A FUNDAMENTALLY TOPOLOGICAL PERSPECTIVE ON GRAPH THEORY

by

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I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

We adopt a novel topological approach for graphs, in which edges are modelled as points as opposed to arcs. The model of "classical" *topologized graphs* translates graph isomorphism into topological homeomorphism, so that *all* combinatorial concepts are expressible in purely topological language. This allows us to extrapolate concepts from finite graphs to infinite graphs equipped with a compatible topology, which, dropping the "classical" requirement, need not be unique. We bring standard concepts from general topology to bear upon questions of a combinatorial inspiration, in an infinite setting.

We show how (possibly finite) graph-theoretic paths are, without any technical subterfuges, a subclass of a broad category of topological spaces, namely "paths", that includes Hausdorff arcs, the real line and all connected orderable spaces (of arbitrary cardinality). We show that all paths, and the topological generalizations of cycles, are topologized graphs. We use feeble regularity to explore relationships between the topologies on the vertex set and the whole space. We employ compactness and weak normality to prove the existence of our analogues for minimal spanning sets and fundamental cycles. In this framework, we generalize theorems from finite graph theory to a broad class of topological structures, including the facts that fundamental cycles are a basis for the cycle space, and the orthogonality between bond spaces and cycle spaces. We show that this can be accomplished in a setup where the set of edges of a cycle has a loose relationship with the cycle itself. It turns out that, in our model, feeble regularity excludes several pathologies, including one identified previously by Diestel and Kühn, in a very different approach which addresses the same issues. Moreover, the spaces surgically constructed by the same authors are feebly regular and, if the original graph is 2-connected, compact. We consider an attractive relaxation of the T_1 separation axiom, namely the S_1 axiom, which leads to a topological universe parallel to the usual one in mainstream topology. We use local connectedness to unify graph-theoretic trees with the dendrites of continuum theory and a more general class of well behaved dendritic spaces, within the class of *ferns*.

We generalize results of Whyburn and others concerning dendritic spaces to ferns, and show how cycles and ferns, in particular paths, are naturally S_1 spaces, and hence may be viewed as topologized hypergraphs. We use topological separation properties with a distinct combinatorial flavour to unify the theory of cycles, paths and ferns. This we also do via a setup involving total orders, cyclic orders and partial orders. The results on partial orders are similar to results of Ward and Muenzenberger and Smithson in the more restrictive setting of Hausdorff dendritic spaces. Our approach is quite different and, we believe, lays the ground for an appropriate notion of completion which links Freudenthal ends of ferns simultaneously with the work of Polat for non-locally-finite graphs and the paper of Allen which recognizes the unique dendritic compactification of a rim-compact dendritic space as its Freudenthal compactification.

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I am deeply grateful for this contribution. Apart from this, I will be forever indebted to my supervisor for his faith in an unorthodox research project, his unfailing availability, unstinting support, amazing generosity and endless patience, and the simple pleasure of working together in the pursuit of beauty in mathematics.

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I dedicate this work to my friend Lavina McClintock, whose tardiness one night several years ago, at the Irish Pub in Pisa, spawned the musings which eventually led to a thesis.

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A note on spelling

This thesis was written mainly in Canada and in New Zealand, both countries where English is an official language. The author is most familiar with British English. With regards to spelling, the intention is to always use a form which is acceptable to the British standard; whenever this constraint does not uniquely determine the possible spellings, we tend to favour the most widely accepted form.

It seems to be a rather common misconception that verbs ending in “-ize” are “American” and the “British” version is the one ending in “-ise”. In fact, British spelling very frequently allows both versions, and Oxford University Press traditionally prefers the “-ize” version for headwords. Since this reduces the conflict with the non-British standards, we adopt the same approach. This also applies to coined terms, such as “topologized graphs”.

Notation

Special Symbols

Most of the assertions which we deign with a dedicated paragraph are followed by some form of justification. This is usually either a proof or a reference to a published book or article. We denote the end of a proof by the symbol ✠. We use the symbol □ whenever this justification takes on the form of a reference, or an argument reproduced or put together from the literature for which we do not claim credit. In these cases the assertion is presented as a "fact". Even on the rare occasion that no justification is given (as in the case of a routine verification which can be done by the reader), the presence of either of these symbols denotes that we consider the truth of the assertion to be established. We also use the symbol □ to mark the end of an intermediate step within a proof.

Binary Relations

A binary relation on a set X is formally a subset of $X \times X$. If $\mathcal{R}$ is such a relation, the notation $a\,\mathcal{R}\,b$ stands for $(a, b) \in \mathcal{R}$. We shall often use symbols similar to $\leq, \preceq, \sqsubseteq, \trianglelefteq$ to denote binary relations, especially transitive, asymmetric binary relations. In this case, the notation $a \prec b$ is an abbreviation for "$a \preceq b$ and $a \neq b$". Also $a \succeq b$ and $a \succ b$ are equivalent to $b \preceq a$ and $b \succ a$ respectively.

Given a binary relation $\preceq$ on X and a point $a \in X$, the sets $\{x \in X \mid x \prec a\}$, $\{x \in X \mid x \preceq a\}$, $\{x \in X \mid x \succ a\}$ and $\{x \in X \mid x \succeq a\}$ are represented by $\alpha(x)$, $\mathtt{A}(x)$, $\omega(x)$ and $\Omega(x)$ respectively.

Topological Operators

Given a subset A of a topological space, $\partial(A)$ denotes the frontier (boundary) of A, $\overline{A}$ its closure, and A° its interior. When convenient, we shall also write $\mathsf{Cl}(A)$ for the closure of a set. In contexts where A is contained in a proper subspace of a topological space, if ambiguity can arise we shall use subscripts to specify the topology according to which the closure operator is being taken e.g. if Y is a subspace of a topological space$X$, $\mathsf{Cl}_Y(A)$

denotes the closure of A with respect to the relative topology on Y, which may differ from $\mathsf{Cl}_X(A)$ if Y is not a closed subset of X. If unspecified, it should be understood that the closure is taken in the larger space.

Singletons

In this thesis, we shall often be dealing with "singletons": sets consisting of a single element. For example, we shall have set operators such as the frontier and closure acting on a singleton; to avoid being cumbersome, in this context we do not distinguish between a singleton and its unique element. Thus we would write, for example, $\overline{x}$ as an abbreviation for $\overline{\{x\}}$ and $\partial(x)$ for $\partial(\{x\})$. We shall also say that a point has a certain attribute, which is usually a property of a *subset* of a topological space, if the corresponding singleton has the attribute in question. So for example we shall speak of open, closed and clopen points.

We shall introduce further notation in the body of the thesis. The following table is given as a quick reference to the relevant definitions.

Table 1: **Summary of Notation**

Combinatorial with topological analogues

Combinatorial	Topological	Mixed	Underlying Concept	Definition
$\mathcal{N}_v$, $A^{\Box}$	$A^{\diamond}$	$A^{\boxdot}$	adding edges	pgs. 8, 89, 31, 101
$\delta(v)$	x', $\rho(v)$, $\partial(A)$	$\delta(A)$	borders, cuts	pgs. 13, 123, 82, xi, 163
$G[e]$	$\mathsf{Cl}(A)$, $\overline{A}$		closure	pgs. 8, xi, xi

Numeric

$\mathbb{Z}$	$\mathbb{N}$	$\mathbb{P}$	$\mathbb{Q}$	$\mathbb{R}$
integers	non-negative integers	positive integers	rational numbers	real numbers

Topological

$\mathbf{K}(x)$, $\mathbf{K}_X(x)$	$\mathbf{B}_c(x)$	A°
component (of X) containing x	component of $X \setminus \{c\}$ containing x	interior of A
pg. 29	pg. 203	pg. 232

Order-theoretic

Generic binary relation	Totally ordered set	Prepath	Underlying Concept
$\Omega(x)$, $\omega(x)$	$[x, \infty)$, (x, ∞)	$\langle x, \infty)$	upper tail, with or without x
$\mathtt{A}(x)$, $\alpha(x)$	$(-\infty, x]$, $(-\infty, x)$	$(-\infty, x\rangle$,	lower tail, with or without x
pg. xi	pg. 45	pg. 45	

Algebraic

$\mathbb{Z}_2^E$	$\mathcal{W}(S)$	$\mathcal{A}(S)$	$\mathcal{S}(S)$	$\mathcal{B}$	$\mathcal{Z}$
power set	weak span	algebraic span	strong span	bond space	cycle space
pg.161	pg. 161	pg. 161	pg. 161	pg. 163	pg. 168

Important Combinatorial Topological Properties

(CSp)	(CSd)	(CSd′), (CI)	CSf	(CSf′)
prepaths	dendritic spaces		ferns	flimsy ferns
pg. 38	pg. 192	pg. 200	pg. 198	pg. 198

Chapter 1

There and Back Again (from combinatorics to topology)

1.1 From Graph Theory to Topology: motivation and basic constructions

In this thesis we adopt a novel topological approach to hypergraphs, focusing especially on graphs. We consider topological models for graphs which are in stark contrast with the traditional one. In this section we explain our motivations, discuss the advantages which make our approach worthwhile, and state our general and specific objectives.

One of the most basic and important building blocks of graph theory is the notion of "connectedness". The same word also has a very important meaning in the field of general topology; indeed, arguably the latter subject grew precisely out of the efforts of several mathematicians[1] to give the right formalization for concepts like "continuity", "convergence", "dimension" and, not least, connectedness. Although formally the two concepts are very different, one depending on finite paths and the other on open sets, the intuition behind the two versions of connectedness is essentially the same, and few will dispute that any link between graph theory and topology should at least reconcile them, if not be entirely dictated by this objective. In fact the usual way of modelling a graph as a topological object does achieve this, albeit in a way which, we feel, is not entirely satisfactory.

Traditionally, a graph is modelled as a one-dimensional cell-complex[2], with open arcs for edges and points for vertices, the neighbourhoods of a "vertex" being the sets containing the vertex itself and a union of corresponding "tails" of every "edge" (arc) incident with the

[1] Incidentally, prominent among these were Kuratowski, Menger and Whitney!

[2] For a precise definition of a cell-complex, the reader is referred to [41, Definitions 1.1, 1.3]

vertex. If the graph is planar, this is equivalent to taking the subspace topology inherited from the Euclidean plane by an appropriate "drawing" of the graph. If the graph is finite, one can always place the vertices in *three*-dimensional Euclidean space, and join up pairs of adjacent vertices by pairwise disjoint open arcs (whose accumulation points are the two adjacent vertices) so that the union of the arcs together with the set of vertices inherits the topology of a cell-complex with the above restriction. Also, in the finite case, this concept coincides with that of a graph in continuum theory (see, for example, [45, Definition 9.1]), and with that of a one-dimensional simplicial cell-complex (a sub-cell structure of a simplex, the convex hull of $n+1$ affinely independent points in Euclidean n-space).

The first objection to this model is a formal one: loosely speaking, a one-dimensional cell-complex is not itself a graph, or at least not one which has any natural combinatorial correspondence with the graph it is modelling. Let us make this statement more precise. In full generality, a graph is often defined (for example, [8, Section 1.1]) as a triple (V, E, f), where V is a set of vertices, E a set of edges disjoint from V, and f the incidence function which associates to every edge a non-empty set of at most two vertices. The incidence function allows for "multiple edges" and those edges whose images are singletons correspond to "loops". Given a graph G, we shall denote the set of vertices, respectively edges, of G by V_G, E_G, and the incidence function by f_G.

On the other hand, a topological space is a pair $(X, \mathcal{T})$, where X is a set and $\mathcal{T}$ is a subset of the power set of X containing X and the empty set, and closed under finite intersections and arbitrary unions. The elements of $\mathcal{T}$ are the open sets. Given a graph (V, E, f) and its corresponding topological space $(X, \mathcal{T})$, the *ground sets* $V \cup E$ and X are entirely different objects, and the combinatorial structure on $V \cup E$ does not exist on X, while the topological structure on X does not exist on $V \cup E$. In fact, if G has at least one edge, X consists of at least continuum many points. Of course, this is not the only context in mathematics where such a discrepancy appears, but at least *a priori* one would prefer to have the same ground set. We shall see that *a posteriori* there is even more reason to do so.

1.1.1 Question: Is it possible to assign, to every graph G, a topology $\mathcal{T}_G$ with ground set $V_G \cup E_G$, so that

- G is graph-theoretically connected if and only if $V_G \cup E_G$ is connected with respect to the topology $\mathcal{T}_G$, and
- given a subgraph H of a graph G, the topology inherited by H from $\mathcal{T}_G$ coincides with $\mathcal{T}_H$?

Note that, any affirmative answer to Question 1.1.1 above would involve modelling edges as *points*, as opposed to arcs. This is perhaps the most obvious (though certainly not the only) difference between our models and the traditional one.

A second objection to the cell-complex model is that the topological structure being employed is much more involved and sophisticated than the combinatorial structure: not only do the cardinalities differ vastly, but the concept of a one-dimensional cell-complex rests upon the notion of an arc, which is often defined as an injection of the unit interval of real numbers into a Hausdorff space. Although these concepts, in particular the real numbers, are commonplace, they are certainly not as elementary as the combinatorial structure of a graph, and appear extraneous to the context of graphs. In fact, in this thesis we hope to make a case for the fact that in some contexts all that matters is that the objects (arcs) corresponding to edges are open and connected (and, of course, their accumulation points).

A third objection is more subjective. Consider, for example, the cycle C_n with n vertices and n edges. The corresponding simplicial complex is a homeomorph of a circle, independently of the value of n. In particular, it is impossible to recover the vertices from the topology, because all points in the circle are topologically identical (for any pair, there exists a homeomorphism of the circle swapping the two points). Thus in this model we are forced to retain the combinatorial information in addition to the topology.

It is arguable that this is not too bad after all, because cycles do have similarities and perhaps the topology is merely capturing something "essential" in the structure of graphs, but we wish to show that it is possible, and indeed very natural, to blur the distinction between the topology and the combinatorics to the point where, not only are the ground sets the same, but the topology even contains *all* the combinatorial information.

In fact, we shall use two different models for graphs which differ on this very point (and in other ways as well). In the case of **topologized graphs** the topology will uniquely determine the combinatorial information, while in the case of **edge spaces** we shall retain the flexibility to look at a given topological space as a graph in more than one possible way, subject to certain compatibility requirements. In the latter case, the only extra information we need to retain on top of the topology is simply which points are *deemed* to be edges.

In both cases the ground *sets* for the topological and combinatorial structures will be the same, and in both cases (when endowed with the corresponding structure) they will be graphs and topological spaces in the strictest sense of the terms, as defined above. The only sense in which our topological spaces will *technically* not be graphs, and our graphs not topological spaces, is that formally a triple (V, E, f) with the requirements of a graph is not a pair $(X, \mathcal{T})$ with the requirements of a topological space.

The crucial way in which our scenario *will* change with respect to the usual context of graph theory lies in the *meaning* of the term "connected". If we restrict ourselves to the class of *finite* graphs, then in the case of topologized graphs the change of model will amount to a change of language, in that the combinatorial structure uniquely determines, as well as being uniquely determined by, the topology. However, throughout this thesis we shall seek to unify the treatment of infinite and finite graphs. In the next section we shall present a very simple example to illustrate how, when one considers infinite graphs, one can very reasonably choose to consider the *same* graph to be "connected" or "not connected". In general, in our models every graph usually considered to be connected will still be so, but we may gain some extra connected structures. This will allow us, for example, to consider infinite cycles.

In order to effect this change in the meaning of the adjective "connected", we shall intentionally pursue a "topologically biased" approach—we shall consider spaces equipped with a pre-assigned topology, and this topology will decree whether a structure is connected or not. Moreover, this information cannot be recovered from the combinatorial structure alone. In particular, at the level of generality of infinite graphs, in *neither* model will the combinatorial information uniquely determine the topology.

1.1.1 A topologically biased philosophy

We hope to explain in Subsection 1.1.2 how the topological bias we mention in the preceding paragraph is, in some sense, a natural choice forced upon us. Once we were led in this direction, at an early stage of this work, we deliberately chose to take the choice one step further and make it an underlying philosophy guiding our investigations.

Throughout this thesis, we shall favour an abstract approach guided by topological "axioms", or "properties": identify a relevant topological property—preferably one which is well understood and proven to be useful, thanks to the work of mathematicians over the last century—and pursue it to where it leads. *A priori*, this may appear as something of a gamble—our grounds for proceeding this way lie in our desire to formally unify graph theory with topology in a fundamental way, based around the concept of connectedness, and in the fact that topology was supposedly created to deal with such concepts.

Beyond this, however, there is little reason to believe that the concepts and tools of traditional point-set topology, which deals with predominantly non-discrete structures, should be relevant and fruitful in a discrete setting.

In fact, we were soon led very naturally to a scenario which, at first sight, appears utterly hopeless from a purely topological standpoint. The reader will recall that a topological space is said to be Hausdorff if any two points can be separated by disjoint open sets. The Hausdorff requirement is usually considered to be a very reasonable, not too restrictive, assumption, because it guarantees that sequences (and nets) have unique limit points. An even weaker assumption—the T_1 axiom—is that all singletons are closed. The vast majority of topological spaces usually considered are T_1 spaces. Two other, fairly restrictive, assumptions of the same character, not usually introduced if not needed, are those of regularity and normality.

Interestingly, one of the reasons why the T_1 assumption is so prevalent in general topology is that it allows one to turn the four properties mentioned above into a hierarchy of successively more restrictive axioms. While it is not true that a regular space is necessarily Hausdorff, or that a normal space is necessarily regular, these implications do hold with the, seemingly innocuous, additional assumption that the space is T_1. For this reason, one refers to T_2, T_3 and T_4 spaces, meaning Haudorff, T_1 regular, and T_1 normal spaces respectively.

Our apparent stumbling block was that the spaces we were led to consider were not even T_1. However, they satisfied a property which was extremely attractive in its own right and appeared to be a very natural relaxation of this: every singleton is closed *or open*! We call such spaces $\mathbf{S_1}$ spaces.[3] Another simple relaxation of a standard separation axiom is the concept of a weakly Hausdorff space. We say that a space is weakly Hausdorff if any two points can be separated by open sets whose intersection is not necessarily empty, but is only finite. This concept led us to a further weakening of the Hausdorff requirement, the assumption that a topological space be feebly Hausdorff. Both these notions lead, by straightforward analogy with the standard counterparts, to "weak" and "feeble" versions of regularity and normality and in either case the S_1 assumption takes on the rôle of the T_1 assumption, giving us a corresponding "weak" and "feeble" hierarchy.

Thus, with respect to the issue of separation axioms, general topology did not immediately furnish the appropriate tools, but it did serve as a good guide, providing the concepts of Hausdorff, regular and normal spaces, which needed fine-tuning to our spaces.

Apart from the above issue of separation axioms, our "topologically biased" choice proved to be distinctly advantageous in several ways. Indeed, in retrospect we feel that the gamble paid off handsomely, perhaps more than we dared believe at the outset. Our faith in the inherent value of standard point-set topology and an abstract axiomatic approach was rewarded not only by the results themselves but also by the surprising way in which standard concepts from mainstream topology came to bear upon questions of a combinatorial inspiration.

Our main results include:

- the fact that finite graph-theoretic (classical) paths are very naturally a specific case of a category of topological objects, namely paths, which include the unit arc, the real line, and connected orderable spaces of arbitrary cardinality (Chapter 2);
- a theory of cycle spaces and bond spaces for compact weakly Hausdorff spaces, including the facts that these spaces are orthogonal and generated by the fundamental cycles and bonds, respectively;

[3] The letter T in the designation "T_i" derives from the German *Trennungsaxiome*, "separation axioms". The letter S not only stands for "separation" in English, and conveniently comes right before T in the alphabet, but is also the first letter in the German *schwach*, meaning "weak" or "feeble".

- a generalization to ferns of the fact that locally connected dendritic spaces are arcwise connected.

More specific illustrations of the rôle of topological concepts in questions of graph-theoretic inspiration are the following facts:

- that feeble regularity, conceived abstractly as "the next step up" from the property of being feebly Hausdorff, in analogy with standard regularity, excludes the pathologies seen in Subsection 3.1.1, and in particular the Diestel-Kühn obstruction (Example 3.1.4);
- that compactness and weak normality are crucial in the arguments guaranteeing the existence of
  - minimal spanning sets (for compact spaces, Corollaries 4.2.14, 4.2.25 and 4.3.22 for weakly Hausdorff edge spaces, feebly Hausdorff topologized graphs and feebly Hausdorff edge spaces respectively, all of which derive from Lemma 4.2.13 and hence Theorem 4.2.13, and, for weakly normal, possibly non-compact, edge spaces, Proposition 4.2.15),
  - edgepaths (Proposition 5.1.17, which depends on 4.3.21 and therefore again on 4.2.13), and
  - fundamental cyclesets (Corollary 5.1.19);
- that the Diestel-Kühn end-quotient of a classical graph G turns out to be weakly Hausdorff and, when G is 2-connected, compact[4] (Theorem 4.4.12);
- that graph-theoretic trees are a specific subclass of a category of topological objects, namely ferns, which includes the dendrites of continuum theory and the locally connected dendritic spaces (Chapter 6).

A couple of other standard topological concepts which unexpectedly turn up for us, albeit only marginally (so far!), are the notions of dimension and upper semicontinuous decompositions.

Other factors which emerged naturally as a result of the choice to pursue a "topologically guided" road, and which make the interplay between topology and combinatorics attractive, are the following:

- that all paths are topologized graphs (Theorem 2.2.7);

[4]But *only* if one models edges as points, as opposed to arcs.

- that all ferns are S_1 spaces, and hence may be viewed as topologized hypergraphs (Corollaries 6.2.9 and 6.4.21);
- that feeble regularity allows one to reconstruct the topology on the whole space from the relative topology on the vertex set, and even relate the dimensions of the two spaces (Proposition 3.4.7), and that this topology is the finest one compatible with the relative topology on the vertex set (Theorem 3.4.3 and Proposition 3.3.1);
- that feeble regularity is a global topological assumption which can be broken down into two "components"—a "combinatorial component" (quasiregularity) in the form of a convergence condition on the edges, and the inherited topological "component" (feeble regularity) on the vertex set, which reduces to usual regularity in the case of topologized graphs (Corollary 3.4.13)—and that neither of these "components" can be discarded, as shown by Examples 3.1.2 and 3.1.7;
- that topological quotients induce an algebraic isomorphism of a "small" cycle space onto a cycle space with a richer set of cycles (Theorem 5.4.3);
- that the decomposition of a topologized hypergraph which identifies parallel edges is upper semicontinuous[4] (Proposition 4.2.21).

Finally, a couple of unusual scenarios, which can probably be regarded as "curiosities" interesting from a purely topological perspective, arise from our approach: the context of decompositions with non-degenerate *open* parts and a number of strong properties, such as upper semicontinuity, or the fact that *any* choice of a system of distinct representatives of the parts is homeomorphic to the induced quotient, and hence is a retract of the whole space (Theorem 4.2.18).

To the combinatorialists, we wish to present a perspective which:

- recasts *all* graph-theoretic concepts in *purely* topological language, often unifying superficially different central concepts from graph theory and topology;
- opens up a variety of interesting questions with a topological flavour which may or may not lead to the solution of purely combinatorial problems, but are nevertheless intriguing for the combinatorialist;
- and provides a way of generalizing well-known theorems from finite graphs to infinite objects, which are also graphs in the strict sense of the term, but for which the meaning of the term "connected" is dictated by some extra structure, namely a preassigned topology, rather than just the combinatorial information.

To the topologists, we wish to exhibit a setting in which the realm between T_0 and T_1 is not simply an exercise in abstraction, but leads to a "parallel" universe in which most of the usual properties can still be fruitfully put to use.

1.1.2 Classical Topologies

In this subsection we address Question 1.1.1, posed at the beginning of Section 1.1. Given a vertex v and an edge e of a graph G, let us denote by $\mathcal{N}_v$ the set consisting of v together with all edges incident with v, and by $G[e]$ the subset consisting of e and the incident vertices. All other technical terms in the following statement are used in their usual graph-theoretic or topological meaning; for precise definitions, we refer the reader to Definitions 1.1.12, 1.1.14 and 1.1.16 or to Appendix B.

1.1.2 ***Proposition:*** *Suppose that G is a complete simple graph, and for every subgraph H of G there exists a topology $\mathcal{T}_H$ with ground set $V_H \cup E_H$, such that:*

(I) *H is graph-theoretically connected if and only if $V_H \cup E_H$ is connected with respect to $\mathcal{T}_H$;*

(II) *the relative topology inherited by $V_H \cup E_H$ from $\mathcal{T}_G$ coincides with $\mathcal{T}_H$;*

(III) *for every two subgraphs H_1, H_2, if H_1 and H_2 are (graph-theoretically) isomorphic then they are also (topologically) homeomorphic.*

Then, with respect to $\mathcal{T}_G$, either:

(A) [**i**] *$\forall v \in V_G$, the singleton $\{v\}$ is closed, $\mathcal{N}_v$ is open, and coincides with the intersection of all the open sets containing v,*

[**ii**] *$\forall e \in E_G$, the singleton $\{e\}$ is open, and $G[e] = \mathsf{Cl}(e)$, and*

[**iii**] *a subset U is open if and only if $\forall v \in V_G, \quad v \in U \;\Rightarrow\; \mathcal{N}_v \subseteq U$;*

or else,

(B) [**i**] *$\forall v \in V_G$, the singleton $\{v\}$ is open and $\mathcal{N}_v = \mathsf{Cl}(v)$,*

[**ii**] *$\forall e \in E_G$, the singleton $\{e\}$ is closed and $G[e]$ is open, and coincides with the intersection of all the open sets containing e, and*

[**iii**] *a subset U is open if and only if $\forall e \in E_G, \quad e \in U \;\Rightarrow\; G[e] \subseteq U$.*

The proof of the above proposition is rather technical, so we relegate it to the end of this section (page 16). A few comments are in order, however.

First, we point out that in Proposition 1.1.2, the graph G is *not* assumed to be finite. Secondly, we recall that no connected topological space contains a proper clopen subset; otherwise it would give, together with its complement, a separation of the whole space. In particular, this implies that, if G contains at least two vertices, then (A) and (B) above cannot both occur. Next, we justify the hypotheses.

1.1.3 Example: Consider the graph consisting of a single vertex v and a single loop e, equipped with the indiscrete topology $\{\emptyset, \{e, v\}\}$. Both subgraphs are trivially graph-theoretically connected, as are the corresponding topological spaces. However, the conclusion of Proposition 1.1.2 fails; for example, no singleton is closed or open.

For this reason we exclude loops in the hypothesis of Proposition 1.1.2. Note, however, that the topology in the above example is not T_0. The following example illustrates why multiple edges are also an issue.

1.1.4 Example: Consider a graph G with 2 vertices, such that all edges are incident with both. Suppose E is equipped with an arbitrary topology. We define a topology on G by declaring a set U to be open if and only if $x \in V_G \cap U \;\Rightarrow E \subseteq U$, and $U \cap E$ is open in E. It is easily verified that this indeed defines a topology, that any subgraph containing an edge is connected, and that the subgraph consisting of the two vertices is not connected. However, the edges are clearly not necessarily open or closed.

At first sight it may appear (it did to us) that condition (III) in the hypothesis of Proposition 1.1.2 is redundant.

1.1.5 Example: Let G be the complete simple graph on three vertices. Let $1, 2, 3$ denote the vertices, and a, b, c the edges, labelled so that $f_G(a) = \{1, 3\}$, $f_G(b) = \{1, 2\}$ and $f_G(c) = \{2, 3\}$. We consider the topology determined by the following base: $\{\{a\}, \{b\}, \{1, a, b\}, \{2, b\}, \{3, a\}, \{2, 3, a, b, c\}\}$. It is routinely verified that assumptions (I) and (II) are satisfied. However, the vertices 2 and 3 are neither open nor closed. Note that they *are* so in $G[e]$ whenever they are incident with the edge e. Reassuringly, (III) fails, since $G[a]$ and $G[c]$ are not homeomorphic: $G[a]$ contains one open, but not closed, point, namely a, and two closed, but not open, points, while c is the only closed, (but not open) point in $G[c]$, the other two being open but not closed.

With regard to the assumption in Proposition 1.1.2 that G be complete, we remark that this can probably be relaxed. Since we are interested in a family of topologies for *every* graph, we do not pursue this further.

Now we point out two properties that are implied by either of the conclusions (A), (B) of Proposition 1.1.2. Both are unusual in that one is usually trivially satisfied and the other usually not satisfied. It turns out that both turn up in the vast literature on topology, although we became aware of this only very late in the writing of this thesis.

The first is that *every singleton is open or closed*, that is, the topology is S_1. This property will recur for us in other contexts, though mostly as a consequence rather than as assumption. It appears that it has been abstractly investigated as a separation axiom in between T_0 and T_1, by Levine in 1970 [40], McSherry in 1974 [42] and Jha in 1977 [34]. We are not aware of any applications of this assumption.

The second property is that *the arbitrary intersection of open sets is open.* This follows from [iii] of (A) or (B). Given a collection $\mathcal{T}$ of subsets of a set X, let us refer to the collection of complements of elements of $\mathcal{T}$ as the dual set system of $\mathcal{T}$. The following proposition is easy to verify.

1.1.6 ***Proposition:*** *Given a topological space, the following are equivalent:*

- *the arbitrary intersection of open sets is open;*
- *every point has a smallest neighbourhood;*
- *the arbitrary union of closed sets is closed;*
- *the dual set system of the topology of H is a topology.* ✠

If the above properties are satisfied for a given topology $\mathcal{T}$ on a ground space X, then X is connected with respect to $\mathcal{T}$ if and only if it is connected with respect to the dual set system of $\mathcal{T}$.

It has recently come to our attention that specialists in low separation axioms call spaces satisfying the above properties Alexandroff discrete—see [31]. In contemporary general topology, the term discrete usually has a different meaning, which renders the topology trivial: all subsets are open. In 1937, Alexandroff [2] investigated the above property in connection with simplicial complexes, much as we are doing here, our "complexes" being one-dimensional. This connection was already made two years earlier, in the book Alexandroff co-authored with Hopf [3]. This was the time when much of topology was still becoming standardized; in fact, this book was probably influential in establishing the "T_i" designation, although the origin seems to lie in an unpublished manuscript of Kolmogoroff.

Our spaces will *not* necessarily be Alexandroff discrete. However, the idea of focusing on the intersection of all the open sets containing a given point, or a given set, will be crucial for us. Note that any *finite* topological space is necessarily Alexandroff discrete. However, an Alexandroff discrete space is not necessarily S_1. This is demonstrated by the following example, which also illustrates the connection to simplicial complexes.

1.1.7 Example: Given any set S, consider the power set of S, equipped with a topology in which a set C is declared to be closed if and only if it is closed under taking subsets, that is, if and only if $A \in C, B \subseteq A \Rightarrow B \in C$. It is easily verified that the only closed points (i.e., subsets of S) are the singletons, and the only open point is S, so if S contains at least three points, the resulting topology is not S_1. Moreover, if S is finite and contains at least two points, then this topological space is homeomorphic to an appropriate quotient of a geometric simplex. More precisely, if T is the convex hull of $|S|$ affinely independent points in $\mathbb{R}^{|S|-1}$, equipped with the topology inherited from the Euclidean one, then one can obtain the above topology by identifying the relative interior of every face of the polyhedron T to a single point.

1.1.8 Remark: Given a graph G and a collection $\mathcal{T}$ of subsets of $V_G \cup E_G$, we have that a $\mathcal{T}$ satisfies (A) in the conclusion of Proposition 1.1.2 if and only if its dual set system satisfies (B).

Now we use the conclusion of Proposition 1.1.2 to define a topology.

1.1.9 Definition: Given a hypergraph H, the classical topology on $V_H \cup E_H$ is the collection of sets U such that, if U contains a vertex v, then it also contains all hyperedges incident with v.

The above definition determines, in particular, a topology $\mathcal{T}_G$ for every (not necessarily simple) graph G. The next proposition essentially says that classical topologies provide one of two answers to Question 1.1.1. We omit the easy proof.

1.1.10 ***Proposition:*** *A graph is graph-theoretically connected if and only if it is connected with respect to the classical topology. The relative topology inherited from the classical topology by a subgraph is its classical topology. A bijection between the vertex sets of two simple graphs is a graph-theoretic isomorphism if and only if it extends to a homemorphism between the two ground sets.* ✠

Since every simple graph occurs as a subgraph of a complete simple graph, from Proposition 1.1.2 we have the following consequence.

Answer (to Question 1.1.1): *There are precisely two operators which assign to every simple graph G a topology $\mathcal{T}_G$ such that the requirements of Proposition 1.1.2 are satisfied. One operator assigns to every graph its the classical topology, the other the dual set system of the classical topology.* ✠

Henceforth, whenever we wish to retain the usual graph-theoretic meaning of "connected", we shall always deal with classical topologies. Their dual set systems will not come up again for us, except briefly in Section 1.2 to justify this seemingly arbitrary choice.

Given that the classical topology is not T_1 (as long as there is at least one edge), it is perhaps surprising that in the following easy proposition, the terms "boundary", "dense", "locally connected", "Lebesgue dimension", and the various forms of compactness can keep their usual topological meaning.

1.1.11 ***Proposition:*** *Let G be a graph and $\mathcal{G}$ the topological space obtained when $V_G \cup E_G$ is equipped with the classical topology. Then:*

- *every vertex is closed, and the set of vertices is discrete;*
- *every edge is open, and the set of edges is discrete;*
- *the set of edges is dense in $\mathcal{G}$ if and only if no vertex of G has degree zero;*
- *the vertices incident with an edge e are precisely the boundary points of $\{e\}$;*
- *$\mathcal{G}$ is locally connected, locally compact and metacompact;*
- *G is finite if and only if $\mathcal{G}$ is compact;*
- *G is locally finite if and only if $\mathcal{G}$ is rim-compact;*
- *$\mathcal{G}$ has Lebesgue dimension 1 if and only if G has at least one edge, and Lebesgue dimension 0 otherwise.* ✠

Before going on to the proof of Proposition 1.1.2, we make more precise some of the terms which we have been using, and introduce some more.

1.1.12 Definition: A hypergraph is an ordered triple (V, E, f) where V is a set of vertices, E is a set of hyperedges and f is an incidence function from E to the set of *non-empty* subsets of V. We also require that V and E be disjoint. Given a hypergraph H we denote by V_H, E_H and f_H respectively the sets of vertices of H, the set of edges of H and the incidence function of H. If $v \in f_H(e)$, then v is an endvertex of e. In this case we also say that v and e are incident. If $|f_H(e)| \leq 2$, then the hyperedge e is an edge. If all hyperedges are edges, then H is a graph. If $|f_H(e)| = 2$, then e is a proper (hyper)edge. If $|f_H(e)| = 1$, then e is a loop. If no edge is a loop and f_H is injective, the hypergraph H is simple.

A simple graph G is complete if f_G induces a bijection between the edges and the unordered pairs of vertices. Given two simple graphs G and H, a bijection $\phi : V_G \to V_H$ is an isomorphism if, for any two vertices $u, v \in V_G$, we have that $\{u, v\}$ are adjacent if and only if $\phi(u)$ and $\phi(v)$ are adjacent in H. Any two simple graphs are isomorphic if there exists an isomorphism between their vertex sets.

The assumption that hyperedges are not "empty", that is, that their image under the incidence function is not the empty set, is in line with that of Berge; see [7].

1.1.13 Technical Note: Isomorphism is clearly an equivalence relation on the class of simple graphs. Although two simple graphs can be isomorphic and yet be formally distinct, we shall not always distinguish an equivalence class from a representative. This will apply to other spaces as well, especially topological spaces, topologized graphs and edge spaces. Thus, for example, two topological spaces may be considered to be the "same" space if they are homeomorphic.

1.1.14 Definition: Let H be a hypergraph. If $e \in E_H$ and $v \in f_H(e)$, then v and e are incident; in this case, we also say that v is an endvertex of e. If two vertices are incident with a common edge, they are adjacent; the same adjective applies to hyperedges incident with a common vertex. If all hyperedges are incident with only finitely many vertex, the hypergraph is finitely incident. If no two vertices are incident with infinitely many edges, the hypergraph is finitely adjacent, and uniquely adjacent if any two vertices are incident with at most *one* common hyperedge. A subset of $V_H \cup E_H$ is spanning if it contains all the vertices.

A subhypergraph of H is a hypergraph H' such that $V_{H'} \subseteq V_H$, $E_{H'} \subseteq E_H$ and, $\forall e \in E_H$, we have that $f_{H'}(e) = f_H(e)$. A subhypergraph is a subgraph if it is a graph. Given a subhypergraph S of a hypergraph H, we denote by $\delta(S)$ the set of edges incident with a vertex of S and a vertex not in S. We also abbreviate $\delta(S) \cup S$ to $\mathcal{N}_S$. If G is a *simple graph*, the degree of a vertex $v \in V_G$ is the cardinality of $\delta(v)$.

Given a set $F \subseteq E_H$ of hyperedges, we denote by $H[F]$ the subhypergraph whose edges are precisely those in F, and whose vertices are those incident with some hyperedge in F.

Note that, in passing to a sub(hyper)graph, vertices may "lose" incident edges, but edges may not "lose" incident vertices.

1.1.15 Technical Note: Given a hypergraph H, there is clearly a one-to-one correspondence between the subhypergraphs of H and the subsets of $V_H \cup E_H$ which contain all the endvertices of any hyperedge they contain. We shall frequently deal with a given subset and the formally distinct corresponding subhypergraph interchangeably. More generally, we shall not always distinguish between a ground set and the space resulting when it is equipped with a given structure, be it that of a topological space, hypergraph, "topologized hypergraph", or "hyperedge space".

Notation: Given a set A of vertices of a hypergraph G, the set $(V_G \setminus A) \cup E_G$ need not be a subhypergraph. Quite frequently in the literature on graph theory (when G is graph), even if this is the case the notation $G \setminus A$ denotes a subgraph, namely the one whose vertices are those of G except for those in A, and whose edges are all those of G except those incident with some vertex in A. In our context, however, the set $V_G \setminus A \cup E_G$ still carries an interesting structure, the relative topology. So *we shall depart from the usual* *graph-theoretic notation* and write $G \setminus A$ for $(V_G \setminus A) \cup E_G$.

The following definition of a graph-theoretic path is perhaps slightly unorthodox. Our reason for adopting it is twofold. Firstly, we feel that, if one wishes to characterize at once finite and infinite paths without a notion of connectedness already available (be it graph-theoretic, that is in terms of finite paths, or topological, that is in terms of the classical topology), then the following definition is not more involved than it needs to be. Secondly, we wish to emphasize the underlying order-theoretic character of paths, which will come up for us in the next Chapter. As defined in Appendix A, a subset S of a totally ordered set is convex if it contains any point with an upper and a lower bound in S.

1.1.16 Definition: A simple graph P is a graph-theoretic path if there is a bijection ϕ between V_P and a convex subset S of the integers, equipped with its usual total order, such that, $\forall a, b \in V_P$, we have that a and b are adjacent if and only if $|\phi(a) - \phi(b)| = 1$. If S has an extremum x, then $\phi^{-1}(x)$ is a terminal vertex of P. Note that a vertex is terminal or not independently of the choice of S or ϕ. If S is bounded from above and below, then P is bounded (note that if S can be chosen to be bounded above and below, then any possible choice for S will also be bounded above and below). In this case, if m and M are respectively the minimum and maximum of S and $a = \phi^{-1}(m), b = \phi^{-1}(M)$, then P is an graph-theoretic ab-path or more simply an ab-path when there is no danger of ambiguity. Note that, composing ϕ with the function $i \mapsto m + M - i$ gives a second bijection with S satisfying the adjacency requirement but inverting the rôles of a and b, so that an ab-path is also a ba-path. If instead S is bounded from above but not from below, or from below but not from above, then P is called a one-way infinite (graph-theoretic) path; if S has no extrema at all, then P is a two-way infinite (graph-theoretic) path.

A graph G is graph-theoretically connected if for any two vertices $a, b \in V_G$, G contains an ab-path as a subgraph. The graph-theoretic connected components, or more simply the graph-theoretic components, of a graph G are the maximal graph-theoretically connected subgraphs of G. Note that the binary relation on V_G defined by $a \sim b$ if and only if there exists an ab-path in G is an equivalence relation, hence the vertex sets of the graph-theoretic components of G are the equivalence classes of this relation. Hence the graph-theoretic components are the subgraphs of G induced by the equivalence classes.

We now recall a few topological facts which we shall use frequently throughout the thesis, in particular for the proof of Proposition 1.1.2.

1.1.17 Fact: Suppose X is a topological space, B a subspace of X, and $A \subseteq B$. If A is closed (open) in B and B is closed (respectively open) in X, then so is A. □

The following definition is perhaps not quite standard; hence we present it here as opposed to Appendix B.

1.1.18 Definition: Given a topological space X, two subsets H, K are mutually separated in X if $\mathsf{Cl}(H) \cap K = H \cap \mathsf{Cl}(K) = \emptyset$. A separation of a topological space is an unordered pair of complementary non-empty open subsets, that is, a non-trivial bipartition into clopen subsets.

1.1.19 Fact: Given a subspace E of a topological space X, a non-trivial bipartition $\{H, K\}$ of E is a separation of E (with respect to the relative topology of E) if and only if H, K are mutually separated in X.

Reference: See [61] General Topology, Willard, Theorem 26.5. □

1.1.20 ***Corollary:*** *If H and K are mutually separated subsets of a topological space X and F is a connected subset of $H \cup K$, then $F \subseteq H$ or $F \subseteq K$.*

Reference: See [61] Corollary 26.6. □

1.1.21 ***Corollary:*** *If $\{A, B\}$ is a separation of a topological space X and K is a component of X, then $K \subseteq A$ or $K \subseteq B$.* □

1.1.22 Fact: Suppose X is a topological space with precisely 2 components. Then $\{A, B\}$ is a separation of X if and only if A, B are the components of X.

Proof: If A, B are the components of X, then $\{A, B\}$ is a non-trivial bipartition of X, and by (1.1.23) A and B are both closed. Hence $\{A, B\}$ is a separation of X. Conversely, if $\{A, B\}$ is a separation of X, let K_1, K_2 be the two components of X. By (1.1.21), each one of K_1, K_2 must be contained in one of A, B. But $K_1 \cup K_2 = X$ and both A and B are non-empty, so we must have $A = K_1, B = K_2$ or $A = K_2, B = K_1$. □

1.1.23 Fact: If X is a topological space and K a component of X, then K is a closed subset of X. If X consists of finitely many components, then every component is clopen and the separations of X are in one-to-one correspondence with the non-trivial bipartitions of the components.

Reference: For the first statement, see [61], Theorem 26.12. The remaining part follows easily from Corollary 1.1.21. □

We are finally in a position to give the proof of Proposition 1.1.2.

Proof of Proposition 1.1.2: Let v be an arbitrary vertex. Since v is a component of the subgraph with all the vertices but no edges, $\mathsf{Cl}(v)$ is disjoint from all the other vertices. For every edge e not incident with v, the singleton $\{v\}$ is a component of the subgraph $G[e] \cup \{v\}$, so $\mathsf{Cl}(v)$ is disjoint from $G[e]$, in particular from e. So $\mathsf{Cl}(v) \subseteq \mathcal{N}_v$.

Now let e be an arbitrary edge, with endvertices v_1, v_2. Then $G[e] = \{v_1, e, v_2\}$ is connected, but $\{v_1, v_2\}$ is not. Since $\{\{v_1\}, \{v_2\}\}$ is a separation of $\{v_1, v_2\}$, which inherits its topology from that of $G[e]$, there exist open sets U_i, $i \in \{1, 2\}$ and closed sets C_i $i \in \{1, 2\}$ in $G[e]$, such that $U_i \cap \{v_1, v_2\} = \{v_i\}$ and $C_i \cap \{v_1, v_2\} = \{v_i\}$. So for $i \in \{1, 2\}$, we have that $\{v_i\} \subseteq C_i, U_i \subseteq \{v_i, e\}$. However, since $G[e]$ is connected, no proper subset is clopen, so for $i \in \{1, 2\}$, $C_i \neq U_i$. Moreover, if $C_i = \{v_i\}$, then $\{e, v_{3-i}\}$, its complement in $G[e]$, is open, and therefore not closed, in $G[e]$, and hence can not coincide with C_{3-i}.

So we have that, with respect to $\mathcal{T}_{G[e]}$, either:

1 for $i \in \{1, 2\}$ $C_i = \{v_i\}$, $U_i = \{v_i, e\}$ and $v_i \in \mathsf{Cl}(e)$; or else

2 for $i \in \{1, 2\}$ $C_i = \{v_i, e\}$, $U_i = \{v_i\}$ and $e \in \mathsf{Cl}(v_i)$.

Note that, if (1) occurs, then $\{e_i\} = U_1 \cap U_2$ is open (but not closed) in $G[e]$, and $\{v_1\}$, $\{v_2\}$ are closed (but not open) in $G[e]$, while if (2) occurs, $\{e\}$ is closed and $\{v_1\}, \{v_2\}$ are open. In particular, the topologies determined by (1) and (2) on $G[e]$ are not homeomorphic.

Now for any two edges e_1, e_2, clearly $G[e_1]$ and $G[e_2]$ are graph-theoretically isomorphic, and therefore should be homeomorphic when equipped with $\mathcal{T}_{G[e_1]}$ and $\mathcal{T}_{G[e_2]}$ respectively. Hence either (1) occurs for all $e \in E_G$, or else (2) does.

Suppose first that (1) occurs. Then for all $v \in V_G$, we have that $\mathsf{Cl}(v)$, which we already know to be contained in $\mathcal{N}_v$, is disjoint from all edges incident with v, and therefore $\{v\}$ is closed with respect to $\mathcal{T}_G$. Moreover, for any fixed v, and any incident edge e, we have that $v \in \mathsf{Cl}(e)$, meaning that every open set containing v contains e. Thus $\mathcal{N}_v$ is contained in the intersection of all open sets containing V.

Now the subgraph $G \backslash \delta(v)$ consists of precisely two components, namely $\{v\}$ and $G \backslash \mathcal{N}_v$. So by Fact 1.1.23 $\{\{v\}, G \setminus \mathcal{N}_v\}$ is a separation of H. Therefore there exists a subset U of $V_G \cup E_G$, open with respect to $\mathcal{G}$, whose intersection with H is precisely $\{v\}$. Hence $U \subseteq \mathcal{N}_v$ and since the open set U contains v we also have $\mathcal{N}_v \subseteq U$. We conclude that $\mathcal{N}_v$ is open and is precisely the intersection of all the open sets containing v. This concludes the proof of (A) [i].

Moreover for any edge e with endvertices u, w, we have that $\{e\} = \mathcal{N}_u \cap \mathcal{N}_w$ and therefore $\{e\}$ is open. Furthermore, the complement of $G[e]$ is the union of $\mathcal{N}_v$ over all $v \in V_G$ not incident with e. This union is open and therefore $G[e]$ is closed. Since u, w belong to $\mathsf{Cl}(e)$, this set is precisely $G[e]$. This concludes the proof of (A)[ii].

Finally, recall that a set is open if and only if it is a neighbourhood of every point it contains. Since the edges are open, this requirement is vacuous when the point is an edge, and since $\mathcal{N}_v$ is the smallest open set containing a given vertex v, we conclude that a set U is open if and only if, for every vertex $v \in U$, the set $\mathcal{N}_v$ is also in U. Thus, in this case (A) holds.

Suppose instead that (2) holds, and let e be an arbitrary edge. Note that $G \setminus \delta(G[e])$ consists precisely of two components, namely $G[e]$ and $G \setminus \mathcal{N}_{G[e]}$. Let us denote the latter by K. By Fact 1.1.23 we have that $\mathsf{Cl}(K)$ is disjoint from $G[e]$. We claim that $G[e]$ is also disjoint from $\mathsf{Cl}(\delta(G[e]))$. Note that $\forall e \in \delta(G[e])$, there exists an endvertex x_e of e in K, and (2) says that $e \in \mathsf{Cl}(x_e)$, that is, every open set containing e contains x_e. So if $y \in G[e] \cap \mathsf{Cl}(\delta(G[e]))$, meaning that any open set containing y contains some edge e in $\delta(\{u, w\})$, then the same set also contains x_e, implying that $y \in \mathsf{Cl}(K)$, a contradiction.

Hence $G \setminus G[e] = \mathsf{Cl}(\delta(G[e])) \cup \mathsf{Cl}(K) = \mathsf{Cl}(G \setminus G[e])$ is closed in G, that is, $G[e]$ is open, where $e \in E_G$ is arbitrary. In addition, every open set containing e contains both incident edges, so $G[e]$ is the intersection of all the open sets containing e. Moreover, for any fixed vertex v and any edge e incident with v, by (2) v is open in $G[e]$ and and therefore, by Fact 1.1.17, in G. Furthermore, $\mathcal{N}_v$ is the complement of the union of $G[e]$ over all $e \in E_G$ not incident with v; the union is open and therefore $\mathcal{N}_v$ is closed.

Also, $e \in \mathsf{Cl}(v)$ means that every open set containing e also contains v; equivalently, every closed set containing v also contains e. Hence $\mathcal{N}_v$ is contained in, and, being closed, coincides with, $\mathsf{Cl}(v)$. Finally, any edge e with endvertices u, w is the unique point in the intersection $\mathcal{N}_u \cap \mathcal{N}_w$, and is therefore closed. This concludes the proof of (B)[i,ii]. The assertion of (B)[iii] follows from the fact that for every point x, either x is open or else it is an edge e with the smallest neighbourhood $G[e]$. Hence in this case (B) holds. ✠

We conclude the treatment of classical topologies by relating them to the usual topological model for graphs.

1.1.24 Definition: Let G be a graph. For any proper edge $e \in E_G$, for the purposes of this definition we arbitrarily designate one of the two vertices incident with e to be the "0-vertex" and the other one to be the "1-vertex"; for a loop, the only endvertex is simultaneously the "0-" and the "1-vertex". For a vertex v incident with e, we denote that v is the 0/1-vertex of e by $e \sim_0 v$ / $e \sim_1 v$ respectively. Suppose that, $\forall e \in E_G$, there exists a set I_e and a bijection ϕ_e from I_e to $(0, 1)$, such that $e \neq e' \Rightarrow I_e \cap I_{e'} = \emptyset$. Also, let

$\{p_v\}_{v\in V_G}$ be an arbitrary set in bijection with V_G. Then we set $X := \left(\bigcup_{e\in E_G} I_e\right) \cup \{p_v\}_{v\in V_G}$ and define a topology on X by the following system $\{\mathcal{N}(x)\}_{x\in X}$ of basic neighbourhoods:

- if $x \in I_e$ for some $e \in E_G$, then $\mathcal{N}(x)$ consists of all sets of the form $\phi^{-1}((a,b))$ with $a < \phi(x) < b$;
- if $x = p_v$ for some $v \in V_G$, then $\mathcal{N}(x)$ consists of all sets of the form

$$\{p_v\} \cup \bigcup_{\substack{e\,:\\ e\sim_0 v}} \phi_e^{-1}\left((0,\varepsilon_e)\right) \cup \bigcup_{\substack{e\,:\\ e\sim_1 v}} \phi_e^{-1}\left((1-\varepsilon_e,1)\right)$$

  where the unions range over all edges e incident with v, and for any such edge ε_e is an arbitrarily small number in $(0,1)$.

Then the resulting topological space is the cell-complex topological space associated with G.

Note that, up to homeomorphism, the above construction does not depend on the nature of the sets I_e and the set $\{p_v\}_{v\in V_G}$, nor on the designation of 0/1-vertices.

1.1.25 ***Proposition:*** *Let G be a graph, and X the cell-complex topological space constructed as in Definition 1.1.24 by means of the sets I_e, with e in E_G. Then the topological space obtained by identifying each set I_e to a point is homeomorphic to the set $V_G \cup E_G$ equipped with the classical topology.* ✠

We wish to emphasize that the above construction is not the way we were led to consider the classical topology. In later chapters we shall use quotients to explore further the issue of how to topologically model edges.

1.1.3 Some graph-theoretic concepts in the language of general topology

The classical topology enables us to characterize any combinatorial concept defined for simple graphs in purely topological language. In this section, we assume some familiarity with graph theory. The reader is referred to [14] for an introduction to any unfamiliar concepts.

In the following easy propositions, G is a graph, and all graphs are equipped with the classical topology.

1.1.26 ***Proposition:*** *Given two vertices a, b, a subset $P \subseteq V_G \cup E_G$ is a graph-theoretic ab-path if and only if it is a minimal connected set containing a and b.* ✠

1.1.27 ***Proposition:*** *Suppose G is connected. Then the following are equivalent:*

- *G is a graph-theoretic path;*
- *among any three points, one separates the other two;*
- *there exists a total order $\preceq$ on $V_G \cup E_G$ such that $x \prec y \prec z$ if and only if y separates x and z.* ✠

1.1.28 ***Proposition:*** *Let C be a connected subset of $V_G \cup E_G$. Then C is a graph-theoretic cycle if and only if no point disconnects C but any two do.* ✠

1.1.29 ***Proposition:*** *Let T be a subset of $V_G \cup E_G$. Then T is a spanning tree of G if and only if it is a minimal connected set containing the vertex set.* ✠

1.1.30 ***Proposition:*** *Let H be a graph. Then H is a graph-theoretic minor of G if and only if it is the image of a subgraph of G under a monotone map.* ✠

The following statement is equivalent to the graph minors theorem of Seymour and Robertson.

1.1.31 **Theorem** (Graph Minors, Robertson & Seymour):

Let $\mathcal{F}$ be any infinite collection of compact connected topological spaces such that, for all members of $\mathcal{F}$:

- *all singletons are either open or closed, and their boundaries consist of at most two points;*
- *the set of closed sets is itself a topology.*

Then there exist A, B in $\mathcal{F}$, a subspace Y of A and a monotone map from Y onto B.

Since the above statement sounds more topological than combinatorial, the following question arises.

1.1.32 Question: Is there a "topological" proof of the Graph Minors Theorem? Can the proof be shortened by recourse to topological methods and concepts? Can the hypotheses of the theorem be relaxed to less stringent, but topologically natural, assumptions? Can the theorem be generalized to a class of possibly infinite topological spaces, in particular one which includes some non-trivial class of infinite classical graphs?

1.2 Relaxing the compatibility requirement

The passage from the usual notion of graph-theoretic connectedness to one based on classical topologies (or their dual set systems) amounts essentially to a change of language. Although the different perspective by itself may provide useful insight, we are aiming for a less rigid interpretation of "connectedness". In this section we make this more precise, and also attempt to justify the choice of the classical topology over its dual set system.

We consider a class of topological spaces which retain some of the properties of the classical topology of a simple graph.

1.2.1 Definition: A topologized graph is a topological space X such that

- every singleton is open or closed;
- $\forall x \in X,\ |\partial(x)| \leq 2$.

Note that, in any S_1 space, the set E of points which are not closed is open, and therefore its complement, V, is closed. Thus the closure of any subset A of E, in particular any singleton, is of the form $E \cup B$ for some $B \subseteq V$, and $\partial(A) = B$.

Of course, a "topologized graph" has an underlying combinatorial structure, as well as a topological one.

1.2.2 Definition: A topological space $S = (X, \mathcal{T})$ is compatible with a hypergraph H if $X = V_H \cup E_H$ and, for all $e \in E_H$, we have that $\{e\}$ is open and $\partial(e) = f_H(e)$; S is strictly compatible if every vertex is also a closed point. We also say that a *topology* $\mathcal{T}$ on $V_H \cup E_H$ is (strictly) compatible if $(V_H \cup E_H, \mathcal{T})$ is a (strictly) compatible topological space.

An arbitrary S_1 topological space always induces a unique strictly compatible hypergraph H, obtained simply by taking the closed points for vertices and the open (but not closed) points for edges, setting $f_H(e) = \partial(e)$ for every edge e. Thus a "topologized graph" is precisely what its name says it is: a graph equipped with a topology—subject to a compatibility requirement, retained from the classical topology, that allows the combinatorial structure to be recovered from the topology. We shall sometimes refer to S_1 spaces as topologized hypergraphs.

Associating a combinatorial structure to an S_1 space allows us to extend notions defined on hypergraphs to topologized hypergraphs: we will talk, for example, of a "vertex" of an S_1 space, of the "degree" of a vertex of a "simple" topologized graph. Also, a topologized sub(hyper)graph of a topologized hypergraph X is a subspace of X such that the induced hypergraph is a sub(hyper)graph of the hypergraph induced by X.

A topologized hypergraph is classical if its topology coincides with the classical topology of the induced hypergraph. We shall also abbreviate the terms "classical topologized (hyper)graph" to classical (hyper)graph.

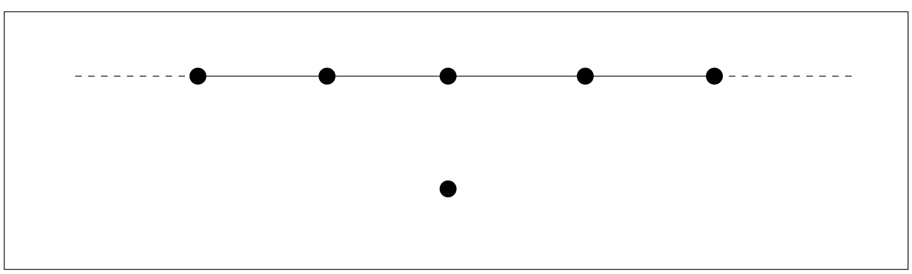
1.2.3 Remark: Let G be a topologized graph and S a subspace of G. If S is closed, then S is a (topologized) subgraph. Conversely, if G is classical and S is a subgraph, then S is closed.

The theme of an open point will be so important for us that it is worth isolating in a purely topological, non-combinatorial setting.

1.2.4 Definition: A hyperedge of a topological space is a point which is open but not closed. A hyperedge of a topological space is an edge if its boundary consists of at most two points. An edge of a topological space is a loop if it has precisely one boundary point, a proper edge otherwise. A point in a topological space is classical if the intersection of all its neighbourhoods is open.

Although a topologized graph can only be compatible with a unique graph, a given graph may be compatible with different topologized graphs. We illustrate this with a motivating example.

Figure 1.1: Two-way infinite path plus vertex at infinity



1.2.5 Example (Two-way infinite path plus vertex at infinity): Consider the simple graph G whose vertex set is $\mathbb{Z} \cup \{a\}$ and in which vertices $i, j \in \mathbb{Z}$ are adjacent if and only if $|i-j|=1$, while a has degree zero; this graph is illustrated in Figure 1.1. We define four different topologies $\mathcal{T}_1$, $\mathcal{T}_2$, $\mathcal{T}_3$, $\mathcal{T}_4$ on $X := V_G \cup E_G$ so that $(X, \mathcal{T}_i)$ $(i = 1 \dots 4)$ is a topologized graph compatible with G. We do this by defining a system $\{\mathcal{N}(p)\}_{p \in X}$ of neighbourhood bases.

In the case of all four topologies, we set, $\forall v \in \mathbb{Z}$, $\mathcal{N}(v) = \{\mathcal{N}_v\}$ and, $\forall e \in E_G$, $\mathcal{N}(e) = \big\{\{e\}\big\}$, while for the neighbourhood basis of a we have the following possibilities:

1. $\mathcal{N}(a) = \big\{\{a\}\big\}$
2. $\mathcal{N}(a) = \big\{ \bigcup_{v \in \alpha(z)} \mathcal{N}_v \mid z \in \mathbb{Z} \big\}$
3. $\mathcal{N}(a) = \big\{ \bigcup_{v \in \omega(z)} \mathcal{N}_v \mid z \in \mathbb{Z} \big\}$
4. $\mathcal{N}(a) = \big\{ \bigcup_{v \in \omega(z) \cup \alpha(-z)} \mathcal{N}_v \mid z \in \mathbb{N} \big\}$

where the notation $\alpha(z)$, $\omega(z)$ refers to the usual order on $\mathbb{Z}$.

It is easy to verify that in all four cases the resulting family $\{\mathcal{N}(x)\}_{x \in X}$ satisfies the properties of a system of basic neighbourhoods and hence defines a topology on X, and that the resulting topologies are distinct and all compatible with G.

Of these four topologies, the first one is the classical topology, and the only one with respect to which the resulting topological space is disconnected. The second and third are homeomorphic, and the fourth is the only one which is compact.

These four topologies do not exhaust all possibilities; we could, for example, obtain finer topologies by replacing $\omega(z)$ by the set $\omega'(z)$ consisting of all *even* integers larger than z. Note also that all four choices give a locally connected topological space, whereas the last example does not.

The fourth topology in Example 1.2.5 produces an appealing topological object. Let us consider this space, $\hat{P}$, as a modification of the two-way infinite path P, and pretend that we are unsure whether to think of the distinguished point as a vertex or an edge. Note that P equipped with the classical topology is homeomorphic to P equipped with its dual set system.

The way we constructed $\hat{P}$, it is a locally connected compactification of the classical two-way infinite path, and seems to be very well-behaved. Note that it also happens to satisfy the topological characterization of cycles we gave in Proposition 1.1.28, except that the topology is not the classical one. One feels compelled to allow the possibility to regard such an object as a “cycle”, alongside the usual cycles of graph theory.

However, none of this would change if we chose to model the two-way infinite path with the dual set system of the classical topology. In that case, the only change would be that the vertices become the open points, so that the distinguished point would naturally be considered to be an edge.

Such a model would be at odds with both our intuition and the usual topological models. Formally, though, the connected sets would be precisely the same. The problem with this alternative model is that the distinguished point is not incident with any other point. The vertices of a graph can be incident with an arbitrary number of edges, while an edge must have precisely one or two endvertices—this is the only asymmetry in the definition of a graph. The alternative model would force us to allow an edge with no endvertices at all.

The following is another easy statement concerning compatibility and the classical topology.

1.2.6 ***Proposition:*** *Given any hypergraph H, the classical topology is the finest topology that is compatible with H. If H is finite, the classical topology is the only compatible topology.* ✠

The setup now seems ideal for an axiomatic topological approach. One approach that has been adopted to extend results from finite to infinite graphs, involving in particular cycle spaces and infinite cycles, is to consider modifications of the corresponding cell-complex, obtained by surgical operations such as adding "points at infinity", also referred to as "ends", and topologically identifying certain points. It has recently emerged from the work of Diestel and Kühn [19, 21] that "such generalizations can, and should, involve the ends of an infinite graph on a par with its other points" [21, Introduction].

The analogous constructions can also be carried out on the corresponding topologized graphs, instead of cell-complexes; in the resulting spaces, the ends are the only non-classical (hence not open) points, and in particular must be considered as vertices. But then why not allow *any* vertex to be non-classical, and consider a more general class of objects endowed with an arbitrary pre-assigned topology? The task then becomes to identify the right topological axioms which make the spaces well-behaved, rather than constructing special spaces starting from a classical graph. We observe that already the simple space $\hat{P}$ does not arise as the Freudenthal compactification, or the Diestel-Kühn end-quotient, of the cell-complex topological space associated with a graph.

1.3 Overview

In Chapter 2 we take the characterizations of classical paths in topological language given in Section 1.1.3 as a starting point to define a topological version of a graph-theoretic path. We approach the issue without any reference to graphs. The class of spaces we obtain can be thought of as a non-Hausdorff generalization of the well-known orderable spaces of general topology, and includes spaces of arbitrary cardinality, such as the unit arc, or the "long line". It turns out that these spaces are naturally topologized graphs, possibly finite graph-theoretic paths. We give various characterizations of these spaces, and prove several results analogous to well-known results for (Hausdorff) orderable spaces, in particular with regard to local connectedness, compactness and order-completeness. We also characterize the compatible orders, and briefly discuss the "cycle-like" version of paths.

In Chapter 3 we introduce our second topological model for hypergraphs, and at the same time develop the theory of weak and feeble separation axioms. We present some pathological examples to justify the imposition of these axioms (with reference to specific issues of graph-theoretic inspiration), explore the parallels with the usual separation axioms of general topology, and develop the technical tools needed in Chapters 4 and 5, such as, for example, the fact that feeble normality is preserved under closed quotient maps. We address the issue of a global topology versus a topology on the vertex set, and focus on the rôle of feeble regularity in this context. We show how this property can be broken down into the inherited property of the topology on the vertex set, and a convergence property (quasiregularity) restricting the behaviour of the edges, still topological but with a certain combinatorial flavour. We begin a deeper discussion of an appropriate topological model for graphs along two main lines: the first one concerning the comparison with the usual model of a cell-complex, especially the specific issue of whether to model an edge by a point or a singleton, the second one with reference to a more generic setup. To address the latter issue, we introduce pre-hyperedge selections and use quotients to relate the more generic model to the one we eventually adopt. We also obtain some attractive results which will be marginal in the overall picture, such as a relationship between the dimensions on the vertex sets and the global space for quasiregular hyperedge spaces, and transitivity of path-connectedness for feebly Hausdorff topologized graphs.

The most important results in Chapter 4 are those guaranteeing the existence of minimal spanning sets (our analogues for spanning trees) and fundamental cyclesets. We give a construction of a non-trivial topologized graph without a minimal spanning set, based on a well-known example from general topology. We use quotients to bootstrap results from weakly Hausdorff to feebly Hausdorff spaces. We conclude our discussion of the choice of an appropriate topological model for graphs. We show that, when the edges in the spaces constructed by Diestel and Kühn [21] are modelled as points, these spaces turn out to be weakly Hausdorff and, with an extra minor assumption, compact. Our investigation of pre-edge selections under the extra assumption of compactness leads us to generalize known facts from general topology to non-Hausdorff spaces, such as the fact that the decomposition of a compact (feebly) Hausdorff space into its components is upper semicontinuous, and also to results concerning quotients which will be useful in Chapter 5. We pose a question concerning Peano continua and formulate a conjecture concerning topologized graphs with a zero-dimensional vertex set. We also show that the analogy of the feeble separation axioms with the standard ones carries over to their relationship with compactness. We illustrate how the question of the existence of a minimal spanning set becomes significantly harder when one considers *hyper*graphs as opposed to graphs.

In Chapter 5 we extend the theory of cycle spaces and bond spaces from finite graphs to compact feebly Hausdorff edge spaces. We define our versions of bonds, cycles, fundamental bonds and cycles, and characterize their orthogonal complements. We show that the orthogonal complement of the bond space coincides with the cycle space and, if the space is 2-edge-connected, the bond space coincides with the orthogonal complement of the cycle space. We use the flexibility of the model of edge spaces, which allows more than one combinatorial structure on the same topological space, to characterize the orthogonal complement of the cycle space in terms of a combinatorial substructure. We consider three different notions of generation by symmetric differences of infinite sets, namely weak, algebraic and strong generation, and define the cycle space and bond space as the strong and weak spans of the (possibly infinite) cyclesets and finite bonds, respectively. We show that the fundamental cyclesets strongly generate the cycle space, and use this to obtain a non-trivial isomorphism between the cycle space of an edge space and that of its clump quotient.

For compact weakly Hausdorff spaces, we show that the elements of the cycle space are precisely the disjoint unions of cycles, that the fundamental cyclesets constitute an algebraic basis for the cycle space, that the algebraic and strong spans of the cyclesets coincide, and that the bond space coincides with the space of edgecuts. For quasiregular, weakly normal spaces, we show that the fundamental bonds constitute a weak basis for the bond space. We also give some partial results towards replacing the compactness assumption, and understanding better the notion of a strong span.

In Chapter 6 we revert to a setting similar to that of Chapter 2. We unify the classical trees of graph theory with the dendrites of continuum theory, within the class of ferns. As in the case of paths we define ferns without any explicit neighbourhood separation axiom, such as Hausdorff, T_1, or S_1, but only with a “disconnecting property” with a combinatorial flavour. It turns out that these spaces are S_1. We show that ferns are path-connected, generalizing a result attributable to Whyburn [60], and give various characterizations of these spaces, in particular a semilattice characterization similar to results of Ward [58] and Muenzenberger, Ward and Smithson [44] (all for Hausdorff spaces). Our construction of the partial order is quite different from that in and [44, 58], and is geared towards a treatment of completions, in future work, meant to unify results of Polat [50] (completion of infinite graphs) and Allen [4] (dendritic compactifications) both of which are linked to the concept of a Freudenthal end.

Chapter 2

Prepaths, Paths and Cycles

Clearly paths and cycles are fundamental concepts in graph theory. In Subsection 1.1.3 we gave some easy characterizations of these objects in terms of the classical topology. These were "intrinsic" characterizations, in the sense that they are verified by a space with a given topology, and the fact that the topology may be inherited from an "ambient" space is irrelevant. The same cannot be said, for example, for the concept of a spanning tree—a tree is "spanning" depending on the graph it lives in. An analogous distinction can be made, for example, between closed and compact subsets of a topological space.

The intrinsic nature of these characterizations allows us to approach the issue of "path-like" and "cycle-like" objects from a generic topological point of view, that is, not with reference to graphs or topologized graphs. Surprisingly, topologized graphs "appear uninvited".

This approach leads us to consider a class of spaces which includes all orderable spaces. Our spaces will be "orderable" as well, in a weaker sense. The usual definition of an orderable space implies the T_2 axiom, while we would like to admit graph-theoretic paths, equipped with the classical topology, among our "orderable" spaces.

The two propositions in Subsection 1.1.3 characterizing paths provide two different starting points, which lead to two slightly different aspects: "orderability" and "minimality". Proposition 1.1.27 corresponds to orderability, Proposition 1.1.26 to minimality. The class of spaces which emerges from the "minimal" approach is contained in the class of "orderable" spaces; it also links up with the well-known " 'theory' of $S[a, b]$"— a few basic facts from general topology concerning the "orderability" of the set of points separating any two given points.

This chapter does not intend to develop a theory of orderability, or to exhaustively explore ways of characterizing "orderable" spaces. This has been achieved by the cumulative work of several mathematicians, albeit in the context of Hausdorff spaces. A good reference on this subject is the tract written by Kok [35], which also contains several original results.

Several of the results in this chapter will parallel facts which are well-known in the context of Hausdorff spaces.

The major contributions of this chapter will be to:

- extend the classes of orderable and non-orderable, cyclically orderable spaces to ones which contain graph-theoretic paths and cycles equipped with the classical topology (Theorem 2.2.1, Propositions 2.2.10, 2.2.40, 2.4.2);
- show how connected "orderable" spaces are naturally topologized graphs (Theorem 2.2.7);
- give characterizations, with a combinatorial flavour, of the orders corresponding to these spaces (Corollary 2.3.14);
- show that these "orderable" spaces behave in a manner similar to their Hausdorff counterparts, in particular with respect to convexity and intervals (Corollary 2.2.31), order-completeness (Proposition 2.2.30), compactness (Theorem 2.3.17) and local connectedness (Proposition 2.3.4).

2.1 Cutpoints and Separations

We begin by recalling a few well-known topological facts.

2.1.1 Fact: If A, B are connected subsets of a topological space with non-empty intersection, then $A \cup B$ is connected.

Reference: This is a special case of Theorem 26.7, part (a), in [61]. □

2.1.2 Fact: If A is a subspace of a topological space B, then every component of A is contained in a (unique) component of B.

Proof: This follows from the fact that every connected subset is contained in a maximal connected subset. □

2.1.3 Fact: If C is a connected subset of a topological space, and A a subset such that $C \subseteq A \subseteq \mathsf{Cl}(C)$, then A is connected.

Reference: See [61], Theorem 26.8. □

2.1.4 Definition: Given a topological space X and a connected subset S, the component of X containing S is denoted by $\mathbf{K}(S)$. Whenever necessary (for example, if S is contained in a subspace of X), we specify the space X by means of a subscript. For example, $\mathbf{K}_V(S)$ denotes the connected component of V containing S. An adherent component of a point $x \in X$ is a component of $\mathbf{K}(x) \setminus \{x\}$. A cutset of a connected topological space X is a subset T such that $X \setminus T$ is disconnected. We also say that T disconnects X, or, if there is no danger of ambiguity, simply that T disconnects. If $X \setminus T$ consists of precisely two connected components, then T is a strong cutset. If a point x is such that $\{x\}$ is a (strong) cutset of $\mathbf{K}(x)$, then we call it a (strong) cutpoint. A cutedge is a cutpoint that is an edge. We shall see that all cutedges are strong cutpoints (Corollary 2.1.17). A non-cutpoint will also be called an endpoint. However, this term will be always refer to non-cutpoints *of the universal space*, that is, if S is a subset of a topological space X, then "S contains an endpoint x" implies that x is a non-cutpoint of X.

If A, T are subsets of a topological space X, then T disconnects A if $A \setminus T$ is *not* contained in a single component of $X \setminus T$. If $A = \{p_i\}_{i \in I}$ is disjoint from T, then T simultaneously disconnects A (or the points in A) if $i \neq j$ implies that p_i and p_j belong to distinct components of $X \setminus T$.

If A, B, C are disjoint subsets of a topological space X and there exists a separation $\{U, V\}$ of $X \setminus A$ with $B \subseteq U$ and $C \subseteq V$ then A separates B and (from) C. Note that any of A, B, C may be singleton, in which case we refer to it as a point, as per our convention (page xi). Thus, for example, a point may separate a set from another point.

2.1.5 Note: For elements x, y, z of a component K of a topological space X, y disconnects x and z with respect to the topology on X if and only if it disconnects x and z with respect to the subspace topology on K.

2.1.6 *Proposition:* *If X is a topological space, $x, y, z \in X$ and y separates x from z, then y disconnects x from z.*

Proof: There is a separation $\{A, B\}$ of $X \setminus \{y\}$ with $x \in A, z \in B$. If the component of $X \setminus \{y\}$ containing x were the same as the one containing z, it would have non-empty intersection with both A and B, contradicting (1.1.21). ✠

2.1.7 Note: The converse of the above proposition is false; in general, it is harder to separate two points, or sets, than to disconnect them. In Section 2.2 we shall see an example of this distinction.

2.1.8 *Proposition:* *If x is a cutpoint of a connected topological space X, there exist nonempty sets C_1, C_2, closed in X, such that $C_1 \cup C_2 \cup \{x\} = X$, $\{C_1 \setminus \{x\}, C_2 \setminus \{x\}\}$ is a separation of $X \setminus \{x\}$ and precisely one of the following is true:*

1. *$\{x\} = (C_1 \cap C_2)$ and is closed;*

2. *$\{C_1, C_2, \{x\}\}$ is a partition of X and $\{x\}$ is open.*

Proof: Since $X \setminus \{x\}$ is disconnected, there must be a separation $(\hat{C}_1, \hat{C}_2)$ with $\hat{C}_1, \hat{C}_2$ nonempty and clopen in $X \setminus \{x\}$. By definition of the relative topology, $\hat{C}_1, \hat{C}_2$ must be of the form $C_1 \cap (X \setminus \{x\}) = C_1 \setminus \{x\}$, $C_2 \cap (X \setminus \{x\}) = C_2 \setminus \{x\}$ respectively, with C_1, C_2 closed in X (and nonempty). Now

$$\begin{aligned} C_1 \cup C_2 \cup \{x\} &\supseteq (C_1 \setminus \{x\}) \cup (C_2 \setminus \{x\}) \cup \{x\} \\ &= \hat{C}_1 \cup \hat{C}_2 \cup \{x\} \\ &= (X \setminus \{x\}) \cup \{x\} \\ &= X. \end{aligned}$$

Also,

$$\begin{aligned} C_1 \cap C_2 &= C_1 \cap C_2 \cap X \\ &= \Big(C_1 \cap C_2 \cap (X \setminus \{x\})\Big) \bigcup \Big(C_1 \cap C_2 \cap \{x\}\Big) \\ &= \Big((C_1 \cap (X \setminus \{x\}) \cap (C_2 \cap (X \setminus \{x\})\Big) \bigcup \Big(C_1 \cap C_2) \cap \{x\}\Big) \\ &= (\hat{C}_1 \cap \hat{C}_2) \bigcup (C_1 \cap C_2) \cap \{x\}) \\ &= (C_1 \cap C_2) \cap \{x\} \\ &\subseteq \{x\}. \end{aligned}$$

So if x lies in both C_1 and C_2, then $\{x\} = C_1 \cap C_2$ and, being the intersection of closed sets, is closed. In this case (1) holds. Suppose instead that $x \notin C_1 \cap C_2$, i.e. $C_1 \cap C_2 = \emptyset$. Then either x lies in one of C_1, C_2 but not the other, or x is in neither. In the former case, we have $C_1 \cup C_2 = C_1 \cup \{x\} \cup C_2 = X$, so (C_1, C_2) is a separation of X, contradicting the fact that X is connected. Hence we must have $x \notin (C_1 \cup C_2)$. But since $C_1 \cup C_2 \cup \{x\} = X$, $\{C_1, C_2, \{x\}\}$ is a partition of X, and $\{x\}$, being the complement in X of $C_1 \cup C_2$ (which is closed because it is the union of two closed sets), is open. In this case (2) holds. ✠

The above proof, with the rôles of open and closed sets reversed, also shows that:

2.1.9 ***Proposition:*** *If x is a cutpoint of a connected topological space X, then there exist nonempty sets U_1, U_2, open in X, such that $U_1 \cup U_2 \cup \{x\} = X$, $\{U_1 \setminus \{x\}, U_2 \setminus \{x\}\}$ is a separation of $X \setminus \{x\}$ and precisely one of the following is true:*

1. *$\{x\} = (U_1 \cap U_2)$ and is open;*

2. $\{U_1, U_2, \{x\}\}$ is a partition of X and $\{x\}$ is closed.

Note that the statements of the above propositions contain the following general fact, which for us is crucial.

2.1.10 Remark: Any cutpoint in any connected topological space is necessarily open or closed.

Also, we observe that an open cutpoint is necessarily a hyperedge, that is it cannot also be closed. While we have already come to S_1 spaces from the starting point of graph theory, we shall see how the above fact makes the concept of S_1 spaces emerge naturally from a purely topological point of view.

Note that conditions (1) in 2.1.8 and 2.1.9 cannot occur simultaneously, and the same is true for conditions (2). These two propositions will be summarized in Corollary 2.1.12. To phrase the corollary neatly, we introduce some terminology and notation to deal with a concept which will be central for us, and has already come up in Chapter 1, in the context of Alexandroff discrete spaces and the classical topology.

2.1.11 Definition: Given a subset A of a topological space, the surrounding set of A, denoted by $A^\diamond$, is the intersection of all the open sets containing A.

2.1.12 *Corollary:* *If x is a cutpoint of a connected topological space X, then, for any separation (A_1, A_2) of $X \setminus \{x\}$, we have that precisely one of the following holds:*

1. *$\{x\}$ is closed, A_i is open in X, and $\mathsf{Cl}(A_i) = A_i \cup \{x\}$ $(i = 1, 2)$;*

2. *$\{x\}$ is open, A_i is closed in X, and $A_i^\diamond = A_i \cup \{x\}$ is open $(i = 1, 2)$.*

2.1.13 *Proposition:* *Let x be a cutpoint of a connected topological spaceX, and suppose $\{S_1, S_2\}$ is a separation of $X \setminus \{x\}$. Then, for $i = 1, 2$, $S_i \cup \{x\}$ is connected.*

Proof: By 2.1.8, there exist sets C_1, C_2, closed in X, with $S_1 = C_1 \setminus \{x\}$, $S_2 = C_2 \setminus \{x\}$ and such that the conclusion of 2.1.8 holds. For $i = 1, 2$, let $Q_i = S_i \cup \{x\}$ and suppose, by way of contradiction, that, for some $i \in \{1, 2\}$, Q_i has a separation (M, N). Since M, N are closed in Q_i, we have $M = \overline{M} \cap Q_i, N = \overline{N} \cap Q_i$.

Without loss of generality, we may assume $x \in M$. This implies $x \notin N$, so $N \subseteq S_i$. But $S_i \subseteq C_i$, so $N \subseteq C_i$. Since C_i is closed, this gives $\overline{N} \subseteq C_i$, and since $C_i \subseteq Q_i$, $\overline{N} \subseteq Q_i$, i.e., $\overline{N} \cap Q_i = \overline{N}$. But $\overline{N} \cap Q_i = N$, so $N = \overline{N}$, i.e. N is closed in X.

Now $\overline{M} \cup C_{3-i}$ is also closed in X, being the union of two closed sets, and $\overline{M} \cup C_{3-i} \cup N \supseteq (M \cup N) \cup C_{3-i} = Q_i \cup C_{3-i} \supseteq Q_i \cup S_{3-i} = X$. Moreover, since $N \subseteq Q_i$, we have $\overline{M} \cap N = \overline{M} \cap N \cap Q_i = M \cap N = \emptyset$, and since $N \subseteq C_i$, $C_{3-i} \cap N = C_{3-i} \cap (N \cap C_i) = (C_1 \cap C_2) \cap N$.

But by 2.1.8 $C_1 \cap C_2 \subseteq \{x\}$, so $(C_{3-i} \cap \overline{N}) \subseteq N \cap \{x\}$, and since $x \notin N$, $C_{3-i} \cap N = \emptyset$. Hence $\overline{M} \cup C_{3-i}$ and N are disjoint and $(\overline{M} \cup C_{3-i}, N)$ is a separation of X, contradicting the fact that X is connected. Hence Q_i must be connected for $i = 1, 2$. ✠

We conclude this section with a couple of basic results about open (but not closed) cutpoints. These will come in useful for us on several occasions.

2.1.14 ***Lemma:*** *Let h be a hyperedge of a connected topological space X, and suppose $\{K_1, K_2, \ldots K_s\}$ is a finite partition of $X \setminus \{h\}$ into non-empty closed subsets. Then every part contains a point in the boundary of x.*

Proof: Let A be the union of the K_i's which contain an endvertex of h, and B the union of the others. Being the finite union of closed sets, both these sets are closed in $X \setminus \{h\}$, which, since $\{h\}$ is open, is closed in X, so A, B are both closed in X. Since h has at least one endvertex, A is non-empty, and if the conclusion is false, B is also non-empty. But then $\{A \cup \overline{\{e\}}, B\}$ is a separation of the connected space X, a contradiction. ✠

2.1.15 ***Corollary:*** *Let x be an open cutpoint of a connected topological space X. Then some two boundary points of x are separated by x.*

Proof: Suppose $\{S_1, S_2\}$ is a separation of $X \setminus \{x\}$. Then by 2.1.14 with $s = 2$ both S_1, S_2 contain at least one point in the boundary of x. ✠

2.1.16 ***Lemma:*** *Let h be a hyperedge of a connected space X with finitely many boundary points. Then every component of $X \setminus \{h\}$ contains a boundary point of h. In particular, if h is an edge, $X \setminus \{h\}$ consists of at most 2 connected components.*

Proof: Let s be the number of boundary points of h. By 2.1.14, every finite partition of $X \setminus \{h\}$ into non-empty closed sets has at most s parts. Since $\{X \setminus \{h\}\}$ is such a partition, we can choose one, say $P := \{C_1, C_2, \ldots, C_t\}$, to maximize the number of parts. Now suppose that C_j is disconnected for some positive $j \leq t$, that is, there exists a separation $\{U, V\}$ of C_j. Since U, V are closed in C_j which is closed in $X \setminus \{h\}$, U, V are both closed in $X \setminus \{h\}$. But then $\{C_1, C_2, \ldots, C_{j-1}, U, V, C_{j+1}, C_{j+2}, \ldots C_t\}$ is a partition of $X \setminus \{h\}$ into $t + 1$ non-empty closed sets, contradicting the choice of P.

Hence all the parts of P are connected, but since they are clopen in $X \setminus \{h\}$, they cannot be strictly contained in a connected component of $X \setminus \{h\}$. Therefore they are the connected components of $X \setminus \{h\}$. ✠

2.1.17 ***Corollary:*** *If e is a cutedge, then $\mathbf{K}(e) \setminus \{e\}$ consists of precisely two connected components, each containing one boundary point of e. In particular, e is not a loop.*

2.2 Orderability

This rather technical section gives results used throughout the rest of this chapter.

2.2.1 ***Theorem:*** *For a topological space X, the following properties are equivalent:*

(A) *there exists a total order $\preceq$ on X such that $\forall\ x, y, z \in X :\ x \prec y \prec z\ \ y$ disconnects x and z;*

(B) *for every triple T of points in X, there exists an element in T which disconnects the other two elements of T;*

(C) *for any point $x \in X$, any adherent component of x contains at most one endpoint, and all cutpoints are strong cutpoints.*

2.2.2 Note: For connected X, property (C) of 2.2.10 is equivalent to:

(C′) $\forall x \in X$, $X \setminus \{x\}$ has at most 2 components and each one contains at most one endpoint.

Proof of 2.2.1:

(A) $\implies$ (B):

Suppose X satisfies (A). Since $\preceq$ is decisive, for any triple T there must exist $y \in T$ such that $x \prec y \prec z$, where x, z are the other two elements of T. Hence by (A) y disconnects x and z. Thus (B) is satisfied.

(B) $\implies$ (C):

Suppose that X satisfies (B). Consider, for any cutpoint x, a separation $\{A, B\}$ of $Y \setminus \{x\}$, where $Y = \mathbf{K}(x)$, and suppose that A is disconnected. Then it must have a separation $\{A_1, A_2\}$. Choose $x_1 \in A_1$, $x_2 \in A_2$ and $x_3 \in B$. Since $A \cap B = \emptyset = A_1 \cap A_2$, x_1, x_2, x_3 are distinct points. Now by (2.1.13) $A \cup \{x\}$ is connected; it also contains x_1 and x_2 and is contained in $Y \setminus \{x_3\}$; this implies that x_1 and x_2 are in the same component of $Y \setminus \{x_3\}$, whence x_3 cannot disconnect x_1 and x_2. Also, applying (2.1.13) once with $A \cup \{x\}$ and once with Y as the connected space, we obtain that $A_1 \cup \{x\}$ and $B \cup \{x\}$ is connected; since both these sets contain x, their union, which contains x_1 and x_3 but not x_2, is also connected. As above, this implies that x_1 and x_3 lie in the same component of $Y \setminus \{x_2\}$, so that x_2 cannot disconnect x_1 and x_3. Similarly x_1 cannot disconnect x_2 and x_3, so property (B) fails for the triple $\{x_1, x_2, x_3\}$. Hence A cannot be disconnected; analogously, neither can B, so $Y \setminus \{x\}$ has precisely two components, i.e. x is a strong cutpoint. Since x is an arbitrary cutpoint, all cutpoints are strong cutpoints.

Suppose now that, for some $x \in X$, a, b are distinct endpoints in the same component of $Y \setminus \{x\}$, where again Y denotes the component of X containing x. Then they are also in the same component of $X \setminus \{x\}$, so x does not disconnect a and b. Since a is an endpoint, $Y \setminus \{a\}$ is connected and contains x and b, whence a does not disconnect x and b, and analogously b does not disconnect a and x. But then (B) fails for the triple $\{a, b, x\}$. Hence an adherent component of $X \setminus \{x\}$ can contain at most one endpoint, so that (C) is satisfied.

(C) $\Longrightarrow$ (A)

Claim 1: We may assume X to be connected.

Proof of Claim 1: Suppose the proposition is true with the additional assumption that X is connected. Then if $\{C_i\}_{i \in I}$ is the set of components of X, for all $i \in I$ we can find a total order $\preceq_i$ on C_i such that $\forall x, y, z \in C_i : \ x \prec_i y \prec_i z$ y disconnects x from z with respect to the subspace topology of C_i. Choosing an arbitrary total order $\sqsubseteq$ on the set of components of X, we can then define a total order $\preceq$ on X by setting $x \preceq y$ whenever $x \preceq_i y$ for some $i \in I$ and whenever $x \in C_i, y \in C_j$ and $C_i \sqsubset C_j$ for some $i, j \in I$.

The resulting binary relation is clearly a total order on X; if $x \prec y \prec z$ and x and z belong to distinct components of X, then they also belong to different components of $X \setminus \{y\}$, i.e. y disconnects x and z, while if they belong to the same component of X, y must belong to the same component (otherwise it would be in the same relation, with respect to $\preceq$, with x and z) so $x \prec_i y \prec_i z$ for some i. By the way $\preceq_i$ was chosen, we have that y disconnects x and z with respect to the subspace topology of C_i, which is equivalent to disconnecting x and z in the topology of X.

Hence we may assume X to be connected (and we will). □

Claim 2: Let x, y be cutpoints, A, B the components of $X \setminus \{x\}$ labelled so that $y \in B$ and C, D the components of $X \setminus \{y\}$ labelled so that $x \in C$. Then $A \subseteq C$, $D \subseteq B$ and $A \cap D = \emptyset$.

Proof of Claim 2: $\{C, D\}$ is a separation of $X \setminus \{y\}$. Therefore by 1.1.19 C and D are mutually separated in X. Since $y \notin A$, $A \subseteq (C \cup D)$, so by 1.1.20 precisely one of $A \cap C$ and $A \cap D$ is nonempty (and is equal to A). Similarly, A and B are mutually separated in X, and since $x \notin D$ precisely one of $D \cap A$ and $D \cap B$ is nonempty (and is equal to D).

Now suppose $A \cap D \neq \emptyset$. Then $A \subseteq D$ and $D \subseteq A$, i.e. $A = D$. Since A, B are mutually separated in X, $\bar{A}$ is disjoint from B, i.e. $\bar{A} \subseteq (A \cup \{x\})$. Similarly $\bar{D} \subseteq (D \cup \{y\})$. So $\bar{A} = \bar{D} \subseteq ((A \cup \{x\}) \cap (D \cup \{y\})) = A = D$, i.e. $A = D$ is closed. Also, $C \cup \{y\} = X \setminus D = X \setminus A = B \cup \{x\}$ so $(X \setminus D) = (X \setminus A) = (B \cup C)$ and $\overline{B \cup C} \subseteq (\bar{B} \cup \bar{C})$

which is disjoint from $A = D$ since $\bar{B}$ is disjoint from A and $\bar{C}$ is disjoint from D, i.e. $\overline{X \setminus A} = \overline{X \setminus D} \subseteq (X \setminus A) = (X \setminus D)$ whence $(X \setminus A) = (X \setminus D)$ is closed. But then $\{A, X \setminus A\}$ is a separation of X, contradicting the fact that X is connected (note $x \in X \setminus A$ implying that $X \setminus A$ is nonempty).

So $A \cap D = \emptyset$, which implies $A \subseteq C$ and $D \subseteq B$. □

We now construct a binary relation on X and show that it satisfies (A). If $|X| \leq 2$, then any total order on X trivially satisfies the requirement of (A), so we may assume $|X| > 2$. Then by (C′) we have that there are at most 2 endpoints, and hence there exists a (strong) cutpoint $\bar{x}$; let A, B be the components of $X \setminus \{\bar{x}\}$. Now for any cutpoint $y \neq \bar{x}$, let C_y denote the connected component of $X \setminus \{y\}$ containing x, and D_y the other one. Then we define $y_1 \preceq y_2$ for any ordered pair (y_1, y_2) such that one of the following holds:

$$\begin{array}{lll}
(1) & y_1 = y_2 & \text{or} \\
(2) & y_1 \in A,\ y_2 \in (X \setminus A) & \text{or} \\
(3) & y_1 \in (X \setminus B), y_2 \in B & \text{or} \\
(4) & y_1 \text{ is an endpoint in } A & \text{or} \\
(5) & y_2 \text{ is an endpoint in } B & \text{or} \\
(6) & y_1, y_2 \text{ are cutpoints, } y_1 \in A,\ y_2 \in A,\ y_2 \in C_{y_1} & \text{or} \\
(7) & y_1, y_2 \text{ are cutpoints, } y_1 \in B,\ y_2 \in B,\ y_2 \in D_{y_1}. &
\end{array}$$

Clearly, the resulting binary relation is reflexive. We need to show prove that it is decisive, antisymmetric and transitive, and that it satisfies (C).

Claim 3:

(a) If x, y are cutpoints in A with $x \preceq y$, then $C_y \subseteq C_x$ and $x \in D_y$.

(b) If x, y are cutpoints in B with $x \preceq y$, then $C_x \subseteq C_y$ and $y \in D_x$.

Proof of Claim 3: Suppose that x, y are cutpoints in A with $x \preceq y$. If $x = y$, then (a) follows trivially; assume then $x \neq y$. (x, y) must be of the form (6), therefore we must have $y \in C_x$. Now either $x \in C_y$ or else $x \in D_y$. If $x \in C_y$, then applying Claim (2) with y for x, x for y, C_y, D_y for A, B respectively and D_x, C_x for C, D respectively, we obtain that $C_x \cap C_y = \emptyset$, contradicting the fact that $\bar{x} \in C_x \cap C_y$. Hence we must have $x \in D_y$, and again applying claim 2, this time with C_x for C and D_x for D, we obtain that $C_y \subseteq C_x$. This proves (a). The proof of (b) is analogous. □

We now proceed to prove that $(\preceq)$ is decisive and antisymmetric. Since we already know it is reflexive, this reduces to showing that for any ordered pair (x, y) with $x \neq y$, either $x \preceq y$ or $y \preceq x$, but not both.

Suppose first that x is an endpoint. If $x \in A$, then (x, y) is of form (4), so $x \preceq y$. To show that $y \not\preceq x$, we must show that (y, x) cannot be of any one of forms (1-7). Form (1) is excluded because we are assuming x and y to be distinct, forms (2,3) and (5) by the fact that $x \in A$, forms (6,7) by the fact that x is an endpoint, and form (4) by the fact that y cannot be an endpoint in A since, by (C), A can contain at most 2 endpoints. Hence $x \in A$ implies $x \preceq y,\ y \not\preceq x$. Analogously, $x \in B$ implies $y \preceq x,\ x \not\preceq y$. So we have dealt with the case when x is an endpoint; similarly we can deal with the case when y is an endpoint.

We may therefore assume that both x and y are cutpoints. Assume in addition that $x = \bar{x}$. If $y \in B$, then (x, y) is of form (3), so $x \preceq y$. Consider now the ordered pair (y, x). Form (1) is again excluded because x and y are distinct, forms (2,3) and (6) because $y \in B$, form (7) because $\bar{x} \notin B$ and forms (4,5) because x, y are cutpoints. So $y \in B$ implies $x \preceq y,\ y \not\preceq x$. Analogously $y \in A$ implies $y \preceq x,\ x \not\preceq y$. This deals with the case that $x = \bar{x}$; the case $y = \bar{x}$ can be dealt with analogously.

So we may assume that x and y are distinct cutpoints in $X \setminus \{\bar{x}\}$. Suppose first that they belong to distinct components of $X \setminus \{\bar{x}\}$. Without loss of generality, we may assume that $x \in A$ and $y \in B$. Then (x, y) is of forms (2,3), so $x \preceq y$. Also, none of forms (1-7) fits (y, x): form (1) is excluded again because x and y are distinct; forms (2,3) and (7) because $x \in A$, form (6) because $y \in B$ and forms $(4, 5)$ because x and y are cutpoints. So again we have that $x \preceq y$ but $y \not\preceq x$. Suppose then that x and y belong to the same component of $X \setminus \{\bar{x}\}$; again, without loss of generality we may assume that this component is A. Then only forms (1) and (6) can apply; form (1) is excluded. If we have $x \preceq y \preceq x$ by virtue of (6), then $x \in C_y$; but by Claim 3, $C_y \subseteq C_x$, a contradiction because $x \notin C_x$.

This concludes the proof of the fact that $\preceq$ is decisive and antisymmetric. It remains to be shown that it is transitive and satisfies property (C). Take any ordered triple $(x, y, z) \subseteq X$ with $x \preceq y \preceq z$; we wish to prove $x \preceq z$, and that if $x \prec y \prec z$, y disconnects x and z. If $x = y$ or $y = z$, then $x \preceq z$ holds trivially, and $x \not\prec y$ or $y \not\prec z$, so that property (A) does not come into play, while if $x = z$ $x \preceq z$ holds by reflexivity, and antisymmetry implies that $y = x = z$, so that again property (A) is trivially satisfied. Hence we may assume that x, y, z are distinct, in particular that $x \prec y \prec z$.

If x is an endpoint, then (x, y) must fall into at least one of the cases (2-5). Cases 2 and 4 trivially imply that $x \in A$; case 3 implies that $x \in A$ or $x = \bar{x}$, but the latter possibility is ruled out by the fact that $\bar{x}$ is a cutpoint. Case 5 implies that y is an endpoint in B, but since B can contain at most one endpoint, again $x \in A$. But then (x, z) is of the form (4), whence $x \preceq z$. Similarly, when z is an endpoint, by considering the pair (y, z), we conclude that $z \in B$ and therefore case (5) applies to (x, z), so that $x \preceq z$. If y were an endpoint, applying the same arguments to both pairs ((y, z) and (x, y)), it follows that y would be in both A and B, a contradiction. Hence we may assume that x, y, z are distinct cutpoints.

Suppose $y = \bar{x}$. Then (x, y) must be of form (2), which implies that $x \in A$, and (y, z) must be of form (3), which implies $z \in B$; but then y disconnects x and z and (x, z) is of form (2) (and (3)), whence $x \preceq z$. Suppose instead that $z = \bar{x}$. Then (y, z) is of the form (2), which implies that $y \in A$, whence (x, y) can only be of form (6), which in turn implies that $x \in A$. But then (x, z) is of form (2), so that $x \preceq z$. Moreover, since both x, z are cutpoints in A, from Claim 3 it follows that $x \in D_y$. Also $z = \bar{x} \in C_y$ by definition of C_y, so y disconnects x and z. Similarly we can deal with the case $x = \bar{x}$.

We may therefore assume that x, y, z are distinct cutpoints in $X \setminus \{\bar{x}\}$. We claim that the case $z \in A, x \in B$ is impossible. Since the pairs $(z, \bar{x})$, $(\bar{x}, x)$ are of forms (2) and (3) respectively, we have that $z \preceq \bar{x} \preceq x$. Now if $y \in A$ then the pair $(y, \bar{x})$ is of form (2), giving $y \preceq \bar{x}$, and since we have already verified transitivity for triples of the form $y \preceq \bar{x} \preceq x$, we can conclude $y \preceq x$, contradicting the assumption $x \preceq y$, since x and y are distinct (and $\preceq$ is antisymmetric). If instead $y \in B$, then we obtain similarly that $\bar{x} \preceq y$ and from $z \preceq \bar{x} \preceq y$ conclude $z \preceq y$, again a contradiction.

In the case $x \in A, z \in B$, (x, z) is of form (2) (and (3)), so $x \preceq z$. Also, if $y \in A$, by Claim 3 we have that $x \in D_y$, and applying Claim 2 with $\bar{x}$ for x, B, A for A, B respectively, y for y and C_y, D_y for C, D respectively, we get that $B \subseteq C_y$. Since $z \in B$, this implies $z \in C_y$ so y disconnects x and z. If instead $y \in B$, the proof proceeds analogously.

So we may assume that x, z belong to the same component of $X \setminus \{\bar{x}\}$. Suppose that this component is A. Now if $y \in B$, then (z, y) would be of the form (2), which would imply $z \preceq y$, again a contradiction, so we must have $y \in A$. But then (y, z) is of form (6), which implies that $z \in C_y$; by (a) of Claim 3, we also have that $C_y \subseteq C_x$ and $x \in D_y$. The latter of these two conclusions implies that y disconnects x and z, and the former that $z \in C_x$ and (x, z) is also of form (6), giving $x \preceq z$. If instead the component of $X \setminus \{\bar{x}\}$ containing x and z is B, we can proceed analogously, showing that $y \in B$ and $x \in C_y$ and then using part (b) of Claim (3) to obtain $C_y \subseteq C_z$.

This concludes the proof of the fact that the binary relation $\preceq$ is transitive and satisfies (A). ✠

A topological space X is usually called orderable if there exists a total order such that for any $x \in X$ the sets $\alpha(x), \omega(x)$ are open in X, and strictly orderable if the sets of this form constitute a subbase for the topology on X. Note that this implies immediately that every singleton is closed, that is, X is a T_1 space.

2.2.3 Definition: A topological space is connection-wise orderable if it satisfies the equivalent statements of Theorem 2.2.1.

In [35] and [36], Kok studies various characterizations of orderable spaces under the assumption that the space be connected and T_1. Although these characterizations do not explicitly require the Hausdorff axiom, the class of spaces characterized inevitably consists of Hausdorff spaces: considering separately the cases whether two given points have a point in between or not, it is easy to see from the definition of an orderable space that it must be Hausdorff.

To our knowledge, "orderable" spaces in the literature are always explicitly assumed to be, or effectively are (according to the definition), T_1 spaces, and therefore Hausdorff. However, this point appears to be have been the source of some confusion in the literature. In what seems to be the seminal work on the subject (1941), Eilenberg [23] introduces "ordered spaces" and immediately proves that, from his definition it follows that ordered spaces are Hausdorff. He then claims (with a purported proof) that a connected topological space X can be ordered if and only if the complement in $X \times X$ of the diagonal $\{(x,x) \mid x \in X\}$ is not connected. In the simple case of a complete simple graph with two vertices equipped with the classical topology (a three-point space), it is clear that the space is not Hausdorff, but the complement of the diagonal is not connected. In a sequel to this paper, Duda [22] adopts the definition of Eilenberg and mentions in a footnote that Mrs. D. Zaremba-Szczepkowicz observed that connected ordered spaces can be characterized by the following property:

(CSp) among any three points, some point separates the other two,

which is *a priori* a slight strengthening of property (B) in Theorem 2.2.1. Again this property is verified in the simple example cited. This accreditation of Duda is reported by Kok [35], who then proves the equivalence for connected T_1 spaces. In [10] Brouwer mentions that it is well-known that connected "weakly orderable" topological spaces are characterized by this property. We could not find a precise definition of the term "weakly orderable"; presumably it means "orderable, but not necessarily strictly orderable". However, it is not clear in the context whether Brouwer is implicitly assuming the T_1 axiom or not.

Property (CSp) is attractive in two different ways: it has a "combinatorial flavour", and by itself it does not imply that the space is T_1. Characterizations of this kind will be important for us, especially in Chapter 6. We shall emphasize the common underlying theme in properties of this kind by the letters "CS" (short for "combinatorial separation") in their designation. In the next section, we shall see that for connected spaces, in our scenario property (CSp) does not need any modification at all.

Given an arbitrary point x in an orderable topological space X, $\{\alpha(x), \omega(x)\}$ is a separation of $X \setminus \{x\}$. Hence orderable spaces satisfy property (A) (and therefore (B) and (C)) of Theorem 2.2.1; that is, orderable spaces are connection-wise orderable.

Property (C) in Theorem 2.2.1 may appear involved and *ad hoc*. It has the advantage that it immediately allows us to deduce that connected connection-wise orderable spaces are S_1 spaces. Moreover, it turns out that several variants of this property have already been deemed worthy of investigation in the literature on orderable spaces.

Kok says that a connected T_1 topological space X "satisfies (H)" if every connected subset of X has at most two non-cutpoints (of itself, not of X), and that it satisfies (B′) if every cutpoint is a strong cutpoint. Let us say that a space satisfies (H′) if every adherent component contains at most one endpoint (of X). Condition (C) in Theorem 2.2.1 is equivalent to (H′) together with (B′), except that our spaces need not be T_1, nor connected.

Brouwer [9] has considered a strengthening of (H) which is even more reminiscent of (H′). Let us say that a space satisfies (H*) if every connected subset has at most one non-cutpoint (again of itself). Property (H) was introduced by Herrlich in his doctoral dissertation [28]. Herrlich proved that a connected space is strictly orderable if and only if it is T_1, connected and satisfies (H). This proof was published in [29].

Clearly (H*) implies (H) and it is not hard (although not entirely trivial) to verify that (H) implies (H′). In [35] Kok proves that a connected T_1 space satisfying (H) and (B′) is orderable (Lemma 8, Chapter 3, Section 3); it follows that for a connected T_1 space these two properties together are equivalent to orderability. Condition (C) of (2.2.1) can be seen as a relaxation of this property.

Property (H) was also studied in its own right, that is, without the additional assumption (B′); in [36] Kok asks whether a connected T_1 space satisfying (H) is necessarily orderable. The question was answered in the negative by Hursch and Verbeek [32], and in Chapter III of [35], dedicated to weakenings of Herrlich's property (all dealing with non-cutpoints of a connected subspace), Kok resolves the issue essentially by showing that for a connected T_1 space, orderability is equivalent to property (H) together with the assumption that the closure of any component K adherent to a cutpoint X is precisely $K \cup \{x\}$.

Kok also shows that this assumption is a relaxation of (B′). So apart from the assumptions of T_1 and connectedness, in comparison with this last characterization of orderability Condition (C) is a trade-off: property (H) is replaced by the weaker property (H′), but the extra assumption is replaced by the stronger property (B′). Interestingly, Kok's extra assumption is itself similar to H′, in that it focuses on adherent components.

We close this section with two examples, the second obtained from the first. At this point, we wish to illustrate how the distinction between separating and disconnecting two given points by a third comes into play in the context of connection-wise orderable spaces. In fact, we consider a slightly larger space and then go to on to consider a subspace. The reason for this is that the larger space contains also other important examples which we shall have occasion to examine later on.

2.2.4 Example: Let P_1 be a one-way infinite graph theoretic path, and P_2 the graph-theoretic path on two vertices. We define a topology on $P_1 \cup P_2$ such that both P_1 and P_2 inherit the classical topologies, but overall the resulting topological space is rather badly behaved. We declare a subset $U \subseteq P_1 \cup P_2$ to be open if and only if $U \cap P_i$ is open with respect to the classical topology on P_i ($i \in \{1,2\}$), and $U \cap P_2 \neq \emptyset$ implies that $P_1 \setminus U$ is finite.

2.2.5 Example: Consider the subspace S of the space constructed in Example 2.2.4 consisting of $V_{P_1} \cup P_2$. Let $\preceq$ be a total order on S such that the only edge of P_2 is the predecessor of one incident edge and the successor of the other. Then the conditions of (2.2.1) are verified, in particular if $x \prec y \prec z$ then y disconnects x and z. However, when x and z are the vertices of P_2, and therefore y is the edge of P_2, y does not separate x and z, because any separation of $S \setminus \{y\}$ would imply the existence of a bipartition of P_1 into two finite sets.

2.2.1 Prepaths

Note that the statements of Theorem 2.2.1 are vacuous on a topological space with two points or less.

2.2.6 Definition: A *connected* topological space X is a prepath if $|X| = 1$, or $|X| = 2$ and the topology on X is not the indiscrete topology, or else $|X| \geq 3$ and X satisfies the statements of Theorem 2.2.1.

2.2.7 Theorem: *A prepath is a topologized graph. Every edge is a cutedge or a loop. Every loop is an endpoint.*

Proof: Let X be a prepath. If X is empty, it is trivially a topologized graphwith the claimed properties. If X is a singleton, its unique point is trivially closed (and open). There are only two connected topologies on two points; the indiscrete topology is excluded in the definition of a prepath and the other one is a topologized graph with one vertex and one loop.

So suppose that $|X| > 2$. We need to show that for any $x \in X$, $\{x\}$ is either open or closed, and if it is open, its boundary consists of at most two points. If x is a cutpoint, from (2.1.12) we have that it is open or closed. Moreover, by (2.2.1) all cutpoints are strong cutpoints. So, if the cutpoint x is open, from (2.1.16) we have that its boundary consists of precisely two points, that is, x is a proper edge. Since x is a cutpoint, it is a cutedge.

Suppose instead that x is an endpoint. Since X has at least three elements and at most two endpoints, there must be at least one cutpoint. For all cutpoints y, let A_y, B_y be the connected components of $X \setminus \{y\}$, labelled so that $x \in A_y$, and define $Z := \bigcap_{y \in C} \overline{A_y}$, where C denotes the set of cutpoints.

Clearly $x \in Z$. Now suppose y is an endpoint in Z distinct from x, and choose any cutpoint z. Since $y \in Z$, we have that $y \in \overline{A_z}$, which by 2.1.8 implies that either $y \in A_z$ or

$y = z$. The former means that x and y are in the same connected component of $X \setminus \{z\}$, contradicting (C′) of (2.2.2), and the latter is impossible because y and z are distinct since z is a cutpoint and y an endpoint. Hence Z contains no endpoints apart from x.

Suppose that y_1, y_2 are distinct cutpoints in Z. Since x is an endpoint, it cannot disconnect y_1 and y_2; by applying (B) of (2.2.1) to $\{y_1, y_2, x\}$, we obtain that either y_1 disconnects x and y_2 or y_2 disconnects y_1 and x; without loss of generality, we may assume the former. But then $y_2 \notin A_{y_1}$, whence $y_2 \notin Z$, a contradiction.

So Z can contain at most one element besides x, and that element, if it exists, is a cutpoint. If $Z = \{x\}$, then $\{x\}$, being the intersection of closed sets, is closed. If instead $Z = \{x, y\}$ for some cutpoint y, then $y \in \overline{A_y}$. So by (2.1.12), $A_y = \{x\}$ is open and $\mathsf{Cl}(x) = \{x, y\}$, that is, x is a loop. ✠

2.2.8 Fact: Let C be a connected subset of a connected topological space X, and K a component of $X \setminus C$. Then $X \setminus K$ is connected.

Reference: See [10], Chapter 0, Lemma 2. □

2.2.9 *Corollary:* *In a connected topological space, among any three points at most one can disconnect the other two.*

Proof: Suppose y disconnects x and z, and let K_x denote the component of $X \setminus \{y\}$ containing x. By Fact 2.2.8, $X \setminus K_x$ is a connected set containing y and z but not x. Hence x can not disconnect y and z. Similarly, z can not disconnect x and y. ✠

The following proposition essentially says that the distinction between separating and disconnecting two points, illustrated by Example 2.2.5 in the context of connection-wise orderable spaces, is lost in the context of (connected) prepaths.

2.2.10 *Proposition:* *For a connected topological space X, the following are equivalent:*

(A) *there exists a total order $\preceq$ on X such that, $\forall\, x, y, z \in X : x \prec y \prec z$, y is a strong cutpoint and disconnects x and z;*

(B) *there exists a total order $\preceq$ on X such that, $\forall\, x, y, z \in X : x \prec y \prec z$, y separates x and z;*

(C) *there exists a total order $\preceq$ on X such that, $\forall\, x, y, z \in X : x \prec y \prec z$, y disconnects x and z;*

(D) *among any three points, there exists one (equivalently, precisely one) which disconnects the other two;*

(E) *among any three points, there exists one (equivalently, precisely one) which separates the other two.*

(F) *among any three points, there exists a strong cutpoint (equivalently, precisely one strong cutpoint) that disconnects the other two.*

Proof: Statements (D), (E) and (F) all assert the existence of a point with certain properties among a given triple of points. *A priori* these statements become stronger with the requirement of uniqueness. In the case of (D) and (F), from 2.2.9 we immediately have that *de facto* this requirement is not more restrictive. Since a cutpoint which separates two points also disconnects them, the same argument applies to the case of (E).

(C) $\Leftrightarrow$ **(D)**: These are two of the conditions in (2.2.1).

(A) $\Rightarrow$ **(B)**: Suppose $x \prec y \prec z$; then $X \setminus \{y\}$ has precisely 2 connected components, K_x and K_z, with $x \in K_x, z \in K_z$. But then by (1.1.22) $\{K_1, K_2\}$ is a separation of $X \setminus \{y\}$ and y separates x and z.

(B) $\Rightarrow$ **(C)**: This is trivial in the light of (2.1.6).

(F) $\Rightarrow$ **(E)** $\Rightarrow$ **(D)**: Similar to (A) $\Rightarrow$ (B) $\Rightarrow$ (C).

(C), (D) $\Rightarrow$ **(A), (F)**: By (2.2.1), every cutpoint is a strong cutpoint. ✠

2.2.11 ***Proposition:*** *Suppose the total order $\preceq$ on the connected topological space X satisfies condition (B) of Proposition 2.2.10. If the point y separates the points x and z, then either $x \prec y \prec z$ or else $z \prec y \prec z$.*

Proof: There exists a separation $\{A, B\}$ of $X \setminus \{y\}$ such that $x \in A$ and $z \in B$. Suppose, by way of contradiction and without loss of generality, that $y \succ x \succ z$. Then x should separate y and z. This is impossible, because by (2.1.13) $B \cup \{y\}$ is a connected subset containing z and y but not x. ✠

2.2.12 ***Lemma:*** *Let $\preceq, \sqsubseteq$ be total orders on a set X, such that, for all $x, a, b \in X$, x is in between a and b with respect to $\preceq$ if and only if it is in between them with respect to $\sqsubseteq$. Then $\preceq$ and $\sqsubseteq$ coincide up to inversion, that is, either $x \preceq y \iff x \sqsubseteq y$ or else $x \preceq y \iff y \sqsubseteq x$.*

Proof: Note that if X consists of fewer than three points, the assumption is vacuous and the conclusion is trivial. So we may assume that X contains at least three points. Let us say that a set T is "inverted" if, whenever $x_1, x_2 \in T$, we have that $x_1 \prec x_2 \Rightarrow x_2 \sqsubset x_1$. The assumption says that all triples are inverted or preserved, but never "scrambled". Since every pair is contained in some triple, it is sufficient to prove that if some triple is inverted, all of X is inverted.

So suppose $a \prec b \prec c$ and $c \sqsubset b \sqsubset a$ and let x_1, x_2 be arbitrary points such that $x_1 \prec x_2$. If x_1 and x_2 are both in $\{a, b, c\}$, we already have $x_2 \sqsubset x_1$. So suppose without loss of generality that $x_1 \notin \{a, b, c\}$ and $x_2 \neq c$. Consider the triple $\{x_1, a, b\}$. This triple cannot be preserved, because then, whichever order these three points fall in with respect to $\preceq$, we would have $a \sqsubset b$ (if x_1 happens to fall in between a and b, we invoke transitivity), contradicting the asymmetry of $\sqsubset$. Hence this triple is inverted.

Now if $x_2 \in \{a, b\}$, we already have that $x_2 \sqsubset x_1$. If not, by an argument similar to the above we conclude the triple $\{x_1, x_2, a\}$ is also inverted, and in particular $x_2 \sqsubset x_1$. ✠

2.2.13 ***Corollary:*** *Let X be a connected topological space and $\preceq$ a total order on X. Then conditions (A), (B) and (C) of Proposition 2.2.10 are equivalent properties of $\preceq$. Moreover, if they are satisfied they uniquely identify $\preceq$ up to inversion; in particular, there are precisely two possible choices for $\preceq$. Furthermore, if $\preceq$ is such a choice, a given point disconnects, or separates, two other points if and only if, with respect to $\preceq$, it lies in between them.*

Proof: Clearly, if $\preceq$ satisfies any one of conditions (A), (B), (C), then so does the binary relation $\subseteq$ defined by $a \subseteq b \iff b \preceq a$. The assertion now follows from (2.2.10), (2.2.11) and (2.2.12). ✠

2.2.14 Definition: Given a prepath P, a total order on P satisfying conditions (A), (B), (C) of Proposition 2.2.10 is said to be compatible with P.

2.2.2 Connected subsets, jumps, gaps and intervals

2.2.15 ***Proposition:*** *A subset of a prepath P is connected if and only if it is a prepath. Given a connected subset, the compatible total orders are precisely those inherited from those of P.*

Proof: By definition of prepath, a disconnected subset is not a prepath. Conversely, if S is a connected subset, let $\preceq_S$ be the total order inherited by S from a compatible total order $\preceq$ on P. If $x, y, z \in S$ and $x \prec_S y \prec_S z$, then $x \prec y \prec z$ and y disconnects x and z, that is, x and z belong to distinct components of $X \setminus \{y\}$. But then *a fortiori* they belong to distinct connected components of $S \setminus \{y\}$. Hence $\prec_S$ on S satisfies (C) of (2.2.10). ✠

2.2.16 ***Proposition:*** *Prepaths have at most two endpoints; moreover, a point is a endpoint if and only if it is a maximum or a minimum with respect to a compatible total order.*

Proof: Suppose a, b, c are distinct endpoints in X. Then one of them must disconnect the other two, and in particular must be a cutpoint, a contradiction. So there can be at most two endpoints.

Suppose x is not a minimum nor a maximum with respect to a compatible total order $\preceq$. Then there exist points a, b such that $a \prec x \prec b$, again implying that x is a cutpoint. So every non-cutpoint is a maximum or a minimum. Conversely, if x is a cutpoint, then x separates some two points a, b and therefore by Proposition 2.2.11 we must have $a \prec x \prec b$ or $b \prec x \prec a$, implying that x is neither a maximum nor a minimum. ✠

2.2.17 Definition: Given a prepath P, a point $p \in P$ is terminal if it is an extremum (equivalently, a maximum or a minimum) with respect to a compatible total order, and internal otherwise. Equivalently (by (2.2.16)), the endpoints are the terminal points and the cutpoints are the internal points.

A prepath is bounded if the compatible total orders are bounded. If some compatible total order is bounded from above but not from below, the prepath is one-sided.

2.2.18 Remark: A graph-theoretic path, as defined in Definition 1.1.16, is a prepath in the sense of Definition 2.2.6 when equipped with the classical topology. Moreover, the classical topology also reconciles the meanings of the terms "bounded" in the two definitions, and the terminal points of a graph-theoretic path in the sense of Definition 2.2.6 are precisely the terminal vertices in the sense of Definition 1.1.16. However, a terminal point of a prepath need not be a vertex.

2.2.19 ***Lemma:*** *Let x be an internal point of a prepath P, and fix a compatible total order. Then precisely one of the following must occur:*

(A) *x is an edge, $\mathtt{A}(x)$ and $\Omega(x)$ are open, $\alpha(x)$ and $\omega(x)$ are closed, $(\alpha(x))^\diamond = \mathtt{A}(x)$ and $(\omega(x))^\diamond = \Omega(x)$;*

(B) *x is a vertex, $\mathtt{A}(x)$ and $\Omega(x)$ are closed, $\alpha(x)$ and $\omega(x)$ are open, $\mathsf{Cl}(\alpha(x)) = \mathtt{A}(x)$ and $\mathsf{Cl}(\omega(x)) = \Omega(x)$.*

Proof: Let x be an internal point, and let $\preceq$ denote the fixed compatible order. By (2.2.10) x is a strong cutpoint; let K_1, K_2 be the two components $P \setminus \{x\}$. By (2.1.12) it is now sufficient to prove that $\{K_1, K_2\} = \{\alpha(x), \omega(x)\}$. If not, there exist $i \in \{1, 2\}$ and points $a, b \in K_i$ such that $a \prec x \prec b$. Then x is supposed to disconnect a and b. This is impossible, since K_i is a connected set containing a and b but not x. ✠

2.2.20 Remark: The above lemma says in particular that if x is a cutpoint, precisely one of $\alpha(x)$ and $\mathtt{A}(x)$ is open, and similarly for $\omega(x)$ and $\Omega(x)$. Also, notice that if (B) occurs, no open subset of $\Omega(x)$ or $\mathtt{A}(x)$ contains x, since its complement necessarily contains $\alpha(x)$ or $\omega(x)$ respectively, whose closure contains x. Hence in both cases there exists a largest open subset of $\Omega(x)$ and $\mathtt{A}(x)$.

2.2.21 Definition: Let a be a point of a prepath P, with a fixed compatible order. The open tail from a upwards is the largest open subset of P contained in $\Omega(a)$, and the open tail from a downwards is the largest open subset of P contained in $\mathtt{A}(a)$.

2.2.22 Note: If a is a minimum with respect to a compatible total order of a prepath P, the open tail from a downwards is $\{a\}$ or the empty set, according to whether $\{a\}$ is open or not. If $\{a\}$ is a maximum, the tail is P. If a is a cutpoint, it is $\alpha(x)$ or $\mathtt{A}(x)$ according to which of the scenarios (A), (B) in (2.2.19) occur. The open tail upwards behaves analogously.

2.2.23 Definition: If $\preceq$ is a total order on a set X, an interval is a subset of X of the form $L \cap U$, where U is of one of the following forms:

1. X;
2. $\omega(a)$ for some $a \in X$;
3. $\Omega(a)$ for some $a \in X$;

and L is of one of the following forms:

1. X;
2. $\alpha(b)$ for some $b \in X$;
3. $\mathtt{A}(b)$ for some $b \in X$.

We denote an interval obtained in this way by the string "s_1, s_2" where s_1 is the substring "$(-\infty$", "$[a$", "$(a$" or according to whether L is of the first, second or third form respectively, and s_2 is "$\infty)$", "$b]$", "$b)$" or according to whether U is of the first, second or third form respectively. Thus for example $(a, b]$ is the set $\{x \in X \mid a \prec x \preceq b\}$.

Clearly, if L is of the second form, then a is a minimum for the interval and if U is of the second form, then b is a maximum for the interval. Note that if $\{x, y\}$ is a jump, then $(-\infty, x] = (-\infty, y)$. Thus an interval may be expressed in more than one way in the above notation. However, the expression is unique if X is unbounded, the interval is non-empty and there are no jumps, or else if we think of $\pm\infty$ as "auxiliary" extrema, discard cases (1) (each one being subsumed into the corresponding case (2)) and only allow a, b to be extrema for the interval. Moreover, it is always possible to (uniquely) express an interval subject to the latter restriction (with this convention, assuming a set is an interval, the extrema always exist, although they may not belong to X).

If X is a prepath and $\preceq$ is a compatible total order on X, then L may be the open tail from b downwards for some $b \in X$ (L will be of form (2) or (3)). In this case, we may denote this by the string "$b\rangle$" for s_2. Similarly, if U is the open tail from a upwards for some $a \in X$, we use the string "$\langle a$". Thus, for example, if e is an edge and v a *non-terminal* vertex of a graph-theoretic path X equipped with the classical topology, then, choosing a compatible total order $\preceq$ such that $e \prec v$, we have that $\langle e, v\rangle = [e, v)$.

The distinction between convex subsets and intervals may seem to be superfluous. It is trivial to see that intervals are convex, but the converse requires the total order to be complete. We shall deal with this in (2.2.28).

2.2.24 Definition: Let H be a topologized hypergraph and p_1, p_2 points in H. Then $\{p_1, p_2\}$ is an edge-vertex incident pair if for some $i \in \{1, 2\}$ we have that p_i is a vertex is p_{3-i} is an incident edge.

2.2.25 ***Proposition:*** *Let P be a prepath. Then*

1. *$\{x, y\}$ is a jump with respect to a compatible total order if and only if it is an edge-vertex incident pair; and*

2. *internal edges have precisely 2 endvertices and terminal edges precisely 1 endvertex.*

Proof: Let $\preceq$ be a compatible total order, and $\{x, y\}$ a jump. Without loss of generality we may assume that $x \prec y$; then P is the disjoint union of $\mathtt{A}(x)$ and $\Omega(y)$. By (2.2.19), if x, y are both vertices, then both these sets are closed, while if they are both edges, then both these sets are open. In either case, $\{\mathtt{A}(x), \Omega(x)\}$ would be a separation of P, contradicting the fact that P is connected.

Conversely, suppose that e is an edge, v is an incident vertex and, by way of contradiction, that $\{e, v\}$ is not a jump. Assuming, without loss of generality, that $e \prec v$, this translates into the existence of an x such that $e \prec x \prec v$. By (2.2.16), y is a cutpoint, and by (2.2.19) and (2.2.22), $\langle x, \infty)$ is an open set containing v and not e, contradicting the fact that e and v are incident. This concludes the proof of (1).

As for (2), from (2.2.7) we know that every edge is a cutedge or a loop. By (2.2.16), an internal edge is a cutedge, and therefore by (2.1.17) has precisely two endvertices. Again by (2.2.16), terminal points are not cutpoints, so they are loops, that is, they are incident with precisely one vertex. ✠

2.2.26 ***Lemma:*** *Let A be a convex subset of a prepath P with at least two points, and suppose $a \in A \setminus A^\circ$. Then a is a maximum or minimum for A, but not an extremum for P.*

Proof: Let $\preceq$ be a compatible total order for P. Suppose there exist $x, y \in A$ such that $x \prec a \prec y$. Then $\langle x, y\rangle$ is open, contains a and, since A is convex, is contained in A. Hence $a \in A^\circ$, a contradiction.

So a is a maximum or a minimum of A. Now let b be another point in A. Note that an extremum of P is a maximum or a minimum. If a is a minimum of P, then $[a, b\rangle = (-\infty, b\rangle$ is an open subset of A containing a, again a contradiction. So a cannot be a minimum for P, and similarly, neither a maximum, that is, a is not an extremum for P. ✠

In the following proposition, the proof of Claim 1 is inspired by that of Kok ([35], Chapter I, Section 2, Theorem 3), in the context of orderable spaces. Note, however, that at this stage we do not know that convex subsets are intervals.

2.2.27 ***Theorem:*** *A subset of a prepath is connected if and only if it is convex.*

Proof: Suppose P is a prepath with compatible total order $\preceq$. If $J \subseteq P$ is not convex, there exist $x, z \in J$ and $y \in P$ such that $x \prec y \prec z$ and $y \notin J$. Then x and z belong to distinct components of $P \setminus \{x\}$, and therefore of J. Hence J is not connected. Conversely, suppose J is convex.

Claim 1: If J is an open convex subset, it is connected.

Proof of Claim 1: Suppose, by way of contradiction, that there exists a separation $\{A, B\}$ of J. We choose $p \in A, q \in B$ and assume, without loss of generality, that $p \prec q$. Since J is open, A and B are open in P. Hence the sets:

$$C := (-\infty, p\rangle \cup (A \cap (-\infty, q\rangle) \qquad D := \langle q, \infty) \cup (B \cap \langle p, \infty))$$

are open in P. Since A and B are disjoint, so are C and D, and since J is convex, $C \cup D$ contains P. Moreover $p \in C$ and $q \in D$. Hence $\{C, D\}$ is a separation of the connected space P, a contradiction. Hence open convex subsets are connected. □

Claim 2: Let J be a convex subset with at least two points. If $a \in J \setminus J^\circ$, then $a \in \mathsf{Cl}(J \setminus \{a\})$, and a is a closed point.

Proof of Claim 2: By Lemma 2.2.26 a is a maximum or minimum for J, but not for P. This implies (by (2.2.16)) that a is a cutpoint. Without loss of generality, let us assume that a is a maximum for J, so that $J \setminus \{a\} \subseteq (-\infty, a)$.

Since $a \notin J^\circ$, a is not an open point; hence by (2.2.19) a is a vertex and $a \in \mathsf{Cl}((-\infty, a))$. Since J contains at least two points, there exists a second point $b \in J$. If b is the[1] predecessor of a, then by (2.2.25) it is an incident edge, and therefore $a \in \mathsf{Cl}(b) \subseteq \mathsf{Cl}(J \setminus \{a\})$. If b is not a predecessor of a, then there exists a point x such that $b \prec x \prec a$. By (2.2.16), x is a cutpoint, so by (2.2.19) and (2.2.22) $\langle x, \infty)$ is an open set containing a but disjoint from $(-\infty, b]$, implying that $a \notin \mathsf{Cl}((-\infty, b])$.

But $a \in \mathsf{Cl}((-\infty, a))$ and $\mathsf{Cl}((-\infty, a)) = \mathsf{Cl}((-\infty, b]) \cup \mathsf{Cl}((b, a))$. This implies that $a \in \mathsf{Cl}((b, a))$. But since J is convex and contains a, b, we have that $(b, a] \subseteq [b, a] \subseteq J$ whence $(b, a) \subseteq J \setminus \{a\}$. Hence $a \in \mathsf{Cl}(J \setminus \{a\})$. □

Now let J be an arbitrary convex subset. If J consists of only one point, it is clearly connected. If J consists of two points, since it is convex it must be a jump, hence by (2.2.25) it is an edge-vertex incident pair, and therefore connected. So we may assume $|J| \geq 3$. Also, if J is open, from Claim 1 J is connected. So we also assume that J is not open, so that there exists a point $a \in J \setminus J^\circ$. By Claim 2, $a \in \mathsf{Cl}(J \setminus \{a\})$ and a is closed. Also, by (2.2.26), a is a maximum or a minimum for J. Without loss of generality, we assume it is a maximum. Note that $J \setminus \{a\}$ is convex.

Suppose first that $J \setminus \{a\}$ is open. Then $J \setminus \{a\}$ is an open convex subset; by Claim 1 it is connected, and since $J \subseteq \mathsf{Cl}(J \setminus \{a\})$ by Fact 2.1.3 J is connected.

Suppose instead that $J \setminus \{a\}$ is not open. Then there exists some point $a' \in (J \setminus \{a\}) \setminus (J \setminus \{a\})^\circ$. Since J contains at least three points, $J \setminus \{a\}$ contains at least two points, as well as being convex. Hence by Claim 2 $a' \in \mathsf{Cl}((J \setminus \{a\}) \setminus \{a'\})$ and a' is closed. Moreover, by (2.2.26) a' is a maximum or a minimum for $J \setminus \{a\}$. If it were a maximum, then $\{a', a\}$ would be a jump, because a is a maximum for J. But since a and a' are both closed points this jump would consist of two vertices, contradicting (2.2.25).

[1] A predecessor may not exist, but if it does it is unique.

Hence a' is a minimum for $J \setminus \{a\}$, and therefore for J, that is, $J = \{a'\} \cup (a', a) \cup \{a\}$. Since no set can have more than one maximum and one minimum, from (2.2.26) we conclude that $(a, a') = J^\circ$ and therefore is open. Of course it is also convex, so by Claim 1 it is connected. Since $(J \setminus a) \setminus a' = (a', a)$ we have that $a' \in \mathsf{Cl}((a, a'))$. Since $J \setminus a = [a', a)$ we have that $a \in \mathsf{Cl}([a', a))$, and since $\mathsf{Cl}(a') = \{a'\}$ we conclude that $a \in \mathsf{Cl}((a, a'))$. Thus $J \subseteq \mathsf{Cl}((a, a'))$, and since (a, a') is connected, by (2.1.3) so is J. ✠

2.2.28 ***Proposition:*** *A subset of a completely totally ordered set is convex if and only if it is an interval.*

Proof: From transitivity it follows that intervals are convex. Conversely, let S be a convex subset. If S has a lower bound, we set $m := \inf S$ and take L to be (m, ∞) or $[m, \infty)$ according to whether $m \notin S$ or $m \in S$, respectively. If S is unbounded from below, we take $L = X$. We choose U analogously so that $S = L \cap U$. ✠

2.2.29 Fact: A total order is order-complete if and only if it has no gaps.

Proof: If (A, B) is a gap, then every element of B is an upper bound for A, and no upper bound for A is in A, because otherwise it would be a maximum of A. Hence the set of upper bounds of A is B. Since B has no minimum, A, which is bounded from above since $B \neq \emptyset$, has no supremum and X is not order-complete.

Conversely, suppose that X is not order-complete, i.e., there exists a nonempty set $S \subseteq X$ such that the set T of upper bounds of S is not empty but does not have a minimum, and let $C := \{x \in S \mid \exists s \in S : x \preceq s\}$. If $x \in X \setminus C$, then x is an upper bound for S, i.e. $C \cup T = X$. Suppose $x \in C \cap T$.

Then by definition of C, T we have that, for some $\overline{s} \in S$, $x \preceq \overline{s} \preceq x$, whence $x = \overline{s} \in S$ and x is a maximum and therefore a supremum of S, contradicting the choice of S. So $C \cap T = \emptyset$. Since C contains S, and S and T are non-empty, $\{C, T\}$ is a non-trivial bipartition of X. Clearly, $\forall\ a \in C, b \in T$, $a \prec b$; moreover, since S does not have a maximum neither does C, and T was assumed not to have a minimum, so (C, T) is a gap. □

2.2.30 ***Proposition:*** *Let P be a prepath and $\preceq$ a compatible total order. Then $\preceq$ is order-complete.*

Proof: By Fact 2.2.29, it is sufficient to show that there are no gaps. By way of contradiction, let (A, B) be a gap. Since in particular (A, B) is a cut, A must be convex. We claim that A is open. If not, there exists some point $a \in A \setminus A^\circ$. By Lemma 2.2.26, a

is a maximum for A, contradicting the definition of a gap. Hence A is open, and similarly so is B. But then $\{A, B\}$ is a separation of the connected space P, a contradiction. ✠

2.2.31 ***Corollary:*** *Let C be a subset of a prepath. The following are equivalent:*

- *C is connected;*
- *C is a prepath;*
- *C is convex;*
- *C is an interval.*

Proof: Follows from (2.2.15), (2.2.27), (2.2.28) (2.2.29) and (2.2.30). ✠

2.2.3 Bounded prepaths

We have defined a bounded prepath as a prepath with two terminal points. The presence of these points allows a slightly different approach to bounded prepaths.

2.2.32 Definition: Let a, b be points of a topological space X; a subset $P \subseteq X$ is an ab-prepath if it is a (bounded) prepath with a and b as the terminal points. Note that a and b need not be distinct. Given an ab-prepath P, the associated total order is the total order compatible with P with respect to which a is a minimum and b is a maximum. A topological space X is prepath connected if for any two points a, b the space X contains an ab-prepath.

2.2.33 ***Proposition:*** *Let a, b be points of a topological space X; then the following two conditions are equivalent:*

(A) $\forall\, x \in X \setminus \{a, b\}$*, x disconnects a and b ;*

(B) *all proper subsets of X containing $\{a, b\}$ are disconnected.*

Proof: **(A) ⇒ (B)**: Assume that (A) holds, let S be any proper subset of X containing $\{a, b\}$, pick $x \in X \setminus S$, and define $Z := X \setminus \{x\}$. Let S_a, S_b be the connected components of S containing respectively a, b (a priori, it might be the case that $S_a = S_b$), and Z_a, Z_b be the connected components of Z containing respectively a, b. Note that $S \subseteq Z$, so from 2.1.2, and from the fact that connected components form a partition of the point set of a topological space, it follows that $S_a \subseteq Z_a$ and $S_b \subseteq Z_b$. Since Z_a and Z_b are disjoint, S_a

and S_b are distinct. In particular, S is disconnected.

(B) ⇒ (A): Let $x \in X \setminus \{a, b\}$. Then $X \setminus \{x\}$ is a proper subset of X containing a, b, hence is disconnected. Now suppose a, b are in the same connected component K of $X \setminus \{x\}$. Then K is a subset of $X \setminus \{x\}$ containing a, b, and yet it is connected by definition, a contradiction. Hence a and b are in distinct components of $X \setminus \{x\}$. ✠

2.2.34 Note: Condition **(A)** in 2.2.33 can not be replaced by:

(A′) $\forall x \in X \setminus \{a, b\}$, $X \setminus \{x\}$ is disconnected

For consider the subspace X of $\mathbb{R}^2$ (with the usual Euclidean topology) given by $Z := ([-1, 1] \times \{0\}) \cup (\{0\} \times (0, 1))$, and let $a := (-1, 0), b := (1, 0)$. Clearly Z satisfies (A′) but not (A).

2.2.35 *Proposition:* *Let a, b be points of a connected topological space X such that condition* (B) *(equivalently, condition* (A) *) of 2.2.33 is satisfied. Then $\forall x \in X \setminus \{a, b\}$, $X \setminus \{x\}$ has precisely two components.*

Proof: Fix $x \in X \setminus \{a, b\}$ and suppose that (A_1, A_2) is a separation of $X \setminus \{x\}$. By 2.1.13, for $i = 1, 2$ $A_i \cup \{x\}$ is connected, and is a proper subset of X, so by (B) of 2.2.33, $A_i \cup \{x\}$ cannot contain both a and b. Without loss of generality we may assume $a \in A_1$, $b \in A_2$.

Now suppose $\exists\, i \in \{1, 2\}$ such that A_i is not connected, say (B_1, B_2) is a separation of A_i such that $B_1 \cap \{a, b\} \neq \emptyset$. Now $A_i = (A_i \cup \{x\}) \setminus \{x\}$, and we have already observed that $A_i \cup \{x\}$ is connected, so by 2.1.13 $B_1 \cup \{x\}$ is connected. Also $A_{3-i} \cup \{x\}$ is connected, and $B_1 \cup A_{3-i} \cup \{x\}$ is a connected proper subset of X (it is disjoint from B_2) containing both a and b, a contradiction. So A_1 and A_2 are connected and hence the two components of $X \setminus \{x\}$. ✠

The following terminology is inspired by that of Willard ([61], Definition 28.5).

2.2.36 Definition: A cutting of a topological space X is an ordered triple (A, x, B) such that $\{A, B\}$ is a separation of $X \setminus \{x\}$. We also say that (A, x, B) is a cutting around x, and if $y \in A$, we refer to A as the y-side of the cutting.

2.2.37 Fact: If $(U, x, O), (A_1, y, A_2)$ are cuttings of a connected topological space and $y \in O$, then one of A_1, A_2 is contained in O.

Proof: Let C_1, C_2 be closed subsets of X such that $C_1 \cap (X \setminus \{y\}) = A_1, C_2 \cap (X \setminus \{y\}) = A_2$. It is sufficient to prove that one of C_1, C_2 is disjoint from the connected set $S := U \cup \{x\}$

(by Proposition 2.1.13). This set is contained in their union, since it is contained in $X \setminus \{y\}$. Moreover, $C_1 \cap C_2 \cap S \subseteq C_1 \cap C_2 \cap (X \setminus \{y\}) = \emptyset$, so if both $C_1 \cap S$, $C_2 \cap S$ were non-empty, they would give a separation of S. □

2.2.38 ***Corollary:*** *If (A_1, a, A_2), (B_1, b, B_2) are cuttings of the same topological space and $a \neq b$, then $A_i \cup \{a\} \subseteq B_j$ and $B_{3-j} \cup \{b\} \subseteq A_{3-i}$ for some $i, j \in \{1, 2\}$.* □

2.2.39 ***Corollary:*** *Let (A_1, x_1, B_1), (A_2, x_2, B_2) be cuttings of the same topological space, $x_1 \neq x_2$ $a \in A_1 \cap A_2$ and $b \in B_1 \cap B_2$. Then, for some $i \in \{1, 2\}$, we have that $(A_i \cup \{x_i\}) \subseteq A_{3-i}$ and $B_{3-i} \cup \{x_{3-i}\} \subseteq B_i$; in particular x_i separates a and x_{3-i}, and x_{3-i} separates x_i and b.* □

2.2.40 ***Proposition:*** *Let a, b be points of a connected topological space X, with $|X| \geq 3$. Then the following are equivalent:*

(A) $\forall x \in X \setminus \{a, b\},$ *x disconnects a and b;*

(B) all proper subsets of X *containing $\{a, b\}$ are disconnected;*

(C) $\forall x \in X \setminus \{a, b\},$ *x separates a and b;*

(D) *X is an ab-prepath*;

(E) *there exists a total order $\preceq$ on X such that:*
- *α) $\forall x \in X$ $a \preceq x \preceq b$ (a is a minimum and b a maximum); and*
- *β) $\forall x, y \in X : x \prec y \prec b,$ x and b lie in distinct components of $X \setminus \{y\}$*

(F) *there exists a total order $\preceq$ on X such that:*
- *α) $\forall x \in X$ $a \preceq x \preceq b$ (a is a minimum and b a maximum); and*
- *β′) $\forall x, y \in X : a \prec x \prec y,$ a and y lie in distinct components of $X \setminus \{x\}$*

(G) *there exists a total order $\preceq$ on X such that:*
- *α) $\forall x \in X$ $a \preceq x \preceq b$ (a is a minimum and b a maximum); and*
- *β″) $\forall x \in X : a \prec x \prec b,$ a and b lie in distinct components of $X \setminus \{x\}$*

Note that here conditions (A),(B) are the same as the ones occurring in 2.2.33.

Proof:

(A) $\Leftrightarrow$ (B): This is 2.2.33 with X connected.

(G) $\Rightarrow$ (A): Since $\preceq$ is antisymmetric, by (α) any $x \in X \setminus \{a, b\}$ satisfies $a \prec x \prec y$, and by (β'') a and b lie in distinct components of $X \setminus \{x\}$.

(E) $\Rightarrow$ (G), **(F) $\Rightarrow$ (G)**: trivial

(A),(B) $\Rightarrow$ (C): By 2.2.35, $\forall x \in X \setminus \{a, b\}$, $X \setminus \{x\}$ has precisely two components, each of which (by condition (A)) contains exactly one of a, b. Hence $\{A, B\}$ is a separation of $X \setminus \{x\}$ and x separates a and b.

(C) $\Rightarrow$ (D): For all $z \in X \setminus \{a, b\}$, by (C) there exists a cutting (A_z, z, B_z) with $a \in A_z, b \in B_z$. First we show that among any triple T of points, one separates the other two. If $\{a, b\} \subseteq T$, this is given by (C). If $T \cap \{a, b\} \neq \emptyset$, we may assume that $T = \{a, x, y\}$, for some two points x, y distinct from b. By (C), both x and y separate a and b. Then by 2.2.39 either $y \in B_x$, implying that x separates y and a, or else $x \in B_y$, implying that y separates x and a.

If $T \cap \{a, b\} = \emptyset$, then by (2.2.39) the elements of T can be labelled x_1, x_2, x_3 so that $A_{x_1} \cup \{x_1\} \subseteq A_{x_2}$, $A_{x_2} \cup \{x_2\} \subseteq A_{x_3}$, $B_{x_3} \cup \{x_3\} \subseteq B_{x_2}$, $B_{x_2} \cup \{x_2\} \subseteq B_{x_1}$; in particular, $x_1 \in A_{x_2}$ and $x_3 \in B_{x_2}$, and x_2 separates x_1 and x_3. Thus X is a prepath.

It remains to be shown that a and b are not cutpoints. Suppose a is a cutpoint. Then it disconnects b from some third point y. However, y disconnects a and b. Thus among the three points a, y, b, there is more than one point which disconnects the other two. This contradicts Corollary 2.2.9.

(D) $\Rightarrow$ (E): Since X is a prepath, by (2.2.1) there exists a total order $\preceq$ on X such that $x \prec y \prec z$ implies that y disconnects x and z. This implies condition (β). Since a and b are the terminal vertices of X, one of them is a minimum and the other a maximum. By (2.2.13), we may choose $\preceq$ so that a is a minimum and b is a maximum.

Conclusion: We have shown the equivalence of all conditions except (F), and also that (F) implies (G), so it is sufficient to show (E) $\Rightarrow$ (F). Since (E) and (A) are equivalent, and (A) is symmetric with respect to a, b, we may assume (E) with the rôles of a, b reversed;

this amounts precisely to (F). ✠

2.2.41 Note: Unlike the case of 2.2.33, without the hypothesis that X is connected the conditions of 2.2.40 are no longer equivalent. In fact, consider the subspace X of $\mathbb{R}^2$ (with the usual Euclidean topology) given by the disjoint union of the open unit disc centred at a and the singleton $\{b\}$. The space X clearly satisfies (A) and (B), but not (F), because for any total order $\preceq$ on X and any two points $x, y \in X \setminus \{a, b\}$ with $x \prec y$, a and y are in the same component of $X \setminus \{x\}$.

2.2.42 Definition: Given points a, b in a connected topological space X, $S(a, b)$ denotes the set of points which separate a and b. Also, $S[a, b) = \{a\} \cup S(a, b)$, $S(a, b] = S(a, b) \cup \{b\}$ and $S[a, b] = S(a, b) \cup \{a, b\}$.

2.2.43 Fact: Let a, b be two points in a connected topological space; then the binary relation on $S[a, b]$ given by

$$x \preceq y \iff \begin{cases} x = y, & \text{or} \\ x = a, & \text{or} \\ x \text{ separates } a \text{ from } y, & \end{cases}$$

is a total order. For any two distinct points $x, y \in S(a, b)$, the following are equivalent:

- $x \prec y$;
- x is on the a-side of every, equivalently some, ab-cutting around y;
- for some, equivalently every, ab-cutting around x, the a-side is contained in the a-side of some, equivalently every, ab-cutting around y;
- for some, equivalently every, ab-cutting around x, the b-side contains the b-side of some, equivalently every, ab-cutting around y.

For any three points $x, y, z \in S[a, b]$, we have that y separates x and z if and only if $x \prec y \prec z$ or $z \prec y \prec x$.

Reference: This appears in most textbooks on topology in one form or another. It is usually proved by means of separations and results such as (2.2.37), (2.2.38), (2.2.39). In fact, the weaker concept of disconnection is sufficient. □

Note that, in general, $S[a, b]$ may be far from connected. In fact, this will be a central point for us in Chapter 6.

2.2.44 ***Corollary:*** *Let X be a connected topological space. Then X is an ab-prepath if and only if $X = S[a, b]$.* ✠

2.3 Paths

2.3.1 The Interval Topology and Local Connectedness

Consider the subset $\mathcal{W}$ of $\mathbb{R}^2$ defined by $\{(0,0)\} \cup \{(x,y) \mid x \in (0,1],\ y = \sin(1/x)\}$, equipped with the relative topology inherited from the Euclidean topology on $\mathbb{R}^2$. Clearly, this space is a prepath and the total order $\preceq$ defined by $a \preceq b \iff p(x) \leq p(y)$, where $p((x,y)) = x$ and $\leq$ is the usual order on the reals, is a compatible total order.

Moreover, the projection p is an order-isomorphism from $\mathcal{W}$ to $[0,1]$ (again with the Euclidean topology), which is itself a prepath with $\leq$ as a compatible order. Topologically, these two spaces differ in that one is locally connected while the other is not, but the compatible orders are essentially the same. This implies that given a total order $\preceq$ on a set X, there may exist very different topologies on X such that X is a prepath compatible with $\preceq$. However, in view of (2.2.31), in this case the collections of connected subsets are the same.

2.3.1 Definition: Let P be a prepath. Then the collection of open[2] intervals of P is a base for a second topology on P. This topology is the interval topology on P.

2.3.2 Remark: Clearly, an equivalent way of obtaining the interval topology is to take for a subbase the collection of sets consisting of X and all sets of the form $(-\infty, v)$, (v, ∞) for some vertex v. We shall use this construction (in a slightly different setting) in (2.3.12). Note that if P consists of no more than two points it is essential to include X in the subbase.

If P is a connected orderable space, the interval topology is usually defined in precisely this way, except that v is an arbitrary point. Since connected orderable spaces are T_1, all points of the topologized graph P are vertices, so in this case the definitions coincide.

Back to the level of generality of prepaths, the sets open with respect to the interval topology are clearly open with respect to the original topology, that is, the interval topology is coarser than the original topology. This fact is also well-known for connected orderable spaces.

2.3.3 Fact: A topological space is locally connected if and only if the components of every open set are open.

Reference: See [61], Theorem 27.9. □

2.3.4 ***Proposition:*** *A prepath is locally connected if and only if the topology coincides with (equivalently, is no finer than) the interval topology.*

[2] We are using "open" here in the topological sense (as always).

Proof: Let P be a prepath, and suppose that P is locally connected. Let $x \in U$ and U be an open set in P containing x. We wish to show that U contains an open interval containing x. Since $\mathbf{K}_U(x)$ is connected, by (2.2.31) it is an interval, and since P is locally connected and U is open, by Fact 2.3.3 $\mathbf{K}(U)x$ is open.

Conversely, suppose P has the interval topology and $U \subseteq P$ is open. Then $\bigcup_{i \in I} J_i$, where the J_i are open intervals. Every component K of U is an open interval and if $x \in K$ there exists an i such that $x \in J_i$. But J_i is a connected subset of U and hence $J_i \subseteq K$. ✠

Again, the content of (2.3.4) is well-known for connected orderable spaces.

2.3.5 Definition: A path is a locally connected prepath. An ab-prepath which is a path is also called an ab-path. A total order $\preceq$ on a set X is (simply) [pre]path compatible if there exists a topology on X such that X is a (simple) [pre]path and $\preceq$ is compatible with X.

2.3.6 *Proposition:* *Let $(X, \mathcal{T})$ be a prepath. When X is endowed with the interval topology, X is a path with the same compatible orders, and therefore the same collection of connected subsets.*

Proof: Since the interval topology $\mathcal{T}'$ is contained in $\mathcal{T}$, a separation in the interval topology would be a separation in the original topology. Hence, with respect to $\mathcal{T}'$, X is connected.

Now let $\preceq$ be a compatible total order on X, with respect to $\mathcal{T}$. Let x, y, z be points such that $x \prec y \prec z$. If y is a vertex, then $\{(-\infty, y), (y, \infty)\}$ is a separation of $X \setminus \{y\}$, and y separates x and z. If y is an (internal) edge, it has a predecessor a and a successor b. Then $\{(-\infty, b) \setminus \{y\}, (a, \infty) \setminus \{y\}\}$ is a separation of $X \setminus \{y\}$. Note that a and b may coincide with x and y respectively, but in all cases y separates x and z. Hence, with respect to $\mathcal{T}'$, X is a prepath, compatible with the *same* total order $\preceq$.

This last point implies that the intervals with respect to $\mathcal{T}'$ (the intervals with repect to the orders compatible with $\mathcal{T}'$) are precisely the intervals with respect to $\mathcal{T}$. Hence the collections of *open* intervals with respect to the two topologies coincide (in one direction, because the open intervals are the basic sets, and therefore open, and in the other because the arbitrary union of open sets is open). In particular, $\mathcal{T}'$ is its own interval topology, and therefore, by (2.3.4), X is locally connected with respect to $\mathcal{T}'$, that is, $(X, \mathcal{T}')$ is a path. ✠

2.3.2 Characterizing prepath compatible orders

The comparison of the space $\mathcal{W}$ with the closed unit interval of real numbers at the beginning of the preceding subsection shows that a compatible total order does not uniquely determine the topology of a prepath. This is still true if we assume that the prepath is a path.

Consider a total order on a set X of three elements. There are two topologies on X such that the resulting topological space is a prepath compatible with the order: in one case, there is one edge, which is proper, with two endvertices, and in the other a single vertex incident with two loops. In both cases the prepath is path, and the two topologies are not homeomorphic.

This example is perhaps not very satisfying because one of the paths in question contains two loops. The set of integers gives another instance of a total order with two compatible path topologies, without the presence of loops: it can be "topologized" in two different ways so that the resulting space is a two-way infinite graph-theoretic path with the classical topology—in one case the vertices are the odd integers and in the other the even integers. However, in this example the two topologies are homeomorphic.

In this section we see how these two examples essentially embody all the possible scenarios regarding the issue of a path topology compatible with a given total order.

2.3.7 Definition: Let X be a totally ordered space. A convex subset S of X is a chain if it is order isomorphic to a convex subset of $\mathbb{Z}$.

2.3.8 Remark: The term "chain" is often used in an order-theoretic context to mean "totally ordered set". Clearly this is a different meaning from the one we intend.

2.3.9 *Lemma:* *Let X be a totally ordered space. Every chain is contained in a maximal chain.*

Proof: Let C be a chain. Clearly a chain maximal subject to containing C is maximal among all chains. So it is sufficient to consider only the chains which contain C. If all chains are empty, then C is empty and maximal. If there exists a non-empty chain, we may assume that C is non-empty.

Consider the set $\mathcal{Z}$ of chains containing C. We show that the union of *all* chains (some of which may be incomparable under set inclusion) in $\mathcal{Z}$ is itself a (maximal) chain. Let $c \in C$, $z \in \mathbb{Z}$. For all $Z \in \mathcal{Z}$, there exists an order-isomorphism ϕ_Z of Z onto a convex subset of $\mathbb{Z}$. By translating (adding the constant $z - \phi_Z(c)$) if necessary, we may assume that, for all $Z \in \mathcal{Z}$, $\phi_Z(c) = z$. This implies that, given any two chains Z_1 and Z_2 in $\mathcal{Z}$, if $x \in Z_1 \cap Z_2$, then $\phi_{Z_1}(x) = \phi_{Z_2}(x)$ (by induction on $|\phi_{Z_1}(x) - z|$).

Now let $\check{Z} = \bigcup_{Z \in \mathcal{Z}} Z$. Since the order-isomorphisms never disagree, we may define the function ϕ on $\check{Z}$ which associates to x the image $\phi_Z(x)$ for any $Z \in \mathcal{Z}$ such that $x \in Z$. Note that since, for all $Z \in \mathcal{Z}$, we have $c \in Z$, for any two points $x, y \in \check{Z}$ with $x \prec y$, either $x, y \in Z$ for some $Z \in \mathcal{Z}$, or else $x \prec c \prec y$ and $[x, c] \subseteq Z_1$, $[c, y] \subseteq Z_2$,

$[x, y] = [x, c] \cup [c, y] \subseteq \check{Z}$ for some $Z_1, Z_2 \in \mathcal{Z}$. Hence $\check{Z}$ is convex and, by the same argument, the union of the images in $\mathbb{Z}$, that is $T := \bigcup_{Z \in \mathcal{Z}} \phi_Z(Z)$, is convex. Since, for all $Z \in \mathcal{Z}$, the mapping ϕ_Z is an order-isomorphism, the same argument also shows that so is ϕ. Note that $T = \phi(\check{Z})$, so $\check{Z}$ is a chain. ✠

2.3.10 ***Proposition:*** *Given a totally ordered set, any two maximal chains are disjoint.*

Proof: Suppose A, B are distinct maximal chains. Since they are both maximal, neither contains the other. So there exist points $a \in A \setminus B$, $b \in B \setminus A$. Since A and B are both convex, we may assume that a is a lower bound for B and b an upper bound for A with respect to $\preceq$.

Suppose, by way of contradiction, that $c \in A \cap B$. We claim that A must have a maximum. Suppose not; then $\{x \in A \mid x \succ c\}$ is infinite. Moreover since B is convex, and contains c and b, which is an upper bound for A, this set is contained in B. Let ϕ_B be an order isomorphism of B onto a convex subset S_B of $\mathbb{Z}$. Then in $\mathbb{Z}$ there exist infinitely many integers in between $\phi_B(c)$ and $\phi_B(b)$, a contradiction. So A has a maximum m.

Since $b \in B$ is an upper bound for A and B is convex, m is in B and has a successor m' not in A. Let ϕ_A be an order isomorphism of A onto a convex subset S_A of $\mathbb{Z}$. Then the mapping ϕ' on $A \cup \{m'\}$ which associates to x the image $\phi_A(x)$ when $x \in A$, and $\phi_A(m)+1$ when $x = m'$, is an order-isomorphism onto the convex subset $S_A \cup \{\phi_A(m) + 1\}$ of $\mathbb{Z}$, implying that $A \cup \{m'\}$ is a chain and contradicting the maximality of A. ✠

2.3.11 Definition: A subset S of a set X equipped with a binary relation on a set is strongly lower-bounded (upper-bounded) if S has a minimum (maximum) m which is itself strictly bounded from below (respectively, from above). A subset is partially strongly bounded if it is strongly lower-bounded *or* strongly upper-bounded, and strongly bounded if it is both strongly lower-bounded *and* strongly upper-bounded.

Clearly any chain with a maximum and a minimum is finite. In particular, strongly bounded maximal chains are finite.

2.3.12 ***Proposition:*** *Suppose $\preceq$ is a complete total order with the property that all strongly bounded (finite) maximal chains are odd. Then*

- *$\preceq$ is path compatible;*
- *if all finite maximal chains are odd, then it is* simply *path compatible.*

Proof: We need to decide which points will be vertices. For every maximal chain C, let ϕ_C be an arbitrary order-isomorphism of C onto a convex subset of $\mathbb{Z}$. If all finite maximal chains are odd, we choose the order isomorphisms so that every maximum (and every minimum) is mapped to an even number.

For the purposes of this proof, given any two neighbourly points x, y belonging to the same maximal chain, let us say that their "distance" is $|\phi_C(x) - \phi_C(y)|$. Also, a point is "neighbourly" if it has a successor or a predecessor, and "lonesome" otherwise. Note that every neighbourly vertex belongs to, and every jump is contained in, some maximal chain; in particular, the maximal chains partition the neighbourly vertices of X. If x belongs to a maximal chain C which is partially strongly bounded, then the minimum and maximum of C (at least one of which exists) are "guardians" for x. The fact that strongly bounded maximal chains are odd implies that whenever a point has two guardians, the parity of the distance from a guardian is independent of the guardian.

Now we declare a point x to be a "vertex" if it satisfies one of the following conditions:

(A) x is lonesome;

(B) x is neighbourly, belongs to a partially strongly bounded maximal chain, and its distance from its guardian(s) is even;

(C) x is neighbourly, belongs to a maximal chain C which is not partially strongly bounded, and $\phi_C(x)$ is even.

Now we define a $\mathcal{T}$ topology on X by taking as a subbase the collection of subsets consisting of X and all subsets of the form $(-\infty, x)$ and (x, ∞) for vertices x; equivalently, a set U is open if and only if $U = X$ or else $x \in U$ implies that there exist vertices v_1, v_2 such that $x \in (v_1, v_2) \subseteq U$. Note that this is the same construction described in Remark 2.3.2, except of course that we do not know yet that X is a prepath.

It is now a routine matter to check that

1. any jump consists of a vertex and a non-vertex;
2. every non-vertex which is bounded from above (below) has a successor (predecessor);
3. for any two points x, y, precisely one of the following occurs:
   - $\{x, y\}$ is a jump;
   - x and y are both vertices, and there exists a unique point in between, which is not a vertex;
   - there exists a vertex in between x and y;
4. $\mathcal{T}$ is indeed a topology;

5. with respect to $\mathcal{T}$, every vertex is a closed point, and every non-vertex is an open point with a boundary consisting precisely of its neighbours;

6. if U is open with respect to $\mathcal{T}$, and $x \in U$ is a vertex bounded from above (below) in X, then U contains some point $y \succ x$ (respectively $y \prec x$) such that $[x, y] \subseteq U$ ($[y, x] \subseteq U$).

To see that the resulting space is connected, let $\{A, B\}$ be a separation and pick $a \in A, b \in B$. Without loss of generality, we may assume $a \prec b$. Let $B' = \{x \in B \mid a \preceq x\}$, $b' = \inf B'$, $A' = \{a \in A \mid a \preceq b'\}$, and $a' = \sup A'$ (the existence of b' and a' is guaranteed by completeness).

Now we have that $a' \preceq b'$ and either $a' \in A$, which implies $a' \in A'$ (since b' is an upper bound for A'), or else $a' \notin A$, which implies $a' \in B$ and $a' = b' \in B'$. So one of $a' \in A'$ or $b' \in B'$ must hold.

Suppose $a' \in A$. Then a' is a maximum for A'. Since A is open, if a' is a vertex, by (6) there exists some point $y \in A$ larger than a' such that $[a', y] \subseteq A$; hence $[a', y] \subseteq A'$ and in particular A' contains some point larger than a', a contradiction. So a' must be a non-vertex, and therefore by (2) has a successor which, by (5), belongs to the same component of X as a' does, and therefore to A and A', again a contradiction. Similarly, $b' \in B'$ leads to a contradiction. Hence $(X, \mathcal{T})$ is connected.

From (3) it follows that among any three points there exists one which separates the other two, and in fact this is the one which lies in between the other two. Hence $(X, \mathcal{T})$ is a prepath, and $\preceq$ is a compatible total order. Moreover, by (5), the points we have been referring to as "vertices" are precisely the vertices of the topologized graph P, and the non-vertices the edges. By Remark 2.3.2, $\mathcal{T}$ is the interval topology, and by (2.3.4) the prepath is locally connected, that is, a path. Finally, if all finite bounded maximal chains are odd, by the choice of the order-isomorphisms and from the fact that the boundary of an edge consists precisely of its neighbours it follows that all non-vertices are proper edges, that is, X is a simple path. ✠

2.3.13 *Proposition:* *Let P be a prepath and $\preceq$ a compatible total order. Then the strongly bounded maximal chains are odd. If P is simple, the finite maximal chains are odd.*

Proof: Since the jumps of a prepath consist of a vertex and an edge, the points in a maximal chain must be alternately edges and vertices. If a maximal chain is, say, strongly upper-bounded, its maximum m must be a vertex, because otherwise it will be a minimum for P or an internal (proper) edge; in both cases there would be a successor, contradicting the fact that m is maximum in the (maximal) chain. Similarly, if a maximal chain is strongly lower-bounded, its minimum is a vertex.

So if a maximal chain is strongly bounded, it has a vertex for minimum and a vertex for maximum. In particular, it is finite. Since vertices and edges alternate, the only way this can happen is if the maximal chain consists of an odd number of points.

Recall that a point in a prepath is a loop if and only if it is a terminal edge. In particular, a terminal point is never a proper edge. A finite chain is either strongly upper-bounded or else its maximum is a terminal point. So if P is simple, the maximum of a finite maximal chain is a vertex. Similarly, so is its minimum, and by the same argument as above the chain is odd. ✠

2.3.14 ***Theorem:*** *A total order $\preceq$ is:*

- *path compatible (equivalently, prepath compatible) if and only if it is complete and the strongly bounded maximal chains are odd;*
- *simply path compatible (equivalently, simply prepath compatible) if and only if it is complete and the finite maximal chains are odd.*

If $\preceq$ is simply (pre)path compatible, it uniquely determines the topology of a compatible simple path up to homeomorphism.

Proof: From (2.2.30) and (2.3.13) we have that a total order compatible with a prepath must be complete, the strongly bounded maximal chains are odd, and if the order is simply prepath compatible, then the finite maximal chains are odd. Conversely, from (2.3.12), if the strongly bounded maximal chains are odd, then the order is path compatible, and simply path compatible if the finite maximal chains are odd.

Now suppose that $\mathcal{T}_1$, $\mathcal{T}_2$ are topologies on X such that the corresponding topological spaces are simple paths compatible with the same total order $\preceq$. Since these spaces are locally connected, the topologies are the interval topologies, and therefore determined by the identity of the vertices.

Let us recall the terminology in the proof of (2.3.12); note that the "lonesome", "neighbourly" and "guardian" character of a point are determined by the order. We know that lonesome points must be vertices (since the edges of a prepath have neighbours), and we saw in the proof of (2.3.13) that in a prepath guardians must be vertices. Thus if a point is lonesome or neighbourly and has a guardian, it is a vertex with respect to $\mathcal{T}_1$ if and only if it is a vertex with respect to $\mathcal{T}_2$. As for the neighbourly vertices which do not have a guardian, they belong to some maximal chain which is not partially strongly bounded. If such a chain has a minimum m, say, then m is also a minimum for X. Since the topologies in question render X a simple prepath, this minimum must be a vertex, and again the edge/vertex character of a point is uniquely determined. The same holds for neighbourly points belonging to a maximal chain which is not partially strongly bounded but has a maximum.

There remain the maximal chains which are not partially strongly bounded and have no maximum and no minimum. For every such chain C, there exists an order-isomorphism ϕ_C of C onto $\mathbb{Z}$. Then with respect to each of $\mathcal{T}_1$ and $\mathcal{T}_2$, the vertices in C are either the points whose images are the even numbers, or the points whose images are the odd numbers, and these sets need not coincide with respect to the two topologies. Let us refer to such a maximal chain C as "shifted" if the vertex sets do not coincide in C.

Now we consider the mapping ψ which is the identity outside the shifted chains, and associates to every point x belonging to a shifted chain C the point $\psi(x) = \phi_C^{-1}(\phi_C(x)+1)$. This is clearly a bijection of X onto itself which maps the vertices with respect to $\mathcal{T}_1$ bijectively to the vertices with respect to $\mathcal{T}_2$, and is therefore a homeomorphism from $(X, \mathcal{T}_1)$ to $(X, \mathcal{T}_2)$. ✠

2.3.3 Compactness

2.3.15 Definition: A collection of subsets of a set X has the finite intersection property if the intersection of every finite subcollection is non-empty.

2.3.16 Fact: Let X be a topological space. The following are equivalent:

- X is compact;
- each family of closed sets with the finite intersection property has non-empty intersection;
- each net has a cluster point.

Reference: See [61], Theorem 17.4. □

The proof of the following theorem is inspired by the ones in [61], 17.2(b) and 28.12(b).

2.3.17 *Theorem:* *Let P be a bounded prepath. Then P is compact if and only if it is locally connected.*

Proof: Let P be a compact ab-prepath, and let $\leq$ denote the associated total order. To show it is locally connected, by (2.3.4) it is sufficient to show that the open intervals are a base for the topology of P.

Take $u \in U \subseteq P$, with U open in P; we wish to prove the existence of an open interval contained in U and containing u. If u is an edge, then $\{u\}$ is itself an open interval satisfying our requirement. Suppose then that it is a vertex. If it is incident with two edges e_1, e_2, then we can take the set $\{e_1, u, e_2\}$ for our interval. At this stage we may assume that u is not incident with any edge. For suppose it is incident with precisely one edge, say a predecessor edge e, and that we know the assertion to be true when u is a (possibly terminal) vertex of degree zero. Then $U \cap [u, b]$ is an open interval of the ub-prepath $[u, b]$, in which u has degree zero. Hence $U \cap [u, b]$ contains some open (in $[u, b]$) interval containing u, which, by definition of the relative topology, is of the form $W \cap [u, b]$. Since $v \in \mathsf{Cl}(e)$, we have that $e \in W$, and since $[e, \infty)$ is open in P (by (2.2.19)) $W \cap [e, \infty)$ is the required open interval in P containing u and contained in U. The case of a successor edge is clearly analogous.

Suppose then that u has no incident edges. Note that this means that it has no successor and no predecessor. Moreover, for every strict upper bound x of u, there exists a second vertex v such that $u < v < x$; for otherwise (u, x), which contains infinitely many points (since u has no successor), would consist entirely of edges, and picking any edge $e \in (u, x)$, we have that $(-\infty, e) = P \setminus [e, \infty) = \bigcup_{c \in (u,e)} (-\infty, c]$ is open as well as closed, and does not contain e. This contradicts the fact that P is connected. Similarly, for every strict lower bound x of u, there is a vertex w such that $x < w < u$.

Now we define $S := \{(x, y) \in V_P \times V_P \mid x \in \{a\} \cup (-\infty, u),\ y \in \{b\} \cup (u, \infty)\}$ and consider the intersection $I := \bigcap_{(x,y) \in S} [x, y]$. By the above observations, $I = \{u\}$.

Now suppose, by way of contradiction, that every open interval containing u is not contained in U. For every $(x, y) \in S$, let $A_{x,y}$ denote the closed set $[x, y] \setminus U$. We claim that $\{A_{x,y}\}_{(x,y) \in S}$ has the finite intersection property. To see this, note that, for any finite $S' \subseteq S$, $\bigcap_{(x,y) \in S'} [x, y] = [\check{x}, \check{y}]$ for some $(\check{x}, \check{y}) \in S'$. But $[\check{x}, \check{y}]$ contains $\langle \check{x}, \check{y} \rangle$, which is an open interval containing u. By assumption $\langle \check{x}, \check{y} \rangle \setminus U$ is non-empty, and therefore $A_{\check{x},\check{y}} = \bigcap_{(x,y) \in S'} A_{x,y} \neq \emptyset$. Since this holds for arbitrary S', the collection $\{A_{x,y}\}_{(x,y) \in S}$ has the finite intersection property.

Since P is compact Fact 2.3.16 implies that $I' := \bigcap_{(x,y) \in S} A_{x,y}$ is non-empty. However, clearly $I' \subseteq I$ and since $u \in U$, we have that $u \notin I'$, while $I = \{u\}$, a contradiction. This concludes the proof that the topology on P is the interval topology, and therefore P is locally connected.

Conversely, suppose P is locally connected. If P consists of a single point, it is clearly compact. So we may assume that $a \neq b$. Let $\mathcal{U}$ be an arbitrary open cover of P. We wish to show that $\mathcal{U}$ has a finite subcover. Let S be the set of points x for which there exists a finite subset of $\mathcal{U}$ covering $[a, x]$. Since a is contained in some element of $\mathcal{U}$, $a \in S$; in particular S is non-empty. Of course S is bounded from above by b. Since the associated total order of P is complete, S has a supremum s. Clearly, it is sufficient to show that s is a maximum for S.

Claim 1: $s \neq a$.

Proof of Claim 1: We show that there exists $U \in \mathcal{U}$ and an interval I consisting of more than one point such that $a \in I \subseteq U$. This implies that $I \subseteq S$. If a is a vertex, we may choose $U \in \mathcal{U}$ arbitrarily such that $a \in U$; for the topology of P coincides with the interval topology, so there exists an open interval I such that $a \in I \subseteq U$, and since a is closed, and therefore not open, $\{a\} \subsetneq I$. If instead a is an edge, it is incident with a successor vertex v, which belongs to some open set $W \in \mathcal{U}$. Since $v \in \mathsf{Cl}(a)$, the neighbourhood W of v contains a; hence W on its own covers $\{a, v\} = [a, v]$ and $s \geq v > a$. □

Claim 2: $s \in S$.

Proof of Claim 2: In general, the supremum of any set S is either a maximum for S (belongs to S) or else has no predecessor, for otherwise its predecessor would be a smaller upper bound for S. So we may assume that s has no predecessor.

Since $s \neq a$, we have that s is a vertex with no smaller incident edge. Pick a set U from $\mathcal{U}$ containing s. Since the topology is the interval topology, there exists some open interval such that $s \in I \subseteq U$. Let $c = \inf I$. Since s is vertex distinct from a and I is open, $c < s$ (otherwise $c = s$, and $(-\infty, s)$ and $I \cup (s, \infty) = [s, \infty)$ are both non-empty and open). Since s has no predecessor, there exists some point $q \in P$ such that $c < q < s$.

Moreover, there must exist some point $\check{s} \in S \cap (q, s]$, otherwise q would be a upper bound for S smaller than s. Since $\check{s} \in S$, there exists a finite subcover $\mathcal{U}'$ of $\mathcal{U}$ covering $[a, \check{s}]$. Also, since $c = \inf I$ and $c < q < \check{s} < s \in I$, both $\check{s}$ and s belong to I. Hence $[\check{s}, s] \subseteq I \subseteq U$, and $\mathcal{U}' \cup \{U\}$ is a finite subcover of $\mathcal{U}$ covering $[a, \check{s}] \cup [\check{s}, s] = [a, s]$, implying that $s \in S$.

Claim 3: $s = b$

Proof of Claim 3: First we show that s has no successor. If t were a successor of s, one of s and t is a vertex contained in the closure of the other. Pick some element U of $\mathcal{U}$ which contains the vertex, and therefore both points; adding U to a finite subcover of $[a, s]$ (which exists because $s \in S$) we obtain a finite subcover covering $[a, t]$, implying $t \in S$ and contradicting the definition of s. So s has no successor. In particular, s is a vertex.

Suppose, by way of contradiction, that $s < b$. As in the proof of Claim 2, any open set which contains s must contain an open interval containing s and some strict upper bound t of s, so any finite cover of $[a, s]$ actually covers $[a, t]$ for some $t > s$, a contradiction. We conclude that $s = b$. ✠

2.3.18 *Corollary:* *Let P be a bounded prepath. The following are equivalent:*

- *P is compact;*
- *P is locally connected;*
- *the topology of P coincides with the interval topology.* ✠

2.4 Cycles

2.4.1 Definition: Given a set X, a ternary relation $S \subseteq X^3$ is a cyclic order on X if

- $\left.\begin{array}{c} a \neq b \neq c \neq a \\ (a,b,c) \notin S \end{array}\right\} \Leftrightarrow (c,b,a) \in S$
- $(a,b,c) \in S \Rightarrow (b,c,a) \in S$
- $\left.\begin{array}{c} (a,b,c) \in S \\ (a,c,d) \in S \end{array}\right\} \Rightarrow (a,b,d) \in S$

If S is a cyclic order on X and $Y \subseteq X$, then Y inherits a cyclic order S' given by all tuples whose entries are all in Y, that is, $S' = S \cap Y^3$. If S is finite, the resulting ordered space will be called a cyclic subsequence of S.

The definition of cyclic order above is taken from [35]. Note that given any set S with a total order $\preceq$, one can construct a cyclic order on S by taking all triples (a,b,c) such that one of $a \preceq b \preceq c$, $c \preceq a \preceq b$ and $b \preceq c \preceq a$ holds. Given a finite ordered tuple $(x_0, x_1, \ldots x_m)$, we denote by $\langle x_0, x_1, \ldots x_m \rangle$ the set $\{x_0, x_1, \ldots x_m\}$ equipped with the cyclic order thus obtained from the total order $\sqsubseteq$ defined by $x_i \sqsubseteq x_j \iff i \leq j$ in $\mathbb{Z}$.

As observed by Kok [35], one can also go in the other direction: given a cyclic order S on a set X and a point $p \in X$, one can define a total order $\preceq$ on $X \setminus \{p\}$ by $a \prec b \iff a = b$ or $(p,a,b) \in S$. This easy to and fro between total and cyclic orders, together with the results of this chapter about prepaths, can be used to deduce the following proposition.

2.4.2 *Proposition:* *Let X be a connected topological space containing at least four points. The following are equivalent:*

1. *among any four points, no one disconnects the other three, but some two disconnect, and are disconnected by, the other two;*

2. *for any four points a, b, c, d, we have that a, b disconnect c, d if and only if a, c do not disconnect b, d and a, d do not disconnect b, c;*

3. *any quadruple can be uniquely partitioned so that some part disconnects the other, and, with this partition, each part disconnects the other*

4. *X is 2-connected, and every quadruple can be uniquely partitioned so that each part disconnects each other part;*

5. *there is a cyclic order on X such that, for any four points a, b, c, d, we have that a, c separate (equivalently, disconnect) b and d if and only if $\langle w, x, y, z\rangle$ is a cyclic subsequence for some choice of $w, y \in \{a, c\}$ and $x, z \in \{b, d\}$;*

6. *X is 2-connected, and there exists a cyclic order on X such that, if $\langle w, x, y, z\rangle$ is a cyclic subsequence, then $\{x, z\}$ separates w from y;*

7. *X is 2-connected, and among any four points, some two disconnect, and are disconnected by, the other two;*

8. *for any two points x, y, there exist prepaths P_1 and P_2 such that $P_1 \cup P_2 \cup \{x\} \cup \{y\} = X$ and $\mathsf{Cl}(P_1) \cap \mathsf{Cl}(P_2) = \mathsf{Cl}(\{x, y\})$;*

9. *X is S_1, and for every finite subset C, the complement of C and C have the same number of connected components;*

10. *X is S_1, and for any two points x, y, the set $\{x, y\}$ and its complement have the same number of components;*

11. *X is S_1, 2-connected and for any two points x, y, the set $\{x, y\}$ and its complement are either both connected or else both disconnected;*

12. *X is S_1 and no point disconnects, but any two non-incident points do;*

13. *X is S_1, 2-connected, and any two non-incident points can be separated by some other two.* ✠

2.4.3 Definition: A connected topological space X (of arbitrary cardinality) is a precycle if it satisfies condition (12) of Proposition (2.4.2). A cycle is a locally connected precycle.

Equivalently, a precycle is a connected topological space such that $|X| = 2$ and X is a topologized graph consisting of a single vertex and a single loop, or else $|X| \geq 4$ and satisfies the conditions of Proposition (2.4.2).

It is easy to verify that graph-theoretic cycles, when equipped with the classical topology, are cycles in the sense of the above definition. The same is true, of course, for the familiar unit circle, that is, the set of points in $\mathbb{R}^2$ at some constant distance from a given point, equipped with the relative topology inherited from the usual topology on $\mathbb{R}^2$. Moreover, from characterization (9) of the above proposition it follows that precycles are topologized graphs. Kok [35] defines a cyclically orderable space to be a topological space X for which there exists a cyclic order S such that the sets of the form $\{x \mid (a, x, b) \in S\}$ for some $a, b \in X$ are open in X. This by itself does not exclude orderable spaces; in fact, Kok discusses "non-orderable, cyclically orderable" spaces. He characterizes these spaces among connected T_1 spaces as those in which no point disconnects but any two do (Chapter 3, Section 4, Theorem 12). From characterization (12) it follows that a T_1 space is a cyclically orderable, non-orderable space in the sense of Kok if and only if it is a precycle in our sense.

One of the most important issues addressed in this thesis is that of "cycle spaces" in compact topological spaces; any treatment of this issue of course entails some concept of "cycle". In spite of the fact that the range of objects characterized by the above definition is extremely broad, it turns out that this definition is unnecessarily restrictive for our purposes! The reason is that so far we have had approximately equal emphasis on vertices and edges, while "cycle spaces" are really about edges. We believe that these objects represent the right concept, for example, in the setting of *locally connected* (as opposed to compact) topological spaces; and we expect that they can also find useful application in a topological treatment of various other graph-theoretic issues (as opposed to cycle spaces).

However, for us the characterizations in the above proposition will serve only as *inspiration* for two other concepts: "edgecycles" (Chapter 5), and "ferns" (Chapter 6). For this reason, we have chosen to present them early on in the thesis; this also happens to respect the chronology of our progress. However, there will be no logical dependence of the treatment of cycle spaces on the results concerning paths and prepaths; in fact, logically (but perhaps not intuitively) the development would suffer no serious loss if paths were to be treated only after cycle spaces.

The "inspiration" mentioned above comes in two different ways. In the context of cycle spaces, we choose the appropriate property from the list in Proposition (2.4.2), while the striking features generally present in these characterizations, and also shared with the characterizations of prepaths and paths, lead us to ferns. These shared features include their combinatorial flavour, the recurring order-theoretic theme, and the fact that these objects are naturally topologized graphs, in spite of the fact that several of the characterizations do not explicitly require that the space be S_1.

The "appropriate property" required for the application to cycle spaces turns out to be the one given by characterization (12). The importance of this property emerged gradually during this work. In Chapter 5 we shall define edgecycles using a variant of this property.

In [53] we used a variant based on cyclic orders, in a setting more restrictive than the one we shall adopt in Chapter 5. Characterization (12) is not the only one which lends itself to an appropriate variant in the context of Chapter 5, as we shall see in Proposition 5.1.6. However, the variant it leads to is arguably the simplest one, and we have found it the easiest to use in order to prove the desired theorems. The connection to the characterization by Kok was only made in the latest stages of the compilation of this document; indeed, the fact that Kok's version occurs in Theorem 12 of his tract and that our version is the twelfth in our list is purely coincidental. Had we made the connection earlier, our work might have been less arduous. We shall remember it as the "twelfth characterization", not least because this number is not a bad estimate of the number of versions we adopted before finally settling on characterization (12).

Ferns will be a direct generalization of paths to "tree-like" objects; thus, paths will be more important for us in the remainder of this thesis than the present "cycles". However, the change from cycles to edgecycles also implies that not even paths will have a rôle in the treatment of cycle spaces.

Chapter 3

Edge Spaces and Separation Axioms

In Chapter 1 we introduced the model of topologized graphs, and we also alluded to the model of "edge spaces". The former emerges naturally as the way to reconcile graph-theoretic with topological connectedness on a ground set consisting of vertices and edges. By contrast, we formulate the latter only with the advantage of hindsight; as we mentioned in Chapter 2, the concept of topologized graphs places an equal emphasis on edges and vertices, while one of our main concerns, namely cycle spaces, is biased in favour of edges.

We have also mentioned our need for separation axioms along the lines of the standard Hausdorff, regular and normal separation axioms of standard topology that we can meaningfully apply to our spaces, which are typically not T_1.

This chapter is mainly dedicated to these two points. Inevitably, it will be rather technical. We introduce, and discuss the merits of, two hierarchies of separation axioms, parallel to each other and to the usual hierarchy. As we go along, we also address the issue of how best to model an edge, with reference to topological quotients.

We aim to justify the most obvious way in which our topological model is at odds with the standard one—the fact that an edge is modelled by a point.

In Chapter 1 we expressed objections *a priori* to the usual model, all of which are resolved by modelling an edge as an open singleton whose boundary consists of the incident vertices, as we do in the case of topologized graphs, in particular the classical topology, and as we shall do in the case of edge spaces. We shall also present three arguments that appear *a posteriori*, that is, arguments which emerge from an attempted proof, the requirements of a proof, or an analysis at some depth. The very nature of such arguments means that we can not discuss them fully at this stage.

In the next section we shall present the first argument together with the necessary background, and throughout this chapter we shall develop the technical tools to expound the other arguments. We present the second argument at the end of this chapter. It seems appropriate to tackle the third point in Chapter 4, and of course the full force of these points can only be appreciated when the change in perspective comes into play in the proofs of specific results, in Chapters 4 and 5.

One of the original objectives of this project was to unify and extend results from finite to infinite graphs. In 1999 Richter posed the question of how one might generalize the well-known fact from graph theory that the fundamental cycles with respect to a fixed spanning tree of a given graph generate its entire cycle space. This question was taken up by Diestel and Kühn [19, 20, 21], who addressed the issue in two different ways, both based on the standard cell-complex topological space associated with an (infinite) graph. These articles also appear to have triggered a whole new line of research—Diestel gives a survey in [13].

This work not only represents one instance of non-trivial extensions of theorems from finite to infinite graphs, but also happens to be heavily topological. Since our topological model is so different from the standard one, the fact that the issue of cycle spaces had already been addressed was more of an incentive than a deterrent for us to pursue the same topic; the subject of cycle spaces seemed an appropriate test for the merits and demerits of the novelty in the model, and also for the philosophy of allowing the topology to "dictate matters".

In fact, the three points we refer to above all arise from comparisons with the work of Diestel and Kühn. In terms of subject matter (issues addressed), the overlap of this thesis with their work lies mainly in the existence of analogues for spanning trees ("topological spanning trees" for them, "minimal spanning sets" for us) and the fact that the fundamental cycles generate the cycle space. With respect to these issues, the main contribution of our work will be in

- the topological model, which differs not just in that edges are singletons as opposed to arcs, but other important ways, such that the fact that the vertex set need not be predominantly discrete, not even totally disconnected, and the fact that the same topological space may model different combinatorial structures;

- a more axiomatic approach and a methodology which draws heavily on the concepts of general topology;

- the level of generality of the nature of topological spaces in question, which differs from that of Diestel and Kühn partly because we consider arbitrary spaces equipped with a pre-assigned, well-behaved topology rather than spaces constructed from a classical graph by adding "points at infinity".

The above points relate specifically to the issue of how to model an edge—singleton versus arc. In fact, we attempt to approach this issue from a more generic standpoint. Clearly the notion of a topological arc for an edge is a convenient tool from standard topology, mainly because it is so familiar and well understood. However, even assuming that one is willing to renounce the simplicity of having a topological and a combinatorial structure cohabiting on a single ground set, the choice of an arc appears to us as rather arbitrary. Some of the properties of the arc appear relevant, others irrelevant. The relevant points certainly include the fact that the closure of the open arc adds precisely the two "endpoints"—this gives the basic relationship between an edge and the two endvertices. Another point which is clearly relevant is the fact that the arc is itself connected—this gives a way in which edges connect vertices.

A third feature, present in the usual model, and also in our models, is that the edge is modelled by an *open* subset. Our arguments will depend on this assumption, and we believe not unnecessarily so. Hence we feel that this feature is relevant at least in that it makes the arguments simpler. However, we are not entirely confident that a viable (more general) theory of cycle spaces can not be developed *without* this feature; we progressively discarded many features present in the usual model as it became clear that they were irrelevant or even obscured matters, but we realize that the crucial rôle of this feature in the results we present on cycle spaces may be due to some degree of "mental prejudice", an unconscious reluctance to renounce such a seemingly basic property.

The last point is quite interesting because it comes up in the connection with some work done in continuum theory. In Sections 3.5 and 4.3 we contemplate a prototype of a topological model different from ours, and use topological quotients to address the issue of how it would relate to our models, in terms of separation axioms and compactness, both crucial aspects in several of our proofs. We settle on the above three features as minimum ingredients for an "edge", and look at a scenario in which edges are modelled by arbitrary *connected*, *open* subsets with at most *two points in their boundary.* The setting is that of an arbitrary topological space, and we assume that these "edges" are pairwise disjoint. It turns out that, by considering the complement of the union of such a collection of "edges", *if the space in question is locally connected* this concept translates into the concept of a T-set, which arises in continuum theory, completely independently of any graph-theoretic motivation. If the space is not locally connected, the snag in the translation lies precisely in that the "edges" obtained from a T-set would not necessarily be open. Since continua are compact, T-sets are usually considered in the context of compact spaces. The spaces we consider in Chapters 4 and 5 are for the most part compact, but not necessarily locally connected. Some results in continuum theory concerning T-sets (some of which may be found in [48]) evoke an intriguing connection to our work in Chapters 4 and 5; the fact that these results are proved for *locally connected* continua suggests that perhaps the assumption that the edges be open can not be dispensed with.

One of the features which makes this chapter rather technical is the presence of topological quotients. Quotients also recur intermittently throughout Chapters 4 and 5. In some places, such as Theorem 5.4.3 (which could not even be phrased without quotients), they are central to the issue at hand. Once we have them, we do not refrain from using them, especially since this is in line with our philosophy of pursuing topological concepts whenever they arise; however, we wish to emphasize that in some instances their rôle is only marginal. The concept of singletons for edges arises naturally from Question 1.1.1 posed in Section 1.1, not from an identification of the arcs in the cell-complex topological space. Whenever we model edges as sets consisting of more than one point, and then identify these sets to a single point, it is only for the purposes of comparing our model to an alternative one, or occasionally as a means of concisely describing our feebly Hausdorff spaces starting from the more familiar Hausdorff spaces. Note, however, that arbitrarily identifying open connected subsets with two boundary points in Hausdorff spaces to edges may result in a non-feebly-Hausdorff space, even if the original space is compact and Hausdorff (Example 4.3.13); hence the well-behaved spaces can be discerned only in the model with singletons for edges.

More general quotients come in handy in several ways. Clump-hyperedge quotients, and in particular clump quotients, allow us to relate the topological model of hyperedge spaces to that of topologized hypergraphs (Proposition 3.5.6, Corollary 4.3.20). We shall also use quotients as a way to bootstrap results from weakly Hausdorff to feebly Hausdorff spaces. Again, the development of the whole theory of cycle spaces for weakly Hausdorff (as opposed to feebly Hausdorff) spaces would remain intact (save cosmetic changes) without the presence of quotients.

We believe that in no case have we applied quotients in a trivial way, although this may not always be obvious. However, we have not been sparing in their use. There may be instances in which their use is not strictly necessary, for example in dealing with the parallel decomposition. We have intentionally pursued the avenue of quotients precisely because one of the objectives of this exercise is to explore the way in which topological concepts relate to combinatorial ones. The reward was that upper semicontinuous decompositions have a bearing on some of the issues relevant to us. In the case of the parallel decomposition, we also obtain a rather unusual scenario which is something of a curiosity, interesting from a purely topological perspective (Theorem 4.2.18, Proposition 4.2.21).

3.1 Separation Axioms

3.1.1 Some pathological examples

We begin with two simple examples of two topologized graphs compatible with the same infinite graph.

3.1.1 Example (Infinite bond plus regular vertex): Consider a graph with countably infinitely many edges incident with the same two vertices, and a third vertex of degree

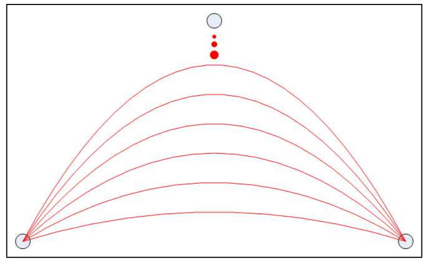


Figure 3.1: The infinite bond plus irregular vertex

zero. The "infinite bond plus regular vertex" is the topologized graph obtained by endowing this graph with the classical topology.

3.1.2 Example (Infinite bond plus irregular vertex): Consider the graph of Example 3.1.1, equipped with the topology in which all points except for the vertex v of degree zero have the same neighbourhoods as in the classical topology, the neighbourhoods of v being those whose complement is finite. We refer to the resulting topologized graph as the "Infinite bond plus irregular vertex". This space is illustrated in Figure 3.1.

The above two examples provide a simple but interesting comparison. They are distinct topologized graphs inducing the same underlying graph (the unique graph strictly compatible with the corresponding topological space). One is connected and the other disconnected. Moreover, the (discrete) relative topologies inherited by the vertex sets are the same. This in particular means that the global topology can not be recovered from the relative topology on the vertex set, even if we also know the "combinatorial" edge-vertex incidence information. In other words, the global topological information amounts to *more than* the combinatorial information *and* the relative topology on V *put together*. Note that all of the above still holds if we replace the single edges with classical paths containing two edges and one (internal) vertex each, so that we have a *simple* graph, and choose an arbitrary sequence of edges, one from each path, to converge to the vertex v in the case of the second example; what does change with this modification is that the space resulting from the second example is not compact.

Intuitively, there seems to be something "wrong" with the second example. The edges are keeping the space connected ("holding the vertices together") in a way which "shouldn't be allowed". Our separation axioms will, in fact, exclude examples of this kind. At this point we can precisely point out two ways in which this example is badly behaved.

In Section 1.1.3 we characterized spanning trees as minimal connected sets containing the vertices. In Chapters 4 and 5, we shall take this characterization as the definition for our analogues for spanning trees. In the example of the infinite bond plus irregular vertex, although the space is connected, there simply is no such object! This phenomenon disappears if we replace the examples with the "simple" versions just described. Finding instances of this phenomenon becomes progressively harder as one imposes more and more structure on the spaces in question. Our next example will show how this can happen for a *simple* topologized graph, and in Section 4.2 we shall give an example (based on a standard, but rather involved, topological counterexample) with a *feebly regular* topology on a simple graph. In Chapter 4 we prove the existence of "minimal spanning sets" for *compact* feebly Hausdorff topologized *graphs* (and the more general "edge spaces"). The remaining gap in our understanding will concern whether this can be extended to *hyper*graphs.

Another anomaly of the infinite bond plus irregular vertex is the following. Suppose we choose to somehow reduce the infinitely many parallel edges to a single edge. If we simply discard all edges but one, we are left with a classical graph, consisting of a copy of K_2 (the complete graph on two vertices) and a vertex of degree zero. The problem with this is that we start out with a connected space but the result is a disconnected space. If instead we identify all the edges by taking a topological quotient, we are left with a topologized *hyper*graph: the space is connected and S_1, but we have a single hyperedge with *three* incident vertices! Both of these situations will be undesirable in our context.

3.1.3 Example (The overcrowded fan): The space we construct in this example is illustrated in Figure 3.2. Let M, N, P be pairwise-disjoint sets such that M and N have the cardinality of the continuum, and P has the cardinality of the integers (we could take M, N, P to be subsets of $\mathbb{R}$, but this would obscure the construction). Consider the simple complete bipartite graph with vertex set $P \cup M$, with every vertex in P adjacent to every vertex in M. Let G be the topologized graph obtained when this graph is endowed with the classical topology.

The intuition may be aided by likening the set $\mathcal{N}_m$ (for every $m \in M$) to a countably infinite "two-dimensional fan" with apex m, and G itself to an uncountably infinite "three-dimensional fan"[1] with P for a "spine".

Now we add N to the vertex set to obtain a larger graph H. The edges of H are precisely the edges of G, so that the vertices in N have degree zero in H. To define the global topology, let $\{v_x\}_{x\in M}$ be a bijection of M onto N. We declare a subset U to be open if and only if $U \cap G$ is open in G with respect to the classical topology and $v_x \in U$ implies that $\delta(x) \setminus U$ is finite.

Note that the relative topology on G remains the classical topology. For every $x \in M$, if we were to enumerate the edges incident with x, we could think of them as a sequence converging to v_x; the vertices v_x and x are, however, mutually separated. It is easy to verify that this space is a topologized graph strictly compatible with H.

[1]Note, however, that the relative topology on M is discrete; hence M could not be embedded, say, as a circle in $\mathbb{R}^3$ perpendicular to a copy of $\mathbb{N}$ for a spine.

Now we claim that H has no "minimal spanning set". Suppose, by way of contradiction, that $Z \subseteq H$ is an (inclusion-wise) minimal connected subset of H containing V_H. For the purposes of this discussion, given a subgraph K of H and a vertex $u \in V_K$, we denote by $\mathcal{A}_K(u)$ the set of vertices adjacent to u in K.

We observe that, for any $x \in M$, if $\mathcal{A}_Z(x)$ is finite, then v_x is clopen in Z. Since Z is connected, for all $x \in M$ we have that $\mathcal{A}_Z(x)$ is infinite. In particular, every vertex in M is adjacent in Z to some vertex in P. This implies that, for some vertex $p \in P$, the set $M' := \mathcal{A}_Z(p)$ is uncountably infinite, for otherwise $M = \bigcup_{v \in P} \mathcal{A}_Z(v)$ would be a countable union of countable sets, contradicting the fact that M is uncountable.

Let P' be the set of vertices in $P \setminus \{p\}$ adjacent to some vertex in M', and Z' the subgraph of Z induced by $M' \cup P'$. Note that, for all $x \in M'$, we have that $\mathcal{A}_{Z'}(x)$ is $\mathcal{A}_Z(x)$ less a single edge (the one joining p and x); in particular, $\mathcal{A}_{Z'}(x)$ is again infinite. Moreover, P' is again only countable, and M', chosen to be uncountable, is again the countable union of $\mathcal{A}_{Z'}(x)$ over x in P'. Hence there exists a vertex $p' \in P'$ such that $\mathcal{A}_{Z'}(p')$ is infinite. In this case, it is sufficient for our purposes that it contains two vertices m_1 and m_2.

For now m_1, m_2, p and p' together induce a graph theoretic cycle C in Z. Since the topology on G is the classical topology, for any edge $e \in C$, the subset $C \setminus \{e\}$ is connected; in particular, the two endvertices of e belong to the same component of $Z \setminus \{e\}$. However, since $Z \setminus e$ is contains all the vertices of H and Z is a minimal connected set containing V_H, the subset $Z \setminus e$ should be disconnected, that is, e should be a cutedge of Z. By (2.1.17), the two endvertices of e should be in different components of $Z \setminus \{e\}$, a contradiction.

3.1.4 Example (The Diestel-Kühn obstruction): Let G be the graph consisting of a one-way infinite graph-theoretic path P, and a vertex v not in P and adjacent to every vertex in P. We add a vertex w of degree zero, and define a topology on $D := G \cup \{w\}$ by declaring a set U to be open if and only if $U \cap G$ is with respect to the classical topology on G and $w \in U$ implies that $V_P \setminus U$ is finite. This space is illustrated in Figure 3.3.

The above configuration was identified by Diestel and Kühn [21] as an essential problem in extending theorems from finite graph theory to apply to spaces consisting of infinite classical graphs together with their ends. They deal with this problem by identifying the vertices v and w to obtain a space that is better behaved; they say that the vertex v dominates the end w.

Interestingly, the subgraph G in this example was also singled out as a minor to be excluded by Higgs [30] in a different, but not unrelated, context. Higgs refers to it as the "Bean graph".

The spaces we shall consider will not be constructed with reference to ends of classical graphs. However, it turns out that the separation axioms we shall impose, which were

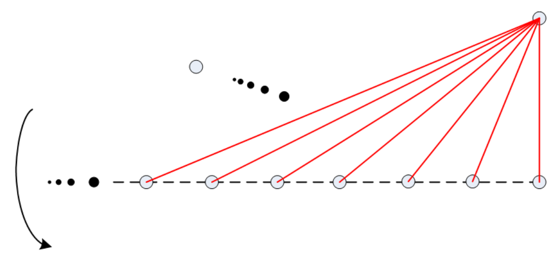


Figure 3.2: The overcrowded fan

originally tailor-made to allow classical topologies and suit the needs of the theorems in Chapter 4, rule out the Diestel-Kühn obstruction, but only when we model edges as points as opposed to arcs. Moreover, the spaces obtained by Diestel and Kühn after identifying ends with their dominating vertices fall within the class of spaces allowed by our separation axioms.

From our perspective, the first problem with this example is the following. Consider the subset $T := V_D \cup \delta(v)$. We shall refer to this substructure as the infinite star plus irregular vertex. It is easy to verify that this topologized subgraph is connected, and that deleting any edge (or any number of edges) will result in a disconnected space. Hence it is a valid candidate as a "minimal spanning set" for D. However, there does not appear to be any kind of "path-like" substructure in D which goes from v to w.

Strictly speaking, the presence of such a path-like substructure will not be crucial for us, so let us place this example more precisely into our context. One of the results we shall deal with in Chapter 5 will concern fundamental cycles: that is, a "cycle-like" structure which appears in $T \cup \{e\}$ when we add an edge $e \notin T$ to a "minimal spanning set" T.

Back to our example, note that T is a valid candidate for a minimal spanning set for any topologized graph H which has the same vertices as D and contains D as a topologized subgraph, that is, any H which consists of D plus some extra edges. Suppose that H

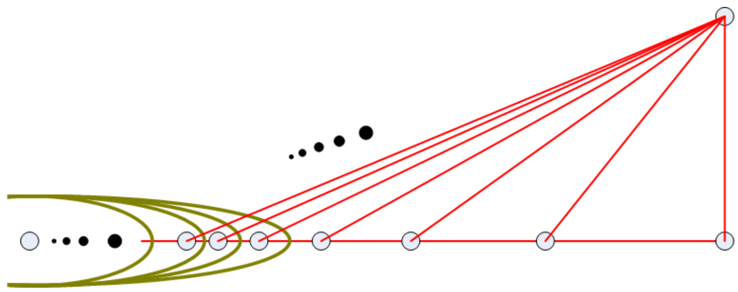


Figure 3.3: The Diestel-Kühn obstruction.

is such a topologized graph, and that it happens to contain an edge e joining v and w. Then adding our e to T will not induce any "cycle-like" structure in $T \cup \{e\}$. Of course, the absence of the cycle-like structure in $T \cup \{e\}$ goes hand in hand with the absence of the path-like structure in T, but we shall see in Chapter 5 that the rôle of the path-like structures is, for us, only marginal.

Diestel and Kühn [21] have also given examples to show how the above configuration leads to various problems, such as failure of Menger's Theorem, and cycles which are not expressible as sums of fundamental cycles. Since "topological spanning trees" in their spaces are restricted versions of our "minimal spanning sets", the considerations regarding cycle spaces remain true for us, that is, we too regard such an object as not well-behaved. However, this configuration provides the first instance in which we feel that the benefits of modelling an edge as a singleton, as opposed to an arc, become apparent. The issue is not that of cycle spaces but that of ends of graphs.

3.1.5 Argument I (Ends): In [18] Diestel and Kühn compare the "graph-theoretic" ends of a graph (introduced by Halin [27] in 1964) with the "topological" ends (introduced by Freudenthal [25] in 1931) of the corresponding cell-complex topological space. They remark that these two concepts agree when the graph is locally finite, in the sense that there is a natural one-to-one correspondence between the Halin ends (defined on graphs, without a need for a topology) and the Freudenthal ends of the associated cell-complex topological space.

They observe that, on the other hand, in the (usual) setup with arcs for edges, when the graph is not locally finite, there may be Halin ends which do not qualify as Freudenthal ends; hence they proceed to give a topological characterization of Halin ends by introducing "directions". For this reason, we shall refer to the standard (Hausdorff) construction obtained by adding the Halin ends (topological directions) as "points at infinity" for the cell-complex topological space associated with a graph G as the direction extension of G; for a precise definition, we refer the reader to [21].

The definition adopted by Diestel and Kühn for a topological "end" is based on the original one given by Freudenthal: an end of a Hausdorff space is essentially an (equivalence class of) decreasing sequence(s) $U_1 \supseteq U_2 \supseteq U_2 \supseteq \cdots$ of *non-empty, open, connected* subsets with *compact frontiers* such that the common intersection of their closures is empty. They immediately remark that, given any sequence $U_1 \supseteq U_2 \supseteq \ldots$ of subsets U_i, the conditions on the U_i (non-empty, open, connected, with compact frontier) are equivalent to them being components of the complement of a compact subset. This alternative approach is not marginal to the concept of a topological end—the idea of separating (or disconnecting) subsets by compact sets seems to be the usual way in which the Freudenthal compactification is constructed in textbooks on topology; see, for example, [61] Exercise 41B, or [33] Chapter 6, Theorem 30.

We wish to point out that with this alternative formulation of a topological end, and *provided that one models edges with singletons*, the discrepancy between the Halin ends and the Freudenthal ends disappears.

For Diestel and Kühn, the setting is that of an arbitrary Hausdorff space. It appears that they overlook the fact that if a Hausdorff space is not locally connected, the components of the complement of a closed set will fail to be open. So in fact, at the level of generality of arbitrary Hausdorff spaces, the two formulations are not equivalent. Consider, for example, the subset of points (x, y) in $\mathbb{R}^2$ such that $\sqrt{x^2 + y^2} < 1$ and either $y = 0$ or else x/y is a positive integer, equipped with the relative topology inherited from the usual topology on $\mathbb{R}^2$. This is a connected Hausdorff space which is not locally connected. The subsets $(x, 1) \times \{0\}$ for $x \in (0, 1)$ can be obtained as components of the complement of the compact subsets $[0, x] \times \{0\}$, and clearly the intersection of their closures over $x \in (0, 1)$ is empty, but they are not open and do not have compact frontiers.

However, it is easy to see that the two formulations *are* equivalent for locally connected Hausdorff spaces. Since the cell-complex topological space associated with a graph, and classical topologized graphs, are always locally connected, this assumption does not really affect our discussion. Also, note that Freudenthal introduced ends for compactification purposes, and the construction of the Freudenthal compactification requires the space to be rim-compact, while the context of graphs which are not locally finite translates into the fact that the topological spaces in question are not rim-compact (whether we model edges as arcs or singletons); hence it is not surprising that some assumption of a "local" character comes in to replace rim-compactness.

Now let us try to convey briefly the mechanics of how the Diestel-Kühn obstruction, or more precisely the (classical) subgraph G of the space D constructed above, is problematic with respect to the Halin-Freudenthal end correspondence. A Halin end corresponds essentially to an (equivalence class of) one-way-infinite path(s). In the subgraph G there is only one such path, hence precisely one Halin end. This "end" fails to arise as a Freudenthal end, according to the above definition—if we attempt to construct a sequence U_i with the required properties, we are faced with two obstacles: in order for the sets U_i to be "nested" and have empty intersection, they must contain a "tail" of the path P, and then either they will all contain the vertex v in their boundary, and therefore in the common intersection of their closures, or else some U_i will contain infinitely many "half edges", implying that its frontier is not compact. Of course, the situation does not improve if we switch to the second formulation: none of the neighbourhoods of v is compact.

However, this last fact changes if we model edges as singletons! In the classical topology, the subset $\mathcal{N}_u$ is always compact for any vertex u. More generally, the union of $\mathcal{N}_u$ over a finite set of vertices u is compact. If we enumerate the (countably infinite) set of vertices of G, starting at v, and take for K_i the union of $\mathcal{N}_u$ as u ranges over the first i vertices ($i \in \mathbb{N}$), then the (connected) complements of K_i give us the decreasing sequence of U_i's representing the unique Halin end of G.

Note that the equivalence between the two formulations for Hausdorff (locally connected, non-rim-compact spaces) is easy, but does not hold without the Hausdorff assumption.

The Diestel-Kühn obstruction is only a specific configuration, but it is easy to see that in general, once one adopts the classical topology, in order to reconcile the Freudenthal ends with the Halin ends of graphs which are not locally finite, one simply needs to adopt the second formulation (given by Diestel and Kühn) and then drop the Hausdorff assumption. Indeed, since finite unions of sets of the form $\mathcal{N}_u$ for vertices u are always compact, thanks to the fact that the classical topology preserves graph-theoretic connectedness the Freudenthal ends reduce to the Halin ends almost *by definition*!—one of the usual ways of defining the Halin ends is to declare one-way infinite paths to correspond to different ends if they can be "separated" by a finite set W of vertices. Here "separating" means deleting W in the usual graph-theoretic sense, that is, when we delete a vertex we also delete all incident edges. But that corresponds precisely to deleting the compact set $\bigcup_{u \in W} \mathcal{N}_u$ from the corresponding topologized graph. Conversely, a compact subset K of a classical topologized graphs may contain only finitely many vertices and finitely many edges not incident with a vertex, so there will be some finite set of vertices whose (graph-theoretic) deletion removes all of K

but not any one-way-infinite path.

We present the next example mainly as a precursor to the last one.

3.1.6 Example (The doubled-up arc): Consider the subset of $\mathbb{R}^2$ given by $S := [0,1] \times \{-1,1\}$, equipped with the relative topology inherited from $\mathbb{R}^2$. We fix $z \in (0,1)$ and pairwise identify the points with x-coordinate larger than z. More precisely, we define the function $p(x,y)$ on S which is the identity if $x \leq z$ and otherwise returns the point $(x,0)$. The doubled-up arc is the image $T = [0,z] \times \{-1,1\} \cup (z,1] \times \{0\}$ equipped with the quotient topology, that is, a subset of T is open if and only if its inverse image is open in S. This space is illustrated in Figure 3.4.

The doubled-up arc is a connected T_1 space which is not Hausdorff: there are no disjoint neighbourhoods for the points $(z,-1)$ and $(z,1)$. The problem with this space is that path-connectedness is not transitive: if we denote by a,b,c respectively the points $(0,-1)$, $(1,0)$ and $(0,1)$, we see that there are ab- and bc-paths (in fact, these are arcs), but no ac-path.

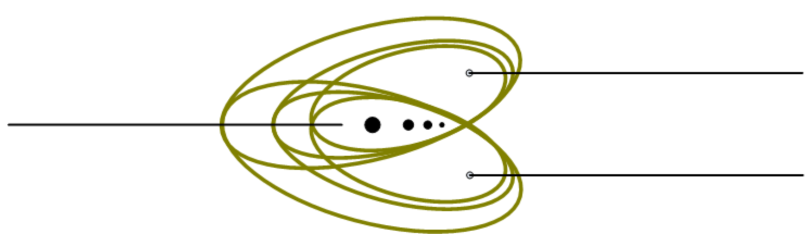


Figure 3.4: The doubled-up arc

Now we give a variant of this example, based on a previous example, which is more relevant to the context of (non-T_1) topologized graphs.

3.1.7 Example (The one-way infinite path with a double end): We recall the construction of Example 2.2.4; the one-way infinite path with a double end is the topologized graph obtained by deleting the single edge of the path P_2. This space is illustrated in Figure 3.5.

This example exhibits essentially the same anomaly as the doubled up-arc: if a,c are the terminal points of P_2 and b the terminal point of P_1, then P_1 together with a or c is an ab- or bc-path respectively, but there is no ac-path.

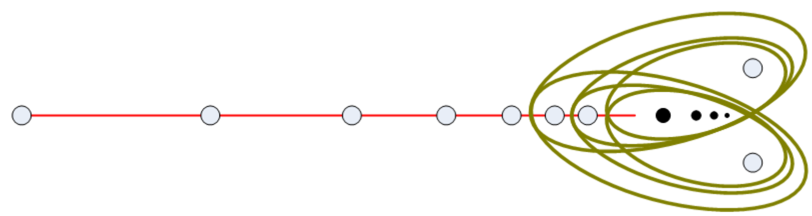


Figure 3.5: The one-way infinite path with a double end

3.1.2 Hierarchical Separation Axioms

Recall that a topological space is Hausdorff if any two points can be "separated" by disjoint (open) neighbourhoods. Several relaxations of this concept in the literature revolve around the *kind* of "separating sets" (semi-open, pre-open, feebly open, β-open) or else restrict the pairs of points which can be separated; see [37]. We use a relaxation which still talks of open sets, and considers all possible pairs, but relaxes the requirement that the "separating" sets should be disjoint.

3.1.8 Definition: A topological space is weakly Hausdorff if, for any distinct points x, y, there exist neighbourhoods U_x, U_y of x, y respectively such that $|U_x \cap U_y|$ is finite. If the two neighbourhoods can always be chosen so that they intersect in at most one element, the topological space is almost Hausdorff.

The following two propositions constitute almost everything we shall say about almost Hausdorff spaces.

3.1.9 *Proposition:* *A prepath is almost Hausdorff.*

Proof: Let x, y arbitrary points in a prepath P, with a given compatible total order. If at least one, say x, is an edge, then $\{x\}$ and P are the required neighbourhoods. Suppose both are vertices. Then there exists an intermediate edge or vertex, say z. In both cases the open sets $(-\infty, z\rangle$ and $\langle z, \infty)$ intersect in at most one point, and may be taken as the required neighbourhoods. ✠

3.1.10 ***Proposition:*** *An almost Hausdorff topological space is a uniquely adjacent topologized hypergraph.*

Proof: Let X be an almost Hausdorff space. First we show that X is a topologized hypergraph. We need to show that, for any $x \in X$, $\{x\}$ is either open or closed. If $\{x\}$ is not closed, there must be some $y \in \mathsf{Cl}(x) \setminus \{x\}$, that is, some point y such that all open sets containing y also contain x (otherwise $X \setminus \{x\}$ would be open). By definition of almost Hausdorff spaces there must exist open sets U_x, U_y containing x and y respectively and such that their intersection consists of at most one point. But by choice of y, the set U_y contains x, so $x \in U_x \cap U_y$ and therefore $\{x\} = U_x \cap U_y$, which, being the intersection of open sets, is open.

In order to show that the topologized hypergraph is uniquely adjacent, let u, v be any two vertices (closed points). If there were two distinct edges incident with both, they would both have to be in any neighbourhoodcontaining either u or v, contradicting the fact that X is almost Hausdorff. ✠

Recall that we have defined the surrounding set $A^\diamond$ of a subset A of a topological space to be the intersection of all the open sets containing A, and that this notion comes up in the context of Alexandroff discrete spaces. It will be crucial for us.

3.1.11 Definition: Given a topological space X and a subset $S \subseteq X$, the envelope $\rho(S)$ of a set S is the intersection of $S^\diamond$ and $(X \setminus S)^\diamond$. A topological space is finitely adjacent if, for all distinct points x, y, we have that $x^\diamond \cap y^\diamond$ is finite and uniquely adjacent if this intersection always consists of at most one element. A topological space is finitely incident if, for any $x \in X$, we have that $\mathsf{Cl}(x)$ is finite.

Recall that for any topological space X and any subset S, the closure $\mathsf{Cl}(S)$ of S is the intersection of all closed sets containing S, and the frontier $\partial(S)$ of S is the intersection of $\mathsf{Cl}(S)$ and $\mathsf{Cl}(X \setminus S)$. Thus the notions of surrounding set and closure and frontier and envelope are dual pairs, in the sense that open sets are replaced in their rôle by closed sets. Note that, for any points x, y in any topological space, $y \in x^\diamond \Leftrightarrow x \in \mathsf{Cl}(y)$ and, if $x \neq y$, then $y \in \rho(x) \Leftrightarrow x \in \partial(y)$.

Just as the closure of a set S is the set of points x such that any open set containing x has non-empty intersection with S, contains S, and is equal to S if S is closed, the surrounding set of S is the set of points x such that any closed set containing x has non-empty intersection with S, contains S, and is equal to S if S is open.[2] Another analogy is that, for any set S, just as $\mathsf{Cl}(S) = S \cup \partial(S)$, we also have $S^\diamond = S \cup \rho(S)$. Also, for disjoint sets A, B, we have $\rho(A) \cap \rho(B) = A^\diamond \cap B^\diamond$.

3.1.12 ***Lemma:*** *For any subset A of a topological space X, we have* $A^\diamond = \bigcup_{a \in A} a^\diamond$.

Proof: Suppose $x \in A^\diamond$. Then every open set containing A also contains x (equivalently, every closed set containing x intersects A). Now if, $\forall a \in A$, $x \notin a^\diamond$, then, for all $a \in A$, there exists an open set U_a containing a but not x, and the union $\bigcup_{a \in A} U_a$ is open and contains A but not x, a contradiction. The converse inclusion is trivial. ✠

In a T_1 space, every singleton is closed and so, for any set S, we have $S^\diamond = S$ and $\rho(S) = \emptyset$. For non-T_1 spaces, however, the surrounding set and envelope operators are non-trivial. In the case of a a topologized hypergraph, for any vertex v, the surrounding set of $\{v\}$ is $\mathcal{N}_v$, while $\rho(v) = \delta(v)$, that is, the set of edges incident with v.[3] The frontier of an edge-singleton is simply the set of its endvertices. Thus, in the case of topologized hypergraphs, the combinatorial and the topological meanings of the terms "uniquely adjacent", "finitely adjacent" and "finitely incident" coincide (Definitions 1.1.14, 3.1.11).

Note that, in a T_1 space, for any two distinct points x, y, we have that $x^\diamond \cap y^\diamond = \emptyset$. In fact, this condition is equivalent to the space being T_1 because, for any fixed point x, since $x \in x^\diamond$, for any other point y we have that $x \notin y^\diamond$, so there exists an open set containing y but not x, and taking the union over $y \neq x$ yields that $X \setminus \{x\}$ is open, that is, $\{x\}$ is closed. Thus, in a sense, uniquely (finitely) adjacent spaces are to T_1 spaces as almost (respectively, weakly) Hausdorff spaces are to Hausdorff spaces. Also, a space is T_1 if and only if for any two distinct points x, y there exist disjoint *closed* sets each containing one of x, y.

The intuition behind the "feeble" separation axioms is that, if the "separating" open sets can not be chosen to be disjoint, that is only because they have to contain the points which belong to *every* open set containing the given points (or sets).

3.1.13 Definition: Let X be a topological space. Then

[2] But $S = S^\diamond$ can happen even if S is *not* open.

[3] However, we shall extend the notation $\delta(A)$ to a meaning which diverges from that of $\rho(A)$.

- X is feebly Hausdorff if for any two distinct points x, y there exist open sets U_x, U_y with $x \in U_x, y \in U_y$ and $U_x \cap U_y \subseteq x^\diamond$;
- X is feebly regular if for any point x and any closed set C not containing x, there exist open sets U_x, U_C with $x \in U_x, y \in U_C$ and $(U_x \cap U_y) \subseteq (x^\diamond \cap C^\diamond)$;
- X is feebly normal if for any two disjoint closed sets C, D, there exist open sets U_C, U_D with $C \subseteq U_C, D \subseteq U_D$ and $U_C \cap U_D \subseteq C^\diamond$.

3.1.14 Remark: An Alexandroff discrete space is feebly Hausdorff, feebly regular, and feebly normal.

Note that the *prima facie* asymmetric conditions in the definitions of feebly Hausdorff and feebly normal spaces can in fact be equivalently phrased in the symmetric forms:

- "$\forall\, x, y\ \exists$ open neighbourhoods U_x, U_y of x, y respectively such that $U_x \cap U_y \subseteq x^\diamond \cap y^\diamond$ $(= \rho(x) \cap \rho(y))$"; and
- "$\forall\, C, D$ disjoint closed subsets $\exists$ open sets U_C, U_D containing C, D respectively such that $U_C \cap U_C \subseteq C^\diamond \cap D^\diamond$ $(= \rho(C) \cap \rho(D))$".

The feebly normal, feebly regular and feebly Hausdorff conditions are respectively slightly less stringent than the analogous standard separation axioms of normality, regularity and of being Hausdorff. It is trivial to see that each one of the three standard axioms implies its "feeble" counterpart; moreover, for T_1-spaces, the envelope of any set is empty, so the notions become pairwise equivalent. The converse implications are, however, not true in general: it is sufficient to consider the complete simple graph on two vertices, equipped with the classical topology.

The analogy with the standard separation axioms can be carried further. It is well-known that in general regular and normal spaces need not be T_0, while a T_0 regular space is necessarily Hausdorff and a T_1 normal space is necessarily regular (see [55]); hence the explicit T_1 requirement in the definition of T_4 (and T_3) spaces.

Similarly, in general a feebly regular (not even a regular) topological space need not be feebly Hausdorff. To see this, consider a countably infinite set S with two distinguished elements a, b, equipped with a topology defined in the following way: all singletons, except for $\{a\}$ and $\{b\}$, are open, while the sets with finite complement and containing a and b are the neighbourhoods of a *and* b. It is easy to verify that these requirements indeed define a topology, which is clearly not T_0.

We have that $a \in \mathsf{Cl}(b)$ and $b \in \mathsf{Cl}(a)$, hence any closed set contains either both a and b or else neither. Moreover, the only closed sets disjoint from $\{a, b\}$ are finite, and therefore clopen. So if x is an arbitrary point and C an arbitrary closed set with $x \notin C$, then one of

$\{x\}$ or C is clopen and, together with its complement, gives the required (disjoint) open sets. Hence S is regular.

However, S is not feebly Hausdorff, because we have $a^\diamond = b^\diamond = \{a, b\}$ but any neighbourhood of a will have infinite intersection with any neighbourhood of b.

Now let us modify the topology of S slightly, so that it becomes T_0: we declare the neighbourhoods of a to be those with finite complement (not necessarily containing b). Now we have $b \in \mathsf{Cl}(a)$ but $a \notin \mathsf{Cl}(b)$. Thus it is impossible to separate $\mathsf{Cl}(b) = \{b\}$ from a according to the requirement of feeble regularity, because the intersection of any open set containing a will still have infinite intersection with any neighbourhood containing b, while $a^\diamond = \{a\}$.

However, the modified topology yields a feebly normal space, because no two disjoint closed sets can each contain one of a, b. Thus this is an example of a T_0, feebly normal space which is not feebly regular.

In the above example, the singleton $\{a\}$ is neither open nor closed, and in fact this is essentially the only possible obstacle preventing a feebly normal space from being feebly regular: a feebly normal S_1 space[4] is feebly regular.

3.1.15 Remark: Given any open point p in any topological space X, and any subset C disjoint from p, there always exist open neighbourhoods of p and C whose intersection is contained in $p^\diamond \cap C^\diamond$. For if $p \in C^\diamond$, then any open set containing C intersects $\{p\}$ (which is a neighbourhood of p) in $C^\diamond$, while if $\mathsf{Cl}(p)$ is disjoint from C, then its complement may be taken as the required neighbourhood of C, whose intersection with $\{p\}$ is empty. Hence, in Definition 3.1.13, in the special case of x being open, the separation conditions for feeble regularity and for being feebly Hausdorff are vacuous.

Thus, to verify that an S_1 space is, say, feebly regular, we need only consider pairs x, C where x is a *closed* point not belonging to the closed set C. If we know the space to be feebly normal, the required open sets are given by the condition defining feeble normality.

It is easy to see that T_0 feebly regular spaces are feebly Hausdorff, but the above example shows that T_0 is not a strong enough assumption for our purposes. On the other hand, S_1 spaces are trivially seen to be T_0. In view of the these observations, we make the following definitions.

3.1.16 Definition: An $\mathbf{S_2}$ space is a feebly Hausdorff S_1 space, an $\mathbf{S_3}$ space is a feebly regular S_1 space, and an $\mathbf{S_4}$ space is a feebly normal S_1 space. For convenience, we also redesignate T_0 spaces to be $\mathbf{S_0}$ spaces.

Thanks to Remark 3.1.15, we have the following easy proposition.

[4]Recall that we have defined (page 5) an S_1 space to be a topological space in which every singleton is open or closed, and that we also refer to S_1 spaces as "topologized hypergraphs".

3.1.17 *Proposition:* *Let X be a topological space, and the symbol $\mathcal{S}_i$ denote the proposition "X is an S_i-space". Then $\mathcal{S}_4 \Rightarrow \mathcal{S}_3 \Rightarrow \mathcal{S}_2 \Rightarrow \mathcal{S}_1 \Rightarrow \mathcal{S}_0$.* ✠

Thus for us the context of topologized hypergraphs has the same rôle played by the T_1 condition in standard topology: it ensures that the topology has a bearing on the singletons, and guarantees an escalation of successively stronger separation axioms.

With respect to subspaces, the behaviour of the "feeble" separation axioms is analogous to that of the standard ones. We omit the proof of the following easy proposition.

3.1.18 *Proposition:* *Let S be any subspace of any topological space$X$. If X is feebly Hausdorff (respecively feebly regular) then so is S. If X is feebly normal and S is closed, then S is feebly normal.* ✠

At this stage we also point out that the variants of the Hausdorff axiom also fall into sequence just as the adjectives, we hope, suggest: "Hausdorff" implies "almost Hausdorff" which implies "weakly Hausdorff" which implies "feebly Hausdorff". Of these three implications, the only one which is perhaps not immediate is the last: given any two distinct points x, y, we can find corresponding open neighbourhoods U, V with finite intersection. For any point $z \in U \cap V$ not in $x^\diamond$, there exists an open set A_z which contains x but not z; now the set $U \cap \bigcap_{z \in (U \cap V) \setminus x^\diamond} A_z$ is contained in $x^\diamond$ and, since $U \cap V$ is finite, is open, and so can be taken as the neighbourhood of X together with V as the neighbourhood of y.

We have already observed that for T_1 spaces "feebly Hausdorff" is equivalent to "Hausdorff", so that in this context the distinction between the four variants is lost, but the simple examples of the classical graphs with two vertices, no loops, and zero, one, two, or infinitely many edges show that in the context of topologized hypergraphs the distinction is meaningful. The following proposition captures the distinctions in terms of adjacency.

3.1.19 *Proposition:* *A feebly Hausdorff topological space is:*

- *weakly Hausdorff if and only if it is finitely adjacent;*
- *almost Hausdorff if and only if it is uniquely adjacent; and*
- *Hausdorff if and only if it is T_1.* ✠

One could also consider a "weak" hierarchy alongside a "feeble" one. We shall only have occasion to consider "weakly normal" spaces.

3.1.20 Definition: A topological space is weakly normal if, for every two disjoint closed sets C, D, there exist neighbourhoods U_C, U_D of C, D respectively such that $U_C \cap U_D$ is finite.

It is easy to see that weakly normal spaces are feebly normal, in the same way we saw that weakly Hausdorff spaces are feebly Hausdorff.

3.2 Feebly Hausdorff Spaces

In Subsection 3.1.1 we saw examples of topological spaces in which path-connectedness fails to be "transitive", that is, there may be three points a, b, c in a topological space, an ab-path and an ac-path but no ac-path. Our first application of the "feeble" separation axioms is to guarantee that this does not happen.

3.2.1 *Proposition:* *Suppose X is a feebly Hausdorff topologized graph, a, b vertices in X and P an ab-path. Then P is closed in X.*

Proof: Suppose, by way of contradiction, that P is not closed, that is, $X \setminus P$ is not open. Since a set is open if and only if it is a neighbourhood of every point it contains, there exists some point $v \in X \setminus P$ such that every neighbourhood of v has non-empty intersection with P. Since any singleton consisting of an edge is a neighbourhood of that edge, v must be a vertex.

Since X is feebly Hausdorff, for every vertex w in P there exist open sets U_w, N_w containing w, v respectively such that $(U_w \cap N_w) \subseteq \rho(v) = \delta(v)$. The set $\{U_w\}_{w \in P \cap V}$ is an open cover of P, since any edge in P must be incident with some vertex in P (otherwise the corresponding singleton would be both open and closed in P). Moreover, by (2.3.17) the bounded prepath P is compact; hence there must be a finite subset W of $V_X \cap P = V_P$ such that $\{U_w\}_{w \in W}$ is still a cover of P.

Consider now the set $N := \bigcap_{w \in W} N_w$. This set clearly contains v and, being the finite intersection of open sets, is open. Hence N is an open neighbourhood of v, and therefore by choice of v, we have that $(N \cap P) \neq \emptyset$. But

$$\begin{aligned}
(N \cap P) &\subseteq N \cap \left(\bigcup_{w \in W} U_w \right) \\
&= \left(\bigcap_{w \in W} N_w \right) \bigcap \left(\bigcup_{w \in W} U_w \right) \\
&\subseteq \bigcup_{w \in W} (N_w \cap U_w) \\
&\subseteq \delta(v)
\end{aligned}$$

so $N \cap P$ contains an edge e incident with v. Since a, b are vertices, e is incident with 2 vertices in P, and since $v \notin P$, e is incident with at least three vertices, contradicting the fact that X is a topologized graph. ✠

3.2.2 ***Proposition:*** *Suppose X is a feebly Hausdorff topologized graph, a, b, c are vertices of X, P is an ab-path in X and Q is a bc-path in X. Then $P \cup Q$ contains an ac-path.*

Proof: Let $\preceq_P, \preceq_Q$ denote the associated total orders of P, Q respectively, chosen so that a is a minimum in P (with respect to $\preceq_P$) and b is a minimum in Q (with respect to $\preceq_Q$). Let z be the infimum with respect to $\preceq_P$ of $P \cap Q$. The point z is well defined and belongs to P because $b \in (P \cap Q)$ and $\preceq_P$ is order-complete.

Claim: The point z is a vertex and belongs to Q.
Proof of Claim: We need to show that an arbitrary neighbourhood N of z has non-empty intersection with $P \cap Q$. All references to order in the proof of this claim are with respect to $\preceq_P$.

Suppose z is an edge. Then it has a successor s in P, which is an incident vertex. Then the only point of P which is larger than or equal to z but not larger than or equal to s is z, so if $z \notin P \cap Q$ then s would still be a lower bound for $P \cap Q$, but it is also bigger than z, contradicting the definition of z. So z belongs to Q, and therefore has two incident vertices in Q. If both were also in P, then one of them would be smaller than z and this would again contradict the definition of z. So one of them must be outside of P. But z already has two incident vertices in P, so it has at least 3 incident vertices, contradicting the fact that X is a topologized graph.

So z must be a vertex; by way of contradiction, suppose that it does not lie in Q. Since X is a feebly Hausdorff topologized graph, by (3.2.1) Q is closed. Therefore z cannot be an accumulation point of Q, that is, there exists a neighbourhood N of z disjoint from Q. Since $z \in P$, we have that $N \cap P$ is a neighbourhood of z in P, and therefore must contain a set of the form $\langle m, M \rangle$ with $m \prec_P z \prec_P M$. Notice now that, for $x \in P$, $M \succ_P x$ implies either that $x \prec_P z$, whence $x \notin Q$ since z is a lower bound for $P \cap Q$, or that $x \in [z, M)$, whence again $x \notin Q$ since (m, M) is disjoint from Q. Hence, $\forall x \in (P \cap Q)$, $M \preceq_P x$, yet $M \succ_P z$, again contradicting the definition of z. □

Henceforth in this proof we shall use subscripts P or Q to denote whether intervals are taken in P with respect to $\preceq_P$ or in Q with respect to $\preceq_Q$ respectively. By the above claim, we have that z is actually a minimum, not just a greatest lower bound. Now we set $R := [a, z]_P \cup [z, c]_Q$. Since it is the union of two connected sets with non-empty intersection, R is connected. We wish to prove that R is an ac-path. Pick any $x \in R \setminus \{a, c\}$; we would like to find a separation of $R \setminus \{x\}$. Without loss of generality we may assume that $x \in (a,z]_P$. Note that most of the assertions in the next two paragraphs will be trivially true in the case of $x = z$.

We claim that $[a,x)_P$ is open in $R \setminus \{x\}$. Suppose not; then there is some point p in $[a,x)_P$ such that every neighbourhood of p contains some point in $(R \setminus \{x\}) \setminus [a,x)_P = (x,z]_P \cup [z,c]_Q$. Since $[a,x)$ is open in $[a,z]_P \setminus \{p\}$, every neighbourhood of p is disjoint from $(x,z]_P$, so every neighbourhood of p contains some point in $[z,c]_Q$. Since $p \preceq_P x \prec z$, we have that $p \notin Q$, whence $p \notin [z,c]_Q$, so p is an accumulation point of $[z,c]_Q$. But since z is a vertex, $[z,c]_Q$ is a path in X and X is a feebly Hausdorff topologized graph, by (3.2.1) $[z,c]_Q$ is closed, a contradiction.

We also claim that $(x,z]_P \cup [z,c]_Q$ is open in $R \setminus \{x\}$. If not, there would be some $q \in (x,z]_P \cup [z,c]_Q$ such that every neighbourhood of q has non-empty intersection with $[a,x)_P$. As above, this is impossible for $q \in (x,z]_P$ since $(x,z]_P$ is open in $[a,z]_P$; this leaves the possibility $z \in (z,c]_Q$, but as above this would imply that q is an accumulation point of $[a,x)_P \subseteq [a,z]_P$, a contradiction since $q \notin [a,z]_P$.

Hence $\{[a,x)_P, (x,z] \cup [z,c]_Q\}$ is a separation of $R \setminus \{x\}$. This shows that R is a prepath. Since $[a,z]_P, [z,c]_Q$ are paths, they are compact, hence their union is compact, and therefore R is a path. ✠

3.3 Compatibility and Edge Spaces

In this section we introduce our second important topological model for graphs, namely the model of edge spaces. We also begin the analysis of the relationship between the global topology on $V \cup E$ and the topology on V.

3.3.1 Extending a topology from V to $V \cup E$

It turns out that the issue of a global topology versus a topology on the vertex set is intimately related to the feeble separation axioms, as we shall see in Section 3.4. However, without any separation axioms we can obtain partial results, such as Propositions 3.3.1 and 3.3.6.

Notation: Given a hypergraph H and a subset $A \subseteq H$, the notation $A^\square$ stands for the smallest subset of H containing A and all edges incident with a vertex in A. Thus, if $A \subseteq V_H$, the $A^\square = \bigcup_{v \in A} \mathcal{N}_v$. More generally, we have that $A^\square = (A \cap E) \cup \bigcup_{v \in A \cap V} \mathcal{N}_v$.

Recall that, given a hypergraph H, a topology on $V_H \cup E_H$ is compatible with H (in the sense of Definition 1.2.2) if all the edges are open and their boundaries are precisely the set of their endvertices.

3.3.1 ***Proposition:*** *Let $H = (V, E, f)$ be a hypergraph and $\mathcal{T}$ a topology on V such that, $\forall e \in E$, $f(e)$ is closed* [5]*. Then the collection of sets*

$$\widehat{\mathcal{T}} = \left\{W^{\square} \cup F \ : \ W \in \mathcal{T},\, F \subseteq E\right\}$$

is the finest topology on $V \cup E$ compatible with H with the property that $\mathcal{T}$ is the relative topology on V.

Proof: Note that, for any element Z of $\widehat{\mathcal{T}}$ obtained by a particular choice of W and E, the intersection $V \cap Z$ is precisely W. First we verify that that $\widehat{\mathcal{T}}$ is indeed a topology. Taking $W = F = \emptyset$ and $W = V, F = E$ gives that both $\emptyset$ and $V \cup E$ are in $\widehat{\mathcal{T}}$.

Now take any subset $\{A_i\}_{i\in I}$ of $\widehat{\mathcal{T}}$. Then, $\forall i \in I$, A_i is of the form $\left(\bigcup_{v\in W_i} \mathcal{N}_v\right) \cup F_i$, with F_i contained in E and $W_i \in \mathcal{T}$. Hence $\bigcup_{i\in I} A_i = \left(\bigcup_{v\in W} \mathcal{N}_v\right) \cup F$ where $W = \bigcup_{i\in I} A_i$ and $F = \bigcup_{i\in I} F_i$. Clearly $F \subseteq E$ and, since $\mathcal{T}$ is closed under arbitrary unions, $W \in \mathcal{T}$. Hence $\widehat{\mathcal{T}}$ is also closed under arbitrary unions.

It remains to be shown that it is also closed under finite intersections. Given $A_1, A_2 \in \mathcal{T}$, using the same notation for the decomposition of A_1, A_2, we have that $A_1 \cap A_2 \cap V = W_1 \cap W_2$ so that $A_1 \cap A_2 = \left(\bigcup_{v\in \widehat{W}} \mathcal{N}_v\right) \cup \widehat{F}$, where $\widehat{W} = W_1 \cap W_2$ and $\widehat{F} = A_1 \cap A_2 \cap E$. Hence $\widehat{\mathcal{T}}$ is indeed a topology.

Now if $S \in \mathcal{T}$, then $\widehat{S} := S^{\square}$ is in $\widehat{\mathcal{T}}$ and therefore $S = \widehat{S} \cap V$ is in the relative topology on V inherited from $\widehat{\mathcal{T}}$. Conversely, if S is in the relative topology, then $S = \widehat{S} \cap V$, where $\widehat{S} = W^{\square} \cup F$, for some $W \in \mathcal{T}, F \subseteq E$. But then $S = W$ and so $S \in \mathcal{T}$. Hence $\mathcal{T}$ is the same as the relative topology on V inherited from $\widehat{\mathcal{T}}$.

To see that that $\widehat{\mathcal{T}}$ is compatible with H, let the edge e be incident with the vertex u. Then all open (with respect to $\widehat{\mathcal{T}}$) sets containing u contain also all of $\mathcal{N}_u$, and therefore e. Hence u lies in the closure of $\{e\}$, and, since it is also is in its complement, u belongs to its frontier. Conversely, suppose e is not incident with u. Then, since $f(e)$ is closed, there exists an open (with respect to $\mathcal{T}$) set U containing u but disjoint from $f(e)$. Then $U^{\square}$ is open with respect to $\widehat{\mathcal{T}}$, contains u and is disjoint from e, again showing that u is not in the closure of $\{e\}$, and hence is not in its frontier. This shows that $\partial(e) = f(e)$. Clearly $\{e\}$ is open with respect to $\widehat{\mathcal{T}}$, so $\widehat{\mathcal{T}}$ is compatible with H.

Now suppose that $\mathcal{S}$ is any topology on $V \cup E$ compatible with H and with the property that $\mathcal{T}$ is the inherited relative topology on V. We wish to prove that $\mathcal{S} \subseteq \widehat{\mathcal{T}}$. Pick any

[5] Note that this condition is automatically satisfied if $\mathcal{T}$ is T_1 and the hypergraph H is finitely incident.

$S \in \mathcal{S}$. Then $S \cap V$ is open with respect to $\widehat{\mathcal{T}}$ because $\widehat{\mathcal{T}}$ is the relative topology inherited from $\mathcal{S}$. Also, since $\mathcal{S}$ is compatible with H, $u \in S \Rightarrow \mathcal{N}_u \subseteq S$ (otherwise there would be some edge e incident with u and outside the open set S containing u, giving that the endvertex u of e is not in the closure of $\{e\}$). Therefore for some set $F \subseteq E$ we have $S = \bigcup_{v \in V \cap S} \mathcal{N}_v \cup F$, whence $S \in \widehat{\mathcal{T}}$. ✠

Terminology: We shall refer to $\widehat{\mathcal{T}}$ as the topology combinatorially induced by $\mathcal{T}$.

3.3.2 Note: The condition that $f(e)$ be closed cannot be dropped. Suppose V is the subspace of $\mathbb{R}$ consisting of 0 and all points of the form 2^{-n} for some positive integer n, and suppose there is only one hyperedge, incident with all vertices except 0. In this case the induced topology on $V \cup E$ is not compatible with H because 0 is in the closure of the singleton consisting of the only edge in H, even though it is not an endvertex.

3.3.2 Edge spaces

We now introduce our second topological model for hypergraphs. The essential difference from the model of topologized (hyper)graphs is that vertices are not required to be closed. This leads not only to a greater generality but to the flexibility of multiple combinatorial structures on the same topological space.

3.3.3 Definition: A hyperedge space is a topological space together with a specified subset of hyperedges (in the sense of Definition 1.2.4, that is, points which are open but not closed). If all the hyperedges of a hyperedge space $\mathcal{H}$ are proper, then $\mathcal{H}$ is an edge space.

There are various technically different, but equivalent, ways of specifying an edge space. The one immediate from the definition is to give a pair $(\mathcal{X}, E)$, where $\mathcal{X}$ is a topological space and E a subset consisting of hyperedges. However, note that strictly speaking $\mathcal{X}$ is itself a pair $(X, \mathcal{T})$, where X is a set and $\mathcal{T}$ a topology. So technically this is not so simple; another drawback for us is that it does not *explicitly* give the bipartition of the ground set into vertices and edges.

A (hyper)edge space $\mathcal{H}$ induces a compatible hypergraph H, obtained simply by taking the hyperedges (open but not closed points) of $\mathcal{H}$ for the hyperedges, all the other points for vertices, and the boundary points of an edge for its incident vertices. Conversely, given a hypergraph $H = (V, E, f)$ and a compatible topology $\mathcal{T}$ on $V \cup E$[6], then the topological space $(V \cup E, \mathcal{T})$, together with E, specifies a hyperedge space in which the hyperedges

[6] "compatible" again in the sense of Definition 1.2.2, that is, a topology such that every edge is an open point and the incidence function agrees with the boundary operator on the edges.

are precisely the hyperedges of H. Hence a second way of specifying a hyperedge space is to give a hypergraph and a compatible topology. This certainly emphasizes the underlying combinatorial structure; note however, that save for the identity of the edges (and vertices), all the combinatorial information is contained in the topology (and more). Thus, specifying the hypergraph apart from the topology is unnecessarily cumbersome.

We attempt to strike a balance between the two approaches by specifying a hyperedge space as a triple (V, E, ∂), where E is the set of hyperedges and ∂ is a frontier operator which defines a topology on $V \cup E$ while rendering all points in E open but not closed. This emphasizes the underlying combinatorial structure and draws a parallel with the triples which define hypergraphs.

3.3.4 Definition: Let $H = (V, E, f)$ be a hypergraph, $\mathcal{T}$ a topology on V, and $\widehat{\mathcal{T}}$ the topology combinatorially induced by $\mathcal{T}$. Then the combinatorial extension of **V** according to **f** is the hyperedge space $\mathcal{H} = (V, E, \partial)$, where ∂ is the frontier operator according to $\widehat{\mathcal{T}}$.

Whenever we use combinatorial concepts or notation in the context of a hyperedge space (V, E, ∂), it will always be with reference to the induced hypergraph, that is, the graph (V, E, f) where f is the restriction of ∂ to the points of (singletons in) E. On the rare occasions that we need to discard the topological structure, we refer to this hypergraph as the underlying hypergraph. Given (hyper)edge spaces M, N, we say that N is a (hyper)edge subspace of M if $V_N \cup E_N$ is a topological subspace of $V_M \cup E_M$ and the underlying hypergraph of N is a sub(hyper)graph of the underlying hypergraph of M; in particular, N contains the closure in M of every hyperedge it contains.

Notation and Terminology: Given a hyperedge space H, a topological property $\mathcal{P}$ and an adjective "$\mathbb{A}$" which means that a given topological space "satisfies property $\mathcal{P}$", the adjective "V-'$\mathbb{A}$'" applied to H means that the relative topology on V_H is "$\mathbb{A}$" (satisfies property $\mathcal{P}$). We shall speak, for example, of V-zero-dimensional hyperedge spaces.

Note that nowhere have we required the vertices of a hyperedge space to be closed points, so a given topological space (even a topologized graph) with at least one open-but-not-closed point can be given the structure of a hyperedge space in more than one way; in some situations, we shall have two or more edge spaces on the same topological space with a given point being a vertex in one and an edge in the other.

However, a topologized hypergraph canonically determines a hyperedge space obtained by keeping the same collection of hyperedges. Just as we use the underlying hypergraph to extend combinatorial notions to hyperedge spaces, we use this canonical transformation to extend to topologized hypergraphs the notation and terminology (such as "quasiregularity", see below) defined for hyperedge spaces.

One standard result about compact topological spaces which does not depend on the Hausdorff axiom, and which will be very important for us, is the following.

3.3.5 Fact: A closed subset of a compact space is compact.

Reference: See, for example, [61], Theorem 17.5. □

Proposition 3.3.1 allows us to deduce the following relationship between the global topology on $V \cup E$ and the relative topology on V.

3.3.6 *Proposition:* *Let H be a hyperedge space. Then $V_H \cup E_H$ is compact if and only if V_H is compact.*

Proof: Suppose $V_H \cup E_H$ is compact. Since E_H is open, V_H is a closed subset of a compact space, hence compact (by Fact 3.3.5). Conversely, suppose V_H is compact. Let $\mathcal{T}$ be the topology on $V_H \cup E_H$, $\mathcal{T}'$ the relative topology on V, and $\widehat{\mathcal{T}'}$ the topology combinatorially induced by $\mathcal{T}'$.

We need to show that every cover of $V_H \cup E_H$ with elements from $\mathcal{T}$ contains a finite subcover. Since $\mathcal{T}$ is compatible with the underlying hypergraph of H, by Proposition 3.3.1 we have that $\mathcal{T} \subseteq \widehat{\mathcal{T}'}$. Hence it is sufficient to prove that $V_H \cup E_H$ is compact with respect to $\widehat{\mathcal{T}'}$.

So let $\mathcal{U} = \{U_i\}_{i \in I} \subseteq \widehat{\mathcal{T}'}$ be an open cover of V_H. Since $\mathcal{T}'$ coincides with the topology inherited from $\widehat{\mathcal{T}'}$, we have that $\{U_i \cap V_H\}_{i \in I}$ is an open cover of V_H with respect to $\mathcal{T}'$. Since V_H is compact, there exists a finite $J \subseteq I$ such that $\{U_j \cap V_H\}_{j \in J}$ is an open cover of V_H. But from the definition of $\widehat{\mathcal{T}'}$ it follows that $\forall j \in J$ we have $(U_j \cap V_H) \subseteq (U_j \cap V_H)^{\Box} \subseteq U_j$. Moreover, since every hyperedge is incident with some vertex, $\{(U_j \cap V_H)^{\Box}\}_{j \in J}$ covers $V_H \cup E_H$. Hence $\{U_j\}_{j \in J}$ is the required open subcover of $\mathcal{U}$. ✠

3.4 Feeble Regularity and Quasiregularity

Among the separation axioms we investigate in this thesis, that of "feeble regularity" is probably the most important. In this section we develop our understanding of this important property, and examine its rôle in the issue of the global topology versus the relative topology on the vertex set. We also obtain an unexpected result relating the Lebesgue dimensions of the two spaces.

3.4.1 Reconstructing the topology on $V \cup E$.

In this subsection, we show that if a hyperedge space is feebly regular, then its global topology is determined by the relative topology on the vertex set together with the incidence information.

The following lemma has an important consequence in the context of topologized hypergraphs or, more generally, hyperedge spaces. Suppose S is a set of hyperedges and Z a set consisting of one endvertex for every edge in S. Since any open set containing a vertex has to contain all incident edges, any accumulation point for Z is also an accumulation point for S. In particular, if x is an accumulation point (vertex) of Z not incident with any edge in S and Z' is *any other* set of endvertices obtained by picking at least one endvertex per hyperedge, the following lemma implies that x is also an accumulation point for Z'.

Recall that, given a subset A of a topological space X, when convenient we use the notation $\overline{A}$ interchangeably with $\mathsf{Cl}(A)$.

3.4.1 *Lemma:* *Let X be a feebly regular topological space, $x \in X$ and S, Z subsets with the property that $\forall s \in S, \overline{s} \cap Z \neq \emptyset$ and $x \notin \overline{s}$. Then $x \in \overline{S}$ implies $x \in \overline{Z}$.*

Proof: If not, there would be open sets U_x, U_Z containing $x, \overline{Z}$ respectively such that $(U_x \cap U_Z) \subseteq x^\diamond \cap (\overline{Z})^\diamond$. For any $s \in S$ there exists $z \in Z$ with $s \in z^\diamond \subseteq U_Z$, so $S \subseteq U_Z$. Also, since $x \in \overline{S}$, every neighbourhood of x must contain some point from S; hence $\exists s \in (S \cap U_x \cap U_Z) = (S \cap U_x)$. By choice of U_x, U_Z, we have that $s \in x^\diamond$, i.e. $x \in \overline{s}$, a contradiction. ✠

This above property by itself is weaker than feeble regularity, and is not strong enough for our purposes, but it captures the "combinatorial component" of feeble regularity in the context of hyperedge spaces.

3.4.2 Definition: Given any set H of hyperedges in a hyperedge space, we say that it satisfies the hyperedge-convergence property if

(A) for every subset $F \subseteq H$, every subset Z containing at least one endvertex of each of the hyperedges in F and every point x in the closure of F but not an endvertex of any edge in F,

(C) we have that x is also in the closure of Z.

Moreover, we say that a hyperedge space X is quasiregular if E_X satisfies the hyperedge-convergence property.

The designations "(A)" and "(C)" in the above definition stand for "assumption" and "conclusion". We shall come across variants of the above condition, namely those given in Definitions 3.5.9 and 3.5.13, and we shall use an analogous format to emphasize the parallels.

3.4.3 *Theorem:* *Let X be a quasiregular hyperedge space with underlying hypergraph (V, E, f). Then X is the combinatorial extension of V_X according to f.*

Proof: It is sufficient to show that an arbitrary subset A of X is open if and only if it is of the form $F \uplus (U \setminus E)^{\Box}$ for some $F \subseteq E$ and some open subset U of X. From Proposition 3.3.1 we have that any open set is of this form.

Conversely, suppose $A = \hat{A} \uplus F$ where $\hat{A} = (U \setminus E)^{\Box}$ for some open set U of X and F is an arbitrary subset of E. Clearly it is sufficient to show that $\hat{A}$ is open. Since U is open and every endvertex belongs to the closure of each incident edge, for any $x \in U \cap V$, we have $\mathcal{N}_x \subseteq U$; hence $\hat{A} \subseteq U$. Also $S := U \setminus \hat{A}$ can only contain edges, and for any $e \in S$, we have $\overline{e} \cap U = \{e\}$. But edge-singletons are not closed, so for every $e \in S$ we may pick $x_e \in \partial(e) \setminus U$ and define the set $Z := \{x_e \,|\, e \in S\}$ (we are using the axiom of choice here).

Now suppose that $\hat{A}$ is not open. Then there exists some point $x \in \hat{A}$ such that every neighbourhood of x has non-empty intersection with $U \setminus \hat{A}$ (U is open), implying that $x \in \overline{S}$. Also since $x \in U$, for all $s \in S$, we have $s \notin x^{\diamond}$, so x, S and Z satisfy the requirements of 3.4.1. Hence $x \in \overline{Z}$, a contradiction because Z is contained in the closed set $X \setminus U$, so that $\overline{Z}$ is disjoint from U, which contains x. ✠

3.4.4 Note: In Proposition 3.4.3, the fact that edges must have at least one endvertex is crucial. Let ∞ respresent a point not in $\mathbb{N}$, and consider the topology on $\mathbb{N} \cup \{\infty\}$ in which all singletons except for $\{\infty\}$ are open, the neighbourhoods of ∞ being the sets containing ∞ with finite complement. This topology is feebly regular. However, if the (clopen) points in $\mathbb{N}$ were allowed to be (all the) hyperedges, so that $V = \{\infty\}$, then the combinatorially induced topology would be discrete, and certainly would not agree with the original topology.

Proposition 3.4.3 yields a couple of corollaries. The first one is immediate and will be very important for us.

3.4.5 *Corollary:* *If X is a quasiregular hyperedge space and A is open in V_X, then $A^{\Box}$ is open in X.* ✠

In order to place the second corollary in its proper context, we recall the following fact.

3.4.6 Fact: If V is a closed subset of a topological space X, then $\dim(V) \leq \dim(X)$.

Reference: See, for example, Fedorchuk's chapter [5], Section 2.6, Proposition 6, in [24]. □

3.4.7 *Corollary:* *Let $H = (V, E, \partial)$ be a quasiregular hyperedge space, and suppose that $d = \dim(V)$ is finite and the positive integer k is an upper bound for the number of vertices incident with any hyperedge. Then $\dim(V \cup E) - \dim(V) \leq (d+1)(k-1)$. In particular, if V is disconnected and zero-dimensional, but H is a connected edge space, then X is one-dimensional.*

Proof: We need to give a bound b such that every finite open cover of $V \cup E$ has a refinement of multiplicity which does not exceed b. As in the proof of 3.3.6, we may assume that H is the combinatorial extension of V, that is, that the topology $\widehat{\mathcal{T}}$ on $V \cup E$ is the topology combinatorially induced by the relative topology $\mathcal{T}$ on V.

Let $\mathcal{U} := \{U\}_{i \in I} \subseteq \widehat{\mathcal{T}}$ be a finite open cover of $V \cup E$. Then $\{U \cap V\}_{i \in I} \subseteq \mathcal{T}$ is a finite open cover of V, and hence has an open refinement of order at most $d+1$, that is, an open cover $\{W_j\}_{j \in J}$ such that for every $j \in J$ there exists $i_j \in I$ with $W_j \subseteq (U_{i_j} \cap U)$, and no $v \in V$ lies in more than $d+1$ distinct W_j's.

From the definition of the combinatorially induced topology, it follows that, for every $j \in J$, we have that $W_j^{\square}$ is open and $W_j \subseteq W_j^{\square} \subseteq U_{i_j}$. In particular, $\{W_j^{\square}\}_{j \in J}$ is an open refinement of $\mathcal{U}$ and again, since every hyperedge is incident with some vertex, this refinement also covers $V \cup E$.

Clearly any vertex belongs to $W_j^{\square}$ if and only if it belongs to W_j. On the other hand, a hyperedge e belongs precisely to the $W_j^{\square}$'s which contain any one of its endvertices, and since $W_j^{\square} \cap V = W_j$, there can be at most $d+1$ such $W_j^{\square}$'s per endvertex of e. Since k is an upper bound on the number of endvertices, we have that $\dim(V \cup E) \leq \max\{d, (d+1)k - 1\} = (d+1)k - 1$. Since $\dim(V) = d$, we have $\dim(V \cup E) - \dim(V) \leq (d+1)(k-1)$. ✠

3.4.8 Fact: If X is a zero-dimensional T_0 topological space, then X is totally disconnected. There exists a totally disconnected T_5 (in fact, perfectly normal) topological space which is not zero-dimensional. However, a compact totally-disconnected space is zero-dimensional.

Reference: See [55], Section 5, and [61], Chapter 29. Note that any compact space is "rim-compact" (as defined in [61]). Moreover, a totally disconnected space is necessarily T_1 (since components are closed)—hence under the assumption that X is compact and totally disconnected we may apply Theorem 29.7 in [61]. Also note that the terminology of [55] regarding the T_i separation axioms differs from ours—a "T_i" space ($i \geq 2$) is not necessarily T_1, but a "normal" or "regular" space is. The Knaster-Kuratowski space, with its dispersion point deleted, is totally disconnected and T_5, but not zero-dimensional. It is discussed in detail in [55], Example 129. We reproduce this construction in Example 4.2.3, as part of the construction of the Knaster-Kuratowski graph (Example 4.2.9). □

3.4.9 ***Corollary:*** *Suppose G is a connected quasiregular V-zero-dimensional topologized graph with at least two vertices. Then G is one-dimensional.*

Proof: From Proposition 3.4.7 with $d = 0$ and $k = 2$ we have that $\dim(G) \leq 1$ and from Fact 3.4.8, since V_G is T_1, we have that V_G is totally disconnected. Since $|V_G| \geq 2$ there exists a separation $\{A, B\}$ of V_G. Since G is feebly regular, from Proposition 3.4.3 we have that $A^\square, B^\square$ are open in G, and since every edge is incident with some vertex, together they cover G. Hence $\{A^\square, B^\square\}$ is a finite open cover of G.

Since A, B are both non-empty, so are $A^\square, B^\square$. Since G is connected, $\{A^\square, B^\square\}$ may not be a separation of $V_G \cup E_G$. Hence there exists some point $p \in A^\square \cap B^\square$. However, A and B are disjoint, so p is an edge incident with some vertex $v \in A$ and some other vertex $w \in B$. Now, for any open cover $\mathcal{U}$ of G, we have that $v \in U \in \mathcal{U} \implies p \in U$ and $v \in U \in \mathcal{U} \implies p \in U$. Of course, each of v, w must be in some $U \in \mathcal{U}$. However, no refinement of $\{A^\square, B^\square\}$ may cover v, w with a single open set. Hence any open refinement of $\{A^\square, B^\square\}$ covering $V_G \cup E_G$ must have multiplicity at least one. ✠

3.4.10 Note: In Corollary 3.4.9, the assumption of quasiregularity can not be dropped. The infinite bond plus irregular vertex (Example 3.1.2) is a V-zero-dimensional topologized graph with dimension two.

3.4.11 Remark: Proposition 3.4.3 says that, for feebly regular topological spaces, the topology on $V \cup E$ is completely determined by the topology on V *together with* the combinatorial information of edge-vertex incidences. Hence, *once we impose feeble regularity*, we may restrict our attention to V in identifying further topological properties which make our spaces well-behaved.

This fact should be interpreted with caution, however. First of all, we have no guarantee that the conditions would be easier to find or more elegant to express. Secondly, the combinatorial information (the incidence function) is contained in the topology on $V \cup E$, but of course not in the topology on V. Most importantly, feeble regularity *does not reduce to regularity* (or any other topological assumption) on V, even if V is T_1. The simple examples of the infinite bond with a regular or irregular vertex (Examples 3.1.1,3.1.2) are indistinguishable from their topology on V, which is discrete and in particular regular, and even from their incidence functions, yet the former is feebly regular while the latter is not.

Moreover, the converse to Proposition 3.4.3 is false, even with strong hypotheses on the topology on V. The infinite star with irregular vertex (the minimal spanning set we examined for the Diestel-Kühn obstruction, page 75) is not feebly regular, yet its topology can be recovered from the relative topology on V in the same way as above. Note that in this example the topology on V is regular and completely metrizable. Thus, in order to construct a feebly regular space, it is not sufficient to take a regular topology on V, an arbitrary set of edges, and then take the combinatorial extension according to an arbitrary incidence function with closed images.

3.4.2 Analysing feeble regularity

In this subsection we show how quasiregularity is the "missing link" between (feeble) regularity on V and feeble regularity on $V \cup E$.

3.4.12 ***Theorem:*** *Let X be a quasiregular hyperedge space. If V is feebly Hausdorff, feebly regular or feebly normal, so is X.*

Note: The following proof is the only instance where we need to distinguish between the surrounding set $A^\diamond$ of a set A taken in a subspace V of a topological space X and the surrounding set taken in X itself. We use, here only, the notation $\mathsf{Surr}_V(A)$ and $\mathsf{Surr}_X(A)$ to distinguish between these two operators.

Proof: We give the proof in the case of the regular separation axioms, since it carries the ingredients of the proofs in both the Hausdorff and the normal cases. Let x be an arbitrary point in X and C an arbitrary closed set in X not containing x. We wish to find open neighbourhoods U_C, U_x of C and x respectively whose intersection is contained in the intersection of the surrounding sets (with respect to the topology on X) of C and x. As observed in Remark 3.1.15, this is trivially achieved if x is a hyperedge. If instead x is a vertex, since V is feebly regular there exist *in* V open neighbourhoods U'_C, U'_x of $C \cap V$ and x respectively such that $(U'_C \cap U'_x) \subseteq (\mathsf{Surr}_V(x) \cap \mathsf{Surr}_V(C \cap V))$. Since $C \cap V$ is closed and disjoint from x, we have that $V \setminus C$ is open and contains x; hence we may assume that U'_x is disjoint from C. Moreover, if $\mathsf{Cl}(x)$ is disjoint from C, we choose U'_C to be disjoint from x.

We claim that, for any open set P of V and any set F of hyperedges incident with some vertex in each of V and $V \setminus P$, the set Q of endvertices of hyperedges in F that are in P is closed in P. For otherwise $\mathsf{Cl}(Q) \setminus Q$ contains some point p in P, which is also in $\mathsf{Cl}(F)$ but not incident with any hyperedge in F. By quasiregularity p should also be in the closure of the set of endvertices of edges of F outside P, contradicting the fact that $V \setminus P$ is closed.

Consider the set T of hyperedges which are incident with some vertex in U'_x *and* some vertex outside U'_x, but not with x itself, and let M_T denote the set of vertices in U'_x incident with some edge in T. Then by the second claim above, $\mathsf{Cl}(M_T) \cap U'_x = M_T$, and in particular $\mathsf{Cl}(M_T)$ is disjoint from x. We set $W_x := U'_x \setminus \mathsf{Cl}(M_T) = U'_x \setminus M_T$, which is open in V.

Similarly, we denote by S the set of hyperedges incident with some vertex in U'_C and some vertex outside U'_C, but not with any vertex in $C \cap V$, and by M_S the set of vertices in U'_C incident with some hyperedge in S. Then M_S is closed in U'_C, and we set W_C to be the set $U'_C \setminus M_S$, which is open in V and contains $C \cap V$.

Finally we claim that we may take $U_x := W_x^\square$ and $U_C := W_C^\square$. By Corollary 3.4.5, they are both open, and obviously U_x contains x. Since C is closed, every hyperedge $h \in C \setminus V$ is incident with some vertex in $C \cap V$, and since U_C is open and contains $C \cap V$, it also contains h. Hence $C \subseteq U_C$. Note that $U_C \cap U_x$ is the disjoint union of $W_C \cap W_x$, which is contained in $(U'_x \cap U'_C) \subseteq (\mathsf{Surr}_V(x) \cap \mathsf{Surr}_V(C \cap V)) \subseteq (\mathsf{Surr}_X(x) \cap \mathsf{Surr}_X(C \cap V)) \subseteq (\mathsf{Surr}_X(x) \cap \mathsf{Surr}_X(C))$, together with set some set F of hyperedges. Hence, it now suffices to show that $F \subseteq \mathsf{Surr}_X(x) \cap \mathsf{Surr}_X(C)$.

We observe that U_C may contain two kinds of hyperedges: those incident with some vertex in C and those whose endvertices are all contained in U'_C. Similarly, U_x may contain hyperedges which either are incident with x or else have all their endvertices in U'_x. Now if $f \in F$ is neither incident with x nor with any vertex in $C \cap V$, then all its endvertices (of which there is at least one) must be in $U'_x \cap U'_C$, and if w is such a vertex, we have $f \in \mathsf{Surr}_X(w)$ and $w \in \mathsf{Surr}_V(x) \subseteq \mathsf{Surr}_X(x)$, implying $f \in \mathsf{Surr}_X(w)$, that is, f is incident with x, a contradiction. If instead f is incident with a vertex in C but not with x, then all its endvertices are contained in U'_x, a contradiction because this is disjoint from C. If f is incident with x but not with any vertex in C, then we consider two cases, according to whether $\mathsf{Cl}_V(x) \cap (C \cap V)$ is empty or not. In the latter case, we have $f \in \mathsf{Surr}_X(x)$ and $x \in \mathsf{Surr}_V(C \cap V) \subseteq \mathsf{Surr}_X\, C \cap V$, which imply that $f \in \mathsf{Surr}_X(C \cap V)$, again a contradiction. Recall that in the former case U'_C was chosen not to contain x; but since f is not incident with any vertex in C, all its endvertices are contained in U'_C, again a contradiction. So f must be incident with both x and some vertex in C, and since $f \in F$ was arbitrary, we conclude that $F \subseteq \mathsf{Surr}_X(x) \cap \mathsf{Surr}_X(C)$. ✠

3.4.13 ***Corollary:*** *A hyperedge space is feebly regular if and only if it is quasiregular and V-feebly-regular.*

Proof: Let $\mathcal{H}$ be a hyperedge space. If $V_\mathcal{H} \cup E_\mathcal{H}$ is feebly regular, then, since $V_\mathcal{H}$ is closed, by Proposition 3.1.18 $V_\mathcal{H}$ is feebly regular and the convergence property in Definition 3.4.2 follows from Proposition 3.4.1. The converse follows from Theorem 3.4.12. ✠

3.4.14 ***Corollary:*** *A topologized hypergraph is feebly regular if and only if it is quasiregular and V-regular.*

Proof: Follows from Corollary 3.4.13 and the facts that the vertex set of a topologized hypergraph is T_1, and that feeble regularity and regularity are equivalent for T_1 spaces. ✠

The above corollaries essentially say that feeble regularity is a global topological assumption that can be "broken down" into two independent "components": a "topological component", namely (feeble) regularity of the relative topology on V, and a "combinatorial component", namely quasiregularity.

3.5 Pre-edges, clumps and quotients

This is one of the two sections in this thesis in which we model edges by sets consisting of more than one point. We do this for the purposes of comparison with alternative models, including the usual cell-complex model. In this section we also discuss another feature of the "edge space" model which we *will* keep throughout the thesis—the fact that the components of the vertex set are not necessarily singletons. We refer to these components as "clumps".

Recall that the partitions of a topological space X are in one-to-one correspondence with the quotients of X: given a partition $\mathcal{P}$ of X, one abstractly constructs a quotient space $X/\mathcal{P}$ whose points are the equivalence classes, and whose topology is the coarsest which renders the projection $p : x \mapsto [x]$ of X onto $X/\mathcal{P}$ continuous, where $[x]$ denotes the equivalence class to which x belongs. Equivalently, a subset U is open in $X/\mathcal{P}$ if and only if $p^{-1}(U)$ is open in X. In this context, partitions usually are referred to as decompositions. We shall refer to the equivalence classes as parts (we do not use the term cells in order to avoid confusion in the context of a CW-complex). Note that parts are non-empty. We shall say a part is degenerate if it consists of only one point. Also, a decomposition is closed if all its parts are closed subsets. The following definition is standard [61, Definition 9.8].

3.5.1 Definition: Given a topological space X and a decomposition $\mathcal{P}$ of X, a subset of X is saturated if it is the union of parts of $\mathcal{P}$. The decomposition $\mathcal{P}$ is upper semicontinuous if, for any part $P \in \mathcal{P}$ and any open set U containing P, there exists a saturated open set V with $P \subseteq V \subseteq U$.

The importance of upper semicontinuous decompositions lies in the following well-known fact [61, Theorem 9.9]. Recall that a function from one topological space into another is closed if and only if the image of a closed set is closed.

3.5.2 Fact: Given a quotient map $p : X \to Y$ induced by a decomposition $\mathcal{P}$ of X, p is a closed map if and only if $\mathcal{P}$ is upper semicontinuous.

Reference: See for example Theorem 9.9 in [61]. □

3.5.3 Definition: An open connected subset of a topological space is a pre-hyperedge if it is not closed, a pre-edge if it has at most two boundary points, and a proper pre-(hyper)edge if it has precisely two boundary points. A pre-(hyper)edge selection $\{U_i\}_{i\in I}$ is a set of pairwise disjoint pre-(hyper)edges. Given a pre-(hyper)edge selection $\mathcal{U}$,

- the pre-complement of $\mathcal{U}$ is the closed subset $X \setminus \bigcup_{U\in\mathcal{U}} U$,
- the induced (hyper)edge decomposition is the partition whose parts are the pre-(hyper)edges and the pre-complement,
- the clumps are the components of the pre-complement,
- the induced clump-(hyper)edge decomposition is the partition whose parts are the pre-(hyper)edges and the clumps,
- and the induced (hyper)edge quotient and clump-(hyper)edge quotient are the quotient spaces induced by the (hyper)edge and clump-(hyper)edge decompositions respectively.

Given any subset U of the precomplement, we denote by $U^{\boxdot}$ the union of U and all pre-edges whose closure intersects U.

Note that given any hyperedge space, one can take the collection of all singletons consisting of a hyperedge as a pre-hyperedge selection. We shall extend the terminology defined with respect to pre-hyperedge selections to hyperedge spaces via this translation. Moreover, in this context we shall refer to the clump-edge quotient more simply as the clump-quotient.

Recall that a **T**-set (short for Treybig set) is a closed subset of a topological space (which is usually assumed to be compact) such that the components of its complement have precisely two points in their boundary. This concept was introduced by Treybig in [57], while the abbreviation to T-set is due to J. Nikiel [46].

By analogy with this notation, we say a subset of a topological space is an **H**-set if it is the union of the pre-hyperedges of some pre-hyperedge selection, and a (proper) **E**-set if all these hyperedges are (proper) edges. Also, a subset is a **V**-set if it is the complement of some H-set (that is, the pre-complement of some pre-hyperedge selection).

3.5.4 Note: The complement of a proper E-set is always a T-set, but the complement of a T-set need not be an H-set, because its components may fail to be open. However, it is true that in a locally connected space a subset is a T-set if and only if its complement is a proper E-set.

Technical Note: Given any H-set in any topological space, the induced hyperedge and clump-hyperedge quotients can be given the structure of a hyperedge space by taking the images of the pre-hyperedges (components of the H-set) for the hyperedges of the quotient. Technically the quotient is a topological space, and will be compatible with different hypergraphs (i.e. can be given the structure of a hyperedge space in different ways), but unless explicitly stated otherwise, *we shall always assume that it has the structure* (V_Q, E_Q, ∂_Q) *of a hyperedge space* Q, with E_Q consisting precisely of the (point) images of the pre-hyperedges.

In the special case of the clump quotient (when the pre-hyperedges of the original hyperedge space are singletons), the restriction of the quotient map to the set of edges of the original space gives a bijection onto the set of edges of the quotient. In this context, **we shall identify the hyperedges of the original space with their images in the quotient**, and refer to a single set of hyperedges, and to a given open point as being simultaneously a hyperedge of two different hyperedge spaces.

We omit the proof of the following easy proposition.

3.5.5 ***Proposition:*** *Let G be a graph, $\hat{G}$ the cell-complex topological space associated with G, and $\mathcal{E}$ the pre-edge selection consisting of the 1-cells of $\hat{G}$. Then the edge-quotient of $\hat{G}$ induced by $\mathcal{E}$ is the classical graph*[7] *associated with G.* ✠

3.5.6 ***Proposition:*** *Let Z be an arbitrary V-set of an arbitrary topological space X. Then the induced clump-hyperedge quotient is a V-totally-disconnected topologized hypergraph.*

Proof: Let Q denote the clump-hyperedge quotient. Since Z is closed, its components, which are closed in Z, are also closed in X. Hence their images in Q (whose inverse images consist precisely of such a component) are closed in Q. Similarly we see that the images of the pre-hyperedges are open points. We conclude that the quotient is an S_1 space, that is, a topologized hypergraph.

Now suppose that C is a subset of V_Q consisting of more than one point. Its inverse image P is the union of at least two components of Z, and therefore is disconnected in Z. Let $\{A, B\}$ be a separation of this set. Since A, B are clopen in P, they are saturated

[7] Recall that a *classical* graph carries a topology.

with respect to the decomposition of P into its components. In particular, they coincide with the inverse images of their images.[8] Hence the images are clopen in C, and give a separation of C, which is therefore not connected. ✠

3.5.7 ***Proposition:*** *Given a pre-hyperedge selection in a topological space X such that all the pre-hyperedges have finite boundaries, let $q : X \to Q$ be the induced quotient map onto the clump-hyperedge quotient Q. Then a vertex $v \in V_Q$ is incident with a hyperedge $e \in Q$ if and only if the pre-hyperedge whose image is e has a boundary point in the clump whose image is v. In particular,*

- *the number of endvertices of e does not exceed the number of boundary points of its pre-image;*
- *if X is a edge space, then Q is a edge space, and e is incident with v in Q if and only if it is incident in X with some vertex in the clump whose image is v.*

Proof: Let v be an arbitrary vertex in Q, and e an arbitrary edge in Q. Then $\{v\} = q(C)$ for some clump C and $\{e\} = q(H)$ for some pre-hyperedge. Suppose $x \in \partial(H) \cap C$ in X, and consider an arbitrary neighbourhood U of v in Q. Then $q^{-1}(U)$ is a neighbourhood of C; in particular it contains x, and therefore must contain some point $p \in U$. But then $q(p) = e \in q(q^{-1}(U)) = U$. Hence every neighbourhood of v contains e, that is, $v \in \partial(e)$.

Conversely, suppose that $\partial(H) \cap C = \emptyset$. Since $\partial(H)$ is finite, and since all clumps, being components of the (closed) pre-complement, are closed in X, we have that the union B of H (equivalently, $\mathsf{Cl}(H)$) with all clumps containing a boundary point of H is closed. Hence $A := X \setminus B$ is open. Since B is disjoint from C, we have that A also contains C. Moreover, the complementary sets A and B are both saturated with respect to the clump-hyperedge decomposition of X. Therefore $q(A), q(B)$ are disjoint and $q^{-1}(q(A)) = A$, so $q(A)$, which contains v and is disjoint from $e \in q(B)$, is open. This implies that $v \notin \partial(e)$. So if $v \in \partial(e)$, then $C \cap \partial(H) \neq \emptyset$. ✠

3.5.8 Note: In Proposition 3.5.7, the assumption that the pre-hyperedges have finite boundaries can not be dropped. Consider the subset $V \subseteq \mathbb{R}^2$ given by the union of $\{(0,0)\}$ with the set of all points (x, y) such that $1/y$ is an integer. We define a hypergraph H with vertex set V and a single hyperedge whose endvertices are all the vertices (x, y) such

[8] Note that, if a subset V of a topological space X is saturated with respect to a given decomposition $\mathcal{D}$, and $p : X \to \widehat{X}$ is the associated quotient map, then $p(V)$, (equipped with the relative topology inherited from $\widehat{X}$), is homeomorphic to the quotient of the subspace V (equipped with the relative topology inherited from X) determined by the decomposition of V whose parts are the parts of $\mathcal{D}$ contained in V.

that $x = 1$. To define a topology on $X := V \cup \{e\}$, let V be equipped with the relative topology inherited from $\mathbb{R}^2$ and X with the combinatorially induced topology.

Now $\{\{e\}\}$ is a pre-hyperedge selection in X whose single pre-hyperedge $\{e\}$ has infinite boundary. The pre-complement is V and the clumps are $\{(0,0)\}$ and all sets of the form $\{(x,y) \in \mathbb{R}^2 \mid 1/y = z\}$ for some $z \in \mathbb{Z}$. The point $(0,0)$ is not in the boundary of the pre-hyperedge, but its image in the clump-hyperedge quotient is an endvertex of the only hyperedge of the quotient.

The assumption that the boundaries of the pre-hyperedges are finite may be replaced by the assumptions that the decomposition of the V-set into clumps is upper semicontinuous, and that the hyperedge selection satisfies the weak pre-hyperedge convergence property (see Definition 3.5.13 below).

3.5.9 Definition: A pre-hyperedge selection $\mathcal{U} = \{U\}_{i \in I}$ satisfies the strong pre-hyperedge convergence property if

(A) $\forall J \subseteq I$, $\{v_j\}_{j \in J} \subseteq X$ and $x \in X$ such that:

$$x \in \mathsf{Cl}\left(\bigcup_{j \in J} U_j\right) \setminus \bigcup_{j \in J} \mathsf{Cl}(U_j) \quad \text{and,} \quad \forall j \in J,\ v_j \in \partial(U_j)$$

(C) we have that $x \in \mathsf{Cl}(\{v_j \mid j \in J\})$.

Note that since pre-edges are open and pairwise disjoint, x must belong to the precomplement.

Note that the above condition reduces to that of quasiregularity (Definition 3.4.2) in the case when the U_j's are all singletons.

3.5.10 *Proposition:* *Let $\mathcal{E}$ be a pre-hyperedge selection in a topological space X. Then the induced hyperedge quotient is quasiregular if and only if $\mathcal{E}$ satisfies the strong hyperedge convergence property, and feebly regular if and only if the pre-complement is feebly regular and $\mathcal{E}$ satisfies the strong pre-hyperedge convergence property.*

Proof: This follows from the definition of quasiregularity and from Corollary 3.4.13 ✠

3.5.11 Remark: *The Diestel-Kühn obstruction is not feebly regular.* We have defined the Diestel-Kühn obstruction by adding a "point at infinity" to a classical graph, in which edges are modelled as singletons. Diestel and Kühn consider this configuration in the context of the usual cell-complex topological space associated with a graph, in which edges are modelled as arcs. In particular, they start from a Hausdorff space; it is easy to see that with this model the space resulting after adding the "point at infinity" to this particular configuration is T_4.

We have already stated (in slightly different language, Proposition 1.1.25) that the edge quotient of the cell-complex topological space associated with a graph with the 1-cells for pre-edges coincides with the graph equipped with the classical model; this easy correspondence does not change with the presence of the extra point at infinity.

Once we model the edges as singletons, the Diestel-Kühn obstruction fails to be feebly Hausdorff, or quasiregular. We find that the easiest way to see this is by working directly in the model with singletons for edges. Let v and w denote respectively the vertex of infinite degree and the vertex of degree zero. Every open neighbourhood of v contains all the incident edges, and every neighbourhood of w must also contain infinitely many of these edges, none of which belong to $w^\diamond$. Hence v and w can not be separated in the sense of Definition 3.1.13.

Working instead in the Hausdorff model, taking the 1-cells for the U_j's in definition 3.5.9, w for x and a point v_j in each of the interiors of the 1-cells, we see that (A) is satisfied but (C) is not satisfied. Thus, the strong pre-hyperedge convergence property is not satisfied and the edge quotient is not feebly regular.

The contrast means that the well-behaved spaces can more easily be discerned in the non-Hausdorff model; while the anomaly in the Diestel-Kühn configuration is captured by feeble regularity, it slips past the usual T_i axioms in the Hausdorff model and one needs the rather involved strong pre-hyperedge convergence condition to discern a well-behaved structure from a badly-behaved one.

Another, more important, difference between the Hausdorff and non-Hausdorff models which we have already alluded to (in the case of the Diestel-Kühn obstruction) concerns the issue of compactness. Since several of our results in Chapters 4 and 5 will rely heavily on the assumption of compactness, this is another point which shows how the well–behaved spaces are more easily captured in the non-Hausdorff model. This point is actually more important than the issue of separation axioms because if one starts out from a compact Hausdorff space, one may in fact fail to capture some well-behaved structures: in the case of the Diestel-Kühn obstruction the Hausdorff space is not compact and the entire configuration is not one we would like to capture because it is badly behaved, but it is possible for a non-compact Hausdorff space to be compact and feebly Hausdorff in the non-Hausdorff model—in Chapter 4 we shall see that this is true in the case of the Diestel-Kühn end-quotient.

Note that any quotient of a compact space will be compact, but it is also possible for a compact Hausdorff space to have a non-feebly-Hausdorff quotient. Our next example will illustrate this fact. Another feature of the Diestel-Kühn obstruction is that, with respect to the pre-edge selection with 1-cells for pre-edges, the clumps happen to be singletons, so the clump-edge quotient coincides with the edge quotient. We now give an example of a pre-edge selection in a compact Hausdorff space which induces an edge quotient which is not quasiregular (hence not feebly regular), and does not trivially coincide with the clump-edge quotient. This example will be important as a precursor to Example 4.3.13.

3.5.12 Example: For $x \in \mathbb{R}$, let I_x denote the "vertical" segment in $\mathbb{R}^2$ given by $\{x\} \times [0,1]$, and consider the Euclidean subspace of $\mathbb{R}^2$ consisting of the union of the segments $I_{2^{-n}}$ (for all non-negative integers n), I_0, and the segment $[0,1] \times \{0\}$. Since this is a subspace of (the regular) Euclidean 2-space, it is regular (and T_3, in fact compact and completely metrizable since it is is closed and bounded). For any $z \in [0,1]$, the sequence of points $\{(2^{-n}, z)\}_{n\in\mathbb{N}}$ has the point $(z, 0)$ as its unique cluster point. We take the pre-edges to be $e'_n := I_{2^{-n}} \setminus \{(0, 2^{-n}), (1, 2^{-n})\}$, for all non-negative integers n, and denote by p_n, q_n the points $(2^{-n}, 1/2), (2^{-n}, 1)$ respectively. The sequence $\{p_n\}_{n\in\mathbb{N}}$ converges to $(0, 1/2)$ but this point is not a cluster point of $\{q_n\}_{n\in\mathbb{N}}$, despite the fact that $p_n \in e'_n$ and $q_n \in \partial(e'_n)$. Thus, the edge quotient is not quasiregular.

Note that in the above example we are not identifying the clumps, that is, we are only discussing the *edge*-quotient, as opposed to the *clump-edge* quotient. It turns out that the clump-edge quotient in the above example is the same space as the quotient obtained from the Diestel-Kühn obstruction by identifying the vertex of degree zero with the vertex of infinite degree. This identification is used by Diestel and Kühn to obtain well-behaved spaces; we shall see that the spaces they obtain, in particular the clump-edge quotient obtained from the above example, are feebly regular. We shall also give a slightly more involved example (Example 4.3.13), based on the above example, of a pre-edge selection in a compact Hausdorff space which leads to a non-feebly-regular clump-edge quotient.

3.5.13 Definition: Given a pre-hyperedge selection $\mathcal{U} = \{U\}_{i\in I}$, suppose that $\mathcal{Z}$ is the collection of components of the pre-complement Z. Then we say that $\mathcal{U}$ satisfies the weak pre-hyperedge convergence property if

(A) $\forall J \subseteq I$, $\{K_j\}_{j\in J} \subseteq \mathcal{Z}$ and $K \in \mathcal{Z}$ such that:

- $K \cap \mathsf{Cl}\left(\bigcup_{j\in J} U_j\right) \neq \emptyset$ and,
- $\forall j \in J$ we have that $\mathsf{Cl}(U_j) \cap K_j \neq \emptyset$ but $K \cap \mathsf{Cl}(U_j) = \emptyset$;

(C) it follows that $\mathsf{Cl}\left(\bigcup_{j\in J} K_j\right) \cap K \neq \emptyset$.

3.5.14 Remark: The above condition is equivalent to the following:

If Λ is a directed set, $\forall\lambda \in \Lambda$ we have points $x_\lambda \in U_{i_\lambda} \in \mathcal{U}$ and $y_\lambda \in \partial(U_{i_\lambda})$, the set K is a component of the pre-complement disjoint from the closure of any one of the U_{i_λ}'s, and the net $(x_\lambda)_{\lambda\in\Lambda}$ converges to a point x in K, then $\forall\lambda \in \Lambda$ there exist z_λ belonging to the same component of Z as y_λ such that the net $(z_\lambda)_{\lambda\in\Lambda}$ has a subnet converging to a point in K (possibly distinct from x).

We observe that a variant of this condition comes up in the study of quotients of metric spaces; see for example [12, Example 18.A.20, (ii)]. In fact, this variant will also come up for us—see Lemma 4.3.10, which we invoke in Theorem 4.3.12. Also, we draw the reader's attention to the similarities between this condition and the strong pre-hyperedge convergence property (Definition 3.5.9) and the hyperedge convergence property (Definition 3.4.2).

3.5.15 ***Proposition:*** *Let $\mathcal{E}$ be a pre-hyperedge selection in a connected topological space X such that the boundary of each pre-hyperedge is finite. Then the induced clump-hyperedge quotient is quasiregular if and only if $\mathcal{E}$ satisfies the weak pre-hyperedge convergence property.*

Proof: This follows from Proposition 3.5.7 and the definition of quasiregularity. ✠

3.6 Feebly Normal Spaces

3.6.1 ***Proposition:*** *Let $p : X \to Y$ be a closed quotient mapping. If X is feebly normal, so is Y.*

Proof: Let C_1, C_2 be arbitrary disjoint closed subsets of Y, and K_1, K_2 the respective inverse images under p. Since X is feebly normal there exist open sets U_1, U_2 containing K_1, K_2 respectively such that $U_1 \cap U_2 \subseteq K_1^\diamond \cap K_2^\diamond$. Since p is closed, the decomposition of X into the fibres of p is closed and therefore, for every $y \in C_i$ ($i = 1, 2$), there exists an open set W_y which is the union of equivalence classes and such that $p^{-1}(y) \subseteq W_y \subseteq U_i$. Setting $W_i = \bigcup_{y\in C_i} W_y$ gives an open set which is the union of fibres and such that $K_i \subseteq W_i \subseteq U_i$.

Since p is closed the set $Y \setminus (p(X \setminus W_i))$ is open in Y and since W_i is the union of fibres this set is precisely $p(W_i)$. Clearly $p(W_i)$ contains C_i. Now suppose $y \in p(W_1) \cap p(W_2)$. Since W_1, W_2 are unions of fibres, $p^{-1}(y) \subseteq W_1 \cap W_2 \subset U_1 \cap U_2$. Hence if A is any open set in

Y containing C_1, the set $p^{-1}(A)$ (which is open because p is continuous) containing K_1 contains $p^{-1}(y)$, and therefore $A = p(p^{-1}(A))$ contains y. Since A is an arbitrary open set containing C_1 and y an arbitrary point in $p(W_1) \cap p(W_2)$, it follows that $p(W_1) \cap p(W_2) \subseteq C_1^{\diamond}$. ✠

3.6.2 ***Proposition:*** *Given a V-set W in a feebly normal topological space, suppose that the decomposition of W into its components is upper semicontinuous, and that the induced pre-hyperedges have finite boundaries. Then the induced clump-hyperedge quotient is feebly normal if and only if the induced pre-hyperedge selection satisfies the weak pre-hyperedge convergence property.*

Proof: Let Q denote the induced clump-hyperedge quotient. Since W is a closed subset of a feebly normal space, it is itself feebly normal (by Proposition 3.1.18). Since its decomposition $\mathcal{D}$ into components is upper semicontinuous, by Fact 3.5.2 the quotient map restricted to W is closed, and by Proposition 3.6.1 we have that $V_Q \cong W/\mathcal{D}$ is feebly normal.

If the pre-hyperedge selection satisfies the weak pre-hyperedge convergence property, then by Proposition 3.5.15, the clump-hyperedge quotient is quasiregular, and therefore, by Theorem 3.4.12, feebly normal. Conversely, if the quotient is not feebly normal, by Theorem 3.4.12 it is not quasiregular, and by Proposition 3.5.15 the induced pre-hyperedge selection does not satisfy the weak pre-hyperedge convergence property. ✠

We now give an example of a pre-edge selection which satisfies the weak pre-edge convergence property but induces a pre-complement whose decomposition into its components is not upper semicontinuous, so that the clump-edge quotient is quasiregular but not feebly regular.

3.6.3 Example: For every non-negative integer n, let R_n denote the rectangular subset of $\mathbb{R}^2$ given by $(\{-n, n\} \times [-a_n, a_n]) \cup ([-n, n] \times \{-a_n, a_n\})$, where a_n denotes $1 - \frac{1}{2^n}$. Let $V := \mathbb{R} \times \{-1, 1\} \cup \bigcup_{n \geq 0} R_n$. We define a hypergraph with vertex set V and, for every positive integer n, an edge e_n with endvertices $(-n+1, 0), (-n, 0)$. Now we define a topology on $V \cup E$ by taking the relative topology on V inherited from $\mathbb{R}^2$ and then taking the combinatorially induced topology on $V \cup E$. Note that the vertex set is a closed subset of $\mathbb{R}^2$, and therefore normal. From this it is easy to verify that the resulting topologized graph is feebly normal.

Note that we could replace the edges with arcs to obtain a T_4 space. The subset V is a standard example from general topology illustrating the distinction between connected components and quasicomponents. The reader is referred to Figure 3.6 for an illustration.[9]

[9]Note that the diagram is not to scale.

If we take V as a proper V-set, the induced pre-edge selection (which has singletons for pre-edges, but could equally well have arcs instead) trivially satisfies the weak pre-hyperedge convergence property, since the closure of the union of any collection of pre-edges coincides with the union of the closures. In fact, the induced quotient is a space which we already know to be not even feebly Hausdorff (hence not feebly regular): it is the one-way infinite path with a double end (Example 3.1.7). It is easy to verify that directly that this quotient space is not feebly Hausdorff, but we have already observed that in this space path-connectedness fails to be transitive, and proved that for feebly Hausdorff spaces this may not happen (Proposition 3.2.2). Note however that the one-way infinite path with a double end is quasiregular.

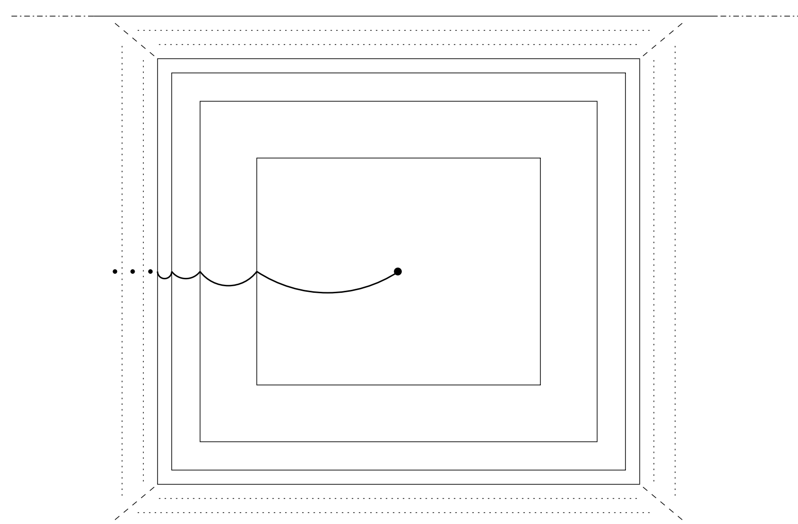


Figure 3.6: A V-set inducing a quasiregular, non-feebly-regular clump-edge quotient.

Examples 3.5.12 and 3.6.3 illustrate how the the effect of contracting the *clumps* can lead to interesting scenarios. The model of hyperedge spaces allows non-degenerate clumps but requires edges to be modelled as singletons. On the other hand, the usual topological model, that is, the cell-complex topological space, has sets for edges and points for vertices. In a sense, the model of edge spaces is the opposite of the usual model for graphs.

In Example 3.6.3, some of the parts of the decomposition (the "pre-edges") are open singletons. This is a rather unusual context for topological quotients.

3.6.4 ***Proposition:*** *Let X be a quasiregular hyperedge space. If the decomposition of V_X into its components is upper semicontinuous, then the clump-hyperedge decomposition of X is upper semicontinuous.*

Proof: Let D be a part of the edge-clump decomposition and U be an open set containing D. We wish to exhibit a saturated open set containing D and contained in U. If D is a singleton consisting of a hyperedge, this is trivial. If not, $U \cap V_X$ is open in V_X and contains D. Since D is a part with respect to the decomposition of V_X into clumps, there exists an open set W containing D, contained in $U \cap V_X$ and saturated with respect to the decomposition of V_X into the clumps. Since X is quasiregular by 3.4.5, $W^{\square}$ is open in X, is contained in any open set containing W, in particular U, and is the union of clumps and singletons consisting of hyperedges, that is, it is saturated with respect to the edge-clump decomposition of X. Hence this decomposition is upper semicontinuous. ✠

Chapter 4

Compact Spaces

We have already alluded to "minimal spanning sets" as our analogues for the spanning trees of graph theory. In this chapter we prove their existence under various conditions. The key ingredient in these proofs will be compactness—in fact, this will be the underlying topological idea in all of this chapter.

We present a (non-compact) modification of a famous example from general topology as an example of a feebly regular space which does not have a minimal spanning set. In our context it is natural to consider spaces which are connected (as well as compact); hence it is not surprising that we find links to continuum theory. We prove feebly Hausdorff (as opposed to Hausdorff) analogues of several results from general general topology, such as the fact that the decomposition of a compact feebly Haudsorff topological space into its components is upper semicontinuous.

In Section 4.4 we consider a topological construction of Diestel and Kühn based on a graph, and show that the spaces considered in [21] essentially fall within our assumptions.

We resume and conclude the various aspects relating to the discussion of a topological model for graphs. We use the comparison with the spaces of Diestel and Kühn mentioned above to present the remaining two arguments relating to the specific issue of whether to model an edge as a point or an arc. In Remark 4.3.18 we address the possibilities of a topological model from a more generic standpoint. We also revisit pre-edge selections under the additional assumption of compactness, and obtain results about quotients which we shall apply in Chapter 5.

We also encounter the "topological curiosity" alluded to before. This arises out of the seemingly innocuous idea of identifying parallel edges, and yields a scenario of upper semicontinuous decompositions with *open* parts (consisting of parallel edges), and with the property that the subspace given by *any* system of distinct representatives is homeomorphic to the induced quotient, and therefore a retract of the original space.

With respect to the parallel decomposition, a comment is in order. If one wishes to extend the existence result of (4.2.13) to allow for infinitely many parallel edges joining a given pair of vertices, by "identifying parallel edges", topological quotients may or may not be the best way to proceed; however, any such argument needs to take into account quasiregularity, as is easily shown in the discussion of the infinite bond plus irregular vertex (Example 3.1.2).

4.1 Separating Closed and Compact Sets

In this section we look at yet another parallel between the standard separation axioms and their "feeble" and "weak" counterparts.

4.1.1 Fact: Let X be a topological space, and for $i = 2, 3, 4$, let $\mathcal{T}_i$ denote the assertion "X is a T_i space". If X is compact, then $\mathcal{T}_2 \Rightarrow \mathcal{T}_3 \Rightarrow \mathcal{T}_4$.

Reference: See [61], Theorem 17.10. □

The following technical lemma about compact subsets in feebly Hausdorff spaces will come in useful for us.

4.1.2 *Lemma:* *Let X be a feebly Hausdorff topological space and K_1, K_2 disjoint compact subsets. Then there exist open sets A_1, A_2 containing K_1, K_2, respectively, and finite subsets $I \subseteq K_1$, $J \subseteq K_2$ such that*

$$A_1 \cap A_2 \subseteq \bigcup_{(i,j)\in I\times J} \rho(i) \cap \rho(j)$$

Proof: Since X is feebly Hausdorff for any $x \in K_1$ and any $y \in K_2$ there exist open sets U_x^y, W_x^y such that $x \in U_x^y$, $y \in W_x^y$ and $U_x^y \cap W_x^y \subseteq \rho(x) \cap \rho(y)$. For a fixed $y \in K_2$, $\{U_x^y\}_{x\in K_1}$ is an open cover of K_1, so there exists a finite subset $I_y \subseteq K_1$ such that $\{U_x^y\}_{x\in I_y}$ is also an open cover of K_1. Now let $W_y = \bigcap_{x\in I_y} W_x^y$ and $U_y = \bigcup_{x\in I_y} U_x^y$. Note that U_y contains K_1, W_y contains y and, since I_y is finite, U_y and W_y are both open. Now again $\{W_y\}_{y\in K_2}$ is an open cover of K_2 so there must be a finite subset $J \subseteq K_2$ such that $\{W_y\}_{y\in J}$ is also an open cover of K_2.

Setting $A_2 = \bigcup_{y\in J} W_y$, $A_1 = \bigcap_{y\in J} U_y$ and $I = \bigcup_{y\in J} I_y$ fulfills the requirements of the assertion: the finiteness of J ensures that I is also finite, and that A_1 and A_2 are both open sets (containing K_1 and K_2 respectively), while $z \in A_1 \cap A_2 \Longrightarrow z \in W_{\check{y}} \cap U_{\check{y}}$ for some $\check{y} \in J$, which in turn implies that $z \in U_{\check{x}}^{\check{y}} \cap W_{\check{x}}^{\check{y}} \subseteq \rho(\check{x}) \cap \rho(\check{y})$ for some $\check{x} \in I_{\check{y}} \subseteq I$. ✠

4.1.3 ***Corollary:*** *Let X be a feebly Hausdorff topological space and K_1, K_2 disjoint compact subsets. Then there exist open sets A_1, A_2 containing K_1, K_2, respectively, such that $U_1 \cap U_2 \subseteq A_1^\diamond \cap A_2^\diamond$.*

The above can be seen as an instance of the rule of thumb "compact sets behave like points" and is analogous to the fact that in Hausdorff spaces compact subsets can be separated by open sets. Another example is the following.

4.1.4 ***Corollary:*** *Let X be a weakly Hausdorff topological space and K_1, K_2 disjoint compact subsets. Then there exist open sets A_1, A_2 containing K_1, K_2, respectively, such that $U_1 \cap U_2$ is finite.* ✠

4.1.5 ***Proposition:*** *Let X be a compact topological space. If X is feebly Hausdorff, then it is feebly normal, and if X is weakly Hausdorff, then it is weakly normal.*

Proof: Let C_1, C_2 be disjoint closed subsets. Since X is compact, C_1, C_2 are compact. Hence by Corollary 4.1.3 in the case of the feeble assumption, or by Corollary 4.1.4 in the case of the weak assumption, there exist open sets U_1, U_2 containing C_1, C_2 respectively such that their intersection is contained in $C_1^\diamond \cap C_2^\diamond$ (or is finite). ✠

4.1.6 ***Proposition:*** *Let X be a topological space, and for $i = 2, 3, 4$, let $\mathcal{S}_i$ denote the assertion "X is an S_i space". If X is compact, then $\mathcal{S}_2 \Rightarrow \mathcal{S}_3 \Rightarrow \mathcal{S}_4$.*

Proof: This follows from Propositions 4.1.5 and 3.1.17. ✠

4.2 Existence Theorems

At this point we introduce the objects which take the rôle of spanning trees in our context.

4.2.1 Definition: A subset S of a hyperedge space H is a *minimal spanning set* if it is minimal with the property of being connected and containing V_H.

4.2.2 ***Proposition:*** *A connected subset of a hyperedge space H containing V_H is a minimal spanning set if and only if every hyperedge is a cutpoint.*

Proof: If S is a minimal spanning set, then, for any hyperedge h, $X \setminus \{h\}$ still contains V, and therefore must be disconnected. Conversely, if T is connected and contains V_H, and every hyperedge is a cutpoint, then any proper subset A of T containing V_H must be contained in some set of the form $T \setminus \{h\}$ for some hyperedge $h \in T$. Since h is a cutpoint, there exists a separation $\{C, D\}$ of $T \setminus \{h\}$, and $C \cap A, D \cap A$ is a separation of Z, implying that A is not connected. ✠

The definition of minimal spanning sets is modelled on the characterization of spanning trees in classical graphs given in Proposition 1.1.29. While it is easy to see that any connected classical graph contains a spanning tree, the corresponding question for hyperedge spaces (or even topologized graphs) appears to be far from trivial. In Subsection 3.1.1 we already saw two examples of topologized graphs which do not have a minimal spanning set, namely the infinite bond plus irregular vertex (Example 3.1.2) and the overcrowded fan (Example 3.1.3). These two spaces are not feebly regular, hence may be considered to be "too pathological" to be interesting. Finding spaces without a minimal spanning tree becomes considerably harder when one restricts to spaces with strong separation axioms.

Specific instances of this problem (modulo the modification of modelling edges as singletons) have been considered by Diestel and Kühn in [21]. They construct a topological space by identifying certain points in the direction extension associated with a graph, to obtain a Hausdorff space that we shall refer to as the Diestel-Kühn end-quotient of the graph (this is the space they denote by $\tilde{G}$). They impose two slightly different assumptions, one of which is implied by the other: the fact that no two vertices are joined by infinitely many independent paths (property (2)), and the stronger requirement that no two vertices are joined by infinitely many edge-disjoint paths (property (3)).

In Theorem 5.2 they show that if a connected classical graph satisfies property (3), then its Diestel-Kühn end-quotient has what they refer to as a topological spanning tree—a path connected subset containing all the vertices and ends (which we would collectively refer to as vertices), does not contain a circle, and contains every edge of which it contains an inner point. We remark that in this instance the entire set of points which is modelling an edge is being taken "all at once". In Theorem 5.3 they also show that if the original graph is countable and satisfies (2), then its Diestel-Kühn end-quotient has the property that every closed connected subset is path-connected.

From the latter theorem (and our Proposition 4.2.2) it follows that, at least in the case when the graph satisfies (2) and is countable, a topological spanning tree in the sense of Diestel and Kühn is a minimal spanning set in our sense,[1] because edges are modelled as open sets.

In Proposition 3.4 Diestel and Kühn also show that there exists a countable graph satisfying (2) for which the Diestel-Kühn end-quotient contains a circle consisting entirely

[1]modulo replacing the arcs with singletons

of vertices and ends (vertices for us). Such a graph is automatically disqualified from having a topological spanning tree. This appears to be one of the main reasons why Diestel and Kühn consider (the stronger) property (3).

One of the ways in which our approach differs from that of Diestel and Kühn is that we do not attempt to exclude a cycle-like structure *within the vertex set* (which is contained in any minimal spanning set). Instead, our approach excludes "edgecycles" from minimal spanning set, although not explicitly (since the definition does not mention edgecycles).

Diestel and Kühn also state that they have been unable to settle the question if the direction extension of a connected classical graph must have a topological spanning tree (Problem 7.7), and show that this must be the case if the graph has only one end (Proposition 7.8). Although the direction extension differs from the Diestel-Kühn end-quotient precisely when the direction extension is not feebly regular, as pointed out by the authors the difficulty of the question (given the current understanding of classical infinite graphs) lies in allowing *topological* spanning trees, that is, spanning sets which are connected thanks to the ends, as opposed to considering only those whose intersection with the graph is connected. In the latter case the intersection would have to be an end-faithful spanning tree (Proposition 7.1 in the same paper), and there are examples in the literature of classical graphs which do not have end-faithful spanning trees [54, 56].

4.2.1 The Knaster-Kuratoswki graph

In this subsection we construct a feebly regular topologized graph with no minimal spanning set. Our construction will be based on a well-known pathological example from general topology, variously referred to as the "Knaster-Kuratowski fan/space", "Cantor's Leaky Tent", and "Cantor's Teepee". A detailed discussion of this space can be found in [55], Example 129. We give the construction of this space in Example 4.2.3.

In various paragraphs in this subsection, we introduce notation which we also subsequently use throughout the remainder of this subsection, such as the notation $L(c)$ and $A(p)$ introduced in Example 4.2.3.

4.2.3 Example (The Knaster-Kuratowski space): We construct the Knaster-Kuratwoski fan as a subspace $Y \subseteq \mathbb{R}^2$, with the inherited Euclidean topology. Let C be the Cantor set situated on $[0,1] \times \{0\}$, that is, the set of points $(x, 0)$ in such that x is a real number in $[0,1]$ and has a (unique, possibly infinite) ternary expansion which does not use the integer 1. Let D denote the subset of C consisting of the "endpoints of the deleted intervals", that is, the points $(x, 0) \in C$ such that x has a finite ternary expansion (possibly using 1). Also, we set $F := C \setminus D$ and $p := \left(\frac{1}{2}, 1\right)$.

Now $\forall\, c \in \mathbb{R} \times \{0\}$, let $A(c)$ denote the half-ray $\{p + \alpha(c - p) \mid \alpha > 0\}$, and for all $c \in C$, let $L(c)$ denote the set $\{(x, y) \in A(c) \mid 0 \leq y \in \mathbb{Q}\}$ if $c \in D$, and $\{(x, y) \in A(c) \mid 0 \leq y \notin \mathbb{Q}\}$

if $c \in F$. We define the set Y^* as the union of $L(c)$ over $c \in C$. The Knaster-Kuratowski space is $\mathbb{K} := Y^* \cup \{p\}$.

4.2.4 Fact: The set C is dense-in-itself.

Reference: This is usually phrased as "the Cantor set is perfect". See [61], immediately after Definition 30.1. □

4.2.5 Fact: The set D is dense in C.

Proof: Let $c = (x, 0) \in C$. The real x has a ternary expansion not using the integer 1. If the expansion is finite, then $c \in D$. If the expansion is infinite, the successively longer truncations of this expansion give a sequence $\{x_i\}_{i\in\mathbb{N}}$ of reals converging to x, so that the points $(c_i, 0) \in D$ converge to c. Thus $\mathsf{Cl}(D) = C$. □

4.2.6 Fact: Let X be a T_1 space. If X is dense-in-itself and D is dense in P, then D is dense-in-itself.

Proof: Pick $x \in D$. If $x \notin \mathsf{Cl}_D(D \setminus \{x\})$, then $x \notin \mathsf{Cl}_X(D \setminus \{x\})$. Since X is T_1, we have that $\mathsf{Cl}(x) = \{x\}$. But $X = \mathsf{Cl}(D) = \mathsf{Cl}(D \setminus \{x\}) \cup \mathsf{Cl}(x)$, so $X \setminus \{x\} = \mathsf{Cl}(D \setminus \{x\})$ is closed. This contradicts the fact that x is an accumulation point for X. ✠

4.2.7 Fact: The space $Y^* = \mathbb{K} \setminus \{p\}$ is totally disconnected. For every $c \in C$, the quasicomponent of c in Y^* is $L(c)$.

Reference: See [55], Example 129, part 3. □

Now we consider topologized graphs with vertex set $V := Y^* \cup \{q\}$, where q is a point not belonging to Y^*. Let G be any such graph in which the only edges are incident with q and some point in Y^*, and let $N(G)$ be the set of vertices in Y^* adjacent with q. Moreover, for $y \in Y^*$, we denote by $Z(G)$ the set of points $c \in C$ such that $N(G) \cap L(c) \neq \emptyset$. We consider V to be the topological adjunction of q to Y^*, that is, a subset of V is open if and only if its intersection with Y^* is open in Y^*. Being a subspace of $\mathbb{R}^2$, Y^* is a T_3 space; hence so is V. In particular, V is T_1. Since every finite subset of a T_1 space is closed, the set of (two) endvertices of every edge is closed. By Proposition 3.3.1 we may equip G with the topology combinatorially induced by V. Note that we are not yet claiming that G is feebly regular.

4.2.8 *Proposition:* *We have that G is connected if and only if $Z(G)$ is dense in C.*

Proof: Suppose first that $Z(G)$ is dense in C and, by way of contradiction, that there exists a separation $\{P, Q\}$ of G. Then $\{P \cap Y^*, Q \cap Y^*\}$ is a separation of Y^*. Without loss of generality, we may assume $q \in Q$. Note that $\mathcal{N}_q \cup N(G)$ is connected. Hence $\mathcal{N}_q \cup N(G) \subseteq Q$. However, for every $y \in N(G)$, there exists a unique $c \in C$ such that $y \in L(c)$, and since $L(c)$ is a quasicomponent of Y^*, we have that $L(c) \subseteq (Q \cap Y^*) \subseteq Q$. Hence $Z(G) \subseteq Q$. Since Q is closed, we have that $C = \mathsf{Cl}_C(Z(G)) \subseteq \mathsf{Cl}_G(Z(G)) \subseteq Q$. Again, $c \in C \cap Q \Rightarrow L(c) \subseteq Q$ and we conclude that $Y^* \subseteq Q$ and therefore $Q = G$, contradicting the fact that $P \neq \emptyset$.

Conversely, suppose that $Z(G)$ is not dense in C. Then there exists $c = (x, 0) \in C$ and an open interval $(a, b) \subseteq \mathbb{R}$ such that $a < x < b$ and $\big((a, b) \times \{0\}\big) \cap Z(G) = \emptyset$.

Now either $x = 0$ or else there exist reals a_1, a_2, a_3 such that $a < a_1 < a_2 < a_3 < x$ and a_1 and a_3 are consecutive points in D, that is, they have finite ternary expansions of the same length and differing precisely in the last "digit", which is 1 for a_1 and 2 for a_3. In particular $(a_1, a_3) \times \{0\}$ is disjoint from C. If $x = 0$, we put $a' = -1$, otherwise $a' = a_2$. Similarly, either $x = 1$ or else there exist reals b_1, b_2, b_3 such that $x < b_1 < b_2 < b_3 < b$ and $(b_1, b_3) \times \{0\}$ is disjoint from C. In the former case we put $b' = 2$, in the latter case $b' = b_2$.

Note that $x \in (a', b')$ and at least one of 0,1 does not belong to (a', b'). Now we consider the open cone S in $\mathbb{R}^2$ given by $\displaystyle\bigcup_{z \in (a', b')} A(z, 0)$. Clearly S is open in $\mathbb{R}^2$; moreover, $\mathsf{Cl}(S) = \{p\} \cup \displaystyle\bigcup_{z \in [a', b']} A(z)$. Hence $S' := S \cap Y^*$ is clopen in Y^*. Since Y^* is closed in G, we have that S' is closed in G.

Since Y^* is open in V, we have that S' is open in V. Moreover, since $(a', b') \subseteq (a, b)$ and $(a, b) \times \{0\}$ is disjoint from $Z(G)$, we have that $S' = (S')^{\square}$, that is, no vertex in S' is incident with an edge. By definition of the combinatorially induced topology, S is open in G. Hence S is a clopen proper non-empty subset of G, implying that G is not connected. ✠

4.2.9 Example (The Knaster-Kuratowski graph): Let $\{c_i\}_{i \in \mathbb{N}}$ be an enumeration of D, and for $i = 1, 2$, let v_i be the unique point $(x, y) \in L(c_i)$ such that $y = 1 - 2^{-i}$. Then we choose G subject to the above restrictions so that $N(G) = \{v_i\}_{i \in \mathbb{N}}$, that is, G is a simple topologized graph with vertex set $V = Y^* \cup \{q\}$, edge set $\{e_i\}_{i \in \mathbb{N}}$, incidence function $f(e_i) = \{v_i, q\}$, and global topology combinatorially induced from V according to f.

4.2.10 *Proposition:* *The Knaster-Kuratowski graph G is a connected feebly regular topologized graph without a minimal spanning set.*

Proof: Note that the Knaster-Kuratowski graph satisfies $Z(G) = D$. By Fact 4.2.5 D is dense in C, by Proposition 4.2.8 G is connected. Suppose G has a minimal spanning set T. Since T is connected, we must have that $Z(T)$ is dense in C, and therefore (by Facts 4.2.4 and 4.2.6) dense-in-itself.

Note that for all $i : e_i \in E_T$, we have $Z(T \setminus \{e_i\}) = Z(T) \setminus \{c_i\}$. Since $Z(T)$ is dense-in-itself, $Z(T) \setminus \{c_i\}$ is dense in $Z(T)$, and hence in C. Thus, again by Proposition 4.2.8, $T \setminus \{e_i\}$ is still connected, contradicting Proposition 4.2.2. We conclude that G has no minimal spanning set.

Finally, we observe that, *in* $\mathbb{R}^2$, the only accumulation point of $N(G)$ in p. Thus $N(G)$ is closed in V and discrete with respect to the relative topology inherited from V. This implies that the edge-convergence condition of quasiregularity (Definition 3.4.2) is trivially satisfied. Since V is T_3, from Corollary 3.4.14 we have that G is feebly regular. ✠

The space Y^* is often provided as an example of a totally-disconnected T_1 space which is not zero-dimensional. This prompts the following question.

4.2.11 Question: Does a connected, V-zero-dimensional, feebly regular topologized graph necessarily have a minimal spanning set?

A spanning tree of a classical graph can be characterized equally well as a *minimal* connected set containing the vertices or a *maximal* acyclic subgraph. For classical (possibly infinite, therefore not compact) graphs, the proof of the existence of a spanning tree becomes easy when one considers maximal cycle-free subgraphs, as opposed to minimal connected spanning sets. It appears that a “maximal” approach (which we shall not pursue) would be more amenable to (possibly non-compact) V-zero-dimensional spaces, while the “minimal” approach is more amenable to a compact setting which does not restrict the dimension of the vertex set.

We shall not address Question 4.2.11, or the wider issue of zero-dimensionality, in this thesis; instead, we go down the road of compactness and “minimality”. Note that the Knaster-Kuratowski graph is neither compact nor V-zero-dimensional, and that neither of these two assumptions implies the other.

It may very well be that the two approaches just mentioned can lead to interesting dual results for cycle spaces. In [61], Willard states that “Compactness and discreteness are, in a sense, dual properties [...]”, and zero-dimensionality appears to be a natural generalization of discreteness. As the example of the Knaster-Kuratowski space shows, zero-dimensionality is a more subtle concept than compactness. The potential of this property emerged slowly during the course of our progress. It is likely that the cycles and paths of Chapter 2 would find their proper place in this alternative setting.

If this turns out to be true, it would be a good illustration of a feature that we feel makes the work of a combinatorialist hard and at the same time prone to criticism. Finite classical graphs are compact and V-zero-dimensional, but in this context these two sides of the same coin (assuming our speculation about zero-dimensional spaces has some foundation) are very hard to identify and sift apart, especially since finite classical graphs also happen to satisfy several other important topological properties, such as local connectedness, discreteness, various forms of compactness, etc., and almost certainly a host of properties which we cannot see here because they are not topological. Local connectedness, for example, will play no rôle in our results on cycle spaces (Chapter 5). In this perspective, it appears that it is not true that there is "not enough structure", as is often claimed, but that there is *too much* structure, where the excess is only a problem in that it makes it harder to discern the relevant properties, and hence to arrive at an "axiomatic" theory.

Indeed, in our view this should not come as a surprise, for the following reason. Non-combinatorial mathematical objects, such as the "real" numbers, are often *conceived* to have a desirable property, and their connection, if any, to the "real world", which exists independently of our "desires", lies in their applicability to merely *model* problems arising in the sciences. On the other hand, combinatorial structures are so "concrete" that it is not clear at first sight which structure they should be equipped with in order to address a given issue, or even which issues are the interesting ones.

4.2.2 Weakly Hausdorff Spaces

4.2.12 Definition: Recall that a continuum is usually defined to be a compact connected Hausdorff space (see, for example, [61, Definition 28.1]); some authors omit the Hausdorff requirement, for example, [59, I §10, pg. 15], others require that the space be metric, for example [45, pg. 3]). By analogy, we define a weak (feeble) continuum to be a compact connected weakly (feebly) Hausdorff space.

4.2.13 *Theorem:* *Let X be a weak continuum and M a subset of X. Then there exists a subset of X which is (inclusion-wise) minimal with respect to the property of being closed, connected and containing M.*

Proof: Consider the set $\mathcal{C}$ of all closed connected subsets of X containing M, partially ordered by inclusion. Then the assertion is equivalent to the statement that $\mathcal{C}$ has a minimal element. Using Zorn's lemma, it is sufficient to prove that for any totally ordered subset $\mathcal{C}'$ of $\mathcal{C}$ there exists an element $\check{C} \in \mathcal{C}$ such that $\check{C} \subseteq C \ \forall C \in \mathcal{C}'$.

We claim that such a $\check{C}$ can be chosen by setting $\check{C} := \bigcap_{C \in \mathcal{C}'} C$. Clearly $\check{C} \subseteq C \ \forall C \in \mathcal{C}'$, and $\check{C}$ is closed and contains M. In order to establish $\check{C} \in \mathcal{C}$, it remains to be proved that $\check{C}$ is connected. Suppose not. Then there exists a separation (K_1, K_2) of $\check{C}$. Since K_1, K_2 are closed in $\check{C}$ and $\check{C}$ is closed, K_1 and K_2 are closed in X, and since X is compact, they are also compact. Since X is weakly Hausdorff, by 4.1.4 there exist open sets M_1, M_2 with finite intersection and containing K_1, K_2 respectively. Since K_1, K_2 are closed we may assume that M_i is disjoint from M_{3-i} $(i = 1, 2)$ and therefore $M_1 \cap M_2$ is disjoint from $\check{C}$.

For any $m \in M_1 \cap M_2$, there exists a $C_m \in \mathcal{C}'$ such that $m \notin C_m$. Since $M_1 \cap M_2$ is finite and $\mathcal{C}'$ is totally ordered with respect to inclusion, there exists an $\overline{m} \in M_1 \cap M_2$ such that $C_{\overline{m}}$ is contained in all of the C_m's as m ranges in $M_1 \cap M_2$. Hence $M_1 \cap M_2$ is disjoint from $C_{\overline{m}}$.

Since $M_1 \cup M_2$ covers all of $\check{C}$, for any $x \in C_{\overline{m}} \setminus (M_1 \cup M_2)$, we have $x \notin \check{C}$ and therefore there exists some $C_x \in \mathcal{C}'$ such that $x \notin C_x$. Since C_x is closed, we can find an open neighbourhood U_x of x disjoint from C_x. But the set $C_{\bar{m}} \setminus (M_1 \cup M_2) = C_{\bar{m}} \cap (X \setminus (M_1 \cup M_2))$ is closed and therefore compact, so the open cover $\{U_x\}_{x \in C_{\bar{m}} \setminus (M_1 \cup M_2)}$ has a subcover $\{U_x\}_{x \in J}$ for some finite $J \subseteq C_{\bar{m}} \setminus (M_1 \cup M_2)$. For all $x \in J$, U_x is disjoint from C for $C \in \mathcal{C}'$ sufficiently "small". Since J is finite, there must be some $\hat{C} \in \mathcal{C}'$ which is disjoint from all of $C_{\overline{m}} \setminus (M_1 \cup M_2)$ (and contained in $C_{\overline{m}}$) , i.e. $\hat{C} \subseteq M_1 \cup M_2$. But $M_1 \cap M_2 \cap \hat{C} \subseteq M_1 \cap M_2 \cap C_{\overline{m}} = \emptyset$, and M_k is non-empty because it contains K_k, which is non-empty. Therefore $(M_1 \cap \hat{C}, M_2 \cap \hat{C})$ gives a separation of $\hat{C}$, contradicting the fact that $\hat{C}$ is connected. ✠

It is a well-known fact from the theory of (Hausdorff) continua that the intersection of a nest of continua is a continuum. The proof of Theorem 4.2.13 could be reasonably broken down into two proofs: some form of an analogous lemma for weak (as opposed to Hausdorff) continua and an application of this lemma in conjunction with Zorn's Lemma. However, this approach is slightly less natural for us than it would be in continuum theory, for the following reasons.

Firstly, it is an important point in continuum theory that the intersection of a nest of continua is non-empty; this is sometimes hidden in the fact that the definition of continuum requires it to be non-empty (see [45]). In our context we do not need to show that the intersection is non-empty. Secondly, in (Hausdorff) continua a subspace is compact if and only if it is a closed subset; in a non-Hausdorff context, a compact space need not be closed, but the argument uses the fact that the intersection is closed, and is contained in a compact space. Thus the "analogous lemma" would be "in a weak continuum, the intersection of a nest of closed connected subsets is connected", and the emphasis shifts from an "intrinsic" property (compactness) to a "relative" property (being closed).

Note that, in a hyperedge space, a subset containing all the vertices is necessarily closed. Thus, we obtain the following corollary.

4.2.14 ***Corollary:*** *A compact connected weakly Hausdorff hyperedge space has a minimal spanning set.* ✠

The proof of Theorem 4.2.13, with the last paragraph essentially omitted, effectively shows the truth of following assertion, in which the assumption of compactness is absent.

4.2.15 ***Proposition:*** *A connected weakly normal hyperedge space has a minimal spanning set.* ✠

Note that, in the above results, we are allowing hyperedges with an arbitrary number of points in their boundary (that is, we are not restricting to edges). In the transition from weakly Hausdorff spaces to feebly Hausdorff, this point leads to an issue with a combinatorial aspect.

4.2.16 Example: Let $V' = \{0\} \cup \{2^{-n}\}_{n\in\mathbb{N}}$ and $V = V' \cup \{v\}$ where the arbitrary point $v \notin V'$. Consider the hypergraph $H = (V, E, f)$ where $E = \{e_n\}_{n\in\mathbb{N}} \cup \{g_n\}_{n\in\mathbb{N}}$ and $f(e_n) = \{2^{-n}, 2^{-(n+1)}\}, f(g_n) = \{0, v, 2^{-n}\}$. Now if V' is equipped with the relative Euclidean topology inherited from $\mathbb{R}$, the set V can be seen as the topological space obtained as the adjunction of $\{v\}$ to V', and the combinatorial extension of this gives a topology on $V \cup E$. It is easy to see that the resulting topologized hypergraph is compact, feebly Hausdorff (therefore feebly normal), simple and V-totally-disconnected, but not weakly Hausdorff. For $m \in \mathbb{N}$, the complements of the sets $\{g_n \mid n \leq m\}$ give a totally-ordered (with respect to inclusion) collection of closed, connected sets containing the closed set V, but their intersection is not connected. Although this example does have a minimal spanning set, it illustrates how the argument in the proof of 4.2.13 fails.

4.2.17 Question: Given a compact feebly Hausdorff *hyper*edge space, does there always exist a minimal spanning set?

Example 4.2.16 shows that the "minimal" approach mentioned at the end of Subsection 4.2.1 would need to become more sophisticated to deal with the Question 4.2.17. In the next subsection we show the existence of minimal spanning sets in compact feebly Hausdorff topologized graphs.

4.2.3 Existence in Feebly Regular Topologized Graphs

This section is motivated by the desire to extend the applicability of the theorems in the previous section to feebly, as opposed to weakly, separated spaces. This is achieved by reducing to a smaller weakly Hausdorff spanning subspace—essentially, we discard extra hyperedges, keeping only one copy of hyperedges with the same endvertices. Although the idea is simple, feeble regularity is essential. It turns out that the related concepts are interesting from a purely topological point of view, so we do not aim for an approach that would be the most efficient for the sole purpose of the reduction.

Systems of Distinct Representatives as Quotients and Retracts

Recall that a continuous surjection q onto a subspace A of X is called a retraction of X onto A if and only if its restriction to A is the identity. When such a retraction exists, A is a retract of X. Since topological spaces in the literature tend to be T_1, it would not be very surprising to learn that the following fact about generic topological quotients has gone unnoticed; in fact, in some books the parts of a decomposition are assumed to be closed (e.g. [38, §19, I]). Given a decomposition $\mathcal{P}$ of a topological space X, we shall denote the induced quotient space by $X/\mathcal{P}$.

4.2.18 ***Theorem:*** *Let $\mathcal{P}$ be an upper-semicontinuous decomposition of a topological space X with the property that all non-degenerate parts are open. Suppose $S \subseteq X$ is a system of distinct representatives for $\mathcal{P}$. Then the quotient $X/\mathcal{P}$ embeds onto S. Thus, S is a retract of X.*

Proof: Let $p : X \to X/\mathcal{P}$ be the quotient map $x \mapsto [x]$, and f the restriction of p to S. We shall show that f is a homeomorphism. Since p is continuous, from this it will follow that the function $x \mapsto f^{-1}(p(x))$, which is the identity on S and maps X onto S, is continuous and therefore a retraction.

For any subset A in $X/\mathcal{P}$, $f^{-1}(A) = p^{-1}(A) \cap S$, so continuity of f follows from continuity of p. Clearly f is bijective. It remains to be shown that f is open. So suppose B is an arbitrary open subset of S, i.e. it is of the form $U \cap S$ for some open subset U of X, and let $\hat{B}$ be the smallest saturated set containing B, that is, the set $\bigcup_{b \in B} [b]$. Then the set $W = U \cup \hat{B}$ is open, because for $w \in U$, we have that U itself is a neighbourhood of w contained in W, and for any $w \in W \setminus U$, the part $[w]$ is an open neighbourhood of w contained in W. Moreover, $W \cap S = B \subseteq \hat{B} \subseteq W$.

As $\mathcal{P}$ is upper semicontinuous, for any part $C \subseteq \hat{B}$, we can find a saturated set W_C such that $C \subseteq W_C \subseteq W$. Then the open set $W' := \bigcup_{C \subseteq \hat{B}} W_C$ is also saturated and satisfies $\hat{B} \subseteq W' \subseteq W$, whence $W' \cap S = B$.

Hence we may write $p^{-1}(f(B)) = p^{-1}(p(B)) = p^{-1}(p(W' \cap S))$. But since W' is saturated, $p(W') = p(W' \cap S)$ and $p^{-1}(f(B)) = p^{-1}(p(W')) = W'$, and therefore $p^{-1}(f(B))$ is open. Since p is a quotient map, this implies that $f(B)$ is open in $X/\mathcal{P}$ and concludes the proof that f is open. ✠

We give a very simple example to illustrate how the situation of Theorem 4.2.18 differs from the usual scenario of quotients of Hausdorff spaces.

4.2.19 Example: Consider the "unit square", that is, the subspace S of $\mathbb{R}^2$ given by $S := [0,1] \times [0,1]$, and the decomposition $\{L_x\}_{x \in [0,1]}$ of S, where $L_x = [0,1] \times \{x\}$. Clearly all the parts in $\mathcal{D}$ are compact. Using this property it is easy to see that $\mathcal{D}$ is upper semicontinuous.[2] Of course, the parts are not open; in fact, the quotient is isomorphic to $[0,1]$, but clearly some systems of distinct representatives in S are not isomorphic to $[0,1]$.

Parallel decompositions and reductions

Recall that, given a subset of A of a topological space X, the derived set of A is the set of accumulation points (as defined in [61], Definition 4.9). For the purposes of this section, given a point x, we denote by x' the derived set of $\{x\}$.

4.2.20 Definition: A topological space is reduced if for any two points $x, y \in X$, $x' = y' \neq \emptyset \Rightarrow x = y$. Given a topological space X, the parallel decomposition of X is the partition of X whose parts are the equivalence classes of the binary relation defined by $x \sim y \Leftrightarrow x' = y'$. The parallel reduction of X, denoted by $\hat{X}$, is the quotient space induced by the parallel decomposition.

Note that T_1 spaces are reduced. With respect to the parallel decomposition of a topologized hypergraph, distinct points are equivalent if and only if they are hyperedges with precisely the same endvertices, while each vertex constitutes an entire equivalence class.

4.2.21 *Proposition:* *The parallel decomposition of a feebly regular topologized hypergraph is upper semicontinuous.*

Proof: Let X be a feebly regular topologized hypergraph, P be any equivalence class of the parallel decomposition and U any open set containing P. We need to show that there exists a saturated open set W with $P \subseteq W \subseteq U$. This is trivial if P consists of hyperedges. Suppose then that $P = \{v\}$ for some vertex v. Since X is feebly regular by (3.4.3) $U = (V_X \cap U)^{\square} \uplus F$ for some $F \subseteq E_X$ and again by (3.4.3) $W := (V_X \cap U)^{\square}$ is also open. Also $P \subseteq v^{\square} \subseteq W$, so it is sufficient to prove that W is saturated, that is, for all $z \in W$, the equivalence class $[z]$ of z is all contained in W. This is trivial is z is a vertex. If instead z is a hyperedge, then $z \in u^{\square}$ for some vertex $u \in U$, and if w is any other hyperedge equivalent to z, then $w \in u^{\square} \subseteq W$. ✠

Proposition 4.2.21 is, in a sense, a dual version of Proposition 3.6.4. Both are refinements of the bipartition into edges and vertices, but the latter takes the vertices in "clumps" and refines the edge set to singletons, whereas the former refines the vertex set to singletons and takes the edges in equivalence classes.

[2] Note we are not claiming that a decomposition of an arbitrary space into compact sets is upper semicontinuous.

4.2.22 Fact: The continuous image of a connected set is connected. The continuous image of a compact space is compact.

Reference: See, for example, [61], Theorems 17.7. and 26.3. □

4.2.23 ***Proposition:*** *Let $p : X \to \widehat{X}$ be the quotient map of a feebly regular topologized hypergraph X onto its parallel reduction $\widehat{X}$. Then p is closed and, for any subset $C \subseteq \widehat{X}$, we have that C is closed and connected if and only if $p^{-1}(C)$ is closed and connected. Moreover, if $S \subseteq X$ is any system of distinct representatives for the parallel decomposition of X, then S is a homeomorph of $\widehat{X}$ as a spanning feebly regular topologized subhypergraph*[3] *of X.*

Proof: There exists a bijection $\phi : \widehat{X} \to S$ such that, $\forall x \in \widehat{X}$, $\phi(x) \in x$. From Proposition 4.2.21 we have that the parallel decomposition is upper semicontinuous, and by Fact 3.5.2 p is closed. Now the non-degenerate parts consist of edges, and are therefore open. By Theorem 4.2.18 we have that ϕ is a homeomorphism, that is, an embedding of $\widehat{X}$ onto S. In particular, since feeble regularity is inherited by subspaces (as we observed in Proposition 3.1.18), S, and therefore $\hat{X}$, is feebly regular. Since all vertex-singletons are degenerate parts, we have $V_X \subseteq S$. Hence S is closed in X, and is a spanning topologized subhypergraph of X.

Now let C be an arbitrary closed subset of $\widehat{X}$. Since p is continuous, and the image of a connected set under a continuous function is connected (Fact 4.2.22), and since $p(p^{-1}(C)) = C$, if $p^{-1}(C)$ is connected, so is C. Conversely, suppose that C is connected and, by way of contradiction, that $p^{-1}(C)$ has a separation $\{A, B\}$. Since C is closed and p is continuous, $p^{-1}(C)$ is also closed, and therefore A, B are both closed in X. Since p is closed, $p(A)$ and $p(B)$, which together cover C, are closed, and therefore they cannot be disjoint. Hence there must be two points $e_1 \in A, e_2 \in B$ which are mapped to the same point by p, that is, they are equivalent with respect to the parallel decomposition. But then e_1, e_2 are hyperedges with the same set of endvertices. But we have $e_1' \subseteq \bar{e}_1 \subseteq A$ and $e_2' \subseteq \bar{e}_2 \subseteq B$, contradicting the fact that A and B are disjoint. ✠

Minimal Connected Sets

4.2.24 ***Proposition:*** *Let X be a compact connected feebly Hausdorff topologized graph and M an arbitrary subset. Then there exists a subset of X which is (inclusion-wise) minimal with respect to the property of being closed, connected and containing M.*

[3] This means that the edges of S have the same endvertices as in X, as defined on page 21.

Proof: Let S be a system of distinct representatives of the parallel decomposition of X such that for all $m \in M$, $M \cap [m] \cap S \neq \emptyset$, that is, the representative of any part represented in M is taken from M. Note that here we are using the axiom of choice. Since the parallel decomposition is upper-semicontinuous, and since the non-trivial parts consist solely of edges and are therefore open, by 4.2.18 S is topologized subgraph homeomorphic to the parallel reduction of X. Let q be the corresponding retraction of X onto S.

Since X is a feebly regular, so is S. We claim that S is almost Hausdorff. By Proposition 3.1.19 it is sufficient to show that S is uniquely adjacent, that is, for any two points u, v, $|u^\diamond \cap v^\diamond| \leq 1$. This is trivial if at least one of u, v is an edge. If u, v are both vertices and $x, y \in u^\diamond \cap v^\diamond$, then x, y are edges incident with u, v, in S and in X. Since no hyperedges in X are incident with more than two vertices, we have $x' = \{u, v\} = y'$. Since S is a system of distinct representatives, $x = y$. So S is almost Hausdorff, in particular weakly Hausdorff.[4]

Since S is the image of the compact connected set X under the continuous function q, S is compact and connected. Hence, by 4.2.13 there exists a subset M' of S which is (inclusion-wise) minimal with respect to the property of being closed, connected and containing $M \cap S$. We claim that $\widehat{M} = M \cup M'$ is the required minimal closed connected set in X.

Since vertices are only equivalent to themselves (with respect to the parallel decomposition), $V_X \subseteq S$ and therefore S is closed. Hence M' and $\widehat{M}$ are closed in X. Moreover, for every $m \in M$, the closure C_m (in M, $\widehat{M}$ or X) of $[m] \cap M$ is connected and contains some vertex, which is also in the connected set M'. Hence $\widehat{M} = M' \cup \bigcup_{m \in M} C_m$ is connected, as well as closed. Clearly it also contains M.

Now suppose C is a closed connected set such that $M \subseteq C \subsetneq \widehat{M}$. Note that, for any part P of the parallel decomposition, $(P \cap M) \subseteq (P \cap \widehat{M})$ and if $|P \cap \widehat{M}| \geq 2$, then $P \cap M = P \cap \widehat{M}$. Hence there exists some $x \in M'$ with $[x]$ disjoint from C, so that $q(C) \subsetneq q(\widehat{M}) = q(M') = M'$. But using again the fact that q is continuous and closed, $q(C)$ is closed and connected, and since $M \subseteq C$, we have $M \cap S = q(M) \subseteq q(C) \subsetneq M'$, contradicting the choice of M'. ✠

Proposition 4.2.24 is a variant of Theorem 4.2.13. In the former, we assume feebly, as opposed to weakly, Hausdorff, but we restrict to topologized graphs. We shall strengthen this to feebly Hausdorff edge spaces (Theorem 4.3.21).

[4] The topological argument dealing with the case of u, v both being vertices can be translated into a combinatorial one, because $u^\diamond = u^\Box$, and $x' = X[x]$ (where x is an edge). However, the former is only true because S is quasiregular. The combinatorial argument would be essentially identical.

4.2.25 ***Corollary:*** *A compact, connected feebly Hausdorff topologized graph has a minimal spanning set.* ✠

4.3 Compactness and Quotients

In this section, we resume the discussion of edge-clump quotients from Subsection 3.5. Not surprisingly, the assumption of compactness enables us to conclude much more.

4.3.1 ***Lemma:*** *Let S be a union of components of a topological space X. Then $S^\diamond = S$.*

Proof: Let $\mathcal{K}$ be the collection of the components contained in S. Using 3.1.12 we have that

$$S^\diamond = \bigcup_{s\in S} s^\diamond = \bigcup_{K\in\mathcal{K}} \bigcup_{s\in K} s^\diamond = \bigcup_{K\in\mathcal{K}} K^\diamond,$$

so if $x \in S^\diamond \setminus S$ we have that $x \in s^\diamond \setminus K$ for some $K \in \mathcal{K}$ and some $s \in K$. But then $\{s, x\}$ is a connected subset of S intersecting, but not contained in, the component K, a contradiction. ✠

Terminology: We say that two sets C, D in a topological space X can be separated if there exists a separation $\{A, B\}$ of X such that $C \subseteq A$, $D \subseteq B$. As usual, we do not necessarily distinguish between a point and the corresponding singleton.

4.3.2 ***Lemma:*** *Let A, B be compact subsets of a topological space X such that, for all $a \in A, b \in B$, the two points a and b can be separated. Then A and B can be separated.*

Proof: For every pair of points $a \in A, b \in B$, let M_{ab} be a clopen set containing b and not a. For a fixed $a \in A$, $\{M_{ab} \cap B\}_{b\in B}$ is an open cover in the relative topology of the compact set B. Hence there exists a finite subset $I_a \subseteq B$ such that $B \subseteq \left(\bigcup_{b\in I_a} M_{ab} \cap B\right) \subseteq \bigcup_{b\in I_a} M_{ab} =: B_a$. Note that B_a is clopen and contains B but not a.

Now if A_a denotes the complement of B_a, we have that $\{A_a\}_{a\in A}$ is a cover of the compact subset A with clopen sets disjoint from B, and as above we obtain a finite clopen subcover and therefore a clopen set containing all of A and still disjoint from B. ✠

4.3.3 ***Theorem:*** *If A, B are distinct components of a compact feebly Hausdorff topological space X, then there exists a separation $\{U, V\}$ of X with $A \subseteq U, B \subseteq V$.*

Proof: It is sufficient to show that the quasicomponents of X coincide with the components. Note that quasicomponents are closed subsets, and disjoint unions of components. Suppose, by way of contradiction, that there exists a quasicomponent Q which is not connected, that is, there exist non-empty disjoint clopen subsets P, R of Q whose union is Q. Since Q is closed, P, R are closed in X, and since X is feebly Hausdorff and compact, and therefore feebly normal, there exist open sets P', R' containing P, R respectively such that $P' \cap R' \subseteq P^\diamond \cap R^\diamond$. Since no component of Q can intersect both P and R, and Q is the union of components of X, P and R are the union of components of X, and by 4.3.1 they coincide with their surrounding set. Since they are also disjoint, we conclude that so are P' and R'.

In particular, R is disjoint from $\mathsf{Cl}(P')$. Together with the fact that P is disjoint from $\mathsf{Cl}(X \setminus P') = X \setminus P'$, this implies that Q is disjoint from $\partial(P')$. Thus, by definition of a quasicomponent, any point in Q can be separated from any point in $\partial(P')$. Since Q and $\partial(P')$ are both closed subsets of the compact space X, they are compact, and since X is also feebly Hausdorff from 4.3.2 we have that they can be separated, that is, there exists a clopen set F containing Q and disjoint from $\partial(P')$.

Now consider the set $K := F \cap P'$. Since both Q and P' contain P, while the latter is disjoint from R, we have that K contains P but is disjoint from R. Also, K is clearly open. Moreover, K is closed because

$$\mathsf{Cl}(F \cap P') \subseteq \mathsf{Cl}(F) \cap \mathsf{Cl}(P') = F \cap \mathsf{Cl}(P') = (F \cap \partial(P')) \cup (F \cap P') = F \cap P'.$$

Hence K is a clopen set separating P and R, contradicting the fact that P, R are contained in the same quasicomponent. ✠

4.3.4 Note: The proofs of 4.3.2, 4.3.3 are based on those of Theorems 1,2 respectively in [39], §47, II. Although in Kuratowski's textbook these theorems are phrased in the context of Hausdorff spaces, the proof of Theorem 1 does not use this property and the one we give here is essentially the same. On the other hand, as the following example shows, Theorem 4.3.3 fails without an appropriate separation axiom. Here we relax Hausdorff to feebly Hausdorff, and use 4.3.1 to supplant the proof of Theorem 2.

4.3.5 Example: Consider the topological space with ground set $\mathbb{N} \cup \{a, b\}$, a set being open if and only if it is disjoint from $\{a, b\}$ or else has a finite complement. This is compact and totally-disconnected (and therefore T_1), but not (feebly) Hausdorff and not totally-separated—Theorem 4.3.3 fails with $U = \{a\}, V = \{b\}$. Note that this space is the vertex set of the one-way infinite path with a double end (Example 3.1.7), and, as we have already observed, may be obtained as the quotient of the V-set in Example 3.6.3 induced by the decomposition into components.

4.3.6 *Corollary:* *Let A_1, A_2 be disjoint closed subsets which are unions of connected components of a compact feebly Hausdorff space X. Then A, B can be separated in X.*

Proof: For $i = 1, 2$, let $\{C_j\}_{j \in J_i}$ be the set of components of A_i. By 4.3.3, for all $j \in J_1, k \in J_2$, there exists a clopen set $M_{j,k}$ containing C_j and disjoint from C_k. Using the same argument as in 4.3.2, we obtain a clopen subset containing A and disjoint from B. ✠

Notation: For the purposes of the remainder of this section, given a V-set Z and a subset $U \subseteq Z$, we denote by $U^{\boxdot}$ the union of U with the induced *pre*-hyperedges having at least one boundary point in U.

4.3.7 *Lemma:* *Suppose Z is a V-set in a topological space X such that the induced pre-hyperedge selection satisfies the weak pre-hyperedge convergence property. If U is clopen in Z, then $U^{\boxdot}$ is open in X.*

Proof: Let C be the complement of U in Z and $\mathcal{F} = \{F_j\}_{j \in J}$ the set of pre-edges with no boundary points in U. If $U^{\boxdot}$ is not open in X, then its complement, which can be expressed as $C \cup \bigcup_{j \in J} F_j$, is not closed. However, C is closed in the closed subset Z, and therefore in X. So there exists a a point z belonging to $U^{\boxdot}$ *and* to the closure of the union of all F over $F \in \mathcal{F}$. Since the (open) pre-hyperedges contained in $U^{\boxdot}$ are disjoint from the pre-hyperedges in $\mathcal{F}$, $z \in U$.

Since pre-hyperedges are not closed, for all $j \in J$, there exists $y_j \in \partial(F_j) \subseteq C$. Now both U and C are clopen in Z, and therefore the disjoint union of components of Z. Since $x \in U$, we must have $\mathbf{K}_Z(x) \subseteq U$ and, for all $c \in C$, in particular for any boundary point of any $F \in \mathcal{F}$, $\mathbf{K}_Z(c) \subseteq C$. Since C and U are disjoint, $\mathbf{K}_Z(x)$ is disjoint from the boundary of, and therefore the closure of, every $F \in \mathcal{F}$. Thus, assumption (A) in the definition of the weak pre-hyperedge convergence property (Definition 3.5.13) is satisfied with the pre-hyperedges F_j for U_j, and the components $\mathbf{K}_Z(y_j)$, $\mathbf{K}_Z(x)$ for K_j and K respectively. Hence $\mathsf{Cl}\left(\bigcup_{j \in J} \mathbf{K}_Z(y_j)\right) \cap \mathbf{K}_Z(x) \neq \emptyset$. But $\forall j \in J$ $\mathbf{K}_Z(y_j)$ is contained in the closed set C, disjoint from U, which contains $\mathbf{K}_Z(x)$, a contradiction. ✠

The above lemma runs a parallel with the fact that in a quasiregular hyperedge space, if U is an open subset of the vertex set, then $U^{\square}$ is open (Corollary 3.4.5). The assumption here is stronger — we are taking U to be *clopen* in the V-set. The following example shows that the additional assumption that U be closed can not be dropped.

4.3.8 Example: For a positive integer n, let H_n be the subset of $\mathbb{R}^2$ given by $\{(x, x^{2n}) \mid x \in (0,1)\}$, and let S be the union of the H_n over all positive integers n together with the two segments $[0,1] \times \{0\}$ and $\{1\} \times [0,1]$. When equipped with the relative Euclidean topology, S is a compact metric space. Now if we choose the pre-edges to be H_n for all positive integers n, the weak pre-edge convergence property is trivially satisfied, since the induced V-set Z is connected. The edge-clump quotient is a classical graph with one vertex and infinitely many loops (in particular it is quasiregular, as it should be, according to Proposition 3.5.15), but the set $U := (0,1) \times \{0\}$ is open in Z, while $U^{\boxdot} = U$ is not open in X.

We recall the following fact from general topology, which summarizes the way in which nets replace sequences for non-metric spaces. The reader is referred to Appendix B for definitions regarding nets.

4.3.9 Fact: Given a subset A of a topological space X, a point $x \in \mathsf{Cl}(A)$ if and only if there is a net in A converging to x. A point y is a cluster point for a net if and only if there is a subnet which converges to y. If a net converges to y, so does every subnet.

Reference: See [61], Theorems 11.5 and 11.7, and Example 11.4(e). □

4.3.10 *Lemma:* *Let $\{C_i\}_{i \in I}$ and $\check{C}$ be connected components of a compact feebly Hausdorff topological space X and suppose that, for all $i \in I$, the points x_i, y_i are in C_i. If $\check{x} \in \mathsf{Cl}(\{x_i\}_{i \in I})$ and $\check{x} \in \check{C}$, then there exists a point $\check{y} \in \check{C} \cap \mathsf{Cl}(\{y_i\}_{i \in I})$.*

Proof: Note that the assertion is clearly true if $\check{C} = C_i$, for some $i \in I$. Let $(x_{i_j})_{j \in J}$ be a net converging to $\check{x}$ (by Fact 4.3.9), with J a directed set such that $i_j \in I$ for all $i \in I$. Since X is compact, by Fact 2.3.16 the net $(y_{i_j})_{j \in J}$ must have a cluster point $\check{y}$ and therefore a subnet $(y_{i_k})_{k \in K}$ converging to $\check{y}$ (again by Fact 4.3.9), where K is a directed set and, $\forall k \in K$, $i_k = i_{\phi(k)}$ for some non-decreasing cofinal function $\phi : K \to J$.

Now let Z be the component of X containing $\check{y}$. If Z and $\check{C}$ are the same component, then the assertion is true. If not, by 4.3.3, there exists a separation $\{A_x, A_y\}$ of X such that $\check{x} \in \check{C} \subseteq A_x$ and $\check{y} \in Z \subseteq A_y$. Since A_x and A_y are open, any net converging to $\check{x}$ must be residually in A_x, and any net converging to $\check{y}$ must be residually in A_y. Note that, for all $i \in I$, since C_i is connected, $y_i \in A_y$ implies that $C_i \subseteq A_y$. Thus for sufficiently large k we have that $C_{i_k} \subseteq A_y$. But $(x_{i_k})_{k \in K}$ is a subnet of $(x_{i_j})_{j \in J}$ and therefore (Fact 4.3.9) converges to $\check{x}$. Thus by the same argument $C_{i_k} \subseteq A_x$ for sufficiently large k. This contradicts the fact that A_x and A_y are disjoint. ✠

4.3.11 Note: The convergence assumption in 4.3.10 occurs in the literature on quotients of metric spaces. It is known that given a decomposition of a metric space into compact subsets, this condition is equivalent to upper semicontinuity; see [12], Exercise 18.A.20. We do not know if this condition has been considered for non-metric spaces.

4.3.12 ***Theorem:*** *Let $\mathcal{E}$ be a pre-hyperedge selection in a compact feebly Hausdorff topological space X. If $\mathcal{E}$ satisfies the weak pre-hyperedge convergence property, the induced clump-hyperedge quotient is a compact, V-zero-dimensional feebly normal topologized hypergraph.*

Proof: From Proposition 3.5.6 we know that the quotient is a V-totally-disconnected topologized hypergraph. Compactness follows from Fact 4.2.22. Since a V-totally-disconnected compact space is zero-dimensional (Fact 3.4.8), and since a compact feebly Hausdorff space is feebly normal (Proposition 4.1.3), it is now sufficient to show that for any two distinct points $u, w \in \widehat{X}$, there exist open sets U_u, U_w with $u \in U_u, w \in U_w$ and $U_u \cap U_w \subseteq u^\diamond$. As remarked in (3.1.15), this is trivial if one of u, w is a hyperedge.

So suppose u, w are both vertices, and let $U = p^{-1}(u), W = p^{-1}(w)$. Since the pre-complement Z is a closed subset of the compact feebly Hausdorff space X, and U, W are distinct components of Z, by Theorem 4.3.3 there exists a separation $\{\widehat{U}, \widehat{W}\}$ of Z with $U \subseteq \widehat{U}, W \subseteq \widehat{W}$. By Lemma 4.3.7 $\widehat{U}^{\boxdot}$ and $\widehat{W}^{\boxdot}$ are open in X. Note that $\widehat{U}^{\boxdot} \cap \widehat{W}^{\boxdot}$ is precisely the union of those pre-hyperedges with a boundary point in each of $\widehat{U}$ and $\widehat{W}$. Let $\mathcal{F}$ be the set of all such pre-hyperedges, except those with a boundary point in U.

Now consider the set A which is the union of the components of $\widehat{U}$ containing a boundary point of some pre-edge in $\mathcal{F}$. We claim that A is closed. If not, by Fact 4.3.9 there exists a net $(x_\lambda)_{\lambda\in\Lambda}$ in A converging to some $x \notin A$. For all $\lambda \in \Lambda$, let C_λ denote the component of A containing x_λ. Since $\widehat{U}$ is closed, it contains x. Let K be the component of $\widehat{U}$ containing x. Since $x \notin A$, K is distinct from C_λ, for all $\lambda \in \Lambda$. Let E_λ be a pre-hyperedge in $\mathcal{F}$ containing a point v_λ in $C_\lambda \cap \partial(E_\lambda)$ (E_λ and v_λ exist by definition of $\mathcal{F}$ and A). Then by Lemma 4.3.10 there exists a point $\hat{x} \in K$ which is contained in $\mathsf{Cl}\left(\{v_\lambda\}_{\lambda\in\Lambda}\right) \subseteq \mathsf{Cl}\left(\bigcup_{F\in\mathcal{F}} F\right)$.

Note that, since A contains *all* the boundary points of E_λ in $\widehat{U}$, for all $\lambda \in \Lambda$ we have that $\mathsf{Cl}(E_\lambda) \cap K = \emptyset$. But then, if w_λ denotes some boundary point of E_λ in $\widehat{W}$ (which again exists by definition of $\mathcal{F}$), assumption (A) in the definition of the weak pre-hyperedge convergence property (Definition 3.5.13) is satisfied by taking E_λ for U_j, W_λ for K_j and K (for K). Thus we have that $K \cap \mathsf{Cl}\left(\bigcup_{\lambda\in\Lambda} W_\lambda\right) \neq \emptyset$. This contradicts the fact that the W_λ's are contained in the closed set $\widehat{W}$, which is disjoint from K. Hence A is closed.

Now the set $A \cup \widehat{W}$ is closed and a union of components of Z. Hence by Corollary 4.3.6 there exists a separation $\{S, T\}$ of Z with $U \subseteq S$ and $A \cup \widehat{W} \subseteq T$. By Corollary 4.3.7, the saturated set $S^{\boxdot}$ is open in X. Note that the only pre-hyperedges with a boundary point in each of $\widehat{U}$ and $\widehat{W}$ contained in $S^{\boxdot}$ must have a boundary point in U, because the ones which do not belong to $\mathcal{F}$. Thus, not only are $S^{\boxdot} \cap \widehat{U}^{\boxdot}$ and $\widehat{W}^{\boxdot}$ open saturated sets containing U and W respectively, but $S^{\boxdot} \cap \widehat{U}^{\boxdot} \cap \widehat{W}^{\boxdot} \subseteq U^{\boxdot}$. Taking the images under p, we have that $p(S^{\boxdot} \cap \widehat{U}^{\boxdot})$ and $p(\widehat{W}^{\boxdot})$ are open sets containing u, w respectively such that $p(S^{\boxdot} \cap \widehat{U}^{\boxdot}) \cap p(\widehat{W}^{\boxdot}) \subseteq p(U^{\boxdot})$.

Now $p(U^{\boxdot})$ consists of u together with all hyperedges $p(H)$ for some pre-hyperedge H with a boundary point in U. Since p is continuous, for every open set Y containing u, $p^{-1}(u)$ is open, and of course contains U, in particular the boundary point(s) of H in U, and therefore some point in H. Hence $p(H) \in p(p^{-1}(Y)) = Y$. We conclude that $p(U^{\boxdot}) \subseteq u^{\diamond}$. ✠

The proof of Theorem 4.3.12 is in a way similar to that of Theorem 3.4.12. The latter uses Corollary 3.4.5 to lift *open* sets in V to open sets in the whole space, while the former uses Corollary 4.3.7 to lift *clopen* sets in the pre-complement to open sets in the whole space. In the case of the latter, the separating open sets are given by the feeble separation axioms; in the case of the former, the separating clopen sets are provided by Corollary 4.3.6, which depends on compactness.

We remark that both these proofs are rendered more delicate by the fact that the number of boundary points of the (pre-)hyperedges is not restricted to two, that is, we are dealing with generic (pre-)hyperedges, not necessarily (pre-)edges. In the proof of Theorem 4.3.12, it is crucial that the set A (T in the case of (3.4.12)) is defined in terms of $\mathcal{F}$ (F in the case of (3.4.12)), not viceversa. That is, A may be smaller than the set of components of $\widehat{U}$ which contain some boundary point of a pre-hyperedge which also has a boundary point in $\widehat{U}$, and there may be hyperedges with boundary points in $\widehat{U}$ and $\widehat{W}$ which are not in $\mathcal{F}$. This is important because it allows one to claim that the closure of each pre-hyperedge in $\mathcal{F}$ is disjoint from K. Of course, this issue does not arise if the pre-(hyper)edges have at most two boundary points, since K is disjoint from A.

This point, together with Example 4.2.16 and the limitations of the technique in the proof of Theorem 4.2.13 with respect to extending the result to cover feebly (as opposed to weakly) Hausdorff spaces, suggests that pursuing further the issue of *hyper*edges may give an additional combinatorial flavour to the issues we are addressing here.

In view of Proposition 3.5.15, the requirement that the pre-hyperedge selection satisfy the weak pre-hyperedge convergence property is necessary in the assumption of Theorem 4.3.12. The Diestel-Kühn obstruction, with arcs for edges, can be taken for an example of a pre-edge selection which does not satisfy the weak pre-hyperedge convergence property. However, this space is not compact (when the edges are modelled as arcs). The question arises as to whether or not this "weak" convergence is actually vacuous under the assumption that the original space is compact and Hausdorff. We now give an example which shows that this is not the case. Hence the weak pre-hyperedge convergence property in the above theorem can not be dropped.

4.3.13 Example: For the topological space, we take the same space as in Example 3.5.12, and we also keep the pre-edges and the notation in this example, but we also add to our pre-edge selection the pre-edges $(2^{-(m+1)}, 2^{-m}) \times \{0\}$ for all non-negative integers m, and $\{0\} \times (2^{-(p+1)}, 2^{-p})\}$ for all non-negative integers p. In this case, the pre-complement is totally disconnected, and the edge-quotient coincides with the clump-edge quotient. Now for any vertex v of the quotient whose first coordinate is zero, any neighbourhood of v must also contain, for sufficiently large n, all the edges which are images of e'_n. Hence for example any neighbourhood of $(0, 1)$ intersects any neighbourhood of $(0, 0)$ in infinitely many of these edges, none of which are incident with either vertex (i.e. they do not belong to either surrounding set). Hence the quotient is not feebly Hausdorff (nor feebly regular or feebly normal).

The topological space in Example 4.3.13 is not locally connected. This leads to the following question:

4.3.14 Question: Does there exist an H-set in a locally connected feeble continuum such that the induced clump-hyperedge quotient is not feebly Hausdorff? Equivalently, does an arbitrary pre-hyperedge selection in a locally connected feeble continuum necessarily satisfy the weak pre-hyperedge convergence property?

Note that Question 4.3.14 is still interesting in the context of standard Hausdorff, locally connected continua (Peano continua). When the pre-hyperedges have finite boundaries, the equivalence of the two formulations follows from our results (Proposition 3.5.15, Theorem 4.3.12 and Corollary 3.4.13). In general, this follows from Theorem 4.3.16 and the fact that Proposition 3.5.7 extends to V-sets with an upper semicontinuous into components and inducing a pre-hyperedge selection satisfying the weak pre-hyperedge convergence property.

Terminology: Given a topological property $\mathcal{P}$ and an adjective "$\mathbb{A}$" which means that a given topological space "satisfies property $\mathcal{P}$", and a topological space space X, the adjective "hereditarily '$\mathbb{A}$'" applied to X means that every connected subspace of X is "$\mathbb{A}$" (satisfies property $\mathcal{P}$).

It may be that there is an easy positive solution to the Question 4.3.14 in the form of an E-set in a space which is locally connected but not *hereditarily* locally connected. If so, the obvious modification to the above question, in the case of compact topologized graphs, becomes equivalent to the following conjecture.

4.3.15 ***Conjecture:*** *Let X be a compact connected V-zero-dimensional topologized graph. Then the following are equivalent:*

- *X is feebly Hausdorff;*
- *X is hereditarily locally connected;*
- *X is hereditarily locally path connected;*
- *X is hereditarily path connected.*

In Subsection 5.1.3 we give an example (Example 5.1.18, Figure 5.2) of a non-compact V-totally-disconnected topological space in which the above conjecture fails.

The following theorem is given in [12] for metric spaces. Our proof will be rather different.

4.3.16 ***Theorem:*** *The decomposition of a compact feebly Hausdorff topological space into its components is upper semicontinuous.*

Proof: Let C be an arbitrary component of X, and U an open set containing C. Since U is open, $\partial(U)$ is disjoint from U. So, for all $x \in \partial(U)$, the component $\mathbf{K}(x)$ is distinct from C. Hence by 4.3.3 there exists a separation $\{A_x, B_x\}$ of X with $C \subseteq A_x$ and $\mathbf{K}(x) \subseteq B_x$.

Now $\{B_x\}_{x\in\partial(U)}$ is an open cover of $\partial(U)$. Since $\partial(U)$ is closed and X is compact, $\partial(U)$ is compact, so there exists a finite subset $I \subseteq \partial(U)$ such that $\{B_i\}_{i\in I}$ is a subcover of $\partial(U)$ (in the relative topology, $\{B_x \cap \partial(U)\}_{x\in\partial(U)}$ is an open cover and $\partial(U) \subseteq \left(\bigcup_{i\in I} B_i \cap \partial(U)\right)$, which set is, in the original space, contained in $\bigcup_{i\in I} B_i$).

We set $A := \bigcap_{i\in I} A_i$, $B := \bigcup_{i\in I} B_i$. Clearly $C \subseteq A$ and $\partial(U) \subseteq B$. Moreover, since A_x is open for all $x \in \partial(U)$ and I is finite, A is open. Also, for all $x \in \partial(U)$, since A_x, B_x are clopen in X, they are disjoint unions of components of X, that is, saturated with respect to the decomposition of X into its components, and therefore so are A, B.

We claim that $A \cap U$ is also saturated. If not, then there exists some component K of X in A such that $K \cap U \neq \emptyset \neq K \setminus U$. Since U is open, $U \cap K$ is open in K, but since K is connected, $U \cap K$ is not clopen in K, and therefore not closed, that is, there exists a point $y \in (\mathsf{Cl}(U) \cap K) \setminus (U \cap K)$. In particular, $y \in \partial(U)$, whence $y \in B$. Since B is saturated, $K = \mathbf{K}(y) \subseteq B$. But B is disjoint from A, contradicting the fact that $K \subseteq A$. Hence $A \cap U$ is saturated, as well as open, and therefore can be taken as the required open saturated set. ✠

4.3.17 *Corollary:* *Let $\mathcal{E}$ be a pre-hyperedge selection in a compact feebly Hausdorff topological space X, and Q the induced clump-hyperedge quotient. Then the following are equivalent:*

(I) *Q is feebly Hausdorff;*

(II) *Q is feebly regular;*

(III) *Q is feebly normal;*

(IV) *Q is quasiregular.*

If all the pre-hyperedges have finite boundaries, then the above conditions are also equivalent to the following:

(V) *$\mathcal{E}$ satisfies the weak pre-hyperedge convergence property.*

Proof: From 3.5.6 we know that Q is an S_1 space, hence conditions (I), (II) and (III) are equivalent to Q being an S_2, S_3 or S_4 space respectively. So from Proposition 3.1.17 we have (III) $\Rightarrow$ (II) $\Rightarrow$ (I). On the other hand, from Fact 4.2.22 we know that Q is compact, hence from 3.3.6 we conclude that these three conditions are equivalent. The equivalence of (IV) and (V) is precisely what Proposition 3.5.15 affirms. If all the pre-hyperedges have finite boundaries, then the equivalence of (III) and (V) follows from Theorem 4.3.16 and Proposition 3.6.2. ✠

Although we have not proved this, it is not hard to see that the assumption that the pre-hyperedges have finite boundaries can be dropped. Again, this follows from Theorem 4.3.16 and the fact that Proposition 3.5.7 still holds if this assumption is replaced by the weak pre-hyperedge convergence property and upper semicontinuity of the decomposition of the V-set into clumps.

4.3.18 Remark: With regard to the choice of topological model for a graph, one may contemplate independently the issues of how to model edges and vertices, and specifically how the connected components of the vertex set and the edge set are "holding the space together". Assuming that each edge is to be modelled by a connected open set, in the set of edges there will inevitably be one component per edge. Thus the issue of whether an edge is a point or a singleton boils down to whether the set of edges is totally-disconnected (in fact, discrete) or not. The analogous question, of course, is whether the set of vertices should be totally-disconnected or not. Although intuitively a totally-disconnected vertex set seems more appropriate, assuming that each vertex is to be modelled by a closed set does not automatically imply that the vertex set is totally disconnected (otherwise there would be no connected T_1 spaces with more than one point!). This suggests that there may be something interesting to be gleaned from a scenario where the "clumps", that is, the components of the vertex set, are in fact not necessarily singletons. We still contemplate all four possibilities, that is, we consider independently whether the vertex set and the edge set should be totally disconnected or not.

The usual model, of course, opts for a totally disconnected vertex set and an edge set with a non-degenerate component per edge. We discuss this option in Arguments 3.1.5, 4.4.4 and 4.4.13; we shall not address this possibility here. This leaves us with three "levels" (scenarios) to work in: a first in which one carries a selection of pre-edges in a (possibly T_1) space, a second in which the edges are points (hyperedge spaces), and a third in which the edges *and* the clumps are singletons. (V-totally-disconnected topologized graphs).

The weak pre-edge convergence property appears to be rather awkward to phrase or use, yet Proposition 3.5.15 says that it is equivalent to quasiregularity in the quotient, a condition which, in the light of the pathological examples seen in Subsection 3.1.1, such as the Diestel-Kühn obstruction (Example 3.1.4), appears essential for our spaces to be well-behaved. Moreover, Corollary 4.3.17 above says that this condition is equivalent to the stronger property of feeble regularity under the additional assumption that the original space is compact and feebly Hausdorff (Example 3.6.3 illustrates how, without compactness, the strong pre-edge convergence property does not guarantee feeble regularity in the quotient, which in the case of this example turns out to be the one-way infinite path with a double end). We are not aware of a natural way to ensure this property when working in the first scenario (pre-edge selections), except perhaps for local connectedness, which leads to a loss of generality.

On the other hand, the strong (and the weak) pre-hyperdege convergence property is trivially satisfied in the case of quasiregular hyperedge spaces (the second scenario), and therefore feebly regular ones. Moreover, although intuitively one may be led to think of an entire clump as corresponding to a single "vertex", not only do feebly regular edge spaces constitute a significantly wider class of objects than feebly regular (equivalently, S_3) V-totally-disconnected topologized graphs, but the transition from a hyperedge space to its clump quotient is not entirely trivial in our context, as will become apparent in the next chapter. The reason for this is that the topology on the vertex set does play a rôle, by dictating how the space is "held together".

Furthermore, in a certain sense the model of hyperedge spaces is more general for us than that of pre-edge selections. In Subsection 4.2.1, immediately following the construction of the Knaster-Kuratowski graph (Example 4.2.9), we discuss why compactness is a reasonable assumption to impose on our spaces; these reasons will be reinforced by the results on the Diestel-Kühn end-quotient in the next section, and also addressed in 4.4.13. However, it is possible for a T_4 space to be non-compact, and yet contain an E-set which induces a compact feebly regular edge quotient, meaning that a well-behaved configuration may be unnecessarily excluded in the first scenario by the assumption of compactness, while it is retained as a legitimate structure in the model of hyperedge spaces. In the next section we shall see how this occurs in the case of the Diestel-Kühn end-quotient.

The comparison with the context of V-totally-disconnected topologized graphs (the “third scenario”), arguably so much closer to one’s intuition of a “graph”, depends of course on whether this restriction is actually necessary or not. One important advantage of the “second” over the “third” scenario is that, since vertices are not necessarily closed points, the same topological space can be given the structure of an edge space in different ways. Doubtless the simplicity and symmetry of the concept of an (S_1) topologized graph are very attractive and warrant investigation, especially with the backdrop of the results of Chapter 2. This setup becomes even more attractive with the stronger assumption of a totally disconnected vertex set, but in the next chapter we propose to show that, at least in the context of cycle spaces and with the assumption of compactness, one can sensibly generalize results from graph theory without resorting to these strong assumptions, and indeed obtain a theory which is not a trivial translation of what one would obtain in the restricted (“third”) scenario.

The concept of a pre-hyperedge selection was introduced mainly to address the issue of which “scenario” to choose. Since we shall not pursue the “first scenario” any further, in Chapters 5 and 6 we shall not consider pre-hyperedges or pre-edges, but only work in a context where edges are actually points, so that the assumption of feeble regularity (and often compactness) is sufficient for our purposes. However, the results in this section do have a bearing on what is to follow.

Firstly, since we do *not* assume that the clumps are singletons, we shall on occasion consider the clump quotient of topological spaces, both because we feel that the relationship between a edge space and its clump quotient is interesting in its own right (see for example Theorem 5.4.3), and as a tool for other results.

Secondly, the results in this section give us the technical tools to obtain a version of the existence results in Subsections 4.2.2 and 4.2.3 which is more useful for our purposes, although this could probably also be achieved in other reasonable ways.

With regard to the former point, we now summarize (in Corollary 4.3.20) the most important implications of this section in the special case of the clump-quotient of a feebly regular edge space (as opposed to the context of pre-edges). In order to do this, we recall yet another fact from general topology.

4.3.19 Fact: Suppose that $\mathcal{D}$ is an upper semicontinuous decomposition of a topological space X into connected subsets. Then, for every subset $C \subseteq Q$, we have that the inverse image $p^{-1}(C)$ in X of C is connected if and only if C is connected.

Reference: See [12], Theorem 18.A.21. Note that since a quotient map is continuous, if $p^{-1}(C)$ is connected, then $p(p^{-1}(C))$ is always connected, independently of whether the decomposition is upper semicontinuous or not. □

4.3.20 *Corollary:* *Let $p : G \to Q$ be the quotient map of a compact feebly Hausdorff hyperedge space onto its clump quotient Q. Then Q is a compact, feebly normal V-zero-dimensional topologized hypergraph. Moreover p is closed, and for every subset $C \subseteq Q$, we have that $p^{-1}(C)$ is connected if and only if C is connected, and closed if and only if C is closed.*

Proof: The claimed properties of Q follow from Theorem 4.3.12 and the fact that the weak pre-hyperedge convergence property is trivially satisfied in quasiregular hyperedge spaces. From Theorem 4.3.16 we have that the decomposition of V_G into its components is upper semicontinuous. From this, Proposition 3.6.4 allows us to conclude that the clump-edge decomposition of P is upper semicontinuous, and therefore apply Facts 4.3.19 and 3.5.2 (so p is a closed map). The fact that $p^{-1}(C)$ is closed if C is closed follows from the continuity of p, while if $p^{-1}(C)$ is closed then $C = p(p^{-1}(C))$ is closed. ✠

We conclude this section with the existence result mentioned above, analogous to Theorem 4.2.13 and Proposition 4.2.24.

Terminology: Given an edge space G and a property $\mathcal{P}$ of the subsets of G (collection of subsets of G), we say that that a subset $S \in \mathcal{P}$ is edge-minimal if every subset of S in $\mathcal{P}$ contains all the edges in S.

4.3.21 *Theorem:* *Let G be a compact connected feebly Hausdorff edge space and M a subset of G. Then there exists an edge-minimal connected subset of G containing M. This subset can also be chosen to be closed.*

Proof: Let $p : G \to Q$ be the quotient map of G onto its clump quotient Q. By Theorem 4.3.12 Q is a compact connected feebly Hausdorff topologized graph, and by Proposition 4.2.24 there exists a minimal closed connected set C' in Q containing $p(M)$. Hence by Corollary 4.3.20 $C := p^{-1}(C')$ is closed and connected, and of course contains M. Now suppose that K is a connected set containing M, contained in C, and such that $E_K \subsetneq E_C$. Let K' denote $\mathsf{Cl}(K)$. Since C is closed, K' is also contained in C. Moreover, by Fact 2.1.3, since K is connected, so is K'. Furthermore, since an open point is never an accumulation point of a set it does not belong to, $E_K = E_{K'}$. Hence K' is a topologized subgraph contained in C whose edgeset is properly contained in that of C.

Now since $M \subseteq K' \subseteq C$, we have that $p(M) \subseteq p(K') \subseteq p(C) = C'$. Also, for any subset A of X, the edges of G contained in A are in one-to-one correspondence with the edges of Q contained in $p(A)$. So the edgeset of $p(K')$ is properly contained in the edgeset of $p(C) = C'$. But since p is continuous and K' is connected, by Fact 4.2.22 $p(K')$ is connected. Moreover, by Corollary 4.3.20 p is closed, so $p(K')$ is also closed. This contradicts the fact that C' is a minimal closed connected set containing $p(M)$. Hence C is an edge-minimal connected set containing M, which also happens to be closed. ✠

4.3.22 ***Corollary:*** *A compact connected feebly Hausdorff edge space has a minimal spanning set.* ✠

4.4 The Diestel-Kühn end-quotient

In this section we consider a construction of Diestel and Kühn, introduced in [21] for the purpose of generalizing results about cycle spaces to non-locally-finite graphs. The construction starts from a graph G and obtains a topological space. We refer to this space as the "Diestel-Kühn end-quotient" of G.

We use this construction to address the issue of whether to model an edge as a point or an arc. We show that, under the assumptions imposed by Diestel and Kühn, modulo the trick of modelling edges as points, the Diestel-Kühn end-quotient is weakly Hausdorff, and, if the original graph is 2-connected, compact. It is an important point for us that in the model with arcs for edges, the first of these assumptions is satisfied in a trivial way (since the space is Hausdorff), while the second is not satisfied, except for the special case when the Diestel-Kühn end-quotient coincides with the usual topological model for a graph with its ends. We address these points in Arguments 4.4.4 and 4.4.13.

Proposition 4.4.3 and Theorem 4.4.12 together imply that when the original graph is 2-connected, our results about minimal spanning sets (Chapter 4) and cycle spaces (Chapter 5) apply directly to the spaces considered in [21], modulo the trick of modelling arcs with edges. In the general case, these results can be derived easily by considering the blocks (maximal 2-connected subgraphs) of the given graph.

We do not assume familiarity with the construction of the Diestel-Kühn end-quotient of a graph. We give the definition of this space below, as given (in slightly different language) in [21]. We do assume familiarity with the graph-theoretic notion of a Halin end, and the direction extension of a graph, that is, the standard construction which, given a graph G, adds the ends of G as "points at infinity" to the cell-complex topological space associated with G. The term "direction extension" is not standard. As we discussed in Argument 3.1.5, we use this term because, as pointed out in [18], in the Hausdorff model, the natural bijection between the Halin ends of a graph and the Freudenthal ends of the associated cell-complex topological space fails, but the notion of a Halin end, which is the one needed to construct the direction extension, can be captured topologically by the concept of a direction, introduced in the same paper.

In this section we shall (explicitly) impose one of the the following two properties on the graphs in question.

(†) Every end is dominated by at most one vertex.

(2) No two vertices are joined by infinitely many pairwise internally disjoint paths.

These are the same properties imposed in [21], and in fact the designations (†) and (2) are taken from this paper. Note that property (2) implies (†).

Following Diestel and Kühn, we say that a vertex u dominates an end ω if there is a one-way infinite path R in ω and infinitely many (u, R)-paths, disjoint except for their one common end u.

4.4.1 Definition: Given a graph G satisfying property (†), and its direction extension $\bar{G}$, the domination decomposition of $\bar{G}$ is the one whose only non-degenerate parts are all the sets consisting of a vertex dominating some end, together with all the ends it dominates. The Diestel-Kühn end-quotient of G is the quotient space induced by the domination decomposition.

Note that property (†) guarantees that the non-degenerate "parts" defined above are actually disjoint. Also, note that no two vertices (non-ends) are identified.

Given a graph G, the "open"[5] 1-cells in the cell-complex topological space $\check{G}$ associated with G are open[6] arcs in the direction extension $\bar{G}$ with precisely the same boundary as in $\check{G}$. We shall simply refer to them as the "open 1-cells" in $\bar{G}$. They constitute a pre-edge selection in $\bar{G}$, and since each one is an open saturated set, their images in the Diestel-Kühn end-quotient $\tilde{G}$ give a pre-edge selection in $\tilde{G}$. In this section we refer to the reduced Diestel-Kühn end-quotient of G, meaning the edge-quotient of the $\tilde{G}$ with respect to this pre-edge selection, and denote this space by $\hat{G}$. We also refer to the reducing map, meaning the natural quotient map of $\bar{G}$ onto $\hat{G}$, the composition of two quotient maps.

Summarizing, there will be five kinds of spaces in this section: a graph G, the associated cell-complex topological space $\check{G}$, the direction extension $\bar{G}$, the Diestel-Kühn end-quotient $\tilde{G}$, and the reduced quotient $\hat{G}$. The last is "our version" of the Diestel-Kühn end-quotient. This chain of derivations, except of course for the last step, is the way the Diestel-Kühn end-quotient is constructed. We consider the reduced Diestel-Kühn end-quotient for the purposes of the comparison of the Hausdorff (1-cells for edges) with the non-Hausdorff (points for edges) models.

[5] In the sense of a cell-complex

[6] topologically

4.4.1 Separation Axioms

4.4.2 *Proposition:* *Let G be classical graph satisfying property (†), and $q : \bar{G} \to \hat{G}$ the reducing map from the direction extension $\bar{G}$ of G onto its reduced Diestel-Kühn end-quotient $\tilde{G}$. Then $\hat{G}$ is a topologized graph. An edge (open point) c is incident with a vertex (closed point) v in $\hat{G}$ if and only if $q^{-1}(v)$ contains a vertex which is a boundary point of $q^{-1}(c)$.*

Proof: Let $\mathcal{D}$ be the domination decomposition of $\bar{G}$ and $p : \bar{G} \to \tilde{G}$ the corresponding quotient map. Since p acts trivially on points which are neither ends nor vertices, we identify a 1-cell e with its image in $\tilde{G}$. We claim that $\mathcal{D}$ is closed. Since $\bar{G}$ is T_1, it is sufficient to verify that every non-degenerate part is closed. Let D be a non-degenerate part, and v the only vertex (non-end) it contains. Pick $x \notin D$. We wish to exhibit a neighbourhood of x disjoint from D.

If x belongs to an open 1-cell e, then e is an open set containing x and disjoint from D. If x is a vertex, then x together with all open 1-cells corresponding to edges incident with x is a neighbourhood of x disjoint from D. If x is an end not dominated by v, then there is a finite set F of vertices in G such that v and x belong to distinct connected components of $G \setminus F$[7], say $v \in C_1$ and $x \in C_2$. Since v cannot dominate any end in C_2, we have that C_2 is disjoint from D. Since $\{v\}$ is closed and $\bar{G}$ is locally connected, C_2 is open. Hence C_2 is an open set containing x and disjoint from D. We conclude that D is closed.

Hence $\tilde{G}$ is T_1. Moreover, for any open 1-cell e with boundary points a and b in $\bar{G}$ (possibly $a = b$), clearly e is open in $\tilde{G}$, and since quotient maps are continuous, the points $p(a)$, $p(b)$ are still boundary points for e in $\tilde{G}$. Since $p^{-1}(a)$ and $p^{-1}(b)$ are closed, the complement of $p^{-1}(a) \cup e \cup p^{-1}(b)$ is open, and since it is also saturated, its image is an open set in $\tilde{G}$ disjoint from e, and therefore from $\mathsf{Cl}_{\tilde{G}}(e)$. Thus e has precisely $p(a), p(b)$ as boundary points in $\tilde{G}$, and since no two vertices (non-ends) are identified, if a, b are distinct so are $p(a)$, $p(b)$.

Now we consider the decomposition of $\tilde{G}$ whose only non-degenerate parts are the 1-cells. Since $\tilde{G}$ is T_1, the corresponding quotient is S_1, and clearly the only boundary points of the images of the 1-cells are precisely the images of the boundary points. In particular, the number of these points is at most two. ✠

4.4.3 *Proposition:* *Let G be a classical graph and $\hat{G}$ its reduced Diestel-Kühn end-quotient. If G satisfies property (2), then $\hat{G}$ is weakly Hausdorff.*

Proof: Let $p : \bar{G} \to \hat{G}$ be the reducing map. The space $\hat{G}$ has four kinds of points: edges, original vertices which have not been identified with an end, original ends which have not

[7] This is topological, not combinatorial, notation: all the edges are still there.

been identified with a vertex, and points which are the identification of a vertex with the ends it dominates. The last three kinds all correspond to closed points (vertices in our model).

Let u and v be distinct points in $\hat{G}$. We need to obtain respective neighbourhoods with finite intersection. If one of u, v is an edge, as remarked in (3.1.15) one can find neighbourhoods which intersect in at most one point. If precisely one of u, v, say u, is an unidentified end, then in $\bar{G}$ there exists a vertex v' such that $p(v') = v$. Since v and u are distinct in $\hat{G}$, there exists a finite set F of vertices in $\bar{G}$ such that $\bar{G} \setminus F$ contains distinct (disjoint, open) components C_1, C_2, containing v' and u' respectively, where u' is the undominated end in $\bar{G}$ such that $p(u') = u$. The same is true if both u and v are unidentified ends, except that v' is an end. Since no vertex in C_1 can dominate an end outside C_1, and no end in C_1 can be dominated by a vertex outside C_1, we have that C_1, and similarly C_2, is saturated. Hence $p(C_1)$ and $p(C_2)$ may be taken as the required neighbourhoods of u and v; in this case, the intersection is empty.

So we may assume that u, v respectively contain (distinct) vertices u', v' in their inverse images. Let $\mathcal{F}$ be the set of open 1-cells with $\{u, v\}$ for boundary, and F their union. Since G satisfies (2), $\mathcal{F}$ is finite. Now we consider the subspace $\bar{G}' := \bar{G} \setminus F$. This is the direction extension of the subgraph G' obtained from G by discarding the edges corresponding to 1-cells in $\mathcal{F}$.

Since G, and therefore G', satisfies (2), by Menger's Theorem there exists a finite set K of vertices in G' such that u' and v' belong to distinct components of $\bar{G}' \setminus K$, say $u' \in C_1$ and $v' \in C_2$. Again, C_1 and C_2 are open in $\bar{G}'$ and saturated with respect to the domination decomposition of $\bar{G}'$. Now for any topological space X, if A is open in X and B is open in $X \setminus A$, then $A \cup B$ is open in B. Hence $F \cup C_1, F \cup C_2$ are open in $\bar{G}$. Since the ends of G and G' are in one-to-one correspondence and no vertex dominates an end in G unless it does so already in G', and since F is itself saturated with respect to the domination decomposition of $\bar{G}$, we have that $F \cup C_1, F \cup C_2$ are open, saturated sets in $\bar{G}$ whose intersection is precisely F. Their images in $\hat{G}$ are open and intersect in finitely many points. Hence they may be taken as the required neighbourhoods of u, v. ✠

4.4.4 Argument II (Separation Axioms): In Example 3.1.4 we discussed the inherent problems with the Diestel-Kühn obstruction, present whether edges are modelled as points or arcs. In the model which takes 1-cells for edges, this configuration is the direction extension of a specific graph (the "Bean graph" discussed on page 75). In general, the direction extension is a Hausdorff space, and therefore an entirely "legitimate" space from the point of view of the standard separation axioms. On the other hand, in Remark 3.5.11 we saw that *when the edges are modelled as singletons*, the Diestel-Kühn obstruction is excluded by the assumption that the space is feebly Hausdorff. More generally, if some vertex of a graph dominates an end, the edge-quotient of the corresponding direction extension is either not compact or not feebly Hausdorff. Thus, modelling edges as singletons allows one to exclude the problems identified by Diestel and Kühn in terms of separation axioms (and compactness). We shall take up the issue of compactness again in Argument 4.4.13.

We have also seen that the inherent problems may be due both to the configuration of the edges (such as in the case of the Diestel-Kühn obstruction), or to the topology on the vertex set (such as in the case of the one-way-infinite path with a double end, Example 3.1.7, for which Theorem 4.2.13 fails). In Corollaries 3.4.13 and 3.4.14 we saw how these two aspects can be seen as independent "components" which together are equivalent to feeble regularity. Thus, feeble regularity is a single global topological assumption which captures these two requirements for well-behaved spaces.

The notion of feeble regularity is meaningful for Hausdorff spaces but it is only useful in our context because we model edges by singletons. The weak and feeble separation axioms arose, once the non-Hausdorff model of topologized graphs was constructed, as relaxations of the standard separation axioms which were the required properties for the proofs guaranteeing the existence of minimal spanning sets (4.2.14, 4.2.25,4.3.22) and, more generally, minimal closed connected sets (4.2.13, 4.2.24, 4.3.21), results which will be useful for us in Chapter 5.

4.4.2 Compactness

Combinatorial Lemmas

Notation and Terminology: Henceforth in this chapter, given a graph G and a subgraph X, we denote by $\mathbf{G} - \mathbf{X}$ the subgraph $G \setminus X^{\square}$. Given two subgraphs X, Y, the notation $\mathbf{G}[\mathbf{X}, \mathbf{Y}]$ stands for the subgraph $X \cup Y \cup (X^{\square} \cap Y^{\square})$. An $(\mathbf{X}, \mathbf{Y})$-path is an ab-path for some vertices a, b with $a \in X$ and $b \in Y$.

4.4.5 ***Lemma:*** *Let G be a connected classical graph and let U be an infinite set of vertices of G. Then either there is a one-way infinite path R and infinitely many totally disjoint (U, R)-paths or there is a vertex u of G such that there are infinitely many (u, U)-paths, disjoint except for their common end u.*

Proof: Let $u_1, u_2, \dots$ be an enumeration of any infinite countable subset of U. We inductively construct an infinite sequence of paths P_i and trees T_i in the following way. We set $T_i := \{u_1\}$ and, given T_{i-1}, choose P_i to be a path from u_i to a vertex of T_{i-1} that is disjoint from T_{i-1} except for its end in T_{i-1}. Then we set $T_i = T_{i-1} \cup P_i$. Note that $T := \bigcup_{i=1}^{\infty} T_i$ is a countable tree.

Suppose T contains a one-way infinite path R. Let $\preceq$ be a compatible total order such that the terminal point is a minimum and $v_1, v_2, \dots$ be an enumeration of the vertices of

R such that $v_i \prec v_j \Leftrightarrow i < j$. If infinitely many of the u_i are on R, then we are done. So we may choose R so that it is disjoint from the u_i's. For each i there is a maximal subpath of R that is contained in T_i, say ending at v_{j_i}. There is some $k_i > i$ such that v_{j_i+1} is on the path P_k. This path contains a subpath Q_{k_i} that connects u_{k_i} to R. Choose $i_1, i_2, \dots$ so that $i_1 < k_{i_1} < i_2 < k_{i_2} < i_3 < \cdots$. Then the paths $Q_{k_{i_j}}$ are pairwise disjoint, as required.

Finally, suppose T contains no one-way infinite path. Then T has a vertex v of infinite degree. There are infinitely many components in $T - v$ and by construction of T every one of them contains at least one vertex from $u_1, u_2, \dots$. If C is any such component, containing the vertex u_j, then there is in $G[C, v]$ a (v, u_j)-path $Q(C)$. Clearly the infinitely many paths $Q(C)$ are pairwise disjoint except for v. ✠

4.4.6 ***Corollary:*** *Let G be a 2-connected classical graph satisfying property (2). If U is any infinite set of vertices, then there is a one-way infinite path R of G and infinitely many totally disjoint (U, R)-paths.*

Proof: By Lemma 4.4.5, the only alternative is that there exists a vertex v of G and an infinite set $\mathcal{P}$ of (v, U) paths which pairwise intersect only in v. Let U' be the (infinite) set of terminal points in U of paths in $\mathcal{P}$.

Since $G - v$ is connected, Lemma 4.4.5 applies to U' in $G - v$. If there were a vertex w of $G - v$ joined to infinitely many vertices in U', then we claim v and w would be joined in G by infinitely many pairwise internally disjoint paths.

To see this, let $\mathcal{Q}$ be a set of pairwise disjoint (w, U')-paths. For each path $Q \in \mathcal{Q}$, let Q' be the minimal subpath that joins w to a vertex in some path in $\mathcal{P}$. Each path in $\mathcal{P}$, being finite, can contain terminal vertices of at most finitely many Q', so there are infinitely many Q' that have their terminal vertices in pairwise distinct paths in $\mathcal{P}$ and now it is obvious that v and w are joined by infinitely many pairwise internally disjoint paths.

We conclude that there is a one-way infinite path R of $G - v$ and infinitely many totally disjoint (U', R)-paths. ✠

4.4.7 ***Lemma:*** *Let A be any set of vertices in a connected classical graph H. Then there is a minimal tree in H containing A.*

Proof: There is a spanning tree of H, so the set of trees in H containing A is nonempty. If $\mathcal{T}$ is any nest of trees all containing A, then we claim $\check{T} := \bigcap_{T \in \mathcal{T}} T$ is a tree containing A. It clearly contains A and clearly contains no cycles, so suppose it is not connected. Then it has two components U and V which are joined in some $T \in \mathcal{T}$ by a path P.

Some internal point x of P does not belong to $\check{T}$. Let P' be a maximal subpath of P containing x and internally disjoint from $\check{T}$. Then the terminal points of P' are vertices in some two components U', V' of $\check{T}'$. Since P' is finite, there exists some $T' \in \mathcal{T}$ disjoint from P', except for the two terminal vertices. Clearly $T' \subset T$. There is a path Q in T' joining U' and V', then Q is a path in T joining U and V and, therefore, T contains a cycle consisting of paths in U and V joining the terminal vertices of P' and Q, together with P' and Q, a contradiction that proves $\check{T}$ is connected. ✠

4.4.8 ***Lemma:*** *Let G be a connected classical graph with countable degrees. Then G is countable.*

Proof: Since the countable union of countable sets is countable, it is sufficient to show that the vertex set is countable. Let v be a vertex, and for every non-negative integer i, let D_i be the set of vertices at distance i from v. Since D_0 is certainly countable, and D_{i+1} is contained in the union of the countable neighbour sets of the vertices in D_i, by induction D_i is countable for every non-negative integer i, and therefore the vertex set, being the countable union of countable sets, is countable. ✠

4.4.9 ***Lemma:*** *Let G be a 2-connected classical graph satisfying property (2). Then G is countable.*

Proof: Suppose not. Then by Lemma 4.4.8 some vertex v of G has uncountable degree. The graph $G-v$ is connected by hypothesis. By Lemma 4.4.7, there is a minimal tree T in $G-v$ containing all the vertices that are adjacent in G to v. Since T is connected and has uncountably many vertices, again by Lemma 4.4.8 it has a vertex u of uncountable degree. Every component of $T-u$ contains a neighbour of v, by minimality. But then there are infinitely many internally disjoint uv-paths in G, a contradiction. ✠

4.4.10 ***Lemma:*** *Let G be a 2-connected classical graph satisfying property (2). Then every vertex of infinite degree dominates some end of G.*

Proof: Let v have infinite degree in G, and U the neighbour set of v. Then $G-v$ is connected. Corollary 4.4.6 implies that there is a one-way infinite path R in $G-v$ and infinitely many totally disjoint (U, R)-paths. Clearly v dominates the end containing R. ✠

4.4.11 ***Lemma:*** *Let G be a 2-connected classical graph satisfying property (2). If A is a finite set of vertices of G, then $G-A$ has only finitely many components.*

Proof: Suppose, by way of contradiction, that there exists an infinite set U of vertices no two of which belong to the same component of $G - A$. By Corollary 4.4.6, there exist in G a one-way infinite path R and infinitely many pairwise totally disjoint (U, R)-paths. But then, in $G - A$, all but finitely many of the vertices in U belong to the same component, a contradiction. ✠

Compactness of the Diestel-Kühn end-quotient

4.4.12 ***Theorem:*** *Let G be a 2-connected classical graph satisfying property (2). Then the reduced Diestel-Kühn end-quotient of G is compact.*

Proof: Since we do not assume anything about the Diestel-Kühn end-quotient (other than Proposition 4.4.2), this proof will make reference to the direction extension $\bar{G}$ of G, as well as the reduced Diestel-Kühn end-quotient $\hat{G}$, and G itself. To manage the correspondence between these spaces, we adopt the following convention: any subgraph S in G determines a corresponding topologized subgraph S in $\hat{G}$ (by Propositon 4.4.2), which we denote by $\hat{S}$, and a sub-cell complex S' of the associated cell-complex topological space, which is embedded in $\bar{G}$. We denote by $\bar{S}$ the closure of S' in $\bar{G}$. Note that, if V_S is finite, then S' is closed, but otherwise $\bar{S}$ may contain ends, while S does not. We also denote by $S^{\boxdot}$ the union of S' with all open 1-cells incident with some point in S'.

Let $\mathcal{U}$ be an arbitrary open cover of $\hat{G}$. We need to show that $\mathcal{U}$ has a finite subcover. If V_G is finite, then $\mathcal{U}$ clearly contains a finite cover of $\hat{V}_G$, and since every edge is incident with some vertex, any such cover is also a cover of $\hat{G}$. So we may assume that V_G is infinite. By Lemma 4.4.9, V_G is countable. Let $v_1, v_2, \ldots$ be an enumeration of $V(G)$ and, for each $i = 1, 2, \ldots$, let $V_i = \{v_1, v_2, \ldots, v_i\}$.

Now we consider the collection $\mathcal{C}$ of components C of $G - V_i$, for some non-negative integer i, such that no element of $\mathcal{U}$ satisfies $\overline{C} \subseteq p^{-1}(U)$, where p is the reducing map from $\bar{G}$ to $\hat{G}$. We refer to the elements of $\mathcal{C}$ as "bad" components. We give this set the structure of a rooted tree T by taking G for the root and, for any other $C \in \mathcal{C}$, we define the parent of C by choosing the largest integer $j(C)$ such that C is strictly contained in a component K of $G - V_{j(C)}$. Note that $\overline{K}$ can not be contained in an element of $\overline{\mathcal{U}}$, that is, $K \in \mathcal{C}$. We take K for the parent of C. Also, either $K = G$ or else $j(K) < j(C)$. Hence after a finite number of iterations the ancestor of C coincides with the root G.

Claim: For some i, no component of $G - V_i$ is a bad component.
Proof of Claim: Suppose that, for every i, there is a "bad" component of $G - V_i$. Since every vertex is eventually in V_i, there are infinitely many bad components, that is, T is infinite. By Lemma 4.4.11, T has only finite degrees. By König's Lemma, there exist an

infinite nest $C_1 \supsetneq C_2 \supsetneq C_3 \supsetneq \cdots$ of bad components and points $u_i \in C_i \setminus C_{i+1}$, for all positive integers i. Note that if C_k is a component of $G - V_i$ and C_{k+1} is a component of $G - V_{i'}$, then $i' > i$, and, for all t such that $i \leq t \leq i'$, one of C_{k+1} or C_k is a component of $G - V_t$. Hence, for every i, there exists k_i such that C_{k_i} is a component of $G - V_i$.

Let $X := \{u_i\}_{i=1}^{\infty}$. By Corollary 4.4.6, there exists a one-way infinite path R in G and infinitely many totally disjoint (X, R)-paths. Let ω be the end of R. There is a $U \in \mathcal{U}$ containing $p(\omega) \in \hat{G}$. Hence, in $\bar{G}$, we have that $\omega \in p^{-1}(U)$. Thus, by construction of the direction extension,[8] there is a finite set W of vertices of G such that the component C of $G - W$ containing the infinite component of $R - W$, satisfies $\overline{C} \subseteq p^{-1}(U)$. Moreover, this component contains all but finitely many of the points in X. However, there exists some i such that $W \subseteq V_i$, while C_{k_i} is a component of $G - V_i$ containing all but finitely many of the points in X (by construction of X). Since the components of $G - V_i$ are contained in the components of $G - W$, we have $C_{k_i} \subseteq C$, whence $\overline{C}_{k_i} \subseteq \overline{C} \subseteq p^{-1}(U)$, a contradiction. □

So there is some integer i such that, if $\mathcal{Z}$ denotes the set of components of $G - V_i$, then for every $C \in \mathcal{Z}$ there exists some $U_C \in \mathcal{U}$ satisfying $\overline{C} \subseteq p^{-1}(U_C)$. Note that $\{V_i^{\boxdot}\} \cup \mathcal{Z}$ is a cover of G, hence $\{\overline{V}_i^{\boxdot}\} \cup \{\overline{C}\}_{C \in \mathcal{Z}}$ is a cover of G. Since V_i is finite, by Lemma 4.4.11 $\mathcal{Z}$ is finite. Moreover, for every $v \in V_i$, there exists some $U_v \in \mathcal{U}$ such that $\hat{v} \in U_v$, whence $\bar{v} \in p^{-1}(U_v)$. Now for any open 1-cell $e \in \bar{G}$ with a boundary point $\bar{u} \in \bar{V}_i$, the image $\hat{e}$ in $\hat{G}$ is an edge (open point) incident with $\hat{u}$ in $\hat{G}$, so $\hat{e} \in U_u$. Hence $\bar{V}^{\boxdot}$ is contained in $\bigcup_{v \in V_i} p^{-1}(U_v)$. Thus $\{p^{-1}(U_v)\}_{v \in V_i} \cup \{p^{-1}(U_C)\}_{C \in \mathcal{Z}}$ is a finite cover of $\bar{G}$, whence $\{U_v\}_{v \in V_i} \cup \{U_C\}_{C \in \mathcal{Z}}$ is a finite subcover of $\hat{G}$. ✠

The above theorem is equivalent to a result of Diestel [16, Corollary 4.3].

4.4.13 Argument III: As we saw in Argument 3.1.5, one crucial difference which arises when one models edges as points as opposed to arcs is that certain open sets become compact, when they would not be so in the Hausdorff model. This fact ensures that the reduced Diestel-Kühn end-quotient is compact. On the other hand, in the Hausdorff model, the Diestel-Kühn end-quotient need not be compact: already in the simple case of the Diestel-Kühn obstruction one can see that this is not the case: taking a countably infinite sequence of points, each contained in some open 1-cell but no two contained in the same 1-cell, it is easy to check that this sequence has no cluster point. Diestel [16] has recently proposed an alternative Hausdorff topology, to be imposed on the direction extension before the identification of ends with their dominating vertices, which renders the Diestel-Kühn end-quotient compact. Although in this case the edges are still modelled by arcs, in the special case of ends the basic neighbourhoods are defined so that for any

[8] The basic open neighbourhoods of ω are defined so that they contain all the ends of C.

such neighbourhood U and any edge e, the set of inner points of e is either disjoint from U or else entirely contained in U. Since this set of points is being taken "all at once", it seems a small step from this to contracting the edges to points.

Example 4.2.9 shows that separation axioms on their own are not sufficient to obtain well-behaved spaces. The ensuing discussion suggests that compactness is not an unreasonable assumption to impose in this context. The fact that the reduced Diestel-Kühn end-quotient turns out to be compact (when the original graph is 2-connected), in spite of the fact that this is not obvious from its construction, reinforces this impression. In Chapter 5 we see that compactness is sufficient (together with the feebly Hausdorff property, but nothing else) for the purpose of generalizing results about cycle spaces.

On the other hand, considering compact *Hausdorff* spaces would be both cumbersome and, more importantly, unnecessarily restrictive.

The unnecessary restriction arises not just because feebly Hausdorff spaces constitute a larger class, but because there may be pre-edge selections in a *non-compact* Hausdorff space which induce a compact feebly Hausdorff edge-quotient, as in the case of the Diestel-Kühn end-quotient. Thus, considering "compact Hausdorff spaces" and modelling edges with arcs we may fail to capture some well-behaved configurations which are easily captured in the non-Hausdorff model.

Moreover, Examples 3.5.12 and 4.3.13 (note that the latter is an edge-quotient *and* a clump-edge quotient) show that configurations we do capture in this way need not necessarily be well-behaved, even if the space is Hausdorff and compact. Proposition 3.5.15 says that excluding the bad behaviour of the edges (the "combinatorial component", failure of quasiregularity) is equivalent to imposing the rather involved weak pre-hyperedge convergence condition (Definition 3.5.13) in the case of the clump-hyperedge quotient, or, in the case of the edge quotient, the slightly less involved strong pre-hyperedge convergence property (Definition 3.5.9, Proposition 3.5.10). The latter property reduces to the less involved condition of quasiregularity (Definition 3.4.2) when the edges are singletons, which property is in turn implied by feeble regularity (Lemma 3.4.1), and is only one of the two components of feeble regularity in edge spaces and topologized graphs (Corollaries 3.4.13 and 3.4.14).

Chapter 5

Cycle Spaces

In this chapter we investigate ways to extend the well-known theory of cycle spaces and bond spaces from classical graphs to edge spaces. In view of the pathological examples seen in Section 3.1.1, we consider almost exclusively feebly Hausdorff spaces. Our results hinge mainly on compactness, and to a lesser extent the weakly Hausdorff, quasiregular and weakly normal properties.

In the first section we introduce cyclesets, edgecycles and uv-edgepaths. An important point in this context is that the edges of an edgecycle ("cyclesets") do not necessarily determine the edgecycle itself. In the second section we introduce edgecuts and bonds, and also consider three varying notions of algebraic generation by symmetric differences of infinite sets, namely the weak, algebraic and strong spans. The first is a trivial extension of the usual concept for finite sets, and the second has been considered by Diestel and Kühn [21] in the setting of cycle spaces (in different language). We define bond and cycle spaces in terms of weak and strong spans, respectively. Using quasiregularity, we develop the working tools to deal with bonds and edgecuts, also with respect to a scenario where two different combinatorial structures sit on the same topological space (Propositions 5.2.12 and 5.2.14). In Section 5.2 we also see one of the few results which do not require compactness, the fact that the elements of the bond space are disjoint unions of bonds (Proposition 5.2.10).

In Sections 5.3 and 5.4 we present our main results on cycle spaces and bond spaces. For feebly Hausdorff spaces, we show that edgecuts and cyclesets are orthogonal (Proposition 5.3.5), that the orthogonal complement of the bond space coincides with the cycle space, and, if the space is 2-edge-connected, the orthogonal complement of the cycle space is the bond space.[1] More generally, we characterize the orthogonal complement of the bond space in terms of the cycle space of a combinatorial substructure (the block restriction, Theorem 5.4.8).

We define fundamental bonds and fundamental cyclesets, and show that the cycles are

[1] In the context, orthogonality of spaces is not necessarily symmetric.

algebraically generated by the fundamental cycles with respect to a given minimal spanning set, and therefore the strong span of the fundamental cyclesets coincides with the cycle space (Corollary 5.4.2). We use this fact to obtain a non-trivial isomorphism between the cycle space of an edge space and that of its clump quotient (Theorem 5.4.3). We also relate the cycle space of an edge space with that of the clump quotient of a combinatorial substructure (the essential quotient, Theorem 5.4.5). Both the block restriction and the essential quotient are defined in terms of an edge space which has the same ground space as the original edge space, but fewer edges.

For compact weakly Hausdorff spaces, we show that the elements of the cycle space are disjoint unions of cyclesets (Theorem 5.3.11), that the fundamental cyclesets with respect to a given minimal spanning set constitute an algebraic basis for the cycle space (Theorem 5.3.16), that the algebraic and strong spans of the cyclesets coincide (Corollary 5.3.12), and that the bond space coincides with the space of edgecuts (Corollary 5.3.9)

For weakly normal, quasiregular (possibly non-compact) spaces, we show that the fundamental bonds with respect to a given minimal spanning set constitute a weak basis for the bond space (Theorem 5.3.18).

In Section 5.5 we present two partial results towards replacing the assumption of compactness and understanding better the notion of a strong span.

5.1 Edgeycles, edgepaths and plants

5.1.1 Edgecycles and cyclic orders

5.1.1 Definition: A connected edge space is 2-edge-connected if for any edge e, $X \setminus \{e\}$ is connected. An edgecycle is a 2-edge-connected edge space X such that, for any two edges e, f, we have that $X \setminus \{e, f\}$ is disconnected. Given a hyperedge space H, a non-empty subset $F \subseteq E_H$ is a cycleset if $F = E_C$ for some edge subspace C which is an edgecycle.

Note that the above definition leads to a situation which is quite different from that of standard graph theory. The edgecycle which determines a given cycleset is not uniquely determined by the cycleset. It may even happen that a cycleset is a disjoint union of smaller cyclesets. The following simple example gives an illustration of this phenomenon.

5.1.2 Example: Let $S := [0,1] \times [0,1]$, and consider the topological subspace of $\mathbb{R}^2$ given by $V := S \cup \{u, v\}$, where $u = (2,1)$ and $v = (2,0)$. We pick four points in S, $w := (1,0)$, $x := (0,0)$, $y := (0,1)$ and $z := (1,1)$. Also, we set $E = \{e_{zu}, e_{uv}, e_{vw}, e_{xy}\}$ and, we define the function f on E by $f(e_{ij}) = \{i, j\}$. Now we consider the edge space G that is the combinatorial extension of V according to f. Refer to Figure 5.1 for an illustration.

Note that $\{e_{zu}, e_{uv}, e_{vw}\}$ and $\{e_{xy}\}$ are both cyclesets, because for either set of edges we have that the union of the closure with S is an edgecycle. Moreover, all of E is itself a cycleset, for if I_{wx} and I_{yz} denote the straight line segments in S joining the pairs $\{w, x\}$ and $\{y, z\}$ respectively, then $\mathsf{Cl}(E) \cup I_{wx} \cup I_{yz}$ is an edgecycle in G whose edgeset is precisely E. The figure also illustrates the effect of contracting the clumps.

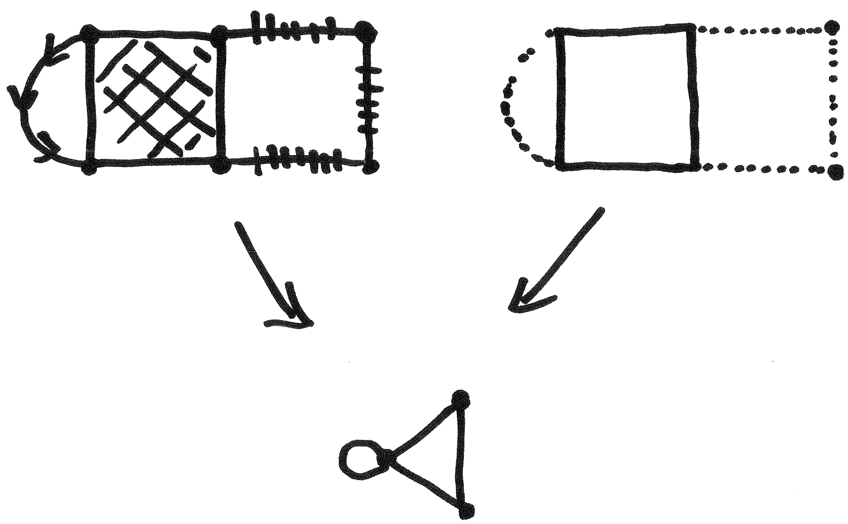


Figure 5.1: The behaviour of cyclesets

5.1.3 Remark: Let X be an edgecycle. Then precisely one of the following situations must occur:

(0) X has no edges;

(1) X has precisely one edge, which may be a loop, and V_X is connected;

(2) X has precisely 2 edges and 2 clumps, each containing precisely one vertex incident with each of the two edges; and

(3) X has 3 or more edges and, among other properties, we have that each edge is incident with two vertices in distinct clumps, but no two edges are "parallel", that is, no two clumps together contain all the endvertices of two different edges.

In particular, an edge in an edgecycle can be a loop only if it is the only edge.

Although Definition 5.1.1 is certainly satisfied by finite cycles in classical graphs, edgecycles thus defined may appear to be potentially very different from the cycles usually occurring in graph theory, and indeed in a sense they are—an individual component of the vertex set could be any connected topological space whatsoever. We shall see that this definition is not only well-suited for the purposes of generalizing results concerning the cycle space, but also leads to several parallels with graph theory, involving not only cycles but also their interaction with analogues of paths and spanning trees, and the existence of fundamental cycles.

Recall that we have defined cyclic orders and cyclic subsequences in Definition 2.4.1.

5.1.4 *Lemma:* *Let X be an edgecycle and let $F \subseteq E_X$ be nonempty and finite. Then $X \setminus F$ has $m = |F|$ components and there is a cyclic order $\langle K_0, K_1, K_2, \dots, K_{m-1}\rangle$ of the components of $X \setminus F$ and a cyclic order $\langle e_0, e_1, e_2, \dots, e_{m-1}\rangle$ of F such that, for $i = 0, 1, 2, \dots, m-1$, the ends of e_i are in K_i and K_{i+1}, where the indices are read modulo m.*

Proof: The proof is by induction on $|F|$. Let $F = \{e_i\}_{i=0}^{m-1}$ In the case $|F| = 1$, the assertion is true by the definition of cycle. For $|F| = 2$, notice that $X \setminus \{e_0\}$ is connected, so by 2.1.16 $X \setminus \{e, f\} = (X \setminus \{e_0\}) \setminus \{e_1\}$ has at most two components. Since it is not connected, it has precisely two components and e_1 has one endvertex in each of the two components. Since the result is symmetric in e_0 and e_1, we have that e_0 also has one endvertex in each of the two components.

Now suppose $|F| \geq 3$. Let $e \in F$ and let $F' = F \setminus \{e\}$. Applying the inductive hypothesis to F', we obtain $m - 1 = |F'|$ components with cyclic order $\langle K_0, K_1 \dots, K_{m-2}\rangle$ and a cyclic order $\langle e_0, e_1, \dots, e_{m-2}\rangle$ of F' so that, for $i = 0 \dots m-2$, the edge e_i has its endvertices in K_i and K_{i+1} (indices modulo $m-1$). We choose the labelling so that $e \in K_{m-2}$. Let a be the endvertex of e_{m-2} in K_{m-2}.

Note that $R = (K_0 \cup K_1 \cup \dots \cup K_{m-3}) \cup (F' \setminus \{e_{m-2}\})$ is a connected subspace of $X \setminus \{e_{m-2}, e\}$ (it is the union of the connected sets $K_i \cup \{e_i\} \cup K_{i+1}$, $i = 0, 1, \dots m-4$ and $K_{m-3} \cup \{e_{m-3}\}$). From the case $m = 2$, we know that $X \setminus \{e_{m-1}, e\}$ has precisely two components M, N; we choose the labelling so that $R \subseteq N$.

Since all of $e_0, e_1, \dots e_{m-3}$ are in N, we have that $M \subseteq X \setminus F$ and M is a component of $X \setminus F = (X \setminus F') \setminus \{e\}$ containing endvertices of precisely e_{m-2} and e. Since K_{m-2} is the only component of $X \setminus F'$ containing an endvertex of e, and e has an endvertex in M, it follows that M is contained in K_{m-2}, and indeed is a component of $K_{m-2} \setminus \{e\}$. Furthermore, no other edge of F has an endvertex in M, so the ends of e_{m-2} and e_{m-3} in K_{m-2} are in different components of $K_{m-2} \setminus \{e\}$. Thus K_{m-2} has precisely two components, namely $K'_{m-1} := M$, containing the end of e_{m-2}, and K'_{m-2}, the one containing the end of e_{m-3}. Thus, we have the cyclic order $\langle K_0, K_1, K_2, \dots, K_{m-3}, K'_{m-2}, K'_{m-1}\rangle$ of the components of

$X \setminus F$ that matches the cyclic order $\langle e_0, e_1, e_2, \dots, e_{m-2}, e, e_{m-1} \rangle$ of the edges of E. ✠

5.1.5 Remark: From the fact that, if $|F| \geq 2$, each edge has precisely 2 endvertices, which belong to distinct components of $X \setminus F$, it follows that if the ternary relation R is a valid choice for the cyclic order in 5.1.4, then so is the ternary relation $\{(x, y, z) \mid (z, y, x) \in R\}$, and that these are the only two different possibilities for the cyclic order. Moreover, given, for any unordered triple, a corresponding ordered triple belonging to the cyclic order, the entire cyclic order is uniquely determined.

Lemma 5.1.4 will come in useful later on, but already it allows us to prove the following equivalence.

5.1.6 ***Theorem:*** *Let X be a connected edge space with $|E_X| \geq 4$. The following are equivalent:*

(I) *X is an edgecycle;*

(II) *there is a cyclic order on E_X such that, if $\langle w, x, y, z \rangle$ is a cyclic subsequence, then x, z separate w, y;*

(III) *X is 2-edge-connected and any set $F \subseteq E_X$ with 4 elements can be partitioned as $F = \{a, b\} \cup \{c, d\}$ so that a, b separate c, d and c, d separate a, b.*

Proof:

(I) $\Rightarrow$ **(II)**: Fix any three distinct edges a, b, c of X and let x, y, z be any three distinct edges of X. Let $F = \{a, b, c, x, y, z\}$. By 5.1.4, there is a cyclic ordering $\langle K_0, \dots, K_m \rangle$ of the components of $X \setminus F$ and a cyclic ordering $\langle e_0, \dots, e_m \rangle$ of F such that each e_i has ends in K_i and K_{i+1} (indices modulo $m+1$). By 5.1.5 there are precisely two ways of choosing the cyclic orders, and only one such that $\langle a, b, c \rangle$ occurs as a cyclic subsequence. We choose this one; to define the cyclic order on E_X, we declare the one of (x, y, z) and (x, z, y) that occurs in $\langle e_0, e_1, \dots, e_m \rangle$ to belong to the ternary relation R.

The facts that exactly one of (x, y, z) and (x, z, y) belongs to R, and that $(x, y, z) \in R \implies (y, z, x) \in R$, follow simply from the fact that $\langle e_0, e_1, \dots, e_m \rangle$ is itself a cyclic ordering. To see that R is a cyclic order, it remains to be shown that (w, x, y) and (w, y, z) both occurring implies (w, x, z) occurs.

So now consider $F = \{a, b, c, w, x, y, z\}$ and apply Lemma 5.1.4 again. Choose the cyclic order S of F so that it has $\langle a, b, c \rangle$ as a cyclic subsequence. Now for any three distinct points $d, e, f \in F$, considering the set $X \setminus \{a, b, c, d, e, f\}$ we see that $(d, e, f) \in S \iff (d, e, f) \in R$. Hence if (w, x, y) and (w, y, z) are both in R, they must also be in S, which is a cyclic order, hence contains (w, x, z), which must therefore also be in R, as required.

Now if $\langle w, x, y, z\rangle$ is a cyclic subsequence, $X \setminus \{w, y\}$ has precisely two connected components, and from 5.1.4 it follows that x and z are in different connected components.

(II) ⇒ (III): Since any 4 edges must fall in *some* cyclic order, this is trivial (note that $\langle a, b, c, d\rangle = \langle b, c, d, a\rangle$).

(III) ⇒ (I): Let e_1, e_2 be arbitrary edges of X. It is sufficient to prove that $X \setminus \{e_1, e_2\}$ is disconnected. Since $|E_X| \geq 4$, there must exist at least two other edges. If e_1, e_2 separate them, the proof is complete. Otherwise, the bipartition of the set of four edges given by (III) has e_1, e_2 in different parts, i.e. there exist separations $\{A_1, B_1\}$ and $\{A_2, B_2\}$ of $X \setminus \{e_2, a\}$ and $X \setminus \{e_1, b\}$ respectively such that $e_1 \in A_1$, $e_2 \in A_2$, $b \in B_1$, $a \in B_2$ and a, b are the two extra edges. Since X is 2-edge-connected, A_1, B_1 are connected and e_2 has an endvertex in A_1. Since A_2 is closed in the closed subset $X \setminus \{b, e_1\}$, and therefore in X, any endvertex of e_2 is in A_2. This shows that $A_1 \cap A_2$ is non-empty. But then $\{A_1 \cap A_2, B_1 \cup B_2\}$ is a separation of $X \setminus \{e_1, e_2\}$. ✠

Note that (II) can be replaced by the following statement:

(II′) there exists a cyclic order on E_X such that, for any four edges a, b, c, d, the two edges a, c separate b, d if and only if $\langle w, x, y, z\rangle$ is a cyclic subsequence for some choice of $w, y \in \{a, c\}$ and $x, z \in \{b, d\}$.

However, (III) can not be replaced by the weaker statement:

(III′) among any four edges there exist two which separate the other two.

For if $X = (V, E, \partial)$ is an edge space with $V_X = \mathbb{R}^+ \cup \{-1, -2\}$, the topology on V_X being the usual Euclidean topology, and $E_X = \{a, b, c, d\}$ with $\partial(a) = \{-1, -2\}$, $\partial(b) = \{-2, 1\}$, $\partial(c) = \{2, -1\}$ and $\partial(d) = \{3, 4\}$, the topology on X being the combinatorial extension, clearly b and c separate a and d, and since E_X is the only set of four distinct edges, (III′) is true. In fact, b, c is the only choice of a pair edges which separate the remaining pair, but a and d do not separate b and c.

Also, since the deletion of any two edges leaves finitely many (at most two!) components, the phrase "p, q separate r, s" in (II), (II′) and (III) can be replaced by "p, q disconnect r, s" .

5.1.2 Edgepaths and uv-edgepaths

One of the main points in the transition from topologized graphs, in particular the paths and cycles considered in Chapter 2, to edge spaces is that the emphasis shifts to edges, as opposed to edges *and* vertices. By analogy with paths, we have the following.

5.1.7 ***Proposition:*** *Let X be a connected hyperedge space. Then the following are equivalent:*

- *there exists a total order $\preceq$ on E_X such that $a \prec b \prec c$ implies that b separates a and c;*
- *there exists a total order $\preceq$ on E_X such that, for any three points a, b, c, b is in between a and b if and only if b separates a and c;*
- *among any three hyperedges, there exists one which separates the other two; and*
- *among any three hyperedges, there exists precisely one which separates the other two.* ✠

We say that an edge space is an edgepath if it satisfies the conditions of Proposition 5.1.7. However, the above characterizations are not strong enough to capture the appropriate "path-like" concept in our context. In Subsection 5.1.3 we give an example of a space (Example 5.1.18) which satisfies the above properties but not the following definition.

5.1.8 Definition: Let P be a connected edge space with $u, v \in V_P$. Then P is a uv-edgepath if every edge of P separates u and v.

Note that if E_P is empty, u and v may be the same point. Also, an edge in a uv-edgepath can never be a loop.

Our application of uv-edgepaths will not require the following proposition. We state it in order to illustrate the analogy with paths.

5.1.9 ***Proposition:*** *Let $X = (V, E, \partial)$ be a connected hyperedge space with $u, v \in V$ and $|E| \geq 3$. The following are equivalent:*

- *X is a uv-edgepath;*
- *for every proper subspace S of X containing V_X, u and v are in different components of S;*
- *there exists a total order $\preceq$ on E such that $x \prec y \prec z$ implies that y separates $\mathbf{K}_V(u) \cup \{x\}$ and $\mathbf{K}_V(v) \cup \{z\}$;*
- *there exists a total order $\preceq$ on E_X such that $x \prec y \prec z$ if and only if y separates $\mathbf{K}_V(u) \cup \{x\}$ and $\mathbf{K}_V(v) \cup \{z\}$.* ✠

It is easy to see that if $\preceq$ is a valid choice for the total order in one of the conditions of 5.1.9, it is also a valid choice for the other, and that the same holds for 5.1.7. Moreover, the conditions in 5.1.9 imply the conditions in 5.1.7, and the total orders may be taken to be the same. Furthermore, if $\preceq$ is a valid choice for any of the four conditions involving total orders, then the total order $\{(a,b) \mid (b,a) \in \preceq\}$ will also be a valid choice, and these are the only possible choices.

5.1.10 ***Proposition:***

(A) *If e is an edge in an edgepath P, then $P \setminus \{e\}$ is the free union of at most two non-empty edgepaths.*

(B) *Let u, v be endvertices of an edge e in an edge space X. Then X is an edgecycle if and only if $X \setminus \{e\}$ is a uv-edgepath.*

Proof: We show part (B), which is the one we need. Suppose X is an edgecyle. Then $X \setminus \{e\}$ is connected but, for any other edge, $X \setminus \{e, f\}$ consists of precisely two components, each containing one endvertex of f. But $X \setminus \{f\}$ is connected, so e is a cutedge of $X \setminus \{f\}$ and u, v belong to distinct components of $X \setminus \{e, f\}$. Since there are only two components, they constitute a separation. Thus, f separates u and v in $X \setminus \{e\}$, which is the required uv-edgepath.

Conversely, suppose $P := X \setminus \{e\}$ is a uv-edgepath. Then for every edge $f \in P$, $X \setminus \{e, f\}$ consists of two components, each one containing one of u, v. The union of these two components with $\mathsf{Cl}(e)$ is connected, and is precisely $X \setminus \{f\}$. Thus no edge disconnects (e itself does not disconnect because P is connected).

Let a_1, a_2 be arbitrary edges; it is now sufficient to show that $X \setminus \{a_1, a_2\}$ is disconnected. This is clear if $e \in \{a_1, a_2\}$. Suppose not; by the above argument, both $\{e, a_1\}$ and $\{e, a_2\}$ disconnect, and there exist closed sets C_1, D_1, C_2, D_2 such that $\{C_1, D_1\}$ is a separation of $X \setminus \{e, a_1\}$ and $\{C_2, D_2\}$ is a separation of $X \setminus \{e, a_2\}$. Each of C_1, C_2, D_1, D_2 contains precisely one endvertex of each of e, a_1, a_2; we may choose the labelling so that $u \in C_1 \cap C_2$.

We claim that, if $A := \mathsf{Cl}(e) \cup (C_1 \cap C_2) \cup (D_1 \cap D_2)$ and $B := (C_1 \cap D_2) \cup (C_2 \cap D_1)\}$, then $\{A, B\}$ is a separation on $X \setminus \{a_1, a_2\}$. Clearly A and B are both closed, A is non-empty, and $\{A, B\}$ is a partition of $X \setminus \{a_1, a_2\}$. To see that B is non-empty, note that a_1 belongs to one of C_2 or D_2. If $a_1 \in C_2$, since A_2 is closed, both endvertices of a_1 are in C_2. But one of these is in D_1. Hence in this case $C_2 \cap D_1$ is non-empty. Similarly, if $a_1 \in D_2$, then $C_1 \cap D_2$ is non-empty. In both cases B is non-empty. ✠

5.1.11 ***Proposition:*** *Let X be a compact connected feebly Hausdorff edge space and u, v points in V_X. Then X contains a closed uv-edgepath.*

Proof: We may assume that $u \neq v$; otherwise the assertion is trivial. By Theorem 4.3.21, X contains an edge-minimal connected set P containing u and v. For any edge $e \in P$, the set $P \setminus \{e\}$ still contains u, v, hence must be disconnected, that is, e is a cutedge of P. Hence by Corollary 2.1.17 $P \setminus e$ consists of two components, which give a separation of $P \setminus e$. Now if u, v belong to the same component, that componentwould be a connected subset of P containing u, v with one edge less than P, a contradiction to the choice of P. So u, v belong to distinct components. Hence e separates u and v, and since $e \in E_P$ is arbitrary, P is a uv-edgepath. From Theorem 4.3.21, P may be taken to b closed. ✠

5.1.3 Plants and Fundamental Cyclesets

5.1.12 Definition: A plant is a connected hyperedge space such that for any hyperedge e, $X \setminus \{e\}$ is disconnected. A plant is a graphic if it is an *edge* space.

In Chapter 6 we shall specialize the notion of a plant to that of a fern. In the following lemma, the equivalence of conditions (A), (B) and (C) is easy, and is essentially the content of Proposition 4.2.2. We shall not make use of condition (D), but we state it for the purposes of comparison with Theorem 6.5.1.

5.1.13 *Lemma:* *Let X be a connected hyperedge space. The following are equivalent:*

(A) *X is a plant;*

(B) *every proper subset of X containing V_X is disconnected (X is its own minimal spanning set);*

(C) *every proper closed subset of X containing V_X is disconnected;*

(D) *there exists a lower-directed binary relation $\trianglelefteq$ on E_X such that:*

- *for any three distinct edges a, b, c, we have that b disconnects a and c if and only if $b \triangleleft x$ holds for precisely one choice of $x \in \{a, c\}$ or else $b = \inf\{a, c\}$.*[2]

Note that a minimal spanning set of a topologized hypergraph is always a plant. The following is an analogue to the graph-theoretic fact that a spanning tree does not contain any cycles.

5.1.14 *Proposition:* *A plant does not contain any cyclesets.*

[2]Note that, contrary to the case of Theorem 6.5.1, we are not claiming the infimum always exists.

Proof: Suppose e is an edge of an edgecycle C, a subgraph of a plant T. Since $T \setminus \{e\}$ is disconnected, by 2.1.16 e must be incident with two vertices u, v in distinct components of $T \setminus \{e\}$. Since C is a subgraph of T, u, v are also vertices in C. But $C \setminus \{e\}$ is a connected subset of $T \setminus \{e\}$ containing u and v, a contradiction. ✠

5.1.15 ***Proposition:*** *Let T be a graphic plant, and suppose that $u, v \in V_T$ belong to a uv-edgepath P. Then the set of edges of P is uniquely determined.*

Proof: Suppose, by way of contradiction, that P_1, P_2 are two uv-edgepaths, and $e \in E_T \cap P_2 \setminus P_1$. Since T is graphic, e is incident with at most two points in P_2. By 2.1.16 $P_2 \setminus \{e\}$ has at most 2 components. Since P is an edgepath, $P_2 \setminus \{e\}$ is disconnected, and so has precisely two connected components, K_u and K_v, containing u and v respectively, and e is incident with precisely two points in P_2, which we shall call u' and v', with $u' \in K_u$ and $v' \in K_v$.

As T is a graphic plant, $T \setminus \{e\}$ is also disconnected, and by the same argument as above has precisely two connected components, U and V, each of which must contain precisely one of u', v'. Since the components of $P_2 \setminus \{e\}$ must be contained in the components of the larger set $T \setminus \{e\}$, we must have $\{u, u'\} \subseteq K_u \subseteq U$ and $\{v, v'\} \subseteq K_v \subseteq V$. However, since $e \notin P_1$ and P_1 is connected, u and v must belong to the same component of $T \setminus \{e\}$, a contradiction. ✠

5.1.16 ***Corollary:*** *Let T be a graphic plant, and suppose that $u, v \in V_T$ belong to a uv-edgepath P. Then the set of edges of P coincides precisely with the set of edges which separate u, v in T.*

Proof: Suppose $e \in E_T$ separates u, v. Then u, v belong to distinct components of $V_T \setminus \{e\}$, and no connected subset of $V_T \setminus \{e\}$ contains u, v. Since P is connected and contains u and v, we have that $e \in P$.

Conversely, suppose $e \in E_P$. Both $T \setminus \{e\}$ and $P \setminus \{e\}$ consist of precisely two distinct components, and since in both cases the two endvertices of e belong to distinct components, each component of $T \setminus \{e\}$ contains one of the components of $P \setminus \{e\}$. But $P \setminus \{e\}$ contains u and v, so if e does not separate u, v in T, that is, u, v belong to the same component of $T \setminus \{e\}$, then they also belong to the same component of $P \setminus \{e\}$, that is, e does not separate u and v in P, a contradiction. So e must separate u and v in T. ✠

5.1.17 ***Corollary:*** *If T is a compact feebly Hausdorff graphic plant and $u, v \in V_T$, then T contains a uv-edgepath P with a uniquely determined edgeset. This edgeset is precisely the set of edges which separate u from v in T. Moreover, P can be chosen to be closed.*

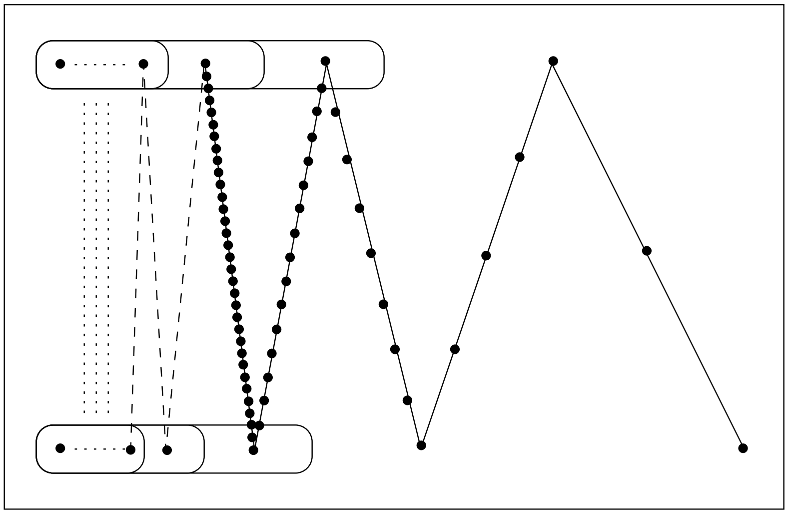


Figure 5.2: "Discrete $\sin(1/x)$". A feebly normal, V-metric, V-zero-dimensional, non-compact, non-locally-connected, non-weakly-normal graphic plant failing Conjecture 4.3.15. A space satisfying the characterizations of Proposition 5.1.7, but not Definition 5.1.8.

Proof: Follows from Proposition 5.1.11, Corollary 5.1.16 and Proposition 5.1.15. ✠

We have already seen an example of a badly behaved plant, in a different context, namely Example 3.6.3. It may appear that the problems with this example are due to the fact that the clumps are allowed to be non-degenerate.

5.1.18 Example: The following construction may be thought of as a "discrete modification" of the well-known "$\sin(1/x)$" construction. The reader is referred to Figure 5.2 for an illustration.

Let P be a simple[3] classical one-way infinite path. Let $\preceq$ be a compatible total order on P, and let $\{v_i\}_{i=0}^{\infty}$ be an enumeration of V_P such that $v_i \prec v_j$ if and only if $i < j$. For $j = 0, 1$, let A_j denote the set $\{v_{2^{2i+j}}\}_{i=0}^{\infty}$. Now we add points a_0, a_1, and define the topology on $T := P \cup \{a_0, a_1\}$ by declaring a subset U to be open if and only if $U \cap P$ is open in P and $a_j \in U$ implies that $U \setminus A_j$ is finite.

It is easy to verify that this defines a feebly normal, V-zero-dimenionsal topologized graph which is not compact and not locally connected. Thus, Conjecture 4.3.15 fails without the assumption of compactness. Moreover, this space is a graphic plant and among any three edges, precisely one separates the other two, that is, this space satisfies

[3]Recall that our definition of "path" allows loops for terminal points.

the conditions of Proposition 5.1.7. However, there is no a_1a_2-edgepath in T and, if T happens to be the minimal spanning set of some larger space which contains an edge e incident with a_1 and a_2, then $T \cup \{e\}$ does not contain an edgecycle (or any apparent cycle-like structure).

The same anomaly (with the unbounded clumps in the rôle of a_1 and a_2) is exhibited by the space of Example 3.6.3. We remark that both these (non-compact) spaces are feebly normal but not weakly normal. This suggests that in a scenario which does not assume compactness, such as the V-zero-dimensional scenario envisaged in the discussion following Example 4.2.9, one needs to resort to the weakly normal, as opposed to feebly Hausdorff, axiom to obtain well-behaved spaces.

5.1.19 ***Corollary:*** *If T is a minimal spanning set of a compact connected feebly Hausdorff edge space and the edge e does not belong to T, then $T \cup \{e\}$ contains a unique cycleset F, and a closed edgecycle C such that $F = E_C$. The cycleset F consists precisely of e and the edges in T which separate the endvertices of e in T.*

Proof: Follows from 5.1.17 and 5.1.10. ✠

5.1.20 Definition: Given a minimal spanning set T of a compact connected feebly Hausdorff edge space and an edge $e \notin T$, the fundamental cycleset induced by e with respect to T is the cycleset determined by 5.1.19. It will be denoted by C_e.

The following is another easy analogy with standard graph theory, which we shall not have occasion to use.

5.1.21 ***Proposition:*** *Given a compact feebly Hausdorff edge space G, a spanning subset T of G is a minimal spanning set if and only if it is maximally acyclic, that is, if every set strictly containing T contains an edgecycle.* ✠

5.2 Cycle Spaces and Bond Spaces

5.2.1 Weak, Algebraic and Strong Spans

Notation: Given a set E any two subsets A, B of E, we denote by $A \Delta B$ the symmetric difference of A and B, that is, the set of points contained in precisely one of A and B. Clearly the Δ operator is associative and commutative. We also denote by $\mathbb{Z}_2^E$ the power set of E.

5.2.1 Definition: Let E be an arbitrary set. A subset S of $\mathbb{Z}_2^E$ will be called boolean if it is closed under taking symmetric differences. We also say that S is a boolean "space". Following Diestel and Kühn [19], we say that a family $F = (A_i)_{i \in I}$ of subsets of E is thin if no point occurs in infinitely many A_i, and in this case we define the linear combination of F to be $\bigwedge_{i \in I} A_i := \{z \in Z \ : \ |\{i \in I \ : \ z \in A_i\}| \text{ is odd}\}$.

It is easy to verify that, if $I = \{1, 2, \ldots, n\}$, then $\bigwedge_{i \in I} A_i = A_1 \,\Delta\, A_2 \,\Delta\, \cdots \,\Delta\, A_n$, and that given any two (index-disjoint) thin families, the linear combination of the union (taken on the index sets) of two thin families is the symmetric difference of the respective linear combinations.

5.2.2 Definition: Given a set E and a subset S of $\mathbb{Z}_2^E$, the algebraic span of S, denoted by $\mathcal{A}(S)$, is the subset of $\mathbb{Z}_2^E$ consisting of linear combinations of thin subfamilies of S. Any subset of the algebraic span of S is said to be algebraically generated by S. If no linear combination of a non-empty, thin subfamily of S consisting of distinct, non-empty subsets of Z is the empty set, then S is algebraically independent, and if this holds for *finite* linear combinations, linearly independent. If S is algebraically independent and the boolean space B coincides with $\mathcal{A}(S)$, then S is an algebraic basis of B.

Note that, if $\{\mathcal{F}_j\}_{j \in J}$ is a collection of subsets of $\mathbb{Z}_2^E$, each closed under $\bigwedge$, then $\bigcap_{j \in J} \mathcal{F}_j$ is closed under $\bigwedge$. The same holds for boolean subsets, that is, for Δ in place for $\bigwedge$. Moreover, $\mathbb{Z}_2^E$ is of course also closed under both operators. Thus, given any set S of subsets of E, there always exists a unique (inclusion-wise) minimal set containing S and closed under the given operator.

We shall call this set the weak span of S, denoted by $\mathcal{W}(S)$, in the case of the Δ operator, and the strong span, denoted by $\mathcal{S}(S)$, in the case of the $\bigwedge$ operator. Any element or subset of the weak (strong) span of S is said to be weakly (strongly) generated by S. If S is linearly independent and the boolean space B coincides with $\mathcal{W}(S)$, then S is a weak basis of B.

Any two subsets of E are orthogonal if their intersection is finite and even. A boolean subset of $\mathbb{Z}_2^E$ is the orthogonal complement of S if it coincides with the set of subsets of E orthogonal with every element of S.[4] Two boolean subsets are an orthogonal pair if they are each other's orthogonal complement.

5.2.3 Note: It is easy to see that $\mathcal{W}(S)$ is the set of linear combinations of finite subsets of S. Moreover, since the symmetric difference of two linear combinations of thin subfamilies of S is a linear combination of a thin subfamily of S, $\mathcal{A}(S)$ is closed under taking symmetric

[4] Note that, if X is the orthogonal complement of Y, then Y is not necessarily the orthogonal complement of X.

differences and therefore contains $\mathcal{W}(S)$. Finally, the fact that $\mathcal{S}(S)$ contains S and is closed under taking linear combinations implies that it contains $\mathcal{A}(S)$. In fact, we have

$$\mathcal{W}(S) \subseteq \mathcal{A}(S) \subseteq \bigcup_{i\geq 1} \mathcal{A}^i(S) = \mathcal{S}(S)$$

where $\mathcal{A}^1(S)$ stands for $\mathcal{A}(S)$ and, for $i \geq 2$, $\mathcal{A}^i(S) = \mathcal{A}(\mathcal{A}^{i-1}(S))$. To see the equality, note that $\mathcal{A}\left(\bigcup_{i\geq 1} \mathcal{A}^i(S)\right) \subseteq \bigcup_{i\geq 1} \mathcal{A}^i(S)$.

Also, note that weak/strong generation is a "transitive relation", that is, if a set A weakly (or strongly) generates a set B, which in turn weakly (respectively, strongly) generates C, then trivially A weakly (strongly) generates C. This, however, fails in the algebraic case. See Example 5.2.4. Thus, while in a sense the notion of strong, or weak, generation of sets is more natural, the statement that a set Z is algebraically generated by another set S is, in general, stronger than the corresponding statement for strong generation; on the other hand, the statement that T algebraically generates $\mathcal{A}(Z)$ is weaker than the corresponding statement for $\mathcal{S}(Z)$.

5.2.4 Example (Infinite bond): Consider the classical graph consisting of two vertices and countably infinitely many edges incident with both vertices. The weak and and algebraic spans of the set of fundamental cyclesets with respect to any spanning tree coincide, and consist precisely of the finite even sets of edges, while the strong span consists of the whole power set of E_B.

On the other hand, if instead we choose the set of all cyclesets as our generating set, then the weak span is the set of all finite even sets of edges, while the algebraic and strong spans coincide with the power set of E_B.

Note that the fundamental cyclesets algebraically generate all the cyclesets, and the cyclesets algebraically generate the power-set of E_B, but this is not algebraically generated by the fundamental cyclesets.

5.2.5 Note: Consider the mapping ϕ defined on $\mathbb{Z}_2^E$ which associates to a subset A of E the characteristic function $\chi_A : Z \to \mathbb{Z}_2$, which is equal to 1 on A and 0 otherwise. The function ϕ is a one-to-one correspondence between $\mathbb{Z}_2^E$ and the vector space U over the field $\mathbb{Z}_2$ of characteristic functions, with the property that $\chi_{A\Delta B} = \chi_A + \chi_B$. In this point of view, the finite linear combinations of 5.2.1 reduce to linear combinations in the usual sense of linear algebra.

Moreover, with respect to the mapping which associates an element $\langle f, g\rangle \in \mathbb{Z}_2$ to a pair (f, g) of characteristic functions whose supports have finite intersection, defined by $\langle f, g\rangle := \sum_{e\in E} f(e)g(e)$, two subsets $A, B \subseteq Z$ are orthogonal (as defined in 5.2.1) if and only if $\langle \chi_A, \chi_B\rangle = 0$. Note that the mapping $(f, g) \mapsto \langle f, g\rangle$ fails to be a non-degenerate bilinear form only in that it not defined on all of $U \times U$.

5.2.2 Bonds and Edgecuts

Recall that, if X is a hyperedge space and A a subset of V_X, then $A^\Box \subseteq A^\diamond$. In general, $A^\diamond$ may be strictly larger than $A^\Box$, but if A is open in V_X, or if V_X is T_1 (X is a topologized graph), then we have that $A^\Box = A^\diamond$. For, if $x \in A^\diamond$, by Lemma 3.1.12 $x \in a^\diamond$ for some $a \in A$. If $x \in E_X$, then x is an edge (combinatorially) incident with a and therefore belongs to $A^\Box$, while if $x \in V_X$ (this can only happen when V_X is not T_1), then the closure of x in V_X contains a, and since A is open in V_X, we have $x \in A$. Moreover, if X is quasiregular and A is open in V_X (but no restriction on V_X), we have that the set $A^\Box = A^\diamond$ is open in X (Corollary 3.4.5), but without quasiregularity, this may fail even if V_X is discrete (Example 3.1.2).

5.2.6 Definition: Given an edge space G and a clopen subset $C \subseteq V_G$, the set $\delta(C)$, consisting of all edges incident with some vertex in C and some vertex not in C, is an edgecut; in view of the above observation, a set of edges is an edgecut if and only if it is either empty or else of the form $A^\Box \cap B^\Box$ (equivalently, $A^\diamond \cap B^\diamond$) for some separation $\{A, B\}$ of V_G. A bond is an (inclusion-wise) minimal non-empty edgecut of a *connected* edge space. The bond space of G, denoted by $\mathcal{B}_G$, is the *weak* span of the *finite* bonds of G.

5.2.7 *Lemma:* *Let X be quasiregular edge space X and $\{A, B\}$ a separation of V_X. If $F = A^\diamond \cap B^\diamond$, we have that $\{A^\diamond \setminus F, B^\diamond \setminus F\}$ is a separation of $X \setminus F$.*

Proof: Since A, B are open in V and X is quasiregular, by 3.4.5 $A^\diamond$ and $B^\diamond$ are open in X, and therefore $A^\diamond \setminus F, B^\diamond \setminus F$ are open in $X \setminus F$. But these sets are also disjoint and non-empty (they contain A, B respectively), hence they give a separation of $X \setminus F$. ✠

5.2.8 *Corollary:* *If F is a non-empty edgecut of a quasiregular edge space X, then $X \setminus F$ is disconnected. Moreover, if X is connected and F is a bond, then $X \setminus F$ has precisely 2 connected components, and every edge in F is incident with some vertex in each of the two components.*

Proof: Since F is non-empty, there exists a separation $\{A, B\}$ of V such that $F = A^\diamond \cap B^\diamond$. Let $\widehat{A}, \widehat{B}$ denote $A^\diamond \setminus F, B^\diamond \setminus F$ respectively. By 5.2.7, $\{\widehat{A}, \widehat{B}\}$ is a separation of $X \setminus F$, which is therefore disconnected.

Now suppose that X is connected, that F is a bond and, by way of contradiction, that $\widehat{B}$ is disconnected, that is, there exists a separation U, W of $\widehat{B}$. Since every edge in F can only be incident with a vertex in one of, but not both, U and W, for one of these two sets (in fact both), say U, the subset $F' \subseteq F$ of edges incident with some vertex in U is strictly contained in F. Let us set $U' := U \cap V$ and $W' := W \cap V$. Note that since U and W are non-empty and any edge must be incident with at least one vertex, U' and W' are both non-empty.

Now since U is clopen in $\widehat{B}$, U' is clopen in $B = \widehat{B} \cap V$, which is clopen in V, so U' is clopen in V. Note that the complement of U' in V is $A \cup W'$, so $\{U', A \cup W'\}$ is a separation of V. Now the set of edges incident with a vertex on either side of this separation is the union of F' together with the set of edges with one endvertex in U' and the other in W'. However, any edge of the latter kind would be contained in $\widehat{B}$ as well as in any open set containing either of $U' \subseteq U$ and $W' \subseteq W$, contradicting the fact that U, W are disjoint. Hence F' is an edgecut strictly contained in F, contradicting the minimality of F. So $\widehat{B}$ must be connected, and similarly we see that $\widehat{A}$ must also be connected. We conclude that $\widehat{A}$ and $\widehat{B}$ are the connected components of $X \setminus F$. Also, $F \subseteq (A^\diamond \cap B^\diamond) \subseteq (\widehat{A}^\diamond \cap \widehat{B}^\diamond)$. ✠

5.2.9 Note: The above corollary says that if a set of edges in a connected edge space is an edgecut, it is a cutset. However, the converse is of course not true, not even in the case of finite classical graphs: in any non-bipartite classical graph, the set of all edges is a cutset but not an edgecut.

Also, if A_1, A_2 are clopen subsets of any topological space X, then $A_1 \Delta A_2$ is also clopen. This simple fact lies behind the Stone duality between Boolean topological spaces and Boolean algebras, but for us it has the simple implication that $\delta(A_1 \Delta A_2) = \delta(A_1)\Delta\delta(A_2)$ is also an edgecut. Thus the set of edgecuts is a boolean space. This makes the proof of the following proposition easy.

5.2.10 ***Proposition:*** *A set of edges of an edge space X is a finite edgecut if and only if it is the disjoint union of finitely many finite bonds. The bond space coincides with the space of finite edgecuts.*

Proof: Suppose $F = \biguplus_{i=1}^{n} B_i$ where, for all $i = 1 \ldots n$, B_i is a finite bond; then $F = B_1 \Delta B_2 \Delta \cdots \Delta B_n$ and therefore is an edgecut. Conversely, suppose F is a finite edgecut; then there exists a minimal edgecut B contained in F. Moreover, $F' := B \Delta F = F \setminus B$ is a finite edgecut of G. If F' is not empty, we can find partition F' into another finite bond and an edgecut within F'; proceeding recursively in this manner (or by induction on the number of edges in F), we can express F as a finite disjoint union of bonds.

Recall that the bond space $\mathcal{B} = \mathcal{W}(\mathcal{F})$, where $\mathcal{F}$ stands for the set of finite bonds, is precisely the set of finite symmetric differences of elements in $\mathcal{F}$. Since a finite disjoint union of sets is trivially a finite symmetric difference of the same sets, from the above we have that any finite edgecut belongs to the bond space. Conversely, since the set of edgecuts is boolean space, any finite symmetric difference of finite bonds is a finite edgecut. ✠

5.2.11 ***Proposition:*** *Let F be a set of edges of a connected quasiregular edge space. Then F is a bond if and only if it is a minimal cutset.*

Proof: Let F be a set of edges in a edge space$X$, and suppose that A is clopen in V_X and $F = \delta(A) = \delta(B)$ is a bond for some separation $\{A, B\}$ of X. Then by 5.2.8 $X \setminus F$ has a separation $\{C, D\}$ with C, D connected, so that in particular F is a cutset. Moreover, for any proper subset F' of F, there exists an edge $e \in F \setminus F'$, which is incident with a vertex in C *and* a vertex in D, so $C \cup D \cup \mathsf{Cl}(e)$ is connected, and therefore so is $X \setminus F' = C \cup D \cup \bigcup_{e \in F} \mathsf{Cl}(e)$. Hence F' is not a cutset and F is a minimal cutset.

Conversely, suppose F is a minimal cutset. Then $X \setminus F$ has a separation $\{C, D\}$, but for every edge $e \in F$, $(X \setminus F) \cup \{e\}$ is connected. So e is a cutedge of $(X \setminus F) \cup \{e\}$, and by 2.1.17 C, D are connected and both contain one endvertex of e. Now since C, D are both closed in $X \setminus F$, $C \cap V_X$ and $D \cap V_X$ are both closed, and therefore clopen, in V_X. Moreover, since C, D are closed, if they contain an edge they contain all its endvertices, and since they are disjoint, every edge containing an endvertex in $C \cap V_X$ and an endvertex in $D \cap V_X$ must be in F. Thus F is precisely the set of edges with one endvertex in $C \cap V_X$ and the one outside; since $C \cap V_X$ is clopen in V_X, F is an edgecut. Now by 5.2.8, any proper non-empty subset of F which is an edgecut would be a cutset, contradicting the fact that F is a minimal cutset. Hence F is a minimal edgecut. ✠

One of the benefits of the above proposition is that it expresses the property of being a bond in the form of a topological condition on the ground space, reducing the rôle of the bipartition into vertices and edges to a bare minimum: apart from the topology, the only "combinatoral" thing we need to know to decide if a given set is a bond is simply whether or not its elements are actually deemed to be edges. This is in contrast with the definition of a bond, which involves edgecuts, and therefore clopen subsets of the vertex set.

This characterization can be useful when we have different edge spaces on the same ground set. The following corollary gives an example of such a context.

5.2.12 ***Corollary:*** *Let G, H be quasiregular edge spaces such that $V_G \cup E_G = V_H \cup E_H$, and suppose $F \subseteq E_H \subseteq E_G$. Then F is a bond of H if and only if it is a bond of G.* ✠

5.2.13 ***Proposition:*** *Let F be a non-empty set of edges in a connected quasiregular edge space G. Then F is an edgecut if and only if there exists a separation $\{A, B\}$ of $G \setminus F$ such that F is precisely the set of edges incident with a vertex in each of A, B.*

Proof: Note that, if F is any set of edges and $\{A, B\}$ is a separation of $X \setminus F$, then $\{A \cap V_G, B \cap V_G\}$ is a separation of V_G (A and B cannot consist solely of edges, since they contain the closure of any edge they contain). Moreover, the set of edges incident with a vertex in each of A and B is precisely the set of edges incident with a vertex in each of $A \cap V_G, B \cap V_G$.

Suppose F is an edgecut of G. Since F is non-empty, there exists a separation $\{C, D\}$ of V_G such that $F = C^\diamond \cap D^\diamond = C^\square \cap D^\square$. We set $A := C^\diamond \setminus F$, $B := D^\diamond \setminus F$. By Lemma 5.2.7 $\{A, B\}$ is a separation of $G \setminus F$. Since F is precisely the set of edges incident with some vertex in each of C, D and $C \cap V_G = A$, $D \cap V_G = B$, we have that F is precisely the set of edges incident with some vertex in each of A, B.

Conversely, suppose there exists a separation $\{A, B\}$ of $G \setminus F$ such that F is precisely the set of edges incident with a vertex in each of A, B. Then $C := A \cap V_G$, $D := B \cap V_G$ is a separation of V_G and $F = \delta(C)$, that is, F is the edgecut consisting of edges incident with a vertex in each of C and D. ✠

The above allows us to deduce the following analogue to Corollary 5.2.12, with just a tiny bit more effort.

5.2.14 ***Corollary:*** *Let G, H be quasiregular edge spaces such that $V_G \cup E_G = X = V_H \cup E_H$, and suppose $F \subseteq E_H \subseteq E_G$. Then F is an edgecut of H if and only if it is an edgecut of G.*

Proof: Clearly the assertion is true if F is empty. If F is a non-empty edgecut of G, by Proposition 5.2.13 there is a separation $\{A, B\}$ of $X \setminus F$ such that F is precisely the set of edges with an endvertex in each of A, B, that is, every edge of F is incident with an endvertex in each of A and B but no other edge of G is. Since H has fewer edges, the latter statement is true *a fortiori* for H. Since X is also the ground space of H, by Proposition 5.2.13 F is an edgecut of H.

The converse follows from the fact that for any separation $\{C, D\}$ of $X \setminus F$, the sets C, D are closed, implying that the (connected) closure of any edge in $E_G \setminus E_H$ is contained entirely in one of C, D. ✠

5.2.15 ***Proposition:*** *Let K be an edge subspace of an edge space G, and suppose C is an edgecut of G. Then $C' \cap K$ is an edgecut of K (possibly empty).*

Proof: Suppose $C' = \delta(S')$ for some clopen subset S' of V_G, and let $C := C' \cap K$. Then $S := S' \cap V_K$ is clopen in V_K. If $e \in C$, then $e \in K$ and since K is subgraph (edge subspace) of G, both endvertices belong to V_K; clearly, the one in S is also in S' and the one not in S is not in S'. Hence $C \subseteq \delta_K(S)$. On the other hand, if $e \in \delta_K(S)$, then the endvertex in S is in S' and the endvertex not in S' is not in S; hence $e \in \delta(S) = C'$ and since obviously $e \in K$ we have $e \in C$. Hence $\delta_K(S) = C$ and C is an edgecut. ✠

5.2.16 ***Proposition:*** *Let F be a set of edges in a compact feebly Hausdorff edge space G, and K a component of $V_G \cup F$. If C is an edgecut of K, then there exists an edgecut C' of G such that $C' \cap F = C$.*

Proof: Note that, since $V_G \cup F$ is closed in G, any component S of the closed subset $V_G \cup F$ is closed in G, whence $V \cap S$ is closed in V_G; in particular V_K is closed in V_G. So if $\{A, B\}$ is a separation of V_K such that $C = \delta_K(A) = \delta_K(B)$, then A and B are closed in V_G. Moreover $x \in V_K \Rightarrow \mathbf{K}_{V_G}(x) \subseteq K$ so A, B are unions of components of V_G which, being closed in G, is itself compact and feebly Hausdorff. By Corollary 4.3.6 there exists a separation $\{\widehat{A}, \widehat{B}\}$ of V_G with $A \subseteq \widehat{A}$ and $B \subseteq \widehat{B}$. Let C' be the edgecut $(\widehat{A}^\diamond \cap \widehat{B}^\diamond) = (\widehat{A}^\Box \cap \widehat{B}^\Box) = \delta(\widehat{A}) = \delta(\widehat{B})$.

For every $e \in F$, let $\mathcal{K}(e)$ denote the component of $V_G \cup F$ containing e. Since $\widehat{A}, \widehat{B}$ are (disjoint) closed, and therefore compact, subsets of the compact feebly Hausdorff space G, by Lemma 4.1.2 there exist open sets $U_{\widehat{A}}$ and $U_{\widehat{B}}$ containing $\widehat{A}, \widehat{B}$ respectively and finite subsets $I \subseteq \widehat{A}, J \subseteq \widehat{B}$ of vertices such that

$$C' = A^\diamond \cap B^\diamond \subseteq U_{\widehat{A}} \cap U_{\widehat{B}} \subseteq \bigcup_{(i,j) \in I \times J} \rho(i) \cap \rho(j).$$

Note that, for every $e \in F \cap C'$, since $\mathcal{K}(e)$ is closed in G, it contains both endvertices of e. It follows that, for every $e_1, e_2 \in C' \cap F, i \in I, j \in J$, we have that $e_1, e_2 \in \rho(i) \cap \rho(j) \Rightarrow i, j \in \mathsf{Cl}(e_1) \cap \mathsf{Cl}(e_2) \Rightarrow \mathcal{K}(e_1) = \mathcal{K}(e_2)$, so the set $\{\mathcal{K}(e)\}_{e \in C' \cap F}$ is finite (since $I \times J$ is finite). Moreover, for every $e \in F \cap C'$, if $\mathcal{K}(e) = K$, then $e \in C$.

Let $R := (C' \cap F) \setminus C$ and $T := \bigcup_{e \in R} \mathcal{K}(e)$. For every $e \in R$, we have that $\mathcal{K}(e) \neq K$. Thus T is a finite union of (closed) components of $V_G \cup F$, each disjoint from K, and is therefore itself closed in $V_G \cup F$ and disjoint from K. But $V_G \cup F$ is compact and feebly Hausdorff, so by Corollary 4.3.6 there exists a clopen subset Z of $V_G \cup F$ containing K and disjoint from T (note T may be empty).

Since Z is clopen in $V \cup F$, for every $e \in F$ we have that $\mathsf{Cl}(e)$ is contained in Z or in $(V \cup F) \setminus Z$. It follows that, if Z' denotes the set $V_G \cap Z$, clopen in V_G, then $\delta(Z')$ is disjoint from F.

Now we claim that we may take L to be $\delta(Z' \cap \widehat{A})$. Clearly $Z' \cap \widehat{A}$ is clopen in V_G and, since both $\widehat{A}$ and Z' contain A but $\widehat{A}$ is disjoint from B, we have that $C \subseteq \delta(Z' \cap \widehat{A})$ (and $C \subseteq F$).

Conversely, suppose $e \in \delta(Z' \cap \widehat{A}) \cap F$. Then e has some endvertex in Z'; since $e \in F$ and $\mathsf{Cl}(e)$ is either contained in, or disjoint from, Z, we have $\mathsf{Cl}(e) \subseteq Z$ and both endvertices of e are in $Z \cap V_G = Z'$. Since e belongs to $\delta(Z' \cap \widehat{A})$, it must have one endvertex in $\widehat{A}$ and the other not in $\widehat{A}$, that is, in $\widehat{B}$, so that $e \in C'$. Hence $e \in C' \cap F$; since $e \in Z$, which is disjoint from R, we have $e \notin R$ and $e \in C$. ✠

5.3 Orthogonality and Cycle Spaces

5.3.1 Weak Orthogonality

5.3.1 Definition: Given an edge space G, the cycle space of G, denoted by $\mathcal{Z}_G$, is the *strong* span of the set of cyclesets.

5.3.2 *Lemma:* *Let G be a compact connected feebly Hausdorff edge space. Then $\forall\ e \in E_G$, either $\{e\}$ is a bond or else $e \in C$ for some cycleset C, but not both.*

Proof: Since G is connected, if $G \setminus \{e\}$ is disconnected for some $e \in E_G$, then $\{e\}$ is a minimal cutset and from Proposition 5.2.11 $\{e\}$ is a bond. So if $\{e\}$ is not a bond, $G \setminus \{e\}$ is connected, and since it is also closed, it is a compact feebly Hausdorff edge subspace; hence if $\{u, v\} = \partial(e)$, by 5.1.11 there exists a uv-edgepath P in $G \setminus \{e\}$, and by 5.1.10 $P \cup \{e\}$ is a cycle in G. Finally, if X is an edgecycle which is an edge subspace of G, and $e \in E_X$, then $X \setminus e$ is connected and therefore so is $G \setminus e$. Thus, e cannot simultaneously constitute a bond and be in a cycleset. ✠

5.3.3 *Corollary:* *A compact connected feebly Hausdorff edge space G is 2-edge-connected if and only if every edge belongs to some cycleset.* ✠

5.3.4 Fact: A classical graph is bipartite if and only if it does not contain any cycle of odd length.

Reference: See [14]. □

5.3.5 *Proposition:* *Let T be an edgecut of a compact feebly Hausdorff edge space G and Z a cycleset. Then T and Z are orthogonal.*

Proof: First we claim that $|T \cap Z|$ is finite. Let $\{K_1, K_2\}$ be a separation of V such that $\delta(K_1) = T$, and let O be a cycle such that $E_O = C$. Being closed subsets in V, which is closed in the compact space G, K_1 and K_2 are compact; hence by 4.1.2 there exist open sets A_1, A_2 containing K_1, K_2 respectively and finite subsets $I \subseteq K_1$, $J \subseteq K_2$ such that

$$T \ = \ K_1^\diamond \cap K_2^\diamond \ \subseteq \ A_1 \cap A_2 \ \subseteq \bigcup_{(i,j)\in I\times J} \rho(i) \cap \rho(j).$$

Note that any clump of O must be contained in a clump of G, and therefore in one of K_1, K_2. From 5.1.3 we see that if $e \in T \cap Z$, then e can not be the only edge of Z and if Z contains only one more edge, then it too must be in T, so that $|T \cap Z| = 2$. So we may assume that $|Z| \geq 3$, so that no two clumps of O together contain all the vertices incident with a given pair of edges. Hence, $\forall\ (i, j) \in I \times J$, $|\rho(i) \cap \rho(j) \cap Z| \leq 1$. From this it follows that $|T \cap Z|$ is finite.

Therefore we may apply Lemma 5.1.4 to O, with $F = T \cap Z$. Let $\mathcal{K}$ be the (finite) set of components of $O \setminus F$. Now by 5.2.7, if F is non-empty, there exists a separation $\{A, B\}$ of $X \setminus F$ such that $\forall\ e \in F$, e is incident with a vertex in both A and B. Moreover, since $O \setminus F \subseteq X \setminus F$, any component in $\mathcal{K}$ is contained in one of A, B. Hence the finite classical graph with $\mathcal{K}$ for the vertices, F for the edges, and with the property that for any $e \in F, v \in \mathcal{K}$, e is incident with v if and only if v contains some vertex incident with e in O, is a bipartite cycle, and therefore by Fact 5.3.4 can only have an even number of edges. ✠

5.3.6 ***Corollary:*** *Suppose Z belongs to the cycle space of a compact feebly Hausdorff edge space. Then Z is orthogonal to every finite edgecut.*

Proof: Recall that, for any subset $\mathcal{C}$ of a given power set, $\mathcal{S}(\mathcal{C}) = \bigcup_{i \geq 0} \mathcal{A}^i(C)$. Taking $\mathcal{C}$ to be the set of cyclesets, there exists an integer i such that $Z \in \mathcal{A}^i(\mathcal{C})$. Our proof is by induction on i.

If $i = 0$, the assertion is a special case of 5.3.5. So suppose the conclusion is true for a specific value of i, and assume that $Z \in \mathcal{A}^{i+1}(\mathcal{C})$, that is, $Z = \bigwedge_{j \in J} Z_j$, for some thin family $(Z_j)_{j \in J}$ of elements of $\mathcal{A}^i(\mathcal{C})$. Let T be any finite edgecut. If we denote by J_e the (finite) subset of indices $j \in J$ such that $e \in Z_j$, and set $J' := \bigcup_{e \in T} J_e$, we have

$$< Z, T > = \sum_{e \in T}^{2} \mathcal{X}_Z(e) = \sum_{e \in T}^{2} \sum_{j \in J_e}^{2} \mathcal{X}_{Z_j}(e) = \sum_{j \in J'}^{2} \sum_{e \in Z_j \cap T}^{2} \mathcal{X}_{Z_j}(e) = \sum_{j \in J'}^{2} < Z_j, T >$$

where the symbol $\sum^2$ denotes addition in $\mathbb{Z}_2$. Note that all sums above, in particular the set J', are finite. By the inductive hypothesis all terms in the last sum are zero, so Z and T are orthogonal. ✠

5.3.7 ***Corollary:*** *For a compact feebly Hausdorff edge space, we have that $\mathcal{Z} \subseteq \mathcal{B}^{\perp}$ and $\mathcal{B} \subseteq \mathcal{Z}^{\perp}$.* ✠

Weakly Hausdorff Cycle and Bond Spaces

5.3.8 ***Lemma:*** *An edgecut of a compact weakly Hausdorff edge space is finite.*

Proof: Let $\{A, B\}$ be a separation of V_X. Since V is closed in X, and A, B are closed in V, they are also closed in X, which is compact. Hence they are compact, and by 4.1.4 there exists open neighbourhoods U_1, U_2 of A, B respectively whose intersection is finite. Hence the edgecut $A^{\diamond} \cap B^{\diamond} \subseteq U_1 \cap U_2$ must be finite. ✠

5.3.9 ***Corollary:*** *The bond space of a compact weakly Hausdorff edge space coincides with the space of edgecuts.* ✠

Another consequence of Lemma 5.3.8 is the following analogue of Lemma 5.3.2.

5.3.10 ***Corollary:*** *Let G be a compact connected weakly Hausdorff edge space. Then $\forall\ e \in E_G$, either $\{e\}$ is a cycleset or else $e \in B$ for some finite bond B, but not both.*

Proof: For any edge e, we have that $\{e\}$ is a cycleset if and only if $\partial(e)$ is contained in some component of V_G. Hence if $\{e\}$ is a cycleset e can not belong to any edgecut. Suppose A, B are components of V_X and e is incident with a vertex in each. Then by Theorem 4.3.3 there exists a separation $\{C, D\}$ of V_G such that $A \subseteq C$ and $B \subseteq D$. Then $e \in \delta(C)$ and by Lemma 5.3.8 $\delta(C)$ is finite. By Proposition 5.2.10 $\delta(C)$ is a disjoint union of bonds and e belongs to one of them. ✠

5.3.11 ***Theorem:*** *Let X be a compact weakly Hausdorff edge space. Then*

(I) *the cycle space of X is the orthogonal complement of the bond space, i.e., $\mathcal{B}^{\perp} = \mathcal{Z}$; and*

(II) *a subset of E_X is in the cycle space if and only if it is a disjoint union of cyclesets.*

Proof: Corollary 5.3.7 implies that $\mathcal{Z} \subseteq \mathcal{B}^{\perp}$. For the converse containment, since a disjoint union of cyclesets is trivially in the cycle space, it is sufficient to show that if Y is an arbitrary set of edges which is orthogonal to every bond (equivalently, every (finite) edgecut), then it is a disjoint union of cyclesets.

Now consider the collection $\mathcal{S}$ of sets of pairwise disjoint cyclesets contained in Y; note that set inclusion gives a partial order on $\mathcal{S}$. Given an arbitrary totally ordered subcollection $\{S_i\}_{i \in I}$ of $\mathcal{S}$, the set $\bigcup_{i \in I} S_i$ is an upper bound in $\mathcal{S}$. By Zorn's Lemma there exists an (inclusion-wise) maximal set $S = \{C_j\}_{j\in J}$ in $\mathcal{S}$. Clearly $Y' := \bigcup_{j\in J} C_j$ is in the cycle space, and therefore (by 5.3.6) orthogonal to every edgecut.

Hence $A := Y\Delta Y' = Y \setminus Y'$ is also orthogonal to every edgecut. We wish to show that A is empty. Suppose there exists an edge $e \in A$, and let $\mathcal{K}_e$ be the component of $V \cup A$ containing e. Being a (closed) component of the closed subset $V \cup A$, $\mathcal{K}_e$ is closed in G, and therefore a compact weakly Hausdorff edge subspace; hence by 5.3.2, either $\{e\}$ is a bond of $\mathcal{K}_e$ or else $e \in C \subseteq A$ for some cycleset C. In the first case, by 5.2.16 there exists an edgecut C' of X such that $C' \cap A = \{e\}$, implying $< C', A > = 1$, a contradiction. In the second case, since $A \subseteq Y$ and C is also a cycleset of X, we would have $S \subsetneq (S \cup \{C\}) \in \mathcal{S}$, contradicting the maximality of S. Hence A is empty and Y is a disjoint union of cyclesets. ✠

5.3.12 ***Corollary:*** *For a compact weakly Hausdorff edge space, the algebraic and strong spans of the cyclesets coincide.* ✠

We shall see that part (I) of 5.3.11, which is a strengthening of 5.3.6, generalizes to feebly Hausdorff edge spaces and, for 2-edge-connected weakly Hausdorff topologized graphs can be further strengthened.

5.3.2 Fundamental Bases

5.3.13 Definition: Let T be minimal spanning set of a quasiregular edge space G. If $f \in E_T$, then by 5.1.13 $T \setminus \{f\}$ is disconnected, and if $\{Y, Z\}$ is a separation of $T \setminus \{f\}$, then $\{Y \cap V_T, Z \cap V_T\}$ is a separation of $V_T = V_X$, which induces an edgecut B_f. Since Y, Z are connected, for any $f' \in B_f$, $Y \cup Z \cup \{f'\}$ is connected, so for any proper subset $F \subseteq B_f$ $X \setminus F'$ is connected, and by 5.2.8 F' is either empty or not an edgecut, meaning that B_f is a bond. By analogy with fundamental cyclesets (defined in 5.1.20), we shall refer to it as the fundamental bond (induced by f, with respect to T), and denote it by B_f.

5.3.14 ***Lemma:*** *Let T be a minimal spanning set of a compact feebly Hausdorff edge space X and e, f arbitrary edges of X with $f \in T, e \notin T$. Then $f \in C_e \Leftrightarrow e \in B_f$.*

Proof: Let A, B be the two connected components of $X \setminus B_f$, and suppose first that $f \in C_e$. Since the two endvertices of f belong to different connected components of $X \setminus B_f$, but also to $C_e \setminus \{f\}$, which is connected, we have that $C_e \setminus \{f\} \nsubseteq X \setminus B_f$. But $C_e \subseteq T \cup \{e\} \subseteq A \cup B \cup \{e, f\}$, so $C_e \setminus \{f\} \subseteq A \cup B \cup \{e\} = (X \setminus B_f) \cup \{e\}$. Hence $e \in B_f$.

Suppose now that $e \in B_f$. Then the endvertices of e are in different components of $X \setminus B_f$, and they also belong to $C_e \setminus \{e\}$, which is connected. Hence, $C_e \setminus \{e\} \nsubseteq (X \setminus B_f)$.

But as above $C_e \setminus \{e\} \subseteq A \cup B \cup \{f\} = (X \setminus B_f) \cup \{f\}$, so $f \in C_e$. ✠

5.3.15 ***Corollary:*** *The family of fundamental cyclesets with respect to a given minimal spanning set T of a compact weakly Hausdorff edge space G is thin and algebraically independent.*

Proof: Any edge $e \notin T$ occurs in precisely one fundamental cycleset, namely C_e. From this it follows that the family of fundamental cyclesets is algebraically independent. If we had $e \in C_f$ for infinitely many $f \in E_G \setminus E_T$, then $e \in T$ and by 5.3.14, $f \in B_e$ for infinitely many $f \in E_G$. But by 5.3.8 all edgecuts, in particular B_e, are finite, a contradiction. ✠

5.3.16 ***Theorem:*** *Given a minimal spanning set T of a compact connected weakly Hausdorff edge space X, the fundamental cyclesets with respect to T constitute an algebraic basis for the cycle space. More precisely, if $Z \subseteq E_X$ is in the cycle space, then*

$$Z = \bigwedge_{e \in Z \setminus T} C_e.$$

Proof: By 5.3.15, the family of fundamental cycles is thin and, for arbitrary Z in the cycle space, $Z' := \bigwedge_{e \in Z \setminus T} C_e$ is well defined. Since Z and Z' both belong to the cycle space, so does $D := Z \Delta Z'$. Note that $\forall e \in Z \setminus T$, the edge e belongs to precisely one of the fundamental cycles $\{C_f\}_{f \in Z \setminus T}$, namely C_e, so that $e \in Z'$. Hence $D \subseteq T$. But by 5.3.11 (II), D is a disjoint union of cyclesets, and by 5.1.14 T does not contain any cyclesets. We conclude that D is empty and $Z' = Z$. ✠

5.3.17 ***Corollary:*** *Given a minimal spanning set of a compact weakly Hausdorff edge space, the algebraic and strong spans of the set of fundamental cyclesets, and the algebraic and strong spans of the set of cyclesets, all coincide.*

Proof: Let F denote the set of fundamental cyclesets, and C the set of cyclesets. Since $F \subseteq C$, we have $\mathcal{A}(F) \subseteq \mathcal{A}(C) \subseteq \mathcal{S}(C)$ and $\mathcal{A}(F) \subseteq \mathcal{S}(F) \subseteq \mathcal{S}(C)$. But Theorem 5.3.16 shows $\mathcal{S}(C) \subseteq \mathcal{A}(F)$. ✠

5.3.18 ***Theorem:*** *Given a minimal spanning set T of a connected quasiregular edge space X, the fundamental bonds with respect to T weakly generate the bond space. More precisely, if $K \subseteq E_X$ is a finite edgecut, then*

$$K = \bigwedge_{e \in K \cap T} B_e.$$

If also X is weakly normal, the fundamental bonds constitute a weak basis for the bond space.

Proof: Let K be a finite edgecut. Then the sum on the right hand side is finite. Let K' be the edgecut $\bigwedge_{e \in K \cap T} B_e$. Note that, for any $e \in E_T$, if $\{A, B\}$ denotes the separation of $T \setminus \{e\}$, then, since A, B are closed, e is the only edge in T incident with a vertex in each of A, B, and therefore the only edge in $B_e \cap T$. Moreover, for any non-empty edgecut F, we must have that $C \cap T \neq \emptyset$, for otherwise T would be a connected subset of $X \setminus F$ containing all the vertices, contradicting Lemma 5.2.7.

Now we claim that, for any $e \in T$, we have that $e \in K \Leftrightarrow e \in K'$. For if $e \in K$, then $e \in B_e$ but $e \notin B_f$ for any other $f \in T$, while if $e \notin K$, then for every $f \in K \cap T$, the edge f is the only one in $B_f \cap T$, whence $e \notin B_f$ and $e \notin K'$.

Thus the edgecut $K \Delta K'$ is disjoint from T, and therefore empty. Hence $K = K'$ is a sum of a (finite) thin family of fundamental bonds; since K is arbitrary, the fundamental bonds weakly generate the bond space, that is, the set of finite edgecuts (by 5.2.10).

If G is weakly normal, the fundamental bonds, being of the form $A^\diamond \cap B^\diamond$ for some separation $\{A, B\}$ of V_G, are finite; hence they are themselves finite edgecuts. Since the bond space is closed under symmetric differences, the weak span of the fundamental bonds is precisely the bond space. Since, for any two fundamental bonds B_e, B_f, we have that $B_e \neq B_f \Rightarrow e \neq f, e \notin B_f, f \notin B_e$, it follows that the set of fundamental bonds is linearly independent. ✠

5.3.3 Orthogonal pairs

5.3.19 ***Theorem:*** *Let X be a compact 2-edge-connected weakly Hausdorff edge space. Then the orthogonal complement of the cycle space coincides with the space of (finite) edgecuts.*

Proof: From Corollary 5.3.7 we have that $\mathcal{B} \subseteq \mathcal{Z}^\perp$. We pick once and for all a minimal spanning set of X (which we know to exist by Corollary 4.3.22). Henceforth in this proof, all references to fundamental cyclesets are with respect to this minimal spanning set; let $\mathcal{F}$ denote the set of these fundamental cycles.

Suppose that A is orthogonal to every element of the cycle space. First we show that A is finite. Suppose, by way of contradiction, that A is infinite.

Consider the simple graph G whose vertex set is $A \cup \mathcal{F}$, and where two vertices v_1, v_2 are adjacent if and only if $v_i \in A$, $v_{3-i} \in \mathcal{F}$ and $v_i \in v_{3-i}$ for some $i \in \{1, 2\}$. Note that $\forall\, e \in A$, $\{e\}$ is not a fundamental cycleset, for otherwise we would have that $< A, \{e\} >= 1$, contradicting the fact that A is orthogonal to every element in the cycle space. Hence A and $\mathcal{F}$ are disjoint. Clearly the bipartition $\{A, \mathcal{F}\}$ of V_G shows G to be a bipartite graph.

Note that, since every element of $\mathcal{F}$ (being a cycleset) is orthogonal to A, in particular $|A \cap v|$ is finite for all $v \in \mathcal{F}$. This means that the degree of v in G is finite. Moreover, since the family of fundamental cyclesets is thin (by Corollary 5.3.15), any edge is contained in only finitely many fundamental cyclesets. This means that the degree in G of all vertices in A is finite, so G is locally finite. Furthermore, since X is 2-edge-connected, by 5.3.3 every edge belongs to some cycleset, and since every cycleset (being in the cycle space) is generated by the fundamental cyclesets (by Theorem 5.3.16), every edge belongs to some fundamental cycleset. Hence the set $\mathcal{F}'$ of vertices in $\mathcal{F}$ adjacent with some vertex in A (the neighbour set of A) is infinite.

Now we inductively construct an infinite sequence $\{w_i\}_{i=1}^{\infty}$ of vertices in $\mathcal{F}'$ and an infinite increasing sequence of finite subsets $\{S_i\}_{i=1}^{\infty}$ of $\mathcal{F}'$ in the following way. We pick an arbitrary vertex w in $\mathcal{F}'$ and set $S_1 = \{w\}$ and $w_1 = w$. Given S_i, we consider the set S_i' of vertices in $\mathcal{F}'$ at distance at most two from S_i (so S_i' is the neighbour set of the neighbour set of S_i). Since all degrees are finite, and S_i is finite, S_i' is also finite. But $\mathcal{F}$ is infinite, so there exists a vertex in $\mathcal{F}' \setminus S_{i'}$; we take w_{i+1} to be any such vertex and set $S_{i+1} := S_i' \cup \{w_{i+1}\}$. Clearly $S_i \subseteq S_{i+1} \subseteq \mathcal{F}'$. Now w_{i+1} does not have any neighbours in common with any vertex in S_i, in particular with any vertex w_j, for $j < i$ (note that $w_i \in S_i$ by definition, not even by inductive hypothesis).

So $\{w_i\}_{i=1}^{\infty}$ is an infinite set of fundamental cyclesets all of which contain some edge in A, but no two of which contain an edge of A in common. Hence their sum (which is well-defined since the family of fundamental cyclesets is thin) has infinite intersection with A, contradicting the fact that A is orthogonal to every element in the cycle space. We conclude that A must be finite.

It now follows from Lemma 2.1.16 that $X \setminus A$ has finitely many components. We consider a second graph H whose vertices are these components and whose edges are the edges of A, the incidence function being that which assigns to $e \in A$ the (set of one or two) components containing its boundary points. We consider the finite graph H to be equipped with the classical topology, so that the term "cycle" (in H) has the usual meaning in graph theory.

We claim that if C is a cycle in H, consisting of vertices $v_0, v_1, \ldots, v_{k-1}$ and edges $e_0, e_1, \ldots, e_{k-1}$, the edge e_i (for $i = 0 \ldots k-1$) being incident with v_i and v_{i+1} (here and henceforth in this proof, all operations on indices should be taken modulo the integer k), then there exists an edgecycle D in the subspace $\left(\bigcup_{i=0}^{k-1} v_i \right) \cup \{e_i\}_{i=0}^{k-1}$ of X such that $E_X \cap D \supseteq \{e_i\}_{i=0}^{k-1}$.

For all $i = 0, 1, 2, \dots k$, let y_i, z_i be the boundary points in v_i of e_{i-1} and e_i respectively. By Proposition 5.1.11, for all $i = 0, \dots k-1$, there exists a $y_i z_i$-edgepath P_i in v_i. We set D to be $\{e_i\}_{i=0}^{k-1} \cup \bigcup_{i=0}^{k-1} P_i$. We need to verify that D is an edgecycle. Since D is the union of a finite sequence of connected sets, with successive pairs of connected sets having non-empty intersection, from Fact 2.1.1 we see, using the fact that the union of an edge with its two boundary points is connected, that D is connected. To see that it is 2-edge-connected, consider any edge $e \in D$. If $e \in P_j$ for some $j \in \{0, 1, \dots, k-1\}$, then $P_j \setminus e$ consists of two components Y_i, Z_i containing y_i and z_i respectively, and applying successively Fact 2.1.1 once more (taking Z_i as the first connected set, then the closures of the e_i's and the v_i's in alternating fashion, until e_{j-1}, and finally Y_j) shows that $D \setminus e$ is still connected. If $e \in A \cap D$, then the same argument (with a more obvious choice of connected sets) again shows that $D \setminus e$ is connected. We also need to show that for any pair e, f of edges, $D \setminus \{e, f\}$ is disconnected. There are three cases: both edges are in A, neither is, or only one is. In all three cases it is easy to express $D \setminus \{e, f\}$ as a disjoint union of two closed sets, using the fact that the P_i's are closed (in $D \setminus A$) and that the union of an edge with its two boundary points is also closed.

So D is indeed an edgecycle and $< E_D, A >= k$. Hence k is even, and since the cycle C in H is arbitrary, from Fact 5.3.4 we conclude that H is bipartite. Note that the above argument covers the case $k = 1$, that is, the possibility that H contains a loop, in which case D would be $P_0 \cup \{e_0\}$, and P_0 could (in principle) be a singleton if $u_0 = v_0$.

Now let $\{M, N\}$ be a bipartition of H such that all edges of A are incident with a vertex in each of M, N. Since we may assume that A is non-empty (otherwise it is trivially in the bond space), we may also assume that M, N are both non-empty. So the subsets $M' := \bigcup_{v \in M} v$, $N' := \bigcup_{v \in N} v$ of X are also non-empty. Since they are finite unions of components of $X \setminus A$, they are both closed in $X \setminus A$, and $\{M', N'\}$ is a separation of $X \setminus A$. Since the endvertex in H of any edge $e \in A$ contains a boundary point of e in X, we have that $M' \cap V_X, N' \cap V_X$ are non-empty, $\{M' \cap V_X, N' \cap V_X\}$ is a separation of V_X and $A \subseteq \delta(M' \cap V_X)$. On the other hand, the closure of any edge not in A is contained in some single component of $X \setminus A$, and in particular can not contain vertices of X in each of $M' \cap V_X, N' \cap V_X$. Hence $A = \delta(M' \cap V_X)$ is an edgecut. Since A is an arbitrary set of edges orthogonal to the cycle space, we conclude that this orthogonal complement is contained in, and therefore coincides with, the space of (finite) edgecuts. ✠

5.3.20 ***Corollary:*** *The cycle space and the space of edgecuts of a compact 2-edge-connected weakly Hausdorff edge space are an orthogonal pair.* ✠

5.4 Quotients and Cycle Spaces in Feebly Hausdorff Spaces

5.4.1 ***Theorem:*** *Given a minimal spanning set T of a compact connected feebly Hausdorff topological space X and a cycleset C in X, the set C is a sum of fundamental cyclesets with respect to T. More precisely, we have*

$$C = \bigwedge_{e \in C \setminus T} F_e.$$

Proof: Since, for all $e \notin T$, we have $F_e \subseteq (T \cup \{e\})$, we have $\bigcup_{e \in C \setminus T} F_e \subseteq (C \cup T)$, and it is sufficient to show that any edge f in $C \cup T$ appears in an odd number of F_e's if and only if it belongs to C.

If $f \in C \setminus T$, f belongs to C_e if and only if $e = f$; hence, f appears only once. Suppose $f \in T$; then by 5.3.14 f appears precisely for those e's in $B_f \cap (C \setminus T)$. Now $B_f \cap T = \{f\}$ and by 5.3.5 $B_f \cap C$ is even. Hence f appears an odd number ($|B_f \cap C| - 1$) of times if it belongs to C (note in particular that in this case $f \in (B_f \cap C) \Rightarrow |B_f \cap C| \geq 2$), and an even number ($|B_f \cap C|$) of times otherwise. ✠

5.4.2 ***Corollary:*** *Given a minimal spanning set T of a compact connected feebly Hausdorff edge space X, the fundamental cyclesets strongly generate the cycle space.* ✠

We now use the above result to obtain information about the relationship between the cycle space of a compact feebly Hausdorff edge space G and that of its clump-quotient Q. Note that, in this context, although the quotient map gives an obvious one-to-one correspondence between the edgesets of G and that of its clump quotient, this does not translate to a one-to-one correspondence between the cyclesets of G and those of Q—although the inverse image of an edgecycle in the quotient is an edgecycle in G (as we shall see in this section), it is *not* true that the image of an edgecycle in G is necessarily an edgecycle in Q. The reader is referred to Example 5.1.2 and Figure 5.1, where the clump quotient is a classical graph consisting of a cycle of length 3 and a loop, for an illustration of this.

5.4.3 ***Theorem:*** *Let $q : G \to Q$ be the quotient map of a compact feebly Hausdorff edge space onto its clump quotient Q. Then Q is a compact feebly Hausdorff topologized graph, and q induces:*

(I) *a one-to-one correspondence ϕ from the minimal spanning sets of G to the minimal spanning sets of Q;*

(II) *for any minimal spanning set T of G, a one-to-one correspondence from the fundamental cyclesets of G with respect to T to those of Q with respect to $\phi(T)$;*

(III) *an injection from the cyclesets of Q to those of G;*

(IV) *an isomorphism between the cycle space of G and that of Q;*

(V) *a one-to-one correspondence between the edgecuts of G and the edgecuts of Q.*

Proof: From Corollary 4.3.20 Q is a compact feebly Hausdorff topologized hypergraph, and from Proposition 3.5.7 it is a topologized graph. To prove (I), we need to show that, for any $F \subseteq E$, the set $V_G \cup F$ is a minimal spanning set of G if and only if $V_Q \cup F$ is a minimal spanning set of Q. For any subset F' of E, we have that $q^{-1}(V_Q \cup F') = V_G \cup F'$, so by Corollary 4.3.20 $V_G \cup F'$ is connected if and only if $V_Q \cup F'$ is connected. This is true in particular for $F' = F$ and F' of the form $F \setminus e$ for $e \in F$. This concludes the proof of (I), and we may take $\phi(V_G \cup F) = V_Q \cup F$.

Now we show (III). Let C be a cycleset in Q, and S an edgecycle in Q such that $E_S = C$. Then C is also the set of edges of $S' := q^{-1}(S)$ in G. We claim that $q^{-1}(S)$ is an edgecycle. Since S is connected, from Corollary 4.3.20 S' is connected. Moreover, for any $e \in C$, since $S \setminus e$ is connected, again from 4.3.20 $q^{-1}(S \setminus \{e\}) = S' \setminus \{e\}$ is connected. Finally, for any two distinct edges $e, f \in C$, since $S \setminus \{e, f\}$ is disconnected, using 4.3.20 once more we see that $S' \setminus \{e, f\} = q^{-1}(S \setminus \{e, f\})$ is also disconnected. So S' is indeed an edgecycle, and C a cycleset in G. So the required injection is in fact the identity (modulo the identification of the edgesets) on the set of cyclesets of Q.

In order to show (II), let F be a set of edges such that $V_G \cup F$, $V_Q \cup F$ are minimal spanning sets of G, Q respectively. Then for all $e \in E \setminus F$, let A_e, B_e be the fundamental cyclesets in $V_G \cup F \cup \{e\}$ and $V_Q \cup F \cup \{e\}$ respectively. Clearly $A_e \mapsto B_e$ is a one-to-one correspondence. We need to show that it is in fact the one induced by q, that is, that $A_e = B_e$. From (III), we know that B_e is a cycleset in G, in fact in $V_G \cup F \cup \{e\}$, but A_e is the unique such cycleset (by Corollary 5.1.19), so $A_e = B_e$.

For (IV), note G is compact and feebly Hausdorff, so from Corollary 4.3.22 we have that G has a minimal spanning set, and the corresponding family of fundamental cyclesets. By (I), Q does as well. Since the one-to-one correspondence of (II) is in fact the identity on the set of fundamental cyclesets, (IV) follows from the fact that the fact that the cycle space is the strong span of the fundamental cycles (Corollary 5.4.2).

To prove (V), we need to show that a subset $F \subseteq E$ is an edgecut of G if and only if it is an edgecut of Q. If F is an edgecut of Q, by Proposition 5.2.13 there exists a separation $\{A, B\}$ of $G \setminus F$ such that F is precisely the set of edges of Q incident with one endvertex in each of A, B. Since A, B are closed in the closed set $Q \setminus F$, they are closed in Q and since q is continuous, $q^{-1}(A), q^{-1}(B)$ are closed in $G \setminus F$ and give a separation of $G \setminus F$. From Proposition 3.5.7 it follows that an edge in G is incident with a vertex in $q^{-1}(A)$ (or $q^{-1}(B)$) in G if and only if it is incident with a vertex in A (respectively B) in Q, so F is the set of edges in G incident with an endvertex in each of $q^{-1}(A), q^{-1}(B)$. Thus, by 5.2.13 F is an edgecut of G.

Conversely, if F is an edgecut of G, by Proposition 5.2.13 there exists a separation $\{C, D\}$ of $G \setminus F$ such that F is precisely the set of edges of Q incident with one endvertex in each of C, D. Again C, D are closed in G, and by Corollary 4.3.20, q is closed, so $q(A), q(B)$ are closed in Q. Moreover, q is injective "across clumps", that is, $q(a) = q(b)$ implies that a and b belong to the same clump, and therefore to the same component of $G \setminus F$. So if $a \in A, b \in B$, we have $q(a) \neq q(b)$; hence $q(A),\ q(B)$ are disjoint, $\{q(A), q(B)\}$ is a separation of $Q \setminus F$ and $A = q^{-1}(q(A)),\ B = q^{-1}(q(B))$. Hence, using Proposition 3.5.7 once more, F is the set of edges of Q incident with a vertex in each of $q(A), q(B)$, so that, again by 5.2.13, F is an edgecut of Q. ✠

5.4.4 Note: Note that the mappings "induced by q" in (III), (IV) and (V) of the above theorem are actually the identity (modulo the usual identification of the edges of an edge space and those of its clump quotient). In particular, from the definition of bond it follows that the one-to-one correspondence of (V) restricts to a one-to-one correpondence between the (finite) bonds of G and of those of Q, and therefore an isomorphism between the bond spaces of G and Q.

5.4.5 Definition: Let G be a feebly Hausdorff edge space, and F denote the set of proper edges of G. Consider the equivalence relation on F given by $e_1 \sim e_2 \iff \partial(e_1) = \partial(e_2)$.[5] An infinite parallel class of G is an infinite equivalence class with respect to this equivalence relation. An edge is inessential if it belongs to some infinite parallel class, and essential otherwise. Now we denote by E' the set of essential edges and by H the edge space (V', E', ∂), where $V' = V_G \cup E \setminus E'$, that is, the complement of E' in G. So H is a edge space with the same topological space as G for the ground space, but fewer edges. The essential quotient of G is the clump quotient of the edge space H.

Recall that, in a compact space, every sequence has a cluster point (Fact 2.3.16). We shall use this in the proof of the following theorem, to show that the quotient is *weakly*, as opposed to feebly, Hausdorff. This could also probably be shown using Lemma 4.1.2.

[5] So this equivalence relation can be thought of as the restriction of the parallel decomposition (as defined in Definition 4.2.20) of X to F, via the usual correspondence between equivalence relations and partitions. Note however that the clumps are unaffected.

5.4.6 ***Theorem:*** *Let G be a compact feebly Hausdorff edge space, Q its essential quotient, and Z_G, Z_Q the cycle spaces of G, Q, respectively. Then Q is a compact weakly Hausdorff topologized graph, and we have that*

$$Z_G \cong Z_Q \oplus \bigoplus_P \mathbb{Z}_2^P = Z_Q \oplus \mathbb{Z}_2^N,$$

where the direct sum $\bigoplus$ ranges over all infinite parallel classes P of G, and N denotes the set of all inessential edges of G. On the other hand, the finite bonds, and bond spaces, of G and Q are precisely the same.

Proof: Let H be the edge space with $V_H = V_G \cup N$ and $E_H = E_G \setminus N$. Then Q is the clump quotient of H. Since H is feebly Hausdorff, from Theorem 5.4.3 Q is a compact feebly Hausdorff topologized graph. From Remark 3.1.19, in order to show that Q is weakly Hausdorff, it is sufficient to show that it is finitely adjacent, that is, that it does not contain an infinite parallel class (note V_Q is T_1).

Suppose that P is an infinite parallel class of Q, and let v_1, v_2 be the endvertices of the edges in P, and K_1, K_2 their respective pre-images in G. Then from Proposition 3.5.7, we have that, for every $e \in P$, there exist vertices $y_e^{(1)} \in (V_1 \cap \partial(e))$, $y_e^{(2)} \in (V_2 \cap \partial(e))$. Since all edges of Q are essential in G, no pair $\{y_e^{(1)}, y_e^{(2)}\}$ occurs infinitely often as e ranges in P. Hence, for some $i \in \{1, 2\}$, there exists an infinite sequence $\{y_{e_j}^{(i)}\}_{j\in\mathbb{P}}$ of distinct points in K_i. But K_i is a component of, and therefore closed in, V_G, which is closed in G; hence K is closed in G, which is compact. So K_i is compact. By Fact 2.3.16 the sequence $\{y_{e_j}^{(i)}\}_{j\in\mathbb{P}}$ must have a cluster point z in K_i. Now we can not exclude that $z = y_{e_k}^{(i)}$ for some $k \in \mathbb{P}$, but this can only happen for one value of k. So we set $J := \{j \in \mathbb{P} \mid y_{e_j}^{(i)} = z\}$ (note J is either empty or a singleton) and $\mathbb{P}' := \mathbb{P} \setminus J$. Then z is still a cluster point of $\{y_{e_j}^{(i)}\}_{j\in\mathbb{P}'}$. Since, $\forall j \in \mathbb{P}'$, it is distinct from y_{e_j}, and since y_{e_j} is the *only* endvertex of the edge e_j in K_i, for all $j \in \mathbb{P}'$ the point z is not incident with the edge e_{i_j}. But Q is S_1 and feebly Hausdorff, that is, an S_2 space, as well as compact, so by quasiregularity (4.1.6 and 3.4.14) z should be in the closure of the set $\{y_{e_j}^{(3-i)}\}_{j\in\mathbb{P}'} \subseteq K_{3-i}$, contradicting the fact that K_{3-i} is closed and disjoint from K_i.

Let Z_H denote the cycle space of H. Thanks to (IV) of 5.4.3, in order to prove the claimed relationship between cycle spaces, it is sufficient to show that a subset $F \subseteq E_G$ is in Z_Q if and only if it is of the form $F_N \cup F_H$ for subsets $F_N \subseteq N$ and $F_H \in Z_H$. One direction is trivial: let S be an edgecycle in G; then it is of course a connected subset of H, and the separation conditions on the edges still hold on the subset of edges of S in H. Hence if C is the cycleset corresponding to S in G, we have that $C = (C \cap N) \cup (C \cap E_H)$ is of the required form. Since this holds for any cycleset C, any element of Z_G can be expressed in the required form.

For the converse, first we observe that, for any two edges e, f belonging to the same infinite parallel class, $\mathsf{Cl}(\{e, f\})$ is an edgecyle, so that $\{e, f\}$ is a cycleset of G. Moreover, for any edge e_0 belonging to an infinite parallel class P, there exists a sequence $\{e_i\}_{i\in\mathbb{P}}$ of distinct edges in P such that $e_0 \neq e_i$ for all $i \in \mathbb{P}$, so that if we take C_i to be the cycleset $\{e_i, e_{i+1}\}$ (for all non-negative integers i), we have that $\bigwedge_{i=0}^{\infty} C_i = \{e_0\}$, which implies that $\{e_0\}$, and similarly $\{e\}$ for any $e \in P$, is in Z_G, whence so is any subset of N.

Since H is compact and feebly Hausdorff, by Corollary 4.3.22 there exists a subset $D \subseteq E'$ such that $T_H := V_H \cup D$ is a minimal spanning set of H. By 5.1.17, for every $e \in E' \setminus D$ there exists a fundamental cycleset Y_e of H with respect to T_H; this set consists precisely of e and all the edges of H in T_H which separate the endvertices of e in T_H. Now consider the subspace T_H as a topologized subgraph T' of G (so all the inessential edges of G are edges of T'). Since its vertex set is precisely V_G, from 3.3.6 we have that T' is compact. Of course, being a minimal spanning set of H, it is connected, and being a subspace of G, it is feebly Hausdorff. Hence from Corollary 4.3.22 there exists a subset $D' \subseteq D \cup N$ such that $R := V_G \cup D'$ is a minimal spanning set of T'. Now for any $e \in D$, since $T_H \setminus \{e\}$ is disconnected, any subset of T_H containing the two boundary points of e, which are in R, is not connected. So $e \in R$ and $D \subseteq D'$.

Note that, since T' is a spanning edge subspace of G (that is, V_G is the vertex set of T' and every edge of G in T' is an edge of T'), R is also a minimal spanning set of G. Using Corollary 5.1.17 again, for every $e \in E' \setminus D$ there exists a fundamental cycleset Y'_e of G with respect to R; this set, apart from e itself, contains precisely the edges of G in R which separate the endvertices of e in R. Now if $f \in Y_e \cap T_H$, then it separates these endvertices in T_H, and since $f \in D \subseteq D' \subseteq R \subseteq T_H$, the edge f also separates these endvertices in R; hence $Y_e \subseteq Y'_e$. Thus the T_H-fundamental cycleset induced by $e \in E' \setminus D$ is contained in the T_G-fundamental cycleset induced by e. The extra edges are all inessential.

In order to conclude the proof of the equality, let $A := A_N \cup A_H$ for some $A_N \subseteq N$ and $A_H \in Z_H$. Since the family $\{Y_e\}_{e\in E'\setminus D}$ strongly generates Z_H and the cyclesets Y'_e belong to Z_G, there exists some set $A' \in Z_G$ such that $A' = A_H \cup B_N$, for some $B_N \subseteq N$. But $W := B_N \Delta A_N \subseteq N$ is also in Z_G and therefore $A'\Delta W = (A_H \cup B_N)\Delta W = A_H \cup (B_N \Delta W) = A_H \cup (B_N \Delta B_N \Delta A_N) = A_H \cup A_N = A$ belongs to Z_G, as required.

Finally, we claim that the bond space of G coincides with the bond space of H. Clearly if an edgecut contains an inessential edge, it contains the entire infinite parallel class it belongs to, so a finite edgecut does not contain any inessential edges. Thus, if K is a finite bond of G, then $K \subseteq E_H$. But by Proposition 5.2.11, a set of edges is a bond if and only if it is a minimal cutset, and the ground sets of G and H are the same, so K is a finite bond of H. Conversely, if K is a finite bond of H, then, since $E_H \subseteq E_G$, by Corollary 5.2.12 K is a finite bond of G. Hence the finite bonds of G and H are precisely the same. But from (V) of Theorem 5.4.3 (and Note 5.4.4) the bonds of H and those of Q are precisely the same. Hence the finite bonds, and bond spaces, of G and Q are precisely the same. ✠

5.4.7 Definition: Given a edge space G, the block restriction of G is the edge space whose ground space is the same as that of G, but whose edges are precisely the edges of G which are not cutedges of G.

Note that given a connected edge space G, the block restriction of G is 2-edge-connected.

5.4.8 ***Theorem:*** *Let G be a feebly Hausdorff edge space. Then the orthogonal complement of the bond space is the cycle space, that is, $\mathcal{B}_G^\perp = \mathcal{Z}_G$, and the bond space is contained in the orthogonal complement of the cycle space, that is, $\mathcal{B}_G \subseteq \mathcal{Z}_G^\perp$. On the other hand, we have that $\mathcal{Z}_G^\perp = \mathcal{B}_S \oplus \mathbb{Z}_2^C$, where C is the set of cutedges of G and S is the block restriction of G. In particular, if G is 2-edge-connected, the bond space and the cycle space are an orthogonal pair.*

Proof: Note that, if $A \subseteq B$, and $\mathcal{L}$ is a set of subsets of A, then the orthogonal complement of $\mathcal{L}$ in A is of course different from the orthogonal complement in B. For the purposes of this proof we shall denote the orthogonal complement of a set by the operator Γ, and specify which space the operator is taken in by means of a subscript, e.g. $\Gamma_A(\mathcal{L})$ denotes the complement taken in A. Note that, when $\mathcal{L}$ consists of subsets of A, we have that $\Gamma_B(\mathcal{L}) \cong \Gamma_A(\mathcal{L}) \oplus \mathbb{Z}_2^{B \setminus A}$.

Let M, N denote the sets of essential and inessential edges of G, respectively, and Q the essential quotient of G. From Theorem 5.4.6 we have that $\mathcal{B}_G = \mathcal{B}_Q$. Moreover, since Q is weakly Hausdorff (again by Theorem 5.4.6), from Theorem 5.3.11 we know that $\Gamma_M(\mathcal{B}_Q) = \mathcal{Z}_Q$. So $\mathcal{B}_G^\perp$, which can be rewritten as $\Gamma_E(\mathcal{B}_Q)$, is isomorphic to $\Gamma_M(\mathcal{B}_Q) \oplus \mathbb{Z}_2^N = Z_Q \oplus \mathbb{Z}_2^N$. But from Theorem 5.4.6 this is isomorphic to $\mathcal{Z}_G$.

The fact that $\mathcal{B}_G \subseteq \mathcal{Z}_G^\perp$ was already observed in Corollary 5.3.7. We need to describe $\mathcal{Z}_G^\perp$. Since $\mathcal{Z}_G = \mathcal{Z}_Q \oplus \mathbb{Z}_2^N$, we have $\mathcal{Z}_G^\perp \subseteq \mathbb{Z}_2^M$. Therefore the orthogonal complement $\mathcal{Z}_G^\perp = \Gamma_E(\mathcal{Z}_G)$ is the same as $\Gamma_M(\mathcal{Z}_Q)$.

Note that, since the finite bonds of G are precisely the finite bonds of Q (Theorem 5.4.6, once more), the cutedges of G and of Q are precisely the same. Let C denote the common set of cutedges. Since a cutedge must separate its two endvertices, all cutedges are essential, that is, $C \subseteq M$. Let $D := M \setminus C$ and R be the block restriction of Q, so that $E_R = D$.

From 5.3.2 we have that no cutedge is in a cycleset (or an edgecycle). Hence a set of edges is a cycleset of Q if and only if it is a cycleset of R. So $\mathcal{Z}_Q = \mathcal{Z}_R$. But R is weakly Hausdorff and 2-edge-connected, so by Theorem 5.3.19 $\Gamma_D(\mathcal{Z}_R) = \mathcal{B}_R$, whence $\Gamma_M(\mathcal{Z}_Q) = \mathcal{B}_R \oplus \mathbb{Z}_2^C$. It is now sufficient to show that $\mathcal{B}_R = \mathcal{B}_S$, where S is the block restriction of G. Recall that the bonds of G and Q are precisely the same (from Theorem 5.4.6).

Note that the edgeset of S, namely $D \cup N$, contains D, the edgeset of R. However, the ground sets of R and S are not the same. Let F be a finite bond of R. Then by Corollary 5.2.12 it is a finite bond of Q, hence of G, and since $F \subseteq E_S$, by Corollary 5.2.12 also of S. Conversely, if F is a finite bond of S, by Corollary 5.2.12 it is a finite bond of G, hence of Q. Using again the fact that the finite edgecuts of G do not contain any inessential edges of G, we conclude that $F \subseteq E_S \setminus N = D = E_R$. So by Corollary 5.2.12 F, being a finite bond of Q, is a finite bond of R. We conclude that $\mathcal{B}_R = \mathcal{B}_S$ and $\mathcal{Z}^\perp = \mathcal{B}_S \oplus \mathbb{Z}_2^C$. If G is 2-edge-connected, C is empty, $S = G$ and $\mathcal{Z}_G^\perp = \mathcal{B}_G$ which, coupled with the fact that $\mathcal{B}_G^\perp = \mathcal{Z}_G$, means that $\mathcal{Z}_G$ and $\mathcal{B}_G$ are an orthogonal pair. ✠

5.5 Partial results

In this section we present two results which address two issues arising from the work in the previous sections: normality versus compactness, and strong versus algebraic spans.

5.5.1 Restrictions on V

Removing compactness

The major problem in replacing the assumption of compactness with some form of normality to obtain results similar to the ones in this chapter lies in the difficulty (or impossibility) of obtaining fundamental cyclesets (minimal spanning sets can easily be obtained, as we observed in 4.2.15).

Let us say that an edge space is clump-totally-separated if for every two distinct components A, B there exists a separation $\{C, D\}$ of V_G such that $A \subseteq C$ and $B \subseteq D$.

5.5.1 *Lemma:* *Let T be a weakly normal, clump-totally-separated, S_1 graphic plant. Then for any two clumps of T, there exists an edge which separates them.*

Proof: Suppose that, for all edges $e \in E_T$, the two clumps U, V belong to the same component of $T \setminus \{e\}$. Since T is clump-totally-separated, there exists a separation $\mathcal{U}, \mathcal{V}$ of V_T with $U \subseteq \mathcal{U}$ and $W \subseteq \mathcal{W}$. From the fact that T is weakly normal and S_1, it follows that T is feebly regular, and in particular it is quasiregular. Since $\mathcal{U}$ and $\mathcal{V}$ are open in V, $\mathcal{U}^\diamond$ and $\mathcal{V}^\diamond$ are open in T. Clearly they are non-empty, and since all edges are incident with some vertex, and $\mathcal{U}, \mathcal{V}$ together cover V_T, they cover T.

Since T is connected, there must be some point x in $\mathcal{U}^\diamond \cap \mathcal{V}^\diamond$. Clearly x is an edge. We now inductively construct a sequence $\{e_i\}_{i \in \mathbb{N}}$ of edges and infinite decreasing chains

$Q_0 \supseteq Q_1 \supseteq Q_2 \dots,$ $C_0 \supseteq C_1 \supseteq C_2 \dots,$ $D_0 \supseteq D_1 \supseteq D_2 \dots$
of closed subspaces of T such that

- $C_0 \cap D_0 = \emptyset$ and
- - for all i, Q_i is a plant containing U, V and e_i with the property that no edge separates U and V,
  - Q_{i+1} is the component of $Q_i \setminus \{e_i\}$ containing U and V,
  - $\{C_i, D_i\}$ is a separation of $Q_i \cap V_T$ with $U \subseteq C_i, V \subseteq D_i$, and
  - $e_i \in C_i^\diamond \cap D_i^\diamond$.

Note that this last property implies that for all i, $e_i \in C_0^\diamond \cap D_0^\diamond$, contradicting the fact that T is weakly normal. We begin by setting $e_0 = x$, $C_0 = \mathcal{U}, D_0 = \mathcal{V}$. Suppose we have constructed e_i, Q_i, C_i, D_i for all $i \le n$. With regard to the following construction, the reader is referred to Figure 5.3.

Since Q_n is a plant, $Q_n \setminus \{e_n\}$ has precisely two components and by inductive hypothesis one of them must contain both U and V. So we choose Q_{n+1} to be this connected component, denote the other one by P_{n+1}, and set $C_{n+1} := \mathcal{U} \cap Q_{n+1} = C_n \cap Q_{n+1}$, $D_{n+1} := \mathcal{V} \cap Q_{n+1} = D_n \cap Q_{n+1}$. Since $\mathcal{U}$ and $\mathcal{V}$ are disjoint, and together cover all of, and are closed in, V_T, it follows that C_{n+1} and D_{n+1} are disjoint, and together cover all of, and are closed in, $Q_{n+1} \cap V_T$. Moreover, since $U \subseteq \mathcal{U} \cap Q_{n+1}$ and $V \subseteq \mathcal{V} \cap Q_{n+1}$, $\{C_{n+1}, D_{n+1}\}$ is a separation of $Q_{n+1} \cap V_T$.

Note that since Q_{n+1} is a component of the closed subset $Q_n \setminus \{e_n\}$ of Q_n, it is closed in Q_n, which is itself closed in T. Hence Q_{n+1} is closed in T, and therefore so are C_{n+1} and D_{n+1}. Also, since Q_n is a plant, Q_{n+1} is itself a plant.

Since Q_{n+1} is closed in T, it is a weakly normal edge subspace of T. In particular it is quasiregular; since C_{n+1}, D_{n+1} are open in $V_{Q_{n+1}}$, $C_{n+1}^\diamond$ and $D_{n+1}^\diamond$ are open in Q_{n+1}, and since C_n, D_n together cover $V_{Q_{n+1}}$ and any edge of Q_{n+1} is incident with some vertex of Q_{n+1}, together they cover Q_{n+1}. But Q_{n+1} is connected, so there must be some point $y \in C_{n+1}^\diamond \cap D_{n+1}^\diamond$.[6] Since C_{n+1} and D_{n+1} are disjoint, y is an edge. So we set $e_{n+1} := y$.

It remains to be proved that Q_{n+1} has the property that for any $e \in E_{Q_{n+1}}$, U and V are contained in the same component of $Q_{n+1} \setminus \{e\}$. By way of contradiction, let X, Y be the components of $Q_{n+1} \setminus \{e\}$, and suppose that $U \subseteq X, V \subseteq Y$.

Now e_n has precisely one endvertex in Q_{n+1}; without loss of generality , we may assume that this endvertex is in X. The other endvertex is in P_{n+1}, the other (closed) component of $Q_n \setminus \{e_n\}$, which is itself closed in Q_n; hence P_{n+1} is closed in Q_n. Similarly X and Y are both closed in Q_{n+1} and therefore in Q_n. But then, if A denotes $X \cup \overline{\{e_n\}} \cup P_{n+1}$, the

[6] The point y belongs to the surrounding sets taken in Q_{n+1}, and therefore the surrounding sets taken in T.

bipartition $\{A, Y\}$ is a separation of $Q_n \setminus \{e_{n+1}\}$ with $U \subseteq A, V \subseteq Y$, contradicting the inductive hypothesis.

Finally, we observe that since $Q_0 \supseteq Q_1 \supseteq Q_2 \supseteq \cdots \supseteq Q_n \supseteq Q_{n+1}$, for all $i = 0 \ldots n$, from the choice of $e_i \in Q_i \setminus Q_{i+1}$ it follows that $e_i \notin Q_{n+1}$, so that $e_i \neq e_{n+1}$. This guarantees that the sequence $\{e_i\}_{i \in \mathbb{N}}$ is indeed an infinite set. ✠

The above result strongly suggests that, in weakly normal, V-zero-dimensional topologized graphs, fundamental cyclesets can be obtained in essentially the same way.

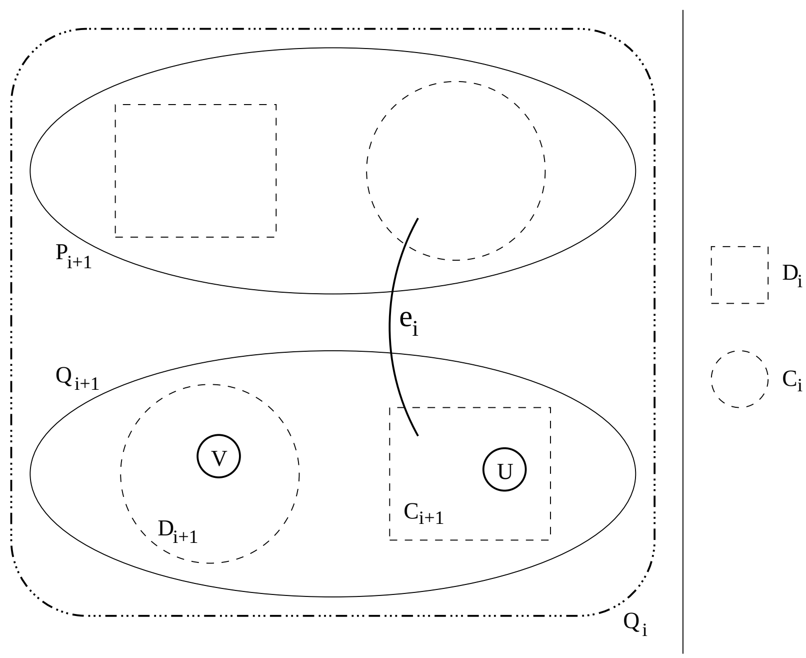


Figure 5.3: The construction in the proof of Lemma 5.5.1. Note that the endvertex of e_n in Q_{n+1} may in fact be in D_{n+1}.

Some characterizations of compact V-totally-disconnected topologized graphs

Let us say that an edge space is edge-separable if for any two vertices u, v there exists a set of edges which separates u and v, and finitely edge-separable if this set of edges can always be chosen to be finite. We conjecture that the following claims are easy to verify.

5.5.2 *Claim:* *Let X be a compact feebly Hausdorff edge space. Then the following are equivalent:*

- *G is edge-separable;*
- *G is V-totally-disconnected;*
- *G is V-zero-dimensional.*

Also, the following are equivalent:

- *G is finitely edge-separable;*
- *G is weakly Hausdorff and V-zero-dimensional.*

5.5.3 ***Claim:*** *Let G be a compact weakly Hausdorff edge space. Then the following are equivalent:*

- *G is edge-separable;*
- *G is finitely edge-separable;*
- *G is V-zero-dimensional.*

5.5.4 ***Claim:*** *Let G be a compact, feebly Hausdorff V-zero-dimensional topologized graph. The following are equivalent:*

- *G is finitely adjacent;*
- *G is finitely edge-separated;*
- *G is weakly Hausdorff;*
- *every edgecut is finite;*
- *every bond is finite;*
- *every proper edge is in a finite bond;*
- *every proper edge is in a finite edgecut.*

5.5.2 Strong vs Algebraic Spans

Another issue which arises in this chapter is the distinction between strong and algebraic spans. Corollary 5.3.12 states that for compact weakly Hausdorff edges spaces, the algebraic and strong spans of the cyclesets coincide. The following result approaches this issue in a somewhat more generic setting.

5.5.5 ***Theorem:*** *Let E be a countable set and $\mathcal{C}$ a subset of $\mathbb{Z}_2^E$ such that the symmetric difference of finitely many elements of $\mathcal{C}$ is a disjoint union of elements of $\mathcal{C}$. Then the algebraic and strong spans of $\mathcal{C}$ coincide.*

Proof: It suffices to show that $\mathcal{A}(C)$ is closed under taking linear combinations of thin families. Let $e_1, e_2, \ldots$ be an enumeration of the elements of $\mathcal{E}$. Let $S = \bigwedge_{i \in I} Z_i$ be the linear combination the arbitrary thin family $\{Z_i\}_{i \in I}$, where each Z_i is, in turn, the linear combination of a thin family of cyclesets, that is, for each $i \in I$, there is an index set J_i and, for every $j \in J_i$, an element $C_{i,j} \in \mathcal{C}$, such that $Z_i = \bigwedge_{j \in J_i} C_{i,j}$.

For any such triple S, $\{Z_i\}_{i \in I}$, and $\big\{\{C_{i,j}\}_{j \in J_i} \mid i \in I\big\}$, we denote by $m(S)$ and $m(\{C_{i,j}\})$ the least integers m, n such that $e_m \in S$ and $e_n \in \bigcup_{i \in I} \bigcup_{j \in J_i} C_{i,j}$, respectively.

Claim: For a fixed S, the sets $\{Z_i\}_{i \in I}$ and $\big\{\{C_{i,j}\}_{j \in J_i} \mid i \in I\big\}$ may be chosen so that $m(\{C_{i,j}\}) = m(S)$.

Proof of Claim: Obviously $e_m \in S$ implies that there exist i, j such that $e_m \in C_{i,j}$, so $m(S) \geq m(\{C_{i,j}\})$, and we may assume $m(S) > m(\{C_{i,j}\})$. It suffices to find a thin family $\{Z'_i\}_{i \in I} \subseteq \mathcal{C}$ and a collection of thin families $\big\{\{D_{i,k}\}_{k \in K_i} \,|\, i \in I\big\}$ each contained in $\mathcal{C}$ such that S is still the linear combination of the Z'_i, each Z'_i is the linear combination of the $D_{i,k}$, and moreover $m(\{D_{i,k}\}) > m(\{C_{i,j}\})$.

Since the collection $\big\{\{C_{i,j}\}_{j \in J_i} \mid i \in I\big\}$ will be fixed for this argument, for ease of notation, let $\check{e}$ denote $e_{m(\{C_{i,j}\})}$. Also, let $\check{C}$ be any one of the $C_{i,j}$ that contain $\check{e}$; clearly there is at least one such and any one will do. Now we consider the subset $I' \subset I$ of indices i such that $\check{e} \in C_{i,j}$ for some $j \in J_i$. Note that for $i \in I'$, the subset $J'_i \subseteq J_i$ of indices j such that $\check{e} \in C_{i,j}$ is finite, and that $\check{e} \in Z_i$ if and only if $|J'_i|$ is odd (while $\forall i \notin I', \check{e} \notin Z_i$. Moreover, since $\check{e} \notin S$, we have that the subset $I'' \subseteq I$ of indices such that $e \in Z_i$ itself has even cardinality.

Now for every $i \in I', j \in J'_i$, we consider $C'_{i,j} := C_{i,j} \Delta \check{C}$. Since $\check{e}$ belongs to both $C_{i,j}$ and $\check{C}$, we have $\check{e} \notin C'_{i,j}$. By assumption, there exists a collection $\{D_{i,j,k}\}_{k \in K_{i,j}} \subseteq \mathcal{C}$ such that $C'_{i,j} = \biguplus_{k \in K_{i,j}} D_{i,j,k}$.

Now, for every $i \in I'$, we replace every $C_{i,j}$ with $j \in J'_i$ with the entire collection $\{D_{i,j,k}\}_{k\in K_{i,j}}$, that is, for a fixed $i \in I'$, we replace the family $\{C_{i,j}\}_{j\in J_i}$ with the family $\{D_{i,k}\}_{k\in K_i}$ where

$$K_i := \{(j,k) \mid j \in J'_i, k \in K_{i,j}\} \cup (J_i \setminus J'_i) \text{ and } D_{i,k} := \begin{cases} C_{i,j} \text{ if } k \in J_i \setminus J'_i \\ D_{i,j,k'} \text{ for } k = (j,k') \in K_i. \end{cases}$$

Note that, for fixed $i \in I', j \in J'_i$, a given $e \in E$ can only occur in $D_{i,j,k}$ for a one $k \in K_{i,j}$ (since the $D_{i,j,k}$ are pairwise disjoint). Since, for every $i \in I'$, J'_i is finite and the family $\{C_{i,j}\}_{j\in J_i}$ is thin, the family $\{D_{i,k}\}_{k\in K_i}$ is still thin.

Now we set, for $i \in I$, $Z'_i := \bigwedge_{k\in K_i} D_{i,k}$, where, for $i \in I\setminus I'$, "everything stays the same", that is, $K_i = J_i$, and for $j \in J_i$ $D_{i,j} = C_{i,j}$, whence $Z_i = Z'_i$. For $i \in I'$, the effect of the replacement in the comparison between Z_i and Z'_i is to take the symmetric difference with $\check{C}$ as many times as there are indices in J'_i, which is odd if $i \in I''$ and even if $i \in I' \setminus I''$. In the latter case, $Z'_i = Z_i$; since I'' is finite, the family $\{Z'_i\}_{i\in I}$ differs from the thin family $\{Z_i\}_{i\in I}$ in finitely many elements, and therefore is itself thin. Moreover, since I'' is even, we have $S = \bigwedge_{i\in I} Z'_i$. Furthermore, since $\check{C}$ was chosen from the $C_{i,j}$'s, no element $e_s \in E$ with $s < m(\{C_{i,j}\})$ occurs in $\check{C}$, and therefore neither can such an element reappear in the $D_{i,k}$'s (since $\forall i \in I', j \in J'_i, \check{e} \notin C_{i,j}$). Finally, we observe that the $C_{i,j}$ which have been removed are all those for which $\check{e} \in C_{i,j}$. Hence $\check{e}$ does not occur in any $D_{i,k}$, and $m(\{D_{i,k}\}) > m(\{C_{i,j}\})$. □

To complete the proof, we need to show that S is a linear combination of a thin family of elements of $\mathcal{C}$. In order to do this, we systematically eliminate elements of E from S. By the above claim, we may assume that $m(S) = m(\{C_{i,j}\})$. Let $\hat{e}$ denote $e_{m(S)}$. We claim that there exists a finite set $\mathcal{C}_S$ of cyclesets such that the least integer m such that $\hat{e}$ occurs in any element of $\mathcal{C}_S$ coincides with $m(S)$.

To see this, let $\hat{I} \subseteq I$ be the set of indices i such that $\hat{e} \in Z_i$. Then $|\hat{I}|$ is finite and odd. For each $i \in \hat{I}$, the set $\hat{J}_i \subseteq J_i$ of indices j such that $\hat{e} \in C_{i,j}$ is finite and odd. Let F_1 be the symmetric difference of the finitely many $C_{i,j}$ for $i \in \hat{I}$ and $j \in \hat{J}_i$.

The set $S_1 := F\Delta S$ is again a linear combination of $\hat{Z}_i$'s, namely those defined by $\hat{Z}_i = Z_i$ if $i \notin \hat{I}$, and $\hat{Z}_i := Z_i\Delta\left(\bigwedge_{j\in\hat{J}_i} C_{i,j}\right)$, each of which is the linear combination of a (thin) subfamily of $\{C_{i,j}\}_{j\in J_i}$. Note that, since $m(S) = m(\{C_{i,j}\})$, we have that $e_m \in F \Rightarrow m \geq m(S)$; moreover, since both F_1 and S contain $\hat{e}$, we also have that $m(S_1) > m(S)$. At this point, by the above claim, we may express S_1 as a linear combination of a thin family of linear combinations of thin families of elements of $\mathcal{C}$ so that the smallest integer m such that e_m occurs in any of the elements of $\mathcal{C}$ coincides with $m(S_1)$. Note that, since F_1 is a disjoint union of elements of $\mathcal{C}$, by the assumption it may be expressed as the disjoint union of a family $\mathcal{C}_1$ of elements of $\mathcal{C}$.

Hence we may inductively construct sequences $\{F_n\}_{n=1}^{\infty}$, $\{\mathcal{C}_n\}_{n=1}^{\infty}$ and $\{S_n\}_{n=0}^{\infty}$ such that $S_0 = S$ and, for all $n = 1, 2, \ldots$ we have that $\mathcal{C}_n$ consists of pairwise disjoint elements of $\mathcal{C}$, $F_n = \biguplus_{C \in \mathcal{C}_i} C$, $S_n = S_{n-1} \Delta F_n$, $m(S_n) > m(S_{n-1})$ and the least integer m_n such that $e_{m_n} \in F_n$ is precisely $m(S_{n-1})$.

In order to express S as a linear combination of a thin family of elements of $\mathcal{C}$, it is now sufficient to show that $\bigcup_{n=0}^{\infty} \mathcal{C}_n$ is thin; this, however, is clear because for a fixed n, no element of E occurs in more than one element of $\mathcal{C}_n$, and for a fixed element e_s we have that, for sufficiently large n, $e_s \notin S_n$. ✠

Chapter 6

Local connectedness and Ferns

6.1 Where we stand

6.1.1 Changing the topological universe

So far, the topological properties underlying the results in this work have been feeble regularity and compactness. The former appears to be essential for any kind of space that we would consider "well-behaved". The latter, which is clearly a very natural assumption from the topological point of view, also relates well to the context of graph theory, in that it leads us to consider a large class of spaces which include, in particular, all finite classical graphs, as well as infinite constructions which have previously arisen (modulo the trick of modelling edges as *points*) with a graph-theoretic motivation, such as the Freudenthal compactification of locally finite graphs (rediscovered in a combinatorial setting by Halin in [27]), and the Diestel-Kühn end-quotient of 2-edge-connected graphs [21]. Despite the fact that these constructions occurring in the literature seem to have little resemblance to a typical compact feebly Hausdorff edge space, such an object retains enough structure in common with finite graphs so that concepts like spanning trees, paths, edgecuts, and (fundamental) cycles and bonds, all have reasonable analogues, the existence of these objects can be guaranteed, and the cycle-bond space generation and orthogonality theorems of Chapter 5 still hold.

On the other hand, it would be desirable to obtain a theory which is not restricted to compact spaces. From the topological point of view, the assumption of compactness is not only very natural, but also quite strong, leading, for example, to the rich theory of continua. The context of graph theory gives us more motivation to do away with the assumption of compactness—suffice it to say that, although graph theorists dealing with infinite graphs have been drawn towards compact or almost compact constructions, an infinite classical graph by itself is not compact.

Another issue that we feel needs to be addressed at this point concerns the S_1 axiom. As we saw in Chapter 1, the question of assigning a topology on a possibly finite graph in such a way as to reconcile the standard topological and graph-theoretic notions of connectedness leads very naturally to a scenario which is extremely attractive in its symmetry and simplicity: that of a topological space in which every point is open or closed. From this topological starting point, the connection back to combinatorics is almost equally simple: take the vertices to be the closed points, the (hyper)edges the open points, and, if desired, impose that no point has more than two other points in its boundary to deal exclusively with graphs.

One cannot help being intrigued by this setup, and feel that there is a wealth of structure and connections to graph theory to be unearthed in this framework, possibly with the additional assumption that the clumps be singletons. Yet as soon as one imposes compactness, the fact that the vertices are closed (or actually components of the vertex set, which is even stronger) becomes redundant in the proofs, at least in the specific context of cycle spaces.

Thus in a sense we have not had the opportunity to exploit the rich structure in such a setup. Of course, there are numerous issues of graph-theoretic inspiration which can be addressed, apart from cycle spaces, and one would expect that some of these would bring out the rôle of these strong assumptions. At this point we do not propose to resolve an issue such as Menger's Theorem, for example, but we shall try to take one small step forward towards understanding S_1 spaces.

Hence we have a double motivation to change the "topological universe" we are working in: renouncing compactness, and bringing in S_1. It turns out that one way of doing this, which also allows us to address our *third* concern at this point (to be discussed below), is to bring into play another standard topological axiom. This assumption, in the context of this chapter, will *imply* S_1.

6.1.2 What should trees be?

Although our *philosophy* is to take abstract topological axioms as a starting point, we need some *strategy* to ensure that our topological wanderings lead to a context which is relevant to graph theory. The strategy we opted for was to characterize graph-theoretic concepts in topological language, and then investigate which descriptions were conducive to interesting results. It was especially useful not to focus just on the properties of the "ambient space", such as regularity (inherited by subspaces) and compactness (inherited by closed subspaces), but also on specific substructures.

Thus graph-theoretic paths and cycles led to "paths" and "cycles" as defined in Chapter 2, and subsequently to edgepaths and edgecycles, while trees became plants and spanning trees minimal spanning sets. Although paths and cycles do not play a direct rôle in the later chapters, it is unlikely that we would have been able to obtain results at the level of generality of compact feebly Hausdorff edge spaces without the insight offered by the early results. Moreover, we feel that paths and cycles should have an important rôle in the context of S_1 topologized graphs, which we are interested in, as discussed above.

Among the three basic concepts of paths, cycles and trees, at present the last stands out in our picture. The first two seem to admit two different versions: a purely topological version in terms of a topological space with certain properties, and a second version as an edge space, which focuses on edges, and retains just enough of the structure of its topological precursor for our purposes. Given that the results about cycle spaces are concerned with edges, it is not surprising that this minimum has to do with edges. Note that the various characterizations of paths and cycles do not even refer to open or closed points, and it is a consequence of the definition that these objects are topologized graphs.

On the other hand, so far we only have one version of "trees", namely plants, which is an "edge space" version and clearly falls in line with edgecycles and edgepaths. Note that the related concepts of spanning trees and minimal spanning sets really have no place in this discussion, because they are not "intrinsic" concepts: given the "spanning" requirement, a tree is a spanning tree only in relation to the graph it is purportedly spanning, just as a minimal spanning set must be considered in the space it lives in. Hence this concept goes hand in hand with that of a cycleset, whereas edgecycles, edgepaths and plants (or their topological counterparts) are spaces with their own structure, even if we may choose the one inherited from a larger space. This distinction is analogous, for example, to that between compact and closed sets in a topological space.

Loosely speaking, in this chapter we would like to do with graph-theoretic trees what we did with graph-theoretic paths in Chapter 2.

Background

It turns out that there is much material in the literature which is relevant to the issue of "tree-like" spaces. We give here a brief overview of what we feel are the most relevant results for us, focusing on the issues of separation axioms (Hausdorff vs non-Hausdorff), and forms of compactness vs alternative assumptions. Before doing this, we introduce the concept of "endpoint".

6.1.1 Definition: We say that a point x of a topological space X is a topological endpoint if every open set containing x contains an open set containing x with precisely one point in its boundary.

Reference: The above definition is taken from [45, Exercise 6.25]. □

We remark that in some texts which are very relevant to this context, e.g. [10] and [35], the term "endpoint" is used synonymously with "non-cutpoint"; we have used the term "endpoint" in Chapter 2 to refer to a non-cutpoint a topological space X which also happens to belong to a subset of X, with the meaning that the point does not disconnect the space X, as opposed to the subset.

6.1.2 Fact: Let X be a continuum (compact, connected metric space). The following are equivalent:

——— X is locally connected and contains no simple closed curve;

(CSd) any two points are separated in X by a third point of X;

(CoE) each point is a cutpoint or an endpoint;

——— each non-degenerate subcontinuum contains uncountably many points;

(INT2) the intersection of any two connected subsets of X is connected;

——— for any point $x \in X$, the component number and the order number of X are equal whenever either is finite.

See [45], Section 10.1. □

The designation "INT2" is taken from [10]; property (INT2) is a strengthening of "hereditary unicoherence", which is important in continuum theory. The letters "CS" in the designation of property (CSd) stand for "combinatorial separation"; we shall justify this later on. This property will be one of several important ones for us. The following definition is standard.

6.1.3 Definition: A topological space is dendritic if it satisfies property (CSd).

It is important for us that (CSd) implies that the space is Hausdorff. For locally connected spaces, this is affirmed by Whyburn in the stronger Theorem 9.1 of [60], and in general it seems that this was first shown by Brouwer [10] (Chapter III, Section 1, Proposition 1).

The first efforts to move away from compactness and metrizability came with the abovementioned paper of Whyburn, published in 1968, and in spite of the ensuing stream of papers, for several reasons we feel that this is still the most relevant for us today.

From our perspective, the most important point of this paper is that if a locally connected space is dendritic, then for any two points a, b, the union of $\{a, b\}$ with the set of points which separate a, b is an ordered continuum. In our case, it will be an ab-path; this fact will be central in the whole setup.

Another important point is that Whyburn had already cast aside any assumption of compactness, and was instead assuming the space to be locally connected. Most of the efforts in the immediate aftermath of this paper revolved around some form of compactness. Between 1969 and 1977 it emerged that the above statement about locally connected dendritic spaces is also true for rim-compact dendritic spaces. According to Brouwer [10], in 1969 Proizvolov [51] ascribed this to a paper by Gurin [26] of the same year, but Gurin's paper does not prove this result. Brouwer observed that Bennett's purported proof [6], published in 1973, implicitly assumed that the space is locally connected. So Brouwer gives his own proof (Theorem 5, Section 3, Chapter III, in [10], 1977), as well as treating separately the locally connected case (Chapter III, Section 2), and giving sufficient conditions for a Hausdorff, locally compact (and therefore rim-compact) space to be dendritic (Chapter III, Section 4). Apart from the gap in the result ascribed to Gurin, Proizvolov [51, 52] showed that a rim-compact dendritic space is hereditarily normal and has a unique dendritic compactification, which was also studied by Allen [4] and Pearson [49].

However, according to Ward [58], the earliest one of the papers mentioned in the above paragraph, namely Gurin's (published in Russian[1]), showed that a rim-compact dendritic space is locally connected! On the other hand, we do not see any compelling way in which rim-compact dendritic spaces are better behaved than locally connected ones. While a dendritic space which is not rim-compact can not be expected to have a dendritic compactification, even from a purely topological standpoint this by itself does not seem to be enough reason to restrict to rim-compact spaces, especially since one can hope for a "completion" of some sort as opposed to a "compactification". Essentially, given a choice between rim-compact and locally connected spaces, the latter option has the effect of making the proofs simpler, and the class of spaces more general. We, of course, have very concrete reasons for which we prefer "locally connected" over "rim-compact": not only our professed desire to do away with compactness, but also the simple fact that a classical tree with infinite degrees is not rim-compact.

6.1.4 Fact: Let X be a connected, locally connected Hausdorff space. The following are equivalent:

– **(INT)** the intersection of an arbitrary collection of connected subsets is connected;

– **(INT2)** the intersection of any two connected subsets is connected;

– **(CSd)** X is dendritic;

[1] A proof in English can be found in [10].

– **(W)** the boundary of each component of the complement of a non-empty connected subset is a singleton.

Reference: The equivalence of (INT), (INT2) and (CSd) was shown by Whyburn [60]. The equivalence with (W) is due to Brouwer [10], and the designations (INT), (INT2) and (W) are taken from the same tract. Brouwer states that the property (W) is inspired by the concept of an A-set, which is also due to Whyburn. It appears that the property (W) was first considered by Brouwer [10] and, with a different formulation, Kok [35]. Property (CSd) is what Brouwer refers to as (S). □

Brouwer made some attempt to investigate non-Hausdorff variants of the above.

6.1.5 Fact: Let X be a locally connected topological space. The following are equivalent:

– **(CSd)** for any two points there exists a third which separates them;

– **(INT′)** X is T_1 and satisfies (INT);

– **(W″)** X is T_2 and satisfies (W);

——— X is T_2 and if C, S are subsets such that C is connected but $C \setminus S$ is not, then S contains some cutpoint of C.

Reference: See [10] Chapter III, Section 2, Theorem 4. □

Note that although two of the above assumptions do not explicitly require the Hausdorff property (or any separation property at all), the class of spaces characterized does consist of Hausdorff spaces, precisely the same ones characterized in Fact 6.1.4.

Brouwer [10, Chapter 2, Section 4, Theorem 9] has also shown that for connected, locally connected spaces T_1 spaces, (W) implies (INT2). However, he states that (W) is a "rather weak property" (pg. 37, Chapter II, Section 0) and gives an example of a connected, locally connected T_1 space satisfying (W) with no cutpoints at all (pg. 47, Chapter II, Section 4, the first example after the statement of Theorem 9). In order to show that that (W) implies (CSd), he requires the Hausdorff property—this he does separately for locally connected and locally compact spaces (Theorems 6, 8 in Chapter II, Section 3). Again, *a posteriori* Hausdorff spaces satisfying (W) are rim-compact dendritic spaces and therefore locally connected.

In 1974, Ward [58] showed that, for a connected, locally connected, Hausdorff space, the following are equivalent:

- **(D1)** S[a,b] is an ordered continuum;

- **(D2)** S[a,b] is connected;
- **(D3)** if A, B are disjoint closed connected susbsets of X, then there exists $z \in X$ such that z separates A and B;
- **(CSd)** X is dendritic;
- **(D5)** if A, B are disjoint closed continua of X, then there exists $z \in X$ such that z separates A and B.
- **(O1)** $\forall x \in X \ \ \trianglelefteq_x$ is closed and dense;
- **(O2)** $\forall x \in X \ \ \trianglelefteq_x$ is dense;
- **O−** X admits a partial order, with minimum, such that for all $x \in X$, $\omega(x)$ is open, and one of the following holds:
  - **(OCSd)** $\preceq$ is dense, upper semiclosed and, $\forall x \neq y, \quad \alpha(x) \neq \alpha(y)$;
  - **(O3)** $\preceq$ is tree-like, closed and dense;
  - **(O4)** $\preceq$ is tree-like, semi-closed and dense;
  - **(OD2)** $\preceq$ is tree-like, upper semi-closed and $\forall x \in X \ \mathtt{A}(x)$ is connected
  - **(OD1)** $\preceq$ is upper semi-closed and $\forall x \in X$, $\mathtt{A}(x)$ is an ordered continuum.

In the above, the partial order $\trianglelefteq_x$ is the cutpoint partial order with basepoint x, the partial order defined by $a \triangleleft b \Leftrightarrow a$ separates x and b. It seems that the partial orders mentioned in the properties $O-$ are all obtained in this way. We shall construct partial orders satisfying properties similar to these, which need not be the cutpoint partial order for any basepoint. The term "upper (lower) semi-closed" means that for every $x \in X$, we have that $\Omega(x)$ (respectively, $\mathtt{A}(x)$) is closed in X, and the term "semi-closed" applied to $\preceq$ means that $\preceq$ is upper and lower semi-closed. The term "dense" means that for any $x, y \in X$, $x \prec y$ there exists a point m such that $x \prec m \prec y$, and the term "tree-like" means that for every $x \in X$, the subset $\mathtt{A}(x)$ is totally ordered.

In the same paper, Ward showed that without the assumption that X is locally connected (but still Hausdorff), the property (OCSd) is equivalent to (CSd), (OD1) is equivalent to (D1), (OD2) is equivalent to (D2), and (D1) is equivalent to the assumption that X is "arcwise" connected (connected by ordered continua) and satisfies any one of (D2), (D3) or (CSd). Moreover, he gives several examples of pathological "tree-like" spaces, including a dendritic space (Example 8) in which, for a certain choice of a, b, $S[a, b]$ is totally disconnected.

Similar results were obtained by Muenzenberger and Smithson [43], who filled in a gap in a proof in [58] by showing that the cutpoint partial order gives X the structure of a lower semilattice. We shall show that the partial orders we construct have the same property.

In a later paper [44] Ward, Muenzenberger and Smithson showed that a space is arcwise connected (in the sense of the preceding paragraph) and dendritic if and only if it is Hausdorff, uniquely arcwise connected and each connected subspace is arcwise connected. The authors also obtained characterizations of other "tree-like" spaces, namely "arboroids", which are compact and arcwise connected by definition, "trees", which are compact dendritic (and therefore Hausdorff) spaces, and "weakly nested" spaces, which are Hausdorff and arcwise connected by definition.

Further work in this area was done by Nikiel [46], who approaches the issue from a purely order-theoretic starting point, considering different topologies on tree-like partially ordered spaces. See [47] for an overview.

6.1.3 Combinatorial Separation Axioms

We now list a striking selection of axioms from among the ones we have encountered so far.

- A *path* is a connected, locally connected space such that:

  (CSp) among any three points, one separates the other two.

- A *cycle* is a connected, locally connected space such that one of the following holds:

  (CSc1) among any four points, no one disconnects the other three, but some two disconnect, and are disconnected by, the other two;

  (CSc2) for any four points a, b, c, d, we have that a, b disconnect c, d if and only if a, c do not disconnect b, d and a, d do not disconnect b, c;

  (CSc3) any quadruple can be uniquely partitioned so that some part disconnects the other, and, with this partition, each part disconnects the other.

- A *dendritic space* is a connected space such that:

  (**CSd**) for any two points, there is a third which separates them.

  Although dendritic spaces are not usually defined to be locally connected, their extremely well-behaved precursors, namely dendrites, which arise in continuum theory and hence come with the additional assumptions of compactness and metrizability, are automatically locally connected. As discussed above, this is also true for rim-compact dendritic spaces, and without any form of compactness, dendritic spaces have been shown to be well-behaved when they are locally connected, but possibly highly ill-behaved without any further assumption, in a way which is not just the failure of a T_i axiom.

In the designation of the properties above, the letters "CS" stand for "combinatorial separation". We find the above properties very appealing, and not just because of their "combinatorial" character. They have one other characteristic in common: they do not explicitly require any T_i axiom, nor any less restrictive "neighbourhood separation" axiom, such as S_1. Instead, they implicitly contain some such separation property: paths and cycles are S_1 spaces, and dendritic spaces are Hausdorff spaces. In a sense, they "get right to the point" and *transcend* the technical issue of separation by open sets. Of course, one can argue that "combinatorial separation" axioms are themselves a form of "neighbourhood separation" axioms, but apart from the fact that they can be phrased in terms of connected components rather than open sets, the examples of the (possibly non-Hausdorff) paths and cycles show that a "perfectly natural" "combinatorial separation" property can be "less natural", or at least unusual, in terms of "neighbourhood separation" axioms.

In general topology, one often proceeds by imposing *a priori* a "neighbourhood separation axiom" such as T_1 or T_2. Usually one has no qualms in imposing the Hausdorff axiom, since it has come to be regarded as a "natural" property which is not too restrictive and determines a well-behaved "topological universe", and rightly so. In our case, however, the class of objects we wish to deal with should contain the classical graphs, which are simply not Hausdorff—they are not even T_1. We feel that "neighbourhood separation axioms" should be a technical help more than decisive criteria in their own right—that is, a space does not become less interesting because it fails a technical "neighbourhood separation axiom", but rather these axioms are conditions which are imposed only when needed and used as technical tools. In this specific context, this feeling is reinforced by the simplicity and attractiveness of topologized graphs.

The problem is that, once we are out of the usual well-studied topological universe of Hausdorff spaces, we hardly have a convincingly "natural" neighbourhood separation assumption to start from. We feel that there already is a case to be made for the S_1 axiom, but this is based solely on the results of Chapter 2 and on the fact that S_1 so seamlessly takes on the rôle of T_1 in ensuring the hierarchy of feeble separation axioms.

Hence the ideal solution in our context would be a class of spaces defined by means of a "combinatorial separation" property similar to the ones above, *without* an explicit neighbourhood separation requirement.

Another intriguing aspect of the issue at hand is the issue of order. Paths can be characterized in terms of total orders and cycles in terms of cyclic orders. Partial orders come to mind, and the work of Ward suggests that there should be a connection in this respect.

6.1.4 Introducing the newcomers

The "newcomers" are intuitively rather different: an abstract topological assumption setting the "topological universe" and a concrete class of spaces living in this universe.

In Subsection 6.1.1 we set ourselves the objective of replacing the assumption of compactness with an alternative topological assumption. We saw in Chapter 2 that a bounded prepath is locally connected if and only if it is compact, and a (possibly unbounded) locally connected prepath (path) is a union of a nest of compact paths, and therefore quite "well-behaved", but not necessarily compact. So in this context local connection seems to have the edge over compactness, but only just.

It may have occurred to the reader that several counterexamples we have given so far have been not locally connected; this is true in the case of the Diestel-Kühn obstruction, the badly-behaved plants of Examples 3.6.3 and 5.1.18, and the example of a pre-edge selection in a continuum inducing a non-feebly-Hausdorff edge-clump quotient (Example 4.3.13). We have even conjectured an equivalence between feeble regularity and a strengthened version of local connectedness (Conjecture 4.3.15).

Finally, in Subsection 6.1.2 we hope to have made a case for the rôle of local connectedness in the issue of finding a "topological version" of trees.

In the light of the above observations, the property of local connectedness appears to be a valid candidate for the desired substitute for compactness. It will be the one topological assumption underlying the present chapter.

Before defining the class of spaces which will be our analogue of trees, we have an easy lemma.

6.1.6 ***Lemma:*** *Suppose c is a cutpoint of a connected, locally connected topological space, and that, for $i \in \{1,2,3\}$, K_i is a component of $X \setminus \{c\}$. Then the following are equivalent:*

- *K_1, K_2, K_3 are pairwise separated;*
- *there exists a partition $\{C_1, C_2, C_3\}$ of $X \setminus \{x\}$ into clopen subsets such that, for $i \in \{1,2,3\}$, we have that $K_i \subseteq C_i$.* ✠

6.1.7 Definition: A topological space satisfies property (CSf) if

(CSf) for any three points, either one disconnects the other two or else there exists a fourth which disconnects all three.

A topological spacesatisfies property (CSf′) if

(CSf′) for any three points, either one separates the other two or else there exists a fourth which separates all three.

A fern is a connected, locally connected topological space with at least three points which satisfies property (CSf). A fern is flimsy if it satisfies property (CSf′).

We now proceed to present our results on ferns. Our main results will be the fact ferns are path-connected (Theorem 6.2.16, generalizing Whyburn's result that locally connected dendritic spaces are path connected) and a partial order characterization (Theorem 6.5.1).

6.2 Ferns

6.2.1 Separation Properties in Locally Connected Spaces

6.2.1 ***Lemma:*** *Let F be a connected topological space and let U be a non-empty connected set in F. Let K be a component of $F \setminus U$ and let L be a union of finitely many components of $F \setminus U$. Then: $F \setminus L$ is connected; if U is open, L is closed in F; and, if U is open and F is locally connected, $K \cap \mathsf{Cl}(U) \neq \emptyset$ and $K \cup U$ is connected.*

Proof: Let $L_1, L_2, \dots L_k$ be the components of L. From Fact 2.2.8 we have that $F \setminus L_1$ is connected. Let $F^{(i)} := F \setminus (L_1 \cup L_2 \cup \dots L_i)$ and, by induction, suppose $F^{(i)}$ is connected. Then L_{i+1} is a component of $F^{(i)} \setminus U$, so again by Fact 2.2.8 $F^{(i+1)} = F^{(i)} \setminus L_{i+1}$ is connected.

Suppose that U is open. We note that all components of $F \setminus U$ are closed in $F \setminus U$ (Fact 1.1.23). Since U is open in F, $F \setminus U$ is closed in F, so L is a union of finitely many closed sets, and therefore closed (in F).

Now suppose that, in addition, F is locally connected, and that $K \cap \mathsf{Cl}(U) = \emptyset$. Then K is a connected subset of $F \setminus \mathsf{Cl}(U)$, so there is a component K' of $F \setminus \mathsf{Cl}(U)$ such that $K \subseteq K'$. In turn, because $U \subseteq \mathsf{Cl}(U)$, K' is a connected subset of $F \setminus U$, so there is a component K'' of $F \setminus U$ such that $K' \subseteq K''$. Now K and K'' are both components of $F \setminus U$ and $K \subseteq K''$. It follows that $K = K''$ and, therefore, $K = K'$. But K is closed in F by the preceding paragraph and, since K' is a component of the open set $F \setminus \mathsf{Cl}(U)$ and F is locally-connected, K' (i.e., K) is open in F. But K is a proper non-empty subset of a connected space, a contradiction.

Since U is connected, so is $U \cup (K \cap \mathsf{Cl}(U))$. Since K is connected and $\mathsf{Cl}(U) \cap K \neq \emptyset$, we conclude that $K \cup U$ is connected. ✠

6.2.2 ***Lemma:*** *Let c be a cutpoint of a connected, locally connected topological space X and K a component of $X \setminus \{c\}$. Then $K \cup \{c\}$ is connected, and one of the following must occur, but not both:*

(A) *c is closed, K is open in X, $\mathsf{Cl}(K) = K \cup \{c\}$, and $\partial(K) = \{c\}$;*

(B) *c is open, K is closed in X, $K^\diamond = K \cup \{c\}$, $\rho(K) = \{c\}$ and c has a boundary point in K.*

Proof: By Remark 2.1.10, c is open or closed, and since X is connected, cannot be clopen. If c is closed, then K is a component of an open set, and since X is locally connected, K is open. Moreover, K is not clopen but since it is closed in $K \setminus \{c\}$, its closure is disjoint from the other components of $K \setminus \{c\}$, whence $\mathsf{Cl}(K) = K \cup \{c\}$. Moreover, $\mathsf{Cl}(X \setminus K) = X \setminus K$, whence $\partial(K) = \{c\}$. Also, since K is connected, $\mathsf{Cl}(K)$ is connected.

If c is open, by 6.2.2 not only are all the components of $X \setminus \{c\}$, in particular K, closed, but $\mathsf{Cl}(c) \cap K \neq \emptyset$. In particular, since K is connected, so is $K \cup \{c\}$. Moreover, $K \cup \{c\} \subseteq K^\diamond$. On the other hand, for every other component K' of $X \setminus \{c\}$, the complement of K' is an open set containing K and disjoint from K'. Hence $K^\diamond = K \cup \{c\}$. Since K is closed, $(X \setminus K)^\diamond = X \setminus K$, whence $\rho(K) = \{c\}$. ✠

6.2.3 ***Lemma:*** *Let F be a flimsy fern and x a cutpoint of X. Then the quasicomponents of $X \setminus \{x\}$ coincide with the components.*

Proof: Let K_1, K_2 be components of $X \setminus \{x\}$. If these are the only components of $X \setminus \{x\}$, then the assertion is trivial. Let K_3 be a third component of $X \setminus \{x\}$. Choose $x_i \in K_i$, for $i = 1, 2, 3$. By definition of a flimsy fern, either one of them separates the other two or else there exists a fourth point x_4 which simultaneously separates them. By Lemma 6.2.2, $K_i \cup \{x\}$ is connected. Hence no x_i can separate the other two. Moreover, no point other than x can simultaneously separate all three x_i. We conclude that $x = x_4$. In particular, there exists a separation $\{A, B\}$ of $X \setminus \{x\}$ with $K_1 \subseteq A$ and $K_2 \subseteq A$. ✠

6.2.4 Note: $F \setminus \{x\}$ need not be locally connected!

6.2.5 Example: Let V be any totally disconnected space. Let $h \notin V$ and define a topology on $V \cup \{h\}$ by declaring U to be open if and only if $U = \emptyset$ or else $h \in U$. It is easy to verify that this is a fern. The quasicomponents of $X \setminus \{h\}$ are precisely the quasicomponents of V. Then F is flimsy if and only if V is totally separated, and $F \setminus \{h\}$ is locally connected if and only if V is discrete.

6.2.6 ***Theorem:*** *Suppose that a connected, locally connected topological space X satisfies property (CSf). Then:*

(CSd′) *for any two non-incident points, there exists a third point which disconnects them; and*

(CI) *among any three points, some two are not incident.*

If X satisfies (CSf′), then X is feebly Hausdorff.

Proof: If X is empty or a singleton, all three assertions are vacuously true. If X consists of two points, then (CI) is vacuously true, and X is either indiscrete or else contains an open singleton and a closed singleton; in both cases, the two points are incident, and it is easy to check that X is feebly Hausdorff. Hence we assume that X contains at least three distinct points.

To show (CI), let x, y, z be distinct points. Note that no point can disconnect two incident points. Thus if, say, y disconnects x and z, then x and z can not be incident. If no one of x, y, z disconnects the other two, there exists a point w disconnecting all three, and no two among x, y, z are incident.

Now let a, b be arbitrary distinct points; we claim that either a, b are incident or else there is some point disconnecting them. For every $c \in X \setminus \{a, b\}$, either there exists some d_c which disconnects a, b, c, or else one of a, b, c disconnects the other two. If the former case occurs for some c, then d_c disconnects a and b. If not, the sets $A = \{c \mid a \text{ disconnects } b \text{ and } c\}$, $B = \{c \mid b \text{ disconnects } a \text{ and } c\}$, and $C = \{c \mid c \text{ disconnects } a \text{ and } b\}$ together cover $X \setminus \{a, b\}$. If some point disconnects a, b, we are done; hence we assume that C is empty.

Now if A is empty, b disconnects a from all the other points of X, so that $\{a\}$ is a component of $X \setminus \{b\}$. By Lemma 6.2.2, if $\{b\}$ is open then $a \in \mathsf{Cl}(b)$, and if b is closed, $b \in \mathsf{Cl}(a)$. In both cases, a, b are incident. Similarly, if B is empty, a and b are incident. So we assume that A, B are both non-empty. In particular, both a and b are cutpoints.

Now consider the components of $X \setminus \{b\}$, and let M be the one containing a. Since $X \setminus M$ is connected and contains b but not a, any point disconnected from b by a is in M, that is, $A \subseteq M$. Since M is connected and contains a but not b, any point disconnected from a by b cannot be in M, that is, B is disjoint from M . But since A, B together cover $X \setminus \{a, b\}$, we have that $M = A \cup \{a\}$ and B is precisely the union of the connected components of $X \setminus \{b\}$ except for M. Moreover, $B \cup \{b\}$ is connected, and since a disconnects all points in A from b, the components of $X \setminus \{a\}$ are $B \cup \{b\}$ and the components of $M \setminus \{a\}$.

Now if $\{a\}$ is open, it has a boundary point in $B \cup \{b\}$. But $\mathsf{Cl}(M)$ is disjoint from B, and contains a, so this boundary point is b, implying that a and b are incident. If $\{b\}$ is open, it contains a boundary point in $M = A \cup \{a\}$. But $\mathsf{Cl}(B \cup \{b\}) \subseteq \mathsf{Cl}(B \cup \{a, b\})$, which is disjoint from A. So this point is a, and again a, b are incident.

So suppose a, b are both closed. Since b is closed, M is open, that is, $B \cup \{b\}$ is closed. But since a is closed, it is the boundary point of $B \cup \{b\}$, a contradiction. This concludes the proof of (CSd′).

Now suppose X satisfies (CSf′), and let a, b again be arbitrary points, and suppose first that they can be separated by some third point c; let $\{P, Q\}$ be a separation of $X \setminus \{c\}$ with

$a \in P, b \in Q$. If $\{c\}$ is closed, then P, Q are disjoint neighbourhoods of a, b respectively. If $\{c\}$ is open and incident with both a, b, then $P \cup \{c\}$, $Q \cup \{c\}$ are neighbourhoods of a, b intersecting only at $c \in a^\diamond \cap b^\diamond$. If $\{c\}$ is open and not incident with at least one of a, b, say b, then by (CSd$'$) there exists a fourth point d separating c and b. Since $Q \cup \{c\}$ is connected, $d \in Q$, and if $\{P', Q'\}$ is a separation of $X \setminus \{d\}$ with $c \in P', b \in Q'$, we have $P \subseteq P'$ and $Q' \subseteq Q$. Now one of $P, P \cup \{c\}$ is open, and one of $Q', Q' \cup \{d\}$ is open; these two open sets are the required neighbourhoods of a, b.

So suppose instead that a, b can not be separated by a third point. Then, by part (CSd$'$), they are incident. In particular, no point separates them. So for any other point c, neither can there be a fourth separating all three, nor can c itself separate them. Let A, B be defined as in the proof of (CSd$'$). If A, B are both non-empty, a, b are both cutpoints. If A is empty, then b is a cutpoint, a is component of $X \setminus \{b\}$, and either b is open and a is closed, or b is closed and a is open. Similarly, if B is empty, both a and b are either closed or open. In fact, this holds in all cases. Since they are incident, one is closed and one is open. But then, by Remark 3.1.15, the feeble separation condition is automatically satisfied. ✠

6.2.7 ***Corollary:*** *A T_1 fern is a dendritic space.*

Proof: Follows from (CSd$'$) of 6.2.6 and the fact that no two closed points can be incident. ✠

6.2.8 ***Lemma:*** *Suppose a connected feebly Hausdorff topological space X satisfies property (CI) of Lemma 6.2.6. If $|X| \geq 3$, then X is an S_2 space.*

Proof: Since X is feebly Hausdorff, it is sufficient to show that X is S_1. Suppose y is a point which is not closed and not open. Since it is not closed, there exists some point $x \in \mathsf{Cl}(y)$. Since y is not open, every neighbourhood of y contains some point other than y. We claim that every neighbourhood of y contains some point other than x (and itself).

Suppose not. Then $\{x, y\}$ is an open proper subset of X. Since X is connected, $\{x, y\}$ is not closed, that is, there exists a point z distinct from x, y in $\mathsf{Cl}(\{x, y\} = \mathsf{Cl}(x) \cup \mathsf{Cl}(y)$. So $z \in \mathsf{Cl}(x)$ or $z \in \mathsf{Cl}(y)$. We consider two cases, according to whether $z \in \mathsf{Cl}(x)$ or not. If not, there exists some open set U_z containing z and disjoint from $\{x\}$. But in this case, $z \in \mathsf{Cl}(y)$, which implies that $y \in U_z$, so that $\{y\} = U_z \cap \{x, y\}$ is open, a contradiction. If $z \in \mathsf{Cl}(x)$, since $x \in \mathsf{Cl}(y)$ we have that $z \in \mathsf{Cl}(y)$ and the three points are pairwise incident, that is, the triple $\{x, y, z\}$ fails assumption (CI).

Since X is feebly Hausdorff we may choose neighbourhoods U_x, U_y of x, y respectively such that $(U_x \cap U_y) \subseteq y^\diamond$. Then $U_y' := U_x \cap U_y$ is also an open neighbourhoodof y, and therefore contains some point w other than x. But w is in both U_x and U_y, so must belong to $y^\diamond$, that is, $y \in \mathsf{Cl}(w)$. But then again, we have $x \in \mathsf{Cl}(y) \subseteq \mathsf{Cl}(w)$, so the triple of points $\{x, y, w\}$ fails assumption (CI). ✠

6.2.9 ***Corollary:*** *A flimsy fern is an S_2 space.*

6.2.2 From $S[a, b]$ to $D[a, b]$

In Subsection 2.2.3 we stated some basic topological facts concerning the set $S(a, b)$ of points separating two given points a, b is a given topological space. The proofs of these facts is usually based on the content of Proposition 2.1.13, but it is easy to see that one can extend these facts to the set of points which *disconnect* a and b. The easy proof depends on Fact 2.2.8.

Notation and Terminology: Given points a, b in a connected topological space X, $D(a, b)$ denotes the set of points which disconnect a and b. Also, $D[a, b) = \{a\} \cup D(a, b)$, $D(a, b] = D(a, b) \cup \{b\}$ and $D[a, b] = D(a, b) \cup \{a, b\}$.

If x, z are distinct points in X, the component of $X \setminus \{x\}$ containing z will be denoted by $\mathbf{B}_x(z)$. Thus $\mathbf{B}_x(z)$ is shorthand for $\mathbf{K}_{X\setminus\{x\}}(z)$.

6.2.10 ***Lemma:*** *Let a, b be two points in a topological space; then the binary relation on $D[a, b]$ given by*

$$x \preceq y \iff \begin{cases} x = y, & \textit{or} \\ x = a, & \textit{or} \\ x \textit{ disconnects } a \textit{ and } y, \end{cases}$$

is a total order. For any two distinct points $x, y \in D(a, b)$, the following are equivalent:

- $x \prec y$;
- $\mathbf{B}_y(x) = \mathbf{B}_y(a)$;
- $\mathbf{B}_x(a) \subseteq \mathbf{B}_y(a)$;
- $\mathbf{B}_x(b) \supseteq \mathbf{B}_y(b)$.

For any three points $x, y, z \in D[a, b]$, we have that y disconnects x and z if and only if $x \prec y \prec z$ or $z \prec y \prec x$. ✠

We refer to the order defined above as the associated total order of $D[a, b]$.

6.2.11 ***Lemma:*** *Let a, b be points of a topological space, and u, v points in $D[a,b]$ such that $u \prec v$ with respect to the associated total order $\preceq$ of $D[a,b]$. Then $D[u,v]$, with its associated total order, is the interval in $D[a,b]$ with extrema u, v.*

Proof: Let $x \in D(u,v)$. If $b \notin \mathbf{B}_x(v)$, then by Fact 2.2.8 $X \setminus \mathbf{B}_x(v)$ is a connected subset of $X \setminus \{v\}$ containing u and b, contradicting $u \prec v$ in $D[a,b]$. So $b \in \mathbf{B}_x(v)$ and symmetrically $a \in \mathbf{B}_x(u)$. Thus $x \in D[a,b]$ and $u \prec x \prec v$ in $D[a,b]$. ✠

6.2.12 ***Lemma:*** *If a, b are points of a topological space X belonging to a connected subset C, then $S[a,b] \subseteq D[a,b] \subseteq C$.*

Proof: Suppose $x \in S[a,b]$. By Corollary 2.1.6 a and b belong to different components of $X \setminus \{x\}$. But if $x \notin C$, they belong to the one containing C. ✠

6.2.13 ***Corollary:*** *Let X be a connected, locally connected topological space and a, b points in X. For every cutpoint $c \in X$, we have that $D[a,b] \subseteq \mathbf{B}_c(a) \cup \mathbf{B}_c(b) \cup \{c\}$.*

Proof: Follows from Lemmas 6.2.12 and 6.2.2. ✠

6.2.14 ***Lemma:*** *Let C be a connected set in a topological space. For any two points a, b, we have that $C \cap D[a,b]$ is an convex.*

Proof: Suppose $c, e \in C \cap D[a,b]$ with $c < e$, where $\leq$ is the associated total order of $D[a,b]$, and suppose $d \in D[a,b]$ is such that $c < d < e$. Then by Lemma 6.2.11, $d \in D[c,e]$ and by Lemma 6.2.12 $D[c,e] \subseteq C$. ✠

6.2.3 $D[a,b]$ in Ferns

6.2.15 ***Lemma:*** *Let U and V be connected subsets of a fern F and a, b points in F. If $U \cap D[a,b]$ and $V \cap D[a,b]$ are disjoint and non-empty, then U and V are disjoint.*

Proof: Suppose, by way of contradiction, that the point $x \in U \cap V$. Let $u \in U \cap D[a,b]$ and $v \in V \cap D[a,b]$. By Lemma 6.2.11 $D[u,v] \subseteq D[a,b]$, and by the assumption $x \notin D[a,b]$, so x does not disconnect u and v. Since u, x are in the connected set U and v is not, by Lemma 6.2.12 v does not disconnect u and x. Similarly, u does not disconnect v and x.

Since a fern satisfies property (CSf) there is a point w of F that simultaneously disconnects all three of u, v, and x. Since w disconnects u and v, $w \in D[a,b]$. Since w disconnects u and x, by Lemma 6.2.12 $w \in U$, and similarly, $w \in V$. But then $w \in U \cap V \cap D[a,b]$, contradicting the assumption. ✠

6.2.16 *Theorem:* *Let F be a fern and let a and b be non-incident points of F. Then $D[a,b]$ is connected.*

Proof: Suppose (U,V) is a separation of $D[a,b]$. Let U' and V' be open sets in F such that $U = U' \cap D[a,b]$ and $V = V' \cap D[a,b]$. For each $c \in D[a,b]$, let K_c be the component of either U' or V' containing c and let L_c denote the union of the components of $F \setminus K_c$ that are disjoint from $D[a,b]$. By Lemma 6.2.14, $K_c \cap D[a,b]$ is convex. Also, since F is locally connected and U', V' are open, K_c is open.

Now we claim that there are at most two components of $F \setminus K_c$ that intersect $D[a,b]$. Choose points $u, v \in D[a,b] \setminus K_c$ such that $u \prec v \prec c$ (where $\preceq$ denotes the associated total order of $D[a,b]$) and suppose that they belong to distinct components of $F \setminus K_c$, say K_u and K_v. By Lemma 6.2.1 $K_c \cup K_u$ is connected, and therefore, by Lemma 6.2.12, should contain $D[u,c]$. But by Lemma 6.2.11 $v \in D[u,c]$ while K_v is disjoint from both K_u and K_c and contains v, a contradiction. Similarly, the upper tail of $D[a,b] \setminus K_c$ is contained in some component of $F \setminus K_c$. Hence the number of components of $F \setminus K_c$ that contain some point of $D[a,b]$ is at most two. This argument also shows that, in the special cases of $c = a$ or $c = b$, this number is at most one.

Lemma 6.2.1 implies that $K_c \cup L_c$ is open and connected in F. Now let $\hat{U} = \bigcup_{c \in U} (K_c \cup L_c)$ and $\hat{V} = \bigcup_{c \in V} (K_c \cup L_c)$. Clearly $\hat{U}$ and $\hat{V}$ are open in F. Let $u \in U$ and $v \in V$. The sets $K_u \cup L_u$ and $K_v \cup L_v$ are connected open sets whose common intersection with $D[a,b]$ is $K_u \cap K_v \cap D[a,b] \subseteq (U' \cap D[a,b]) \cap (V' \cap D[a,b]) = U \cap V = \emptyset$. By Lemma 6.2.15, $K_u \cup L_u$ and $K_v \cup L_v$ are disjoint, and, since u, v are arbitrary points in U, V respectively, $\hat{U}$ and $\hat{V}$ are disjoint.

Finally, we show that $F = \hat{U} \cup \hat{V}$. Let $z \in F$. If $z \in D[a,b]$, then $z \in U \cup V$, whence $z \in \hat{U} \cup \hat{V}$. If $z \notin D[a,b]$, then by property (CSf) either one of a and b disconnects z from the other, or there is a fourth point c that simultaneously disconnects all three of a, b, and z. If, say, a disconnects z from b, then by Lemma 6.2.12 z cannot belong to the same component of $F \setminus K_a$ as b, so $z \in K_a \cup L_a$, and $z \in \hat{U} \cup \hat{V}$. If c simultaneously disconnects a, b, and z, then $c \in D[a,b] \cap D[a,z] \cap D[b,z]$ and z cannot belong to the same component of $F \setminus K_c$ as either of a, b, so $z \in K_c \cup L_c$, and again $z \in \hat{U} \cup \hat{V}$.

Hence $\{\hat{U}, \hat{V}\}$ is a separation of F. We conclude that $D[a,b]$ is connected, for otherwise F could not be connected. ✠

6.2.17 *Corollary:* *A fern is hereditarily locally connected.*

Proof: Let C be an arbitrary connected subset of the fern F, and x an arbitrary point in C and U an (arbitrary) open set in C containing x. Then $U = U' \cap C$ for some subset U' of F, open in F. Since F is locally connected and U' contains x, there exists an open connected subset W' of F containing x. By definition of the relative topology, $W := W' \cap C$ is open in C.

Now for every $a, b \in C$, by Lemma 6.2.12 we have that $D[a, b] \subseteq C$; similarly, for every $a, b \in W'$, since W' is connected $D[a, b] \subseteq W'$. Hence for every $a, b \in W$, we have that $D[a, b] \subseteq W$. But Theorem 6.2.16 says that $D[a, b]$ is connected, so a, b belong to the same component of W. Since $a, b \in W$ are arbitrary, W is connected, and may be taken as the required open connected neighbourhood of x contained in U. ✠

The above is analogous to the following.

6.2.18 Fact: A dendrite is hereditarily locally connected.

Reference: See [45], Corollary 10.5. □

6.2.19 ***Corollary:*** *For any two points a, b of a fern, $D[a, b]$ is an ab-path.*

Proof: Theorem 6.2.16 says that $D[a, b]$ is connected, and Corollary 6.2.17 implies that it is locally connected. In the light of Lemma 2.2.43, this means that $D[a, b]$ is an ab-path. ✠

6.2.20 ***Corollary:*** *Let F be a fern, and C a subset of F containing at least three points. Then C is a fern if and only if it is connected. If F is flimsy and C is connected, then C is a flimsy fern.*

Proof: If C is not connected, clearly it is not a fern, because ferns are connected by definition. Suppose C is connected (and contains at least three points). By Corollary 6.2.17, C is locally connected. Let x, y, z be distinct points in C. If any one among these three points disconnects (separates) the other two in F, it does so also in C. If not, there exists a fourth point c which simultaneously disconnects (separates) all three points. Now $c \in D[x, y]$, and since C is connected, by Lemma 6.2.12, $D[x, y] \subseteq C$. So $c \in C$, and since it disconnects (separates) all three points in F, it does so also in the smaller set C. Hence C is a (flimsy) fern. ✠

6.2.21 ***Corollary:*** *The set of cutpoints of a fern F is connected.*

Proof: Let C be the set of cutpoints, and a, b points in C. Then $D[a, b]$ consists entirely of cutpoints, and is therefore a (connected) ab-path in C. Hence a and b belong to the same component of C. Since a, b are arbitrary points in C, the set C is connected. ✠

6.2.22 ***Corollary:*** *A fern is hereditarily uniquely path-connected.*

Proof: Let F be a fern. Thanks to Corollary 6.2.20, it is sufficient to show that F is uniquely path-connected. Corollary 6.2.19 says that, for any two points a, b, $D[a, b]$ is an ab-path. To see uniqueness, let P be an arbitrary ab-path. Since P is connected, from Lemma 6.2.12, we have $D[a, b] \subseteq P$. If $x \in P \setminus D[a, b]$, then x is supposed to separate a and b in P, but $D[a, b]$ is a connected subset of $P \setminus \{x\}$, containing a and b, a contradiction. Hence $P = D[a, b]$. ✠

6.3 Topological Characterizations of Ferns

6.3.1 ***Proposition:*** *Let X be a connected topological space. The following are equivalent:*

(INT) *the arbitrary intersection of a collection of connected subsets is connected;*

(INT ∗)
- **(INT′)** *the intersection of a nest of connected sets is connected, and*
- **(INT2)** *the intersection of any two connected sets is connected;*

(P1) *X is uniquely and hereditarily prepath connected;*

(Dab1) *for any two points a, b, $D[a, b]$ is an ab-prepath;*

(Dab2) *for any two points a, b, $D[a, b]$ is connected.*

Proof: The implication (INT) $\Rightarrow$ (INT$*$) is trivial. To see that (INT$*$) $\Rightarrow$ (P1), let a, b be arbitrary points in X. We claim that every connected subset C of X containing a and b contains a minimal connected subset containing a and b.

Consider the collection of all connected subsets of X containing a, b, partially ordered by inclusion. Since the intersection of an arbitrary nest N of such subsets is connected, and obviously contains a, b, it is a lower bound for N. By Zorn's Lemma there exists a minimal connected set containing a and b, and this set is an ab-prepath. Since X itself connected, X is prepath connected.

To see uniqueness, suppose P_1, P_2 are two ab-prepaths, and $x \in P_1 \setminus P_2$. Then $P_1 \setminus \{x\}$ consists of precisely two components, the one containing a and the one containing b. Since P_2 is connected and contains a and b but not x, $P_2 \cup (P_1 \setminus \{x\})$ is also connected. But $(P_2 \cup (P_1 \setminus \{x\})) \cap P_1 = P_1 \setminus \{x\}$ is not connected, contradicting (INT2).

Since (INT′) and (INT2) are clearly hereditary properties, the above argument applied to an arbitrary connected subset of X shows that X is hereditarily uniquely prepath connected.

To see (P1) $\Rightarrow$ (Dab1), let a, b be arbitrary points of X, and P the unique ab-prepath in X. Since P is connected, by Lemma 6.2.12, $D[a,b] \subseteq P$. If $x \in P \setminus D[a,b]$, then a, b belong to the same component of $X \setminus x$. Since X is hereditarily prepath connected, this component contains a second ab-prepath, which is distinct from P because it does not contain x, and is obviously contained in X, contradicting the fact that X is uniquely prepath connected. Hence $D[a,b] = P$ is an ab-prepath.

The implication (Dab1) $\Rightarrow$ (Dab2) is trivial. To see (Dab2) $\Rightarrow$ (INT), let $\mathcal{C}$ be an arbitrary collection of connected subsets of X, let K denote the intersection $\bigcap_{C \in \mathcal{C}} C$, and choose arbitrary points $a, b \in K$. Then $\forall\ C \in \mathcal{C}$, by Lemma 6.2.12 $D[a,b] \subseteq C$, hence $D[a,b] \subseteq K$, and since $D[a,b]$ is connected, a and b belong to the same component of K. Since a and b are arbitrary, K is connected. ✠

6.3.2 ***Conjecture:*** *Let X be a set and $\mathcal{T}$ a topology on X such that $(X, \mathcal{T})$ is a connected topological space satisfying the conditions of Proposition 6.3.1. Then there exists a coarser locally connected topology which preserves the collection of connected subsets. More precisely, let $\mathcal{T}' \subseteq \mathcal{T}$ be the subcollection of open subsets U such that, for any two points $a, b \in X$, the components of $D[a,b] \cap U$ are open in $D[a,b]$. Then $(X, \mathcal{T}')$ is a connected, locally connected topological space satisfying the conditions of Proposition 6.3.1 and such that the collection of connected subsets of $(X, \mathcal{T})$ coincides with the collection of connected subsets of $(X, \mathcal{T}')$.*

6.3.3 ***Lemma:*** *Let X be a connected, locally connected topological space. If X satisfies (INT2), X is hereditarily locally connected.*

Proof: Let C be a connected subset of X, and choose arbitrarily a point $c \in C$ and a subset U of C containing c and open in the relative topology on C. Then U is of the form $U' \cap C$, where U' is open in X. Since $c \in U'$ and X is locally connected, we have that there exists an open connected subset W such that $u \in W \subseteq U$. Now by (INT2), $W \cap C$ is connected and, by definition of the relative topology, open in C. Since it obviously contains c, it is the required open connected neighbourhood of c in C. ✠

6.3.4 ***Proposition:*** *Let X be a connected, locally connected topological space. Then the following are equivalent:*

—— *X satisfies the conditions of Proposition 6.3.1;*

(P1′) *X is uniquely and hereditarily path-connected;*

(Dab1′) *for any two points a, b, $D[a, b]$ is an ab-path.*

If the above conditions are satisfied, X is hereditarily locally connected.

Proof: By Lemma 6.3.3 we have that, when X is locally connected, the conditions of Proposition 6.3.1 imply that X is hereditarily locally connected. Therefore, if X satisfies the conditions of Proposition 6.3.1, a subset of X is a prepath if and only if it is a path; hence X is uniquely and hereditarily path-connected. The proof of the implication (P1′) ⇒ (Dab1′) is the same as that of the implication (P1) ⇒ (Dab1), and clearly (Dab1′) implies (Dab1). ✠

6.3.1 Bringing in S_1

We now aim to characterize ferns in terms of the S_1 axiom and property (CSd′). Recall that in Proposition 6.2.6 we introduced property (CI), which says that among any three points, some two are not incident. Note that any S_1 space satisfies this property.

6.3.5 Remark: Let F be a topological space satisfying properties (CI) and (CSd′) and let u, v be distinct points of F both incident with a third point w. Then: u and v are not incident; and the only point of F that disconnects u and v is w.

6.3.6 *Lemma:* *Let F be a connected, locally connected topological space satisfying properties (CI) and (CSd′), and let $w \in F$ be open. Then:*

- *every component K of $F \setminus \{w\}$ contains precisely one point of $\mathsf{Cl}(w)$;*
- *if $K \cap \mathsf{Cl}(w) = \{x\}$, then every component of $K \setminus \{x\}$ is a component of $F \setminus \mathsf{Cl}(w)$; and*
- *x separates every point of $K \setminus \{x\}$ from w.*

Proof: By Lemma 6.2.1, $K \cap \mathsf{Cl}(w) \neq \emptyset$. By Remark 6.3.5, w disconnects any two points of $\mathsf{Cl}(w)$. Thus, K contains precisely one point of $\mathsf{Cl}(w)$. Let L denote the union of all the components of $F \setminus \{w\}$ except K, together with w. By Lemma 6.2.1 (with $U = \{w\}$), L is connected and open in F. We now show that $\mathsf{Cl}(L) = L \cup \{x\}$.

If $y \in \mathsf{Cl}(L) \setminus (L \cup \{x\})$, then, since $F = K \cup L$, $y \in K \setminus \{x\}$. Since $y \in \mathsf{Cl}(L)$ and L is connected, $L \cup \{y\}$ is connected. On the other hand, $K \cup \{w\}$ is also connected.

Notice that $y \notin \mathsf{Cl}(w)$ and, since w is open, $w \notin \mathsf{Cl}(y)$. Therefore, y and w are not incident, so $D(y, w) \neq \emptyset$. Any connected set containing y and w contains $D(y, w)$. This applies to $L \cup \{y\}$ and $K \cup \{w\}$. So $D(y, w) \subseteq (L \cup \{y\}) \cap (K \cup \{w\})$. But $(L \cup \{y\}) \cap (K \cup \{w\}) = \{y, w\}$, a contradiction.

It follows that $L \cup \{x\}$ is closed in F, so L is both open and closed in $F \setminus \{x\}$. Since it is connected, it is a component of $F \setminus \{x\}$. The remaining components of $F \setminus \{x\}$ are the components of $K \setminus \{x\}$, as required. ✠

For the purposes of the following two proofs, we use the notation $\mathbf{B}_a(b)$ without requiring that $a \neq b$; for convenience, if $a = b$, we set $\mathbf{B}_a(b) = \emptyset$.

6.3.7 ***Theorem:*** *Let X be a connected, locally connected S_1 topological space satisfying property (CSd′). Then, for any two points a and b, $\mathsf{Cl}(D[a,b]) = \bigcup_{c \in D[a,b]} \mathsf{Cl}(c)$.*

Proof: Note that the assertion is trivial if a, b are incident (in particular, if they are the same point). So we may assume that a and b are not incident, that is, that $D(a,b) \neq \emptyset$. One containment is trivial. For the other, let $\bar{D}[a,b] = \bigcup_{c \in D[a,b]} \mathsf{Cl}(c)$ and suppose there exists $z \in \mathsf{Cl}(D[a,b]) \setminus \bar{D}[a,b]$. Since F is S_1, z is closed in F.

For each open $c \in D(a,b)$, let c_a and c_b be the unique points in $\mathsf{Cl}(c) \cap \mathbf{B}_c(a)$ and $\mathsf{Cl}(c) \cap \mathbf{B}_c(b)$, respectively. Note that, for example, c_a is either a or in $D(a,b)$.

Suppose first that there is some $c \in D[a,b]$ such that $z \notin \mathbf{B}_c(a) \cup \mathbf{B}_c(b)$. If c is closed, then $\mathbf{B}_c(z)$ is open and disjoint from $\mathbf{B}_c(a) \cup \mathbf{B}_c(b) \cup \{c\}$ and so, by Corollary 6.2.13, is disjoint from $D[a,b]$, a contradiction. If c is open, then, by the choice of z, $z \notin \mathsf{Cl}(c)$, so letting c' denote the unique point in $\mathsf{Cl}(c) \cap \mathbf{B}_c(z)$, we see from Lemma 6.3.6 and Corollary 6.2.13 that $\mathbf{B}_{c'}(z)$ is open and disjoint from $D[a,b]$, again a contradiction.

We conclude that $z \in \mathbf{B}_c(a) \cup \mathbf{B}_c(b)$. Let A denote the set of $c \in D[a,b]$ such that $z \in \mathbf{B}_c(b)$ and let B denote the $c \in D[a,b]$ such that $z \in \mathbf{B}_c(a)$. We notice that A and B partition $D[a,b]$, and that A and B are intervals in $D[a,b]$.

We may choose the labelling of a and b so that $z \in \mathsf{Cl}(A)$.

Our first step is to show that A has no maximum. By way of contradiction, suppose m is a maximum for A. If m is closed, then $\mathbf{B}_m(b)$ is open, contains z and is disjoint from A, a contradiction. If m is open, then $z \in \mathbf{B}_m(b)$ and $z \in \mathbf{B}_{m_b}(a)$. By Lemma 6.3.6 these sets are disjoint, another contradiction. So A has no maximum.

We now show that the net $(a)_{a \in A}$ converges to z, where the order on A is the one inherited from $D[a,b]$. Let U be an open set containing z and let V be the component of U containing z. Because F is locally-connected, V is open in F. Since $z \in \mathsf{Cl}(A)$, $V \cap A$ is not empty. Let $c \in V \cap A$ and let $c' \in A$ be such that $c' > c$. Then $z \in \mathbf{B}_{c'}(b)$ and $c \in \mathbf{B}_{c'}(a)$. Thus, c' disconnects z and c in F and, therefore, does so in U. In particular, $c' \in U$, as required.

Let A' denote the set of points of A that are closed in F. Let $\hat{A} = \bigcup_{c \in A'} \mathbf{B}_c(a)$. We claim that $A \subseteq \hat{A}$ and that $\mathsf{Cl}(\hat{A}) = \hat{A} \cup \{z\}$.

For the former, let $c \in A$. Since A has no maximum, there is a $c' \in A$ such that $c' > c$. Let d be c', if c is closed, and c'_b if c' is open. Since $c' \in A$ and A has no maximum, $d \in A$ and d is closed, i.e., $d \in A'$. Also, $c \in \mathbf{B}_d(a)$, as required.

For the latter, there is only one containment that is not trivial. Let $y \in \mathsf{Cl}(\hat{A})$. Furthermore, we may assume that $y \notin \hat{A}$. We wish to show $y = z$. We start by showing that $y \in \mathsf{Cl}(A)$ and $(a)_{a\in A}$ converges to y.

Let U be a connected open set in F containing y and let $c \in A'$ be such that $U \cap \mathbf{B}_c(a) \neq \emptyset$. Since $y \in \mathbf{B}_c(b)$, c disconnects $\mathbf{B}_c(a)$ from y in F and therefore does so in U, that is, $c \in U$. If $c' \in A$ is such that $c' > c$, then $c \in \mathbf{B}_{c'}(a)$ and $y \in \mathbf{B}_{c'}(b)$, so c' disconnects c and y in F and, therefore, in U, that is, $c' \in U$, as required.

Finally, we wish to show $y = z$. If not, notice that y and z are both closed in F and, therefore, are not incident in F. Thus, there is a point w that disconnects them. Using Lemma 6.3.6, we may assume that either w is closed or both y and z are in $\mathsf{Cl}(w)$. In the first case, $\mathbf{B}_w(y)$ and $\mathbf{B}_w(z)$ are disjoint open sets containing y and z, respectively. Since $(a)_{a\in A}$ converges to both y and z, both sets must contain a tail of A; this implies they are not disjoint, a contradiction.

In the latter case, notice that if $w \notin A$, then $A \subseteq \mathbf{B}_w(a)$, while if $w \in A$, then a tail of A is contained in $\mathbf{B}_w(b)$. In either case, it is only one component K of $F \setminus \{w\}$ that contains a tail of A. But then Lemma 6.2.1 implies $F \setminus K$ is open in F, does not contain a tail of A and does contain either y or z, contradicting the fact that $(a)_{a\in A}$ converges to both. ✠

6.3.8 ***Theorem:*** *Let F be a connected, locally connected S_1 topological space satisfying property (Csd′). Then F satisfies property (CSf).*

Proof: Suppose a, b, z are three distinct points of F such that no one disconnects the other two and no point of F simultaneously disconnects all three. Then, for every $c \in D[a, b]$, $z \in \mathbf{B}_c(a) \cup \mathbf{B}_c(b)$. Also, $z \in \mathbf{B}_a(b) \cap \mathbf{B}_b(a)$.

Set $L_z = \{c \in D[a, b] \mid z \in \mathbf{B}_c(b)\}$ and $R_z = \{c \in D[a, b] \mid z \in \mathbf{B}_c(a)\}$. Clearly, L_z and R_z partition $D[a, b]$ into two non-empty intervals.

For any partition of $D[a, b]$ into two intervals L and R, with $a \in L$ and $b \in R$, let $Z(L, R) = \{z \in F \mid L_z = L \text{ and } R_z = R\}$. We aim to prove that $Z(L, R)$ is open. Let $z \in Z(L, R)$.

We first show that $z \notin \mathsf{Cl}(D[a, b])$. Otherwise, by Lemma 6.3.7, there is a $c \in D[a, b]$ such that $z \in \mathsf{Cl}(c)$. Since $z \notin D[a, b]$, we see that c simultaneously disconnects a, b, z, a contradiction. Hence $z \notin \mathsf{Cl}(D[a, b])$.

Let K be the component of $F \setminus \mathsf{Cl}(D[a, b])$ containing z. We claim that $K \subseteq Z(L, R)$. Let $y \in K$. First suppose some $c \in D[a, b]$ simultaneously disconnects a, b, y. Since K is a connected subset of $F \setminus \{c\}$, $K \subseteq \mathbf{B}_c(y)$ and, by the choice of c, neither a nor b is in $\mathbf{B}_c(y)$. But then $z \in \mathbf{B}_c(y)$, so c simultaneously disconnects a, b, and z, a contradiction.

Thus, for every $y \in K$, no point of a, b, y disconnects the other two and no other point simultaneously disconnects these three.

It follows that $y \in Z(L_y, R_y)$. We claim that $L_y = L$. By the symmetry we now have between y and z, it suffices to show that $L \setminus L_y = \emptyset$. The alternative is that there is a $c \in L \setminus L_y$. It follows that $y \in \mathbf{B}_c(a)$ and $z \in \mathbf{B}_c(b)$, so that c disconnects y and z in F. But K is a connected subset of $F \setminus \{c\}$ containing both y and z, a contradiction. Hence $L_y = L$ and $R_y = R$, so $K \subseteq Z(L, R)$.

Since K is a component of $F \setminus \mathsf{Cl}(D[a, b])$, K is open in F, whence $Z(L, R)$ is open in F, as required. Furthermore, $Z(L, R)$ is the union of components of $F \setminus \mathsf{Cl}(D[a, b])$, so that $F \setminus (Z(L, R) \cup \mathsf{Cl}(D[a, b]))$ is a union of components of $F \setminus \mathsf{Cl}(D[a, b])$ and, therefore, is open in F. That is, $Z(L, R) \cup \mathsf{Cl}(D[a, b])$ is closed in F. If follows that $\mathsf{Cl}(Z(L, R)) \subseteq Z(L, R) \cup \mathsf{Cl}(D[a, b])$.

We aim to show that $\mathsf{Cl}(Z(L, R)) \cap \mathsf{Cl}(D[a, b])$ consists of a single point, which is either a maximum for L or a minimum for R. To this end, let $y \in \mathsf{Cl}(Z, R) \cap \mathsf{Cl}(D[a, b])$. By Lemma 6.3.7, there is a $c \in D[a, b]$ such that $y \in \mathsf{Cl}(c)$. We may choose the labelling so that $c \in L$. This implies that $Z(L, R) \subseteq \mathbf{B}_c(b)$.

First suppose c is open. If $c_b \in L$, then c, and therefore y, is contained in the open set $\mathbf{B}_{c_b}(a)$, which is disjoint from $Z(L, R)$, which is contained in $\mathbf{B}_{c_b}(b)$. Otherwise, $c_b \in R$, in which case $Z(L, R)$ is contained in both $\mathbf{B}_c(b)$ and $\mathbf{B}_{c_b}(a)$. But these sets are disjoint, a contradiction.

Therefore, c is closed. Then $y = c$ and $\mathbf{B}_c(b)$ is open in F. If c is not a maximum for L, then there is a $c' \in L$ such that $c' > c$. If c' is closed, then $\mathbf{B}_{c'}(a)$ is open, contains c and is disjoint from $Z(L, R)$, a contradiction to the fact that $c = y \in \mathsf{Cl}(Z(L, R))$. If c' is open, then either c'_b is in L, and the preceding sentence holds with c'_b in place of c', or $c'_b \in R$. In the latter case, $Z(L, R)$ is contained in $\mathbf{B}_{c'_b}(a)$ and in $\mathbf{B}_{c'}(b)$. But these sets are disjoint, a contradiction.

We conclude that c is the maximum for L. The only possibilities, then, for points in $\mathsf{Cl}(Z(L, R)) \cap \mathsf{Cl}(D[a, b])$ are a maximum of L and a minimum for R. If L has a maximum ℓ *and* R has a minimum r, then let $x \in F$ disconnect ℓ and r. Then x disconnects a and b, so $x \in D[a, b]$ satisfies $\ell \prec x \prec r$, which cannot happen since L and R are the intervals $D[a, \ell]$ and $D[r, b]$, whose union is $D[a, b]$. Thus, $\mathsf{Cl}(Z(L, R)) \cap \mathsf{Cl}(D[a, b])$ has at most one point.

If $\mathsf{Cl}(Z(L, R)) \cap \mathsf{Cl}(D[a, b])$ is empty, then $Z(L, R)$ is a non-empty proper subset of F that is both open and closed, contradicting the fact that F is connected. We deduce that $\mathsf{Cl}(Z(L, R)) \cap \mathsf{Cl}(D[a, b])$ has a unique point c, which is closed and we may take to be the maximum of L.

Since $Z(L, R)$ is open in F, it is open in $F \setminus \{c\}$. Furthermore, $\mathsf{Cl}(Z(L, R)) = Z(L, R) \cup \{c\}$, so that $Z(L, R)$ is closed in $F \setminus \{c\}$. But this implies c disconnects a and b from every point of $Z(L, R)$, as required. ✠

6.3.9 ***Corollary:*** *Let F be a connected, locally-connected S_1 space with at least three points. The following are equivalent:*

A *X satisfies the conditions of Propositions 6.3.1 and 6.3.4;*

B *X satisfies (CSd′);*

C *X satisfies (CSf).*

Proof: (A) $\Rightarrow$ (B): Let a, b be two non-incident points; then $\{a, b\}$ is not connected. By (Dab2) of Proposition 6.3.1, $D[a, b]$ is connected. Hence $D(a, b) \neq \emptyset$.

(B) $\Rightarrow$ (C): This is precisely Theorem 6.3.8.

(C) $\Rightarrow$ (A): From Theorem 6.2.16, for any two points x, y, $D[x, y]$ is connected. Since x, y are arbitrary, X satisfies the properties of Proposition 6.3.1, and since X is locally connected, also those of Proposition 6.3.4. ✠

6.4 Cutends

In this section we approach ferns from an entirely different perspective. The immediate objective is to obtain a partial order characterization of ferns (Theorem 6.5.1). The long-term aim (which we do not address in this thesis) is to move towards a theory of end completions in locally connected spaces.

6.4.1 ***Lemma:*** *Let P be a set equipped with a binary relation. Then every upper-directed (lower-directed) subset, in particular every singleton, is contained in a maximal upper-directed (lower-directed) subset.*

Proof: In the following proof, the terms "directed" and "common bound" should be read alternately "upper-directed" and "common upper bound" respectively, or else "lower-directed" and "common lower bound" respectively.

Suppose D is a directed subset, and consider the collection $\mathcal{D}$ of directed subsets of P containing D. Clearly set inclusion is a partial order on $\mathcal{D}$. Let M be the union of a nest of elements of $\mathcal{D}$. By Zorn's Lemma, it is sufficient to show that $M \in \mathcal{D}$. If $x, y \in M$, then $x \in D_x$ and $y \in D_y$ for some $D_x, D_y \in \mathcal{D}$. Without loss of generality, $D_x \subseteq D_y$, so $x, y \in D_y$ and since D_y is directed, there exists some common bound z in D_y, which is contained in M. Hence M is directed and of course contains D. So $M \in \mathcal{D}$. ✠

6.4.2 Definition: Given a connected topological space X and a cutpoint $x \in X$, an x-branch is a component of $X \setminus \{x\}$. A branch of X is an x-branch for some cutpoint x. Thus a branch is of the form $\mathbf{B}_x(z)$ for some cutpoint x and some point $z \neq x$.

If $\mathcal{B}$ denotes the set of branches of X, the branching partial order of X is the binary relation of set inclusion on $\mathcal{B}$. A cutend of a connected topological space is a maximal lower-directed subset of $\mathcal{B}$, equipped with the branching partial order. A cutend $\mathcal{E}$ is free if $\bigcap_{B \in \mathcal{E}} B = \emptyset$, and fixed otherwise.

In the context of ferns, one can think of free "cutends" as "directions in which the fern escapes to infinity". One of the reasons we introduce them is to be able to "*root* ferns at infinity". Much as we would like to "be positive" and speak of "upper-directed" as opposed to "lower-directed" cutends, there are reasons for us not to do so. One is simply compatibility with the work of Ward, for whom the "downward" choice was more natural, since his "rooting" was occurring at a base point, rather than "at infinity". The other is that real-life ferns and trees are "rooted" in the ground!

It follows from 6.2.2 that given a branch B of a connected, locally connected topological space, the cutpoint which determines it is uniquely determined by B itself, that is, B can be an x-branch for only one possible cutpoint x. For B must be open or closed, but is not clopen; if B is closed, the cutpoint is the unique boundary point of B, and if B is open, the cutpoint is the unique point in $\rho(B)$.

6.4.3 ***Lemma:*** *Let C be a maximal upper-directed (lower-directed) subset of a set equipped with a transitive binary relation. Then C is closed under lower (upper) bounds.*

Proof: Let $\preceq$ denote the binary relation. Suppose C is a maximal upper-directed subset, $c \in C$, $y \notin C$, and $y \preceq c$. Consider the set $C' := C \cup \{y\}$, and an arbitrary pair $P := \{p_1, p_2\}$ of points in C'. If $P \subseteq C$, since C is upper-directed there is a common upper bound in C for p_1 and p_2, which is also in C'. If $p_i = y$ for some $i \in \{1, 2\}$, then there exists a common upper bound u for c and p_{3-i} in C, and therefore in C', which is a common upper bound also for $\{y, p_{3-i}\}$ since $y \preceq c \preceq u$ implies that $y \preceq u$. Hence C' is a directed subset, and yet strictly contains C, contradicting the maximality of C. The proof for maximal lower-directed subsets is entirely symmetrical. ✠

6.4.4 ***Corollary:*** *Let B_1, B_2 be branches in a topological space, and suppose B_1 belongs to a cutend $\mathcal{C}$. If $B_1 \subseteq B_2$, then $B_2 \in \mathcal{C}$.* ✠

6.4.5 Fact: Any non-empty collection of non-empty subsets of a given set S which is lower-directed under set-inclusion is a filter base for some filter on S.

Reference: This is stated in [61], Definition 12.1, and follows immediately from the definitions of filter and filter bases (see Appendix B). □

Thus a cutend of a topological space X "generates" (is a filter base for) a filter on X. Clearly the filter is fixed (free) if and only if the cutend is. This motivates the use of the adjectives "fixed" and "free" for cutends.

The following easy lemmas are the "disconnection analogues" for the properties of cuttings given in Corollaries 2.2.38 and 2.2.39. Again, the proof depends on Fact 2.2.8. We make them explicit because they will be recurrent working tools henceforth.

6.4.6 ***Lemma:*** *Let z, c be cutpoints in a connected topological space X. Then all the z-branches which do not contain c are contained in the c-branch $\mathbf{B}_c(z)$.* ✠

6.4.7 ***Lemma:*** *Let c, c' be cutpoints in a connected topological space X and z a point distinct from both. If c' disconnects c and z, then $\{c'\} \cup \mathbf{B}_{c'}(z) \subseteq \mathbf{B}_c(z)$.* ✠

6.4.8 ***Lemma:*** *Let $E := \{B_\lambda\}_{\lambda\in\Lambda}$ be a cutend of a topological space. Then $B := \bigcap_{\lambda\in\Lambda} B_\lambda$ does not contain any cutpoints.*

Proof: Suppose there exists a cutpoint $z \in B$ and, for ease of notation, $\forall\lambda \in \Lambda$, let λ be a cutpoint which determines B_λ.

From Lemma 6.4.6 we know that $\forall\lambda \in \Lambda$, all the z-branches which do not contain λ (of which there is at least one) are contained in B_λ. We wish to choose one which is contained in all of them, that is, in B. We claim that Λ is itself contained in a single z-branch, so that any other z-branch is appropriate for our purposes.

First we show that if λ, λ' are cutpoints such that $B_\lambda \subseteq B_{\lambda'}$, then λ and λ' belong to the same z-branch. To see this, note that $\lambda' \notin B_\lambda$, simply because $\lambda' \notin X \setminus \{\lambda'\} \supseteq B_{\lambda'} \supseteq B_\lambda$. So $X \setminus B_\lambda$, which is connected by Lemma 6.2.1, contains λ and λ' but not z. Hence $\mathbf{B}_z(\lambda) = \mathbf{B}_z(\lambda')$.

Now let λ_1, λ_2 be arbitrary cutpoints in Λ. Since a cutend is directed, there exists a third cutpoint, λ_3, such that $B_{\lambda_3} \subseteq (B_{\lambda_1} \cap B_{\lambda_2})$. Then by the above argument $\mathbf{B}_z(\lambda_1) = \mathbf{B}_z(\lambda_3) = \mathbf{B}_z(\lambda_2)$.

So there exists a z-branch Z which is properly contained in B_λ, independently of the choice of $\lambda \in \Lambda$ (note $z \in B_\lambda \setminus Z$). But then $Z \cup \{B_\lambda\}_{\lambda\in\Lambda}$ is a set of branches, lower-directed with respect to the branching partial order, and strictly containing the cutend $\{B_\lambda\}_{\lambda\in\Lambda}$, a contradiction to the maximality of a cutend.

We conclude that B does not contain any cutpoints. ✠

6.4.9 Definition: A marginal point of a connected topological space X is a point belonging to the common intersection of all the branches in some cutend of X.

6.4.10 Question: If K is a component of the set of marginal points, is it true that $|\partial(K)| \leq 1$?

6.4.11 ***Lemma:*** *Let $E := \{B_\lambda\}_{\lambda\in\Lambda}$ be a cutend of a fern. Then $B := \bigcap_{\lambda\in\Lambda} B_\lambda$ consists of at most one point.*

Proof: Once more we denote, for all $\lambda \in \Lambda$, the cutpoint which determines B_λ by λ. First we claim that no two points in B can not be disconnected by a cutpoint of X.

Let x, y be distinct points in A, and suppose that the cutpoint z disconnects them. For all $\lambda \in \Lambda$, since $B_\lambda \cup \{\lambda\}$ is connected and B_λ contains both x and y, we must have $z \in B_\lambda \cup \{\lambda\}$. However, λ itself does not disconnect x, y (since these two points belong to the same λ-branch, namely B_λ). So $z \in B_\lambda$, and since $\lambda \in \Lambda$ is arbitrary, the cutpoint $z \in B$, contradicting Lemma 6.4.8.

Hence no two points in B can be disconnected by a cutpoint. Since any two non-incident points in a fern can be disconnected by a cutpoint, any two points in B are incident, and since no three points are pairwise incident, B consists of at most two incident points. Suppose again that x, y are distinct points in B.

Since they can not be disconnected, from property (CSf) we have that for every other point z one of x, y disconnects the other from z. Since a fern contains at least three points, there exists such a point z. But then one of x, y is a cutpoint in B, again contradicting Lemma 6.4.8. ✠

6.4.12 Fact: A filter on a set X is a fixed ultrafilter if and only if it is precisely the collection of sets containing a fixed point.

Reference: See [61], Example 12.13 a). □

6.4.13 ***Corollary:*** *A fixed cutend of a fern is a filter base for a fixed ultrafilter.* ✠

6.4.14 ***Corollary:*** *No two marginal points of a fern are incident. The set of marginal points of a fern is totally disconnected.*

Proof: Suppose x, y are two incident marginal points, and let $\{B_\lambda\}_{\lambda\in\Lambda}$ be a cutend such that $x \in B := \bigcap_{\lambda\in\Lambda} B_\lambda$. Since incident points can not be disconnected, and since y is not itself a cutpoint (by Lemma 6.4.8), for all $\lambda \in \Lambda$, we have that $y \in B_\lambda$, that is, $y \in B$.

So B contains two points, contradicting Lemma 6.4.11. Hence no two marginal points are incident.

So suppose that $x, y \in C$, where C is a connected set of marginal points. Since x, y are not incident, there is some cutpoint which disconnects them. Since C is connected, this point is in C. But all points in C are marginal, and marginal points are not cutpoints. Hence the only connected sets consisting exclusively of marginal points are the singletons, that is, the set of marginal points is totally disconnected. ✠

The above result is analogous to the following fact.

6.4.15 Fact: Let D be a rim-compact dendritic space. Then the set of non-cutpoints is a zero-dimensional.

Reference: Follows from Theorem 5, Section 3, Chapter III in [10]. □

6.4.16 *Conjecture:* *If a fern is flimsy, the set of marginal points is zero-dimensional.*

6.4.17 *Lemma:* *Let z be a non-cutpoint of a fern, and C the set of cutpoints. Then $\mathcal{E} := \{\mathbf{B}_c(z)\}_{c \in C}$ is a cutend of F.*

Proof: First we show that $\mathcal{E}$ is lower-directed. Let c_1, c_2 be arbitrary cutpoints. We wish to show that there exists some cutpoint c for which $\mathbf{B}_c(z) \subseteq (\mathbf{B}_{c_1}(z) \cap \mathbf{B}_{c_2}(z))$. This is trivial if $\mathbf{B}_{c_1}(z)$ and $\mathbf{B}_{c_2}(z)$ are comparable under set inclusion.

From Lemma 6.4.7, we have that if c_1 disconnects c_2 and z, then $\mathbf{B}_{c_1}(z) \subseteq \mathbf{B}_{c_2}(z)$, and if c_2 disconnects c_1 and z, then $\mathbf{B}_{c_2}(z) \subseteq \mathbf{B}_{c_1}(z)$. So if $\mathbf{B}_{c_1}(z)$ and $\mathbf{B}_{c_2}(z)$ are incomparable, no one point among c_1, c_2, z disconnects the other two (z is a non-cutpoint), so by property (CSf) there must be a fourth point c which simultaneously disconnects all three. But then by the same lemma $\mathbf{B}_c(z) \subseteq (\mathbf{B}_{c_1}(z) \cap \mathbf{B}_{c_2}(z))$. This concludes the proof that $\mathcal{E}$ is lower-directed.

As for maximality, let B be a branch not in $\mathcal{E}$, that is, a c-branch not containing z, for some cutpoint c. Then $B \cap \mathbf{B}_c(z) = \emptyset$, implying that, since branches are non-empty, B and $\mathbf{B}_c(z)$ can have no common lower bound. This shows that no set of branches strictly containing $\mathcal{E}$ can be lower-directed. Hence $\mathcal{E}$ is a cutend. ✠

6.4.18 *Corollary:* *Every point of a fern is a cutpoint or a marginal point, but not both.*

The above property is analogous to property (CoE) of dendrites (Fact 6.1.2). However, it is not sufficient to characterize ferns.

6.4.19 Example: Let $\{A_x\}_{x\in[0,2\pi]}$ be a collection of pairwise disjoint arcs (equipped with a topology), and, for $x \in [0, 2\pi]$, let a_x denote a fixed terminal point of A_x. Let S_1 denote the set $\{a_x\}_{x\in[0,2\pi]}$, equipped with a topology such that S_1 is homeomorphic to the unit circle. Now we define a topology on $J := \bigcup_{x\in[0,2\pi]} A_x$ by declaring a subset U to be open if and only if $U \cap S_1$ is open in S_1 and, $\forall x \in [0, 2\pi]$, $U \cap A_x$ is open in A_x. Then J is a Hausdorff space which is not a fern, but every point is a cutpoint or a marginal point.

6.4.20 ***Lemma:*** *Let x be a marginal point of a fern. Then x is open or closed, and is incident with at most one point.*

Proof: Let $\mathcal{E} := \{B_\lambda\}_{\lambda\in\Lambda}$ be a cutend such that $\{x\} = \bigcap_{\lambda\in\Lambda} B_\lambda$, where, $\forall\ \lambda \in \Lambda$, λ is the cutpoint which determines B_λ, and consider the set $C := \bigcap_{\lambda\in\Lambda} \mathsf{Cl}(B_\lambda)$.

Suppose first that $\mathcal{E}$ has no minimum. Since a cutend is lower-directed, $\forall\lambda \in \Lambda$ there exists $\lambda' \in \Lambda$ such that $B_{\lambda'} \subsetneq B_\lambda$. This implies that $\lambda' \in B_\lambda$, for if λ' were in any other λ-branch, say B, then $F \setminus B$ would be a connected subset of F containing B_λ but not λ', and in particular B_λ would be entirely contained in a single λ'-branch, so that no λ'-branch could be strictly contained in B_λ, as $B_{\lambda'}$ is. But then $\mathsf{Cl}(B_{\lambda'}) \subseteq B_{\lambda'} \cup \{\lambda'\} \subseteq B_\lambda$. So $\lambda \notin C$, and since $\lambda \in \Lambda$ is arbitrary, $C = \{x\}$. But C, being the intersection of closed sets, is closed. Hence $\{x\}$ is closed. Moreover, for any other point y, there exists some cutpoint λ for which $y \notin B_\lambda$, and a second cutpoint λ' for which $B'_\lambda \subsetneq B_\lambda$ and the set $X \setminus B_\lambda$, which contains y and, by Lemma 6.2.1, is connected, is contained in $\mathbf{B}_{\lambda'}(\lambda)$. Hence $\mathsf{Cl}(y) \subseteq \mathsf{Cl}(X \setminus B_\lambda) \subseteq \{\lambda'\} \cup \mathbf{B}_{\lambda'}(\lambda)$, which is disjoint from $B_{\lambda'}$ and in particular from x. Thus x is not in the closure of y, and since x is closed, neither is y in the closure of x. So in this case x is not incident with any other point.

Suppose instead that $\mathcal{E}$ does have a minimum. This means that $\{x\}$ is a component of $F\setminus\{\lambda\}$ for some cutpoint λ. By Lemma 6.2.2, if λ is open, x is closed and $\rho(x) = \{\lambda\}$, and if λ is closed, x is open and $\partial(x) = \{\lambda\}$. In both cases, x is incident with, and only with, λ. ✠

6.4.21 ***Corollary:*** *A fern is an S_1 space, that is, a topologized hypergraph.*

Proof: From Corollary 6.4.18, every point is a cutpoint or a marginal point. From Lemma 6.4.20, marginal points are open or closed, and from Remark 2.1.10, cutpoints are open or closed. ✠

6.4.22 Example: This example is illustrated in Figure 6.1. Let D denote the subset of $\mathbb{R}$ consisting of the reciprocals of the positive integers. Consider the subspace of $\mathbb{R}^2$ (with the usual Euclidean topology) given by $C := (D \times [0, 1]) \cup ([0, 1] \times \{0\})$. It is easy to verify that C is a fern, and that $p := (0,0)$ is a marginal point. However, p is not a topological endpoint, that is, there exist neighbourhoods of p such that no smaller neighbourhood of p has a boundary consisting of one point.

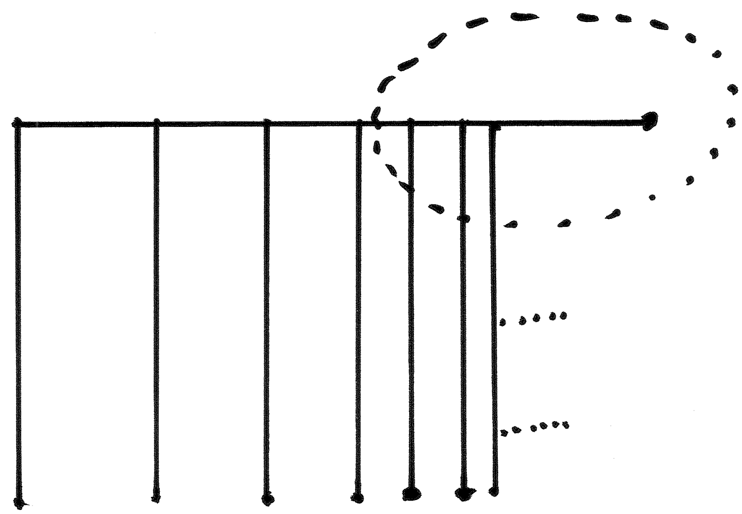


Figure 6.1: A marginal point which is not a topological endpoint.

6.4.23 Question: Given a T_1 fern (equivalently, a locally connected dendritic space), or, more generally, a flimsy fern, is it always possible to replace the topology with a rim-compact, or locally-compact, topology while preserving the collection of connected sets?

6.4.24 *Corollary:* *Let F be a loopless fern. Then the set of cutpoints of F is dense in F.*

Proof: First we claim that every open point is a cutpoint. Let e be an open point. Since F is loopless, e is incident with at least two (necessarily closed) points, v_1 and v_2. Since no three points are pairwise incident (or simply because no two closed points can be incident), v_1 and v_2 are not themselves incident. So there must some point which disconnects them. Since $\{e, v_1, v_2\}$ is connected, this point can only be e itself. Hence e is a cutpoint.

Let U be a non-empty open subset of F. We wish to show that U contains a cutpoint. This holds if U contains an open point. So we assume that U consists entirely of closed points. Since F is locally connected, there exists a connected open subset W of U. The open set W cannot consist of a single (closed) point, since then it would be a clopen singleton, contradicting the fact that a fern is a connected space with more than one point. So W contains at least two closed points, which can not be incident, and hence can be disconnected by a cutpoint c. Since W is connected, by Lemma 6.2.12, $c \in W \subseteq U$. ✠

6.4.25 ***Lemma:*** *Let c be an arbitrary cutpoint of a fern and $\mathcal{E}$ an arbitrary cutend. Then precisely one c-branch belongs to $\mathcal{E}$.*

Proof: Let $\mathcal{E} = \{B_\lambda\}_{\lambda\in\Lambda}$, where B_λ is a λ-branch. Note that no two disjoint branches could belong to a cutend of any space, since they could never have a common lower bound, branches being non-empty. So it is sufficient to show that some c-branch is in $\mathcal{E}$. This is trivial if $c \in \Lambda$. So suppose that this is not the case.

First we claim that there exists some $B \in \mathcal{E}$ which is entirely contained in one of the c-branches. By Lemma 6.4.6, for every $\lambda \in \Lambda$, the only λ-branch which is not contained in B_λ is the one which contains c. So if our claim were not to be true, the cutpoint c would belong to all the branches in $\mathcal{E}$, contradicting Lemma 6.4.8.

So let B denote a c-branch which contains some branch B' in $\mathcal{E}$, and suppose $B \notin \mathcal{E}$. Then for any pair $\{B, B''\}$ of branches with $B'' \in \mathcal{E}$, any common lower bound of B', B'' in $\mathcal{E}$ is also a common lower bound for B, B'', so $\mathcal{E} \cup \{B\}$ is lower-directed, and yet strictly contains $\mathcal{E}$, a contradiction. We conclude that the c-branch $B \in \mathcal{E}$. ✠

6.4.26 ***Corollary:*** *Let C be the set of cutpoints of a fern F. A set of branches of F is a fixed cutend if and only if it is of the form $\{\mathbf{B}_c(z)\}_{c\in C}$ for some $z \notin C$. The fixed cutends of F are in one-to-one correspondence with the marginal points.*

6.5 A Partial Order Characterization of Ferns

6.5.1 ***Theorem:*** *Suppose X is a connected, locally-connected topological space. The following are equivalent:*

A *for any three points, either one disconnects the other two or else there is a fourth which disconnects all three;*

B *X can be given the structure $(X, \trianglelefteq)$ of a lower semilattice such that:*

- *for any three distinct points a, b, c, we have that b disconnects a and c if and only if $b \lhd x$ holds for precisely one choice of $x \in \{a, c\}$ or else $b = \inf\{a, c\}$.*

If (A) is satisfied, then the binary relation $\trianglelefteq$ in (B) can always be chosen so that:

($*$) *if x is a minimum, then x is not a cutpoint.*

Proof: (A) $\Rightarrow$ (B):

Since a fern has at least three points and satisfies properties (CI) and (CSd), it has at least one cutpoint, and therefore a non-empty cutend. Let $\mathcal{C}$ be a cutend of X. For the purposes of this proof, given a cutpoint c, we shall denote by $\mathbf{B}_c(\mathcal{C})$ the c-branch in $\mathcal{C}$ (which we know to exist by Lemma 6.4.25). Also, for another point x, we shall say that "c disconnects x from the cutend" if $x \notin \mathbf{B}_c(\mathcal{C})$.

Definition of the binary relation

Given an arbitrary ordered pair (a, b) of points of X, we define $a \trianglelefteq b$ if and only if one of the following occurs:

(1) $a \in \bigcap_{B \in \mathcal{C}} B$;

(2) $a = b$;

(3) a is a cutpoint which disconnects b from the cutend.

Note that $\mathcal{C}$ need not be a fixed cutend; if not, Condition 1 is vacuous. Indeed, there may not be any fixed cutends. If $\mathcal{C}$ is fixed, then the corresponding marginal point is a minimum, and this is the only marginal point which has any upper bounds at all.

Verification of the properties of a partial order

We need to verify that this binary relation is reflexive, antisymmetric and transitive. Reflexivity is guaranteed by the definition. To see that $\trianglelefteq$ is antisymmetric, suppose that $x \trianglelefteq y$, $y \trianglelefteq x$ but $x \neq y$. Condition (2) is thus excluded. If $x \trianglelefteq y$ holds by virtue of (1) with $x = a$, then since $x \neq y$ from Lemma 6.4.11 it follows that $y \trianglelefteq x$ can not hold also by virtue of (1) (with $a = y$), so it would have to by virtue of (3). That is, y would be a cutpoint disconnecting x from the cutend, contradicting the fact that $\mathbf{B}_y(x)$, which obviously contains x, is the y-branch in $\mathcal{C}$, precisely because it contains x and $x \in \bigcap_{B \in \mathcal{C}} B$. Hence $x \trianglelefteq y$ cannot occur as (1), and similarly neither can $y \trianglelefteq x$.

The remaining possibility is that $x \trianglelefteq y$ and $y \trianglelefteq x$ both hold by virtue of (3), that is, x and y are both cutpoints, $\mathbf{B}_x(y) \neq \mathbf{B}_x(\mathcal{C})$ and $\mathbf{B}_y(x) \neq \mathbf{B}_y(\mathcal{C})$. Then from Lemma 6.4.6

$\mathbf{B}_x(\mathcal{C}) \subseteq \mathbf{B}_y(x)$ and therefore from Lemma 6.4.4 $\mathbf{B}_y(x) \in \mathcal{C}$, contradicting the fact that y disconnects x from the cutend. This shows that $\trianglelefteq$ is antisymmetric.[2]

To see that $\trianglelefteq$ is transitive, suppose $x \trianglelefteq y \trianglelefteq z$. We wish to show to that $x \trianglelefteq z$. This is trivial if $x = y$ or $y = z$, and, by reflexivity, also if $x = z$. So we may assume that all three points are distinct. If $x \in \bigcap_{B \in \mathcal{C}} B$, then $x \trianglelefteq z$ holds by virtue of (1). So we also assume that this is not the case, so that $x \trianglelefteq y$ can only hold by virtue of (3), implying that x is a cutpoint and $\mathbf{B}_x(\mathcal{C}) \neq \mathbf{B}_x(y)$. If $z \notin \mathbf{B}_x(\mathcal{C})$, then x disconnects z from the cutend and $x \trianglelefteq z$ holds by virtue of (3).

Now since $x \lhd y$, and we have already shown $\trianglelefteq$ to be antisymmetric, y is not the minimum, and $y \trianglelefteq z$ can not hold by virtue of (1). Therefore it must hold by virtue of (3). In particular, y is a cutpoint. But by Lemma 6.4.6 the y-branch containing x also contains $\mathbf{B}_x(\mathcal{C})$, and therefore by Lemma 6.4.4 also belongs to the cutend $\mathcal{C}$. So if $z \in \mathbf{B}_x(\mathcal{C})$, y does not disconnect z from the cutend, and $y \trianglelefteq z$ can not hold by virtue of (3), a contradiction. This concludes the proof of transitivity.

Existence of the infimum

At this point we observe that every point p which is not a minimum can be disconnected from the cutend. For otherwise $p \in \bigcap_{B \in \mathcal{C}} B$ and p is the minimum.

To show the existence of the infimum, suppose a and b are arbitrary points. If they are comparable, then the smaller one is the required infimum (reflexivity guarantees that it is a common lower bound, the fact that it is a *greatest* lower bound is trivial, and antisymmetry guarantees uniqueness). So suppose a and b are not comparable. Then neither a nor b is a minimum, and by the above remark both can be disconnected from the cutend, that is, there exist cutpoints c_a and c_b, a c_a-branch B_a and a c_b-branch B_b such that $a \notin B_a$, $b \notin B_b$ but $B_a, B_b \in \mathcal{C}$. Since $\mathcal{C}$ is downward-directed there exists a c-branch B_c for some cutpoint c such that $B_c \in \mathcal{C}$ and $B_c \subseteq B_a \cap B_b$.

Clearly neither a nor b belong to B_c, that is, c disconnects both a and b from the cutend. We claim that c may be chosen so that it also disconnects a and b. Suppose that $\mathbf{B}_c(a) = \mathbf{B}_c(b)$, and let us denote this set by B_{ab}. First we show that neither of a, b disconnects the other from c. Suppose b disconnects a and c, that is, b is a cutpoint and $a \notin \mathbf{B}_b(c)$. From Lemma 6.4.6 $B_c \subseteq \mathbf{B}_b(c)$ and, since $B_c \in \mathcal{C}$, from Corollary 6.4.4 $\mathbf{B}_b(c) \in \mathcal{C}$, implying that b disconnects a and the cutend, contradicting the fact that a and b are incomparable. Similarly, a does not disconnect b and c.

[2] Note that for any antisymmetric binary relation, minima and maxima, and consequently greatest lower bounds and least upper bounds, are unique when they exist.

Since we are assuming $\mathbf{B}_c(a) = \mathbf{B}_c(b)$, the cutpoint c does not disconnect a and c either. Hence no one of a, b, c disconnects the other three, and by property (CSf) there exists a fourth point c' which simultaneously disconnects all three. By Lemma 6.2.12, since in particular c' disconnects a and b, both of which belong to the connected set B_{ab}, we have that $c' \in B_{ab}$. In particular, $\mathbf{B}_c(c') \neq \mathbf{B}_c(\mathcal{C})$ and from Lemma 6.4.6 $\mathbf{B}_c(\mathcal{C}) \subseteq \mathbf{B}_{c'}(c)$ and again by Corollary 6.4.4 $\mathbf{B}_{c'}(c) \in \mathcal{C}$. Hence c' disconnects a and b and simultaneously disconnects each one from the cutend. So we may assume $c = c'$, that is, $\mathbf{B}_c(a) \neq \mathbf{B}_c(b)$. The following claim shows that c is then the required infimum.

Claim: Suppose c is a cutpoint and $\mathbf{B}_c(a)$, $\mathbf{B}_c(b)$ and $\mathbf{B}_c(\mathcal{C})$ are distinct. Then $\inf\{a, b\}$ exists and is the cutpoint c.

Proof of Claim: By virtue of (3), we have that $c \trianglelefteq a$ and $c \trianglelefteq b$. We need to show that, if x is another common lower bound for a, b, then $x \trianglelefteq c$. This is trivial if x is the minimum. So we assume that x is not the minimum, so that $x \trianglelefteq a, b$ can not hold by virtue of (1).

Next we show that a, b are incomparable[3]. Since they do not belong to $\mathbf{B}_c(\mathcal{C})$, neither a nor b is a minimum, so that $a \trianglelefteq b$ could only hold by virtue of (3), with a being a cutpoint. But in that case by Lemma 6.4.6 $\mathbf{B}_c(\mathcal{C}) \cup \mathbf{B}_c(b) \cup \{c\} \subseteq \mathbf{B}_a(c)$ and from Lemma 6.4.4 $\mathbf{B}_a(c)$, which contains b, is the a-branch contained in $\mathcal{C}$, and a does not disconnect b from the cutend. Hence $a \ntrianglelefteq b$, and similarly $b \ntrianglelefteq a$.

Therefore $x \trianglelefteq a, b$ can not hold by virtue of (2) either. Hence they occur by virtue of (3), and x is a cutpoint.

Now if $x \notin (\mathbf{B}_c(\mathcal{C}) \cup \mathbf{B}_c(a))$, then by Lemma 6.4.6 both $\mathbf{B}_c(\mathcal{C})$ and $\mathbf{B}_c(a)$ (and in particular a) are contained in the same x-branch, namely $\mathbf{B}_x(c)$. But since this x-branch contains $\mathbf{B}_c(\mathcal{C})$, again by Lemma 6.4.4 it is the one which belongs to $\mathcal{C}$, whence x does not disconnect a from the cutend, contradicting the fact that $x \trianglelefteq a$ holds by virtue of (3). Similarly, if $x \notin (\mathbf{B}_c(\mathcal{C}) \cup \mathbf{B}_c(b))$, then $x \trianglelefteq b$ can not hold. Since $x \trianglelefteq a, b$, we have that $x \in \mathbf{B}_c(\mathcal{C})$.

Now by Lemma 6.4.6 we have that $\mathbf{B}_c(a) \cup \mathbf{B}_c(b) \cup \{c\} \subseteq \mathbf{B}_x(c)$. Since $a \in \mathbf{B}_c(a)$ and $x \triangleleft a$ holds by virtue of (3), that is, x disconnects a from the cutend, the x-branch in $\mathcal{C}$ is not $\mathbf{B}_x(c)$. Hence x disconnects c from the cutend, so by (3) $x \trianglelefteq c$. □

We now go on to show that $\trianglelefteq$ has the claimed interaction with the topology. Again, let a, b, c be distinct points and suppose $b \trianglelefteq a$ but $b \ntrianglelefteq c$. Since b can not be a minimum, $b \trianglelefteq a$ can only hold by virtue of (3), so b is a cutpoint disconnecting a from the cutend. Now if $c \notin \mathbf{B}_b(\mathcal{C})$ we would also have $b \trianglelefteq a$, so $c \in \mathbf{B}_b(\mathcal{C})$ and b disconnects a from c.

We showed in the construction of the infimum above that any two incomparable points are disconnected by their (unique) infimum. So if $b = \inf\{a, c\}$ (which implies that a, c are incomparable), b disconnects a and c, as required.

[3] We already know this in the context above, but we shall invoke this claim once more.

Conversely, suppose b disconnects a and c. Suppose first that one of $\mathbf{B}_b(a)$ and $\mathbf{B}_b(c)$, say $\mathbf{B}_b(c)$, is the c-branch in $\mathcal{C}$. By virtue of (3) $b \trianglelefteq a$. Since b is a cutpoint, $b \triangleleft c$ can not hold by virtue of (1) and since $\mathbf{B}_b(\mathcal{C}) = \mathbf{B}_b(c)$, neither can it hold by virtue of (3). Hence $b \ntrianglelefteq c$ and $b \triangleleft x$ holds for precisely one of choice of $x \in \{a, c\}$.

Alternatively, $\mathbf{B}_b(\mathcal{C})$ is neither $\mathbf{B}_b(a)$ nor $\mathbf{B}_b(c)$. Then by the claim above $b = \inf\{a, c\}$.

(B) $\Rightarrow$ (A):

Note that, for any two points x, y, we have that $\inf\{x, y\} \in \{x, y\}$ if and only if x and y are comparable. Let a, b, c be arbitrary points in X. Suppose first that some two are comparable, say $a \trianglelefteq b$. Then $a \ntrianglelefteq c$ implies that a disconnects b and c, while $a \trianglelefteq c$ implies that a is a common lower bound for b, c. Then if b and c are comparable, the smaller one disconnects a from the larger one, and if not either $a = \inf\{b, c\}$ and a disconnects b and c, or else $a \triangleleft \inf\{b, c\} \triangleleft b, c$ and $\inf\{b, c\}$ simultaneously disconnects a, b, c. So we may assume that no two points in $\{a, b, c\}$ are comparable, which implies that any two can be disconnected by their infimum, and that this cutpoint is distinct from the third point.

So there exists a point $y = \inf\{a, b\} \neq c$ which disconnects a and b. If a, b, c belong to distinct y-branches, we are done. So suppose, without loss of generality, that $b, c \in B_{bc} := \mathbf{B}_y(b)$. Let $y' := \inf\{b, c\}$. Since y' disconnects b and c and B_{bc} is connected and contains b and c, we have that $y' \in B_{bc}$. We claim that in fact y' simultaneously disconnects a, b, c. This could only fail if $\mathbf{B}_{y'}(c) = \mathbf{B}_{y'}(a)$ or $\mathbf{B}_{y'}(b) = \mathbf{B}_{y'}(a)$.

Suppose first that $\mathbf{B}_{y'}(c) = \mathbf{B}_{y'}(a)$. Then y' disconnects a and b. Since $y' \neq y = \inf\{a, b\}$ and $y' = \inf\{b, c\} \trianglelefteq b$, we must have that $y' \ntrianglelefteq a$. But $y' \trianglelefteq c$, so y' must disconnect a and c, a contradiction.

Suppose instead that $\mathbf{B}_{y'}(b) = \mathbf{B}_{y'}(a)$. Then y' disconnects a and c. Also, $y' = \inf\{b, c\} \trianglelefteq c$, so either $y' \ntrianglelefteq a$ or $y' = \inf\{a, c\}$. Since $y' \trianglelefteq b$, if $y' \ntrianglelefteq a$ then y' should disconnect a and b, a contradiction. So $y' = \inf\{a, c\}$.

Now since y disconnects a and c and $y = \inf\{a, b\} \trianglelefteq a$ but $y \neq y' = \inf\{a, c\}$, we must have $y \ntrianglelefteq c$. But $y \trianglelefteq b$, so y should disconnect b and c, contradicting the assumption that $b, c \in B_{bc} = \mathbf{B}_y(b)$. We conclude that $\mathbf{B}_{y'}(a)$, $\mathbf{B}_{y'}(b)$ and $\mathbf{B}_{y'}(c)$ are distinct, that is, y' simultaneously disconnects a, b and c. ✠

We remark that cutpoint partial orders always have a minimum, and in those spaces with no marginal points (such as the real line), condition $(*)$ is not verified by any cutpoint partial order.

We summarize the most important characterizations of ferns in the following corollary.

6.5.2 ***Corollary:*** *Let X be a connected, locally connected topological space with at least three points. The following are equivalent:*

- *X is a fern (for any three points, either one disconnects the other two or else there is a fourth which disconnects all three);*
- *X is S_1 and for every two non incident points, there exists a third which disconnects them;*
- *for any two points a, b, $D[a, b]$ is an ab-path;*
- *X can be given the structure $(X, \trianglelefteq)$ of a lower semilattice such that:*
  - *for any three distinct points a, b, c, we have that b disconnects a and c if and only if $b \lhd x$ holds for precisely one choice of $x \in \{a, c\}$ or else $b = \inf\{a, c\}$, and*
  - *if x is a minimum, then x is not a cutpoint;*
- *the arbitrary intersection of connected subsets is connected;*
- *X is uniquely and hereditarily path-connected;*
- *for any two points a, b, $D[a, b]$ is connected.* ✠

Chapter 7

Directions for further research

In this chapter we briefly describe a few topics for further research which emerge from this work, and which, we hope, will be tackled in the future.

7.1 The Jordan Curve Theorem

The Jordan Curve Theorem states that a simple closed curve in the plane partitions its complement into two connected components. Since the plane is the cartesian product of the real line with itself, and since the real line is an unbounded path (in the sense of Definition 2.3.5), the following question arises naturally.

Suppose that P is an arbitrary unbounded path, and that the subspace C of $P \times P$ is a cycle in the sense of Definition 2.4.3. Is it still true that the complement of C in $P \times P$ consists of precisely two components A, B, such that $\partial(A) = \partial(B) = C$ and precisely one of $\mathsf{Cl}(A)$, $\mathsf{Cl}(B)$ is compact? We conjecture that the answer to this question is yes.

The usual metric on $\mathbb{R}^2$ can be a useful tool in the proof of the Jordan Curve Theorem. Note, however, that in general the cartesian product $P \times P$ need not even be Hausdorff.

7.2 Menger's Theorem

Currently one of the areas of most active research in infinite graph theory relates to Menger's Theorem. Erdős conjectured that, given an infinite graph and vertex sets A, B, there always exist a set S of vertices S which disconnects A and B, a set $\mathcal{P}$ of disjoint (A, B)-paths, and a bijection $\phi : S \to \mathcal{P}$ such that, for every $s \in S$, we have that $s \in \phi(s)$. The conjecture has been verified under various assumptions by various authors; see [1] for a survey. In particular, this question has recently been considered in a topological setup by Diestel [15, 17] and Bruhn, Diestel and Stein [11].

The question arises as to what topological assumptions are required for the assertion to hold in the general context of V-totally-disconnected topologized graphs.

7.3 Ends and Completions

Ends of topological spaces are used to obtain the Freudenthal compactification of *rim-compact* topological spaces. The problems associated with spaces which are not rim-compact are illustrated for example by the Diestel-Kühn obstruction, which may arise in the direction extension of a non-locally-finite (non-rim-compact) graph.

Polat [50] has used the concepts of uniformities and completions to obtain results concerning classical (possibly non-locally-finite) graphs together with their ends. Uniformizable spaces are necessarily completely regular, and in particular Hausdorff. Polat obviates this obstruction by defining a uniform structure on the *vertex set* of a graph, in terms of the entire structure of graph (including edges).

We believe that it is possible to meaningfully construct a larger space from an arbitrary locally connected, possibly non-rim-compact, topological space X by adding its ends "at infinity". Let us refer to such this space as the Freudenthal-Polat completion of X. Since non-trivial classical topologized graphs are not Hausdorff, we would like to construct this space without assuming that X is Hausdorff. This entails the use of quasi-uniformities, as opposed to uniformities. Although any topological space is quasi-uniformizable, this point complicates the issue because the theory of completions for quasi-uniform spaces appears to be not as tidy as that for uniform spaces.

Ferns seem to offer a good class of spaces in which to explore these ideas. The partial order in the proof of Theorem 6.5.1 was constructed with these issues in mind.

7.3.1 ***Conjecture:*** *Let F be a fern. The following are in one-to-one correspondence:*

- *the cutends of F;*
- *the partial orders on F satisfying conditions (B) and ($*$) of Theorem 6.5.1*

The free cutends correspond to partial orders without a minimum, and the fixed cutends to partial orders with minimum.

If F is T_1 (equivalently, T_2), the following are in one-to-one correspondence:

- *the free cutends of F;*
- *the Freudenthal ends of F.*

Gurin [26] has shown that a rim-compact dendritic space is locally connected. Let us refer to a connected topological space satisfying the conditions of Proposition 6.3.1 as a pre-fern.

7.3.2 ***Conjecture:*** *Let F be a rim-compact pre-fern. Then F is locally connected and is compact if and only if all the cutends are fixed. If F is not compact, the Freudenthal-Polat completion of F coincides with the Freudenthal compactification of F.*

Proizvolov [51] has shown that a rim-compact dendritic space admits a unique dendritic compactification, and Allen [4] recognized it as the Freudenthal compactification.

7.3.3 ***Conjecture:*** *Let F be a fern. The Freudenthal-Polat completion of F is a fern, and coincides with F if and only if F all the cutends are fixed.*

7.3.4 ***Conjecture:*** *If F is a classical graph, the vertex set of the Freudenthal-Polat completion coincides with the Polat completion of the vertex set of F.*

7.4 Graph Minors

In (1.1.32) we raise the issue of a "topological proof" of the Graph Minors Theorem of Robertson and Seymour. This may be a overly ambitious aim, but already when one restricts one's attention to specific classes of compact topological spaces this appears to be an interesting non-trivial question. The question arises, for example, whether or not compact paths are well-quasi-ordered; in our opinion, this question is also interesting in the light of the fact that paths come with a total order, and that the ordinals are well-quasi-ordered. The same question can be posed, for example, for compact ferns, or compact weakly Hausdorff V-totally-disconnected topologized graphs.

Appendix A: Binary Relations

A binary relation on a set X is a subset $R \subseteq X \times X$. One often writes (especially in the context of *transitive* binary relations, see below) $x \leq y$ synonymously with $(x, y) \in R$, and $x < y$ as an abbreviation for $x \leq y$ and $x \neq y$. A binary relation is:

- reflexive if $x \leq x$ for all $x \in X$;
- transitive if, for all $x, y, z \in X$, whenever $x \leq y$ and $y \leq z$ we also have $x \leq z$;
- symmetric if, for any two elements $x, y \in X$, whenever $x \leq y$ we also have that $y \leq x$;
- antisymmetric if, for any two elements $x, y \in X$, whenever $x \leq y$ and $y \leq x$ we have that $x = y$; and
- decisive if for any two elements $x, y \in X$ at least one of $x \leq y$ and $y \leq x$ holds.

An equivalence relation is a reflexive, symmetric, transitive binary relation. Given a set $S \subseteq X$, an upper bound for S is an element $p \in X$ such that $x \leq p$ for all $x \in S$; lower bounds are defined analogously. An element of X is maximal (minimal) if it is not bounded from above (below) by any other element.

A binary relation is upper-directed (lower-directed) if for any two elements there exists a common upper (lower) bound. A partial order is a transitive, reflexive, antisymmetric binary relation. A total order, also sometimes referred to as a linear order in the literature, is a decisive partial order. Totally ordered subsets are also referred to as nests or chains. We avoid using the latter term with this connotation, and use the term "chain" with a different meaning (page 57). A directed set is a set equipped with a reflexive, transitive, upper-directed binary relation.

Zorn's Lemma states that if a non-empty set X is equipped with a binary relation such that every nest has an upper bound, then there exists a maximal element in X. Zorn's Lemma is equivalent to the Axiom of Choice.

Given a set S, an upper bound for S is the supremum of S if it is also a lower bound for the set of upper bounds of S. Infima are defined analogously. An extremum is a supremum or an infimum.

A partial order is a lower-semilattice (upper-semilattice) if for every points two x, y, there exists an infimum (supremum) for $\{x, y\}$. A binary relation is order-complete if every subset bounded from above has a supremum, and every subset bounded from below has a supremum.

A maximum (minimum) of a totally ordered subset S is a lower (upper) bound which belongs to S. A subset S of the totally ordered set X is convex if, whenever $a < x < b$ and $a, b \in S, x \in X$, we also have that $x \in S$. A jump is a convex subset consisting of two points. A cut is an ordered pair (A, B) such that $\{A, B\}$ is a non-trivial bipartition and, for all $a \in A, b \in B$, we have that $a < b$. If A has no maximum and B has no minimum, then (A, B) is a gap. A point c is in between two other points a, b if $a < b < c$ or $c < b < a$.

Given sets A, B equipped with respective binary relations, a function $f : A \to B$ is increasing if, whenever $x, y \in A$ and $x \leq y$, then we also have that $f(x) \leq f(y)$, and cofinal if, for any $b \in B$, there exists an $a \in A$ such that $b \leq f(a)$.

Appendix B: General Topology

A topology on a set X is a subset of the power set of X which is closed under arbitrary unions and finite intersections, and contains the empty set and X itself. A topological space is a pair $(X, \mathcal{T})$ where X is a set and $\mathcal{T}$ is a topology on X. An element of $\mathcal{T}$ is said to be an open set. A function from one topological space X to another topological space Y is continuous if the inverse image of an open set is open. A homeomorphism from X to Y is a continuous bijection with continuous inverse. Two topological spaces are homeomorphic if there exists a homeomorphism from X to Y (equivalently, from Y to X).

A closed set is the complement of an open set. A set is clopen if it is open and closed. The closure of a set A, denoted by $\mathsf{Cl}(A)$ or $\overline{A}$, is the intersection of all the closed sets containing A. A subset is dense in the topological space X if its closure coincides with X. The boundary, or frontier, of a set A, denoted by $\partial(A)$, is the intersection of the closures of A and its complement. Given a subset S of a topological space X, an accumulation point for S, also referred to as a cluster point, is a point $x \in X$ such that every neighbourhood of x contains some point in S other than x itself. The derived set of S, sometimes denoted by S', is the set accumulation points of S. A subset is dense-in-itself if it coincides with its derived set. A topological space is perfect if it is a dense-in-itself subset of itself.

Given a subset S of X and a topology $\mathcal{T}$ on X, the relative topology inherited by S is the collection $\{A \cap S \mid A \in \mathcal{T}\}$. Given a point x, or a set S, in X, a neighbourhood of x (or S) is a set containing an open set containing x (respectively, S). Given a subset A, the interior of A is the set of points $a \in A$ for which there exists a neighbourhood of a contained in A. A base for a topology is a subcollection F of open sets such that every open set is a union of elements of F. A subbase is a subcollection H of open sets such that the collection of a finite intersections of elements of H is a base. Given a point x in a topological space X, a system of basic neighbourhoods (also referred to as a neighbourhood base in the literature) is a collection $\mathcal{S}$ of neighbourhoods of x such that every neighbourhood of x contains some element of $\mathcal{S}$. A topological space is connected if it is not the union of two disjoint open non-empty subsets. A space is locally connected if for any point x and any neighbourhood A of x there exists an open connected neighbourhood of x contained in A.

A topological space is Hausdorff (respectively, regular, normal) if for any two distinct points x, y (respectively, for any closed set x and any point $y \notin x$, or for any two disjoint closed sets x, y), there exist disjoint neighbourhoods U_x, U_y of x, y respectively.

A cover of a set Z is a collection of subsets of Z whose union contains V. An open cover of a topological space is a cover consisting of open sets. A topological space is compact if every open cover has a finite subcover. A space is rim-compact if for every point x and every neighbourhood U of x there exists a neighbourhood of x contained in U with compact boundary. Given a cover $\mathcal{A}$, a second cover $\mathcal{B}$ is a refinement if for every $B \in \mathcal{B}$ there exists an $A \in \mathcal{A}$ such that $B \subseteq A$.

The order of a cover $\mathcal{C}$ is the largest integer such that $\mathcal{C}$ contains $n+1$ sets with non-empty intersection. The Lebesgue dimension of a topological space X, denoted by $\dim(X)$, is defined inductively by setting $\dim(X) \leq n$ if every finite open cover is refined by a finite open cover of order at most $n+1$. If, for some integer n (which is necessarily at least -1), we have $\dim(X) \leq n$ but $\dim(X) \not\leq (n-1)$, then $\dim(X) = n$. This definition is taken from [24].

A net in a set X is a function $P : \Lambda \to X$, where Λ is some directed set. A subnet is the composition $P \circ \phi$, where $\phi : M \to \Lambda$ is an increasing cofinal function from a directed set M to Λ (see Appendix A for the definitions of directed sets and increasing and cofinal functions). The net P is often also denoted by $(x_\lambda)_{\lambda \in \Lambda}$, and x_λ stands for $P(\lambda)$.

Given a net $(x_\lambda)_{\lambda \in \Lambda}$ in a topological space X, a cluster point for $(x_\lambda)_{\lambda \in \Lambda}$ is a point x such that, for every neighbourhood U of x, x_λ is cofinally in U, that is, for every $\lambda \in \Lambda$ there exists some $\lambda' \geq \lambda$ such that $x_{\lambda'} \in U$. The net converges to x if, for every neighbourhood U of x, $(x_\lambda)_{\lambda \in \Lambda}$ lies residually in U, that is, there exists some $\lambda_0 \in \Lambda$ such that $\lambda \geq \lambda_0$ implies that $x_\lambda \in U$.

A filter $\mathcal{F}$ on a set X is a non-empty collection of non-empty subsets of X, closed under pairwise intersections, and such that, whenever $F \in \mathcal{F}$ and $F \subseteq F'$, we also have that $F' \in \mathcal{F}$. A subcollection $\mathcal{F}' \subseteq \mathcal{F}$ is a filter base if each element of $\mathcal{F}$ contains some element of $\mathcal{F}'$. A filter $\mathcal{F}$ is fixed is the intersection over all $F \in \mathcal{F}$ is non-empty, and free otherwise.

A distance function d on a set X induces a topology obtained by declaring a subset U to be open whenever $x \in U$ implies that there exists a positive real number ϵ such that all points of distance strictly less than ϵ from x are also in U. A topological space is metrizable if its topology is induced by some distance function. For any non-negative integer n, the Euclidean topology on $\mathbb{R}^n$ is the topology induced by the Euclidean distance. An arc is a topological space homeomorphic to the subset of $\mathbb{R}$ consisting of zero, one and all reals between zero and one, equipped with the relative topology inherited from the Euclidean topology on $\mathbb{R}$.

A continuous function from one topological space to another is monotone if the inverse image of every point is connected. A continuous function from a topological space X onto a subspace A is a retraction of X onto A if the restriction to A is the identity. When such a retraction exists, A is a retract of X. Given a topological space Y, an embedding of Y into X is a homeomorphism from Y onto a subspace of X.

Given an equivalence relation on a topological space X, the quotient topology on the set of equivalence classes is obtained by declaring a set to be open if and only if its inverse image under p is open, where p is the function which associates to a point the equivalence class to which it belongs. The quotient space induced by an equivalence relation is the set of equivalence classes equipped with the quotient topology. In this context, the partition of X into equivalence classes is referred to as a decomposition. We refer to the individual equivalence classes as parts. A part is degenerate if it consists of only one point.

A decomposition is closed if every part is closed. Given a decomposition $\mathcal{D}$ of a topological space X, a subset S is saturated if it is the union of parts of $\mathcal{D}$. A decomposition $\mathcal{D}$ is upper semicontinuous if, for every part $P \in \mathcal{D}$ and every open set U containing P, there exists an open saturated set containing P and contained in U.

A point x of a topological space X is a topological endpoint if every open set containing x contains an open set containing x with precisely one point in its boundary.

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