would be non-2-knitted and this would contradict the minimality of (D, ω). From Proposition (5.2) a must be at the end of an s-path of length at least two. Therefore, $(D/a, \omega/a)$ has the same number of s-paths as (D, ω) (notice that the trace is still well-defined even if an s-path has length one). Therefore, since $\mathbb{T}(D, \omega) \nsubseteq \mathbb{T}(D/a, \omega/a)$ then there must exist $x \in \mathbb{T}(D, \omega)$ where every crossing cut d with $x \in s(q(d))$ also has $a \in d$. □

Lemma 5.14. *If (D, ω) is minimally non-2-knitted and a_1, a_2 are adjacent arcs in a maximal s-path in (D, ω) then there exists $x \in \mathbb{T}$ such that if d is a crossing cut where $x \in s(q(d))$ then either $a_1 \in d$ or $a_2 \in d$.*

Proof. If $\mathbb{T}(D, \omega) \subseteq \mathbb{T}(D/\{a_1, a_2\}, \omega/\{a_1, a_2\})$ then by Proposition (5.11) we have that, $(D/\{a_1, a_2\}, \omega/\{a_1, a_2\})$ would be non-2-knitted and this would contradict the minimality of (D, ω). If a_1 or a_2 is at the end of a maximal s-path then by Lemma (5.13) there will be a crossing cut of (D, ω) containing a_1 or a_2. Otherwise, the maximal s-path containing a_1 and a_2 has length at least four, and so $(D/\{a_1, a_2\}, \omega/\{a_1, a_2\})$ has the same number of s-paths as (D, ω). Therefore, since $\mathbb{T}(D, \omega) \nsubseteq \mathbb{T}(D/\{a_1, a_2\}, \omega/\{a_1, a_2\})$ then there must exist $x \in \mathbb{T}(D, \omega)$ where every crossing cut d with $x \in s(q(d))$ also has $a_1 \in d$ or $a_2 \in d$. □

One may hope that the technique used in the previous lemmas could be used to show that *every* arc of non-zero weight is contained in each crossing cut of a particular type. However, a simple extension of these arguments is not possible. For example, consider the leftmost graph in Figure (5.6). Notice that arc a has weight 1 and is not in any crossing cut. Furthermore, contracting a results in a knitted weighted directed graph, found in the center of Figure (5.6). The contraction has had the effect of increasing the number of s-paths from three to four. (The trace changed from $\{000, 011, 101, 110\}$ to $\{0000, 1000, 0011, 0111, 0101, 1101, 1010, 1110\}$ which allows the packing 1001, 0110.)

In general, contracting an arc at the end of an s-path, or contracting two consecutive arcs on an s-path, shortens an s-path but does not split it into two s-paths. On the other hand, contracting a single arc in the middle of an s-path does split the s-path into two s-paths, and this is essentially what prevents the discussed extension of Lemma (5.13) and (5.14). This is illustrated in the rightmost graphic in Figure (5.6), where the results of contracting b_1, contracting f_1 and b_2, and contracting f_1 are shown.

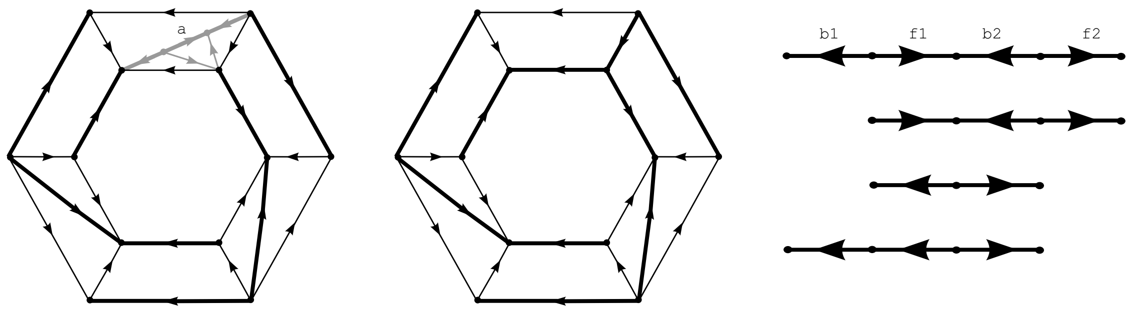

Figure 5.6: Contracting a gives the four s-path weighted directed graph in the middle.

5.3.1 $D_{1,3}$ Family

In the next two chapters, we will see several techniques for how to uncover every non-2-knitted s-path graph with a particular trace. Before advancing to these chapters, it is interesting to point out a bit of the back story for this thesis.

For $p \geq 3$, let $\mathbb{T}_p \in 2^{\{0,1\}^p}$ be the following trace

$$\mathbb{T}_p = 0^p \bigcup_{i=0}^{p-1} q(Cyc_i(011\pm^{p-3}))$$

where 0^p represents the binary vector with 0 repeated p times, and $Cyc_i(110\pm^{p-3})$ represents the string $1110\pm^{p-3}$ with a right cyclic shift of i positions.

Remark 5.15. *For any odd p, $\mathbb{T}_p$ is the trace of (D_1, ω_1) generalized to p s-paths.*

Remark 5.16. *For any even p, $(01)^{\frac{p}{2}} \notin \mathbb{T}_p$ and $(10)^{\frac{p}{2}} \notin \mathbb{T}_p$.*

Something surprising occurred to the author while trying to uncover all of the minimally non-2-knitted weighted directed graphs that have $\mathbb{T}_p$ for a trace. At one point

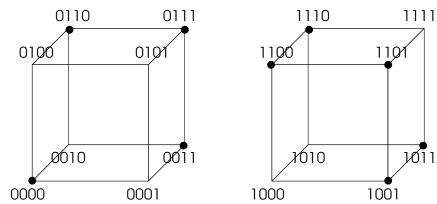

Figure 5.7: The trace $\mathbb{T}_4$. Notice an opposite pair of nodes is uncovered.

during the excavation, it become apparent that every potential candidate had to look very similar to one of the generalizations of (D_1, ω_1). In fact, each of the candidates could be formed from a generalization of (D_1, ω_1) by a simple *twist* of two s-paths. It was this observation that led the author to the new embedding of (D_3, ω_3) and the new infinite $D_{1,3}$ family. Incidentally, no other minimally non-2-knitted weighted directed graphs were found.

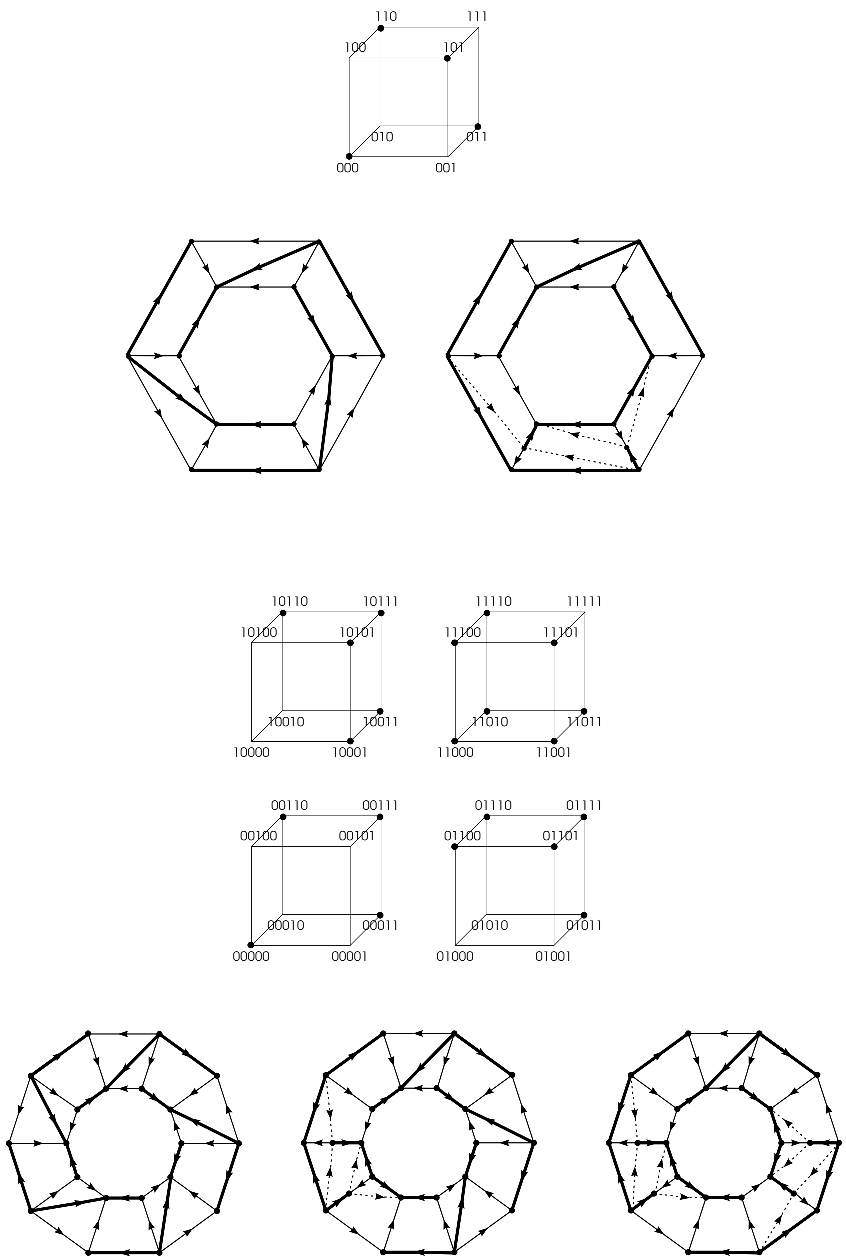

Figure 5.8: The $D_{1,3}$ family members of size three and five together with $\mathbb{T}_3$ and $\mathbb{T}_5$.

Chapter 6

Three S-Paths

In this chapter we restrict our attention to minimally non-2-knitted s-path graphs with three paths. We find that any such s-path graph must have special crossing cuts, and in fact, they must have the same trace. This trace leads us to a proof that (D_1, ω_1), $(D_2, \omega_2)^+$, and $(D_3, \omega_3)^+$ are the only s-path graphs with three paths that are minimally non-2-knitted.

The next remark follows from the first remark of the previous chapter, which told us that every crossing cut must have weight at least 3, and every arc in an s-path has weight 1.

Remark 6.1. *Every crossing cut intersects all three s-paths.*

Lemma 6.2. *If (D, ω) is an s-path graph with three paths that is minimally non-2-knitted then $\mathbb{T}(D, \omega)$ is reversible.*

Proof. If $x \in \mathbb{T}$ then there exists a crossing cut d where $x \in s(q(d))$. From Remark (6.1), it must be that d intersects each s-path, so we have equality, $x = s(q(d))$. Therefore, every term of size one is reversible. Suppose that $x \subseteq \mathbb{T}$ is a term of size at least two. Therefore, x contains two terms x_0 and x_1 of size one that differ in only one bit. Therefore, there exist two crossing cuts d_0 and d_1 that intersect only one s-path in opposite directions. Therefore, the union or intersection of d_0 and d_1 will intersect only two s-paths. This contradicts Remark (6.1). Hence, the only terms in $\mathbb{T}$ have size one, and $\mathbb{T}$ is reversible. □

Proposition 6.3. *If (D, ω) is an s-path graph with three paths that is minimally non-2-knitted then, without loss of generality, $\mathbb{T}(D, \omega) = \{000, 011, 101, 110\}$.*

Proof. Without loss of generality, suppose that $000 \in \mathbb{T}$. From Remark (6.1) and Lemma (6.2), there cannot be any terms of size two in $\mathbb{T}$. Therefore, 100, 010, $001 \notin \mathbb{T}$. By Proposition (5.11), this implies that 011, 101, $110 \in \mathbb{T}$. In particular, since $110 \in \mathbb{T}$ and the size two term, $s(11\pm) \notin \mathbb{T}$ then $111 \notin \mathbb{T}$. □

It is interesting to briefly return to clutters and the Q_6 property. By pushing forward on an s-path P, two directed joins J_1, J_2 are found that intersect P in the forward direction, and intersect the other two s-paths in opposite directions. Likewise, by pushing backward on P, two directed joins J_3, J_4 are found that intersect P in the backward direction and intersect the other two s-paths in opposite directions. Furthermore, suppose that J_a and J_b intersected two s-paths in the same direction, for $a \in \{1, 2\}$ and $b \in \{3, 4\}$. Since J_a and J_b are not accommodating, there must exist crossing cuts d_a and d_b where $d_a \subseteq J_a$, and $d_b \subseteq J_b$. However, this implies that d_a and d_b intersect only one s-path in opposite directions, and thus, their union or intersection contradicts Remark (6.1). Therefore, for $a \in \{1, 2\}$ and $b \in \{3, 4\}$, the pair J_a and J_b share directions on exactly one s-path. By considering the six columns of Q_6 as the six s-path directions, the four directed joins J_1, J_2, J_3, J_4, can be seen as the four rows of Q_6. In other words, the trace found in Proposition (5.9) can be seen as a direct consequence of the Q_6 property.

We will denote this special trace by $\mathbb{T}_3$. Figure (6.1) represents trace $\mathbb{T}_3$ by colouring the appropriate nodes of a three dimensional hyper-cube.

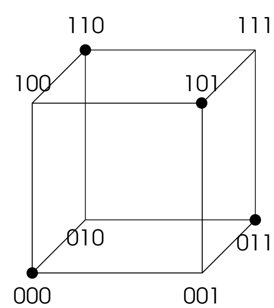

Figure 6.1: The trace $\mathbb{T}_3$.

Now let us consider the crossing cuts implied by $\mathbb{T}_3$. Let d_0, d_1, d_2, and d_3 be crossing cuts of a minimally non-2-knitted s-tree graph (D, ω) with three paths, P_1, P_2, and P_3, where $q(d_0) = 000$, $q(d_1) = 011$, $q(d_2) = 101$, and $q(d_3) = 110$. Notice that each pair of the crossing cuts d_0, d_1, d_2, d_3 intersect one of the s-paths P_1, P_2, P_3 in the same direction, and the other two s-paths in the opposite direction. Therefore, if the union or intersection of any two of these crossing cuts is also a crossing cut, then the resulting crossing cut would intersect one s-path, and skip the other two. This cannot happen, so we have the following remark.

Remark 6.4. *In $\mathbb{T}$ each pair of the crossing cut d_0, d_1, d_2, d_3 must be together on one s-path and apart on another s-path.*

Individually, each of the four cuts is involved in three together and three apart pairs, and collectively the six distinct pairs of these crossing cuts produce s-paths that have six together pairs and six apart pairs. Therefore, we need only to consider sets of s-paths and crossing cuts where the six together, six apart criterion can be matched.

Notice that $\mathbb{T}$ ensures that each s-path in P_1, P_2, P_3 has two of d_0, d_1, d_2, d_3 intersecting it in the forwards direction and the other two intersecting it in the backwards direction. Therefore, each s-path the crossing cuts contribute a total of $2 \cdot 2 = 4$ together or apart pairs. Hence, for any set of s-paths and crossing cuts, we can classify each s-path based on how many together pairs it contributes. We will call an s-path *Type i* if it contributes i together pairs and $4 - i$ apart pairs. Recall that Remark (5.13) and Remark (5.14) ensure that no arc at the end of an s-path, and no two adjacent arcs in an s-path are absent from all of d_0, d_1, d_2, d_3. Figure (6.2) shows every possible s-path and crossing configuration given these restrictions.

Up to taking a directional dual, by reversing the direction of every arc, there are only four possible combinations of Types that will result in six together and six apart pairs. Since the Type numbers in each of these combinations must sum to six, the possibilities are: [Type 2, Type 2, Type 2], [Type 3, Type 3, Type 0] (dual to [Type 1, Type 1, Type 4]), [Type 3, Type 2, Type 1], and [Type 4, Type 2, Type 0].

We will see that the first three combinations lead to (D_1, ω_1), $(D_2, \omega_2)^+$, and $(D_3, \omega_3)^+$ respectively, while the fourth combination does not yield any minimally non-2-knitted examples.

Before proceeding to this analysis, an additional remark needs to be made. For contradiction, suppose that some node n was included in two of the s-paths, say P_1 and P_2. Since internal nodes of s-paths induce directed cuts of weight 2, this node cannot be internal to P_1 or P_2. Therefore, the node is at the end of both of the s-paths, and the P_1 plus P_2 form a single path P. Without loss of generality, label forward and backward on the two s-paths so that the forward arcs of P are the forward arcs of P_1 and P_2. By Remark (3.4) this implies that every crossing cut either intersects P_1 in the forward

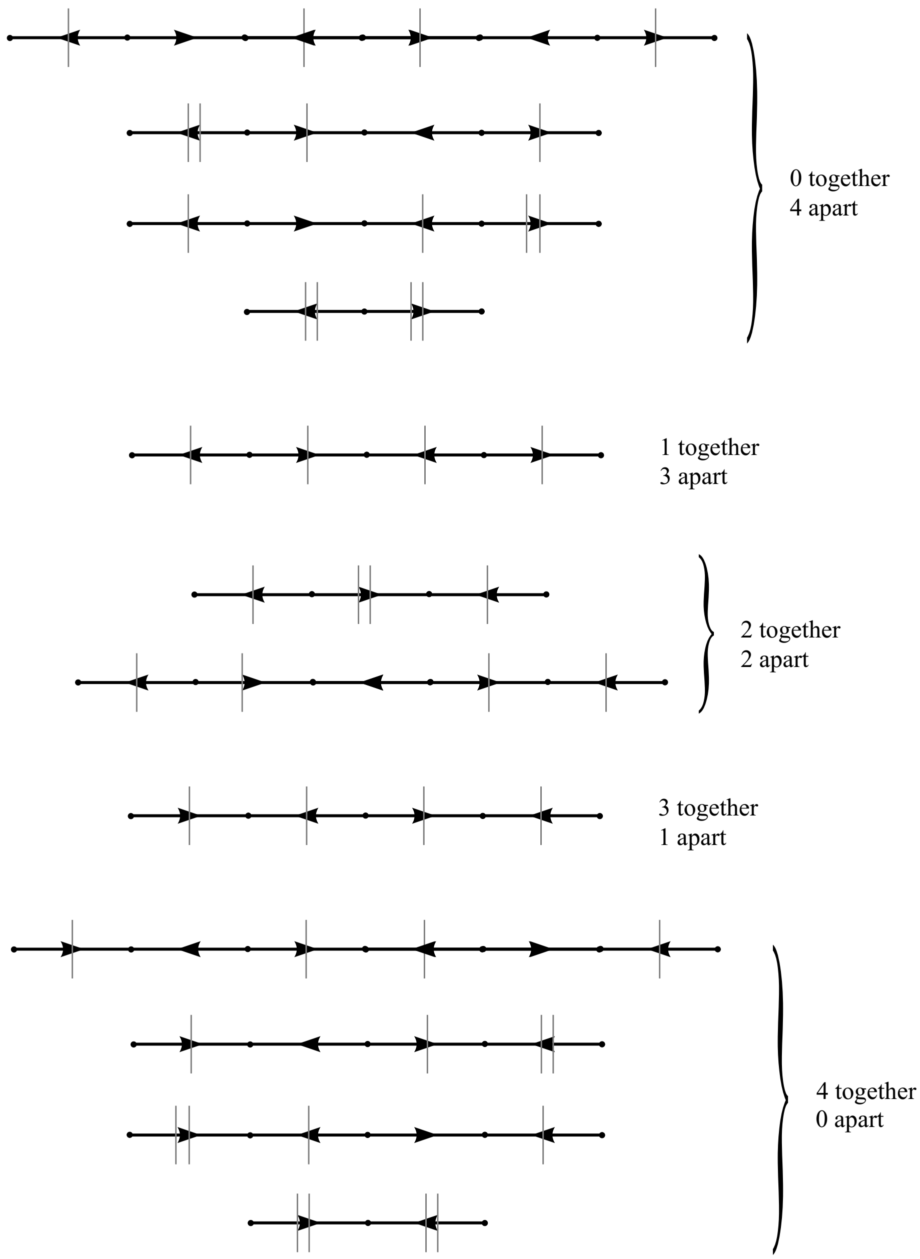

Figure 6.2: Trace $\mathbb{T}_3$, Remark (5.13), and Remark (5.14) limit each s-path and crossing cut intersections to the above possibilities.

direction and P_2 in the backward direction, or P_1 in the backward direction and P_2 in the forward direction. However, this contradicts the trace $\mathbb{T}_3$ because the four crossing cuts d_0, d_1, d_2, d_3 intersect P_1 and P_2 in the four possible forward/backward combinations. Therefore, no node in (D, ω) is present in both P_1 and P_2. The same argument works for any pair of P_1, P_2, P_3. Combined with Proposition (5.2), this implies the following remark.

Remark 6.5. *Every node of (D, ω) is present in exactly one of P_1, P_2, P_3.*

6.1 Minimally non-Knitted D_1

The first combination uses three s-paths of Type 2. Assume for the moment that every arc in an s-path is included in one of d_0, d_1, d_2, d_3. This assumption, which will be justified, implies that we have the three s-paths of length three found in Figure (6.3).

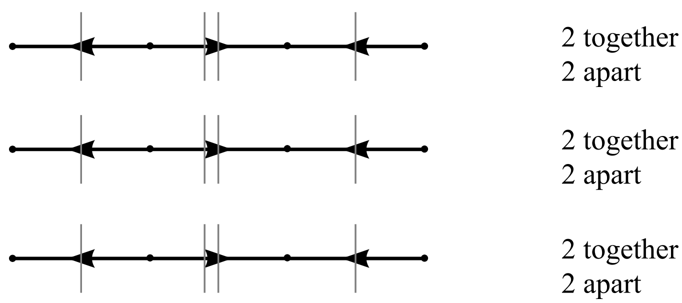

Figure 6.3: The s-paths of (D_1, ω_1).

During this discussion, for any s-path, the *side* arcs refer to the arcs for which there are two in the same direction, and the *middle* arc refers to the arc that is the only one in its direction. S-paths of odd length have less symmetry than s-paths of even length, since the two directions are not equivalent. For this reason $\mathbb{T}_3$, could actually represent two different groups of crossing cuts. The first option is that there is a crossing cut using all middle arcs, and three different crossing cuts using one middle arc and two side arcs. The second option is that there is a crossing cut using all side arcs, and three different crossing cuts using one side arc and two middle arcs.

However, this second option is not possible. As discussed, in general, there is a global restriction, over all pairs of crossing cuts, of six together, and six apart pairs. In particular, each crossing cut is both together and apart with the other three crossing cuts, so it must have a total of three together, and three apart pairs. However, each side arc either contributes two together, or two apart pairs. Therefore, it is not possible to have a directed cut that uses three side arcs.

Once the 000 crossing cut is placed along the middle arcs, the remaining crossing cuts have a unique placement, since each must use two different sides in order to be together and apart with 000.

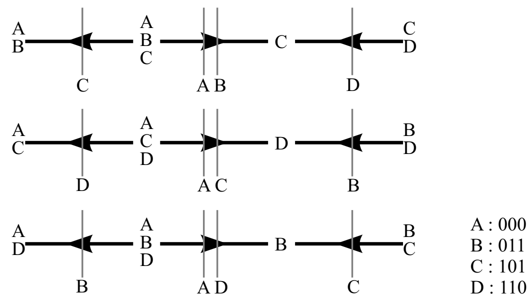

Figure 6.4: The crossing cuts of (D_1, ω_1).

The final step is to add arcs of weight zero. In particular, any arc of weight zero that does not enter the four crossing cuts, or the six trivial directed cuts, can be added without changing the fact that there will be no two packing of directed joins. In particular, it must be that these zero weight arcs remove every directed cut of weight 0 and 1. Remarkably, this is exactly what happens.

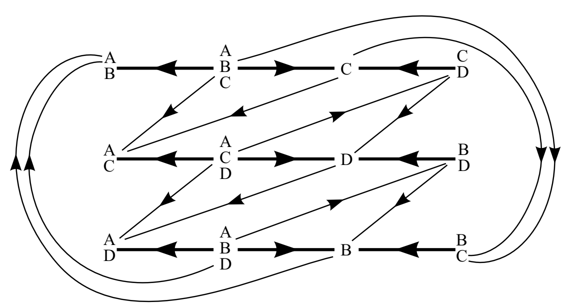

Figure 6.5: An embedding of (D_1, ω_1).

In Figure (6.5) each letter represents a crossing cut, and every node lists the crossing cuts that it is inside. A zero weight arc can be directed from u to v, if and only if, u is not a sink v is not a source and the list of crossing cuts in u contains the list of crossing cuts in v.

Notice that transitive weight zero arcs are implied between three pairs of sources and sinks. For example, the arc ACD to C, is implied by ACD to CD and CD to C. Adding such arcs are not necessary, since they do not change any of the directed cuts.

Now let us justify our previous assumption that each arc of weight 1 is in a crossing cut. From Figure (6.2) we must also consider that any of the s-paths could have length five. The upcoming arguments are valid regardless of the number of s-paths of length five. Let us then concentrate on the first s-path, P_1. Notice that crossing cuts A and B intersect the middle arc of P_1. Therefore, when using an s-path of length five we can either take A to be inside of B (Figure (6.6)), or B to be inside of A (Figure (6.7)). In both cases, it is possible to add arcs of weight 0 to create non-2-knitted weighted directed graphs. However, we will show that, by contraction or folding, the results are not minimal.

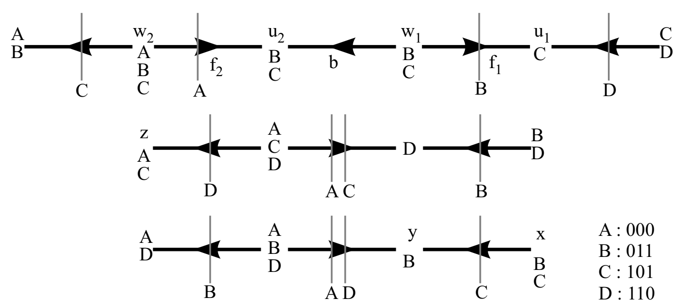

Figure 6.6: Extensions of these s-paths and crossing cuts can be folded.

Let us attempt to extend the s-paths and crossing cuts in Figure (6.6) to minimally non-2-knitted (D, ω). To ensure $\tau(D, \omega) > 1$, the node z must have arcs of weight 0 entering and leaving it. The only possible weight 0 arc entering z is (w_2, z), and the only possible weight 0 arc leaving z is (z, u_1). Therefore, there is a directed path from w_2 to u_1 in (D, ω). This is condition (F4) for folding.

Next, the only arc that can enter u_2 (and does not originate from w_1 or w_2) is (x, u_2). The only two arcs that can leave w_1 (and do not terminate at u_1 or u_2) are (w_1, x) and (w_1, y). Since (x, y) is an arc, we have also satisfied condition (F5) for folding. Since (F1), (F2), (F3) are also satisfied, we can fold on the first s-path. Therefore, (D, ω) cannot be minimal.

By appropriately relabeling the nodes x, y, z a similar argument shows that the s-paths and crossing cuts in Figure (6.7) do not extend to a minimally non-2-knitted (D, ω). The difference in this case is there are two possible arcs entering u_2 and one leaving w_1.

Finally, notice that these arguments

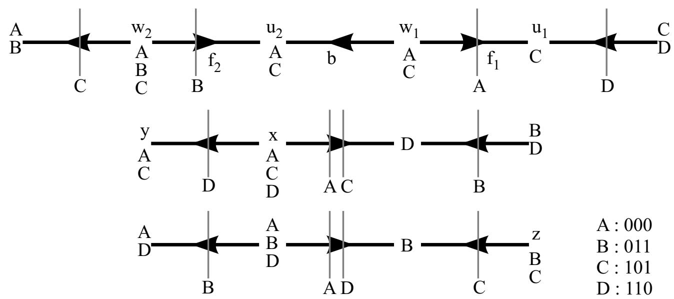

Figure 6.7: Extensions of these s-paths and crossing cuts can be folded.

6.2 Minimally non-Knitted D_2

The second combination uses two s-paths of Type 3, and one of Type 0. Again, we assume for the moment that every arc in each s-path is included in one of d_0, d_1, d_2, d_3. This implies that we have the two s-paths of length four and the one s-path of length two found in Figure (6.8).

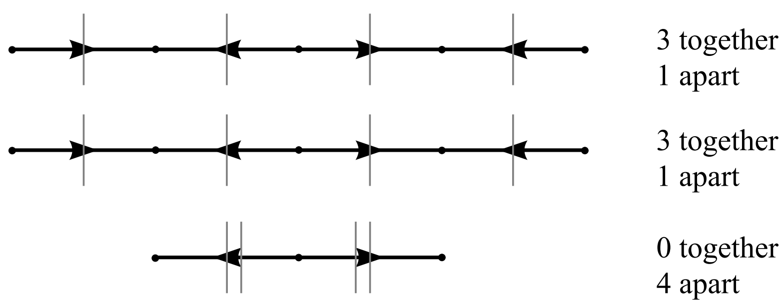

Figure 6.8: The s-paths of (D_2, ω_2).

Up to symmetry, there is a unique way to place the crossing cuts on the s-paths. In particular, two crossing cuts share the 0 direction on the s-path of length two, and two crossing cuts share the 1 direction on the s-path of length two. Each of these pairs must have an apart pair on one of the remaining s-paths. Since there is only one apart pair on each of the s-paths of length four, their positioning is determined, and this, in turn determines the positioning of the other crossing cuts.

Of particular note in this example is that the middle node in the path of length two

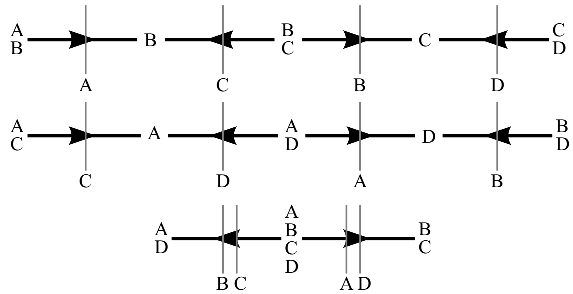

Figure 6.9: The crossing cuts of (D_2, ω_2).

is forced to be a super-source, connected by a directed path to every sink via an arc to the ends of each s-path.

Also of note, the list of crossing cuts on the end nodes of the path of length two are supersets of the list of crossing cuts contained on two sink nodes apiece. For example, the path end AD, is a superset of both sinks A and D on the second s-path.

Therefore, for both path ends, an arc of weight zero can be directed from the path end, to either of the sinks, or both. In fact, in both cases, all such choices are independently sufficient for ensuring that there are no directed cuts of weight one. Therefore, before isomorphism, there are nine possible variations of D_2, and these constitute the members of $(D_2, \omega_2)^+$.

Now let us justify our previous assumption that each arc of weight 1 is in a crossing cut. Figure (6.2) shows that such an arc cannot be present in an s-path of Type 3. However, there are several choices for the s-path of Type 0. Although there are a large number of possible configurations, we can quickly argue that none can extend to minimally non-2-knitted weighted directed graphs. For illustrative purposes, consider the configuration given in Figure (6.10).

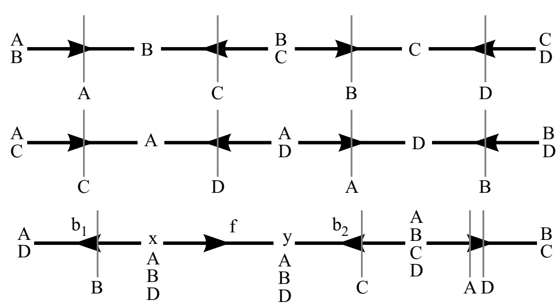

Figure 6.10: Extensions of these s-paths and crossing cuts can be contracted.

Notice that altering the third s-path, P_3, has resulted in a sink node y that is inside three out of the four crossing cuts. There is no arc of weight 0 that can be added that enters this sink. Therefore, if $\delta(X)$ is a crossing cut with $b_2 \in \delta(X)$ then $\delta(X')$ for $X = X \cup \{x, y\})$ is also a crossing cut. In particular,

$$\delta(X') \cap A_1 = \delta(X) \cap A_1 \backslash b_1 \cup b_2.$$

This contradicts Remark (5.14) since contracting f and b_2 will not eliminate any crossing cuts. This argument illustrates why the combination of two Type 3 s-paths, and one Type 0 s-path, only results in the minimally non-2-knitted $(D_2, \omega_2)^+$.

6.3 Minimally non-Knitted D_3

The third combination uses s-paths of Type 3, 2, and 1. Again, we assume for the moment that every arc in each s-path is included in one of d_0, d_1, d_2, d_3. This implies that we have the two s-paths of length four and the one s-path of length three found in Figure (6.11).

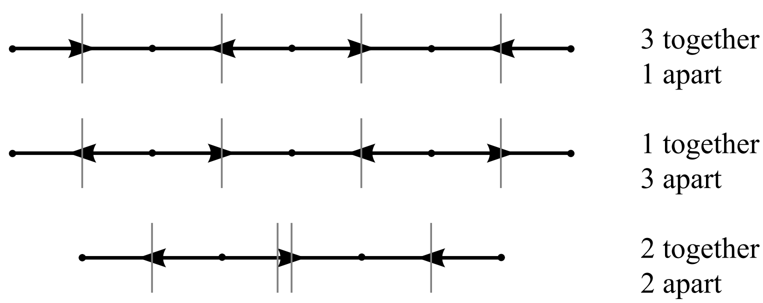

Figure 6.11: The s-paths of (D_3, ω_3).

Again there is a unique way, up to symmetry, to place the crossing cuts on the s-paths. In particular, without loss of generality, suppose that the third s-path is as presented. $D = 011$ and $B = 110$ are together on the third s-path, so they must be apart on the first s-path. Therefore, D and B use the two center arcs in the first s-path. $D = 011$ and $C = 101$ are apart on the third s-path, so they must be together on the second s-path. Therefore, D and C use the two center arcs in the second s-path. The remaining choices are then fixed.

As in the previous combination, there is some choice for placing arcs of weight zero.

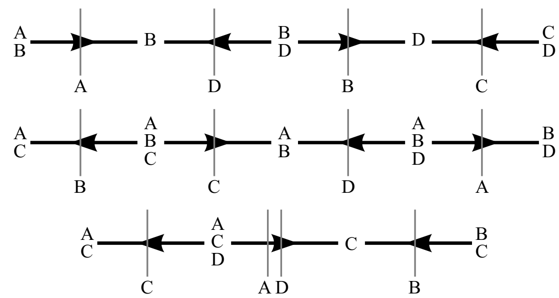

Figure 6.12: The crossing cuts of (D_3, ω_3).

In particular, path end BD of the second s-path, can have arcs directed out to sink B or sink D in the first s-path. Additionally, path end AB of the first s-path, can have arcs directed in from source ABC or source ABD on the second s-path.

Again, the condition that needs to be satisfied, reduces to the fact that every path end requires one arc of weight zero entering it, and one arc of weight zero exiting it. This is sufficient for removing the potential directed cuts of weight one involving the path end.

Now let us justify our previous assumption that each arc of weight 1 is in a crossing cut. Figure (6.2) shows that such an arc cannot be present in an s-path of Type 3 or Type 1. However, there is a choice for the s-path of Type 2.

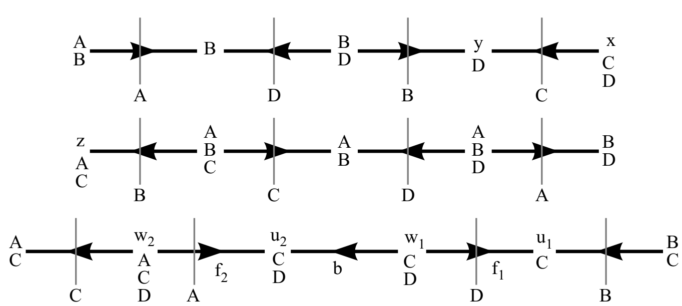

Figure 6.13: Extensions of these s-paths and crossing cuts can be folded.

It is not difficult to verify that any extension of Figure (6.13) or Figure (6.14) can be folded. The proof is the same as for the modifications of (D_1, ω_1).

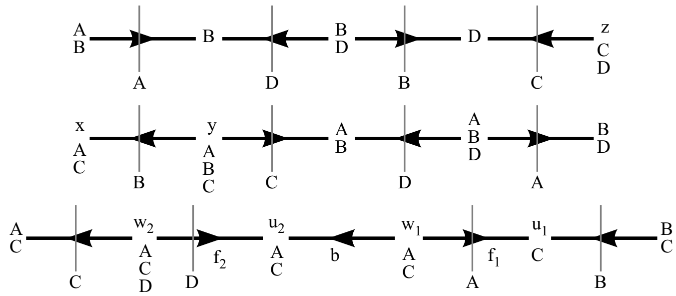

Figure 6.14: Extensions of these s-paths and crossing cuts can be folded.

6.4 Invalid

The final combination uses s-paths of Type 4, 2, and 0. However, there is an immediate problem with this combination, and this problem is irrespective of the length of each s-path. In order to visualize the situation, refer to Figure (6.15) which uses one s-path of length three, and two s-paths of length two.

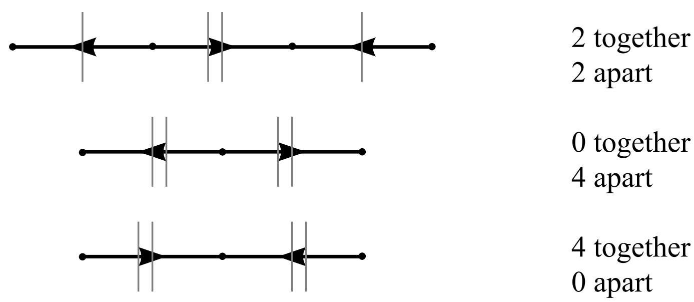

Figure 6.15: This configuration does not extend to a non-knitted weighted directed graph.

The problem is the restriction that, individually, each crossing cut must be involved in three together pairs and three apart pairs. However, each crossing cut accumulates two together pairs from one of the s-paths of Type 4, and two apart pairs from the s-path of Type 0. Therefore, each crossing cut must accumulate one together pair and one apart pair from the s-path of Type 2. However, this is not possible on an s-path.

Of particular note, if this combination did yield a non-knitted weighted directed graph, then the directed graph would contain a super-source, and a super-sink.

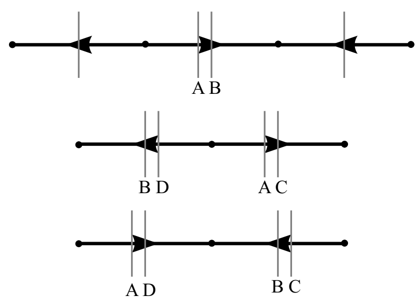

Figure 6.16: Individually the crossing cuts cannot have three together and apart pairs.

Chapter 7

Four S-Paths

In this chapter we turn our attention to s-path graphs with four s-paths. When extending from three to four s-paths we introduce the possibility that crossing cuts can skip one of the s-paths. For this reason, we use the notation $P[d] \in \{0, 1, +, -\}$ to represent how crossing cut d intersects with s-path P. Also, we use $P[d_1, d_2] = +$ if $\delta(X) = d_1 \cup d_2$ where X contains all of the nodes of P. Likewise, we use $P[d_1, d_2] = -$ if $\delta(X) = d_1 \cap d_2$ where X contains none of the nodes of P.

The main result of this chapter is that there are no minimally non-knitted s-path graphs with four s-paths. The proof works by showing that there are no appropriate traces. We begin by showing that any potential trace is reversible.

Lemma 7.1. *If (D, ω) is minimally non-2-knitted and has four s-paths, then $\mathbb{T}(D, \omega)$ is reversible. Furthermore, every term in $\mathbb{T}(D, \omega)$ has size one or two.*

Proof. From Proposition (5.2) each crossing cut must intersect at least three out of the four s-paths. Let us show that if t is a term of size two in $\mathbb{T}$ then there exists a crossing cut d where $s(q(d)) = t$. Without loss of generality, assume $t = s(\pm 000) = \{0000, 1000\}$ is a term in $\mathbb{T}$. Since every crossing cut intersects at least three s-paths, there cannot exist a crossing cut d where $t \subsetneq s(q(d))$. Therefore, if there does not exist a crossing cut d where $t = s(q(d))$ then there exist two crossing cuts, d_0 and d_1, where $0000 \subseteq s(q(d_0))$ and $1000 \subseteq s(q(d_1))$. Suppose that $P_1[d_0, d_1] = +$. Since d_0 and d_1 can skip at most one s-path each, at least one of P_2, P_3, P_4 has the property that both d_0 and d_1 intersect that s-path in the 0 direction. Without loss of generality, suppose that $P_4[d_0] = P_4[d_1] = 0$. Furthermore, since $P_2[d_0, d_1] \neq -$ and $P_3[d_0, d_1] \neq -$, then $d = d_0 \cup d_1$ must be a crossing cut. In particular, since $P_1[d] = P_1[d_0, d_1] = +$ then it must be that $P_2[d_0, d_1] = P_3[d_0, d_1] = P_4[d_0, d_1] = 0$. Hence, we have found a crossing cut d where $t = s(q(d))$.

Therefore, every term of size two is reversible. Now we argue, via an argument similar to one found above, that $\mathbb{T}$ cannot contain a term of size four. Without loss of generality, suppose $\mathbb{T}$ contained the term $t = s(\pm\pm 00)$. Therefore, we have adjacent terms of size two, $t_0 = s(0 \pm 00)$ and $t_1 = s(1 \pm 00)$. Since t_0 and t_1 are reversible then there exist crossing cuts d_0 and d_1 where $t_0 = s(q(d_0))$ and $t_1 = s(q(d_1))$. However, either $d = d_0 \cup d_1$ or $d = d_0 \cap d_1$ gives us a contradiction because d skips the s-paths P_1 and P_2. Therefore, $\mathbb{T}$ contains no terms of size larger than two.

In conclusion, we have that $\mathbb{T}$ has no maxterms of size larger than two, every maxterm of size two is reversible, and every maxterm of size one is always reversible. Therefore, $\mathbb{T}$ is reversible. ☐

Since the remaining arguments are visual in nature, it is useful to borrow graph theoretic terminology for the traces. A term of size one in $\mathbb{T}$ is a *vertex* and a term of size two in $\mathbb{T}$ is an *edge*. Define $G(\mathbb{T}) = (V, E)$ as follows:

$$
\begin{aligned}
V &= \{x | x \in \{0,1\}^4 \text{ and } x \in \mathbb{T}\} \\
E &= \{(x,y) | x, y \in \{0,1\}^4 \text{ and } x \neq y \text{ and } (x \cup y) \subseteq \mathbb{T}\}
\end{aligned}
$$

Remark 7.2. *If (D, ω) is minimally non-2-knitted with four s-paths then the degree of every vertex in $G(\mathbb{T}(D, \omega)) = (V, E)$ is at most two.*

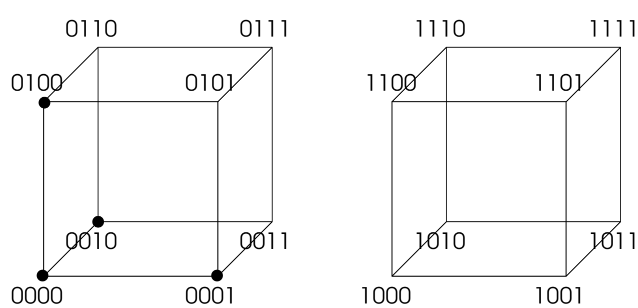

Figure 7.1: A vertex in $G(\mathbb{T})$ cannot be incident to more than two edges.

Proof. For contradiction, suppose that there is some vertex $v \in V$ incident to three edges in E. Without loss of generality, suppose that $x = 0000$ and the edges are

$e_1 = (0000, 1000)$, $e_2 = (0000, 0100)$, $e_3 = (0000, 0010) \in E$. Therefore, $s(\pm 000), s(0 \pm 00), s(00 \pm 0) \in \mathbb{T}$ and by Lemma (7.1) there exist crossing cuts d_1, d_2, d_3 where $s(q(d_1)) = s(\pm 000)$, $s(q(d_2)) = s(0 \pm 00)$, $s(q(d_3)) = s(00 \pm 0)$. Now of the values $P_1[d_1]$, $P_2[d_2]$, $P_3[d_3]$, at least two are $+$, or at least two are $-$. Without loss of generality, suppose that $P_1[d_1] = P_2[d_2] = +$. However, this is a contradiction since then $d = d_1 \cup d_2$ is a crossing cut skipping two of the s-paths. □

Remark 7.3. *If (D, ω) is minimally non-2-knitted with four s-paths then $G(\mathbb{T}(D, \omega)) = (V, E)$ is acyclic.*

Proof. Suppose otherwise. Without loss of generality, suppose $(0000, 0001)$ and $(0001, 0011)$ are edges in a cycle. Hence, $0000, 0001, 0011 \in \mathbb{T}$ (Figure (7.2)).

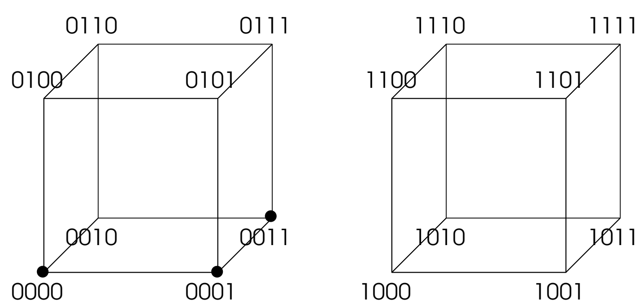

Figure 7.2:

By Lemma (7.1) $s(00 \pm \pm) \not\subseteq \mathbb{T}$ so $0010 \notin \mathbb{T}$. Also, since vertex 0001 already has degree two, by Remark (7.2) we have $1001, 0101 \notin \mathbb{T}$. Therefore, by Remark (5.11) on opposite pairs, these observations imply $1101, 0110, 1010 \in \mathbb{T}$. At this point we have $0000, 0001, 0011, 1101, 0110, 1010 \in \mathbb{T}$ and $00001, 1001, 0101 \notin \mathbb{T}$ (Figure (7.3)).

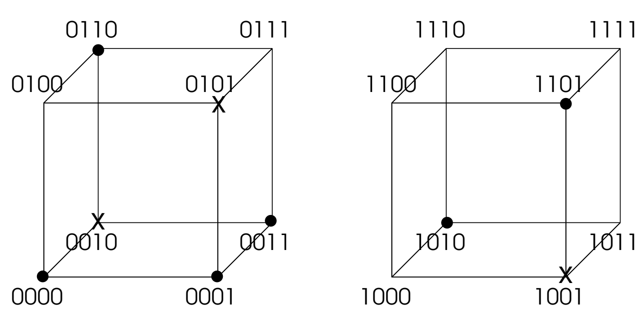

Figure 7.3:

The cycle now must extend with either (0011, 1011) or (0011, 0111). However, notice that $\mathbb{T}$ is currently symmetric on P_1 and P_2. Therefore, without loss of generality, the cycle also contains edge (0011, 0111). Hence, $0111 \in \mathbb{T}$. Since vertex 0011 and now has degree two, by Remark (7.2) $1011 \notin \mathbb{T}$. Therefore, by Remark (5.11) on opposite pairs, $0100 \in \mathbb{T}$ (Figure (7.4)).

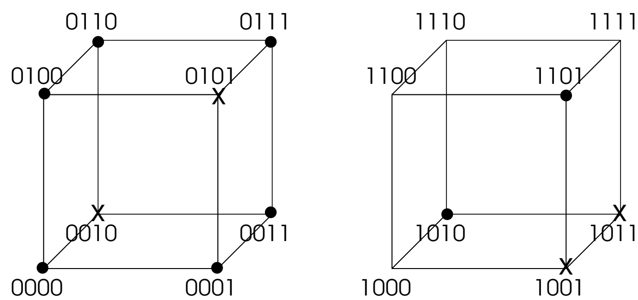

Figure 7.4:

Therefore, we have a cycle of length six. For completion, we will finish the argument to show that this trace candidate is unique. Since the vertices 0111, 0110, 0100, 0000 have degree two, then by Remark (7.2) we have $1111, 1110, 1100, 1000 \notin \mathbb{T}$ (Figure (7.5)).

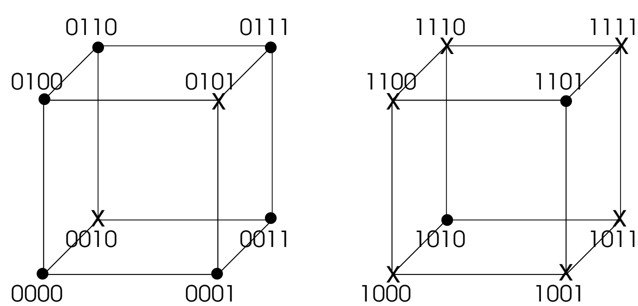

Figure 7.5:

Now we show that $\mathbb{T}$ is not valid. Since $s(0 \pm 00), s(01 \pm 0), s(011\pm), s(0 \pm 11), s(00 \pm 1), s(000\pm) \in \mathbb{T}$ then by Lemma (7.1) there exist crossing cuts $d_1, d_2, d_3, d_4, d_5, d_6$ such that

$$\begin{aligned} s(q(d_1)) &= s(0{\pm}00) \\ s(q(d_2)) &= s(01{\pm}0) \\ s(q(d_3)) &= s(011{\pm}) \\ s(q(d_4)) &= s(0{\pm}11) \\ s(q(d_5)) &= s(00{\pm}1) \\ s(q(d_6)) &= s(000{\pm}) \end{aligned}$$

Let us represent the skip value of d_i by $x_i \in \{+,-\}$. That is, let $x_1 = P_2[d_1]$, $x_2 = P_3[d_2]$, $x_3 = P_4[d_3]$, $x_4 = P_2[d_4]$, $x_5 = P_3[d_5]$, and $x_6 = P_4[d_6]$. We will index the crossing cuts d and the values of x cyclically, so that $d_7 = d_1$ and $x_7 = x_1$. If $x_i = x_{i+1}$ then either $d = d_i \cup d_{i+1}$ or $d = d_i \cap d_{i+1}$ gives a contradiction since d would be a crossing cut skipping two s-paths. Therefore, without loss of generality assume that $x_1 = x_3 = x_5 = +$ and $x_2 = x_4 = x_6 = -$. Now consider $d = d_1 \cup d_3 \cup d_5$. Since $P_2[d_1] = P_3[d_5] = P_4[d_3] = +$ and $P_1[d_1] = P_1[d_3] = P_1[d_5] = 0$ then d is a crossing cut skipping three s-paths. Hence, $\mathbb{T}$ is not valid. □

Therefore, $G(\mathbb{T})$ is acyclic and from Remark (7.2) each vertex has degree at most two. Hence, $G(\mathbb{T})$ is a collection of paths and isolated vertices. We will conclude our proof by showing that $G(\mathbb{T})$ is not a collection of isolated vertex, and cannot contain a path.

Remark 7.4. *If (D, ω) is minimally non-2-knitted with four s-paths then $G(\mathbb{T}(D, \omega)) = (V, E)$ is not a collection of isolated vertices.*

Proof. For contradiction assume otherwise. That is, assume that $E = \emptyset$ so every maxterm of $\mathbb{T}$ is size one. Without loss of generality suppose that $0000 \in \mathbb{T}$. Therefore, $1000, 0100, 0010, 0001 \notin \mathbb{T}$. By Remark (5.11) for opposite pairs, we must have that $0111, 1011, 1101, 1110 \in \mathbb{T}$ (Figure (7.6)).

From Remark (7.2), 0111 has degree at most two in $G(\mathbb{T})$ so at least one of 0011, 0101, 0110 is not in $\mathbb{T}$. Since $\mathbb{T}$ is completely symmetric at this point, without loss of generality, suppose that $0101 \notin \mathbb{T}$. Therefore, by Remark (5.11) on opposite pairs, we have that $1010 \in \mathbb{T}$ (Figure (7.7)).

However, this contradicts our assumption on $G(\mathbb{T})$ since $\{1010, 1110\} \subseteq \mathbb{T}$ so $E \neq \emptyset$. □

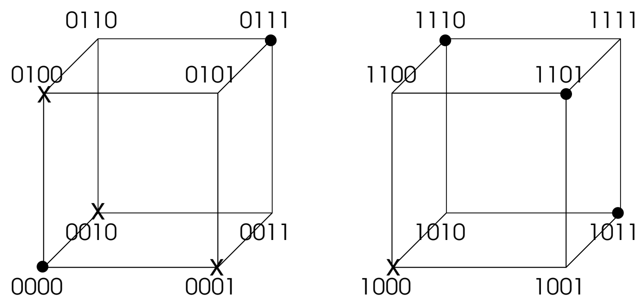

Figure 7.6:

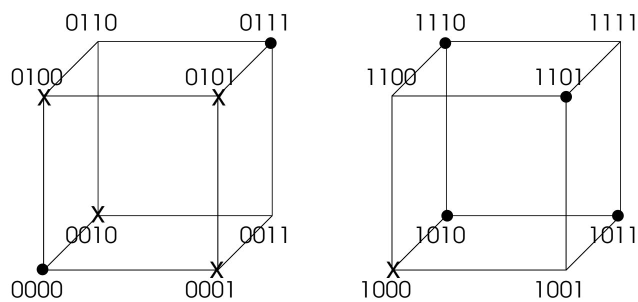

Figure 7.7:

Remark 7.5. *If* (D, ω) *is minimally non-2-knitted with four s-paths then* $G(\mathbb{T}(D, \omega)) = (V, E)$ *does not contain a path.*

Proof. If $G(\mathbb{T})$ contains a path then it must have a node at the end of the path. Without loss of generality, suppose that $0000 \in V$ is the end of a path, and that $(0000, 0001) \in E$ is an edge in the path. Hence, $0000, 0001 \in \mathbb{T}$ and since 0000 is the end of a path then $1000, 0100, 0010 \notin \mathbb{T}$. Therefore, by Remark (5.11) on opposite pairs, $0111, 1011, 1101 \in \mathbb{T}$ (Figure (7.8)).

From Remark (7.2), 0101, 0011, 1001, and 1111 have degree at most two in $G(\mathbb{T})$ so 0101, 0011, 1001, $1111 \notin \mathbb{T}$. Therefore, by opposite pairs, 1010, 1100, $0110 \in \mathbb{T}$. Finally, 1110 has degree at most two and we see that $1110 \notin \mathbb{T}$ (Figure (7.9)).

However, our result is a trace in which P_1 plays no significant role. By contracting the entire s-path we are left with the trace $\mathbb{T}_3$. Hence, the trace cannot produce minimally non-2-knitted weighted directed graphs. □

From the above line of reasoning, we have proven the following result. The natural question is if this result can be extended to higher numbers of s-paths.

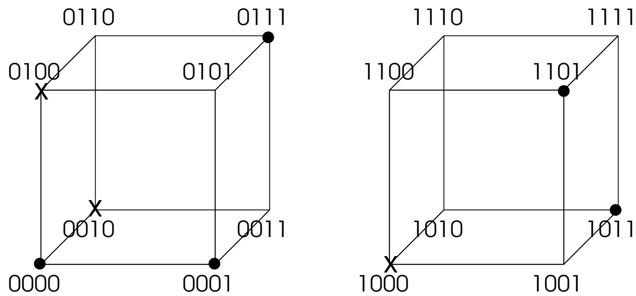

Figure 7.8: 0000 is the end of a path with edge $(0000, 0001)$.

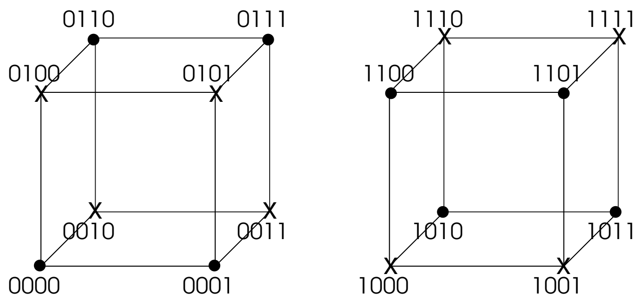

Figure 7.9: The trace is fully determined since degree three nodes are forbidden.

Proposition 7.6. *No minimally non-knitted weighted directed graph has four s-paths.*

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