Hyperpfaffians in Algebraic Combinatorics

by

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I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

The pfaffian is a classical tool which can be regarded as a generalization of the determinant. The hyperpfaffian, which was introduced by Barvinok in [3], generalizes the pfaffian to higher dimension. This was further developed by Luque and Thibon in [15] and Abdesselam in [1]. There are several non-equivalent definitions for the hyperpfaffian, which are discussed in the introduction of this thesis. Following this we examine the extension of the Matrix-Tree theorem to the Hyperpfaffian-Cactus theorem by Abdesselam, proving it without the use of the Grassman-Berezin Calculus and with the new terminology of the non-uniform hyperpfaffian. Next we look at the extension of pfaffian orientations for counting matchings on graphs to hyperpfaffian orientations for counting matchings on hypergraphs. Finally pfaffian rings and ideals are extended to hyperpfaffian rings and ideals, but we show that under reasonable assumptions the algebra with straightening law structure of these rings cannot be extended.

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Chapter 1

Introduction

1.1 Outline

The pfaffian is the signed sum over perfect matchings in the way the determinant is the signed sum over permutations. It is in fact a generalization of the determinant, as will be described in the next section. It has been used in many fields, including combinatorics. The hyperpfaffian generalizes the pfaffian to partitions of $\{1, \ldots, n\}$ with blocks of size other than two. These were first introduced by Barvinok in [3] who created a hyperpfaffian for partitions into blocks of size k where k is even. A new definition was created by Luque and Thibon in [15] which worked for even and odd k. Abdesselam in [1] proved a result which used a structure which is effectively a hyperpfaffian over partitions into sets of non-uniform even size. In this thesis we shall attempt to generalize some results of the pfaffian in combinatorics to results for the hyperpfaffian. These different definitions of the hyperpfaffian will be described later on in this introduction.

The first result we examine is the Hyperpfaffian-Cactus theorem, proved by Abdesselam in [1]. It is a generalization of Masbaum and Vaintrob's Pfaffian-Tree theorem in [16], which is in turn based on the classical Matrix-Tree theorem. This theorem constructs a structure, similar to the Kirchoff matrix , from which the hyperpfaffian generates all the spanning trees (or cacti) of a hypergraph. In chapter two we prove this result without the use of the Grassman-Berezin Calculus used by Abdeselam.

The second topic is the construction of Hyperpfaffian orientations. The pfaffian orientation has seen significant study. The idea is to orient a graph so that the signed sum of its perfect matching becomes the unsigned sum of its perfect matchings. For reasons of computational efficiency, this is a preferable method of counting the number of such matchings. Only certain graphs admit such orientations, such as planar graphs. In fact many of the results in this subject have to do with embedding graphs in surfaces. In chapter three we adapt these results on the embedding of graphs in general to finding orientations of hypergraphs which make the hyperpfaffian an unsigned sum.

The final topic we examine is that of pfaffian rings and ideals. The idea of chapter four is if one takes the standard antisymmetric matrix of invariants, and creates an ideal generated by pfaffian minors of this matrix in the k-algebra generated by the invariants, one can get a quotient ring with interesting combinatorial structure. Many of the results relating to this are based around the algebra with straightening law structure of the pfaffian minors, or at least slightly weaker versions of this. In this thesis we prove that there is no "reasonable" way of putting this structure on the similarly defined hyperpfaffian rings with the uniform hyperpfaffian, and that the structures of the non-uniform hyperpfaffian rings are trivial.

1.2 The Pfaffian

The pfaffian is the lesser known cousin of the determinant. It was first invented by Johann Friedrich Pfaff in 1815, who used it to solve systems of partial differential equations. It was given its name (after Pfaff) by Cayley in 1849, at which time Cayley also proved its relation to the square root of the determinant (which we will see shortly). It has applications to many subjects, such as the enumeration of plane partitions, trees, and perfect matchings.

There are many ways to define the pfaffian. Here we first consider one given by Stembridge in [20]. In this definition we work with a sum over the set of perfect matchings or one-factors on $\{1, \ldots, n\}$. This is the set of partitions of $\{1, \ldots, n\}$ into two-element sets. For example $\{1, 3\}, \{4, 7\}, \{2, 5\}, \{6, 10\}, \{8, 9\}$ is a perfect matching on $\{1, \ldots, 10\}$. Note that no such matchings exist if n is not even. Throughout this thesis we shall assume n is even.

Two elements of the matching, $\{i, j\}$ and $\{k, l\}$, are said to *cross* if $i < k < j < l$. This can be visualized by writing out the numbers in order on a line, and connecting paired elements with half circles above the line. These half circles cross if and only if the pairs they represent cross. Another way to see these crossings visually is to place the numbers in order around a circle and draw straight lines connecting matched pairs. The lines again cross exactly when the pairs they represent cross. The crossing number of a perfect matching is the number of crossing pairs, and the *signum* of the matching is then $(-1)^c$ where c is the crossing number of the matching. We denote this signum by sgn.

Considering the matching above, we can represent it in either of the ways given in Figure 1.1. From these diagrams we see that it has three crossings. Thus its signum is negative.

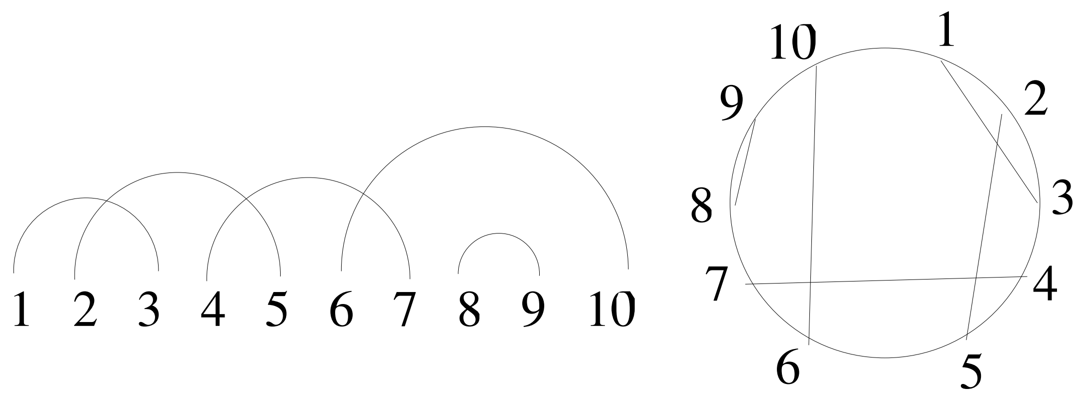

Figure 1.1: Two types of crossing diagrams for a matching

Our first definition of the pfaffian acts on an upper triangle, which we shall define as follows:

Definition 1.1. An *upper triangle*, Λ, of order n is a function on the two element subsets of $\{1, \ldots, n\}$.

This can be considered to be the strict upper triangle of a matrix, where the ij entry of the matrix is $\Lambda(\{i, j\})$ for all $i < j$ and zero otherwise.

If we take a graph with vertices $1, \ldots, n$ and no multiple edges or loops, then it can represent the upper triangle which maps sets of two vertices to the edge that connects them, or a label given to it (zero if there is no edge connecting the two vertices). The *perfect matchings* on this graph correspond to the *perfect matchings* in our sum (with zero terms corresponding to perfect matchings involving missing edges). For this reason we shall often talk of an upper triangle as a graph (as any upper triangle can be represented this way). We shall then refer to the pfaffian of a graph and to the pairs in matchings as edges.

Definition 1.2. Let $\mathcal{M}_n$ be the set of perfect matchings on $\{1, \ldots, n\}$ and Λ be our upper triangle. Then the *pfaffian* of Λ is:

$$pf\left(\Lambda\right) = \sum_{M \in \mathcal{M}_n} \operatorname{sgn}(M) \prod_{\{a,b\} \in M} \Lambda\left(\{a, b\}\right).$$

Consider the example where we set $n = 6$ and define $\Lambda(\{i, j\}) = x_{ij}$ for $i < j$. This corresponds to the complete graph on six vertices where the edge connecting i to j for $i < j$ is labelled x_{ij}. We can see that there are fifteen perfect matchings on six elements:

$$\begin{aligned}&\left(\{1,2\},\{3,4\},\{5,6\}\right), \left(\{1,2\},\{3,5\},\{4,6\}\right), \left(\{1,2\},\{3,6\},\{4,5\}\right),\\ &\quad\left(\{1,3\},\{2,4\},\{5,6\}\right), \left(\{1,3\},\{2,5\},\{4,6\}\right), \left(\{1,3\},\{2,6\},\{4,5\}\right),\\ &\quad\left(\{1,4\},\{2,3\},\{5,6\}\right), \left(\{1,4\},\{2,5\},\{3,6\}\right), \left(\{1,4\},\{2,6\},\{3,5\}\right),\\ &\quad\left(\{1,5\},\{2,3\},\{4,6\}\right), \left(\{1,5\},\{2,4\},\{3,6\}\right), \left(\{1,5\},\{2,6\},\{3,4\}\right),\\ &\qquad\left(\{1,6\},\{2,3\},\{4,5\}\right), \left(\{1,6\},\{2,4\},\{3,5\}\right), \left(\{1,6\},\{2,5\},\{3,4\}\right).\end{aligned}$$

If one checks the signs of these matching one sees that they alternate between positive and negative. In fact when Pfaff first defined the pfaffian this is how he defined the sign of a matching, he put the terms in lexicographical order and assigned signs alternating along this order. The pfaffian is then the following:

$$\begin{aligned}&x_{12}x_{34}x_{56} - x_{12}x_{35}x_{46} + x_{12}x_{36}x_{45} - x_{13}x_{24}x_{56} + x_{13}x_{25}x_{46}\\ &\quad - x_{13}x_{26}x_{45} + x_{14}x_{23}x_{56} - x_{14}x_{25}x_{36} + x_{14}x_{26}x_{35} - x_{15}x_{23}x_{46}\\ &\qquad + x_{15}x_{24}x_{36} - x_{15}x_{26}x_{34} + x_{16}x_{23}x_{45} - x_{16}x_{24}x_{35} + x_{16}x_{25}x_{34}.\end{aligned}$$

Another common definition of the pfaffian comes from the theorem proved by Cayley. Here we consider the pfaffian to act on an *antisymmetric matrix* (recall this is a matrix such that $x_{ij} = -x_{ji}$ and thus $x_{ii} = 0$). Though this sounds different from the upper triangle we used in the previous definition, if we start by using our upper triangle as the strict upper triangle of our matrix, we determine the rest of the matrix by antisymmetry. Thus we define the antisymmetric matrix A from the upper triangle Λ in the following way:

$$A_{ij} = \begin{cases} \Lambda(\{i,j\}) & i < j \\ -\Lambda(\{i,j\}) & i > j \\ 0 & i = j \end{cases}.$$

So the antisymmetric matrix from the previous example is:

$$\begin{bmatrix} 0 & x_{12} & x_{13} & x_{14} & x_{15} & x_{16} \\ -x_{12} & 0 & x_{23} & x_{24} & x_{25} & x_{26} \\ -x_{13} & -x_{23} & 0 & x_{34} & x_{35} & x_{36} \\ -x_{14} & -x_{24} & -x_{34} & 0 & x_{45} & x_{46} \\ -x_{15} & -x_{25} & -x_{35} & -x_{45} & 0 & x_{56} \\ -x_{16} & -x_{26} & -x_{36} & -x_{46} & -x_{56} & 0 \end{bmatrix}.$$

Definition 1.3. The *pfaffian* of an antisymmetric matrix A is defined by the following relation:

$$pf(A)^2 = \det(A),$$

where we choose the positive or negative square root to match our previous definition.

We shall next prove a lemma from [20] that will be useful in proving that this definition is consistent.

Lemma 1.4. *Take a perfect matching π such that $\{i,j\}$ and $\{k,l\}$ are in π and not equal. If neither of i nor l are between j and k, then the matching created by replacing these two pairs in π with $\{i,k\}$ and $\{j,l\}$ has the opposite signum from π.*

Proof. To prove this consider which edges gain or lose crossings by this switch. Since only two edges change, only crossings with at least one of these two edges can change. Aside from the edges $\{i,j\}$ and $\{k,l\}$ themselves, any other edge which gains or loses crossings must have one end between j and k, and the other not. An edge $\{e, f\}$ with both e and f between j and k, does not cross either the new or the old edges. This is because none of i, j, k, l are between e and f, so they can't cross it. Only ends in between j and k have a different order relative to $\{i, j\}$ and $\{i, k\}$ or $\{j, l\}$ and $\{k, l\}$, so edges without such an end cannot gain or lose crossings.

Take an edge $\{e, f\}$ where e is between j and k, and f is not. This edge crosses only one of $\{i, j\}$ or $\{i, k\}$. Since e is between j and k, it is between exactly one of the pairs $\{i, j\}$ and $\{i, k\}$. Since the end f is not between j and k, it is either between both or neither of these pairs. Thus $\{e, f\}$ crosses either $\{i, j\}$ or $\{i, k\}$, not both. Similarly it crosses exactly one of $\{k, l\}$ and $\{j, l\}$. This means our crossing number has changed by two or zero, and either way its parity remains the same.

The last remaining crossing to check is between $\{i, j\}$ and $\{k, l\}$. It is routine to check that $\{i, j\}$ and $\{k, l\}$ cross if and only if $\{i, k\}$ and $\{j, l\}$ do not cross. This gives us a change of one in the crossing number, thus a change in parity, and a reversal of the signum of the matching. □

Equipped with this lemma, we are now prepared to prove the following theorem:

Theorem 1.5. *If Λ is an upper triangle, and A is the antisymmetric matrix it defines, using our original definition of pfaffian (Definition 1.2), then:*

$$pf(\Lambda)^2 = \det(A).$$

Proof. Here we give a proof based on that in [20]. Our first step will be to show that in the standard permutation expansion of this determinant, the contribution of those permutations containing odd cycles is zero.

We shall construct a sign reversing involution on the terms coming from permutations with odd cycles. Given such a permutation choose the odd cycle which has the smallest minimal element. Now replace that odd cycle with its inverse. For example if we had the permutation (125)(34678) we get the permutation (521)(34678). Since the elements in the cycles have not changed, if we apply the involution again we choose the same cycle, and thus we return to our original permutation.

The sign of the permutation does not change, since its cycle type remains the same. Taking the inverse of the cycle replaces $A_{i\sigma(i)}$ with $A_{i\sigma^{-1}(i)}$ for each i in the cycle. Reordering the product by σ we see that this is the same as $A_{\sigma(i)i}$ for each i in the cycle. The antisymmetry of the matrix implies we have replaced $A_{i\sigma(i)}$ with $-A_{i\sigma(i)}$ for each i in the cycle. Since there is an odd number of elements in the cycle, this reverses the sign of the term. Note that aside from the sign the term has not changed, and thus we have a sign reversing involution. For example, the contribution of the permutation (125)(34678) to the pfaffian of A is:

$$A_{12}A_{25}A_{51}A_{34}A_{46}A_{67}A_{78}A_{83}.$$

After applying our involution we get:

$$A_{52}A_{21}A_{15}A_{34}A_{46}A_{67}A_{78}A_{83} = (-A_{25})(-A_{12})(-A_{51})A_{34}A_{46}A_{67}A_{78}A_{83}.$$

We can see this is the negative of our first term, and thus they cancel.

Next we shall show that there is a bijection between the set of permutations containing no odd cycles and the set of ordered pairs of perfect matchings. From any even cycle of σ we construct two matchings of its elements. For a cycle C of length d with lowest element i we construct our matchings in the following way:

$$\pi_{C,1} = \left\{ \{\sigma^{2j-2}(i), \sigma^{2j-1}(i)\} : 1 \leq j \leq \frac{d}{2} \right\},$$

$$\pi_{C,2} = \left\{ \{\sigma^{2j-1}(i), \sigma^{2j}(i)\} : 1 \leq j \leq \frac{d}{2} \right\}.$$

Each element of the cycle occurs exactly once in each of the two matchings (noting, $\sigma^0(i) = \sigma^d(i) = i$). Thus these are indeed perfect matchings. We combine the matchings for the cycles of the permutation to get a pair of matchings for the whole permutation, setting $\pi_1 = \cup_C \pi_{C,1}$ and $\pi_2 = \cup_C \pi_{C,2}$. This gives us a pair of matchings on the whole set. For example, for the permutation $\sigma = (3481)(27)(56)$ we obtain the matchings:

$$\pi_1 = (\{1,3\}, \{4,8\}, \{2,7\}, \{5,6\}), \pi_2 = (\{3,4\}, \{1,8\}, \{2,7\}, \{5,6\}).$$

To show this is a bijection we shall give the inverse function. All of the vertices of the union of these two matchings (π_1, π_2), considered as a graph, have degree 2.

Thus it is a union of disjoint cycles. Since each vertex must have exactly one edge from each matching, all the cycles must be even. These cycles can then be formed into the cycles of the permutations by mapping each element to the next element in the cycle. The only problem is that there are two ways of traveling around a cycle, and thus two ways of defining the permutation for each cycle. Fortunately we know that if we take the lowest element in the cycle we must travel in the direction of the edge from π_1 attached to this vertex. It is then easy to check this is the inverse of our previous function.

Since each ordered pair of perfect matchings (π_1, π_2) represents a term in the square of the pfaffian and each permutation represents a term in the expansion of the determinant, we would like to show that the terms corresponding under our bijection are the same. Specifically we would like to show:

$$\mathrm{sgn}(\pi_1)\mathrm{sgn}(\pi_2) \prod_{\{i,j\}\in\pi_1} \Lambda(\{i,j\}) \prod_{\{i,j\}\in\pi_2} \Lambda(\{i,j\}) = \mathrm{sgn}(\sigma) \prod_{i=1}^{n} A_{i,\sigma(i)},$$

for $(\pi_1, \pi_2) \to \sigma$ in our bijection.

Every pair $i, \sigma(i)$ appears as $\{i, \sigma(i)\}$ in exactly one of π_1 and π_2. We also know that if $i < \sigma(i)$ then $A_{i\sigma(i)} = \Lambda(\{i, \sigma(i)\})$ and otherwise $-A_{i\sigma(i)} = \Lambda(\{i, \sigma(i)\})$. From this we see:

$$(-1)^d \mathrm{sgn}(\pi_1)\mathrm{sgn}(\pi_2) \prod_{i=1}^{n} A_{i,\sigma(i)} = \mathrm{sgn}(\sigma) \prod_{i=1}^{n} A_{i,\sigma(i)},$$

where d is the number of pairs i and $\sigma(i)$ such that $\sigma(i) < i$. So we need only show that

$$(-1)^d \mathrm{sgn}(\pi_1)\mathrm{sgn}(\pi_2) = \mathrm{sgn}(\sigma).$$

To do this we shall first show that both sides of this equation are invariant under conjugation of σ, and then that the equation holds for at least one element of each conjugacy class.

Since any permutation can be written as the product of cycles of the form $(i, i+1)$, we need only show that this relation is invariant under conjugation by permutations of this form. Since conjugation preserves cycle type, we know it does not change the signum of the right side. Conjugation by $(i, i+1)$ replaces all occurrences of i with $i+1$ and vice versa in the permutation, and thus also in the matchings. Take σ', π_1', π_2'

to be the conjugated permutation and corresponding matchings. On the left side we have three cases to consider.

Case 1: The pair $\{i, i+1\}$ is in neither matching.

This means that we can apply Lemma 1.4 to both π_1 and π_2. Since $\{i, i+1\}$ is not in either, they both give distinct edges, and since nothing can be between i and $i+1$ the other condition holds. This gives us that $\text{sgn}(\pi_1) = -\text{sgn}(\pi_1')$ and $\text{sgn}(\pi_2) = -\text{sgn}(\pi_2')$, and so $\text{sgn}(\pi_1)\text{sgn}(\pi_2) = \text{sgn}(\pi_1')\text{sgn}(\pi_2')$.

Also since $\{i, i+1\}$ is not in either matching it means $\sigma(i) \neq i+1$ and $\sigma(i+1) \neq i$. Thus $\sigma(i) < i$ if and only if $\sigma'(i+1) < i+1$ since there is nothing between i and $i+1$. Similarly $\sigma(i+1) < i+1$ if and only if $\sigma'(i) < i$. Thus d remains the same. Therefore the left side remains the same.

Case 2: The pair $\{i, i+1\}$ is in exactly one of the matchings.

This means we can apply Lemma 1.4 to only one of the matchings. The other remains unchanged as the edge $\{i, i+1\}$ goes to $\{i+1, i\}$. So $\text{sgn}(\pi_1)\text{sgn}(\pi_2) = -\text{sgn}(\pi_1')\text{sgn}(\pi_2')$.

However this means either $\sigma(i) = i+1$ or $\sigma(i+1) = i$, but not both. In the former case we know $i < \sigma(i)$ but $\sigma'(i+1) < i+1$ and $i+1 < \sigma(i+1)$ if and only if $i < \sigma'(i)$. So the parity of d is reversed. The latter works in the same way. Thus we obtain:

$$(-1)^{d'}\text{sgn}(\pi_1')\text{sgn}(\pi_2') = (-1)^{d+1}(-\text{sgn}(\pi_1)\text{sgn}(\pi_2)) = (-1)^{d}\text{sgn}(\pi_1)\text{sgn}(\pi_2).$$

Thus our left side remains unchanged.

Case 3: The pair $\{i, i+1\}$ is in both matchings.

This means that $\pi_1' = \pi_1$ and $\pi_2' = \pi_2$. It also means that $\sigma(i) = i+1$ and $\sigma(i+1) = i$ and thus $\sigma = \sigma'$. Thus the conjugation had no effect, and so this certainly remains the same.

All that remains to be shown is that our equation holds for one element of each conjugacy class. Conjugacy classes are determined by cycle type, and we have restricted ourselves to permutations whose cycles are all even. We choose the representative of each conjugacy class in the form $(1,,,n_1)(n_1+1,\ldots,n_2),\ldots(n_{m-1}+1,\ldots,n_m)$. Note that there are then m cycles, and $n_m = n$. Since all even cycles have negative signum, the right side of our expression is $(-1)^m$

Looking at the left side, $(-1)^d \mathrm{sgn}(\pi_1)\mathrm{sgn}(\pi_2)$, we see that $d = m$ since $i > \sigma(i)$ only for $i = n_j$ for some $1 \leq j \leq m$. The matchings for any $(n_{j-1}+1,\ldots,n_j)$ are:

$$\pi_1 = (\{n_{j-1}+1, n_{j-1}+2\}, \{n_{j-1}+3, n_{j-1}+4\}, \ldots, \{n_j - 1, n_j\})$$

and

$$\pi_2 = (\{n_j, n_{j-1}+1\}, \{n_{j-1}+2, n_{j-1}+3\}, \ldots, \{n_j - 2, n_j - 1\}).$$

The only pairs in these matchings which do not consist of adjacent elements are of the form $\{n_j, n_{j-1}+1\}$. Thus these are the only edges which can cross. However none of these cross each other, as there is no element of the form n_j or $n_j + 1$ between $n_{k-1}+1$ and n_k. Thus there is no crossing and $\mathrm{sgn}(\pi_1) = \mathrm{sgn}(\pi_2) = 1$. Thus the left side of the expression is $(-1)^d \mathrm{sgn}(\pi_1)\mathrm{sgn}(\pi_2) = (-1)^m$.

The left and right sides match, and our statement holds. □

Since there are well established ways of computing the determinant, this theorem gives us ways to compute the pfaffian.

Another way the pfaffian can be defined is as follows:

Definition 1.6. For an $2n \times 2n$ antisymmetric matrix A the *pfaffian* is:

$$pf(A) = \frac{1}{n!2^n} \sum_{\sigma \in \mathfrak{S}_n} \mathrm{sgn}(\sigma) \prod_{i=1}^{n} A_{\sigma(2i-1)\sigma(2i)}.$$

This definition is equivalent to our previous two, if we work over a field of characteristic zero. In this case we are representing the matchings as permutations. If a permutation is written as a sequence, then the first and second elements of the sequence are paired, followed by the third and fourth, and so on, to obtain a matching. For example if we had the sequence 123654, then we would get the matching $\{1,2\}, \{3,6\}, \{4,5\}$. Aside from the coefficient, the term given by this permutation is the same as the term given by the matching formed this way (including sign). On the other hand, there are many permutations which represent the same matching. For example the permutation given by the sequence 215436 gives the same matching as the one mentioned above. For a given matching there is one permutation for each choice of order for the edges and for the vertices of each edge. Since each edge has two orders for its vertices, and there are n edges, we have 2^n possible vertex orders for the set of edges. Since there are n edges there are $n!$ ways to order them. This gives us $n!2^n$ permutations representing any given matching. This is why we place $\frac{1}{n!2^n}$ as a coefficient of each term. As a side effect we have to restrict this definition to work only when the characteristic of the field is zero.

A very similar definition is the following:

Definition 1.7. Define $\mathfrak{E}_{2n} \subseteq \mathfrak{S}_{2n}$ to be all $\sigma \in \mathfrak{S}_{2n}$ such that $\sigma(i) < \sigma(i+1)$ and $\sigma(i) < \sigma(i+2)$ for all odd i. For an $2n \times 2n$ upper triangle Λ the *pfaffian* is:

$$pf(\Lambda) = \sum_{\sigma \in \mathfrak{E}_n} \operatorname{sgn}(\sigma) \prod_{i=0}^{n-1} \Lambda\left(\{\sigma(2i+1), \sigma(2i+2)\}\right).$$

This is like Definition 1.6, except that we have chosen a canonical representation of each matching. Our canonical representation has the vertices of each edge in increasing order (enforced by $\sigma(i) < \sigma(i+1)$) and the edges in increasing order of minimal element (through $\sigma(i) < \sigma(i+2)$). By using this canonical order we have one permutation for each matching. This means we no longer need to divide by $2^n n!$, and so this removes the restriction on the characteristic of our field.

We have shown that the pfaffian can be expressed as the square root of a determinant, but it is also true that any determinant can be expressed in terms of a pfaffian. If we have an $n \times n$ matrix M, then we can create a $2n \times 2n$ upper triangle Λ to represent it. To make this easier to see we replace the usual $1, \ldots, 2n$ labelling for the elements of the upper triangle with $1_1, 2_1, \ldots, n_1, n_2, (n-1)_2, \ldots, 1_2$. Then we set:

$$\Lambda(e) = \begin{cases} M_{ij} & e = \{i_1, j_2\} \\ 0 & e = \{i_1, j_1\} \text{ or } \{i_2, j_2\} \end{cases}.$$

For example if we wanted to express the determinant:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix},$$

as a pfaffian, we would use the following upper triangle:

$$\begin{bmatrix} 0 & b & a \\ & d & c \\ & & 0 \\ & & \end{bmatrix}.$$

Checking this we see the pfaffian is:

$$ad - bc.$$

Proposition 1.8. *For a matrix M and upper triangle Λ defined from M as above, we have* $\det M = pf\Lambda$.

Proof. Let X be the set of indices with subscript one and Y be the set with subscript two.

Any matching in the expansion of the pfaffian which gives a non-zero term must have all its edges connecting an element of X to an element of Y. If we think of the graph corresponding to this upper triangle, it is the bipartite graph with bipartition X and Y.

Take a matching π which gives a non-zero term. Let π' be the associated permutation in the form of Definition 1.7 (i.e. $\pi'(2i-1) < \pi'(2i)$ and $\pi'(2i-1) < \pi'(2i+1)$ for all $1 \leq i \leq n$, and if a and b are matched in π, $a \leq b$ then $\pi'(2i-1) = a$ and $\pi'(2i) = b$). The term from this matching is then:

$$\mathrm{sgn}(\pi') \prod_{i=1}^{n} \Lambda(\{\pi'(2i-1), \pi'(2i)\}).$$

Since any member of X is less than or equal to any member of Y, and each matching edge contains one of each, then $\pi'(2i-1) \in X$ and $\pi'(2i) \in Y$ for each $1 \leq i \leq n$.

Since each $\pi'(2i-1) < \pi'(2i+1)$ and both are in X we know that $\pi(2i-1) = i_1$ for all $1 \leq i \leq n$. So we can reduce this term to:

$$\mathrm{sgn}(\pi') \prod_{i=1}^{n} \Lambda(\{i_1, \pi'(2i)\}).$$

We would like to show that a matching (which gives a non-zero term) corresponds to the permutation which maps i to j for each i_1 matched to j_2. Let σ be the permutation corresponding to π in this way. Then we can rewrite our term as:

$$\operatorname{sgn}(\pi') \prod_{i=1}^{n} \Lambda(\{i_1, \sigma(i)_2\}).$$

Through our choice of Λ this reduces to:

$$\operatorname{sgn}(\pi') \prod_{i=1}^{n} M_{i\sigma(i)}.$$

If $\operatorname{sgn}(\pi') = \operatorname{sgn}(\sigma)$ then this is the term from the determinant.

Since π' considered as a sequence is $1_1\sigma(1)_2 2_1\sigma(2)_2 \ldots n_1\sigma(n)_2$ the same rearrangement which converts σ to the identity converts π' to $1_1 1_2 2_1 2_2 \ldots n_1 n_2$. Denoting this by e', we have $\operatorname{sgn}(\pi') = \operatorname{sgn}(\sigma)\operatorname{sgn}(e')$.

The permutation e' can be rearranged to the identity by starting with n_2 and then descending down Y, shifting each element to the end in turn. Each i_2 has to cross each j_1 and j_2 for each $j > i$. Since this is an even number of elements, doing so does not change the signum. So $\operatorname{sgn}(e) = 1$ and thus $\operatorname{sgn}(\pi') = \operatorname{sgn}(\sigma)$ and so we have a map of each non-zero term in our pfaffian to a term in the determinant.

To check that this is a bijective correspondence, note that every permutation σ can be represented uniquely in the form $1_1\sigma(1)_2 2_1\sigma(2)_2 \ldots n_1\sigma(n)_2$. □

If we consider the pfaffian to act on a graph, then we can now consider the determinant to be the special case where that graph is bipartite. For this reason it is often suggested that pfaffians are in fact more general than determinants (such as in [14]).

1.3 Hyperpfaffians

We are now ready to start looking at generalizations of these concepts to higher dimensions. The determinant and the pfaffian are both restricted to two dimensional arrays. Here we would like to consider extending them to higher dimensions. The focus of this thesis will be on the hyperpfaffian, which is the analogue of the pfaffian for higher dimensional arrays.

Before we can generalize the pfaffian, we shall first need to define our analogues of the antisymmetric matrix and the upper triangle. We can generalize Definition 1.1 of the upper triangle fairly easily.

Definition 1.9. A k-dimensional *upper triangle* of order n is a function on the k-element subsets of $\{1, \ldots, n\}$.

The analogue of the antisymmetric matrix is only slightly more complicated.

Definition 1.10. A $k-$dimensional *alternating tensor* of order n, is a function Λ on $\{1, \ldots, n\}^k$ with the following restriction:

$$\Lambda(i_1, \ldots, i_k) = \operatorname{sgn}(\sigma)\Lambda(i_{\sigma(1)}, \ldots, i_{\sigma(k)}),$$

for any $\sigma \in \mathfrak{S}_k$ and $1 \leq i_1, \ldots, i_k \leq n$.

One can check that this is exactly the restriction on an antisymmetric matrix when $k = 2$. One can also see that if $i_a = i_b$ for any $a \neq b$ then this will be zero.

As with the antisymmetric matrix and upper triangle, we obtain a k-dimensional alternating tensor Λ from a k-dimensional upper triangle Γ as follows:

$$\Lambda(i_1, \ldots, i_k) = \begin{cases} \operatorname{sgn}(i_1, \ldots, i_k)\Gamma(\{i_1, \ldots, i_k\}) & i_a \neq i_b, \forall a \neq b \in \{1, \ldots, k\} \\ 0 & \text{otherwise} \end{cases}.$$

Once again we can describe these as graphs, or in this case hypergraphs. We shall examine hypergraphs more carefully in the next section, but for now we just need to know that they are graphs where edges can have more than two vertices. So our hypergraph with edges containing k vertices corresponds to our k-dimensional upper triangle just as it did in the two dimensional case.

As in the two dimensional case, our hyperpfaffian will be a sum over perfect matchings of this graph. It is harder to describe the sign in this sum in terms of crossing number, as hyperedges may cross in many different ways; some we do not want to consider to be crossings, but others we do. We instead use the analogue of the permutation form of the signum from Definitions 1.6 and 1.7, which we shall see when we define the hyperpfaffian.

Like the pfaffian, there are several ways of defining the hyperpfaffian. Unlike the pfaffian, they are not all equivalent. The definition which will be considered the standard definition of the *uniform hyperpfaffian* in this thesis is that given by Luque and Thibon in [15].

Definition 1.11. [Hyperpfaffian, Luque-Thibon [15]] Let $\mathfrak{E}_{km,k} \subseteq \mathfrak{S}_{km}$ be the set of permutations σ such that $\sigma(ki+j) < \sigma(ki+j+1)$ and $\sigma(ki+1) < \sigma(k(i+1)+1)$ for all $0 \leq i < m$ and $1 \leq j \leq k$. Then for a $k-$dimensional upper triangle Λ of order km , we define the *hyperpfaffian* of Λ to be:

$$pf_k(\Lambda) = \sum_{\sigma \in \mathfrak{E}_{mk,k}} \operatorname{sgn}(\sigma) \prod_{i=0}^{m-1} \Lambda(\{\sigma(ki+1), \ldots, \sigma(ki+k)\}).$$

This definition most closely resembles Definition 1.7 of the pfaffian. One can check that it is in fact the same if $k = 2$. As in that definition it chooses a canonical permutation to represent the matching so that there is only one permutation per matching. As before it orders the vertices in each edge in increasing order, and it orders the edges by increasing lowest element. Note that the k-dimensional hyperpfaffian is denoted by pf_k. We shall refer to this definition of the hyperpfaffian as the Luque-Thibon definition of the hyperpfaffian, or the *Luque-Thibon hyperpfaffian*.

This was not the first definition of the hyperpfaffian. Luque and Thibon refer to [3] where Barvinok introduced the hyperpfaffian. However Barvinok's definition was slightly different, and not equivalent. We shall refer to this as the Barvinok definition of the hyperpfaffian or the *Barvinok hyperpfaffian*.

Definition 1.12. [Hyperpfaffian, Barvinok [3]] For a $k-$dimensional alternating tensor Λ of order n, where $n = mk$, we define the *hyperpfaffian* of Λ to be:

$$pfb_k(\Lambda) = \frac{1}{m!} \sum_{\sigma \in \mathfrak{S}_n} \operatorname{sgn}(\sigma) \prod_{i=0}^{m-1} \Lambda(\sigma(ki+1), \ldots, \sigma(ki+k)).$$

This is very similar to Definition 1.6 of the pfaffian. Unlike in the Luque-Thibon definition, the Barvinok hyperpfaffian differs from the pfaffian in the case of $k = 2$. It only differs by a constant factor (depending on the size of the matrix). For a $2m \times 2m$ antisymmetric matrix, the relation is:

$$\frac{1}{2^m} pfb_2(A) = pf(A).$$

While the Luque-Thibon hyperpfaffian can be non-zero for any dimension $k \geq 2$, the Barvinok hyperpfaffian is zero for all odd dimensions. We can see this by putting a sign reversing involution on the terms of its expansion. By switching the first pair of edges represented by the permutation, since each has an odd number of vertices, we reverse the signum of the permutation. Aside from that, the term remains unchanged, and thus these all cancel. For even k, switching these edges has no effect on the signum.

For example, consider the four dimensional alternating tensor of order eight defined by

$$\Lambda(i, j, k, l) = x_{ijkl}, \quad 1 \leq i < j < k < l \leq 8.$$

The rest of the entries are defined by antisymmetry. Using the Luque-Thibon definition we get:

$$\begin{aligned} x_{1234}x_{5678} - x_{1235}x_{4678} + x_{1236}x_{4578} - x_{1237}x_{4568} + x_{1238}x_{4567} \\ + x_{1245}x_{3678} - x_{1246}x_{3578} + x_{1247}x_{3568} - x_{1248}x_{3567} + x_{1256}x_{3478} \\ - x_{1257}x_{3468} + x_{1258}x_{3467} + x_{1267}x_{3458} - x_{1268}x_{3457} + x_{1278}x_{3456} \\ - x_{1345}x_{2678} + x_{1346}x_{2578} - x_{1347}x_{1258} + x_{1348}x_{2567} - x_{1356}x_{2478} \\ + x_{1357}x_{2468} - x_{1358}x_{2467} - x_{1367}x_{2458} + x_{1368}x_{2457} - x_{1378}x_{2456} \\ + x_{1456}x_{2378} - x_{1457}x_{2368} + x_{1458}x_{2367} - x_{1467}x_{2358} + x_{1468}x_{2357} \\ + x_{1478}x_{2356} - x_{1567}x_{2348} + x_{1568}x_{2347} - x_{1578}x_{2346} + x_{1678}x_{2345} \end{aligned}$$

The same terms appear in the Barvinok definition, but they each appear multiple times. More precisely they appear $2 \cdot 4!^2$ times each, corresponding to the two ways of ordering the edges and 4! ways of ordering the vertices within each of the two edges. Each of these terms has a coefficient of $\frac{1}{2}$, meaning, when all $2 \cdot 4!^2$ are summed, we end up with a result that is $4!^2$ times the Luque-Thibon hyperpfaffian. For any even k we see that the Barvinok hyperpfaffian is $k!^m$ times the Luque-Thibon hyperpfaffian, where k is the dimension and mk is the order.

Barvinok comments that the hyperpfaffian of an integer valued alternating tensor is an integer under his definition. This is true, but in fact the Barvinok hyperpfaffian of an integer valued alternating tensor is a multiple of $(k!)^m$. We instead prefer to add the stronger condition that the hyperpfaffian of the general alternating tensor of indeterminates is a monic polynomial. It would also be preferable to bring this more in line with both the standard pfaffian and the Luque-Thibon hyperpfaffian, so we introduce the reduced Barvinok definition of the hyperpfaffian:

Definition 1.13. [Hyperpfaffian, Reduced Barvinok] For a k-dimensional alternating tensor Λ of order $n = mk$, we define the *reduced Barvinok hyperpfaffian* of Λ by

$$pf_k(\Lambda) = \frac{1}{k!^m} pfb_k(\Lambda).$$

This has value zero for any odd k, but when k is even it agrees with the Luque-Thibon hyperpfaffian (hence we will use the same notation), and thus with the pfaffian when $k = 2$. It is worth noting that the Barvinok and reduced Barvinok hyperpfaffian will only be well defined when dealing with a field of characteristic zero, because of the coefficient used to normalize them. The Luque-Thibon hyperpfaffian is not restricted this way.

At first it seems very important that the Luque-Thibon hyperpfaffian can handle odd k. However this has some problems. In the even case the Luque-Thibon definition agrees with the reduced Barvinok definition of the hyperpfaffian because for any choice of order for the edges, the permutation representing them will give the same signum. While different orders of the vertices in the edges will change the signum of the permutation, this will be countered by the antisymmetry of the tensor. On the other hand, edge ordering does affect the signum for odd k. Selecting an order for it is somewhat arbitrary. Having to use this order makes the definition harder to work with.

For this reason, despite its apparent improvement, it is rare that the k odd case yields interesting results. Even so, the Luque-Thibon is our most flexible definition, as it does not restrict characteristic or parity of k and it agrees with the reduced Barvinok definition whenever the latter is defined.

Like the pfaffian, the determinant can also be generalized to higher dimension. The hyperdeterminant is a much older definition, introduced by Cayley in the nineteenth century. It acts on a k-dimensional tensor, like the Barvinok hyperpfaffian, but without the antisymmetric restriction. The tensor is just a function on $\{1, \ldots, n\}^k$.

Definition 1.14. For a k-dimensional tensor $\Lambda$of order n, the *hyperdeterminant* of

Λ is:

$$\det_k \Lambda = \frac{1}{r!} \sum_{\sigma_1,\ldots,\sigma_k \in \mathfrak{S}_r} \left(\prod_{i=1}^{k} \operatorname{sgn}(\sigma_i) \right) \left(\prod_{i=1}^{r} \Lambda(\sigma_1(i), \ldots, \sigma_k(i)) \right).$$

One can note that if $k = 2$ then Λ is a matrix and this becomes:

$$\begin{aligned} \det_2 \Lambda &= \frac{1}{r!} \sum_{\sigma_1,\sigma_2 \in \mathfrak{S}_r} \operatorname{sgn}(\sigma_1)\operatorname{sgn}(\sigma_2) \left(\prod_{i=1}^{r} \Lambda(\sigma_1(i), \sigma_2(i)) \right) \\ &= \sum_{\sigma_2 \in \mathfrak{S}_r} \operatorname{sgn}(\sigma_2) \left(\prod_{i=1}^{r} \Lambda(i, \sigma_2(i)) \right), \end{aligned}$$

which is the determinant of the matrix.

Like the Barvinok Hyperpfaffian, this is zero if k is even. If instead of switching the first pair of edges one switches $\sigma_i(1)$ and $\sigma_i(2)$ for each i it has the same effect. Since k is odd this has an effect of reversing the signum once. The terms remain otherwise unchanged, and so they cancel. As with the pfaffian and the determinant, we can represent any hyperdeterminant as a hyperpfaffian. If we take a k-dimensional order r tensor Γ, where k is even, then we need to create a k-dimensional order kr alternating tensor. We index our alternating tensor in the following way:

$$1_1, 2_1, \ldots r_1, r_2, (r-1)_2, \ldots, 1_2, 1_3, 2_3, \ldots \ldots r_k, (r-1)_k, \ldots, 1_k.$$

Let X_i be the set of indices with subscript i. Define:

$$\Lambda(x_1, \ldots, x_k) = \begin{cases} \operatorname{sgn}(\sigma)\Gamma(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(k)}) & \exists \sigma \in \mathfrak{S}_k, x_{\sigma(i)} \in X_i, \forall 1 \le i \le k \\ 0 & \text{otherwise} \end{cases},$$

where we assume the arguments of Γ ignore subscripts. Here we have set each entry which does not have a member from each of the sets X_i, $i = 1, \ldots, k$ to zero. The entries which do have one of each are given the entry from Γ corresponding to their indices ordered by subscript.

Proposition 1.15. *The hyperpfaffian (reduced Barvinok or Luque-Thibon) of* Λ *equals the hyperdeterminant of* Γ.

Proof. Start by looking at the reduced Barvinok expansion of the hyperpfaffian:

$$pf_k(\Lambda) = \frac{1}{r!(k!)^r} \sum_{\sigma \in \mathfrak{S}_{rk}} \operatorname{sgn}(\sigma) \prod_{i=0}^{r-1} \Lambda(\sigma(ki+1), \ldots, \sigma(ki+k)).$$

To make this a little simpler, we can canonically order the vertices within the edges, so let $\mathfrak{F}_{rk}$ be the permutations where $\sigma(ki+1) < \sigma(ki+1) < \cdots < \sigma(ki+k)$ for all $0 \le i \le r-1$. Each of these terms covers $(k!)^r$ terms in our original expansion (all of which were equal), so we get:

$$pf_k(\Lambda) = \frac{1}{r!} \sum_{\sigma \in \mathfrak{F}_{rk}} \operatorname{sgn}(\sigma) \prod_{i=0}^{r-1} \Lambda(\sigma(ki+1), \ldots, \sigma(ki+k)).$$

For a term to be non-zero in this expansion, we need each edge in the corresponding matching to have one vertex from each of the $X_1, \ldots, X_k$. Since each edge is represented with vertices in increasing order, that means $\sigma(ki+j) \in X_j$ for any $i = 0, \ldots, r-1$ and $j = 1, \ldots, k$. Let $\mathfrak{F}'_{kr}$ be the permutations restricted in this way (thus merely eliminating zero terms). Then for any $\sigma \in \mathfrak{F}'_{kr}$, define $\sigma_j(i+1) = l$ so that $\sigma(ki+j) = l_j$ for each $i = 0, \ldots, r-1$ and $j = 1, \ldots, k$. Our expansion now becomes:

$$pf_k(\Lambda) = \frac{1}{r!} \sum_{\sigma \in \mathfrak{F}'_{rk}} \operatorname{sgn}(\sigma) \prod_{i=0}^{r-1} \Gamma(\sigma_1(i+1), \ldots, \sigma_k(i+1)).$$

Next we shall show that $\operatorname{sgn}(\sigma) = \prod_{j=1}^{k} \operatorname{sgn}(\sigma_j)$. The subsequence of σ composed of the jth element of each k-tuple is the sequence σ_j (with a subscript of j). Thus reordering σ to $1_1 1_2 \ldots, 1_k, 2_1, \ldots r_k$ is equivalent to reordering each σ_j to the identity. Thus $\operatorname{sgn}(\sigma)$ times the signum of this permutation equals $\prod_{j=1}^{k} \operatorname{sgn}(\sigma_j)$. Now we need only show that this has the same signum as the identity. This can be checked fairly quickly by first moving the elements of X_k to the end starting with r_k and continuing down to 1_k. Then the X_{k-1} are moved to just before the block of X_k we have just created at the end. For these we start with 1_{k-1} and move up to r_k. We continue this process until the sequence is sorted. Each step of this moves each element past an even block, and thus has no effect on the signum. So $1_1 1_2 \ldots, 1_k, 2_1, \ldots r_k$ has signum 1.

Since the the signum is the same, we can observe that the permutations in $\mathfrak{F}'_{rk}$ are completely and uniquely described by the sets of σ_is. Thus we get the final expansion:

$$\frac{1}{r!} \sum_{\sigma_1, \ldots, \sigma_k \in \mathfrak{S}_r} \left(\prod_{i=1}^{k} \operatorname{sgn}(\sigma_i) \right) \left(\prod_{i=1}^{r} \Gamma(\sigma_1(i), \ldots, \sigma_k(i) \right).$$

This is the expression for the hyperdeterminant. $\square$

1.4 The Non-Uniform Hyperpfaffian

Our definitions of the hyperpfaffian from the previous section work only over partitions of uniform size (hence we referred to them as *uniform hyperpfaffians*). When considered as a hypergraph, using these partitions means we are dealing only with hypergraphs whose edges all have the same number of ends (which are called uniform hypergraphs). This seems like a reasonable generalization of the pfaffian, but in many circumstances we could use an even more flexible tool. Our goal in this section will be to develop a hyperpfaffian which acts on non-uniform upper triangles, which we define as follows:

Definition 1.16. A *non-uniform upper triangle* of order n is a function on the non-empty subsets of $\{1, \ldots, n\}$.

We saw that k-dimensional upper triangles, from Definition 1.9, correspond to hypergraphs whose edges all have the same number of vertices. This upper triangle corresponds to arbitrary hypergraphs. In a similar way we can define a non-uniform analogue of an alternating tensor:

Definition 1.17. An *antisymmetric tensor algebra* Λ of order n is a function on finite non-empty sequences of elements of $\{1, \ldots, n\}$ such that

$$\Lambda(i_1, \ldots, i_k) = \operatorname{sgn}(\sigma)\Lambda(i_{\sigma(1)}, \ldots, i_{\sigma(k)}),$$

for any $\sigma \in \mathfrak{S}_k$.

As with alternating tensors and k-dimensional upper triangles, we can define an antisymmetric tensor algebra Λ from a non-uniform upper triangle Γ in the following way:

$$\Lambda(i_1, \ldots, i_k) = \begin{cases} \operatorname{sgn}(i_1, \ldots, i_k)\Gamma(\{i_1, \ldots, i_k\}) & i_j \neq i_l \forall 1 \leq j, l \leq k \\ 0 & \text{otherwise} \end{cases}.$$

Similarly we define an *even non-uniform upper triangle* to be an non-uniform upper triangle acting only on the even cardinality subsets of $\{1, \ldots, n\}$. We define an *even antisymmetric tensor algebra* to be one which acts only on even length sequences.

Based on the results in [1] using the Grassmann-Berezin calculus, we can create a hyperpfaffian which works over these objects.

We begin by letting $\mathcal{P}_n$ be the set of partitions of $\{1, \ldots, n\}$ into non-empty disjoint subsets. Let $\mathcal{E}_n \subseteq \mathcal{P}_n$ be the set of such partitions of $\{1, \ldots, n\}$ into non-empty disjoint even cardinality subsets. Note this is empty if n is not even.

As with our perfect matchings, we can define sequences or permutations to represent these partitions in $\mathcal{E}_n$. As with matchings we have a sequence represent a partition if it can be divided up into blocks which are the sets of the partition. We say it strictly represents a partition if within the blocks the elements are in increasing order. Thus if we have the partition $\{1, 3, 5, 6\}, \{2, 10\}, \{4, 7, 8, 9\}$, then the sequences $1, 3, 5, 6, 2, 10, 4, 7, 8, 9$ and $4, 7, 8, 9, 1, 3, 5, 6, 2, 10$ would both strictly represent this, and $3, 4, 6, 1, 4, 7, 8, 9, 2, 10$ would represent it, but not strictly. Note, unlike with matchings, a sequence may represent more than one partition. For example the first sequence mentioned above also strictly represents the partition $\{1, 3\}, \{5, 6\}, \{2, 10\}, \{4, 7, 8, 9\}$.

Using this representation we can define the signum of a partition in $\mathcal{E}_n$.

Definition 1.18. For a partition $\pi \in \mathcal{E}_n$, and sequence σ which strictly represents it, we define the *signum* of π to be $\operatorname{sgn}(\sigma)$.

Note that since $\pi \in \mathcal{E}_n$ this is well defined. All sequences which strictly represent a partition are the same up to the order of the sets in the partition. Since all the sets are of even size, this does not change the signum of the sequence. Looking at the example used above, we see that $\{1, 3, 5, 6\}, \{2, 10\}, \{4, 7, 8, 9\}$ has signum negative one.

So we can now define our non-uniform hyperpfaffian:

Definition 1.19. Given an even non-uniform upper triangle Λ of order n then the *non-uniform hyperpfaffian* of Λ is:

$$pf_*(\Lambda) = \sum_{\pi \in \mathcal{E}_n} \operatorname{sgn}(\pi) \prod_{X \in \pi} \Lambda(X).$$

This definition is very similar to our first definition of the pfaffian, Definition 1.2. We have extended it to partitions of all even sizes, not just size k. If we would prefer to use an antisymmetric tensor algebra, the non-uniform hyperpfaffian can be defined equivalently as follows:

Definition 1.20. Let $\mathfrak{N}_n$ be the set of strictly increasing sequences of even numbers between 1 and n ending with n. Given an even antisymmetric tensor algebra Λ of order n, the *non-uniform hyperpfaffian* of Λ is:

$$pf_*(\Lambda) = \sum_{\sigma \in \mathfrak{S}_n} \sum_{q \in \mathfrak{N}_n} \operatorname{sgn}(\sigma) \frac{1}{|q|!} \prod_{j=1}^{|q|} \frac{1}{(q_i - q_{i-1})!} \Lambda(i_{\sigma(q_{i-1}+1)}, \ldots, i_{\sigma(q_i)}).$$

where $q_1, \ldots, q_{|q|}$ is the sequence q, $q_0 = 0$ and $|q|$ is the length of the sequence q.

Here we use the sequence of even numbers to represent the different ways of breaking up a sequence into even blocks. Thus the second sum is effectively over all members of $\mathcal{E}_n$ which σ represents (non-strictly).

This definition resembles Definition 1.6 of the pfaffian or the reduced Barvinok hyperpfaffian (Definition 1.13). In fact if one restricts the even sequence to the sequence $k, 2k, 3k, \ldots, n$ one gets exactly the reduced Barvinok definition of the hyperpfaffian. We can then see that if we restrict our upper triangle or antisymmetric tensor algebra to uniform even size sets we have the Luque-Thibon or reduced Barvinok hyperpfaffian.

One can adjust our definitions to cover the odd case. Our antisymmetric tensor algebra form can be modified simply by removing the restriction that we use even valued sequences. However, like the reduced Barvinok hyperpfaffian, this gives zero for all non-even partitions. If we use our non-uniform upper triangle version we need only find a way to define a signum on elements of $\mathcal{P}_n$.

To do this we simply need to further restrict what sequences represent a given partition. In addition to the requirements of the strict representation before, that within the blocks the elements of the sequence are increasing, we also require that the blocks are ordered so their first elements are increasing. This could also be phrased as ordering the edges in increasing order of lowest element. We can then define the signum of $\pi \in \mathcal{P}_n$ to be the signum of the permutation representing it in this manner (note there is now only one permutation for each partition, though there may be more than one partition for a given permutation).

We can then define the non-uniform hyperpfaffian in the following way:

Definition 1.21. Given an non-uniform upper triangle Λ of order n then the *non-uniform hyperpfaffian* of Λ is:

$$pf_*(\Lambda) = \sum_{\pi \in \mathcal{P}_n} \operatorname{sgn}(\pi) \prod_{X \in \pi} \Lambda(X).$$

This suffers from the same problems as the odd case for the Luque-Thibon hyperpfaffian, with which it agrees if we restrict to a partition of uniform size. The need for specific edge order means we shall almost always restrict to the even case.

1.5 Hypergraphs

Since hyperpfaffians are taken over matchings in a hypergraph, hypergraphs will come up many times in this document, mostly in Chapters 2 and 3. For this reason it will help to give a little background on hypergraphs before we begin.

Definition 1.22. A *hypergraph* is a pair of sets (V, E), where V is the set of vertices and E is the set of edges and where each edge has a corresponding set of vertices.

Let us denote the set of vertices corresponding to an edge e by $V(e)$. This set of vertices is the set of ends of the edge, which in the case of a standard graph is a set of size two. So we have defined this to be a graph where edges may have any number of ends. Note that if $|V(e)| = 2$ we have a standard multigraph with no loops. If we only require $|V(e)| \leq 2$ then we have a multigraph which allows loops. We shall refer to $|V(e)|$ as the degree of the edge e.

Definition 1.23. A *simple hypergraph* is a hypergraph (V, E) where if for any $e, f \in E$ $V(e) = V(f)$ then $e = f$.

Note then that a simple graph is a simple hypergraph with all edges of degree two.

Definition 1.24. A *uniform hypergraph* of degree k is a hypergraph (V, E) such that for every $e \in E$ then $|V(e)| = k$.

Thus all graphs (without loops) are uniform hypergraphs of degree two. The uniform hyperpfaffian pf_k acts on a hypergraph which is uniform of degree k.

We draw hypergraphs in one of two ways. Vertices are represented by dots as with graphs, but edges have two forms, depending on our purposes at the time. The first is to draw lines from each vertex to all meet at a point, like standard edges to an imaginary vertex. The other is to draw arcs connecting the vertices of this edge in a cycle. These enclose a face which we are not allowed to draw any further edges through. Figure 1.2 shows an example of these. In Section 3.2 we shall discuss more formally how to embed hypergraphs in surfaces.

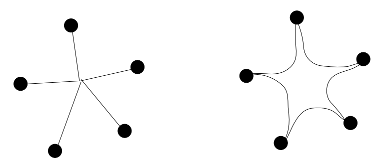

Figure 1.2: Representations of a hyperedge of degree 5

Definition 1.25. Define a *walk* of length n in a hypergraph (V, E) to be a sequence $v_1, e_1, v_2, e_2, \ldots, e_n, v_{n+1}$ where $v_1, \ldots, v_{n+1} \in V$ and $e_1, \ldots, e_n \in E$, where $v_i, v_{i+1} \in V(e_i)$ for all $1 \le i \le n$.

This corresponds exactly to a walk in a graph if we restrict to standard graphs. From this we can define a path:

Definition 1.26. We define a *path* of length n in a hypergraph (V, E) to be a walk, where none of the vertices or edges are repeated.

Note, unlike in the case of a graph, we must specify that both the edges and vertices are distinct, since neither condition implies the other. We can now use this to define cycles, connectedness, and trees much as we have before:

Definition 1.27. A *cycle* of length n of a hypergraph is a walk $v_1, e_1, v_2, e_2, \ldots, e_n, v_{n+1}$ where all the edges and vertices are distinct, except that $v_{n+1} = v_1$.

Definition 1.28. For a hypergraph (V, E) a pair of vertices $v, w \in V$ are *connected* if there is a path with first vertex v and last vertex w. We call a hypergraph *connected* if every pair of its vertices are connected.

Definition 1.29. A hypergraph is a *tree* if it has no cycles and is connected.

These are all defined in essentially the same way as they are for graphs, and they have many of the same properties. Next we list a few convenient results about hypergraphs (taken from [4]), which are similar to those for graphs.

Proposition 1.30. *For a hypergraph (V, E) with c components, n vertices, and no cycles we have:*

$$\sum_{e \in E} (\deg(e) - 1) = n - c.$$

Proposition 1.31. *A hypergraph with c components has exactly one cycle if and only if:*

$$\sum_{e \in E} (\deg(e) - 1) = n - c + 1.$$

Definition 1.32. A *simple cycle* in a hypergraph is a cycle where for any edge e in the cycle it has only two vertices which are in other edges of the cycle. We call these vertices the *shared vertices* of the edge.

This means that if we take the subgraph which contains only the edges of the cycle and the associated vertices there is only one cycle. It also means the walk generating the cycle is uniquely defined up to direction and starting point. For example in Figure 1.3 we see a simple cycle and one that is not. Using Proposition 1.5 we see:

Proposition 1.33. *A cycle in a hypergraph of length l, composed of edges $e_1, \ldots, e_l$ is a simple cycle if and only if:*

$$\sum_{i=1}^{l} (\deg(e_i) - 1) = n$$

Equipped with these basics, we are now ready to proceed.

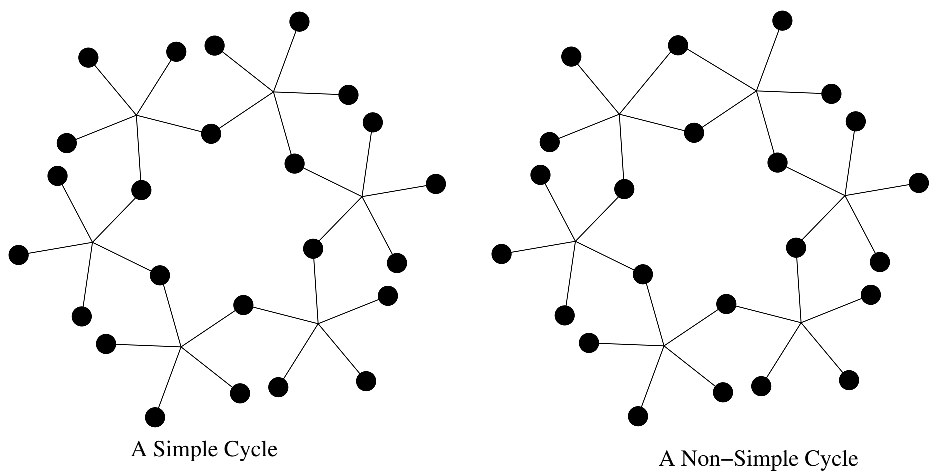

Figure 1.3: A simple cycle and a non-simple cycle

Chapter 2

The Hyperpfaffian-Cactus Theorem

2.1 The Matrix-Tree Theorem

The Matrix-Tree theorem is a very important classical result in combinatorics. To begin let us define the matrix involved (called the *Kirchhoff matrix*), for the undirected graph version of the theorem. For a graph G with vertices labelled from 1 to n, assign each edge e of the graph an indeterminate y_e. Define an $n \times n$ matrix $K(G)$ by assigning each of its each non-diagonal entries as follows:

$$K(G)_{ij} = -\sum_{\substack{e \in E(G) \\ \{i,j\} = V(e)}} y_e.$$

This is the negative sum of all the edges connecting i and j. If we do not allow multiple edges then $K(G)_{ij} = -y_{ij}$. For the diagonal entries we assign the following:

$$K(G)_{ii} = \sum_{\substack{e \in E(G) \\ i \in V(e)}} y_e.$$

This is the sum of all the edges which have i as an end. Denote the minor of this matrix obtained by removing the row and column for vertex v by $K_v(G)$.

The Matrix-Tree theorem (from [21]) states:

Theorem 2.1 (Matrix-Tree Theorem, undirected version). *For a graph G with Kirchhoff matrix $K(G)$, let $\mathcal{T}$ be the set of spanning trees of G. Then we have:*

$$\det(K_v(G)) = \sum_{T \in \mathcal{T}} \prod_{e \in T} y_e.$$

For the directed version of the theorem, we define a very similar matrix. For a digraph D with n labelled vertices, we define an $n \times n$ matrix $K(D)$. We set its non-diagonal entries to be:

$$K(D)_{ij} = - \sum_{\substack{e \in E(D) \\ i=h(e), j=t(e)}} y_e.$$

This is basically the same as in the undirected case, except that we require i to be the head (and thus j the tail). We define the diagonal entries similarly:

$$K(D)_{ii} = \sum_{\substack{e \in E(D) \\ i=h(e)}} y_e.$$

Once again our only change is to restrict it so that i is the head. We shall again use the minor of $K(D)$ taken by removing the row and column representing the vertex v, and denote it $K_v(D)$.

An *arborescence* rooted at v is defined to be a tree with all its edges directed away from v. Denote the set of all spanning arborescences rooted at v by $\mathcal{T}_v$. We can now state the directed version of the theorem:

Theorem 2.2 (Matrix-Tree Theorem, directed version). *For a digraph D with Kirchhoff matrix $K(D)$,*

$$\det(K_v(D)) = \sum_{T \in \mathcal{T}_v} \prod_{e \in T} y_e.$$

The undirected version follows from the directed version by considering the digraph which has an edge oriented each way corresponding to each edge of the undirected graph. Then there is an arborescence directed away from each vertex for each spanning tree.

These theorems give us an easy way to get the generating series for all the spanning trees (or arborescences) for a graph (or digraph). By substituting in 1 for every y_e we get the number of such trees.

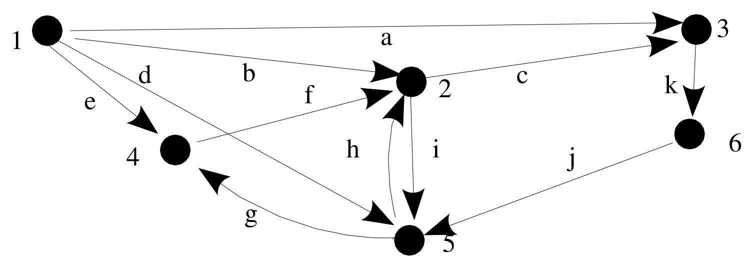

Figure 2.1:

Let us now look at an example. Take the graph D given in Figure 2.1. We can construct the Kirchoff matrix for this as follows:

$$K(D) \quad = \quad \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ -y_b & y_b + y_f + y_h & 0 & -y_f & -y_h & 0 \\ -y_a & -y_c & y_a + y_c & 0 & 0 & 0 \\ -y_e & 0 & 0 & y_e + y_g & -y_g & 0 \\ -y_d & -y_i & 0 & 0 & y_d + y_j + y_i & -y_j \\ 0 & 0 & -y_k & 0 & 0 & y_k \end{bmatrix}.$$

Eliminating the first row and column we get:

$$K_1(D) \quad = \quad \begin{bmatrix} y_b + y_f + y_h & 0 & -y_f & -y_h & 0 \\ -y_c & y_a + y_c & 0 & 0 & 0 \\ 0 & 0 & y_e + y_g & -y_g & 0 \\ -y_i & 0 & 0 & y_d + y_j + y_i & -y_j \\ 0 & -y_k & 0 & 0 & y_k \end{bmatrix}.$$

Expanding the determinant we get:

$$\begin{aligned} \det(K_1(D)) = {} & (y_b + y_f + y_h)(y_a + y_c)(y_e + y_g)(y_d + y_j + y_i)y_k \\ & - y_f(-y_c)(-y_g)(-y_j)(-y_k) - y_f(y_a + y_c)(-y_g)(-y_i)y_k \\ & - (-y_h)(-y_c)(y_e + y_g)(-y_j)(-y_k) - (-y_h)(y_a + y_c)(y_e + y_g)(-y_i)y_k. \end{aligned}$$

Substituting 1 into the above expression we get $3 \cdot 2 \cdot 2 \cdot 3 - 1 - 2 - 4 - 2 = 27$, so there are 27 spanning arborescences rooted at vertex 1. If instead of substituting we reduce the full expression we get:

$$\begin{aligned}
&= y_ky_by_ay_ey_i + y_ky_by_ay_ey_d + y_ky_by_ay_ey_j + y_ky_by_ay_gy_i + y_ky_by_ay_gy_d + y_ky_by_ay_gy_j \\
&+ y_ky_fy_ay_ey_i + y_ky_fy_ay_ey_d + y_ky_fy_ay_ey_j + y_ky_fy_ay_gy_d + y_ky_fy_ay_gy_j + y_ky_hy_ay_ey_d \\
&+ y_ky_hy_ay_ey_j + y_ky_hy_ay_gy_d + y_ky_hy_ay_gy_j + y_ky_by_cy_ey_i + y_ky_by_cy_ey_d + y_ky_by_cy_ey_j \\
&+ y_ky_by_cy_gy_i + y_ky_by_cy_gy_d + y_ky_by_cy_gy_j + y_ky_fy_cy_ey_i + y_ky_fy_cy_ey_d + y_ky_fy_cy_ey_j \\
&+ y_ky_fy_cy_gy_d + y_ky_hy_cy_ey_d + y_ky_hy_cy_gy_d.
\end{aligned}$$

Each of these terms represents one of the 27 spanning arborescences of the graph rooted at vertex 1.

2.2 The Pfaffian-Tree Theorem

We shall next discuss an extension of the Matrix-Tree theorem that was proved by Masbaum and Vaintrob in [16], called the Pfaffian-Tree theorem. This uses the pfaffian of a matrix to obtain the generating series for all spanning subtrees of a hypergraph of uniform degree three. Before we can understand this extension, there are two things we should examine. The first is the orientation of hypergraphs.

Definition 2.3. An *orientation* on a hypergraph is an assignment of an ordering of the vertices of each edge up to even permutation.

The first thing to note about this definition is that it specializes to our definition of an orientation on a standard graph. In a standard graph we assign one vertex of each edge to be the head and the other to be the tail. This effectively assigns an order to the vertices, and the only even permutation on two elements is the identity. Thus, in our standard case we have effectively set one ordering to be positive and one to be negative. For hypergraphs we have chosen a whole set of orderings to be positive and another set to be negative.

We can define an orientation on any set in the same way, by giving the set an order up to even permutation. The orientation of a set with odd cardinality can be represented in another way, as a cyclic permutation. For an order of the set we use the cyclic permutation on this set which maps each element to the next element in the order, and the last one to the first. Since the number of elements is odd, any other ordering which gives the same cyclic permutation (i.e. one that has the same order but starts in a different place) is an even permutation of this ordering and thus is equivalent. Permuting the original order by π is equivalent to conjugating this cyclic permutation σ by π, so we consider these cyclic representations to be equivalent up to even conjugation. Thus we are representing our orientation as an assignment of one of two conjugacy classes of the group of even permutations (noting that our original element was an odd cycle and thus in this group, and that conjugation preserves cycle type).

For example, if we have the ordering $3, 2, 1, 5, 4$ we denote this by the permutation which in cycle form is (32154). Our orientation is then the conjugacy class containing this in the group of even permutations A_5.

Next we must define the orientation of a tree induced by the orientation of its edges. Like the edges, an orientation on a tree is an ordering of its vertices up to even permutation.

Definition 2.4. The *orientation* of a tree induced by its edges (all of which must have odd degrees) is represented by the cyclic permutation given by the product of the cyclic permutations representing the orientations of its edges.

There are three things to check to prove that this is well defined: that this product gives a cyclic permutation, that the order of the product does not change the orientation, and that the representative of the conjugacy class for each edge does not affect the conjugacy class of the product. These are our next three results.

Lemma 2.5. *For a tree composed of odd degree edges in an oriented hypergraph there is a choice of order for the edges $e_1, \dots, e_n$ with cyclic representations $\pi_{e_1}, \dots, \pi_{e_n}$ so that $\prod_{i=1}^{n} \pi_{e_i}$ is a cyclic permutation on all the vertices of the tree.*

Proof. We shall use an order which starts with an arbitrary edge, then is chosen so that each subsequent edge shares a vertex with an edge prior to it in the order. Since the tree is connected such an order exists.

We can think of this as constructing the tree by adding each edge in this order. This builds our tree so that each step is a connected graph. The result is that no edge shares more than one vertex with the edges added prior to it. This is because any two vertices already added are connected, so connecting them again with this edge creates a cycle. Since trees have no cycles this isn't possible.

Since the product of two cycles which have only one element in common is a cycle of the union of their elements, we can see by induction that each partial product in this order is a cycle of the vertices thus far added. So our final product is a cycle of all the vertices of the tree. □

Lemma 2.6. *If the choice of representatives of the orientations of the edges of a hypergraph does not change the conjugacy class of our product, then the order of the product also does not affect its conjugacy class.*

Proof. If we have our product $A\pi_e\pi_f B$ for some edges e and f we see that:

$$A\pi_e\pi_f B = A\pi_f\pi_{f^{-1}}\pi_e\pi_f B.$$

Noting that $\pi_{f^{-1}}\pi_e\pi_f$ is a conjugate of π_e, and since conjugation of an edge does not change the conjugacy class of the product, $A\pi_f\pi_e B$ must be a conjugate of $A\pi_e\pi_f B$.

Thus we can switch any adjacent pair of edges, and reorder however we like. □

Proposition 2.7. *The orientation of a tree induced by that of its edges is well defined.*

Proof. We shall show by induction on the number of edges in the tree that our choice of representative of the conjugacy class for each edge does not alter our tree orientation. This holds trivially in the case of one edge.

For our induction hypothesis we assume that for any tree of fewer than k edges, choice of representative of the edge does not change the conjugacy class of the product. From Lemmas 2.5 and 2.6 above we see that this further implies that the order chosen does not affect the conjugacy class either, and that the permutation is cyclic.

Take any pair of cyclic permutations sharing only one vertex c. Represent each cycle as $\pi_a = (c, a_1, \ldots, a_n)$ and $\pi_b = (c, b_1, \ldots, b_m)$. Then we have:

$$\pi_a\pi_b = (c, b_1, \ldots, b_m, a_1, \ldots, a_n).$$

If these are both odd cycles (making their product also an odd cycle), then their conjugacy class is determined by the signum of the order in that representation.

Conjugating π_a, where we restrict to permutations of $\{c, a_1, \dots, a_n\}$, gives us another permutation $\pi'_a = (c, a_{\sigma(1)}, \dots, a_{\sigma(n)})$ for some even σ. The order given by the new product $\pi'_a \pi_b = (c, b_1, \dots, b_m, a_{\sigma(1)}, \dots, a_{\sigma(n)})$ is an even permutation of that given by $\pi_a \pi_b$, and thus they are in the same conjugacy class. This works for π_b in exactly the same way. Thus the conjugacy class of the product of two odd cycles sharing only one element is invariant under the conjugation of the original cycles (as long as the conjugation is only amongst elements in the cycles).

Take a tree T with k edges. Choose an order for the edges $e_1, \dots, e_k$, and representatives $\pi_1, \dots, \pi_k$ for the orientations of these edges. Then choose an arbitrary edge e_j, for which we choose a second representative π'_j. We need to show that:

$$\prod_{i=1}^{k} \pi_i$$

is a conjugate of:

$$\left(\prod_{i=1}^{j-1} \pi_i\right) \pi'_j \left(\prod_{i=j+1}^{k} \pi_i\right).$$

To do this we choose a leaf e_l (since we dealt with the base case we can safely assume there are at least two leaves and thus $l \neq j$). For convenience we assume that $j < l$ but the proof works in the same way if $l < j$.

Then we can write the first product as $A\pi_j B\pi_l C$ and the second as $A\pi'_j B\pi_l C$ where A, B, and C are even permutations. Since

$$A\pi_j B\pi_l C = C^{-1} C A\pi_j B\pi_l C,$$

this is a conjugate of $CA\pi_j B\pi_l$. Let T' be the tree created by removing e_l from T. Since this has only $k-1$ edges, the induction hypothesis implies that any ordering and choice of representatives of its edges gives a cyclic permutation of the same conjugacy class in $A_{V(T')}$. Two such choices are $CA\pi_j B$ and $CA\pi'_j B$.

Since e_l is a leaf it shares only one vertex with the rest of the tree, and thus π_l shares only one element with $CA\pi_k B$. By the above paragraph this means that since $CA\pi_j B$ and $CA\pi'_j B$ are conjugates and cyclic, so are $CA\pi_j B\pi_l$ and $CA\pi'_j B\pi_l$. We have already shown the former is a conjugate of $A\pi_j B\pi_l C$, and in the same manner the latter is a conjugate of $A\pi'_j B\pi_l C$.

Thus by induction we prove our choice of representative of the orientation of the edges does not affect the conjugacy class of the product. Applying Lemmas 2.5 and 2.6 again finishes our proof that the definition of tree orientation works.

□

Using this definition of tree orientation we can refer to a tree as positive or negative with respect to an ordering of the vertices in the hypergraph. We say the *sign* of the tree is positive if the order of the vertices agrees with the orientation of the tree, and negative otherwise. As an example, consider the tree in Figure 2.2.

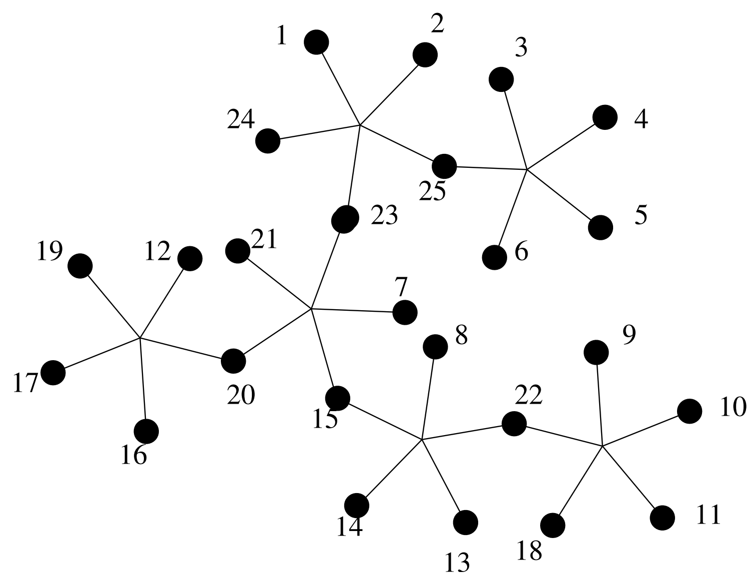

Figure 2.2: Example of Tree Orientation

We orient each edge according to its label order. The product of cyclic permutations representing the orientation of the edges is:

$$(1, 2, 23, 24, 25)(3, 4, 5, 6, 25)(7, 15, 20, 21, 23)$$
$$(9, 10, 11, 18, 22)(8, 13, 14, 15, 22)(12, 16, 17, 19, 20).$$

Carrying out this multiplication we get:

$$(1, 2, 23, 7, 15, 9, 10, 11, 18, 22, 8, 13, 14, 20, 12, 16, 17, 19, 21, 24, 25, 3, 4, 5, 6).$$

Since the signum of this sequence (not this permutation) is negative, it disagrees with the vertex order, so the tree is negatively oriented with respect to the vertex labelling. Note that this is not the signum of the cyclic permutation, as that is always even (since all odd cycles are even). Also note that it is not an even permutation of the labelling of the vertices despite the fact that it was created from edges which were all oriented with that labelling.

We are now ready to set up the matrix which we are going to use for the Pfaffian-Tree theorem. Take a uniform degree three hypergraph $G = (V, E)$ with n vertices labelled from 1 to n. Represent each edge by a variable y_e. Define:

$$(\Lambda)_{ij} = \sum_{k \in V \setminus \{i,j\}} \sum_{\substack{e \in E \\ V(e) = \{i,j,k\}}} \operatorname{sgn}(ijk) y_e, \quad i \neq j, 1 \leq i, j \leq n.$$

Since we want an antisymmetric matrix we must let $(\Lambda)_{ii} = 0$ for all $1 \leq i \leq n$.

This is an $n \times n$ antisymmetric matrix. Its entries are the signed sums of the edges containing the two vertices represented by its column and row. If there are any spanning trees at all, then the graph must have an odd number of vertices, since a tree in a uniform degree three hypergraph must have one more than twice as many vertices as edges (noting that each edge adds two vertices, and we start with one). To have a non-trivial pfaffian we must have an even dimensional anti-symmetric matrix. As in the Matrix-Tree theorem, we delete the pth row and column for some p. Denote this minor by Λ_p. This also means that the terms of our pfaffian have the desired number of edges.

Theorem 2.8. *Consider a uniform degree three hypergraph oriented according to its vertex labels and the antisymmetric matrix Λ described above. Let $\mathcal{T}$ be the set of spanning trees of the graph. Then:*

$$(-1)^{p-1} pf(\Lambda_p) = \sum_{T \in \mathcal{T}} \mathfrak{o}(T) \prod_{e \in T} y_e,$$

where $\mathfrak{o}(T)$ is the sign of the orientation of the tree T with respect to the vertex labels.

We shall not prove this here, as its proof will follow from the more general version, Theorem 2.9 in Section 2.3.

This theorem has one major property which is not as convenient as the Matrix-Tree theorem; the generating series here is signed. While the generating series in the Pfaffian-Tree theorem does give us every spanning tree, because it is signed we cannot count them by substituting 1 for the y_es as we did with the result of the Matrix-Tree theorem.

Let us try an example using the graph shown in Figure 2.3. Start by orienting

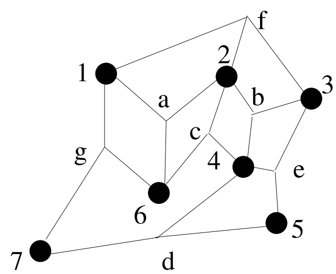

Figure 2.3: Example Graph

each edge according to the vertex labels. We can then define our $\Lambda(G)$ to be:

$$\Lambda(G) = \begin{bmatrix} 0 & y_a + y_f & -y_f & 0 & 0 & -y_a + y_g & -y_g \\ -y_a - y_f & 0 & y_f + y_b & -y_b + y_c & 0 & y_a - y_c & 0 \\ y_f & -y_b - y_f & 0 & y_b + y_e & -y_e & 0 & 0 \\ 0 & y_b - y_c & -y_b - y_e & 0 & y_e + y_d & y_c & -y_d \\ 0 & 0 & y_e & -y_e - y_d & 0 & 0 & y_d \\ y_a - y_g & -y_a + y_c & 0 & -y_c & 0 & 0 & y_g \\ y_g & 0 & 0 & y_d & -y_d & -y_g & 0 \end{bmatrix}.$$

Then if we choose to eliminate the first vertex we get:

$$\Lambda_1(G) = \begin{bmatrix} 0 & y_f + y_b & -y_b + y_c & 0 & y_a - y_c & 0 \\ -y_b - y_f & 0 & y_b + y_e & -y_e & 0 & 0 \\ y_b - y_c & -y_b - y_e & 0 & y_e + y_d & y_c & -y_d \\ 0 & y_e & -y_e - y_d & 0 & 0 & y_d \\ -y_a + y_c & 0 & -y_c & 0 & 0 & y_g \\ 0 & 0 & y_d & -y_d & -y_g & 0 \end{bmatrix}.$$

Taking the pfaffian of this gives us:

$$pf(\Lambda_1(G)) = y_f y_e y_g + y_e y_g y_c - y_f y_c y_d - y_b y_d y_a + y_f y_g y_d + y_g y_d y_b$$

We can see this agrees with the only six possible spanning trees of the graph.

2.3 The Hyperpfaffian-Cactus Theorem

Abdelmalek Abdesselam generalized the pfaffian tree theorem in [1] using the hyperpfaffian to handle trees (which he calls cacti) of hypergraphs whose edges all have odd degree. While his result is given in terms of the Grassmann-Berezin Calculus, he does note it specifies to the hyperpfaffian in the uniform case. We now look at this in terms of the non-uniform hyperpfaffian (whose definition is equivalent to his Grassmann-Berezin result).

Initially we should define the antisymmetric tensor algebra that corresponds to the Kirchhoff matrix or the antisymmetric matrix for the Pfaffian-Tree theorem. Take a hypergraph $G = (V, E)$ on n vertices with edges all of odd degree greater than one, oriented by $\mathfrak{o}$. As usual we label the vertices from 1 to n and assign the indeterminate y_e to each edge e.

For any sequence σ of distinct elements of the set $\{1, \ldots, n\}$, define our antisymmetric tensor algebra $\Lambda(G)$ of order n as follows:

$$\Lambda(G)(\sigma) = \sum_{\substack{p \in V \\ p \notin \sigma}} \sum_{\substack{e \in E \\ \sigma \cup \{p\} = V(e)}} \operatorname{sgn}_{\mathfrak{o}}(\sigma + p) y_e,$$

where $\sigma + p$ is the sequence σ with the vertex p appended to the end. The signum denoted by $\operatorname{sgn}_{\mathfrak{o}}(\sigma + p)$ is the sequence's signum with respect to the orientation of the edge (i.e. negative if it agrees with the orientation and positive if not). It is easy to see that this tensor algebra is antisymmetric, as reordering σ only alters the signum, and thus works exactly as required. Denote the antisymmetric tensor algebra restricted to disallow the use of v in any sequence by $\Lambda_v(G)$.

Theorem 2.9 (The Hyperpfaffian-Cactus theorem). *Consider an oriented hypergraph all of whose edges have odd degree greater than one, and the antisymmetric tensor algebra $\Lambda(G)$ described above. If $\mathcal{T}$ is the set of spanning trees of the hypergraph, then*

$$pf_*(\Lambda_v(G)) = (-1)^{v-1} \sum_{T \in \mathcal{T}} \mathfrak{o}(T) \prod_{e \in T} y_e,$$

where $\mathfrak{o}(T)$ *is the sign of the tree orientation with respect to the vertex labels.*

Proof. To prove this result we are going to put a sign reversing involution on all the terms of the expansion of the hyperpfaffian (using Definition 1.20) which represent graphs containing cycles. Let us look at what types of terms are produced by this hyperpfaffian. The first thing to note is that for each term composed of q indeterminates $e_1, \ldots, e_q$, $\sum_{i=1}^{q}(\deg(e_i)-1) = n-1$. This is because the partition of $n-1$ that generates that term is composed of q sets corresponding to the edges, each having one fewer element than the degree of that edge. By Proposition 1.30 all spanning trees have this property. Also any graph with this property which is not a spanning tree is not connected, and has at least one cycle.

Each term derives from a particular partition of $\{1, \ldots, v-1, v+1, \ldots, n\}$. For each set of size k in this partition there is a $(k+1)-$edge from our graph containing the k vertices from the set. We can then refer to an edge in a term with respect to a partition as corresponding to a set in the partition. We also refer to its other vertex as its non-partition vertex.

This means that any component of a graph generated by a term in our hyperpfaffian must contain at most one cycle. In fact each component not containing v contains exactly one cycle and the component containing v is a tree. The latter can be seen quickly from the fact that if this component contains edges $e_1, \ldots, e_q$, it must have at least $\sum_{i=1}^{q}(\deg(e_i)-1)+1$ vertices (this is because each partition set must contribute its edge's degree minus one vertices plus it contains v). Since it is a component it is connected and thus cannot have more than this many vertices. Thus by Proposition 1.30 it is a tree.

Any other component is composed of only the vertices of the partition sets of its edges, since if it contains another vertex it would be connected to any edge containing that vertex, and thus the other vertices of its partition set. This means it must have $\sum_{i=1}^{q}(\deg(e)i)-1)$ vertices, and so cannot be a tree, which implies it has a cycle. Take that cycle and ignore the other edges. If there are q' edges in the cycle, $e_1, \ldots, e_{q'}$ then it must contain at least $\sum_{i=1}^{q'}(\deg(e_i-1)$ vertices. If it had more vertices it would be a tree since it is connected. Thus it has exactly this many vertices, and thus is a simple cycle (Definition 1.32).

Take any component $e_1, \ldots, e_j$ of the subgraph created by taking our original component and removing the edges of the cycle. It must have at least $\sum_{i=1}^{j}(\deg(e_i) - 1) + 1$ vertices, since it must have its own partition sets and at least one more vertex from the cycle. This is the most it can have and be connected, and thus it must be a tree. Therefore all the components not containing v are simple cycles with trees attached.

To define our sign reversing involution first select one component with a cycle of each non-tree graph arising as a term in our hyperpfaffian. It does not matter how this cycle is chosen, only that it is a unique choice for any given graph. Refer to this cycle as the *special cycle* of that graph.

Consider any term $t = \pm \frac{1}{(\deg(e_1)-1)!\ldots(\deg(e_m)-1)!m!} y_{e_1} \ldots y_{e_m}$ not representing a spanning tree in the expansion of our hyperpfaffian. Let σ be the sequence and π be the partition from which it arises. Let H be the graph it represents. Look at the special cycle of the H. There are exactly two different partitions of that cycle which can give rise to it. The partition is determined by the choice of non-partition vertex for any given edge. The non-partition vertex of each edge of the cycle must be a shared vertex, otherwise the non-partition vertex is not in any partition set, since it is in no other edge in the cycle. This means the other shared vertex is a partition vertex for that edge. This forces the choice of partition vertices for the edge sharing that vertex. Each forced choice then forces the next choice along the cycle. This continues until we return to the first edge. Since there are two initial choices there are two different partitions.

For an example of this, see Figure 2.4. Looking at the left partition, by choosing $\{1, 2, 3, 4\}$ we must pick $\{21, 22, 23, 24\}$, those are all the remaining vertices for that edge. This in turn forces the choice $\{17, 18, 19, 20\}$. This process continues giving us $\{13, 14, 15, 16\}$, $\{9, 10, 11, 12\}$, and $\{5, 6, 7, 8\}$. The right partition shows the other choice, $\{1, 2, 3, 5\}$, which forces a different partition. These are the only two possibilities.

Let π' be the partition which is similar to π, but where we replace the sets covering the special cycle of H with the other choice for this cycle. There are many sequences which correspond to π', but we choose σ' to be the one which is identical to σ except for at the shared vertices of the cycle. For each block of the sequence representing a

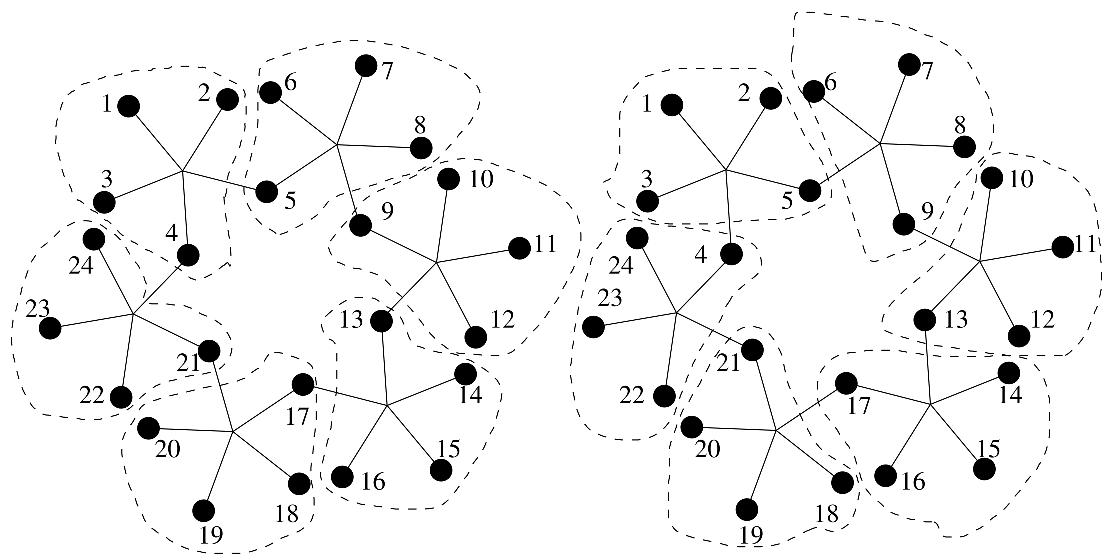

Figure 2.4: The two possible partitions for a simple cycle.

partition set corresponding to an edge e, replace the non-partition vertex of e with respect to π' in σ with the non-partition vertex of e with respect to π. Thus σ differs from σ' in only one vertex for each partition set corresponding to an edge in the special cycle.

Note that our choice of π' means that σ' gives rise to a term t' which represents the same graph as t. We define our involution to map t to t'. Since t was arbitrary this defines our involution. To check that it is indeed an involution, note since t' represents the same graph it has the same special cycle. This means that changing our choice of partition π' along this cycle gives us π. The reverse replacements on σ' are then made to give us back σ and thus the term t.

Next, to prove that this involution reverses sign, note that applying the permutation which maps each vertex to the next vertex along the cycle (the direction is from the one in π to the one in π' for the first edge, and then so on) to σ gives σ'. Thus the change in the signum in front of the term from the hyperpfaffian corresponds to the number of edges in the cycle. An even cycle changes this signum, an odd cycle does not. For each y_e we have switched the vertex which is not in the sequence with the other shared vertex, and otherwise left the order the same. This is a single transposition to the sequence, which then reverses its signum. Thus in our new term y_e is replaced by $-y_e$. Since this is done to each edge in the cycle it reverses the sign of the term if the cycle is odd, and not if it is even. Thus if the cycle is odd the sign is reversed by the change of sign of the y_e's, and if it is even the sign is reversed by the signum of the sequence. Thus in both cases the sign is reversed.

Before we continue with the proof, consider an example of how this involution works. Let us say we have a term:

$$-\frac{1}{(24)^{12}(12)!}y_a y_b y_c y_d y_e y_f y_g y_h y_i y_j y_k y_l$$

generating the graph in Figure 2.5, with the edges oriented according to their labels. Let us say we have chosen $v = 1$ and this term came from the sequence (which we'll group in fours representing the partition):

$$(7,15,20,21),(12,16,17,19),(13,8,14,22),(9,10,11,18),\\(3,4,5,6),(2,23,25,26),(28,27,30,49),(29,31,32,33),\\(34,35,36,37),(38,39,42,40),(41,43,44,45),(24,46,48,47).$$

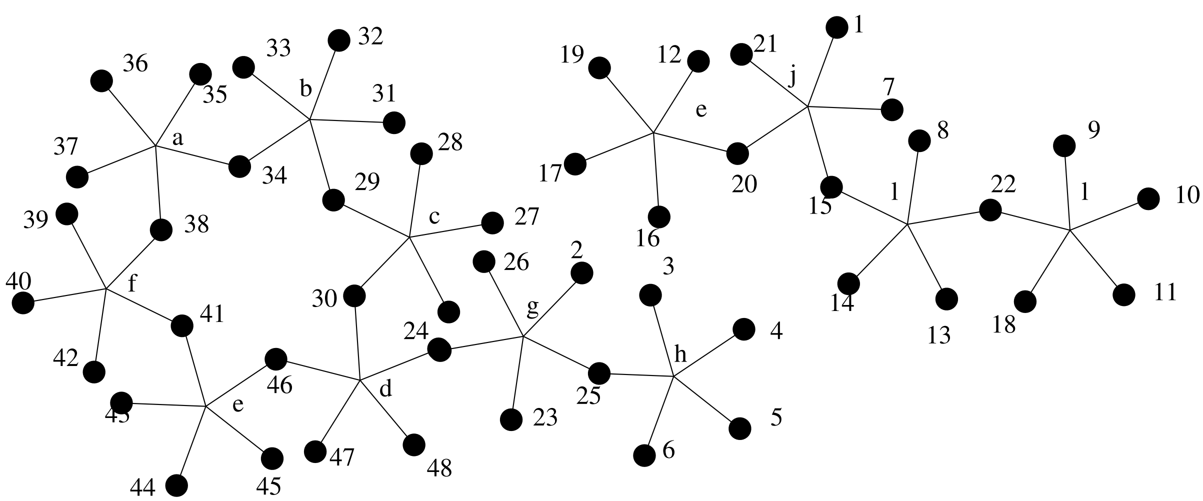

Figure 2.5: Example graph

This sequence has negative parity, and each edge is positive with respect to it. Since there is only one cycle, we know that this is our special cycle. We switch each non-partition vertex in each edge of the cycle with its other shared vertex resulting in the sequence:

$$(7, 15, 20, 21), (12, 16, 17, 19), (13, 8, 14, 22), (9, 10, 11, 18), \\ (3, 4, 5, 6), (2, 23, 25, 26), (28, 27, 29, 49), (34, 31, 32, 33), \\ (38, 35, 36, 37), (41, 39, 42, 40), (46, 43, 44, 45), (24, 30, 47, 48).$$

This has been an application of the permutation $(46, 30, 29, 34, 38, 41)$ to our original sequence. Since this is an even cycle, it changes the parity of the sequence to positive. For the edge d we use the sequence $(24, 30, 48, 47, 46)$ which does not match the orientation of the edge, so this contributes $-y_d$ to our product. The same thing happens to each other edge in the cycle. From this sequence we get the term:

$$\frac{1}{(24)^{12}(12)!}(-y_a)(-y_b)(-y_c)(-y_d)(-y_e)(-y_f)y_g y_h y_i y_j y_k y_l.$$

This term is identical to our original one, except with opposite sign.

Equipped with our sign reversing involution we can now return to the proof. This involution causes all our unwanted terms to cancel, but we must now show that we have all the required terms. To do this we shall use the expansion of the non-uniform hyperpfaffian from Definition 1.19, which uses canonical representations of partitions as sequences. We have established that we have no extraneous terms, so we need only show that all the terms we want remain and have the correct sign.

To do this we shall show that there is only one partition that gives rise to any given tree. Construct our tree from the vertex v down. Clearly any edge which contains v must choose its other vertices for the partition. This defines all partition sets for these edges. Then any edge connected to these first edges must have all of its other vertices chosen as its partition set. We can continue these forced choices for all edges. We can always choose these partition sets, since if more than one vertex was already taken we would get a cycle. An example of this is shown in Figure 2.6. Since we then have a unique partition for each spanning tree, we have a unique term representing each spanning tree. Thus all the required terms do appear.

Finally we need to show that the sign matches that of the tree orientation in the appropriate way. Here we return to the Definition 1.20 expansion of the non-uniform

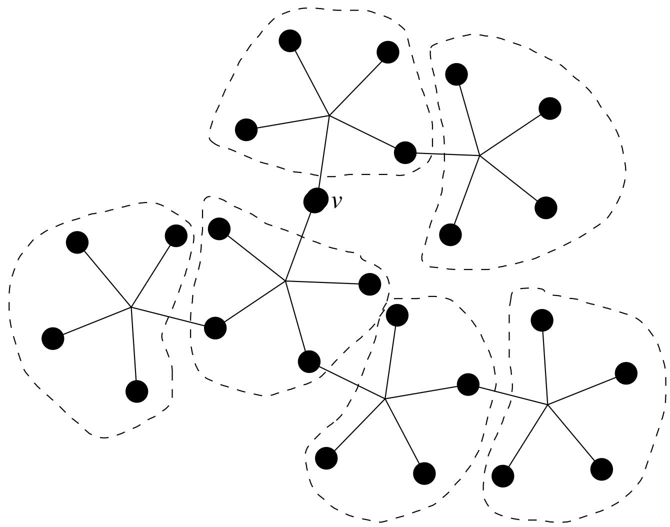

Figure 2.6: The forced matching on a tree

hyperpfaffian. Using this we see that each term representing a given tree has the same sign, so we need only check one. Once again we construct our tree edge by edge.

From its definition we can construct a representative of our tree's orientation by starting with an edge containing v, and write it as $(v, a_1, \ldots, a_k)$, then at each stage we add an edge which contains exactly one vertex already added. For this choose a representative of the orientation of each edge so that its non-partition vertex is last (note when this is added the non-partition vertex is the already added vertex). Each time this inserts $b_1, \ldots, b_j$ somewhere within the cycle. We can shift this even block without changing conjugacy class, since the blocks are even (since its conjugacy class is determined by the signum of the sequence representing the cycle). We move this block to the end (note that these blocks correspond to the matching component of the edge).

As a result of this, we end up with v inserted at the beginning of a sequence which corresponds to the partition for this tree. Thus this partition gives a term with the signum of this sequence, which is thus $(-1)^v$ times the signum of the orientation of the tree. Since each block was chosen so that when its final vertex is added it is a representative of the orientation of the edge, the term is composed of the product of positive signed indeterminates y_e. Thus the sign of the term is the same as the signum of the sequence. $\square$

Now for an example of the Hyperpfaffian-Cactus theorem we examine Figure 2.7. Noting that only two partitions give non-zero terms, we get the following for the

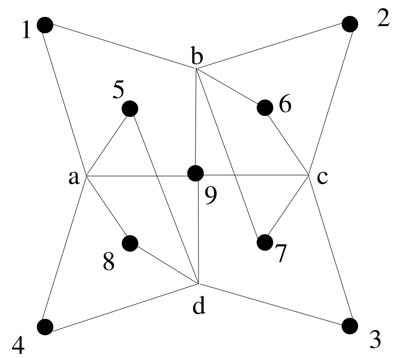

Figure 2.7: Example Graph

hyperpfaffian on the alternating tensor with the vertex 1 eliminated:

$$pf_*(\Lambda_1(G)) = (y_c y_a + y_c y_d) + (-y_c y_d + y_b y_d)$$
$$= y_c y_a + y_b y_d.$$

Here we can see that these correspond to the only two spanning trees.

2.4 The Even Case

A weakness of the Hyperpfaffian-Cactus theorem is that it does not cover hypergraphs containing edges of even degree. Here we shall discuss a few methods I have considered unsuccessfully to solve this problem. For simplicity we restrict to the uniform case, allowing us to use the Luque-Thibon and Barvinok definitions of the hyperpfaffian. Certainly if it works for the non-uniform case it must work for the uniform case.

The two basic approaches are either to use the Luque-Thibon definition of the hyperpfaffian (Definition 1.11) which allows for odd dimensioned hyperpfaffians or to use the hyperdeterminant, whose dimension matches the degree of the graph it describes.

We first look at using the Luque-Thibon hyperpfaffian. It would seem fairly natural to define our tensor in the same way as before, only with odd dimension, and apply this hyperpfaffian. Our proof that all the terms representing spanning trees appear in this expression still holds. However we need to be more careful with the sign reversing involution. The proof we gave of the hyperpfaffian-cactus theorem used the version of the non-uniform hyperpfaffian which resembles the Barvinok definition of the hyperpfaffian (Definition 1.13), however with only slight modification this could be done using the Luque-Thibon definition, except for one clause. The two factors which allow the involution to be sign reversing (the signum of the edge, and the sign of the matching) still acts in an identical way and thus leave the involution sign reversing. However there is now a third factor in determining the sign of the term; the order of the edges in the partition or matching. As mentioned when discussing the differences in our definition of the hyperpfaffian, in the odd dimensional case the order of the matching matters. When we apply the cycle to our matching, we may change which vertex is first, and thus the order of the edges, thus further changing the sign. If it does cause a change, our involution is no longer sign reversing.

There is clearly no way around this problem in general, since we are stuck with extra terms when applying the hyperpfaffian at least to the complete graph working over a field of characteristic zero. However there are two ways of restricting our problem to make this work. We can either restrict our graph to a class of graphs which do not have this problem, or restrict our field.

First we look at which graphs give no extra terms. The most precise characterization of these graphs can be made by taking the hyperpfaffian of Λ for the complete graph on an appropriate set of vertices. If our graph does not contain at least one edge from every one of the extraneous terms of this hyperpfaffian then the theorem holds for this graph. While this is a precise characterization of which graphs work, it is one basically by definition, so we would like to find a more useful condition.

To find this it is important to note that the extra terms of a graph depend on its labelling. As such we may try to find which graphs can be labelled in such a way that they have no residual terms. What we find is that any cycle can be labelled in a "*good*" way or a "*bad*" way (see Figure 2.8). A "good" way is one where if we switch partitions representing the cycle our involution is sign reversing. Looking to the example on the right we see that the two valid sequences are:

$$(1,2,13),(3,12,18),(4,5,14),(6,7,15),(8,9,16),(10,11,17)$$

and

$$(1,3,12),(2,13,14),(4,5,15),(6,7,16),(8,9,17),(10,11,18).$$

Note that since we are using the Luque-Thibon definition, Definition 1.11, there is only one sequence per matching. Now because the order of the first two edges is reversed, we have an additional sign change. Before this change we had applied an even cycle to our sequence, thus changing its sign, and made a single flip on the six edges. Any change in sign from reordering the vertices of a matching (to make them increasing as per the Luque-Thibon definition) is manifested in a sign change in both the sequence's signum and that of the edge, and thus has no effect. This means we have exactly one too many sign reversals, leaving us with a term of the same sign as our original.

If we instead look at the left side of Figure 2.8 we see that no edge has its lowest labelled vertex as a shared vertex. This means that changing these vertices does not change the edge order. If a cycle does not have this property it is hard to tell if it is "good". Even if the change from taking a different vertex does not upset the order of the cycle, it may be interfered with by other edges in the graph. If there are multiple cycles, flipping them may change whether the cycle is "good" or "bad".

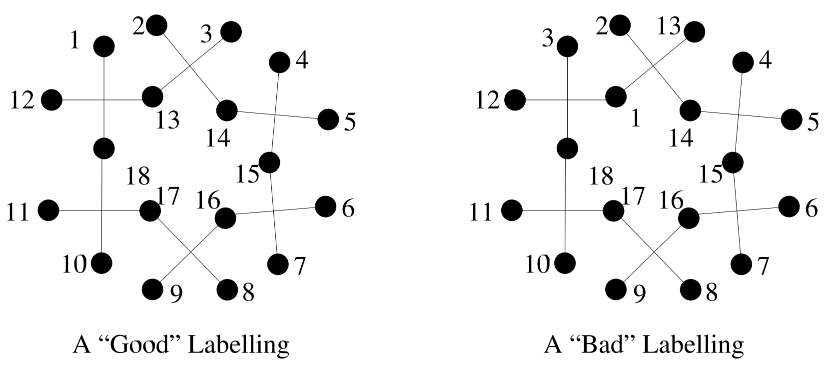

Figure 2.8: "Good" and "Bad" labellings of a cycle

We need only know that any subgraph generated by partitions (i.e. those ones with the appropriate number of edges, and components described above) contains at least one "good" cycle. We can then choose this as the special cycle, and our theorem will be proved in the same manner as before, with the exception of the requirements on the sign of the terms. Note that we only have tree orientation defined for odd degree hypergraphs, so we cannot expect ours to match this. The easiest class of graphs to describe with this property are those in which each edge contains at least one vertex which is not a shared vertex in any cycle. For each edge label this vertex lowest among the edge. This makes every cycle good.

As mentioned before, instead of restricting our graph we can instead restrict to a field of characteristic two. The involution is still an involution, and thus any of our bad terms has an even coefficient, while the terms we want remain with coefficient one. Thus we would get the answer we desire, but certainly we have lost a lot by this restriction (note since we must be using the Luque-Thibon definition, it is not a problem to use non-zero characteristic). One does see in general that this theorem looks simpler in characteristic two, as we are no longer troubled by the differing signs of the terms, and so we can drop our orientation.

Our second approach is to use the hyperdeterminant. In this case we look to extend the Matrix-Tree theorem instead of the Pfaffian-Tree theorem. Since in the determinant case we use graphs of the same degree as the dimension of the tensor, it seems as if it might work. However this idea quickly shows its failing when we note that the number of edges does not scale in the appropriate way. If we construct our matrix as before, with $n-1$ of the n vertices, we see that the hyperdeterminant of dimension k gives us terms of degree $n-1$, representing hypergraphs of $n-1$ edges. On the other hand to form a tree in a degree k uniform hypergraph we need to have only $(n-1)/(k-1)$ vertices. So this certainly does not work.

Chapter 3

Hyperpfaffian Orientations

3.1 Pfaffian Orientations

Since the pfaffian is a sum over perfect matchings, it is natural to think that one might use it to count the perfect matchings of a graph. However it quickly becomes apparent that the signs in the sum interfere with this. In fact the direct tool to use for this is called the hafnian.

Definition 3.1. Let $\mathcal{M}_n$ be the set of perfect matchings on $\{1,\ldots,n\}$, and take an upper triangle Λ on $\{1,\ldots,n\}$. Then the *hafnian* is:

$$hf(\Lambda) = \sum_{M\in\mathcal{M}_n} \prod_{\{a,b\}\in M} \Lambda(\{a,b\}).$$

This definition resembles Definition 1.2 for the pfaffian. It is the unsigned sum over perfect matchings, rather than the signed sum. Introduced by Caianiello in 1953, the hafnian is to the pfaffian as the permanent is to the determinant (recall that the permanent of a matrix is the unsigned sum over the permutations in the way that the determinant is the signed sum). The hafnian of a matrix is not the square root of the permanent of a matrix, as with the pfaffian in Theorem 1.5, but the permanent can be expressed as a hafnian, as the determinant is in Proposition 1.8. The proof of this is exactly the same as for Proposition 1.8, only without the need to check that the signs match.

The hafnian is exactly the tool to count the perfect matchings of a graph. Unfortunately, as pointed out in [17], there is no known way of calculating the hafnian efficiently. In fact it is known that calculation of the permanent is #P complete, even when restricted to matrices containing only zeros and ones. As we have mentioned, we can calculate the permanent from the hafnian, so it must also be at least #P complete.

To find a better way of counting perfect matchings, we turn to Kasteleyn who in [13] created the pfaffian orientation. In a pfaffian orientation the graph is directed in such a way that all the terms of the pfaffian are positive, like the hafnian. Since the pfaffian is the square root of the determinant of the antisymmetric matrix it can be computed efficiently (for numeric valued matrices).

For an undirected graph $G = (V, E)$ we have defined its antisymmetric matrix to be:

$$A_{ij}(G) = \begin{cases} \sum\limits_{\substack{e \in E \\ V(e)=\{i,j\}}} y_e & i < j \\ -\sum\limits_{\substack{e \in E \\ V(e)=\{i,j\}}} y_e & i > j \\ 0 & \text{otherwise} \end{cases}.$$

For a directed graph $D = (V, E)$ we now define its antisymmetric matrix to be:

$$A_{ij}(D) = \begin{cases} \sum\limits_{\substack{e \in E \\ h(e)=j \\ t(e)=i}} y_e - \sum\limits_{\substack{e \in E \\ t(e)=j \\ h(e)=i}} y_e & i \neq j \\ 0 & \text{otherwise} \end{cases}.$$

Here we use $h(e)$ to denote the head of e, and $t(e)$ to the tail.

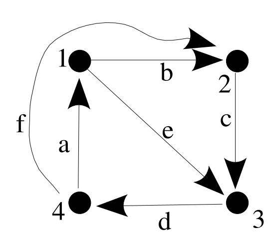

Figure 3.1:

For example if we look at the graph in Figure 3.1, considered as an undirected graph G, its corresponding antisymmetric matrix is:

$$A(G) = \begin{bmatrix} 0 & y_b & y_e & y_a \\ -y_b & 0 & y_c & y_f \\ -y_e & -y_c & 0 & y_d \\ -y_a & -y_f & -y_d & 0 \end{bmatrix}.$$

When considered as a digraph D, it corresponds to:

$$A(D) = \begin{bmatrix} 0 & y_b & y_e & -y_a \\ -y_b & 0 & y_c & -y_f \\ -y_e & -y_c & 0 & y_d \\ y_a & y_f & -y_d & 0 \end{bmatrix}.$$

The matrix for our undirected graph G would then be the same as the matrix for the digraph D obtained by directing every edge of G from lower to higher labelled vertex. We can define this for an upper triangle in the same way, using the upper triangle corresponding to this matrix. In terms of the upper triangle, we have used a negative value for each edge directed from higher to lower labelled vertex. Thus we can think of edges as positive or negative, based on whether the orientation agrees with the vertex order or not. We define the pfaffian of a digraph to be the pfaffian of this antisymmetric matrix.

So the pfaffian of the digraph in Figure 3.1 is:

$$pf(A(D)) = \text{sgn}(1234)y_by_d + \text{sgn}(1324)y_e(-y_f) + \text{sgn}(1423)(-y_a)y_f$$

$$= y_by_d + y_ey_f - y_ay_c.$$

If we look at the pfaffian of this considered as an undirected graph we would get instead:

$$pf(A(G)) = \text{sgn}(1234)y_by_d + \text{sgn}(1324)y_ey_f + \text{sgn}(1423)y_ay_c$$

$$= y_by_d - y_ey_f + y_ay_c.$$

Definition 3.2. A permutation σ which represents a matching π in the manner of Definition 1.6 is said to *respect orientation* if in it each adjacent pair representing a matching edge is ordered tail first, head second.

For example if we again refer to Figure 3.1, and consider the matching $\{1, 2\}, \{3, 4\}$. Then 1243 or 4312 are permutations for this matching which respect orientation. On the other hand 2143 does not, since the edge 12 must be given in that order.

For a matching π of a digraph D we see that the sign of the term corresponding to π in the pfaffian of D is the same as the signum of any permutation σ which represents π and respects orientation. If we look at the expansion of the hyperpfaffian in Definition 1.6, we see that each edge e in the matching π is taken in the term of the expansion corresponding to σ as $\Lambda(t(e), h(e))$ and thus we get y_e not $-y_e$. Thus the term's sign is determined only by the signum of σ. Since all terms in the expansion representing a given matching have the same sign, this is the same as the sign of the complete term representing the matching.

Definition 3.3. A *pfaffian orientation* of a graph is an assignment of direction to each edge of the graph so that every term of the pfaffian of the digraph is positive.

If all the terms are positive we can count the matchings, as we would with the hafnian (thus the pfaffian of the directed graph would equal the hafnian of the undirected graph). Since the sign of a term corresponding to a matching π has the same sign as the signum of any permutation representing it and respecting orientation, to show something is a pfaffian orientation we need only show that each matching has such a permutation with positive signum.

For an example of a pfaffian orientation, look at the left side of Figure 3.2. The

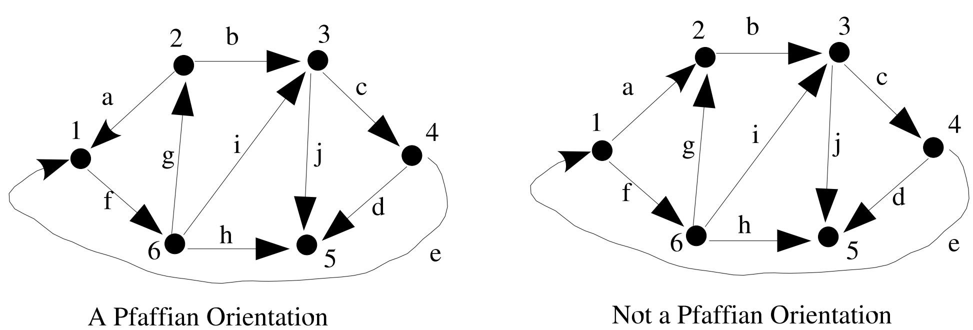

Figure 3.2: An example of a graph oriented two ways, one pfaffian

antisymmetric matrix which this defines is:

$$A(G) = \begin{bmatrix} 0 & -y_a & 0 & -y_e & 0 & y_f \\ y_a & 0 & y_b & 0 & 0 & -y_g \\ 0 & -y_b & 0 & y_c & y_j & -y_i \\ y_e & 0 & -y_c & 0 & y_d & 0 \\ 0 & 0 & -y_j & -y_d & 0 & -y_h \\ -y_f & y_g & y_i & 0 & y_h & 0 \end{bmatrix}.$$

If we take the pfaffian of this we get:

$$y_g y_e y_j + y_d y_b y_f + y_a y_d y_i + y_h y_b y_e + y_a y_h y_c.$$

All terms of this pfaffian are positive, so this is indeed a pfaffian orientation. If we look at the graph on the right of Figure 3.2 we see that the only difference is that the edge a is oriented oppositely. This means by replacing y_a with $-y_a$ in the pfaffian of the graph on the left we obtain the pfaffian of the graph on the right:

$$y_g y_e y_j + y_d y_b y_f - y_a y_d y_i + y_h y_b y_e - y_a y_h y_c.$$

Here we see a mixture of positive and negative signs, so this is not a pfaffian orientation.

We would like to know when it is possible to find such an orientation. One of the most fundamental results on this subject was proven by Kasteleyn in [13], which is:

Theorem 3.4. *Any planar graph has a pfaffian orientation.*

Here we shall give a proof based on that from [11].

Proof. We orient the graph so that every internal face has an odd number of counterclockwise edges (note with respect to a cycle or face in a planar embedding of a graph each edge has a counterclockwise and a clockwise direction). We shall prove that all planar graphs can be oriented this way by induction on the number of edges. Since a graph with no edges has no internal face the base case is trivial.

If we take any planar graph and remove an edge e on the outer face, we obtain a graph of fewer edges which is still planar. By the induction hypothesis we can orient this graph in the manner described above. When we add e again, we know e is in at most two faces. By our choice of e one of these is not internal, so it is in at most one internal face. We direct e so that its internal face has an odd number of counterclockwise edges, or arbitrarily if there is no such face. All other internal faces remain unchanged, and thus have an odd number of counterclockwise edges. So by induction an orientation of this type exists. Next we need to show that this orientation is a pfaffian orientation.

To prove this is a pfaffian orientation, we only need to show that with this orientation any two matchings give terms of the same sign. If all terms are negative we can reverse all the edges around a given vertex. Since exactly one of these edges is in any matching, this reverses the sign of every term, giving us all positive terms.

To show that any two terms have the same sign, start by observing that the symmetric difference between any two perfect matchings is a graph where each vertex has degree either two or zero. This means that it is a collection of isolated points and disjoint cycles. We also know that the edges are two colourable (coloured by which matching they came from), and thus these are even cycles. We call these cycles *transition cycles*.

Taking the symmetric difference of a perfect matching and one of its transition cycles gives a new perfect matching. Applying each transition cycle of the symmetric difference of two perfect matchings M and M' in sequence creates a sequence of perfect matchings each one transition cycle different from the previous one, starting with M and ending with M'. Thus it suffices to prove that the terms corresponding to any two matchings whose symmetric difference is a single transition cycle have the same sign.

For example if we have the graph on vertices $\{1, \ldots, 10\}$ and the two matchings:

$$\pi_1 = \{1,6\}, \{2,7\}, \{3,8\}, \{4,9\}, \{5,10\},$$

$$\pi_3 = \{1,2\}, \{3,4\}, \{6,7\}, \{8,9\}, \{5,10\}.$$

Then their symmetric difference is the two transition cycles 1,2,6,7 and 3,4,8,9. Applying the first cycle to π_1 gives the perfect matching:

$$\pi_2 = \{1,2\}, \{6,7\}, \{3,8\}, \{4,9\}, \{5,10\}.$$

Applying the other transition cycle to π_2 we get π_3. In two steps we can go from π_1 to π_3 where each step is a perfect matching one transition cycle away. If π_1 gives a term of the same sign as π_2 which gives a term the same sign as π_3 then π_1 has the same sign as π_3.

We are next going to show that every transition cycle has an odd number of counterclockwise edges in a graph oriented in this way. To show this we shall show that for an even cycle C and G' the subgraph obtained by removing everything outside of C, the number of counterclockwise edges in C has opposite parity of the number of vertices in G'. Let f be the number of faces enclosed by C (the number of internal faces of G'). Let v and e be the number of vertices and edges of G', respectively, and let m be the length of C.

Number the internal faces from 1 to f. Let c_i be the number of counterclockwise edges for face i. Let c_0 be the number of counterclockwise edges of C. Note that every internal edge of G' is counterclockwise in one face and clockwise in one face. Thus the sum of the the counterclockwise edges of the internal faces is the sum of the internal edges, plus the counterclockwise edges of C. Then we have:

$$\sum_{i=1}^{f} c_i = e - m + c_0.$$

Since c_i is odd for each $i = 1, \ldots, f$ and m is even, then taken modulo two this is:

$$f \equiv e + c_0.$$

By Euler's formula we know (since f does not include the outer face)

$$f - e + v = 1.$$

Combining these equations we see that

$$v + 1 \equiv c_0.$$

Thus we see that the number of counterclockwise edges in C has opposite parity of the number of vertices of G'.

For example if we look at Figure 3.3 we have that our sequence of c_i's starting with c_1 is $3, 1, 1, 1, 1, 1, 3, 1, 3$. There are 17 edges, and the cycle is of length 6 with 3 counterclockwise edges, so we see:

$$\sum_{i=1}^{f} c_i = 14 = 17 - 6 + 3 = e - m + c_0.$$

The sum of the c_i is 14, which equals the 8 internal faces mod two. There are three counterclockwise edges and ten vertices, which we see have opposite parity. By the

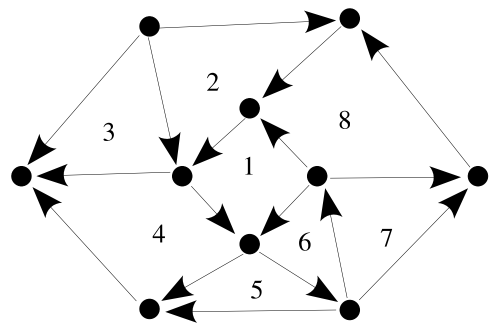

Figure 3.3:

planarity of the graph, we know that no edge can cross C. Thus internal vertices of G' cannot be connected to vertices outside of C. This means in any perfect matching internal vertices of G' must be matched to vertices of G'. If C is a transition cycle of M and M', then the vertices of C are all matched to each other, and thus the internal vertices must all be matched with each other in both M and M'. Thus G' must have an even number of vertices. This means that any transition cycle has an odd number of counterclockwise edges.

Consider two perfect matchings M and M' of G with a symmetric difference of a transition cycle C. Take a permutation σ representing the matching M which respects the orientation. We need to show that σ has the same signum as such a permutation for M'. Let σ_1 be the permutation which is the same as σ, except that all edges of $C \cap M$ are ordered clockwise (relative to C) instead of tail before head. Let a be the number of edges in $C \cap M$ which were oriented counterclockwise. Then $\text{sgn}(\sigma_1) = (-1)^a \text{sgn}(\sigma)$.

Let γ be the permutation that is the clockwise cycle C (i.e. it maps each vertex of C to the vertex next in counterclockwise order in C, and leaves the other vertices unchanged).

Then $\gamma\sigma_1$ is a permutation representing M'. This is because if we have an edge in M which is not in M', then it must be in C. If it is u, v ordered clockwise, then $\sigma_1(2i-1) = u$ and $\sigma_1(2i) = v$ for some i. Since these are in clockwise order $\gamma(u) = v$ and $\gamma(v) = w$ where w is the vertex clockwise from v in C. This means that vw is in M'. Since $\gamma(\sigma_1(2i-1)) = v$ and $\gamma(\sigma_1(2i)) = w$, these are matched in $\gamma\sigma_1$. This is true for all edges of C. The other edges remain unchanged, and thus remain ordered correctly.

In $\gamma\sigma_1$, every edge is in orientation order except those edges in $C \cap M'$, which are in clockwise order. Define σ_2 to be the same as $\gamma\sigma_1$, except that the edges of $C \cap M'$ (and thus all edges) are in orientation order. Let b be the number of counterclockwise edges of $C \cap M'$, then $\text{sgn}(\sigma_2) = (-1)^b \text{sgn}(\gamma\sigma_1)$. Since σ_2 is now a permutation which represents the matching M' and respects orientation, it only remains to be shown that $\text{sgn}(\sigma_2) = \text{sgn}(\sigma)$.

Since γ is an even cycle, we know that $\text{sgn}(\gamma) = -1$ and thus we see:

$$\text{sgn}(\sigma_2) = (-1)^b \text{sgn}(\gamma\sigma_1) = (-1)^{b+1} \text{sgn}(\sigma_1) = (-1)^{a+b+1} \text{sgn}(\sigma).$$

Since a is the number of counterclockwise edges in $M \cap C$ and b the number in $M' \cap C$, and since every edge of C is in exactly one of M or M', $a + b$ is the number of counterclockwise edges of C. Since all transition cycles have an odd number of counterclockwise edges in this orientation, $\text{sgn}(\sigma_2) = (-1)^{a+b+1} \text{sgn}(\sigma) = \text{sgn}(\sigma)$. Therefore this is a pfaffian orientation. □

One can check that the graph from Figure 3.2 is oriented in the manner described in this proof. In fact the inductive argument can be converted into an algorithm for constructing a pfaffian orientation. The method of showing that every transition cycle has an odd number of counterclockwise edges is so common that it is often used as a definition of pfaffian orientation. In [18] they define a pfaffian orientation to be one where every central even cycle is oddly oriented. A subgraph H of a graph G is said to be *central* if there is a perfect matching of the graph $G \backslash H$.

It is worth showing that not all graphs have pfaffian orientations. An example of a graph that does not is $K_{3,3}$.

Proposition 3.5. *There is no pfaffian orientation of $K_{3,3}$.*

Proof. If we label one partition $1, 2, 3$ and the other $4, 5, 6$ we can see the pfaffian of the undirected graph is:

$$pf(K_{3,3}) = -y_{14}y_{25}y_{36} + y_{14}y_{26}y_{35} + y_{15}y_{24}y_{36} - y_{15}y_{26}y_{34} - y_{16}y_{24}y_{35} + y_{16}y_{25}y_{34}.$$

A pfaffian orientation on this is a choice of sign for each indeterminate. Since each edge, and thus each indeterminate, is in exactly two matchings, each indeterminate chosen as negative reverses the sign in exactly two terms. This means no matter how many we choose, it results in an even number of sign changes (note sign changes can cancel each other, but only in even pairs, and thus the number of terms with different signs is always even). However to make all these terms positive, we need exactly three sign changes, which is not even. □

It is stated in [18] that:

Theorem 3.6. *A bipartite graph has a pfaffian orientation if and only if it does not "contain" $K_{3,3}$.*

Define G to *contain* H if G has a central subgraph H' such that H' is an even subdivision. By an *even subdivision* of H we mean we can replace the edges of H with vertex disjoint odd length paths (thus adding an even number of vertices) to get H'.

This reminds us of Kuratowski's theorem:

Theorem 3.7 (Kuratowski). *A graph is non-planar if and only if it has a graph that is an edge subdivision of K_5 or $K_{3,3}$*

This shows a link between planarity and pfaffian orientability, since if a graph contains $K_{3,3}$ it has a subgraph which is an edge subdivision of $K_{3,3}$. "Contains" has the two additional clauses, that the subgraph is central, and that the subdivision is even, so the implication is one way. The graph K_5 does not cause the same problem, since it has an odd number of vertices. Thus it has no perfect matchings, and its pfaffian is trivial. The exact relation is unclear, but many of our results on pfaffian orientations are related to embeddings of the graphs in surfaces. For example in [10] they proved that a graph which can be embedded in a genus g surface can be oriented so that the hafnian is a linear combination of 4^g pfaffians.

3.2 Hypergraph Embeddings and Orientations

Our goal is to extend the theorems about pfaffian oriented graphs to hyperpfaffian oriented hypergraphs. To do so we should first discuss hypergraph embeddings, as most of our results on pfaffian orientations are based on embeddings of the graph. Following that we shall examine how to represent hypergraph orientations in these embeddings.

As we mentioned in Section 1.5, we use two methods of drawing hypergraphs, as shown in Figure 3.4. We should now discuss more formally how these work. Naturally

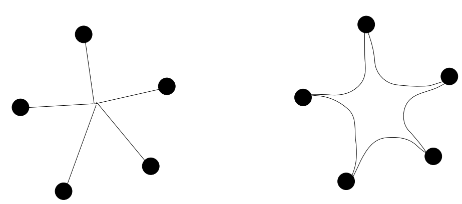

Figure 3.4: Representations of a hyperedge of degree 5

we embed vertices in a surface in the same way as for a standard graph in either method. For the style on the left, we embed an edge of k vertices in a surface by adding an imaginary vertex for the edge. Then we draw a standard edge connecting each of the k vertices of the hyperedge to this imaginary vertex. As usual we do not allow lines to cross. Call this the *point embedding* of the hypergraph.

The form on the right is slightly more complicated. Here we we start by embedding a standard k cycle of the vertices of the hyperedge. The cycle must enclose a standard face. This means we may not put another part of an edge through this region and the region enclosed must be homotopic to a disc. It may make more sense to fill in this face, as nothing may cross it, as in Figure 3.5. Call this the *cycle embedding* of the hypergraph.

We define planarity or embeddability into any surface in the same way as we have

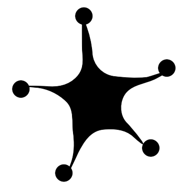

Figure 3.5: Filled cycle embedding of a hyperedge

done with standard graphs using these techniques. It is easy to check that these methods are equivalent, as the first comes from shrinking the face of the second down to a point.

Moving on we shall now discuss how to represent orientations of these graphs. As mentioned in Chapter 2 in Definition 2.3, we consider the orientation of an edge of a hypergraph to be an order of its vertices up to even permutation. In that chapter we dealt almost exclusively with odd degree hypergraphs, and represented the orientation with cyclic permutations. In this chapter we shall instead be dealing almost exclusively with even degree edges. Here we cannot use the cyclic permutation representation, as even cycles are odd permutations. This means that the ordering 1234 is not an even permutation of 2341. These give identical cyclic permutations.

Instead we use the cycle embedding to give us a representation. If we look at this embedding, we can direct the edges of the cycle representing the hyperedge. Let us start by only directing every other edge in the cycle, as shown in Figure 3.6. Thus every vertex is adjacent to one directed edge of the cycle. This specifies an orientation

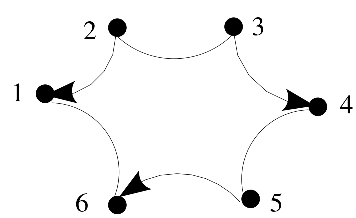

Figure 3.6: Half directed cycle embedding of a hyperedge

for the hyperedge, by ordering each pair of its vertices. Each directed edge gives an order for its vertices. By placing the pairs in any order, but requiring the defined order within the pair, we define an orientation on the hyperedge. Choosing a different order of pairs only alters this by an even permutation, so this gives us a well defined orientation on the hyperedge. Thus Figure 3.6 represents the orders 215634 or 562134, or several others. Note that not all representatives of this orientation are given by this drawing, but all orders given by this drawing are representatives of the same orientation. For example 251364 is a representative of the same order, but it does not correspond to the diagram.

We may want to direct every edge in the cycle, in which case we need the restriction that the cycle has an odd number of counterclockwise edges (and thus an odd number of clockwise as well). Figure 3.7 shows an example of doing this correctly, and of doing this incorrectly, from the graph in Figure 3.6. Using this representation we can then

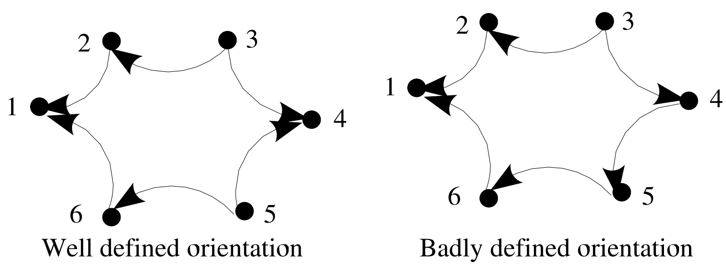

Figure 3.7: Fully directed cycle embedding of a hyperedge, correct and incorrect

take our choice of matching for the cycle to generate our order. Our restrictions on the parity of counterclockwise edges ensures that we get a consistent orientation from both choices. Note in Figure 3.7 on the left we see that if we take the other matching we generate the order 325461, which one can check is an even permutation of the order 215634 we had previously acquired. On the other hand if we look at the one on the right of Figure 3.7, choosing the other matching gives us the order 324561, which is not an even permutation of the first. Thus this does not consistently define an orientation.

To see that the restriction on the number of odd cycles ensures consistency, start by defining σ to be the sequence listing the vertices in clockwise order starting at a given vertex v. Let M be the matching of this cycle containing vu where u is the next vertex clockwise around the cycle from v. Let M' be the other matching, which starts with the edge clockwise from u. Thus both M and M' define an orientation, and we want to show these are the same orientations. Note the signum of the sequence σ differs from that of the orientation given by M by a factor of $(-1)^c$ where c is the number of counterclockwise edges in M. Then if we apply the even cycle moving the vertices one step clockwise to σ, we obtain the clockwise sequence starting from u. Similarly it differs from the sequence given by M' by a factor of $(-1)^{c'}$ where c' is the number of counterclockwise edges in M'. Applying the even cycle reverses the signum. This means the signum of the sequence for M differs from that of M' by $(-1)^d$ where d is the number of counterclockwise edges in the cycle plus one. So the two orders differ by an even permutation if and only if there is an odd number of counterclockwise edges.

Using this representation, there is only one orientation from a given representation, but there are certainly many representatives of any given orientation. Note that any two edges in the same matching of a cycle representing a hyperedge can be reversed without changing the orientation.

Based on our need for only half the edges of the cycle embedding to represent the orientation of a graph, we now define the pair-graph of a hypergraph.

Definition 3.8. A *pair-graph* $P = (V, E_p)$ of a hypergraph $G = (V, E)$ whose edges have all even degree is defined to be a multigraph on the same set of vertices, where for each edge e of degree $2k$ in E there are k edges in E_p which cover all the vertices of e.

A hypergraph does not have a unique pair-graph, as for every even hyperedge of degree greater than two there are multiple choices of how to pair the vertices of this edge. A pair-graph with a fixed grouping of edges by hyperedge defines a unique hypergraph. Figure 3.8 shows a hypergraph and one of its pair-graphs.

We can then represent an orientation of our hypergraph in its pair-graph by directing the pair-graph. As with the cycle embedding, putting an order on pairs of vertices of a hyperedge defines an ordering up to even permutation of those vertices. Any choice of direction for the pair-graph corresponds to some orientation of the hypergraph, and any orientation of the hypergraph gives such an orientation of the pair-graph (though not a unique one).

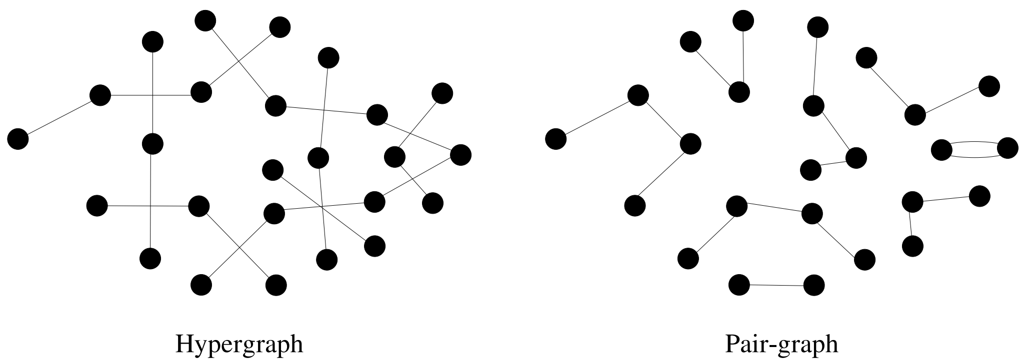

Figure 3.8: A hypergraph and a corresponding pair-graph

Figure 3.9 shows a hyperedge and two directed pair-graphs of it inducing the same orientation on it. That orientation is 16325478.

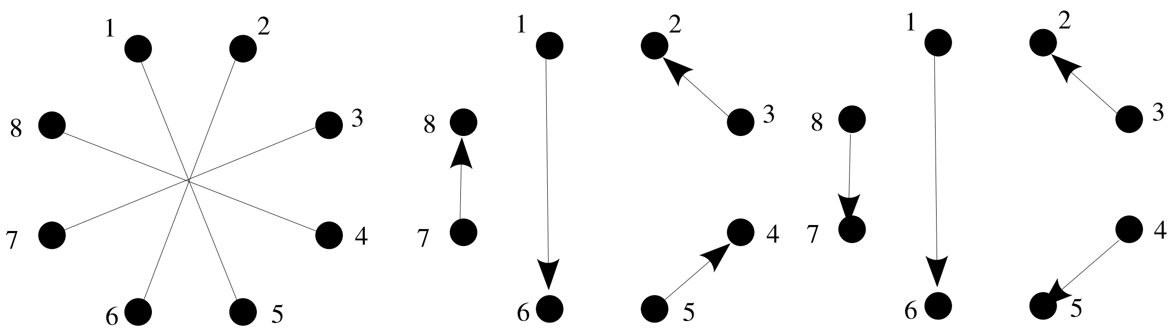

Figure 3.9: A hyperedge and two oriented pair-graphs of it

We call a hypergraph *pair-embeddable* in a surface if it has a pair-graph which is embeddable in this surface (and *pair-planar* if it can be embedded into a plane). It is easy to check that this is a strictly weaker condition than embeddability of the hypergraph.

3.3 Hyperpfaffian Orientations

Now that we have described how to relate orientations and embeddings of hypergraphs, let us see how they affect hyperpfaffians. In Section 3.1 we saw how to define the pfaffian for a directed graph. We shall now do the same for an oriented hypergraph. For a non-oriented hypergraph $G = (V, E)$ whose edges all have even degree, we have defined its antisymmetric tensor algebra Λ to be:

$$\Lambda_G(i_1, \ldots, i_k) = \sum_{\substack{e \in E \\ V(e)=\{i_1,\ldots,i_k\}}} \operatorname{sgn}(i_1 \ldots i_k) y_e.$$

For an oriented hypergraph $D = (V, E)$ whose edges all have even degree, we can define its alternating tensor algebra as follows:

$$\Lambda_D(i_1, \ldots, i_k) = \sum_{\substack{e \in E \\ V(e)=\{i_1,\ldots,i_k\}}} \operatorname{sgn}_{\mathfrak{o}(e)}(i_1 \ldots i_k) y_e,$$

where $\operatorname{sgn}_{\mathfrak{o}(e)}$ of a sequence is positive if the sequence is a representative of the orientation of e, and negative otherwise. This is exactly the same as replacing y_e with $-y_e$ when the orientation of the edge is against the order of the vertices. We define the hyperpfaffian of an oriented hypergraph to be the hyperpfaffian of this alternating tensor algebra.

Definition 3.9. We say an orientation of a hypergraph G is a *hyperpfaffian orientation* if the non-uniform hyperpfaffian of the oriented hypergraph G has all positive terms.

It was established in [3] that the k dimensional hyperpfaffian on a tensor of order n can be computed in $O(2^n n^{k+1})$ arithmetic operations (for fixed k). This is exponential, so unfortunately it may not represent a great improvement in efficiency. The computation of the non-uniform hyperpfaffian and the hyperhafnian have not been studied. However the problem is still interesting for its own sake.

Once again we can define what it means for a permutation representing a matching to respect orientation.

Definition 3.10. A permutation σ representing a matching π on an oriented hypergraph whose edges all have even degree is said to *respect orientation* if each block of σ corresponding to an edge in π has its vertices ordered in a representative of the orientation of that edge.

For example if we have the matching $\{1, 2, 4, 6\}, \{3, 8\}, \{5, 7\}$ with orientations 1462, 83, 57, then the permutation 83416257 is a permutation which represents this and respects orientation. On the other hand 12463857 is a permutation which represents the matching, but it does not respect orientation.

The sign of the term given by this matching agrees with the signum of any permutation representing it which respects orientation. To see this, examine the terms of the expansion:

$$pf_*(\Lambda) = \sum_{\sigma \in \mathfrak{S}_n} \sum_{q \in \mathfrak{N}_n} \mathrm{sgn}(\sigma) \frac{1}{|q|!} \prod_{j=1}^{|q|} \frac{1}{(q_i - q_{i-1})!} \Lambda(i_{\sigma(q_{i-1}+1)}, \ldots, i_{\sigma(q_i)}).$$

If σ respects orientation then each $\Lambda(i_{\sigma(q_{i-1}+1)}, \ldots, i_{\sigma(q_i)}) = y_e$, since

$$\mathrm{sgn}_{\mathfrak{o}(e)}(i_{\sigma(q_{i-1}+1)}, \ldots, i_{\sigma(q_i)}) = 1.$$

This means the sign of the term is determined only by the signum of σ. So to show that an orientation is a hyperpfaffian orientation we need only show that every perfect matching has a permutation representing it which respects orientation with positive signum.

The following theorem will allow us to adapt most of our pfaffian orientation results to work for hyperpfaffian orientations.

Theorem 3.11. *Take a hypergraph G whose edges all have even degree. If its pair-graph P has a pfaffian orientation then G has a hyperpfaffian orientation, and a pfaffian orientation of P defines a hyperpfaffian orientation of G (though not necessarily the other way around).*

Proof. Let $e_1, \ldots, e_n$ be the edges of G. Then let $e_{i,1}, \ldots, e_{i,k_i}$ be the edges of P corresponding to e_i. Take a pfaffian orientation of the pair-graph P. Orient G as defined by this orientation on P. If we take any perfect matching $\pi = \{e_{i_1}, \ldots, e_{i_m}\}$ of G, this translates to a perfect matching $\pi' = \{e_{i_1,1}, \ldots, e_{i_1,k_{i_1}}, e_{i_2,1}, \ldots, e_{i_m,k_{i_m}}\}$.

Let σ be a permutation representing π' which respects orientation and has the edges ordered so that the edges are grouped by the hyperedges they represent. Since we are using a pfaffian orientation of P, the signum of σ is one.

Since the edges of P are grouped according to their edges of G, this is also a permutation representing π. Since each pair of vertices in π is ordered according to the orientation of P, they are in the order defined by P on G. Thus σ respects the orientation of G.

Since $\text{sgn}(\sigma) = 1$, each perfect matching of G has a permutation representing it which respects orientation and has positive signum. Therefore this is a hyperpfaffian orientation. □

Corollary 3.12. *Any pair-planar, and thus any planar hypergraph whose edges all have even degree has a pfaffian orientation.*

Using Theorem 3.11 we can convert almost any result about pfaffian orientations using embeddings to results about hyperpfaffian orientations (on hypergraphs of even degree edges) with pair-embeddings.

Chapter 4

Hyperpfaffian Rings and Ideals

4.1 Pfaffian and Determinantal Rings and Ideals

Determinantal and pfaffian ideals and rings are objects whose study has been important in commutative algebra, algebraic geometry, and combinatorics. For this reason we are going to look at extending these concepts to the hyperpfaffian.

Let us start by considering the k-algebra $k[x_{ij}]_{1\leq i<j\leq n}$. Let A be the $n\times n$ antisymmetric matrix defined by $A_{ij} = x_{ij}, i < j$ and $A_{ij} = -x_{ji}, j < i$, $A_{ii} = 0$. We denote a *minor* of A by $[q_1, q_2, \ldots, q_t]$, $1 \leq q_1, q_2, \ldots, q_t \leq n$ defined by $[q_1, \ldots, q_t]_{ij} = A_{q_i q_j}$, where $1 \leq i, j \leq t$. We say a minor is represented in *standard form* if $q_1 < q_2 < \cdots < q_t$.

If $q_i = q_j$ for some $i \neq j$ then the pfaffian is zero, because it implies the ith and jth rows of $[q_1, \ldots, q_t]$ are the same. This implies that the determinant of the matrix is zero, and thus so is the pfaffian (its square root). Also of note is that for any $\sigma \in \mathfrak{S}_t$,

$$pf([q_1, \ldots, q_t]) = \operatorname{sgn}(\sigma) pf([q_{\sigma(1)}, \ldots, q_{\sigma(t)}]).$$

This can be verified fairly easily from Definition 1.6 for the pfaffian. This means that any non-zero pfaffian minor is the same as a minor in standard form, up to sign.

Definition 4.1. A *pfaffian ideal* of even order t, denoted by P_t, is the ideal of $k[x_{ij}]_{1\leq i<j\leq n}$ generated by the set:

$$\{pf([q_1, \ldots, q_t]) : q_1, \ldots, q_t \in \{1, \ldots, n\}\}.$$

One could add the restriction that the minors be in standard form without changing the ideal. We define the pfaffian ring in the following way:

Definition 4.2. The *pfaffian ring* of order t is the quotient ring $R_t = k[x_{ij}]_{1\leq i<j\leq n}/P_t$.

For example, if we look at the pfaffian ideal of order two, we see that our set is $\{x_{ij}, i < j\}$, which generates the ideal of the elements of $k[x_{ij}]_{1\leq i<j\leq n}$ with no constant term. Thus R_2 is k. We define the empty minor $[\emptyset]$ to have a pfaffian of 1. This means $P_0 = k[x_{ij}]_{1\leq i<j\leq n}$ and $R_0 = \{0\}$.

We define the determinantal ideal in a similar manner to the pfaffian ideal. For determinants we work over the k-algebra $k[x_{ij}]_{1\leq i,j\leq n}$ and we use the $n \times n$ matrix M defined by $M_{ij} = x_{ij}$. We then denote a *minor* by $[a_1, \ldots, a_t|b_1, \ldots, b_t]$. This is defined by $[a_1, \ldots, a_t|b_1, \ldots, b_t]_{ij} = M_{a_i b_j}$. Once again we define a minor to be in standard form if $a_1 < a_2 < \cdots < a_t$ and $b_1 < b_2 \cdots < b_t$. As before we see that if any $a_i = a_j$ or $b_i = b_j$ for $i \neq j$, then the minor has determinant zero. Also it is well known with determinants that for any $\sigma_a, \sigma_b \in \mathfrak{S}_t$:

$$\det[a_1, \ldots, a_t|b_1, \ldots, b_t] = \operatorname{sgn}(\sigma_a)\operatorname{sgn}(\sigma_b)\det[a_{\sigma_a(1)}, \ldots, a_{\sigma_a(t)}|b_{\sigma_b(1)}, \ldots, b_{\sigma_b(t)}].$$

So again any non-zero determinant minor is the same as one in standard form, up to sign.

Definition 4.3. A *determinantal ideal* of order t, denoted by D_t, is the ideal in $k[x_{ij}]_{1\leq i,j\leq n}$ generated by the set:

$$\{\det[a_1, \ldots, a_t|b_1, \ldots, b_t], a_1, \ldots, a_t, b_1, \ldots, b_t \in \{1, \ldots, n\}\}.$$

As before we can restrict our set to those in standard form. We define the determinantal ring in a similar fashion to the pfaffian ring.

Definition 4.4. The *determinantal ring* of order t is the quotient ring

$$S_t = k[x_{ij}]_{1\leq i<j\leq n}/D_t.$$

For example, if we take $t = 1$, D_1 is the set of all members of $k[x_{ij}]_{1\leq i,j\leq n}$ without a constant term. Thus $S_1 = k$. This can be shown in the same way as for P_2 and R_2.

Many interesting results relating these pfaffian and determinantal rings to combinatorics have been studied. For example in [12] and [9] they relate the dimension of the homogeneous degree n elements of the determinantal and pfaffian rings (known as the Hilbert function of these rings) to sets of non-intersecting lattice paths. Most of these results are based around the structure put on these rings, referred to as an algebra with straightening law or an ordinal hodge algebra. So in the next section we shall review these concepts.

4.2 Algebras with Straightening Law

A very important property of determinantal and pfaffian rings is that they are what is referred to in [6] as an *algebra with straightening law*, or *ASL*. This is referred to as an *ordinal hodge algebra* in [7].

Definition 4.5. A *graded ring* R is a ring which can be decomposed into the direct sum of abelian groups (additive) as follows:

$$R = R_0 \oplus R_1 \oplus \dots,$$

with the condition that $r_i \in R_i$ and $r_j \in R_j$ implies $r_i r_j \in R_{ij}$.

One of the most common examples of a graded ring is the ring of polynomials, where R_i is the set of homogeneous polynomials of degree i. We define the *homogeneous elements* of grade i to be the elements of R_i.

Definition 4.6. For a k-algebra A, and a partially ordered finite subset Π of A, we say A is an *algebra with straightening law* if the following conditions hold:

1. $A = A_0 \oplus A_1 \oplus \dots$ is graded with $A_0 = k$. A is generated by Π as a k-algebra. Every element of Π is homogeneous of positive grade.

2. We define the *standard monomials*, M, to be the product of chains in Π (with respect to its partial order), where the empty chain is $1 \in k$. The standard monomials must be linearly independent.

3. For any $a, b \in \Pi$, then ab can be expressed as a linear combination (over k) of standard monomials, where each monomial has a factor which is less than or equal to both a and b (note this is trivial if a and b are comparable).

It is easy to check using condition 1 and 3 that the set of standard monomials spans A as a module, and by the second condition the standard monomials are linearly independent. Thus the standard monomials form a basis of A as a module. ASLs are often interesting because they let us examine the structure of an algebra in terms of a finite poset.

A very simple example of an ASL is the ring $A = k[x_1, \ldots, x_n]$, graded by degree, and $\Pi = \{x_1, \ldots, x_n\}$ equipped with a total order. Then condition 1 is trivial, as the algebra is defined to be generated by Π. Condition 2 is also easy to check, since the standard monomials are the monomials of this algebra. Condition 3 is again trivial, as there are no incomparable elements, so the product of any two elements of Π is a standard monomial.

It is shown in [6] that the algebra $k[x_{ij}]_{1 \le i,j \le n}$ with the set Π of determinant minors in standard form is an ASL. Here the partial order is defined by the relation $[a_1, \ldots, a_t | b_1, \ldots, b_t] \le [c_1, \ldots, c_u | d_1, \ldots, d_u]$ if and only if $t \ge u$ and $a_i \le c_i$ and $b_i \le d_i$ for all $1 \le i \le u$. Note it is important that the minors are in standard form for this comparison.

Using this partial order we can find a correspondence between the standard monomials and Young bitableaux.

Definition 4.7. A *standard Young tableau* is an assignment of positive integers to:

$$\begin{array}{llll} a_{1,1} & a_{1,2} & \cdots & \quad a_{1,k_1} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,k_2} \\ \vdots & & & \\ a_{m,1} & a_{m,2} \cdots & a_{m,k_m} & \end{array} .$$

with the property that $k_1 \ge k_2 \ge \cdots \ge k_m$ and $a_{i,1} < a_{i,2} < \cdots < a_{i,k_i}$ for all $1 \le i \le m$ and $a_{1,i} \le a_{2,i} \le \ldots$ for all $1 \le i \le k_1$.

A standard Young bitableau is an ordered pair of standard Young tableaux of the same shape. The *shape* of a Young tableau is the sequence $k_1, \ldots, k_m$. A bitableau

$(\{a_{ij}\},\{b_{ij}\})$ corresponds to the standard monomial:

$$\prod_{i=1}^{m}[a_{i,1},\ldots,a_{i,k_i}|b_{i,1},\ldots,b_{i,k_i}].$$

One can check that the restrictions $k_1 \geq k_2 \geq \cdots \geq k_m$ and $a_{1,i} \leq a_{2,i} \leq \ldots$ for all $1 \leq i \leq k_1$ corresponds exactly to the restriction of comparability under our partial order. Young bitableaux have seen a great deal of study in combinatorics, so this relation leads to many interesting results.

Using the straightening law relation, we can see more of the structure of the ring S_t. The ideal D_t is then the span of all the standard monomials which contain minors $[a_1,\ldots,a_m|b_1,\ldots,b_m]$ where $m \geq t$. Thus S_t is spanned by the standard monomials which are the product of minors of the form $[a_1,\ldots,a_{m'}|b_1,\ldots,b_{m'}]$ with $m' < t$. Considered as bitableaux these can be counted using the Gessel-Viennot method, which relates this basis to sets of non-intersecting paths.

In the case of the pfaffian minors in standard form, it is claimed in various papers that these also form an ASL, under the very similar partial order $[a_1,\ldots,a_t] \leq [b_1,\ldots,b_u]$ if and only if $t \geq u$ and $a_i \leq b_i$ for all $1 \leq i \leq u$. However I have been unable to find a proof of this. In [8] they do prove that the standard monomials defined by this order form a basis for the algebra. This property is enough to get many useful results, even without the full straightening properties. To prove that the standard monomials form a basis we begin with a few lemmas on pfaffian minors.

Lemma 4.8.

$$[a_1,\ldots,a_t][b_1,\ldots,b_u] = \sum_{h=1}^{t}[a_1,\ldots,a_{h-1},b_1,a_{h+1},\ldots,a_t][a_h,b_2,\ldots,b_u] + \sum_{k=2}^{t}(-1)^k[b_k,b_1,a_1,\ldots,a_t][b_2,\ldots,b_{k-1},b_{k+1},\ldots,b_u]$$

This is proved in [8] as Lemma 6.1 through elementary properties of the pfaffian.

Lemma 4.9. *For integers $a_1,\ldots,a_i$, $x_1,\ldots,x_{u+1}$, and $b_{i+2},\ldots,b_t$, then*

$$\sum_{\sigma\in\mathfrak{S}_{u+1}} sgn(\sigma)[a_1,\ldots,a_i,x_{\sigma(1)},\ldots,x_{\sigma(u-i)}][x_{\sigma(u-i+1)},\ldots,x_{\sigma(u+1)},b_{i+2},\ldots,b_t]$$

where $u > i$, can be expressed as a linear combinations of terms $\mu\psi$ where μ and ψ are minors and μ has more than u indices.

Proof. This proof is taken from [8].

We shall prove this by induction on i. The base case of $i = 0$ is Lemma 4.8.

Let us assume this holds for $i-1$, and then use Lemma 4.8 to see:

$$\begin{aligned}
&\sum_{\sigma\in\mathfrak{S}_{u+1}} \operatorname{sgn}(\sigma)[a_1,\dots,a_i,x_{\sigma(1)},\dots,x_{\sigma(u-i)}][x_{\sigma(u-i+1)},\dots,x_{\sigma(u+1)},b_{i+2},\dots,b_t] \\
&\quad = \sum_{\sigma\in\mathfrak{S}_{u+1}} \operatorname{sgn}(\sigma)\Bigg(\sum_{h=1}^{i}[a_1,\dots,a_{h-1},x_{\sigma(u-i+1)},a_{h+1},\dots,a_i,x_{\sigma(1)},\dots,x_{\sigma(u-i)}] \\
&\qquad\qquad \cdot [a_h,x_{\sigma(u-i+2)},\dots,x_{\sigma(u+1)},b_{i+2},\dots,b_t] \\
&\qquad + \sum_{h=i+1}^{u}[a_1,\dots,a_i,x_{\sigma(1)},\dots,x_{\sigma(h-i-1)},x_{\sigma(u-i+1)},x_{\sigma(h-i+1)},\dots,x_{\sigma(u-i)}] \\
&\qquad\qquad \cdot [x_{\sigma(h-i)},x_{\sigma(u-i+2)},\dots,x_{\sigma(u+1)},b_{i+2},\dots,b_t] \\
&\qquad + \sum_{k=2}^{i+1}(-1)^{k-1}[x_{\sigma(u-i+k)},x_{\sigma(u-i+1)},a_1,\dots,a_i,x_{\sigma(1)},\dots,x_{\sigma(u-i)}] \\
&\qquad\qquad \cdot [x_{\sigma(u-i+1)},\dots,x_{\sigma(u-i+k-1)},x_{\sigma(u-i+k+1)},\dots,x_{\sigma(u+1)}] \\
&\qquad + \sum_{k=i+2}^{t}(-1)^{k-1}[b_k,x_{\sigma(u-i+1)},a_1,\dots,a_i,x_{\sigma(1)},\dots,x_{\sigma(u-i)},b_{i+2},\dots,b_t] \\
&\qquad\qquad \cdot [x_{\sigma(u-i+1)},\dots,x_{\sigma(u+1)},b_{i+2},\dots,b_{k-1},b_{k+1},\dots,b_t]\Bigg).
\end{aligned}$$

To clean this up a little, let R be the terms above which contain a minor of more than u elements and let S be the left hand side. This leaves us with:

$$\begin{aligned}
S = &\sum_{\sigma\in\mathfrak{S}_{u+1}} \operatorname{sgn}(\sigma)\Bigg(\sum_{h=1}^{i}[a_1,\dots,a_{h-1},x_{\sigma(u-i+1)},a_{h+1},\dots,a_i,x_{\sigma(1)},\dots,x_{\sigma(u-i)}] \\
&\qquad\qquad \cdot [a_h,x_{\sigma(u-i+2)},\dots,x_{\sigma(u+1)},b_{i+2},\dots,b_t] \\
&\qquad + \sum_{h=i+1}^{u}[a_1,\dots,a_i,x_{\sigma(1)},\dots,x_{\sigma(h-i-1)},x_{\sigma(u-i+1)},x_{\sigma(h-i+1)},\dots,x_{\sigma(u-i)}] \\
&\qquad\qquad \cdot [x_{\sigma(h-i)},x_{\sigma(u-i+2)},\dots,x_{\sigma(u+1)},b_{i+2},\dots,b_t]\Bigg) + R.
\end{aligned}$$

Let us next examine the group of terms:

$$S = \sum_{\sigma \in \mathfrak{S}_{u+1}} \operatorname{sgn}(\sigma) \sum_{h=1}^{i} [a_1, \ldots, a_{h-1}, x_{\sigma(u-i+1)}, a_{h+1}, \ldots, a_i, x_{\sigma(1)}, \ldots, x_{\sigma(u-i)}]$$

$$\cdot [a_h, x_{\sigma(u-i+2)}, \ldots, x_{\sigma(u+1)}, b_{i+2}, \ldots, b_t].$$

$$= \sum_{h=1}^{i} (-1)^{u-h}(-1)^{t-1} \sum_{\sigma \in \mathfrak{S}_{u+1}} \operatorname{sgn}(\sigma)[a_1, \ldots, a_{h-1}, a_{h+1}, \ldots, a_i, x_{\sigma(1)}, \ldots, x_{\sigma(u-i+1)}]$$

$$\cdot [x_{\sigma(u-i+2)}, \ldots, x_{\sigma(u+1)}, b_{i+2}, \ldots, b_t, a_h],$$

since we can reverse the order of the summations and reorder the elements in the minor.

We can see this is of the same form as our original summation, but with i replaced by $i-1$, and still having the same value for u. Thus by the induction hypothesis we can express this as a linear combination of terms each containing a minor of size greater than u. Call this R'. Thus we are left with:

$$S = \sum_{\sigma \in \mathfrak{S}_{u+1}} \operatorname{sgn}(\sigma) \sum_{h=i+1}^{u} [a_1, \ldots, a_i, x_{\sigma(1)}, \ldots, x_{\sigma(h-i-1)}, x_{\sigma(u-i+1)}, x_{\sigma(h-i+1)}, \ldots, x_{\sigma(u-i)}]$$

$$\cdot [x_{\sigma(h-i)}, x_{\sigma(u-i+2)}, \ldots, x_{\sigma(u+1)}, b_{i+2}, \ldots, b_t] + R + R'.$$

If we reverse the order of the sums, and let γ_h be the permutation which is the transposition of $h-i$ and $u-i+1$, we get:

$$S = \sum_{h=i+1}^{u} \sum_{\sigma\gamma_h \in \mathfrak{S}_{u+1}} \operatorname{sgn}(\sigma\gamma_h)[a_1, \ldots, a_i, x_{\sigma(1)}, \ldots, x_{\sigma(h-i-1)}, x_{\sigma(h-i)}, x_{\sigma(h-i+1)}, \ldots, x_{\sigma(u-i)}]$$

$$\cdot [x_{\sigma(u-i+1)}, x_{\sigma(u-i+2)}, \ldots, x_{\sigma(u+1)}, b_{i+2}, \ldots, b_t] + R + R'.$$

$$= -(u-i) \sum_{\sigma \in \mathfrak{S}_{u+1}} \operatorname{sgn}(\sigma)[a_1, \ldots, a_i, x_{\sigma(1)}, \ldots, x_{\sigma(u-i)}]$$

$$\cdot [x_{\sigma(u-i+1)}, \ldots, x_{\sigma(u+1)}, b_{i+2}, \ldots, b_t] + R + R'.$$

Since γ_h is a single transposition, its signum is -1.

Then we can see this summation is a multiple of our original expression, so moving it to the other side we see that our final expression is:

$$\sum_{\sigma \in \mathfrak{S}_{u+1}} \operatorname{sgn}(\sigma)[a_1, \ldots, a_i, x_{\sigma(1)}, \ldots, x_{\sigma(u-i)}] \cdot [x_{\sigma(u-i+1)}, \ldots, x_{\sigma(u+1)}, b_{i+2}, \ldots, b_t] = \frac{1}{u-i+1}(R + R').$$

This proves the lemma by induction (since $u > i$ implies that $u - i + 1 \neq 0$). ☐

Lemma 4.10. *The standard monomials of pfaffian minors are linearly independent.*

Proof. This proof is adapted from that in [6] for the determinantal case. Using the terminology of that proof we refer to a pair of positive integers (i, j) as a *special pair* for a minor in standard form $[a_1, \ldots, a_m]$ if $i < j$ and there exists $a_k = i$ but no $a_k = j$. The pair is called *extra special* if it is the lexicographically least special pair for that minor. We then define a pair to be special for a standard monomial μ if it is special for some minor which is a factor of μ. We define a pair to be extra special for μ if it is the lexicographically least special pair for μ. We further define a pair to be extra special for a set of standard monomials if it is the lexicographically least extra special pair of the standard monomials of the set.

We next prove the linear independence of the standard monomials by downward induction on the extra special pairs for sets of standard monomials. Here we do not allow the empty minor. Since every other standard monomial is homogeneous of degree greater than zero it is linearly independent from the rest.

The highest possible extra special pair for a standard monomial is $(n-1, n+1)$, since we must have at least two indices in some minor between 1 and n, and $n+1$ is not a valid index for a minor. This is then the index for our base case. The only minor for which this is extra special is $[n-1, n]$ and thus a set of standard monomials for which $(n-1, n+1)$ is extra special must be a set of powers of $[n-1, n]$. Since this is clearly linearly independent, our base case holds.

For our induction hypothesis, assume that any set of standard monomials with extra special pair greater than (i', j') is linearly independent.

Let us define the function $\Phi_{i,j}$ to act on a minor by replacing i with j if (i, j) is special for the minor and as the identity otherwise. Then have it act on a standard monomial by acting on each of the minors composing it. For a standard monomial μ, define the function $v_{i,j}(\mu)$ to be the number of factors of μ for which (i,j) is a special pair.

We claim that if we have minors $\alpha \leq \beta$ for which (i,j), with $i < j$, is less than or equal to their extra special pairs, then $\Phi_{i,j}(\alpha) \leq \Phi_{i,j}(\beta)$. To prove this claim note that if (i,j) is not special for α then this is trivial. If it is special for α then it must be extra special, since it is less than or equal to its extra special pair. Let $\alpha = [\alpha_1, \ldots, \alpha_t]$ and $\beta = [\beta_1, \ldots, \beta_u]$ in standard form. Since $\alpha \leq \beta$ then $t \geq u$. As (i,j) is extra special for α we know that $\alpha_1 = i, \alpha_2 = i+1, \ldots, \alpha_{j-i} = j-1$ since there cannot be a lower index than i and there can't be a missing index between i and j. Since it is missing j then $\alpha_{j-i+1} > j$ (or does not exist).

Case 1: i is in β.

Since (i,j) is also less than or equal to the extra special pair of β, then if β contains i it must also have the elements $\beta_1 = i, \beta_2 = i+1, \ldots, \beta_{j-i} = j-1$. Since $\alpha \leq \beta$ then $\beta_{j-i+1} \geq \alpha_{j-i+1} > j$ (if $j-i+1 \leq t$, otherwise $u \leq t < j-i+1$ so there is a β_{j-i+1}). Thus (i,j) is special for β. Thus $\Phi_{i,j}(\alpha)_1 = \Phi_{i,j}(\beta)_1 = i+2$ and this continues up to $\Phi_{i,j}(\alpha)_{j-i} = \Phi_{i,j}(\beta)_{j-i} = j$. Beyond that $\Phi(\alpha)_k = \alpha_k \leq \beta_k = \Phi(\beta)_k$, so $\Phi_{i,j}(\alpha) \leq \Phi_{i,j}(\beta)$.

Case 2: i is not in β.

Then $\beta_1 \geq i+1$, since otherwise it would have a lower special pair than (i,j). Thus $\beta_q \geq i+q = \Phi(\alpha)_q$ for all $1 \leq q \leq i-j$ and $\beta_q \geq \alpha_q = \Phi(\beta)_q$ for all other q. Since (i,j) is not special for β, $\Phi(\beta) = \beta$. Thus $\Phi_{i,j}(\alpha) \leq \Phi_{i,j}(\beta)$.

Using this result we see that if we choose (i,j), $i < j$ less than or equal to the extra special pair for a standard monomial μ then $\Phi_{i,j}(\mu)$ is also standard.

Next we claim that if we have standard monomials $\mu \neq \psi$ and (i,j) which is less than or equal to the extra special pairs of μ and ψ such that $v_{i,j}(\mu) = v_{i,j}(\psi)$ then $\Phi_{i,j}(\mu) \neq \Phi_{i,j}(\psi)$. Let $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_r$ be the minors composing μ and $\psi_1 \leq \psi_2 \leq \cdots \leq \psi_s$ be those composing ψ.

As before, note that since (i,j) is less than or equal to the extra special pairs of μ and ψ, if (i,j) is special for any ψ_i or μ_i they are extra special for it. If (i,j)

is not special for any minor in μ or ψ then $\Phi_{i,j}$ is the identity on μ and ψ, thus $\Phi_{i.j}(\mu) \neq \Phi_{i,j}(\psi)$.

Let μ_t be the first (lowest t value) so that (i, j) is extra special for μ_t, and u for ψ_u be similarly defined. Let $q = v_{i,j}(\mu) = v_{i,j}(\psi)$. Now if $\mu_p \geq \mu_t$ and (i, j) is not special for μ_p, then the first $i - j$ elements of μ_p must be greater than or equal to those of μ_t, which are $i, i + 1, \dots, j - 1$. If μ_p starts with i it must have a j (since (i, j) is not special for it), but this is forbidden by $\mu_p \geq \mu_t$ as in the first case of our previous claim. So the first index of μ_p must be greater than i, which implies that nothing greater than μ_p may have (i, j) special. Thus the minors for which (i, j) is special must occur in blocks of the chain. So they are $\mu_t, \dots, \mu_{t+q-1}$ and $\psi_u, \dots, \psi_{u+q-1}$.

One can check that restricted to minors for which (i, j) is special $\Phi_{i,j}$ is a bijection, since it is restricted to the complement of this set. Thus if $t = u$ and $\mu \neq \psi$ then there is some $\mu_k \neq \psi_k$, which means $\Phi_{i,j}(\mu_k) \neq \Phi_{i,j}(\psi_k)$, and thus $\Phi_{i,j}(\mu) \neq \Phi(i, j)(\psi)$.

If $t \neq u$ then without loss of generality assume $t < u$. Now $\Phi_{i,j}(\mu_t)$ must start with an $i + 1$ since μ_t started with an i and (i, j) is special for it. However since $\psi_t < \psi_u$ it must start with i (if it started with something lower, it would have a lower extra special pair than (i, j)). Since (i, j) is not special for ψ_t (as $t < u$), $\Phi_{i,j}(\psi_t) = \psi_t \neq \Phi_{i,j}(\mu_t)$. Thus $\Phi_{i,j}(\mu) \neq \Phi_{i,j}(\psi)$. This proves our claim.

Now consider the algebra homomorphism $\Gamma_{i,j} : k[x_{ij}]_{1 \leq i < j \leq n} \to k[x_{ij}]_{1 \leq i < j \leq n}[w]$ defined by the following:

$$\Gamma_{i,j}(x_{st}) = \begin{cases} x_{st}, & s \neq i, t \neq i \\ x_{st} + w x_{jt}, & s = i \\ x_{st} + w x_{tj}, & t = i \end{cases} .$$

If the above gives x_{jt} where $j < t$ then we mean $-x_{tj}$ and if $t = j$ then we mean zero. Now if we consider this as acting on a minor $[a_1, \dots, a_m]$ then if $i \neq a_k$ for all k it acts as the identity. If $a_k = i$ then it maps this to $[a_1, \dots, a_{k-1}, j, a_{k+1}, \dots, a_k]$. Note this is zero if j is also in the minor. Thus if (i, j) is less than or equal to the extra special pair for a minor a then $\Gamma_{i,j}(a) = a \pm w\Phi_{i,j}(a)$ if (i, j) is special for a and $\Gamma_{i,j}(a) = a$ otherwise.

Consider the set S of standard monomials, where (i', j') is extra special for the set. For any $\mu \in S$ we see $\Gamma_{i',j'}(\mu) = w^{v_{i',j'}(\mu)}\Phi_{i',j'}(\mu) + R$ where R is of lower degree in q than $w^{v_{i',j'}}$.

Assume for contradiction that S is linearly dependent. Then we have non-zero c_q such that $\sum_{q=1}^{m} c_q\mu_q = 0$, for $\mu_q \in S$. Then $\Gamma_{i',j'}\left(\sum_{q=1}^{m} c_q\mu_q\right) = 0$. Then consider $v = \max\{v_{i',j'}(\mu_q)\}$. The term of degree v in w of $\Gamma_{i',j'}\left(\sum_{q=1}^{m} c_q\mu_q\right)$ is

$$\sum_k \pm c_{q_k}\Phi_{i',j'}(\mu_{q_k}) = 0,$$

where the q_k are those such that $v_{i',j'}(\mu_{q_k}) = v$. This implies the set of $\Phi_{i',j'}(\mu_{q_k})$ are linearly dependent. However, $\Phi_{i',j'}(\mu_{i,j})$ altered all the minors for which (i',j') was extra special to have a higher extra special pair, and the rest already did. Thus the set must have an extra special pair (i'',j'') such that $(i'',j'') \geq (i',j')$. This means we can apply the induction hypothesis to get a contradiction. □

Theorem 4.11. *The standard monomials of pfaffian minors of order n form a basis for* $k[x_{ij}]_{1\leq i<j\leq n}$.

Proof. To prove this we need only show that the standard monomials span this space, since Lemma 4.10 shows they are linear independent.

Since it is clear the minors generate the algebra, we need only show that any pair of incomparable minors can be expressed as a linear combination of standard monomials.

If we have two minors, $a = [a_1,\ldots,a_s]$ and $b = [b_1,\ldots,b_t]$ where $s \geq t$, then we refer to their *incomparability index* as the ordered pair $(s-t,i)$ where i is the lowest value such that $a_i > b_i$. If none such exists then use $n+1$ for i. We shall then use downward induction on incomparability index, ordered lexicographically, to show we can express these minors as a linear combination of standard monomials.

For our base case, note the highest possible incomparability index is $(n,n+1)$. In this case the two minors must be comparable, since they in fact are for any (n,i) or $(i,n+1)$, since in the former there is only one minor and in the latter the larger minor has all corresponding elements lower (and thus by definition is lower itself). For our induction hypothesis, assume that any product of minors ab with incomparability index greater than (u,v) is expressible as a linear combination of standard monomials.

Then take ab with incomparability index (u,v). By Lemma 4.9 we see that if we let $x_i = a_{i+v-1}$ for $1 \le i \le s-v+1$ and $x_i = b_{i-s+v-1}$ for $s-v+2 \le i \le s+1$, then we obtain:

$$\sum_{\sigma \in \mathfrak{S}_{s+1}} \operatorname{sgn}(\sigma)[a_1,\dots,a_i,x_{\sigma(1)},\dots,x_{\sigma(s-v+1)}][x_{\sigma(s-v+2)},\dots,x_{\sigma(s+1)},b_{v+1},\dots,b_t] = R,$$

where R is a sum of terms composed of one factor with size greater than a and the other of size less than b. The terms of R then have a higher incomparability index, since their first coordinate will be larger. Any permutation in the sum which can be decomposed into a permutation on $1,\dots,s-v+1$ and one on $s-v+2,\dots,s$ gives a term of ab. Thus we have:

$$\begin{aligned}&(s-v+1)!v!ab = -R\\ &\quad - \sum_{\sigma \in \mathfrak{S}'_{s+1}} \operatorname{sgn}(\sigma)[a_1,\dots,a_i,x_{\sigma(1)},\dots,x_{\sigma(s-v+1)}][x_{\sigma(s-v+2)},\dots,x_{\sigma(s+1)},b_{v+1},\dots,b_t],\end{aligned}$$

where $\mathfrak{S}'_{s+1}$ is the set of permutations on $s+1$ which map at least one element from $1,\dots,s-v+1$ to $s-v+2,\dots,s+1$. Note that $b_1 < \cdots < b_v < a_v < \cdots < a_s$. Take the term

$$[a_1,\dots,a_{v-1},x_{\sigma(1)},\dots,x_{\sigma(s-v+1)}][x_{\sigma(s-v+2)},\dots,x_{\sigma(s+1)},b_{v+1},\dots,b_t],$$

and put in standard form as

$$[a'_1,\dots,a'_u][b'_1,\dots,b_t].$$

Then $a'_i \le a_i$ for all $1 \le i \le v-1$, since $a_1,\dots,a_{v-1}$ are all in the left minor. Each a'_i must be chosen from $a_1,\dots,a_s$ and $b_1,\dots,b_v$. The only members of this set which are larger or equal to a_v are $a_v,\dots,a_s$. Since this permutation is in $\mathfrak{S}'_{v+1}$ we know at least one of the $a_v,\dots,a_s$ must not be in the left minor. Thus $a'_v < a_v$, and since $a_{v-1} \le b_{v-1} < b_v$ this means $a'_v \le b_v$. The indices $b'_1,\dots,b'_v$ are taken from $b_1,\dots,b_t$ and $a_v,\dots,a_s$. The lowest possible values for these are $b_1,\dots,b_v$, and so $b'_i \ge b_i$ for $1 \le i \le v$. Putting this together we see $a'_i \le a_i \le b_i \le b'_i$ for all $1 \le i < v$ and $a'_v \le b_v \le b'_v$. Since the size of our minors has not changed, our incomparability index is then (u,v'), where $v' > v$.

Applying our induction hypothesis we can replace the right side with a linear combination of standard monomials and our proof is complete. $\square$

This is very similar to the proof in [6] that the determinantal minors form an ASL. For determinantal minors there is a result similar to Lemma 4.9 which has zero instead of R. This allows us to keep the additional requirement that each term in the expansion of ab has a factor less than or equal to a, since this is true of every one of the other terms. Unfortunately this is not necessarily true with the terms of R so we cannot retain this information.

This does at least give us the result that the standard Young tableaux with rows of even length form a basis in this manner. This property was used in [12] and [2] to prove some interesting results.

4.3 Hyperpfaffian Rings

The definitions of pfaffian ring and pfaffian ideal can be extended fairly easily to the hyperpfaffian. The minor of an alternating tensor can be defined in the same way as an antisymmetric matrix. The *minor* $[q_1, \ldots, q_t]$ of Λ is the map which takes $(i_1, \ldots, i_k)$ to $\Lambda(q_{i_1}, \ldots, q_{i_k})$. We define standard form as before.

Definition 4.12. For k and t a multiple of k, the *uniform hyperpfaffian ideal* of dimension k and order t, denoted $P_{k,t}$ is the ideal in $A = k[x_{i_1,\ldots,x_k}]_{1 \le i_1 < i_2 < \cdots < i_k \le n}$ generated by the set

$$\{pf_k([q_1, \ldots, q_t]) : q_1 < \cdots < q_t \in \{1, \ldots, n\}\}.$$

Definition 4.13. For k and t a multiple of k, we define the *uniform hyperpfaffian ring* of dimension k and order t to be $R_{k,t} = A/P_{k,t}$.

Since most results on pfaffian rings and ideals are based around the ASL structure of the pfaffian minors in standard from, or at least the fact that they are a basis for the algebra, we would like to prove a similar result for the hyperpfaffian ring. Since minors can be described in the same way (though with the restriction that they have a multiple of k elements, rather than two, to be non-trivial), our first thought is to use the same partial order. Unfortunately it is easy to show this does not work. If we take the minors $[1, 4, 5, \ldots, k+2]$ and $[2, 3, 4, \ldots, k+2]$ we see these are incomparable, yet they are the only possible way of producing the term $x_{1,4,\ldots,k+2}x_{2,3,\ldots,k+2}$.

So our next thought is to try a different partial order. Unfortunately for sufficiently large n, no "reasonable" partial order exists that makes the hyperpfaffian minors in standard form of dimension $k > 2$ form basis of A. We now discuss what we mean by "reasonable".

Definition 4.14. We define a partial order $\preccurlyeq$ on the set of hyperpfaffian minors in standard form to be *label independent* if the relation $[i_1, \dots, i_l] \preccurlyeq [j_1, \dots, j_m]$ depends only on the order on the indices $i_1, \dots, i_l, j_1, \dots, j_m$, not on their actual values.

For example if we have a label independent partial order $\preccurlyeq$ then $[1, 2, 3, 4] \preccurlyeq [5, 6, 7, 8]$ if and only if $[5, 6, 7, 8] \preccurlyeq [9, 10, 11, 12]$, since the second pair is the same as the third pair shifted by four. It does not have to be a uniform shift, in fact $[1, 2, 3, 4] \preccurlyeq [5, 6, 7, 8]$ if and only if $[i_1, i_2, i_3, i_4] \preccurlyeq [j_1, j_2, j_3, j_4]$ for every choice of $i_1, \dots, i_4, j_1, \dots, j_4$ such that $i_a < j_b$ for all $a, b = 1, \dots, 4$, as these are all relabellings of the same order (it is assumed that $i_1 < \cdots < i_4$ and $j_1 < \cdots < j_4$ since we are dealing only with minors in standard form). On the other hand, $[1, 2, 3, 4] \preccurlyeq [5, 6, 7, 8]$ implies nothing about the relation of $[1, 2, 3, 5]$ and $[4, 6, 7, 8]$ as in the former $i_4 < j_1$, whereas in the latter $i_4 > j_1$.

One can see that our order on the standard pfaffian minors has this property, as it is defined only in terms of the order on the indices and the number of indices. This restriction ensures that all the subalgebras of A defined by restricting the indices are identical (for a given size) with respect to the order of their minors.

A partial order which is label independent can be described by defining the relation on diagrams which gives the order of the indices in the minors. Our diagrams has rows representing the minors being compared, with columns representing the indices in increasing order. For example if we have the minors $[1, 4, 5, 6, 8]$ and $[2, 3, 8, 9]$ we would use the following diagram:

$$\begin{array}{cccccccc} a & & & a & a & a & a & \\ & b & b & & & & b & b \end{array}.$$

An a has been placed in the first row for each column representing an index of $[1, 4, 5, 6, 8]$ and a b has been placed in the second row correspondingly for $[2, 3, 8, 9]$. Through label independence, we know that $[1, 4, 5, 6, 8] \preccurlyeq [2, 3, 8, 9]$ if and only if

$[i_1, \ldots, i_5] \preccurlyeq [j_1, \ldots, j_4]$ for all choices of $i_1, \ldots, i_5, j_1, \ldots, j_4$ so that:

$$i_1 < j_1 < j_2 < i_2 < i_3 < i_4 < i_5 = j_3 < j_4.$$

This is exactly the information in the diagram. Any choice of labelling of the columns of the diagram, such that the labels increase from left to right, gives a pair of minors with this relation. We can then think of a partial order as an assignment of $\prec, \succ, =$,or $\bowtie$ to each diagram of this type (note we use $\bowtie$ to denote incomparable elements). Certainly not all such assignments are partial orders, but all label independent partial orders define such an assignment. If such an assignment does define a partial order, then that partial order is label independent.

Definition 4.15. For a monomial $\prod_{i=1}^{m} x_{a_{i,1},\ldots,a_{i,k}} \in A$ we define its *content* to be an index of the number of $a_{ij} = q$ for each integer q.

So for example $x_{1,2,3,4}^2 x_{3,5,7,8}$ has content $1^2, 2^2, 3^3, 4^2, 5^1, 7^1, 8^1$ since it has two 1's, 2's, and 4's, three 3's, and one 5,7, and 8.

Definition 4.16. For a product of minors $\prod_{i=1}^{m} [a_{i,1}, \ldots, a_{i,k}] \in A$ we define its *content* to be an index of the number of $a_{ij} = q$ for each integer q.

So if we had $[1, 2, 3, 4]^2[1, 2, 3, 5, 7, 8]$ its content is $1^3, 2^3, 3^3, 4^2, 5^1, 7^1, 8^1$. A product of hyperpfaffian minors with content C is homogeneously composed of monomials of content C.

Theorem 4.17. *There is no label independent partial order on the hyperpfaffian minors of dimension $k \geq 3$ so that the standard monomials under that order form a basis for A.*

Proof. Assume for contradiction that $\preccurlyeq$ is such a label independent partial order.

Let us first look at the space spanned by the monomials of content $1^1, 2^1, \ldots, (2k)^1$. The monomials that span this can be identified with partitions of $\{1, \ldots, 2k\}$ into two sets A and B of size k by $x_A x_B$. The products of minors with this content (which have non-trivial hyperpfaffian) are either of the form $[A][B]$ above, or $[1, \ldots, 2k]$. Thus there is exactly one more minor with this content than the dimension of the space, so the partial order must exclude exactly one of these. This cannot be $[1, \ldots, 2k]$ since any single element is a chain in any partial order.

If we look at any $x_{i_1,\dots,i_k}x_{j_1,\dots,j_k}$ where $|\{i_1,\dots,i_k,j_1,\dots,j_k\}| < 2k$ (i.e. one where there is at least one pair a, b such that $i_a = j_b$), then any product of minors with the same content must be the product of at least two minors. Since there are only $2k$ elements, it must be the product of exactly two. Thus we can see specifically that the only product of minors generating this term is $[i_1,\dots,i_k][j_1,\dots,j_k]$. We see that any two minors of length k sharing at least one element must be comparable in $\preccurlyeq$.

In terms of our diagrams, that means that any diagram with fewer than $2k$ columns and k elements in each row must be comparable. Only one diagram (and the diagram which is a vertical flip of it) with exactly k columns is incomparable. Our goal is to show that the transitivity of $\preccurlyeq$ does not allow this diagram to exist.

To do so we are going to construct an algorithm to place a minor between the two incomparable minors. To do so first construct the incomparable diagram, with one minor labelled a the other c. In this diagram we refer to positions as points between columns, and at the beginning or end. We then place new elements b in positions (inserting a column between two existing columns) to form a new minor. The algorithm is as follows:

Step 1 : Set positions p_1 and p_2 to be the start.

Step 2 : Set x to be a and y to c if the element following p_1 is an a and x to c and y to a otherwise.

Step 3 : Let q be the number of x's following p_1 before the first y. Set p_2 to be the position prior to that first y.

Step 4 : Add q b's at p_2, set p_1 to immediately follow the b's.

Step 5 : Set p_2 to be the first position such that there are q y's after p_1 and before p_2.

Step 6 : let q be the number of x's between p_1 and p_2.

Step 7 : Add q b's at p_2, set p_1 to immediately follow these b's (or to p_2 if $q = 0$).

Step 8 : If p_2 is the end of the diagram STOP.

Step 9 : If $q = 0$ go back to **Step 2**, otherwise go back to **Step 5**.

This algorithm places a new minor b of length k between a and c in this diagram. The new diagram restricted to a over b is the same as the diagram restricted to b over c. We added k b's, as they are always added in correspondence with both the a's and the c's. We always place as many b's in front of the next batch of c's as the number of a's we had just passed, and then continued past that many b's. This ensures that there are as many a's before the first b as there are b's before the first c. It also ensures there are as many c's after the first block of b's as there are b's after the first block of a's. Then by adding as many b's as a's that were passed, we ensure that the second block of b's with respect to the c's is the same size as the second block of a's with respect to b's. Continuing in this way we ensure that the relation is as we desire.

For example, consider applying the algorithm to the diagram:

$$\begin{array}{cccccccc} a & a & & a & & & & a \\ & & c & & c & c & c & \end{array}.$$

Step 1 has us place p_1 and p_2 at the start. Then in step 2 we set $x = a$ and $y = c$, and p_1 is then placed after the second a. There are two leading a's, so q is set to 2 in step 3 and p_2 is set to p_1. In step 4 we then insert two b's after the first two a's and before the first c giving us the diagram

$$\begin{array}{cccccccccc} a & a & & & & a & & & & a \\ & & b & b & & & & & & \\ & & & & c & & c & c & c & \end{array}.$$

We then progress past two c's and place p_1 at the end of these for step 5. In doing so we move past one a, so in step 6 we set $q = 1$. So for step 7 we insert one b after the second c, giving us

$$\begin{array}{ccccccccccc} a & a & & & & a & & & & & a \\ & & b & b & & & & b & & & \\ & & & & c & & c & & c & c & \end{array}.$$

We then return to step 5, and move past one c. We have then passed no a's on the way, so $q = 0$ and we go back to step 2. This time $x = c$ and $y = a$. There is one leading c, so we set $q = 1$ and advance past it, and insert a b, giving us

$$\begin{array}{cccccccccccc} a & a & & & & a & & & & & & a \\ & & b & b & & & & b & & & b & \\ & & & & c & & c & & c & c & & \end{array}.$$

We then advance past our one a, and stop as we are at the end of the diagram. Now if we look at the two restrictions of our diagram we get

$$\begin{array}{cccccccc} a & a & & & a & & & a \\ & & b & b & & b & b & \end{array},$$

$$\begin{array}{cccccccc} b & b & & & b & & & b \\ & & c & c & & c & c & \end{array}.$$

We see these are the same diagrams.

Since these relations are the same, we know that if $a \prec b$ then $b \prec c$ and thus by transitivity $a \prec c$. Similarly if $a \succ b$ then $a \succ c$. But since $a \bowtie c$ we know $a \bowtie b$ and $b \bowtie c$. Thus this the digram for a over c must be the same as the diagrams for a over b and b over c, since there is only one diagram of this size which is incomparable. It cannot be the reverse since the first b comes before the first y and the first x comes before the first y.

Assuming this has happened, we can further adjust our b's if there are two or more b's in a row. In that case take the first b in this block and switch it with the index before it. Take the last b in the block and switch it with the one after it. Moving the b back a step exchanges the order of the first b in a group with that of the last a (or c but WLOG choose a and c so that the group is preceded by a's), and since we have the same relation all the way down, we know that this group of a's corresponds to the group of b's with relation to c ending with this group of a's, thus advancing the b one provides the corresponding change to the other diagram. Since this is a new diagram, it implies $a \prec b$ or $a \succ b$ and thus by transitivity as before $a \prec c$ and $a \succ c$. Since we haven't changed the relationship of a and c this contradicts $a \bowtie c$.

For example take the diagram

$$\begin{array}{cccccccc} a & a & a & & & & & a \\ & & & c & c & c & c & \end{array}.$$

Then applying the algorithm we get

$$\begin{array}{cccccccccccc} a & a & a & & & & & & & & & a \\ & & & b & b & b & & & & & b & \\ & & & & & & c & c & c & c & & \end{array}.$$

We see that all three relations are then the same so we have to adjust this. Moving the first and third b back and forward gives us

$$\begin{array}{cccccccccccc}
a & a & & a & & & & & & & & a \\
& & b & & b & & b & & & & b & \\
& & & & & c & & c & c & c & &
\end{array}.$$

We can see here that the restricted diagrams are

$$\begin{array}{cccccccc}
a & a & & a & & & & a \\
& & b & & b & b & b &
\end{array}$$

and

$$\begin{array}{cccccccc}
b & b & & b & & & & b \\
& & c & & c & c & c &
\end{array}.$$

These are clearly the same diagram, and not the same as a over c.

Our final case is that we have the same diagram for a over c, a over b, and b over c and there are no two b's in a row. For this to happen q must always be one or zero. Thus there must be one leading term followed by a b inserted, then the other term, then repeat. Thus it must be some sequence of the following diagrams:

$$\begin{array}{ccccccc}
a & & & & & & a \\
& b & & \text{or} & & b & \\
& & c & & c & &
\end{array}$$

It is here we are going to rely on $k > 3$. We have shown that a partial order of this type does exist for $k = 2$, so we must use this fact somewhere. The incomparable diagram for $k = 2$ is

$$\begin{array}{cccc}
a & & & a \\
& b & b &
\end{array}.$$

When our algorithm is applied to this diagram we get more of the same diagram, with no more than one b in a row. We now examine the following diagram, to show that this final case does not lead to a partial order if $k \geq 3$:

$$\begin{array}{cccccccccccc}
a & & & & a & & & & & & & a \\
& b & & & & b & & & & & b & \\
& & c & & & & c & & & c & & \\
& & & d & & & & d & d & & &
\end{array}. \tag{4.1}$$

This is composed of three diagonal blocks. The direction of the diagonal does not affect us, and if there are more treat them exactly like the last block. We start by choosing new b and c minors, as in the following diagram:

$$\begin{array}{ccccccccc} a & & a & & & & & & a \\ b & & & b & & & & b & \\ & c & & c & & & c & & \\ & d & & & d & d & & & \end{array} . \tag{4.2}$$

The diagram of a over b is the same as that of c over d, and since this relation has fewer than k indices it is not incomparable. Similarly the diagram of b over c cannot be incomparable. To avoid transitivity causing a and d to be comparable, the relation of b over c must be opposite to a over b. Let us say that $a \prec b$ WLOG, and thus $b \succ c$.

While keeping the relation of a over b and c over d the same, we can adjust the relation of b over c in the following way:

$$\begin{array}{ccccccccc} a & & a & & & & & & a \\ b & & & b & & & & b & \\ & c & & & c & & & c & \\ & d & & & & d & d & & \end{array} . \tag{4.3}$$

Since their relations are unchanged, we know $a \prec b$ and $c \prec d$. This means to maintain the incomparability of $a \bowtie d$ we must have $b \succ c$ in this diagram. Readjusting a and d with respect to b and c respectively we get the following diagram:

$$\begin{array}{ccccccccc} a & & & & a & & & & a \\ & b & & & b & & & b & \\ & & c & & & c & & c & \\ & & & d & & d & d & & \end{array} . \tag{4.4}$$

This diagram restricted to a over b is the same as the relation of b over c in diagram 4.2. This implies $a \succ b$ and $c \succ d$. From diagram 4.3, which had the same diagram of b over c as this one, we know $b \succ c$. Thus by transitivity $a \succ d$, so we have a contradiction, and no such order exists. $\square$

The unfortunate result of this is that very little of what has been shown about pfaffian rings can be converted to the hyperpfaffian case. There may still be interesting structure here, since it is clearly different from the pfaffian ring, but it is less obvious how to approach it.

4.4 Non-Uniform Hyperpfaffian Rings

We can also define hyperpfaffian rings and ideals in terms of the non-uniform hyperpfaffian. Here we use $[q_1, \ldots, q_t]$ to denote the *minor* of the antisymmetric tensor algebra Λ to be the map on the sequences $i_1, \ldots, i_k$ which map to $\Lambda(q_{i_1}, \ldots, q_{i_k})$.

Definition 4.18. The *non-uniform hyperpfaffian ideal* of order t, denoted by $P_{*,t}$ is the ideal in $k[x_I]_{I \subseteq \{1,\ldots,n\}}$ generated by the set

$$\{pf_*([q_1, \ldots, q_t]) : q_1 < \cdots < q_t \in \{1, \ldots, n\}\}.$$

Note that here we use the non-uniform hyperpfaffian that includes the odd sized tensor algebra of Definition 1.21. The results hold in exactly the same way if we use the even dimension only tensor algebra.

Definition 4.19. We define the *non-uniform hyperpfaffian ring* of order t to be $R_{*,t} = k[x_I]_{I \subseteq \{1,\ldots,n\}} / P_{*,t}$.

Unlike the uniform hyperpfaffian, the non-uniform hyperpfaffian minors do form an ASL. Unfortunately it is a rather trivial one. Our grading of $k[x_I]_{I \subseteq \{1,\ldots,n\}}$ is going to have to be slightly different. Instead of grading by degree we are going to give the term $\prod_{i=1}^m x_{I_i}$ the grade of $\sum_{i=1}^n |I_i|$. Note for the uniform case this is like assigning a grading of k times the degree instead of the degree. This grading makes the hyperpfaffian minors homogeneous, whereas degree does not.

Theorem 4.20. *The k=algebra $k[x_I]_{I \subseteq \{1,\ldots,n\}}$ graded as above with the set Π of non-uniform hyperpfaffian minors totally ordered forms an ASL*

Proof. Let us first check the first condition, that the algebra is graded, Π generates it, and the elements of Π are homogeneous. We have just established the grading, and the homogeneity of Π, so we need only check Π generates the algebra. If we look at the algebra generated by Π, we see first that it must contain all the single content monomials, since $x_{\{i\}} = [i]$. Then it must contain any two element monomial since $x_{\{i,j\}} + R = [ij]$, where R is composed of monomials composed of products of single content monomials. We can continue this inductively to show we have all the indeterminates and thus the entire algebra.

The second condition requires that the standard monomials are linearly independent. To see this we construct a linear operator which maps the monomials $\prod_{i=1}^{m} x_{I_i}$ to the standard monomials. Note that

$$\prod_{i=1}^{m} [I_m] = \prod_{i=1}^{m} x_{I_i} + R,$$

where R is composed of higher degree (not grade) terms. We order the basis of monomials of the form $\prod_{i=1}^{m} x_{I_i}$ by degree, ordering monomials of the same degree arbitrarily. Consider the linear transformation which maps $\prod_{i=1}^{m} x_{I_i}$ to $\prod_{i=1}^{m} [I_i]$. With respect to this basis, it is lower triangular with 1s down the diagonal. Thus this is invertible, and so the standard monomials must be linearly independent.

Finally the third condition holds trivially, as there are no incomparable elements in Π. $\square$

From this we get a good description of the structure of the non-uniform hyperpfaffian ring. We see that $R_{*,t}$ is isomorphic to $k[x_I]_{I \subseteq \{1,\ldots,n\}}/X_t$ where X_t is the ideal generated by $\{x_I : |I| = t\}$. This means that little of the structure of the pfaffian is involved. For this reason it is unlikely we shall find any interesting results using this ring.

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