# General Quadratic Risk Minimization: a Variational Approach

by

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I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

**Abstract**

Mean-variance portfolio selection and mean-variance hedging are mainstream research topics in mathematical finance, which can be subsumed within the framework of a general problem of quadratic risk minimization. We study this quadratic risk minimization problem in the setting of an Itô process market model with random market parameters. Our particular contribution is to introduce a combination of constraints on both the trading strategy (i.e. portfolio) and the wealth process, which includes in particular *portfolio insurance* in the form of a stipulated lower-bound on the wealth process over *the entire trading interval* (this is also called an *American wealth constraint*). The result is a stochastic control problem which includes the combination of a portfolio constraint (i.e. a "control constraint") and a wealth constraint over the trading interval (i.e. a "state constraint"). The goal of the present thesis is to address this stochastic control problem. Even in the setting of deterministic (or non-random) optimal control it is well known that a combination of control constraints and state constraints over the control interval presents some particular challenges, and of course these challenges increase considerably for stochastic control problems with the same combination of constraints. In this thesis we shall take advantage of the *convexity* of the problem and apply a powerful *variational method* of Rockafellar which has proved to be very effective in the deterministic optimal control of partial differential equations, convex optimization in continuum mechanics, and stochastic convex programming over finite dimensional spaces. The variational approach of Rockafellar enables one to systematically construct an appropriate vector space of *dual variables*, together with a *dual problem* on this space of dual variables, and gives conditions which ensure that there is *zero duality gap* (i.e. the values of the primal and dual problems are equal) as well as existence of a solution of the dual problem (i.e. existence of Lagrange multipliers for the constraints in the problem). The key to applying the Rockafellar variational approach to the stochastic control problem outlined above turns out to be a mild feasibility condition on the wealth process which is very reminiscent of "Slater-type" conditions familiar from convex optimization. With this condition in place we are able to construct an associated dual problem, and establish existence of a solution of the dual problem, together with *Kuhn-Tucker optimality conditions* which relate putative solutions of the primal and dual problems. We then use these optimality conditions to construct an optimal portfolio in terms of the solution of the dual problem.

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Last but not least, a special appreciation should be given to my parents and Ms. Jiale Li, who have constantly supported my academic journey with great patience over the years.

## Dedication

This is dedicated to my parents, Linwu Zhu and Li Feng.

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|---|---|---|
| $(\Omega, \mathcal{F}, P)$ | complete probability space | 5, 6 |
| $W$ | $\mathbb{R}^N$-valued standard Brownian motion | 5 |
| $\{\mathcal{F}_t \mid t \in [0,T]\}$ | filtration generated by $\{W(s) \mid s \leqslant t\}$ | 5 |
| $\mathcal{N}(P)$ | collection of $P$-null events in $\mathcal{F}$ | 5, 173, 206 |
| $S_0$ | money market account | 5 |
| $S_n$ | the $n$-th security | 5 |
| $r$ | interest rate | 5 |
| $\mu_n$ | appreciation rate for the $n$-th security $S_n$ | 5 |
| $\sigma_{nm}$ | the $m$-th dimension volatility for the $n$-th security $S_n$ | 5 |
| $'$ | transposition of $z \in \mathbb{R}^n$ | 6 |
| $\lVert\cdot\rVert$ | Pythagorean norm | 6 |
| $\mathcal{F}^*$ | $\mathcal{F}_t$-progressively measurable $\sigma$-algebra | 6 |
| $\mathfrak{B}([0,T])$ | Borel $\sigma$-algebra on $[0,T]$ | 6, 136 |
| $\lambda$ | Lebesgue measure | 6 |
| $\mu$ | drift vector | 6 |
| $\sigma$ | volatility matrix | 6 |
| $\theta$ | market price of risk | 7 |
| $H$ | state price density | 7 |
| $\pi_n$ | trading strategy: the dollar amount invested in the $n$-th security $S_n$ for $n = 0, 1, 2, ..., N$ | 7 |
| $X$ | wealth process of a trading strategy $\{\pi_n, n = 0, 1, 2, ..., N\}$ | 7, 8 |
| $\pi$ | portfolio process $\pi = (\pi_1, ..., \pi_N)'$ | 8 |
| $X^\pi$ | wealth process corresponding to a portfolio process $\pi$ and initial wealth $x_0$ | 8, 9 |
| $\Pi$ | set of portfolio processes | 8, 59 |
| $J(\cdot)$ | risk criterion function or objective function | 10, 25, 62 |
| $\underline{l}$ | uniform lower bound of the expected risk | 10, 11, 63, 70, 109, 156 |
| $A$ | region of a convex closed $\mathbb{R}^N$ set for regulated portfolio | 11, 70, 131 |
| $\mathcal{A}$ | set of regulated portfolio processes | 11, 70, 131 |

| Notation | Description | Page |
|---|---|---|
| $\mathbb{X}$ | vector space of primal variables | 19 |
| $f(x)$ | primal function | 19 |
| $\mathbb{U}$ | vector space of perturbations | 20 |
| $F(x,u)$ | perturbation function | 20 |
| $\mathbb{Y}$ | vector space of dual variables | 20 |
| $K(x,y)$ | Lagrangian function | 21 |
| $g(y)$ | dual function | 21 |
| $\tau(\mathbb{U},\mathbb{Y})$ | Mackey topology on $\mathbb{U}$ corresponding to the given duality system $(\mathbb{U},\mathbb{Y},\langle\cdot,\cdot\rangle)$ | 23 |
| $J^*(\cdot)$ | convex conjugate function of $J$ | 25, 62 |
| $\varkappa(\cdot)$ | support functional (of a convex set) | 27, 37, 73, 101, 113, 142, 161 |
| $(S,\Sigma,\mu)$ | measure space | 41, 216 |
| $\mathcal{L}_\infty(S,\Sigma,\mu)$ | normed vector space of $\mathbb{R}$-valued $\mu$-essentially bounded and $\Sigma$-measurable functions on $S$ | 41 |
| $\lVert\cdot\rVert_\infty$ | $\mu$-essential supremum norm | 41 |
| $\mathcal{L}_p(S,\Sigma,\mu)$ | normed vector space of $\mathbb{R}$-valued functions $f$ on $S$ such that $f$ is $\Sigma$-measurable and $\lvert f\rvert^p$ is $\mu$-integrable with $1\leqslant p<\infty$ | 41 |
| $\lVert\cdot\rVert_p$ | $\mathcal{L}_p(S,\Sigma,\mu)$-norm with $1\leqslant p<\infty$ | 41 |
| $\mathcal{L}_p^*(S,\Sigma,\mu)$ | vector space of all $\lVert\cdot\rVert_p$-continuous linear functionals on $\mathcal{L}_p(S,\Sigma,\mu)$ with $1\leqslant p\leqslant\infty$ (norm-dual of $\mathcal{L}_p(S,\Sigma,\mu)$) | 41 |
| $\lVert\cdot\rVert_p^*$ | $\mathcal{L}_p^*(S,\Sigma,\mu)$ functional norm with $1\leqslant p\leqslant\infty$ | 41 |
| $\mathbb{1}_E(\cdot)$ | indictor function of a set $E\in\Sigma$ for a measure space $(S,\Sigma,\mu)$ (equal to one if contained in $E$ and zero otherwise) | 44, 136, 217 |
| $\mathcal{Z}(S,\Sigma,\mu)$ | set of all singular functionals on $\mathcal{L}_\infty(S,\Sigma,\mu)$ | 44 |
| $L_{21}$ | vector space of all $\mathcal{F}^*$-measurable processes $\xi$ such that $E\left[\left(\int_0^T\lvert\xi(t)\mathrm{d}t\rvert\right)^2\right]<\infty$ | 59 |
| $\mathbb{B}$ | vector space of Itô processes and composed as $\mathbb{R}\times L_{21}\times\Pi$ | 59 |
| $\mathbb{B}_1$ | vector subspace of $\mathbb{B}$ and composed as $\mathbb{R}\times\Pi$ | 60 |
| $\partial J^*(\cdot)$ | derivative function of convex conjugate $J^*(\cdot)$ | 62 |
| $\delta_{\mathcal{X}}(\cdot\lvert D)$ | characteristic function of set $D$ on vector space $\mathcal{X}$ | 75 |
| $\delta^*_{\mathbb{R}^N}\{\cdot\lvert A\}$ | support function of set $A\in\mathbb{R}^N$ | 75 |
| $\mathcal{A}_1$ | subset of regulated portfolio processes $\mathcal{A}$ | 112 |
| $\mathcal{L}_2(\{\mathcal{F}_t\};\mathcal{C})$ | vector space of squared integrable continuous processes | 134 |
| $\mathcal{L}_2(\{\mathcal{F}_t\};\mathcal{BV}_0^r)$ | vector space of squared integrable right continuous bounded variation processes | 135 |

| Notation | Description | Page |
|---|---|---|
| $\|\cdot\|_T$ | total variation norm | 135, 220 |
| $\mathbb{B}_2$ | vector space of stochastic processes and composed as $\mathbb{R} \times L_{21} \times \mathcal{L}_2(\{\mathcal{F}_t\}; \mathcal{BV}_0^r)$ | 137 |
| $\hat{M}(X, Y)$ | $\mathcal{F}_t$-martingale component of $XY$ for $X \in \mathbb{B}$ and $Y \in \mathbb{B}_2$ | 139 |
| $\mathcal{L}_\infty(\{\mathcal{F}_t\}; \mathcal{C})$ | vector space of all essentially bounded continuous processes | 156 |
| $\|\cdot\|_{u(\infty)}$ | essential supremum norm | 156 |
| $\mathcal{L}_\infty^*(\{\mathcal{F}_t\}; \mathcal{C})$ | norm-dual space of $\mathcal{L}_\infty(\{\mathcal{F}_t\}; \mathcal{C})$ | 156 |
| $\mathcal{A}_2$ | subset of regulated portfolio processes $\mathcal{A}$ | 159 |
| $M(X, Y)$ | $\mathcal{F}_t$-martingale component of $XY$ for $X, Y \in \mathbb{B}$ | 189, 221 |
| $\mathcal{C}[0, T]$ | vector space of all continuous functions on $[0, T]$ | 219 |
| $\|\cdot\|_u$ | uniform norm | 219 |
| $(\mathcal{C}[0, T])^*$ | vector space of all $\|\cdot\|_u$-continuous linear functionals on $\mathcal{C}[0, T]$ (norm-dual of $\mathcal{C}[0, T]$) | 219 |
| $\|\cdot\|_u^*$ | functional norm on $(\mathcal{C}[0, T])^*$ | 219 |
| $\mathcal{P}[0, t]$ | collection of all partitions of $[0, t]$ | 219 |
| $T_\phi$ | total variation function of $\phi$ | 219 |
| $P_\phi$ | positive variation function of $\phi$ | 219 |
| $N_\phi$ | negative variation function of $\phi$ | 219 |
| $\mathcal{BV}_0[0, T]$ | vector space of bounded variation functions | 219 |
| $\mathcal{BV}_0^r[0, T]$ | vector space of right continuous bounded variation functions | 220 |

# Chapter 1

# Introduction

Mean-variance portfolio selection and mean-variance hedging are both mainstream research topics in mathematical finance. Mean-variance portfolio selection is concerned with the allocation of wealth among a basket of securities to achieve optimal trade-off between the expected return on investment and risk measured by the variance of the terminal wealth over a fixed time period. Mean-variance portfolio selection in a static one-period setting was pioneered more than fifty years ago in the seminal paper of Markowitz [23], and recently formulated by Lim and Zhou [20] in a dynamic setting as an optimal stochastic linear-quadratic problem. Somewhat related to, but different from, mean-variance portfolio selection is the problem of mean-variance hedging, which is concerned with approximating a contingent claim under mean squared error by the terminal wealth of a self-financed trading strategy. This problem was introduced in 1992 by Schweizer [36], who solved an unconstrained mean-variance hedging problem in an incomplete market by projecting the contingent claim, which is a given square integrable random variable, onto a space of square integrable stochastic integrals.

Of considerable importance in both mean-variance hedging and mean-variance portfolio selection are convex constraints on the portfolio process. These constraints model trading restrictions such as prohibition on short selling or prohibition on investment in designated securities. Needless to say, mean-variance hedging and mean-variance portfolio selection become significantly more challenging when portfolio constraints are present, and in particular the methods used in the preceding works of Lim and Zhou [20] and Schweizer [36], which rely in an essential way on the portfolios being unconstrained, are ruled out. Convex portfolio constraints were first addressed in the setting of utility maximization by Xu and Shreve [41], specifically for the constraint of prohibition on short selling, and then by Cvitanic and Karatzas [7] for completely general convex portfolio constraints. In particular Cvitanic and Karatzas [7] introduced the method of an *auxiliary market* for dealing with convex portfolio constraints. The basic idea is to formulate a so-called *auxiliary market*, which is a complete market model having the property that *unconstrained* optimization in this auxiliary market amounts to *constrained* optimization in the actual market. This is an extremely powerful approach for utility maximization, but it is far from clear how to formulate an analogous auxiliary market for problems of mean-variance portfolio selection and mean-variance hedging. Portfolio constraints for such problems were addressed by

Labbé and Heunis [18], who instead built upon a duality synthesis for convex stochastic control problems due to Bismut [4]. The basic approach of Labbé and Heunis [18] gives a unified method for dealing with portfolio constraints for both mean-variance portfolio selection and hedging (as well as utility maximization) and avoids the need to define an auxiliary market.

Even with the generality allowed by portfolio constraints, a major disadvantage of the mean-variance formulation is that the wealth process may well take negative values over the trading interval, and one cannot rule out the possibility that at optimality one could end up with negative wealth at close of trade with strictly positive probability (this drawback does not arise in problems of utility maximization, since the structure of the utility function typically ensures that the wealth is inherently and naturally non-negative over the entire trading interval). In order to overcome this drawback one must of course impose a constraint which requires that the wealth at close of trade be almost surely non-negative. The problem of mean variance hedging, with such a constraint on the wealth at close of trade and with unconstrained portfolios, was first addressed by Korn [17], who side-stepped the constraint of non-negative wealth by defining the portfolio as the *fraction* of total wealth in each risky asset, so that the wealth process is consequently strictly positive over the trading interval, and the constraint on wealth at close of trade is therefore automatically satisfied and never an active constraint. For problems of mean variance hedging it is nevertheless quite unnatural to denominate the portfolio in terms of fraction of total wealth, since this seriously distorts the quadratic structure of the problem and leads to a rather restricted class of portfolios. It is more usual and certainly more interesting to define the portfolio as the *monetary amount* invested in each risky asset, and with this denomination of the portfolio non-negativity of the wealth process is no longer assured. Mean variance hedging was addressed in this case, again with unconstrained portfolios, by Bielecki, Jin, Pliska and Zhou [3]. The problem of characterizing and constructing an optimal portfolio is considerably more challenging in this case since the constraint of non-negative wealth at close of trade will generally be an active constraint, and consequently there arises the whole question of appropriate Lagrange multipliers for this constraint. The optimal portfolio is nevertheless completely characterized and constructed in Bielecki *et al.* [3] using a clever adaptation of the risk-neutral approach.

The preceding works of Korn [17] and Bielecki *et al.* [3] address the problem of mean variance hedging with stipulated non-negative wealth at close of trade and *unconstrained* portfolios. Given the importance of portfolio constraints for modeling restrictions on trade, it is of clear interest to add a convex portfolio constraint to the constraint of non-negative wealth at close of trade. It is also useful to (slightly) generalize this latter constraint (of the form $X^{\pi}(T) \geqslant 0$ a.s., in which $X^{\pi}(T)$ denotes the total wealth at close of trade $T$ corresponding to a portfolio process $\pi$) to a constraint of the more general form $X^{\pi}(T) \geqslant b$ a.s., in which $b$ is a given random variable which stipulates a *floor-level* of wealth at close of trade, not necessarily equal to zero. With these modifications one ends up with a stochastic control problem which involves the combination of a portfolio constraint (or "control constraint") over the trading interval, together with a wealth constraint (or "state constraint") at close of trade. For this problem it turns out that one cannot just adapt or extend the methods used in Korn [17], Bielecki *et al.* [3], and Labbé and Heunis [18].

Indeed, the works of Korn [17] and Bielecki *et al.* [3] rely in an essential way on the absence of portfolio constraints and do not extend to include such constraints, while the approach of Labbé and Heunis [18] is likewise limited to problems with portfolio constraints only, without any constraints on the wealth process. It is well known that even deterministic optimal control problems with a combination of control and state constraints constitute a definite challenge, as is clearly evident from the works of Dubovitskii and Mil'yutin [9] and Makowski and Neustadt [22], and of course this challenge is substantially increased for problems of stochastic control with the same combination of constraints. In particular, one sees from Dubovitskii and Mil'yutin [9] and Makowski and Neustadt [22] that this combination of constraints typically calls upon Lagrange multipliers which are "singular" or "degenerate" in the sense of being members of the adjoint space $\mathcal{L}_\infty^*$ of some space $\mathcal{L}_\infty$ of essentially bounded functions, and can therefore involve *finitely-additive* measures. On the other hand, the problem of mean variance hedging with convex portfolio constraints together with a constraint of the form $X^\pi(T) \geqslant b$ a.s., has the immensely valuable property of being a *convex* optimization problem, and this makes available the powerful tools of convex optimization, and in particular a variational approach of Rockafellar [31] (also known as the *Rockafellar-Moreau* approach) for addressing abstract convex optimization problems of very general structure. This approach has been used with considerable effect for convex deterministic optimal control problems in which the dynamics are described by partial differential equations, convex optimization in continuum mechanics, and stochastic convex programming on finite dimensional spaces (numerous applications are given in Ekeland and Temam [10], Rockafellar [31] and Rockafellar and Wets [32]). The Rockafellar variational approach also appears to be ideally suited to convex stochastic optimal control problems, and was in fact used in Heunis [12] to address a general problem of quadratic risk minimization (which includes mean variance hedging and mean variance portfolio selection as special cases) with the combination of constraints indicated above, that is a convex portfolio constraint together with a stipulated a.s. lower-bound on the wealth at close of trade (i.e. $X^\pi(T) \geqslant b$ a.s.)

In the preceding discussion the state constraint on the wealth process has always been an a.s. inequality constraint on the wealth *at close of trade*, having the form $X^\pi(T) \geqslant b$ a.s., in which $b$ is a stipulated random variable (borrowing terminology from Mnif and Pham [24] this will be dubbed a *European wealth constraint*); non-negative $b$ guarantees a form of *portfolio insurance* at close of trade, while $b := 0$ corresponds to prohibition of bankruptcy, and negative $b$ amounts to a specified limit on debt at close of trade. One can of course similarly introduce an inequality constraint on the wealth process $X^\pi$, corresponding to a portfolio process $\pi$, which applies over the *entire control interval* $t \in [0, T]$, instead of just at close of trade (i.e. $t = T$), to get an inequality state constraint of the form $X^\pi(t) \geqslant B(t),\ t \in [0, T]$ a.s., in which $\{B(t),\ t \in [0, T]\}$ is a given $\mathbb{R}$-valued process which effectively stipulates a floor-level of wealth *over the full trading interval* (this will be termed an *American wealth constraint*, again borrowing terminology from Mnif and Pham [24]). This constraint represents a form of *continuous portfolio insurance*, at least when the process $B$ is non-negative, which guarantees a specified level of wealth over the full trading interval, instead of just at close of trade. It is evident that an American wealth constraint over the full interval $t \in [0, T]$ is significantly more complex, and likely to be substantially more challenging, than a European wealth constraint at close of trade. The goal of the present thesis is to address a general problem of quadratic risk minimization, which includes mean variance hedging and mean variance portfolio selection as special cases, with convex portfolio constraints and an American wealth constraint over the full trading interval. We shall see that the variational approach of Rockafellar indicated previously is very well suited to this problem, and more than equal to the challenges posed by this rather difficult combination of constraints.

The structure of the thesis is as follows. In Chapter 2 we define the market model and formulate in precise terms the canonical problem of interest, which includes mean-variance portfolio selection and mean-variance hedging as special cases. Chapter 3 focuses on the technical background, and in particular gives a systematic introduction to the variational method of Rockafellar with several illustrative examples. As noted above, this method is at the heart of the basic approach adopted in the thesis. The goal of Chapter 4 is two-fold: First, we set forth further technical background, mainly drawn from the classic work of Bismut [4], which will be essential tools for addressing the problem of this thesis. Second, in order to illustrate how one applies the Rockafellar variational method to convex problems of stochastic optimal control, we shall use this approach to derive *in a unified manner* the main results of the works indicated in the previous discussion, namely the results of Lim and Zhou [20], Labbé and Heunis [18] and Bielecki *et al.* [3], all of which were originally established by a variety of rather problem-specific methods. In the course of discussing these problems we shall acquire valuable insight on how to use the variational approach for stochastic optimal control problems of increasing complexity. This will serve us well in Chapter 5 in which we address the main problem of interest, namely quadratic risk minimization with convex portfolio constraints and an American wealth constraint over the full trading interval. Finally, in Chapter 6, we briefly indicate some possibilities for continuing the work of this thesis. Several appendices follow Chapter 6. In an attempt to keep the main lines of development reasonably clear we have relegated to Appendix A the proofs of a number of technical results occurring in the main body of the thesis, while Appendices B - F are meant to enhance readability of the thesis by including for easy reference a miscellany of useful mathematical tools, ideas and results. It is suggested that these appendices be consulted for reference only, and not read in their entirety.

# Chapter 2

# Continuous Time Market Model and Problem Formulation

In this chapter we shall formulate the market model used throughout the thesis and define the problem to be addressed in the thesis.

## 2.1 Market Model and Portfolio Wealth

Throughout this thesis the following conditions are always in force:

**Condition 2.1.1.** We are given a finite "horizon" $T \in (0, \infty)$ which determines a fixed interval $[0, T]$ over which all trades take place. We are also given a complete probability space $(\Omega, \mathcal{F}, P)$ on which is defined some $\mathbb{R}^N$-valued standard Brownian motion $W = (W_1(t), ..., W_N(t))'$, $t \in [0, T]$. The information available to the investor is assumed to be given by the filtration

$$\{\mathcal{F}_t \mid t \in [0, T]\} := \sigma\{W(s),\ s \in [0, t]\} \vee \mathcal{N}(P), \quad t \in [0, T], \tag{2.1.1}$$

in which $\mathcal{N}(P)$ denotes the collection of all $P$-null events in $\mathcal{F}$.

**Condition 2.1.2.** The market comprises $N + 1$ continuously tradable assets, namely a **money market account** with price $S_0$, and $N$ **securities** with prices $S_n$, $n = 1, 2, \ldots, N$, which are modeled by continuous stochastic processes, satisfying the following *stochastic differential equations* (SDE)

$$\mathrm{d}S_0(t) = r(t)S_0(t)\mathrm{d}t, \quad S_0(0) = 1, \tag{2.1.2}$$

and

$$\mathrm{d}S_n(t) = S_n(t) \left[ \mu_n(t)\mathrm{d}t + \sum_{m=1}^{N} \sigma_{nm}(t)\mathrm{d}W_m(t) \right], \tag{2.1.3}$$

where the **initial values** $S_n(0)$ are strictly positive constants. Moreover, the **interest rate** $r$ as well as the **appreciation rates** $\mu_n$ and the **volatilities** $\sigma_{nm}$ are all $\mathbb{R}$-valued uniformly bounded, $\mathcal{F}_t$-progressively measurable processes, and the interest rate $r$ is non-negative.

**Remark 2.1.3.** The $\mathcal{F}_t$-progressively measurable processes $r$, $\mu_n$ and $\sigma_{nm}$ stipulated in Condition 2.1.2 are *specified* in advance as part of the *market model*. These processes are typically referred to as **market parameters** or **market settings**. In particular, since $r$ is a $\mathbb{R}$-valued non-negative uniformly bounded process, i.e., there exists some $\tilde{r} \in (0, \infty)$ such that

$$\tilde{r} := \underset{\omega \in \Omega}{\text{P-ess-sup}} \left\{ \sup_{0 \leqslant t \leqslant T} |r(\omega; t)| \right\}, \tag{2.1.4}$$

and then

$$0 < e^{-\tilde{r}t} \leqslant \frac{1}{S_0(t)} \leqslant 1 \leqslant S_0(t) \leqslant e^{\tilde{r}t} \qquad \text{a.s.}, t \in [0, T]. \tag{2.1.5}$$

**Notation 2.1.4.** The notation defined here is used throughout this work.

(1) The prime symbol $'$ denotes transposition, and $\|z\|$ denotes the Pythagorean norm, of $z \in \mathbb{R}^n$ (the dimension $n$ being clear from the context).

(2) $\mathcal{F}^*$ denotes the $\mathcal{F}_t$-progressively measurable $\sigma$-algebra on $\Omega \times [0, T]$. If $\{\eta(t),\ t \in [0, T]\}$ is an $\mathbb{R}^n$-valued process on $(\Omega, \mathcal{F}, P)$ then the notation $\eta \in \mathcal{F}^*$ indicates that the mapping $\eta : \Omega \times [0, T] \to \mathbb{R}^n$ is $\mathcal{F}^*$-measurable, that is the process $\eta$ is $\mathcal{F}_t$-progressively measurable.

(3) "a.s." means "almost surely" with respect to the probability $P$ on $(\Omega, \mathcal{F})$, that is "a.s." really means "$P$-a.s.". Similarly "a.e." refers to the measure $P \otimes \lambda$ on $\mathcal{F} \otimes \mathfrak{B}([0, T])$, where $\mathfrak{B}([0, T])$ denotes the Borel $\sigma$-algebra on $[0, T]$ and $\lambda$ denotes Lebesgue measure on $\mathfrak{B}([0, T])$. For example, for a Borel set $A \subset \mathbb{R}^N$ and $\mathcal{F}^*$-measurable mapping $\pi : \Omega \times [0, T] \to \mathbb{R}^N$, the statement "$\pi(t) \in A$ a.e." means that $\pi(\omega; t) \in A$ for $P \otimes \lambda$-almost all $(\omega; t) \in \Omega \times [0, T]$.

We also write the vector form of the above parameters, namely the **drift vector** $\mu(\omega; t) := (\mu_1(\omega; t), ..., \mu_N(\omega; t))'$, the **volatility matrix** $\sigma(\omega; t) := (\sigma_{nm}(\omega; t))_{N \times N}$ and always assume the following *strong non-degeneracy condition*:

**Condition 2.1.5.** The **covariance matrix** $\sigma'\sigma$ satisfies

$$z'\sigma'\sigma z \geqslant \kappa_0 \|z\|^2, \qquad \forall\, z \in \mathbb{R}^N, \forall\, t \in [0, T], \ \text{ and } \forall \omega \in \Omega, \tag{2.1.6}$$

for some constant $\kappa_0 \in (0, \infty)$.

This implies (see Xu and Shreve [41, (2.4), (2.5), p.90]) that there is a constant $\kappa_1 \in (1, +\infty)$ such that

$$\frac{1}{\kappa_1} \max\left\{\left\|(\sigma(t))^{-1}z\right\|, \left\|(\sigma'(t))^{-1}z\right\|\right\} \leqslant \|z\| \leqslant \kappa_1 \min\left\{\left\|(\sigma(t))^{-1}z\right\|, \left\|(\sigma'(t))^{-1}z\right\|\right\} \tag{2.1.7}$$

for all $(z, t) \in \mathbb{R}^N \times [0, T]$, a.s.. This bound is used repeatedly throughout the thesis.

Define the so-called *market price of risk* and the strictly positive *state price density process* in the usual way as follows.

**Definition 2.1.6.** Define the **market price of risk**

$$\theta := \sigma(t)^{-1}\left[\mu(t) - r(t)\mathbb{1}\right], \quad \forall t \in [0,T], \tag{2.1.8}$$

where $\mathbb{1} \in \mathbb{R}^N$ has all unit entries, and the **state price density process**

$$H := [S_0(t)]^{-1} \exp\left\{-\int_0^t \theta'(s)\mathrm{d}W(s) - \frac{1}{2}\int_0^t \|\theta(s)\|^2 \mathrm{d}s\right\}, \quad t \in [0,T]. \tag{2.1.9}$$

**Remark 2.1.7.** (1) By Condition 2.1.2 and (2.1.7), we know $\theta \in \mathcal{F}^*$ and is a uniformly bounded process.
(2) It is well-known from Itô's formula that $H$ satisfies the SDE

$$\mathrm{d}H(t) = H(t)\left[-r(t)\mathrm{d}t - \theta'(t)\mathrm{d}W(t)\right], \qquad H(0) = 1. \tag{2.1.10}$$

and by Doob's $L^2$-inequality, we have

$$E\left[\sup_{t\in[0,T]} |H(t)|^2\right] < \infty. \tag{2.1.11}$$

(3) $H$ is a continuous process, and from (2.1.5) and Definition 2.1.6, $\inf_{t\in[0,T]} H(t) > 0$, a.s. Therefore, $G := 1/H$ is well-defined and follows the SDE (as seen from Itô's formula)

$$\mathrm{d}G(t) = G(t)\left[(r(t) + \theta'(t)\theta(t))\mathrm{d}t + \theta'(t)\mathrm{d}W(t)\right], \qquad G(0) = 1. \tag{2.1.12}$$

**Definition 2.1.8.** An investor's **trading strategy** is defined by the *dollar amount* $\pi_n$ invested in the $n$-th security with price $S_n(t)$ at time $t \in [0,T]$, such that

(1) $\pi_n$ is a $\mathcal{F}_t$-progressively measurable process, or for short, $\pi_n \in \mathcal{F}^*$,

(2) $\int_0^T |\pi_n(t)|^2 \mathrm{d}t < \infty$, a.s.,

for $n = 0, 1, 2, ..., N$. The **wealth process** for this trading strategy $\{\pi_n, n = 0, 1, ..., N\}$ is then defined by

$$X(t) := \sum_{n=0}^{N} \pi_n(t), \qquad \text{for all } t \in [0,T] \text{ a.s..} \tag{2.1.13}$$

To develop an SDE for the wealth process, we further require that the trading strategy $\{\pi_n, n = 0, 1, ..., N\}$ is *self-financed* .

**Definition 2.1.9.** A trading strategy $\{\pi_n, n = 0, 1, ..., N\}$ and the corresponding wealth process $X$ given by (2.1.13) are called **self-financed** if the following equation holds:

$$\mathrm{d}X(t) \overset{(2.1.13)}{=} \mathrm{d}\sum_{n=0}^{N} \pi_n(t) = \sum_{n=0}^{N} \frac{\pi_n(t)}{S_n(t)}\mathrm{d}S_n(t) \qquad \text{for all } t \in [0,T]. \tag{2.1.14}$$

Intuitively, a self-financed strategy does not allow any addition or withdrawal of capital after the initial investment. Combining (2.1.14), (2.1.3), and (2.1.13), we can write a SDE for the wealth process

$$\mathrm{d}X(t) = \left\{ r(t)X(t) + \sum_{n=1}^{N} \pi_n(t)[\mu_n(t) - r(t)] \right\} \mathrm{d}t \\ + \sum_{n=1}^{N} \sum_{m=1}^{N} \pi_n(t)\sigma_{nm}(t)\mathrm{d}W_m(t). \qquad (2.1.15)$$

With the help of the market price of risk (recall (2.1.8)) and denoting the last $N$ entries of a self-financed trading strategy by

$$\pi(t) := (\pi_1(t), ..., \pi_N(t))', \qquad t \in [0, T], \qquad (2.1.16)$$

we can rewrite (2.1.15) in the vector form

$$\mathrm{d}X(t) = [r(t)X(t) + \pi'(t)\sigma(t)\theta(t)]\,\mathrm{d}t + \pi'(t)\sigma(t)\mathrm{d}W(t). \qquad (2.1.17)$$

From now on we shall always restrict attention to self-financed trading strategies, in which the dollar amounts invested in the securities are *square integrable* in the following sense:

**Definition 2.1.10.** A **portfolio process** $\pi = (\pi_1, ..., \pi_N)'$ consists of the last $N$ entries of a self-financed trading strategy and is defined as a $\mathbb{R}^N$-valued $\mathcal{F}^*$-measurable process such that

$$\int_0^T \|\pi(t)\|^2 \,\mathrm{d}t < +\infty \quad \text{a.s.} \qquad (2.1.18)$$

We note that (2.1.17) has a pathwise-unique solution for every such portfolio process $\pi$. Throughout this thesis we shall mainly be concerned with square-integrable portfolios, that is portfolio processes $\pi$ such that

$$E\left[\int_0^T \|\pi(t)\|^2 \,\mathrm{d}t\right] < +\infty, \qquad (2.1.19)$$

and denote the set of all square-integrable portfolio processes by

$$\Pi := \left\{ \pi : \Omega \times [0, T] \to \mathbb{R}^N \;\middle|\; \pi \in \mathcal{F}^* \text{ and } E\left[\int_0^T \|\pi(t)\|^2 \,\mathrm{d}t\right] < +\infty \right\}. \qquad (2.1.20)$$

Square integrable portfolio processes in the sense of the preceding definition are very tractable, and it turns out that $\Pi$ is certainly large enough to include solutions of the problems addressed in this thesis.

We denote by $X^\pi$ the **wealth process of a portfolio** $\pi = (\pi_1, \pi_2, ..., \pi_N)'$, and assume the following condition throughout this thesis:

**Condition 2.1.11.** [**Initial Investment Condition**] The investor begins trading at $t = 0$ with a given *non-random* strictly positive *initial wealth* denoted by $x_0$.

The initial wealth in the preceding condition together with the SDE (2.1.17) fully describe a wealth process $X^\pi$ of a portfolio $\pi$ as:

$$\mathrm{d}X^\pi(t) = [r(t)X^\pi(t) + \pi'(t)\sigma(t)\theta(t)]\,\mathrm{d}t + \pi'(t)\sigma(t)\mathrm{d}W(t), \quad X^\pi(0) = x_0, \tag{2.1.21}$$

By Itô's formula this equation can be solved explicitly in terms of the portfolio $\pi$ as follows:

$$X^\pi(t) = S_0(t)\left\{x_0 + \int_0^t \frac{\pi'(s)\sigma(s)\theta(s)}{S_0(s)}\mathrm{d}s + \int_0^t \frac{\pi'(s)\sigma(s)}{S_0(s)}\mathrm{d}W(s)\right\}, \quad t \in [0,T]. \tag{2.1.22}$$

For later use we demonstrate that the wealth process $X^\pi(t)$ is square integrable for every portfolio $\pi \in \Pi$ (see Definition 2.1.10). To this end observe from (2.1.22) that

$$\begin{aligned}|X^\pi(t)|^2 &= |S_0(t)|^2\left\{\left|x_0 + \int_0^t \frac{\pi'(s)\sigma(s)\theta(s)}{S_0(s)}\mathrm{d}s + \int_0^t \frac{\pi'(s)\sigma(s)}{S_0(s)}\mathrm{d}W(s)\right|^2\right\}\\ &\leqslant 3\,|S_0(t)|^2\left\{x_0^2 + \left|\int_0^t \frac{\pi'(s)\sigma(s)\theta(s)}{S_0(s)}\mathrm{d}s\right|^2 + \left|\int_0^t \frac{\pi'(s)\sigma(s)}{S_0(s)}\mathrm{d}W(s)\right|^2\right\}.\end{aligned} \tag{2.1.23}$$

By Condition 2.1.2, $r$ is non-negative and uniformly bounded, then $S_0$ from (2.1.2) is uniformly bounded and lower bounded by 1. Applying Doob's $L^2$-inequality to the d$W$ term in (2.1.23), we easily obtain the bound

$$E\left[\sup_{t\in[0,T]} |X^\pi(t)|^2\right] < \infty, \qquad \text{for each } \pi \in \Pi. \tag{2.1.24}$$

In particular, $X^\pi(T)$ is a $\mathcal{F}_T$-measurable random variable with $E\left[|X^\pi(T)|^2\right] < \infty$.

We also record the following standard result which is a straightforward consequence of the state price density $H$ at (2.1.9) and the wealth equation at (2.1.17) (the proof is included for completeness in Appendix A):

**Remark 2.1.12.** The random variable $H(T)X^\pi(T)$ is integrable for each $\pi \in \Pi$, as follows from (2.1.24) and (2.1.11).

**Proposition 2.1.13. [Wealth Structure]** *For any wealth process $X^\pi$ of a portfolio $\pi \in \Pi$,*

$$X^\pi(t) = H^{-1}(t)E\left[H(T)X^\pi(T)\,\middle|\,\mathcal{F}_t\right], \qquad \textit{for all } t \in [0,T], \tag{2.1.25}$$

*and in particular,*

$$x_0 = E\left[H(T)X^\pi(T)\right]. \tag{2.1.26}$$

## 2.2 Overview of Optimization Problems

In its most elementary form the classic $L^2$-*hedging problem* is as follows: for a stipulated square integrable $\mathcal{F}_T$-measurable random variable $\gamma$ (called a *contingent claim*) one must determine a portfolio $\pi \in \Pi$ which *minimizes* the discrepancy

$$E\left[|X^\pi(T) - \gamma|^2\right], \tag{2.2.1}$$

in which $X^\pi$ is given by the relation (2.1.22).

One can generalize this problem in several ways, namely by adopting a more general risk criterion to be minimized, and by adding appropriate *constraints* to the minimization problem. In this thesis we shall address the following basic problem

$$\text{minimize } E\left[J(X^\pi(T))\right] \quad \text{subject to appropriate problem constraints,} \tag{2.2.2}$$

in which the **risk criterion function** is always defined by

$$J(x,\omega) := \frac{1}{2}\left[a(\omega)x^2 + 2c(\omega)x\right] + q, \qquad (x,\omega) \in \mathbb{R} \times \Omega, \tag{2.2.3}$$

and the following condition holds:

**Condition 2.2.1.** $q \in \mathbb{R}$ is a constant, and $a, c$ are $\mathcal{F}_T$-measurable random variables such that

$$\bar{a} := \sup_{\omega\in\Omega} a(\omega) < \infty, \quad \underline{a} := \inf_{\omega\in\Omega} a(\omega) > 0 \quad \text{and} \quad E\left[c^2\right] < \infty. \tag{2.2.4}$$

It is clear that the risk criterion $E\left[J(X^\pi(T))\right]$ at (2.2.2) generalizes the $L^2$-hedging criterion at (2.2.1).

**Remark 2.2.2.** The constant $q \in \mathbb{R}$ on the right of (2.2.3) is of course redundant as far as the formulation of problem (2.2.2) is concerned. We nevertheless keep this constant in the definition of $J$ since it will be useful in Section 4.4.1 when we address mean variance portfolio selection with the constraint of a stipulated *expected wealth* at close of trade, and must introduce a (scalar) Lagrange multiplier for this constraint.

**Remark 2.2.3.** We see from (2.1.24) that $X^\pi(T)$ is square integrable for each $\pi \in \Pi$. It then follows from Condition 2.2.1 that $E\left[J(X^\pi(T))\right]$ is defined and $E\left[J(X^\pi(T))\right] \in \mathbb{R}$ for every $\pi \in \Pi$. In fact, from (2.2.3) and (2.2.4), we have

$$J(x,\omega) \geqslant -\frac{c^2(\omega)}{2a(\omega)} + q \geqslant -\frac{c^2(\omega)}{2\underline{a}} + q, \quad (x,\omega) \in \mathbb{R} \times \Omega, \tag{2.2.5}$$

and therefore

$$-\infty < \underline{l} \leqslant E\left[J(X^\pi(T))\right] < \infty, \quad \text{for all } \pi \in \Pi, \text{ in which } \underline{l} := q - \tfrac{E\left[c^2\right]}{2\underline{a}}. \tag{2.2.6}$$

Under the formulation of the quadratic optimization problem (2.2.2), we generally want to discuss two types of constraints: *portfolio constraints* and *wealth constraints*, which are, in the terminology of optimal control, respectively *control constraints* and *state constraints*. Problem (2.2.2) becomes quite challenging when either portfolio or wealth constraints are included, and becomes even more challenging when both types of constraints are present together.

The definitive work on problems which involve **portfolio constraints** is that of Cvitanic and Karatzas [7], which involves the maximization of utility from both investment and consumption subject to the portfolio $\pi$ taking values within a specified non-empty convex closed set $A \subset \mathbb{R}^N$, i.e., (recall Notation 2.1.4)

$$\pi(t) \in A, \qquad \text{a.e..} \tag{2.2.7}$$

We denote

$$\mathcal{A} := \{\pi \in \Pi \mid \pi(t) \in A, \text{ a.e.}\}, \tag{2.2.8}$$

and any portfolio $\pi \in \mathcal{A}$ is called **a regulated portfolio**. The risk-free portfolio, i.e. $\pi(t) \equiv 0$ a.e., is usually regarded as *regulated*, because in this case $X^\pi(t) = \pi_0(t) = x_0 e^{\int_0^t r(s)\mathrm{d}s}$ for all $t \in [0,T]$, a.s.(recall Definition 2.1.8 and (2.1.16)). This effectively amounts to the assumption that all risky assets are completely liquid so that it is admissible for the investor not to hold any risky assets at all. In common with most works dealing with portfolio constraints we shall also make this assumption, that is we shall always suppose

**Condition 2.2.4.** The set $A \subset \mathbb{R}^N$ in the definition of regulated portfolios $\mathcal{A}$ at (2.2.8) is a closed and convex set with $0 \in A$.

The basic approach of Cvitanic and Karatzas [7] involves the *a-priori* introduction of a so-called *auxiliary market* which is formulated in such a way that unconstrained utility maximization amounts to utility maximization with convex portfolio constraints in the actual market. The Lagrange multipliers which "enforce" the portfolio constraint are obtained from the solution of an associated *dual optimization problem* and these Lagrange multipliers effectively determine the auxiliary market.

In the same way that Cvitanic and Karatzas [7] added portfolio constraints to the basic problem of utility maximization one can of course similarly add a convex portfolio constraint to the quadratic minimization problem (2.2.2) to obtain

$$\text{minimize } E\left[J(X^\pi(T))\right] \quad \text{such that } \pi \in \mathcal{A}, \tag{2.2.9}$$

i.e., to determine an optimal portfolio $\bar{\pi} \in \mathcal{A}$ such that

$$E\left[J(X^{\bar{\pi}}(T))\right] = \inf_{\pi \in \mathcal{A}} \{E\left[J(X^\pi(T))\right]\}. \tag{2.2.10}$$

In review of Remark 2.2.3 and the non-emptiness of $A$, we see at once that the value of problem (2.2.10) is finite (recall (2.2.6)):

$$\vartheta := \inf_{\pi \in \mathcal{A}} \{E\left[J(X^\pi(T))\right]\} \in \mathbb{R} \quad \text{with } \vartheta \geqslant \underline{l}. \tag{2.2.11}$$

Problem (2.2.9) is quite analogous to the problem of Cvitanic and Karatzas [7], except that it involves the minimization of a quadratic criterion rather than utility maximization. A consequence of this difference is that it is not at all easy to formulate in advance an auxiliary market for problem (2.2.10) which plays a role analogous to the auxiliary market in [7], namely unconstrained quadratic minimization in the auxiliary market should amount to quadratic minimization with the convex constraints. Accordingly, Labbé and Heunis [18] addressed problem (2.2.10) by a duality approach which is motivated by the seminal work of Bismut [4], which enables one to dispense with auxiliary markets and to synthesize an associated dual problem together with optimality relations (the approach of [18] can also be used to recover the main results of [7] without having to introduce an auxiliary market).

The given portfolio constraint set $A$ can be chosen to reflect various trading restrictions as follows (see Karatzas and Shreve [15, Example 5.4.1, p.206-207]):

**Example 2.2.5.** We give some examples of trading constraints specified by a closed and convex set $A$.

(i) No short-selling: here $A := [0, +\infty)^N$.

(ii) Prohibition of investment in some designated securities: here $A := \left\{\pi \in \mathbb{R}^N \mid \pi_i \equiv 0, i \in I\right\}$ for a subset $I \subset \{1, 2, ..., N\}$ denoting the prohibited securities.

(iii) Combining the above two cases or even more generally: $A$ is a closed convex cone in $\mathbb{R}^N$.

(iv) Limit control of common trading floor risk management: here $A := \Pi_{n=1}^N I_n$, where $I_n, n = 1, ..., N$ are intervals of $\mathbb{R}$ containing the origin.

□

We have so far discussed only convex portfolio constraints, and we next introduce **wealth constraints**. Effectively these are constraints on the wealth process, and therefore amount to *state constraints* in the terminology of stochastic optimal control. Of particular interest are wealth constraints which specify a *lower floor value* for the wealth process, either at close of trade (i.e. maturity) or over the entire trading interval, that is these constraints are typically of the form

$$X^\pi(T) \geqslant b, \text{ a.s.} \qquad \text{for some given } \mathcal{F}_T\text{-measurable random variable } b, \tag{2.2.12}$$

so that $b$ specifies the lower floor level at maturity; or of the form

$$X^\pi(t) \geqslant B(t), \ t \in [0, T], \text{ a.s.} \quad \text{for some given } \mathcal{F}_t\text{-adapted continuous process } \{B(t), t \in [0, T]\}, \tag{2.2.13}$$

in which case the process $\{B(t), t \in [0, T]\}$ stipulates a floor level over the trading interval. Following terminology due to Mnif and Pham [24] we shall refer to constraints of the form

(2.2.12) as **European wealth constraints**, while constraints of the form (2.2.13) will be referred to as **American wealth constraints**. It should be evident that American wealth constraints are likely to be significantly more challenging than European wealth constraints. With a *European wealth constraint* of the form (2.2.12) included, but without any portfolio constraint, problem (2.2.2) becomes

$$\text{minimize } E\left[J(X^{\pi}(T))\right] \quad \text{such that } \pi \in \Pi \text{ and } X^{\pi}(T) \geqslant b \text{ a.s.}, \tag{2.2.14}$$

i.e., the goal is to determine an optimal portfolio $\bar{\pi} \in \Pi$ such that

$$E\left[J(X^{\bar{\pi}}(T))\right] = \inf_{\pi \in \Pi} \left\{ E\left[J(X^{\pi}(T))\right] \;\middle|\; X^{\pi}(T) \geqslant b, \text{ a.s.} \right\}. \tag{2.2.15}$$

On the other hand, with an *American wealth constraint* of the form (2.2.13) included, but again without any portfolio constraint, problem (2.2.2) becomes

$$\text{minimize } E\left[J(X^{\pi}(T))\right] \quad \text{such that } \pi \in \Pi \text{ and } X^{\pi}(t) \geqslant B(t),\ t \in [0,T] \text{a.s.}, \tag{2.2.16}$$

i.e., the goal is to determine an optimal portfolio $\bar{\pi} \in \Pi$ such that

$$E\left[J(X^{\bar{\pi}}(T))\right] = \inf_{\pi \in \Pi} \left\{ E\left[J(X^{\pi}(T))\right] \;\middle|\; X^{\pi}(t) \geqslant B(t), \quad t \in [0,T] \text{ a.s.} \right\}. \tag{2.2.17}$$

We note that, for the problem (2.2.14) to even make sense, the floor-level random variable $b$ must be stipulated such that

$$X^{\hat{\pi}}(T) \geqslant b \quad \text{a.s.} \qquad \text{for some } \hat{\pi} \in \Pi, \tag{2.2.18}$$

since without this condition one can never satisfy the wealth constraint. Similarly, for the problem (2.2.16) to make sense, the floor-level random process $B$ must be stipulated such that

$$X^{\hat{\pi}}(t) \geqslant B(t) \ \ t \in [0,T], \text{a.s} \qquad \text{for some } \hat{\pi} \in \Pi. \tag{2.2.19}$$

The earliest contributions to problems with wealth constraints of which we are aware are due to Korn [17] and Bielecki *et al.* [3], both of whom address a special case of (2.2.16) involving $L^2$ hedging and specifically no-bankruptcy over the trading interval. That is, the goal is to determine some $\bar{\pi} \in \Pi$ such that

$$E\left[\left|X^{\bar{\pi}}(T) - \gamma\right|^2\right] = \inf_{\pi \in \Pi} \left\{ E\left[|X^{\pi}(T) - \gamma|^2\right] \;\middle|\; X^{\pi}(t) \geqslant 0, \quad t \in [0,T] \text{ a.s.} \right\}. \tag{2.2.20}$$

Effectively, in Korn [17] the American wealth constraint with $B(t) = 0, \quad t \in [0,T]$ is side-stepped by denominating the investment in securities in terms of *fractions* of the total wealth so that the wealth process is necessarily strictly positive and therefore the wealth constraint becomes irrelevant. In contrast, Bielecki *et al.* [3] denominate the investment in securities in terms of the *monetary amount* in each security (exactly as at the wealth equation (2.1.21)). This presents a significantly more challenging problem, since the wealth process can in fact become negative over the trading interval and therefore the wealth constraint can bind at optimality. It should be noted that although [3] addresses the

American wealth constraint with the no-bankruptcy floor level $B(t) = 0, \quad t \in [0, T]$, the very simplicity of this constraint, together with the absence of any portfolio constraints, means that it just amounts to a European constraint of the form (2.2.12) with $b = 0$ (this reduction is used in [3] and extended in Remark 5.1.2 to the case where portfolio constraints are present as well)).

**Remark 2.2.6.** The purpose of the American wealth constraint of the form (2.2.19) is to *limit intertemporal risks* over the entire trading interval. This represents a type of "continuously applied" portfolio insurance over the trading interval, in contrast to the European wealth constraint of the form (2.2.18) in which portfolio insurance is applied only at close of trade and not anywhere else over the trading interval. A simple but important example of an American wealth constraint is a so-called *minimum guarantee*, for which the wealth process never drops below some fixed level, i.e., $B(t) \equiv c$, for some $c \in (-\infty, x_0)$. In the case where $c \in (0, x_0)$, this is a genuine guarantee on investor wealth throughout the trading interval, whereas when $c < 0$ this becomes a constraint on debt over the trading interval.

The goal of this thesis is to address the quadratic minimization problem (2.2.2) with a combination of both portfolio constraints and American wealth constraints, that is we shall focus on the following *canonical problem*, which clearly includes problems (2.2.15) and (2.2.17) as special cases:

**Problem 2.2.7.** Given a fixed initial investment $0 < x_0 < \infty$, the **canonical problem** is

$$\text{minimize } E\left[J(X^{\pi}(T))\right] \quad \text{such that } \pi \in \mathcal{A} \text{ and } X^{\pi}(t) \geqslant B(t), \; t \in [0, T]; \text{a.s.}, \tag{2.2.21}$$

that is determine an optimal portfolio $\bar{\pi} \in \mathcal{A}$ such that (recall (2.2.3) and (2.2.8))

$$E\left[J(X^{\bar{\pi}}(T))\right] = \inf\left\{E\left[J(X^{\pi}(T))\right] \;\middle|\; X^{\pi} \geqslant B \text{ and } \pi \in \mathcal{A}\right\}, \tag{2.2.22}$$

where $X^{\pi} \geqslant B$ is short for $X^{\pi}(t) \geqslant B(t), t \in [0, T]$ a.s..

From now on we will always suppose that the stipulated floor-level wealth process $B$ in the canonical problem 2.2.7 satisfies the following:

**Condition 2.2.8.** The floor level process $B$ is continuous and $\mathcal{F}_t$-adapted.

Of course, for the canonical problem (2.2.21) to make sense, we must also impose a *feasibility condition* similar to, (2.2.19), namely

**Condition 2.2.9.** There is some $\hat{\pi} \in \mathcal{A}$ such that

$$X^{\hat{\pi}}(t) \geqslant B(t), \qquad t \in [0, T] \text{ a.s..} \tag{2.2.23}$$

Indeed without this condition there would not exist any portfolio $\pi$ satisfying the joint portfolio and wealth constraints in Problem 2.2.7, and this problem would then not even make any sense. Actually, we shall see later that Condition 2.2.9 by itself is not quite strong enough to apply convex duality to Problem 2.2.7, and we shall need a mild but essential strengthening of this condition to a *Slater-type* feasibility condition in order to secure existence of solutions of an associated *dual problem* (see Condition 5.3.2 and Remark 5.3.3 in Section 5.3). The essential role of Slater-type feasibility conditions will become clear in the next chapter, in which we introduce a powerful *variational approach* essentially due to Rockafellar [31], which gives a systematic approach for solving convex optimization problems of great generality. In particular, we shall see that this variational approach is the key thing for dealing with the challenges posed by the canonical problem (2.2.21).

**Remark 2.2.10.** Since the main focus of this thesis is on the canonical problem (2.2.21), which is an optimal control problem with control and state constraints, some general remarks on this type of problem seem to be appropriate. There is in fact a rich literature on optimal control problems with state constraints, most of it addressing *deterministic optimal control*. We have already noted some earlier works in the Introduction, in particular Dubovitskii and Mil'yutin [9] and Makowski and Neustadt ([22], both of which focus on *necessary conditions* for optimality, particularly in the form of Pontryagin-type *maximum principles*. These are motivated by the classical maximum principle of Pontryagin, Bolt'yanskii, Gamkrelidze and Mischenko [27], and assert, as a necessary consequence of optimality, the existence of *co-state arcs* (effectively Lagrange multipliers) together with a complementary slackness relation in the form of a so-called "maximum condition" (or "generalized Weierstrass condition"). Chapter 9 of the book of Vinter [37] gives a thorough and comprehensive account of the main results on maximum principles for deterministic optimal control problems with state constraints. These results have furthermore been extended to the optimal control of deterministic *differential inclusions*, the most significant contributions to this problem being the works of Clarke [5], Loewen and Rockafellar [21] and Vinter and Zheng [38]. Again, a comprehensive account of the main results on optimal control of differential inclusions with state constraints can be found in Chapter 10 of Vinter [37]. Despite the clear significance of the preceding works, we shall say nothing about them here, for these are concerned almost entirely with the question of necessary conditions for optimality in the deterministic setting, with very general dynamics, typically nonlinear, and the methods used do not in any way pertain to our canonical problem (2.2.21). This is because, in the first instance, our problem is stochastic and not deterministic, and this has a huge impact on the appropriate methodology and approach. Secondly, our problem is *convex* with state constraints that are simple a.s. *inequalities*, and our goal is to exploit these special properties as much as possible. Indeed, these properties are key to application of the variational approach of Rockafellar, already mentioned in the Introduction, which will be at the core of this thesis. Finally, one of our main goals, in common with most works on portfolio optimization in mathematical finance, is to address *existence of solutions*, for both the given (i.e. primal) problem, as well as an associated *dual problem*, and to study the relation between these solutions in terms of *optimality relations*. This is very different from the elucidation of necessary conditions for optimality which is the main objective of the works [5], [9], [21], [22] and [38] in the preceding discussion, and consequently there is essentially no overlap between the goals and methods of the present thesis and the questions and methods which motivate these works.

**Remark 2.2.11.** The canonical problem (2.2.21) amounts to a stochastic optimal control problem with a control constraint (in the form of the portfolio constraint $\pi \in \mathcal{A}$), together with a state constraint over the full control interval $t \in [0, T]$ (in the form of the American wealth constraint $X^\pi(t) \geqslant B(t), \ \ t \in [0, T]$). With the exception of Mnif and Pham [24], we are not aware of any works dealing with stochastic optimal control problems that feature a combination of control and state constraints over the full control interval. The problem addressed by Mnif and Pham [24] involves the maximization of *expected utility of wealth at close of trade* in the abstract setting of general semimartingale market models. The state constraint is a stipulated lower-bound on the wealth process over the trading interval (much like the constraint $X^\pi \geqslant B$ in Problem 2.2.7) and the candidate wealth processes are members of a stipulated predictably convex set of semimartingales closed in the semimartingale topology (this constitutes an indirect form of convex portfolio constraint). It is difficult to make a detailed comparison of the results in Mnif and Pham [24] with the results to be established in the present thesis, and we confine ourselves to the following observations:

1. The approach of Mnif and Pham [24] relies on the one-sided monotonicity of the utility function, with the corresponding non-negativity of wealth, and does not apply to the problems of quadratic minimization with non-monotonic risk criterion function that we shall address in this thesis. Moreover, the approach of [24]) is somewhat non-synthetic or non-constructive, meaning that essential entities such as the space of dual variables (defined by the unnumbered equation for $\mathcal{Y}^0_{loc}$ at the foot of page 165 of [24]) and the dual functional (defined by equation (5.6) on page 166 of [24]) are introduced *a-priori* without much clear motivation.

2. The duality analysis in [24] is essentially incomplete, in that existence of solutions of the dual problem is not established and must simply be *assumed* (see items (2) and (3) of Theorem 5.1 on page 167 of [24]). As discussed on page 167 of [24], this means that the a-priori defined space of dual variables is possibly not correctly formulated in the sense that it might be "to small" to host a solution of the dual problem, and the formulation of a "correct" larger space of dual variables (as well as the extension of the a-priori defined dual functional to the larger space of dual variables) is left as an open problem and is not resolved in [24].

3. There is nothing in [24] resembling a systematic construction of *Kuhn-Tucker optimality conditions* (to be discussed in the next chapter) which are a familiar tool in problems of convex optimization, and in particular there are no clearly formulated feasibility conditions on the primal and dual variables, nor are there any complementary slackness conditions relating the primal variables, the dual variables, and the problem constraints.

In contrast, in this thesis we shall use as the basic tool a powerful variational approach of Rockafellar (already noted in the introductory discussion of Chapter 1, and to be introduced in detail in the following Chapter 3) which enables one to *synthetically construct* such essential entities as vector spaces of dual variables, dual functionals and Kuhn-Tucker optimality relations. Furthermore, we shall establish a clear line of descent from very simple problems of convex optimization in finite-dimensional Euclidean space to the stochastic optimal control problems outlined in the present chapter. In particular, we shall see that the fundamental methodology involved in using the Rockafellar variational approach for simple finite-dimensional problems of convex optimization carries over essentially without change to the stochastic control problems just outlined, although the latter class of problems will obviously involve a lot more technical effort. Finally, we shall see that a powerful theorem that is an essential part of the Rockafellar variational approach (see Theorem 3.1.7 in the following chapter) guides the construction of the space of dual variables and dual functional so as to ensure that a solution of the dual problem actually exists and does not have to be assumed.

## 2.3 Conclusion

In this chapter, we set up a standard market model with a money market account and several risky assets with random market parameters. We formulate the canonical problem (2.2.21), which involves minimization of a quadratic risk criterion subject to a combination of a convex portfolio constraint and an American-type wealth constraint over the trading interval. It is the combination of these constraints which constitutes the real challenge of the present work.

# Chapter 3

# A Variational Method of Rockafellar

One of the main challenges in dealing with a given convex optimization problem (usually called the "primal problem") is to formulate an appropriate vector space of *dual variables*, together with a *dual function* for an associated *dual optimization problem* defined on the vector space of dual variables. Solutions of the dual optimization problem constitute the "Lagrange multipliers" for the constraints in the problem. The structure of the Lagrange multipliers, as well as the form of the associated dual function, is often far from evident *a-priori* just based on the given primal problem. This is certainly true of the canonical problem (2.2.21), for which appropriate Lagrange multipliers for the joint portfolio and wealth constraints, as well as a dual function, is not at all clear at the outset.

The goal of this chapter is to present a powerful *variational approach* due to Rockafellar [31] which enables one to synthetically construct *dual variables* which are appropriate for the constraints in the problem, together with a *dual functional* and *optimality conditions* which relate putative solutions of the primal and dual problems, as well as to establish existence of solutions to the dual problem (that is, existence of Lagrange multipliers for the problem constraints). The method of Rockafellar has been applied very effectively to deterministic infinite dimensional convex optimization problems in calculus of variations, optimal control of partial differential equations, and continuum mechanics. This is amply clear from the book of Ekeland and Temam [10], in which these and other applications are discussed in some considerable detail. Regarding this approach, Ekeland and Temam remark (see the comments for Chap.III on p.xii of [10]) "This very flexible abstract theory can be adapted to a wide variety of situations". Despite the inherent power in Rockafellar's variational approach, it has seemingly not been much used for the sort of stochastic convex optimization problems that arise in mathematical finance. In the present work we shall demonstrate that the basic approach of Rockafellar is in fact very well suited to the canonical problem (2.2.21), which is the main goal of the thesis.

The present chapter is organized as follows: In section 3.1 we present the basic outlines of Rockafellar's variational approach. Then, in Section 3.2 we shall illustrate the use of this approach on some simple "tutorial" problems of convex optimization.

## 3.1 Basic Outlines of the Variational Approach

In this section, we summarize Rockafellar's variational approach to convex optimization. Detailed accounts, with numerous illustrative examples, are given by Rockafellar [31] and Ekeland and Temam [10], but the short summary of the present section will be ample for this thesis.

A given convex optimization problem, called the *primal problem*, is usually complicated by the presence of problem constraints. The general approach of *conjugate duality* is to somehow construct an associated *dual optimization problem*, together with *optimality relations* (frequently called *the Kuhn-Tucker relations*) which relate putative solutions of the primal and dual optimization problems. One expects that it may be easier to "solve" the dual optimization problem than to directly solve the given primal problem; if one can secure a solution of the dual problem then the optimality relations can be used to construct a solution of the primal problem in terms of the solution of the dual problem. The solution of the dual problem then constitutes the Lagrange multipliers for the constraints in the primal problem. In short, one *first* determines the Lagrange multipliers and *then* obtains the solution of the primal problem in terms of the Lagrange multipliers. This general approach has been used with considerable success for finite dimensional convex optimization problems, such as linear and quadratic programming problems. In such problems the basic structure of the Lagrange multipliers, and therefore the vector space of dual variables over which the dual optimization problem is defined, is usually *a-priori* clear, and substantial experience has been accumulated on appropriate dual functionals for the dual optimization problem. This is far from being the case for *infinite-dimensional* problems of convex optimization (such as convex optimal control problems), where, not only is the form of the dual functional usually not *a-priori* evident, but even the vector space of dual variables on which the dual functional should be defined is itself also not evident at the outset. In fact, the formulation of an appropriate vector space of dual variables presents a particular challenge in implementing conjugate duality on infinite dimensional convex primal problems, since solutions of the dual optimization will not exist if this vector space (which is itself typically infinite dimensional) is "too small". The variational approach of Rockafellar constitutes a systematic method for addressing these challenges, and in the remainder of this section we present the main outlines of this approach.

Suppose that we are given a real vector space of **primal variables $\mathbb{X}$**, together with a real-valued convex objective functional $f_0 : \mathbb{X} \to \mathbb{R}$ and a convex subset $E \subset \mathbb{X}$, which is regarded as a **constraint set**. The *primal problem* is to minimize the objective function $f_0$ over the set $E$. We can formally simplify this problem by defining a so-called **primal function $f$** $: \mathbb{X} \to \mathbb{R} \cup \{+\infty\}$ as follows

$$f(x) := \begin{cases} f_0(x), & \text{when } x \in E, \\ +\infty, & \text{otherwise}, \end{cases} \tag{3.1.1}$$

in which case the **primal problem** formally reduces to the minimization of the primal function $f$ over the entire vector space $\mathbb{X}$.

In barest outline the variational approach of Rockafellar comprises the following steps:

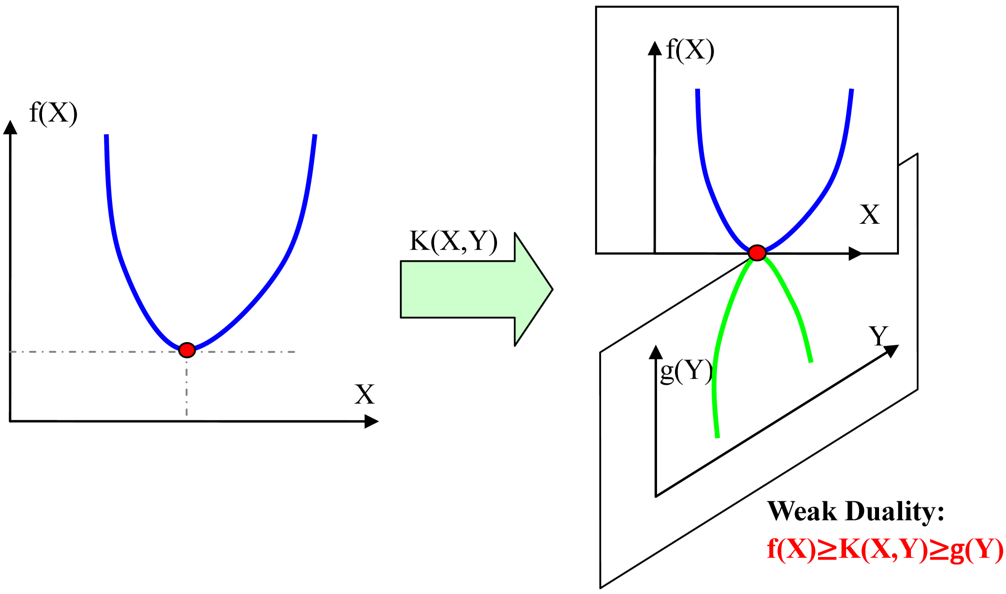



Figure 3.1: Primal Problem v.s. Dual Problem

1. *perturb* the primal function;
2. specify a candidate vector space of dual variables;
3. synthesize a Lagrangian functional and a dual functional by calculating concave conjugates.

We shall now elaborate on these steps in more detail.

**Step 3.1.1.** To perturb the primal function one chooses a real vector space $\mathbb{U}$ of **perturbations** together with a so-called **perturbation function** $F : \mathbb{X} \times \mathbb{U} \to [-\infty, \infty]$ which is *convex* on the vector space $\mathbb{X} \times \mathbb{U}$, and such that the following *consistency relation* is satisfied:

$$F(x, 0) = f(x), \quad x \in \mathbb{X}, \tag{3.1.2}$$

(cf. Rockafellar [31, (4.1), p.18]). Effectively, this means that we recover the primal function in the case of *zero perturbation*. Observe that there is considerable freedom of choice when formulating the space of perturbations $\mathbb{U}$ and perturbation mapping $F$; the only conditions to be satisfied are convexity of $F$ on $\mathbb{X} \times \mathbb{U}$ and the consistency relation (3.1.2).

**Step 3.1.2.** To specify a candidate vector space of dual variables one chooses another real vector space $\mathbb{Y}$ called the **space of dual variables** together with some real-valued bilinear form $\langle \cdot, \cdot \rangle$ on $\mathbb{U} \times \mathbb{Y}$ (see Definition B.0.5 in the background Appendix B).

**Step 3.1.3.** With the perturbation space $\mathbb{U}$, the space of dual variables $\mathbb{Y}$, and the bilinear form $\langle\cdot,\cdot\rangle$ fixed in the preceding steps, we next define a **Lagrangian function** $K : \mathbb{X}\times\mathbb{Y} \to [-\infty,+\infty]$ given by the concave conjugate

$$K(x,y) := \inf_{u\in\mathbb{U}} [\langle u,y\rangle + F(x,u)], \quad (x,y) \in \mathbb{X}\times\mathbb{Y}, \tag{3.1.3}$$

together with a **dual function** $g : \mathbb{Y} \to [-\infty,\infty]$ given by

$$g(y) := \inf_{x\in\mathbb{X}} K(x,y) = \inf_{(x,u)\in\mathbb{X}\times\mathbb{U}} [\langle u,y\rangle + F(x,u)], \quad y \in \mathbb{Y}, \tag{3.1.4}$$

(cf. Rockafellar [31, (4.2) and (4.6), p.19]).

Observe that the Lagrangian function $K(\cdot,\cdot)$ and dual function $g(\cdot)$ defined in Step 3.1.3 above are *completely determined* by the choices of perturbation space, perturbation function, space of dual variables and bilinear form in Step 3.1.1 and Step 3.1.2.

**Remark 3.1.4.** It is immediate from (3.1.2) to (3.1.3) that $g(\cdot)$ is *concave* on $\mathbb{Y}$ (being the point-wise infimum of a collection of affine functionals on $\mathbb{Y}$), and the following **weak duality** relation is immediate (see Figure 3.1):

$$f(x) \geqslant K(x,y) \geqslant g(y), \quad (x,y) \in \mathbb{X}\times\mathbb{Y}. \tag{3.1.5}$$

The maximization of $g(y)$ on $y \in \mathbb{Y}$ (when the supremum of $g$ over $\mathbb{Y}$ is attained) is referred as the **dual problem**, and maximizers $\bar{y} \in \mathbb{Y}$ of the dual function $g$ are called **Lagrange multipliers** (the reason for this terminology will become clear when we look at the *optimality relations* to be discussed later). From (3.1.5) we clearly have the following:

$$\begin{cases} \text{if } f(\bar{x}) = g(\bar{y}) \text{ for some } (\bar{x},\bar{y}) \in \mathbb{X}\times\mathbb{Y} \text{ then} \\ \bar{x} \text{ solves the primal problem of minimizing } f \text{ on } \mathbb{X} \\ \text{while } \bar{y} \text{ solves the dual problem of maximizing } g \text{ on } \mathbb{Y}. \end{cases} \tag{3.1.6}$$

The statement at (3.1.6) is essentially the crux of the method of conjugate duality. From (3.1.5) we see that

$$\inf_{x\in\mathbb{X}} f(x) - \sup_{y\in\mathbb{Y}} g(y) \geqslant 0. \tag{3.1.7}$$

This quantity is called the **duality gap**, and is of course associated with our choices of perturbation space, perturbation function, space of dual variables and bilinear form in Step 3.1.1 and Step 3.1.2 above. If the duality gap is *non-zero*, that is strictly positive, then of course there fails to exist any pair $(\bar{x},\bar{y}) \in \mathbb{X}\times\mathbb{Y}$ such that $f(\bar{x}) = g(\bar{y})$. This means that one or more of the items specified in Step 3.1.1 and Step 3.1.2 has been inappropriately chosen and we cannot extract anything useful from the resulting dual problem based on the dual function $g(\cdot)$ that we have synthesized. Furthermore, even supposing that the duality gap is equal to zero, if there fails to exist some maximizer $\bar{y}$ in the space of dual variables $\mathbb{Y}$ of the dual function $g(\cdot)$, then again there will not exist any pair $(\bar{x},\bar{y}) \in \mathbb{X}\times\mathbb{Y}$ such that $f(\bar{x}) = g(\bar{y})$, that is our choices are again inappropriate and we cannot extract anything useful from the dual problem. We are now going to state a crucial theorem due to Rockafellar [31, Theorem 17(a) and Theorem 18(a), p.41] and Moreau [25] giving very useful sufficient conditions on the choice of perturbation space $\mathbb{U}$ and perturbation function $F(\cdot,\cdot)$ at Step 3.1.1, and the bilinear form $\langle\cdot,\cdot\rangle$ pairing $\mathbb{U}$ with the space of dual variables $\mathbb{Y}$ at Step 3.1.2, which ensure that there is *zero duality gap* and that there actually exist maximizers in the space of dual variables $\mathbb{Y}$ of the dual function $g(\cdot)$. To state this theorem we require the notion of a compatible locally convex topology on the perturbation space $\mathbb{U}$. This is formulated as Definition B.0.9 in the background Appendix B, but we repeat it here for convenience:

**Definition 3.1.5.** A locally convex linear topology $\mathcal{U}$ on the perturbation space $\mathbb{U}$ is called **compatible** with the bilinear form $\langle\cdot,\cdot\rangle$ on $\mathbb{U}\times\mathbb{Y}$ (or $\langle\mathbb{U},\mathbb{Y}\rangle$**-compatible** for short) when

(1) the mapping $u\to\langle u,y\rangle:\mathbb{U}\to\mathbb{R}$ is $\mathcal{U}$-continuous for each $y\in\mathbb{Y}$,

(2) each $\mathcal{U}$-continuous linear functional $\Phi:\mathbb{U}\to\mathbb{R}$ is necessarily given by $\Phi(u)=\langle u,y\rangle$, $u\in\mathbb{U}$, for some kernel $y\in\mathbb{Y}$.

**Example 3.1.6.** In many applications the perturbation space $\mathbb{U}$ is a normed vector space, and a common choice of a space of dual variables is the *norm-dual space* $\mathbb{Y}=\mathbb{U}^*$, that is the set of all norm-bounded $\mathbb{R}$-valued linear functionals on $\mathbb{U}$, together with the usual bilinear form

$$\langle u,y\rangle := y(u) \qquad (u,y)\in\mathbb{U}\times\mathbb{Y}. \tag{3.1.8}$$

Then it is immediate that the corresponding norm-topology on $\mathbb{U}$ is $\langle\mathbb{U},\mathbb{Y}\rangle$-compatible. □

The following theorem is due to Rockafellar [31, Theorem 17(a) and Theorem 18(a), p.41] and Moreau [25]:

**Theorem 3.1.7.** *Fix some $\langle\mathbb{U},\mathbb{Y}\rangle$-compatible topology $\mathcal{U}$ on $\mathbb{U}$ (see Definition 3.1.5). If there is some $x_1\in\mathbb{X}$, and some $\mathcal{U}$-neighborhood $G$ of $0\in\mathbb{U}$, such that*

$$\sup_{u\in G} F(x_1,u) < +\infty, \tag{3.1.9}$$

*then there exists some $\bar{y}\in\mathbb{Y}$ such that*

$$\inf_{x\in\mathbb{X}} f(x) = \sup_{y\in\mathbb{Y}} g(y) = g(\bar{y}). \tag{3.1.10}$$

**Remark 3.1.8.** Theorem 3.1.7 gives conditions ensuring that one has a duality gap equal to zero *as well as* existence of a maximizer $\bar{y}$ in the space of dual variables $\mathbb{Y}$ of the dual function $g(\cdot)$ (i.e. existence of a Lagrange multiplier). Theorem 3.1.7 guides the choice of the perturbation space $\mathbb{U}$, the space of dual variables $\mathbb{Y}$, and the bilinear form $\langle\cdot,\cdot\rangle$ in Step 3.1.1 and Step 3.1.2, since we clearly want to choose these to satisfy the hypotheses of the theorem. It is most important to understand the role of the compatible topology $\mathcal{U}$ in the theorem. In the background Appendix B, we have briefly summarized the structure of the $\langle\mathbb{U},\mathbb{Y}\rangle$-compatible topologies on $\mathbb{U}$. In particular, there is a *weakest* such topology, namely the usual weak topology $\mathfrak{S}(\mathbb{U},\mathbb{Y})$ (see Remark B.0.8), and a *strongest* such topology, namely the **Mackey topology** $\tau(\mathbb{U},\mathbb{Y})$ (see Remark B.0.12), and every $\langle\mathbb{U},\mathbb{Y}\rangle$-compatible topology $\mathcal{U}$ on $\mathbb{U}$ is between these two extremes, in the sense that

$$\mathfrak{S}(\mathbb{U},\mathbb{Y}) \subset \mathcal{U} \subset \tau(\mathbb{U},\mathbb{Y}). \tag{3.1.11}$$

Clearly, in the verification of (3.1.9), we should look for sets $G$ in the *strongest* $\langle\mathbb{U},\mathbb{Y}\rangle$-compatible topologies on $\mathbb{U}$, namely the Mackey topology (since it is easier to find neighbourhoods $G$ in a strong topology), that is we should take $\mathcal{U} := \tau(\mathbb{U},\mathbb{Y})$. Unfortunately, the Mackey topology is usually difficult to characterize for specific duality systems $(\mathbb{U},\mathbb{Y},\langle\cdot,\cdot\rangle)$, and in applications of Theorem 3.1.7 one may have to make a careful choice of a compatible topology $\mathcal{U}$ whose neighbourhood base is fairly simple. Fortunately, in many applications of this theorem it suffices to take the perturbation space $\mathbb{U}$ to be a *normed vector space*, with the space of dual variables $\mathbb{Y}$ and bilinear form given in Example 3.1.6. It then follows from Theorem B.0.14 that the Mackey topology $\tau(\mathbb{U},\mathbb{U}^*)$ coincides with the norm-topology on $\mathbb{U}$, so that we can take $\mathcal{U}$ to be the norm-topology on $\mathbb{U}$ when verifying the condition (3.1.9), that is we verify the condition for some $G$ which is open in the norm topology on $\mathbb{U}$. In fact we shall see that Theorem 3.1.7 is usually very easy to use, and that a simple and natural **Slater condition** on the problem constraints will suffice to give both some $x_1 \in \mathbb{X}$ and some norm-open neighbourhood $G$ of $0 \in \mathbb{U}$ for which (3.1.9) holds.

**Remark 3.1.9.** From (3.1.6) we see that minimization of $f(\cdot)$ on $\mathbb{X}$ boils down to constructing some pair $(\bar{x},\bar{y}) \in \mathbb{X}\times\mathbb{Y}$ such that $f(\bar{x}) = g(\bar{y})$, for then $\bar{x}$ is a minimizer of $f(\cdot)$ on $\mathbb{X}$ (and $\bar{y}$ is a maximizer of $g(\cdot)$ on $\mathbb{Y}$). How can such a pair $(\bar{x},\bar{y})$ be constructed? If the conditions of Theorem 3.1.7 are in force, then we already have at our disposal a maximizer $\bar{y} \in \mathbb{Y}$ of the dual function $g$, and the task is then to somehow construct $\bar{x}$ *in terms of the maximizer* $\bar{y}$ such that $f(\bar{x}) = g(\bar{y})$. This task can be accomplished as follows: For *arbitrary* $(x,y) \in \mathbb{X}\times\mathbb{Y}$ we must establish an equivalence of the form

$$f(x) = g(y) \quad \Longleftrightarrow \quad \begin{cases} \text{some useful optimality relations on } (x,y) \in \mathbb{X}\times\mathbb{Y} \text{ hold} \\ \text{e.g. complementary slackness conditions,} \\ \text{feasibility conditions, transversality conditions.} \end{cases} \tag{3.1.12}$$

The relations on the right side of this equivalence are called **Kuhn-Tucker optimality relations**. If such a set of optimality relations can be established, then our task becomes one of constructing an $\bar{x} \in \mathbb{X}$ in terms of the maximizer $\bar{y} \in \mathbb{Y}$ such that the pair $(\bar{x},\bar{y})$ satisfies the Kuhn-Tucker optimality relations, for then it follows from the equivalence at (3.1.12) that we have $f(\bar{x}) = g(\bar{y})$, as required to show that $\bar{x}$ is the minimizer of the primal function $f$ on $\mathbb{X}$. We shall see that the construction of $\bar{x}$ in terms of $\bar{y}$ so that the pair $(\bar{x},\bar{y})$ satisfies the Kuhn-Tucker relations is closely tied to being able to establish *necessary conditions* which result from the known optimality of $\bar{y}$ for the dual function $g$. In fact these necessary conditions for the optimality of $y$ turn out to be the essential tool for verifying the Kuhn-Tucker optimality relations.

**Remark 3.1.10.** In Step 3.1.1 and Step 3.1.2 above there is complete freedom of choice of the vector spaces $\mathbb{U}$ and $\mathbb{Y}$, the perturbation function $F(\cdot,\cdot)$ on $\mathbb{X}\times\mathbb{U}$, and the bilinear form $\langle \cdot, \cdot \rangle$ on $\mathbb{U} \times \mathbb{Y}$, provided only that the perturbation function $F(\cdot, \cdot)$ be convex on $\mathbb{X} \times \mathbb{U}$, and that the consistency relation (3.1.2) hold. Different choices of these items will lead to correspondingly different Lagrangian and dual functions. These items should be chosen to ensure the following:

(a) The conditions of Theorem 3.1.7 must hold. This establishes that the duality gap is zero and that we have a maximizer $\bar{y} \in \mathbb{Y}$ of the dual function $g$. Our goal is to construct a solution of the primal problem in terms of the optimal dual solution $\bar{y}$ obtained from Theorem 3.1.7.

(b) The dual function $g$ should be reasonably tractable, so that we can get necessary conditions from the optimality of $\bar{y}$. These necessary conditions will be essential when we construct a solution of the primal problem in terms of the optimal dual solution $\bar{y}$.

(c) One must be able to write the condition $f(x) = g(y)$ (for an *arbitrary* pair $(x, y) \in \mathbb{X} \times \mathbb{Y}$) in the form of *Kuhn-Tucker* optimality relations familiar from convex optimization. These relations should comprise (i) feasibility conditions on the primal and dual variables; (ii) complementary slackness conditions which relate the primal variable $x$, the dual variable $y$, and the constraints in the primal problem; and (iii) transversality conditions which relate the primal variable $x$, the dual variable $y$ and the objective function of the primal problem. The Kuhn-Tucker relations, together with the necessary conditions for the optimality of $\bar{y}$ (recall (b)), will then be used to construct an $\bar{x} \in \mathbb{X}$ in terms of the maximizer $\bar{y} \in \mathbb{Y}$ given by Theorem 3.1.7, such that $f(\bar{x}) = g(\bar{y})$.

In the following Section 3.2 we are going to illustrate the general approach summarized above on some simple deterministic convex optimization problems.

## 3.2 The Variational Approach Illustrated on Simple Deterministic Problems

In Section 3.1 we formulated Rockafellar's variational approach as a sequence of steps (see Step 3.1.1 - Step 3.1.3) which involve the choice of a real vector space $\mathbb{U}$ of perturbations, together with a perturbation function $F(\cdot, \cdot)$, a real vector space $\mathbb{Y}$ of dual variables and a bilinear form on $\mathbb{U} \times \mathbb{Y}$. With these specified the Lagrangian function and dual function necessarily follow from the defining relations (3.1.3) and (3.1.4).

In the present section we are going to illustrate the variational approach on some very simple finite-dimensional deterministic convex optimization problems. The results of this section are not new, but we believe that this section will give a feel for the general methodology of the variational approach, which will be of definite help when we apply this approach to the much more challenging stochastic control problems of this thesis, in particular the canonical problem (2.2.21).

We begin with a finite-dimensional deterministic static convex programming problem which is in fact a very simple yet instructive precursor of the problem at (2.2.9), namely the minimization of a quadratic criterion with a convex portfolio constraint. Throughout this section we are given an **objective function**

$$J : \mathbb{R}^n \to \mathbb{R} \tag{3.2.1}$$

which is some smooth convex function (e.g. a quadratic function). The **convex conjugate function** $J^*(\cdot)$ of the objective function $J(\cdot)$ is defined in the usual way as

$$J^*(y) := \sup_{x \in \mathbb{R}^n} \{x'y - J(x)\}, \qquad \text{for } y \in \mathbb{R}^n. \tag{3.2.2}$$

**Problem 3.2.1.**

$$\text{minimize } J(x) \text{ subject to } x \in A, \tag{3.2.3}$$

where $A \subset \mathbb{R}^n$ is some non-empty closed convex set, and we shall suppose that

$$\inf_{x \in A} J(x) > -\infty. \tag{3.2.4}$$

The real vector space of primal variables for this problem is of course

$$\mathbb{X} := \mathbb{R}^n. \tag{3.2.5}$$

There is of course no guarantee that Problem (3.2.3) even has a solution (e.g. take $\mathbb{X} := \mathbb{R}$, $J(x) := e^{-x}$ for all $x \in \mathbb{X}$, $A := [0, \infty)$). However, at this point our goal is just to apply the Rockafellar variational approach to synthesize a Lagrangian, a dual problem, and optimality relations, a construction which can be carried out irrespective of whether or not the problem has a solution. Later we shall see that there is a clear connection between resolving the optimality relations and solvability of the problem (see Remark 3.2.25 which follows).

Following the general form of (3.1.1) we define the *primal function* $f : \mathbb{X} \to (-\infty, +\infty]$ as

$$f(x) := \begin{cases} J(x), & \text{when } x \in A, \\ +\infty, & \text{otherwise,} \end{cases} \tag{3.2.6}$$

for each $x \in \mathbb{X}$. The *primal problem* is then to determine a $\bar{x} \in \mathbb{X}$ minimizing $f$, i.e.,

$$f(\bar{x}) = \inf_{x \in \mathbb{X}} f(x) \in \mathbb{R}, \tag{3.2.7}$$

where the set membership at (3.2.7) is immediate from (3.2.6) and (3.2.4). We shall now follow the steps outlined in Section 3.2, that is we shall choose a real vector space $\mathbb{U}$ of perturbations, a perturbation function $F(\cdot, \cdot)$, a real vector space $\mathbb{Y}$ of dual variables and a bilinear form on $\mathbb{U} \times \mathbb{Y}$, and then construct a Lagrangian function and a dual function.

1. For Step 3.1.1 there are basically two possible entities in problem (3.2.3) that we can "perturb", namely the objective function $J$ and the constraint set $A$. Numerous perturbations of these entities are of course possible. One of the simplest and most effective is to perturb just the *objective function* $J$ with elements $u$ in a space of perturbations $\mathbb{U}$ defined by

$$\mathbb{U} := \mathbb{R}^n, \tag{3.2.8}$$

and then to define a perturbation function $F : \mathbb{X} \times \mathbb{U} \to (-\infty, +\infty]$ as

$$F(x,u) := \begin{cases} J(x-u), & \text{when } x \in A, \\ +\infty, & \text{otherwise.} \end{cases} \tag{3.2.9}$$

**Remark 3.2.2.** We have borrowed the choice of perturbation at (3.2.9) from Rockafellar [31] and Ekeland and Temam [10]; these works give numerous examples drawn from deterministic convex optimal control and calculus of variations which clearly illustrate the effectiveness of this type of perturbation for dealing with convex constraints on the primal variable $x \in \mathbb{X}$. The equation (3.2.9) represents the very simplest instance of such a perturbation. We shall see later (in Section 4.2 and Section 4.3) how a very similar perturbation of the risk criterion function is also the key for dealing with convex stochastic control problems without any constraints as well as with convex portfolio constraints.

From (3.2.6) and (3.2.9) we have the consistency relation of the form (3.1.2), namely

$$F(x,0) = f(x), \quad x \in \mathbb{X}. \tag{3.2.10}$$

2. For Step 3.1.2, we need to pair the perturbations $\mathbb{U}$ with another appropriate vector space $\mathbb{Y}$ of dual variables through a bilinear form $\langle \cdot, \cdot \rangle$ on $\mathbb{U} \times \mathbb{Y}$. In view of (3.2.8), the only natural choice of pairing is

$$\mathbb{Y} := (\mathbb{U})^* = \mathbb{R}^n, \quad \text{with} \quad \langle u, y \rangle := u'y \quad \text{for } (u,y) \in \mathbb{U} \times \mathbb{Y}. \tag{3.2.11}$$

This completes all the choices required for Steps 3.1.1 - 3.1.2, and it remains to synthesize a Lagrangian and a dual function.

3. For Step 3.1.3, in view of (3.1.3) define the *Lagrangian function* $K : \mathbb{X} \times \mathbb{Y} \to [-\infty, +\infty]$ as follows:

$$K(x,y) := \inf_{u \in \mathbb{U}} \{ \langle u, y \rangle + F(x,u) \}, \qquad x \in \mathbb{X}, \ \ y \in \mathbb{Y}. \tag{3.2.12}$$

We can easily calculate this Lagrangian explicitly as follows. From (3.2.9) and (3.2.11) we obtain

$$
\begin{aligned}
K(x,y) &= \begin{cases} \inf\limits_{u\in\mathbb{R}^n} \{u'y + J(x-u)\}, & \text{when } x \in A, \\ +\infty, & \text{otherwise,} \end{cases} \\
&\overset{v=x-u}{=} \begin{cases} \inf\limits_{v\in\mathbb{R}^n} \{(x-v)'y + J(v)\}, & \text{when } x \in A, \\ +\infty, & \text{otherwise,} \end{cases} \\
&= \begin{cases} x'y + \inf\limits_{v\in\mathbb{R}^n} \{J(v) - v'y\}, & \text{when } x \in A, \\ +\infty, & \text{otherwise,} \end{cases} \\
&= \begin{cases} x'y - \sup\limits_{v\in\mathbb{R}^n} \{v'y - J(v)\}, & \text{when } x \in A, \\ +\infty, & \text{otherwise,} \end{cases} \\
&= \begin{cases} x'y - J^*(y), & \text{when } x \in A, \\ +\infty, & \text{otherwise,} \end{cases} \qquad (3.2.13)
\end{aligned}
$$

for all $x \in \mathbb{X}$ and $y \in \mathbb{Y}$, and where $J^*$ is given by (3.2.2). In view of (3.1.4), we can now define the *dual function* $g : \mathbb{Y} \to [-\infty, +\infty)$ as

$$
\begin{aligned}
g(y) &:= \inf_{x\in\mathbb{R}^n} K(x,y) = \inf_{x\in A}\{x'y\} - J^*(y) \\
&= -\varkappa(y) - J^*(y), \qquad y \in \mathbb{Y}, \qquad (3.2.14)
\end{aligned}
$$

where

$$
\varkappa(y) := \sup_{x\in A}\{-x'y\}, \qquad y \in \mathbb{Y}, \qquad (3.2.15)
$$

**Remark 3.2.3.** The function $\varkappa(\cdot)$ at (3.2.15) is the **support functional** of the convex set $-A$, and is an essential entity in convex optimization. Notice that the support functional as well as the convex conjugate $J^*$ appear very naturally in the preceding calculation. This suggests that the choice of perturbation function at (3.2.9) is indeed an appropriate one.

By the weak duality relation (3.1.5), it follows that

$$
f(x) \geqslant K(x,y) \geqslant g(y), \qquad \text{all } (x,y) \in \mathbb{X} \times \mathbb{Y}, \qquad (3.2.16)
$$

and the *dual problem* is then to maximize $g(y)$ over all $y \in \mathbb{Y}$, i.e., to determine a $\bar{y} \in \mathbb{Y}$ such that

$$
g(\bar{y}) = \sup_{y\in\mathbb{Y}} \{g(y)\}. \qquad (3.2.17)
$$

Existence of a maximizer $\bar{y} \in \mathbb{Y} := \mathbb{R}^n$ at (3.2.17) is not immediately clear. However, it is very easy to establish existence of $\bar{y}$ on the basis of Theorem 3.1.7 as follows. Fix arbitrary $x_1 \in A$, fix some (small) $\epsilon \in (0, \infty)$, put

$$
G := \{u \in \mathbb{U} \ : \ \|u\| < \epsilon\}, \qquad (3.2.18)
$$

and let $\mathcal{U}$ be the usual Euclidean topology on $\mathbb{Y} := \mathbb{R}^n$. Then $G$ is a $\mathcal{U}$-neighbourhood of $0 \in \mathbb{U}$. Since $J : \mathbb{X} := \mathbb{R}^n \to \mathbb{R}$ is assumed to be continuous, it is immediate that

$$\sup_{u \in G} F(x_1, u) = \sup_{u \in G} J(x_1 - u) < +\infty, \tag{3.2.19}$$

where the equality follows from (3.2.9) and $x_1 \in A$. From (3.2.19) and Theorem 3.1.7 we obtain

$$\inf_{x \in \mathbb{X}} f(x) = \sup_{y \in \mathbb{Y}} g(y) = g(\bar{y}) \qquad \text{for some } \bar{y} \in \mathbb{Y}. \tag{3.2.20}$$

We therefore secure existence of a maximizer $\bar{y}$ of the dual function $g(\cdot)$, as well as the fact that the duality gap is zero (see Remark 3.1.4). As noted in Remark 3.1.4, if the duality gap is strictly positive, then one or more of the entities specified in Steps 3.1.1 - 3.1.2 of the variational approach has been inappropriately chosen, and the fact this gap is actually zero suggests that the above choices are correct. Notice that, although we have secured existence of a solution of the dual problem, we do not as yet know anything about the existence of solutions of the primal problem. We address this question next.

From the weak duality at (3.2.16) we of course have

$$f(x) \geqslant g(y), \qquad \text{all } (x, y) \in \mathbb{X} \times \mathbb{Y}, \tag{3.2.21}$$

from which we immediately get the following equivalence: for each $(x, y) \in \mathbb{X} \times \mathbb{Y}$, we have

$$f(x) = g(y) \iff f(x) = \inf_{\tilde{x} \in \mathbb{X}} f(\tilde{x}) = \sup_{\tilde{y} \in \mathbb{Y}} g(\tilde{y}) = g(y). \tag{3.2.22}$$

The equality $f(x) = g(y)$ is therefore equivalent to $x$ being a solution of the primal problem and $y$ being a solution of the dual problem, with zero duality gap. To use the equivalence at (3.2.22) we shall next establish so-called *Kuhn-Tucker optimality relations* which are logically equivalent to the assertion $f(x) = g(y)$ for arbitrary $(x, y) \in \mathbb{X} \times \mathbb{Y}$. These optimality relations typically take the form of feasibility conditions, complementary slackness conditions and transversality conditions on the primal and dual variables (recall Remark 3.1.9 and Remark 3.1.10(c)).

To this end, observe from (3.2.6) and (3.2.14) that, for each $(x, y) \in \mathbb{X} \times \mathbb{Y}$, we have $f(x) \in (-\infty, +\infty]$ and $g(y) \in [-\infty, +\infty)$. Then, by the weak duality relation (3.2.16), we must have

$$f(x) = g(y) \iff f(x) = K(x, y) \in \mathbb{R} \text{ and } g(y) = K(x, y) \in \mathbb{R}, \tag{3.2.23}$$

for each $(x, y) \in \mathbb{X} \times \mathbb{Y}$. We next deal separately with each of the two statements on the right side of the equivalence at (3.2.23). From (3.2.6) and (3.2.13), for each $(x, y) \in \mathbb{X} \times \mathbb{Y}$ we have

$$f(x) = K(x, y) \in \mathbb{R} \iff (1)\ x \in A,\ (2)\ J(x) = x'y - J^*(y). \tag{3.2.24}$$

From elementary convex analysis theory (see (C.4)), there is the following equivalence: for arbitrary $(x, y) \in \mathbb{X} \times \mathbb{Y}$ we have

$$J(x) + J^*(y) = x'y \iff y = \partial J(x), \tag{3.2.25}$$

in which $\partial J(\cdot)$ denotes the derivative function of the objective function $J(\cdot)$. Upon combining (3.2.25) and (3.2.24) we get the equivalence

$$f(x) = K(x, y) \in \mathbb{R} \quad \Longleftrightarrow \quad (1)\ x \in A,\ (2)\ y = \partial J(x), \tag{3.2.26}$$

for each and every $(x, y) \in \mathbb{X} \times \mathbb{Y}$. As for the second statement on the right side of the equivalence at (3.2.23), we see from (3.2.13) and (3.2.14) that

$$g(y) = K(x, y) \in \mathbb{R} \quad \Longleftrightarrow \quad (1)\ x \in A,\ (2)\ x'y + \varkappa(y) = 0, \tag{3.2.27}$$

for each $(x, y) \in \mathbb{X} \times \mathbb{Y}$. Now combine (3.2.23), (3.2.27) and (3.2.24) to obtain

$$f(x) = g(y) \iff (1)\ x \in A,\ (2)\ x'y + \varkappa(y) = 0,\ (3)\ y = \partial J(x), \tag{3.2.28}$$

for each $(x, y) \in \mathbb{X} \times \mathbb{Y}$.

**Remark 3.2.4.** Items (1) - (3) on the right side of the equivalence at (3.2.28) are the *Kuhn-Tucker optimality conditions* (recall Remark 3.1.9, Remark 3.1.10(c), and in particular the equivalence (3.1.12)). The condition (3.2.28)(1) is an obvious *feasibility condition* on the primal variable $x$, and simply means that the constraint in the primal problem must be satisfied. On the other hand, the condition (3.2.28)(2) can be stated as follows

$$x'y + \sup_{x \in A}\{-x'y\} = 0,$$

(recall (3.2.15)), which is a relation between the primal variable $x$, the dual variable $y$ and the constraint set $A$. Such relations between the primal and dual variables and the constraints are termed *complementary slackness conditions*. Finally, (3.2.28)(3) is a relation between the primal variable $x$, the dual variable $y$, and the *objective function* $J(\cdot)$. Such relations are termed *transversality conditions*.

**Remark 3.2.5.** In view of the equivalence at (3.2.28), if we can somehow construct a pair $(x, y) \in \mathbb{X} \times \mathbb{Y}$ which satisfies the optimality relations (3.2.28)(1)-(3), then we have $f(x) = g(y)$, and it follows from (3.2.22) that $x$ is a solution of the primal problem (and of course $y$ is a solution of the dual problem). It is at this point that we can use existence of a solution $\bar{y}$ of the dual problem that we have already established at (3.2.20), namely

$$g(y) \leqslant g(\bar{y}), \qquad \text{for all } y \in \mathbb{Y}. \tag{3.2.29}$$

This suggests the possibility of extracting *necessary conditions* resulting from the optimality of $\bar{y}$ at (3.2.29), and then using these necessary conditions to *construct* some $\bar{x}$ in terms of $\bar{y}$ such that the pair $(\bar{x}, \bar{y})$ satisfies the optimality relations (3.2.28)(1)-(3), that is satisfy

$$(1)\ \bar{x} \in A,\ (2)\ \bar{x}'\bar{y} + \varkappa(\bar{y}) = 0,\ (3)\ \bar{y} = \partial J(\bar{x}), \tag{3.2.30}$$

for then we have $f(\bar{x}) = g(\bar{y})$ from (3.2.28), and therefore $\bar{x}$ indeed solves the primal problem. If the objective function $J$ is such that the conjugate transform $J^*$ is $\mathbb{R}$-valued and smooth on $\mathbb{R}^n$ (certainly this is the case when $J$ is a strictly positive definite quadratic function on $\mathbb{R}^n$) then (3.2.30)(3) is equivalent to

$$\bar{x} = \partial J^*(\bar{y}), \tag{3.2.31}$$

(see (C.5)). Now we can use (3.2.31) to actually *define* $\bar{x}$ in terms of the optimal dual solution $\bar{y}$, and then it is enough to use necessary conditions from the optimality of $\bar{y}$ at (3.2.29) to verify that (3.2.30)(1)(2) hold. We shall not, however, work through this verification in the present simple example, since we are going to encounter exactly the same sort of construction in considerable detail later on, in Chapter 4, in the technically more challenging setting of problem (2.2.9), which is a "stochastic control analogue" of problem (3.2.3).

**Remark 3.2.6.** Later we shall see that the very simple approach established above for the elementary Problem 3.2.1 (i.e. (3.2.3)) actually generalizes very directly to genuine stochastic control problems, in particular the unconstrained quadratic minimization problem addressed by Lim and Zhou [20] (see Section 4.2), as well as the quadratic minimization problem with convex portfolio constraints addressed by Labbé and Heunis [18] (see Section 4.3).

Several of the problems outlined in Chapter 2 involve constraints on the *wealth process*, of either the European or American type, as well as possible portfolio constraints (see in particular problems (2.2.14), (2.2.16), and (2.2.21)). In the parlance of stochastic control theory these constraints on the wealth process constitute *state constraints*, in contract to constraints on the portfolio process, which constitute *control constraints*. We are going to see in Chapter 4 that control constraints can be dealt with by perturbation functions which are formally a direct generalization of the perturbation function at (3.2.9) for the simple problem at (3.2.3). On the other hand, state constraints require a very different kind of perturbation function. Although general state constraints present a definite challenge, the state constraints in the problems (2.2.14), (2.2.16) and (2.2.21), nevertheless have one very nice feature, namely they are all simple *inequality constraints* in which a lower bound is stipulated for the wealth process. We can get a feel for these inequality constraints by looking at a particular case of Problem 3.2.1 in which the convex constraint set $A$ has the structured form of an inequality on the primal variable $x$ of the form

$$A = \{x \in \mathbb{X} := \mathbb{R}^n \mid x \geqslant b\}, \tag{3.2.32}$$

for some $b \in \mathbb{R}^n$. We shall establish a perturbation function which takes advantage of the special inequality structure at (3.2.32) which will serve us well when we come to stochastic control problems with inequality constraints on the wealth process. We therefore look at the following problem with an objective function $J$ given by (3.2.1) and a real vector space of primal variables given by

$$\mathbb{X} := \mathbb{R}^n. \tag{3.2.33}$$

**Problem 3.2.7.**

$$\text{minimize } J(x) \text{ subject to } x \geqslant b \quad \text{for some fixed vector } b \in \mathbb{X} = \mathbb{R}^n, \tag{3.2.34}$$

where $x \geqslant b$ in $\mathbb{X}$ means the $i$-th entry $x_i \geqslant b_i$ in $\mathbb{R}$ for all $i = 1, 2, ..., n$. We shall suppose

$$\inf_{x \geqslant b} J(x) > -\infty. \tag{3.2.35}$$

**Remark 3.2.8.** Of course, the problem (3.2.34) is nothing but problem (3.2.3) in which the constraint set has the specially simple form at (3.2.32). Our intent is to use the Rockafellar variational approach to see how the dual function and Kuhn-Tucker optimality relations simplify with this simpler constraint set. This will serve us well when we later address European and American-type wealth constraints.

Similar to (3.2.6), we first define a *primal function* $f : \mathbb{X} \to (-\infty, +\infty]$ as

$$f(x) := \begin{cases} J(x), & \text{when } x \geqslant b, \\ +\infty, & \text{otherwise}, \end{cases} \tag{3.2.36}$$

for each $x \in \mathbb{X}$. The *primal problem* is then to determine a $\bar{x} \in \mathbb{X}$ minimizing $f$, i.e.,

$$f(\bar{x}) = \inf_{x \in \mathbb{X}} f(x) \in \mathbb{R}, \tag{3.2.37}$$

where the set-membership at (3.2.37) is immediate from (3.2.36) and (3.2.35). Following the steps outlined in Section 3.2, we next choose a real vector space $\mathbb{U}$ of perturbations, a perturbation function $F(\cdot, \cdot)$, a real vector space $\mathbb{Y}$ of dual variables and a bilinear form on $\mathbb{U} \times \mathbb{Y}$, and then construct a Lagrangian function and a dual function.

1. For Step 3.1.1 we shall choose a perturbation different from that adopted in (3.2.9), even though this perturbation would work perfectly well for the present problem with $A$ defined by (3.2.32). The perturbation we choose here takes advantage of the fact that the convex constraint set is particularly simple, being defined only by inequalities. We perturb the constraint $x \geqslant b$ with elements $v$ in the space of perturbation $\mathbb{U}$ defined by

$$\mathbb{U} := \mathbb{R}^n, \tag{3.2.38}$$

and the perturbation function $F : \mathbb{X} \times \mathbb{U} \to (-\infty, +\infty]$ is now defined as

$$F(x, v) := \begin{cases} J(x), & \text{when } x + v \geqslant b, \\ +\infty, & \text{otherwise}. \end{cases} \tag{3.2.39}$$

**Remark 3.2.9.** The perturbation at (3.2.39) should be compared with the perturbation at (3.2.9)), in which only the objective function was perturbed. Here, in contrast, we perturb only the inequality constraint.

From (3.2.36) and (3.2.39) we have the consistency relation of the form (3.1.2):

$$F(x, 0) = f(x), \quad x \in \mathbb{X}. \tag{3.2.40}$$

2. For Step 3.1.2, we follow (3.2.11) and pair the space of perturbations $\mathbb{U}$ with the same vector space $\mathbb{Y}$ of dual variables through a bilinear form $\langle \cdot, \cdot \rangle$ on $\mathbb{U} \times \mathbb{Y}$ as

$$\mathbb{Y} := (\mathbb{U})^* = \mathbb{R}^n, \quad \text{with} \quad \langle v, y \rangle := v'y \quad \text{for } (v, y) \in \mathbb{U} \times \mathbb{Y}. \tag{3.2.41}$$

It then remains to synthesize a Lagrangian and a dual function corresponding to the perturbation (3.2.39) and the duality pairing at (3.2.41).

3. For Step 3.1.3, in view of (3.1.3) define the *Lagrangian function* $K : \mathbb{X} \times \mathbb{Y} \to [-\infty, +\infty]$ as:

$$K(x,y) := \inf_{v \in \mathbb{U}} \{\langle v, y\rangle + F(x,v)\}, \qquad x \in \mathbb{X}, \ \ y \in \mathbb{Y}. \tag{3.2.42}$$

From (3.2.39) and (3.2.41) we obtain

$$\begin{aligned} K(x,y) &= \inf_{v \in \mathbb{U}} \{v'y + J(x) \mid x + v \geqslant b\}, \\ &= \inf_{v \geqslant b - x} \{v'y\} + J(x), \\ &= \begin{cases} (b-x)'y + J(x), & \text{when } y \geqslant 0, \\ -\infty, & \text{otherwise,} \end{cases} \end{aligned} \tag{3.2.43}$$

for all $x \in \mathbb{X}$ and $y \in \mathbb{Y}$.

**Remark 3.2.10.** The function at (3.2.43) is the Lagrangian for the perturbation at (3.2.39). Observe the "Lagrange weighting" $(b-x)'y$ for the constraint $x \geqslant b$ that is present on the right side of (3.2.43) (for $y \geqslant 0$). This term is of course familiar from elementary optimization.

In view of (3.1.4), define the *dual function* $g : \mathbb{Y} \to [-\infty, +\infty)$ as

$$\begin{aligned} g(y) &:= \inf_{x \in \mathbb{X}} K(x,y) \overset{(3.2.43)}{=} \begin{cases} b'y - \sup_{x \in \mathbb{R}^n} \{x'y - J(x)\}, & \text{when } y \geqslant 0, \\ -\infty, & \text{otherwise,} \end{cases} \\ &= \begin{cases} b'y - J^*(y), & \text{when } y \geqslant 0, \\ -\infty, & \text{otherwise,} \end{cases} \end{aligned} \tag{3.2.44}$$

for all $y \in \mathbb{Y}$, and where $J^*$ is given by (3.2.2).

By the weak duality relation (3.1.5), it follows that

$$f(x) \geqslant K(x,y) \geqslant g(y), \qquad (x,y) \in \mathbb{X} \times \mathbb{Y}, \tag{3.2.45}$$

and the *dual problem* is then to maximize $g(y)$ over all $y \in \mathbb{Y}$. We shall now use Theorem 3.1.7 to establish

$$\inf_{x \in \mathbb{X}} f(x) = \sup_{y \in \mathbb{Y}} g(y) = g(\bar{y}) \qquad \text{for some } \ \bar{y} \in \mathbb{Y}. \tag{3.2.46}$$

Existence of a maximizer $\bar{y} \in \mathbb{Y} := \mathbb{R}^n$ at (3.2.46) can be easily established with Theorem 3.1.7 as follows. Fix arbitrary $\hat{x} \in \mathbb{X} = \mathbb{R}^n$ such that $\hat{x} > b$, and then define

$$\varepsilon := \min\{\hat{x}_i - b_i \mid i = 1, 2, ..., n\} \in (0, \infty). \tag{3.2.47}$$

Put

$$G := \{v \in \mathbb{U} \ : \ \|v\| < \varepsilon\}, \tag{3.2.48}$$

where $\|\cdot\|$ indicates the Euclidean norm on $\mathbb{U}$. From (3.2.47) and (3.2.48), it follows that

$$\hat{x} + v \geqslant b, \qquad \text{for all } v \in G. \tag{3.2.49}$$

Let $\mathcal{U}$ be the usual Euclidean topology on $\mathbb{Y} := \mathbb{R}^n$. Then $G$ is a $\mathcal{U}$-neighbourhood of $0 \in \mathbb{U}$. Since $J : \mathbb{X} := \mathbb{R}^n \to \mathbb{R}$, it is immediate from (3.2.39) and (3.2.49) that

$$\sup_{v \in G} F(\hat{x}, v) = J(\hat{x}) < +\infty. \tag{3.2.50}$$

From (3.2.50) and Theorem 3.1.7 we obtain

$$\inf_{x \in \mathbb{X}} f(x) = \sup_{y \in \mathbb{Y}} g(y) = g(\bar{y}) \qquad \text{for some } \bar{y} \in \mathbb{Y}. \tag{3.2.51}$$

We therefore secure existence of a maximizer $\bar{y}$ of the dual function $g(\cdot)$, as well as the fact that the duality gap is zero (see Remark 3.1.4).

We next establish the Kuhn-Tucker optimality conditions, which are equivalent to the duality relation $f(x) = g(y)$ (for general $(x, y) \in \mathbb{X} \times \mathbb{Y}$). To this end, observe from (3.2.36) and (3.2.44) that, for each $(x, y) \in \mathbb{X} \times \mathbb{Y}$, we have $f(x) \in (-\infty, +\infty]$ and $g(y) \in [-\infty, +\infty)$, so that, by the weak duality relation (3.2.45), we must have

$$f(x) = g(y) \iff f(x) = K(x, y) \in \mathbb{R} \text{ and } g(y) = K(x, y) \in \mathbb{R}, \tag{3.2.52}$$

for each $(x, y) \in \mathbb{X} \times \mathbb{Y}$. By (3.2.36) and (3.2.42) we have

$$f(x) = K(x, y) \in \mathbb{R}, \iff (1)\ x \geqslant b,\ (2)\ y \geqslant 0,\ (3)\ (b - x)'y = 0; \tag{3.2.53}$$

and from (3.2.42) and (3.2.44),

$$\begin{aligned} g(y) = K(x, y) \in \mathbb{R} \quad &\iff \quad (1)\ y \geqslant 0,\ (2)\ J(x) = x'y - J^*(y); \\ &\iff \quad (1)\ y \geqslant 0,\ (2)\ y = \partial J(x), \end{aligned} \tag{3.2.54}$$

where the second equivalence at (3.2.54) follows from (C.4). Combine (3.2.52), (3.2.54) and (3.2.53) to obtain the Kuhn-Tucker optimality relations

$$f(x) = g(y) \iff (1) x \geqslant b,\ (2)\ y \geqslant 0,\ (3)\ (b - x)'y = 0,\ (4)\ y = \partial J(x). \tag{3.2.55}$$

for each $(x, y) \in \mathbb{X} \times \mathbb{Y}$.

**Remark 3.2.11.** Items (1) - (4) of (3.2.55) are Kuhn-Tucker optimality conditions. In particular (3.2.55) - (1) is the *feasibility condition* on the primal variable $x$ and (3.2.55) - (2) is the *feasibility condition* on the dual variable $y$. The relation (3.2.55) - (3) is a *complementary slackness condition* relating the primal variable $x$ and the dual variable $y$. Effectively, this condition dictates that, at optimality, the "Lagrange weighting" $(b - x)'y$ on the right side of the Lagrangian at (3.2.43) (when $y \geqslant 0$) must be zero (see Remark 3.2.10). Finally, (3.2.55) - (4) is a *transversality condition*, which relates the optimal primal and dual values $x$ and $y$ to the *objective function* $J(\cdot)$.

**Remark 3.2.12.** The optimality relation (3.2.55)(3) between the primal solution $x$ and the corresponding Lagrange multiplier $y$ can of course be equivalently stated as

$$(b_i - x_i)y_i = 0, \qquad \text{for all } i = 1, 2, \ldots, n, \tag{3.2.56}$$

as is immediate from (3.2.55)(1)(2) which together imply that

$$(b_i - x_i)y_i \leqslant 0, \qquad \text{for all } i = 1, 2, \ldots, n. \tag{3.2.57}$$

**Remark 3.2.13.** We could have obtained the Lagrangian at (3.2.43), the dual function at (3.2.44) and the optimality relations at (3.2.55), just by appropriately simplifying the Lagrangian at (3.2.13), the dual function at (3.2.14) and the optimality relations at (3.2.28), taking into account the special structure of the constraint set at (3.2.32). The reason that we have chosen instead to work through all the details is that in later problems of stochastic convex optimization it will be essential to take advantage of the special structure of inequality constraints by using an appropriate generalization of the sort of perturbation that we used at (3.2.39).

**Remark 3.2.14.** From (3.2.35) and (3.2.36) we have

$$\inf_{x \in \mathbb{X}} f(x) \in \mathbb{R}, \tag{3.2.58}$$

and then it follows from (3.2.58), together with (3.2.46), that

$$g(\bar{y}) \in \mathbb{R}. \tag{3.2.59}$$

In view of (3.2.59) and (3.2.44) we get the following feasibility condition on the dual variable $\bar{y}$:

$$\bar{y} \geqslant 0, \tag{3.2.60}$$

that is the Kuhn-Tucker relation (3.2.55)(2) is satisfied. It remains to construct some $\bar{x} \in \mathbb{X}$ in terms of $\bar{y}$ such that the Kuhn-Tucker relations (3.2.55)(1)(3)(4) hold, that is

$$(1)\ \bar{x} \geqslant b, \qquad (2)\ (b - \bar{x})'\bar{y} = 0, \quad (3)\ \bar{y} = \partial J(\bar{x}), \tag{3.2.61}$$

for then we see from (3.2.61), (3.2.60) and the equivalence at (3.2.55), that

$$f(\bar{x}) = g(\bar{y}), \tag{3.2.62}$$

as required to establish the optimality of $\bar{x}$ (in view of the weak duality at (3.2.45)). Exactly as at Remark 3.2.5, if the convex conjugate $J^*$ is $\mathbb{R}$-valued and smooth on $\mathbb{R}^n$, then (3.2.61)(3) is equivalent to

$$\bar{x} = \partial J^*(\bar{y}). \tag{3.2.63}$$

Now we can use the relation (3.2.63) to *define* $\bar{x}$ in terms of $\bar{y}$, and then use necessary conditions resulting from the optimality of $\bar{y}$ (recall (3.2.46)) to verify (3.2.61)(1)(2). As at Remark 3.2.5 we shall not give the details here, since these are quite straightforward for the simple finite-dimensional problem (3.2.34), and we shall later see in Chapter 4 the same type of construction in the technically more demanding context of problem (2.2.14), which is a "stochastic control analogue" of problem (3.2.34).

We next look at the case where we have a *combination* of the constraints in Problem 3.2.7 and Problem 3.2.3, namely

**Problem 3.2.15.**

$$\text{minimize } J(x) \text{ subject to } x \in C \text{ and } x \geqslant b, \tag{3.2.64}$$

for some vector $b \in \mathbb{R}^n$ and convex set $C \subset \mathbb{R}^n$. Here the *objective function* $J : \mathbb{R}^n \to \mathbb{R}$ is some smooth convex function (recall (3.2.1)) and inequality $\geqslant$ are the same as in Problem 3.2.7, and we shall suppose that

$$\inf_{x \in C,\ x \geqslant b} J(x) > -\infty. \tag{3.2.65}$$

Of course, for Problem 3.2.15 to even make sense the set

$$A_1 := C \cap \{x \in \mathbb{R}^n \mid x \geqslant b\}. \tag{3.2.66}$$

must be non-empty, for otherwise the problem has no solution! We therefore assume a *feasibility condition* as follows:

**Condition 3.2.16.** The set $A_1$ is non-empty.

In view of (3.2.65) and Condition 3.2.16 we see that

$$\inf_{x \in C,\ x \geqslant b} J(x) \in \mathbb{R}. \tag{3.2.67}$$

Observe that Problem 3.2.15 just amounts to Problem 3.2.3 in which the convex constraint set $A$ is replaced with the convex constraint set $A_1$ at (3.2.66), so we could just adapt the Lagrangian, dual function and optimality relations obtained for Problem 3.2.3 to the "compound" constraint set $A_1$. However, we are not going to take advantage of this, and are instead going to introduce *separate* perturbations for the constraints $x \in C$ and $x \geqslant b$ along the lines used in Problem 3.2.7 and Problem 3.2.3. The reason for this is that, when dealing with *infinite dimensional* problems of convex optimization (such as the Canonical Problem 2.2.7) we shall *not* be able to unify or combine constraints along the lines of (3.2.66) and will be compelled to introduce separate perturbations. We are in fact going to see that Problem 3.2.15 is a sort of miniature template of the Canonical Problem 2.2.7, and that the approach we adopt for the Canonical Problem 2.2.7 will be a direct, although technically more involved, generalization of the approach that we shall now illustrate for Problem 3.2.15.

The real vector space of primal variables for Problem 3.2.15 is of course

$$\mathbb{X} := \mathbb{R}^n. \tag{3.2.68}$$

To implement Rockafellar's variational approach, we define the *primal function* $f : \mathbb{R}^n \to (-\infty, +\infty]$ following the general form of (3.1.1), namely

$$f(x) := \begin{cases} J(x), & \text{when } x \in C \text{ and } x \geqslant b, \\ +\infty, & \text{otherwise,} \end{cases} \tag{3.2.69}$$

for each $x \in \mathbb{X}$. The *primal problem* is then to determine a $\bar{x} \in \mathbb{X}$ minimizing $f$, i.e.,

$$f(\bar{x}) = \inf_{x \in \mathbb{X}} f(x) \in \mathbb{R}, \tag{3.2.70}$$

where the set-membership at (3.2.70) is immediate from (3.2.69) and (3.2.67). Following the steps outlined in Section 3.2, we shall choose a real vector space $\mathbb{U}$ of perturbations, a perturbation function $F(\cdot,\cdot)$, a real vector space $\mathbb{Y}$ of dual variables and a bilinear form on $\mathbb{U} \times \mathbb{Y}$, and then construct a Lagrangian function and a dual function.

1. For Step 3.1.1, we combine the perturbation variables $u$ given in (3.2.9) for Problem 3.2.1 and $v$ given in (3.2.39) for Problem 3.2.7. Define a vector space $\mathbb{U}$ of perturbations as

   $$\mathbb{U} := \mathbb{R}^n \times \mathbb{R}^n, \tag{3.2.71}$$

   with generic member $(u, v) \in \mathbb{U}$ for $u \in \mathbb{R}^n$ and $v \in \mathbb{R}^n$, and define a perturbation function $F : \mathbb{X} \times \mathbb{U} \to (-\infty, +\infty)$ by

   $$F(x, (u, v)) := \begin{cases} J(x-u), & \text{when } x \in C \text{ and } x + v \geqslant b, \\ +\infty, & \text{otherwise,} \end{cases} \tag{3.2.72}$$

   for all $(x, (u, v)) \in \mathbb{X} \times \mathbb{U}$.

   **Remark 3.2.17.** Notice how the perturbations at (3.2.72) are just a natural combination of the perturbations at (3.2.9) (for the convex constraint set $A$ in Problem 3.2.1) and the perturbations at (3.2.39) (for the inequality constraint $x \geqslant b$ in Problem 3.2.7).

   From (3.2.69) and (3.2.72) we have the consistency relation of the form (3.1.2), that is

   $$F(x, 0) = f(x), \quad x \in \mathbb{X}. \tag{3.2.73}$$

2. For Step 3.1.2, we need to pair the perturbations $\mathbb{U}$ with another appropriate vector space $\mathbb{Y}$ of dual variables through a bilinear form $\langle \cdot, \cdot \rangle$ on $\mathbb{U} \times \mathbb{Y}$. Similar to (3.2.11), (3.2.41), and in view of (3.2.71), the only natural choice of pairing is

   $$\mathbb{Y} := (\mathbb{U})^* = \mathbb{R}^n \times \mathbb{R}^n, \tag{3.2.74}$$

   with

   $$\langle (u, v), (y, z) \rangle := u'y + v'z \qquad \text{for} \quad (u, v) \in \mathbb{U},\ (y, z) \in \mathbb{Y}. \tag{3.2.75}$$

3. For Step 3.1.3, in view of (3.1.3) define the *Lagrangian* $K : \mathbb{X} \times \mathbb{Y} \to [-\infty, +\infty]$ as follows:

   $$K(x, (y, z)) := \inf_{(u,v) \in \mathbb{U}} \{\langle (u, v), (y, z) \rangle + F(x, (u, v))\}, \quad x \in \mathbb{X}, (y, z) \in \mathbb{Y}. \tag{3.2.76}$$

We next evaluate this Lagrangian explicitly. From (3.2.76), (3.2.75) and (3.2.72)

$$K(x,(y,z)) = +\infty, \qquad x \notin C, \ \ (y,z) \in \mathbb{Y}, \tag{3.2.77}$$

and

$$K(x,(y,z)) = \inf_{u\in\mathbb{R}^n} \{u'y + J(x-u)\} + \inf_{v\in\mathbb{R}^n} \{v'z \mid x+v \geqslant b\}, \ \ x \in C, \ \ (y,z)\in\mathbb{Y}, \tag{3.2.78}$$

that is, from (3.2.78) and (3.2.77),

$$K(x,(y,z)) = \begin{cases} \inf_{u\in\mathbb{R}^n} \{u'y + J(x-u)\} + \inf_{v\in\mathbb{R}^n} \{v'z \mid x+v \geqslant b\}, & x \in C, \\ +\infty, & x \notin C, \end{cases} \tag{3.2.79}$$

for all $(x,(y,z)) \in \mathbb{X}\times\mathbb{Y}$. We now evaluate the right side of (3.2.79) explicitly. Fix any $x \in C$ and observe that

$$\begin{aligned} & \inf_{u\in\mathbb{R}^n} \{u'y + J(x-u)\} \\ = \ & x'y - \sup_{u\in\mathbb{R}^n} \{(x-u)'y - J(x-u)\} \\ \overset{v=x-u}{=} \ & x'y - \sup_{v\in\mathbb{R}^n} \{v'y - J(v)\} \\ \overset{(3.2.2)}{=} \ & x'y - J^*(y), \end{aligned} \tag{3.2.80}$$

and

$$\inf_{v\in\mathbb{R}^n} \{v'z \mid x+v \geqslant b\} = \begin{cases} z'(b-x), & z \geqslant 0, \\ -\infty, & z \ngeqslant 0. \end{cases} \tag{3.2.81}$$

Combining (3.2.79), (3.2.80) and (3.2.81), we have

$$K(x,(y,z)) \overset{(3.2.13)}{=} \begin{cases} x'y - J^*(y) + z'(b-x), & x\in C \text{ and } z \geqslant 0, \\ -\infty, & x\in C \text{ and } z \ngeqslant 0, \\ +\infty, & x \notin C, \end{cases} \tag{3.2.82}$$

for all $x \in \mathbb{X}$ and $(y,z)\in\mathbb{Y}$ (recall that $J^*$ is defined by (3.2.2)). In view of (3.1.4), the *dual function* $g : \mathbb{Y} \to [-\infty, +\infty)$ is defined as

$$g(y,z) := \inf_{x\in\mathbb{R}^n} K(x,(y,z)) = \begin{cases} -\varkappa(y,z) - J^*(y), & z \geqslant 0, \\ -\infty, & \text{otherwise}, \end{cases} \tag{3.2.83}$$

for all $(y,z)\in\mathbb{Y}$, in which we define

$$\varkappa(y,z) := \sup_{x\in C} \{-x'y + z'(x-b))\}, \quad (y,z) \in \mathbb{Y} := \mathbb{R}^n \times \mathbb{R}^n, \tag{3.2.84}$$

(the equality on the right of (3.2.83) is immediate from (3.2.82)).

By the weak duality relation (3.1.5), it follows that

$$f(x) \geqslant K(x,(y,z)) \geqslant g(y,z), \tag{3.2.85}$$

and the *dual problem* is then to maximize $g(y,z)$ over all $(y,z) \in \mathbb{Y} := \mathbb{R}^n \times \mathbb{R}^n$, i.e. to establish that

$$g(\bar{y},\bar{z}) = \sup_{(y,z)\in\mathbb{Y}} \{g(y,z)\} \qquad \text{for some } (\bar{y},\bar{z}) \in \mathbb{Y}. \tag{3.2.86}$$

In order to use Theorem 3.1.7 to secure existence of an optimizer $(\bar{y},\bar{z}) \in \mathbb{Y} := \mathbb{R}^n \times \mathbb{R}^n$ we will need to strengthen the feasibility Condition 3.2.16 to the following *Slater condition:*

**Condition 3.2.18.** The set

$$A_2 := \{x \in C \mid x_i \geqslant b_i + \varepsilon \text{ for all } i = 1,2,\ldots,n\} \tag{3.2.87}$$

is nonempty for some $\varepsilon \in (0,\infty)$.

**Remark 3.2.19.** Notice that we have strengthened Condition 3.2.16 to the "Slater" Condition 3.2.18 by "working" some (small) $\varepsilon \in (0,\infty)$ into the inequality constraints $x \geqslant b$. This is the essence of all Slater-type conditions, namely the persistence of feasibility under a slight strengthening of an inequality constraint $x \geqslant b$ to $x_i \geqslant b_i + \varepsilon$, $i = 1,...,n$, for some (small) $\varepsilon \in (0,\infty)$, and is what secures existence of a solution of the corresponding dual problem. We develop this idea next for the very simple Problem 3.2.15, but this Slater-type condition will be a recurring theme throughout this thesis. Indeed, this type of condition is essential for verifying the condition (3.1.9) of Theorem 3.1.7 when we have the combination of constraints $x \in C$ and $x \geqslant b$ at problem (3.2.64). Later, in Chapter 4 and Chapter 5, the constraint $x \in C$ will be generalized to a convex portfolio constraint while the constraint $x \geqslant b$ will be generalized to a European or American wealth constraint, and we will end up dealing with a natural generalization of the Slater Condition 3.2.18, again order to verify the conditions of Theorem 3.1.7.

In view of Condition 3.2.18 that $A_2$ is nonempty for some $\varepsilon \in (0,\infty)$, we can fix some $\hat{x} \in A_2$. Let $\mathcal{U}$ be the usual Euclidean topology on $\mathbb{Y} := \mathbb{R}^n$ and $\|\cdot\|$ indicate the Euclidean norm on $\mathbb{U}$, then

$$\{(u,v) \in \mathbb{U} \mid \|(u,v)\| \leqslant \varepsilon\}, \tag{3.2.88}$$

is a $\mathcal{U}$-neighbourhood of $0 \in \mathbb{U}$. Since $J(\cdot)$ is continuous on $\mathbb{R}^n$ it follows from (3.2.87) and (3.2.72) that condition (3.1.9) holds:

$$\sup_{\|(u,v)\|\leqslant\varepsilon} F(\hat{x},(u,v)) = \sup_{\|(u,v)\|\leqslant\varepsilon} J(\hat{x}-u) < +\infty. \tag{3.2.89}$$

Existence of a maximizer $(\bar{y},\bar{z}) \in \mathbb{Y}$ which satisfies (3.2.86) is now an immediate consequence of Theorem 3.1.7, that is one has

$$\inf_{x\in\mathbb{X}} f(x) = \sup_{(y,z)\in\mathbb{Y}} g(y,z) = g(\bar{y},\bar{z}) \qquad \text{for some } (\bar{y},\bar{z}) \in \mathbb{Y}. \tag{3.2.90}$$

The weak duality relation (3.2.85) suggests that we can also establish the Kuhn-Tucker optimality conditions, which are equivalent to the stipulated equality $f(x) = g(y, z)$ (for general $(x, (y, z)) \in \mathbb{X} \times \mathbb{Y}$). We establish these in the following result the proof of which is in Appendix A:

**Proposition 3.2.20.** *For each $(x, (y, z)) \in \mathbb{X} \times \mathbb{Y}$, we have*

$$f(x) = g(y, z) \iff \begin{cases} (1)\ x \in C,\ (2)\ x \geqslant b,\ (3)\ z \geqslant 0, \\ (4)\ z'(b - x) = 0,\ (5)\ x'y + \varkappa(y, z) = 0, \\ (6)\ y = \partial J(x). \end{cases} \tag{3.2.91}$$

*Here $f(\cdot)$ is defined on $\mathbb{X}$ by (3.2.69) and $\varkappa(\cdot)$ and $g(\cdot)$ are defined on $\mathbb{Y}$ by (3.2.84) and (3.2.83).*

**Remark 3.2.21.** Suppose that the conjugate transform $J^*(\cdot)$ is $\mathbb{R}$-valued and smooth on $\mathbb{R}^n$. Then the condition (3.2.91)(6) is equivalent to the condition $x = \partial J^*(y)$ (in view of (C.5)), and in this case Proposition 3.2.20 takes the following more convenient form: For each $(x, (y, z)) \in \mathbb{X} \times \mathbb{Y}$, we have

$$f(x) = g(y, z) \iff \begin{cases} (1)\ x \in C,\ (2)\ x \geqslant b,\ (3)\ z \geqslant 0, \\ (4)\ z'(b - x) = 0,\ (5)\ x'y + \varkappa(y, z) = 0, \\ (6)\ x = \partial J^*(y). \end{cases} \tag{3.2.92}$$

**Remark 3.2.22.** The proof of the equivalence at (3.2.91) is a simple generalization of the arguments used for Problem 3.2.1 and Problem 3.2.7. For this reason, we relegate the proof to Appendix A.

**Remark 3.2.23.** In view of the set-membership at (3.2.70), together with (3.2.90), we obtain

$$g(\bar{y}, \bar{z}) \in \mathbb{R}, \tag{3.2.93}$$

and then it follows from (3.2.93) and (3.2.83) that

$$\bar{z} \geqslant 0. \tag{3.2.94}$$

Now suppose that the convex conjugate $J^*(\cdot)$ is $\mathbb{R}$-valued and smooth on $\mathbb{R}^n$. We then have the equivalence at (3.2.92), and, motivated by (3.2.92)(6), we define $\bar{x}$ in terms of the first member $\bar{y}$ of the dual solution $(\bar{y}, \bar{z})$, that is

$$\bar{x} := \partial J^*(\bar{y}). \tag{3.2.95}$$

It then remains to verify that $\bar{x}$ and $(\bar{y}, \bar{z})$ satisfy the relations

$$(1)\ \bar{x} \in C,\quad (2)\ \bar{x} \geqslant b,\quad (3)\ \bar{z}'(b - \bar{x}) = 0,\quad (4)\ \bar{x}'\bar{y} + \varkappa(\bar{y}, \bar{z}) = 0, \tag{3.2.96}$$

since it follows from (3.2.96), (3.2.95) and (3.2.94), together with the equivalence at (3.2.92) that

$$f(\bar{x}) = g(\bar{y}, \bar{z}), \tag{3.2.97}$$

as required to establish optimality of $\bar{x}$ for problem (3.2.64) (in view of the weak duality (3.2.85)). In order to verify (3.2.96) we need to use necessary conditions resulting from the optimality of $(\bar{y}, \bar{z})$ at (3.2.90). We shall not give the details here, since these are quite straightforward for the simple finite-dimensional problem (3.2.64), and we shall later see in Chapter 4 the full details of this construction for the technically more demanding problem (4.5.2), which is a "stochastic control analogue" of problem (3.2.64).

**Remark 3.2.24.** The optimality relations (3.2.91) (1)-(6) have a definite structure which is characteristic of all Kuhn-Tucker conditions. In particular, (3.2.91) (1) - (2) are *feasibility conditions* on the primal variable $x$, (3.2.91) (3) is a *feasibility condition* on the dual variable $z$ (the Lagrange multiplier for the inequality constraint $x \geqslant b$). On the other hand, (3.2.91) (4) is a *complementary slackness condition* for the inequality constraint, which relates the primal variable $x$, the dual variable $z$, and the inequality constraint. Similarly, (3.2.91) (5) is another *complementary slackness condition* which again relates the primal variable $x$, the dual variables $(y, z)$, and the constraint $x \in C$. Finally, (3.2.91) (6) is a *transversality condition* which relates the primal variable $x$, the dual variable $y$, and the objective function $J(\cdot)$ in problem (3.2.64). Optimality relations with a structure analogous to the relations (3.2.91) (1) - (6) and (3.2.92) (1) - (6) will be seen again later in the thesis in regard to the stochastic control problem (4.5.2) (see Remark 4.5.20), and in regard to the canonical problem (2.2.21) (see Remark 5.2.23).

**Remark 3.2.25.** There is no guarantee that a given convex optimization problem necessarily has a solution. For example, take

$$\mathbb{X} := \mathbb{R}, \qquad J(x) := e^{-x}, \qquad x \in \mathbb{X}, \qquad b := 0, \tag{3.2.98}$$

in problem (3.2.34). We cannot minimize $e^{-x}$ subject to $x \geqslant 0$, so that we have a convex optimization problem without any solution. In such problems it must always be the case that the Kuhn-Tucker optimality relations are *mutually inconsistent*, that is there fails to exist any pair of primal and dual variables which satisfy the Kuhn-Tucker relations. Thus, for problem (3.2.34) in the particular case of (3.2.98), we get the following concrete Kuhn-Tucker relations from (3.2.55):

$$f(x) = g(y) \iff (1) x \geqslant 0, \ (2)\ y \geqslant 0, \ (3)\ xy = 0, \ (4)\ y = -e^{-x}. \tag{3.2.99}$$

for each $(x, y) \in \mathbb{X} \times \mathbb{Y} = \mathbb{R} \times \mathbb{R}$. From (3.2.99)(3)(4) one sees that $x$ must satisfy

$$xe^{-x} = 0. \tag{3.2.100}$$

The unique root of (3.2.100) is $x = 0$, and then (3.2.99)(4) gives $y = -1$. This however contradicts (3.2.99)(2). It follows that the Kuhn-Tucker relations (3.2.99)(1) - (4) are inconsistent and cannot be satisfied by any pair of primal-dual variables $(x, y) \in \mathbb{X} \times \mathbb{Y} = \mathbb{R} \times \mathbb{R}$. This is a consequence of non-existence of a solution of the problem defined by (3.2.34) and (3.2.98).

## 3.3 A Stochastic Problem Illustrating the Variational Approach

This section has two main goals. We shall look at a very simple static convex stochastic optimization problem involving *only* a single inequality constraint, which illustrates yet another rather important aspect of applying the Rockafellar variational approach (see Problem 3.3.12 which follows). In the course of addressing this problem we shall, for the first time in this thesis, come across Lagrange multipliers which are not scalars or members of a finite dimensional Euclidean space (as was the case for all the problems of Section 3.2), and are not functions in any "ordinary" sense either, but instead are "singular" in a very specific sense. The origin of these singular multipliers is in the celebrated *Yosida-Hewitt* decomposition theorem. Thus, the second main goal of this section is to clearly define what these singular elements are and to state the most basic version of the Yosida-Hewitt decomposition theorem.

**Notation 3.3.1.** Let $(S, \Sigma, \mu)$ be a measure space, and as usual let $\mathcal{L}_\infty(S, \Sigma, \mu)$ denote the normed vector space of all $\mathbb{R}$-valued, $\mu$-essentially bounded and $\Sigma$-measurable equivalent classes of functions on $S$ with the usual $\mu$-essential supremum norm $\|\cdot\|_\infty$. Similarly, for each $p \in [1, \infty)$ let $\mathcal{L}_p(S, \Sigma, \mu)$ denote the normed vector space of all $\mathbb{R}$-valued and $\Sigma$-measurable equivalent classes of functions on $S$ such that $|f|^p$ is $\mu$-integrable, with norm defined by

$$\|f\|_p := \left(\int_S |f|^p \,\mathrm{d}\mu\right)^{\frac{1}{p}}, \qquad f \in \mathcal{L}_p(S, \Sigma, \mu), \tag{3.3.1}$$

(that $\mathcal{L}_p(S, \Sigma, \mu)$ is a vector space is immediate from Minkowski's inequality). For each $p \in [1, \infty]$ (this includes $p = \infty$), let $\mathcal{L}_p^*(S, \Sigma, \mu)$ denote the usual *norm-dual* or *adjoint* space of $\mathcal{L}_p(S, \Sigma, \mu)$, that is the vector space of all $\|\cdot\|_p$-continuous linear functionals on $\mathcal{L}_p(S, \Sigma, \mu)$. It is a standard result of elementary functional analysis that, for each $p \in [1, \infty]$,

$$\|l\|_p^* := \sup_{\substack{f \in \mathcal{L}_p(S,\Sigma,\mu), \\ \|f\|_p \leqslant 1}} |l(f)|, \qquad l \in \mathcal{L}_p^*(S, \Sigma, \mu). \tag{3.3.2}$$

defines a norm on $\mathcal{L}_p^*(S, \Sigma, \mu)$, and that furthermore $\mathcal{L}_p^*(S, \Sigma, \mu)$ is a Banach space with this norm.

Fix $1 < p < \infty$, define $1 < q < \infty$ by

$$\frac{1}{p} + \frac{1}{q} = 1, \tag{3.3.3}$$

and fix some $g \in \mathcal{L}_q(S, \Sigma, \mu)$ Then it is immediate from Holder's inequality that the mapping

$$L_g(f) := \int_S g f \mathrm{d}\mu, \qquad f \in \mathcal{L}_p(S, \Sigma, \mu), \tag{3.3.4}$$

is well-defined and gives a member of $\mathcal{L}_p^*(S,\Sigma,\mu)$. The celebrated *Riesz Representation Theorem* asserts that *every* member of $\mathcal{L}_p^*(S,\Sigma,\mu)$ is given by the integral on the right side of (3.3.4) for some unique $g \in \mathcal{L}_q(S,\Sigma,\mu)$:

**Theorem 3.3.2.** *Fix $1 < p < \infty$, and define $1 < q < \infty$ by (3.3.3). Corresponding to each $l \in \mathcal{L}_p^*(S,\Sigma,\mu)$, there exists some unique $g \in \mathcal{L}_q(S,\Sigma,\mu)$ such that*

$$l(f) = \int_S gf\,d\mu, \qquad f \in \mathcal{L}_p(S,\Sigma,\mu). \tag{3.3.5}$$

*Moreover,*

$$\|l\|_p^* = \|g\|_q\,. \tag{3.3.6}$$

**Remark 3.3.3.** Consider the case where $p = \infty$. For some $g \in \mathcal{L}_1(S,\Sigma,\mu)$, define

$$l(f) := \int_S fg\mathrm{d}\mu, \qquad f \in \mathcal{L}_\infty(S,\Sigma,\mu). \tag{3.3.7}$$

Then it is immediate from the dominated convergence theorem that $l \in \mathcal{L}_\infty^*(S,\Sigma,\mu)$. Motivated by Theorem 3.3.2, the natural question is the following: is *every* member of $\mathcal{L}_\infty^*(S,\Sigma,\mu)$ necessarily given by the integral on the right hand side of (3.3.7) for some $g \in \mathcal{L}_1(S,\Sigma,\mu)$? The following example shows that this is not the case:

**Example 3.3.4.** (from Exercise 13, Chap.6, on page 134 of Rudin [34])
The fact that there exist linear functionals $l \in \mathcal{L}_\infty^*(S,\Sigma,\mu)$ which cannot be represented in the form of (3.3.7) for some $g \in \mathcal{L}_1(S,\Sigma,\mu)$ can be seen as follows. Put

$$S := [0,1], \quad \Sigma := \mathfrak{B}([0,1]), \quad \mu := \text{Lebesgue measure}, \tag{3.3.8}$$

and abbreviate $\mathcal{L}_p(S,\Sigma,\mu)$ ($\mathcal{L}_p^*(S,\Sigma,\mu)$) with $\mathcal{L}_p[0,1]$ ($\mathcal{L}_p^*[0,1]$) for $p = 1,\infty$. Define the linear functional $l : \mathcal{C}[0,1] \to \mathbb{R}$ as follows:

$$l(f) := f(0), \qquad f \in \mathcal{C}[0,1], \tag{3.3.9}$$

where $\mathcal{C}[0,1]$ denotes the vector space of all continuous functions on $[0,1]$ equipped with the *uniform norm* given as

$$\|f\|_u := \sup_{t\in[0,1]} |f(t)|\,. \tag{3.3.10}$$

Then, under the functional norm $\|\cdot\|_u^*$ on the norm-dual space $(\mathcal{C}[0,1])^*$,

$$\|l\|_u^* := \sup_{\substack{f\in\mathcal{C}[0,1],\\ \|f\|_u\leqslant 1}} |l(f)| = 1, \tag{3.3.11}$$

thus $l$ is a bounded linear functional on $\mathcal{C}[0,1]$. But $\mathcal{C}[0,1]$ is a $\|\cdot\|_\infty$-norm-closed linear subspace of $\mathcal{L}_\infty[0,1]$, and by the Hahn-Banach extension theorem, $l$ can be extended to a bounded linear functional $\tilde{l}$ on $\mathcal{L}_\infty[0,1]$, that is there exists some

$$\tilde{l} \in \mathcal{L}_\infty^*[0,1] \qquad \text{such that} \qquad \tilde{l}(f) = l(f) \qquad \text{for all } f \in \mathcal{C}[0,1]. \tag{3.3.12}$$

Suppose there exists some $g \in \mathcal{L}_1[0,1]$ such that

$$\tilde{l}(f) = \int_0^1 f(t)g(t)\mathrm{d}t, \qquad \text{all } f \in \mathcal{L}_\infty[0,1]. \tag{3.3.13}$$

Define the sequence of continuous function $\{f_n; n \in \mathbb{N}\} \subset \mathcal{C}[0,1]$ as follows

$$f_n(t) := 1 - n\min\{t, 1/n\}, \qquad t \in [0,1]. \tag{3.3.14}$$

Then $f_n \in \mathcal{L}_\infty[0,1]$ with $\|f_n\|_\infty^* = 1$ for all $n \in \mathbb{N}$, and

$$\lim_{n\to\infty} f_n(t) = 0, \qquad t \in (0,1]. \tag{3.3.15}$$

By (3.3.13), (3.3.15) and dominated convergence, we have

$$\lim_{n\to\infty} \tilde{l}(f_n) \overset{(3.3.13)}{=} \lim_{n\to\infty} \int_0^1 f_n(t)g(t)\mathrm{d}t = \int_0^1 \lim_{n\to\infty} f_n(t)g(t)\mathrm{d}t \overset{(3.3.15)}{=} 0. \tag{3.3.16}$$

But

$$\lim_{n\to\infty} \tilde{l}(f_n) \overset{(3.3.9)}{=} \lim_{n\to\infty} f_n(0) \overset{(3.3.14)}{=} 1, \qquad n \in \mathbb{N}. \tag{3.3.17}$$

In view of the contradiction of (3.3.17) and (3.3.16) we see that the assumed representation (3.3.13) for $\tilde{l}$ cannot hold. □

**Remark 3.3.5.** From Example 3.3.4, we see that in general

$$\mathcal{L}_1(S,\Sigma,\mu) \subset \mathcal{L}_\infty^*(S,\Sigma,\mu) \qquad \text{but} \qquad \mathcal{L}_1(S,\Sigma,\mu) \neq \mathcal{L}_\infty^*(S,\Sigma,\mu). \tag{3.3.18}$$

It follows from Remark 3.3.5 that $\mathcal{L}_1(S,\Sigma,\mu)$ must be "complemented" by a non-trivial vector subspace of $\mathcal{L}_\infty^*(S,\Sigma,\mu)$ in order to get a full description of $\mathcal{L}_\infty^*(S,\Sigma,\mu)$, that is there must exist some non-trivial vector subspace $M \subset \mathcal{L}_\infty^*(S,\Sigma,\mu)$ such that

$$\mathcal{L}_1(S,\Sigma,\mu) \oplus M = \mathcal{L}_\infty^*(S,\Sigma,\mu). \tag{3.3.19}$$

The celebrated *Yosida-Hewitt decomposition theorem* completely characterizes the subspace $M$ and establishes that every $l \in \mathcal{L}_\infty^*(S,\Sigma,\mu)$ is the sum of a unique "regular" member $l_r \in \mathcal{L}_1(S,\Sigma,\mu)$ and a unique "singular" member $l_s$ in the subspace $M$. We next formulate the characteristic property of these singular elements:

**Definition 3.3.6.** In a measure space $(S,\Sigma,\mu)$, a sequence $\{E_n; n \in \mathbb{N}\} \subset \Sigma$ is said to **decrease to a $\mu$-null set** when $E_{n+1} \subset E_n$ with $\mu(E_n) > 0$ for all $n \in \mathbb{N}$, and $\mu(E_n) \downarrow 0$ as $n \uparrow \infty$. The functional $l \in \mathcal{L}_\infty^*(S,\Sigma,\mu)$ is called a **singular** linear functional on $\mathcal{L}_\infty(S,\Sigma,\mu)$, if there exists some sequence $\{E_n; n \in \mathbb{N}\} \subset \Sigma$ decreasing to a $\mu$-null set such that

$$l(f) = l(f\mathbb{1}_{E_n}), \qquad \text{for all } f \in \mathcal{L}_\infty(S,\Sigma,\mu) \text{ and } n \in \mathbb{N}, \tag{3.3.20}$$

where $\mathbb{1}_E\colon S \to \{0,1\}$ is the *indicator function* of the set $E \subset S$ defined for every $x \in S$ by

$$\mathbb{1}_E(x) := \begin{cases} 1 & x \in E, \\ 0 & x \notin E. \end{cases} \tag{3.3.21}$$

We denote the set of all *singular* linear functionals on $\mathcal{L}_\infty(S,\Sigma,\mu)$ by $\mathcal{Z}(S,\Sigma,\mu)$, that is $\mathcal{Z}(S,\Sigma,\mu) \subset \mathcal{L}_\infty^*(S,\Sigma,\mu)$.

**Remark 3.3.7.** If $l \in \mathcal{Z}(S,\Sigma,\mu)$ then there certainly exists some sequence $\{E_n; n \in \mathbb{N}\} \subset \Sigma$ decreasing to a $\mu$-null set such that (3.3.20) holds. However, this sequence is of course not necessarily unique, and there may be many such sequences. It is straightforward to verify that $\mathcal{Z}(S,\Sigma,\mu)$ is a *vector subspace* of $\mathcal{L}_\infty^*(S,\Sigma,\mu)$.

Clearly the zero member of $\mathcal{L}_\infty^*(S,\Sigma,\mu)$ is an element of $\mathcal{Z}(S,\Sigma,\mu)$, and the following question arises: do there actually exist *non-trivial* members of $\mathcal{L}_\infty^*(S,\Sigma,\mu)$ which are singular in the sense of Definition 3.3.6? If not then this definition is certainly without interest! The following example demonstrates such a non-trivial linear functional:

**Example 3.3.8.** (motivated by Yoside and Hewitt [42])
Let $(S,\Sigma,\mu)$ be a probability space which is not finitely atomic, so that there exists some sequence $\{E_n; n \in \mathbb{N}\} \subset \Sigma$ which decreases to a $\mu$-null set. Define a sequence of linear functionals $\{l_n; n \in \mathbb{N}\} \subset \mathcal{L}_\infty^*(S,\Sigma,\mu)$ by

$$l_n(f) := E\left[f\mathbb{1}_{E_n}\right]/\mu(E_n), \qquad \text{for all } f \in \mathcal{L}_\infty(S,\Sigma,\mu) \text{ and } n \in \mathbb{N}. \tag{3.3.22}$$

Then $\{l_n; n \in \mathbb{N}\}$ is a subset of the unit ball of $\mathcal{L}_\infty^*(S,\Sigma,\mu)$, hence by the Alaoglu theorem must have a $\sigma(\mathcal{L}_\infty^*, \mathcal{L}_\infty)$-accumulation point $l_0 \in \mathcal{L}_\infty^*(S,\Sigma,\mu)$. Since $l_n(1) = 1$ for all $n \in \mathbb{N}$ we must have

$$l_0(1) = 1. \tag{3.3.23}$$

It follows that $l_0$ is non-trivial. To see that $l_0$ is singular, fix some $m \in \mathbb{N}$. Since the sequence $\{E_n; n \in \mathbb{N}\}$ is decreasing we have

$$l_n(f\mathbb{1}_{E_m}) = l_n(f), \qquad \text{for all } f \in \mathcal{L}_\infty(S,\Sigma,\mu) \text{ and } n \geqslant m, \tag{3.3.24}$$

and therefore, since $l_0$ is a $\sigma(\mathcal{L}_\infty^*, \mathcal{L}_\infty)$-accumulation point of $\{l_n; n \in \mathbb{N}\}$, we must have

$$l_0(f\mathbb{1}_{E_m}) = l_0(f), \qquad \text{for all } f \in \mathcal{L}_\infty(S,\Sigma,\mu). \tag{3.3.25}$$

By the arbitrary choice of $m \in \mathbb{N}$ we see from (3.3.25) that $l_0 \in \mathcal{Z}(S,\Sigma,\mu)$. □

The next result establishes that the complementary subspace $M$ at (3.3.19) is the vector space $\mathcal{Z}(S,\Sigma,\mu)$:

**Theorem 3.3.9. (Yosida-Hewitt [42])** *Suppose* $(S, \Sigma, \mu)$ *is a finite measure space. For any* $l \in \mathcal{L}_\infty^*(S, \Sigma, \mu)$*, there exists some unique* $l_r \in \mathcal{L}_1(S, \Sigma, \mu)$ *and* $l_s \in \mathcal{Z}(S, \Sigma, \mu)$ *such that*

$$l(f) = E[l_r f] + l_s(f), \qquad f \in \mathcal{L}_\infty(S, \Sigma, \mu). \tag{3.3.26}$$

*That is,* $\mathcal{L}_\infty^*(S, \Sigma, \mu)$ *can be written as a direct sum:*

$$\mathcal{L}_\infty^*(S, \Sigma, \mu) = \mathcal{L}_1(S, \Sigma, \mu) \oplus \mathcal{Z}(S, \Sigma, \mu). \tag{3.3.27}$$

**Remark 3.3.10.** We usually write the decomposition of $l \in \mathcal{L}_\infty^*(S, \Sigma, \mu)$ given at (3.3.26) in the abbreviated form $l = (l_r, l_s)$, calling $l_r \in \mathcal{L}_1(S, \Sigma, \mu)$ the *regular part* , and $l_s \in \mathcal{Z}(S, \Sigma, \mu)$ the *singular part*, of the functional $l$.

**Remark 3.3.11.** A functional $l \in \mathcal{L}_\infty^*(S, \Sigma, \mu)$ is said to be *non-negative* when

$$l(f) \geqslant 0 \qquad \text{for all} \qquad f \in \mathcal{L}_\infty(S, \Sigma, \mu) \qquad \text{such that} \qquad f \geqslant 0 \;\; \mu\text{-a.e.}, \tag{3.3.28}$$

and this is denoted by $l \geqslant 0$. For future reference we state the following simple result on non-negativity of members of $\mathcal{L}_\infty^*(S, \Sigma, \mu)$: for each $l \in \mathcal{L}_\infty^*(S, \Sigma, \mu)$ with Yosida-Hewitt decomposition $l = (l_r, l_s)$ (in the notation of Remark 3.3.10) we have the equivalence

$$l \geqslant 0 \qquad \Longleftrightarrow \qquad l_r \geqslant 0 \;\; \mu\text{-a.e. and } \; l_s \geqslant 0, \tag{3.3.29}$$

(see page 514 of Rockafellar and Wets [32]).

With the above tools in place we can now give another illustration of the Rockafellar variational approach on a very simple convex stochastic optimization problem. Despite the simplicity of this problem, in the course of applying the Rockafellar variational approach we shall learn a valuable method of perturbations which is well suited to the use of Theorem 3.1.7 for ensuring zero duality gap together with existence of Lagrange multipliers for the problem constraints. These insights will serve us well when we address the canonical problem (2.2.21). The example also illustrates a case in which the Lagrange multiplier appropriate to the constraints in the problem turns out very naturally to be a singular linear functional in the sense of Definition 3.3.6.

**Problem 3.3.12.** (motivated by Rockafellar and Wets [32])
We are given the deterministic objective function

$$J(x) := \frac{x^2}{2}, \qquad x \in \mathbb{R}, \tag{3.3.30}$$

together with a random variable $\xi$ on the probability space $(\Omega, \mathcal{F}, P)$, which is uniformly distributed over the unit interval $[0, 1]$. We address the following problem:

$$\text{minimize } J(x) \text{ subject to the constraint } x \geqslant \xi \text{ a.s.} \tag{3.3.31}$$

**Remark 3.3.13.** Since $\xi$ is uniformly distributed over $[0,1]$ we have

$$\mathrm{P}\text{-ess-sup}_{\omega\in\Omega}\, \xi(\omega) = 1, \tag{3.3.32}$$

and therefore problem (3.3.31) is of course identical to the trivial deterministic problem

$$\text{minimize } J(x) \text{ subject to the constraint } x \geqslant 1, \tag{3.3.33}$$

with minimizer

$$x = 1. \tag{3.3.34}$$

However, the precise character of the dual variables which "enforce" the "a.s."-constraint on $x$ at (3.3.31) is not entirely clear, and neither is the associated dual functional and corresponding optimality relations. In fact it is quite instructive to implement the Rockafellar variational approach to ascertain the form of these entities for this very simple problem, and we address this next.

The real vector space of primal variables for the problem (3.3.31) is just the real line

$$\mathbb{X} := \mathbb{R}. \tag{3.3.35}$$

Following the general form (3.1.1), we define the *primal function* $f : \mathbb{X} \to (-\infty, +\infty]$ as

$$f(x) := \begin{cases} J(x), & \text{when } x \geqslant \xi \quad \text{a.s.,} \\ +\infty, & \text{otherwise,} \end{cases} \tag{3.3.36}$$

for all $x \in \mathbb{R}$. The *primal problem* is then to minimize $f(x)$ over all $x \in \mathbb{X}$, i.e. to determine some $\bar{x} \in \mathbb{X}$ such that

$$f(\bar{x}) = \inf_{x\in\mathbb{X}} f(x) \in \mathbb{R}. \tag{3.3.37}$$

We shall now follow the steps outlined in Section 3.1, that is we shall choose a real vector space $\mathbb{U}$ of perturbations, a perturbation function $F(\cdot,\cdot)$, a real vector space $\mathbb{Y}$ of dual variables and a bilinear form on $\mathbb{U} \times \mathbb{Y}$, and then construct a Lagrangian function and a dual function. Notice in particular that problem (3.3.31) involves only an *inequality* constraint. We shall therefore be guided by the approach that was followed for Problem 3.2.7, which also involved just an inequality constraint. However, Problem 3.2.7 involves just a *finite dimensional* constraint, whereas the constraint in the present problem is *infinite dimensional*. In fact the inequality constraint of Problem 3.2.7 really amounts to the finite set of constraints

$$x_i \geqslant b_i, \tag{3.3.38}$$

indexed by $i = 1, 2, \ldots, n$ with corresponding complementary slackness relations

$$(x_i - b_i)y_i = 0, \qquad \text{for all } i = 1, 2, \ldots, n, \tag{3.3.39}$$

(see Remark 3.2.12), that is the Lagrange multiplier is really the $\mathbb{R}$-valued function defined on the index set $\{1, 2, \dots, n\}$ as

$$i \to y_i : \{1, 2, \dots, n\} \to [0, \infty). \tag{3.3.40}$$

On the other hand, in Problem (3.3.31) we really have an "infinite" set of inequality constraints

$$x \geqslant \xi(\omega) \qquad \text{for } P\text{-almost all } \omega \in \Omega. \tag{3.3.41}$$

that is, the index set of the constraints is now the set $\Omega$. By analogy with (3.3.40) we expect the Lagrange multiplier for Problem (3.3.31) to be an $\mathbb{R}$-valued function on the index set of the constraints (that is $\Omega$) i.e. a random variable

$$\omega \to y(\omega) : \Omega \to [0, \infty), \tag{3.3.42}$$

(that is $y$ is a non-negative random variable), and, by analogy with (3.3.39), it is reasonable to expect that the corresponding complementary slackness relation between the primal solution $x = 1$ (see (3.3.34)) and the Lagrange multiplier $y$ should be

$$[x - \xi(\omega)]y(\omega) = 0, \qquad \text{for } P\text{-almost all } \omega \in \Omega. \tag{3.3.43}$$

But, since $\xi$ is *uniformly* distributed on the unit interval $[0, 1]$, we must have

$$x = 1 > \xi(\omega), \qquad \text{for } P\text{-almost all } \omega \in \Omega, \tag{3.3.44}$$

and (3.3.44) together with (3.3.43) then gives

$$y(\omega) = 0, \qquad \text{for } P\text{-almost all } \omega \in \Omega, \tag{3.3.45}$$

that is, the Lagrange multiplier $y$ is necessarily equal to zero! However, this clearly cannot be the case since the constraint in Problem (3.3.31) binds at optimality (removal of the constraint causes the primal solution to shift from $x = 1$ to $x = 0$). The preceding difficulty arises from the fact that, while the constraint in Problem (3.3.31) clearly binds, it nevertheless binds *only* on a set of $P$-measure equal to zero, that is

$$P\{x = \xi\} = 0 \quad (\text{with } x = 1). \tag{3.3.46}$$

One cannot expect a Lagrange multiplier given by a random variable to "enforce" a constraint which is active only on a set of $P$-measure zero, since random variables are insensitive to change on sets of $P$-measure zero. A fundamental insight, introduced by Rockafellar and Wets [32] for static problems of stochastic convex optimization (of which Problem (3.3.31) is a special case), is that Lagrange multipliers corresponding to constraints which bind on a set of $P$-measure zero must necessarily be *singular* linear functionals in $\mathcal{L}^*_\infty(\Omega, \mathcal{F}, P)$ (recall Definition 3.3.6). We are now going to see how this arises very naturally from the Rockafellar variational approach which we now implement in a way which closely parallels the use of this approach for Problem 3.2.7.

1. For Step 3.1.1 we must fix a vector space of perturbations $\mathbb{U}$ and define a perturbation function. In Problem 3.2.7 the perturbation space is a finite dimensional vector space (see (3.2.38)) as is appropriate to the finite number of constraints in this problem. In contrast, we have seen that Problem (3.3.31) has infinitely many constraints indexed by $\omega \in \Omega$ so it is reasonable to fix a vector space of *functions* defined on $\Omega$ as the space of perturbations. Of course there are many possible choices of function space. We shall shortly see that a particularly appropriate choice, which will allow us to use Theorem 3.1.7 to secure existence of Lagrange multipliers, is

$$\mathbb{U} := \mathcal{L}_\infty(\Omega, \mathcal{F}, P). \tag{3.3.47}$$

We now define a perturbation function $F : \mathbb{X}\times\mathbb{U} \to (-\infty, +\infty]$ very much by analogy with (3.2.39) as

$$F(x, v) := \begin{cases} J(x), & \text{when } x + v \geqslant \xi, \text{ a.s.,} \\ +\infty, & \text{otherwise,} \end{cases} \tag{3.3.48}$$

for all $(x, v) \in \mathbb{R} \times \mathbb{U}$ (with $\mathbb{U}$ of course being given by (3.3.47)). From (3.3.36) and (3.3.48) we have the consistency relation of the form (3.1.2), namely

$$F(x, 0) = f(x), \quad x \in \mathbb{X}. \tag{3.3.49}$$

2. According to Step 3.1.2 we must pair the space of perturbations at (3.3.47) with a vector space $\mathbb{Y}$ through a bilinear form $\langle\cdot,\cdot\rangle$ on $\mathbb{U} \times \mathbb{Y}$, and the natural choice is

$$\mathbb{Y} := (\mathbb{U})^* = \mathcal{L}^*_\infty(\Omega, \mathcal{F}, P), \quad \text{with} \quad \langle v, y\rangle := y(v) \quad \text{for } (v, y) \in \mathbb{U} \times \mathbb{Y}. \tag{3.3.50}$$

This completes all the choices required for Steps 3.1.1 - 3.1.2, and it remains to synthesize a Lagrangian and a dual function.

3. According to Step 3.1.3 define the *Lagrangian function* $K : \mathbb{X} \times \mathbb{Y} \to [-\infty, +\infty]$ as:

$$K(x, y) := \inf_{u\in\mathbb{U}} [\langle v, y\rangle + F(x, v)] \quad (x, y) \in \mathbb{R} \times \mathbb{Y}, \tag{3.3.51}$$

(c.f. (3.1.3)). From (3.3.48) and (3.3.50) we obtain

$$\begin{aligned} K(x, y) &= \inf_{v\in\mathcal{L}_\infty(\Omega,\mathcal{F},P)} \{y(v) + J(x) \mid v \geqslant \xi - x\} \\ &\overset{u=v-(\xi-x)}{=} \inf_{0\leqslant u\in\mathcal{L}_\infty(\Omega,\mathcal{F},P)} \{y(u)\} + y(\xi - x) + J(x), \\ &= \begin{cases} y(\xi - x) + J(x), & \text{when } y \geqslant 0, \\ -\infty, & \text{otherwise,} \end{cases} \end{aligned} \tag{3.3.52}$$

for all $x \in \mathbb{X}$ and $y \in \mathbb{Y}$. The notation $y \geqslant 0$ at (3.3.52) indicates that the linear functional $y \in \mathbb{Y}$ is non-negative in the sense that

$$y(\eta) \geqslant 0 \text{ for all } \eta \in \mathcal{L}_\infty(\Omega, \mathcal{F}, P) \text{ such that } \eta \geqslant 0 \ P\text{-a.s.} \tag{3.3.53}$$

At the final equality of (3.3.52) we have also used the elementary fact that

$$\inf_{0\leqslant u\in\mathcal{L}_\infty(\Omega,\mathcal{F},P)}\{y(u)\} = \begin{cases} 0, & \text{when } y\geqslant 0,\\ -\infty, & \text{otherwise.}\end{cases} \qquad (3.3.54)$$

In view of (3.1.4), define the *dual function* $g:\mathbb{Y}\to[-\infty,+\infty)$ as

$$g(y) := \inf_{x\in\mathbb{X}} K(x,y) \overset{(3.3.52)}{=} \begin{cases} \inf_{x\in\mathbb{X}}\{y(\xi)-y(x)+J(x)\}, & \text{when } y\geqslant 0,\\ -\infty, & \text{otherwise,}\end{cases} \qquad (3.3.55)$$

for each $y\in\mathbb{Y}$. Note that $y(x)=xy(1)$ for all $(x,y)\in\mathbb{X}\times\mathbb{Y}:=\mathbb{R}\times\mathcal{L}_\infty^*(\Omega,\mathcal{F},P)$, and therefore

$$\begin{aligned} &\inf_{x\in\mathbb{X}}\{y(\xi)-y(x)+J(x)\} = y(\xi)-\sup_{x\in\mathbb{X}}\{y(x)-J(x)\}\\ = \; & y(\xi)-\sup_{x\in\mathbb{X}}\{xy(1)-J(x)\} = y(\xi)-J^*(y(1)). \end{aligned} \qquad (3.3.56)$$

Notice from (3.3.30) and (3.2.2) that

$$J^*(\alpha)=\frac{\alpha^2}{2}, \qquad \alpha\in\mathbb{R}. \qquad (3.3.57)$$

Combining (3.3.55) and (3.3.56), we get

$$g(y) = \begin{cases} y(\xi)-J^*(y(1)), & \text{when } y\geqslant 0,\\ -\infty, & \text{otherwise,}\end{cases} \qquad (3.3.58)$$

for all $y\in\mathbb{Y}:=\mathcal{L}_\infty^*(\Omega,\mathcal{F},P)$.

The *dual problem* is then to maximize $g(y)$ over all $y\in\mathbb{Y}:=\mathcal{L}_\infty^*(\Omega,\mathcal{F},P)$, i.e., to establish

$$g(\bar{y}) = \sup_{y\in\mathbb{Y}}\{g(y)\} \qquad \text{for some } \bar{y}\in\mathbb{Y}. \qquad (3.3.59)$$

Existence of such a maximizing $\bar{y}$ is by no means clear from the form of $g(\cdot)$ given by (3.3.58). It is at this point that Theorem 3.1.7 plays an essential role. In order to use this theorem we need to fix some $\langle\mathbb{U},\mathbb{Y}\rangle$-compatible topology $\mathcal{U}$ on the perturbation space $\mathbb{U}$. Since $\mathbb{U}=\mathcal{L}_\infty(\Omega,\mathcal{F},P)$ (see (3.3.47)) is a normed vector space and the space of dual variables $\mathbb{Y}$ is the norm-dual of $\mathbb{U}$ (recall (3.3.50)), it follows at once from Example 3.1.6 that the usual norm-topology on $\mathbb{U}$ serves as the $\langle\mathbb{U},\mathbb{Y}\rangle$-compatible topology $\mathcal{U}$. In order to use Theorem 3.1.7 we must now establish that

$$\sup_{u\in G} F(x_1,u) < +\infty, \qquad (3.3.60)$$

for some $\mathcal{U}$-neighbourhood $G$ of $0\in\mathbb{U}$ and for some $x_1\in\mathbb{X}:=\mathbb{R}$ (see (3.3.35)), with $F$ defined by (3.3.48). To this end fix any

$$x_1\in(1,+\infty), \qquad \text{and fix some} \qquad \varepsilon\in(0,x_1-1). \qquad (3.3.61)$$

From (3.3.48) and the $U[0,1]$-distribution of $\xi$ we have

$$\sup_{v\in\mathbb{U},\|v\|_\infty<\varepsilon} F(x_1,v) = J(x_1) < +\infty, \tag{3.3.62}$$

that is (3.3.60) holds with $G := \{v \in \mathbb{U} \ : \ \|v\|_\infty < \varepsilon\}$. It is now immediate from Theorem 3.1.7 that

$$\inf_{x\in\mathbb{R}} f(x) = \sup_{y\in\mathbb{Y}} g(x) = g(\bar{y}) \quad \text{for some } \bar{y} \in \mathbb{Y}, \tag{3.3.63}$$

that is $\bar{y}$ is an optimal solution of the dual problem.

**Remark 3.3.14.** Application of Theorem 3.1.7 to get (3.3.63) relies on choosing a duality pairing $\langle\mathbb{U},\mathbb{Y}\rangle$ as well as a $\langle\mathbb{U},\mathbb{Y}\rangle$-compatible topology $\mathcal{U}$ such that (3.1.9) holds for some $x_1 \in \mathbb{X} := \mathbb{R}$ and for some $\mathcal{U}$-neighbourhood $G$ of $0 \in \mathbb{U}$. It is immediate from (3.3.48) and (3.3.62) that the appropriate choice of $\mathbb{U}$ is that given by (3.3.47) and that the appropriate topology $\mathcal{U}$ on $\mathbb{U}$ must be the norm-topology. With $\mathbb{U}$ and $\mathcal{U}$ chosen in this way the space of dual variables $\mathbb{Y}$ and duality pairing at (3.3.50) are essentially mandatory. Of course one might try to pair $\mathbb{U}$, not with the very "large" space $\mathbb{Y} := \mathcal{L}^*_\infty(\Omega,\mathcal{F},P)$, but perhaps with the smaller (and more manageable) space $\mathbb{Y} := \mathcal{L}_1(\Omega,\mathcal{F},P)$. In fact, as we shall see shortly, we really do need the very large space $\mathbb{Y} := \mathcal{L}^*_\infty(\Omega,\mathcal{F},P)$, since the Lagrange multiplier for the a.s. inequality constraint in 3.3.31 turns out to be a *singular* element of this space, that is a member of $\mathcal{Z}(\Omega,\mathcal{F},P)$ (recall Definition 3.3.6). Static convex optimization problems with almost-sure inequality constraints (of which Problem 3.3.12 is a very special case) were first addressed by Rockafellar and Wets [32], and one of the main insights of this work is that the space of essentially bounded functions (together with the norm topology on this space) is the correct choice of perturbations for dealing with such constraints. We shall see later that essentially bounded perturbations are equally appropriate for the *dynamic* problems of convex optimization addressed in this thesis, and in particular are key for applying Theorem 3.1.7 to the almost-sure American wealth constraint $X(t) \geqslant B(t)$, $t \in [0,T]$ in the Canonical Problem 2.2.7.

Continuing with problem (3.3.31), we next construct the Kuhn-Tucker optimality conditions which are equivalent to the condition $f(x) = g(y)$ (for general $(x,y)$). To this end, observe from (3.3.36) and (3.3.55) that, for each $(x,y) \in \mathbb{X}\times\mathbb{Y}$, we have $f(x) \in (-\infty,+\infty]$ and $g(y) \in [-\infty,+\infty)$, and by the weak duality relation (3.1.5),

$$f(x) \geqslant K(x,y) \geqslant g(y), \quad (x,y) \in \mathbb{R}\times\mathbb{Y}. \tag{3.3.64}$$

Therefore we have

$$f(x) = g(y) \quad \iff \quad f(x) = K(x,y) \in \mathbb{R} \text{ and } g(y) = K(x,y) \in \mathbb{R}, \tag{3.3.65}$$

for each $(x,y) \in \mathbb{X}\times\mathbb{Y}$. Put

$$(\partial J^*)(\alpha) := \frac{\mathrm{d}J^*(\alpha)}{\mathrm{d}\alpha} \overset{(3.3.57)}{=} \alpha, \qquad \alpha \in \mathbb{R}. \tag{3.3.66}$$

For arbitrary $(x, y) \in \mathbb{X} \times \mathbb{Y}$ we have the equivalence

$$\begin{aligned} & g(y) = K(x, y) \in \mathbb{R} \\ \overset{(3.3.52)}{\underset{(3.3.58)}{\iff}} \quad & (1)\ y \geqslant 0,\ (2)\ J(x) + J^*(y(1)) = xy(1) \\ \iff \quad & (1')\ y \geqslant 0,\ (2')\ y(1) = \partial J(x). \end{aligned} \tag{3.3.67}$$

The equivalence of (2) and (2') at (3.3.67) follows from the equivalence at (C.4) of Remark C.0.5. Moreover, from (3.3.36) and (3.3.51) we also have

$$f(x) = K(x, y) \in \mathbb{R} \iff (1)\ x \geqslant \xi \ \text{ a.s.},\ (2)\ y \geqslant 0,\ (3)\ y(\xi - x) = 0. \tag{3.3.68}$$

Combining (3.3.65), (3.3.67) and (3.3.68), we then obtain the following *Kuhn-Tucker optimality relations* for the problem (3.3.31): For each $(x, y) \in \mathbb{R} \times \mathbb{Y}$ we have the equivalence

$$f(x) = g(y) \iff \begin{cases} (1)\ x \geqslant \xi \ \text{ a.s.}, & (2)\ y \geqslant 0, \\ (3)\ y(\xi - x) = 0, & (4)\ y(1) = \partial J(x) \end{cases} \tag{3.3.69}$$

**Remark 3.3.15.** Items (1) - (4) of (3.3.69) are the Kuhn-Tucker optimality conditions. In particular (3.3.69)(1)(2) are *feasibility conditions* on the primal variable $x$ (these are just the constraints in the primal problem) and the dual variable $y$ respectively. The relation (3.3.69)(3) is a *complementary slackness condition*, and as such relates the primal variable $x$, the dual variable $y$ and the constraints in the primal problem. Finally, (3.3.69)(4) is a *transversality relation*, which relates the primal variable $x$, the dual variable $y$ and the risk criterion function $J$ in the primal problem. It is worthwhile comparing the Kuhn-Tucker relations (3.3.69)(1)-(4) for problem (3.3.31) and the Kuhn-Tucker relations (3.2.55)(1)-(4) for the optimization problem (3.2.34), for these are clearly very similar in form. This is not a surprise, since problem (3.3.31) is just an "infinite-dimensional" version of problem (3.2.34), both problems exhibiting specifically just inequality constraints. It is for this reason that the dual variables are *non-negative* at optimality in both cases (see (3.3.69)(2) and (3.2.55)(2), and the complementary slackness relation (3.3.69)(3) is an infinite-dimensional analogue of the complementary slackness relation (3.2.55)(3).

**Remark 3.3.16.** From (3.3.57) one sees that the convex conjugate $J^*(\cdot)$ is $\mathbb{R}$-valued and smooth, and therefore we can rewrite the transversality relation (3.3.69)(4) in the equivalent form $x = (\partial J^*)(y(1))$ (see (C.4) of Remark C.0.5). The Kuhn-Tucker conditions (3.3.69) therefore take the following form: For each $(x, y) \in \mathbb{X} \times \mathbb{Y}$ we have the equivalence

$$f(x) = g(y) \iff \begin{cases} (1)\ x \geqslant \xi \ \text{ a.s.}, & (2)\ y \geqslant 0, \\ (3)\ y(\xi - x) = 0, & (4)\ x = (\partial J^*)(y(1)). \end{cases} \tag{3.3.70}$$

In the remainder of this example we are going to construct some $\bar{x} \in \mathbb{X}$ in terms of the maximizer $\bar{y}$ of $g$, the existence of which has been established using Theorem 3.1.7 (see (3.3.63)), such that the pair $(\bar{x}, \bar{y}) \in \mathbb{X} \times \mathbb{Y}$ satisfies the optimality relations (3.3.70)(1)-(4), for then we conclude from the equivalence at (3.3.70) that $f(\bar{x}) = g(\bar{y})$. With this established it then follows from (3.1.6) that $\bar{x}$ is optimal for the primal problem (3.3.31).

The transversality relation (3.3.70)(4) dictates that the only sensible definition of $\bar{x}$ in terms of the maximizer $\bar{y}$ is

$$\bar{x} := (\partial J^*)(\bar{y}(1)) \stackrel{(3.3.66)}{=} \bar{y}(1). \tag{3.3.71}$$

We now have a pair $(\bar{x}, \bar{y}) \in \mathbb{U} \times \mathbb{Y}$, and we are going to verify that this pair satisfies the remaining optimality relations (3.3.70)(1) - (3):

1. We note from the primal problem that $\inf_{x \in \mathbb{R}} f(x) \in \mathbb{R}$, hence from (3.3.63) we have $g(\bar{y}) \in \mathbb{R}$, and then, from (3.3.55), we obtain

$$\bar{y} \geqslant 0, \tag{3.3.72}$$

so that (3.3.70) - (2) is verified.

2. We next verify that $\bar{x}$ satisfies the feasibility relation (3.3.70)(1). To this end we first obtain a necessary condition resulting from the known optimality of $\bar{y} \in \mathbb{Y}$ for the dual problem (recall (3.3.63) and the discussion of Remark 3.1.9). From (3.3.63) we have

$$\frac{1}{\varepsilon}\left[g(\bar{y}) - g(\bar{y} + \varepsilon y)\right] \geqslant 0, \quad \text{for all } \varepsilon > 0 \text{ and } y \in \mathbb{Y}, \tag{3.3.73}$$

(the difference is well-defined because of $g(\bar{y}) \in \mathbb{R}$). Fix some $y \in \mathbb{G}$ for

$$\mathbb{G} := \{y \in \mathbb{Y} \ : \ y \geqslant 0\}; \tag{3.3.74}$$

since $\bar{y} \geqslant 0$, we find $\bar{y} + \varepsilon y \geqslant 0$ for $\varepsilon > 0$, so that

$$\frac{1}{\varepsilon}\left[J^*(\bar{y}(1) + \varepsilon y(1)) - J^*(\bar{y}(1))\right] - y(\xi) \geqslant 0, \tag{3.3.75}$$

for all $\varepsilon > 0$ (as follows from (3.3.73) and (3.3.58)). Taking $\varepsilon \to 0$ at (3.3.75) we obtain

$$(\partial J^*)(\bar{y}(1))y(1) - y(\xi) \geqslant 0, \qquad \text{for all } y \in \mathbb{G}. \tag{3.3.76}$$

We note that (3.3.76) is a *necessary condition* obtained from the optimality of $\bar{y}$. From (3.3.76) and (3.3.71) we get

$$\bar{x}y(1) - y(\xi) \geqslant 0, \qquad \text{for all } y \in \mathbb{G}, \tag{3.3.77}$$

which can be written as

$$y(\bar{x} - \xi) \geqslant 0, \qquad \text{for all } y \in \mathbb{G}, \tag{3.3.78}$$

(since $\bar{x}y(1) = y(\bar{x})$). Now define

$$\eta(\omega) := \begin{cases} \xi(\omega) - \bar{x} & \text{for } \omega \in \{\xi > \bar{x}\} \subset \Omega, \\ 0 & \text{otherwise,} \end{cases} \tag{3.3.79}$$

so that $\eta \in \mathcal{L}_\infty(\Omega, \mathcal{F}, P)$ (since $\xi$ is uniformly distributed so that $\xi \in \mathcal{L}_\infty(\Omega, \mathcal{F}, P)$), and define

$$\hat{y}(u) := E[u\eta], \qquad u \in \mathcal{L}_\infty(\Omega, \mathcal{F}, P). \tag{3.3.80}$$

It follows that $\hat{y} \in \mathcal{L}^*_\infty(\Omega, \mathcal{F}, P)$. Furthermore, since $\eta \geqslant 0$, we also have $\hat{y}(u) \geqslant 0$ for all $P$-a.s. non-negative $u \in \mathcal{L}_\infty(\Omega, \mathcal{F}, P)$, so that in fact $\hat{y} \in \mathbb{G}$.

We now use (3.3.78) with $y := \hat{y}$ to get

$$0 \leqslant \hat{y}(\bar{x} - \xi) \overset{(3.3.79)}{=} -E\left[(\xi - \bar{x})^2; \xi > \bar{x}\right], \tag{3.3.81}$$

from which we then get

$$\bar{x} \geqslant \xi \text{ a.s.}, \tag{3.3.82}$$

so that (3.3.70) - (1) is verified.

3. As for showing (3.3.70) - (3), taking $y = -\bar{y}$ and $\varepsilon \in (0,1)$ at (3.3.73), and since $(1-\varepsilon)\bar{y} \geqslant 0$ for all $\varepsilon \in (0,1)$ by $\bar{y} \geqslant 0$, together with (3.3.58), it follows that

$$\frac{1}{\varepsilon}\left[J^*((1-\varepsilon)\bar{y}(1)) - J^*(\bar{y}(1))\right] + \bar{y}(\xi) \geqslant 0 \tag{3.3.83}$$

for all $\varepsilon \in (0,1)$. Taking $\varepsilon \to 0$ at (3.3.83) and using (3.3.71) then gives $\bar{y}(\xi - \bar{x}) \geqslant 0$. But because $\bar{x} \geqslant \xi$ and $\bar{y} \geqslant 0$, we also have $\bar{y}(\xi - \bar{x}) \leqslant 0$, which gives

$$\bar{y}(\xi - \bar{x}) = 0, \tag{3.3.84}$$

as required to verify (3.3.70) - (3).

**Remark 3.3.17.** To summarize, we have used Theorem 3.1.7 to establish that (3.3.63) holds. We then defined $\bar{x}$ in terms of $\bar{y}$ at (3.3.71) so that (3.3.70)(4) holds, and have verified that the pair $(\bar{x}, \bar{y})$ thus constructed satisfies the remaining optimality relations (3.3.70)(1) - (3) (see items 2,3 and 4 above). From the equivalence at (3.3.70) we conclude that

$$f(\bar{x}) = g(\bar{y}), \tag{3.3.85}$$

and therefore $\bar{x}$ must be the minimizer for the primal function $f$ given by (3.3.36) (recall Remark 3.1.9), and therefore $\bar{x}$ is a solution of the problem (3.3.31).

**Remark 3.3.18.** From the complementary slackness relation (3.3.84) we see that $\bar{y}$ is the Lagrange multiplier which enforces the constraint $x \geqslant \xi$ a.s. in the primal problem (3.3.31). It is of interest to discuss the structure of this Lagrange multiplier. According to Remark 3.3.10 the Lagrange multiplier $\bar{y} \in \mathbb{Y} = \mathcal{L}^*_\infty(\Omega, \mathcal{F}, P)$ has the Yosida-Hewitt decomposition $\bar{y} = (\bar{y}_r, \bar{y}_s)$ for the *regular* part $\bar{y}_r \in \mathcal{L}_1(\Omega, \mathcal{F}, P)$ and the *singular* part $\bar{y}_s \in \mathcal{Z}(\Omega, \mathcal{F}, P)$. We are now going to see that $\bar{y}_r = 0$, so that in fact $\bar{y} = \bar{y}_s$, that is the Lagrange multiplier

$\bar{y}$ is a *purely singular* member of $\mathbb{Y} = \mathcal{L}_\infty^*(\Omega, \mathcal{F}, P)$! To see this, observe that $\bar{y} \geqslant 0$ (from (3.3.72)) so that (recall Remark 3.3.11) we have

$$\bar{y}_r \geqslant 0 \ \text{ a.s.} \qquad \text{and} \qquad \bar{y}_s \geqslant 0. \tag{3.3.86}$$

This, together with (3.3.82), gives (recall Remark 3.3.11)

$$\bar{y}_r(\xi - \bar{x}) \leqslant 0 \ \text{ a.s.} \qquad \text{and} \qquad \bar{y}_s(\xi - \bar{x}) \leqslant 0. \tag{3.3.87}$$

On the other hand, from (3.3.84) we have

$$E\left[\bar{y}_r(\xi - \bar{x})\right] + \bar{y}_s(\xi - \bar{x}) = \bar{y}(\xi - \bar{x}) = 0, \tag{3.3.88}$$

(the first equality is from the Yosida-Hewitt decomposition of $\bar{y}$, see (3.3.26)). Since each of the summands on the left of (3.3.88) is non-positive, as follows from (3.3.87), we have

$$E\left[\bar{y}_r(\xi - \bar{x})\right] = 0 \qquad \text{and} \qquad \bar{y}_s(\xi - \bar{x}) = 0. \tag{3.3.89}$$

From the first inequality at (3.3.87) and the first relation at (3.3.89) we obtain

$$\bar{y}_r(\xi - \bar{x}) = 0 \ \text{ a.s.} \tag{3.3.90}$$

By inspection, we already know $\bar{x} = 1$, thus $\bar{x} > \xi$ a.s. since $\xi$ is $U[0,1]$-distributed, whence (3.3.90) yields $\bar{y}_r = 0$ a.s., i.e., $\bar{y} = \bar{y}_s$. We conclude that the Lagrange multiplier $\bar{y}$ is non-trivial and a *singular* member of $\mathbb{Y}$. The reason for this is that at optimality the constraint $x \geqslant \xi$ "binds" only on a set of zero measure (since $\bar{x} > \xi$ a.s.), and to enforce a constraint of this kind one needs a singular Lagrange multiplier.

**Remark 3.3.19.** Problem 3.3.12 is a very special instance of a class of convex stochastic programming problems studied by Rockafellar and Wets [32]with almost-sure inequality constraints in which *singular Lagrange multipliers* (that is members of $\mathcal{Z}(S, \Sigma, \mu)$) occur quite naturally as Lagrange multipliers for these constraints. The essential insight of Rockafellar and Wets is that inequality constraints which are *active* (that is, removal of the constraints strictly reduces the value of the primal problem) but "bind" only on a set of measure zero (as in Problem 3.3.12) necessarily call for Lagrange multipliers which are singular, that is "singularly binding" constraints necessarily involve singular Lagrange multipliers. In the setting of Problem 3.3.12 the reason for this is clearly evident in the complementary slackness relation (3.3.69)(3), which is used in Remark 3.3.18 to establish that $\bar{y} \in \mathcal{Z}(\Omega, \mathcal{F}, P)$.

**Remark 3.3.20.** As a complement to the discussion of Remark 3.3.15 we shall now explore in some detail the consequences of choosing a space of perturbations different from that at (3.3.47), such as the seemingly "more natural" space

$$\mathbb{U} := \mathcal{L}_2(\Omega, \mathcal{F}, P), \tag{3.3.91}$$

and we shall see what can "go wrong" when we use the space of perturbations (3.3.91). We shall keep the form of the perturbation function (3.3.48) (which is motivated by analogy with the perturbation function (3.2.39) for the finite-dimensional problem (3.2.34), with the obvious modification that the perturbation variable $v$ is now a member of $\mathcal{L}_2(\Omega, \mathcal{F}, P)$ rather than $\mathcal{L}_\infty(\Omega, \mathcal{F}, P)$:

$$F(x,v) := \begin{cases} J(x), & \text{when } x + v \geqslant \xi, \text{ a.s.,} \\ +\infty, & \text{otherwise,} \end{cases} \tag{3.3.92}$$

for all $(x,v) \in \mathbb{X} \times \mathbb{U} = \mathbb{R} \times \mathcal{L}_2(\Omega, \mathcal{F}, P)$. Exactly as at (3.3.49) we have the consistency relation

$$F(x,0) = f(x), \quad x \in \mathbb{X} := \mathbb{R}. \tag{3.3.93}$$

In accordance with Example 3.1.6 and Step 3.1.2, we shall pair the space of perturbations at (3.3.91) with the vector space $\mathbb{Y}$ taken to be the norm-dual of $\mathbb{U} := \mathcal{L}_2(\Omega, \mathcal{F}, P)$, and with the bilinear form $\langle \cdot, \cdot \rangle$ on $\mathbb{U} \times \mathbb{Y}$ defined by

$$\mathbb{Y} := (\mathbb{U})^* = \mathcal{L}_2(\Omega, \mathcal{F}, P), \quad \text{with} \quad \langle v, y \rangle := E[yv] \quad \text{for } (v,y) \in \mathbb{U} \times \mathbb{Y}. \tag{3.3.94}$$

(c.f. (3.3.50)). In accordance with Step 3.1.3 we define the *Lagrangian* $K : \mathbb{X} \times \mathbb{Y} \to [-\infty, +\infty]$ as follows:

$$K(x,y) := \inf_{v \in \mathcal{L}_2(\Omega, \mathcal{F}, P)} \{E[yv] + F(x,v)\}, \quad (x,y) \in \mathbb{R} \times \mathbb{Y}, \tag{3.3.95}$$

(c.f. (3.3.51)). From (3.3.92) and (3.3.94) we easily obtain (c.f. (3.3.52))

$$K(x,y) = \begin{cases} E[y(\xi - x)] + J(x), & \text{when } y \geqslant 0, \\ -\infty, & \text{otherwise,} \end{cases} \tag{3.3.96}$$

for all $(x,y) \in \mathbb{X} \times \mathbb{Y} := \mathbb{R} \times \mathcal{L}_2(\Omega, \mathcal{F}, P)$ (the notation $y \geqslant 0$ at (3.3.96) indicates that $y \in \mathbb{Y} := \mathcal{L}_2(\Omega, \mathcal{F}, P)$ is a.s. non-negative). In view of (3.1.4), define the *dual function* $g : \mathbb{Y} \to [-\infty, +\infty)$ as follows:

$$g(y) := \inf_{x \in \mathbb{R}} K(x,y), \qquad y \in \mathbb{Y} := \mathcal{L}_2(\Omega, \mathcal{F}, P). \tag{3.3.97}$$

Now it follows from (3.3.97), (3.3.96) and (3.3.30) that

$$g(y) = \begin{cases} E[y\xi] - J^*(E[y]), & \text{when } y \geqslant 0, \\ -\infty, & \text{otherwise,} \end{cases} \tag{3.3.98}$$

for all $y \in \mathbb{Y} := \mathcal{L}_2(\Omega, \mathcal{F}, P)$, in which $J^*(\cdot)$ is defined by (3.3.57) (c.f. (3.3.58)). With $g$ given by (3.3.98) and $f$ defined (as usual) by (3.3.36), one easily establishes the *Kuhn-Tucker optimality relations* in the form of the following equivalence which holds for each $(x,y) \in \mathbb{X} \times \mathbb{Y} := \mathbb{R} \times \mathcal{L}_2(\Omega, \mathcal{F}, P)$:

$$f(x) = g(y) \quad \Longleftrightarrow \quad \begin{cases} (1)\ x \geqslant \xi \ \text{ a.s.,} & (2)\ y \geqslant 0, \ \text{ a.s.,} \\ (3)\ y(\xi - x) = 0 \ \text{ a.s.,} & (4)\ x = (\partial J^*)(E[y]). \end{cases} \tag{3.3.99}$$

(c.f. (3.3.70)). We next observe that there *fails to exist* any $(\bar{x}, \bar{y}) \in \mathbb{X} \times \mathbb{Y} := \mathbb{R} \times \mathcal{L}_2(\Omega, \mathcal{F}, P)$ which satisfies (3.3.99)(1) - (4), that is which satisfies the conditions

$$\begin{cases} (1)\ \bar{x} \geqslant \xi \ \text{ a.s.}, & (2)\ \bar{y} \geqslant 0, \ \text{ a.s.}, \\ (3)\ \bar{y}(\xi - \bar{x}) = 0 \ \text{ a.s.}, & (4)\ \bar{x} = (\partial J^*)(E\,[\bar{y}]). \end{cases} \tag{3.3.100}$$

In fact, suppose that there exists a pair $(\bar{x}, \bar{y}) \in \mathbb{X} \times \mathbb{Y}$ such that (3.3.100)(1) - (4) holds. Then, from the equivalence (3.3.99), we get

$$f(\bar{x}) = g(\bar{y}), \tag{3.3.101}$$

and, in view of the universal inequality (3.1.7), from (3.3.101) we get

$$f(\bar{x}) = \inf_{x \in \mathbb{X}} f(x) = \sup_{y \in \mathbb{Y}} g(y) = g(\bar{y}). \tag{3.3.102}$$

From (3.3.36), together with the fact that $\xi$ is uniformly distributed over the unit interval $[0, 1]$ (see Problem 3.3.12), and the first equality at (3.3.102), we must have

$$\bar{x} = 1 \qquad \text{thus} \qquad f(\bar{x}) = \frac{1}{2}. \tag{3.3.103}$$

Moreover, from the uniform distribution of $\xi$ over $[0, 1]$ we have $P[\xi = 1] = 0$, and therefore

$$\bar{x} - \xi > 0 \qquad \text{a.s.}, \tag{3.3.104}$$

and from (3.3.104), together with the complementary slackness relation (3.3.100)(3), we find

$$\bar{y} = 0 \qquad \text{a.s.} \tag{3.3.105}$$

In view of (3.3.105), (3.3.98), and (3.3.57),

$$g(\bar{y}) = 0. \tag{3.3.106}$$

The contradiction resulting from (3.3.106), (3.3.103), and (3.3.101), establishes that there fails to exist any $(\bar{x}, \bar{y}) \in \mathbb{X} \times \mathbb{Y} = \mathbb{R} \times \mathcal{L}_2(\Omega, \mathcal{F}, P)$ which satisfies the optimality relations (3.3.100), that is, with the choice of perturbation space at (3.3.91), the Kuhn-Tucker optimality relations (3.3.99)(1)-(4) *can never be verified.* This of course is a consequence of the fact that the space of dual variables $\mathbb{Y}$ at (3.3.94), resulting from the perturbation space at (3.3.91), cannot possibly contain the Lagrange multiplier for the constraint $x \geqslant \xi$ in Problem 3.3.12, since we have already seen at Remark 3.3.18 that this Lagrange multiplier is a singular element in the space $\mathcal{Z}(\Omega, \mathcal{F}, P)$. As a final comment, it should be noted that it is impossible to verify the conditions of Theorem 3.1.7 when the perturbation space is defined by (3.3.91), since one cannot secure the condition (3.1.9). In fact, since $\mathbb{U} = \mathcal{L}_2(\Omega, \mathcal{F}, P)$ is a normed vector space, and the space of dual variables $\mathbb{Y}$ is the norm-dual of $\mathbb{U}$ (recall (3.3.94)), it follows at once from Example 3.1.6 that the topology from the norm $\|\cdot\|_2$ on $\mathbb{U}$ serves as the $\langle \mathbb{U}, \mathbb{Y} \rangle$-compatible topology $\mathcal{U}$, and verification of (3.1.9) then reduces to establishing

$$\sup \{F(x_1, v) \mid v \in \mathbb{U} := \mathcal{L}_2(\Omega, \mathcal{F}, P) \text{ with } \|v\|_2 < \alpha\} < +\infty, \tag{3.3.107}$$

for some $x_1 \in \mathbb{X} := \mathbb{R}$ and some $\alpha \in (0, \infty)$. Fix any $x_1 \in \mathbb{X}$. Since $\xi$ is essentially bounded, from elementary integration theory it follows that

$$\begin{cases} \text{for each } \alpha > 0 \text{ there is some } v \in \mathbb{U} := \mathcal{L}_2(\Omega, \mathcal{F}_T, P) \\ \text{such that } \|v\|_2 < \alpha \text{ and } P\{x_1 + v < \xi\} > 0. \end{cases} \tag{3.3.108}$$

In view of (3.3.108), (3.3.92) and the arbitrary choice of $x_1 \in \mathbb{X}$, we obtain

$$\sup_{\substack{v \in \mathbb{U} \\ \|v\|_2 < \alpha}} F(x_1, v) = +\infty \qquad \text{for each } x_1 \in \mathbb{X} \text{ and } \alpha > 0, \tag{3.3.109}$$

so that it is impossible to verify (3.3.107) for some $x_1 \in \mathbb{X}$ and $\alpha \in (0, \infty)$, and therefore one cannot verify condition (3.1.9) of Theorem 3.1.7.

# Chapter 4

# Quadratic Risk Minimization with Portfolio and European Wealth Constraints

## 4.1 Introduction

In this chapter we shall apply the Rockafellar variational approach to study problem (2.2.2) with a variety of different constraints on the portfolio and on the wealth at close of trade.

In contrast to the problems examined in Chapter 3, all of which involved minimization over *finite-dimensional* space, in the present chapter we are dealing with a *control problem*, the essence of which is to minimize a function $E[J(X^\pi(T))]$ of the portfolio process $\pi$, that is the problem is *infinite dimensional*. Despite this, we shall see that the Rockafellar approach applies much as in the problems of Chapter 3.

In the course of this chapter we shall address several of the stochastic control problems that were outlined in Chapter 2 (but not the Canonical Problem 2.2.7 which is reserved for Chapter 5). We shall see in particular that the variational approach of Rockafellar outlined in Chapter 3 establishes a *unified method* for addressing several problems of quadratic minimization which have already been studied by a variety of rather disparate methods. This includes, in particular, an *unconstrained* problem of quadratic minimization, addressed by Lim and Zhou [20] by classical stochastic linear quadratic control (see Section 4.2); a problem of quadratic minimization with convex portfolio constraints only, addressed by Labbé and Heunis [18] using a rather problem-specific application of conjugate duality which relies completely on the absence of any wealth constraints (see Section 4.3); and a problem of quadratic minimization with a European wealth constraint (but without portfolio constraints), a particular case of which was addressed by Bielecki *et al.* [3] using a problem-specific application of the risk-neutral method which relies on the absence of portfolio constraints (see Section 4.4). Finally, in Section 4.5, we look at a problem of quadratic minimization with a combination of convex portfolio constraints and a European wealth constraint, which seemingly cannot be addressed by any of the special methods indicated above and which calls for the full power of the Rockafellar variational approach. This problem was addressed in Heunis [12]. Besides illustrating another non-trivial application of the Rockafellar method, a major goal of Section 4.5 is to thoroughly re-work, simplify and streamline the use of Rockafellar's method in the work Heunis [12], so that the resulting application of this method generalizes as smoothly as possible to the Canonical Problem 2.2.7 which is addressed in Chapter 5.

In this chapter we shall convert the stochastic control problems that we address to a "primal form", in which we are guided by a classic work of Bismut [4] on convex stochastic calculus of variations. The work [4] has been, and remains, a continuing influence on much work in portfolio optimization. In particular, we follow Bismut [4] and introduce a real vector space of Itô processes which is large enough to include the wealth process $X^\pi$ given by (2.1.21) of a portfolio $\pi \in \Pi$, the definition of which we recall for convenience (see Notation 2.1.4 and (2.1.20)):

$$\Pi := \left\{ \pi : \Omega \times [0,T] \to \mathbb{R}^N \;\middle|\; \pi \in \mathcal{F}^* \text{ and } E\left[\int_0^T \|\pi(t)\|^2 \, \mathrm{d}t\right] < +\infty \right\}. \tag{4.1.1}$$

Following Bismut [4] define the real vector spaces $L_{21}$ and $\mathbb{B}$ as follows:

$$L_{21} := \left\{ \xi : \Omega \times [0,T] \to \mathbb{R} \;\middle|\; \xi \in \mathcal{F}^* \text{ and } E\left[\left(\int_0^T |\xi(t)| \, \mathrm{d}t\right)^2\right] < \infty \right\}, \tag{4.1.2}$$

(see Notation 2.1.4(2)) and

$$\mathbb{B} := \mathbb{R} \times L_{21} \times \Pi. \tag{4.1.3}$$

We write $X \equiv (X(0), \dot{X}, \Lambda_X) \in \mathbb{B}$ to indicate that $\{X(t), t \in [0,T]\}$ is the $\mathbb{R}$-valued $\mathcal{F}_t$-adapted continuous Itô process given by

$$X(t) := X(0) + \int_0^t \dot{X}(s) \mathrm{d}s + \int_0^t \Lambda_X'(s) \mathrm{d}W(s), \tag{4.1.4}$$

for some triplet $(X(0), \dot{X}, \Lambda_X) \in \mathbb{B}$. By the following Lemma 4.1.1, the "integrand processes" $\dot{X} \in L_{21}$ and $\Lambda_X \in \Pi$ in the representation (4.1.4) are a.e.-uniquely determined on $\Omega \times [0,T]$. That is, every triplet $(X_0, \dot{X}, \Lambda_X)$ in $\mathbb{B}$ corresponds to an $\mathcal{F}_t$-Itô process $X$ defined by (4.1.4), and the notation $X \in \mathbb{B}$ indicates that $X$ is an $\mathcal{F}_t$-Itô process defined by (4.1.4) for some $X(0) \in \mathbb{R}$ and some (necessarily a.e.-unique) integrands $\dot{X} \in L_{21}$ and $\Lambda_X \in \Pi$. The elementary proof of Lemma 4.1.1 is included in Appendix A.

**Lemma 4.1.1.** *If $X \in \mathbb{B}$, then there exists a.e.-uniquely a triplet $(X_0, \dot{X}, \Lambda_X) \in \mathbb{R} \times L_{21} \times \Pi$ such that (4.1.4) holds.*

**Remark 4.1.2.** The spaces of processes $L_{21}$ and $\mathbb{B}$ were introduced by Bismut [4]. The space $L_{21}$ (of which the integrand processes $\dot{X}$ are members) turns out to be exactly the "right" space for the optimal control problems that we shall address in this chapter. Do note that the integrand process $\dot{X}$ on the right of (4.1.4) does *not* denote the $t$-derivative of the process $X$ (this would be the case if $\Lambda_X = 0$); the "dot" over $X$ just serves to remind us that the process $\dot{X}$ goes naturally with a Lebesgue (or d$s$) integral to give the process of finite variation in the semimartingale $X$.

**Remark 4.1.3.** Comparing (4.1.4) and (2.1.21), we expect to have $X^\pi \in \mathbb{B}$ and

$$X^\pi(0) = x_0, \quad \dot{X}^\pi = rX^\pi + \pi'\sigma\theta \quad \text{and} \quad \Lambda'_{X^\pi} = \pi'\sigma, \qquad \text{for all } \pi \in \Pi. \tag{4.1.5}$$

In fact we have the following proposition, the elementary proof of which is given in Appendix A.

**Proposition 4.1.4.** *Assume Condition 2.1.1, 2.1.2 and 2.1.5 on market settings. If $X$ is an $\mathbb{R}$-valued $\mathcal{F}_t$-adapted Itô process given by the SDE at (2.1.17), that is*

$$dX(t) = [r(t)X(t) + \pi'(t)\sigma(t)\theta(t)]\, dt + \pi'(t)\sigma(t)\, dW(t), \quad \text{for some } \pi \in \Pi, \tag{4.1.6}$$

*with $X(0) \in \mathbb{R}$ non-random, then*

$$X \in \mathbb{B} \qquad \text{with} \quad \dot{X} = rX + \pi'\sigma\theta \text{ and } \Lambda'_X = \pi'\sigma. \tag{4.1.7}$$

From Doob's maximal $L^2$-inequality one obtains the following result (the proof is in Appendix A).

**Proposition 4.1.5.** *If $X \in \mathbb{B}$, then*

$$E\left[\max_{t\in[0,T]} |X(t)|^2\right] < \infty. \tag{4.1.8}$$

Closely related to Proposition 4.1.5 is the following result, the proof of which is in Appendix A:

**Proposition 4.1.6.** *Assume Condition 2.1.1, 2.1.2, 2.1.5 and 2.1.11 on market settings. If $\pi$ is a portfolio process in the sense of Definition 2.1.10 (that is $\pi$ is $\mathcal{F}^*$-measurable and subject to (2.1.18)), and $X^\pi$ is the corresponding wealth process defined by the wealth equation (2.1.21), then we have the equivalence*

$$X^\pi \in \mathbb{B} \qquad \Longleftrightarrow \qquad \pi \in \Pi. \tag{4.1.9}$$

We also introduce the following vector subspace of $\mathbb{B}$, which is important for the construction of the space of dual variables $\mathbb{Y}$ for problem (2.2.2):

$$\mathbb{B}_1 := \left\{ Y \equiv (Y(0), \dot{Y}, \Lambda_Y) \in \mathbb{B} \mid \dot{Y}(t) = -r(t)Y(t) \quad \text{a.e.} \right\}. \tag{4.1.10}$$

**Remark 4.1.7.** (a) If $Y = (Y(0), \dot{Y}, \Lambda_Y) \in \mathbb{B}_1$, we have from (4.1.10) and (4.1.4) that

$$\mathrm{d}Y(t) = -r(t)Y(t)\mathrm{d}t + \Lambda'_Y(t)\mathrm{d}W(t), \tag{4.1.11}$$

and it follows at once from the integration-by-parts formula that

$$Y(t) = \frac{1}{S_0(t)} \left\{ Y(0) + \int_0^t S_0(\tau)\Lambda'_Y(\tau)\mathrm{d}W(\tau) \right\} \quad t \in [0, T]. \tag{4.1.12}$$

The relation (4.1.12) motivates the following definition:

$$\Xi(y,\gamma)(t) := \frac{1}{S_0(t)}\left\{y + \int_0^t S_0(\tau)\gamma'(\tau)\mathrm{d}W(\tau)\right\}, \quad t\in[0,T], \quad (y,\gamma)\in\mathbb{R}\times\Pi. \tag{4.1.13}$$

It is easily verified that

$$\Xi : \mathbb{R}\times\Pi \to \mathbb{B}_1 \text{ is a linear bijection,} \tag{4.1.14}$$

and it follows from (4.1.12) that

$$\Xi(Y(0),\Lambda_Y) = Y \qquad \text{for all } Y\in\mathbb{B}_1. \tag{4.1.15}$$

We write $Y = (Y(0),\Lambda_Y)$ (for some $Y(0)\in\mathbb{R}$ and $\Lambda_Y\in\Pi$) to denote that $Y\in\mathbb{B}_1$ and is given by (4.1.15), equivalently given by (4.1.12).

(b) Recalling Definition 2.1.6 and (2.1.10), we see that the *state price density process* $H$ is an element of $\mathbb{B}_1$, more precisely

$$H\in\mathbb{B}_1 \qquad \text{with} \qquad H(0)=1 \qquad \text{and} \qquad \Lambda_H = -H\theta, \tag{4.1.16}$$

where $\theta$ is the *market price of risk* (recall Definition 2.1.6). Furthermore, if $Y\in\mathbb{B}_1$ and $\Lambda_Y = -Y\theta$, we have from (4.1.11) and (4.1.4) that

$$Y(t) = Y(0)H(t), \qquad t\in[0,T] \quad \text{a.s.} \tag{4.1.17}$$

The following result, which is proved in Appendix A, will be used numerous times in this thesis:

**Lemma 4.1.8.** *Assume Condition 2.1.1, 2.1.2, 2.1.5, and recall $\theta$ at (2.1.8). Then, for any $y\in\mathbb{R}$ and $\nu\in\Pi$, there exists some $Y\in\mathbb{B}_1$ such that*

$$Y(0) = y \quad \textit{and} \quad \Lambda_Y + \theta Y = \nu \quad \textit{a.e.} \tag{4.1.18}$$

The next result establishes that any given element of $\mathcal{L}_2(\Omega,\mathcal{F}_T,P)$ defines a unique element of $\mathbb{B}_1$ as follows:

**Proposition 4.1.9.** *Suppose Condition 2.1.1 and 2.1.2. Then, for each $\eta\in\mathcal{L}_2(\Omega,\mathcal{F}_T,P)$, there is a unique $Y\in\mathbb{B}_1$ such that $Y(T)=\eta$ a.s..*

The proof of Proposition 4.1.9 is an elementary application of the martingale representation theorem, and is included in Appendix A. Since $\mathbb{B}_1\subset\mathbb{B}$, we have that $Y(T)\in\mathcal{L}_2(\Omega,\mathcal{F}_T,P)$ for all $Y\in\mathbb{B}_1$ (by Proposition 4.1.5), and the following is immediate:

**Corollary 4.1.10.** *$\mathbb{B}_1$ and $\mathcal{L}_2(\Omega,\mathcal{F}_T,P)$ are isomorphic in the sense that the mapping $\Upsilon:\mathbb{B}_1\to\mathcal{L}_2(\Omega,\mathcal{F}_T,P)$, defined by $\Upsilon(Y)=Y(T)$ for all $Y\in\mathbb{B}_1$, is a linear bijection.*

In the remainder of this chapter, we shall address different versions of problem (2.2.2) with constraint structures of increasing complexity. To this end, recall the *risk criterion function* $J$ at (2.2.3), that is

$$J(x,\omega) := \frac{1}{2}\left[a(\omega)x^2 + 2c(\omega)x\right] + q, \qquad (x,\omega) \in \mathbb{R} \times \Omega, \tag{4.1.19}$$

for some $q \in \mathbb{R}$, $0 < a \in \mathcal{L}_\infty(\Omega, \mathcal{F}_T, P)$ and $c \in \mathcal{L}_2(\Omega, \mathcal{F}_T, P)$ satisfying Condition 2.2.1. The *convex conjugate function* $J^*(\cdot)$ with respect to the variable $x$ (with $\omega$ fixed) is calculated in the usual way:

$$J^*(y,\omega) := \sup_{x\in\mathbb{R}} \{xy - J(x,\omega)\} = \frac{(y - c(\omega))^2}{2a(\omega)} - q, \qquad (y,\omega) \in \mathbb{R} \times \Omega, \tag{4.1.20}$$

and the derivative function of the convex conjugate $J^*(\cdot)$ is of course:

$$\partial J^*(y,\omega) := \frac{y - c(\omega)}{a(\omega)}, \qquad (y,\omega) \in \mathbb{R} \times \Omega. \tag{4.1.21}$$

**Remark 4.1.11.** It is immediate from (4.1.21) that $J^*(\cdot,\omega)$ is $\mathbb{R}$-valued and smooth on $\mathbb{R}$ for each $\omega \in \Omega$. Therefore, for each $x, y \in \mathbb{R}$ and $\omega \in \Omega$, we have the equivalence that

$$J(x,\omega) + J^*(y,\omega) = xy \quad \Longleftrightarrow \quad x = \partial J^*(y,\omega), \tag{4.1.22}$$

(see (C.6)). The following proposition will be useful for establishing Kuhn-Tucker relations in this chapter as well as Chapter 5. The elementary proof is in Appendix A.

**Proposition 4.1.12.** *For $\xi, \eta \in \mathcal{L}_2(\Omega, \mathcal{F}, P)$, we have the following equivalence:*

$$E\left[J(\xi) + J^*(\eta) - \xi\eta\right] = 0 \quad \Longleftrightarrow \quad \xi = \partial J^*(\eta) \ a.s. \tag{4.1.23}$$

## 4.2 The Unconstrained Case

In this section, we first look at problem (2.2.2) with no constraints, namely

$$\text{minimize } E\left[J(X^\pi(T))\right] \quad \text{such that } \pi \in \Pi, \tag{4.2.1}$$

that is, determine an optimal portfolio process $\bar{\pi} \in \Pi$, recall (4.1.1), such that

$$E\left[J(X^{\bar{\pi}}(T))\right] = \inf_{\pi\in\Pi}\{E\left[J(X^\pi(T))\right]\}. \tag{4.2.2}$$

**Remark 4.2.1.** We recall that for each $\pi \in \Pi$, the corresponding wealth process $X^\pi$ starting from the initial wealth $x_0$ stipulated in Condition 2.1.11 is given by the stochastic differential equation (2.1.21).

First, write problem (4.2.2) in the form of *primal problem* by taking the space of primal variables $\mathbb{X}$ to be the vector space $\Pi$, that is

$$\mathbb{X} = \Pi. \tag{4.2.3}$$

The unconstrained problem is then

**Problem 4.2.2.** Determine an optimal portfolio process $\bar{\pi} \in \Pi$ such that

$$f(\bar{\pi}) = \inf_{\pi \in \Pi} f(\pi), \tag{4.2.4}$$

where the *primal function* $f(\cdot) : \Pi \to (-\infty, +\infty]$ is defined following the general form (3.1.1) as

$$f(\pi) := E\left[J(X^{\pi}(T))\right], \qquad \text{for } \pi \in \Pi, \tag{4.2.5}$$

and $J$ is defined by (2.2.3) (see also (4.1.19)). From Remark 2.2.3

$$\vartheta := \inf_{\pi \in \Pi}\{f(\pi)\} = \inf_{\pi \in \Pi}\{E\left[J(X^{\pi}(T))\right]\} \in \mathbb{R} \quad \text{ with } \vartheta \geqslant \underline{l}. \tag{4.2.6}$$

**Remark 4.2.3.** Problem 4.2.2 is addressed in Section 3.2 of Lim and Zhou [20] using classical stochastic linear quadratic control. This requires extensive use of BSDEs (backwards stochastic differential equations), the introduction of a *stochastic Riccati equation* (a particular BSDE), and a solvability theory for this BSDE. The approach that we take in the present section does not require the introduction of any of these entities, but instead relies on the underlying convexity of the problem, together with an extremely simple application of the Rockafellar variational approach summarized in Chapter 3.

We next follow the steps of Rockafellar's variational approach outlined in Section 3.1, that is we shall choose a real vector space $\mathbb{U}$ of perturbations, a perturbation function $F(\cdot, \cdot)$, a real vector space $\mathbb{Y}$ of dual variables and a bilinear form on $\mathbb{U} \times \mathbb{Y}$, and then construct a Lagrangian function and a dual function.

1. For Step 3.1.1, we must fix a vector space of perturbations $\mathbb{U}$ and define a perturbation function. Since $X^{\pi}(T)$ is square integrable (recall (2.1.24)), we will start with the "natural" choice of perturbation space

$$\mathbb{U} := \mathcal{L}_2(\Omega, \mathcal{F}_T, P), \tag{4.2.7}$$

and define the *perturbation function* $F : \mathbb{X} \times \mathbb{U} \to (-\infty, +\infty]$ as

$$F(\pi, u) := E\left[J(X^{\pi}(T) - u)\right] \qquad (\pi, u) \in \Pi \times \mathbb{U}. \tag{4.2.8}$$

The convexity of $F$ on $\Pi \times \mathbb{U}$ follows since $\pi \to X^{\pi}(T) : \Pi \mapsto \mathcal{L}_2(\Omega, \mathcal{F}_T, P)$ is an affine mapping, together with (4.1.19) and (4.2.8). From (4.2.5) it follows that the consistency relation (3.1.2) holds, i.e.,

$$F(\pi, 0) = f(\pi) \quad \text{ for all } \pi \in \Pi. \tag{4.2.9}$$

**Remark 4.2.4.** Problem 4.2.2 is said to be *unconstrained* because there are no direct constraints on the portfolio process $\pi$. There is nevertheless a "hidden" constraint present in Problem 4.2.2, namely the relation between the "primal variable" $\pi$ and the corresponding wealth $X^{\pi}(T)$ determined by the wealth relation (2.1.21). It is to deal with this hidden constraint that we introduce the perturbation by the variable $u \in \mathbb{U} := \mathcal{L}_2(\Omega, \mathcal{F}_T, P)$ at (4.2.8). The perturbation at (4.2.8) is rather analogous to the perturbation at (3.2.9) for Problem 3.2.1, in which the "perturbational variable" $u \in \mathbb{U} := \mathbb{R}^n$ accounts for the direct constraint $x \in A$ at (3.2.3) (see Remark 3.2.2).

2. Following Step 3.1.2, we must pair the space of perturbations at (4.2.7) with a vector space $\mathbb{Y}$ of dual variables through a bilinear form $\langle\cdot,\cdot\rangle$ on $\mathbb{U}\times\mathbb{Y}$, and the natural choice is the adjoint $(\mathcal{L}_2(\Omega,\mathcal{F}_T,P))^* = \mathcal{L}_2(\Omega,\mathcal{F}_T,P)$ (recall Remark 3.1.8). But Corollary 4.1.10 establishes a linear isomorphism between $\mathcal{L}_2(\Omega,\mathcal{F}_T,P)$ and $\mathbb{B}_1$, and we therefore define the space of dual variables (recall (4.1.10))

$$\mathbb{Y} := \mathbb{B}_1, \tag{4.2.10}$$

together with the "natural" bilinear form on $\mathbb{U}\times\mathbb{Y}$ defined by

$$\langle u, Y\rangle := E\left[uY(T)\right] \qquad \text{for all } u\in\mathbb{U} \text{ and } Y\in\mathbb{Y}. \tag{4.2.11}$$

This completes all the choices required for Steps 3.1.1 - 3.1.2, and it remains to synthesize a Lagrangian and a dual function.

3. According to Step 3.1.3 and (3.1.3), define the *Lagrangian function* $K:\Pi\times\mathbb{Y}\to[-\infty,+\infty]$ as follows:

$$K(\pi,Y) := \inf_{u\in\mathcal{L}_2(\Omega,\mathcal{F}_T,P)}\{\langle u,Y\rangle + F(\pi,u)\}. \tag{4.2.12}$$

From (4.2.12) (with $u:=0$) and (4.2.9) we have the basic inequality

$$f(\pi)\geqslant K(\pi,Y), \qquad \text{for all } (\pi,Y)\in\Pi\times\mathbb{Y}. \tag{4.2.13}$$

We shall now explicitly evaluate the right-hand side of (4.2.12):

$$K(\pi,Y) \overset{(4.2.8)}{\underset{(4.2.11)}{=}} \inf_{u\in\mathcal{L}_2(\Omega,\mathcal{F}_T,P)}\left\{E\left[uY(T)+J(X^\pi(T)-u)\right]\right\}, \tag{4.2.14}$$

for $(\pi,Y)\in\Pi\times\mathbb{Y}$. Since $X^\pi\in\mathbb{B}$ for all $\pi\in\Pi$ and $Y\in\mathbb{Y}=\mathbb{B}_1\subset\mathbb{B}$ (see (4.2.10) and (4.1.10)), we have from (4.1.8) that

$$X^\pi(T), Y(T)\in\mathcal{L}_2(\Omega,\mathcal{F}_T,P) \qquad \text{for all } \pi\in\Pi \text{ and } Y\in\mathbb{Y}. \tag{4.2.15}$$

Therefore, we can apply Proposition D.0.8 to (4.2.14) by replacing $\xi,\eta$ in (D.6) with $X^\pi(T), Y(T)$ and get the Lagrangian function in evaluated form:

$$K(\pi,Y) = E\left[X^\pi(T)Y(T)\right] - E\left[J^*(Y(T))\right] \qquad (\pi,Y)\in\Pi\times\mathbb{Y}. \tag{4.2.16}$$

In view of (3.1.4), we can now define the *dual function* $g:\mathbb{Y}\to[-\infty,+\infty)$ as:

$$g(Y) := \inf_{\pi\in\Pi} K(\pi,Y). \tag{4.2.17}$$

From (4.2.17)

$$K(\pi,Y)\geqslant g(Y), \qquad \text{for all } (\pi,Y)\in\Pi\times\mathbb{Y}. \tag{4.2.18}$$

Upon combining (4.2.18) and (4.2.13) we get the fundamental *weak duality principle:* (c.f. (3.2.16))

$$f(\pi)\geqslant K(\pi,Y)\geqslant g(Y), \qquad \text{for all } (\pi,Y)\in\Pi\times\mathbb{Y}. \tag{4.2.19}$$

To evaluate the dual function $g$ at (4.2.17), we need the following result which is proved in Appendix A (this result is also used again later in this chapter, in Section 4.4):

**Proposition 4.2.5.** *Suppose Condition 2.1.1, 2.1.2, 2.1.5 and 2.1.11. For each $Y \in \mathbb{B}_1$, we have*

$$\inf_{\pi\in\Pi} \{E[X^\pi(T)Y(T)]\} = \begin{cases} x_0 Y(0) & \text{if } Y = Y(0)H \text{ a.e.,} \\ -\infty & \text{otherwise.} \end{cases} \tag{4.2.20}$$

Combining (4.2.17), (4.2.16) and (4.2.20), we have the following closed form for the dual functional:

$$\begin{aligned} g(Y) &\overset{(4.2.17)}{\underset{(4.2.16)}{=}} -E[J^*(Y(T))] + \inf_{\pi\in\Pi}\{E[X^\pi(T)Y(T)]\} \\ &\overset{(4.2.20)}{=} \begin{cases} x_0Y(0) - E[J^*(Y(0)H(T))] & \text{if } Y = Y(0)H \text{ a.e.,} \\ -\infty & \text{otherwise,} \end{cases} \end{aligned} \tag{4.2.21}$$

for all $Y \in \mathbb{Y}$.

The *dual problem* is then to maximize $g(Y)$ over all $Y \in \mathbb{Y} := \mathbb{B}_1$, i.e., to establish that there exists some $\bar{Y} \in \mathbb{Y}$ such that

$$g(\bar{Y}) = \sup_{Y\in\mathbb{Y}}\{g(Y)\}. \tag{4.2.22}$$

The dual problem in (4.2.22) is defined over the space of dual variables $\mathbb{Y}$, but by the form of the dual function $g$ given in (4.2.21) it just reduces to a one-dimensional optimization problem over $\mathbb{R}$. To see this, define the $\mathbb{R}$-valued function $g_1$ on $\mathbb{R}$ as

$$g_1(y) := x_0 y - E[J^*(yH(T))], \qquad y \in \mathbb{R}. \tag{4.2.23}$$

From (4.2.21), for each $Y \in \{Y \in \mathbb{Y} \mid Y \neq \alpha H, \alpha \in \mathbb{R}\}$ we have $g(Y) = -\infty$, and therefore

$$g_1(y) = g(yH) \geqslant g(Y) = -\infty, \quad y \in \mathbb{R}. \tag{4.2.24}$$

Since $yH \in \mathbb{Y}$ for all $y \in \mathbb{R}$, it follows from (4.2.24) that

$$\sup_{Y\in\mathbb{Y}}\{g(Y)\} = \sup_{y\in\mathbb{R}}\{g(yH)\} = \sup_{y\in\mathbb{R}}\{g_1(y)\}. \tag{4.2.25}$$

Therefore, to solve the dual problem in (4.2.22) is equivalent to determine some $\bar{y} \in \mathbb{R}$ such that

$$g_1(\bar{y}) = \sup_{y\in\mathbb{R}}\{g_1(y)\}, \tag{4.2.26}$$

for it then follows from (4.2.26), (4.2.23) and (4.2.21) that

$$g(\bar{Y}) = \sup_{Y\in\mathbb{Y}}\{g(Y)\} \qquad \text{for} \qquad \bar{Y} := \bar{y}H. \tag{4.2.27}$$

We next establish that the maximizer $\bar{y} \in \mathbb{R}$ asserted at (4.2.26) indeed exists. Using (4.1.20) in (4.2.23), we get

$$\begin{aligned} g_1(y) &= x_0 y - E\left[J^*(yH(T))\right] = x_0 y - E\left[\frac{(yH(T)-c)^2}{2a} - q\right] \\ &= -y^2 E\left[\frac{H(T)^2}{2a}\right] + y\left(x_0 + E\left[\frac{cH(T)}{a}\right]\right) - E\left[\frac{c^2}{2a} - q\right], \quad y \in \mathbb{R}. \end{aligned} \tag{4.2.28}$$

Then $g_1$ is the quadratic function

$$g_1(y) = -\left(y - \frac{B}{2A}\right)^2 + \frac{B^2 - 4AC}{4A^2}, \tag{4.2.29}$$

with $A := E\left[\frac{H(T)^2}{2a}\right] > 0$, $B := \left(x_0 + E\left[\frac{cH(T)}{a}\right]\right)$ and $C := E\left[\frac{c^2}{2a} - q\right]$. Thus, the maximizer of problem (4.2.26) is

$$\bar{y} := \arg\max_{y \in \mathbb{R}}\{g_1(y)\} = \frac{B}{2A} = \left(x_0 + E\left[\frac{cH(T)}{a}\right]\right) / E\left[\frac{H(T)^2}{a}\right], \tag{4.2.30}$$

and by (4.2.27) the maximizer of the dual problem (4.2.22) is

$$\bar{Y} := \bar{y}H \in \mathbb{Y} := \mathbb{B}_1. \tag{4.2.31}$$

We have therefore established that there exists some $\bar{Y} \in \mathbb{Y}$ which maximizes the dual function $g$ over the space of dual variables $\mathbb{Y}$, that is (4.2.22) holds, and furthermore this maximizing element $\bar{Y}$ is given by (4.2.30) and (4.2.31).

**Remark 4.2.6.** Because of the simplicity of the unconstrained problem (4.2.1) the dual problem essentially boils down to minimizing a non-degenerate quadratic function defined on $\mathbb{R}$ (see (4.2.29)), and existence of a solution of the dual problem is immediate. In particular, in this case there is no need to use Theorem 3.1.7 to secure existence of a solution of the dual problem (although an essentially trivial use of Theorem 3.1.7 would have also yielded this result). Later, when we introduce genuine constraints, we will begin to see the value of Theorem 3.1.7 for establishing existence of dual solutions. In fact, the very simple problem (3.3.31) already gives a hint of how useful Theorem 3.1.7 can be when constraints are present; this theorem immediately establishes the existence of a maximizer $\bar{y} \in \mathcal{L}^*_\infty(\Omega, \mathcal{F}, P)$ of the dual function $g(\cdot)$ defined on $\mathbb{Y} := \mathcal{L}^*_\infty((\Omega, \mathcal{F}, P))$ by (3.3.58).

From Remark 3.1.9, together with the *weak duality* relation (4.2.19), we see that Problem 4.2.2 (i.e. the *primal problem*) reduces to the construction of some pair $(\bar{\pi}, \bar{Y}) \in \mathbb{X} \times \mathbb{Y}$ such that

$$f(\bar{\pi}) = g(\bar{Y}), \tag{4.2.32}$$

for it then follows from the weak duality relation (4.2.19) that $\bar{\pi}$ is a minimizer of $f(\cdot)$ on $\mathbb{X} = \Pi$ and $\bar{Y}$ is a maximizer of $g(\cdot)$ on $\mathbb{Y} = \mathbb{B}_1$. However, we have already secured such a maximizer $\bar{Y}$, so the solution of Problem 4.2.2 actually reduces to the construction of some $\bar{\pi} \in \Pi$ such that (4.2.32) holds for the $\bar{Y}$ given by (4.2.30) and (4.2.31). To facilitate construction of this $\bar{\pi}$ we next establish a set of *Kuhn-Tucker optimality relations* which are fully equivalent to the equality $f(\pi) = g(Y)$ for *arbitrary* $(\pi, Y) \in \Pi \times \mathbb{Y}$ (recall Remark 3.1.10 (c)):

**Proposition 4.2.7. [Kuhn-Tucker Optimal Conditions]**
*Suppose Condition 2.1.1, 2.1.2, 2.1.5, 2.1.11 on the market settings and Condition 2.2.1 on the risk criterion function $J$ given in (4.1.19). Then we have the following equivalence for each $(\pi, Y) \in \Pi \times \mathbb{Y}$*

$$f(\pi) = g(Y) \iff (1)\ Y = Y(0)H \ a.e.,\ (2)\ X^{\pi}(T) = \partial J^{*}(Y(T)), \tag{4.2.33}$$

*where $\partial J^{*}(\cdot)$ denotes the derivative function of the convex conjugate function $J^{*}(\cdot)$ given in (4.1.21).*

The proof of Proposition 4.2.7 is given in Appendix A.

**Remark 4.2.8.** Condition (1) on the right hand side of (4.2.33) is a *feasibility condition* on the dual variable $Y$, while Condition (2) on the right hand side of (4.2.33) is a so-called *transversality condition* which explicitly states a relationship between the primal variable $\pi \in \mathbb{X}$, the dual variable $Y \in \mathbb{Y}$, and the risk criterion functional $J(\cdot)$ (problem (4.2.1) is so simple that no complementary slackness conditions naturally occur among the Kuhn-Tucker relations). For the maximizer $\bar{Y} \in \mathbb{Y}$ given by (4.2.31) we already know that $\bar{Y}$ satisfies the feasibility condition

$$\bar{Y} = \bar{Y}(0)H, \tag{4.2.34}$$

(from (4.2.31) and $H(0) = 1$), so it remains to construct $\bar{\pi} \in \Pi$ such that the following transversality condition holds

$$X^{\bar{\pi}}(T) = \partial J^{*}(\bar{Y}(T)), \tag{4.2.35}$$

in order to secure (4.2.32) from Proposition 4.2.7.

To construct a portfolio process $\bar{\pi} \in \Pi$ such that (4.2.35) holds we need the following technical result from Labbé and Heunis [18, Lemma 5.1 & 5.2, p.88], a tool which will be used several times during this thesis:

**Proposition 4.2.9.** *For any $\mathcal{F}_T$-measurable random variable $\xi$ such that $E\left[\xi^2\right] < \infty$, define*

$$X(t) := H^{-1}(t)E\left[H(T)\xi \,\middle|\, \mathcal{F}_t\right], \qquad t \in [0, T]. \tag{4.2.36}$$

*Then there exists some a.e. unique $\mathbb{R}^N$-valued $\mathcal{F}_t$-progressively measurable process $\psi$ such that*

$$\int_0^T \|\psi(t)\|^2\, dt < \infty \quad a.s. \qquad and \qquad H(t)X(t) = X(0) + \int_0^t \psi'(s)dW(s).$$

*Moreover, for the $\mathbb{R}^N$-valued $\mathcal{F}_t$-progressively measurable process*

$$\pi(t) := (\sigma'(t))^{-1}\left[H^{-1}(t)\psi(t) + X(t)\theta(t)\right], \tag{4.2.37}$$

*we have*

- $\pi \in \Pi$ *(recall Definition 2.1.10),*
- $X$ *satisfies the relation*

$$dX(t) = [r(t)X(t) + \pi'(t)\sigma(t)\theta(t)]\,dt + \pi'(t)\sigma(t)dW(t), \tag{4.2.38}$$

*with the initial value*

$$X(0) = E\left[H(T)\xi\right]. \tag{4.2.39}$$

**Remark 4.2.10.** Notice that the square-integrability postulated for $\xi$ in Proposition 4.2.9 ensures that $H(T)\xi$ is *integrable*, so that the conditional expectation at (4.2.36) is well defined.

Now, from Condition 2.2.1, we know

$$\partial J^*(\bar{Y}(T)) \overset{(4.1.20)}{=} \frac{\bar{Y}(T) - c}{a} \overset{(4.2.31)}{=} \frac{(\bar{y}H(T) - c)}{a} \in \mathcal{L}_2(\Omega, \mathcal{F}_T, P), \tag{4.2.40}$$

with $\bar{y} \in \mathbb{R}$ given in (4.2.30). Motivated by (4.2.36) define the process $\bar{X}$ as

$$\bar{X}(t) := H^{-1}(t)E\left[H(T)\partial J^*(\bar{Y}(T))\,\middle|\,\mathcal{F}_t\right] \qquad t \in [0, T]. \tag{4.2.41}$$

Then, from Proposition 4.2.9, there exists some a.e. unique $\mathbb{R}^N$-valued $\mathcal{F}_t$-progressively measurable process $\bar{\psi}$ such that

$$\int_0^T \left\|\bar{\psi}(t)\right\|^2 \mathrm{d}t < \infty \quad \text{a.s.} \qquad \text{and} \qquad H(t)\bar{X}(t) = \bar{X}(0) + \int_0^t \bar{\psi}'(s)\mathrm{d}W(s).$$

Now define an $\mathbb{R}^N$-valued $\mathcal{F}_t$-progressively measurable process

$$\bar{\pi}(t) := (\sigma'(t))^{-1}\left[H^{-1}(t)\bar{\psi}(t) + \bar{X}(t)\theta(t)\right]. \tag{4.2.42}$$

It follows from Proposition 4.2.9 again that

$$\bar{\pi} \in \Pi, \tag{4.2.43}$$

and

$$\mathrm{d}\bar{X}(t) = \left[r(t)\bar{X}(t) + \bar{\pi}'(t)\sigma(t)\theta(t)\right]\mathrm{d}t + \bar{\pi}'(t)\sigma(t)\mathrm{d}W(t), \tag{4.2.44}$$

with the initial value

$$\bar{X}(0) = E\left[H(T)\partial J^*(\bar{Y}(T))\right]. \tag{4.2.45}$$

Combine (4.2.45), (4.2.40) and (4.2.30), it then follows from straightforward calculation

$$\begin{aligned}\bar{X}(0) &= E\left[H(T)\frac{(\bar{y}H(T)-c)}{a}\right] = \bar{y}E\left[\frac{H(T)^2}{a}\right] - E\left[\frac{H(T)c}{a}\right] \\ &= \frac{\left(x_0 + E\left[\frac{cH(T)}{a}\right]\right)}{E\left[\frac{H(T)^2}{a}\right]}E\left[\frac{H(T)^2}{a}\right] - E\left[\frac{H(T)c}{a}\right] = x_0. \end{aligned} \tag{4.2.46}$$

With (4.2.44) and (4.2.46), we can define a wealth process of the portfolio $\bar{\pi} \in \Pi$ in the sense of (2.1.21) as

$$X^{\bar{\pi}} := \bar{X}. \tag{4.2.47}$$

It is also clear from (4.2.47) and (4.2.41) that the transversality condition (4.2.35) holds, and we therefore conclude that $\bar{\pi} \in \Pi$ given by (4.2.42) is the optimal portfolio process solving the Problem 4.2.2 (recall Remark 4.2.8).

**Remark 4.2.11.** Verification of the Kuhn-Tucker optimality conditions at the right hand side of (4.2.33) here is more or less trivial for this unconstrained problem 4.2.2 because the dual variable essentially reduces to a real number i.e. the initial value $Y(0) \in \mathbb{R}$ (see (4.2.26)). We shall see that verification of the optimality relations when portfolio constraints are included follows much the same basic approach but is technically considerably more involved because the dual variable is now more complex than just a real scalar. We take this up in the next section.

**Remark 4.2.12.** Lim and Zhou [20] address the unconstrained problem (4.2.1) by means of a stochastic linear quadratic approach which involves the construction of a Riccati equation in the form of a BSDE (backward stochastic differential equation) for which it is necessary to establish existence and uniqueness of solutions. Furthermore, the approach of [20] relies in an essential way on the absence of constraints and cannot be extended to constraints on either the portfolio or the wealth process. In contrast, the variational approach of Rockafellar used above involves a very simple one-dimensional dual problem and does not require any use of BSDEs. Most importantly, this approach extends naturally to problems which include constraints. We take this up in the next section, in which we address a problem with convex portfolio constraints.

## 4.3 Problem with Portfolio Constraint Only

In Section 3.2, we solved a static (or non-dynamic) problem with a convex constraint on the primal variable $x$ (see Problem 3.2.1) by Rockafellar's method introduced in Section 3.1. In this section we are going to add a convex constraint to the portfolio process $\pi \in \Pi$ in problem (4.2.1), leading to a stochastic control problem with convex "control" constraint. We shall see that the essence of the approach used for Problem 3.2.1 effectively carries over to the constrained control problem of this section.

We are given a non-empty convex closed set $A \subset \mathbb{R}^N$ (recall (2.2.7) and (2.2.8)) satisfying Condition 2.2.4 and introduce the set of *constrained* or *regulated* portfolio processes defined by (recall Notation 2.1.4(3))

$$\mathcal{A} := \{\pi \in \Pi \mid \pi(t) \in A, \text{ a.e.}\}. \tag{4.3.1}$$

With this *regulated portfolio set*, problem (2.2.2) becomes

$$\text{minimize } E\left[J(X^\pi(T))\right] \quad \text{such that } \pi \in \mathcal{A}, \tag{4.3.2}$$

i.e. determine a portfolio $\bar{\pi} \in \mathcal{A}$ such that

$$E\left[J(X^{\bar{\pi}}(T))\right] = \inf_{\pi \in \mathcal{A}} \{E\left[J(X^\pi(T))\right]\}. \tag{4.3.3}$$

(recall (2.2.10)). At this point, we recall Remark 4.2.1, which applies equally well in the present problem with portfolio constraints.

**Remark 4.3.1.** The problem defined at (4.3.2) was addressed by Labbé and Heunis [18] using a "stochastic calculus of variations" approach due Bismut [4], whereas in the present Section we shall address the same problem using the Rockafellar variational approach (see Remark 4.3.23 for further discussion of this).

**Remark 4.3.2.** Comparing problem (4.3.2) with problem (4.2.1), it is clear that we now have a definite convex constraint in the "primal variable" $\pi$. We shall address this problem by means of the Rockafellar variational approach outlined in Chapter 3, by an approach which is formally very reminiscent of the way in which we used this variational approach in the non-dynamic problem (3.2.3). This is because the primal variable constraint $x \in A$ in problem (3.2.3) is very analogous to the primal variable constraint $\pi \in \mathcal{A}$ in the present problem. Of course, more technical effort will be involved since we are now dealing with a dynamic control problem, but the outlines of the approach are much the same.

To match the notation with that of Chapter 3, define the space of *primal variables*

$$\mathbb{X} = \Pi, \tag{4.3.4}$$

and reformulate the problem (4.3.3) in the form of a *primal problem*:

**Problem 4.3.3.** Determine a portfolio process $\bar{\pi} \in \Pi$ such that

$$f(\bar{\pi}) = \inf_{\pi \in \Pi} f(\pi), \tag{4.3.5}$$

in which the *primal function* $f(\cdot) : \Pi \to (-\infty, +\infty]$ is defined following the general form (3.1.1) as follows

$$f(\pi) := \begin{cases} E\left[J(X^\pi(T))\right], & \text{for } \pi \in \mathcal{A}, \\ +\infty, & \text{otherwise,} \end{cases} \tag{4.3.6}$$

and $J$ is given in (2.2.3) (compare (4.3.6) with the primal function at (4.2.5) for the unconstrained case, as well as with the primal function at (3.2.6) for the non-dynamic but constrained Problem 3.2.1).From (2.2.11), and recalling (2.2.6),

$$\vartheta := \inf_{\pi \in \Pi} \{f(\pi)\} = \inf_{\pi \in \mathcal{A}} \{E\left[J(X^\pi(T))\right]\} \in \mathbb{R} \quad \text{with } \vartheta \geqslant \underline{l}. \tag{4.3.7}$$

We next follow the steps of Rockafellar's approach outlined in Section 3.1, that is we shall choose a real vector space $\mathbb{U}$ of perturbations, a perturbation function $F(\cdot,\cdot)$, a real vector space $\mathbb{Y}$ of dual variables and a bilinear form on $\mathbb{U}\times\mathbb{Y}$, and then construct a Lagrangian function and a dual function. We shall also keep in mind the summary Remark 3.1.10 on making the above choices, and we shall see the power of Theorem 3.1.7 in obtaining existence of an optimal dual solution.

1. For Step 3.1.1, we must fix a vector space of perturbations $\mathbb{U}$ and define a perturbation function. Since $X^{\pi}(T)$ is square integrable (recall (2.1.24)), we use the same "natural" choice of perturbation as at (4.2.7), namely

$$\mathbb{U} := \mathcal{L}_2(\Omega, \mathcal{F}_T, P), \tag{4.3.8}$$

and define the *perturbation function* $F : \mathbb{X}\times\mathbb{U} \to (-\infty, +\infty]$ as

$$F(\pi, u) := \begin{cases} E\left[J(X^{\pi}(T) - u)\right], & \text{when } \pi \in \mathcal{A}, \\ +\infty, & \text{otherwise,} \end{cases} \quad \text{for all } (\pi, u) \in \Pi\times\mathbb{U}. \tag{4.3.9}$$

The convexity of $F$ on $\Pi\times\mathbb{U}$ follows since $\pi \to X^{\pi}(T) : \Pi \mapsto \mathcal{L}_2(\Omega, \mathcal{F}_T, P)$ is an affine mapping, together with (4.1.19) and (4.3.9). It is also clear that the consistency relation (3.1.2) is satisfied, i.e.

$$F(\pi, 0) = f(\pi) \quad \text{for all } \pi \in \Pi. \tag{4.3.10}$$

**Remark 4.3.4.** Notice the structural similarity of the perturbation functions at (3.2.9) and (4.3.9). In fact, Remark 3.2.2 motivates the definition of the perturbation (4.3.9) in exactly the same way that it motivated the perturbation function (3.2.9).

**Remark 4.3.5.** Problem (4.3.2) amounts to problem (4.2.1) but with the portfolio constraint $\pi \in \mathcal{A}$ included. Also implicit in problem (4.3.2) is a further "hidden" constraint dictated by the relation between the "primal variable" $\pi$ and the corresponding wealth $X^{\pi}(T)$ through the wealth equation (2.1.21) (exactly as was the case for problem (4.2.1), see Remark 4.2.4), so that problem (4.3.2) really involves two constraints, namely the constraint implicit in the wealth dynamics (2.1.21) together with the portfolio constraint $\pi \in \mathcal{A}$. The perturbation by the variable $u \in \mathbb{U} := \mathcal{L}_2(\Omega, \mathcal{F}_T, P)$ at (4.3.9) is introduced to deal with this pair of constraints in exactly the same way that the perturbation by the variable $u \in \mathbb{U} := \mathcal{L}_2(\Omega, \mathcal{F}_T, P)$ at (4.2.8) is introduced to deal with the single "hidden constraint" in problem (4.2.1) (see Remark 4.2.4). Of course, the portfolio constraint $\pi \in \mathcal{A}$ in problem (4.3.2) explicitly shows up in the condition on the right hand side of the perturbation function (4.3.9) (there is no comparable condition in the perturbation function at (4.2.8) for problem (4.2.1) without portfolio constraints).

2. Following Step 3.1.2, we must pair the space of perturbations at (4.3.8) with a vector space $\mathbb{Y}$ of *dual variables* through a bilinear form $\langle\cdot,\cdot\rangle$ on $\mathbb{U}\times\mathbb{Y}$. Exactly as at (4.2.10) define

$$\mathbb{Y} := \mathbb{B}_1, \tag{4.3.11}$$

(see (4.1.10)) together with the "natural" bilinear form on $\mathbb{U} \times \mathbb{Y}$ given by:

$$\langle u, Y\rangle := E\left[uY(T)\right] \qquad \text{for all } u \in \mathbb{U} \text{ and } Y \in \mathbb{Y}. \tag{4.3.12}$$

This completes all the choices required for Steps 3.1.1 - 3.1.2, and it remains to synthesize a Lagrangian and a dual function.

3. According to Step 3.1.3 and (3.1.3), define the *Lagrangian function* $K : \Pi \times \mathbb{Y} \to [-\infty, +\infty]$ as:

$$K(\pi, Y) := \inf_{u \in \mathcal{L}_2(\Omega, \mathcal{F}_T, P)} \{\langle u, Y\rangle + F(\pi, u)\}. \tag{4.3.13}$$

It immediately follows from (4.3.10) and (4.3.13) that

$$f(\pi) \geqslant K(\pi, Y), \qquad \text{for all } (\pi, Y) \in \Pi \times \mathbb{Y}. \tag{4.3.14}$$

We now evaluate the right-hand side at (4.3.13): From (4.2.14) and (4.2.16), for each $(\pi, Y) \in \Pi \times \mathbb{Y}$ we have:

$$K(\pi, Y) \overset{(4.3.9)}{\underset{(4.3.12)}{=}} \begin{cases} \inf_{u \in \mathcal{L}_2(\Omega, \mathcal{F}_T, P)} \{E\left[uY(T) + J(X^\pi(T) - u)\right]\}, & \text{if } \pi \in \mathcal{A}, \\ +\infty, & \text{otherwise.} \end{cases} \tag{4.3.15}$$

Since $X^\pi \in \mathbb{B}$ for all $\pi \in \Pi$ and $Y \in \mathbb{Y} = \mathbb{B}_1 \subset \mathbb{B}$ (see (4.3.11) and (4.1.10)), we have from (4.1.8) that

$$X^\pi(T), Y(T) \in \mathcal{L}_2(\Omega, \mathcal{F}_T, P) \quad \text{for all } \pi \in \mathcal{A} \text{ (see (4.3.1)) and } Y \in \mathbb{Y}. \tag{4.3.16}$$

Therefore, we can apply Proposition D.0.8 to the right side of (4.3.15) and get

$$K(\pi, Y) = \begin{cases} E\left[X^\pi(T)Y(T)\right] - E\left[J^*(Y(T))\right], & \text{if } \pi \in \mathcal{A}, \\ +\infty, & \text{otherwise.} \end{cases} \tag{4.3.17}$$

**Remark 4.3.6.** Notice that the Lagrangian at (4.3.17) reduces to the Lagrangian at (4.2.16) in the unconstrained case, where effectively $\mathcal{A}$ reduces to $\Pi$. It is also instructive to compare the Lagrangian at (4.3.17) with the Lagrangian at (3.2.13) in the non-dynamic but constrained Problem 3.2.1. We see that there are clear structural similarities between these Lagrangian functions, in particular the Lagrangians take the value $+\infty$ when the primal constraint ($x \in A$ in the case of Problem 3.2.1 and $\pi \in \mathcal{A}$ in the case of the present problem) is *not* satisfied. Furthermore, $E\left[X^\pi(T)Y(T)\right]$ on the right side of (4.3.17) is an obvious "dynamic" analog of the inner product $x'y$ on the right side of (3.2.13), and $E\left[J^*(Y(T))\right]$ is again a clear dynamic analog of $J^*(y)$. In other words, the Lagrangian (4.3.17) is just a "natural" extension to the dynamic setting of the Lagrangian at (3.2.13) in the static case.

In view of (3.1.4), define the *dual function* $g : \mathbb{Y} \to [-\infty, +\infty)$ as:

$$g(Y) := \inf_{\pi \in \Pi} K(\pi, Y). \tag{4.3.18}$$

Combining (4.3.18) and (4.3.14), we get the fundamental *weak duality principle*: (c.f. (3.2.16))

$$f(\pi) \geqslant K(\pi, Y) \geqslant g(Y), \qquad \text{for all } (\pi, Y) \in \Pi \times \mathbb{Y}. \tag{4.3.19}$$

With (4.3.17) , the *dual function* $g$ at (4.3.18) can be rewritten as:

$$\begin{aligned} g(Y) &\overset{(4.3.17)}{=} -E\left[J^*(Y(T))\right] + \inf_{\pi \in \mathcal{A}}\{E\left[X^\pi(T)Y(T)\right]\} \\ &= -E\left[J^*(Y(T))\right] - \varkappa(Y), \end{aligned} \tag{4.3.20}$$

where $\varkappa$ denotes the *support functional* of $-\mathcal{A}$, that is

$$\varkappa(Y) := \sup_{\pi \in \mathcal{A}}\{-E\left[X^\pi(T)Y(T)\right]\}, \tag{4.3.21}$$

for all $Y \in \mathbb{Y}$.

**Remark 4.3.7.** Exactly as at Remark 4.3.6, we see again that the dual function defined at (4.3.20) - (4.3.21) is just an analog for the present dynamic control problem of the dual function at (3.2.14) - (3.2.15) for the static Problem 3.2.1.

The *dual problem* is then to maximize $g(Y)$ over all $Y \in \mathbb{Y} := \mathbb{B}_1$, i.e., to establish that

$$g(\bar{Y}) = \sup_{Y \in \mathbb{Y}}\{g(Y)\} \qquad \text{for some} \qquad \bar{Y} \in \mathbb{Y}. \tag{4.3.22}$$

We are going to see that existence of the maximizing $\bar{Y}$ is secured by Theorem 3.1.7, in very much the same way that this theorem was used to get existence of a dual solution in the case of the static Problem 3.2.1 (see (3.2.18) - (3.2.20)).

**Theorem 4.3.8.** *There is some $\bar{Y} \in \mathbb{Y}$ such that*

$$g(\bar{Y}) = \sup_{Y \in \mathbb{Y}}\{g(Y)\} = \inf_{\pi \in \Pi}\{f(\pi)\} = \vartheta \in \mathbb{R}. \tag{4.3.23}$$

**Proof.** Denote by $\mathcal{U}$ the usual norm-topology on $\mathbb{U}$ corresponding to the $\mathcal{L}_2(\Omega, \mathcal{F}_T, P)$-norm defined by

$$\|u\|_2 := \left(E\left[u^2\right]\right)^{1/2}, \qquad u \in \mathbb{U}. \tag{4.3.24}$$

Then (see Remark 3.1.8) the topology $\mathcal{U}$ is $\langle \mathbb{U}, \mathbb{Y}\rangle$-compatible for the duality pairing given by (4.3.8), (4.3.11) and (4.3.12) (recall Definition 3.1.5). To see this, observe that the mapping $u \to \langle u, Y\rangle$ is clearly $\mathcal{U}$-continuous for each $Y \in \mathbb{Y}$; this verifies (1) of Definition 3.1.5. To verify (2) of Definition 3.1.5 suppose that $u \to \phi(u)$ is norm-continuous on $\mathcal{L}_2(\Omega, \mathcal{F}_T, P)$. Then the Riesz representation theorem gives some $\xi \in \mathcal{L}_2(\Omega, \mathcal{F}_T, P)$ such that $\phi(u) = E\left[u\xi\right]$ for all $u \in \mathbb{U} = \mathcal{L}_2(\Omega, \mathcal{F}_T, P)$. In view of Proposition 4.1.9, there is a unique $Y \in \mathbb{Y} := \mathbb{B}_1$ such that $\phi(u) = E\left[uY(T)\right]$ for all $u \in \mathcal{L}_2(\Omega, \mathcal{F}_T, P)$. This verifies (2) of Definition 3.1.5, so it follows that the topology $\mathcal{U}$ is $\langle \mathbb{U}, \mathbb{Y}\rangle$-compatible, as asserted. It remains to verify that (3.1.9) holds for the perturbation function $F$ defined by (4.3.9).

Fix any $\pi_1 \in \mathcal{A} \subset \Pi$. Then the function

$$h(u) := E\left[J(X^{\pi_1} - u)\right], \qquad u \in \mathcal{L}_2(\Omega, \mathcal{F}_T, P), \tag{4.3.25}$$

is $\|\cdot\|_2$-norm continuous, as easily follows from (4.1.19) and Condition 2.2.1 (i.e. (2.2.4)). Thus there exists some $\varepsilon \in (0, \infty)$ such that

$$|h(u) - h(0)| < 1 \qquad \text{for all} \quad u \in G := \{u \in \mathcal{L}_2(\Omega, \mathcal{F}_T, P) \mid \|u\|_2 < \varepsilon\}. \tag{4.3.26}$$

Then, since $h(0) \in \mathbb{R}$, it follows from (4.3.25) and (4.3.26) that

$$F(\pi_1, u) := E\left[J(X^{\pi_1} - u)\right] < 1 + h(0) < \infty, \qquad \text{for all } u \in G, \tag{4.3.27}$$

which verifies (3.1.9).

From (4.3.27), the $\langle \mathbb{U}, \mathbb{Y}\rangle$-compatibility of the norm-topology $\mathcal{U}$ on $\mathbb{U}$, the fact that $F(\cdot)$ is convex on $\Pi \times \mathbb{U}$ and satisfies a consistency relation of the form (4.3.10), and Theorem 3.1.7, we obtain the existence of some $\bar{Y} \in \mathbb{Y}$ such that

$$g(\bar{Y}) = \sup_{Y \in \mathbb{Y}} \{g(Y)\} = \inf_{\pi \in \Pi} \{f(\pi)\} \overset{(4.3.7)}{\in} \mathbb{R}. \tag{4.3.28}$$

□

**Remark 4.3.9.** In the proof of Theorem 4.3.8 we used Theorem 3.1.7 as the tool for establishing existence of an optimal dual solution (i.e. that (4.3.22) holds). Actually, the dual functional defined by (4.3.20) - (4.3.21) is simple enough that one can also establish existence of an optimal dual solution directly, without application of Theorem 3.1.7. This was the method used by Labbé and Heunis (see Proposition 5.4 of [18]). This method of direct verification relies on Proposition II-1-2 of Ekeland and Témam [10], which in turn depends on the fact that the closed unit ball in a reflexive Banach space is weakly sequentially compact. It is much simpler and more elegant to establish existence of an optimal dual solution by means of Theorem 3.1.7 than it is to use the direct approach of [18]. In later problems we shall see that the use of Theorem 3.1.7 becomes essential.

**Remark 4.3.10.** We see from the first equality in (4.3.23) that (4.3.22) holds. Moreover, we also see from the second equality in (4.3.23) that there is a zero *duality gap* between the values of the primal and dual problems. Moreover, from $g(\bar{Y}) \in \mathbb{R}$ (see (4.3.23)) together with (4.3.20) and the fact that $E\left[J^*(Y(T))\right] \in \mathbb{R}$ for all $Y \in \mathbb{B}_1$, we see that

$$\varkappa(\bar{Y}) \in \mathbb{R}, \tag{4.3.29}$$

that is $\varkappa(\bar{Y})$ *never* takes the values $+\infty$ or $-\infty$. We shall need this fact shortly.

Notice that, although Theorem 4.3.8 secures existence of a solution $\bar{Y}$ of the dual problem (recall (4.3.22)), we do not as yet know anything about the existence of solutions $\bar{\pi}$ of the primal Problem 4.3.3. We address this question next by establishing Kuhn-Tucker optimality relations which are a "stochastic control" analogue of the optimality relations (3.2.28) for the simple finite-dimensional Problem 3.2.1.

**Proposition 4.3.11. [Kuhn-Tucker Optimal Conditions]** *Suppose Condition 2.1.1, 2.1.2, 2.1.5, 2.1.11 on the market settings and Condition 2.2.1 on the quadratic criterion function $J$ given in (4.1.19). Then, for each $(\pi, Y) \in \Pi \times \mathbb{Y}$, we have the following equivalence:*

$$f(\pi) = g(Y) \quad \Longleftrightarrow \quad \begin{cases} 1)\ \pi \in \mathcal{A}, \\ 2)\ E\left[X^{\pi}(T)Y(T)\right] + \varkappa(Y) = 0, \\ 3)\ X^{\pi}(T) = \partial J^*(Y(T)). \end{cases} \tag{4.3.30}$$

*Here, $\partial J^*(\cdot)$ denotes the derivative function of the convex conjugate function $J^*(\cdot)$ given in (4.1.21).*

**Remark 4.3.12.** Notice that (4.3.30)-(1)(2)(3) are exact analogs of (3.2.28)-(1)(2)(3).

The proof of Proposition 4.3.11 is given in Appendix A.

**Remark 4.3.13.** We are now going to use the optimality relations of Proposition 4.3.11 to construct some $\bar{\pi} \in \Pi$, in terms of the Lagrange multiplier $\bar{Y}$, whose existence is established by Theorem 4.3.8, such that

$$f(\bar{\pi}) = g(\bar{Y}). \tag{4.3.31}$$

Recalling the *weak duality principle* (4.3.19), namely

$$f(\pi) \geqslant g(Y), \qquad \text{for all } (\pi, Y) \in \Pi \times \mathbb{Y}, \tag{4.3.32}$$

it follows from (4.3.31) and (4.3.32) that $\bar{\pi}$ is a solution of problem (4.3.2) .

In view of Proposition 4.3.11 we must construct some $\bar{\pi} \in \Pi$ such that

$$(1)\ \bar{\pi} \in \mathcal{A}, \quad (2)\ E\left[X^{\bar{\pi}}(T)\bar{Y}(T)\right] + \varkappa(\bar{Y}) = 0, \quad (3)\ X^{\bar{\pi}}(T) = \partial J^*(\bar{Y}(T)), \tag{4.3.33}$$

in which $\bar{Y}$ is given by Theorem 4.3.8. To this end, we introduce the following notation which will be used both here and later in the thesis.

**Notation 4.3.14.** (a) For a general vector space $\mathcal{X}$ and a set $D \subset \mathcal{X}$, define the *characteristic function* of $D$ on $\mathcal{X}$ as the two-valued mapping $x \to \delta_{\mathcal{X}}\{x|D\} : \mathcal{X} \to \{0, \infty\}$ given by

$$\delta_{\mathcal{X}}\{x|D\} := \begin{cases} 0, & \text{when } x \in D, \\ \infty, & \text{when } x \notin D. \end{cases} \tag{4.3.34}$$

(b) For any set $A \subset \mathbb{R}^N$ define the *support function* of $A$ as follows:

$$\delta^*_{\mathbb{R}^N}\left\{x \middle| A\right\} := \sup_{z \in A}\{z'x\}, \qquad \text{for all } x \in \mathbb{R}^N. \tag{4.3.35}$$

**Remark 4.3.15.** $D$ is a convex set if and only if $\delta_{\mathcal{X}}\{\cdot|D\}$ is a convex function on $\mathcal{X}$. Characteristic functions play a fundamental role in convex analysis, and should not be confused with the familiar indicator functions of real analysis (see (3.3.21)). We shall always refer to (4.3.34) as a characteristic function and (3.3.21) as an indicator function.

**Remark 4.3.16.** In this Remark, we develop an identity which will be used several times in the thesis. Referring to Theorem D.0.6, we take

$$S := \Omega \times [0,T], \quad \Sigma := \mathcal{F}^*, \quad \mu := P \otimes \lambda, \quad \mathbb{L} = \mathbb{M} = \Pi. \tag{4.3.36}$$

Put

$$\phi(u) := \delta_{\mathbb{R}^N}\{u|A\} \quad u \in \mathbb{R}^N, \text{ so that } \quad \phi^*(y) = \sup_{v \in A}\{y'v\}, \quad y \in \mathbb{R}^N, \tag{4.3.37}$$

(recall Condition 2.2.4). Then $\phi(\cdot)$ is a normal convex integrand (by Lemma 1 of [29], p.528). Put $\vartheta(\omega;t) = 0$ for $(\omega;t) \in S$, i.e. $\vartheta(\cdot)$ is the zero element of $\Pi$. Since $0 \in A$ (see Condition 2.2.4) we have

$$\phi(\vartheta(\omega;t)) = 0 \quad \text{and} \quad \phi^*(\vartheta(\omega;t)) = 0, \quad \text{for all } (\omega;t) \in S, \tag{4.3.38}$$

i.e. $\phi(\vartheta(\cdot))$ and $\phi^*(\vartheta(\cdot))$ are (trivially) $\mu$-integrable. Therefore, since $\Pi = \mathcal{L}_2(\Omega,\mathcal{F},P)$ is decomposable, from Theorem D.0.6, we have

$$\sup_{\pi \in \Pi}\left\{\int_{\Omega\times[0,T]} \pi'(\omega;s)\vartheta(\omega;s)\mathrm{d}\mu - \int_{\Omega\times[0,T]} \phi(\pi(\omega;s))\mathrm{d}\mu\right\} = \int_{\Omega\times[0,T]} \phi^*(\vartheta(\omega;s))\mathrm{d}\mu \tag{4.3.39}$$

for all $\vartheta \in \Pi$, or

$$\sup_{\pi \in \Pi} E\left[\int_0^T [\pi'(s)\vartheta(s) - \delta_{\mathbb{R}^N}\{\pi(s)|A\}]\,\mathrm{d}s\right] = E\left[\int_0^T \delta^*_{\mathbb{R}^N}\{\vartheta(s)|A\}\,\mathrm{d}s\right] \quad \vartheta \in \Pi. \tag{4.3.40}$$

We shall use (4.3.40) as a tool later in the thesis.

Another tool for later use is the following Proposition 4.3.17, the proof of which is in Appendix A.

**Proposition 4.3.17.** *Suppose Condition 2.1.1, 2.1.2, 2.1.5, 2.1.11, 2.2.1 and 2.2.4, and recall (2.1.8), Remark 2.1.7 - (1), the set of constrained portfolio processes at (4.3.1), the definition of $\varkappa(\cdot)$ at (4.3.21), and Notation 4.3.14-(b). For each $Y \in \mathbb{B}_1$ (see (4.1.10)) we have*

$$\begin{aligned}&\sup_{\pi \in \mathcal{A}}\{-E[X^\pi(T)Y(T)]\}\\ &= -x_0Y(0) + E\left[\int_0^T \delta^*_{\mathbb{R}^N}\{-\sigma(s)[\theta(s)Y(s) + \Lambda_Y(s)]\big|A\}\,ds\right].\end{aligned} \tag{4.3.41}$$

Proposition 4.3.17 gives the following representation of the function $\varkappa(\cdot)$ at (4.3.21):

$$\varkappa(Y) = -x_0 Y(0) + E\left[\int_0^T \delta^*_{\mathbb{R}^N}\left\{-\sigma(s)[\theta(s)Y(s) + \Lambda_Y(s)]\big|A\right\} \mathrm{d}s\right], \quad Y \in \mathbb{B}_1. \tag{4.3.42}$$

The next result establishes a *necessary condition* resulting from the optimality of $\bar{Y}$ given by Theorem 4.3.8. This necessary condition will be essential for the later construction of a $\bar{\pi} \in \Pi$ which satisfies the conditions (4.3.33)-(1)(2)(3).

**Proposition 4.3.18.** *Suppose Condition 2.1.1, 2.1.2, 2.1.5, 2.1.11, 2.2.1 and 2.2.4, and recall the derivative function $\partial J^*(\cdot)$ at (4.1.21) and the support functional $\varkappa(\cdot)$ at (4.3.42). For each $Y \in \mathbb{Y} \overset{(4.3.11)}{=} \mathbb{B}_1$ (see (4.1.10)) we have*

$$\varkappa(Y) + E\left[\partial J^*(\bar{Y}(T))Y(T)\right] \geqslant 0, \tag{4.3.43}$$

*and in particular,*

$$\varkappa(\bar{Y}) + E\left[\partial J^*(\bar{Y}(T))\bar{Y}(T)\right] = 0, \tag{4.3.44}$$

*where $\bar{Y} \in \mathbb{Y}$ is given by Theorem 4.3.8.*

**Proof.** From Theorem 4.3.8 we have

$$g(\bar{Y} + \varepsilon Y) \leqslant g(\bar{Y}) \in \mathbb{R}, \qquad \text{for all } \varepsilon \in (0,\infty) \text{ and } Y \in \mathbb{B}_1. \tag{4.3.45}$$

From (4.3.45), (4.3.20) and (4.3.21), for all $\varepsilon \in (0,\infty)$ and $Y \in \mathbb{B}_1$, we have

$$\varkappa(\bar{Y} + \varepsilon Y) - \varkappa(\bar{Y}) + E\left[J^*(\bar{Y}(T) + \varepsilon Y(T)) - J^*(\bar{Y})\right] \geqslant 0, \tag{4.3.46}$$

and

$$\varkappa(\bar{Y} + \varepsilon Y) \leqslant \varepsilon \sup\nolimits_{\pi \in \mathcal{A}} E\left[-X^\pi(T)Y(T)\right] + \varkappa(\bar{Y}). \tag{4.3.47}$$

From (4.3.46), (4.3.47) and $\varkappa(\bar{Y}) \in \mathbb{R}$ (see (4.3.29)), we obtain

$$\sup_{\pi \in \mathcal{A}} E\left[-X^\pi(T)Y(T)\right] + E\left[\frac{J^*(\bar{Y}(T) + \varepsilon Y(T)) - J^*(\bar{Y}(T))}{\varepsilon}\right] \geqslant 0, \tag{4.3.48}$$

for all $\varepsilon \in (0,\infty)$ and $Y \in \mathbb{B}_1$. From (4.1.20) and Condition 2.2.1, it is easy to use dominated convergence to evaluate the limit on the left of (4.3.48) as $\varepsilon \to 0$, to get

$$\sup_{\pi \in \mathcal{A}} E\left[-X^\pi(T)Y(T)\right] + E\left[\partial J^*(\bar{Y}(T))Y(T)\right] \geqslant 0, \qquad \text{for all } Y \in \mathbb{B}_1. \tag{4.3.49}$$

Then, from (4.3.49) and (4.3.21), we get (4.3.43), and in particular

$$\varkappa(\bar{Y}) + E\left[\partial J^*(\bar{Y}(T))\bar{Y}(T)\right] \geqslant 0. \tag{4.3.50}$$

To establish (4.3.44), we again use the optimality of $\bar{Y}$ given by Theorem 4.3.8, namely

$$g(\bar{Y} - \varepsilon\bar{Y}) \leqslant g(\bar{Y}), \qquad \varepsilon \in [0,1]. \tag{4.3.51}$$

From (4.3.20) and (4.3.51) we have

$$-E\left[J^*(\bar{Y}(T)-\varepsilon\bar{Y}(T))\right]-\varkappa(\bar{Y}-\varepsilon\bar{Y})\leqslant -E\left[J^*(\bar{Y}(T))\right]-\varkappa(\bar{Y}), \tag{4.3.52}$$

i.e.,

$$E\left[J^*(\bar{Y}(T)-\varepsilon\bar{Y}(T))\right]+\varkappa(\bar{Y}-\varepsilon\bar{Y})\geqslant E\left[J^*(\bar{Y}(T))\right]+\varkappa(\bar{Y}). \tag{4.3.53}$$

From (4.3.21) and $\varepsilon\in[0,1]$ we have

$$\varkappa(\bar{Y}-\varepsilon\bar{Y})=(1-\varepsilon)\varkappa(\bar{Y}), \tag{4.3.54}$$

and, upon combining (4.3.54) and (4.3.53), we get

$$-\varkappa(\bar{Y})+\frac{1}{\varepsilon}\left\{E\left[J^*(\bar{Y}(T)-\varepsilon\bar{Y}(T))\right]-E\left[J^*(\bar{Y}(T))\right]\right\}\geqslant 0, \qquad \varepsilon\in[0,1]. \tag{4.3.55}$$

From $\varepsilon\to 0$ in (4.3.55) and dominated convergence,

$$-\varkappa(\bar{Y})-E\left[\partial J^*(\bar{Y}(T))\bar{Y}(T)\right]\geqslant 0, \tag{4.3.56}$$

i.e.,

$$\varkappa(\bar{Y})+E\left[\partial J^*(\bar{Y}(T))\bar{Y}(T)\right]\leqslant 0. \tag{4.3.57}$$

From (4.3.57) and (4.3.50), we have verified (4.3.44). □

Motivated by the transversality condition (4.3.33)-(3) and Proposition 4.2.9 with

$$\xi:=\partial J^*(\bar{Y}(T))\overset{(4.1.21)}{=}\frac{\bar{Y}(T)-c}{a}\underset{Condition 2.2.1}{\overset{(4.5.27)}{\in}}\mathcal{L}_2(\Omega,\mathcal{F}_T,P), \tag{4.3.58}$$

define the process $\bar{X}$ as

$$\bar{X}(t):=H^{-1}(t)E\left[H(T)\partial J^*(\bar{Y}(T))\,\middle|\,\mathcal{F}_t\right] \qquad t\in[0,T]. \tag{4.3.59}$$

Then, from Proposition 4.2.9, there exists some a.e. unique $\mathbb{R}^N$-valued $\mathcal{F}_t$-progressively measurable process $\bar{\psi}$ such that

$$\int_0^T\left\|\bar{\psi}(t)\right\|^2\mathrm{d}t<\infty \quad \text{a.s.} \qquad \text{and} \qquad H(t)\bar{X}(t)=\bar{X}(0)+\int_0^t\bar{\psi}'(s)\mathrm{d}W(s).$$

Motivated by (4.2.37), define the $\mathbb{R}^N$-valued $\mathcal{F}_t$-progressively measurable process

$$\bar{\pi}(t):=(\sigma'(t))^{-1}\left[H^{-1}(t)\bar{\psi}(t)+\bar{X}(t)\theta(t)\right]. \tag{4.3.60}$$

From Proposition 4.2.9 again, it follows that

$$\bar{\pi}\in\Pi, \tag{4.3.61}$$

and

$$d\bar{X}(t) = \left[r(t)\bar{X}(t) + \bar{\pi}'(t)\sigma(t)\theta(t)\right] dt + \bar{\pi}'(t)\sigma(t)dW(t), \tag{4.3.62}$$

with the initial value

$$\bar{X}(0) = E\left[H(T)\partial J^*(\bar{Y}(T))\right], \tag{4.3.63}$$

(see (4.3.58) and (4.2.39)). In the following Proposition 4.3.20, we will show that the portfolio $\bar{\pi}$ at (4.3.60) is a member of $\mathcal{A}$ that is $\bar{\pi}$ satisfies portfolio constraints, that $\bar{X}(0)$ at (4.3.63) is equal to the initial wealth $x_0$ stipulated in Condition 2.1.11, and that $\bar{X} = X^{\bar{\pi}}$ (recall (2.1.21)). To this end, the necessary conditions established at Proposition 4.3.18 will be essential.

**Remark 4.3.19.** In Proposition 4.3.20 which follows, we will show that $\bar{\pi}$ satisfies the portfolio constraints $\bar{\pi} \in \mathcal{A}$ and that $\bar{X}$ is the wealth process corresponding to the portfolio $\bar{\pi}$.

**Proposition 4.3.20.** *Suppose Condition 2.1.1, 2.1.2, 2.1.5, 2.1.11, 2.2.1 and 2.2.4, and recall $\bar{X}$ and $\bar{\pi}$ defined at (4.3.59) - (4.3.60) in terms of $\bar{Y} \in \mathbb{B}_1$ given by Theorem 4.3.8. Then*

(1) $$\bar{X}(0) = x_0, \tag{4.3.64}$$

(2) $$\bar{\pi} \in \mathcal{A}, \tag{4.3.65}$$

(3) $$\bar{X} = X^{\bar{\pi}}. \tag{4.3.66}$$

**Proof.** From (4.3.62), (4.3.61) and Proposition 4.1.4 we get

$$\bar{X} \in \mathbb{B} \quad \text{with} \quad \dot{\bar{X}} = r\bar{X} + \bar{\pi}'\sigma\theta \text{ and } \Lambda_{\bar{X}} = \sigma'\bar{\pi}, \tag{4.3.67}$$

while, from (4.1.10),

$$\dot{Y} = -rY \qquad \text{for each} \qquad Y \in \mathbb{B}_1. \tag{4.3.68}$$

In view of Proposition F.0.1, together with $\bar{X} \in \mathbb{B}$ (see (4.3.67)) and $Y \in \mathbb{B}_1 \subset \mathbb{B}$, it follows

$$\begin{aligned} M(\bar{X}, Y)(T) &= \bar{X}(T)Y(T) - \bar{X}(0)Y(0) \\ &\quad - \int_0^T \{\bar{X}(s)\dot{Y}(s) + \dot{\bar{X}}(s)Y(s) + \Lambda'_{\bar{X}}(s)\Lambda_Y(s)\}ds, \end{aligned} \tag{4.3.69}$$

and

$$E\left[M(\bar{X}, Y)(T)\right] = 0, \qquad \text{for each } Y \in \mathbb{B}_1. \tag{4.3.70}$$

Using (4.3.67) and (4.3.68) in (4.3.69), we get

$$M(\bar{X}, Y)(T) = \bar{X}(T)Y(T) - \bar{X}(0)Y(0) - \int_0^T \bar{\pi}'(s)\sigma(s)\left[Y(s)\theta(s) + \Lambda_Y(s)\right] ds, \tag{4.3.71}$$

From (4.3.70) and (4.3.71), we get

$$E\left[\bar{X}(T)Y(T)\right] = \bar{X}(0)Y(0) + E\left[\int_0^T \bar{\pi}'(s)\sigma(s)\left[Y(s)\theta(s) + \Lambda_Y(s)\right]\mathrm{d}s\right],\ Y \in \mathbb{B}_1. \quad (4.3.72)$$

From (4.3.59), we have $\bar{X}(T) = \partial J^*(\bar{Y}(T))$, and combining this with (4.3.72), (4.3.42) and the necessary condition (4.3.43) gives

$$\begin{aligned} &E\left[\int_0^T \bar{\pi}'(s)\sigma(s)\left[Y(s)\theta(s) + \Lambda_Y(s)\right]\mathrm{d}s + \delta^*_{\mathbb{R}^N}\left\{-\sigma(s)[\theta(s)Y(s) + \Lambda_Y(s)]\big|A\right\}\right] \\ &+ \left(\bar{X}(0) - x_0\right)Y(0) \geqslant 0, \qquad\qquad Y \in \mathbb{B}_1. \end{aligned} \quad (4.3.73)$$

Note that (4.3.73) is a necessary condition resulting from the optimality of $\bar{Y}$ given by Theorem 4.3.8. To use this inequality to verify (4.3.64) - (4.3.66), we have to construct appropriate $Y \in \mathbb{B}_1$ by Lemma 4.1.8.

(a) By Lemma 4.1.8, for each $y \in \mathbb{R}$ there is some $Y \in \mathbb{B}_1$ such that

$$Y(0) = y \quad \text{and} \quad \theta Y + \Lambda_Y = 0 \quad \text{a.e.} \quad (4.3.74)$$

Then, for this $Y$, we see that the quantity in square brackets on the left side of (4.3.73) equals zero (since clearly $\delta^*_{\mathbb{R}^N}\left\{0\big|A\right\} = 0$), and (4.3.73) is reduced to

$$\left(\bar{X}(0) - x_0\right)y \geqslant 0, \qquad \text{for all } y \in \mathbb{R}, \quad (4.3.75)$$

from which (4.3.64) follows.

(b) It remains to establish (4.3.65). First define a set

$$O := \{(\omega;t) \in \Omega \times [0,T] \mid \bar{\pi}(\omega;t) \in A\}. \quad (4.3.76)$$

From Lemma F.0.3 (also see Lemma 5.4.2 of [15, p.207]), corresponding to $\bar{\pi} \in \mathcal{F}^*$ there exists some $\mathbb{R}^N$-valued $\bar{\nu} \in \mathcal{F}^*$ such that

$$\begin{cases} \|\bar{\nu}(t)\| \leqslant 1, \left|\delta^*_{\mathbb{R}^N}\left\{-\bar{\nu}(t)\big|A\right\}\right| \leqslant 1, & \text{a.e. on } \Omega \times [0,T], \\ \bar{\pi}'(t)\bar{\nu}(t) + \delta^*_{\mathbb{R}^N}\left\{-\bar{\nu}(t)\big|A\right\} = 0, & \text{a.e. on } O, \\ \bar{\pi}'(t)\bar{\nu}(t) + \delta^*_{\mathbb{R}^N}\left\{-p\bar{\nu}(t)\big|A\right\} < 0, & \text{a.e. on } (\Omega \times [0,T]) \setminus O. \end{cases} \quad (4.3.77)$$

From $\|\bar{\nu}(t)\| \leqslant 1$ and (2.1.7) we have

$$\sigma^{-1}\bar{\nu} \in \Pi, \quad (4.3.78)$$

and, from (4.3.78) and Lemma 4.1.8, there is some $Y \in \mathbb{B}_1$ such that

$$Y(0) = 0 \quad \text{and} \quad \sigma^{-1}\bar{\nu} = [Y\theta + \Lambda_Y] \quad \text{a.e.} \quad (4.3.79)$$

Using this $Y$ in (4.3.73), together with (4.3.64), we have

$$E\left[\int_0^T \{\bar{\pi}'(s)\bar{\nu}(s) + \delta^*_{\mathbb{R}^N}\left\{-\bar{\nu}(s)\big|A\right\}\}\mathrm{d}s\right] \geqslant 0, \quad (4.3.80)$$

which, together with (4.3.77), gives

$$P \otimes \lambda\left(O^{c}\right) = 0, \tag{4.3.81}$$

i.e.

$$\bar{\pi}(t) \in A \quad \text{a.e.} \qquad \text{therefore} \qquad \bar{\pi} \in \mathcal{A} \qquad (\text{see } (4.3.61)). \tag{4.3.82}$$

We have established (4.3.65). Now, (4.3.66) is immediate from (4.3.64) and (4.3.65) together with (4.3.62). ☐

**Remark 4.3.21.** It is now clear that $\bar{\pi} \in \Pi$ defined by (4.3.60) satisfies all the conditions in (4.3.33). Indeed, (4.3.33)-(1) follows from (4.3.65). Also,

$$X^{\bar{\pi}}(T) \overset{(4.3.66)}{=} \bar{X}(T) \overset{(4.3.59)}{=} \partial J^{*}(\bar{Y}(T)), \tag{4.3.83}$$

which verifies (4.3.33)-(3). Using (4.3.83) in (4.3.44), we also verify the complementary slackness condition at (4.3.33)-(2). Thus (4.3.33)-(1)(2)(3) have been verified, and therefore Proposition 4.3.11 gives (4.3.31). In view of Remark 4.3.13 we see that $\bar{\pi}$ is an optimal portfolio for problem (4.3.2).

**Remark 4.3.22.** Combining (4.3.20), (4.1.20) and (4.3.42), we obtain the dual functional $g(\cdot)$ in the following form:

$$\begin{aligned} g(Y) &= x_0 Y(0) - E\left[\frac{(Y(T)-c)^2}{2a}\right] \\ &- E\left[\int_0^T \delta^{*}_{\mathbb{R}^N}\left\{-\sigma(s)[\theta(s)Y(s) + \Lambda_Y(s)] \middle| A\right\} \mathrm{d}s\right] + q, \end{aligned} \tag{4.3.84}$$

for all $Y \in \mathbb{B}_1$. Modulo some inessential sign-changes, this is exactly the dual function obtained by Labbé and Heunis for problem (4.3.2) (see (5.15), (5.35) and (5.37) in [18]) by a different approach (see the following Remark 4.3.23). In fact, the dual problem in [18] is one of *minimization* (rather than maximization as at (4.3.22)), and the dual function in [18] amounts to $-g(-Y)$ for all $Y \in \mathbb{B}_1$ (taking into account the difference between (5.15) of [18] and (4.3.35)).

**Remark 4.3.23.** In the above we have essentially established the main result of Labbé and Heunis for problem (4.3.2) (see Proposition 5.6 of [18]). The approach of [18] is based on a convex "stochastic calculus of variations" due to Bismut [4], whereas in the preceding we have instead used the variational method of Rockafellar summarized in Section 3.1. This is to illustrate the great generality inherent in Rockafellar's variational method, since it has been used as a unified approach to recover the main results of Lim and Zhou [20] for unconstrained portfolios (see Remark 4.2.12) and of Labbé and Heunis [18] for convex portfolio constraints. The convex stochastic calculus of variations approach of Bismut [4] is extremely powerful when the structure of the constraints is such that the dual variables are *semimartingales*, as is the case in the present section (see (4.3.11)). Indeed, this approach has been extended to problems similar to (4.3.2) but with "regime switching" also included in the market model (see Donnelly and Heunis [8]), where again the appropriate dual variables are semimartingales. However, later in the thesis we shall address problems with constraints which demand dual variables more general than just semimartingales. This is the case when there are constraints not only on the portfolio but also on the wealth process itself, that is there are "state constraints". For such problems the variational method of Rockafellar constitutes not just a pleasing unified approach but becomes an essential tool. In the following section we shall use the Rockafellar variational method to establish and extend the main results of one of the earliest and simplest works involving such "wealth constraints", namely quadratic risk minimization without bankruptcy (see Bielecki *et al.* [3]).

## 4.4 Problem with European Wealth Constraint Only

In this section, we consider a quadratic minimization problem with *unconstrained portfolios* and a *European wealth constraint* (2.2.12), that is, the terminal wealth must exceed a specified square integrable random variable

$$b \in \mathcal{L}_2(\Omega, \mathcal{F}_T, P). \tag{4.4.1}$$

The problem we address is (2.2.14), that is

$$\text{minimize } E\left[J(X^{\pi}(T))\right] \quad \text{such that } \pi \in \Pi \text{ and } X^{\pi}(T) \geqslant b \text{ a.s.}, \tag{4.4.2}$$

i.e. determine a portfolio process $\bar{\pi} \in \Pi$ (recall (4.1.1)) such that $X^{\bar{\pi}}(T) \geqslant b$ a.s. and

$$E\left[J(X^{\bar{\pi}}(T))\right] = \inf_{\pi \in \Pi} \left\{ E\left[J(X^{\pi}(T))\right] \,\middle|\, X^{\pi}(T) \geqslant b \text{ a.s.} \right\}. \tag{4.4.3}$$

**Remark 4.4.1.** Notice that problem (4.4.2) involves only a *constraint on the wealth at close of trade* (hence the designation *European wealth constraint*) and, unlike problem (4.3.2), does not include any portfolio constraint. The problem (4.4.2) was addressed by Bielecki *et al.* [3] in the special case where $b := 0$, by an adaptation of the risk-neutral method. The approach used in [3] relies in an essential way on the absence of portfolio constraints, and cannot be extended to work for problems which involve a combination of convex portfolio constraints (of the kind occurring in problem (4.3.2)) and the European wealth constraint of problem (4.4.2) (we shall address a problem with precisely this combination of constraints in Section 4.5). As is the case throughout this thesis, in the present section we shall approach the problem by the variational method of Rockafellar outlined in Section 3.1, namely the approach which has already been systematically used as a unifying tool in all the problems addressed above. In particular, we can regard problem (4.4.2) as a "dynamic analogue" of the static problem (3.2.34), and in basic outline we shall approach the present problem (4.4.2) along very much the same lines used for addressing problem (3.2.34).

**Remark 4.4.2.** As mentioned in Section 2.2, there is a "hidden condition" built into the formulation of problem (4.4.2), namely the floor-level random variable $b$ must be stipulated such that $X^{\pi}(T) \geqslant b$ a.s., for some $\pi \in \Pi$, since otherwise it is impossible to even satisfy the constraint on wealth at close of trade, in which case problem (4.4.2) does not even make sense! In such a case we have been "too greedy" in stipulating the random variable $b$ as the lower-bound on wealth at close of trade, since this lower-bound cannot be satisfied by *any* $\pi \in \Pi$, and we need to make a "more reasonable" choice of random variable $b$ when defining the problem. In fact, exactly as at Remark 3.2.19, we are going to very slightly strengthen this implied condition on the choice of $b$ by "working" some small $\varepsilon \in (0, \infty)$ into the constraint in the form of the following Slater-type Condition 4.4.3. Typically Slater-type conditions are needed for existence of dual solutions, and we shall see that Condition 4.4.3 which follows will be needed to secure existence of a solution of the *dual problem* that we shall construct in due course.

**Condition 4.4.3.** There is some $\hat{\pi} \in \Pi$ and non-random constant $\varepsilon \in (0, \infty)$ such that $X^{\hat{\pi}}(T) \geqslant b + \varepsilon$ a.s..

**Remark 4.4.4.** As we have already observed, the "correct" stipulation of the random variable $b$ when defining problem (4.4.2) is all-important; if $b$ is set too large then there may fail to exist any $\pi \in \Pi$ such that $X^{\pi}(T) \geqslant b$, and the problem does not make sense. Condition 4.4.3 just forces one to fix a "reasonable" lower-bound $b$ on the wealth at close of trade, in the sense that there exists some $\hat{\pi} \in \Pi$ which not only satisfies $X^{\hat{\pi}}(T) \geqslant b$ but is such that $X^{\hat{\pi}}(T)$ *exceeds* $b$ by some (small) "margin" $\varepsilon \in (0, \infty)$. Condition 4.4.3 therefore amounts to a very mild restriction on the choice of $b$ when defining problem (4.4.2). In the particular case of $b = 0$ (addressed by Bielecki *et al.* [3]) it is clear that Condition 4.4.3 is automatically satisfied. Indeed, we can just take $\hat{\pi} = 0$ (i.e. no investment in any risky assets), for then

$$X^{\hat{\pi}}(T) = S_0(T)x_0 \geqslant x_0 > b = 0, \tag{4.4.4}$$

(see (2.1.22)), where the first inequality follows since $r$ is assumed non-negative (see Condition 2.1.2), and the second inequality follows since $x_0 \in (0, \infty)$ (see Condition 2.1.11). Thus, in the case where $b = 0$, it follows from (4.4.4) that Condition 4.4.3 is satisfied when $\hat{\pi} = 0$ and $\varepsilon \in (0, x_0)$.

**Remark 4.4.5.** In this remark we elaborate further on the choice of the random variable $b$ in problem (4.4.2) so as to ensure that Condition 4.4.3 holds. The *gains process* $\{G^{\pi}(t)\}$, corresponding to a self-financed portfolio process $\pi \in \Pi$, is defined by

$$G^{\pi}(t) := S_0(t) \left\{ \int_0^t \frac{\pi'(s)\sigma(s)\theta(s)}{S_0(s)} \mathrm{d}s + \int_0^t \frac{\pi'(s)\sigma(s)}{S_0(s)} \mathrm{d}W(s) \right\}, \quad t \in [0, T], \tag{4.4.5}$$

(see (1.2.9) and (1.2.11) of Karatzas and Shreve [[15], page 7]), and therefore

$$X^{\pi}(t) = S_0(t)x_0 + G^{\pi}(t), \quad t \in [0, T], \tag{4.4.6}$$

for each $\pi \in \Pi$, as follows from (4.4.5) and (2.1.22). Now suppose that the random variable $b$ in problem (4.4.2) is stipulated such that

$$b \geqslant x_0 S_0(T), \qquad \text{a.s.} \tag{4.4.7}$$

If Condition 4.4.3 holds for this choice of $b$ then

$$X^{\hat{\pi}}(T) \geqslant \epsilon + x_0 S_0(T), \qquad \text{a.s.,} \tag{4.4.8}$$

for some $\hat{\pi} \in \Pi$ and some non-random constant $\epsilon \in (0, \infty)$, and therefore, in view of (4.4.8) and (4.4.6), we have

$$G^{\hat{\pi}}(T) \geqslant \epsilon > 0, \qquad \text{a.s.,} \tag{4.4.9}$$

for some $\hat{\pi} \in \Pi$ and some non-random constant $\epsilon \in (0, \infty)$. We now observe that $\hat{\pi}$ is a *tame* portfolio process (see Definition 1.2.4 of Karatzas and Shreve [15]). The argument which establishes this is fairly standard but for completeness we repeat the details here. Define the usual ingredients

$$W_0(t) := W(t) + \int_0^t \theta(\tau) \mathrm{d}\tau, \qquad t \in [0, T], \tag{4.4.10}$$

$$Z_0(t) := \exp\left\{-\int_0^t \theta'(\tau) dW(\tau) - \frac{1}{2}\int_0^t \|\theta(\tau)\|^2 \, \mathrm{d}\tau\right\}, \qquad t \in [0, T], \tag{4.4.11}$$

$$P_0(A) := E\left[Z_0(T)\mathbb{1}_A\right], \qquad A \in \mathcal{F}_T. \tag{4.4.12}$$

In view of Remark 2.1.7(1), together with Remark 1.5.2 of Karatzas and Shreve [15], we know that $Z_0$ is a $\mathcal{F}_t$-martingale on $(\Omega, \mathcal{F}, P)$, and therefore in particular

$$P_0 \text{ is a probability measure on } (\Omega, \mathcal{F}). \tag{4.4.13}$$

Then, in view of (4.4.10) - (4.4.13) and the Girsanov theorem, it follows that

$$\{(W_0(t), \mathcal{F}_t),\ t \in [0, T]\} \text{ is a standard } \mathbb{R}^N\text{-Brownian motion on } (\Omega, \mathcal{F}, P_0), \tag{4.4.14}$$

(see Remark 1.5.3 of Karatzas and Shreve [15]). From (4.4.5) and (4.4.10) one has

$$\frac{G^{\hat{\pi}}(t)}{S_0(t)} = \int_0^t \frac{\hat{\pi}'(\tau)\sigma(\tau)}{S_0(\tau)} \mathrm{d}W_0(\tau), \qquad t \in [0, T]. \tag{4.4.15}$$

In view of (4.4.15), (4.4.14), and (2.1.2), together with the uniform-boundedness of the market parameters $\sigma$ and $r$ (recall Condition 2.1.2), and $\hat{\pi} \in \Pi$ (recall (2.1.20)), we get

$$\{S_0^{-1}(t)G^{\hat{\pi}}(t),\ t \in [0, T]\} \text{ is a } \mathcal{F}_t\text{-martingale on } (\Omega, \mathcal{F}, P_0). \tag{4.4.16}$$

Upon taking $P_0$-conditional expectations with respect to $\mathcal{F}_t$ and using (4.4.16) we find

$$\frac{G^{\hat{\pi}}(t)}{S_0(t)} = E_0\left[\frac{G^{\hat{\pi}}(T)}{S_0(T)}\right] \geqslant \epsilon E_0\left[\frac{1}{S_0(T)}\right] > 0, \qquad t \in [0, T], \tag{4.4.17}$$

in which the first inequality of (4.4.17) follows from (4.4.9). Since the probability measures $P$ and $P_0$ are equivalent (from (4.4.12)) it follows from (4.4.17) that $\hat{\pi}$ is a tame portfolio process, and hence, in view of (4.4.9), one sees that $\hat{\pi}$ is an *arbitrage opportunity* (see Definition 1.4.1 of Karatzas and Shreve [15]). To summarize, if Condition 4.4.3 holds when $b$ is stipulated subject to (4.4.7), then $\hat{\pi}$ is an arbitrage opportunity. However, from (4.4.12) - (4.4.13), it follows that $E[Z_0(T)] = 1$, and then it is immediate from Theorem 1.4.2 of Karatzas and Shreve [15] that the market model stipulated by Condition 2.1.1, Condition 2.1.2 and Condition 2.1.5 is *viable*, that is does not have any arbitrage opportunities. We therefore conclude that Condition 4.4.3 can *never* be satisfied if the random variable $b$ is stipulated according to (4.4.7), and hence, at the very least, we must always fix $b$ such that

$$b \ngeqslant x_0 S_0(T), \qquad \text{a.s.} \tag{4.4.18}$$

**Remark 4.4.6.** In Remark 4.4.5 we saw that Condition 4.4.3 cannot be satisfied if one chooses the random variable $b$ in problem (4.4.2) too far on the "upside" (in the sense that (4.4.7) holds). It is important to understand that the constraint $X^{\pi}(T) \geqslant b$ can also be used to limit "downside loss" at close of trade. If one simply minimized the quadratic loss criterion without any constraints (as in Section 4.2) then there is no natural lower-bound to the wealth $X^{\pi}(T)$ even when $\pi$ is optimal (as noted in Section 1 this is one of the main drawbacks of quadratic loss minimization, in contrast to utility maximization where the structure of the problem forces a natural non-negativity on the wealth process). With this in mind, one can argue that an entirely valid choice of $b$ is to stipulate $b \leqslant 0$ a.s., since this represents quadratic loss minimization subject to a permissible level of *debt* specified by the random variable $b$. Of course, Condition 4.4.3 automatically holds when $b \leqslant 0$ a.s., since this condition has already been seen to hold when $b = 0$ (recall Remark 4.4.4).

Having discussed the implications of the choice of the random variable $b$ and Condition 4.4.3 in Remark 4.4.4, Remark 4.4.5 and Remark 4.4.6, we turn now to addressing problem (4.4.2). As usual, we first write this in the form of a *primal problem* by taking the space of primal variables

$$\mathbb{X} := \Pi, \tag{4.4.19}$$

and define the *primal problem* as follows:

**Problem 4.4.7.** Determine a portfolio process $\bar{\pi} \in \Pi$ such that

$$f(\bar{\pi}) = \inf_{\pi \in \Pi} f(\pi), \tag{4.4.20}$$

in which the *primal function* $f(\cdot) : \Pi \to (-\infty, +\infty]$ is defined as

$$f(\pi) := \begin{cases} E[J(X^{\pi}(T))], & \text{for } X^{\pi}(T) \geqslant b \text{ a.s.}, \\ +\infty, & \text{otherwise}, \end{cases} \tag{4.4.21}$$

and $J$ is given in (2.2.3) subject to Condition 2.2.1. From Remark 2.2.3 and (2.2.6)

$$\vartheta := \inf_{\pi \in \Pi} \{f(\pi)\} = \inf_{\pi \in \Pi} \{E[J(X^{\pi}(T))] \mid X^{\pi}(T) \geqslant b \text{ a.s.}\} \in \mathbb{R} \quad \text{with } \vartheta \geqslant \underline{l}, \tag{4.4.22}$$

(note that Condition 4.4.3 ensures that the set $\{\pi \in \Pi \mid X^{\pi}(T) \geqslant b \text{ a.s.}\}$ is non-empty).

We shall follow Rockafellar's variational approach outlined in Section 3.1 to choose a real vector space $\mathbb{U}$ of perturbations, a perturbation function $F(\cdot,\cdot)$, a real vector space $\mathbb{Y}$ of dual variables and a bilinear form on $\mathbb{U}\times\mathbb{Y}$, and then use these to construct a Lagrangian function and a dual function:

1. According to Step 3.1.1, we must fix a vector space of perturbations $\mathbb{U}$ and define a perturbation function. To this end, we put

$$\mathbb{U} := \mathcal{L}_2(\Omega, \mathcal{F}_T, P) \times \mathcal{L}_2(\Omega, \mathcal{F}_T, P), \tag{4.4.23}$$

and define the *perturbation function* $F : \mathbb{X} \times \mathbb{U} \to (-\infty, +\infty]$ as

$$F(\pi, (u, v)) := \begin{cases} E\left[J(X^\pi(T) - u)\right], & \text{when } X^\pi(T) + v \geqslant b \text{ a.s.}, \\ +\infty, & \text{otherwise}, \end{cases} \tag{4.4.24}$$

for all $\pi \in \Pi$ and $(u, v) \in \mathbb{U}$. The convexity of $F$ on $\Pi \times \mathbb{U}$ follows from (4.1.19) and (4.4.24), and the fact that $\pi \to X^\pi(T) : \Pi \mapsto \mathcal{L}_2(\Omega, \mathcal{F}_T, P)$ is an affine mapping. It is also clear from (4.4.21) that the consistency relation (3.1.2) is satisfied, i.e.,

$$F(\pi, 0) = f(\pi), \qquad \text{for all } \pi \in \Pi. \tag{4.4.25}$$

**Remark 4.4.8.** We observe that Problem 4.4.7 is really just Problem 4.2.2 with the constraint $X^\pi(T) \geqslant b$ included. The perturbation variable $u \in \mathcal{L}_2(\Omega, \mathcal{F}_T, P)$ at (4.4.24) plays exactly the same role as the perturbation variable $u \in \mathcal{L}_2(\Omega, \mathcal{F}_T, P)$ at (4.2.8), namely to account for the "hidden constraint" represented by the wealth dynamics (see Remark 4.2.4). On the other hand, the perturbation variable $v \in \mathcal{L}_2(\Omega, \mathcal{F}_T, P)$ appearing in the perturbation function at (4.4.24) is included to account for the wealth constraint $X^\pi(T) \geqslant b$ present in Problem 4.4.7. This perturbation variable plays a role quite analogous to that of the perturbation variable $v \in \mathbb{R}^n$ at (3.2.39) which accounts for the inequality constraint $x \geqslant b$ present in Problem 3.2.7.

2. According to Step 3.1.2, we must pair the space of perturbations at (4.4.23) with a vector space $\mathbb{Y}$ through a bilinear form $\langle \cdot, \cdot \rangle$ on $\mathbb{U} \times \mathbb{Y}$. Exactly as at (4.2.10) we shall pair the perturbation variables $u \in \mathcal{L}_2(\Omega, \mathcal{F}_T, P)$ with elements of $\mathbb{B}_1$, and we pair the perturbation variable $v \in \mathcal{L}_2(\Omega, \mathcal{F}_T, P)$ with members of the adjoint $(\mathcal{L}_2(\Omega, \mathcal{F}_T, P))^*$ of $\mathcal{L}_2(\Omega, \mathcal{F}_T, P)$, which is $\mathcal{L}_2(\Omega, \mathcal{F}_T, P)$ itself (recall Remark 3.1.8). We therefore define the space of dual variables

$$\mathbb{Y} := \mathbb{B}_1 \times \mathcal{L}_2(\Omega, \mathcal{F}_T, P), \tag{4.4.26}$$

(see (4.1.10)) together with the "natural" bilinear form on $\mathbb{U} \times \mathbb{Y}$ as

$$\langle (u, v), (Y, \xi) \rangle := E\left[uY(T)\right] + E\left[v\xi\right], \tag{4.4.27}$$

for $(u, v) \in \mathbb{U} = \mathcal{L}_2(\Omega, \mathcal{F}_T, P) \times \mathcal{L}_2(\Omega, \mathcal{F}_T, P)$ and $(Y, \xi) \in \mathbb{Y} = \mathbb{B}_1 \times \mathcal{L}_2(\Omega, \mathcal{F}_T, P)$. This completes all the choices required for Steps 3.1.1 - 3.1.2, and it remains to synthesize a Lagrangian and a dual function.

3. According to Step 3.1.3, define (recall (3.1.3)) the *Lagrangian function* $K : \Pi \times \mathbb{Y} \to [-\infty, +\infty]$ as follows

$$K(\pi, (Y, \xi)) := \inf_{(u,v) \in \mathbb{U}} \{ \langle (u, v), (Y, \xi) \rangle + F(\pi, (u, v)) \}. \tag{4.4.28}$$

From (4.4.25) and (4.4.28), we have the basic inequality

$$f(\pi) \geqslant K(\pi, (Y, \xi)), \qquad \text{for all } (\pi, (Y, \xi)) \in \Pi \times \mathbb{Y}. \tag{4.4.29}$$

Now, the Lagrangian at (4.4.28) can be explicitly evaluated as follows: from (4.4.27) and (4.4.24) we obtain

$$K(\pi, (Y, \xi)) = \inf_{u \in \mathcal{L}_2(\Omega, \mathcal{F}_T, P)} \{ E[uY(T) + J(X^\pi(T) - u)] \} + \inf_{\substack{v \in \mathcal{L}_2(\Omega, \mathcal{F}_T, P) \\ X^\pi(T) + v \geqslant b \text{ a.s.}}} \{ E[v\xi] \},$$
$$\text{for all } (\pi, (Y, \xi)) \in \Pi \times \mathbb{Y}. \tag{4.4.30}$$

We next evaluate each of the terms on the right of (4.4.30). Since $X^\pi \in \mathbb{B}$ for all $\pi \in \Pi$ and $Y \in \mathbb{B}_1 \subset \mathbb{B}$ (see (4.4.26) and (4.1.10)), we have from (4.1.8) that

$$X^\pi(T), Y(T) \in \mathcal{L}_2(\Omega, \mathcal{F}_T, P) \qquad \text{for all } \pi \in \Pi \text{ and } Y \in \mathbb{B}_1. \tag{4.4.31}$$

Therefore, we can apply Proposition D.0.8 to the first term on the right side of (4.4.30) and get

$$\inf_{u \in \mathcal{L}_2(\Omega, \mathcal{F}_T, P)} \{ E[uY(T) + J(X^\pi(T) - u)] \} = E[X^\pi(T)Y(T)] - E[J^*(Y(T))] \tag{4.4.32}$$

Also, from (4.4.1) and (4.4.31), it follows that

$$\inf_{\substack{v \in \mathcal{L}_2(\Omega, \mathcal{F}_T, P) \\ X^\pi(T) + v \geqslant b \text{ a.s.}}} \{ E[v\xi] \} = \begin{cases} E[(b - X^\pi(T))\xi] & \text{if } \xi \geqslant 0 \text{ a.s.}, \\ -\infty & \text{otherwise}. \end{cases} \tag{4.4.33}$$

Therefore, upon combining (4.4.30), (4.4.32) and (4.4.33), we get an explicit evaluation of the Lagrangian as follows: for all $\pi \in \Pi$ and $(Y, \xi) \in \mathbb{Y}$ we have

$$\begin{aligned} & K(\pi, (Y, \xi)) \\ &= \begin{cases} E[X^\pi(T)Y(T)] - E[J^*(Y(T))] + E[(b - X^\pi(T))\xi] & \text{if } \xi \geqslant 0 \text{ a.s.}, \\ -\infty & \text{otherwise}, \end{cases} \\ &= \begin{cases} E[X^\pi(T)(Y(T) - \xi)] - E[J^*(Y(T))] + E[b\xi] & \text{if } \xi \geqslant 0 \text{ a.s.}, \\ -\infty & \text{otherwise}. \end{cases} \end{aligned} \tag{4.4.34}$$

In the view of (3.1.4), the *dual function* $g : \mathbb{Y} \to [-\infty, \infty)$ is defined as:

$$\begin{aligned} & g(Y, \xi) := \inf_{\pi \in \Pi} K(\pi, (Y, \xi)) \\ &= \begin{cases} \inf_{\pi \in \Pi} \{ E[X^\pi(T)(Y(T) - \xi)] \} - E[J^*(Y(T))] + E[b\xi] & \text{if } \xi \geqslant 0 \text{ a.s.} \\ -\infty & \text{otherwise}, \end{cases} \\ &= \begin{cases} \inf_{\pi \in \Pi} \{ E[X^\pi(T)Y(T)] - E[\xi(X^\pi(T) - b)] \} \\ \qquad - E[J^*(Y(T))] & \text{if } \xi \geqslant 0 \text{ a.s.} \\ -\infty & \text{otherwise}, \end{cases} \end{aligned} \tag{4.4.35}$$

for all $(Y,\xi) \in \mathbb{Y}$. Again, the *weak duality* relation (c.f. (3.2.16)) holds from (4.4.29) and (4.4.35), that is:

$$f(\pi) \geqslant K(\pi,(Y,\xi)) \geqslant g(Y,\xi), \qquad \text{for all } (\pi,(Y,\xi)) \in \Pi \times \mathbb{Y}. \tag{4.4.36}$$

The *dual problem* is then to maximize $g(Y,\xi)$ over all $(Y,\xi) \in \mathbb{Y} = \mathbb{B}_1 \times \mathcal{L}_2(\Omega,\mathcal{F}_T,P)$. Define the value of this dual problem, namely

$$\bar{\vartheta} := \sup_{(Y,\xi)\in\mathbb{Y}} \{g(Y,\xi)\}, \tag{4.4.37}$$

i.e., the dual problem is to establish that

$$g(\bar{Y},\bar{\xi}) = \bar{\vartheta}, \qquad \text{for some } (\bar{Y},\bar{\xi}) \in \mathbb{Y}. \tag{4.4.38}$$

**Remark 4.4.9.** We will see shortly that the dual problem at (4.4.38) can actually be simplified to a problem which involves the maximization of a concave function defined over the real line $\mathbb{R}$ (see (4.4.65), Proposition 4.4.14 and Proposition 4.4.16 which follow).

**Remark 4.4.10.** Recalling Condition 4.4.3 and the weak duality (4.4.36), we find that

$$\bar{\vartheta} \overset{(4.4.37)}{=} \sup_{(Y,\xi)\in\mathbb{Y}} \{g(Y,\xi)\} \leqslant \inf_{\pi\in\Pi} \{f(\pi)\} \leqslant f(\hat{\pi}) < \infty. \tag{4.4.39}$$

From (4.4.35) we obtain

$$\begin{aligned}
&\sup_{(Y,\xi)\in\mathbb{Y}} \{g(Y,\xi)\} \\
&\overset{(4.4.35)}{=} \sup_{\substack{Y\in\mathbb{B}_1,\xi\geqslant 0\\ \xi\in\mathcal{L}_2(\Omega,\mathcal{F}_T,P)}} \left\{ \inf_{\pi\in\Pi} \{E[X^\pi(T)(Y(T)-\xi)]\} - E[J^*(Y(T))] + E[b\xi] \right\} \\
&= \sup_{Y,R\in\mathbb{B}_1,R(T)\geqslant 0} \left\{ \inf_{\pi\in\Pi} \{E[X^\pi(T)(Y(T)-R(T))]\} - E[J^*(Y(T))] + E[bR(T)] \right\}, \\
&= \sup_{\substack{Y,\tilde{Y}\in\mathbb{B}_1\\ Y(T)-\tilde{Y}(T)\geqslant 0}} \left\{ \inf_{\pi\in\Pi} \left\{ E\left[X^\pi(T)\tilde{Y}(T)\right]\right\} - E[J^*(Y(T))] + E\left[b(Y(T)-\tilde{Y}(T))\right] \right\},
\end{aligned} \tag{4.4.40}$$

where, at the second equality of (4.4.40), we have used Proposition 4.1.9 to replace $\xi \in \mathcal{L}_2(\Omega,\mathcal{F}_T,P)$ with $R \in \mathbb{B}_1$, and at the third equality of (4.4.40), we have replaced the variable $R \in \mathbb{B}_1$ with the variable $\tilde{Y} \in \mathbb{B}_1$ defined by

$$\tilde{Y} := Y - R \in \mathbb{B}_1. \tag{4.4.41}$$

From (4.4.41) and Proposition 4.2.5, we have

$$\inf_{\pi\in\Pi} \left\{ E\left[X^\pi(T)\tilde{Y}(T)\right]\right\} = \begin{cases} x_0\tilde{Y}(0) & \text{if } \tilde{Y} = \tilde{Y}(0)H \text{ a.e.,} \\ -\infty & \text{otherwise,} \end{cases} \tag{4.4.42}$$

for all $\tilde{Y} \in \mathbb{B}_1$. We then get

$$\begin{aligned}
&\sup_{(Y,\xi)\in\mathbb{Y}} \{g(Y,\xi)\} \\
&\overset{(4.4.40)}{\underset{(4.4.42)}{=}} \sup_{\substack{Y,\tilde{Y}\in\mathbb{B}_1,\tilde{Y}=\tilde{Y}(0)H\\ Y(T)-\tilde{Y}(T)\geqslant 0}} \left\{x_0\tilde{Y}(0) - E\left[J^*(Y(T))\right] + E\left[b(Y(T)-\tilde{Y}(T))\right]\right\}, \\
&= \sup_{\substack{Y\in\mathbb{B}_1,\alpha\in\mathbb{R}\\ Y(T)\geqslant\alpha H(T)}} \left\{x_0\alpha - E\left[J^*(Y(T))\right] + E\left[b(Y(T)-\alpha H(T))\right]\right\}, \qquad (4.4.43)
\end{aligned}$$

where in the second equality of (4.4.43) we have replaced $\tilde{Y}(0) \in \mathbb{R}$ with $\alpha \in \mathbb{R}$. From (4.4.43) it follows that

$$\begin{aligned}
&\sup_{(Y,\xi)\in\mathbb{Y}} \{g(Y,\xi)\} \\
&= \sup_{\alpha\in\mathbb{R}} \left\{\alpha(x_0 - E\left[bH(T)\right]) + \sup_{\substack{Y\in\mathbb{B}_1\\ \alpha H(T)\leqslant Y(T)}} \left\{E\left[bY(T) - J^*(Y(T))\right]\right\}\right\}. \qquad (4.4.44)
\end{aligned}$$

**Remark 4.4.11.** Since $H(T) \in \mathcal{L}_2(\Omega, \mathcal{F}_T, P)$ (recall (2.1.11)), it follows from (4.4.44) that

$$\begin{aligned}
\bar{\vartheta} &\overset{(4.4.37)}{=} \sup_{(Y,\xi)\in\mathbb{Y}} \{g(Y,\xi)\} \overset{(4.4.44)}{\geqslant} \alpha(x_0 - E\left[bH(T)\right]) + E\left[b(\alpha H(T)) - J^*(\alpha H(T))\right]\} \\
&\geqslant \alpha x_0 - E\left[J^*(\alpha H(T))\right] \in \mathbb{R}, \qquad \text{for all } \alpha \in \mathbb{R}, \qquad (4.4.45)
\end{aligned}$$

and (4.4.45) together with (4.4.39) gives

$$\bar{\vartheta} \in \mathbb{R}. \qquad (4.4.46)$$

We next evaluate the supremum inside the braces on the right side of (4.4.44) for each fixed $\alpha \in \mathbb{R}$. To this end, recalling Corollary 4.1.10, we define

$$\begin{aligned}
\Gamma(\alpha) &:= \sup_{\substack{Y\in\mathbb{B}_1\\ \alpha H(T)\leqslant Y(T)}} \left\{E\left[bY(T) - J^*(Y(T))\right]\right\} \\
&= \sup_{\substack{\eta\in\mathcal{L}_2(\Omega,\mathcal{F}_T,P)\\ \alpha H(T)\leqslant\eta}} \left\{E\left[b\eta - J^*(\eta)\right]\right\}, \qquad \alpha \in \mathbb{R}. \qquad (4.4.47)
\end{aligned}$$

Motivated by the expression $[b\eta - J^*(\eta)]$ on the right side of (4.4.47), for each $(y,\omega) \in \mathbb{R}\times\Omega$ define

$$\begin{aligned}
h(y,\omega) &:= b(\omega)y - J^*(y,\omega) \overset{(4.1.20)}{=} b(\omega)y - \frac{(y-c(\omega))^2}{2a(\omega)} - q \\
&= -\frac{(y-(c+ab)(\omega))^2}{2a(\omega)} + \left(b(c+\frac{ab}{2})\right)(\omega) + q, \quad (y,\omega) \in \mathbb{R}\times\Omega. \quad (4.4.48)
\end{aligned}$$

Then, combining (4.4.48) and (4.4.47), we have

$$\Gamma(\alpha) = \sup_{\substack{\eta\in\mathcal{L}_2(\Omega,\mathcal{F}_T,P)\\ \alpha H(T)\leqslant\eta}} \{E\left[h(\eta)\right]\}, \qquad \alpha\in\mathbb{R}. \tag{4.4.49}$$

Indeed, it will be shown in Proposition 4.4.12 which follows that the supremum in (4.4.49) is achieved for each given $\alpha\in\mathbb{R}$ at some $\eta_\alpha\in\mathcal{L}_2(\Omega,\mathcal{F}_T,P)$ defined as:

$$\begin{aligned}\eta_\alpha(\omega) \;&:=\; \begin{cases} c(\omega)+a(\omega)b(\omega) & \text{if } \alpha H(\omega;T)<(c+ab)(\omega),\\ \alpha H(\omega;T) & \text{if } \alpha H(\omega;T)\geqslant(c+ab)(\omega),\end{cases}\\ &=\; \max\{\alpha H(\omega;T), c(\omega)+a(\omega)b(\omega)\}, \qquad \omega\in\Omega,\end{aligned} \tag{4.4.50}$$

where the second equality of (4.4.50) follows since the function $y\to h(y,\omega):\mathbb{R}\to\mathbb{R}$ defined at (4.4.48) attains it maximum at $y=(c+ab)(\omega)$. Note that we also have

$$\eta_\alpha\in\mathcal{L}_2(\Omega,\mathcal{F}_T,P) \qquad \text{and} \qquad \alpha H(T)\leqslant\eta_\alpha \quad \text{a.s.} \tag{4.4.51}$$

for each $\alpha\in\mathbb{R}$ (as follows from (2.1.11) and Condition 2.2.1).

**Proposition 4.4.12.** *Suppose Condition 2.2.1, and recall (4.4.1), (4.4.48), (4.4.49) and (4.4.50). Then*

$$\Gamma(\alpha)=E\left[h(\eta_\alpha)\right]=E\left[h^{opt}(\alpha H(T))\right], \qquad \textit{for all } \alpha\in\mathbb{R}, \tag{4.4.52}$$

*where*

$$h^{opt}(y,\omega) \;:=\; \begin{cases} h((c+ab)(\omega),\omega) & \textit{if } y<(c+ab)(\omega),\\ h(y,\omega) & \textit{if } y\geqslant(c+ab)(\omega),\end{cases} \qquad (y,\omega)\in\mathbb{R}\times\Omega. \tag{4.4.53}$$

*In particular,*

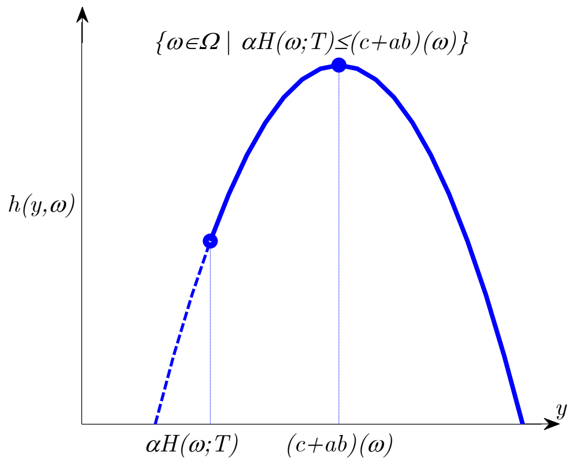
$$h(\eta_\alpha)=h^{opt}(\alpha H(T)) \quad \textit{a.s.} \qquad \textit{for all } \alpha\in\mathbb{R}. \tag{4.4.54}$$

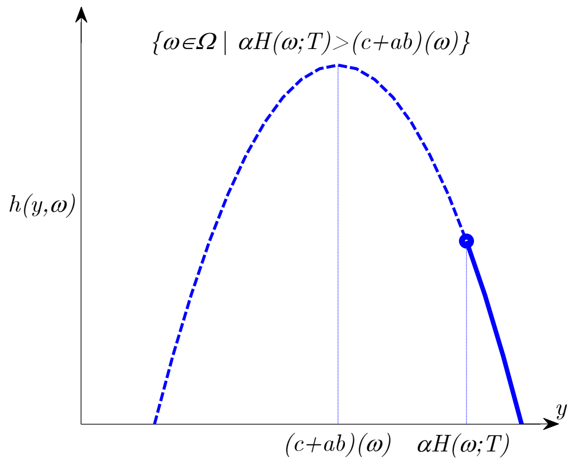
**Proof.** Fix some $\alpha\in\mathbb{R}$. From (4.4.51) and (4.4.49), it directly follows that

$$E\left[h(\eta_\alpha)\right] \overset{(4.4.51)}{\leqslant} \sup_{\substack{\eta\in\mathcal{L}_2(\Omega,\mathcal{F}_T,P)\\ \alpha H(T)\leqslant\eta}} \{E\left[h(\eta)\right]\} \overset{(4.4.49)}{=} \Gamma(\alpha). \tag{4.4.55}$$

On the other hand, we see from (4.4.48) and Condition 2.2.1 that $y\to h(y,\omega)$ is an inverted quadratic function for each fixed $\omega\in\Omega$, and therefore it holds (see figure)

$$\begin{aligned} h(\eta_\alpha(\omega),\omega) \;&\overset{(4.4.50)}{=}\; \begin{cases} h((c+ab)(\omega),\omega) & \text{if } \alpha H(\omega;T)<(c+ab)(\omega),\\ h(\alpha H(\omega;T),\omega) & \text{if } \alpha H(\omega;T)\geqslant(c+ab)(\omega),\end{cases}\\ &\geqslant\; h(y,\omega), \quad \text{for } (y,\omega)\in\mathbb{R}\times\Omega \text{ such that } y\geqslant\alpha H(\omega;T).\end{aligned} \tag{4.4.56}$$






Given any $\eta \in \mathcal{L}_2(\Omega, \mathcal{F}_T, P)$ such that $\eta \geqslant \alpha H(\omega; T)$ a.s., from (4.4.56) we have

$$h(\eta_\alpha(\omega), \omega) \overset{(4.4.56)}{\geqslant} h(\eta(\omega), \omega), \qquad \omega \in \Omega, \tag{4.4.57}$$

i.e.,

$$h(\eta_\alpha) \overset{(4.4.57)}{\geqslant} h(\eta) \quad \text{a.s.} \quad \text{for any } \eta \in \mathcal{L}_2(\Omega, \mathcal{F}_T, P) \text{ such that } \eta \geqslant \alpha H(T) \text{ a.s..} \tag{4.4.58}$$

Thus,

$$E\left[h(\eta_\alpha)\right] \overset{(4.4.58)}{\geqslant} \sup_{\substack{\eta \in \mathcal{L}_2(\Omega, \mathcal{F}_T, P) \\ \alpha H(T) \leqslant \eta}} \left\{E\left[h(\eta)\right]\right\} \overset{(4.4.49)}{=} \Gamma(\alpha). \tag{4.4.59}$$

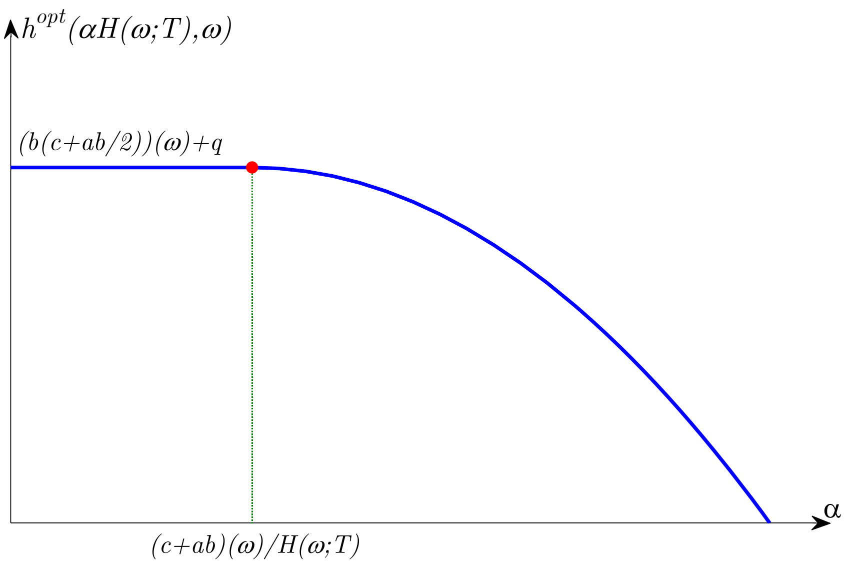
By the arbitrary choice of $\alpha \in \mathbb{R}$, the first equality in (4.4.52) follows from (4.4.55) and (4.4.59). By comparing (4.4.56) with (4.4.53), we also have

$$h(\eta_\alpha(\omega), \omega) = h^{opt}(\alpha H(\omega; T), \omega), \quad \text{for } \omega \in \Omega \text{ and } \alpha \in \mathbb{R}, \tag{4.4.60}$$

i.e., (4.4.54) holds and the second equality in (4.4.52) follows by taking expectation on both sides of (4.4.60). □

**Remark 4.4.13.** By Proposition 4.4.12 and (4.4.47), we see that $\Gamma(\alpha) = E\left[h^{opt}(\alpha H(T))\right]$ is equal to the supremum in braces on the right side of (4.4.44) as a function of $\alpha \in \mathbb{R}$. Fix some $\omega \in \Omega$: from (4.4.54), (4.4.48) and (4.4.50), we have for all $\alpha \in \mathbb{R}$ that

$$\begin{aligned} h^{opt}(\alpha H(\omega; T), \omega) &\overset{(4.4.54)}{=} h(\eta_\alpha(\omega), \omega) \\ &\overset{(4.4.48)}{\underset{(4.4.50)}{=}} \begin{cases} \left(b(c + \dfrac{ab}{2})\right)(\omega) + q, & \text{if } \alpha H(\omega; T) < (c + ab)(\omega), \\ -\dfrac{(\alpha H(\omega; T) - (c + ab)(\omega))^2}{2a(\omega)} + \left(b(c + \dfrac{ab}{2})\right)(\omega) + q & \\ & \text{if } \alpha H(\omega; T) \geqslant (c + ab)(\omega), \end{cases} \end{aligned} \tag{4.4.61}$$




and $\alpha \to h^{opt}(\alpha H(\omega;T),\omega)$ is an "inverted half-quadratic" function on $\mathbb{R}$ (see the figure which follows). We further define

$$\Psi(\alpha) := \alpha(x_0 - E\left[bH(T)\right]) + E\left[h^{opt}(\alpha H(T))\right], \quad \alpha \in \mathbb{R}. \tag{4.4.62}$$

Then, from (4.4.47), (4.4.52) and (4.4.62), we have

$$\Psi(\alpha) = \alpha(x_0 - E\left[bH(T)\right]) + \sup_{\substack{Y \in \mathbb{B}_1 \\ \alpha H(T) \leqslant Y(T)}} \{E\left[bY(T) - J^*(Y(T))\right]\}, \tag{4.4.63}$$

and by (4.4.63) we see that (4.4.44) reduces to

$$\sup_{(Y,\xi)\in\mathbb{Y}} \{g(Y,\xi)\} = \sup_{\alpha\in\mathbb{R}}\{\Psi(\alpha)\}. \tag{4.4.64}$$

As we shall see shortly (at Proposition 4.4.16) the dual problem (4.4.38) effectively simplifies to the following one-dimensional problem:

$$\text{determine some } \bar{\alpha} \in \mathbb{R} \text{ such that } \Psi(\bar{\alpha}) = \sup_{\alpha\in\mathbb{R}}\{\Psi(\alpha)\}.. \tag{4.4.65}$$

Of course, it is not immediately clear that such a maximizing $\bar{\alpha}$ exists. The following Proposition 4.4.14, which is proved in Appendix A, ensures that this is the case:

**Proposition 4.4.14.** *Suppose Condition 4.4.3. Then $\Psi(\cdot)$, given by* (4.4.62) *and* (4.4.53), *is concave and continuous on $\mathbb{R}$, and* (recall (4.4.37) and (4.4.46))

$$\Psi(\bar{\alpha}) = \sup_{\alpha\in\mathbb{R}}\{\Psi(\alpha)\} = \bar{\vartheta} \overset{(4.4.46)}{\in} \mathbb{R}, \quad \textit{for some } \bar{\alpha} \in \mathbb{R}. \tag{4.4.66}$$

**Remark 4.4.15.** One sees from the proof of Proposition 4.4.14 in Appendix A that the Slater-type Condition 4.4.3 plays an essential role in securing existence of $\bar{\alpha}$.

We shall now use the maximizer $\bar{\alpha}$ given by Proposition 4.4.14 to construct a solution $(\bar{Y}, \bar{\xi}) \in \mathbb{Y}$ of the dual problem (4.4.38):

**Proposition 4.4.16.** *Let $\bar{\alpha} \in \mathbb{R}$ be the maximizer given by Proposition 4.4.14. Then $\eta_{\bar{\alpha}} \in \mathcal{L}_2(\Omega, \mathcal{F}_T, P)$ (recall (4.4.50) and (4.4.51)). Define*

$$\bar{\xi} := \eta_{\bar{\alpha}} - \bar{\alpha} H(T). \tag{4.4.67}$$

*Then*

$$\bar{\xi} \in \mathcal{L}_2(\Omega, \mathcal{F}_T, P) \quad \text{and} \quad \bar{\xi} \geqslant 0. \tag{4.4.68}$$

*Fix $\bar{Y} \in \mathbb{B}_1$ such that $\bar{Y}(T) = \eta_{\bar{\alpha}}$ (recall Proposition 4.1.9). Then $(\bar{Y}, \bar{\xi}) \in \mathbb{Y}$ and*

$$g(\bar{Y}, \bar{\xi}) = \bar{\vartheta} \overset{(4.4.37)}{=} \sup_{(Y,\xi)\in\mathbb{Y}} \{g(Y, \xi)\}. \tag{4.4.69}$$

**Proof.** In view of Proposition 4.4.14 there exists some $\bar{\alpha} \in \mathbb{R}$ such that

$$\begin{aligned}
\bar{\vartheta} &\overset{(4.4.66)}{=} \Psi(\bar{\alpha}) \\
&\overset{(4.4.62)}{=} \bar{\alpha}(x_0 - E[bH(T)]) + E\left[h^{opt}(\bar{\alpha}H(T))\right] \\
&\overset{(4.4.52)}{=} \bar{\alpha}(x_0 - E[bH(T)]) + E[h(\eta_{\bar{\alpha}})] \\
&\overset{(4.4.48)}{=} \bar{\alpha}(x_0 - E[bH(T)]) + E[b\eta_{\bar{\alpha}} - J^*(\eta_{\bar{\alpha}})] \\
&= \bar{\alpha}x_0 - E[J^*(\eta_{\bar{\alpha}})] - E[b(\bar{\alpha}H(T) - \eta_{\bar{\alpha}})].
\end{aligned} \tag{4.4.70}$$

Since $\eta_{\bar{\alpha}} \in \mathcal{L}_2(\Omega, \mathcal{F}_T, P)$ (recall (4.4.51)), from Proposition 4.1.9, we see that

$$\text{there exists some } \bar{Y} \in \mathbb{B}_1 \text{ such that } \bar{Y}(T) = \eta_{\bar{\alpha}} \overset{(4.4.50)}{=} \max\{\bar{\alpha}H(T), c + ab\}, \tag{4.4.71}$$

and we can define

$$\bar{\xi} := \eta_{\bar{\alpha}} - \bar{\alpha}H(T). \tag{4.4.72}$$

Then, (4.4.72) together with (4.4.51) and (2.1.11) gives

$$\bar{\xi} \in \mathcal{L}_2(\Omega, \mathcal{F}_T, P) \quad \text{and} \quad \bar{\xi} \geqslant 0. \tag{4.4.73}$$

In view of (4.4.71), (4.4.72) and (4.4.73), we see that

$$(\bar{Y}, \bar{\xi}) \in \mathbb{Y} \quad \text{with} \quad \bar{Y}(T) = \eta_{\bar{\alpha}} \quad \text{and} \quad \bar{\xi} := \eta_{\bar{\alpha}} - \bar{\alpha}H(T), \tag{4.4.74}$$

where $\eta_{\bar{\alpha}}$ is defined by (4.4.50). We shall now verify that $(\bar{Y}, \bar{\xi})$ is the optimal solution of the *dual problem* (4.4.38). Actually, from (4.4.35) and (4.4.73), it follows that

$$\begin{aligned}
g(\bar{Y}, \bar{\xi}) &= \inf_{\pi\in\Pi}\{E\left[X^\pi(T)(\bar{Y}(T) - \bar{\xi})\right]\} - E\left[J^*(\bar{Y}(T))\right] + E\left[b\bar{\xi}\right] \\
&\overset{(4.4.74)}{=} \inf_{\pi\in\Pi}\{E[X^\pi(T)(\bar{\alpha}H(T))]\} - E[J^*(\eta_{\bar{\alpha}})] + E[b(\eta_{\bar{\alpha}} - \bar{\alpha}H(T))] \\
&\overset{(2.1.26)}{=} \bar{\alpha}x_0 - E[J^*(\eta_{\bar{\alpha}})] - E[b(\bar{\alpha}H(T) - \eta_{\bar{\alpha}})] \\
&\overset{(4.4.70)}{=} \bar{\vartheta} := \sup_{(Y,\xi)\in\mathbb{Y}} \{g(Y, \xi)\}.
\end{aligned} \tag{4.4.75}$$

We have therefore established existence of some $(\bar{Y}, \bar{\xi}) \in \mathbb{Y}$ which maximizes the dual function $g$ over the space of dual variables $\mathbb{Y}$, and furthermore this maximizing element $(\bar{Y}, \bar{\xi})$ is given by (4.4.74). $\square$

**Remark 4.4.17.** In light of Proposition 4.4.14 and Proposition 4.4.16 we see the dual problem essentially reduces to the maximization of a concave function $\Psi(\cdot)$ defined on the real line, that is the dual problem is effectively one-dimensional. This is really a consequence of the simplicity of problem (4.4.2), which involves only an a.s. wealth constraint $X^\pi(T) \geq b$ a.s. and does not include a portfolio constraint. Because of this, we can establish existence of an optimal dual solution directly, without recourse to Theorem 3.1.7 (much as was the case with the unconstrained problem 4.2.1, recall Remark 4.2.6). In the Section 4.5 which follows we shall add a portfolio constraint the problem (4.4.2), and application of Theorem 3.1.7 to get existence of an optimal dual solution will be essential.

**Remark 4.4.18.** Recalling Remark 3.1.9 and the *weak duality* relation of (4.4.36), we see that Problem 4.4.7 (*primal problem*) amounts to the construction of some pair $(\pi, (Y, \xi)) \in \mathbb{X} \times \mathbb{Y}$ such that

$$f(\pi) = g(Y, \xi), \tag{4.4.76}$$

in which case $\pi$ is a minimizer of $f(\cdot)$ on $\mathbb{X} = \Pi$, that is $\pi$ is the optimal portfolio, and $(Y, \xi)$ is a maximizer of $g(\cdot)$ on $\mathbb{Y} = \mathbb{B}_1 \times \mathcal{L}_2(\Omega, \mathcal{F}_T, P)$. However, we have already secured such a maximizer $(\bar{Y}, \bar{\xi})$, so the solution of Problem 4.4.7 actually reduces to the construction of some $\bar{\pi} \in \Pi$ such that

$$f(\bar{\pi}) = g(\bar{Y}, \bar{\xi}), \tag{4.4.77}$$

for the optimal $(\bar{Y}, \bar{\xi})$ given by Proposition 4.4.16. To facilitate construction of this $\bar{\pi}$, we next establish a set of *Kuhn-Tucker optimality relations* which are fully equivalent to the equality at (4.4.76) for *arbitrary* $(\pi, (Y, \xi)) \in \Pi \times \mathbb{Y}$ (recall Remark 3.1.10 (c)).

**Proposition 4.4.19. [Kuhn-Tucker Optimality Conditions]**
*Suppose Condition 2.1.1, 2.1.2, 2.1.5, 2.1.11 on the market settings and Condition 2.2.1 on the quadratic criterion function $J$. Then we have the following equivalence for each $(\pi, (Y, \xi)) \in \Pi \times \mathbb{Y}$:*

$$f(\pi) = g(Y, \xi) \iff \begin{cases} (1)\ X^\pi(T) \geqslant b, \quad (2)\ \xi \geqslant 0, \\ (3)\ Y(T) - \xi = \alpha H(T) \text{ for some } \alpha \in \mathbb{R}, \\ (4)\ (X^\pi(T) - b)\xi = 0, \quad (5)\ X^\pi(T) = \partial J^*(Y(T)), \end{cases} \tag{4.4.78}$$

*where $\partial J^*(\cdot)$ denotes the derivative function of the convex conjugate function $J^*(\cdot)$ given in (4.1.20).*

The proof of Proposition 4.4.19 is in Appendix A.

**Remark 4.4.20.** The Kuhn-Tucker conditions at (4.4.78) can be compared with the Kuhn-Tucker conditions at (3.2.55) for Problem 3.2.7. In particular, (4.4.78) - (1) is a *feasibility condition* on the primal variable $\pi$ and (4.4.78) - (2)(3) are *feasibility conditions* on the dual variable $(Y, \xi)$. By Proposition 4.4.16, the maximizer $(\bar{Y}, \bar{\xi})$ satisfies the feasibility condition:

$$(2)\ \bar{\xi} \geqslant 0, \qquad (3)\ \bar{Y}(T) - \bar{\xi} = \bar{\alpha} H(T). \tag{4.4.79}$$

The relation (4.4.78) - (4) is a *complementary slackness condition* relating the primal variable $\pi$ and the dual variable $\xi$. This condition indicates that, at optimality, the "Lagrange weighting" $(b - X^{\pi}(T))\xi$ for the inequality constraint $X^{\pi} \geqslant b$ on the right hand side of the first equality at (4.4.34) must be zero a.s.(when $\xi \geqslant 0$). Finally, (4.4.78) - (5) is a *transversality condition*, which relates the optimal primal and dual values $\bar{\pi}$ and $\bar{Y}$ to the *risk criterion function* $J$. In view of (4.4.79) the remaining task is to construct a portfolio process $\bar{\pi} \in \Pi$ such that (4.4.78)-(1)(4)(5) hold, i.e.

$$(1) \quad X^{\bar{\pi}}(T) \geqslant b, \quad (4) \quad (X^{\bar{\pi}}(T) - b)\bar{\xi} = 0, \quad (5)\ X^{\bar{\pi}}(T) = \partial J^*(\bar{Y}(T)), \tag{4.4.80}$$

with $(\bar{Y}, \bar{\xi})$ given Proposition 4.4.16, for it then follows from Proposition 4.4.19, together with (4.4.80) and (4.4.79), that

$$f(\bar{\pi}) = g(\bar{Y}, \bar{\xi}), \qquad \bar{\pi} \in \Pi, \qquad (\bar{Y}, \bar{\xi}) \in \mathbb{Y}, \tag{4.4.81}$$

and $\bar{\pi} \in \Pi$ must be the optimal portfolio for Problem 4.4.7 (as follows from Remark 4.4.18).

In order to construct some $\bar{\pi} \in \Pi$ such that (4.4.80) holds we shall require the following Proposition 4.4.21, which gives a necessary condition resulting from the optimality of $(\bar{Y}, \bar{\xi}) \in \mathbb{Y}$ for the dual problem (see (4.4.38) and (4.4.69)).

**Proposition 4.4.21.** *Suppose Condition 2.1.1, 2.1.2, 2.1.5, 2.1.11, 2.2.1 and 4.4.3. Then, for each $Y \in \mathbb{B}_1$ (see (4.1.10)), we have*

$$E\left[\partial J^*(\bar{Y}(T))Y(T)\right] \geqslant \inf_{\pi \in \Pi}\{E\left[X^{\pi}(T)Y(T)\right]\}, \tag{4.4.82}$$

*where $\partial J^*(\cdot)$ denotes the derivative function of the convex conjugate function $J^*(\cdot)$ given in (4.1.20) and $\bar{Y}$ is given by Proposition 4.4.16.*

**Proof.** From (4.4.75), we know

$$g(\bar{Y}, \bar{\xi}) = \sup_{(Y,\xi) \in \mathbb{Y}}\{g(Y, \xi)\} \in \mathbb{R}. \tag{4.4.83}$$

For all $\varepsilon \in (0, +\infty)$, it holds from (4.4.83) that

$$g(\bar{Y} + \varepsilon Y, \bar{\xi}) \leqslant g(\bar{Y}, \bar{\xi}) \in \mathbb{R}. \tag{4.4.84}$$

From (4.4.84) and (4.4.35) we find

$$\begin{aligned} &\inf_{\pi \in \Pi}\{E\left[X^{\pi}(T)(\bar{Y}(T) + \varepsilon Y(T) - \bar{\xi})\right]\} - E\left[J^*(\bar{Y}(T) + \varepsilon Y(T))\right] \\ \leqslant\ &\inf_{\pi \in \Pi}\{E\left[X^{\pi}(T)(\bar{Y}(T) - \bar{\xi})\right]\} - E\left[J^*(\bar{Y}(T))\right], \qquad \text{for all } Y \in \mathbb{B}_1. \end{aligned} \tag{4.4.85}$$

Now rearrange (4.4.85),

$$\begin{aligned}
& E\left[J^*(\bar{Y}(T)+\varepsilon Y(T))\right]-E\left[J^*(\bar{Y}(T))\right] \\
\geqslant\ & \inf_{\pi\in\Pi}\{E\left[X^\pi(T)(\bar{Y}(T)+\varepsilon Y(T)-\bar{\xi})\right]\}-\inf_{\pi\in\Pi}\{E\left[X^\pi(T)(\bar{Y}(T)-\bar{\xi})\right]\} \\
\geqslant\ & \inf_{\pi\in\Pi}\{E\left[X^\pi(T)(\bar{Y}(T)-\bar{\xi})\right]\}+\varepsilon\inf_{\pi\in\Pi}\{E\left[X^\pi(T)Y(T)\right]\}-\inf_{\pi\in\Pi}\{E\left[X^\pi(T)(\bar{Y}(T)-\bar{\xi})\right]\} \\
\geqslant\ & \varepsilon\inf_{\pi\in\Pi}\{E\left[X^\pi(T)Y(T)\right]\},
\end{aligned} \tag{4.4.86}$$

and divide both sides of (4.4.86) by $\varepsilon\in(0,+\infty)$:

$$E\left[\frac{J^*(\bar{Y}(T)+\varepsilon Y(T))-J^*(\bar{Y}(T))}{\varepsilon}\right]-\inf_{\pi\in\Pi}\{E\left[X^\pi(T)Y(T)\right]\}\geqslant 0,\quad Y\in\mathbb{B}_1. \tag{4.4.87}$$

From (4.1.20) and Condition 2.2.1, we can use dominated convergence to evaluate the limit on the left of (4.4.87) as $\varepsilon\to 0$, to get

$$E\left[\partial J^*(\bar{Y}(T))Y(T)\right]\geqslant\inf_{\pi\in\Pi}\{E\left[X^\pi(T)Y(T)\right]\},\qquad Y\in\mathbb{B}_1. \tag{4.4.88}$$

□

Now, from Condition 2.2.1, we know

$$\partial J^*(\bar{Y}(T))\overset{(4.1.20)}{=}\frac{\bar{Y}(T)-c}{a}\overset{(4.4.71)}{=}\frac{\eta_{\bar{\alpha}}-c}{a}\in\mathcal{L}_2(\Omega,\mathcal{F}_T,P), \tag{4.4.89}$$

where $\eta_\alpha:=\max\{\alpha H(T),c+ab\}$ is given in (4.4.50) (recall Proposition 4.4.14 for the existence of $\bar{\alpha}$). Motivated by (4.2.36) define the process $\bar{X}$ as

$$\bar{X}(t):=H^{-1}(t)E\left[H(T)\partial J^*(\bar{Y}(T))\,\middle|\,\mathcal{F}_t\right]\qquad t\in[0,T]. \tag{4.4.90}$$

Then, from Proposition 4.2.9, there exists some a.e. unique $\mathbb{R}^N$-valued $\mathcal{F}_t$-progressively measurable process $\bar{\psi}$ such that

$$\int_0^T\left\|\bar{\psi}(t)\right\|^2\mathrm{d}t<\infty\quad\text{a.s.}\qquad\text{and}\qquad H(t)\bar{X}(t)=\bar{X}(0)+\int_0^t\bar{\psi}'(s)\mathrm{d}W(s).$$

Now define an $\mathbb{R}^N$-valued $\mathcal{F}_t$-progressively measurable process

$$\bar{\pi}(t):=(\sigma'(t))^{-1}\left[H^{-1}(t)\bar{\psi}(t)+\bar{X}(t)\theta(t)\right]. \tag{4.4.91}$$

It follows from Proposition 4.2.9 again that

$$\bar{\pi}\in\Pi, \tag{4.4.92}$$

and

$$\mathrm{d}\bar{X}(t)=\left[r(t)\bar{X}(t)+\bar{\pi}'(t)\sigma(t)\theta(t)\right]\mathrm{d}t+\bar{\pi}'(t)\sigma(t)\mathrm{d}W(t), \tag{4.4.93}$$

with the initial value

$$\bar{X}(0)=E\left[H(T)\partial J^*(\bar{Y}(T))\right]. \tag{4.4.94}$$

We next use Proposition 4.4.21 as a tool for establishing the following Proposition 4.4.22, which gives properties of $\bar{X}$ and $\bar{\pi}$ defined at (4.4.90) - (4.4.91):

**Proposition 4.4.22.** *Suppose Condition 2.1.1, 2.1.2, 2.1.5, 2.1.11, 2.2.1 and 4.4.3, and recall $\bar{X}$ and $\bar{\pi}$ defined at (4.4.90) - (4.4.91) in terms of $\bar{Y} \in \mathbb{B}_1$ given by Proposition 4.4.16. Then*

$$\text{(1)} \qquad \bar{X}(0) = x_0, \tag{4.4.95}$$
$$\text{(2)} \qquad \bar{X}(T) \geqslant b, \tag{4.4.96}$$
$$\text{(3)} \qquad \bar{X} = X^{\bar{\pi}}. \tag{4.4.97}$$

**Proof.** Fix some $y \in \mathbb{R}$, and put

$$\hat{Y} := yH. \tag{4.4.98}$$

By Remark 4.1.7 (b), it follows $\hat{Y} \in \mathbb{B}_1$. With Proposition 4.4.21 in hand, we replace $Y$ with $\hat{Y}$ in (4.4.82) and get

$$\begin{aligned} y\bar{X}(0) &\overset{(4.4.94)}{=} yE\left[\partial J^*(\bar{Y}(T))H(T)\right] \overset{(4.4.98)}{=} E\left[\partial J^*(\bar{Y}(T))\hat{Y}(T)\right] \\ &\overset{(4.4.82)}{\geqslant} \inf_{\pi\in\Pi}\{E\left[X^\pi(T)\hat{Y}(T)\right]\} = y\inf_{\pi\in\Pi}\{E\left[X^\pi(T)H(T)\right]\} \\ &\overset{(2.1.26)}{=} yx_0. \end{aligned} \tag{4.4.99}$$

By the arbitrary choice of $y \in \mathbb{R}$, we obtain from (4.4.99)

$$\bar{X}(0) = x_0. \tag{4.4.100}$$

Comparing (4.4.93) and (4.4.100) with (2.1.21), we get $\bar{X}$ is the wealth process of the portfolio $\bar{\pi}$, i.e.,

$$\bar{X} = X^{\bar{\pi}}. \tag{4.4.101}$$

Finally, from (4.4.90), (4.4.89), (4.4.50) and Condition 2.2.1, we have

$$\begin{aligned} \bar{X}(T) &\overset{(4.4.90)}{:=} \partial J^*(\bar{Y}(T)) \overset{(4.4.89)}{=} \frac{\eta_{\bar{\alpha}} - c}{a} \overset{(4.4.50)}{=} \frac{\max\{\bar{\alpha}H(T), c + ab\} - c}{a} \\ &\overset{(2.2.4)}{=} \max\{\frac{\bar{\alpha}H(T) - c}{a}, b\} \geqslant b. \end{aligned} \tag{4.4.102}$$

□

**Remark 4.4.23.** From Proposition 4.4.22, we see that $\bar{\pi} \in \Pi$ given by (4.4.91) satisfies the optimality relations (4.4.80)-(1)(4)(5). Indeed, (4.4.80)-(1) follows from Proposition 4.4.22-(2)(3). Also

$$X^{\bar{\pi}}(T) \overset{(4.4.97)}{=} \bar{X}(T) \overset{(4.4.90)}{=} \partial J^*(\bar{Y}(T)), \tag{4.4.103}$$

so (4.4.80)-(5) is verified. To verify (4.4.80)-(4) we use Condition 2.2.1 to see that

$$\begin{aligned} &(X^{\bar{\pi}}(T) - b)\bar{\xi} \overset{(4.4.103)}{=} \left(\partial J^*(\bar{Y}(T)) - b\right)\bar{\xi} \overset{(4.4.89)}{=} \left(\frac{\eta_{\bar{\alpha}} - c}{a} - b\right)(\eta_{\bar{\alpha}} - \bar{\alpha}H(T)) \\ &= \frac{1}{a}(\eta_{\bar{\alpha}} - (c + ab))(\eta_{\bar{\alpha}} - \bar{\alpha}H(T)) \overset{(4.4.50)}{=} 0, \end{aligned} \tag{4.4.104}$$

which verifies (4.4.80)-(4). Note that the last equality in (4.4.104) is a consequence of (4.4.105) below, which is established as follows:

$$
\begin{aligned}
&(\eta_{\bar{\alpha}}(\omega) - (c+ab)(\omega))(\eta_{\bar{\alpha}}(\omega) - \bar{\alpha}H(\omega;T)) \\
&\overset{(4.4.50)}{=} \begin{cases} 0 \cdot ((c+ab)(\omega) - \bar{\alpha}H(\omega;T)) & \text{if } \alpha H(\omega;T) \leqslant c(\omega) + a(\omega)b(\omega), \\ (\bar{\alpha}H(\omega;T) - (c+ab)(\omega)) \cdot 0 & \text{if } \alpha H(\omega;T) > c(\omega) + a(\omega)b(\omega), \end{cases} \\
&= 0,
\end{aligned} \tag{4.4.105}
$$

for all $\omega \in \Omega$. We have therefore established that $\bar{\pi} \in \Pi$ given by (4.4.91) satisfies the optimality relations (4.4.80)-(1)(4)(5). Now it follows from Remark 4.4.20 that $\bar{\pi}$ is the optimal portfolio for Problem 4.4.7.

We summarize the solution for problem (4.4.2) in the following proposition.

**Proposition 4.4.24.** *Suppose Condition 2.1.1, 2.1.2, 2.1.5, 2.1.11, 2.2.1 and 4.4.3. There exists a "Lagrange multiplier" $(\bar{Y}, \bar{\xi}) \in \mathbb{Y} := \mathbb{B}_1 \times \mathcal{L}_2(\Omega, \mathcal{F}_T, P)$ which maximizes the dual function $g(\cdot)$ on $\mathbb{Y}$ (recall (4.4.35), (4.4.26), (4.1.20), (4.1.10) and Proposition 4.4.16). Define the $\mathbb{R}$-valued process $\bar{X}$ (in terms of $\bar{Y}$ and state price density $H$ given by (2.1.9))*

$$\bar{X}(t) := H^{-1}(t) E\left[\partial J^*(\bar{Y}(T)) H(T) \,\middle|\, \mathcal{F}_t\right], \quad t \in [0,T], \tag{4.4.106}$$

*and define the $\mathbb{R}^N$-valued process $\bar{\pi} \in \mathcal{F}^*$ (see Notation 2.1.4-(2)) by*

$$\bar{\pi}(t) := (\sigma'(t))^{-1} \left[H^{-1}(t)\bar{\psi}(t) + \bar{X}(t)\theta(t)\right], \qquad t \in [0,T], \tag{4.4.107}$$

*with $\bar{\psi} \in \mathcal{F}^*$ being the a.e.-unique $\mathbb{R}^N$-valued process on $\Omega \times [0,T]$ such that*

$$\int_0^T \left\|\bar{\psi}(t)\right\|_2 dt < \infty \ \text{ a.s., and } \bar{X}(t)H(t) = \bar{X}(0) + \int_0^t \bar{\psi}'(s)\,dW(s), \ t \in [0,T], \tag{4.4.108}$$

*given by the martingale representation theorem and (2.1.1). Then (recall (4.1.1) and (2.1.21))*

$$\bar{\pi} \in \Pi, \quad \bar{X} = X^{\bar{\pi}}, \quad X^{\bar{\pi}}(T) \geqslant b \text{ a.s.}, \tag{4.4.109}$$

*and*

$$
\begin{aligned}
&E\left[J(X^{\bar{\pi}}(T))\right] = \inf_{\pi \in \Pi} \left\{ E\left[J(X^{\pi}(T))\right] \,\middle|\, X^{\pi}(T) \geqslant b \text{ a.s.} \right\} \\
&= \sup_{(Y,\xi) \in \mathbb{Y}} \{g(Y,\xi)\} = g(\bar{Y}, \bar{\xi}) \in \mathbb{R}.
\end{aligned} \tag{4.4.110}
$$

*In particular, $\bar{\pi}$ is an optimal portfolio for the problem (4.4.2).*

### 4.4.1 Mean Variance Portfolio Selection with European Wealth Constraint

In this subsection we shall add an *equality constraint* on the expected wealth at close of trade to problem (4.4.2) and minimize specifically the variance of the wealth at close of trade in place of the general quadratic risk criterion function at (4.4.2). That is, we shall address the following problem:

$$\text{minimize } Var(X^\pi(T)) \quad \text{such that } \pi \in \Pi,\ E\left[X^\pi(T)\right] = d \text{ and } X^\pi(T) \geqslant b \text{ a.s.} \tag{4.4.111}$$

Here $Var(X^\pi(T))$ is the *variance* of the terminal wealth, and $d \in (0, \infty)$ is a stipulated expected terminal wealth, that is the stipulation of $d$ is part of the specification of the problem.

**Remark 4.4.25.** Problems such as 4.4.111, which involve seeking a portfolio to minimize the *variance* of the wealth $X^\pi(T)$ at close of trade subject to the condition $E\left[X^\pi(T)\right] = d$ (for a specified value of $d \in (0, \infty)$), and possibly other constraints as well, are known as problems of *mean-variance portfolio selection*, and have been addressed in several works (see e.g. Lim and Zhou [20], Li, Zhou and Lim [19]). Problem 4.4.111 is an example of such a problem with the additional constraint $X^\pi(T) \geqslant b$ a.s. included. This problem was addressed by Bielecki *et al.* [3] in the particular case where $b = 0$ (to ensure non-negative wealth at close of trade). The approach adopted in [3] is an ingenious but nevertheless highly problem-specific adaptation of the risk-neutral method. In the present section we shall demonstrate that the solution of problem (4.4.2) is easily extended to include the additional constraint on the expected terminal wealth (i.e. $E\left[X^\pi(T)\right] = d$) in problem (4.4.111) by the introduction of a Lagrange multiplier in $\mathbb{R}$ for this constraint. Alternatively, we could account for this additional constraint by the introduction of a further perturbation (in $\mathbb{R}$) for the constraint, extending the space of dual variables and the perturbation function at (4.4.23) and (4.4.24) accordingly, and then simply re-work the approach that we followed for problem (4.4.2). However, this would be rather lengthy and repetitive. Instead, we are going to use an "off-the-shelf" result on scalar Lagrange multipliers (see Theorem C.0.6) to simply augment our solution of problem (4.4.2) to include the additional constraint in problem (4.4.111). In the course of this, we shall recover, and non-trivially extend, the results of Bielecki *et al.* [3] on problem (4.4.111). The ideas and constructions of the present section are quite specific to variance minimization with the additional constraint $E\left[X^\pi(T)\right] = d$ in problem (4.4.111) and are not used in the remainder of the thesis. This section may therefore be bypassed by readers not interested in this problem.

To formulate the problem (4.4.111) precisely, we define (recall (4.1.1))

$$G(\pi) := E\left[X^\pi(T)\right] - d, \qquad \pi \in \Pi; \tag{4.4.112}$$

$$\hat{J} := x^2/2, \qquad x \in \mathbb{R}; \tag{4.4.113}$$

$$\hat{\mathcal{A}} := \left\{\pi \in \Pi \mid X^\pi(T) \geqslant b \text{ a.s.}\right\}. \tag{4.4.114}$$

Then,

$$Var(X^\pi(T)) = 2E\left[\hat{J}(X^\pi(T))\right] - d^2 \qquad \text{for all } \pi \in \Pi \text{ such that } G(\pi) = 0, \tag{4.4.115}$$

thus the problem (4.4.111) can be stated in the following form:

**Problem 4.4.26.** Determine some $\bar{\pi} \in \hat{\mathcal{A}}$ such that

$$G(\bar{\pi}) = 0 \quad \text{and} \quad \hat{\vartheta} = E\left[\hat{J}(X^{\bar{\pi}}(T))\right], \tag{4.4.116}$$

where

$$\hat{\vartheta} := \inf_{\pi \in \hat{\mathcal{A}}} \left\{ E\left[\hat{J}(X^{\pi}(T))\right] \,\middle|\, G(\pi) = 0 \right\}. \tag{4.4.117}$$

For Problem 4.4.26 to make sense we must of course have $G(\pi) = 0$ for some $\pi \in \hat{\mathcal{A}}$, that is we must have

$$0 \in \left\{ G(\pi) \mid \pi \in \hat{\mathcal{A}} \right\} \subset \mathbb{R}, \tag{4.4.118}$$

since, if (4.4.118) fails to hold, then there fails to exist a $\pi \in \Pi$ which satisfies the constraints of Problem 4.4.26, and hence the problem does not make sense. In fact, in order to secure existence of a Lagrange multiplier for the additional constraint $E[X^{\pi}(T)] = d$ at (4.4.111) (that is the constraint $G(\pi) = 0$ in Problem 4.4.26) we are going to strengthen (4.4.118) to the following Slater-type condition:

**Condition 4.4.27.** The interior of $\left\{ G(\pi) \mid \pi \in \hat{\mathcal{A}} \right\} \subset \mathbb{R}$ is non-empty and includes $0 \in \mathbb{R}$.

**Remark 4.4.28.** Condition 4.4.27 will be essential when we use Theorem C.0.6 to get existence of a Lagrange multiplier for constraint $G(\pi) = 0$ in Problem 4.4.26. Observe that this is a very natural and mild condition in the following sense: The set $\hat{\mathcal{A}}$ is convex, as follows from (4.4.114) and the fact that $X^{\pi}(T)$ is affine in $\pi \in \mathbb{X}$ (see (2.1.22)), and consequently $\left\{ G(\pi) \in \mathbb{R} \mid \pi \in \hat{\mathcal{A}} \right\}$ is a convex set in $\mathbb{R}$, and hence must be an interval in $\mathbb{R}$. We then have (from (4.4.112))

$$\begin{aligned} &\text{interior} \left\{ G(\pi) \mid \pi \in \hat{\mathcal{A}} \right\} \\ &\quad = \left( \inf \left\{ E[X^{\pi}(T)] \mid \pi \in \hat{\mathcal{A}} \right\} - d, \sup \left\{ E[X^{\pi}(T)] \mid \pi \in \hat{\mathcal{A}} \right\} - d \right). \end{aligned} \tag{4.4.119}$$

Condition 4.4.27 then amounts to the reasonable condition that $d$ should be fixed such that

$$\inf \left\{ E[X^{\pi}(T)] \mid \pi \in \hat{\mathcal{A}} \right\} < d < \sup \left\{ E[X^{\pi}(T)] \mid \pi \in \hat{\mathcal{A}} \right\}, \tag{4.4.120}$$

in the formulation of Problem 4.4.26.

To solve Problem 4.4.26, define the usual Lagrangian function in terms of a "multiplier" $\lambda \in \mathbb{R}$ for the constraint $G(\pi) = 0$ as:

$$\hat{K}(\lambda; \pi) := E[J_1(\lambda; X^{\pi}(T))], \quad (\lambda, \pi) \in \mathbb{R} \times \Pi, \tag{4.4.121}$$

where

$$J_1(\lambda;x) := \hat{J}(x) + \lambda(x-d) \overset{(4.4.113)}{=} \frac{1}{2}(x^2+2\lambda x) - \lambda d, \quad \lambda, x \in \mathbb{R}. \tag{4.4.122}$$

Recalling (2.1.22), (4.1.1), (4.1.19) and (4.4.112) - (4.4.114), it follows from Condition 2.1.2, Condition 2.1.5 and Condition 2.1.11 that $\hat{\mathcal{A}} \subset \Pi$ is convex and $G$ is an affine functional on $\Pi$, while $\pi \to E\left[\hat{J}(X^\pi(T))\right]$ is $\mathbb{R}$-valued and convex on $\Pi$. Therefore, we can apply Theorem C.0.6 with Condition 4.4.27 to Problem 4.4.26 with the Lagrangian given by (4.4.121), and get

$$\text{there exists some } \bar{\lambda} \in \mathbb{R} \text{ such that} \\ \hat{\vartheta} = \inf_{\pi \in \hat{\mathcal{A}}} \hat{K}(\bar{\lambda};\pi) = \sup_{\lambda \in \mathbb{R}} \inf_{\pi \in \hat{\mathcal{A}}} \hat{K}(\lambda;\pi). \tag{4.4.123}$$

By (4.4.123), Problem 4.4.26 is reduced to the following set of problems parameterized by $\lambda \in \mathbb{R}$:

$$\text{minimize } \pi \to \hat{K}(\lambda;\pi) \text{ over } \pi \in \hat{\mathcal{A}} \text{ (see (4.4.114)) for each } \lambda \in \mathbb{R}, \tag{4.4.124}$$

that is, in view of (4.4.121) and (4.4.114) the preceding problem becomes:

$$\text{minimize } \pi \to E\left[J_1(\lambda;x)\right] \text{ over } \pi \in \{\pi \in \Pi \mid X^\pi(T) \geqslant b \text{ a.s.}\} \text{ for each } \lambda \in \mathbb{R}. \tag{4.4.125}$$

That is, we have a family of minimization problems over $\pi \in \Pi$ parameterized by $\lambda \in \mathbb{R}$, and each such problem exactly matches problem (4.4.2) when we make the following substitutions at (4.1.19) (see (4.4.122)):

$$a := 1, \quad c := \lambda; \quad q := -\lambda d. \tag{4.4.126}$$

**Remark 4.4.29.** Note that the constant $q$ in the risk criterion function $J$ defined by (4.1.19) has been somewhat redundant thus far in the present chapter, since it has no effect on the minimization of $J$. However, we see from (4.4.126) that the constant $q$ plays an essential role as it involves the parameter $\lambda \in \mathbb{R}$, and the value $\inf_{\pi \in \hat{\mathcal{A}}} \hat{K}(\lambda;\pi)$ relies crucially on the definition of $q$ at (4.4.126).

Therefore, for each $\lambda \in \mathbb{R}$, we get from (4.4.35) the dual function for problem (4.4.124) as:

$$g_1(\lambda;Y,\xi) = \begin{cases} -\varkappa(Y,\xi) - E\left[J_1^*(\lambda;Y(T))\right] & \text{if } \xi \geqslant 0 \text{ a.s.}, \\ -\infty & \text{otherwise}, \end{cases} \quad (Y,\xi) \in \mathbb{Y}, \tag{4.4.127}$$

where we have defined

$$\varkappa(Y,\xi) := -\inf_{\pi \in \Pi}\{E\left[X^\pi(T)(Y(T)-\xi) - b\xi\right]\} \qquad (Y,\xi) \in \mathbb{Y}, \tag{4.4.128}$$

and the function

$$J_1^*(\lambda;y) := \frac{1}{2}(y-\lambda)^2 + \lambda d, \qquad y \in \mathbb{R}, \tag{4.4.129}$$

is the convex conjugate of $x \to J_1(\lambda; x)$ at (4.4.122).

The dual problem is then to establish that, for each fixed $\lambda \in \mathbb{R}$,

$$g_1(\lambda; \bar{Y}_\lambda, \bar{\xi}_\lambda) = \sup_{(Y,\xi)\in\mathbb{Y}} g_1(\lambda; Y, \xi), \qquad \text{for some } (\bar{Y}_\lambda, \bar{\xi}_\lambda) \in \mathbb{Y}, \tag{4.4.130}$$

and by Remark 4.4.13 this can be further simplified to a problem which involves the maximization of a concave function defined over the real line $\mathbb{R}$. We shall use the values of the coefficients given by (4.4.126) at (4.4.48) and (4.4.53) to give "$\lambda$-parameterized" versions of $h(y,\omega)$ and $h^{opt}(y,\omega)$, namely (recall (4.4.129))

$$h_1(\lambda; y, \omega) \;:=\; b(\omega)y - J_1^*(\lambda; y) = b(\omega)y - \lambda d - \frac{(y-\lambda)^2}{2}, \tag{4.4.131}$$

$$h_1^{opt}(\lambda; y, \omega) \;:=\; \begin{cases} h_1(\lambda; \lambda + b(\omega), \omega) & \text{if } y < \lambda + b(\omega), \\ h_1(\lambda; y, \omega) & \text{if } y \geqslant \lambda + b(\omega), \end{cases} \tag{4.4.132}$$

for all $\lambda \in \mathbb{R}$, $(y,\omega) \in \mathbb{R} \times \Omega$. Motivated by the definition of $\Psi(\cdot)$ at (4.4.62), for each $\lambda \in \mathbb{R}$ define

$$\Psi_1(\lambda; \alpha) := \alpha(x_0 - E[bH(T)]) + E\left[h_1^{opt}(\lambda; \alpha H(T))\right], \quad \alpha \in \mathbb{R}. \tag{4.4.133}$$

Proposition 4.4.14 and Proposition 4.4.16 apply to problem (4.4.125), and for each $\lambda \in \mathbb{R}$ gives some $\bar{\alpha}_\lambda$ such that (recall (4.4.133) and (4.4.127))

$$\sup_{(Y,\xi)\in\mathbb{Y}} g_1(\lambda; Y, \xi) = g_1(\lambda; \bar{Y}_\lambda, \bar{\xi}_\lambda) = \Psi_1(\lambda; \bar{\alpha}_\lambda) = \sup_{\alpha\in\mathbb{R}} \Psi_1(\lambda; \alpha) \in \mathbb{R}, \tag{4.4.134}$$

where $(\bar{Y}_\lambda, \bar{\xi}_\lambda) \in \mathbb{Y}$ is defined as (recall (4.4.50) and (4.4.126))

$$\bar{Y}_\lambda(T) \;:=\; \eta_{\bar{\alpha}_\lambda} = \max\{\bar{\alpha}_\lambda H(T), \lambda + b\}, \tag{4.4.135}$$

$$\bar{\xi}_\lambda \;:=\; \eta_{\bar{\alpha}_\lambda} - \bar{\alpha}_\lambda H(T) = \max\{\bar{\alpha}_\lambda H(T), \lambda + b\} - \bar{\alpha}_\lambda H(T). \tag{4.4.136}$$

Motivated by Proposition 4.4.24, for each $\lambda \in \mathbb{R}$ define the $\mathbb{R}$-valued continuous process $\bar{X}_\lambda$ as:

$$\begin{aligned} \bar{X}_\lambda(t) \;&:=\; H^{-1}(t)E\left[H(T)\partial J_1^*(\lambda; \bar{Y}_\lambda(T)) \,\middle|\, \mathcal{F}_t\right] \\ &\overset{(4.4.129)}{=} H^{-1}(t)E\left[H(T)(\bar{Y}_\lambda(T) - \lambda) \,\middle|\, \mathcal{F}_t\right] \\ &\overset{(4.4.135)}{=} H^{-1}(t)E\left[H(T)\max\{\bar{\alpha}_\lambda H(T) - \lambda, b\} \,\middle|\, \mathcal{F}_t\right], \qquad t \in [0,T], \end{aligned} \tag{4.4.137}$$

with $\bar{\alpha}_\lambda \in \mathbb{R}$ (recall (4.4.134)) is the maximizer given by Proposition 4.4.14 with the substitutions at (4.4.126). For each $\lambda \in \mathbb{R}$ define the $\mathbb{R}^N$-valued process $\bar{\pi}_\lambda \in \mathcal{F}^*$ by

$$\bar{\pi}_\lambda(t) := (\sigma'(t))^{-1}\left[H^{-1}(t)\bar{\psi}_\lambda(t) + \bar{X}_\lambda(t)\theta(t)\right], \qquad t \in [0,T], \tag{4.4.138}$$

with $\bar{\psi}_\lambda \in \mathcal{F}^*$ being the a.e.-unique $\mathbb{R}^N$-valued process such that

$$\int_0^T \left\|\bar{\psi}_\lambda(t)\right\|_2 \mathrm{d}t < \infty \;\text{ a.s., and } \bar{X}_\lambda(t)H(t) = \bar{X}_\lambda(0) + \int_0^t \bar{\psi}_\lambda'(s)\mathrm{d}W(s), \; t \in [0,T]. \tag{4.4.139}$$

Notice that, in contrast to (4.4.109) and (4.4.110), the processes $\bar{X}_\lambda$ and $\bar{\pi}_\lambda$ at (4.4.137) and (4.4.139) are *completely determined* by the pair of scalar parameters $(\lambda, \bar{\alpha}_\lambda)$; this is a consequence of the portfolios being unconstrained.

We are going to show that

$$\bar{\pi}_{\bar{\lambda}} \text{ (for } \bar{\lambda} \text{ at (4.4.123)) is the optimal portfolio for problem (4.4.111).} \tag{4.4.140}$$

For each $\lambda \in \mathbb{R}$, Proposition 4.4.24 establishes that

$$\bar{\pi}_\lambda \in \Pi, \quad \bar{X}_\lambda = X^{\bar{\pi}_\lambda}, \quad X^{\bar{\pi}_\lambda}(T) \geqslant b \text{ a.s.}, \tag{4.4.141}$$

thus, in particular, we have (see (4.4.114))

$$\bar{\pi}_\lambda \in \hat{\mathcal{A}}, \tag{4.4.142}$$

and

$$E\left[J_1(\lambda; X^{\bar{\pi}_\lambda}(T))\right] = \inf_{\pi \in \hat{\mathcal{A}}} \hat{K}(\lambda; \pi) = \sup_{(Y,\xi) \in \mathbb{Y}} \{g_1(\lambda; Y, \xi)\} = g_1(\lambda; \bar{Y}_\lambda, \bar{\xi}_\lambda) \in \mathbb{R}, \tag{4.4.143}$$

(where we have identified $J(\cdot)$ and $g(\cdot)$ at (4.4.110) with $J_1(\lambda; \cdot)$ and $g_1(\lambda; \cdot)$ respectively, and used (4.4.121) and (4.4.134)). We next show that

$$G(\bar{\pi}_{\bar{\lambda}}) = 0. \tag{4.4.144}$$

From (4.4.143) and (4.4.123), we have

$$\begin{aligned} \sup_{\lambda \in \mathbb{R}} \sup_{(Y,\xi) \in \mathbb{Y}} \{g_1(\lambda; Y, \xi)\} &\overset{(4.4.143)}{=} \sup_{\lambda \in \mathbb{R}} \inf_{\pi \in \hat{\mathcal{A}}} \hat{K}(\lambda; \pi) \\ &\overset{(4.4.123)}{=} \inf_{\pi \in \hat{\mathcal{A}}} \hat{K}(\bar{\lambda}; \pi) \overset{(4.4.143)}{=} g_1(\bar{\lambda}; \bar{Y}_{\bar{\lambda}}, \bar{\xi}_{\bar{\lambda}}) \in \mathbb{R}, \end{aligned} \tag{4.4.145}$$

In particular, (4.4.145) establishes that

$$g_1(\lambda; \bar{Y}_{\bar{\lambda}}, \bar{\xi}_{\bar{\lambda}}) \leqslant g_1(\bar{\lambda}; \bar{Y}_{\bar{\lambda}}, \bar{\xi}_{\bar{\lambda}}) \in \mathbb{R}, \quad \text{for all } \lambda \in \mathbb{R}, \tag{4.4.146}$$

that is, by (4.4.127),

$$E\left[J_1^*(\lambda; \bar{Y}_{\bar{\lambda}}(T))\right] \geqslant E\left[J_1^*(\bar{\lambda}; \bar{Y}_{\bar{\lambda}}(T))\right], \quad \text{for all } \lambda \in \mathbb{R}. \tag{4.4.147}$$

Combining (4.4.147) and (4.4.129), we get that

$$\inf_{\lambda \in \mathbb{R}} E\left[\frac{1}{2}(\bar{Y}_{\bar{\lambda}}(T) - \lambda)^2 + \lambda d\right] = E\left[\frac{1}{2}(\bar{Y}_{\bar{\lambda}}(T) - \bar{\lambda})^2 + \bar{\lambda} d\right], \tag{4.4.148}$$

and then

$$\inf_{\lambda \in \mathbb{R}} \left\{\frac{1}{2}\lambda^2 - \lambda\left(E\left[\bar{Y}_{\bar{\lambda}}(T)\right] - d\right)\right\} = \frac{1}{2}\bar{\lambda}^2 - \bar{\lambda}\left(E\left[\bar{Y}_{\bar{\lambda}}(T)\right] - d\right). \tag{4.4.149}$$

From (4.4.149) we see that $\bar{\lambda}$ is the minimizer for the $\mathbb{R}$-valued quadratic function

$$\frac{1}{2}\lambda^2 - \lambda\left(E\left[\bar{Y}_{\bar{\lambda}}(T)\right] - d\right) \qquad \text{for all } \lambda \in \mathbb{R},$$

therefore

$$\bar{\lambda} = E\left[\bar{Y}_{\bar{\lambda}}(T)\right] - d. \tag{4.4.150}$$

On the other hand, in view of (4.4.137) and (4.4.141), we also have

$$X^{\bar{\pi}_{\bar{\lambda}}}(T) = \bar{X}_{\bar{\lambda}}(T) = \bar{Y}_{\bar{\lambda}}(T) - \bar{\lambda}. \tag{4.4.151}$$

Therefore, (4.4.144) follows from (4.4.150) and (4.4.151). From (4.4.144) and (4.4.142), one sees that

$$\hat{\pi} := \bar{\pi}_{\bar{\lambda}} \text{ meets all constraints for Problem 4.4.26, i.e. } \hat{\pi} \in \hat{\mathcal{A}} \text{ and } G(\hat{\pi}) = 0. \tag{4.4.152}$$

Moreover, from (4.4.122)

$$E\left[J_1(\lambda; X^{\pi}(T)) = E\left[\hat{J}(X^{\pi}(T))\right]\right] \quad \text{for } \pi \in \{\pi \in \Pi \mid G(\pi) = 0\}. \tag{4.4.153}$$

Therefore, it follows from the first equality of (4.4.143) (with $\lambda := \bar{\lambda}$), alone with (4.4.153) and (4.4.122), that

$$E\left[\hat{J}(X^{\bar{\pi}_{\bar{\lambda}}}(T)) \leqslant E\left[\hat{J}(X^{\pi}(T))\right]\right] \quad \text{for each } \pi \in \left\{\pi \in \hat{\mathcal{A}} \mid G(\pi) = 0\right\}. \tag{4.4.154}$$

Thus we get the portfolio $\hat{\pi} := \bar{\pi}_{\bar{\lambda}}$, defined by (4.4.123), (4.4.137) and (4.4.138) is the optimal portfolio for problem (4.4.111) (formally Problem 4.4.26).

**Remark 4.4.30.** The optimal portfolio $\bar{\pi}_{\bar{\lambda}}$ and the corresponding optimal wealth process $X^{\bar{\pi}_{\bar{\lambda}}}$ for problem (4.4.111) are given as (recall (4.4.141), (4.4.137) and (4.4.138)):

$$\begin{cases} X^{\bar{\pi}_{\bar{\lambda}}}(t) = H^{-1}(t)E\left[H(T)\max\{\bar{\alpha}_{\bar{\lambda}}H(T) - \bar{\lambda}, b\} \,\middle|\, \mathcal{F}_t\right], & t \in [0,T], \\ \bar{\pi}_{\bar{\lambda}}(t) = (\sigma'(t))^{-1}\left[H^{-1}(t)\bar{\psi}_{\bar{\lambda}}(t) + X^{\bar{\pi}_{\bar{\lambda}}}(t)\theta(t)\right], & t \in [0,T], \end{cases} \tag{4.4.155}$$

where $\bar{\psi}_{\bar{\lambda}} \in \mathcal{F}^*$ is the a.e.-unique $\mathbb{R}^N$-valued process such that,

$$\int_0^T \left\|\bar{\psi}_{\bar{\lambda}}(t)\right\|_2 \mathrm{d}t < \infty \text{ a.s., and } X^{\bar{\pi}_{\bar{\lambda}}}(t)H(t) = X^{\bar{\pi}_{\bar{\lambda}}}(0) + \int_0^t \bar{\psi}'_{\bar{\lambda}}(s)\mathrm{d}W(s),\ t \in [0,T]. \tag{4.4.156}$$

That is, the solution $(\bar{\pi}_{\bar{\lambda}}, X^{\bar{\pi}_{\bar{\lambda}}})$ is now completely determined by the pair of scalar parameter $(\bar{\lambda}, \bar{\alpha}_{\bar{\lambda}})$. We shall see that $(\bar{\lambda}, \bar{\alpha}_{\bar{\lambda}})$ maximizes the function $(\lambda, \alpha) \to \Psi_1(\lambda;\alpha) : \mathbb{R}^2 \to \mathbb{R}$ defined by (4.4.133), (4.4.132) and (4.4.131), and also satisfies a pair of nonlinear algebraic equations (see (4.4.162) and (4.4.167) which follow).

From Condition 4.4.3, we have

$$x_0 \overset{(2.1.26)}{=} E\left[H(T)X^{\hat{\pi}}(T)\right] \geqslant E\left[H(T)(b+\varepsilon)\right], \quad \text{for some } \varepsilon \in (0,+\infty), \tag{4.4.157}$$

and by (4.4.157) and (2.1.9), it follows that

$$x_0 > E\left[bH(T)\right]. \tag{4.4.158}$$

Therefore, from (4.4.158) the function $\Psi_1(\lambda;\cdot)$ is *strictly concave* on $\mathbb{R}$ (recall (4.4.133) and (4.4.132)), thus the maximizer $\bar{\alpha}(\lambda)$ at the third equality of (4.4.134) is *unique* for each $\lambda \in \mathbb{R}$, so that we must have

$$\partial_\alpha \Psi_1(\lambda;\alpha) = 0 \qquad \text{at } \alpha = \bar{\alpha}(\lambda) \text{ for each } \lambda \in \mathbb{R}. \tag{4.4.159}$$

From (4.4.132), we get the following $y$-partial derivative:

$$\partial_y h_1^{opt}(\lambda;y,\omega) = \min\{0, b(\omega)+\lambda-y\}. \tag{4.4.160}$$

Therefore, (4.4.159), together with (4.4.133), dominated convergence, and (4.4.160), gives that for each $\lambda \in \mathbb{R}$ the scalar $\bar{\alpha}(\lambda) \in \mathbb{R}$ at (4.4.134) is the *unique* root of the following equation in $\alpha$:

$$E\left[H(T)\max\{0, \alpha H(T)-\lambda-b\}\right] = x_0 - E\left[bH(T)\right]. \tag{4.4.161}$$

Moreover, from (4.4.145) and (4.4.134), we have

$$\sup_{\lambda\in\mathbb{R}}\sup_{\alpha\in\mathbb{R}} \Psi_1(\lambda;\alpha) \overset{(4.4.134)}{=} \sup_{\lambda\in\mathbb{R}}\sup_{(Y,\xi)\in\mathbb{Y}} g_1(\lambda;Y,\xi) \overset{(4.4.145)}{=} g_1(\bar{\lambda};\bar{Y}_{\bar{\lambda}},\bar{\xi}_{\bar{\lambda}}) \overset{(4.4.134)}{=} \Psi_1(\bar{\lambda};\bar{\alpha}_{\bar{\lambda}}). \tag{4.4.162}$$

In particular, from (4.4.162) we have

$$\Psi_1(\lambda;\bar{\alpha}_{\bar{\lambda}}) \leqslant \Psi_1(\bar{\lambda};\bar{\alpha}_{\bar{\lambda}}), \quad \text{for all } \lambda \in \mathbb{R}, \tag{4.4.163}$$

and then

$$\partial_\lambda \Psi_1(\lambda;\bar{\alpha}_{\bar{\lambda}}) = 0 \qquad \text{at } \lambda = \bar{\lambda}. \tag{4.4.164}$$

From (4.4.132) and (4.4.131), we obtain the $\lambda$-partial derivative

$$\partial_\lambda h_1^{opt}(\lambda;y,\omega) = b(\omega) - d + \max\{0, y-\lambda-b(\omega)\}. \tag{4.4.165}$$

Therefore, (4.4.164), together with (4.4.133), dominated convergence, and (4.4.165), gives

$$E\left[\max\{0, \bar{\alpha}_{\bar{\lambda}}H(T)-\lambda-b\}\right] = d - E\left[b\right]. \tag{4.4.166}$$

Combining (4.4.166) and (4.4.161), we find that the pair $(\bar{\lambda},\bar{\alpha}_{\bar{\lambda}}) \in \mathbb{R}^2$, which completely determines the optimal portfolio $\bar{\pi}_{\bar{\lambda}}$ (recall Remark 4.4.30), satisfies the following nonlinear algebraic equations (in the variables $(\alpha,\lambda) \in \mathbb{R}^2$)

$$\begin{cases} E\left[H(T)\max\{0, \alpha H(T)-\lambda-b\}\right] = x_0 - E\left[bH(T)\right], \\ E\left[\max\{0, \alpha H(T)-\lambda-b\}\right] = d - E\left[b\right]. \end{cases} \tag{4.4.167}$$

**Remark 4.4.31.** In the case where $b := 0$ one sees that, modulo a sign-change for the variables $\lambda$ and $\alpha$, the system (4.4.167) is identical to the pair of equations (4.3) of Bielecki *et al.* ([3], p.226) the solution of which completely determines the optimal portfolio (see Theorem 4.1 and Theorem 2.1 of [3]); furthermore, this optimal portfolio is clearly identical to the portfolio $\bar{\pi}_{\bar{\lambda}}$ given by (4.4.155) with $(\bar{\lambda}, \bar{\alpha}_{\bar{\lambda}}) \in \mathbb{R}^2$ determined by solution of (4.4.167). Notice that the system of equations (4.4.167) has been obtained from the dual functional $\Psi_1(\cdot;\cdot)$ at (4.4.133) through the conditions (4.4.159) and (4.4.164), which themselves result from the duality relation (4.4.134). This should be compared with the approach adopted in Bielecki *et al.* [3], which is not based on the use of conjugate duality, and which does not involve any dual functional such as (4.4.133). In fact, the system (4.3) of [3] (that is, system (4.4.167) with $b := 0$), along with the optimal portfolio, are obtained in [3] by a rather problem-specific approach which relies in an essential way on the absence of portfolio constraints. The essence of this approach is to combine the risk-neutral method and an application of the convex separation theorem in $\mathbb{R}^n$ (see Proposition 4.1 and Theorem 4.1 of [3]). In contrast, in the present section we have merely borrowed the main results established for the general problem (4.4.2) (in particular Proposition 4.4.14, Proposition 4.4.16 and Proposition 4.4.24 applied to the $\lambda$-parametrized problem (4.4.125)) which have been established on the basis of the Rockafellar variational approach that has been used as the basic and unifying tool throughout this chapter. We have combined these results with a "Lagrange duality theorem" for $\mathbb{R}$-valued affine constraints (Theorem C.0.6) to deal with the further constraint $E[X^\pi(T)] = d$ at the mean variance problem (4.4.111). Finally, it should be noted that the duality approach of the present section, which yields the dual function at (4.4.133), provides additional insight into solving the rather tightly coupled pair of algebraic equations at (4.4.167), and (at least in principle) suggests how the solution of (4.4.167) may be *decoupled* into two simpler single-dimensional problems. In fact, we have $\Psi_1(\lambda, \bar{\alpha}_\lambda) \leqslant \Psi_1(\bar{\lambda}, \bar{\alpha}_{\bar{\lambda}})$ for each $\lambda \in \mathbb{R}$ (see (4.4.134) and (4.4.162)), that is

$$\bar{\lambda} \text{ (given at (4.4.123)) maximizes the function } \lambda \to \Psi_1(\lambda; \bar{\alpha}_\lambda) \text{ on } \mathbb{R}, \tag{4.4.168}$$

in which $\bar{\alpha}_\lambda$ is determined for each $\lambda \in \mathbb{R}$ by the root-finding problem at (4.4.161). In this way determination of the pair $(\bar{\lambda}, \bar{\alpha}_{\bar{\lambda}})$ reduces to the one-dimensional maximization at (4.4.168) with $\bar{\alpha}_\lambda$ being determined for each $\lambda \in \mathbb{R}$ by the one-dimensional root-finding problem (4.4.161).

## 4.5 Problem with Combined Portfolio and European Wealth Constraints

In Sections 4.3 and 4.4, we looked at quadratic risk minimization with portfolio constraints and European wealth constraints applied *separately* (see (4.3.2) and (4.4.2)). In this section, we shall consider quadratic risk minimization with portfolio constraints and European wealth constraints applied *together*, that is, we shall apply the portfolio constraint (2.2.7) together with the European constraint (2.2.12). We shall see that the joint action of these constraints demands a Lagrange multiplier for the European wealth constraint which is a member of the adjoint space of $\mathcal{L}_\infty(\Omega, \mathcal{F}_T, P)$, something that is not needed in the simpler problem (4.4.2) without portfolio constraints, for which the Lagrange multiplier that "enforces" the European wealth constraint is effectively just a member of the real line, as we have seen in Section 4.4. In this section we continue to denote the set of *admissible* or *regulated* portfolio processes by

$$\mathcal{A} := \{\pi \in \Pi \mid \pi(t) \in A, \text{ a.e.}\}, \tag{4.5.1}$$

(as in Section 4.3), and the *terminal wealth floor* by $b$ (as in Section 4.4). The problem we address is therefore

$$\text{minimize } E\left[J(X^\pi(T))\right] \quad \text{such that } \pi \in \mathcal{A} \text{ and } X^\pi(T) \geqslant b \text{ a.s.}, \tag{4.5.2}$$

that is, our goal is to determine a portfolio process $\bar{\pi} \in \mathcal{A}$ such that

$$E\left[J(X^{\bar{\pi}}(T))\right] = \inf_{\pi \in \mathcal{A}} \left\{ E\left[J(X^\pi(T))\right] \,\middle|\, X^\pi(T) \geqslant b \text{ a.s.} \right\}. \tag{4.5.3}$$

For the constraints in problem (4.5.2) we assume the following:

**Condition 4.5.1.** All the regulated portfolio processes $\pi(\cdot) \in \mathcal{A}$ take values in a subset $A \subset \mathbb{R}^N$ for all $t \in [0, T]$ a.s. such that (also recall Condition 2.2.4)

$$A \text{ is a closed convex set with } 0 \in A. \tag{4.5.4}$$

The terminal wealth is lower bounded by a given essentially bounded random variable $b$:

$$b \in \mathcal{L}_\infty(\Omega, \mathcal{F}_T, P). \tag{4.5.5}$$

**Remark 4.5.2.** Observe that problem (3.2.64) constitutes a sort of static and finite dimensional precursor of the stochastic control problem (4.5.2). We shall in fact implement the Rockafellar variational approach for the problem (4.5.2) in a way that is formally very reminiscent of how the variational method was used for problem (3.2.64).

**Remark 4.5.3.** Problem (4.5.2) amounts to a stochastic optimal control problem which includes a *control* (or portfolio) constraint (namely $\pi \in \mathcal{A}$) together with an a.s. *state* constraint namely ($X^\pi(T) \geqslant b$ a.s.). Stochastic control problems of this kind constitute a particular challenge, and have received very little attention in the established literature. It is nevertheless amply clear from the works of Dubovitskii and Mil'yutin ([9], Section 4), Makowski and Neustadt ([22], Theorem 11.1) and Neustadt ([26], Theorem (V.3.44)), which are devoted to problems of *deterministic* optimal control with control and state (or "phase") constraints applied together, that this combination of constraints naturally gives rise to Lagrange multipliers which are "singular" or "degenerate" in the sense of being members of the adjoint $\mathcal{L}_\infty^*$ of some space $\mathcal{L}_\infty$ of essentially bounded functions, and can therefore involve *finitely-additive* measures. In exactly the same way, in this section we shall see that the dual variables appropriate for problem (4.5.2) involve members of the particular adjoint space $\mathcal{L}_\infty^*(\Omega, \mathcal{F}_T, P)$. Of course, we have already seen how dual variables in an adjoint space $\mathcal{L}_\infty^*$ arise very naturally in even the simplest optimization problems with almost-sure constraints which are "singularly binding" (recall problem (3.3.31) and Remark 3.3.19).

**Remark 4.5.4.** For problem (4.5.2) to make sense, there must of course exist some $\pi \in \mathcal{A}$ such that $X^{\pi}(T) \geqslant b$ a.s., since, if such a $\pi$ fails to exist, then the constraints in the problem can never be satisfied and the problem is ill-defined and makes no sense. This is completely analogous to what we saw at Remark 4.4.4 for the problem without portfolio constraints. Thus, in defining problem 4.5.2 we must at least stipulate the random variable $b$ such that $\{\pi \in \mathcal{A} \mid X^{\pi}(T) \geqslant b \text{ a.s.}\}$ is non-empty, and this means that we cannot be "too greedy" in specifying the floor terminal wealth $b$ in the formulation of problem (4.5.2). In fact, exactly as at Condition 4.4.3 we are going to impose a Slater-type condition which is slightly stronger than non-emptiness of the set $\{\pi \in \mathcal{A} \mid X^{\pi}(T) \geqslant b \text{ a.s.}\}$, namely

**Condition 4.5.5.** There is some $\hat{\pi} \in \mathcal{A}$ and non-random constant $\varepsilon \in (0, \infty)$ such that

$$X^{\hat{\pi}}(T) \geqslant b + \varepsilon \quad \text{a.s.} \tag{4.5.6}$$

**Remark 4.5.6.** The conclusions of Remark 4.4.5 apply equally to the specification of the random variable $b$ for the wealth constraint in problem (4.5.2). Indeed, in Remark 4.4.5 we saw that, when $b$ is specified such that

$$b \geqslant x_0 S_0(T), \qquad \text{a.s.}, \tag{4.5.7}$$

then Condition 4.4.3 cannot hold, that is there fails to exist some $\hat{\pi} \in \Pi$ and some $\epsilon \in (0, \infty)$ such that $X^{\hat{\pi}}(T) \geqslant b + \varepsilon$ a.s. Since $\mathcal{A} \subset \Pi$, if $b$ is such that (4.5.7) holds, then Condition 4.5.5 certainly cannot be satisfied either. Thus, exactly as at (4.4.18), $b$ must necessarily be chosen such that

$$b \ngeqslant x_0 S_0(T), \qquad \text{a.s.}. \tag{4.5.8}$$

In particular, when $b \leqslant 0$ a.s. then we can take $\hat{\pi} \in \mathcal{A}$ defined by $\hat{\pi} := 0$ (since $0 \in A$ - see (4.5.4)) to get

$$X^{\hat{\pi}}(T) = S_0(T)x_0 \geqslant x_0 > 0 \geqslant b, \tag{4.5.9}$$

(exactly as at (4.4.4)), so that Condition 4.5.5 holds with $\hat{\pi} := 0$ and $\epsilon \in (0, x_0)$. The choice of $b \leqslant 0$ a.s. corresponds to quadratic loss minimization with the portfolio constraint $\pi \in \mathcal{A}$, together with a permissible level of debt specified by the random variable $b$ (exactly as at Remark 4.4.6).

We shall see that Condition 4.5.5 is essential for securing the existence of a solution to an associated dual problem that we shall later construct. To match the notation with that of Chapter 3, define the space of *primal variables*

$$\mathbb{X} = \Pi, \tag{4.5.10}$$

and reformulate the problem (4.5.3) in the following *primal* form:

**Problem 4.5.7. [primal problem]** Determine an optimal portfolio process $\bar{\pi} \in \Pi$ such that

$$f(\bar{\pi}) = \inf_{\pi \in \Pi} f(\pi), \tag{4.5.11}$$

in which the *primal function* $f(\cdot) : \Pi \to (-\infty, +\infty]$ is defined as (recall (3.1.1))

$$f(\pi) := \begin{cases} E\left[J(X^\pi(T))\right], & \text{when } \pi \in \mathcal{A}, \text{ and } X^\pi(T) \geqslant b \text{ a.s.}, \\ +\infty, & \text{otherwise}, \end{cases} \tag{4.5.12}$$

and $J$ is given in (2.2.3) subject to Condition 2.2.1. From Remark 2.2.3

$$\vartheta := \inf_{\pi \in \Pi} \{f(\pi)\} = \inf_{\pi \in \mathcal{A}} \{E\left[J(X^\pi(T))\right] \mid X^\pi(T) \geqslant b \text{ a.s.}\} \in \mathbb{R} \quad \text{ with } \vartheta \geqslant \underline{l}, \tag{4.5.13}$$

(note that Condition 4.5.5 ensures that the set $\{\pi \in \mathcal{A} \mid X^\pi(T) \geqslant b \text{ a.s.}\}$ is non-empty).

We next introduce *order relations* "$\geqslant$" ("$\leqslant$") on the vector space $\mathcal{L}_\infty(\Omega, \mathcal{F}_T, P)$ and its adjoint space $\mathcal{L}_\infty^*(\Omega, \mathcal{F}_T, P)$ as follows:

**Notation 4.5.8.** (1) For any $v, v_1, v_2 \in \mathcal{L}_\infty(\Omega, \mathcal{F}_T, P)$, we write

$$v \geqslant 0 \;\text{ if }\; v(\omega) \geqslant 0 \;\text{ a.s.}, \quad \text{ and }\; v_1 \leqslant v_2 \text{ if }\; v_2 - v_1 \geqslant 0. \tag{4.5.14}$$

In particular, denote

$$(\mathcal{L}_\infty(\Omega, \mathcal{F}_T, P))^+ := \{v \in \mathcal{L}_\infty(\Omega, \mathcal{F}_T, P) \mid v \geqslant 0\}. \tag{4.5.15}$$

(2) For any $z, z_1, z_2 \in \mathcal{L}_\infty^*(\Omega, \mathcal{F}_T, P)$, we write

$$z \geqslant 0 \text{ if } z(v) \geqslant 0 \;\text{ for all } v \in (\mathcal{L}_\infty(\Omega, \mathcal{F}_T, P))^+, \;\text{ and }\; z_1 \leqslant z_2 \text{ if }\; z_2 - z_1 \geqslant 0. \tag{4.5.16}$$

In particular, denote

$$(\mathcal{L}_\infty^*(\Omega, \mathcal{F}_T, P))^+ := \{z \in \mathcal{L}_\infty^*(\Omega, \mathcal{F}_T, P) \mid z \geqslant 0\}. \tag{4.5.17}$$

(3) In this section, we shall sometimes use the abbreviations $\mathcal{L}_2$, $\mathcal{L}_\infty$, $\mathcal{L}_\infty^+$, $\mathcal{L}_\infty^*$, $(\mathcal{L}_\infty^*)^+$ for $\mathcal{L}_2(\Omega, \mathcal{F}_T, P)$, $\mathcal{L}_\infty(\Omega, \mathcal{F}_T, P)$, $(\mathcal{L}_\infty(\Omega, \mathcal{F}_T, P))^+$, $\mathcal{L}_\infty^*(\Omega, \mathcal{F}_T, P)$ and $(\mathcal{L}_\infty^*(\Omega, \mathcal{F}_T, P))^+$.

**Remark 4.5.9.** We have used the Rockafellar variational method outlined in Section 3.1 as a unified approach for addressing all of the optimization problems of this thesis. These include the "static" problems (3.2.3), (3.2.34) and (3.2.64) of Section 3.2, and problem (3.3.31) of Section 3.3, as well as the dynamic quadratic minimization problem (4.2.1) of Section 4.2 (unconstrained problem), problem (4.3.2) of Section 4.3 (problem with portfolio constraint), and problem (4.4.2) of Section 4.4 (problem with "European" a.s. state constraint on wealth at close of trade). In the course of these developments we saw that the Rockafellar variational approach was a general tool which effectively recovered results obtained earlier by Lim and Zhou [20] (for the unconstrained problem (4.2.1)), Labbé and Heunis [18] (for problem (4.3.2) with portfolio constraints), and Bielecki *et al.* [3] (for problem (4.4.2) with a.s. state constraint on wealth at close of trade). The methods used in [20], [18] and [3] are all, to a greater or lesser extent, rather problem-specific, and in particular none of these methods seems to generalize to the combination of control and a.s. state constraints exhibited by problem (4.5.2). We shall now see that the Rockafellar variational approach, already used above for the quadratic minimization problems (4.2.1), (4.3.2), and (4.4.2), generalizes in a fairly straightforward way to problem (4.5.2). This problem was first addressed by means of the Rockafellar variational approach in Heunis [12]. Our goal in the present section is to re-work, simplify and streamline the way in which the Rockafellar method was used in Heunis [12], so that the resulting application of this method generalizes as smoothly as possible to the Canonical Problem 2.2.7 which is addressed in Chapter 5.

1. To implement Step 3.1.1, we must fix a vector space of perturbations $\mathbb{U}$ and define a perturbation function. Define the vector space of *perturbations* by

$$\mathbb{U} := \mathcal{L}_2(\Omega, \mathcal{F}_T, P) \times \mathcal{L}_\infty(\Omega, \mathcal{F}_T, P), \tag{4.5.18}$$

with generic element $(u, v)$ for $u \in \mathcal{L}_2(\Omega, \mathcal{F}_T, P)$ and $v \in \mathcal{L}_\infty(\Omega, \mathcal{F}_T, P)$, and define the **perturbation function** $F : \Pi \times \mathbb{U} \to (-\infty, +\infty]$ as

$$F(\pi, (u, v)) := \begin{cases} E\left[J(X^\pi(T) - u)\right], & \text{where } \pi \in \mathcal{A} \text{ and } X^\pi(T) + v \geqslant b, \\ +\infty, & \text{otherwise.} \end{cases} \tag{4.5.19}$$

The convexity of $F$ on $\Pi \times \mathbb{U}$ follows from (4.1.19) and (4.5.19), and the fact that $\pi \to X^\pi(T) : \Pi \mapsto \mathcal{L}_2(\Omega, \mathcal{F}_T, P)$ is an affine mapping. The *consistency relation* (3.1.2) between the primal function and perturbation function is clearly satisfied, that is

$$F(\pi, (0, 0)) = f(\pi), \quad \pi \in \Pi, \tag{4.5.20}$$

as is immediate from (4.5.19) and (4.5.12).

**Remark 4.5.10.** In this remark we discuss the space of perturbations at (4.5.18) and the perturbation function at (4.5.19). Problem (4.5.2) is essentially problem (4.3.2), but with the a.s. state constraint $X^\pi(T) \geqslant b$ added. Effectively then, problem (4.5.2) really involves three constraints, namely the "hidden constraint" always implicit in the wealth dynamics (2.1.21) (recall the discussion in Remark 4.2.4 for problem (4.2.1)), together with the portfolio constraint $\pi \in \mathcal{A}$, and the state constraint $X^\pi(T) \geqslant b$. The perturbation by the variable $u \in \mathbb{U} := \mathcal{L}_2(\Omega, \mathcal{F}_T, P)$ at (4.5.19) is introduced to deal with the combination of the "hidden constraint" and the portfolio constraint, exactly as was the case in problem (4.3.2) (see Remark 4.3.5). As for the perturbation variable $v \in \mathcal{L}_\infty(\Omega, \mathcal{F}_T, P)$ appearing in the perturbation function at (4.5.19), its role is to account for the wealth constraint $X^\pi(T) \geqslant b$ in Problem 4.5.2, in exactly the same way that the perturbation variable $v \in \mathcal{L}_2(\Omega, \mathcal{F}_T, P)$ appearing in the perturbation function at (4.4.24) accounts for the same wealth constraint $X^\pi(T) \geqslant b$ in problem (4.4.2). The obvious question arises: why is the perturbation variable $v$ a member of the space $\mathcal{L}_\infty(\Omega, \mathcal{F}_T, P)$ in the perturbation (4.5.19) for problem (4.5.2), when this variable is a member of $\mathcal{L}_2(\Omega, \mathcal{F}_T, P)$ in the perturbation at (4.4.24) for problem (4.4.2)? We shall see later that taking $v \in \mathcal{L}_\infty(\Omega, \mathcal{F}_T, P)$ is essential for applying Theorem 3.1.7 to secure the existence of a solution of an associated *dual problem* that we will shortly construct.

**Remark 4.5.11.** Notice that problem (4.5.2) is really just an infinite dimensional "stochastic control analogue" of the finite dimensional static problem (3.2.64). The underlying similarity in the two problems shows up in the strong structural similarity between the perturbations (3.2.72) (for the problem (3.2.64)) and the perturbations (4.5.19) (for the problem (4.5.2)).

2. To implement Step 3.1.2 we must pair the space of perturbations at (4.5.18) with a vector space $\mathbb{Y}$ through a bilinear form $\langle \cdot, \cdot \rangle$ on $\mathbb{U} \times \mathbb{Y}$. We pair the perturbation variables $u \in \mathcal{L}_2(\Omega, \mathcal{F}_T, P)$ with elements of $\mathbb{B}_1$ (see (4.1.10)), much as we did for problem (4.2.1) (see (4.2.10)), as well as problem (4.3.2) (see (4.3.11)), and problem (4.4.2) (see (4.4.26)). As for the European wealth constraint perturbation variable $v \in \mathcal{L}_\infty(\Omega, \mathcal{F}_T, P)$, it is natural to pair this with the members $z \in \mathcal{L}_\infty^*(\Omega, \mathcal{F}_T, P)$ of the norm-dual space by the natural bilinearity $\langle v, z \rangle \to z(v)$ (much as we did at (3.3.50) for problem (3.3.31)). We therefore define the space of dual variables

$$\mathbb{Y} := \mathbb{B}_1 \times \mathcal{L}_\infty^*(\Omega, \mathcal{F}_T, P), \tag{4.5.21}$$

with generic element $(Y, z)$ for $Y \in \mathbb{B}_1$ and $z \in \mathcal{L}_\infty^*(\Omega, \mathcal{F}_T, P)$, and the bilinear form on $\mathbb{U} \times \mathbb{Y}$ is defined by

$$\langle (u, v), (Y, z) \rangle := E\left[uY(T)\right] + z(v), \qquad \text{for } (u, v) \in \mathbb{U} \text{ and } (Y, z) \in \mathbb{Y}, \tag{4.5.22}$$

(c.f. the bilinear form at (4.4.27) for the pairing of the spaces at (4.4.23) and (4.4.26) in problem (4.4.2)). This completes all the choices required for Steps 3.1.1 - 3.1.2, and we shall now synthesize a Lagrangian and a dual function.

**Remark 4.5.12.** It is instructive to compare the space of dual variables defined at (4.5.21) for problem (4.5.2) with the space of dual variables defined at (4.4.26) for the simpler problem (4.4.2) (which does not involve portfolio constraints). For each $\eta \in \mathcal{L}_2(\Omega, \mathcal{F}_T, P)$ the mapping $\xi \to E\left[\xi\eta\right] : \mathcal{L}_\infty(\Omega, \mathcal{F}_T, P) \to \mathbb{R}$ is norm-continuous (by the dominated convergence theorem), and is therefore a member of $\mathcal{L}_\infty^*(\Omega, \mathcal{F}_T, P)$, so that each $\eta \in \mathcal{L}_2(\Omega, \mathcal{F}_T, P)$ is effectively an element of $\mathcal{L}_\infty^*(\Omega, \mathcal{F}_T, P)$ i.e. we have

$$\mathcal{L}_2(\Omega, \mathcal{F}_T, P) \subset \mathcal{L}_\infty^*(\Omega, \mathcal{F}_T, P). \tag{4.5.23}$$

It follows from (4.5.23) that the space of dual variables at (4.4.26) is a *subset* of the space of dual variables at (4.5.21), that is the space of dual variables for problem (4.5.2) is *larger* than the space of dual variables for problem (4.4.2). Now $\mathcal{L}_2(\Omega, \mathcal{F}_T, P)$ is $w^*$-dense (or $\sigma(\mathcal{L}_\infty^*, \mathcal{L}_\infty)$-dense) subspace of $\mathcal{L}_\infty^*(\Omega, \mathcal{F}_T, P)$, as follows easily from standard functional analysis (see e.g. Theorem 6.24(3) of Aliprantis and Border [1]). It is in this rather technical sense that the space of dual variables at (4.5.21) is only a "slight" enlargement (or "relaxation") of the space of dual variables at (4.4.26). Nevertheless, we shall see that this "slightly enlarged" space of dual variables turns out to be large enough to contain solutions for the dual problem that we will soon construct. Notice that the space of dual variables at (4.5.21) for problem (4.5.2) is consistent with what we would expect on the basis of Remark 4.5.3.

3. According to Step 3.1.3, define the *Lagrangian function* $K : \Pi \times \mathbb{Y} \to [-\infty, +\infty]$ as follows (recall (3.1.3)):

$$K(\pi, (Y, z)) := \inf_{(u,v)\in\mathbb{U}} \{\langle (u, v), (Y, z)\rangle + F(\pi, (u, v))\}. \tag{4.5.24}$$

From (4.5.20) and (4.5.24), we have the basic inequality

$$f(\pi) \geqslant K(\pi, (Y, z)), \qquad \text{for all } (\pi, (Y, z)) \in \Pi \times \mathbb{Y}. \tag{4.5.25}$$

The Lagrangian at (4.5.24) can be partially evaluated as follows: from (4.5.19) and (4.5.22) we obtain, for all $(\pi, (Y, z)) \in \Pi \times \mathbb{Y}$,

$$K(\pi, (Y, z)) = \begin{cases} \inf_{u\in\mathcal{L}_2} \{E\left[uY(T) + J(X^\pi(T) - u)\right]\} & \\ \quad + \inf_{v\in\mathcal{L}_\infty} \left\{z(v) \;\middle|\; X^\pi(T) + v \geqslant b\right\}, & \text{if } \pi \in \mathcal{A}, \\ +\infty, & \text{otherwise.} \end{cases} \tag{4.5.26}$$

Since $X^\pi \in \mathbb{B}$ for all $\pi \in \Pi$ and $Y \in \mathbb{B}_1 \subset \mathbb{B}$ (see (4.5.21) and (4.1.10)), we have from (4.1.8) that

$$X^\pi(T), Y(T) \in \mathcal{L}_2(\Omega, \mathcal{F}_T, P) \quad \text{ for all } \pi \in \Pi \text{ and } Y \in \mathbb{B}_1. \tag{4.5.27}$$

Therefore, we can apply Proposition D.0.8 to the first term on the right side of (4.5.26) and get

$$\inf_{u\in\mathcal{L}_2(\Omega,\mathcal{F}_T,P)} \{E\left[uY(T) + J(X^\pi(T) - u)\right]\} = E\left[X^\pi(T)Y(T)\right] - E\left[J^*(Y(T))\right]. \tag{4.5.28}$$

Next, define the following subset $\mathcal{A}_1$ of the set $\mathcal{A}$ of admissible portfolio processes:

$$\mathcal{A}_1 := \{\pi \in \mathcal{A} \mid \text{there is a constant } \hat{\alpha} \in \mathbb{R} \text{ such that } X^\pi(T) + \hat{\alpha} \geqslant b \text{ a.s.}\}. \tag{4.5.29}$$

**Remark 4.5.13.** Effectively $\mathcal{A}_1$ is the set of all $\pi \in \mathcal{A}$ such that $X^\pi(T) - b$ is a.s. essentially lower-bounded. From Condition 4.5.5, we know

$$\mathcal{A}_1 \neq \emptyset \quad \text{and} \quad \mathcal{A}_1 \subset \mathcal{A}. \tag{4.5.30}$$

The significance of the set $\mathcal{A}_1$ is that, when $\pi \in \Pi \backslash \mathcal{A}_1$, then $K(\pi, (Y, z)) = +\infty$ for all $(Y, z) \in \mathbb{Y}$ (see Proposition 4.5.14). In this sense, portfolios $\pi \in \Pi$ outside $\mathcal{A}_1$ are somewhat pathological, and will not be of any interest in the following analysis. In the next proposition we complete the partial evaluation of the Lagrangian at (4.5.26) in terms of the set of portfolios $\mathcal{A}_1$. The proof of this proposition is in Appendix A.

**Proposition 4.5.14.** *Recall* (4.1.20) *and* (4.5.29)*. For each* $\pi \in \Pi$ *(see* (4.1.1)*) and* $(Y, z) \in \mathbb{Y}$ *(see* (4.5.21) *and* (4.1.10)*) the Lagrangian* $K(\pi, (Y, z))$ *at* (4.5.24) *is given by*

$$K(\pi, (Y, z)) = \begin{cases} E\left[X^\pi(T)Y(T)\right] - E\left[J^*(Y(T))\right] & \\ \quad + \inf_{v\in\mathcal{L}_\infty} \{z(v) \mid X^\pi(T) + v \geqslant b\}, & \text{if } \pi \in \mathcal{A}_1,\ z \geqslant 0, \\ -\infty, & \text{if } \pi \in \mathcal{A}_1,\ z \ngeqslant 0, \\ +\infty, & \text{if } \pi \in \Pi\backslash\mathcal{A}_1. \end{cases} \tag{4.5.31}$$

In view of (3.1.4), the *dual function* $g : \mathbb{Y} \to [-\infty, \infty)$ is defined as:

$$
\begin{aligned}
g(Y,z) \quad &:= \quad \inf_{\pi\in\Pi} K(\pi,(Y,z)) \\
&\overset{(4.5.31)}{=} \quad \begin{cases} -\varkappa(Y,z) - E\left[J^*(Y(T))\right], & \text{if } z \geqslant 0, \\ -\infty, & \text{otherwise,} \end{cases}
\end{aligned}
\tag{4.5.32}
$$

in which we have defined

$$
\varkappa(Y,z) \quad := \quad \sup_{\pi\in\mathcal{A}_1} \left\{ -E\left[X^\pi(T)Y(T)\right] - \inf_{v\in\mathcal{L}_\infty} \left\{ z(v) \mid X^\pi(T) + v \geqslant b \right\} \right\}, \tag{4.5.33}
$$

for all $(Y,z) \in \mathbb{Y}$. From (4.5.25) and (4.5.32) we get the usual *weak duality* relation (c.f. (3.2.16)):

$$
f(\pi) \geqslant K(\pi,(Y,z)) \geqslant g(Y,z), \qquad \text{for all } (\pi,(Y,z)) \in \Pi\times\mathbb{Y}. \tag{4.5.34}
$$

The *dual problem* is then to maximize $g(Y,z)$ over all $(Y,z) \in \mathbb{Y} = \mathbb{B}_1 \times \mathcal{L}_\infty(\Omega,\mathcal{F}_T,P)$, that is, to establish

$$
g(\bar{Y},\bar{z}) = \sup_{(Y,z)\in\mathbb{Y}} \left\{ g(Y,z) \right\}, \qquad \text{for some } (\bar{Y},\bar{z}) \in \mathbb{Y}. \tag{4.5.35}
$$

In the following proposition we shall see that Theorem 3.1.7 is essential for securing existence of a maximizer $(\bar{Y},\bar{z})$:

**Proposition 4.5.15.** *Assume the Slater-type Condition 4.5.5. Then, there exists some* $(\bar{Y},\bar{z}) \in \mathbb{Y}$ *such that*

$$
\inf_{\pi\in\Pi}\{f(\pi)\} = \sup_{(Y,z)\in\mathbb{Y}} \{g(Y,z)\} = g(\bar{Y},\bar{z}) \in \mathbb{R}. \tag{4.5.36}
$$

**Proof.** Define the norm $\|\cdot\|_{\mathbb{U}}$ on $\mathbb{U}$ as follows:

$$
\|(u,v)\|_{\mathbb{U}} := \max\{\|u\|_2, \|v\|_\infty\}, \quad (u,v) \in \mathbb{U} = \mathcal{L}_2(\Omega,\mathcal{F}_T,P) \times \mathcal{L}_\infty(\Omega,\mathcal{F}_T,P), \tag{4.5.37}
$$

(recall (4.5.18)), and denote $\|\cdot\|_{\mathbb{U}}$-norm topology on $\mathbb{U}$ by $\mathcal{U}$. We need to verify that the locally convex topology $\mathcal{U}$ on $\mathbb{U}$ is $\langle\mathbb{U},\mathbb{Y}\rangle$-compatible (recall Definition 3.1.5) for the duality pairing given by (4.5.18), (4.5.21) and (4.5.22), that is

$$
\begin{cases} \mathbb{U} := \mathcal{L}_2(\Omega,\mathcal{F}_T,P) \times \mathcal{L}_\infty(\Omega,\mathcal{F}_T,P), & \mathbb{Y} := \mathbb{B}_1 \times \mathcal{L}_\infty^*(\Omega,\mathcal{F}_T,P), \\ \langle (u,v),(Y,z)\rangle := E\left[uY(T)\right] + z(v), & \text{for } (u,v) \in \mathbb{U} \text{ and } (Y,z) \in \mathbb{Y}. \end{cases} \tag{4.5.38}
$$

Indeed, the mapping $(u,v) \to \langle (u,v),(Y,z)\rangle$ is clearly $\mathcal{U}$-continuous on $\mathbb{U}$ for each $(Y,z) \in \mathbb{Y}$. Next, fix any $\mathcal{U}$-continuous linear functional $\phi^*$ on $\mathbb{U}$.

1) Since $\phi^*$ is linear on $\mathbb{U}$ we have

$$
\phi^*(u,v) = \phi^*(u,0) + \phi^*(0,v), \quad (u,v) \in \mathbb{U}. \tag{4.5.39}
$$

2) Since $\phi^*$ is a linear functional continuous in the $\|\cdot\|_{\mathbb{U}}$-norm topology on $\mathbb{U}$ it follows that $v \to \phi^*(0,v)$ is linear and norm-continuous on $\mathcal{L}_\infty(\Omega, \mathcal{F}_T, P)$, and there exists a $z \in \mathcal{L}_\infty^*(\Omega, \mathcal{F}_T, P)$ such that

$$\phi^*(0,v) = z(v), \quad v \in \mathcal{L}_\infty(\Omega, \mathcal{F}_T, P). \tag{4.5.40}$$

3) Again, since $\phi^*$ is a linear functional continuous in the $\|\cdot\|_{\mathbb{U}}$-norm topology on $\mathbb{U}$, it follows that $u \to \phi^*(u,0)$ is linear and norm-continuous on $\mathcal{L}_2(\Omega, \mathcal{F}_T, P)$. It then follows from the classical Riesz representation theorem applied to the Hilbert space $\mathcal{L}_2(\Omega, \mathcal{F}_T, P)$ that there exists some $\xi \in \mathcal{L}_2(\Omega, \mathcal{F}_T, P)$ such that

$$\phi^*(u,0) = E[u\xi], \quad u \in \mathcal{L}_2(\Omega, \mathcal{F}_T, P). \tag{4.5.41}$$

From Proposition 4.1.9, there is a unique $Y \in \mathbb{B}_1$ such that $Y(T) = \xi$ a.s., and then

$$\phi^*(u,0) = E[uY(T)], \quad u \in \mathcal{L}_2(\Omega, \mathcal{F}_T, P). \tag{4.5.42}$$

Combining (4.5.39), (4.5.40) and (4.5.42), we conclude that there exists some $Y = (Y, z) \in \mathbb{Y}$ such that

$$\phi^*(u,v) = E[uY(T)] + z(v) = \langle (u,v), (Y,z) \rangle, \quad (u,v) \in \mathbb{U}. \tag{4.5.43}$$

Therefore, the locally convex topology $\mathcal{U}$ on $\mathbb{U}$ is $\langle \mathbb{U}, \mathbb{Y} \rangle$-compatible.

Having established the $\langle \mathbb{U}, \mathbb{Y} \rangle$-compatibility of $\mathcal{U}$ we are now going to use Theorem 3.1.7 to establish the stated existence of an optimal dual solution $(\bar{Y}, \bar{z})$: We shall use Condition 4.5.5 to see that there exists some $\alpha \in (0,\infty)$ and some $\hat{\pi} \in \Pi$ such that

$$\sup \{ F(\hat{\pi}, (u,v)) \mid (u,v) \in \mathbb{U} \text{ with } \|(u,v)\|_{\mathbb{U}} < \alpha \} < +\infty, \tag{4.5.44}$$

that is, the condition (3.1.9) needed to apply Theorem 3.1.7 is satisfied by the perturbation function defined at (4.5.19). For the portfolio $\hat{\pi}$ and $\varepsilon \in (0,\infty)$ asserted in Condition 4.5.5 we clearly have

$$\hat{\pi} \in \mathcal{A} \text{ and } X^{\hat{\pi}}(T) + v \geqslant b, \quad \text{for all } v \in \mathcal{L}_\infty(\Omega, \mathcal{F}_T, P) \text{ with } \|v\|_\infty \leqslant \varepsilon. \tag{4.5.45}$$

Thus,

$$F(\hat{\pi}, (u,v)) \overset{(4.5.19)}{\underset{(4.5.45)}{=}} E\left[J(X^{\hat{\pi}} - u)\right], \quad v \in \mathcal{L}_\infty(\Omega, \mathcal{F}_T, P) \text{ with } \|v\|_\infty \leqslant \varepsilon, \tag{4.5.46}$$

for all $u \in \mathcal{L}_2(\Omega, \mathcal{F}_T, P)$. Moreover, since $X^{\hat{\pi}}(T) \in \mathcal{L}_2(\Omega, \mathcal{F}_T, P)$, by Condition 2.2.1 and (2.2.3), we have

$$u \to E\left[J(X^{\hat{\pi}} - u)\right] \quad \text{is norm-continuous on } (\mathcal{L}_2(\Omega, \mathcal{F}_T, P), \|\cdot\|_2), \tag{4.5.47}$$

and it is immediate from (4.5.47) that there exists some $\varepsilon_1 \in (0,\infty)$ such that

$$\left|E\left[J(X^{\hat{\pi}} - u)\right]\right| \leqslant \left|E\left[J(X^{\hat{\pi}})\right]\right| + 1 \in \mathbb{R}, \quad u \in \mathcal{L}_2(\Omega, \mathcal{F}_T, P) \text{ with } \|u\|_2 \leqslant \varepsilon_1. \tag{4.5.48}$$

Put $\alpha := \min\{\varepsilon, \varepsilon_1\}$; then (4.5.44) follows from (4.5.46), (4.5.48) and (4.5.37), and we have therefore verified condition (3.1.9) of Theorem 3.1.7 when $\mathcal{U}$ is the $\|\cdot\|_{\mathbb{U}}$-norm topology on $\mathbb{U}$ (see (4.5.37)) and the perturbation function $F$ is defined at (4.5.19). From $\langle\mathbb{U}, \mathbb{Y}\rangle$-compatibility of the norm-topology $\mathcal{U}$ on $\mathbb{U}$, the fact that $F(\cdot)$ is convex on $\Pi \times \mathbb{U}$ and satisfies a consistency relation of the form (4.5.20), and Theorem 3.1.7, we obtain the existence of some $(\bar{Y}, \bar{z}) \in \mathbb{Y}$ such that

$$\inf_{\pi\in\Pi} f(\pi) = \sup_{Y\in\mathbb{Y}} g(Y, z) = g(\bar{Y}, \bar{z}), \quad \text{for some } \bar{Y} \in \mathbb{Y}. \tag{4.5.49}$$

Moreover, from (4.5.13), we see that $\inf_{\pi\in\Pi} f(\pi) \in \mathbb{R}$, i.e., (4.5.49) takes a real value. □

**Remark 4.5.16.** It is clear from the proof of Proposition 4.5.15 that existence of the optimal dual solution $(\bar{Y}, \bar{z}) \in \mathbb{Y}$ is established on the basis of Theorem 3.1.7, the use of which relies on establishing (4.5.44) in order to verify condition (3.1.9). The keys to establishing (4.5.44) are the use of the *norm-topology* $\mathcal{U}$, together with (4.5.45), which itself is a consequence of Condition 4.5.5 and the fact that the second factor in the perturbation space $\mathbb{U}$ at (4.5.18) comprises *essentially bounded* random variables. This second point is of particular importance, and in Remark 4.5.32 at the end of the present section we shall explore the consequences of using *square-integrable* random variables in place of essentially bounded random variables in the second factor of $\mathbb{U}$ at (4.5.18).

**Remark 4.5.17.** From (4.1.20) we know that $J^*(Y(T))$ is $P$-integrable for all $Y \in \mathbb{B}_1$, and Proposition 4.5.15 ensures that $g(\bar{Y}, \bar{z}) \in \mathbb{R}$, therefore it follows from (4.5.32) that

$$\bar{z} \geqslant 0 \qquad \text{and} \qquad \varkappa(\bar{Y}, \bar{z}) \in \mathbb{R}. \tag{4.5.50}$$

This fact will be used several times in the sequel.

**Remark 4.5.18.** We know from Remark 3.1.9, together with the weak duality relation of (4.5.34), that it is enough to establish

$$f(\pi) = g(Y, z) \qquad \text{for some } \pi \in \Pi \text{ and } (Y, z) \in \mathbb{Y}, \tag{4.5.51}$$

(recalling (4.5.12), (4.5.21), and (4.5.32)), for then $\pi$ is the minimizer of $f(\cdot)$ on $\Pi$ (hence an optimal portfolio for problem (4.5.2), recall the associated Primal Problem 4.5.7), while $(Y, z)$ is a maximizer of $g(\cdot)$ on $\mathbb{Y} = \mathbb{B}_1 \times \mathcal{L}^*_\infty(\Omega, \mathcal{F}_T, P)$. However, we have already secured such a maximizer $(\bar{Y}, \bar{z}) \in \mathbb{Y}$ from Proposition 4.5.15, so the solution of problem 4.5.2 reduces to the construction of some $\bar{\pi} \in \Pi$ in terms of the maximizer $(\bar{Y}, \bar{z})$ such that

$$f(\bar{\pi}) = g(\bar{Y}, \bar{z}). \tag{4.5.52}$$

To this end, we first establish a set of *Kuhn-Tucker optimality relations* which are fully equivalent to the equality $f(\pi) = g(Y, z)$ for *arbitrary* $\pi \in \Pi$ and *arbitrary* $(Y, z) \in \mathbb{Y} = \mathbb{B}_1 \times \mathcal{L}^*_\infty(\Omega, \mathcal{F}_T, P)$ (recall Remark 3.1.10 (c)). Proposition 4.5.19 which follows is to be compared with Proposition 4.4.19 (for problem 4.4.2), as well as Proposition 4.3.11 (for problem 4.3.2) and Proposition 4.2.7 (for problem 4.2.1).

**Proposition 4.5.19. [Kuhn-Tucker Optimal Conditions]** *Suppose Condition 2.1.1, 2.1.2, 2.1.5, 2.1.11 on the market settings and Condition 2.2.1 on the quadratic criterion function $J$ given in* (4.1.19). *Then, for each $(\pi, (Y, z)) \in \Pi \times \mathbb{Y}$, we have the following equivalence:*

$$f(\pi) = g(Y, z) \iff \begin{cases} 1)\ \pi \in \mathcal{A},\ 2)\ X^\pi(T) \geqslant b \ a.s.,\ 3)\ z \geqslant 0, \\ 4)\ \inf_{v \in \mathcal{L}_\infty} \{z(v) \mid X^\pi(T) + v \geqslant b \ a.s.\} = 0, \\ 5)\ E\left[X^\pi(T)Y(T)\right] + \varkappa(Y, z) = 0,\ 6)\ X^\pi(T) = \partial J^*(Y(T)). \end{cases} \tag{4.5.53}$$

*Here, $\partial J^*(\cdot)$ denotes the derivative function of the convex conjugate function $J^*(\cdot)$ given in* (4.1.21).

The proof of Proposition 4.5.19 is in Appendix A.

**Remark 4.5.20.** Problem (3.2.64) constitutes a sort of static and finite dimensional precursor of the stochastic control problem (4.5.2), and there is a corresponding close similarity between the Kuhn-Tucker relations (3.2.92) for problem (3.2.64) and the Kuhn-Tucker relations (4.5.53) for problem (4.5.2) (recall Remark 3.2.24). This should not be a surprise considering that our approach to these problems involves a basically very similar application of the Rockafellar variational approach, although of course problem (4.5.2) involves much more technical effort. In particular (4.5.53)-(1)(2) and (3.2.92)-(1)(2) are feasibility conditions on the primal variable ($\pi \in \Pi$ in the case of problem (4.5.2), and $x \in \mathbb{R}^n$ in the case of problem (3.2.64)), while (4.5.53)-(3) and (3.2.92)-(3) are feasibility conditions on the dual variable $z$ (effectively the Lagrange multiplier which enforces the inequality constraint, namely $x \geqslant b$ in the case of problem (3.2.64) and $X^\pi(T) \geqslant b$ in the case of problem (4.5.2)). Likewise, (4.5.53)-(5) is an obvious analogue of (3.2.92)-(5), and in each case is a complementary slackness condition relating the primal variables and the dual variables to the constraints $\pi \in \mathcal{A}$ and $x \in C$ (for problems (4.5.2) and (3.2.64) respectively), while (4.5.53)-(6) and (3.2.92)-(6) are obviously comparable transversality conditions. Finally, (4.5.53)-(4) and (3.2.92)-(4) are complementary slackness conditions relating the primal variables and the dual variables to the inequality constraints $X^\pi(T) \geqslant b$ and $x \geqslant b$ (for problems (4.5.2) and (3.2.64) respectively). Indeed, if it were the case that $X^\pi(T) \in \mathcal{L}_\infty(\Omega, \mathcal{F}_T, P)$, then we would have $X^\pi(T) - b \in \mathcal{L}_\infty(\Omega, \mathcal{F}_T, P)$ (recall (4.5.5)), and it is then immediate that

$$\inf_{v \in \mathcal{L}_\infty} \{z(v) \mid X^\pi(T) + v \geqslant b \text{ a.s.}\} = z(b - X^\pi(T)), \tag{4.5.54}$$

for all $z \in \mathcal{L}^*_\infty(\Omega, \mathcal{F}_T, P)$ with $z \geqslant 0$, so that we could write (4.5.53)-(4) as

$$z(b - X^\pi(T)) = 0, \tag{4.5.55}$$

which very closely resembles (3.2.92)-(4). Of course, we do not generally have $X^\pi(T) \in \mathcal{L}_\infty(\Omega, \mathcal{F}_T, P)$, and so we must settle for the indirect expression of this complementary slackness condition in the form of (4.5.53)-(4).

**Remark 4.5.21.** We shall now construct a portfolio $\bar{\pi} \in \Pi$, in terms of the dual solution $(\bar{Y}, \bar{z}) \in \mathbb{Y}$ given by Proposition 4.5.15, to satisfy the Kuhn-Tucker relations (4.5.53)(1) - (6), that is

$$\begin{cases} 1)\ \bar{\pi} \in \mathcal{A},\ 2)\ X^{\bar{\pi}}(T) \geqslant b \text{ a.s.},\ 3)\ \bar{z} \geqslant 0, \\ 4)\ \inf\limits_{v \in \mathcal{L}_\infty} \{\bar{z}(v) \mid X^{\bar{\pi}}(T) + v \geqslant b \text{ a.s.}\} = 0, \\ 5)\ E\left[X^{\bar{\pi}}(T)\bar{Y}(T)\right] + \varkappa(\bar{Y}, \bar{z}) = 0,\ 6)\ X^{\bar{\pi}}(T) = \partial J^*(\bar{Y}(T)). \end{cases} \tag{4.5.56}$$

It then follows from Proposition 4.5.19 that

$$f(\bar{\pi}) = g(\bar{Y}, \bar{z}), \tag{4.5.57}$$

as required to establish that $\bar{\pi}$ is the optimal portfolio for problem (4.5.2) (see Remark 4.5.18). We already know from Remark 4.5.17 that (4.5.56)-(3) holds. We next construct $\bar{\pi} \in \Pi$ such that the remaining conditions at (4.5.56) hold. Our basic tool will be the following technical proposition, which is really a consequence of the optimality of the pair $(\bar{Y}, \bar{z})$ given by Proposition 4.5.15:

**Proposition 4.5.22.** *Suppose Condition 2.1.1, 2.1.2, 2.1.5, 2.1.11, 2.2.1 and 4.5.1, and recall the derivative function $\partial J^*(\cdot)$ at (4.1.21) and $\varkappa(\cdot)$ at (4.5.33). We have*

$$\varkappa(Y, z) + E\left[\partial J^*(\bar{Y}(T))Y(T)\right] \geqslant 0, \quad \textit{for each } (Y, z) \in \mathbb{Y} \textit{ with } z \geqslant 0, \tag{4.5.58}$$

*and in particular,*

$$\varkappa(\bar{Y}, \bar{z}) + E\left[\partial J^*(\bar{Y}(T))\bar{Y}(T)\right] = 0, \tag{4.5.59}$$

*where $(\bar{Y}, \bar{z}) \in \mathbb{Y}$ is given by Proposition 4.5.15.*

**Proof.** We use the optimality of the dual solution $(\bar{Y}, \bar{z})$ given at Proposition 4.5.15:

$$g(\bar{Y} + \varepsilon Y, \bar{z} + \varepsilon z) \leqslant g(\bar{Y}, \bar{z}) \quad \text{for all } (Y, z) \in \mathbb{Y} \text{ and } \varepsilon \in (0, \infty). \tag{4.5.60}$$

If $z \geqslant 0$, we have (recall (4.5.50) and (4.5.16))

$$\bar{z} + \varepsilon z \geqslant 0, \quad \text{for all } \varepsilon \in (0, \infty). \tag{4.5.61}$$

Combining (4.5.60), (4.5.61) and (4.5.32), we get

$$\varkappa(\bar{Y} + \varepsilon Y, \bar{z} + \varepsilon z) + E\left[J^*(\bar{Y}(T) + \varepsilon Y(T))\right] \geqslant \varkappa(\bar{Y}, \bar{z}) + E\left[J^*(\bar{Y}(T))\right], \tag{4.5.62}$$

for all $(Y, z) \in \mathbb{Y}$ with $z \geqslant 0$ and $\varepsilon \in (0, \infty)$. On the other hand, from (4.5.33), one sees

$$
\begin{aligned}
& \varkappa(\bar{Y} + \varepsilon Y, \bar{z} + \varepsilon z) \\
\overset{(4.5.33)}{=} & \sup_{\pi \in \mathcal{A}_1} \left\{ -E\left[X^\pi(T)(\bar{Y}(T) + \varepsilon Y(T))\right] - \inf_{v \in \mathcal{L}_\infty} \left\{ (\bar{z} + \varepsilon z)(v) \,\Big|\, X^\pi(T) + v \geqslant b \right\} \right\} \\
\leqslant & \sup_{\pi \in \mathcal{A}_1} \Big\{ -E\left[X^\pi(T)\bar{Y}(T)\right] - E\left[\varepsilon X^\pi(T)Y(T)\right] \\
& \qquad - \inf_{v \in \mathcal{L}_\infty} \left\{ \bar{z}(v) \,\Big|\, X^\pi(T) + v \geqslant b \right\} - \inf_{v \in \mathcal{L}_\infty} \left\{ \varepsilon z(v) \,\Big|\, X^\pi(T) + v \geqslant b \right\} \Big\} \\
\leqslant & \sup_{\pi \in \mathcal{A}_1} \left\{ -E\left[X^\pi(T)\bar{Y}(T)\right] - \inf_{v \in \mathcal{L}_\infty} \left\{ \bar{z}(v) \,\Big|\, X^\pi(T) + v \geqslant b \right\} \right\} \\
& \qquad + \sup_{\pi \in \mathcal{A}_1} \left\{ -\varepsilon E\left[X^\pi(T)Y(T)\right] - \varepsilon \inf_{v \in \mathcal{L}_\infty} \left\{ z(v) \,\Big|\, X^\pi(T) + v \geqslant b \right\} \right\} \\
= & \varkappa(\bar{Y}, \bar{z}) + \varepsilon \varkappa(Y, z) \qquad \text{for all } (Y, z) \in \mathbb{Y} \text{ with } z \geqslant 0 \text{ and } \varepsilon \in (0, \infty).
\end{aligned} \tag{4.5.63}
$$

From (4.5.63) and (4.5.62), together with $\varkappa(\bar{Y}, \bar{z}) \in \mathbb{R}$ (recall (4.5.50)), we get

$$
\varkappa(Y, z) + E\left[\frac{J^*(\bar{Y}(T) + \varepsilon Y(T)) - J^*(\bar{Y}(T))}{\varepsilon}\right] \geqslant 0, \qquad \varepsilon \in (0, \infty), \; z \geqslant 0. \tag{4.5.64}
$$

It follows from (4.1.20), Condition 2.2.1 and dominated convergence, that

$$
\lim_{\varepsilon \to 0} E\left[\frac{J^*(\bar{Y}(T) + \varepsilon Y(T)) - J^*(\bar{Y}(T))}{\varepsilon}\right] = E\left[\partial J^*(\bar{Y}(T))Y(T)\right], \quad \text{for } Y \in \mathbb{B}_1. \tag{4.5.65}
$$

Therefore, (4.5.64) and (4.5.65) give (4.5.58), and in particular

$$
\varkappa(\bar{Y}, \bar{z}) + E\left[\partial J^*(\bar{Y}(T))\bar{Y}(T)\right] \geqslant 0. \tag{4.5.66}
$$

To establish (4.5.59), we next use the optimality of the dual solution $(\bar{Y}, \bar{z})$ given at Proposition 4.5.15 again to get

$$
g((1-\varepsilon)\bar{Y}, (1-\varepsilon)\bar{z}) \leqslant g(\bar{Y}, \bar{z}) \quad \text{for all } \varepsilon \in [0, 1). \tag{4.5.67}
$$

Since $\bar{z} \geqslant 0$ (recall Remark 4.5.17), it follows from (4.5.67) and (4.5.32) that

$$
\varkappa((1-\varepsilon)\bar{Y}, (1-\varepsilon)\bar{z}) + E\left[J^*((1-\varepsilon)\bar{Y}(T))\right] \geqslant \varkappa(\bar{Y}, \bar{z}) + E\left[J^*(\bar{Y}(T))\right], \tag{4.5.68}
$$

for all $\varepsilon \in [0, 1)$. From (4.5.33), one also sees that

$$
\varkappa((1-\varepsilon)\bar{Y}, (1-\varepsilon)\bar{z}) = (1-\varepsilon)\varkappa(\bar{Y}, \bar{z}), \quad \text{for all } \varepsilon \in [0, 1). \tag{4.5.69}
$$

Using (4.5.69) in (4.5.68), together with $\varkappa(\bar{Y}, \bar{z}) \in \mathbb{R}$ (recall (4.5.50)), we have

$$
\varkappa(\bar{Y}, \bar{z}) + E\left[\frac{J^*(\bar{Y}(T)) - J^*((1-\varepsilon)\bar{Y}(T))}{\varepsilon}\right] \leqslant 0 \quad \text{for all } \varepsilon \in [0, 1). \tag{4.5.70}
$$

From (4.1.20), Condition 2.2.1 and dominated convergence, we get

$$\lim_{\varepsilon\to 0} E\left[\frac{J^*(\bar{Y}(T)) - J^*((1-\varepsilon)\bar{Y}(T))}{\varepsilon}\right] = E\left[\partial J^*(\bar{Y}(T))\bar{Y}(T)\right]. \tag{4.5.71}$$

Combining (4.5.71) with (4.5.70), we have

$$\varkappa(\bar{Y}, \bar{z}) + E\left[X^{\bar{\pi}}(T)\bar{Y}(T)\right] \leqslant 0, \tag{4.5.72}$$

and (4.5.59) follows from (4.5.66) and (4.5.72). □

With Proposition 4.5.22 established, we are now ready to construct the optimal portfolio $\bar{\pi}$ for the problem (4.5.2). Motivated by the transversality condition (4.5.53)-6) and Proposition 4.2.9 with

$$\partial J^*(\bar{Y}(T)) \overset{(4.1.20)}{=} \frac{\bar{Y}(T) - c}{a} \underset{Condition 2.2.1}{\overset{(4.5.27)}{\in}} \mathcal{L}_2(\Omega, \mathcal{F}_T, P), \tag{4.5.73}$$

define the process $\bar{X}$ as

$$\bar{X}(t) := H^{-1}(t)E\left[H(T)\partial J^*(\bar{Y}(T)) \,\middle|\, \mathcal{F}_t\right] \qquad t \in [0, T]. \tag{4.5.74}$$

Then, from Proposition 4.2.9, there exists some a.e. unique $\mathbb{R}^N$-valued $\mathcal{F}_t$-progressively measurable process $\bar{\psi}$ such that

$$\int_0^T \left\|\bar{\psi}(t)\right\|^2 \mathrm{d}t < \infty \quad \text{a.s.} \qquad \text{and} \qquad H(t)\bar{X}(t) = \bar{X}(0) + \int_0^t \bar{\psi}'(s)\mathrm{d}W(s). \tag{4.5.75}$$

Motivated by (4.2.37), define the $\mathbb{R}^N$-valued $\mathcal{F}_t$-progressively measurable process $\bar{\pi}$ in terms of the integrand process $\bar{\psi}$ given by (4.5.75):

$$\bar{\pi}(t) := (\sigma'(t))^{-1}\left[H^{-1}(t)\bar{\psi}(t) + \bar{X}(t)\theta(t)\right]. \tag{4.5.76}$$

From Proposition 4.2.9 again, it follows that

$$\bar{\pi} \in \Pi, \tag{4.5.77}$$

and

$$\mathrm{d}\bar{X}(t) = \left[r(t)\bar{X}(t) + \bar{\pi}'(t)\sigma(t)\theta(t)\right]\mathrm{d}t + \bar{\pi}'(t)\sigma(t)\mathrm{d}W(t), \tag{4.5.78}$$

with the initial value

$$\bar{X}(0) = E\left[H(T)\partial J^*(\bar{Y}(T))\right], \tag{4.5.79}$$

(see (4.5.73) and (4.2.39)).

**Remark 4.5.23.** Observe that the random variable $\bar{Y}(T) \in \mathcal{L}_2(\Omega, \mathcal{F}_T, P)$, resulting from the pair $(\bar{Y}, \bar{z})$ given by Proposition 4.5.15, effectively determines the whole process $\bar{X}(\cdot)$ through (4.5.74), and therefore also determines the whole process $\bar{\psi}$ through (4.5.75), and the whole process $\bar{\pi}$ through (4.5.76). Our goal is now to show $\bar{\pi}$ is the optimal portfolio process for problem (4.5.2).

We shall now establish that $\bar{X}$ is the wealth process corresponding to $\bar{\pi}$ (see Proposition 4.5.26 which follows) and then show that $\bar{\pi}$ is an admissible portfolio process (see Proposition 4.5.27 which follows). As as tool in the proof of Proposition 4.5.26 we shall need Proposition 4.3.17 established earlier in connection with problem (4.3.2). For convenience we restate this proposition here:

**Proposition 4.5.24.** *Suppose Condition 2.1.1, 2.1.2, 2.1.5, 2.1.11, 2.2.1 and 4.5.1, and recall definitions of $\theta$, $\mathbb{B}_1$ at (2.1.8), (4.1.10), Remark 2.1.7 - (1), and Notation 4.3.14-(b). Then we have*

$$\begin{aligned}&\inf_{\pi\in\mathcal{A}}\{E\left[X^{\pi}(T)Y(T)\right]\}\\&= x_0Y(0) - E\left[\int_0^T \delta^*_{\mathbb{R}^N}\left\{-\sigma(s)[\theta(s)Y(s)+\Lambda_Y(s)]\middle|A\right\}ds\right], \quad Y\in\mathbb{B}_1. \qquad (4.5.80)\end{aligned}$$

**Remark 4.5.25.** Proposition 4.3.17 was established as a tool for addressing problem (4.3.2) in Section 4.3. This proposition concerns only the portfolio constraint set $\mathcal{A}$, the dynamical equation for $X^{\pi}$, and processes $Y\in\mathbb{B}_1$. Since these entities are unchanged in problem (4.5.2) we are entitled to use this result (now stated as Proposition 4.5.24) in the present section as well.

**Proposition 4.5.26.** *Suppose Condition 2.1.1, 2.1.2, 2.1.5, 2.1.11, 2.2.1 and 4.5.1, and recall $\bar{X}$ and $\bar{\pi}$ defined at (4.5.74) and (4.5.76) in terms of $\bar{Y}\in\mathbb{B}_1$ given by Proposition 4.5.15. Then, we have the identity*

$$E\left[\bar{X}(T)Y(T)\right] = \bar{X}(0)Y(0) + E\left[\int_0^T \bar{\pi}'(s)\sigma(s)\left[Y(s)\theta(s)+\Lambda_Y(s)\right]ds\right], \qquad (4.5.81)$$

*for all $Y\in\mathbb{B}_1$, and $\bar{X}$ is the wealth process corresponding to $\bar{\pi}$, that is*

$$X^{\bar{\pi}} = \bar{X} \quad \text{so that} \quad X^{\bar{\pi}}(T) = \bar{X}(T) \overset{(4.5.74)}{=} \partial J^*(\bar{Y}(T)). \qquad (4.5.82)$$

**Proof.** From (4.5.78) and Proposition 4.1.4, we find

$$\bar{X}\in\mathbb{B} \quad \text{with} \quad \dot{\bar{X}} = r\bar{X} + \bar{\pi}'\sigma\theta \ \text{ and } \ \Lambda_{\bar{X}} = \sigma'\bar{\pi}, \qquad (4.5.83)$$

while

$$\dot{Y} = -rY \qquad \text{for each} \qquad Y\in\mathbb{B}_1. \qquad (4.5.84)$$

Then, from (F.1) together with (4.5.83) and (4.5.84), we have

$$\begin{aligned}&M(\bar{X},Y)(T)\\&= \bar{X}(T)Y(T) - \bar{X}(0)Y(0) - \int_0^T\{\bar{X}(s)\dot{Y}(s) + \dot{\bar{X}}(s)Y(s) + \Lambda'_{\bar{X}}(s)\Lambda_Y(s)\}\mathrm{d}s\\&\underset{(4.5.84)}{\overset{(4.5.83)}{=}} \bar{X}(T)Y(T) - \bar{X}(0)Y(0) - \int_0^T \bar{\pi}'(s)\sigma(s)\left[Y(s)\theta(s)+\Lambda_Y(s)\right]\mathrm{d}s, \qquad (4.5.85)\end{aligned}$$

for each $Y \in \mathbb{B}_1$. In view of Proposition F.0.1 together with $\bar{X} \in \mathbb{B}$ (see (4.5.83)) and $Y \in \mathbb{B}_1 \subset \mathbb{B}$, we get

$$E\left[M(\bar{X}, Y)(T)\right] = 0, \qquad \text{for each } Y \in \mathbb{B}_1. \tag{4.5.86}$$

Now (4.5.86) and (4.5.85) give

$$E\left[\bar{X}(T)Y(T)\right] = \bar{X}(0)Y(0) + E\left[\int_0^T \bar{\pi}'(s)\sigma(s)\left[Y(s)\theta(s) + \Lambda_Y(s)\right] \mathrm{d}s\right], \; Y \in \mathbb{B}_1, \tag{4.5.87}$$

which is (4.5.81). To establish (4.5.82), we fix some $y \in \mathbb{R}$. From Lemma 4.1.8,

$$\text{there exists some } Y \in \mathbb{B}_1 \text{ such that } Y(0) = y \text{ and } \theta Y + \Lambda_Y = 0 \text{ a.e.} \tag{4.5.88}$$

and then (4.5.87) is reduced to

$$E\left[\bar{X}(T)Y(T)\right] \overset{(4.5.87)}{\underset{(4.5.88)}{=}} \bar{X}(0)y. \tag{4.5.89}$$

On the other hand, the necessary condition (4.5.58) holds for $(Y, 0) \in \mathbb{Y}$ where $Y$ is given by (4.5.88), i.e.

$$\varkappa(Y, 0) + E\left[\partial J^*(\bar{Y}(T))Y(T)\right] \geqslant 0. \tag{4.5.90}$$

From (4.5.80), (4.5.33) and (4.5.88), we have

$$\begin{aligned} \varkappa(Y, 0) &\overset{(4.5.33)}{\leqslant} \sup_{\pi \in \mathcal{A}_1} \left\{-E\left[X^\pi(T)Y(T)\right]\right\} \\ &\overset{(4.5.30)}{\leqslant} \sup_{\pi \in \mathcal{A}} \left\{-E\left[X^\pi(T)Y(T)\right]\right\} \overset{(4.5.80)}{\underset{(4.5.88)}{=}} -x_0 y. \end{aligned} \tag{4.5.91}$$

From (4.5.74), $\bar{X}(T) = \partial J^*(\bar{Y}(T))$, and combining this with (4.5.89), (4.5.90) and (4.5.91) gives

$$(\bar{X}(0) - x_0)y \geqslant 0. \tag{4.5.92}$$

By arbitrary choice of $y \in \mathbb{R}$ in (4.5.92), we get

$$\bar{X}(0) = x_0. \tag{4.5.93}$$

Therefore, (4.5.82) follows from (4.5.93), (4.5.78), (2.1.21) and (4.5.74). □

In the next proposition it is established that the portfolio $\bar{\pi}$ at (4.5.76) satisfies the constraints in problem (4.5.2):

**Proposition 4.5.27.** *Suppose Condition 2.1.1, 2.1.2, 2.1.5, 2.1.11, 2.2.1 and 4.5.1, and recall $\bar{\pi}$ defined at (4.5.76). Then*

$$\bar{\pi} \in \mathcal{A}, \tag{4.5.94}$$

*and*

$$X^{\bar{\pi}}(T) \geqslant b \quad a.s., \tag{4.5.95}$$

*i.e. $\bar{\pi}$ satisfies (4.5.56)-1), 2).*

**Proof.** First define a set

$$O := \{(\omega;t) \in \Omega \times [0,T] \mid \bar{\pi}(\omega;t) \in A\}. \tag{4.5.96}$$

From Lemma F.0.3 (also see Lemma 5.4.2 of [15, p.207]), corresponding to $\bar{\pi} \in \mathcal{F}^*$ there exists some $\mathbb{R}^N$-valued $\bar{\nu} \in \mathcal{F}^*$ such that

$$\begin{cases} \|\bar{\nu}(t)\| \leqslant 1, \left|\delta^*_{\mathbb{R}^N}\left\{-\bar{\nu}(t)\middle|A\right\}\right| \leqslant 1, & \text{a.e. on } \Omega \times [0,T], \\ \bar{\pi}'(t)\bar{\nu}(t) + \delta^*_{\mathbb{R}^N}\left\{-\bar{\nu}(t)\middle|A\right\} = 0, & \text{a.e. on } O, \\ \bar{\pi}'(t)\bar{\nu}(t) + \delta^*_{\mathbb{R}^N}\left\{-\bar{\nu}(t)\middle|A\right\} < 0, & \text{a.e. on } (\Omega \times [0,T]) \setminus O. \end{cases} \tag{4.5.97}$$

It is clear from $\|\bar{\nu}(t)\| \leqslant 1$ a.e. and Condition 2.1.5 that

$$\sigma^{-1}\bar{\nu} \in \Pi. \tag{4.5.98}$$

Thus, from (4.5.98) and Lemma 4.1.8,

$$\text{there exists some } Y \in \mathbb{B}_1 \text{ such that } Y(0) = 0 \text{ and } \theta Y + \Lambda_Y = \sigma^{-1}\bar{\nu} \text{ a.e.} \tag{4.5.99}$$

With the $Y \in \mathbb{B}_1$ given by (4.5.99), we certainly have (4.5.58) holds for $(Y,0) \in \mathbb{Y}$, i.e.,

$$\varkappa(Y,0) + E\left[\partial J^*(\bar{Y}(T))Y(T)\right] \geqslant 0. \tag{4.5.100}$$

From (4.5.99) and (4.5.33), we have

$$\varkappa(Y,0) = \sup_{\pi \in \mathcal{A}_1} \{-E[X^\pi(T)Y(T)]\}. \tag{4.5.101}$$

Combining (4.5.82) with (4.5.100) and (4.5.101), we get

$$\sup_{\pi \in \mathcal{A}_1} \{-E[X^\pi(T)Y(T)]\} + E\left[\bar{X}(T)Y(T)\right] \geqslant 0. \tag{4.5.102}$$

From (4.5.99) and Proposition 4.5.24 we get

$$\begin{aligned} \sup_{\pi \in \mathcal{A}_1} \{-E[X^\pi(T)Y(T)]\} &\overset{(4.5.30)}{\leqslant} \sup_{\pi \in \mathcal{A}} \{-E[X^\pi(T)Y(T)]\} \\ &\underset{(4.5.99)}{\overset{(4.5.80)}{=}} E\left[\int_0^T \{\delta^*_{\mathbb{R}^N}\left\{-\bar{\nu}(s)\middle|A\right\}\}\mathrm{d}s\right], \end{aligned} \tag{4.5.103}$$

and from (4.5.99) and Proposition 4.5.26, it follows that

$$E\left[\bar{X}(T)Y(T)\right] = E\left[\int_0^T \{\bar{\pi}'(s)\bar{\nu}(s)\mathrm{d}s\right]. \tag{4.5.104}$$

Combining (4.5.102), (4.5.103) and (4.5.104), we have

$$E\left[\int_0^T \{\bar{\pi}'(s)\bar{\nu}(s) + \delta^*_{\mathbb{R}^N}\left\{-\bar{\nu}(s)\middle|A\right\}\}\mathrm{d}s\right] \geqslant 0. \tag{4.5.105}$$

This, together with (4.5.97), gives

$$P \otimes \lambda (O^c) = 0, \tag{4.5.106}$$

From (4.5.106) and (4.5.96), we get $\bar{\pi}(t) \in A$ a.e., and then $\bar{\pi} \in \mathcal{A}$ follows from (4.5.77).

To establish (4.5.95), define some $\xi \in \mathcal{L}_2(\Omega, \mathcal{F}_T, P)$ as follows:

$$\xi := \mathbb{1}_{\{X^{\bar{\pi}}(T)<b\}} \overset{(3.3.21)}{=} \begin{cases} 1 & \text{if } X^{\bar{\pi}}(T) < b, \\ 0 & \text{otherwise.} \end{cases} \tag{4.5.107}$$

Then

$$\xi \in \mathcal{L}^*_\infty(\Omega, \mathcal{F}_T, P) \quad \text{and} \quad \xi \geqslant 0, \tag{4.5.108}$$

where we define

$$\xi(v) := E\left[v\xi\right] \quad \text{for all } v \in \mathcal{L}_\infty(\Omega, \mathcal{F}_T, P). \tag{4.5.109}$$

(recall (4.5.16)). From Proposition 4.1.9, we also have

$$Y_\xi(T) = \xi \quad \text{for some } Y_\xi \in \mathbb{B}_1. \tag{4.5.110}$$

From (4.5.110) and (4.5.108), it follows that (4.5.58) holds for $(Y_\xi, \xi) \in \mathbb{Y}$, i.e.,

$$\varkappa(Y_\xi, \xi) + E\left[\partial J^*(\bar{Y}(T))Y_\xi(T)\right] \geqslant 0. \tag{4.5.111}$$

From (4.5.110), (4.5.33) and (4.5.109), we have

$$\begin{aligned} \varkappa(Y_\xi, \xi) &\overset{(4.5.33)}{=} \sup_{\pi\in\mathcal{A}_1} \left\{ -E\left[X^\pi(T)Y_\xi(T)\right] - \inf_{v\in\mathcal{L}_\infty} \{\xi(v) \mid X^\pi(T) + v \geqslant b\} \right\} \\ &\underset{(4.5.110)}{\overset{(4.5.109)}{=}} \sup_{\pi\in\mathcal{A}_1} \left\{ -E\left[X^\pi(T)\xi\right] - \inf_{v\in\mathcal{L}_\infty} \{E\left[v\xi\right] \mid X^\pi(T) + v \geqslant b\} \right\} \\ &= \sup_{\pi\in\mathcal{A}_1} \left\{ -E\left[X^\pi(T)\xi\right] + \sup_{v\in\mathcal{L}_\infty} \{-E\left[v\xi\right] \mid X^\pi(T) + v \geqslant b\} \right\} \\ &= \sup_{\pi\in\mathcal{A}_1} \left\{ \sup_{v\in\mathcal{L}_\infty} \{-E\left[\xi(X^\pi(T) + v)\right] \mid X^\pi(T) + v \geqslant b\} \right\} \\ &= \sup_{\substack{\pi\in\mathcal{A}_1, v\in\mathcal{L}_\infty \\ X^\pi(T)+v\geqslant b}} \{-E\left[\xi(X^\pi(T) + v)\right]\}. \end{aligned} \tag{4.5.112}$$

Combining (4.5.82) with (4.5.112) and (4.5.111), we have

$$\begin{aligned} &\sup_{\substack{\pi\in\mathcal{A}_1, v\in\mathcal{L}_\infty \\ X^\pi(T)+v\geqslant b}} \{-E\left[\xi(X^\pi(T) + v - b)\right]\} + E\left[(X^{\bar{\pi}}(T) - b)\xi\right] \\ &= \sup_{\substack{\pi\in\mathcal{A}_1, v\in\mathcal{L}_\infty \\ X^\pi(T)+v\geqslant b}} \{-E\left[\xi(X^\pi(T) + v)\right]\} + E\left[X^{\bar{\pi}}(T)\xi\right] \\ &\underset{(4.5.110)}{\overset{(4.5.112)}{=}} \varkappa(Y_\xi, \xi) + E\left[X^{\bar{\pi}}(T)Y_\xi(T)\right] \\ &\overset{(4.5.82)}{=} \varkappa(Y_\xi, \xi) + E\left[\partial J^*(\bar{Y}(T))Y_\xi(T)\right] \overset{(4.5.111)}{\geqslant} 0. \end{aligned} \tag{4.5.113}$$

Now suppose (4.5.95) does not hold, that is we have

$$P\left(X^{\bar{\pi}}(T) < b\right) > 0. \tag{4.5.114}$$

From (4.5.114) and (4.5.107) we obtain

$$E\left[(X^{\bar{\pi}}(T) - b)\xi\right] < 0. \tag{4.5.115}$$

On the other hand, from $\xi \geqslant 0$ (see (4.5.108)), it is immediate that

$$\sup_{\substack{\pi \in \mathcal{A}_1, v \in \mathcal{L}_\infty \\ X^\pi(T)+v \geqslant b}} \{-E\left[\xi(X^\pi(T) + v - b)\right]\} \leqslant 0. \tag{4.5.116}$$

Combining (4.5.116), (4.5.115) and (4.5.108), we have

$$\sup_{\substack{\pi \in \mathcal{A}_1, v \in \mathcal{L}_\infty \\ X^\pi(T)+v \geqslant b}} \{-E\left[\xi(X^\pi(T) + v - b)\right]\} + E\left[(X^{\bar{\pi}}(T) - b)\xi\right] < 0, \tag{4.5.117}$$

which contradicts (4.5.113). Therefore (4.5.114) cannot hold, and we have (4.5.95). □

**Remark 4.5.28.** Having established Proposition 4.5.22, Proposition 4.5.26 and Proposition 4.5.27, we have constructed a pair $(\bar{\pi}, (\bar{Y}, \bar{z})) \in \mathbb{X} \times \mathbb{Y}$ such that

$$\begin{cases} (1)\ \bar{z} \geqslant 0 \text{ (recall (4.5.50) of Remark 4.5.17)}, \\ (2)\ X^{\bar{\pi}}(T) = \partial J^*(\bar{Y}(T)) \text{ (recall (4.5.82))}, \\ (3)\ E\left[X^{\bar{\pi}}(T)\bar{Y}(T)\right] + \varkappa(\bar{Y}, \bar{z}) = 0 \text{ (from (4.5.59) and (2))}, \\ (4)\ \bar{\pi} \in \mathcal{A} \text{ and } X^{\bar{\pi}}(T) \geqslant b \quad \text{a.s. (recall Proposition 4.5.27)} \end{cases} \tag{4.5.118}$$

so that (4.5.56) - 1), 2), 3), 5), 6) are verified. Moreover, (4.5.56) - 4) follows immediately from the following Corollary 4.5.29. Therefore, by Remark 4.5.21, we have verified that

$$\bar{\pi} \text{ is an optimal portfolio for problem (4.5.2).}$$

**Corollary 4.5.29.** *Suppose Condition 2.1.1, 2.1.2, 2.1.5, 2.1.11, 2.2.1 and 4.5.1, and recall $(\bar{Y}, \bar{z})$ at Proposition 4.5.15, and $X^{\bar{\pi}} = \partial J^*(\bar{Y}(T))$ at (4.5.82). Then*

$$\inf_{v \in \mathcal{L}_\infty} \{\bar{z}(v) \mid X^{\bar{\pi}}(T) + v \geqslant b \ a.s.\} = 0, \tag{4.5.119}$$

*i.e. $(\bar{\pi}, (\bar{Y}, \bar{z}))$ satisfies the complementary slackness relations (4.5.56)-4).*

**Proof.** From (4.5.95), we clearly have $X^{\bar{\pi}}(T) + v \geqslant b$ when $v = 0$, and thus

$$\inf_{v \in \mathcal{L}_\infty} \{\bar{z}(v) \mid X^{\bar{\pi}}(T) + v \geqslant b\} \leqslant \bar{z}(0) = 0. \tag{4.5.120}$$

On the other hand, from (4.5.95) again and (4.5.29), we see that

$$\bar{\pi} \in \mathcal{A}_1. \tag{4.5.121}$$

(4.5.121) and (4.5.33) immediately give

$$\varkappa(\bar{Y}, \bar{z}) + E\left[X^{\bar{\pi}}(T)\bar{Y}(T)\right] + \inf_{v \in \mathcal{L}_\infty} \{\bar{z}(v) \mid X^{\bar{\pi}}(T) + v \geqslant b\} \geqslant 0. \tag{4.5.122}$$

Combining (4.5.59) with (4.5.82), we have

$$\varkappa(\bar{Y}, \bar{z}) + E\left[X^{\bar{\pi}}(T)\bar{Y}(T)\right] = 0. \tag{4.5.123}$$

From (4.5.123) and (4.5.122), it follows that

$$\inf_{v \in \mathcal{L}_\infty} \{\bar{z}(v) \mid X^{\bar{\pi}}(T) + v \geqslant b\} \geqslant 0. \tag{4.5.124}$$

Finally, (4.5.119) is given by combining (4.5.124) and (4.5.120). □

**Remark 4.5.30.** The proof of Corollary 4.5.29 uses (4.5.95) and the *necessary condition* (4.5.59) given by Proposition 4.5.22.

**Remark 4.5.31.** Motivated by Problem 3.3.12 and Remark 3.3.19 we can argue as follows that the "singular component" $\bar{z}_s$ of the Lagrange multiplier $\bar{z}$ plays a natural and reasonable role: From the Yosida-Hewitt decomposition of $\bar{z} \in \mathcal{L}^*_\infty(\Omega, \mathcal{F}_T, P)$ into the components $\bar{z} = (\bar{z}_r, \bar{z}_s)$ for unique $\bar{z}_r \in \mathcal{L}_1(\Omega, \mathcal{F}_T, P)$ and $\bar{z}_s \in \mathcal{Z}(\Omega, \mathcal{F}, P)$ (see Remark 3.3.10) one easily establishes the identity

$$\begin{aligned} \inf_{v \in \mathcal{L}_\infty} \{\bar{z}(v) \mid X^{\bar{\pi}}(T) + v \geqslant b\} &= E\left[\bar{z}_r(b - X^{\bar{\pi}}(T))\right] \\ &+ \inf_{v \in \mathcal{L}_\infty} \{\bar{z}_s(v) \mid X^{\bar{\pi}}(T) + v \geqslant b\}. \end{aligned} \tag{4.5.125}$$

From $\bar{z} \geqslant 0$ (see (4.5.118)-(1)) together with the equivalence (3.3.29) one gets

$$\bar{z}_r \geqslant 0 \quad \text{a.s.} \qquad\qquad \bar{z}_s \geqslant 0, \tag{4.5.126}$$

so that, in particular, we have

$$\bar{z}_r(b - X^{\bar{\pi}}) \leqslant 0 \quad \text{a.s.}, \tag{4.5.127}$$

since $X^{\bar{\pi}}(T) \geqslant b \quad$ a.s., as follows from (4.5.118)-(4). Then the expectation on the right hand side of (4.5.125) is defined, with

$$E\left[\bar{z}_r(b - X^{\bar{\pi}})\right] \leqslant 0, \tag{4.5.128}$$

and moreover

$$\inf_{v \in \mathcal{L}_\infty} \{\bar{z}_s(v) \mid X^{\bar{\pi}}(T) + v \geqslant b\} \overset{(4.5.95)}{\leqslant} \inf_{v \in \mathcal{L}_\infty} \{\bar{z}_s(v) \mid v \geqslant 0\} \overset{(4.5.126)}{=} 0. \tag{4.5.129}$$

In view of (4.5.129), (4.5.128), (4.5.127) and (4.5.125), we have the equivalence

$$\inf_{v \in \mathcal{L}_\infty} \{\bar{z}(v) \mid X^{\bar{\pi}}(T) + v \geqslant b\} = 0 \iff \begin{cases} 1)\ \bar{z}_r(b - X^{\bar{\pi}}(T)) = 0 \ \text{ a.s.}, \\ \\ 2)\ \inf_{v \in \mathcal{L}_\infty} \{\bar{z}_s(v) \mid X^{\bar{\pi}}(T) + v \geqslant b\} = 0, \end{cases} \tag{4.5.130}$$

and it then follows from Corollary 4.5.29 together with (4.5.130) that

$$\bar{z}_r(b - X^{\bar{\pi}}(T)) = 0 \ \text{ a.s.}, \qquad \inf_{v \in \mathcal{L}_\infty} \{\bar{z}_s(v) \mid X^{\bar{\pi}}(T) + v \geqslant b\} = 0. \tag{4.5.131}$$

From (4.5.131) we infer the following: Suppose that $\bar{z} \neq 0$, so that the constraint $X^{\pi}(T) \geqslant b$ is active, and that this constraint furthermore "binds" only on a set of $P$-measure zero, that is

$$X^{\bar{\pi}}(T) > b \ \text{ a.s.} \tag{4.5.132}$$

It then follows from (4.5.132) and (4.5.131) that $\bar{z}_r = 0$ a.s., so that the Lagrange multiplier $\bar{z}$ for the constraint $X^{\pi}(T) \geqslant 0$ comprises only the singular part $\bar{z}_s$. This of course is very consistent with what we have seen at Problem 3.3.12 (recall Remark 3.3.18) as well as with the general message of Rockafellar and Wets [32] (recall Remark 3.3.19).

**Remark 4.5.32.** The second factor in the perturbation space $\mathbb{U}$ at (4.5.18) is the vector space $\mathcal{L}_\infty(\Omega, \mathcal{F}_T, P)$ of *essentially bounded* random variables. It was explained at Remark 4.5.16 that this factor space was the key thing when we used the Slater Condition 4.5.5 to verify the conditions of Theorem 3.1.7 (and especially (3.1.9)) in the course of establishing Proposition 4.5.15. Furthermore, we have argued at Remark 4.5.31 that dual variables $z \in \mathcal{L}_\infty^*(\Omega, \mathcal{F}_T, P)$, which are the consequence of using the space of perturbations at (4.5.18), are in fact quite reasonable. It is nevertheless instructive to explore the consequences of replacing the space $\mathcal{L}_\infty(\Omega, \mathcal{F}_T, P)$ in the second factor of (4.5.18) with the space $\mathcal{L}_2(\Omega, \mathcal{F}_T, P)$, since this will result in the space of dual variables $\mathbb{Y} := \mathbb{B}_1 \times \mathcal{L}_2(\Omega, \mathcal{F}_T, P)$ (see (4.5.136) to follow) which seems to be "more natural" than the space of dual variables that was used above (see (4.5.21)). This is the goal of the present remark. We shall see in particular how this choice of perturbation space essentially vitiates the use of Theorem 3.1.7. We therefore define the space of perturbations

$$\mathbb{U} := \mathcal{L}_2(\Omega, \mathcal{F}_T, P) \times \mathcal{L}_2(\Omega, \mathcal{F}_T, P), \tag{4.5.133}$$

in place of the space of perturbations at (4.5.18). Of course we keep the form of the perturbation function (4.5.19), with the obvious modification that the variable $v$ is now a member of $\mathcal{L}_2(\Omega, \mathcal{F}_T, P)$ (and not $\mathcal{L}_\infty(\Omega, \mathcal{F}_T, P)$), that is

$$F(\pi, (u, v)) := \begin{cases} E\left[J(X^{\pi}(T) - u)\right], & \text{where } \pi \in \mathcal{A} \text{ and } X^{\pi}(T) + v \geqslant b, \\ +\infty, & \text{otherwise}, \end{cases} \tag{4.5.134}$$

for all $\pi \in \Pi$ and $(u, v) \in \mathbb{U} := \mathcal{L}_2(\Omega, \mathcal{F}_T, P) \times \mathcal{L}_2(\Omega, \mathcal{F}_T, P)$. Exactly as at (4.5.20) we have the consistency relation

$$F(\pi, (0, 0)) = f(\pi), \quad \pi \in \Pi, \tag{4.5.135}$$

It is natural to pair the space of perturbations at (4.5.133) with the space of dual variables

$$\mathbb{Y} := \mathbb{B}_1 \times \mathcal{L}_2(\Omega, \mathcal{F}_T, P), \tag{4.5.136}$$

through the "natural" duality pairing

$$\langle (u,v),(Y,\xi)\rangle := E\left[uY(T)\right] + E\left[v\xi\right], \tag{4.5.137}$$

for $(u,v) \in \mathbb{U} = \mathcal{L}_2(\Omega,\mathcal{F}_T,P) \times \mathcal{L}_2(\Omega,\mathcal{F}_T,P)$ and $(Y,\xi) \in \mathbb{Y} = \mathbb{B}_1 \times \mathcal{L}_2(\Omega,\mathcal{F}_T,P)$, (compare with (4.5.21) and (4.5.22)). We note in passing that we are now using exactly the space of perturbation variables, the space of dual variables and the duality pairing that we previously used for problem (4.4.2) (without portfolio constraints, see (4.4.26) and (4.4.27)) while the perturbation function at (4.5.134) is just an obvious modification of the perturbation function at (4.4.24) to include the portfolio constraint $\pi \in \mathcal{A}$.

The Lagrangian associated with the definitions at (4.5.133) - (4.5.137) is easily seen to be

$$K(\pi,(Y,z)) = \begin{cases} \inf_{u\in\mathcal{L}_2} \{E\left[uY(T) + J(X^\pi(T) - u)\right]\} \\ \quad + \inf_{v\in\mathcal{L}_2} \left\{E\left[zv\right] \,\middle|\, X^\pi(T) + v \geqslant b\right\}, & \text{if } \pi \in \mathcal{A}, \\ +\infty, & \text{otherwise}, \end{cases} \tag{4.5.138}$$

for all $\pi \in \Pi$ and $(Y,z) \in \mathbb{Y} := \mathbb{B}_1 \times \mathcal{L}_2(\Omega,\mathcal{F}_T,P)$ (compare with (4.5.26)). The infima on the right side of (4.5.138) are easily evaluated explicitly, and this gives

$$K(\pi,(Y,z)) = \begin{cases} E\left[X^\pi(T)Y(T)\right] - E\left[J^*(Y(T))\right] \\ \qquad + E\left[z(b - X^\pi(T))\right], & \text{if } \pi \in \mathcal{A},\ z \geqslant 0, \\ -\infty, & \text{if } \pi \in \mathcal{A},\ z \ngeqslant 0, \\ +\infty, & \text{if } \pi \in \Pi\backslash\mathcal{A}, \end{cases} \tag{4.5.139}$$

for all $\pi \in \Pi$ and $(Y,z) \in \mathbb{Y} := \mathbb{B}_1 \times \mathcal{L}_2(\Omega,\mathcal{F}_T,P)$ (compare with (4.5.31)).

As an aside we note that the "reduced set" of portfolios $\mathcal{A}_1$ (see (4.5.29)) arises very naturally as an entity necessary for writing out the Lagrangian in explicit form at (4.5.31) because of the use of essentially bounded perturbations in the perturbation space $\mathbb{U}$ defined by (4.5.18). In contrast, in the computation of the Lagrangian at (4.5.139) based on the square-integrable perturbations in the perturbation space defined by (4.5.133), the set of admissible portfolios $\mathcal{A}$ itself suffices for writing out the Lagrangian in explicit form.

The dual function $g$ is defined in terms of the Lagrangian at (4.5.139) in the usual way (see (3.1.4)) namely

$$g(Y,z) := \inf_{\pi\in\Pi} K(\pi,(Y,z)), \qquad (Y,z) \in \mathbb{Y} := \mathbb{B}_1 \times \mathcal{L}_2(\Omega,\mathcal{F}_T,P). \tag{4.5.140}$$

Upon using (4.5.138) in (4.5.140) we obtain the dual function

$$g(Y,z) = \begin{cases} -\varkappa(Y,z) - E\left[J^*(Y(T))\right], & \text{if } z \geqslant 0, \\ -\infty, & \text{otherwise}, \end{cases} \tag{4.5.141}$$

where

$$\varkappa(Y,z) \ := \ \sup_{\pi\in\mathcal{A}} \{-E\left[X^\pi(T)Y(T)\right] + E\left[z(X^\pi(T) - b)\right]\}, \tag{4.5.142}$$

for all $(Y, z) \in \mathbb{Y} := \mathbb{B}_1 \times \mathcal{L}_2(\Omega, \mathcal{F}_T, P)$ with $z \geq 0$ (compare (4.5.141) and (4.5.142) with (4.5.32) and (4.5.33)). We must now establish existence of a maximizer for the dual function (4.5.141) - (4.5.142), that is we must establish

$$g(\bar{Y}, \bar{z}) = \sup_{(Y,z)\in\mathbb{Y}} \{g(Y, z)\}, \qquad \text{for some } (\bar{Y}, \bar{z}) \in \mathbb{Y} := \mathbb{B}_1 \times \mathcal{L}_2(\Omega, \mathcal{F}_T, P), \quad (4.5.143)$$

(compare with (4.5.35)), and to this end we would like to use an approach similar to that used to establish Proposition 4.5.15. Define the norm $\|\cdot\|_{\mathbb{U}}$ on the perturbation space $\mathbb{U}$ at (4.5.133) by

$$\|(u, v)\|_{\mathbb{U}} := \max\{\|u\|_2, \|v\|_2\}, \quad (u, v) \in \mathbb{U} = \mathcal{L}_2(\Omega, \mathcal{F}_T, P) \times \mathcal{L}_2(\Omega, \mathcal{F}_T, P), \quad (4.5.144)$$

(compare with (4.5.37)) and let $\mathcal{U}$ be the $\|\cdot\|_{\mathbb{U}}$-norm topology on $\mathbb{U}$. By an obvious modification of the argument used in the proof of Proposition 4.5.15 we can show that the locally convex topology $\mathcal{U}$ on $\mathbb{U}$ is $\langle\mathbb{U}, \mathbb{Y}\rangle$-compatible, and it therefore remains to verify condition (3.1.9) in order to use Theorem 3.1.7. That is, we must establish that there exists some $\pi_1 \in \Pi$ and $\alpha \in (0, \infty)$ such that

$$\sup\{F(\pi_1, (u, v)) \mid (u, v) \in \mathbb{U} \text{ with } \|(u, v)\|_{\mathbb{U}} < \alpha\} < +\infty, \quad (4.5.145)$$

(with $\mathbb{U}$ and $\|\cdot\|_{\mathbb{U}}$ given by (4.5.133) and (4.5.144) respectively, and $F$ being defined by (4.5.134)). Thus, fix some arbitrary $\pi_1 \in \Pi$. Since $X^{\pi_1}(T) - b \in \mathcal{L}_2(\Omega, \mathcal{F}_T, P)$, from elementary measure theory it is clear that

$$\begin{cases} \text{for each (small) } \alpha > 0 \text{ there is some } v \in \mathcal{L}_2(\Omega, \mathcal{F}_T, P) \text{ such} \\ \text{that } \|v\|_2 < \alpha \text{ and } P\{v < b - X^{\pi_1}(T)\} > 0. \end{cases} \quad (4.5.146)$$

In view of (4.5.146) and (4.5.134) we obtain

$$\sup_{\|v\|_2<\alpha} F(\pi_1, (u, v)) = \infty \qquad \text{for each } \alpha > 0 \text{ and } u \in \mathcal{L}_2(\Omega, \mathcal{F}_T, P), \quad (4.5.147)$$

and it follows from (4.5.147) and (4.5.144) that

$$\sup\{F(\pi_1, (u, v)) \mid (u, v) \in \mathbb{U} \text{ with } \|(u, v)\|_{\mathbb{U}} < \alpha\} = +\infty, \quad (4.5.148)$$

for all $\pi_1 \in \Pi$ and $\alpha \in (0, \infty)$. This means that, with the definitions (4.5.133)- (4.5.137), we can *never* verify condition (3.1.9) associated with Theorem 3.1.7. In direct contrast, when we use the perturbation space $\mathbb{U}$ defined at (4.5.18), then, as is clear from the proof of Proposition 4.5.15, we can use Condition 4.5.5 as the means to verify (3.1.9). What the preceding calculation really indicates is that the space of dual variables at (4.5.136), which is a consequence of the perturbation space at (4.5.133), is *too small* to include any solution of the dual problem at (4.5.143), whereas we know from Proposition 4.5.15 that the *larger* space of dual variables at (4.5.21) (recall Remark 4.5.12) resulting from the perturbation space at (4.5.18) does include a solution of the dual problem (4.5.35). Much the same message is in fact already clear from the very simple Problem 3.3.12: If we had used the perturbation space $\mathbb{U} := \mathcal{L}_2(\Omega, \mathcal{F}, P)$ (in place of (3.3.47)), with corresponding space of dual variables $\mathbb{Y} := \mathcal{L}_2(\Omega, \mathcal{F}, P)$ (in place of (3.3.50)), then the dual solution (or Lagrange multiplier) would necessarily have to "live" in the space of dual variables $\mathcal{L}_2(\Omega, \mathcal{F}, P)$, whereas we know from Remark 3.3.18 that the dual solution is in fact a *singular* element in the larger space $\mathcal{L}^*_\infty(\Omega, \mathcal{F}, P)$.

## 4.6 Conclusion

In this chapter we have applied the Rockafellar variational approach, as outlined in Section 3.1, to study the quadratic minimization problem (2.2.2) with a variety of constraints on the portfolio and on the wealth at close of trade. In particular, in Section 4.2 we address the basic problem (2.2.2) with no constraints on the portfolio or the wealth, that is we study problem (4.2.1). Then, in Section 4.3, we add a convex portfolio constraint to the unconstrained problem (4.2.1), resulting in the problem (4.3.2), while in Section 4.4 we add a European wealth constraint to the basic unconstrained problem (4.2.1), resulting in the problem (4.4.2). We note that these problems have all been previously addressed by a variety of problem-specific methods. In particular, Lim and Zhou [20] address the unconstrained problem (4.2.1) by a classical stochastic linear quadratic approach which relies in an essential way on the absence of constraints and involves the construction of a Riccati equation in the form of a BSDE (recall Remark 4.2.12). Likewise, problem (4.3.2) with convex portfolio constraints is addressed by Labbé and Heunis [18] by a powerful convex stochastic calculus of variations approach of Bismut [4] which nevertheless relies on the absence of constraints on the wealth process and seemingly cannot be extended to include such constraints (recall Remark 4.3.23). Again, a variant of problem (4.4.2) is addressed by Bielecki *et al.* [3] by an ingenious but highly problem-specific adaptation of the risk-neutral method which relies in an essential way on the absence of portfolio constraints. Finally, in Section 4.5, we address problem (4.5.2) which involves an amalgam of the constraints in problem (4.3.2) and problem (4.4.2). None of the special methods and approaches in the works [20], [18] and [3] discussed above seems to extend to problem (4.5.2), but, as is clearly demonstrated in Section 4.5, the Rockafellar variational approach does carry over to this problem. Furthermore, the variational approach constitutes a *unified method* for all of the problems addressed in this chapter. Indeed, this basic approach has been applied in the same way for each of the problems (4.2.1), (4.3.2), (4.4.2) and (4.5.2), with progressive but clear generalization for the increasingly complex structure of the constraints. This is particularly evident in the choice of the space of perturbations and the perturbation functions in each case: this is evident from equations (4.2.7) - (4.2.8) for problem (4.2.1), equations (4.3.8) - (4.3.9) for problem (4.3.2), equations (4.4.23) - (4.4.24) for problem (4.4.2), and equations (4.5.18) - (4.5.19) for problem (4.5.2). Finally, it is worthwhile to note that the problems addressed in the present chapter have natural finite-dimensional precursors. In particular, problem (3.2.3) is a static and finite-dimensional precursor of the stochastic control problem (4.3.2) (see Remark 4.3.2), while problem (3.2.34) is a static and finite dimensional precursor of the stochastic control problem (4.4.2) (see Remark 4.4.1), and problem (3.2.64) is a static and finite dimensional precursor of the stochastic control problem (4.5.2) (see Remark 4.5.2). These finite dimensional precursors are extremely useful for obtaining in the simplest possible context the outlines of a general “road-map” to be followed when addressing the actual corresponding stochastic control problems, which of course involve much more technical effort.

# Chapter 5

# Quadratic Risk Minimization with Portfolio and American Wealth Constraints

## 5.1 Introduction

In the previous chapters of this thesis, we have seen that the Rockafellar variational approach is a powerful tool for dealing with general problems of convex optimization. In Chapter 4, we applied this approach to several problems of quadratic risk minimization subject to various constraints. In particular, in Section 4.5 we addressed a problem which involved a combination of convex portfolio constraints and an almost-sure constraint on the wealth at close of trade, that is a European constraint on the wealth process. This problem amounts to a stochastic optimal control problem with a combination of a control constraint (that is the portfolio constraint $\pi \in \mathcal{A}$ at (4.5.2)) together with an almost-sure state constraint (in the form of the stipulated lower bound on terminal wealth $X^{\pi}(T) \geqslant b$ at (4.5.2)).

In the present chapter we address the canonical problem (2.2.21) , which formally resembles (4.5.2), except that we now stipulate an almost-sure lower bound on the wealth process over the *entire interval* $0 \leqslant t \leqslant T$ in place of the constraint $X^{\pi}(T) \geqslant b$ stipulated at close of trade in problem (4.5.2). We shall suppose that $\{B(t); t \in [0, T]\}$ is a given continuous $\mathcal{F}_t$-adapted process which represents the allowed lower bound on wealth over the trading interval, that is we shall address an *American wealth constraint*, which is an almost-sure state constraint of the form (2.2.13), namely the portfolio process $\pi$ must satisfy

$$X^{\pi}(t) \geqslant B(t), \quad t \in [0, T], \text{ a.s.} \tag{5.1.1}$$

The problem that we are going to address is therefore a stochastic optimal control problem which involves both a control constraint together with an almost-sure state constraint over the entire control interval. Such problems are well known to be extremely challenging.

Some results on deterministic optimal control problems have been established for such constraints, typically necessary conditions for optimality in the form of Pontryagin-type "maximum principles" (see Dubovitskii and Mil'yutin [9], Makowski and Neustadt [22], and Neustadt [26]). Of particular interest are necessary and sufficient conditions for optimality in *convex* deterministic optimal control problems with control and state constraints over the entire control interval established by Rockafellar [30]. However, with the exception of the partial results of Mnif and Pham (see Remark 2.2.11) we are not aware of any results for stochastic optimal control problems with constraints of this kind. As was the case in Chapter 4, we shall exploit the inherently convex character of the problem, arising from the quadratic objective function (see (2.2.3)) and linear dynamics (see (2.1.21)), to apply the Rockafellar variational approach in a way that is formally very similar to the use of this approach for problem (4.5.2). However, as we shall see, the constraint on the state process $X^\pi$ at (5.1.1) stipulated over the entire interval $[0, T]$ presents significantly greater technical challenges than the corresponding constraint $X^\pi(T) \geqslant b$ stipulated only at close of trade in problem (4.5.2).

We next define precisely the problem to be addressed in the present chapter. Exactly as in Section 4.3 and Section 4.5 we are given a non-empty convex closed set $A \subset \mathbb{R}^N$ (recall (2.2.7) and (2.2.8)) satisfying Condition 2.2.4, and we define the set of *constrained* or *regulated* portfolio processes in the usual way namely

$$\mathcal{A} := \{\pi \in \Pi \mid \pi(t) \in A, \text{ a.e.}\}. \tag{5.1.2}$$

We next put the following condition on the so-called "floor-level" of wealth at the American-type state constraint (5.1.1):

**Condition 5.1.1.** The floor level process $B = \{B(t); t \in [0, T]\}$ at (5.1.1) is a given $\mathcal{F}_t$-adapted uniformly essentially bounded continuous process, that is $B : \Omega \to \mathcal{C}[0, T]$ such that $B \in \mathcal{F}^*$ and

$$\hat{b} := \text{P-ess-sup}_{\omega\in\Omega} \sup_{t\in[0,T]} |B(\omega; t)| \in (0, \infty). \tag{5.1.3}$$

The problem to be addressed in this chapter is then

$$\text{minimize } E\left[J(X^\pi(T))\right] \quad \text{such that } \pi \in \mathcal{A} \text{ and } X^\pi(t) \geqslant B(t),\ t \in [0, T] \text{ a.s.}, \tag{5.1.4}$$

in which the risk criterion $J$ satisfies the usual conditions (see (2.2.3) and Condition 2.2.1).

**Remark 5.1.2.** The *admissible wealth* condition $X^\pi(t) \geqslant 0$, $t \in [0, T]$, is clearly a very special case of the general *American wealth constraint* at (5.1.1), with the floor level process $B$ defined by $B := 0$, and in this case problem (5.1.4) simplifies to

$$\text{minimize } E\left[J(X^\pi(T))\right] \quad \text{such that } \pi \in \mathcal{A} \text{ and } X^\pi(t) \geqslant 0,\ t \in [0, T] \text{ a.s.} \tag{5.1.5}$$

Problem (5.1.5) is particularly simple in that the solution of this problem just amounts to the solution of problem (4.5.2) with $b := 0$. To see this suppose that $\bar{\pi}$ is a solution of problem (4.5.2) with $b := 0$, that is

$$\begin{cases} \text{(i) } \bar{\pi} \in \mathcal{A}, \qquad \text{(ii) } X^{\bar{\pi}}(T) \geqslant 0, \\ \text{(iii) } E\left[J(X^{\bar{\pi}}(T))\right] = \vartheta_1 := \inf_{\pi\in\mathcal{A}} \left\{ E\left[J(X^\pi(T))\right] \,\middle|\, X^\pi(T) \geqslant 0 \text{ a.s.} \right\}. \end{cases} \tag{5.1.6}$$

Of course we also have

$$\vartheta_2 := \inf_{\pi\in\mathcal{A}} \left\{ E\left[J(X^\pi(T))\right] \,\middle|\, X^\pi(t) \geqslant 0,\; t\in[0,T] \text{ a.s.} \right\} \geqslant \vartheta_1, \tag{5.1.7}$$

and it follows from (5.1.6)(iii) and (5.1.7) that

$$\vartheta_2 \geqslant \vartheta_1 = E\left[J(X^{\bar\pi}(T))\right]. \tag{5.1.8}$$

Moreover, from (5.1.6)(ii), together with Proposition 2.1.13 and $H > 0$ (see Remark 2.1.7 (3)), we find

$$X^{\bar\pi}(t) = H^{-1}(t)E\left[H(T)X^{\bar\pi}(T) \,\middle|\, \mathcal{F}_t\right] \geqslant 0, \qquad \text{for all } t\in[0,T], \tag{5.1.9}$$

and, from (5.1.9), the definition of $\vartheta_2$ at (5.1.7), and (5.1.6)(i), we have

$$E\left[J(X^{\bar\pi}(T))\right] \geqslant \vartheta_2. \tag{5.1.10}$$

Upon combining (5.1.10), (5.1.8), (5.1.9), and (5.1.6)(i) we obtain

$$\begin{cases} \text{(i) } \bar\pi\in\mathcal{A}, \qquad \text{(ii) } X^{\bar\pi}(t)\geqslant 0, \qquad \text{for all } t\in[0,T], \\ \text{(iii) } E\left[J(X^{\bar\pi}(T))\right] = \vartheta_2. \end{cases} \tag{5.1.11}$$

We then see from (5.1.11) and (5.1.7) that, if $\bar\pi$ is the optimal portfolio for problem (4.5.2) with $b := 0$, then $\bar\pi$ is also the optimal portfolio for problem (5.1.5). We conclude that solution of problem (5.1.5) just boils down to the solution of problem (4.5.2) with $b := 0$.

**Remark 5.1.3.** Bielecki *et-al* [3] use the reduction in Remark 5.1.2 to address problem (5.1.5) in the special case where $\mathcal{A} = \Pi$ (i.e. without portfolio constraints).

**Remark 5.1.4.** We can generalize the idea in Remark 5.1.2 to the case where the floor level wealth process in problem (5.1.4) is defined by

$$B(t) := H^{-1}(t)E\left[H(T)b \,\middle|\, \mathcal{F}_t\right], \qquad t\in[0,T], \tag{5.1.12}$$

in which $b\in\mathcal{L}_\infty(\Omega,\mathcal{F}_T,P)$ is some given random variable. Indeed we have the following elementary result:

**Proposition 5.1.5.** *Given a real-valued random variable $b\in\mathcal{L}_\infty(\Omega,\mathcal{F}_T,P)$, define the floor level process $B$ at (5.1.12). If $\pi\in\Pi$ is such that $X^\pi(T)\geqslant b$ a.s., then $X^\pi(t)\geqslant B(t)$ for all $t\in[0,T]$ a.s..*

**Proof.** Since $X^\pi(T)\geqslant b$, from Remark 2.1.7 (3) we have

$$H^{-1}(t)E\left[H(T)(X^\pi(T)-b) \,\middle|\, \mathcal{F}_t\right] \geqslant 0, \quad \text{for all} \quad t\in[0,T], \tag{5.1.13}$$

and then it follows from Proposition 2.1.13 that

$$X^\pi(t) = H^{-1}(t)E\left[H(T)X^\pi(T) \,\middle|\, \mathcal{F}_t\right] \overset{(5.1.13)}{\geqslant} H^{-1}(t)E\left[H(T)b \,\middle|\, \mathcal{F}_t\right] \overset{(5.1.12)}{=} B(t), \tag{5.1.14}$$

for all $t\in[0,T]$. ☐

From Proposition 5.1.5, together with an argument identical to that used for Remark 5.1.2, we conclude the following: if $\bar\pi$ is an optimal portfolio for problem (4.5.2) then $\bar\pi$ is also an optimal portfolio for problem (5.1.4) with the floor level process defined by (5.1.12).

**Remark 5.1.6.** With the floor level defined at (5.1.12) in terms of some specified random variable $b \in \mathcal{L}_\infty(\Omega, \mathcal{F}_T, P)$, the state process (2.1.21) is such that the constraint (5.1.1) in problem (5.1.4) "binds" (at optimality) only at the close of trade $t = T$, and accordingly the Lagrange multiplier for the constraint $X^\pi(T) \geqslant b$ in problem (4.5.2) serves also for the Lagrange multiplier for the constraint (5.1.1) in problem (5.1.4). For general floor level processes $B$, not having the special form in Remark 5.1.4, one will of course not have this very favourable state of affairs, and the constraint (5.1.1) can bind at intermediate instants in the interval $[0, T]$. One of the main challenges in addressing problem (5.1.1) is to construct a space of dual variables which is large enough to contain Lagrange multipliers which enforce such binding at intermediate times.

## 5.2 The Canonical Problem - Combination of Portfolio and American Wealth Constraints: Part I

**Remark 5.2.1.** In the present section we shall establish a *tentative* approach to the main problem of this thesis, namely the canonical problem (2.2.21), which we repeat here for convenience:

$$\text{minimize } E\left[J(X^\pi(T))\right] \quad \text{such that } \pi \in \mathcal{A} \text{ and } X^\pi(t) \geqslant B(t),\ t \in [0,T] \text{ a.s.} \tag{5.2.1}$$

Our approach is tentative in the sense that we do not completely solve the problem (5.2.1) above, but what we learn from this tentative approach turns out to be an essential step towards the following Section 5.3, in which we do solve (5.2.1) completely. The present section is therefore a sort of preliminary *mathematical experiment*, which will illustrate, among other matters, just how useful the Rockafellar variational approach of Section 3.1 can be as a tool for experimenting with choices of perturbation spaces $\mathbb{U}$, and the effect that the choice of perturbation space has on the space of dual variables $\mathbb{Y}$ and the dual function $g$. Specifically, in this section we shall use a *square integrable* perturbation for the American Wealth constraint $X^\pi \geqslant B$ when we implement Step 3.1.1 of the Rockafellar variational approach. With this perturbation we shall construct a space of dual variables, a Lagrangian and a dual function, as well as Kuhn-Tucker optimality relations, much as we have done throughout Chapter 4. However, with this perturbation we shall *not* be able to establish existence of a solution of the dual problem. We shall nevertheless *assume* existence of a dual solution of the dual problem, and we shall then construct an optimal portfolio of problem (5.2.1) in terms of the assumed dual solution. The situation in this section is therefore rather similar to that in Remark 4.5.32 where we discussed a square integrable perturbation for the constraint on the wealth at close of trade (see the second factor on the right hand side of (4.5.133)). In view of the non-existence of solutions of the dual problem when square integrable perturbations are used, one may well question why we even experiment with these perturbations. In fact, in this section we shall gain valuable insight into several aspects of the mechanics of dealing with American type wealth constraints, which will stand us in good stead in the following Section 5.3 when we use the technically more demanding *essentially bounded* perturbations to overcome the problem of non-existence of a dual solution. Thus the present section is a sort of "halfway-house" on the way to the more complete solution of problem (5.2.1) that we shall give in Section 5.3.

Our goal is therefore to determine an optimal portfolio process $\bar{\pi} \in \mathcal{A}$, such that

$$E\left[J(X^{\bar{\pi}}(T))\right] = \inf_{\pi \in \mathcal{A}} \left\{ E\left[J(X^{\pi}(T))\right] \,\middle|\, X^{\pi}(t) \geqslant B(t), t \in [0,T] \text{ a.s.} \right\}, \tag{5.2.2}$$

(recall (4.1.1) and (4.1.19)). Of course, for problem (5.2.1) to make sense we must assume

**Condition 5.2.2.** There exists some $\hat{\pi} \in \mathcal{A}$ such that $X^{\hat{\pi}}(t) \geqslant B(t)$ for all $t \in [0,T]$ a.s.

Following the general approach established in Chapter 4 we shall write problem (5.2.1) in the form of a *primal problem* over the vector space of primal variables

$$\mathbb{X} := \Pi, \tag{5.2.3}$$

as follows (see (4.1.1)):

**Problem 5.2.3.** Determine an optimal portfolio process $\bar{\pi} \in \Pi$ such that

$$f(\bar{\pi}) = \inf_{\pi \in \Pi} f(\pi), \tag{5.2.4}$$

where the *primal function* $f(\cdot) : \Pi \to (-\infty, +\infty]$ is defined as

$$f(\pi) := \begin{cases} E\left[J(X^{\pi}(T))\right], & \text{for } \pi \in \mathcal{A} \text{ and } X^{\pi}(t) \geqslant B(t), t \in [0,T] \text{ a.s.}, \\ +\infty, & \text{otherwise,} \end{cases} \tag{5.2.5}$$

and $J$ is given in (4.1.19) and subject to Condition 2.2.1.

To implement Step 3.1.1 when we have an American wealth constraint $X^{\pi} \geqslant B$, we shall need a space of perturbations $\mathbb{U}$ which plays a role rather similar to that of the space of perturbations at (4.5.18) for the problem with a European wealth constraint $X^{\pi}(T) \geqslant b$ a.s. To define this space of perturbations $\mathbb{U}$, we shall need to formulate some spaces of stochastic processes defined over the interval $[0,T]$ as follows:

**Notation 5.2.4.** Denote by $\mathcal{C}[0,T]$ the vector space of all $\mathbb{R}$-valued *continuous functions* over $[0,T]$ with the *uniform norm* $\|\cdot\|_u$ (see (E.1) for the full definition). Similarly denote by $\mathcal{BV}_0^r[0,T]$ the vector space of all $\mathbb{R}$-valued *right continuous bounded variation functions* over $[0,T]$ with the *total variational norm* $\|\cdot\|_T$ (see (E.9) and (E.10) for the full definitions).

(1) The vector space of square integrable continuous and $\mathcal{F}_t$-adapted processes is denoted by $\mathcal{L}_2(\{\mathcal{F}_t\};\mathcal{C})$ and defined by (recall Notation 2.1.4(2))

$$\mathcal{L}_2(\{\mathcal{F}_t\};\mathcal{C}) := \left\{ V : \Omega \to \mathcal{C}[0,T] \,\middle|\, V \in \mathcal{F}^* \text{ and } E\left[\|V\|_u^2\right] < \infty \right\}. \tag{5.2.6}$$

Thus, members of $\mathcal{L}_2(\{\mathcal{F}_t\};\mathcal{C})$ are processes $V$ such that

$$\|V(\omega)\|_u := \sup_{t \in [0,T]} |V(\omega;t)|, \tag{5.2.7}$$

is a *square-integrable* function of $\omega$. A more detailed notation for $\mathcal{L}_2(\{\mathcal{F}_t\};\mathcal{C})$ might be $\mathcal{L}_2((\Omega,\mathcal{F}_T,\{\mathcal{F}_t\},P);\mathcal{C})$, but this notation is much too cumbersome for us to use. We recall that the notation $V \in \mathcal{F}^*$ at (5.2.6) indicates that the mapping $V : \Omega \times [0,T] \to \mathbb{R}$ is $\mathcal{F}^*$-measurable, that is the process $\{V(t), t \in [0,T]\}$ is $\mathcal{F}_t$-progressively measurable (see Notation 2.1.4-(2)).

We introduce an *order* "$\geqslant$" ("$\leqslant$") on $\mathcal{L}_2(\{\mathcal{F}_t\};\mathcal{C})$ by writing

$$V \geqslant 0 \text{ iff } V(t) \geqslant 0 \text{ for all } t \in [0,T] \text{ a.s.,} \quad \text{and } V_1 \leqslant V_2 \text{ iff } V_2 - V_1 \geqslant 0. \tag{5.2.8}$$

In particular, denote

$$(\mathcal{L}_2(\{\mathcal{F}_t\};\mathcal{C}))^+ := \{V \in \mathcal{L}_2(\{\mathcal{F}_t\};\mathcal{C}) \mid V \geqslant 0\}\,. \tag{5.2.9}$$

Recall from Condition 5.1.1 and (2.1.24) that

$$B, X^\pi \in \mathcal{L}_2(\{\mathcal{F}_t\};\mathcal{C}), \tag{5.2.10}$$

and the *American wealth constraints* of (5.1.1) will usually be written in the abbreviated form

$$X^\pi \geqslant B. \tag{5.2.11}$$

(2) The vector space of square integrable right continuous and $\mathcal{F}_t$-adapted bounded variation processes is denoted by $\mathcal{L}_2(\{\mathcal{F}_t\};\mathcal{BV}_0^r)$ and defined by

$$\mathcal{L}_2(\{\mathcal{F}_t\};\mathcal{BV}_0^r) := \left\{\rho : \Omega \to \mathcal{BV}_0^r[0,T] \,\middle|\, \rho \in \mathcal{F}^* \text{ and } E\left[\|\rho\|_T^2\right] < \infty\right\}, \tag{5.2.12}$$

where the *total variation norm* $\|\cdot\|_T$ on $\mathcal{BV}_0^r[0,T]$ is given by (E.7). Thus, members $\rho$ of $\mathcal{L}_2(\{\mathcal{F}_t\};\mathcal{BV}_0^r)$ are such that the total variation $\|\rho(\omega)\|_T$ of the bounded-variation function $t \to \rho(\omega,t) : [0,T] \to \mathbb{R}$ is a *square-integrable* function of $\omega$.

We also introduce an *order* "$\geqslant$" ("$\leqslant$") on $\mathcal{L}_2(\{\mathcal{F}_t\};\mathcal{BV}_0^r)$ by writing

$$\rho \geqslant 0 \qquad \text{iff} \qquad E\left[\int_0^T V(t)\rho(\mathrm{d}t)\right] \geqslant 0 \quad \text{for all } V \in (\mathcal{L}_2(\{\mathcal{F}_t\};\mathcal{C}))^+. \tag{5.2.13}$$

**Remark 5.2.5.** Recall from Remark E.0.2 that $\int_0^T V(\omega;t)\rho(\omega;\mathrm{d}t)$ is given as a *Lebesgue-Stieltjes* integral for each $\omega \in \Omega$, therefore $\int_0^T V(t)\rho(\mathrm{d}t)$ is a well defined real-valued $\mathcal{F}_T$-measurable random variable for all $(V,\rho) \in \mathcal{L}_2(\{\mathcal{F}_t\};\mathcal{C}) \times \mathcal{L}_2(\{\mathcal{F}_t\};\mathcal{BV}_0^r)$. Furthermore, from (E.15) and Holder's Inequality (recall (E.7) for definition of $\|\cdot\|_T$), we have

$$E\left[\left|\int_0^T V(t)\rho(\mathrm{d}t)\right|\right] \leqslant E\left[\|V\|_u \cdot \|\rho\|_T\right] \leqslant E\left[\|V\|_u^2\right]^{\frac{1}{2}} E\left[\|\rho\|_T^2\right]^{\frac{1}{2}} < \infty, \tag{5.2.14}$$

for all $(V,\rho) \in \mathcal{L}_2(\{\mathcal{F}_t\};\mathcal{C}) \times \mathcal{L}_2(\{\mathcal{F}_t\};\mathcal{BV}_0^r)$.

**Example 5.2.6.** Denote by $\gamma_\tau : \mathfrak{B}([0,T]) \to \{0,1\}$ the Dirac measure for some $\tau \in [0,T]$, that is

$$\gamma_\tau(A) := \begin{cases} 1 & \text{if } \tau \in A, \\ 0 & \text{if } \tau \in A^c, \end{cases} \qquad \text{for any measurable set } A \in \mathfrak{B}([0,T]), \tag{5.2.15}$$

where $\mathfrak{B}([0,T])$ is the Borel $\sigma$-algebra on $[0,T]$ (see Notation 2.1.4 - (3)). Fix some $\pi \in \Pi$ and $\tau \in [0,T]$. Recalling Condition 5.1.1, we define a mapping $\rho^\pi_\tau : \Omega \to \mathcal{BV}^r_0[0,T]$ as:

$$\rho^\pi_\tau(\omega;t) := \int_0^t \mathbb{1}_{\{X^\pi<B\}}(\omega;s)\gamma_\tau(\mathrm{d}s) = \begin{cases} 0 & \text{if } 0 \leqslant t < \tau,\ \omega \in \Omega, \\ \mathbb{1}_{\{X^\pi<B\}}(\omega;\tau) & \text{if } \tau \leqslant t \leqslant T,\ \omega \in \Omega, \end{cases} \tag{5.2.16}$$

where $\mathbb{1}_{\{X^\pi<B\}}$ is the indicator function for a set $\{(\omega;s) \in \Omega \times [0,T] \mid X^\pi(\omega;s) < B(\omega;s)\}$ on $\Omega \times [0,T]$ (recall (3.3.21)), i.e.

$$\mathbb{1}_{\{X^\pi<B\}}(\omega;s) := \begin{cases} 1 & \text{if } X^\pi(\omega;s) < B(\omega;s), \\ 0 & \text{if } X^\pi(\omega;s) \geqslant B(\omega;s). \end{cases} \tag{5.2.17}$$

From (5.2.16), it follows

$$E\left[\|\rho^\pi_\tau\|^2_T\right] = E\left[\mathbb{1}_{\{X^\pi<B\}}(\tau)\right] \leqslant 1, \tag{5.2.18}$$

and then (recall (5.2.12))

$$\rho^\pi_\tau \in \mathcal{L}_2(\{\mathcal{F}_t\}; \mathcal{BV}^r_0). \tag{5.2.19}$$

Moreover, because

$$E\left[\int_0^T V(t)\rho^\pi_\tau(\mathrm{d}t)\right] \overset{(5.2.16)}{=} E\left[V(\tau)\mathbb{1}_{\{X^\pi<B\}}(\tau)\right] \geqslant 0, \quad \text{all } V \in (\mathcal{L}_2(\{\mathcal{F}_t\};\mathcal{C}))^+, \tag{5.2.20}$$

we have (recall (5.2.13))

$$\rho^\pi_\tau \geqslant 0. \tag{5.2.21}$$

□

**Remark 5.2.7.** Recall from Remark 4.1.7 that

$$Y \in \mathbb{B}_1 \iff Y(t) := \frac{1}{S_0(t)}\left\{Y(0) + \int_0^t S_0(s)\Lambda'_Y(s)\mathrm{d}W(s)\right\} \quad t \in [0,T], \tag{5.2.22}$$

for some unique pair $(Y(0), \Lambda_Y) \in \mathbb{R} \times \Pi$. That is, we could have defined $\mathbb{B}_1$ to be the vector space of all processes $Y$ satisfying the identity on the right side of (5.2.22), in other words we can just as well define

$$\mathbb{B}_1 := \left\{\Xi(y,\gamma) \,\middle|\, (y,\gamma) \in \mathbb{R} \times \Pi\right\}. \tag{5.2.23}$$

We must now formulate a space of *dual processes* $Y$ which "extend" the dual processes $Y$ in $\mathbb{B}_1$ in such a way as to account for the state constraint represented by the American wealth constraint $X^\pi \geqslant B$. In the work of Rockafellar [30] which is devoted to deterministic convex optimal control problems, one learns that the way to deal with state constraints over the control interval $[0,T]$ is to add a function of bounded variation over the interval $[0,T]$ to the ordinary differential equation which defines the dual arcs without state constraints (see (6.9) and (6.10) in [30]). For the optimal control problems of this thesis, the natural analogues of the dual arcs without state constraints are processes $Y \in \mathbb{B}_1$, that is processes $Y$ defined by

$$\mathrm{d}Y(t) = -r(t)Y(t)\mathrm{d}t + \gamma'(t)\mathrm{d}W(t), \quad Y(0) = y \in \mathbb{R}, \tag{5.2.24}$$

for some $\gamma \in \Pi$. Motivated by this approach of Rockafellar [30], we are going to modify the dynamics of $Y$ given by (5.2.24) as follows:

$$\mathrm{d}Y(t) = -r(t)Y(t)\mathrm{d}t + \gamma'(t)\mathrm{d}W(t) + \varrho(\mathrm{d}t), \quad Y(0) = y \in \mathbb{R}, \tag{5.2.25}$$

in which $\varrho$ is a "driving process" with paths of bounded variation over the interval $[0,T]$, the exact properties of which we shall now formulate. In the deterministic problems of optimal control addressed in Rockafellar [30], there is of course no term corresponding to the $\mathrm{d}W(t)$ term in (5.2.25), that is the (non-random) dual arcs are effectively of the form

$$\mathrm{d}Y(t) = -r(t)Y(t)\mathrm{d}t + \varrho(\mathrm{d}t), \quad Y(0) = y \in \mathbb{R}, \tag{5.2.26}$$

in which $\varrho \in \mathcal{BV}_0^r[0,T]$ is a non-random "driving term" of bounded variation over the control interval $[0,T]$, that is the dual arcs are "parametrized" by $y \in \mathbb{R}$ and $\varrho \in \mathcal{BV}_0^r[0,T]$, and given by (5.2.26). In our stochastic problem, for equation (5.2.25) to make sense, the third term $\varrho$ must be an adapted process with paths of bounded variation over $[0,T]$ for each $\omega \in \Omega$. Furthermore, in our square integrable setting, we are going to require that the total variation of $\rho$ over the interval $[0,T]$ be square integrable as a function of $\omega$, that is we are going to take $\varrho \in \mathcal{L}_2(\{\mathcal{F}_t\}; \mathcal{BV}_0^r)$ in the equation of (5.2.25). It follows from (5.2.25) and Itô's formula (see Proposition 8.19 of R. Cont and P. Tankov [6]) that the process $Y$ which satisfies (5.2.25) is given by

$$Y(t) := \frac{1}{S_0(t)}\left\{y + \int_0^t S_0(\tau)\gamma'(\tau)\mathrm{d}W(\tau) + \int_0^t S_0(s)\varrho(\mathrm{d}s)\right\}, \quad t \in [0,T]. \tag{5.2.27}$$

Motivated by (4.1.13) and (5.2.27), we are going to define

$$\widetilde{\Xi}(y,\gamma,\varrho)(t) := \frac{1}{S_0(t)}\left\{y + \int_0^t S_0(\tau)\gamma'(\tau)\mathrm{d}W(\tau) + \int_0^t S_0(s)\varrho(\mathrm{d}s)\right\}, \quad t \in [0,T], \tag{5.2.28}$$

for $(y,\gamma,\varrho) \in \mathbb{R} \times \Pi \times \mathcal{L}_2(\{\mathcal{F}_t\}; \mathcal{BV}_0^r)$, and motivated by (5.2.23) we define the set of dual processes as follows:

$$\mathbb{B}_2 := \left\{\widetilde{\Xi}(y,\gamma,\varrho) \;\middle|\; (y,\gamma,\varrho) \in \mathbb{R} \times \Pi \times \mathcal{L}_2(\{\mathcal{F}_t\}; \mathcal{BV}_0^r)\right\}. \tag{5.2.29}$$

We write $Y = (Y(0), \Lambda_Y, \varrho_Y) \in \mathbb{B}_2$ to indicate that $\{Y(t), t \in [0,T]\}$ is the $\mathbb{R}$-valued $\mathcal{F}_t$-adapted stochastic process given by (recall (2.1.2))

$$Y(t) := \frac{1}{S_0(t)} \left\{ Y(0) + \int_0^t S_0(s)\Lambda_Y'(s)\mathrm{d}W(s) + \int_0^t S_0(s)\varrho_Y(\mathrm{d}s) \right\} \quad t \in [0,T], \tag{5.2.30}$$

for some triplet $(Y(0), \Lambda_Y, \varrho_Y) \in \mathbb{R} \times \Pi \times \mathcal{L}_2(\{\mathcal{F}_t\}; \mathcal{BV}_0^r)$. By the following Lemma 5.2.8, the "integrand process" $\Lambda_Y \in \Pi$ and the "integrator process" $\varrho_Y \in \mathcal{L}_2(\{\mathcal{F}_t\}; \mathcal{BV}_0^r)$ in the representation (5.2.30) are a.e.-uniquely determined on $\Omega \times [0,T]$. That is, every triplet $(Y(0), \Lambda_Y, \varrho_Y)$ in $\mathbb{B}_2$ corresponds to an $\mathcal{F}_t$-adapted process $Y$ defined by (5.2.30), and the notation $Y \in \mathbb{B}_2$ indicates that $Y$ is an $\mathcal{F}_t$-adapted process defined by (5.2.30) for some $Y(0) \in \mathbb{R}$ and some (necessarily a.e.-unique) integrand $\Lambda_Y \in \Pi$ and some (necessarily a.e.-unique) integrator $\varrho_Y \in \mathcal{L}_2(\{\mathcal{F}_t\}; \mathcal{BV}_0^r)$. The elementary proof of Lemma 5.2.8 is included in Appendix A.

**Lemma 5.2.8.** *If $Y \in \mathbb{B}_2$, then there exists a.e.-unique a triplet $(Y_0, \Lambda_Y, \varrho_Y) \in \mathbb{R} \times \Pi \times \mathcal{L}_2(\{\mathcal{F}_t\}; \mathcal{BV}_0^r)$ such that (5.2.30) holds.*

**Remark 5.2.9.** In view of (5.2.30), we can turn $\mathbb{B}_2$ into a real vector space by defining a vector addition and scalar multiplication in $\mathbb{B}_2$ as follows: if $Y_i \in \mathbb{B}_2$ are given by $Y_i := \widetilde{\Xi}(y_i, \gamma_i, \varrho_i)$, $i = 1, 2$ and $c_1, c_2 \in \mathbb{R}$, define

$$c_1 Y_1 + c_2 Y_2 := \widetilde{\Xi}(c_1 y_1 + c_2 y_2, c_1\gamma_1 + c_2\gamma_2, c_1\varrho_1 + c_2\varrho_2) \in \mathbb{B}_2. \tag{5.2.31}$$

It is then immediately clear that

$$\widetilde{\Xi} : \mathbb{R} \times \Pi \times \mathcal{L}_2(\{\mathcal{F}_t\}; \mathcal{BV}_0^r) \to \mathbb{B}_2, \tag{5.2.32}$$

is a linear bijection (compare with (4.1.14)).

**Remark 5.2.10.** Comparing (5.2.30) with (4.1.12), we see $Y = (Y_0, \Lambda_Y, \varrho_Y) \in \mathbb{B}_2$ is an extension of $(Y(0), \Lambda_Y) \in \mathbb{B}_1$ with an additional entry of $\varrho_Y \in \mathcal{L}_2(\{\mathcal{F}_t\}; \mathcal{BV}_0^r)$. That is, for any fixed $\widetilde{Y} \in \mathbb{B}_1$ and $\varrho \in \mathcal{L}_2(\{\mathcal{F}_t\}; \mathcal{BV}_0^r)$, define

$$Y(t) := \widetilde{Y}(t) + \frac{1}{S_0(t)} \int_0^t S_0(s)\rho(\mathrm{d}s), \quad t \in [0,T], \tag{5.2.33}$$

and then $Y \in \mathbb{B}_2$. In particular, choosing $\varrho \equiv 0$ in (5.2.33), we have $\widetilde{Y} = Y \in \mathbb{B}_2$ for any $\widetilde{Y} \in \mathbb{B}_1$, i.e.

$$\mathbb{B}_1 \subset \mathbb{B}_2. \tag{5.2.34}$$

As a special example (recall Remark 4.1.7-(b)), the *state price density process* $H$ is also an element of $\mathbb{B}_2$ with $H(0) = 1$, $\Lambda_H = -H\theta$ and $\varrho_H = 0$, where $\theta$ is the *market price of risk*. It is immediate from (5.2.30) that $Y = (Y(0), \Lambda_Y, \varrho_Y) \in \mathbb{B}_2$ is given by the stochastic integral relation:

$$Y(t) \quad = \quad Y(0) - \int_0^t r(s)Y(s)\mathrm{d}s + \int_0^t \Lambda_Y'(s)\mathrm{d}W(s) + \varrho_Y(t) \qquad t \in [0,T]. \tag{5.2.35}$$

From Doob's maximal $L^2$-inequality we obtain the following result (see Appendix A for the proof):

**Proposition 5.2.11.** *If $Y \in \mathbb{B}_2$, then*

$$E\left[\max_{t\in[0,T]} |Y(t)|^2\right] < \infty. \tag{5.2.36}$$

Throughout Chapter 4 we made repeated use of Proposition I-1 of Bismut [4, p.387] (stated also as Proposition F.0.1). In order to deal with the American wealth constraints over the entire interval $[0, T]$ we need to slightly extend this result. By a very simple modification of the argument of Proposition I-1 of Bismut [4, p.387], we can establish the following result (see Appendix A for the elementary proof):

**Proposition 5.2.12.** *Given any $X \equiv (X(0), \dot{X}, \Lambda_X) \in \mathbb{B}$ and $Y \equiv (Y(0), \Lambda_Y, \varrho_Y) \in \mathbb{B}_2$ (recall Lemma 4.1.1 and Lemma 5.2.8), define*

$$\begin{aligned}\hat{M}(X,Y)(t) \;&:=\; X(t)Y(t) - X(0)Y(0) - \int_0^t X(s)\varrho_Y(ds) \\ &\quad - \int_0^t \left[\dot{X}(s)Y(s) - r(s)X(s)Y(s) + \Lambda'_X(s)\Lambda_Y(s)\right] ds,\end{aligned} \tag{5.2.37}$$

*for all $t \in [0, T]$. Then $\hat{M}(X, Y)$ is a $\mathcal{F}_t$-martingale with*

$$\hat{M}(X,Y)(0) = 0. \tag{5.2.38}$$

We shall follow the variational approach of Rockafellar outlined in Section 3.1 to construct a real vector space $\mathbb{U}$ of perturbations, a perturbation function $F(\cdot,\cdot)$, a real vector space $\mathbb{Y}$ of dual variables and a bilinear form on $\mathbb{U} \times \mathbb{Y}$, and then construct a Lagrangian function and a dual function. Since an *American wealth constraint* is an "extended version" of a *European wealth constraint*, we will be generally guided by the approach that we used in Chapter 4, but now taking into account the special structure of American wealth constraints.

1. By Step 3.1.1, we must fix a vector space of perturbations $\mathbb{U}$ and define a perturbation function $F$. We know from Chapter 4 that, whenever one has a European wealth constraint in place, then the perturbation space $\mathbb{U}$ is the *product* of the vector space $\mathcal{L}_2(\Omega, \mathcal{F}_T, P)$, containing elements $u \in \mathcal{L}_2(\Omega, \mathcal{F}_T, P)$ which contribute perturbations of the cost functional of the form $E[J(X^\pi(T) - u)]$, together with some vector space containing perturbations $v$ of the wealth constraint (see (4.4.23), (4.4.24), (4.5.18) and (4.5.19)). We are going to make a similar choice in the case of an American wealth constraint, but in this case the second of the two factor spaces must comprise *processes* rather than the *random variables* which sufficed for European wealth constraints. In the first instance, *as a learning exercise*, we fix this second factor space to be the vector space $\mathcal{L}_2(\{\mathcal{F}_t\}; \mathcal{C})$ of *square integrable* continuous adapted processes much as we did in Remark 4.5.32 for European wealth constraints. In this

way we postpone having to deal with the more mathematically challenging *essentially bounded* perturbations (which will turn out to be essential) until Section 5.3. We shall find that what we learn in the present section for square integrable perturbations will provide valuable guidance for tackling the more challenging case of essentially bounded perturbations in Section 5.3. We thus define the real vector space of perturbations as

$$\mathbb{U} := \mathcal{L}_2(\Omega, \mathcal{F}_T, P) \times \mathcal{L}_2(\{\mathcal{F}_t\}; \mathcal{C}), \tag{5.2.39}$$

and define the *perturbation function* $F : \Pi \times \mathbb{U} \to (-\infty, +\infty]$ as:

$$\begin{aligned} F(\pi, (u, V)) \;&:= \begin{cases} E\left[J(X^\pi(T) - u)\right], & \text{if } \pi \in \mathcal{A} \text{ and} \\ & X^\pi(t) + V(t) \geqslant B(t), t \in [0,T] \text{ a.s.}, \\ +\infty, & \text{otherwise,} \end{cases} \\ &\overset{(5.2.8)}{=} \begin{cases} E\left[J(X^\pi(T) - u)\right], & \text{if } \pi \in \mathcal{A} \text{ and } X^\pi + V \geqslant B, \\ +\infty, & \text{otherwise,} \end{cases} \end{aligned} \tag{5.2.40}$$

(compare with (4.5.133) and (4.5.134) for European wealth constraints). The convexity of $F$ on $\Pi \times \mathbb{U}$ follows since the mapping $\pi \to X^\pi(T) : \Pi \mapsto \mathcal{L}_2(\Omega, \mathcal{F}_T, P)$ is affine, together with (4.1.19) and (5.2.40). It is also apparent that the consistency relation (3.1.2) is satisfied (recall (5.2.5)), i.e.,

$$F(\pi, (0, \mathbf{0})) = f(\pi), \quad \text{for all } \pi \in \Pi. \tag{5.2.41}$$

**Remark 5.2.13.** Observe that problem (3.2.64) is a very simple finite dimensional precursor of the infinite dimensional stochastic control problem (5.2.1), and that the perturbation (5.2.40) (for problem (5.2.1)) is a clear generalization of the perturbation (3.2.72) (for the problem (3.2.64)).

**Remark 5.2.14.** We see from (5.2.39) and (5.2.40) that the American wealth constraint $X^\pi \geqslant B$ acting over the whole interval $[0, T]$ is perturbed by *processes* $V \in \mathcal{L}_2(\{\mathcal{F}_t\}; \mathcal{C})$, in much the same way that the European wealth constraint, acting only at the final instant $t = T$, is perturbed by $\mathcal{F}_T$-measurable *random variables* (see (4.5.133)). In view of what we learned in Section 4.5 it would seem more natural to use perturbation processes $V$ which are *essentially bounded* instead of square integrable, and we will see later (in Section 5.3) that this is indeed the case. For now, we shall see that the "simpler" choice of square integrable perturbing processes nevertheless provides insight which will be very useful in Section 5.3.

2. Following Step 3.1.2, we must pair the space of perturbations at (5.2.39) with a vector space $\mathbb{Y}$ through a bilinear form $\langle \cdot, \cdot \rangle$ on $\mathbb{U} \times \mathbb{Y}$. We accordingly define a real vector space $\mathbb{Y}$ of dual variables and a bilinear form $\langle \cdot, \cdot \rangle$ on $\mathbb{U} \times \mathbb{Y}$ by modifying the space of dual variables $\mathbb{B}_1 \times \mathcal{L}_2(\Omega, \mathcal{F}_T, P)$ in (4.5.136) and the bilinear form given in (4.5.137). Formally, we define the space of dual variables as

$$\mathbb{Y} := \mathbb{B}_2 \times \mathcal{L}_2(\{\mathcal{F}_t\}; \mathcal{BV}_0^r), \tag{5.2.42}$$

(see (5.2.29) and (5.2.12)) together with the bilinear form on $\mathbb{U} \times \mathbb{Y}$ given by

$$\langle (u, V), (Y, \rho) \rangle := E\left[ uY(T) + \int_0^T V(t)\rho(\mathrm{d}t) \right], \tag{5.2.43}$$

for $(u, V) \in \mathbb{U}$ and $(Y, \rho) \in \mathbb{Y}$. This completes all the choices required for Steps 3.1.1 - 3.1.2, and it remains to synthesize a Lagrangian and a dual function.

**Remark 5.2.15.** Comparing the space of dual variables at (5.2.42) with the space of dual variables at (4.5.136), we see again that the second factor space of square integrable random variables in $\mathcal{L}_2(\Omega, \mathcal{F}_T, P)$ is now replaced by a factor space comprising square integrable processes in $\mathcal{L}_2(\{\mathcal{F}_t\}; \mathcal{BV}_0^r)$. This of course is because the European wealth constraint is a constraint on the *terminal wealth random variable* $X^\pi(T)$, whereas the American wealth constraint is a constraint on the *whole wealth process* $X^\pi$. Moreover, the first factor space $\mathbb{B}_1$ of processes in (4.5.136) is replaced by the first factor space $\mathbb{B}_2$ of processes in (5.2.42). This reflects what we noted earlier in Remark 5.2.10 that $\mathbb{B}_2$ is an extension of $\mathbb{B}_1$.

**Remark 5.2.16.** Observe that elements of the space $\mathcal{L}_2(\{\mathcal{F}_t\}; \mathcal{BV}_0^r)$ occur *twice* in the dual variable $(Y, \rho) \in \mathbb{Y} := \mathbb{B}_2 \times \mathcal{L}_2(\{\mathcal{F}_t\}; \mathcal{BV}_0^r)$ (see (5.2.42)). Of course the second member $\rho$ of the pair $(Y, \rho)$ is an element in $\mathcal{L}_2(\{\mathcal{F}_t\}; \mathcal{BV}_0^r)$. In addition, associated with the first member $Y \in \mathbb{B}_2$ of the pair $(Y, \rho)$ is a unique element $\varrho_Y \in \mathcal{L}_2(\{\mathcal{F}_t\}; \mathcal{BV}_0^r)$ (recall Lemma 5.2.8). The dual variable $(Y, \rho)$ therefore involves the elements $\rho$ and $\varrho_Y$ in $\mathcal{L}_2(\{\mathcal{F}_t\}; \mathcal{BV}_0^r)$. It is important not to confuse $\rho$ and $\varrho_Y$.

3. According to Step 3.1.3, define (recall (3.1.3)) the *Lagrangian function* $K : \Pi \times \mathbb{Y} \to [-\infty, +\infty]$ as:

$$K(\pi, (Y, \rho)) := \inf_{(u,V) \in \mathbb{Y}} \{ \langle (u, V), (Y, \rho) \rangle + F(\pi, (u, V)) \}. \tag{5.2.44}$$

From (5.2.41) and (5.2.44), we have the basic inequality

$$f(\pi) \geqslant K(\pi, (Y, \rho)), \qquad \text{for all } (\pi, (Y, \rho)) \in \Pi \times \mathbb{Y}. \tag{5.2.45}$$

Now, from (5.2.44), (5.2.43) and (5.2.40), the Lagrangian can be evaluated as follows: for all $(\pi, (Y, \rho)) \in \Pi \times \mathbb{Y}$,

$$K(\pi, (Y, \rho)) \overset{(5.2.40)}{\underset{(5.2.43)}{=}} \begin{cases} \inf_{u \in \mathcal{L}_2(\Omega, \mathcal{F}_T, P)} \{ E[uY(T) + J(X^\pi(T) - u)] \} \\ \qquad + \inf_{V \in \mathcal{L}_2(\{\mathcal{F}_t\}; \mathcal{C})} \left\{ E\left[ \int_0^T V(t)\rho(\mathrm{d}t) \right] \middle| X^\pi + V \geqslant B \right\}, & \text{if } \pi \in \mathcal{A}, \\ +\infty, & \text{otherwise.} \end{cases} \tag{5.2.46}$$

From (2.1.24) and (5.2.36), we have

$$X^{\pi}(T), Y(T) \in \mathcal{L}_2(\Omega, \mathcal{F}_T, P) \quad \text{ for all } \pi \in \Pi \text{ and } Y \in \mathbb{B}_2. \tag{5.2.47}$$

Therefore, we can apply Proposition D.0.8 to the first term on the right side of (5.2.46) and get

$$\inf_{u \in \mathcal{L}_2(\Omega, \mathcal{F}_T, P)} \{E[uY(T) + J(X^{\pi}(T) - u)]\} = E[X^{\pi}(T)Y(T)] - E[J^*(Y(T))]. \tag{5.2.48}$$

Using (5.2.48), we get an explicit formula for the Lagrangian in the following proposition:

**Proposition 5.2.17.** *For each* $\pi \in \Pi$ *(see (2.1.20)) and* $(Y, \rho) \in \mathbb{Y} = \mathbb{B}_2 \times \mathcal{L}_2(\{\mathcal{F}_t\}; \mathcal{BV}_0^r)$ (see (5.2.42), (5.2.29) and (5.2.12)), *the Lagrangian* $K(\pi, (Y, \rho))$ *at* (5.2.44) *is given by* (recall (4.1.20), (5.1.2) and (5.2.13))

$$K(\pi, (Y, \rho)) = \begin{cases} E[X^{\pi}(T)Y(T)] - E[J^*(Y(T))] - E\left[\int_0^T [X^{\pi}(t) - B(t)]\rho(dt)\right] \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \textit{if } \pi \in \mathcal{A} \textit{ and } \rho \geqslant 0, \\ -\infty \qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \textit{if } \pi \in \mathcal{A} \textit{ and } \rho \not\geqslant 0, \\ +\infty \qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \textit{if } \pi \notin \mathcal{A}. \end{cases} \tag{5.2.49}$$

The proof of Proposition 5.2.17 is in Appendix A. In view of (3.1.4) and (5.2.49), the *dual function* $g : \mathbb{Y} \to [-\infty, \infty)$ is defined as:

$$g(Y, \rho) := \inf_{\pi \in \Pi} K(\pi, (Y, \rho)) = \begin{cases} -\varkappa(Y, \rho) - E[J^*(Y(T))] & \text{if } \rho \geqslant 0, \\ -\infty & \text{otherwise.} \end{cases} \tag{5.2.50}$$

in which we have defined

$$\varkappa(Y, \rho) := \sup_{\pi \in \mathcal{A}} \left\{-E[X^{\pi}(T)Y(T)] + E\left[\int_0^T [X^{\pi}(t) - B(t)]\,\rho(\mathrm{d}t)\right]\right\}, \tag{5.2.51}$$

(compare (5.2.51) with (4.5.142) for the case of European wealth constraint). The *weak duality* relation (c.f. (3.2.16)) holds from (5.2.45) and (5.2.50), that is:

$$f(\pi) \geqslant K(\pi, (Y, \rho)) \geqslant g(Y, \rho), \qquad \text{for all } (\pi, (Y, \rho)) \in \Pi \times \mathbb{Y}. \tag{5.2.52}$$

The *dual problem* is then to maximize $g(Y, \rho)$ over all $(Y, \rho) \in \mathbb{Y} = \mathbb{B}_2 \times \mathcal{L}_2(\{\mathcal{F}_t\}; \mathcal{BV}_0^r)$, that is, we must establish

$$g(\bar{Y}, \bar{\rho}) = \sup_{(Y, \rho) \in \mathbb{Y}} \{g(Y, \rho)\}, \quad \text{for some } (\bar{Y}, \bar{\rho}) \in \mathbb{Y}. \tag{5.2.53}$$

**Remark 5.2.18.** Upon comparing (5.2.50) with (4.5.141), we see that these dual functions have a very similar form. However, in the case of (5.2.50) the dual variable $Y$ is a member of $\mathbb{B}_2$, whereas in the case of (4.5.141) the dual variable $Y$ is a member of smaller space $\mathbb{B}_1$. Moreover, in the case of (5.2.50) the second of the dual variables, namely $\rho$, is a member of the space $\mathcal{L}_2(\{\mathcal{F}_t\}; \mathcal{BV}_0^r)$ of processes, whereas in the case of (4.5.141) the second of the dual variables, namely $z$, is a member of space $\mathcal{L}_2(\Omega, \mathcal{F}_T, P)$ of random variables. Furthermore, upon comparing (5.2.51) with (4.5.142), we see that the second term in braces of (5.2.51) is a Lagrange weighting of the American wealth constraint $X^\pi \geqslant B$ by the dual variable $\rho$, whereas the second term in braces of (4.5.142) is a Lagrange weighting of the European wealth constraint $X^\pi(T) \geqslant b$ by the dual variable $z$. Again, this reflects the fact that we need *processes* to enforce American wealth constraints over the whole trading interval, whereas we only need *random variables* to enforce European wealth constraints at the close of trade.

**Remark 5.2.19.** We would like to use the Rockafellar-Moreau Theorem 3.1.7 to establish that (5.2.53) holds. However, exactly as we saw in Remark 4.5.32 (see (4.5.144) to (4.5.148)), with square integrable perturbations in the second factor space at (5.2.39) we cannot verify the conditions of Theorem 3.1.7. In fact, in order to verify the conditions of Theorem 3.1.7, we are going to have to use *essentially bounded* perturbations in place of square integrable perturbations in the second factor space at (5.2.39). We shall take up the study of these essentially bounded perturbations in Section 5.3. In the meantime, **we are just going to assume that** (5.2.53) **holds**, and we are going to construct an optimal portfolio for the problem (5.2.1) in terms of the assumed optimal dual solution $(\bar{Y}, \bar{\rho})$. This construction will give us a valuable road map to follow when we address the case of essentially bounded perturbations in Section 5.3. In general spirit this is not unlike the path followed by Mnif and Pham [24], who also *assumed* existence of an optimal dual solution and then constructed an optimal wealth process in terms of the assumed optimal dual solution (recall item 2 of Remark 2.2.11). In Section 5.3 we shall no longer need to assume existence of optimal dual solutions, and will establish existence of such solutions on the basis of Theorem 3.1.7, but at the expense of essentially bounded perturbations.

**Remark 5.2.20.** Thus, in the rest of this section, we shall just assume the existence of a Lagrange multiplier $(\bar{Y}, \bar{\rho}) \in \mathbb{Y} \overset{(5.2.42)}{=} \mathbb{B}_2 \times \mathcal{L}_2(\{\mathcal{F}_t\}; \mathcal{BV}_0^r)$, that is we assume existence of a maximizer $(\bar{Y}, \bar{\rho})$ of the dual function $g$ on the space of dual variables $\mathbb{Y}$: To be precise, for the remainder of this section we shall suppose

$$\sup_{Y \in \mathbb{Y}} \{g(Y, \rho)\} = g(\bar{Y}, \bar{\rho}) \in \mathbb{R} \qquad \text{for some } (\bar{Y}, \bar{\rho}) \in \mathbb{Y} := \mathbb{B}_2 \times \mathcal{L}_2(\{\mathcal{F}_t\}; \mathcal{BV}_0^r). \quad (5.2.54)$$

**Remark 5.2.21.** From (4.1.20) and (5.2.47), we know that $J^*(Y(T))$ is $P$-integrable for all $Y \in \mathbb{B}_2$, and Remark 5.2.20 ensures that $g(\bar{Y}, \bar{\rho}) \in \mathbb{R}$, therefore it follows from (5.2.50) that

$$\bar{\rho} \geqslant 0 \quad (\text{recall } (5.2.13)) \qquad \text{and} \qquad \varkappa(\bar{Y}, \bar{\rho}) \in \mathbb{R}. \quad (5.2.55)$$

The proof of the following proposition can be found in Appendix A.

**Proposition 5.2.22. [Kuhn-Tucker Optimal Conditions]** *Suppose Condition 2.1.1, 2.1.2, 2.1.5, 2.1.11 on the market settings and Condition 2.2.1 on the quadratic criterion function $J$ given in (4.1.19). Then, for each $(\pi, (Y, \rho)) \in \Pi \times \mathbb{Y}$, we have the following equivalence:*

$$f(\pi) = g(Y, \rho) \iff \begin{cases} 1)\ \pi \in \mathcal{A}, \quad 2)\ X^\pi \geqslant B, \quad 3)\ \rho \geqslant 0, \\ 4)\ \displaystyle\int_0^T [X^\pi(t) - B(t)]\rho(dt) = 0 \quad a.s., \\ 5)\ E\left[X^\pi(T)Y(T)\right] + \varkappa(Y, \rho) = 0, \\ 6)\ X^\pi(T) = \partial J^*(Y(T)) \quad a.s., \end{cases} \tag{5.2.56}$$

*where $\partial J^*(\cdot)$ denotes the derivative function of the convex conjugate function $J^*(\cdot)$ given in (4.1.20).*

**Remark 5.2.23.** Problem (3.2.64) constitutes a sort of static and finite dimensional precursor of the stochastic control problem (5.2.1). Indeed, upon comparing problems (3.2.64) and (5.2.1), we observe the following: the primal variable in problem (3.2.64) is $x \in \mathbb{X} := \mathbb{R}^n$ while the primal variable in problem (5.2.1) is $\pi \in \mathbb{X} := \Pi$, and the constraint $\pi \in \mathcal{A}$ in problem (5.2.1) is an analogue of the constraint $x \in C$ in problem (3.2.64). In much the same way, the constraint $X^\pi \geqslant B$ at problem (5.2.1) is a clear analogue of the constraint $x \geqslant b$ in problem (3.2.64), although it is clearly a much more "complex" constraint since it involves the dynamic dependence of the wealth process $X^\pi$ on the primal variable $\pi$ through the SDE (2.1.21), whereas there is no such dynamic dependence in the simple static problem (3.2.64). Finally, the objective function $E\left[J(X^\pi(T))\right]$, to be minimized as a function of the primal variable $\pi$ in problem (5.2.1), is a clear analogue of the objective function $J(x)$, to be minimized as a function of the primal variable $x$ in problem (3.2.64). In problem (3.2.64) the dual variable is the pair $(y, z) \in \mathbb{Y} := \mathbb{R}^n \times \mathbb{R}^n$ (see (3.2.74)), while the dual variable in problem (5.2.1) is the pair $(Y, \rho) \in \mathbb{Y} := \mathbb{B}_2 \times \mathcal{L}_2(\{\mathcal{F}_t\}; \mathcal{BV}_0^r)$ (see (5.2.42)). In particular, the dual variable $\rho \in \mathcal{L}_2(\{\mathcal{F}_t\}; \mathcal{BV}_0^r)$ "enforces" the inequality constraint $X^\pi \geqslant B$ in problem (5.2.1) in just the same way that the dual variable $z \in \mathbb{R}^n$ "enforces" the inequality constraint $x \geqslant b$ in problem (3.2.64), so that the dual variables $\rho$ and $z$ play directly analogous roles. Likewise, the dual variable $Y \in \mathbb{B}_2$ in problem (5.2.1) is a direct analogue of the dual variable $y \in \mathbb{R}^n$ in problem (3.2.64). This similarity in the problems (3.2.64) and (5.2.1) shows up in a corresponding similarity in the Kuhn-Tucker relations (see (3.2.92) 1) - 6) for problem (3.2.64), and see (5.2.56) 1) - 6) for problem (5.2.1)). Indeed, (5.2.56) 1) - 2) are feasibility conditions on the primal variable $\pi \in \Pi$ in the same way that (3.2.92) 1) - 2) are feasibility conditions on the primal variable $x \in \mathbb{R}^n$ in problem (3.2.64), while the relations (5.2.56) 3) and (3.2.92) 3) are simply the usual non-negativity required of dual variables which enforce inequality constraints. Again, (5.2.56) 4) is a complementary slackness relation between the primal variable $\pi$ and the dual variable $\rho$ for the inequality constraint $X^\pi \geqslant B$ in problem (5.2.1), in the same way that (3.2.92) 4) is the usual complementary slackness relation between the primal variable $x$ and the dual variable $z$ for the inequality constraint $x \geqslant b$ in problem (3.2.64), while the transversality relation (5.2.56) 6) between the primal variable $\pi$, the dual variable $Y$ and the objective function in problem (5.2.1) is a clear analogue of the transversality relation

(3.2.92) 6) between the primal variable $x$, the dual variable $y$ and the objective function in problem (3.2.64). Finally, upon observing the similarity between (5.2.51) and (3.2.84), we see a clear similarity between the complementary slackness relations (5.2.56) 5) and (3.2.92) 5).

**Remark 5.2.24.** In view of Proposition 5.2.22, we must construct some $\bar{\pi} \in \Pi$ in terms of $(\bar{Y}, \bar{\rho}) \in \mathbb{Y}$ (recall Remark 5.2.20) such that

$$\begin{cases} 1)\ \bar{\pi} \in \mathcal{A}, \quad 2)\ X^{\bar{\pi}} \geqslant B, \quad 3)\ \bar{\rho} \geqslant 0, \\ 4)\ \displaystyle\int_0^T [X^{\bar{\pi}}(t) - B(t)]\bar{\rho}(\mathrm{d}t) = 0 \quad \text{a.s.}, \\ 5)\ E\left[X^{\bar{\pi}}(T)\bar{Y}(T)\right] + \varkappa(\bar{Y}, \bar{\rho}) = 0, \\ 6)\ X^{\bar{\pi}}(T) = \partial J^*(\bar{Y}(T)) \quad \text{a.s.}, \end{cases} \tag{5.2.57}$$

for then it follows from Proposition 5.2.22 that

$$f(\bar{\pi}) = g(\bar{Y}, \bar{\rho}). \tag{5.2.58}$$

From (5.2.58) and (5.2.52), we obtain

$$f(\bar{\pi}) = \inf_{\pi \in \Pi}\{f(\pi)\}, \tag{5.2.59}$$

that is $\bar{\pi} \in \Pi$ is optimal. Of course, we know from (5.2.55) that (5.2.57) - 3) is already satisfied, so it remains to construct $\bar{\pi} \in \mathcal{A}$ such that the remaining conditions of (5.2.57) are satisfied. In the next few propositions, we develop some tools for constructing this $\bar{\pi}$:

**Proposition 5.2.25.** *Assume Condition 2.1.1, 2.1.2 and 2.1.5 on the market settings. If $X$ is an $\mathbb{R}$-valued $\mathcal{F}_t$-adapted Itô process that satisfies the SDE at (2.1.17), that is*

$$dX(t) = [r(t)X(t) + \pi'(t)\sigma(t)\theta(t)]\, dt + \pi'(t)\sigma(t)\, dW(t), \quad \textit{for some } \pi \in \Pi, \tag{5.2.60}$$

*then*

$$\begin{aligned} &E\left[X(T)Y(T) - \int_0^T X(s)\varrho_Y(ds)\right] \\ &\quad = X(0)Y(0) + E\left[\int_0^T \pi'(s)\sigma(s)\left[Y(s)\theta(s) + \Lambda_Y(s)\right] ds\right], \quad \textit{for all } Y \in \mathbb{B}_2. \end{aligned} \tag{5.2.61}$$

**Proof.** Fix some $Y \in \mathbb{B}_2$. From (5.2.60) and Proposition 4.1.4, we have

$$X \in \mathbb{B} \qquad \text{with} \quad \dot{X} = rX + \pi'\sigma\theta \text{ and } \Lambda_X = \pi'\sigma. \tag{5.2.62}$$

From (5.2.62) and (5.2.37), we get

$$\begin{aligned} \hat{M}(X, Y)(T) \;&=\; X(T)Y(T) - X(0)Y(0) - \int_0^T X(s)\varrho_Y(\mathrm{d}s) \\ &\qquad - \int_0^T \pi'(s)\sigma(s)\left[\theta(s)Y(s) + \Lambda_Y(s)\right] \mathrm{d}s. \end{aligned} \tag{5.2.63}$$

From Proposition 5.2.12 we get

$$E\left[\hat{M}(X,Y)(T)\right] = 0. \tag{5.2.64}$$

It then follows from (5.2.64) and (5.2.63) that

$$\begin{aligned} &E\left[X(T)Y(T) - \int_0^T X(s)\varrho_Y(\mathrm{d}s)\right] \\ &\quad = X(0)Y(0) + E\left[\int_0^T \pi'(s)\sigma(s)\left[\theta(s)Y(s) + \Lambda_Y(s)\right]\mathrm{d}s\right]. \end{aligned} \tag{5.2.65}$$

□

We can now use Proposition 5.2.25 to establish:

**Proposition 5.2.26.** *Suppose Condition 2.1.1, 2.1.2, 2.1.5, 2.1.11 and 2.2.4, and recall (2.1.8), Remark 2.1.7 - (1), the set of constrained portfolio processes at (5.1.2), and Notation 4.3.14-(b). For each $Y \in \mathbb{B}_2$ (see (5.2.29)) we have*

$$\begin{aligned} &\inf_{\pi\in\mathcal{A}}\left\{E\left[X^\pi(T)Y(T) - \int_0^T X^\pi(s)\varrho_Y(ds)\right]\right\} \\ &= x_0 Y(0) - E\left[\int_0^T \delta^*_{\mathbb{R}^N}\left\{-\sigma(s)[\theta(s)Y(s) + \Lambda_Y(s)]\big|A\right\} ds\right]. \end{aligned} \tag{5.2.66}$$

**Proof.** Fix some $Y = (Y(0), \Lambda_Y, \varrho_Y) \in \mathbb{B}_2$. Comparing (5.2.60) with (2.1.21), we have from Proposition 5.2.25 that

$$\begin{aligned} &E\left[X^\pi(T)Y(T) - \int_0^T X^\pi(s)\varrho_Y(\mathrm{d}s)\right] \\ &\quad = x_0 Y(0) + E\left[\int_0^T \pi'(s)\sigma(s)\left[\theta(s)Y(s) + \Lambda_Y(s)\right]\mathrm{d}s\right]. \end{aligned} \tag{5.2.67}$$

From (5.1.2) and (5.2.67), we get (recall Notation 4.3.14 - (a))

$$\begin{aligned} &\sup_{\pi\in\mathcal{A}} E\left[-X^\pi(T)Y(T) + \int_0^T X^\pi(s)\varrho_Y(\mathrm{d}s)\right] \\ &= \sup_{\pi\in\Pi}\left\{E\left[-X^\pi(T)Y(T) + \int_0^T X^\pi(s)\varrho_Y(\mathrm{d}s) - \int_0^T \delta_{\mathbb{R}^N}\{\pi(s)|A\}\mathrm{d}s\right]\right\} \\ &= -x_0 Y(0) + \sup_{\pi\in\Pi}\left\{E\left[\int_0^T \pi'(s)\sigma(s)\left[-\theta(s)Y(s) - \Lambda_Y(s)\right] - \delta_{\mathbb{R}^N}\{\pi(s)|A\}\mathrm{d}s\right]\right\}. \end{aligned} \tag{5.2.68}$$

We next evaluate the supremum at the right of (5.2.68). Put

$$\vartheta(\omega;t) := -\sigma(\omega;t)\left[\theta(\omega;t)Y(\omega;t) + \Lambda_Y(\omega;t)\right], \quad (\omega;t) \in \Omega\times[0,T]; \tag{5.2.69}$$

since $\Lambda_Y \in \Pi$ (see (4.1.10) and (4.1.3)), it follows from the boundedness of $\theta$ (see Remark 2.1.7(1)), the bound given by (2.1.7), and Proposition 4.1.5, that $\vartheta \in \Pi$. Using $\vartheta$ given by (5.2.69) in (4.3.40), we can write the supremum at the right of (5.2.68) as

$$\begin{aligned}
&\sup_{\pi\in\Pi}\left\{E\left[\int_0^T \pi'(s)\sigma(s)\left[-\theta(s)Y(s)-\Lambda_Y(s)\right]-\delta_{\mathbb{R}^N}\{\pi(s)|A\}\mathrm{d}s\right]\right\}\\
\overset{(5.2.69)}{=}\ &\sup_{\pi\in\Pi} E\left[\int_0^T \left[\pi'(s)\vartheta(s)-\delta_{\mathbb{R}^N}\{\pi(s)|A\}\right]\mathrm{d}s\right]\\
\overset{(4.3.40)}{=}\ &E\left[\int_0^T \delta^*_{\mathbb{R}^N}\left\{\vartheta(s)\middle|A\right\}\mathrm{d}s\right]\\
\overset{(5.2.69)}{=}\ &E\left[\int_0^T \delta^*_{\mathbb{R}^N}\left\{-\sigma(s)[\theta(s)Y(s)+\Lambda_Y(s)]\middle|A\right\}\mathrm{d}s\right].
\end{aligned}\tag{5.2.70}$$

Combining (5.2.70) with (5.2.68), we obtain (5.2.66):

$$\begin{aligned}
&\inf_{\pi\in\mathcal{A}}\left\{E\left[X^\pi(T)Y(T)-\int_0^T X^\pi(s)\varrho_Y(\mathrm{d}s)\right]\right\}\\
&=x_0Y(0)-E\left[\int_0^T \delta^*_{\mathbb{R}^N}\left\{-\sigma(s)[\theta(s)Y(s)+\Lambda_Y(s)]\middle|A\right\}\mathrm{d}s\right].
\end{aligned}\tag{5.2.71}$$

□

We then have the following result which establishes a useful *necessary condition* resulting from the optimality assumed at (5.2.53) (this necessary condition is (5.2.72) which follows), as well as verifying the complementary slackness relation at (5.2.57) - 5) (see (5.2.73)):

**Proposition 5.2.27.** *Suppose Condition 2.1.1, 2.1.2, 2.1.5, 2.2.1 and recall the pair $(\bar{Y},\bar{\rho})\in\mathbb{Y}$ given by (5.2.53). Then, we have (recall the derivative function $\partial J^*(\cdot)$ at (4.1.21) and $\varkappa(\cdot)$ at (5.2.51))*

$$\varkappa(Y,\rho)+E\left[\partial J^*(\bar{Y}(T))Y(T)\right]\geqslant 0,\quad \textit{for each } (Y,\rho)\in\mathbb{Y} \textit{ with } \rho\geqslant 0,\tag{5.2.72}$$

*and in particular,*

$$\varkappa(\bar{Y},\bar{\rho})+E\left[\partial J^*(\bar{Y}(T))\bar{Y}(T)\right]=0.\tag{5.2.73}$$

**Proof.** We use the optimality of the dual solution $(\bar{Y},\bar{\rho})$ (see (5.2.53)) to get:

$$g(\bar{Y}+\varepsilon Y,\bar{\rho}+\varepsilon\rho)\leqslant g(\bar{Y},\bar{\rho})\quad\text{for all } (Y,\rho)\in\mathbb{Y} \text{ and } \varepsilon\in(0,\infty).\tag{5.2.74}$$

If $\rho\geqslant 0$, we have (recall (5.2.55) and (5.2.13))

$$\bar{\rho}+\varepsilon\rho\geqslant 0,\quad \text{for all } \varepsilon\in(0,\infty).\tag{5.2.75}$$

Combining (5.2.74), (5.2.75) and (5.2.50), we get

$$\varkappa(\bar{Y}+\varepsilon Y,\bar{\rho}+\varepsilon\rho)+E\left[J^*(\bar{Y}(T)+\varepsilon Y(T))\right]\geqslant\varkappa(\bar{Y},\bar{\rho})+E\left[J^*(\bar{Y}(T))\right],\tag{5.2.76}$$

for all $(Y,\rho)\in\mathbb{Y}$ with $\rho\geqslant 0$ and $\varepsilon\in(0,\infty)$. On the other hand, from (5.2.51), one sees

$$\begin{aligned}
&\varkappa(\bar{Y}+\varepsilon Y,\bar{\rho}+\varepsilon\rho)\\
\overset{(5.2.51)}{=}\;& \sup_{\pi\in\mathcal{A}}\left\{E\left[\int_0^T\left[X^\pi(t)-B(t)\right](\bar{\rho}+\varepsilon\rho)(\mathrm{d}t)\right]-E\left[X^\pi(T)(\bar{Y}(T)+\varepsilon Y(T))\right]\right\}\\
=\;& \sup_{\pi\in\mathcal{A}}\left\{E\left[\int_0^T\left[X^\pi(t)-B(t)\right]\bar{\rho}(\mathrm{d}t)\right]-E\left[X^\pi(T)\bar{Y}(T)\right]\right.\\
&\qquad\left.+\varepsilon E\left[\int_0^T\left[X^\pi(t)-B(t)\right]\rho(\mathrm{d}t)\right]-\varepsilon E\left[X^\pi(T)Y(T)\right]\right\}\\
\leqslant\;& \sup_{\pi\in\mathcal{A}}\left\{E\left[\int_0^T\left[X^\pi(t)-B(t)\right]\bar{\rho}(\mathrm{d}t)\right]-E\left[X^\pi(T)\bar{Y}(T)\right]\right\}\\
&\qquad+\varepsilon\sup_{\pi\in\mathcal{A}}\left\{E\left[\int_0^T\left[X^\pi(t)-B(t)\right]\rho(\mathrm{d}t)\right]-E\left[X^\pi(T)Y(T)\right]\right\}\\
=\;& \varkappa(\bar{Y},\bar{\rho})+\varepsilon\varkappa(Y,\rho)\qquad\text{for all }(Y,\rho)\in\mathbb{Y}\text{ with }\rho\geqslant 0\text{ and }\varepsilon\in(0,\infty).
\end{aligned}\tag{5.2.77}$$

From (5.2.77) and (5.2.76), together with $\varkappa(\bar{Y},\bar{\rho})\in\mathbb{R}$ (recall (5.2.55)), we get

$$\varkappa(Y,\rho)+E\left[\frac{J^*(\bar{Y}(T)+\varepsilon Y(T))-J^*(\bar{Y}(T))}{\varepsilon}\right]\geqslant 0,\qquad \varepsilon\in(0,\infty),\ \rho\geqslant 0.\tag{5.2.78}$$

It follows from (4.1.20), Condition 2.2.1 and dominated convergence, that

$$\lim_{\varepsilon\to 0}E\left[\frac{J^*(\bar{Y}(T)+\varepsilon Y(T))-J^*(\bar{Y}(T))}{\varepsilon}\right]=E\left[\partial J^*(\bar{Y}(T))Y(T)\right],\quad\text{for }Y\in\mathbb{B}_2.\tag{5.2.79}$$

Therefore, (5.2.78) and (5.2.79) give (5.2.72), and in particular

$$\varkappa(\bar{Y},\bar{\rho})+E\left[\partial J^*(\bar{Y}(T))\bar{Y}(T)\right]\geqslant 0.\tag{5.2.80}$$

To establish (5.2.73), we next use the optimality of the dual solution $(\bar{Y},\bar{\rho})$ given at (5.2.54) again to get

$$g((1-\varepsilon)\bar{Y},(1-\varepsilon)\bar{\rho})\leqslant g(\bar{Y},\bar{\rho})\quad\text{for all }\varepsilon\in[0,1).\tag{5.2.81}$$

Since $\bar{\rho}\geqslant 0$ (recall Remark 5.2.21), it follows from (5.2.81) and (5.2.50) that

$$\varkappa((1-\varepsilon)\bar{Y},(1-\varepsilon)\bar{\rho})+E\left[J^*((1-\varepsilon)\bar{Y}(T))\right]\geqslant\varkappa(\bar{Y},\bar{\rho})+E\left[J^*(\bar{Y}(T))\right],\tag{5.2.82}$$

for all $\varepsilon\in[0,1)$. From (5.2.51), one also sees that

$$\varkappa((1-\varepsilon)\bar{Y},(1-\varepsilon)\bar{\rho})=(1-\varepsilon)\varkappa(\bar{Y},\bar{\rho}),\quad\text{for all }\varepsilon\in[0,1).\tag{5.2.83}$$

Using (5.2.83) in (5.2.82), together with $\varkappa(\bar{Y},\bar{\rho})\in\mathbb{R}$ (recall (5.2.55)), we have

$$\varkappa(\bar{Y},\bar{\rho})+E\left[\frac{J^*(\bar{Y}(T))-J^*((1-\varepsilon)\bar{Y}(T))}{\varepsilon}\right]\leqslant 0\quad\text{for all }\varepsilon\in[0,1).\tag{5.2.84}$$

From (4.1.20), Condition 2.2.1 and dominated convergence, we get

$$\lim_{\varepsilon\to 0} E\left[\frac{J^*(\bar{Y}(T)) - J^*((1-\varepsilon)\bar{Y}(T))}{\varepsilon}\right] = E\left[\partial J^*(\bar{Y}(T))\bar{Y}(T)\right]. \tag{5.2.85}$$

Combining (5.2.85) with (5.2.84), we have

$$\varkappa(\bar{Y},\bar{\rho}) + E\left[\partial J^*(\bar{Y}(T))\bar{Y}(T)\right] \leqslant 0, \tag{5.2.86}$$

and (5.2.73) follows from (5.2.80) and (5.2.86). □

With Proposition 5.2.27 established, we are now ready to construct the optimal portfolio $\bar{\pi}$ for the problem (5.2.1). Motivated by the transversality condition (5.2.56)-6) and Proposition 4.2.9 with

$$\partial J^*(\bar{Y}(T)) \overset{(4.1.20)}{=} \frac{\bar{Y}(T)-c}{a} \underset{Condition 2.2.1}{\overset{(5.2.47)}{\in}} \mathcal{L}_2(\Omega,\mathcal{F}_T,P), \tag{5.2.87}$$

define the process $\bar{X}$ as

$$\bar{X}(t) := H^{-1}(t)E\left[H(T)\partial J^*(\bar{Y}(T)) \,\middle|\, \mathcal{F}_t\right] \qquad t\in[0,T]. \tag{5.2.88}$$

Then, from Proposition 4.2.9, there exists some a.e. unique $\mathbb{R}^N$-valued $\mathcal{F}_t$-progressively measurable process $\bar{\psi}$ such that

$$\int_0^T \left\|\bar{\psi}(t)\right\|^2 \mathrm{d}t < \infty \quad \text{a.s.} \qquad \text{and} \qquad H(t)\bar{X}(t) = \bar{X}(0) + \int_0^t \bar{\psi}'(s)\mathrm{d}W(s). \tag{5.2.89}$$

Motivated by (4.2.37), define the $\mathbb{R}^N$-valued $\mathcal{F}_t$-progressively measurable process $\bar{\pi}$ in terms of the integrand process $\bar{\psi}$ given by (5.2.89):

$$\bar{\pi}(t) := (\sigma'(t))^{-1}\left[H^{-1}(t)\bar{\psi}(t) + \bar{X}(t)\theta(t)\right]. \tag{5.2.90}$$

From Proposition 4.2.9 again, it follows that

$$\bar{\pi}\in\Pi, \tag{5.2.91}$$

and

$$\mathrm{d}\bar{X}(t) = \left[r(t)\bar{X}(t) + \bar{\pi}'(t)\sigma(t)\theta(t)\right]\mathrm{d}t + \bar{\pi}'(t)\sigma(t)\mathrm{d}W(t), \tag{5.2.92}$$

with the initial value

$$\bar{X}(0) = E\left[H(T)\partial J^*(\bar{Y}(T))\right], \tag{5.2.93}$$

(see (5.2.87) and (4.2.39)).

We shall now establish that $\bar{X}$ is the wealth process corresponding to $\bar{\pi}$ (see Proposition 5.2.28 which follows) and then show that $\bar{\pi}$ is an admissible portfolio process which satisfies the American wealth constraint (see Proposition 5.2.31 which follows).

**Proposition 5.2.28.** *Assume Condition 2.1.1, 2.1.2, 2.1.5, 2.2.1 and the existence of a pair* $(\bar{Y}, \bar{\rho}) \in \mathbb{Y}$ *such that* (5.2.54) *holds. Recall* $\bar{X}$ *and* $\bar{\pi}$ *defined at* (5.2.88) - (5.2.90) *in terms of* $\bar{Y} \in \mathbb{B}_2$. *Then,* $\bar{X}$ *is the wealth process corresponding to* $\bar{\pi}$, *that is*

$$X^{\bar{\pi}} = \bar{X} \quad \text{so that} \quad X^{\bar{\pi}}(T) = \bar{X}(T) \overset{(5.2.88)}{=} \partial J^*(\bar{Y}(T)). \tag{5.2.94}$$

**Proof.** Comparing (5.2.92) and (5.2.60), we have from Proposition 5.2.25 that

$$\begin{aligned} &E\left[\bar{X}(T)Y(T) - \int_0^T \bar{X}(s)\varrho_Y(\mathrm{d}s)\right] \\ &\quad = \bar{X}(0)Y(0) + E\left[\int_0^T \bar{\pi}'(s)\sigma(s)\left[Y(s)\theta(s) + \Lambda_Y(s)\right]\mathrm{d}s\right], \quad Y \in \mathbb{B}_2. \end{aligned} \tag{5.2.95}$$

To establish (5.2.94), we fix some $y \in \mathbb{R}$. From Lemma 4.1.8,

$$\text{there exists some } Y \in \mathbb{B}_1 \text{ such that } Y(0) = y \text{ and } \theta Y + \Lambda_Y = 0 \ \text{ a.e.}, \tag{5.2.96}$$

and (5.2.96) together with Remark 5.2.10 give that

$$\text{there exists some } Y \in \mathbb{B}_2 \text{ such that } Y(0) = y,\ \theta Y + \Lambda_Y = 0 \text{ a.e. and } \varrho_Y \equiv 0. \tag{5.2.97}$$

Substituting $Y$ given by (5.2.97) in (5.2.95), we have

$$E\left[\bar{X}(T)Y(T)\right] \underset{(5.2.96)}{\overset{(5.2.95)}{=}} \bar{X}(0)y. \tag{5.2.98}$$

On the other hand, the necessary condition (5.2.72) holds for $(Y, 0) \in \mathbb{Y}$ where $Y$ is given by (5.2.97), i.e.

$$\varkappa(Y, 0) + E\left[\partial J^*(\bar{Y}(T))Y(T)\right] \geqslant 0. \tag{5.2.99}$$

From (5.2.66), (5.2.51) and (5.2.97), we have

$$\begin{aligned} \varkappa(Y,0) &\overset{(5.2.51)}{=} \sup_{\pi \in \mathcal{A}} \{-E\left[X^\pi(T)Y(T)\right]\} \\ &\overset{(5.2.97)}{=} -\inf_{\pi \in \mathcal{A}} \left\{E\left[X^\pi(T)Y(T) - \int_0^T X^\pi(s)\varrho_Y(\mathrm{d}s)\right]\right\} \quad (\text{where } \varrho_Y = 0) \\ &\overset{(5.2.66)}{=} -x_0Y(0) + E\left[\int_0^T \delta^*_{\mathbb{R}^N}\left\{-\sigma(s)[\theta(s)Y(s) + \Lambda_Y(s)]\middle| A\right\}\mathrm{d}s\right] \\ &\overset{(5.2.97)}{=} -x_0 y. \end{aligned} \tag{5.2.100}$$

From (5.2.88), $\bar{X}(T) = \partial J^*(\bar{Y}(T))$, and combining this with (5.2.98), (5.2.99) and (5.2.100) gives

$$(\bar{X}(0) - x_0)y \geqslant 0. \tag{5.2.101}$$

By arbitrary choice of $y \in \mathbb{R}$ in (5.2.101), we get

$$\bar{X}(0) = x_0. \tag{5.2.102}$$

Therefore, (5.2.94) follows from (5.2.102), (5.2.92), (2.1.21) and (5.2.88). □

**Remark 5.2.29.** From (5.2.73) and (5.2.94), we see

$$\varkappa(\bar{Y},\bar{\rho}) + E\left[X^{\bar{\pi}}(T)\bar{Y}(T)\right] = 0. \tag{5.2.103}$$

The following Lemma is proved in Appendix A.

**Lemma 5.2.30.** *Suppose Condition 2.1.1, 2.1.2, 2.1.5, 2.1.11, 2.2.1, 5.1.1, and fix some $\pi \in \Pi$. If*

$$E\left[\int_0^T \{X^\pi(s) - B(s)\}\rho(ds)\right] \geqslant 0 \tag{5.2.104}$$

*for all $\rho \in \mathcal{L}_2(\{\mathcal{F}_t\};\mathcal{BV}_0^r)$ such that $\rho \geqslant 0$, then*

$$X^\pi(t) \geqslant B(t), t \in [0,T] \ a.s. \tag{5.2.105}$$

We can now use Lemma 5.2.30 to establish the following:

**Proposition 5.2.31.** *Suppose Condition 2.1.1, 2.1.2, 2.1.5, 2.1.11, 2.2.1, 2.2.4, and 5.1.1, and recall $\bar{\pi}$ defined at (5.2.90). Then*

$$\bar{\pi} \in \mathcal{A}, \tag{5.2.106}$$

*and*

$$X^{\bar{\pi}}(t) \geqslant B(t) \quad t \in [0,T], \ a.s., \tag{5.2.107}$$

*i.e. $\bar{\pi}$ satisfies (5.2.56)-1), 2).*

**Proof.** The argument to establish (5.2.106) is similar to the arguments found in Proposition 4.3.20 and Proposition 4.5.27. Nevertheless, we present the argument again for completeness. First define a set

$$O := \{(\omega;t) \in \Omega \times [0,T] \mid \bar{\pi}(\omega;t) \in A\}. \tag{5.2.108}$$

From Lemma F.0.3 (also see Lemma 5.4.2 of [15, p.207]), corresponding to $\bar{\pi} \in \mathcal{F}^*$ there exists some $\mathbb{R}^N$-valued $\bar{\nu} \in \mathcal{F}^*$ such that

$$\begin{cases} \|\bar{\nu}(t)\| \leqslant 1, \left|\delta^*_{\mathbb{R}^N}\left\{-\bar{\nu}(t)\middle|A\right\}\right| \leqslant 1, & \text{a.e. on } \Omega\times[0,T], \\ \bar{\pi}'(t)\bar{\nu}(t) + \delta^*_{\mathbb{R}^N}\left\{-\bar{\nu}(t)\middle|A\right\} = 0, & \text{a.e. on } O, \\ \bar{\pi}'(t)\bar{\nu}(t) + \delta^*_{\mathbb{R}^N}\left\{-\bar{\nu}(t)\middle|A\right\} < 0, & \text{a.e. on } (\Omega\times[0,T])\setminus O. \end{cases} \tag{5.2.109}$$

It is clear from $\|\bar{\nu}(t)\| \leqslant 1$ a.e. and Condition 2.1.5 that

$$\sigma^{-1}\bar{\nu} \in \Pi. \tag{5.2.110}$$

Thus, from (5.2.110) and Lemma 4.1.8,

$$\text{there exists some } Y \in \mathbb{B}_1 \text{ such that } Y(0) = 0 \text{ and } \theta Y + \Lambda_Y = \sigma^{-1}\bar{\nu} \text{ a.e.,} \tag{5.2.111}$$

and (5.2.111) together with Remark 5.2.10 give that

$$\text{there exists some } Y \in \mathbb{B}_2 \text{ such that } Y(0) = 0,\ \theta Y + \Lambda_Y = \sigma^{-1}\bar{\nu} \text{ a.e.and } \varrho_Y \equiv 0. \quad (5.2.112)$$

With the $Y \in \mathbb{B}_2$ given by (5.2.112), we certainly have (5.2.72) holds for $(Y, 0) \in \mathbb{Y}$, i.e.,

$$\varkappa(Y, 0) + E\left[\partial J^*(\bar{Y}(T))Y(T)\right] \geqslant 0. \quad (5.2.113)$$

From (5.2.112) and (5.2.51), we have

$$\varkappa(Y, 0) = \sup_{\pi \in \mathcal{A}} \left\{-E\left[X^\pi(T)Y(T)\right]\right\}. \quad (5.2.114)$$

Combining (5.2.94) with (5.2.113) and (5.2.114), we get

$$\sup_{\pi \in \mathcal{A}} \left\{-E\left[X^\pi(T)Y(T)\right]\right\} + E\left[\bar{X}(T)Y(T)\right] \geqslant 0. \quad (5.2.115)$$

From Proposition 5.2.25, (5.2.92) and (5.2.112), we have

$$E\left[\bar{X}(T)Y(T)\right] = E\left[\int_0^T \pi'(s)\bar{\nu}(s)\mathrm{d}s\right], \quad (5.2.116)$$

and from Proposition 5.2.26 and (5.2.112), it follows that

$$\sup_{\pi \in \mathcal{A}} \left\{-E\left[X^\pi(T)Y(T)\right]\right\} = E\left[\int_0^T \delta^*_{\mathbb{R}^N}\left\{-\bar{\nu}(s)\middle|A\right\}\mathrm{d}s\right]. \quad (5.2.117)$$

Combining (5.2.115), (5.2.116) and (5.2.117), we get

$$E\left[\int_0^T \left(\bar{\pi}'(s)\bar{\nu}(s) + \delta^*_{\mathbb{R}^N}\left\{-\bar{\nu}(s)\middle|A\right\}\right)\mathrm{d}s\right] \geqslant 0, \quad (5.2.118)$$

which, together with (5.2.109), gives

$$P \otimes \lambda\left(O^c\right) = 0, \quad (5.2.119)$$

i.e. $\bar{\pi}(t) \in A$ a.e., or equivalently $\bar{\pi} \in \mathcal{A}$ as required (recall (5.2.91) and (5.1.2)).

To establish (5.2.107), fix some

$$\rho \in \mathcal{L}_2(\{\mathcal{F}_t\}; \mathcal{BV}_0^r) \qquad \text{such that} \qquad \rho \geqslant 0. \quad (5.2.120)$$

From Lemma 4.1.8 and Remark 5.2.10, there is some $Y \in \mathbb{B}_2$ such that

$$Y(0) = 0, \quad \Lambda_Y + \theta Y = 0, \quad \text{and} \quad \varrho_Y = \rho. \quad (5.2.121)$$

From (5.2.121) and (5.2.61) (with $X := X^\pi$, compare (5.2.60) and (2.1.21)), we have

$$E\left[X^\pi(T)Y(T)\right] = E\left[\int_0^T X^\pi(s)\rho(\mathrm{d}s)\right] \quad \pi \in \Pi. \quad (5.2.122)$$

Now, evaluate $\varkappa(\cdot)$ at $(Y, \rho)$ with $\rho$ given by (5.2.120) and $Y$ defined by (5.2.121):

$$\begin{aligned}\varkappa(Y,\rho) &\overset{(5.2.51)}{=} \sup_{\pi\in\mathcal{A}}\left\{E\left[\int_0^T [X^\pi(s)-B(s)]\,\rho(\mathrm{d}s)\right]-E\left[X^\pi(T)Y(T)\right]\right\}\\ &\overset{(5.2.122)}{=} -E\left[\int_0^T B(s)\rho(\mathrm{d}s)\right]. \end{aligned}\tag{5.2.123}$$

Then, (5.2.123) and (5.2.122), together with (5.2.94) and (5.2.72), give

$$E\left[\int_0^T \{X^{\bar\pi}(s)-B(s)\}\rho(\mathrm{d}s)\right] \underset{(5.2.123)}{\overset{(5.2.122)}{=}} \varkappa(Y,\rho)+E\left[X^{\bar\pi}(T)Y(T)\right] \underset{(5.2.94)}{\overset{(5.2.72)}{\geqslant}} 0. \tag{5.2.124}$$

By the arbitrary choice of $\rho \geqslant 0$ at (5.2.120) and Lemma 5.2.30, we get

$$X^{\bar\pi}(t) \geqslant B(t), t\in[0,T] \text{ a.s.} \tag{5.2.125}$$

□

**Remark 5.2.32.** Having established Proposition 5.2.27, Proposition 5.2.28 and Proposition 5.2.31, we have constructed a pair $(\bar\pi,(\bar Y,\bar\rho)) \in \mathbb{X}\times\mathbb{Y}$ such that

$$\begin{cases}\bar\rho \geqslant 0 \text{ (recall (5.2.55) of Remark 5.2.21)},\\ E\left[X^{\bar\pi}(T)\bar Y(T)\right]+\varkappa(\bar Y,\bar\rho)=0 \text{ (recall (5.2.103))},\\ X^{\bar\pi}(T)=\partial J^*(\bar Y(T)) \text{ (recall (5.2.94))},\\ \bar\pi\in\mathcal{A} \text{ and } X^{\bar\pi}(t)\geqslant B(t), t\in[0,T] \text{ a.s. (recall Proposition 5.2.31)}\end{cases}\tag{5.2.126}$$

so that (5.2.57) - 1), 2), 3), 5), 6) are satisfied by $(\bar\pi,(\bar Y,\bar\rho))$. The following Corollary 5.2.33 establishes the remaining complementary slackness condition (5.2.57) - 4), so that all conditions in (5.2.57) have been verified. Therefore, we have established that

$$\bar\pi \text{ is an optimal portfolio for problem (5.2.1).}$$

**Corollary 5.2.33.** *Suppose Condition 2.1.1, 2.1.2, 2.1.5, 2.1.11, 2.2.1, 2.2.4, 5.1.1, and the existence of a pair $(\bar Y,\bar\rho)\in\mathbb{Y}$ such that (5.2.54) holds. Recall $X^{\bar\pi}=\partial J^*(\bar Y(T))$ at (5.2.94). Then*

$$\int_0^T [X^{\bar\pi}(t)-B(t)]\bar\rho(dt)=0 \quad a.s., \tag{5.2.127}$$

*i.e. $(\bar\pi,(\bar Y,\bar\rho))$ satisfies the complementary slackness relations (5.2.56)-4).*

**Proof.** From (5.2.107) and $\bar\rho\geqslant 0$ (see (5.2.55)), we clearly have

$$\int_0^T [X^{\bar\pi}(t)-B(t)]\bar\rho(\mathrm{d}t)\geqslant 0 \quad \text{a.s.} \tag{5.2.128}$$

On the other hand, (5.2.51) gives

$$\begin{aligned}\varkappa(\bar{Y},\bar{\rho}) &\overset{(5.2.51)}{=} \sup_{\pi\in\mathcal{A}}\left\{-E\left[X^{\pi}(T)\bar{Y}(T)\right]+E\left[\int_0^T\left[X^{\pi}(t)-B(t)\right]\bar{\rho}(\mathrm{d}t)\right]\right\}\\ &\overset{(5.2.106)}{\geqslant} -E\left[X^{\bar{\pi}}(T)\bar{Y}(T)\right]+E\left[\int_0^T\left[X^{\bar{\pi}}(t)-B(t)\right]\bar{\rho}(\mathrm{d}t)\right],\end{aligned} \tag{5.2.129}$$

thus,

$$\varkappa(\bar{Y},\bar{\rho})+E\left[X^{\bar{\pi}}(T)\bar{Y}(T)\right] \overset{(5.2.129)}{\geqslant} E\left[\int_0^T\left[X^{\bar{\pi}}(t)-B(t)\right]\bar{\rho}(\mathrm{d}t)\right] \overset{(5.2.128)}{\geqslant} 0. \tag{5.2.130}$$

Combining (5.2.103) with (5.2.130), we have

$$E\left[\int_0^T\left[X^{\bar{\pi}}(t)-B(t)\right]\bar{\rho}(\mathrm{d}t)\right]=0. \tag{5.2.131}$$

Therefore, from (5.2.131) and (5.2.128), we get (5.2.127). □

## 5.3 The Canonical Problem - Combination of Portfolio and American Wealth Constraints: Part II

**Remark 5.3.1.** In Section 5.2 we established a tentative approach to address problem (5.2.1), that is the canonical problem (2.2.21). This approach relied on the assumption that there exists a solution of the dual problem of maximizing the dual function defined by (5.2.50) over the set of dual variables $\mathbb{Y}$ defined by (5.2.42) (see (5.2.53), Remark 5.2.19 and Remark 5.2.20). The situation in Section 5.2 is not unlike that which we saw in Remark 4.5.32 for problem (4.5.2) with a European wealth constraint, in which we experimented briefly with the space of square integrable random variables $\mathcal{L}_2(\Omega,\mathcal{F}_T,P)$ as the second factor in the perturbation space defined at (4.5.133). Our motivation for experimenting with the second choice of factor space at (4.5.133) is that this results in the space of dual variables $\mathbb{Y}$ at (4.5.136), in which the members of the second factor space $\mathcal{L}_2(\Omega,\mathcal{F}_T,P)$ are the dual variables which correspond to the European wealth constraint in problem (4.5.2). Certainly, elements of $\mathcal{L}_2(\Omega,\mathcal{F}_T,P)$ are much more natural mathematical entities than are the elements of the space $\mathcal{L}_\infty^*(\Omega,\mathcal{F}_T,P)$, which turn out to be the actual dual variables for the European wealth constraint (see the second factor space of (4.5.21)). Of course, our reason for using the perturbation space at (4.5.18) with $\mathcal{L}_\infty(\Omega,\mathcal{F}_T,P)$ as the second factor, is that this is key to exploiting the Slater Condition 4.5.5 for verifying the conditions of Theorem 3.1.7 in order to establish existence of an optimal dual solution in the space of dual variables at (4.5.21) (recall the essential role of Condition 4.5.5 in the proof of Proposition 4.5.15). In Remark 4.5.32, we saw that it was not possible to verify condition (3.1.9) of Theorem 3.1.7 on the basis of a Slater-type condition when using the perturbation space at (4.5.133), and argued that the space of dual variables that we get from the perturbation space at (4.5.133), that is the space of dual variables at (4.5.136), is *too small* to include any solution of the dual problem at (4.5.143). We have an exactly analogous situation in Section 5.2: we used the space of perturbations $\mathbb{U}$ given by (5.2.39), with the second factor $\mathcal{L}_2(\{\mathcal{F}_t\};\mathcal{C})$, because this leads to the space of dual variables given by (5.2.42) in which the second factor space $\mathcal{L}_2(\{\mathcal{F}_t\};\mathcal{BV}_0^r)$ gives the dual variables which correspond to the American wealth constraint in problem (5.2.1). However, it turns out to be impossible to verify condition (3.1.9) of Theorem 3.1.7 with the perturbation space at (5.2.39), and for reasons very similar to what we discovered in Remark 4.5.32, and on this basis we assert that the space of dual variables at (5.2.42) is too small for the assertion at (5.2.53) to be true.

In the present section, we shall circumvent the difficulty outlined in Remark 5.3.1 by following an approach to problem (5.2.1) which closely parallels the approach that we followed in Section 5.2. However, in place of the second factor space $\mathcal{L}_2(\{\mathcal{F}_t\};\mathcal{C})$ which we used in the space of perturbations given by (5.2.39), we are going to use a space of *essentially bounded* continuous adapted processes (see (5.3.23) which follows). This will allow us to use the following Slater-type Condition 5.3.2 when verifying the conditions of Theorem 3.1.7 in order to obtain an analogue of Proposition 4.5.15 (see Proposition 5.3.15). Accordingly, from now on, we shall assume the Slater-type condition:

**Condition 5.3.2.** There is some $\hat{\pi} \in \mathcal{A}$ and some nonrandom constant $\varepsilon \in (0, \infty)$ such that

$$X^{\hat{\pi}}(t) \geqslant B(t) + \varepsilon, \qquad \text{all } t \in [0, T], \text{ a.s.,} \tag{5.3.1}$$

(Recall (5.1.2) for $\mathcal{A}$ and Condition 5.1.1 for $B$).

**Remark 5.3.3.** Condition 5.3.2 is an obvious adaptation to the American wealth constraint in problem (5.2.1) of the Slater-type Condition 4.5.5 that we assumed when dealing with the European wealth constraint in problem (4.5.2). Notice that Condition 5.3.2 is really just a mild strengthening of the condition

$$\text{there is some } \hat{\pi} \in \mathcal{A} \text{ such that } X^{\hat{\pi}}(t) \geqslant B(t), \text{ all } t \in [0, T], \text{ a.s.,} \tag{5.3.2}$$

without which the constraints of problem (5.2.1) could never be satisfied, in which case problem (5.2.1) would not even make any sense (recall Remark 4.5.4 with respect to problem (4.5.2)). In fact, exactly as in Remark 4.5.4, Condition 5.3.2 just forces us to make a "non-greedy" stipulation of floor wealth $B$ over the trading interval $[0, T]$ in the formulation of problem (5.2.1).

We now formulate the Canonical Problem in term of a *primal function* $f$ (this is (5.2.5), repeated here for convenience):

**Problem 5.3.4. [primal problem]** Determine an optimal portfolio process $\bar{\pi} \in \mathcal{A}$ such that

$$f(\bar{\pi}) = \inf_{\pi \in \Pi} f(\pi), \tag{5.3.3}$$

where the *primal function* $f(\cdot) : \Pi \to (-\infty, +\infty]$ is defined as

$$f(\pi) := \begin{cases} E\left[J(X^\pi(T))\right], & \text{when } \pi \in \mathcal{A} \text{ and } X^\pi(t) \geqslant B(t), t \in [0,T] \text{ a.s.}, \\ +\infty, & \text{otherwise}, \end{cases} \tag{5.3.4}$$

and $J$ is defined in (4.1.19) subject to Condition 2.2.1.

From Remark 2.2.3 (note $X^\pi, B \in \mathcal{L}_2(\{\mathcal{F}_t\}; \mathcal{C})$ and recall (5.2.11) and (2.2.6))

$$\vartheta := \inf_{\pi \in \Pi} \{f(\pi)\} = \inf_{\pi \in \mathcal{A}} \{E\left[J(X^\pi(T))\right] \mid X^\pi \geqslant B\} \in \mathbb{R} \quad \text{with } \vartheta \geqslant \underline{l}, \tag{5.3.5}$$

(note that Condition 5.3.2 ensures that the set $\{\pi \in \mathcal{A} \mid X^\pi(t) \geqslant B(t), t \in [0,T] \text{ a.s.}\}$ is non-empty).

For notional convenience, we next define another vector space of stochastic processes and its norm-dual as follows.

**Notation 5.3.5.** (1) The vector space of all essentially bounded $\mathcal{F}_t$-adapted continuous processes is denoted by $\mathcal{L}_\infty(\{\mathcal{F}_t\}; \mathcal{C})$, i.e.

$$\mathcal{L}_\infty(\{\mathcal{F}_t\}; \mathcal{C}) := \left\{ V : \Omega \to \mathcal{C}[0,T] \;\middle|\; V \in \mathcal{F}^* \text{ and } \|V\|_{u(\infty)} < \infty \right\}, \tag{5.3.6}$$

where the essential supremum norm $\|\cdot\|_{u(\infty)}$ on $\mathcal{L}_\infty(\{\mathcal{F}_t\}; \mathcal{C})$ is defined by

$$\|V\|_{u(\infty)} := \operatorname*{P\text{-}ess\text{-}sup}_{\omega \in \Omega} \|V(\omega; \cdot)\|_u = \operatorname*{P\text{-}ess\text{-}sup}_{\omega \in \Omega} \sup_{t \in [0,T]} |V(\omega; t)|. \tag{5.3.7}$$

Similar to the case of $\mathcal{L}_2(\{\mathcal{F}_t\}; \mathcal{C})$ at (5.2.6), a more detailed notation for $\mathcal{L}_\infty(\{\mathcal{F}_t\}; \mathcal{C})$ might be $\mathcal{L}_\infty((\Omega, \mathcal{F}_T, \{\mathcal{F}_t\}, P); \mathcal{C})$, but this notation is much too cumbersome for us to use. We recall that the notation $V \in \mathcal{F}^*$ at (5.3.6) indicates that the mapping $V : \Omega \times [0,T] \to \mathbb{R}$ is $\mathcal{F}^*$-measurable, that is the process $\{V(t), t \in [0,T]\}$ is $\mathcal{F}_t$-progressively measurable (see Notation 2.1.4-(2)).

We introduce an *order* "$\geqslant$" ("$\leqslant$") on $\mathcal{L}_\infty(\{\mathcal{F}_t\}; \mathcal{C})$ by writing

$$V \geqslant 0 \text{ if } V(t) \geqslant 0 \text{ for all } t \in [0,T] \text{ a.s.}, \quad \text{and } V_1 \leqslant V_2 \text{ if } V_2 - V_1 \geqslant 0. \tag{5.3.8}$$

In particular, denote

$$(\mathcal{L}_\infty(\{\mathcal{F}_t\}; \mathcal{C}))^+ := \{V \in \mathcal{L}_\infty(\{\mathcal{F}_t\}; \mathcal{C}) \mid V \geqslant 0\}. \tag{5.3.9}$$

(2) The norm-dual space of $\mathcal{L}_\infty(\{\mathcal{F}_t\}; \mathcal{C})$ is denoted by $\mathcal{L}_\infty^*(\{\mathcal{F}_t\}; \mathcal{C})$, and we write for all $Z, Z_i \in \mathcal{L}_\infty^*(\{\mathcal{F}_t\}; \mathcal{C}), i = 1, 2$ that

$$Z \geqslant 0 \text{ if } Z(V) \geqslant 0 \text{ for all } V \in (\mathcal{L}_\infty(\{\mathcal{F}_t\}; \mathcal{C}))^+, \quad \text{and } Z_1 \leqslant Z_2 \text{ if } Z_2 - Z_1 \geqslant 0, \tag{5.3.10}$$

and denote

$$(\mathcal{L}_\infty^*(\{\mathcal{F}_t\}; \mathcal{C}))^+ := \{Z \in \mathcal{L}_\infty^*(\{\mathcal{F}_t\}; \mathcal{C}) \mid Z \geqslant 0\}. \tag{5.3.11}$$

**Remark 5.3.6.** From (5.3.6) and (5.2.6), we have the set inclusion

$$\mathcal{L}_\infty(\{\mathcal{F}_t\};\mathcal{C}) \subset \mathcal{L}_2(\{\mathcal{F}_t\};\mathcal{C}), \tag{5.3.12}$$

and the order relations introduced at (5.3.8) and(5.2.8) are consistent. Recalling Condition 5.1.1, we have

$$B \in \mathcal{L}_\infty(\{\mathcal{F}_t\};\mathcal{C}). \tag{5.3.13}$$

**Example 5.3.7.** Fix a $\rho \in \mathcal{L}_2(\{\mathcal{F}_t\};\mathcal{BV}_0^r)$ and define a mapping $Z$ on $\mathcal{L}_\infty(\{\mathcal{F}_t\};\mathcal{C})$ by

$$Z(V) := E\left[\int_0^T V(t)\rho(\mathrm{d}t)\right], \qquad V \in \mathcal{L}_\infty(\{\mathcal{F}_t\};\mathcal{C}). \tag{5.3.14}$$

In view of (5.3.12) and (5.2.14) one has

$$\begin{aligned} |Z(V)| \leqslant E\left[\left|\int_0^T V(t)\rho(\mathrm{d}t)\right|\right] &\overset{(5.2.14)}{\leqslant} E\left[\|V\|_u \cdot \|\rho\|_T\right] \\ &\overset{(5.2.14)}{\leqslant} E\left[\|V\|_u^2\right]^{\frac{1}{2}} E\left[\|\rho\|_T^2\right]^{\frac{1}{2}} \\ &\leqslant \|V\|_{u(\infty)} E\left[\|\rho\|_T^2\right]^{\frac{1}{2}}, \end{aligned} \tag{5.3.15}$$

for all $V \in \mathcal{L}_\infty(\{\mathcal{F}_t\};\mathcal{C})$. Since $E\left[\|\rho\|_T^2\right] < \infty$, it follows from (5.3.14) and (5.3.15) that $Z : \mathcal{L}_\infty(\{\mathcal{F}_t\};\mathcal{C}) \to \mathbb{R}$ is linear and $\|\cdot\|_{u(\infty)}$-continuous, that is

$$Z \in \mathcal{L}_\infty^*(\{\mathcal{F}_t\};\mathcal{C}). \tag{5.3.16}$$

Therefore each $\rho \in \mathcal{L}_2(\{\mathcal{F}_t\};\mathcal{BV}_0^r)$ gives an element of $\mathcal{L}_\infty^*(\{\mathcal{F}_t\};\mathcal{C})$ through (5.3.14), so that we have

$$\mathcal{L}_2(\{\mathcal{F}_t\};\mathcal{BV}_0^r) \subset \mathcal{L}_\infty^*(\{\mathcal{F}_t\};\mathcal{C}). \tag{5.3.17}$$

**Example 5.3.8.** Fix some $\pi \in \Pi$ and $\tau \in [0,T]$. Recalling $\rho_\tau^\pi$ at (5.2.16), we define a linear mapping $Z_\tau^\pi : \mathcal{L}_\infty(\{\mathcal{F}_t\};\mathcal{C}) \to \mathbb{R}$ as

$$Z_\tau^\pi(V) := E\left[\int_0^T V(t)\rho_\tau^\pi(\mathrm{d}t)\right] \overset{(5.2.16)}{=} E\left[V(\tau)\mathbb{1}_{\{X^\pi < B\}}(\tau)\right]. \tag{5.3.18}$$

Then

$$|Z_\tau^\pi(V)| \overset{(5.3.18)}{=} \left|E\left[V(\tau)\mathbb{1}_{\{X^\pi < B\}}(\tau)\right]\right| \leqslant \|V\|_{u(\infty)}, \tag{5.3.19}$$

i.e.

$$Z_\tau^\pi \in \mathcal{L}_\infty^*(\{\mathcal{F}_t\};\mathcal{C}). \tag{5.3.20}$$

From (5.2.20) and the fact that $(\mathcal{L}_\infty(\{\mathcal{F}_t\};\mathcal{C}))^+ \subset (\mathcal{L}_2(\{\mathcal{F}_t\};\mathcal{C}))^+$ (from (5.3.12)), we have

$$E\left[\int_0^T V(t)\rho_\tau^\pi(\mathrm{d}t)\right] \overset{(5.2.20)}{\geqslant} 0, \quad \text{all } V \in (\mathcal{L}_\infty(\{\mathcal{F}_t\};\mathcal{C}))^+, \tag{5.3.21}$$

and (5.3.21) together with (5.3.18) gives

$$Z_\tau^\pi \geqslant 0. \tag{5.3.22}$$

□

We shall now follow the Rockafellar approach outlined in Section 3.1 to choose a real vector space $\mathbb{U}$ of perturbations, a perturbation function $F(\cdot,\cdot)$, a real vector space $\mathbb{Y}$ of dual variables and a bilinear form on $\mathbb{U} \times \mathbb{Y}$, and then construct a Lagrangian function and a dual function. The following steps are formally very similar to the steps 1 - 3 in Section 5.2 except that we use essentially bounded perturbations instead of square integrable perturbations for the constraint $X^\pi \geqslant B$.

1. To implement Step 3.1.1, we must fix a vector space of perturbations $\mathbb{U}$ and define a perturbation function. Define a vector space of *perturbations* as

$$\mathbb{U} := \mathcal{L}_2(\Omega, \mathcal{F}_T, P) \times \mathcal{L}_\infty(\{\mathcal{F}_t\}; \mathcal{C}), \tag{5.3.23}$$

and define the *perturbation function* $F : \Pi \times \mathbb{U} \to (-\infty, +\infty]$ as

$$\begin{aligned} F(\pi,(u,V)) &:= \begin{cases} E\left[J(X^\pi(T) - u)\right], & \text{if } \pi \in \mathcal{A} \text{ and} \\ & X^\pi(t) + V(t) \geqslant B(t), t \in [0,T] \text{ a.s.,} \\ +\infty, & \text{otherwise,} \end{cases} \\ &\overset{(5.2.11)}{=} \begin{cases} E\left[J(X^\pi(T) - u)\right], & \text{if } \pi \in \mathcal{A} \text{ and } X^\pi + V \geqslant B, \\ +\infty, & \text{otherwise.} \end{cases} \end{aligned} \tag{5.3.24}$$

The convexity of $F$ on $\Pi \times \mathbb{U}$ follows since $\pi \to X^\pi(T) : \Pi \mapsto \mathcal{L}_2(\Omega, \mathcal{F}_T, P)$ is an affine mapping, together with (4.1.19) and (5.3.24). The *consistency relation* (3.1.2) between the primal function and perturbation function is clearly satisfied, that is,

$$F(\pi,(0,0)) = f(\pi), \quad \pi \in \Pi, \tag{5.3.25}$$

as is immediate from (5.3.24) and (5.3.4).

**Remark 5.3.9.** Comparing the spaces of perturbations $\mathbb{U}$ defined at (5.3.23) and (5.2.39), we see that the second factor space at (5.3.23) comprises essentially bounded functions whereas the second factor space at (5.2.39) comprises square integrable functions. The use of essentially bounded perturbations at (5.3.23) will be indispensable for using the Slater Condition 5.3.2 when verifying the conditions of Theorem 3.1.7 to secure zero duality gap and existence of a dual solution. This will be seen in the proof of Proposition 5.3.15. The perturbation function $F$ at (5.3.24) is just the restriction of the perturbation function $F$ at (5.2.40) to perturbations $V \in \mathcal{L}_\infty(\{\mathcal{F}_t\}; \mathcal{C})$

2. Following Step 3.1.2, we must pair the space of perturbations at (5.3.23) with a vector space $\mathbb{Y}$ through a bilinear form $\langle \cdot, \cdot \rangle$ on $\mathbb{U} \times \mathbb{Y}$. We therefore define the space of dual variables as (compare (5.2.42), (4.5.21) and Remark 3.1.8):

$$\mathbb{Y} := \mathbb{B}_2 \times \mathcal{L}^*_\infty(\{\mathcal{F}_t\}; \mathcal{C}), \tag{5.3.26}$$

(see (5.2.29) and Notation 5.3.5 - (2)) together with the bilinear form on $\mathbb{U} \times \mathbb{Y}$ as:

$$\langle (u,V),(Y,Z) \rangle := E\left[uY(T)\right] + Z(V), \quad \text{for } (u,V) \in \mathbb{U} \text{ and } (Y,Z) \in \mathbb{Y}, \tag{5.3.27}$$

(c.f. the bilinear form at (4.5.22) for the pairing of the spaces at (4.5.18) and (4.5.21) in Problem 4.5.7). This completes all the choices required for Steps 3.1.1 - 3.1.2, and it remains to synthesize a Lagrangian and a dual function.

**Remark 5.3.10.** From the set-inclusion (5.3.17) we see that the space of dual variables at (5.3.26), resulting from essentially bounded perturbations (recall (5.3.23)), is larger than the space of dual variables at (5.2.42) which resulted from the use of square integrable perturbations (recall (5.2.39)). As we shall see from Proposition 5.3.15 which follows, the space of dual variables at (5.3.26) is large enough to contain a maximizer of the dual function $g(\cdot)$ that we will shortly construct (see (5.3.40) - (5.3.41)).

3. According to Step 3.1.3, define (recall (3.1.3)) the *Lagrangian function* $K : \Pi \times \mathbb{Y} \to [-\infty, +\infty]$ as:

$$K(\pi, (Y, Z)) := \inf_{(u,V)\in\mathbb{U}} \{\langle (u, V), (Y, Z)\rangle + F(\pi, (u, V))\}. \tag{5.3.28}$$

From (5.3.25) and (5.3.28), we have the basic inequality

$$f(\pi) \geqslant K(\pi, (Y, Z)), \qquad \text{for all } (\pi, (Y, Z)) \in \Pi \times \mathbb{Y}. \tag{5.3.29}$$

Now, the Lagrangian at (5.3.28) can be partially evaluated as follows: from (5.3.24) and (5.3.27) we obtain

$$K(\pi, (Y, Z)) = \begin{cases} \inf_{u\in\mathcal{L}_2(\Omega,\mathcal{F}_T,P)} \{E\left[uY(T) + J(X^\pi(T) - u)\right]\} \\ \quad + \inf_{V\in\mathcal{L}_\infty(\{\mathcal{F}_t\};\mathcal{C})} \left\{ Z(V) \;\middle|\; X^\pi + V \geqslant B \right\}, & \text{if } \pi \in \mathcal{A}, \\ +\infty, & \text{otherwise.} \end{cases} \tag{5.3.30}$$

From (2.1.24) and (5.2.36), we have

$$X^\pi(T), Y(T) \in \mathcal{L}_2(\Omega, \mathcal{F}_T, P) \quad \text{ for all } \pi \in \Pi \text{ and } Y \in \mathbb{B}_2. \tag{5.3.31}$$

Therefore, we can apply Proposition D.0.8 to the first term on the right side of (5.3.30) and get

$$\inf_{u\in\mathcal{L}_2(\Omega,\mathcal{F}_T,P)} \{E\left[uY(T) + J(X^\pi(T) - u)\right]\} = E\left[X^\pi(T)Y(T)\right] - E\left[J^*(Y(T))\right]. \tag{5.3.32}$$

In order to evaluate the right side of (5.3.30) define

$$\mathcal{A}_2 := \left\{ \pi \in \mathcal{A} \;\middle|\; \begin{array}{l} \text{there is some constant } \hat{\alpha} \in \mathbb{R} \text{ such that} \\ X^\pi(t) + \hat{\alpha} \geqslant B(t), \text{ all } t \in [0, T] \text{ a.s.} \end{array} \right\} \tag{5.3.33}$$

**Remark 5.3.11.** Effectively $\mathcal{A}_2$ is the set of all $\pi \in \mathcal{A}$ such that $X^\pi(t) - B(t)$ is a.s. essentially and uniformly in $t$ lower-bounded. From Condition 5.3.2, we know

$$\mathcal{A}_2 \neq \emptyset \qquad \text{and of course} \qquad \mathcal{A}_2 \subset \mathcal{A}. \tag{5.3.34}$$

The set $\mathcal{A}_2$ has a significance very similar to that of the set $\mathcal{A}_1$ that we introduced for the problem 4.5.2 with European wealth constraint (recall (4.5.29) and Remark 4.5.13). In particular, we shall see at Proposition 5.3.12 that for any $\pi \in \Pi \backslash \mathcal{A}_2$, we have $K(\pi,(Y,Z)) = +\infty$ for all dual variables $(Y,Z) \in \mathbb{Y}$. In this sense, portfolios $\pi \in \Pi$ outside $\mathcal{A}_2$ are somewhat pathological, and will turn out not be of any interest.

From (5.3.30) and (5.3.32) we get an explicit formula for the Lagrangian in the following proposition, whose proof is included in Appendix A:

**Proposition 5.3.12.** *For each $\pi \in \Pi$ (see (2.1.20)) and $(Y,Z) \in \mathbb{Y}$ (see (5.3.26)), the Lagrangian $K(\pi,(Y,Z))$ at (5.3.28) is given by* (recall (4.1.20), (5.3.33) and (5.3.10))

$$K(\pi,(Y,Z)) = \begin{cases} E\left[X^{\pi}(T)Y(T)\right] - E\left[J^{*}(Y(T))\right] & \\ + \inf_{V \in \mathcal{L}_{\infty}(\{\mathcal{F}_t\};\mathcal{C})} \{Z(V) \mid X^{\pi} + V \geqslant B\}, & \textit{if } \pi \in \mathcal{A}_2,\ Z \geqslant 0, \\ -\infty, & \textit{if } \pi \in \mathcal{A}_2,\ Z \ngeqslant 0, \\ +\infty, & \textit{if } \pi \in \Pi \backslash \mathcal{A}_2. \end{cases} \tag{5.3.35}$$

**Remark 5.3.13.** Suppose

$$X^{\pi} \in \mathcal{L}_{\infty}(\{\mathcal{F}_t\};\mathcal{C}) \quad \text{for all } \pi \in \Pi. \tag{5.3.36}$$

Then from (5.3.36) and (5.3.13), we see that

$$X^{\pi} - B \in \mathcal{L}_{\infty}(\{\mathcal{F}_t\};\mathcal{C}), \tag{5.3.37}$$

and then it follows from (5.3.37) that, when $Z \geqslant 0$, we have

$$\inf_{V \in \mathcal{L}_{\infty}(\{\mathcal{F}_t\};\mathcal{C})} \{Z(V) \mid X^{\pi} + V \geqslant B\} = Z(B - X^{\pi}). \tag{5.3.38}$$

Accordingly, from (5.3.38) and Proposition 5.3.12, when (5.3.36) holds, we have

$$K(\pi,(Y,Z)) = E\left[X^{\pi}(T)Y(T)\right] - E\left[J^{*}(Y(T))\right] + Z(B - X^{\pi}), \tag{5.3.39}$$

for all $\pi \in \mathcal{A}_2$ and $Z \geqslant 0$ (compare with the first line on the right side of (5.3.35)). The relation at (5.3.39) formally resembles the first line on the right side of (3.2.79), which is the Lagrangian for problem (3.2.64), and clearly identifies the functional $Z \in \mathcal{L}^{*}_{\infty}(\{\mathcal{F}_t\};\mathcal{C})$ as the "Lagrange" weighting for the inequality constraint $X^{\pi} \geqslant B$ in problem (5.2.1). Of course, (5.3.36) does not hold in general, and we only have $X^{\pi} \in \mathcal{L}_2(\{\mathcal{F}_t\};\mathcal{C})$ for each $\pi \in \Pi$. Accordingly, $Z(B - X^{\pi})$ is undefined when $Z \in \mathcal{L}^{*}_{\infty}(\{\mathcal{F}_t\};\mathcal{C})$. Therefore we must represent this Lagrange weighting indirectly by the infimum on the left side of (5.3.38).

**Remark 5.3.14.** In this remark, we compare the Lagrangian given by Proposition 5.3.12, which results from the perturbations at (5.3.23) and (5.3.24), with the Lagrangian given by Proposition 5.2.17, which results from the perturbations at (5.2.39) and (5.2.40). In the latter case, we see from Proposition 5.2.17 that there is no need to introduce a set resembling $\mathcal{A}_2$, and we can write the Lagrangian directly in terms of the set $\mathcal{A}$ at (5.2.49). It is the use of essentially bounded perturbation in the second factor space of (5.3.23), which requires the introduction of the subset $\mathcal{A}_2 \subset \mathcal{A}$ in order to write out the Lagrangian at (5.3.35). Furthermore, for the perturbations at (5.2.39) and (5.2.40), we see from (5.2.49) that

$$E\left[\int_0^T [B(t) - X^\pi(t)]\rho(\mathrm{d}t)\right]$$

explicitly gives the Lagrange weighting for the inequality constraint $X^\pi \geqslant B$ in terms of the dual variable $\rho$ when $\pi \in \mathcal{A}$ and $\rho \geqslant 0$. As we have noted at Remark 5.3.13, there is no such explicit Lagrange weighting for the inequality constraint $X^\pi \geqslant B$ when we use the perturbations at (5.3.23) and (5.3.24).

In the view of (3.1.4), the *dual function* $g : \mathbb{Y} \to [-\infty, \infty)$ is defined as:

$$\begin{aligned} g(Y,Z) \quad &:= \quad \inf_{\pi\in\Pi} K(\pi,(Y,Z)) \\ &\overset{(5.3.35)}{=} \begin{cases} -\varkappa(Y,Z) - E\left[J^*(Y(T))\right], & \text{if } Z \geqslant 0, \\ -\infty, & \text{otherwise,} \end{cases} \end{aligned} \tag{5.3.40}$$

where we have defined

$$\varkappa(Y,Z) := \sup_{\pi\in\mathcal{A}_2}\left\{-E\left[X^\pi(T)Y(T)\right] - \inf_{V\in\mathcal{L}_\infty(\{\mathcal{F}_t\};\mathcal{C})}\{Z(V) \mid X^\pi + V \geqslant B\}\right\}, \tag{5.3.41}$$

for all $(Y,Z) \in \mathbb{Y}$. We then have the following *weak duality* relation (c.f. (3.2.16)), which holds from (5.3.29) and (5.3.40):

$$f(\pi) \geqslant K(\pi,(Y,Z)) \geqslant g(Y,Z), \qquad \text{for all } (\pi,(Y,Z)) \in \Pi\times\mathbb{Y}. \tag{5.3.42}$$

The *dual problem* is then to maximize $g(Y,Z)$ over all $(Y,Z) \in \mathbb{Y} = \mathbb{B}_2 \times \mathcal{L}_\infty^*(\{\mathcal{F}_t\};\mathcal{C})$, that is, to establish

$$g(\bar{Y},\bar{Z}) = \sup_{(Y,Z)\in\mathbb{Y}}\{g(Y,Z)\}, \qquad \text{for some } (\bar{Y},\bar{Z}) \in \mathbb{Y}. \tag{5.3.43}$$

In the following proposition we shall see that Theorem 3.1.7 and Condition 5.3.2 are essential for securing existence of a maximizer $(\bar{Y},\bar{Z}) \in \mathbb{Y}$:

**Proposition 5.3.15.** *Assume the Slater-type Condition 5.3.2. Then, there exists some* $(\bar{Y},\bar{Z}) \in \mathbb{Y}$ *such that*

$$\inf_{\pi\in\Pi}\{f(\pi)\} = \sup_{(Y,Z)\in\mathbb{Y}}\{g(Y,Z)\} = g(\bar{Y},\bar{Z}) \in \mathbb{R}. \tag{5.3.44}$$

**Proof.** Define the norm $\|\cdot\|_{\mathbb{U}}$ on $\mathbb{U}$ as follows:

$$\|(u,V)\|_{\mathbb{U}} := \max\{\|u\|_2, \|V\|_{u(\infty)}\} \qquad \text{for all } (u,V) \in \mathbb{U}, \tag{5.3.45}$$

(recall (5.3.7) and (5.3.23)), and denote $\|\cdot\|_{\mathbb{U}}$-norm topology on $\mathbb{U}$ by $\mathcal{U}$. We need to verify that the locally convex topology $\mathcal{U}$ is $\langle \mathbb{U}, \mathbb{Y} \rangle$-compatible (recall Definition 3.1.5) for the duality pairing given by (5.3.23), (5.3.26) and (5.3.27), that is

$$\begin{cases} \mathbb{U} := \mathcal{L}_2(\Omega, \mathcal{F}_T, P) \times \mathcal{L}_\infty(\{\mathcal{F}_t\}; \mathcal{C}), \qquad \mathbb{Y} := \mathbb{B}_2 \times \mathcal{L}^*_\infty(\{\mathcal{F}_t\}; \mathcal{C}), \\ \langle (u, V), (Y, Z) \rangle := E\left[uY(T)\right] + Z(V), \quad \text{for } (u, V) \in \mathbb{U} \text{ and } (Y, Z) \in \mathbb{Y}. \end{cases} \tag{5.3.46}$$

Indeed, the mapping $(u, V) \to \langle (u, V), (Y, Z) \rangle$ is clearly $\mathcal{U}$-continuous on $\mathbb{U}$ for each $(Y, Z) \in \mathbb{Y}$. Next, fix any $\mathcal{U}$-continuous linear function $\phi^*$ on $\mathbb{U}$, we have

1) Since $\phi^*$ is linear on $\mathbb{U}$ we have

$$\phi^*(u, V) = \phi^*(u, 0) + \phi^*(0, V), \quad (u, V) \in \mathbb{U}. \tag{5.3.47}$$

2) Since $\phi^*$ is a linear functional continuous in the $\|\cdot\|_{\mathbb{U}}$-norm topology on $\mathbb{U}$ it follows that $V \to \phi^*(0, V)$ is linear and norm-continuous on $\mathcal{L}_\infty(\{\mathcal{F}_t\}; \mathcal{C})$, and there exists a $Z \in \mathcal{L}^*_\infty(\{\mathcal{F}_t\}; \mathcal{C})$ such that

$$\phi^*(0, V) = Z(V), \quad V \in \mathcal{L}_\infty(\{\mathcal{F}_t\}; \mathcal{C}). \tag{5.3.48}$$

3) Again, since $\phi^*$ is a linear functional continuous in the $\|\cdot\|_{\mathbb{U}}$-norm topology on $\mathbb{U}$, it follows that $u \to \phi^*(u, 0)$ is linear and norm-continuous on $\mathcal{L}_2(\Omega, \mathcal{F}_T, P)$. It then follows from the classical Riesz representation theorem applied to the Hilbert space $\mathcal{L}_2(\Omega, \mathcal{F}_T, P)$ that there exists some $\xi \in \mathcal{L}_2(\Omega, \mathcal{F}_T, P)$ such that

$$\phi^*(u, 0) = E\left[u\xi\right], \quad u \in \mathcal{L}_2(\Omega, \mathcal{F}_T, P). \tag{5.3.49}$$

From Proposition 4.1.9, there is a unique $Y \in \mathbb{B}_1$ such that $Y(T) = \xi$ a.s., and by (5.2.34), we also have $Y \in \mathbb{B}_2$. Thus, from (5.3.49),

$$\phi^*(u, 0) = E\left[uY(T)\right], \quad u \in \mathcal{L}_2(\Omega, \mathcal{F}_T, P). \tag{5.3.50}$$

Combining (5.3.47), (5.3.48) and (5.3.50), we conclude that there exists some $(Y, Z) \in \mathbb{Y}$ such that

$$\phi^*(u, V) = E\left[uY(T)\right] + Z(V) = \langle (u, V), (Y, Z) \rangle, \quad (u, V) \in \mathbb{U}. \tag{5.3.51}$$

Therefore, the locally convex topology $\mathcal{U}$ on $\mathbb{U}$ is $\langle \mathbb{U}, \mathbb{Y} \rangle$-compatible.

Having established the $\langle \mathbb{U}, \mathbb{Y} \rangle$-compatibility of $\mathcal{U}$ we are now going to use Theorem 3.1.7 to establish the stated existence of an optimal dual solution $(\bar{Y}, \bar{Z})$: We shall first use Condition 5.3.2 to establish that there exists some $\alpha \in (0, \infty)$ and some $\hat{\pi} \in \Pi$ such that

$$\sup\left\{F(\hat{\pi}, (u, V)) \mid (u, V) \in \mathbb{U} \text{ with } \|(u, V)\|_{\mathbb{U}} < \alpha\right\} < +\infty. \tag{5.3.52}$$

For the portfolio $\hat{\pi}$ and $\varepsilon \in (0, \infty)$ asserted in Condition 5.3.2 we clearly have

$$\hat{\pi} \in \mathcal{A} \text{ and } X^{\hat{\pi}} + V \geqslant B, \quad \text{for all } V \in \mathcal{L}_\infty(\{\mathcal{F}_t\}; \mathcal{C}) \text{ with } \|V\|_{u(\infty)} \leqslant \varepsilon. \tag{5.3.53}$$

Thus,

$$F(\hat{\pi},(u,V)) \overset{(5.3.24)}{\underset{(5.3.53)}{=}} E\left[J(X^{\hat{\pi}}-u)\right], \tag{5.3.54}$$

for all $V \in \mathcal{L}_\infty(\{\mathcal{F}_t\};\mathcal{C})$ with $\|V\|_{u(\infty)} \leqslant \varepsilon$ and all $u \in \mathcal{L}_2(\Omega,\mathcal{F}_T,P)$. Moreover, since $X^{\hat{\pi}}(T) \in \mathcal{L}_2(\Omega,\mathcal{F}_T,P)$, by Condition 2.2.1 and (2.2.3), we have

$$u \to E\left[J(X^{\hat{\pi}}-u)\right] \quad \text{is norm-continuous on } (\mathcal{L}_2(\Omega,\mathcal{F}_T,P),\|\cdot\|_2). \tag{5.3.55}$$

and it is immediate from (5.3.55) that there exists some $\varepsilon_1 \in (0,\infty)$ such that

$$\left|E\left[J(X^{\hat{\pi}}-u)\right]\right| \leqslant \left|E\left[J(X^{\hat{\pi}})\right]\right| + 1 \in \mathbb{R}, \quad u \in \mathcal{L}_2(\Omega,\mathcal{F}_T,P) \text{ with } \|u\|_2 \leqslant \varepsilon_1. \tag{5.3.56}$$

Put $\alpha := \min\{\varepsilon,\varepsilon_1\}$; then (5.3.52) follows from (5.3.54), (5.3.56) and (5.3.45), and we have therefore verified condition (3.1.9) of Theorem 3.1.7 when $\mathcal{U}$ is the $\|\cdot\|_{\mathbb{U}}$-norm topology on $\mathbb{U}$ (see (5.3.45)) and the perturbation function $F$ is defined at (5.3.24).

From $\langle \mathbb{U},\mathbb{Y}\rangle$-compatibility of the norm-topology $\mathcal{U}$ on $\mathbb{U}$, the fact that $F(\cdot)$ is convex on $\Pi \times \mathbb{U}$ and satisfies a consistency relation of the form (5.3.25), and Theorem 3.1.7, we obtain

$$\inf_{\pi\in\Pi} f(\pi) = \sup_{Y\in\mathbb{Y}} g(Y,Z) = g(\bar{Y},\bar{Z}), \quad \text{for some } (\bar{Y},\bar{Z}) \in \mathbb{Y}. \tag{5.3.57}$$

Moreover, from (5.3.5), we see that $\inf_{\pi\in\Pi} f(\pi) \in \mathbb{R}$, i.e., (5.3.57) takes a real value. □

**Remark 5.3.16.** From (5.2.36), (4.1.20) and Condition 2.2.1,

$$J^*(Y(T)) \text{ is } P\text{-integrable for all } Y \in \mathbb{B}_2, \tag{5.3.58}$$

and Proposition 5.3.15 ensures that $g(\bar{Y},\bar{Z}) \in \mathbb{R}$, therefore it follows from (5.3.40) that

$$\bar{Z} \geqslant 0 \qquad \text{and} \qquad \varkappa(\bar{Y},\bar{Z}) \in \mathbb{R}. \tag{5.3.59}$$

These conditions on $(\bar{Y},\bar{Z})$ will be useful when we construct the optimal portfolio in terms of the dual solution $(\bar{Y},\bar{Z})$ given by Proposition 5.3.15. The first step in the construction of the optimal portfolio is to establish the usual Kuhn-Tucker optimality conditions:

**Proposition 5.3.17. [Kuhn-Tucker Optimal Conditions]** *Suppose Condition 2.1.1, 2.1.2, 2.1.5, 2.1.11 on the market settings and Condition 2.2.1 on the quadratic criterion function $J$ given in (2.2.3). Then, for each $(\pi,(Y,Z)) \in \Pi \times \mathbb{Y}$, we have the following equivalence:*

$$f(\pi) = g(Y,Z) \iff \begin{cases} 1)\ \pi \in \mathcal{A}, \quad 2)\ X^\pi \geqslant B, \quad 3)\ Z \geqslant 0, \\ 4)\ \inf_{V\in\mathcal{L}_\infty(\{\mathcal{F}_t\};\mathcal{C})} \{Z(V) \mid X^\pi + V \geqslant B\} = 0, \\ 5)\ E\left[X^\pi(T)Y(T)\right] + \varkappa(Y,Z) = 0, \\ 6)\ X^\pi(T) = \partial J^*(Y(T)) \quad a.s., \end{cases} \tag{5.3.60}$$

*Here, $\partial J^*(\cdot)$ denotes the derivative function of the convex conjugate function $J^*(\cdot)$ given in (4.1.20), that is $\partial J^*(y,\omega) = (y - c(\omega))/a(\omega)$ for all $(y,\omega) \in \mathbb{R}\times\Omega$.*

**Remark 5.3.18.** It is instructive to compare the optimality conditions (5.3.60) 1) - 6) which result from the perturbations (5.3.23) and (5.3.24) with the optimality conditions (5.2.56) 1) - 6) which result from the perturbations (5.2.39) and (5.2.40), for we see that the structure of these conditions is almost identical. Indeed, conditions (5.3.60) 1) 2) 6) are identical to conditions (5.2.56) 1) 2) 6). Furthermore, conditions (5.3.60) 3) and (5.2.56) 3) insist on the usual non-negativity of the dual variable which enforces the inequality constraint $X^\pi \geqslant B$: this variable is $\rho \in \mathcal{L}_2(\{\mathcal{F}_t\}; \mathcal{BV}_0^r)$ for the perturbations at (5.2.39) and (5.2.40), and is $Z \in \mathcal{L}_\infty^*(\{\mathcal{F}_t\}; \mathcal{C})$ for the perturbations at (5.3.23) and (5.3.24). Condition (5.3.60) 4) is a complementary slackness condition for the inequality constraint $X^\pi \geqslant B$; if we knew that (5.3.36) held then from (5.3.38) we could write this complementary slackness condition in the more familiar form

$$Z(B - X^\pi) = 0. \tag{5.3.61}$$

However, (5.3.36) does not hold in general, so we must settle for the indirect statement of complementary slackness at (5.3.60) 4). Of course, condition (5.2.56) 4) states the comparable complementary slackness condition when $\rho \in \mathcal{L}_2(\{\mathcal{F}_t\}; \mathcal{BV}_0^r)$ is a dual variable for the inequality constraint $X^\pi \geqslant B$. Finally, conditions (5.3.60) 5) and (5.2.56) 5) are clearly similar complementary slackness conditions for the portfolio constraint $\pi \in \mathcal{A}$, bearing in mind the definitions of $\varkappa(Y, Z)$ at (5.3.41) and $\varkappa(Y, \rho)$ at (5.2.51).

**Proof of Proposition 5.3.17:**
From (5.3.4), we know

$$f(\pi) \in (-\infty, \infty] \quad \text{for all } \pi \in \Pi. \tag{5.3.62}$$

Also from (5.3.34) and (5.3.41), we have $\varkappa(Y, Z) \in (-\infty, \infty]$ for all $(Y, Z) \in \mathbb{Y}$, and together with (5.3.65),

$$g(Y, Z) \in [-\infty, \infty) \quad \text{for all } (Y, Z) \in \mathbb{Y}. \tag{5.3.63}$$

Thus, from (5.3.63), (5.3.62) and (5.3.42), for arbitrary $(\pi, (Y, Z)) \in \Pi \times \mathbb{Y}$ it follows that

$$f(\pi) = g(Y, Z) \iff \begin{cases} f(\pi) = K(\pi, (Y, Z)) \in \mathbb{R}, \\ g(Y, Z) = K(\pi, (Y, Z)) \in \mathbb{R}. \end{cases} \tag{5.3.64}$$

Now, fix some $(\pi, (Y, Z)) \in \Pi \times \mathbb{Y}$. Recalling (5.3.31) and (5.3.58), we see that

$$\text{both } E\left[X^\pi(T)Y(T)\right] \text{ and } E\left[J^*(Y(T))\right] \text{ are real-valued.} \tag{5.3.65}$$

From (5.3.65), together with (5.3.35) and (5.3.40), we get

$$\begin{aligned} & g(Y, Z) = K(\pi, (Y, Z)) \in \mathbb{R} \\ \iff & \begin{cases} 1)\ \pi \in \mathcal{A}_2 \text{ (recall (5.3.33))},\ 2)\ Z \geqslant 0,\ 3)\ \varkappa(Y, Z) \in \mathbb{R}, \\ 4)\ \inf\limits_{V \in \mathcal{L}_\infty(\{\mathcal{F}_t\}; \mathcal{C})} \{Z(V) \mid X^\pi + V \geqslant B\} \in \mathbb{R}, \\ 5)\ E\left[X^\pi(T)Y(T)\right] + \varkappa(Y, Z) + \inf\limits_{V \in \mathcal{L}_\infty(\{\mathcal{F}_t\}; \mathcal{C})} \{Z(V) \mid X^\pi + V \geqslant B\} = 0. \end{cases} \end{aligned} \tag{5.3.66}$$

Moreover, from (5.3.35) and (5.3.4), we have

$$
\begin{aligned}
& f(\pi) = K(\pi, (Y, Z)) \in \mathbb{R} \\
\iff & \begin{cases} 1)\ \pi \in \mathcal{A}_2 \text{ (recall (5.3.33))},\ 2)\ X^\pi \geqslant B,\ 3)\ Z \geqslant 0, \\ 4)\ E\left[J(X^\pi(T))\right] - E\left[X^\pi(T)Y(T)\right] + E\left[J^*(Y(T))\right] \\ \qquad = \inf_{V \in \mathcal{L}_\infty(\{\mathcal{F}_t\};\mathcal{C})} \{Z(V) \mid X^\pi + V \geqslant B\}. \end{cases}
\end{aligned} \tag{5.3.67}
$$

Next, we check both sides of equation (5.3.67)-4). On the right side, from (5.3.67)-2), we have $0 \geqslant B - X^\pi$ a.s., and therefore it follows that

$$
\{V \in \mathcal{L}_\infty(\{\mathcal{F}_t\};\mathcal{C}) \mid V \geqslant B - X^\pi\} \supset \{V \in \mathcal{L}_\infty(\{\mathcal{F}_t\};\mathcal{C}) \mid V \geqslant 0\}. \tag{5.3.68}
$$

Together with (5.3.67)-3), this gives

$$
\inf_{V \in \mathcal{L}_\infty(\{\mathcal{F}_t\};\mathcal{C})} \{Z(V) \mid X^\pi + V \geqslant B\} \overset{(5.3.68)}{\leqslant} \inf_{V \in \mathcal{L}_\infty(\{\mathcal{F}_t\};\mathcal{C})} \{Z(V) \mid V \geqslant 0\} \overset{(5.3.67)-3)}{=} 0. \tag{5.3.69}
$$

On the left side, recall (4.1.20) that $J^*(y) := \sup\{xy - J(x) \mid x \in \mathbb{R}\}$ for $y \in \mathbb{R}$, then

$$
J(X^\pi(T)) - X^\pi(T)Y(T) + J^*(Y(T)) \geqslant 0 \quad \text{a.s.} \tag{5.3.70}
$$

Therefore, by (5.3.69) and (5.3.70), we see that (5.3.67)-4) holds if and only if both sides equal zero:

$$
\begin{aligned}
& E\left[J(X^\pi(T))\right] - E\left[X^\pi(T)Y(T)\right] + E\left[J^*(Y(T))\right] \\
& = \inf_{V \in \mathcal{L}_\infty(\{\mathcal{F}_t\};\mathcal{C})} \{Z(V) \mid X^\pi + V \geqslant B\} = 0.
\end{aligned} \tag{5.3.71}
$$

Moreover, (5.3.71) together with (5.3.70) gives

$$
J(X^\pi(T)) - X^\pi(T)Y(T) + J^*(Y(T)) = 0 \quad \text{a.s.} \tag{5.3.72}
$$

and (5.3.72) is equivalent to (recall Remark 4.1.11)

$$
X^\pi(T) = \partial J^*(Y(T)) \quad \text{a.s.} \tag{5.3.73}
$$

Hence, from (5.3.67), (5.3.71) and (5.3.73), we get

$$
\begin{aligned}
& f(\pi) = K(\pi, (Y, Z)) \in \mathbb{R} \\
\iff & \begin{cases} 1)\ \pi \in \mathcal{A}_2,\ 2)\ X^\pi \geqslant B,\ 3)\ Z \geqslant 0, \\ 4)\ \inf_{V \in \mathcal{L}_\infty(\{\mathcal{F}_t\};\mathcal{C})} \{Z(V) \mid X^\pi + V \geqslant B\} = 0, \\ 5)\ X^\pi(T) = \partial J^*(Y(T)). \end{cases}
\end{aligned} \tag{5.3.74}
$$

From (5.3.33), we have

$$
\{\pi \in \mathcal{A}_2 \mid X^\pi \geqslant B\} = \{\pi \in \mathcal{A} \mid X^\pi \geqslant B\}, \tag{5.3.75}
$$

and then (5.3.75) together with (5.3.74) gives

$$\begin{aligned} & f(\pi) = K(\pi, (Y, Z)) \in \mathbb{R} \\ \iff & \begin{cases} 1) \ \pi \in \mathcal{A}, \ 2) \ X^{\pi} \geqslant B, \ 3) \ Z \geqslant 0, \\ 4) \ \inf\limits_{V \in \mathcal{L}_{\infty}(\{\mathcal{F}_t\};\mathcal{C})} \{Z(V) \mid X^{\pi} + V \geqslant B\} = 0, \\ 5) \ X^{\pi}(T) = \partial J^*(Y(T)). \end{cases} \end{aligned} \tag{5.3.76}$$

Finally, the equivalence (5.3.64), together with (5.3.76) and (5.3.66), gives (5.3.60). □

**Remark 5.3.19.** We shall now construct a portfolio $\bar{\pi} \in \Pi$, in terms of the dual solution $(\bar{Y}, \bar{Z}) \in \mathbb{Y}$ given by Proposition 5.3.15, to satisfy the Kuhn-Tucker relations (5.3.60)(1) - (6), that is

$$\begin{cases} 1) \ \bar{\pi} \in \mathcal{A}, \quad 2) \ X^{\bar{\pi}} \geqslant B, \quad 3) \ \bar{Z} \geqslant 0, \\ 4) \ \inf\limits_{V \in \mathcal{L}_{\infty}(\{\mathcal{F}_t\};\mathcal{C})} \left\{\bar{Z}(V) \mid X^{\bar{\pi}} + V \geqslant B\right\} = 0, \\ 5) \ E\left[X^{\bar{\pi}}(T)\bar{Y}(T)\right] + \varkappa(\bar{Y}, \bar{Z}) = 0, \\ 6) \ X^{\bar{\pi}}(T) = \partial J^*(\bar{Y}(T)). \end{cases} \tag{5.3.77}$$

It then follows from Proposition 5.3.17 that

$$f(\bar{\pi}) = g(\bar{Y}, \bar{Z}), \tag{5.3.78}$$

as required to establish that $\bar{\pi}$ is the optimal portfolio for Problem 5.3.4. We already know from Remark 5.3.16 that (5.3.77)-(3) holds. We next construct $\bar{\pi} \in \Pi$ such that the remaining conditions at (5.3.77) hold.

The next result, a generalization of Proposition 4.3.18, establishes a *necessary condition* resulting from the optimality of $\bar{Y}$ given by Proposition 5.3.15. This necessary condition will be essential for the later construction of a $\bar{\pi} \in \Pi$ which satisfies the conditions (5.3.77)-(1)(2)(4)(5)(6).

**Proposition 5.3.20.** *Suppose Condition 2.1.1, 2.1.2, 2.1.5, 2.2.1 and 2.2.4. Then, we have (recall the derivative function $\partial J^*(\cdot)$ at (4.1.21) and $\varkappa(\cdot)$ at (5.2.51))*

$$\varkappa(Y, Z) + E\left[\partial J^*(\bar{Y}(T))Y(T)\right] \geqslant 0, \quad \textit{for each } (Y, Z) \in \mathbb{Y} \textit{ with } Z \geqslant 0, \tag{5.3.79}$$

*and in particular,*

$$\varkappa(\bar{Y}, \bar{Z}) + E\left[\partial J^*(\bar{Y}(T))\bar{Y}(T)\right] = 0, \tag{5.3.80}$$

*where $(\bar{Y}, \bar{Z}) \in \mathbb{Y}$ is given by Proposition 5.3.15.*

**Proof.** From Proposition 5.3.15 and $\bar{Z} \geqslant 0$ (recall Remark 5.3.16), we have

$$g(\bar{Y} - \varepsilon\bar{Y}, \bar{Z} - \varepsilon\bar{Z}) \leqslant g(\bar{Y}, \bar{Z}) \ \text{ and } \ \bar{Z} - \varepsilon\bar{Z} \in (\mathcal{L}_{\infty}^*(\Omega, \mathcal{F}_T, P))^+, \tag{5.3.81}$$

for all $\varepsilon \in [0,1)$. In view of the first inequality of (5.3.81) and (5.3.40), we get

$$\varkappa(\bar{Y}-\varepsilon\bar{Y},\bar{Z}-\varepsilon\bar{Z})+E\left[J^*(\bar{Y}(T)-\varepsilon\bar{Y}(T))\right]\geqslant\varkappa(\bar{Y},\bar{Z})+E\left[J^*(\bar{Y}(T))\right], \tag{5.3.82}$$

for all $\varepsilon \in [0,1)$. From (5.3.41), one also sees that

$$\varkappa(\bar{Y}-\varepsilon\bar{Y},\bar{Z}-\varepsilon\bar{Z})=(1-\varepsilon)\varkappa(\bar{Y},\bar{Z}),\quad \text{for all } \varepsilon\in[0,1). \tag{5.3.83}$$

Using (5.3.83) in (5.3.82), together with $\varkappa(\bar{Y},\bar{Z})\in\mathbb{R}$ (recall (5.3.59)), we have

$$\varkappa(\bar{Y},\bar{Z})+E\left[\frac{J^*(\bar{Y}(T))-J^*(\bar{Y}(T)-\varepsilon\bar{Y}(T))}{\varepsilon}\right]\leqslant 0\quad \text{for all } \varepsilon\in[0,1). \tag{5.3.84}$$

From Remark 4.1.11, Condition 2.2.1 and dominated convergence, we get

$$\lim_{\varepsilon\to 0}E\left[\frac{J^*(\bar{Y}(T))-J^*(\bar{Y}(T)-\varepsilon\bar{Y}(T))}{\varepsilon}\right]=E\left[\partial J^*(\bar{Y}(T))\bar{Y}(T)\right], \tag{5.3.85}$$

and (5.3.85) with (5.3.84) gives (5.3.79):

$$\varkappa(\bar{Y},\bar{Z})+E\left[\partial J^*(\bar{Y}(T))\bar{Y}(T)\right]\leqslant 0. \tag{5.3.86}$$

Next, we again use the optimality of the dual solution $(\bar{Y},\bar{Z})$ given at Proposition 5.3.15:

$$g(\bar{Y}+\varepsilon Y,\bar{Z}+\varepsilon Z)\leqslant g(\bar{Y},\bar{Z})\quad \text{for all } (Y,Z)\in\mathbb{Y} \text{ and } \varepsilon\in(0,\infty). \tag{5.3.87}$$

If $Z\geqslant 0$, we have from (5.3.59) that

$$\bar{Z}+\varepsilon Z\in(\mathcal{L}^*_\infty(\Omega,\mathcal{F}_T,P))^+,\quad \text{for all } \varepsilon\in(0,\infty). \tag{5.3.88}$$

From (5.3.87), (5.3.88) and (5.3.40), we get

$$\varkappa(\bar{Y}+\varepsilon Y,\bar{Z}+\varepsilon Z)+E\left[J^*(\bar{Y}(T)+\varepsilon Y(T))\right]\geqslant\varkappa(\bar{Y},\bar{Z})+E\left[J^*(\bar{Y}(T))\right], \tag{5.3.89}$$

for all $(Y,Z)\in\mathbb{Y}$ with $Z\geqslant 0$ and $\varepsilon\in(0,\infty)$. On the other hand, from (5.3.41), one sees

$$\begin{aligned}
\varkappa(\bar{Y}+\varepsilon Y,\bar{Z}+\varepsilon Z) &= \sup_{\pi\in\mathcal{A}_2}\Big\{-E\left[X^\pi(T)(\bar{Y}(T)+\varepsilon Y(T))\right] \\
&\qquad\qquad - \inf_{V\in\mathcal{L}_\infty(\{\mathcal{F}_t\};\mathcal{C})}\Big\{(\bar{Z}+\varepsilon Z)(V)\;\Big|\; X^\pi+V\geqslant B\Big\}\Big\} \\
\leqslant\ & \sup_{\pi\in\mathcal{A}_2}\Big\{-E\left[X^\pi(T)\bar{Y}(T)\right]-E\left[\varepsilon X^\pi(T)Y(T)\right] \\
&\quad - \inf_{V\in\mathcal{L}_\infty(\{\mathcal{F}_t\};\mathcal{C})}\Big\{\bar{Z}(V)\;\Big|\; X^\pi+V\geqslant B\Big\} - \inf_{V\in\mathcal{L}_\infty(\{\mathcal{F}_t\};\mathcal{C})}\Big\{\varepsilon Z(V)\;\Big|\; X^\pi+V\geqslant B\Big\}\Big\} \\
\leqslant\ & \sup_{\pi\in\mathcal{A}_2}\Big\{-E\left[X^\pi(T)\bar{Y}(T)\right] - \inf_{V\in\mathcal{L}_\infty(\{\mathcal{F}_t\};\mathcal{C})}\Big\{\bar{Z}(V)\;\Big|\; X^\pi+V\geqslant B\Big\}\Big\} \\
&\quad + \sup_{\pi\in\mathcal{A}_2}\Big\{-\varepsilon E\left[X^\pi(T)Y(T)\right]-\varepsilon\inf_{V\in\mathcal{L}_\infty(\{\mathcal{F}_t\};\mathcal{C})}\Big\{Z(V)\;\Big|\; X^\pi+V\geqslant B\Big\}\Big\} \\
=\ & \varkappa(\bar{Y},\bar{Z})+\varepsilon\varkappa(Y,Z),\qquad \text{for all } (Y,Z)\in\mathbb{Y} \text{ with } Z\geqslant 0 \text{ and } \varepsilon\in(0,\infty).
\end{aligned} \tag{5.3.90}$$

Combining (5.3.90) and (5.3.89), together with $\varkappa(\bar{Y}, \bar{Z}) \in \mathbb{R}$ (recall (5.3.59)), we get

$$\varkappa(Y, Z) + E\left[\frac{J^*(\bar{Y}(T) + \varepsilon Y(T)) - J^*(\bar{Y}(T))}{\varepsilon}\right] \geqslant 0, \qquad \varepsilon \in (0, \infty),\ Z \geqslant 0. \tag{5.3.91}$$

It follows from (4.1.20), Condition 2.2.1 and dominated convergence, that

$$\lim_{\varepsilon \to 0} E\left[\frac{J^*(\bar{Y}(T) + \varepsilon Y(T)) - J^*(\bar{Y}(T))}{\varepsilon}\right] = E\left[\partial \bar{Y}(T) Y(T)\right], \quad \text{for } Y \in \mathbb{B}_2, \tag{5.3.92}$$

Combining (5.3.92) with (5.3.91), we have

$$\varkappa(Y, Z) + E\left[\partial J^*(\bar{Y}(T)) Y(T)\right] \geqslant 0, \quad \text{for each } (Y, Z) \in \mathbb{Y} \text{ with } Z \geqslant 0, \tag{5.3.93}$$

which is (5.3.79) as required. Then, (5.3.80) follows from (5.3.86) and (5.3.93). □

Motivated by the transversality condition (5.3.60)-6) and Proposition 4.2.9 with

$$\xi := \partial J^*(\bar{Y}(T)) \overset{(4.1.20)}{=} \frac{\bar{Y}(T) - c}{a} \underset{Condition 2.2.1}{\overset{(5.3.31)}{\in}} \mathcal{L}_2(\Omega, \mathcal{F}_T, P), \tag{5.3.94}$$

define the process $\bar{X}$ as

$$\bar{X}(t) := H^{-1}(t) E\left[H(T) \partial J^*(\bar{Y}(T)) \,\middle|\, \mathcal{F}_t\right] \qquad t \in [0, T], \tag{5.3.95}$$

(of course $\bar{Y}$ at (5.3.95) and (5.3.94) is the first member of the optimal dual solution $(\bar{Y}, \bar{Z}) \in \mathbb{Y}$, existence of which is established at Proposition 5.3.15). Then, from Proposition 4.2.9, there exists some a.e. unique $\mathbb{R}^N$-valued $\mathcal{F}_t$-progressively measurable process $\bar{\psi}$ such that

$$\int_0^T \left\|\bar{\psi}(t)\right\|^2 \mathrm{d}t < \infty \quad \text{a.s.} \qquad \text{and} \qquad H(t)\bar{X}(t) = \bar{X}(0) + \int_0^t \bar{\psi}'(s) \mathrm{d}W(s).$$

Motivated by (4.2.37), define the $\mathbb{R}^N$-valued $\mathcal{F}_t$-progressively measurable process

$$\bar{\pi}(t) := (\sigma'(t))^{-1}\left[H^{-1}(t)\bar{\psi}(t) + \bar{X}(t)\theta(t)\right]. \tag{5.3.96}$$

From Proposition 4.2.9, it follows that

$$\bar{\pi} \in \Pi, \tag{5.3.97}$$

and

$$\mathrm{d}\bar{X}(t) = \left[r(t)\bar{X}(t) + \bar{\pi}'(t)\sigma(t)\theta(t)\right] \mathrm{d}t + \bar{\pi}'(t)\sigma(t)\mathrm{d}W(t), \tag{5.3.98}$$

with the initial value

$$\bar{X}(0) = E\left[H(T)\partial J^*(\bar{Y}(T))\right], \tag{5.3.99}$$

(see (5.3.94) and (4.2.39)). We shall now establish that $\bar{\pi}$ is an admissible portfolio and $\bar{X}$ is the corresponding wealth process.

**Proposition 5.3.21.** *Suppose Condition 2.1.1, 2.1.2, 2.1.5, 2.1.11, 2.2.1 and 2.2.4, and recall $\theta$, $\bar{X}$ and $\bar{\pi}$ defined at* (2.1.8), (5.3.95) *and* (5.3.96). *Then $\bar{X}$ is the wealth process corresponding to $\bar{\pi}$, that is*

$$X^{\bar{\pi}} = \bar{X} \quad \text{so that} \quad X^{\bar{\pi}}(T) = \bar{X}(T) \overset{(5.2.88)}{=} \partial J^*(\bar{Y}(T)). \tag{5.3.100}$$

**Proof.** Comparing (5.3.98) and (5.2.60), we have from Proposition 5.2.25 that

$$\begin{aligned} & E\left[\bar{X}(T)Y(T) - \int_0^T \bar{X}(s)\varrho_Y(\mathrm{d}s)\right] \\ = \ & \bar{X}(0)Y(0) + E\left[\int_0^T \bar{\pi}'(s)\sigma(s)\left[Y(s)\theta(s) + \Lambda_Y(s)\right]\mathrm{d}s\right], \quad Y \in \mathbb{B}_2. \end{aligned} \tag{5.3.101}$$

We fix some $y \in \mathbb{R}$. From Lemma 4.1.8,

$$\text{there exists some } Y \in \mathbb{B}_1 \text{ such that } Y(0) = y \text{ and } \theta Y + \Lambda_Y = 0 \text{ a.e.} \tag{5.3.102}$$

and (5.3.102) together with Remark 5.2.10 give that

$$\text{there exists some } Y \in \mathbb{B}_2 \text{ such that } Y(0) = y,\ \theta Y + \Lambda_Y = 0 \text{ a.e. and } \varrho_Y \equiv 0. \tag{5.3.103}$$

Substituting $Y$ given by (5.3.103) in (5.3.101), we have

$$E\left[\bar{X}(T)Y(T)\right] \underset{(5.3.102)}{\overset{(5.3.101)}{=}} \bar{X}(0)y. \tag{5.3.104}$$

On the other hand, the necessary condition (5.3.79) holds for $(Y, 0) \in \mathbb{Y}$ where $Y$ is given by (5.3.103), i.e.

$$\varkappa(Y, 0) + E\left[\partial J^*(\bar{Y}(T))Y(T)\right] \geqslant 0. \tag{5.3.105}$$

From (5.2.66), (5.3.41) and (5.3.103), we have

$$\begin{aligned} \varkappa(Y, 0) & \overset{(5.3.41)}{=} \sup_{\pi \in \mathcal{A}_2} \left\{-E\left[X^\pi(T)Y(T)\right]\right\} \\ & \overset{(5.3.34)}{\leqslant} \sup_{\pi \in \mathcal{A}} \left\{-E\left[X^\pi(T)Y(T)\right]\right\} \underset{(5.3.103)}{\overset{(5.2.66)}{=}} -x_0 y. \end{aligned} \tag{5.3.106}$$

From (5.3.95), we have $\bar{X}(T) = \partial J^*(\bar{Y}(T))$, and combining this with (5.3.104), (5.3.105) and (5.3.106) gives

$$(\bar{X}(0) - x_0)y \geqslant 0. \tag{5.3.107}$$

By arbitrary choice of $y \in \mathbb{R}$ in (5.3.107), we get

$$\bar{X}(0) = x_0. \tag{5.3.108}$$

□

**Remark 5.3.22.** From (5.3.80) and (5.3.100), we see

$$\varkappa(\bar{Y},\bar{Z}) + E\left[X^{\bar{\pi}}(T)\bar{Y}(T)\right] = 0. \tag{5.3.109}$$

This verifies (5.3.77)-5).

**Proposition 5.3.23.** *Suppose Condition 2.1.1, 2.1.2, 2.1.5, 2.1.11, 2.2.1 and 2.2.4, and recall $\bar{\pi}$ defined at (5.3.96). Then*

$$\bar{\pi} \in \mathcal{A}, \tag{5.3.110}$$

*i.e. $\bar{\pi}$ satisfies (5.3.77)-1).*

**Proof.** First define a set

$$O := \left\{(\omega;t) \in \Omega \times [0,T] \mid \bar{\pi}(\omega;t) \in A\right\}. \tag{5.3.111}$$

From Lemma F.0.3 (also see Lemma 5.4.2 of [15, p.207]), corresponding to $\bar{\pi} \in \mathcal{F}^*$ there exists some $\mathbb{R}^N$-valued $\bar{\nu} \in \mathcal{F}^*$ such that

$$\begin{cases} \|\bar{\nu}(t)\| \leqslant 1, \left|\delta^*_{\mathbb{R}^N}\left\{-\bar{\nu}(t)\middle|A\right\}\right| \leqslant 1, & \text{a.e. on } \Omega \times [0,T], \\ \bar{\pi}'(t)\bar{\nu}(t) + \delta^*_{\mathbb{R}^N}\left\{-\bar{\nu}(t)\middle|A\right\} = 0, & \text{a.e. on } O, \\ \bar{\pi}'(t)\bar{\nu}(t) + \delta^*_{\mathbb{R}^N}\left\{-\bar{\nu}(t)\middle|A\right\} < 0, & \text{a.e. on } (\Omega \times [0,T]) \setminus O. \end{cases} \tag{5.3.112}$$

It is clear from $\|\bar{\nu}(t)\| \leqslant 1$ a.e. and Condition 2.1.5 that

$$\sigma^{-1}\bar{\nu} \in \Pi. \tag{5.3.113}$$

Thus, from (5.3.113) and Lemma 4.1.8,

$$\text{there exists some } Y \in \mathbb{B}_1 \text{ such that } Y(0) = 0 \text{ and } \theta Y + \Lambda_Y = \sigma^{-1}\bar{\nu} \text{ a.e.,} \tag{5.3.114}$$

and (5.3.114) together with Remark 5.2.10 give that

$$\text{there exists some } Y \in \mathbb{B}_2 \text{ such that } Y(0) = 0,\ \theta Y + \Lambda_Y = \sigma^{-1}\bar{\nu} \text{ a.e.and } \varrho_Y \equiv 0. \tag{5.3.115}$$

With the $Y \in \mathbb{B}_2$ given by (5.3.115), we certainly have (5.3.79) holds for $(Y, 0) \in \mathbb{Y}$, i.e.,

$$\varkappa(Y,0) + E\left[\partial J^*(\bar{Y}(T))Y(T)\right] \geqslant 0. \tag{5.3.116}$$

From (5.3.115) and (5.3.41), we have

$$\varkappa(Y,0) \overset{(5.3.41)}{=} \sup_{\pi\in\mathcal{A}_2}\left\{-E\left[X^{\pi}(T)Y(T)\right]\right\} \overset{(5.3.34)}{\leqslant} \sup_{\pi\in\mathcal{A}}\left\{-E\left[X^{\pi}(T)Y(T)\right]\right\}. \tag{5.3.117}$$

Combining (5.3.100) with (5.3.116) and (5.3.117), we get

$$\sup_{\pi\in\mathcal{A}}\left\{-E\left[X^{\pi}(T)Y(T)\right]\right\} + E\left[\bar{X}(T)Y(T)\right] \geqslant 0. \tag{5.3.118}$$

From Proposition 5.2.25, (5.3.98) and (5.3.115), we have

$$E\left[\bar{X}(T)Y(T)\right] = E\left[\int_0^T \pi'(s)\bar{\nu}(s)\mathrm{d}s\right], \tag{5.3.119}$$

and from Proposition 5.2.26 and (5.3.115), it follows that

$$\sup_{\pi\in\mathcal{A}}\{-E\left[X^{\pi}(T)Y(T)\right]\} = E\left[\int_0^T \delta^*_{\mathbb{R}^N}\left\{-\bar{\nu}(s)\middle|A\right\}\mathrm{d}s\right]. \tag{5.3.120}$$

Combining (5.3.119), (5.3.120) with (5.3.118), we get

$$E\left[\int_0^T \left(\bar{\pi}'(s)\bar{\nu}(s) + \delta^*_{\mathbb{R}^N}\left\{-\bar{\nu}(s)\middle|A\right\}\right)\mathrm{d}s\right] \geqslant 0, \tag{5.3.121}$$

which, together with (5.3.112), gives

$$P\otimes\lambda\left(O^c\right) = 0, \tag{5.3.122}$$

From (5.3.122) and (5.3.111), we get $\bar{\pi}(t)\in A$ a.e., and then $\bar{\pi}\in\mathcal{A}$ follows from (5.3.97). □

**Remark 5.3.24.** Having established Proposition 5.3.21 and Proposition 5.3.23, we have constructed a pair $(\bar{\pi},(\bar{Y},\bar{Z}))\in\mathbb{X}\times\mathbb{Y}$ such that

$$\bar{\pi}\in\mathcal{A},\quad \bar{Z}\geqslant 0 \text{ (recall (5.3.59) of Remark 5.3.16)},\quad \text{and } X^{\bar{\pi}}(T) = \partial J^*(\bar{Y}(T)), \tag{5.3.123}$$

so that (5.3.77) - 1), 3), 6) are verified. Moreover (5.3.77) - 5) is also verified (see Remark 5.3.22). We next verify (5.3.77) - 2), 4), again using the *necessary condition* given by Proposition 5.3.20.

**Proposition 5.3.25.** *Suppose Condition 2.1.1, 2.1.2, 2.1.5, 2.1.11, 2.2.1 and 5.1.1, and recall $\bar{\pi}$ defined at* (5.3.96)*. Then*

$$X^{\bar{\pi}}(t)\geqslant B(t), t\in[0,T], a.s., \tag{5.3.124}$$

*i.e. $\bar{\pi}$ satisfies* (5.3.77)-2).

**Proof.** Fix some arbitrary $\tau\in[0,T]$, and recalling Example 5.3.8 for $Z^{\bar{\pi}}_{\tau}$ at (5.3.18) we have from (5.3.20) and (5.3.22) that

$$Z^{\bar{\pi}}_{\tau}\in\mathcal{L}^*_{\infty}(\{\mathcal{F}_t\};\mathcal{C}) \quad \text{with} \quad Z^{\bar{\pi}}_{\tau}\geqslant 0. \tag{5.3.125}$$

From Lemma 4.1.8 and Remark 5.2.10, there is some $Y\in\mathbb{B}_2$ such that

$$Y(0) = 0,\quad \Lambda_Y + \theta Y = 0,\quad \text{and}\quad \varrho_Y = \rho^{\bar{\pi}}_{\tau}\quad \text{(recall (5.2.16))}. \tag{5.3.126}$$

From (5.3.126) and (5.2.61) (with $X := X^\pi$, compare (5.2.60) and (2.1.21)), we have

$$\begin{aligned} E\left[X^\pi(T)Y(T)\right] &= E\left[\int_0^T X^\pi(s)\rho_\tau^{\bar\pi}(\mathrm{d}s)\right] \\ &\overset{(5.2.16)}{=} E\left[X^\pi(\tau)\mathbb{1}_{\{X^{\bar\pi}<B\}}(\tau)\right], \quad \text{for all } \pi\in\Pi. \end{aligned} \tag{5.3.127}$$

From (5.3.127) and (5.3.97) we get

$$E\left[X^{\bar\pi}(T)Y(T) - B(\tau)\mathbb{1}_{\{X^{\bar\pi}<B\}}(\tau)\right] = E\left[\{X^{\bar\pi}(\tau) - B(\tau)\}\mathbb{1}_{\{X^{\bar\pi}<B\}}(\tau)\right]. \tag{5.3.128}$$

Now, evaluate $\varkappa(\cdot)$ at $(Y, Z_\tau^{\bar\pi})$ with $Z_\tau^{\bar\pi}$ defined by (5.3.18) and $Y$ given by (5.3.126):

$$\begin{aligned} \varkappa(Y, Z_\tau^{\bar\pi}) &\overset{(5.3.41)}{=} \sup_{\pi\in\mathcal{A}_2}\left\{-E\left[X^\pi(T)Y(T)\right] - \inf_{V\in\mathcal{L}_\infty(\{\mathcal{F}_t\};\mathcal{C})}\{Z_\tau^{\bar\pi}(V) \mid X^\pi + V \geqslant B\}\right\} \\ &\underset{(5.3.127)}{\overset{(5.3.18)}{=}} \sup_{\substack{\pi\in\mathcal{A}_2 \\ V\in\mathcal{L}_\infty(\{\mathcal{F}_t\};\mathcal{C})}} \left\{-E\left[\left(X^\pi(\tau)+V(\tau)\right)\mathbb{1}_{\{X^{\bar\pi}<B\}}(\tau)\right] \,\middle|\, X^\pi + V \geqslant B\right\}. \end{aligned} \tag{5.3.129}$$

Given any $\pi\in\mathcal{A}_2$ and $V\in\mathcal{L}_\infty(\{\mathcal{F}_t\};\mathcal{C})$ such that $X^\pi + V \geqslant B$, we have

$$B(\tau) - X^\pi(\tau) - V(\tau) \leqslant 0 \quad \text{a.s.}, \tag{5.3.130}$$

and then

$$\sup_{\substack{\pi\in\mathcal{A}_2 \\ V\in\mathcal{L}_\infty(\{\mathcal{F}_t\};\mathcal{C})}} \left\{E\left[\left(B(\tau) - X^\pi(\tau) - V(\tau)\right)\mathbb{1}_{\{X^{\bar\pi}<B\}}(\tau)\right] \,\middle|\, X^\pi + V \geqslant B\right\} \leqslant 0. \tag{5.3.131}$$

On the other hand, from (5.3.129), it follows that

$$\begin{aligned} &E\left[X^{\bar\pi}(T)Y(T) - B(\tau)\mathbb{1}_{\{X^{\bar\pi}<B\}}(\tau)\right] \\ &\quad + \sup_{\substack{\pi\in\mathcal{A}_2 \\ V\in\mathcal{L}_\infty(\{\mathcal{F}_t\};\mathcal{C})}} \left\{E\left[\left(B(\tau) - X^\pi(\tau) - V(\tau)\right)\mathbb{1}_{\{X^{\bar\pi}<B\}}(\tau)\right] \,\middle|\, X^\pi + V \geqslant B\right\} \\ &= E\left[X^{\bar\pi}(T)Y(T)\right] + \sup_{\substack{\pi\in\mathcal{A}_2 \\ V\in\mathcal{L}_\infty(\{\mathcal{F}_t\};\mathcal{C})}} \left\{-E\left[\left(X^\pi(\tau)+V(\tau)\right)\mathbb{1}_{\{X^{\bar\pi}<B\}}(\tau)\right] \,\middle|\, X^\pi + V \geqslant B\right\} \\ &\overset{(5.3.129)}{=} E\left[X^{\bar\pi}(T)Y(T)\right] + \varkappa(Y, Z_\tau^{\bar\pi}) \overset{(5.3.100)}{=} E\left[\partial J^*(\bar Y(T))Y(T)\right] + \varkappa(Y, Z_\tau^{\bar\pi}) \underset{(5.3.79)}{\overset{(5.3.125)}{\geqslant}} 0. \end{aligned} \tag{5.3.132}$$

Combining (5.3.132) and (5.3.131), we get

$$E\left[X^{\bar\pi}(T)Y(T) - B(\tau)\mathbb{1}_{\{X^{\bar\pi}<B\}}(\tau)\right] \geqslant 0. \tag{5.3.133}$$

Then, (5.3.133), together with (5.3.128), gives

$$E\left[\{X^{\bar\pi}(\tau) - B(\tau)\}\mathbb{1}_{\{X^{\bar\pi}<B\}}(\tau)\right] \geqslant 0. \tag{5.3.134}$$

Now,

$$P\left(X^{\bar{\pi}}(\tau) < B(\tau)\right) > 0 \quad \Rightarrow \quad E\left[\{X^{\bar{\pi}}(\tau) - B(\tau)\}\mathbb{1}_{\{X^{\bar{\pi}}<B\}}(\tau)\right] < 0. \tag{5.3.135}$$

From (5.3.134) and (5.3.135), we have

$$X^{\bar{\pi}}(\tau) \geqslant B(\tau) \quad \text{a.s.} \tag{5.3.136}$$

Fix a dense sequence of $\{\tau_n; n = 1, 2, 3, ...\} \subset [0, T]$. From (5.3.136) we have

$$X^{\bar{\pi}}(\omega; \tau_n) \geqslant B(\omega; \tau_n) \quad \omega \in \mathcal{N}_n^c, \quad \text{where } \mathcal{N}_n \in \mathcal{N}(P) \text{ is a } P\text{-null event.} \tag{5.3.137}$$

for $n = 1, 2, 3, ...$ Put

$$\mathcal{N} := \cup_{n=1}^{\infty} \mathcal{N}_n \in \mathcal{N}(P), \tag{5.3.138}$$

then, combining (5.3.137) with (5.3.138), we have

$$X^{\bar{\pi}}(\omega; \tau_n) \geqslant B(\omega; \tau_n) \quad \text{for all } n = 1, 2, 3, ... \text{ and } \omega \in \mathcal{N}^c. \tag{5.3.139}$$

From (5.3.139), the density of $\{\tau_n; n = 1, 2, 3, ...\} \subset [0, T]$ and the continuity of $X^{\bar{\pi}}$ and $B$ over $[0, T]$, we get (5.3.25) as required:

$$X^{\bar{\pi}}(t) \geqslant B(t) \quad t \in [0, T], \text{ a.s.} \tag{5.3.140}$$

□

**Remark 5.3.26.** From Proposition 5.3.25 and (5.3.33) we see that in fact $\bar{\pi} \in \mathcal{A}_2$. This observation will be needed shortly.

Finally, we verify that the complementary slackness condition (5.3.77) - 4):

**Proposition 5.3.27.** *Suppose Condition 2.1.1, 2.1.2, 2.1.5, 2.1.11, 2.2.1 and 5.1.1, and recall $\varkappa(\cdot)$ defined at (5.3.41), $(\bar{Y}, \bar{Z})$ at Proposition 5.3.15, and $X^{\bar{\pi}} = \bar{X}$ at (5.3.100) with $\bar{X}$ defined at (5.3.95). Then*

$$\inf_{V \in \mathcal{L}_\infty(\{\mathcal{F}_t\}; \mathcal{C})} \left\{\bar{Z}(V) \mid X^{\bar{\pi}} + V \geqslant B\right\} = 0, \tag{5.3.141}$$

*i.e. $(\bar{\pi}, (\bar{Y}, \bar{Z}))$ satisfies the complementary slackness relation (5.3.77)-4).*

**Proof.** From Proposition 5.3.25 we clearly have $X^{\bar{\pi}}(T) + V \geqslant b$ when $V = 0$, so it follows that

$$\inf_{V \in \mathcal{L}_\infty(\{\mathcal{F}_t\}; \mathcal{C})} \left\{\bar{Z}(V) \mid X^{\pi} + V \geqslant B\right\} \leqslant \bar{Z}(0) = 0. \tag{5.3.142}$$

Combining (5.3.100), (5.3.80) and (5.3.142), we get

$$\varkappa(\bar{Y}, \bar{Z}) + E\left[X^{\bar{\pi}}(T)\bar{Y}(T)\right] + \inf_{V \in \mathcal{L}_\infty(\{\mathcal{F}_t\}; \mathcal{C})} \left\{\bar{Z}(V) \mid X^{\pi} + V \geqslant B\right\} \leqslant 0. \tag{5.3.143}$$

On the other hand, (5.3.41) together with $\bar{\pi} \in \mathcal{A}_2$ (recall Remark 5.3.26) immediately gives the opposite direction of the above inequality, and therefore

$$\varkappa(\bar{Y}, \bar{Z}) + E\left[X^{\bar{\pi}}(T)\bar{Y}(T)\right] + \inf_{V \in \mathcal{L}_\infty(\{\mathcal{F}_t\}; \mathcal{C})} \left\{\bar{Z}(V) \mid X^{\pi} + V \geqslant B\right\} = 0. \tag{5.3.144}$$

Combining (5.3.109) and (5.3.144), we get (5.3.141). □

**Remark 5.3.28.** In view of Remark 5.3.24, together with Proposition 5.3.25 and Proposition 5.3.27, all conditions in (5.3.77) of Remark 5.3.19 have now been verified. We therefore have the following:

$$\begin{cases} \bar{\pi} \text{ defined in terms of the optimal dual solution } (\bar{Y}, \bar{Z}) \\ \text{(see Proposition 5.3.15) by (5.3.96) and (5.3.95) is an} \\ \text{optimal portfolio for the canonical problem (5.2.1).} \end{cases} \tag{5.3.145}$$

# Chapter 6

# Conclusions and Further Developments

In this thesis we concentrate on problems of *quadratic risk minimization* subject to various constraints (see (2.2.2)), with particular attention devoted to the *canonical problem* (2.2.21) which includes the combination of a convex portfolio constraint (that is a "control constraint") together with an American wealth constraint (that is a "state constraint") over the trading interval. Even in the case of deterministic optimal control it is well known that the combination of control constraints and state constraints presents clear challenges, and these challenges are of course correspondingly magnified in the setting of stochastic optimal control. The canonical problem (2.2.21) nevertheless has two very nice special properties which we exploit in this thesis, namely the problem is *convex* and the American wealth constraint is a simple a.s. *inequality constraint*. These special properties are key to applying a *variational approach* of Rockafellar (summarized in Chapter 3), which is a constructive method for synthesizing an appropriate vector space of *dual variables* together with a *dual function* defined on the space of dual variables. In synthesizing these entities the all-important thing to keep in mind is that the space of dual variables must be *large enough* to contain a maximizer of the dual function, since it is the maximizers of the dual function which constitute the *Lagrange multipliers* for the constraints. The variational method of Rockafellar enables one to "experiment" with "candidate" spaces of dual variables and dual functions in order to attain this goal, by appropriately choosing a vector space of *perturbations* and a *perturbation function* (see Step 3.1.1 in Chapter 3), together with a *bilinear pairing* of the space of perturbations with a vector space of dual variables (see Step 3.1.2 in Chapter 3). Associated with the variational method of Rockafellar is an extremely powerful tool, in the form of Theorem 3.1.7, which guides the choice of these entities in order to secure existence (in the space of dual variables) of a maximizer of the dual function.

We emphasize in particular the *constructive* or *synthetic* aspect of the Rockafellar variational approach. All too often, in the applications of convex duality to problems of mathematical finance, the space of dual variables and the dual functional are seemingly "pulled out of thin air", with no clear origin or motivation, and then shown retrospectively to work. This point is discussed in some detail in the introductory comments of Rogers

[33] and of Klein and Rogers [16]. In fact, the works [16] and [33] are concerned primarily with establishing a synthetic or constructive approach for arriving at a dual formulation which reduces this guesswork, the focus being on problems of utility maximization instead of the quadratic minimization with which the present thesis is concerned. For this reason, detailed comparisons of the constructive approach in this thesis with that in [16] and [33] are rather difficult to make. Nevertheless, the following comments seem to be in order: The constructive approach of [16] and [33] cannot be used to address problems with portfolio constraints, as is noted by Klein and Rogers (see "Remark: Convex Constraints" on page 239 of [16]). This is in direct contrast to the variational approach used in the present thesis, which applies not only to problems with portfolio constraints (see problem (4.3.2)) but also to problems with a combination of portfolio and wealth constraints (see problem (4.5.2) and the canonical problem (2.2.21)). Furthermore, in [16] and [33], the dual variables are assumed *a-priori* to be semimartingales (referred to in these works as "Lagrangian semimartingales"). Consequently, the approach established in [16] and [33] cannot be used to address problems with involve constraints on the wealth process, since these constraints necessarily require dual variables which are not semimartingales. Indeed, for problem (4.5.2) we know that the space of dual variables comprises *pairs* $(Y, z) \in \mathbb{Y} := \mathbb{B}_1 \times \mathcal{L}^*_\infty(\Omega, \mathcal{F}_T, P)$ (see (4.5.21)) in which the first dual variable $Y \in \mathbb{B}_1$ is a semimartingale necessary for the constraints of the wealth dynamics together with the portfolio constraint, while the second dual variable $z \in \mathcal{L}^*_\infty(\Omega, \mathcal{F}_T, P)$ (which is of course not a semimartingale) is essential for dealing with the a.s. constraint $X^\pi(T) \geqslant b$ applied to the wealth at close of trade. Similarly, for the canonical problem (2.2.21), the space of dual variables again comprises pairs $(Y, Z) \in \mathbb{Y} := \mathbb{B}_2 \times \mathcal{L}^*_\infty(\{\mathcal{F}_t\}; \mathcal{C})$ (see (5.3.26)), in which in particular the second dual variable $Z \in \mathcal{L}^*_\infty(\{\mathcal{F}_t\}; \mathcal{C})$ (not a semimartingale) is needed for the a.s. American wealth constraint $X^\pi \geqslant B$.

An important "sub-theme" of the thesis is the unifying aspect of the Rockafellar variational approach. To make this point clear we have demonstrated in Chapter 4 how this approach can be used to recover several results in quadratic risk minimization which were previously obtained by a miscellany of rather problem-specific methods (see the discussion of Section 4.6). We also wish to emphasize the role of simple finite-dimensional convex optimization problems as "precursors" for the stochastic control problems addressed in the thesis, for these simple problems often encapsulate in a very simple setting essential ideas such as Lagrangian functions and Kuhn-Tucker relations, which preserve their basic form when one generalizes to problems of stochastic control (see Remark 4.3.6, Remark 4.5.2, Remark 4.5.20 and Remark 5.2.23). In particular, finite dimensional problems are an ideal setting within which one can acquire some understanding of how to "perturb" a given primal problem. It is for this reason that we have given a fairly comprehensive treatment of several simple finite dimensional convex optimization problems from the standpoint of the Rockafellar variational approach in Section 3.2 (as well as because such an account seems to be lacking anywhere in the published literature). As is noted in Remark 4.2.4, Remark 4.3.4, Remark 4.4.8, Remark 4.5.11 and Remark 5.2.13, there is a clear line of descent from the perturbations used in the finite dimensional problems to the perturbations used in the stochastic control problems.

As regards future developments, there are clearly a number of directions in which the work of this thesis can be continued. We briefly indicate two such possibilities:

**(I)** The space of dual variables for the canonical problem (2.2.21) (that is, the problem (5.2.1) addressed in Chapter 5) is

$$\mathbb{Y} := \mathbb{B}_2 \times \mathcal{L}_\infty^*(\{\mathcal{F}_t\}; \mathcal{C}), \tag{6.1}$$

(see (5.3.26), (5.2.29), (5.2.28) and (5.2.12)). Proposition 5.3.15 establishes the existence of an optimal dual solution $(\bar{Y}, \bar{Z}) \in \mathbb{Y}$ subject to the Slater-type Condition 5.3.2, and an optimal portfolio $\bar{\pi}$ for the canonical problem (2.2.21) is then constructed in terms of $(\bar{Y}, \bar{Z})$ (see (5.3.96) and (5.3.95)). In principle this construction solves the canonical problem (2.2.21). There is, nevertheless, an element of understanding which remains to be cleared up in this solution, since a concrete representation of $\bar{Z} \in \mathcal{L}_\infty^*(\{\mathcal{F}_t\}; \mathcal{C})$ (which is the Lagrange multiplier for the American wealth constraint $X^\pi \geqslant B$ in the canonical problem (2.2.21), recall Remark 5.3.13) is lacking. One expects elements of $\mathcal{L}_\infty^*(\{\mathcal{F}_t\}; \mathcal{C})$ to have a structure which is "analogous" to the structure of members of $\mathcal{L}_\infty^*(S, \Sigma, \mu)$ (the norm-adjoint of the space of essentially bounded random variables $\mathcal{L}_\infty(S, \Sigma, \mu)$, recall Notation 3.3.1) given by the Yosida-Hewitt decomposition Theorem 3.3.9. In the present case, in place of the norm-adjoint of the space $\mathcal{L}_\infty(S, \Sigma, \mu)$ of essentially bounded random variables, we are instead dealing with the norm-adjoint of the space $\mathcal{L}_\infty(\{\mathcal{F}_t\}; \mathcal{C})$, the vector space of essentially bounded, continuous and $\mathcal{F}_t$-adapted processes (see (5.3.6)). There is in fact a highly suggestive result of Ioffe and Levin (Theorem 3 on page 57 of [13]), which extends the Yosida-Hewitt decomposition theorem to members of the norm-adjoint of the Banach space of all essentially bounded mappings from a finite measure space $(S, \Sigma, \mu)$ into a separable Banach space. Unfortunately, the requirement that members of $\mathcal{L}_\infty(\{\mathcal{F}_t\}; \mathcal{C})$ be $\mathcal{F}_t$-adapted completely ruins any attempt to tailor or extend the result of [13] to get a decomposition of members of $\mathcal{L}_\infty^*(\{\mathcal{F}_t\}; \mathcal{C})$. Despite this setback, there is something very valuable to be learned from the extended Yosida-Hewitt decomposition due to Ioffe and Levin [13], for this result strongly suggests that a linear functional $Z \in \mathcal{L}_\infty^*(\{\mathcal{F}_t\}; \mathcal{C})$ has a "regular part" of the form

$$Z_r(V) := E\left[\int_0^T V(t)\rho(\mathrm{d}t)\right], \qquad V \in \mathcal{L}_\infty(\{\mathcal{F}_t\}; \mathcal{C}), \tag{6.2}$$

for some "kernel" $\rho \in \mathcal{L}_1(\{\mathcal{F}_t\}; \mathcal{BV}_0^r)$, where

$$\mathcal{L}_1(\{\mathcal{F}_t\}; \mathcal{BV}_0^r) := \left\{\rho : \Omega \to \mathcal{BV}_0^r[0,T] \;\middle|\; \rho \in \mathcal{F}^* \text{ and } E\left[\|\rho\|_T\right] < \infty\right\}, \tag{6.3}$$

(compare with (5.2.12)). It is easily seen from (6.2)-(6.3) that $Z_r \in \mathcal{L}_\infty^*(\{\mathcal{F}_t\}; \mathcal{C})$, and that the kernel $\rho \in \mathcal{L}_1(\{\mathcal{F}_t\}; \mathcal{BV}_0^r)$ plays a role analogous to that of the kernel $l_r \in \mathcal{L}_1(S, \Sigma, \mu)$ in the Yosida-Hewitt decomposition of $l \in \mathcal{L}_\infty^*(S, \Sigma, \mu)$ given by (3.3.26). In light of this, we expect each $Z \in \mathcal{L}_\infty^*(\{\mathcal{F}_t\}; \mathcal{C})$ to have the unique decomposition

$$Z := Z_r + Z_s, \tag{6.4}$$

where $Z_r$ is given by (6.2) for some kernel $\rho \in \mathcal{L}_1(\{\mathcal{F}_t\}; \mathcal{BV}_0^r)$, and $Z_s$ is a member of $\mathcal{L}_\infty^*(\{\mathcal{F}_t\}; \mathcal{C})$, which is "singular" in some sense. To establish a decomposition of this kind, we plan to imitate modern proofs of the classical Yosida-Hewitt decomposition Theorem 3.3.9, which we briefly summarize next (see section 27 of Zaanen [43] for the full details). Define an order relation on $\mathcal{L}_\infty^*(S,\Sigma,\mu)$ as follows: for $l_1, l_2 \in \mathcal{L}_\infty^*(S,\Sigma,\mu)$, we put

$$l_1 \geqslant l_2 \quad \Longleftrightarrow \quad l_1 - l_2 \geqslant 0 \qquad \text{(recall Remark 3.3.11)}. \tag{6.5}$$

An easily established property of $\mathcal{L}_\infty^*(S,\Sigma,\mu)$ is that it is a *Riesz space* in the following sense: given $l_1, l_2 \in \mathcal{L}_\infty^*(S,\Sigma,\mu)$, there exists a unique $l_3 \in \mathcal{L}_\infty^*(S,\Sigma,\mu)$ such that

- $l_3 \geqslant l_1$ and $l_3 \geqslant l_2$;
- for each $l \in \mathcal{L}_\infty^*(S,\Sigma,\mu)$ such that $l \geqslant l_1$ and $l \geqslant l_2$, it necessarily follows that $l \geqslant l_3$.

That is, there exists a *least upper bound* $l_3 \in \mathcal{L}_\infty^*(S,\Sigma,\mu)$ of the pair $l_1, l_2 \in \mathcal{L}_\infty^*(S,\Sigma,\mu)$. We denote this least upper bound by $l_1 \vee l_2$. In an analogous way, there exists a *greatest lower bound* for $l_1, l_2 \in \mathcal{L}_\infty^*(S,\Sigma,\mu)$, which we denote by $l_1 \wedge l_2$. For each $l \in \mathcal{L}_\infty^*(S,\Sigma,\mu)$, we define the *lattice absolute value* of $l$ to be the element of $\mathcal{L}_\infty^*(S,\Sigma,\mu)$ given by

$$|l| := l \vee (-l). \tag{6.6}$$

The elements $l_1, l_2 \in \mathcal{L}_\infty^*(S,\Sigma,\mu)$ are said to be *mutually disjoint* (or *mutually alien*) when $|l_1| \wedge |l_2| = 0$, which is denoted by $l_1 \perp l_2$, that is

$$l_1 \perp l_2 \quad \Longleftrightarrow \quad |l_1| \wedge |l_2| = 0, \qquad \text{for each } l_1, l_2 \in \mathcal{L}_\infty^*(S,\Sigma,\mu). \tag{6.7}$$

A further property of $\mathcal{L}_\infty^*(S,\Sigma,\mu)$ is that it is *Dedekind complete* in the following sense: if a subset $D \subset \mathcal{L}_\infty^*(S,\Sigma,\mu)$ is upper bounded in the sense that there is some $\tilde{l} \in \mathcal{L}_\infty^*(S,\Sigma,\mu)$ such that $\tilde{l} \geqslant l$ for each $l \in D$, then there exists a unique $l^* \in \mathcal{L}_\infty^*(S,\Sigma,\mu)$ with the following properties

- $l^* \geqslant l$ for each $l \in D$ (i.e. $l^*$ is an upper bound of D);
- if $\hat{l} \in \mathcal{L}_\infty^*(S,\Sigma,\mu)$ is any upper bound of $D$ (that is $\hat{l} \geqslant l$ for each $l \in D$) then $\hat{l} \geqslant l^*$.

In other words, each upper-bounded subset $D \in \mathcal{L}_\infty^*(S,\Sigma,\mu)$ has a *least upper bound* in $\mathcal{L}_\infty^*(S,\Sigma,\mu)$, which is denoted by $\sup\{D\}$. With these order-theoretic preliminaries cleared up we can now summarize the essential idea for establishing the Yosida-Hewitt decomposition Theorem 3.3.9. Define

$$\mathfrak{M} := \left\{ l \in \mathcal{L}_\infty^*(S,\Sigma,\mu) \;\middle|\; \begin{array}{l} l(f) = E\left[l_r f\right], f \in \mathcal{L}_\infty(S,\Sigma,\mu), \\ \text{for some kernel } l_r \in \mathcal{L}_1(S,\Sigma,\mu) \end{array} \right\}, \tag{6.8}$$

and put

$$\mathfrak{M}^d := \{ l_s \in \mathcal{L}_\infty^*(S,\Sigma,\mu) \mid l_s \perp l \qquad \text{for all } l \in \mathfrak{M} \}, \tag{6.9}$$

($\mathfrak{M}^d$ is called the *lattice disjoint complement* of $\mathfrak{M}$). Then $\mathfrak{M}$ and $\mathfrak{M}^d$ are vector subspaces of $\mathcal{L}_\infty^*(S,\Sigma,\mu)$, and $\mathfrak{M}$ can be shown to be a *band* in $\mathcal{L}_\infty^*(S,\Sigma,\mu)$ (see Definition 7.1 on page 27 of Zaanen [43]). The essential property of bands is given by the *projection band theorem* of Riesz, which essentially states the following (see Theorem 11.4 on page 57 of Zaanen [43]): if $\mathfrak{M}$ is any band in a Dedekind complete Riesz space $\mathfrak{R}$, then $\mathfrak{R}$ is the *direct sum* of the band $\mathfrak{M}$ and its lattice disjoint complement: $\mathfrak{R} = \mathfrak{M} \oplus \mathfrak{M}^d$. Applying this result to the Dedekind complete Riesz space $\mathfrak{R} := \mathcal{L}_\infty^*(S,\Sigma,\mu)$ with the band $\mathfrak{M}$ defined by (6.8) and disjoint complement at (6.9), gives

$$\mathcal{L}_\infty^*(S,\Sigma,\mu) = \mathfrak{M} \oplus \mathfrak{M}^d. \tag{6.10}$$

From (6.10), (6.9) and (6.8) we obtain the following: for each $l \in \mathcal{L}_\infty^*(S,\Sigma,\mu)$ there exists a unique kernel $l_r \in \mathcal{L}_1(S,\Sigma,\mu)$ and a unique $l_s \in \mathfrak{M}^d$ such that

$$l(f) = E\left[l_r f\right] + l_s(f), \qquad f \in \mathcal{L}_\infty(S,\Sigma,\mu). \tag{6.11}$$

Finally, using the notion of mutually disjoint defined at (6.7) together with (6.9) and some tools of measure theory, one can establish that

$$\mathfrak{M}^d = \mathcal{Z}(S,\Sigma,\mu), \tag{6.12}$$

(recall Definition 3.3.6). We see that (6.11) and (6.12) give the Yosida-Hewitt decomposition Theorem 3.3.9.

Our goal is to generalize the "roadmap" summarized above to get a decomposition of the form (6.4) for members $Z \in \mathcal{L}_\infty^*(\{\mathcal{F}_t\};\mathcal{C})$. We have defined a notion of order $\geqslant$ on $\mathcal{L}_\infty^*(\{\mathcal{F}_t\};\mathcal{C})$ at (5.3.10), which is analogous to the notion of order defined on $\mathcal{L}_\infty^*(S,\Sigma,\mu)$ at (6.5). With this notion of order one can establish that $\mathcal{L}_\infty^*(\{\mathcal{F}_t\};\mathcal{C})$ is a Riesz space which is Dedekind complete, so that in particular the least upper bound $Z_1 \vee Z_2$ and greatest lower bound $Z_1 \wedge Z_2$ exist in $\mathcal{L}_\infty^*(\{\mathcal{F}_t\};\mathcal{C})$ for each $Z_1, Z_2 \in \mathcal{L}_\infty^*(\{\mathcal{F}_t\};\mathcal{C})$. Again, for $Z \in \mathcal{L}_\infty^*(\{\mathcal{F}_t\};\mathcal{C})$ we can define the lattice absolute value

$$|Z| := Z \vee (-Z) \in \mathcal{L}_\infty^*(\{\mathcal{F}_t\};\mathcal{C}), \tag{6.13}$$

and we can define the notion of *mutually disjoint* exactly as at (6.7):

$$Z_1 \perp Z_2 \quad \iff \quad |Z_1| \wedge |Z_2| = 0, \quad \text{for each } Z_1, Z_2 \in \mathcal{L}_\infty^*(\{\mathcal{F}_t\};\mathcal{C}). \tag{6.14}$$

Motivated by the plausible regular elements defined at (6.2) - (6.3), together with (6.8), we now define the vector space of "regular" elements of $\mathcal{L}_\infty^*(\{\mathcal{F}_t\};\mathcal{C})$ namely

$$\begin{aligned}\mathfrak{N} := \Big\{ Z \in \mathcal{L}_\infty^*(\{\mathcal{F}_t\};\mathcal{C}) \;\Big|\; Z(V) = E\left[\int_0^T V(t)\rho(\mathrm{d}t)\right],\ V \in \mathcal{L}_\infty(\{\mathcal{F}_t\};\mathcal{C}),\\ \text{for some kernel } \rho \in \mathcal{L}_1(\{\mathcal{F}_t\};\mathcal{BV}_0^r)\Big\},\end{aligned} \tag{6.15}$$

together with its lattice disjoint complement

$$\mathfrak{N}^d := \{Z_s \in \mathcal{L}_\infty^*(\{\mathcal{F}_t\};\mathcal{C}) \mid Z_s \perp Z \text{ for all } Z \in \mathfrak{N}\}. \tag{6.16}$$

If it can be shown that $\mathfrak{N}$ is a band in the Dedekind complete Riesz space $\mathcal{L}_\infty^*(\{\mathcal{F}_t\};\mathcal{C})$, then again from the projection band theorem of Riesz, we will have

$$\mathcal{L}_\infty^*(\{\mathcal{F}_t\};\mathcal{C}) = \mathfrak{N} \oplus \mathfrak{N}^d, \tag{6.17}$$

and then, from (6.17) and (6.15), we will have the following: for each $Z \in \mathcal{L}_\infty^*(\{\mathcal{F}_t\};\mathcal{C})$, there exists a unique kernel $\rho \in \mathcal{L}_1(\{\mathcal{F}_t\};\mathcal{BV}_0^r)$ and a unique $Z_s \in \mathfrak{N}^d$ such that

$$Z(V) = E\left[\int_0^T V(t)\rho(\mathrm{d}t)\right] + Z_s(V). \quad V \in \mathcal{L}_\infty(\{\mathcal{F}_t\};\mathcal{C}), \tag{6.18}$$

This result is close to a Yosida-Hewitt decomposition of $Z \in \mathcal{L}_\infty^*(\{\mathcal{F}_t\};\mathcal{C})$. The critical and most challenging step is to prove that $\mathfrak{N}$ is a band in $\mathcal{L}_\infty^*(\{\mathcal{F}_t\};\mathcal{C})$. This mirrors the fact that proving that $\mathfrak{M}$ is a band in $\mathcal{L}_\infty^*(S,\Sigma,\mu)$ is also the most technically involved step in establishing the classical Yosida-Hewitt decomposition. The tools for showing that $\mathfrak{N}$ is a band in $\mathcal{L}_\infty^*(\{\mathcal{F}_t\};\mathcal{C})$ seem to rely on measure-theoretic results for Banach space-valued random variables, in particular a Radon-Nikodym theorem for these random variables. Such results are available in the literature but at this point we have not yet been able to show that $\mathfrak{N}$ is a band. Although proving this result is turning out to be rather challenging, we do not believe that it is beyond hope, especially since the result itself is quite plausible. There also remains the task of ascertaining the structure of members of $\mathfrak{N}^d$, that is proving an "analogue" of (6.12).

**(II)** Another possible future development is to address the problem of *utility maximization* subject to a combination of convex portfolio constraints together with an American wealth constraint over the whole trading interval, that is we effectively substitute utility maximization for quadratic minimization in the canonical problem (2.2.21). As should be clear from Chapter 3, the Rockafellar variational approach relies only on convexity of the primal problem, and problems of utility maximization with this combination of constraints are convex after a sign change in the object function. In principle, therefore, the Rockafellar variational approach should apply to utility maximization. However, it is important not to underestimate the challenges involved. These arise from the fact that problems of utility maximization are formulated over wealth processes which are general semimartingales without any integrability properties, in contrast, to the canonical problem (2.2.21), for which the wealth processes are *square integrable* Itô processes. This square integrability greatly simplified application of the Rockafellar variational approach. For example, we used this in the perturbation $E\left[J(X^\pi(T)-u)\right]$ for $u \in \mathcal{L}_2(\Omega,\mathcal{F}_T,P)$ at (5.3.24). This perturbation is clearly no longer appropriate if we replace the quadratic function $J$ with a "sign-changed" utility function $-U$, and the only perturbation which seems to make sense in this case involves $u \in \mathcal{L}_\infty(\Omega,\mathcal{F}_T,P)$, which suggests that we should use the space of perturbations

$$\mathbb{U} := \mathcal{L}_\infty(\Omega,\mathcal{F}_T,P) \times \mathcal{L}_\infty(\{\mathcal{F}_t\};\mathcal{C}), \tag{6.19}$$

(compare with (5.3.23)). This of course means that we can no longer use the bilinear pairing at (5.3.26) and (5.3.27), and must somehow define afresh both a space of dual variables $\mathbb{Y}$

and a bilinear form on $\mathbb{U} \times \mathbb{Y}$. These constitute challenging problems which remain to be addressed, but we nevertheless believe that these can be overcome, and that the Rockafellar variational approach will prove to be the same powerful tool for utility maximization with a combination of portfolio and wealth constraints that it has proved to be for quadratic minimization with the same combination of constraints. In this regard, the ability to "experiment" with various choices of perturbation functions, spaces of dual variables and bilinear pairings which one has when using the Rockafellar variational approach, is likely to be particularly important.

# APPENDICES

# Appendix A

# Technical Proofs

We gather in this appendix the proofs of a number of technical results which appear in the main body of the thesis. For the most part these technical results involve either fairly routine arguments or rather standard ideas, and therefore, to keep the main lines of development clear, we have placed the proofs of these results in this Appendix, retaining within the main body of the thesis only the proofs of those results which are of central importance.

**Proof of Proposition 2.1.13**:

Fix some $\pi \in \Pi$. It is enough to show that $H(t)X^\pi(t)$ is a $\mathcal{F}_t$-martingale. To this end, we apply "Integration-by-parts" formula to $\mathrm{d}H(t)X^\pi(t)$ together with (2.1.10) and (2.1.17):

$$\begin{aligned} & \mathrm{d}H(t)X^\pi(t) = H(t)\mathrm{d}X^\pi(t) + X^\pi(t)\mathrm{d}H(t) + \mathrm{d}\left\langle H, X^\pi \right\rangle(t) \\ = \; & H(t)\{[r(t)X^\pi(t) + \pi'(t)\sigma(t)\theta(t)]\,\mathrm{d}t + \pi'(t)\sigma(t)\mathrm{d}W(t)\} \\ & + X^\pi(t)H(t)\left[-r(t)\mathrm{d}t - \theta'(t)\mathrm{d}W(t)\right] - H(t)\pi'(t)\sigma(t)\theta(t)\mathrm{d}t \\ = \; & H(t)\left[\pi'(t)\sigma(t) - X^\pi(t)\theta'(t)\right]\mathrm{d}W(t). \end{aligned} \tag{A.1}$$

By (2.1.1), $W$ is a Brownian motion with respect to $\{\mathcal{F}_t\}$, and thus, from (A.1), we see that $H(t)X^\pi(t)$ is a $\mathcal{F}_t$-local martingale. It remains to show that $\{H(t)X^\pi(t), t \in [0,T]\}$ is a $\mathcal{F}_t$-martingale, and for this to hold it is enough to show

$$E\left[\sup_{t\in[0,T]} |H(t)X^\pi(t)|\right] < \infty, \tag{A.2}$$

(see Theorem 51 on page 38 of P. Protter [28]).

Since

$$\sup_{t\in[0,T]} |H(t)X^\pi(t)| \leqslant \left(\sup_{t\in[0,T]} |H(t)|\right)\left(\sup_{t\in[0,T]} |X^\pi(t)|\right), \tag{A.3}$$

from (A.3) and Cauchy-Schwarz inequality, we get

$$E\left[\sup_{t\in[0,T]} |H(t)X^\pi(t)|\right] \leqslant \sqrt{E\left[\sup_{t\in[0,T]} |H(t)|^2\right]}\sqrt{E\left[\sup_{t\in[0,T]} |X^\pi(t)|^2\right]} < \infty, \tag{A.4}$$

where the final strict inequality of (A.4) follows from (2.1.11) and (2.1.24). □

**Proof of Proposition 3.2.20**:
Fix arbitrary $x \in \mathbb{X}$, $(y,z) \in \mathbb{Y}$. From (3.2.69) and (3.2.83), we have $f(x) \in (-\infty, +\infty]$ and $g(y,z) \in [-\infty, +\infty)$. This, together with the weak duality relation (3.2.85), then gives

$$f(x) = g(y,z) \iff f(x) = K(x,(y,z)) \in \mathbb{R} \text{ and } g(y,z) = K(x,(y,z)) \in \mathbb{R}. \tag{A.5}$$

By (3.2.69) and (3.2.82) we have

$$\begin{aligned} f(x) = K(x,(y,z)) \in \mathbb{R} &\iff \begin{cases} (1)\ x \in C,\ (2)\ x \geqslant b,\ (3)\ z \geqslant 0, \\ (4)\ J(x) = x'y - J^*(y) + z'(b-x). \end{cases} \\ &\iff \begin{cases} (1)\ x \in C,\ (2)\ x \geqslant b,\ (3)\ z \geqslant 0, \\ (4)\ J(x) + J^*(y) - x'y = z'(b-x). \end{cases} \\ &\iff \begin{cases} (1)\ x \in C,\ (2)\ x \geqslant b,\ (3)\ z \geqslant 0, \\ (4)\ z'(b-x) = 0,\ (5)\ J(x) + J^*(y) - x'y = 0. \end{cases} \end{aligned} \tag{A.6}$$

Observe that the third equivalence at (A.6) follows because $z'(b-x) \leqslant 0$ (indicated by (2) and (3)) and $J(x) + J^*(y) - x'y \geqslant 0$ (by the definition of the convex conjugate function $J^*(\cdot)$). Also, from (C.4), we have

$$J(x) + J^*(y) - x'y = 0 \iff y = (\partial J)(x). \tag{A.7}$$

Combining (A.6) with (A.7), we get

$$f(x) = K(x,(y,z)) \in \mathbb{R} \iff \begin{cases} (1)\ x \in C,\ (2)\ x \geqslant b,\ (3)\ z \geqslant 0, \\ (4)\ z'(b-x) = 0,\ (5)\ y = (\partial J)(x). \end{cases} \tag{A.8}$$

On the other hand, from (3.2.82) and (3.2.83) with $J^*(y) \in \mathbb{R}$,

$$\begin{aligned} & g(y,z) = K(x,(y,z)) \in \mathbb{R} \\ \iff & \begin{cases} (1)\ x \in C,\ (2)\ z \geqslant 0, \\ (3)\ x'y - J^*(y) + z'(b-x) = -\varkappa(y,z) - J^*(y), \end{cases} \\ \iff & \begin{cases} (1)\ x \in C,\ (2)\ z \geqslant 0, \\ (3)\ x'y + z'(b-x) + \varkappa(y,z)x = 0, \end{cases} \end{aligned} \tag{A.9}$$

Now, (3.2.91) follows from (A.8) and (A.9). □

**Proof of Lemma 4.1.1**:
Suppose $X \in \mathbb{B}$ and for some $(X_0, \dot{X}, \Lambda_X), (Y_0, \dot{Y}, \Lambda_Y) \in \mathbb{R} \times L_{21} \times \Pi$,

$$\begin{aligned} X(t) &= X_0 + \int_0^t \dot{X}(s)\mathrm{d}s + \int_0^t \Lambda_X'(s)\mathrm{d}W(s) \\ &= Y_0 + \int_0^t \dot{Y}(s)\mathrm{d}s + \int_0^t \Lambda_Y'(s)\mathrm{d}W(s) \qquad \text{a.s.}, t \in [0,T]. \end{aligned} \tag{A.10}$$

From $t = 0$ in (A.10), we get $X_0 = Y_0$. Then, by reorganizing (A.10), we also have

$$\int_0^t \left[\dot{X}(s) - \dot{Y}(s)\right] \mathrm{d}s = \int_0^t [\Lambda'_Y(s) - \Lambda'_X(s)]\, \mathrm{d}W(s) \quad \text{a.s.}, t \in [0, T], \tag{A.11}$$

where the left hand side of (A.11) is a continuous finite variation process while the right hand side of (A.11) is a continuous local martingale. The only continuous local martingale that has finite variation path is the constant zero, therefore

$$\int_0^t \left[\dot{X}(s) - \dot{Y}(s)\right] \mathrm{d}s = 0 \quad \text{a.s.}, t \in [0, T], \tag{A.12}$$

$$\int_0^t [\Lambda'_Y(s) - \Lambda'_X(s)]\, \mathrm{d}W(s) = 0 \quad \text{a.s.}, t \in [0, T]. \tag{A.13}$$

It directly follows from (A.12) that

$$\dot{X} = \dot{Y} \quad \text{a.e.}, \tag{A.14}$$

and by using Itô isometry we also have from (A.13) that

$$0 = E\left[\int_0^t [\Lambda'_Y(s) - \Lambda'_X(s)]\, \mathrm{d}W(s)\right]^2 = E\left[\int_0^t [\Lambda'_Y(s) - \Lambda'_X(s)]^2\, \mathrm{d}s\right], \tag{A.15}$$

which gives

$$\Lambda_X = \Lambda_Y \quad \text{a.e.} \tag{A.16}$$

□

**Proof of Proposition 4.1.4:**
By solving (4.1.6), we have explicitly

$$X(t) = S_0(t)\left\{X(0) + \int_0^t \frac{\pi'(s)\sigma(s)\theta(s)}{S_0(s)}\mathrm{d}s + \int_0^t \frac{\pi'(s)\sigma(s)}{S_0(s)}\mathrm{d}W(s)\right\}, \quad t \in [0, T]. \tag{A.17}$$

From (A.17), it follows that

$$\begin{aligned} |X(t)|^2 &= |S_0(t)|^2\left\{\left|X(0) + \int_0^t \frac{\pi'(s)\sigma(s)\theta(s)}{S_0(s)}\mathrm{d}s + \int_0^t \frac{\pi'(s)\sigma(s)}{S_0(s)}\mathrm{d}W(s)\right|^2\right\} \\ &\leqslant 3\,|S_0(t)|^2\left\{x_0^2 + \left|\int_0^t \frac{\pi'(s)\sigma(s)\theta(s)}{S_0(s)}\mathrm{d}s\right|^2 + \left|\int_0^t \frac{\pi'(s)\sigma(s)}{S_0(s)}\mathrm{d}W(s)\right|^2\right\}. \end{aligned} \tag{A.18}$$

At the last line of (A.18) we used the inequality $(x + y + z)^2 \leqslant 3(x^2 + y^2 + z^2)$ for all $x, y, z \in \mathbb{R}$. Moreover, by the Cauchy-Schwarz inequality, we have

$$\sup_{t\in[0,T]}\left|\int_0^t \frac{\pi'(s)\sigma(s)\theta(s)}{S_0(s)}\mathrm{d}s\right|^2 \leqslant T\int_0^T \left|\frac{\pi'(s)\sigma(s)\theta(s)}{S_0(s)}\right|^2 \mathrm{d}s. \tag{A.19}$$

By Condition 2.1.2, $S_0$ is uniformly bounded and lower bounded by 1 (see (2.1.5)). Since $\pi \in \Pi$ and $\sigma\theta$ is uniformly bounded (from Remark 2.1.7 - (1) and Condition 2.1.2), we have from (A.19) that

$$E\left[\sup_{t\in[0,T]}\left|\int_0^t \frac{\pi'(s)\sigma(s)\theta(s)}{S_0(s)}\mathrm{d}s\right|^2\right] \leqslant TE\left[\int_0^T \left|\frac{\pi'(s)\sigma(s)\theta(s)}{S_0(s)}\right|^2 \mathrm{d}s\right] < \infty. \tag{A.20}$$

Applying Doob's $L^2$-inequality to the d$W$ term in (A.18), and again using $\pi \in \Pi$ together with the uniform boundedness of $\sigma$, we obtain

$$E\left[\sup_{t\in[0,T]}|X(t)|^2\right] < \infty, \qquad \text{for each } \pi \in \Pi. \tag{A.21}$$

From Condition 2.1.5 the boundedness of $\sigma$ and $\pi \in \Pi$, it directly follows

$$\pi'\sigma \in \Pi. \tag{A.22}$$

From Remark 2.1.7 - (1) and the boundedness of $\sigma$ again, we have some upper bounds $\kappa_r, \kappa_\theta, \kappa_\sigma$ for $r, \theta, \sigma$ such that

$$\begin{aligned}
& E\left[\left(\int_0^T |r(t)X(t) + \pi'(t)\sigma(t)\theta(t)|\,\mathrm{d}t\right)^2\right] \\
\leqslant\ & E\left[2\left(\int_0^T |r(t)X(t)|\,\mathrm{d}t\right)^2 + 2\left(\int_0^T |\pi'(t)\sigma(t)\theta(t)|\,\mathrm{d}t\right)^2\right] \\
\leqslant\ & TE\left[2\int_0^T |r(t)X(t)|^2\,\mathrm{d}t + 2\int_0^T |\pi'(t)\sigma(t)\theta(t)|^2\,\mathrm{d}t\right] \\
\leqslant\ & 2T^2\kappa_r^2 E\left[\sup_{t\in[0,T]}|X(t)|^2\right] + 2T\kappa_\theta^2\kappa_\sigma^2 E\left[\int_0^T \|\pi(t)\|^2\,\mathrm{d}t\right],
\end{aligned} \tag{A.23}$$

(we have used Cauchy-Schwarz inequality in the second inequality of (A.23)). Of course, $rX + \pi'\sigma\theta \in \mathcal{F}^*$, it therefore follows from (A.23), (A.21) and $\pi \in \Pi$ that

$$rX + \pi'\sigma\theta \in L_{21}. \tag{A.24}$$

Therefore, (4.1.7) follows from (A.22) and (A.24). □

**Proof of Proposition 4.1.5**:
From (4.1.4) and the inequality $(x+y+z)^2 \leqslant 3(x^2+y^2+z^2)$ for all $x, y, z \in \mathbb{R}$,

$$|X(t)|^2 \leqslant 3\left[|X(0)|^2 + \left|\int_0^t \dot{X}(s)\mathrm{d}s\right|^2 + \left|\int_0^t \Lambda_X'(s)\mathrm{d}W(s)\right|^2\right] < \infty \quad t \in [0,T]. \tag{A.25}$$

Therefore, observing $\left|\int_0^t \dot{X}(s)\mathrm{d}s\right| \leqslant \int_0^t \left|\dot{X}(s)\right| \mathrm{d}s, t \in [0,T]$, we have

$$E\left[\sup_{t\in[0,T]}(X(t))^2\right] \leqslant 3E\left[|X(0)|^2 + \left(\int_0^T \left|\dot{X}(s)\right|\mathrm{d}s\right)^2 + \sup_{t\in[0,T]}\left|\int_0^t \Lambda_X'(s)\mathrm{d}W(s)\right|^2\right]. \tag{A.26}$$

Since $\Lambda_X \in \Pi$ (from (4.1.3)), we can apply Doob's maximal $L^2$-inequality to the last item in (A.26), and together with (4.1.1) and (4.1.2), it follows that

$$E\left[\max_{t\in[0,T]} |X(t)|^2\right] < \infty. \tag{A.27}$$

$\square$

**Proof of Proposition 4.1.6:**
Let $X^\pi$ be a wealth process defined by the wealth equation (2.1.21) for a portfolio process $\pi$ given by the Definition 2.1.10 (that is $\pi$ is $\mathcal{F}^*$-measurable and subject to (2.1.18)).

First, suppose $X^\pi \in \mathbb{B}$. It follows from (4.1.4) and (2.1.21) that

$$(x_0, rX^\pi + \pi'\sigma\theta, \pi'\sigma) \in \mathbb{R} \times L_{21} \times \Pi. \tag{A.28}$$

Therefore, because of the uniform boundedness of $\sigma$ (Condition 2.1.5 and (2.1.7)) and the fact that $\pi'\sigma \in \Pi$, we get

$$\pi \in \Pi. \tag{A.29}$$

Next, suppose $\pi \in \Pi$. We have from (2.1.21) and Proposition 4.1.4 that $X^\pi \in \mathbb{B}$ with $\dot{X}^\pi = rX^\pi + \pi'\sigma\theta \in L_{21}$ and $\Lambda_{X^\pi} = \pi'\sigma \in \Pi$. $\square$

**Proof of Lemma 4.1.8:**
Fix some $y \in \mathbb{R}$ and $\nu \in \Pi$. Define

$$\eta := S_0\nu - y\theta \in \Pi, \tag{A.30}$$

where the set membership follows since $\theta, S_0 \in \mathcal{F}^*$ are uniformly bounded (recall Remark 2.1.7 (1)) and $\nu \in \Pi$. From Lemma F.0.2 and (A.30), there is some $\xi \in \Pi$ such that

$$\eta(t) = \xi(t) + \theta(t)\int_0^t \xi'(s)\mathrm{d}W(s) \qquad \text{a.e.} \tag{A.31}$$

Combining (A.30) and (A.31), we have

$$S_0(t)\nu(t) = y\theta(t) + \eta(t) = y\theta(t) + \xi(t) + \theta(t)\int_0^t \xi'(s)\mathrm{d}W(s) \quad \text{a.e.} \tag{A.32}$$

Now multiply both sides of (A.32) by $[S_0(t)]^{-1}$, and put

$$\gamma(t) := [S_0(t)]^{-1}\xi(t), \qquad t \in [0,T]. \tag{A.33}$$

It then follows that

$$\nu(t) = \frac{y\theta(t)}{S_0(t)} + \gamma(t) + \frac{\theta(t)}{S_0(t)}\int_0^t S_0(s)\gamma'(s)\mathrm{d}W(s) \quad \text{a.e.} \tag{A.34}$$

From Condition 2.1.2, it follows that $\gamma \in \Pi$, and therefore, in view of (4.1.13) and (A.34), we have

$$\nu(t) = \gamma(t) + \theta(t)\Xi(y,\gamma)(t) \quad \text{a.e.} \qquad \text{for some } \gamma \in \Pi. \tag{A.35}$$

Now define $Y := \Xi(y,\gamma)$. From (4.1.14) and $\gamma \in \Pi$, it follows that $Y \in \mathbb{B}_1$. Moreover, by Remark 4.1.7 (a), we see that $Y(0) = y$ and $\Lambda_Y = \gamma$, and therefore from (A.35) we obtain

$$Y(0) = y \qquad \text{and} \qquad \Lambda_Y + \theta Y = \nu \quad \text{a.e.}, \tag{A.36}$$

which is (4.1.18). □

**Proof of Proposition 4.1.9**:

Fix some $\eta \in \mathcal{L}_2(\Omega, \mathcal{F}_T, P)$. By Condition 2.1.2, $S_0(t) = \exp\{\int_0^t r(s)\mathrm{d}s\}, t \in [0,T]$ is uniformly bounded. Thus $S_0(T)\eta \in \mathcal{L}_2(\Omega, \mathcal{F}_T, P)$. By the Itô martingale representation theorem, there is a unique $\phi \in \Pi$ such that

$$S_0(T)\eta = E\left[S_0(T)\eta\right] + \int_0^T \phi(t)\mathrm{d}W(t). \tag{A.37}$$

Put

$$Y(0) := E\left[S_0(T)\eta\right] \qquad \text{and} \qquad \Lambda_Y := S_0^{-1}\phi \in \Pi, \tag{A.38}$$

(we have $S_0^{-1}\phi \in \Pi$ since $S_0(t) \geq 1$), so that $(Y(0), \Lambda_Y) \in \mathbb{B}_1$. It follows from (A.37) and (A.38) that

$$S_0(T)\eta = Y(0) + \int_0^T S_0(t)\Lambda_Y(t)\mathrm{d}W(t), \tag{A.39}$$

and together with (4.1.12), we have $S_0(T)Y(T) = S_0(T)\eta$, i.e.

$$Y(T) = \eta. \tag{A.40}$$

Uniqueness of $Y = (Y(0), \Lambda_Y) \in \mathbb{B}_1$ is immediate from (4.1.14). □

**Proof of Proposition 4.1.12** :

Suppose $\xi = \partial J^*(\eta)$ a.s. Then from (4.1.22), we have

$$J(\xi) + J^*(\eta) = \xi\eta \quad \text{a.s.} \tag{A.41}$$

and therefore $E\left[J(\xi) + J^*(\eta) - \xi\eta\right] = 0$. For the converse, observe from the definition of $J^*$ at (4.1.20) that

$$J(\xi) + J^*(\eta) - \xi\eta \geqslant 0 \quad \text{a.s.} \tag{A.42}$$

From (A.42) and (4.1.22),

$$\begin{aligned} & E\left[J(\xi) + J^*(\eta) - \xi\eta\right] = 0 \\ \overset{(A.42)}{\Rightarrow} \quad & J(\xi) + J^*(\eta) - \xi\eta = 0 \quad \text{a.s.} \\ \overset{(4.1.22)}{\Rightarrow} \quad & \xi = \partial J^*(\eta) \quad \text{a.s.} \end{aligned} \tag{A.43}$$

□

**Proof of Proposition 4.2.5**:
Define

$$M(X^\pi, Y)(t) := X^\pi(t)Y(t) - x_0 Y(0) - \int_0^t \left[X^\pi(s)\dot{Y}(s) + \dot{X}^\pi(s)Y(s) + \Lambda'_{X^\pi}(s)\Lambda_Y(s)\right] \mathrm{d}s, \quad t \in [0,T], \tag{A.44}$$

for all $\pi \in \Pi$ and $Y \in \mathbb{B}_1$. From Proposition 4.1.4 we have

$$X^\pi \in \mathbb{B} \text{ for all } \pi \in \Pi \text{ with } \dot{X}^\pi = rX^\pi + \pi'\sigma\theta \text{ and } \Lambda_{X^\pi} = \sigma'\pi. \tag{A.45}$$

By (4.1.10), we also have

$$Y \in \mathbb{B}_1 \subset \mathbb{B} \text{ with } \dot{Y} = rY. \tag{A.46}$$

With (A.45) and (A.46), we can write $M(X^\pi, Y)$ defined by (A.44) as

$$\begin{aligned} M(X^\pi, Y)(t) &= X^\pi(t)Y(t) - x_0 Y(0) \\ &\quad - \int_0^t \pi'(s)\sigma(s)[\theta(s)Y(t) + \Lambda_Y(s)]\mathrm{d}s, \quad t \in [0,T]. \end{aligned} \tag{A.47}$$

In view of Proposition F.0.1 (given as Proposition I-1 in Bismut [4, p.387]) it follows that

$$\begin{aligned} &M(X^\pi, Y) \text{ is a continuous } \mathcal{F}_t\text{-martingale with} \\ &\quad M(X^\pi, Y)(0) = 0, \text{ for all } \pi \in \Pi \text{ and } Y \in \mathbb{B}_1. \end{aligned} \tag{A.48}$$

Therefore, we have $E\left[M(X^\pi, Y)(T)\right] = M(X^\pi, Y)(0) = 0$ or by (A.44)

$$E\left[X^\pi(T)Y(T)\right] = x_0 Y(0) + E\left[\int_0^T \pi'(t)\sigma(t)[\theta(t)Y(t) + \Lambda_Y(t)]\mathrm{d}t\right], \tag{A.49}$$

for all $\pi \in \Pi$ and $Y \in \mathbb{B}_1$. Then

$$\begin{aligned} &\inf_{\pi\in\Pi} \left\{E\left[X^\pi(T)Y(T)\right]\right\} \\ &\overset{(A.49)}{=} x_0 Y(0) + \inf_{\pi\in\Pi}\left\{E\left[\int_0^T \pi'(t)\sigma(t)[\theta(t)Y(t) + \Lambda_Y(t)]\mathrm{d}t\right]\right\}, \end{aligned} \tag{A.50}$$

for all $Y \in \mathbb{B}_1$. Next, put

$$\pi_\alpha(t) := \alpha\sigma(t)[\theta(t)Y(t) + \Lambda_Y(t)], \qquad t \in [0,T], \qquad \alpha \in \mathbb{R}. \tag{A.51}$$

We observe that

(1) all entries of $\sigma$ and $\theta$ are uniformly bounded (by Remark 2.1.7 and Condition 2.1.2);

(2) $Y \in \mathbb{B}_1 \subset \mathbb{B}$, which, together with Proposition 4.1.5, gives

$$E\left[\max_{t\in[0,T]} |Y(t)|^2\right] < \infty; \tag{A.52}$$

(3) $\Lambda_Y \in \Pi$ by (4.1.14).

From (1), (2) and (3) we get

$$E\left[\int_0^T \|\pi_\alpha(t)\|^2 \,\mathrm{d}t\right] < +\infty, \text{ i.e. } \pi_\alpha \in \Pi \text{ for all } \alpha \in \mathbb{R}. \tag{A.53}$$

We then have from $\{\pi_\alpha; \alpha \in \mathbb{R}\} \subset \Pi$ that

$$\begin{aligned}
&\inf_{\pi\in\Pi}\left\{E\left[\int_0^T \pi'(t)\sigma(t)[\theta(t)Y(t)+\Lambda_Y(t)]\mathrm{d}t\right]\right\} \\
&\leqslant \inf_{\alpha\in\mathbb{R}}\left\{E\left[\int_0^T \pi'_\alpha(t)\sigma(t)[\theta(t)Y(t)+\Lambda_Y(t)]\mathrm{d}t\right]\right\} \\
&\overset{(A.51)}{=} \inf_{\alpha\in\mathbb{R}}\left\{\alpha E\left[\int_0^T [\theta(t)Y(t)+\Lambda_Y(t)]'\sigma'(t)\sigma(t)[\theta(t)Y(t)+\Lambda_Y(t)]\mathrm{d}t\right]\right\}.
\end{aligned} \tag{A.54}$$

By Condition 2.1.5, we see

$$[\theta Y + \Lambda_Y]'\sigma'\sigma[\theta Y + \Lambda_Y] \geqslant \kappa_0 \|\theta Y + \Lambda_Y\|^2, \qquad \forall\, t \in [0,T] \text{ and } \forall \omega \in \Omega, \tag{A.55}$$

for some $\kappa_0 \in (0,\infty)$, and then, from (A.54) and (A.55) we obtain

$$\begin{aligned}
&\inf_{\pi\in\Pi}\left\{E\left[\int_0^T \pi'(t)\sigma(t)[\theta(t)Y(t)+\Lambda_Y(t)]\mathrm{d}t\right]\right\} \\
&= \begin{cases} 0 & \text{if } \theta Y + \Lambda_Y = 0 \text{ a.e.,} \\ -\infty & \text{otherwise.} \end{cases} \\
&= \begin{cases} 0 & \text{if } Y = Y(0)H \text{ a.e.,} \\ -\infty & \text{otherwise,} \end{cases}
\end{aligned} \tag{A.56}$$

for all $Y = (Y(0), \Lambda_Y) \in \mathbb{B}_1$. The last equality in (A.56) follows from Remark 4.1.7 - (b). Combining (A.50) and (A.56), we have (4.2.20). □

**Proof of Proposition 4.2.7**:
Fix any $(\pi, Y) \in \Pi \times \mathbb{Y}$. From (4.2.5) and (4.2.17), we have $f(\pi) \in (-\infty, +\infty]$ and $g(Y) \in [-\infty, +\infty)$. Then, in view of the weak duality relation (3.1.5), it follows that

$$f(\pi) = g(Y) \iff f(\pi) = K(\pi, Y) \in \mathbb{R} \text{ and } g(Y) = K(\pi, Y) \in \mathbb{R}. \tag{A.57}$$

By (4.2.5) and (4.2.16) we have

$$\begin{aligned}
&f(\pi) = K(\pi, Y) \in \mathbb{R} \\
\iff\ & E[J(X^\pi(T))] = E[X^\pi(T)Y(T) - J^*(Y(T))], \\
\overset{(4.1.23)}{\iff}\ & X^\pi(T) = \partial J^*(Y(T)).
\end{aligned} \tag{A.58}$$

On the other hand, from (4.2.16) and (4.2.21),

$$g(Y) = K(\pi, Y) \in \mathbb{R} \iff Y = Y(0)H \text{ a.e..} \tag{A.59}$$

(4.2.33) follows from (A.58) and (A.59). □

**Proof of Proposition 4.3.11**:
From (4.3.6), we know

$$f(\pi) \in (-\infty, +\infty], \quad \text{for all } \pi \in \Pi. \tag{A.60}$$

Also from (4.3.21), we have $\varkappa(Y) \in (-\infty, +\infty]$ for all $Y \in \mathbb{Y}$, and together with (A.63),

$$g(Y) \in [-\infty, +\infty), \quad \text{for all } Y \in \mathbb{Y}. \tag{A.61}$$

Thus, from (A.60), (A.61) and (4.3.19), it follows that

$$f(\pi) = g(Y) \iff f(\pi) = K(\pi, Y) \in \mathbb{R} \text{ and } g(Y) = K(\pi, Y) \in \mathbb{R}. \tag{A.62}$$

Now, fix some arbitrary pair $(\pi, Y) \in \Pi \times \mathbb{Y}$. From (4.3.16), (4.1.20) and Condition 2.2.1,

$$\text{both } E\left[X^\pi(T)Y(T)\right] \text{ and } E\left[J^*(Y(T))\right] \text{ are real-valued.} \tag{A.63}$$

From (A.63), together with (4.3.6) and (4.3.17), we have

$$\begin{aligned} & f(\pi) = K(\pi, Y) \in \mathbb{R} \\ \iff & 1)\ \pi \in \mathcal{A}, \quad 2)\ E\left[J(X^\pi(T))\right] = E\left[X^\pi(T)Y(T)\right] - E\left[J^*(Y(T))\right], \\ \overset{(4.1.23)}{\iff} & 1)\ \pi \in \mathcal{A}, \quad 2)\ X^\pi(T) = \partial J^*(Y(T)). \end{aligned} \tag{A.64}$$

Moreover, from (A.63) again, together with (4.3.17) and (4.3.20), we get

$$g(Y) = K(\pi, Y) \in \mathbb{R} \iff 1)\ \pi \in \mathcal{A},\ 2)\ E\left[X^\pi(T)Y(T)\right] + \varkappa(Y) = 0. \tag{A.65}$$

We conclude from (A.64), (A.65) and (A.62) that

$$f(\pi) = g(Y) \iff \begin{cases} 1)\ \pi \in \mathcal{A}, \\ 2)\ E\left[X^\pi(T)Y(T)\right] + \varkappa(Y) = 0, \\ 3)\ X^\pi(T) = \partial J^*(Y(T)). \end{cases} \tag{A.66}$$

□

**Proof of Proposition 4.3.17**:
In view of Proposition 4.1.4 together with (2.1.21), for each $\pi \in \Pi$ we obtain

$$X^\pi \in \mathbb{B}, \tag{A.67}$$

along with (recall (4.1.4)),

$$X^\pi(0) = x_0, \quad \dot{X}^\pi = rX^\pi + \pi'\sigma\theta, \text{ and } \Lambda_{X^\pi} = \sigma'\pi. \tag{A.68}$$

Define

$$M(X^\pi, Y)(t) := X^\pi(t)Y(t) - x_0 Y_0 - \int_0^t \{X^\pi(s)\dot{Y}(s) + \dot{X}^\pi(s)Y(s) + \Lambda'_{X^\pi}(s)\Lambda_Y(s)\}\mathrm{d}s, \quad t \in [0,T], \tag{A.69}$$

for all $\pi \in \Pi$ and $Y \in \mathbb{B}_1$. From (A.69), (A.68) and (4.1.10), we get

$$M(X^\pi, Y)(T) = X^\pi(T)Y(T) - x_0 Y(0) - \int_0^T \pi'(s)\sigma(s)\left[\theta(s)Y(s) + \Lambda_Y(s)\right]\mathrm{d}s, \tag{A.70}$$

for all $\pi \in \Pi$ and $Y \in \mathbb{B}_1$. From (A.67) and Proposition F.0.1, we get

$$E\left[M(X^\pi, Y)(T)\right] = 0, \qquad \text{for all } \pi \in \Pi \text{ and } Y \in \mathbb{B}_1. \tag{A.71}$$

It then follows from (A.71) and (A.70) that

$$E\left[X^\pi(T)Y(T)\right] = x_0 Y(0) + E\left[\int_0^T \pi'(s)\sigma(s)\left[\theta(s)Y(s) + \Lambda_Y(s)\right]\mathrm{d}s\right], \tag{A.72}$$

for all $\pi \in \Pi$ and $Y \in \mathbb{B}_1$.

From (4.3.1), Notation 4.3.14 (a), and (A.72), we get

$$\begin{aligned} &\sup_{\pi \in \mathcal{A}} E\left[-X^\pi(T)Y(T)\right] = \sup_{\pi \in \Pi}\left\{E\left[-X^\pi(T)Y(T)\right] - E\left[\int_0^T \delta_{\mathbb{R}^N}\{\pi(s)|A\}\mathrm{d}s\right]\right\} \\ = \quad & -x_0 Y(0) + \sup_{\pi \in \Pi}\left\{E\left[\int_0^T \pi'(s)\sigma(s)\left[-\theta(s)Y(s) - \Lambda_Y(s)\right] - \delta_{\mathbb{R}^N}\{\pi(s)|A\}\mathrm{d}s\right]\right\}, \end{aligned} \tag{A.73}$$

for each $Y \in \mathbb{B}_1$. We next evaluate the supremum at the right of (A.73). Now fix $Y \in \mathbb{B}_1$, and put

$$\vartheta(\omega;t) := -\sigma(\omega;t)\left[\theta(\omega;t)Y(\omega;t) + \Lambda_Y(\omega;t)\right], \quad (\omega;t) \in \Omega \times [0,T]; \tag{A.74}$$

since $\Lambda_Y \in \Pi$ (see (4.1.10) and (4.1.3)), it follows from the boundedness of $\theta$ (see Remark 2.1.7(1)), the bound given by (2.1.7), and Proposition 4.1.5, that $\vartheta \in \Pi$. We next use the identity (4.3.40), with $\vartheta$ defined by (A.74), to evaluate the supremum on the right side of (A.73):

$$\begin{aligned} &\sup_{\pi \in \Pi}\left\{E\left[\int_0^T \pi'(s)\sigma(s)\left[-\theta(s)Y(s) - \Lambda_Y(s)\right] - \delta_{\mathbb{R}^N}\{\pi(s)|A\}\mathrm{d}s\right]\right\} \\ \overset{(A.74)}{=} \quad & \sup_{\pi \in \Pi} E\left[\int_0^T \left[\pi'(s)\vartheta(s) - \delta_{\mathbb{R}^N}\{\pi(s)|A\}\right]\mathrm{d}s\right] \\ \overset{(4.3.40)}{=} \quad & E\left[\int_0^T \delta^*_{\mathbb{R}^N}\left\{\vartheta(s)\middle|A\right\}\mathrm{d}s\right] \\ \overset{(A.74)}{=} \quad & E\left[\int_0^T \delta^*_{\mathbb{R}^N}\left\{-\sigma(s)[\theta(s)Y(s) + \Lambda_Y(s)]\middle|A\right\}\mathrm{d}s\right]. \end{aligned} \tag{A.75}$$

Now (4.3.41) follows from (A.75) and (A.73). □

**Proof of Proposition 4.4.14**:
From (4.4.61), we see for each fixed $\omega \in \Omega$ that

$$\text{the mapping } \alpha \to h^{opt}(\alpha H(\omega;T),\omega) \text{ is continuous and concave for all } \alpha \in \mathbb{R}. \tag{A.76}$$

(A.76) together with *dominated/monotone convergence theorem* gives

$$\text{the mapping } \alpha \to E\left[h^{opt}(\alpha H(T))\right] : \mathbb{R} \to \mathbb{R} \text{ is continuous and concave,} \tag{A.77}$$

and then by (4.4.62),

$$\alpha \to \Psi(\alpha) : \mathbb{R} \to \mathbb{R} \text{ is continuous and concave.} \tag{A.78}$$

Also by (4.4.61),

$$h^{opt}(\alpha H(T)) \leqslant b(c + \frac{ab}{2}) + q \quad \text{a.s.,} \qquad \text{for all } \alpha \in \mathbb{R}, \tag{A.79}$$

therefore, from (A.79), for each $\alpha \in \mathbb{R}$ we have

$$\begin{aligned} \Psi(\alpha) &= \alpha(x_0 - E\left[bH(T)\right]) + E\left[h^{opt}(\alpha H(T))\right] \\ &\leqslant \alpha(x_0 - E\left[bH(T)\right]) + E\left[b(c + \frac{ab}{2})\right] + q \in \mathbb{R}. \end{aligned} \tag{A.80}$$

By Condition 4.4.3, there is some $\hat{\pi} \in \Pi$ and constant $\varepsilon \in (0, \infty)$ such that

$$X^{\hat{\pi}}(T)H(T) \geqslant bH(T) + \varepsilon H(T) \quad \text{a.s.,} \tag{A.81}$$

and by (2.1.26), we know

$$E\left[X^{\hat{\pi}}(T)H(T)\right] = x_0. \tag{A.82}$$

Therefore, since $H(T) > 0$ a.s., it holds from (A.81) and (A.82) that

$$x_0 - E\left[bH(T)\right] \geqslant \varepsilon E\left[H(T)\right] > 0. \tag{A.83}$$

Combining (A.80) and (A.83), we have

$$\begin{aligned} \lim_{\alpha \to -\infty} \Psi(\alpha) &= \lim_{\alpha \to -\infty} \{\alpha(x_0 - E\left[bH(T)\right]) + E\left[h^{opt}(\alpha H(T))\right]\} \\ &\leqslant \varepsilon \lim_{\alpha \to -\infty} \{\alpha E\left[H(T)\right]\} + E\left[b(c + \frac{ab}{2})\right] + q = -\infty. \end{aligned} \tag{A.84}$$

On the other hand, again from $H(T) > 0$ a.s., we always have

$$P\left(\alpha_0 H(T) > c + ab\right) > 0, \qquad \text{for some } \alpha_0 > 0, \tag{A.85}$$

and for all $\alpha \geqslant \alpha_0$,

$$\{\omega \in \Omega \mid \alpha_0 H(\omega;T) > (c+ab)(\omega)\} \subset \{\omega \in \Omega \mid \alpha H(\omega;T) > (c+ab)(\omega)\}. \tag{A.86}$$

Thus, for all $\alpha \geqslant \alpha_0$,

$$\begin{aligned}
& E\left[h^{opt}(\alpha H(T))\right] \\
\overset{(4.4.61)}{=} \quad & -E\left[\frac{(\alpha H(T)-(c+ab))^2}{2a}\mathbb{1}_{\{\alpha H(T)>c+ab\}}\right] + E\left[b(c+\frac{ab}{2})\right] + q \\
\overset{(A.86)}{\leqslant} \quad & -E\left[\frac{(\alpha H(T)-(c+ab))^2}{2a}\mathbb{1}_{\{\alpha_0 H(T)>c+ab\}}\right] + E\left[b(c+\frac{ab}{2})\right] + q, \\
= \quad & -\alpha^2 E\left[\frac{(H(T))^2}{2a}\mathbb{1}_{\{\alpha_0 H(T)>c+ab\}}\right] + \alpha E\left[H(T)(\frac{c}{a}+b)\mathbb{1}_{\{\alpha H(T)>c+ab\}}\right] \\
& \qquad + E\left[b\left(c+\frac{ab}{2}\right)\mathbb{1}_{\{\alpha_0 H(T)\leqslant c+ab\}} - \frac{c^2}{2a}\mathbb{1}_{\{\alpha_0 H(T)>c+ab\}}\right] + q,
\end{aligned} \tag{A.87}$$

and then

$$\begin{aligned}
\Psi(\alpha) \overset{(4.4.62)}{=} \quad & \alpha(x_0 - E\left[bH(T)\right]) + E\left[h^{opt}(\alpha H(T))\right] \\
\overset{(A.87)}{\leqslant} \quad & -\alpha^2 E\left[\frac{(H(T))^2}{2a}\mathbb{1}_{\{\alpha_0 H(T)>c+ab\}}\right] + \alpha k_1 + k_0,
\end{aligned} \tag{A.88}$$

where

$$\begin{aligned}
k_0 &:= E\left[b\left(c+\frac{ab}{2}\right)\mathbb{1}_{\{\alpha_0 H(T)\leqslant c+ab\}} - \frac{c^2}{2a}\mathbb{1}_{\{\alpha_0 H(T)>c+ab\}}\right] + q, \\
k_1 &:= E\left[H(T)\left(\frac{c}{a}\mathbb{1}_{\{\alpha_0 H(T)>c+ab\}} - b\mathbb{1}_{\{\alpha_0 H(T)\leqslant c+ab\}}\right)\right] + x_0.
\end{aligned} \tag{A.89}$$

Since

$$k_2 := E\left[\frac{(H(T))^2}{2a}\mathbb{1}_{\{\alpha_0 H(T)>c+ab\}}\right] \overset{(2.2.4)}{\geqslant} \frac{1}{2\underline{a}} E\left[(H(T))^2\mathbb{1}_{\{\alpha_0 H(T)>c+ab\}}\right] \underset{(A.85)}{\overset{(2.2.4)}{>}} 0, \tag{A.90}$$

we have

$$\begin{aligned}
\lim_{\alpha\to+\infty}\Psi(\alpha) \quad = \quad & \lim_{\alpha\to+\infty}\left\{\alpha(x_0 - E\left[bH(T)\right]) + E\left[h^{opt}(\alpha H(T))\right]\right\} \\
\overset{(A.88)}{\leqslant} \quad & \lim_{\alpha\to+\infty}\{-\alpha^2 k_2 + \alpha k_1 + k_0\} \overset{(A.90)}{=} -\infty.
\end{aligned} \tag{A.91}$$

The existence of a maximizer $\bar{\alpha} \in \mathbb{R}$ of the mapping $\alpha \to \Psi(\alpha)$ follows from (A.78), (A.84) and (A.91). □

**Proof of Proposition 4.4.19**:
From (4.4.21), we know

$$f(\pi) \in (-\infty, +\infty], \quad \text{for all } \pi \in \Pi. \tag{A.92}$$

Also from (A.95) and (4.4.35), we have

$$g(Y,\xi)\in[-\infty,+\infty),\quad \text{for all } Y\in\mathbb{Y}. \tag{A.93}$$

It follows from (A.92), (A.93) and (4.4.36) that

$$f(\pi)=g(Y,\xi)\iff f(\pi)=K(\pi,(Y,\xi))\in\mathbb{R}\ \text{ and }\ g(Y,\xi)=K(\pi,(Y,\xi))\in\mathbb{R}. \tag{A.94}$$

Now, fix arbitrary pair $(\pi,(Y,\xi))\in\Pi\times\mathbb{Y}$. Recalling (4.4.31), (4.1.20) and Condition 2.2.1,

$$\text{both } E\left[X^\pi(T)Y(T)\right] \text{ and } E\left[J^*(Y(T))\right] \text{ are real-valued.} \tag{A.95}$$

From (A.95), together with (4.4.21) and (4.4.34), we have

$$\begin{aligned}
&f(\pi)=K(\pi,(Y,\xi))\in\mathbb{R}\\
\iff&\begin{cases}1)\ X^\pi(T)\geqslant b,\quad 2)\ \xi\geqslant 0,\\ 3)\ E\left[J(X^\pi(T))\right]=E\left[X^\pi(T)(Y(T)-\xi)-J^*(Y(T))+b\xi\right],\end{cases}\\
\iff&\begin{cases}1)\ X^\pi(T)\geqslant b,\quad 2)\ \xi\geqslant 0,\\ 3)\ E\left[J(X^\pi(T))-X^\pi(T)Y(T)+J^*(Y(T))\right]+E\left[(X^\pi(T)-b)\xi\right]=0,\end{cases}\\
\iff&\begin{cases}1)\ X^\pi(T)\geqslant b,\quad 2)\ \xi\geqslant 0,\quad 3')\ (X^\pi(T)-b)\xi=0,\\ 4)\ J(X^\pi(T))-X^\pi(T)Y(T)+J^*(Y(T))=0.\end{cases}
\end{aligned} \tag{A.96}$$

Observe that the third equivalence at (A.96) follows because

$$\begin{cases}(X^\pi(T)-b)\xi\geqslant 0,\quad \text{indicated by (1) and (2) of (A.96)},\\ J(X^\pi(T))-X^\pi(T)Y(T)+J^*(Y(T))\geqslant 0,\quad \text{by definition of } J^*(\cdot),\end{cases} \tag{A.97}$$

and then

$$\begin{aligned}
&3)\ E\left[J(X^\pi(T))-X^\pi(T)Y(T)+J^*(Y(T))\right]+E\left[(X^\pi(T)-b)\xi\right]=0,\\
\iff&\begin{cases}3')\ (X^\pi(T)-b)\xi=0,\\ 4)\ J(X^\pi(T))-X^\pi(T)Y(T)+J^*(Y(T))=0.\end{cases}
\end{aligned} \tag{A.98}$$

We also have from (4.1.22) that

$$4)\ J(X^\pi(T))-X^\pi(T)Y(T)+J^*(Y(T))=0\iff 4')\ X^\pi(T)=\partial J^*(Y(T)), \tag{A.99}$$

Combining (A.96) and (A.99), we get

$$f(\pi)=K(\pi,(Y,\xi))\in\mathbb{R}\iff\begin{cases}1)\ X^\pi(T)\geqslant b,\quad 2)\ \xi\geqslant 0,\\ 3')\ (X^\pi(T)-b)\xi=0,\\ 4')\ X^\pi(T)=\partial J^*(Y(T)).\end{cases} \tag{A.100}$$

Moreover, from (A.95), together with (4.4.34) and (4.4.35),

$$g(Y,\xi) = K(\pi,(Y,\xi)) \in \mathbb{R} \iff \begin{cases} 1)\ \xi \geqslant 0, \\ 2)\ E\left[X^{\pi}(T)(Y(T)-\xi)\right] \\ \qquad = \inf_{\hat{\pi}\in\Pi}\{E\left[X^{\hat{\pi}}(T)(Y(T)-\xi)\right]\} \in \mathbb{R}, \end{cases} \tag{A.101}$$

From Proposition 4.1.9, there is a unique $\hat{Y} \in \mathbb{B}_1$ such that

$$\hat{Y}(T) = Y(T) - \xi \in \mathcal{L}_2(\Omega, \mathcal{F}_T, P). \tag{A.102}$$

Therefore, from (A.102) and (4.2.20), we have

$$\begin{aligned} & E\left[X^{\pi}(T)(Y(T)-\xi)\right] = \inf_{\hat{\pi}\in\Pi}\{E\left[X^{\hat{\pi}}(T)(Y(T)-\xi)\right]\} \in \mathbb{R} \\ \iff\ & Y(T) - \xi = \alpha H(T) \quad \text{for some } \alpha \in \mathbb{R}. \end{aligned} \tag{A.103}$$

Combining (A.103) and (A.101), we get

$$g(Y,\xi) = K(\pi,(Y,\xi)) \in \mathbb{R} \iff \begin{cases} 1)\ \xi \geqslant 0, \\ 2)\ Y(T) - \xi = \alpha H(T) \text{ for some } \alpha \in \mathbb{R}. \end{cases} \tag{A.104}$$

We conclude from (A.94), (A.104) and (A.100) that

$$f(\pi) = g(Y,\xi) \iff \begin{cases} (1)\ X^{\pi}(T) \geqslant b, \quad (2)\ \xi \geqslant 0, \\ (3)\ Y(T) - \xi = \alpha H(T) \text{ for some } \alpha \in \mathbb{R}, \\ (4)\ (X^{\pi}(T) - b)\xi = 0, \quad (5)\ X^{\pi}(T) = \partial J^*(Y(T)). \end{cases} \tag{A.105}$$

□

**Proof of Proposition 4.5.14**: (a) Fix some $\pi \in \mathcal{A}\backslash\mathcal{A}_1$. From (4.5.29) we have

$$P\left(\{X^{\pi}(T) + \alpha \geqslant b\}\right) < 1, \qquad \text{for all } \alpha \in \mathbb{R}. \tag{A.106}$$

Then for every $v \in \mathcal{L}_\infty(\Omega, \mathcal{F}_T, P)$,

$$P\left(\{X^{\pi}(T) + v \geqslant b\}\right) \leqslant P\left(\{X^{\pi}(T) + \|v\|_\infty \geqslant b\}\right) \overset{(A.106)}{<} 1. \tag{A.107}$$

From (A.107) together with (4.5.19), we have

$$F(\pi,(u,v)) = +\infty, \qquad \text{for all } (u,v) \in \mathbb{U}. \tag{A.108}$$

From (4.5.22) it is clear that

$$\langle (u,v),(Y,z)\rangle \in \mathbb{R}, \qquad \text{for } (u,v) \in \mathbb{U} \text{ and } (Y,z) \in \mathbb{Y}, \tag{A.109}$$

and then, from (4.5.24), (A.108) and (A.109), together with the arbitrary $\pi \in \mathcal{A}\backslash\mathcal{A}_1$, we conclude

$$K(\pi,(Y,z)) = +\infty, \qquad \text{for all } \pi \in \mathcal{A}\backslash\mathcal{A}_1 \text{ and } (Y,z) \in \mathbb{Y}. \tag{A.110}$$

(b) Now fix some $\pi \in \mathcal{A}_1$. Then, from (4.5.29),

$$X^\pi(T) + \hat{\alpha} \geqslant b \qquad \text{a.s.} \qquad \text{for some } \hat{\alpha} \in \mathbb{R}, \tag{A.111}$$

and then

$$\{v \in \mathcal{L}_\infty \mid v \geqslant \hat{\alpha}\} \overset{\text{(A.111)}}{\subset} \{v \in \mathcal{L}_\infty \mid X^\pi(T) + v \geqslant b\}. \tag{A.112}$$

Furthermore, fix some $z \in \mathcal{L}_\infty^*(\Omega, \mathcal{F}_T, P)$ such that $z \ngeqslant 0$, i.e. (see Notation 4.5.8-(2)),

$$z(\tilde{v}) < 0 \quad \text{for some } \tilde{v} \in (\mathcal{L}_\infty(\Omega, \mathcal{F}_T, P))^+, \tag{A.113}$$

then we have

$$\{\alpha\tilde{v} \in \mathcal{L}_\infty(\Omega, \mathcal{F}_T, P) \mid \alpha \in (0, \infty)\} \overset{\text{(A.113)}}{\subset} (\mathcal{L}_\infty(\Omega, \mathcal{F}_T, P))^+, \tag{A.114}$$

and

$$\inf_{v \in \mathcal{L}_\infty^+} \{z(v)\} \overset{\text{(A.114)}}{\leqslant} \inf_{\alpha \in (0,\infty)} \{z(\alpha\tilde{v})\} = \inf_{\alpha \in (0,\infty)} \{\alpha z(\tilde{v})\} \overset{\text{(A.113)}}{=} -\infty. \tag{A.115}$$

Since $z(\hat{\alpha}) \in \mathbb{R}$ and $v - \hat{\alpha} \in \mathcal{L}_\infty$ for all $v \in \mathcal{L}_\infty$, we therefore obtain, for $\pi \in \mathcal{A}_1$,

$$\begin{aligned} &\inf_{v \in \mathcal{L}_\infty} \{z(v) \mid X^\pi(T) + v \geqslant b\} \overset{\text{(A.112)}}{\leqslant} \inf_{v \in \mathcal{L}_\infty} \{z(v) \mid v \geqslant \hat{\alpha}\} \\ &= z(\hat{\alpha}) + \inf_{v \in \mathcal{L}_\infty} \{z(v - \hat{\alpha}) \mid v - \hat{\alpha} \geqslant 0\} = z(\hat{\alpha}) + \inf_{\hat{v} \in \mathcal{L}_\infty^+} \{z(\hat{v})\} \overset{\text{(A.115)}}{\leqslant} -\infty. \end{aligned} \tag{A.116}$$

Combining (A.116) and (4.5.26), together with the general identity established at (D.6), we get

$$K(\pi, (Y, z)) = -\infty \ \text{ for all } \pi \in \mathcal{A}_1 \text{ and } (Y, z) \in \mathbb{Y} \text{ subject to } z \ngeqslant 0. \tag{A.117}$$

Therefore, from (4.5.28), (4.5.26), (A.110) and (A.117), for all $(\pi, (Y, z)) \in \Pi \times \mathbb{Y}$, it follows that

$$K(\pi, (Y, z)) = \begin{cases} E\left[X^\pi(T)Y(T)\right] - E\left[J^*(Y(T))\right] & \\ \quad + \inf_{v \in \mathcal{L}_\infty} \{z(v) \mid X^\pi(T) + v \geqslant b\}, & \text{if } \pi \in \mathcal{A}_1,\ z \geqslant 0, \\ -\infty, & \text{if } \pi \in \mathcal{A}_1,\ z \ngeqslant 0, \\ +\infty, & \text{if } \pi \in \Pi \backslash \mathcal{A}_1. \end{cases} \tag{A.118}$$

□

**Proof of Proposition 4.5.19**:
From (4.5.12), we know

$$f(\pi) \in (-\infty, \infty] \quad \text{for all } \pi \in \Pi. \tag{A.119}$$

Also from (4.5.30) and (4.5.33), we have $\varkappa(Y, z) \in (-\infty, \infty]$ for all $(Y, z) \in \mathbb{Y}$, and together with (A.122),

$$g(Y, z) \in [-\infty, \infty) \quad \text{for all } (Y, z) \in \mathbb{Y}. \tag{A.120}$$

Thus, from (A.120), (A.119) and (4.5.34), it follows that

$$f(\pi) = g(Y, z) \iff \begin{cases} f(\pi) = K(\pi, (Y, z)) \in \mathbb{R}, \\ g(Y, z) = K(\pi, (Y, z)) \in \mathbb{R}, \end{cases} \tag{A.121}$$

Now, fix some $(\pi, (Y, z)) \in \Pi \times \mathbb{Y}$. Recalling (4.5.27), (4.1.20) and Condition 2.2.1, we see that

$$\text{both } E\left[X^\pi(T)Y(T)\right] \text{ and } E\left[J^*(Y(T))\right] \text{ are real-valued.} \tag{A.122}$$

From (A.122), together with (4.5.31) and (4.5.32), we get

$$\begin{aligned} & g(Y, z) = K(\pi, (Y, z)) \in \mathbb{R} \\ \iff & \begin{cases} 1)\ \pi \in \mathcal{A}_1 \text{ (recall (4.5.29))},\ 2)\ z \geqslant 0,\ 3)\ \varkappa(Y, z) \in \mathbb{R}, \\ 4)\ \inf_{v \in \mathcal{L}_\infty} \{z(v) \mid X^\pi(T) + v \geqslant b\} \in \mathbb{R}, \\ 5)\ E\left[X^\pi(T)Y(T)\right] + \varkappa(Y, z) + \inf_{v \in \mathcal{L}_\infty} \{z(v) \mid X^\pi(T) + v \geqslant b\} = 0. \end{cases} \end{aligned} \tag{A.123}$$

Moreover, from (4.5.31) and (4.5.12), we have

$$\begin{aligned} & f(\pi) = K(\pi, (Y, z)) \in \mathbb{R} \\ \iff & \begin{cases} 1)\ \pi \in \mathcal{A}_1 \text{ (recall (4.5.29))},\ 2)\ X^\pi(T) \geqslant b \text{ a.s.},\ 3)\ z \geqslant 0, \\ 4)\ E\left[J(X^\pi(T)) - X^\pi(T)Y(T) + J^*(Y(T))\right] = \inf_{v \in \mathcal{L}_\infty} \{z(v) \mid X^\pi(T) + v \geqslant b\}. \end{cases} \end{aligned} \tag{A.124}$$

Next, we evaluate both sides of equation (A.124)-4). On the right side, from (A.124)-2), $0 \geqslant b - X^\pi(T)$ a.s., and then

$$\{v \in \mathcal{L}_\infty \mid v \geqslant b - X^\pi(T)\} \supset \{v \in \mathcal{L}_\infty \mid v \geqslant 0\}. \tag{A.125}$$

From (A.125) and (A.124)-3), we have

$$\inf_{v \in \mathcal{L}_\infty} \{z(v) \mid X^\pi(T) + v \geqslant b\} \overset{(A.125)}{\leqslant} \inf_{v \in \mathcal{L}_\infty} \{z(v) \mid v \geqslant 0\} \overset{(A.124)-3)}{=} 0. \tag{A.126}$$

Moreover, from (4.1.20) that $J^*(y) := \sup\{xy - J(x) \mid x \in \mathbb{R}\}$ for $y \in \mathbb{R}$, we get

$$J(X^\pi(T)) - X^\pi(T)Y(T) + J^*(Y(T)) \geqslant 0 \quad \text{a.s.} \tag{A.127}$$

Therefore, by (A.126) and (A.127), we see that (A.124)-4) holds if and only if each side is equal to zero:

$$E\left[J(X^\pi(T)) - X^\pi(T)Y(T) + J^*(Y(T))\right] = \inf_{v \in \mathcal{L}_\infty} \{z(v) \mid X^\pi(T) + v \geqslant b\} = 0. \tag{A.128}$$

From (A.128) and Proposition 4.1.12, we have

$$X^{\pi}(T) = \partial J^*(Y(T)) \quad \text{a.s.} \tag{A.129}$$

Hence, from (A.129), (A.128) and (A.124), we get

$$\begin{aligned} & f(\pi) = K(\pi, (Y, z)) \in \mathbb{R} \\ \iff & \begin{cases} 1)\ \pi \in \mathcal{A}_1,\ 2)\ X^{\pi}(T) \geqslant b \text{ a.s.},\ 3)\ z \geqslant 0, \\ 4)\ \inf_{v \in \mathcal{L}_\infty} \{z(v) \mid X^{\pi}(T) + v \geqslant b\} = 0,\ 5)\ X^{\pi}(T) = \partial J^*(Y(T)). \end{cases} \end{aligned} \tag{A.130}$$

From (4.5.29), we have

$$\{\pi \in \mathcal{A}_1 \mid X^{\pi}(T) \geqslant b \text{ a.s.}\} = \{\pi \in \mathcal{A} \mid X^{\pi}(T) \geqslant b \text{ a.s.}\}, \tag{A.131}$$

and then (A.131) together with (A.130) gives

$$\begin{aligned} & f(\pi) = K(\pi, (Y, z)) \in \mathbb{R} \\ \iff & \begin{cases} 1)\ \pi \in \mathcal{A},\ 2)\ X^{\pi}(T) \geqslant b \text{ a.s.},\ 3)\ z \geqslant 0, \\ 4)\ \inf_{v \in \mathcal{L}_\infty} \{z(v) \mid X^{\pi}(T) + v \geqslant b\} = 0,\ 5)\ X^{\pi}(T) = \partial J^*(Y(T)). \end{cases} \end{aligned} \tag{A.132}$$

Finally, the equivalence (A.121), together with (A.132) and (A.123), gives (4.5.53). □

**Proof of Lemma 5.2.8**:
Suppose $Y \in \mathbb{B}_2$ and for some $(Y_0, \Lambda_Y, \varrho_Y), (\hat{Y}_0, \hat{\Lambda}_Y, \hat{\rho}_Y) \in \mathbb{R} \times \Pi \times \mathcal{L}_2(\{\mathcal{F}_t\}; \mathcal{BV}_0^r)$,

$$\begin{aligned} Y(t) &= \frac{1}{S_0(t)} \left\{ Y(0) + \int_0^t S_0(s) \Lambda_Y'(s) \mathrm{d}W(s) + \int_0^t S_0(s) \varrho_Y(\mathrm{d}s) \right\} \\ &= \frac{1}{S_0(t)} \left\{ \hat{Y}(0) + \int_0^t S_0(s) \hat{\Lambda}_Y'(s) \mathrm{d}W(s) + \int_0^t S_0(s) \hat{\rho}_Y(\mathrm{d}s) \right\} \quad \text{a.s.}, t \in [0, T]. \end{aligned} \tag{A.133}$$

From $t = 0$ in (A.133), we get $Y_0 = \hat{Y}_0$. Then, by reorganizing (A.133), we also have

$$\int_0^t S_0(s) \left(\varrho_Y - \hat{\rho}_Y\right)(\mathrm{d}s) = \int_0^t S_0(s) \left(\hat{\Lambda}_Y'(s) - \Lambda_Y'(s)\right) \mathrm{d}W(s) \quad \text{a.s.}, t \in [0, T], \tag{A.134}$$

where the left hand side of (A.134) is a continuous finite variation process while the right hand side of (A.134) is a continuous local martingale. The only continuous local martingale that has finite variation path is the constant zero, therefore

$$\int_0^t S_0(s) \left(\varrho_Y - \hat{\rho}_Y\right)(\mathrm{d}s) = 0 \quad \text{a.s.}, t \in [0, T], \tag{A.135}$$

$$\int_0^t S_0(s) \left(\hat{\Lambda}_Y'(s) - \Lambda_Y'(s)\right) \mathrm{d}W(s) = 0 \quad \text{a.s.}, t \in [0, T]. \tag{A.136}$$

It directly follows from (A.135) and (2.1.5) that

$$\varrho_Y = \hat{\rho}_Y \quad \text{a.e.}, \tag{A.137}$$

and by using Itô isometry we also have from (A.136) that

$$\begin{aligned} 0 &= E\left[\int_0^t S_0(s)\left(\hat{\Lambda}'_Y(s) - \Lambda'_Y(s)\right) \mathrm{d}W(s)\right]^2 \\ &= E\left[\int_0^t \left\{S_0(s)\left(\hat{\Lambda}'_Y(s) - \Lambda'_Y(s)\right)\right\}^2 \mathrm{d}s\right], \end{aligned} \tag{A.138}$$

together with (2.1.5), it follows

$$\hat{\Lambda}_Y = \Lambda_Y \quad \text{a.e.} \tag{A.139}$$

□

**Proof of Proposition 5.2.11**:

From (5.2.30), (2.1.4), (2.1.5) and the inequality $(x+y+z)^2 \leqslant 3(x^2+y^2+z^2)$ for all $x, y, z \in \mathbb{R}$,

$$\begin{aligned} |Y(t)|^2 &\leqslant \left|Y(0) + \int_0^t S_0(s)\Lambda'_Y(s)\mathrm{d}W(s) + \int_0^t S_0(s)\varrho_Y(\mathrm{d}s)\right|^2 \\ &\leqslant 3e^{2\tilde{r}T}\left[|Y(0)|^2 + \left|\int_0^t \Lambda'_X(s)\mathrm{d}W(s)\right|^2 + |\varrho_Y(t)|^2\right] < \infty \quad t \in [0,T]. \end{aligned} \tag{A.140}$$

Therefore, observing $|\varrho_Y(t)| \leqslant \|\varrho_Y\|_T, t \in [0,T]$, we have

$$E\left[\sup_{t\in[0,T]} (Y(t))^2\right] \leqslant 3e^{2\tilde{r}T} E\left[|Y(0)|^2 + \|\varrho_Y\|_T^2 + \sup_{t\in[0,T]}\left|\int_0^t \Lambda'_Y(s)\mathrm{d}W(s)\right|^2\right]. \tag{A.141}$$

Apply Doob's maximal $L^2$-inequality to the last item in (A.141), together with (4.1.1) and (5.2.12), it follows

$$E\left[\max_{t\in[0,T]} |Y(t)|^2\right] < \infty. \tag{A.142}$$

□

**Proof of Proposition 5.2.12**:

By the definition of *quadratic co-variance* $[X,Y]$ (see Definition I.4.45 of Jacod and Shiryaev [14]), we have

$$X(t)Y(t) - X(0)Y(0) = \int_0^t Y(s-)\mathrm{d}X(s) + \int_0^t X(s-)\mathrm{d}Y(s) + \int_0^t [X,Y](s). \tag{A.143}$$

From Theorem I.4.52 of Jacod and Shiryaev [14], it follows that

$$[X,Y](t) = \langle X^c, Y^c\rangle(t) + \int_0^t (X(s) - X(s-))(Y(s) - Y(s-))\mathrm{d}s, \tag{A.144}$$

where $X^c$ and $Y^c$ denote the continuous martingale parts for $X$ and $Y$. Recalling (4.1.4) and (5.2.35), we have

$$X^c(t) = \int_0^t \Lambda'_X(s)\mathrm{d}W(s) \quad \text{and} \quad Y^c(t) = \int_0^t \Lambda'_Y(s)\mathrm{d}W(s), \tag{A.145}$$

and therefore

$$\langle X^c, Y^c\rangle(t) = \int_0^t \Lambda'_X(s)\Lambda_Y(s)\mathrm{d}s. \tag{A.146}$$

Combining (A.146), (A.144), (A.143) and the fact that $X$ is continuous, we have

$$X(t)Y(t) - X(0)Y(0) = \int_0^t Y(s-)\mathrm{d}X(s) + \int_0^t X(s)\mathrm{d}Y(s) + \int_0^t \Lambda'_X(s)\Lambda_Y(s)\mathrm{d}s. \tag{A.147}$$

Combining (A.147) with (4.1.4) and (5.2.35), we get

$$\begin{aligned} &X(t)Y(t) - X(0)Y(0) \\ &\quad = \int_0^t \left[\dot{X}(s)Y(s-) - r(s)X(s)Y(s) + \Lambda'_X(s)\Lambda_Y(s)\right]\mathrm{d}s \\ &\qquad + \int_0^t X(s)\varrho_Y(\mathrm{d}s) + \int_0^t \left[Y(s-)\Lambda'_X(s) + X(s)\Lambda'_Y(s)\right]\mathrm{d}W(s). \end{aligned} \tag{A.148}$$

Also, $Y$ has at most *countably many* jumps over $[0,T]$ so that

$$\int_0^t \dot{X}(s)Y(s-)\mathrm{d}s = \int_0^t \dot{X}(s)Y(s)\mathrm{d}s \qquad t \in [0,T]. \tag{A.149}$$

Combining (A.149), (A.148) and (5.2.37), we see

$$\hat{M}(X,Y)(t) = \int_0^t \left[Y(s-)\Lambda'_X(s) + X(s)\Lambda'_Y(s)\right]\mathrm{d}W(s), \tag{A.150}$$

and then $\hat{M}(X,Y)$ is a local martingale with

$$\hat{M}(X,Y)(0) = 0. \tag{A.151}$$

By (Theorem 51 on page 38 of P. Protter [28]), the local martingale $\hat{M}(X,Y)$ is uniformly integrable when the following sufficient condition holds:

$$E\left[\sup_{t\in[0,T]}\left|\hat{M}(X,Y)(t)\right|\right] < \infty. \tag{A.152}$$

To verify (A.152), we first observe from (5.2.37) that

$$\begin{aligned} \sup_{t\in[0,T]}\left|\hat{M}(X,Y)(t)\right| &\leqslant \sup_{t\in[0,T]}|X(t)Y(t)| + |X(0)Y(0)| + \sup_{t\in[0,T]}\left|\int_0^t X(s)\varrho_Y(\mathrm{d}s)\right| \\ + \sup_{t\in[0,T]}\left|\int_0^t \dot{X}(s)Y(s)\mathrm{d}s\right| &+ \sup_{t\in[0,T]}\left|\int_0^t r(s)X(s)Y(s)\mathrm{d}s\right| + \sup_{t\in[0,T]}\left|\int_0^t \Lambda'_X(s)\Lambda_Y(s)\mathrm{d}s\right|. \end{aligned} \tag{A.153}$$

Next, the following inequalities hold from the Holder inequality

$$\begin{aligned} E\left[\sup_{t\in[0,T]}|X(t)Y(t)|\right] &\leqslant E\left[\sup_{t\in[0,T]}|X(t)|\sup_{t\in[0,T]}|Y(t)|\right] \\ &\leqslant E\left[\sup_{t\in[0,T]}|X(t)|^2\right]^{1/2} E\left[\sup_{t\in[0,T]}|Y(t)|^2\right]^{1/2} \overset{(4.1.8)}{\underset{(5.2.36)}{<}} \infty, \end{aligned} \tag{A.154}$$

and

$$\begin{aligned} E\left[\sup_{t\in[0,T]}\left|\int_0^t X(s)\varrho_Y(\mathrm{d}s)\right|\right] &\leqslant E\left[\sup_{t\in[0,T]}\int_0^t |X(s)|\,|\varrho_Y|\,(\mathrm{d}s)\right] \\ = E\left[\int_0^T |X(s)|\,|\varrho_Y|\,(\mathrm{d}s)\right] &\leqslant E\left[(\sup_{s\in[0,T]}|X(s)|)\,|\varrho_Y|\,(T)\right] \\ &\leqslant E\left[\sup_{s\in[0,T]}|X(s)|^2\right]^{1/2} E\left[\|\varrho_Y\|_T^2\right]^{1/2} \overset{(4.1.8)}{\underset{(5.2.12)}{<}} \infty. \end{aligned} \tag{A.155}$$

Also, recall (4.1.2) and $\dot{X} \in L_{21}$ that

$$E\left[\left(\int_0^T \left|\dot{X}(t)\right|\mathrm{d}t\right)^2\right] < \infty, \tag{A.156}$$

then

$$\begin{aligned} E\left[\sup_{t\in[0,T]}\left|\int_0^t \dot{X}(s)Y(s)\mathrm{d}s\right|\right] &\leqslant E\left[\sup_{t\in[0,T]}\int_0^t \left|\dot{X}(s)\right||Y(s)|\,\mathrm{d}s\right] \\ = E\left[\int_0^T \left|\dot{X}(s)\right||Y(s)|\,(\mathrm{d}s)\right] &\leqslant E\left[\int_0^T \left|\dot{X}(s)\right|\mathrm{d}s\,\sup_{s\in[0,T]}|Y(s)|\right] \\ &\leqslant E\left[\left(\int_0^T \left|\dot{X}(s)\right|\mathrm{d}s\right)^2\right]^{1/2} E\left[\sup_{s\in[0,T]}|Y(s)|^2\right]^{1/2} \overset{(A.156)}{\underset{(5.2.36)}{<}} \infty. \end{aligned} \tag{A.157}$$

Recalling (2.1.4) for $\tilde{r}$, we have

$$\begin{aligned} E\left[\sup_{t\in[0,T]}\left|\int_0^t r(s)X(s)Y(s)\mathrm{d}s\right|\right] &\leqslant E\left[\sup_{t\in[0,T]}\int_0^t |r(s)|\,|X(s)|\,|Y(s)|\,\mathrm{d}s\right] \\ \overset{(2.1.4)}{\leqslant} \tilde{r}E\left[\int_0^T |X(s)|\,|Y(s)|\,\mathrm{d}s\right] &\leqslant \tilde{r}TE\left[\sup_{s\in[0,T]}|X(s)|\sup_{s\in[0,T]}|Y(s)|\right] \\ \leqslant \tilde{r}TE\left[\sup_{t\in[0,T]}|X(t)|^2\right]^{1/2} &E\left[\sup_{t\in[0,T]}|Y(t)|^2\right]^{1/2} \overset{(4.1.8)}{\underset{(5.2.36)}{<}} \infty. \end{aligned} \tag{A.158}$$

Finally, recall (4.1.1) and the fact that $\Lambda_X, \Lambda_Y \in \Pi$, it follows

$$E\left[\int_0^T \|\Lambda_X(t)\|^2 \, \mathrm{d}t\right] < +\infty, \qquad E\left[\int_0^T \|\Lambda_Y(t)\|^2 \, \mathrm{d}t\right] < +\infty, \tag{A.159}$$

also observe from Holder inequality that

$$\int_0^t |\Lambda_X'(s)\Lambda_Y(s)| \, \mathrm{d}s \leqslant \left(\int_0^t \|\Lambda_X(s)\|^2 \, \mathrm{d}s\right)^{1/2} \left(\int_0^t \|\Lambda_Y(s)\|^2 \, \mathrm{d}s\right)^{1/2}, \tag{A.160}$$

then

$$\begin{aligned} E\left[\sup_{t\in[0,T]} \left|\int_0^t \Lambda_X'(s)\Lambda_Y(s)\mathrm{d}s\right|\right] &\leqslant E\left[\int_0^T |\Lambda_X'(s)\Lambda_Y(s)| \, \mathrm{d}s\right] \\ &\overset{(A.160)}{\leqslant} E\left[\left(\int_0^T \|\Lambda_X(s)\|^2 \, \mathrm{d}s\right)^{1/2} \left(\int_0^T \|\Lambda_Y(s)\|^2 \, \mathrm{d}s\right)^{1/2}\right] \\ &\leqslant \frac{1}{2}\left(E\left[\int_0^T \|\Lambda_X(s)\|^2 \, \mathrm{d}s\right] + E\left[\int_0^T \|\Lambda_Y(s)\|^2 \, \mathrm{d}s\right]\right) \overset{(A.159)}{<} \infty. \end{aligned} \tag{A.161}$$

Therefore, combining (A.153), (A.154), (A.155), (A.157), (A.158) and (A.161), we show (A.152) holds. □

**Proof of Proposition 5.2.17:**

**Proof.** Fix some $\pi \in \Pi$ and $(Y, \rho) \in \mathbb{Y}$. Recalling (5.2.13) and (5.2.9), if $\rho \ngeqslant 0$, that is

$$E\left[\int_0^T V(t)\rho(\mathrm{d}t)\right] \geqslant 0, \quad \text{for all } V \in (\mathcal{L}_2(\{\mathcal{F}_t\}; \mathcal{C}))^+. \tag{A.162}$$

From (A.162), (5.2.10) and (5.2.8),

$$E\left[\int_0^T V(t)\rho(\mathrm{d}t)\right] \geqslant E\left[\int_0^T [B(t) - X^\pi(t)]\,\rho(\mathrm{d}t)\right] \quad \text{for all } V \geqslant B - X^\pi, \tag{A.163}$$

and then

$$\begin{aligned} \inf_{V\in\mathcal{L}_2(\{\mathcal{F}_t\};\mathcal{C})} &\left\{E\left[\int_0^T V(t)\rho(\mathrm{d}t)\right] \,\middle|\, X^\pi + V \geqslant B\right\} \\ &\overset{(A.163)}{=} E\left[\int_0^T [B(t) - X^\pi(t)]\,\rho(\mathrm{d}t)\right]. \end{aligned} \tag{A.164}$$

If $\rho \ngeqslant 0$ (recall (5.2.13) and (5.2.9)), that is

$$E\left[\int_0^T \tilde{V}(t)\rho(\mathrm{d}t)\right] < 0, \quad \text{for some } \tilde{V} \in (\mathcal{L}_2(\{\mathcal{F}_t\}; \mathcal{C}))^+. \tag{A.165}$$

then

$$\left\{\alpha\tilde{V} \mid 0 \leqslant \alpha \in \mathbb{R}\right\} \subset (\mathcal{L}_2(\{\mathcal{F}_t\}; \mathcal{C}))^+. \tag{A.166}$$

Put (recall (5.2.10))

$$U := X^\pi + V - B \in \mathcal{L}_2(\{\mathcal{F}_t\};\mathcal{C}), \tag{A.167}$$

and it follows

$$\begin{aligned}
&\inf_{V\in\mathcal{L}_2(\{\mathcal{F}_t\};\mathcal{C})} \left\{ E\left[\int_0^T V(t)\rho(\mathrm{d}t)\right] \,\middle|\, X^\pi + V \geqslant B \right\} \\
\overset{(A.167)}{=}\ & \inf_{U\in\mathcal{L}_2(\{\mathcal{F}_t\};\mathcal{C})} \left\{ E\left[\int_0^T [U(t)+B(t)-X^\pi(t)]\rho(\mathrm{d}t)\right] \,\middle|\, U \geqslant 0 \right\} \\
=\ & \inf_{U\in(\mathcal{L}_2(\{\mathcal{F}_t\};\mathcal{C}))^+} \left\{ E\left[\int_0^T U(t)\rho(\mathrm{d}t)\right] \right\} + E\left[\int_0^T [B(t)-X^\pi(t)]\rho(\mathrm{d}t)\right] \\
\overset{(A.166)}{\leqslant}\ & \inf_{0\leqslant\alpha\in\mathbb{R}} \left\{ E\left[\int_0^T \alpha\tilde{V}(t)\rho(\mathrm{d}t)\right] \right\} + E\left[\int_0^T [B(t)-X^\pi(t)]\rho(\mathrm{d}t)\right] \\
=\ & \inf_{0\leqslant\alpha\in\mathbb{R}} \left\{ \alpha E\left[\int_0^T \tilde{V}(t)\rho(\mathrm{d}t)\right] \right\} + E\left[\int_0^T |B(t)-X^\pi(t)|\,\rho(\mathrm{d}t)\right] \\
\underset{(5.2.14)}{\overset{(A.165)}{=}}\ & -\infty.
\end{aligned} \tag{A.168}$$

Combining (A.168), (A.164), (5.2.48) and (5.2.46), we get (5.2.49) as required. □

**Proof of Proposition 5.2.22**:
From (5.2.5), we know

$$f(\pi) \in (-\infty, \infty] \quad \text{for all } \pi \in \Pi. \tag{A.169}$$

Also from (5.2.51), we have $\varkappa(Y,\rho) \in (-\infty,\infty]$ for all $(Y,\rho) \in \mathbb{Y}$, and together with (A.172),

$$g(Y,\rho) \in [-\infty, \infty) \quad \text{for all } (Y,\rho) \in \mathbb{Y}. \tag{A.170}$$

Thus, from (A.170), (A.169) and (5.2.52), it follows that

$$f(\pi) = g(Y,\rho) \iff \begin{cases} f(\pi) = K(\pi,(Y,\rho)) \in \mathbb{R}, \\ g(Y,\rho) = K(\pi,(Y,\rho)) \in \mathbb{R}, \end{cases} \tag{A.171}$$

Now, fix some $(\pi,(Y,\rho)) \in \Pi\times\mathbb{Y}$. Recalling (5.2.47), (4.1.20) and Condition 2.2.1, we see that

$$\text{both } E\left[X^\pi(T)Y(T)\right] \text{ and } E\left[J^*(Y(T))\right] \text{ are real-valued.} \tag{A.172}$$

From (A.172), together with (5.2.49) and (5.2.50), we get

$$\begin{aligned}
&g(Y,\rho) = K(\pi,(Y,\rho)) \in \mathbb{R} \iff \\
&\begin{cases} 1)\ \pi\in\mathcal{A}, \quad 2)\ \rho \geqslant 0, \\ 3)\ E\left[X^\pi(T)Y(T) - \displaystyle\int_0^T [X^\pi(t)-B(t)]\,\rho(\mathrm{d}t)\right] + \varkappa(Y,\rho) = 0. \end{cases}
\end{aligned} \tag{A.173}$$

Moreover, from (5.2.49) and (5.2.5), we have

$$f(\pi) = K(\pi,(Y,\rho)) \in \mathbb{R} \qquad \Longleftrightarrow$$
$$\begin{cases} 1)\ \pi \in \mathcal{A}, \quad 2)\ X^\pi \geqslant B, \quad 3)\ \rho \geqslant 0, \\ 4)\ E\left[J(X^\pi(T))\right] = E\left[X^\pi(T)Y(T) - J^*(Y(T)) - \int_0^T [X^\pi(t) - B(t)]\rho(\mathrm{d}t)\right]. \end{cases} \tag{A.174}$$

From (A.174) - 2)3) and recalling (5.2.13), we can have

$$\int_0^T [X^\pi(t) - B(t)]\rho(\mathrm{d}t) \geqslant 0 \qquad \text{a.s..} \tag{A.175}$$

To see this, assume (A.175) does not hold and then there is some $\Omega$-subset $O \in \mathcal{F}_T$ such that $\Pr\{O\} > 0$ and

$$\int_0^T [X^\pi(\omega;t) - B(\omega;t)]\rho(\mathrm{d}\omega;t) < 0 \quad \text{for all } \omega \in O. \tag{A.176}$$

From (A.174) - 2) and (5.2.10), we see (recall (5.2.9))

$$\hat{V} := (X^\pi - B)\mathbb{1}_O \in \left(\mathcal{L}_2(\{\mathcal{F}_t\};\mathcal{C})\right)^+, \tag{A.177}$$

and from (A.176),

$$E\left[\int_0^T \hat{V}(t)\rho(\mathrm{d}t)\right] < 0. \tag{A.178}$$

Therefore, recalling (5.2.13), (A.175) holds from the contradiction between (A.178) and (A.174) - 3). Also recall (4.1.20) that $J^*(y) := \sup\{xy - J(x) \mid x \in \mathbb{R}\}$ for $y \in \mathbb{R}$, then

$$J(X^\pi(T)) - X^\pi(T)Y(T) + J^*(Y(T)) \geqslant 0 \quad \text{a.s.} \tag{A.179}$$

Therefore, by (A.175) and (A.179), (A.174)-4) holds if and only if:

$$E\left[J(X^\pi(T)) - X^\pi(T)Y(T) + J^*(Y(T))\right] = E\left[\int_0^T [B(t) - X^\pi(t)]\rho(\mathrm{d}t)\right] = 0, \tag{A.180}$$

and (A.180) together with (A.179) and (A.175) gives

$$J(X^\pi(T)) - X^\pi(T)Y(T) + J^*(Y(T)) = 0 \text{ and } \int_0^T [X^\pi(t) - B(t)]\rho(\mathrm{d}t) = 0 \quad \text{a.s.} \tag{A.181}$$

The first equation of (A.181) is equivalent to (recall (C.4))

$$Y(T) = \partial J(X^\pi(T)) \quad \text{a.s.} \tag{A.182}$$

(A.182) and (4.1.21) gives (recall (C.5))

$$X^\pi(T) = \partial J^*(Y(T)) \quad \text{a.s.} \tag{A.183}$$

Hence, from (A.183), (A.180) and (A.174), we get

$$
\begin{aligned}
&f(\pi) = K(\pi, (Y,\rho)) \in \mathbb{R} \quad \Longleftrightarrow \\
&\begin{cases} 1)\ \pi \in \mathcal{A}, \quad 2)\ X^{\pi} \geqslant B, \quad 3)\ \rho \geqslant 0, \\ 4)\ \displaystyle\int_0^T [X^{\pi}(t) - B(t)]\rho(\mathrm{d}t) = 0, \text{a.s.} \quad 5)\ X^{\pi}(T) = \partial J^*(Y(T)) \text{ a.s.} \end{cases}
\end{aligned}
\tag{A.184}
$$

Finally, the equivalence of (A.171), together with (A.184) and (A.173), gives (5.2.56). □

**Proof of Lemma 5.2.30**:
To establish (5.2.105), fix some arbitrary $\tau \in [0,T]$, and recalling Example 5.2.6 for $\rho_\tau^{\bar{\pi}}$ at (5.2.16) we have from (5.2.19) and (5.2.19) that

$$\rho_\tau^{\bar{\pi}} \in \mathcal{L}_2(\{\mathcal{F}_t\}; \mathcal{BV}_0^r) \quad \text{with} \quad \rho_\tau^{\bar{\pi}} \geqslant 0. \tag{A.185}$$

From (A.185) and (5.2.104), we have

$$E\left[(X^{\bar{\pi}}(\tau) - B(\tau))\,\mathbb{1}_{\{X^{\bar{\pi}}<B\}}(\tau)\right] = E\left[\int_0^T \{X^{\bar{\pi}}(s) - B(s)\}\rho_\tau^{\bar{\pi}}(\mathrm{d}s)\right] \geqslant 0. \tag{A.186}$$

On the other hand, it is clear that

$$(X^{\bar{\pi}}(\tau) - B(\tau))\,\mathbb{1}_{\{X^{\bar{\pi}}<B\}}(\tau) \leqslant 0 \quad \text{a.s.}, \tag{A.187}$$

Combining (A.187) and (A.186), we get

$$(X^{\bar{\pi}}(\tau) - B(\tau))\,\mathbb{1}_{\{X^{\bar{\pi}}<B\}}(\tau) = 0 \quad \text{a.s.}, \tag{A.188}$$

and then

$$\mathbb{1}_{\{X^{\bar{\pi}}<B\}}(\tau) = 0 \quad \text{a.s.}, \qquad \text{i.e.} \qquad X^{\bar{\pi}}(\tau) \geqslant B(\tau) \quad \text{a.s.} \tag{A.189}$$

By the arbitrary choice of $\tau \in [0,T]$, it follows that (A.189) holds on a dense subset $\{\tau_n; n = 1,2,3,...\} \subset [0,T]$, i.e. for $n = 1,2,3,...$,

$$X^{\bar{\pi}}(\omega;\tau_n) \geqslant B(\omega;\tau_n) \quad \omega \in \mathcal{N}_n^c, \quad \text{where } \mathcal{N}_n \in \mathcal{N}(P) \text{ is a } P\text{-null event.} \tag{A.190}$$

Put

$$\mathcal{N} := \cup_{n=1}^{\infty} \mathcal{N}_n \in \mathcal{N}(P), \tag{A.191}$$

then it follows from (A.191) and (A.190) that

$$P\left(\mathcal{N}^c\right) = 1. \tag{A.192}$$

Combining (A.190) with (A.191), we have

$$X^{\bar{\pi}}(\omega;\tau_n) \geqslant B(\omega;\tau_n) \quad \text{for all } n = 1,2,3,... \text{ and } \omega \in \mathcal{N}^c. \tag{A.193}$$

Since $\{\tau_n; n = 1, 2, 3, ...\} \subset [0, T]$ is a dense subset and $X^{\bar{\pi}}(\omega; \cdot)$ and $B(\omega; \cdot)$ are continuous over $[0, T]$ for each $\omega \in \Omega$, (A.193) gives

$$X^{\bar{\pi}}(\omega; t) \geqslant B(\omega; t) \quad \text{for all } n = 1, 2, 3, ... \text{ and } \omega \in \mathcal{N}^c. \tag{A.194}$$

From (A.194) and (A.192), we get (5.2.105) as required. □

<u>**Proof of Proposition 5.3.12**</u>:
(a) Fix some $\pi \in \mathcal{A} \backslash \mathcal{A}_2$. From (5.3.33), we have

$$P\left(\{X^{\pi}(t) + \alpha \geqslant B(t), \text{ all } t \in [0, T]\}\right) < 1, \quad \text{for all } \alpha \in \mathbb{R}. \tag{A.195}$$

Then for every $V \in \mathcal{L}_\infty(\{\mathcal{F}_t\}; \mathcal{C})$,

$$\begin{aligned} & P\left(\{X^{\pi}(t) + V(t) \geqslant B(t), \text{ all } t \in [0, T]\}\right) \\ & \overset{(5.3.7)}{\leqslant} P\left(\{X^{\pi}(t) + \|V\|_{u(\infty)} \geqslant B(t), \text{ all } t \in [0, T]\}\right) \overset{(A.195)}{<} 1. \end{aligned} \tag{A.196}$$

From (A.195) together with (5.3.24), we have

$$F(\pi, (u, V)) = +\infty, \qquad \text{for all } (u, V) \in \mathbb{U}. \tag{A.197}$$

From (5.3.27) it is clear that

$$\langle (u, V), (Y, Z) \rangle \in \mathbb{R}, \qquad \text{for } (u, V) \in \mathbb{U} \text{ and } (Y, Z) \in \mathbb{Y}, \tag{A.198}$$

and then, from (5.3.28), (A.197) and (A.198), together with the arbitrary $\pi \in \mathcal{A} \backslash \mathcal{A}_2$, we conclude

$$K(\pi, (Y, Z)) = +\infty, \qquad \text{for all } \pi \in \mathcal{A} \backslash \mathcal{A}_2 \text{ and } (Y, Z) \in \mathbb{Y}. \tag{A.199}$$

(b) Now fix some $\pi \in \mathcal{A}_2$. Then, from (5.3.33),

$$X^{\pi}(t) + \hat{\alpha} \geqslant B(t), \text{ all } t \in [0, T] \text{ a.s.}, \quad \text{for some } \hat{\alpha} \in \mathbb{R}, \tag{A.200}$$

and then

$$\{V \in \mathcal{L}_\infty(\{\mathcal{F}_t\}; \mathcal{C}) \mid V \geqslant \hat{\alpha}\} \overset{(A.200)}{\subset} \{V \in \mathcal{L}_\infty(\{\mathcal{F}_t\}; \mathcal{C}) \mid X^{\pi} + V \geqslant B\}. \tag{A.201}$$

Furthermore, fix some $Z \in \mathcal{L}^*_\infty(\{\mathcal{F}_t\}; \mathcal{C})$ such that $Z \ngeqslant 0$ i.e. (see (5.3.10) in Notation 5.3.5),

$$Z(\tilde{V}) < 0 \quad \text{for some } \tilde{V} \in (\mathcal{L}_\infty(\{\mathcal{F}_t\}; \mathcal{C}))^+, \tag{A.202}$$

then we have

$$\left\{\gamma \tilde{V} \in \mathcal{L}_\infty(\{\mathcal{F}_t\}; \mathcal{C}) \,\middle|\, \gamma \in (0, \infty)\right\} \overset{(A.202)}{\subset} (\mathcal{L}_\infty(\{\mathcal{F}_t\}; \mathcal{C}))^+, \tag{A.203}$$

and

$$\inf_{V\in(\mathcal{L}_\infty(\{\mathcal{F}_t\};\mathcal{C}))^+}\{Z(V)\} \overset{(A.203)}{\leqslant} \inf_{\gamma\in(0,\infty)}\{Z(\gamma\tilde{V})\} = \inf_{\gamma\in(0,\infty)}\{\gamma Z(\tilde{V})\} \overset{(A.202)}{=} -\infty. \tag{A.204}$$

Since $Z(\hat{\alpha}) \in \mathbb{R}$ and $V - \hat{\alpha} \in \mathcal{L}_\infty(\{\mathcal{F}_t\};\mathcal{C})$ for all $V \in \mathcal{L}_\infty(\{\mathcal{F}_t\};\mathcal{C})$, we therefore obtain, for all $\pi \in \mathcal{A}_2$,

$$\begin{aligned}
&\inf_{V\in\mathcal{L}_\infty(\{\mathcal{F}_t\};\mathcal{C})}\{Z(V) \mid X^\pi + V \geqslant B\} \overset{(A.201)}{\leqslant} \inf_{V\in\mathcal{L}_\infty(\{\mathcal{F}_t\};\mathcal{C})}\{Z(V) \mid V \geqslant \hat{\alpha}\} \\
&= Z(\hat{\alpha}) + \inf_{V\in\mathcal{L}_\infty(\{\mathcal{F}_t\};\mathcal{C})}\{Z(V-\hat{\alpha}) \mid V - \hat{\alpha} \geqslant 0\} = Z(\hat{\alpha}) + \inf_{\hat{V}\in(\mathcal{L}_\infty(\{\mathcal{F}_t\};\mathcal{C}))^+}\{Z(\hat{V})\} \\
&\overset{(A.204)}{\leqslant} -\infty.
\end{aligned} \tag{A.205}$$

Combining (A.205) and (A.199), together with the general identity established at (D.6), we get

$$K(\pi,(Y,Z)) = -\infty \quad \text{for all } \pi \in \mathcal{A}_2 \text{ and } (Y,Z) \in \mathbb{Y} \text{ subject to } Z \ngeqslant 0. \tag{A.206}$$

Therefore, from (5.3.30), (5.3.32), (A.199) and (A.206), for all $(\pi,(Y,Z)) \in \Pi \times \mathbb{Y}$, it follows that

$$K(\pi,(Y,Z)) = \begin{cases} E\left[X^\pi(T)Y(T)\right] - E\left[J^*(Y(T))\right] & \\ + \inf_{V\in\mathcal{L}_\infty(\{\mathcal{F}_t\};\mathcal{C})}\{Z(V) \mid X^\pi + V \geqslant B\}, & \text{if } \pi \in \mathcal{A}_2,\ Z \geqslant 0, \\ -\infty, & \text{if } \pi \in \mathcal{A}_2,\ Z \ngeqslant 0, \\ +\infty, & \text{if } \pi \in \Pi\backslash\mathcal{A}_2. \end{cases} \tag{A.207}$$

□

# Appendix B

# Topology Background

In this Appendix we summarize for completeness some elementary ideas from general topology and topological vector spaces which are needed in this thesis.

**Definition B.0.1.** A *topology* on a set $\mathbb{S}$ is a designated collection $\mathcal{T}$ of subsets of $\mathbb{S}$ with the following properties.

1. $\emptyset$ and $\mathbb{S}$ are members of $\mathcal{T}$,
2. The union of the members of any subcollection of $\mathcal{T}$ is a member of $\mathcal{T}$,
3. The intersection of the members of any finite subcollection of $\mathcal{T}$ is a member of $\mathcal{T}$.

By definition a *topological space* is an ordered pair $(\mathbb{S}, \mathcal{T})$ comprising a set $\mathbb{S}$ together with a topology $\mathcal{T}$ on $\mathbb{S}$. Every member of $\mathcal{T}$ is called a $\mathcal{T}$-open (or open) subset of $\mathbb{S}$, and a subset of $\mathbb{S}$ which is the complement of a member of $\mathcal{T}$ is called $\mathcal{T}$-closed. A collection $\mathcal{T}' \subset \mathcal{T}$ is a *base* for the topology $\mathcal{T}$ when every member of $\mathcal{T}$ is a union of members of $\mathcal{T}'$.

**Definition B.0.2.** Suppose that $(\mathbb{S}, \mathcal{T})$ is a topological space, and $A \subset \mathbb{S}$. Then the $\mathcal{T}$*-interior* of $A$ , denoted by $A^\circ$, is by definition the union of all subsets of $A$ which are $\mathcal{T}$-open, and the $\mathcal{T}$*-closure of* $A$, denoted by $\bar{A}$, is by definition the intersection of all $\mathcal{T}$-closed supersets of $A$ (it is clear that $A^\circ$ is the largest open subset of $A$ and $\bar{A}$ is the smallest closed superset of $A$). Given some $x \in \mathbb{S}$, a set $U \subset \mathbb{S}$ is said to be a $\mathcal{T}$*-neighborhood of* $\mathbb{S}$ when $U^\circ \neq \phi$ and $x \in U^\circ$ (it is not required that $U \in \mathcal{T}$). A collection $\mathcal{B}$ of $\mathcal{T}$-neighborhoods of a point $x \in \mathbb{S}$ is a *local base* at $x$ if every neighborhood of $x$ contains a member of $\mathcal{B}$.

**Definition B.0.3.** A topological space $(\mathbb{S}, \mathcal{T})$ is called a *Hausdorff space*, and $\mathcal{T}$ is a *Hausdorff topology*, when distinct points of $\mathbb{S}$ have disjoint neighborhoods.

**Definition B.0.4.** Suppose $(\mathbb{U}, \mathcal{T})$ is a topological space with the set $\mathbb{U}$ being specifically a *vector space*. If the vector space operations of vector addition and scalar multiplication are continuous with respect to the topology $\mathcal{T}$ then $\mathcal{T}$ is said to be a *linear topology* on $\mathbb{U}$, and $(\mathbb{U}, \mathcal{T})$ is called a *topological vector space*. Moreover, the vector topology $\mathcal{T}$ is called *locally convex* when for each $\mathcal{T}$-neighborhood $G$ of $0 \in \mathbb{U}$ there exists a convex $\mathcal{T}$-neighborhood $N$ of $0 \in \mathbb{U}$ such that $N \subset G$. That is, $\mathcal{T}$ has a local base of *convex sets* (Definition C.0.1) at $0 \in \mathbb{U}$.

**Definition B.0.5.** Given two vector spaces $\mathbb{U}$ and $\mathbb{Y}$, a mapping $\alpha : \mathbb{U} \times \mathbb{Y} \to \mathbb{R}$ is a *bilinear form* on $\mathbb{U} \times \mathbb{Y}$ when

(a) The mapping $u \to \alpha(u, y) : \mathbb{U} \to \mathbb{R}$ is linear for each fixed $y \in \mathbb{Y}$,

(b) The mapping $y \to \alpha(u, y) : \mathbb{Y} \to \mathbb{R}$ is linear for each fixed $u \in \mathbb{U}$.

**Remark B.0.6.** When there is just one designated bilinear form $\alpha$ on $\mathbb{U} \times \mathbb{Y}$ then the notation $\langle \cdot, \cdot \rangle$ is typically used to denote $\alpha$, so that, in particular, $\alpha(u, y)$ is denoted by $\langle u, y \rangle$. With a fixed bilinear form $\langle \cdot, \cdot \rangle$ the triple $(\mathbb{U}, \mathbb{Y}, \langle \cdot, \cdot \rangle)$ is called a *duality system*.

**Example B.0.7.** A particularly important and useful duality system is as follows: Suppose that $\mathbb{U}$ is a normed vector space, and put $\mathbb{Y} := (\mathbb{U})^*$, so that $y \in \mathbb{Y}$ if and only if $y$ is a norm-continuous linear functional on $\mathbb{U}$. Define

$$\langle u, y \rangle := y(u) \qquad (u, y) \in \mathbb{U} \times \mathbb{Y}. \tag{B.1}$$

Then (B.1) defines a bilinear form on $\mathbb{U} \times \mathbb{Y}$ and it is immediate that $(\mathbb{U}, \mathbb{Y}, \langle \cdot, \cdot \rangle)$ is a duality system.

**Remark B.0.8.** Suppose that $(\mathbb{U}, \mathbb{Y}, \langle \cdot, \cdot \rangle)$ is a duality system in the sense of Remark B.0.6. We do not as yet have any natural topology on either of the vector spaces $\mathbb{U}$ or $\mathbb{Y}$. We denote by $\mathfrak{S}(\mathbb{U}, \mathbb{Y})$ the topology on $\mathbb{U}$ which is generated by the mappings

$$u \to \langle u, y \rangle : \mathbb{U} \to \mathbb{R} \qquad \text{for all } y \in \mathbb{Y}, \tag{B.2}$$

and call $\mathfrak{S}(\mathbb{U}, \mathbb{Y})$ the *weak topology* generated on $\mathbb{U}$ through the bilinear form $\langle \cdot, \cdot \rangle$. In an exactly symmetric way we can define $\mathfrak{S}(\mathbb{Y}, \mathbb{U})$, the *weak topology* generated on $\mathbb{Y}$ through the bilinear form $\langle \cdot, \cdot \rangle$.

**Definition B.0.9.** Suppose that $(\mathbb{U}, \mathbb{Y}, \langle \cdot, \cdot \rangle)$ is a duality system in the sense of Remark B.0.6. A locally convex linear topology $\mathcal{U}$ on the vector space $\mathbb{U}$ is called *compatible* with the bilinear form $\langle \cdot, \cdot \rangle$ on $\mathbb{U} \times \mathbb{Y}$ (or $\langle \mathbb{U}, \mathbb{Y} \rangle$-*compatible* for short) when

(1) the mapping $u \to \langle u, y \rangle : \mathbb{U} \to \mathbb{R}$ is $\mathcal{U}$-continuous for each $y \in \mathbb{Y}$,

(2) each $\mathcal{U}$-continuous linear functional $\Phi : \mathbb{U} \to \mathbb{R}$ is necessarily given by $\Phi(u) = \langle u, y \rangle$, $u \in \mathbb{U}$, for some kernel $y \in \mathbb{Y}$,

(see Rockafellar ([31], page 13) or Definition 8-2-8 in Wilansky [39]).

**Remark B.0.10.** Suppose that $(\mathbb{U}, \mathbb{Y}, \langle \cdot, \cdot \rangle)$ is a duality system in the sense of Remark B.0.6. Then it is a consequence of the Hahn-Banach theorem that the weak topology $\mathfrak{S}(\mathbb{U}, \mathbb{Y})$ (see Remark B.0.6) is $\langle \mathbb{U}, \mathbb{Y} \rangle$-compatible (see Theorem 8-2-12 of Wilansky [39]). Furthermore, if $\mathcal{U}$ is some $\langle \mathbb{U}, \mathbb{Y} \rangle$-compatible topology on $\mathbb{U}$, then it is immediate that $\mathfrak{S}(\mathbb{U}, \mathbb{Y}) \subset \mathcal{U}$, that is $\mathfrak{S}(\mathbb{U}, \mathbb{Y})$ is the weakest (or coarsest) among all the $\langle \mathbb{U}, \mathbb{Y} \rangle$-compatible topologies on $\mathbb{U}$.

**Remark B.0.11.** Suppose that $(\mathbb{U}, \mathbb{Y}, \langle\cdot,\cdot\rangle)$ is a duality system according to Remark B.0.6, which is *separating on* $\mathbb{U}$ in the following sense: corresponding to arbitrary *distinct* members $u_1$ and $u_2$ in $\mathbb{U}$ (i.e. $u_1 \neq u_2$) there exists some $y \in \mathbb{Y}$ such that $\langle u_1, y\rangle \neq \langle u_2, y\rangle$. Then the weak topology $\mathfrak{S}(\mathbb{U}, \mathbb{Y})$ on $\mathbb{U}$ is a Hausdorff topology (as follows from Theorem 6.3.2 of Wilansky [40]). In view of Remark B.0.10 is then follows that, if $\mathcal{U}$ is some $\langle\mathbb{U}, \mathbb{Y}\rangle$-compatible topology on $\mathbb{U}$, then $\mathcal{U}$ is a Hausdorff topology.

**Remark B.0.12.** In light of Remark B.0.10 it is natural to ask if there is a strongest (or finest) $\langle\mathbb{U}, \mathbb{Y}\rangle$-compatible topology on $\mathbb{U}$. Indeed, one can show that the weakest locally convex topology on $\mathbb{U}$ which is strong enough to contain (i.e. is stronger than) *each and every* $\langle\mathbb{U}, \mathbb{Y}\rangle$-compatible topology on $\mathbb{U}$ is itself a $\langle\mathbb{U}, \mathbb{Y}\rangle$-compatible topology on $\mathbb{U}$, and therefore must be the strongest such topology (see Theorem 8-2-14 in Wilansky [39]). This strongest $\langle\mathbb{U}, \mathbb{Y}\rangle$-compatible topology on $\mathbb{U}$ is called the *Mackey topology on* $\mathbb{U}$ *corresponding to the given duality system* $(\mathbb{U}, \mathbb{Y}, \langle\cdot,\cdot\rangle)$, and is denoted by $\tau(\mathbb{U}, \mathbb{Y})$. We see, therefore, that

$$\mathfrak{S}(\mathbb{U}, \mathbb{Y}) \subset \mathcal{U} \subset \tau(\mathbb{U}, \mathbb{Y}), \tag{B.3}$$

for each and every $\langle\mathbb{U}, \mathbb{Y}\rangle$-compatible topology $\mathcal{U}$ on $\mathbb{U}$. It is important to understand that both the weak topology $\mathfrak{S}(\mathbb{U}, \mathbb{Y})$ and the Mackey topology $\tau(\mathbb{U}, \mathbb{Y})$ are *determined entirely* by the given duality system $(\mathbb{U}, \mathbb{Y}, \langle\cdot,\cdot\rangle)$.

**Remark B.0.13.** While a neighbourhood base is easy to write down for the weak topology $\mathfrak{S}(\mathbb{U}, \mathbb{Y})$, the same is certainly not true for the Mackey topology $\tau(\mathbb{U}, \mathbb{Y})$. There is in fact a characterization of the Mackey topology (called the *Mackey-Arens characterization*) for general duality systems $(\mathbb{U}, \mathbb{Y}, \langle\cdot,\cdot\rangle)$ which we shall not state here since this will not be needed for the present thesis (see Theorem 3.2 of Schaefer and Wolff [35, p.131] or Theorem 9-2-3 of Wilansky [39] for this characterization). The Mackey-Arens characterization of the strongest possible compatible topology $\tau(\mathbb{U}, \mathbb{Y})$ is not very easy to apply to concrete duality systems. For example, suppose that $(\Omega, \mathcal{F}, P)$ is a probability space, define the vector spaces

$$\mathbb{U} = \mathcal{L}_\infty(\Omega, \mathcal{F}, P), \qquad \mathbb{Y} = \mathcal{L}_1(\Omega, \mathcal{F}, P), \tag{B.4}$$

and put

$$\langle u, y\rangle := \int_\Omega uy \mathrm{d}P, \qquad u \in \mathbb{U}, \qquad y \in \mathbb{Y}. \tag{B.5}$$

Then it is easily verified that (B.4) - (B.5) define a duality system $(\mathbb{U}, \mathbb{Y}, \langle\cdot,\cdot\rangle)$ in the sense of Remark B.0.6, but the Mackey topology $\tau(\mathbb{U}, \mathbb{Y})$ for this concrete duality system is quite complicated and not especially easy to use. One particularly important duality pairing for which the Mackey topology turns out to be very simple is that given by Example B.0.7. Indeed we have the following result (see Theorem 3.4 of [35, p.132] or Theorem 8-4-9 together with Example 8-4-10 of Wilansky [39]):

**Theorem B.0.14.** **[Mackey-Arens]** *Suppose that* $\mathbb{U}$ *is a* normed vector space *with the norm-topology denoted by* $\mathcal{U}$*, and that* $\mathbb{Y}$ *is the* norm-dual space*, that is* $\mathbb{Y} := \mathbb{U}^*$*, with the canonical duality pairing*

$$\langle u, y\rangle := y(u) \qquad (u, y) \in \mathbb{U} \times \mathbb{Y}. \tag{B.6}$$

*Then the norm-topology* $\mathcal{U}$ *on* $\mathbb{U}$ *is* $\langle \mathbb{U}, \mathbb{Y}\rangle$*-compatible, and is furthermore the Mackey topology for the duality system* $(\mathbb{U}, \mathbb{U}^*, \langle\cdot,\cdot\rangle)$*, that is* $\mathcal{U} = \tau(\mathbb{U}, \mathbb{U}^*)$*.*

The difficulty with the duality system defined by (B.4) - (B.5) is that, although $\mathbb{U} = \mathcal{L}_\infty(\Omega, \mathcal{F}, P)$ clearly carries the usual "essential-supremum" norm topology, we have paired this only with $\mathbb{Y} = \mathcal{L}_1(\Omega, \mathcal{F}, P)$ which is a *strict* subspace of the norm-dual space $\mathcal{L}^*_\infty(\Omega, \mathcal{F}, P)$. If we instead pair $\mathbb{U} = \mathcal{L}_\infty(\Omega, \mathcal{F}, P)$ with the space $\mathbb{Y} = (\mathbb{U})^* = \mathcal{L}^*_\infty(\Omega, \mathcal{F}, P)$ of *all* norm-continuous linear functionals on $\mathbb{U}$ through the pairing (B.1), then, from Theorem B.0.14, the Mackey topology $\tau(\mathbb{U}, \mathbb{Y})$ on $\mathbb{U} = \mathcal{L}_\infty(\Omega, \mathcal{F}, P)$ is just the norm-topology on $\mathbb{U}$.

# Appendix C

# Elementary Convex Analysis Theory

**Definition C.0.1.** Let $\mathbb{U}$ be an arbitrary vector space over $\mathbb{R}$. A set $A \subset \mathbb{U}$ is said to be *convex* if and only if for every finite subset of elements $a_1, ..., a_n$ of $A$ and real positive numbers $\alpha_1, ..., \alpha_n$ with sum unity, we have

$$\sum_{i=1}^{n} \alpha_i a_i \in A. \tag{C.1}$$

**Definition C.0.2.** Let $A$ be a convex subset of $\mathbb{U}$, and $\phi$ a mapping of $A$ into $\bar{\mathbb{R}} := [-\infty, +\infty]$. Then $\phi$ is said to be *convex* when we have

$$\phi(\alpha u + (1-\alpha)v) \leqslant \alpha\phi(u) + (1-\alpha)\phi(v), \tag{C.2}$$

for all $\alpha \in [0,1]$ and for all $u$ and $v$ in $A$ such that the right side of (C.2) is defined (i.e. $u$ and $v$ are such that we never get $\infty - \infty$).

**Definition C.0.3.** Fix an arbitrary duality system $(\mathbb{U}, \mathbb{Y}, \langle\cdot,\cdot\rangle)$ (see Remark B.0.6) and let $\phi$ be a function of $\mathbb{U}$ into $\bar{\mathbb{R}}$ (not necessarily convex). Then

$$\phi^*(y) := \sup_{u \in \mathbb{U}} \{\langle u, y\rangle - \phi(u)\} \qquad y \in \mathbb{Y}, \tag{C.3}$$

defines a function from $\mathbb{Y}$ into $\bar{\mathbb{R}}$, which is called the *conjugate function* of $\phi$.

**Remark C.0.4.** It is immediate from Definition C.0.3 that the mapping $\phi^* : \mathbb{Y} \to \bar{\mathbb{R}}$ is convex, irrespective of whether or not the function $\phi$ is convex.

**Remark C.0.5.** A particularly important special case of Definition C.0.3 is when $\mathbb{U} = \mathbb{Y} = \mathbb{R}^n$, with the usual duality pairing $\langle u, y\rangle := u'y$, and when $\phi$ is a *continuously differentiable* $\mathbb{R}$*-valued* function i.e. $\phi : \mathbb{R}^n \to \mathbb{R}$, and $\phi$ is continuously differentiable on $\mathbb{R}^n$ with derivative denoted by $\partial\phi$. Then, as a very special consequence of Proposition 5.1 of [10, p.21-22], there is the following equivalence: For each $(u, y) \in \mathbb{R}^n \times \mathbb{R}^n$ we have

$$\phi(u) + \phi^*(y) = u'y \qquad \Longleftrightarrow \qquad y = \partial\phi(u). \tag{C.4}$$

Suppose, in addition, that the conjugate function $\phi^*$ is also a *continuously differentiable* $\mathbb{R}$*-valued* function i.e. $\phi^* : \mathbb{R}^n \to \mathbb{R}$, and $\phi^*$ is continuously differentiable with derivative denoted by $\partial\phi^*$ (this is certainly the case when $\phi$ is a strictly positive definite quadratic function on $\mathbb{R}^n$). Then, again as a very special consequence of Corollary 5.2 of [10, p.21-22], there is the following equivalence: For each $(u, y) \in \mathbb{R}^n \times \mathbb{R}^n$ we have

$$y = \partial\phi(u) \qquad \Longleftrightarrow \qquad u = \partial\phi^*(y). \tag{C.5}$$

Upon combining (C.5) and (C.4) we obtain the following: For each $(u, y) \in \mathbb{R}^n \times \mathbb{R}^n$ we have

$$\phi(u) + \phi^*(y) = u'y \qquad \Longleftrightarrow \qquad u = \partial\phi^*(y). \tag{C.6}$$

We shall now reformulate the following "Lagrange duality theorem" which is really just an elementary consequence of Proposition 2.6.1 and Theorem 2.6.1 in Section 2.6 of Aubin [2]:

**Theorem C.0.6.** *Let* $\mathbb{U}$ *be a general vector space. Suppose that*

*(1)* $A$ *is a convex subset of* $\mathbb{U}$*;*

*(2)* $f : A \to \mathbb{R}$ *is a convex function;*

*(3)* $h : \mathbb{U} \to \mathbb{R}$ *is an affine functional;*

*(4)* $d \in \mathbb{R}$ *is a constant in the* interior *of the convex set (i.e. interval)* $\{h(u) \mid u \in A\} \subset \mathbb{R}$*,*

*and consider the constrained convex optimization problem*

$$\textit{minimize } f(u) \textit{ subject to } u \in A \textit{ and } h(u) = d. \tag{C.7}$$

*Define the* Lagrangian function *for the constraint* $h(u) = d$ *in problem* (C.7) *as follows:*

$$\hat{K}(\lambda; u) := f(u) + \lambda(h(u) - d), \qquad \textit{for any } \lambda \in \mathbb{R} \textit{ and } u \in A. \tag{C.8}$$

*Then there exists a constant* $\bar{\lambda} \in \mathbb{R}$ *(called the Lagrange multiplier for the constraint* $h(u) = d$ *in problem* (C.7)*) such that*

$$\inf_{u\in A} \hat{K}(\bar{\lambda}; u) = \sup_{\lambda\in\mathbb{R}} \inf_{u\in A} \hat{K}(\lambda; u) = \inf_{\substack{u\in A\\ h(u)=d}} f(u). \tag{C.9}$$

**Proof.** From Conditions (1)-(3) and Proposition 2.6.1 of Aubin [2], we get

$$\sup_{\lambda\in\mathbb{R}} \hat{K}(\lambda; u) = \begin{cases} f(u), & \text{if } u \in A \text{ and } h(u) = d, \\ +\infty, & \text{otherwise.} \end{cases} \tag{C.10}$$

Then,

$$\sup_{\lambda\in\mathbb{R}} \inf_{u\in A} \hat{K}(\lambda; u) \leqslant \inf_{\substack{u\in A\\ h(u)=d}} f(u). \tag{C.11}$$

Again, by Conditions (1)-(4) and Theorem 2.6.1 of Aubin [2], there exists some $\bar{\lambda} \in \mathbb{R}$ such that

$$\inf_{u \in A} \hat{K}(\bar{\lambda}; u) = \inf_{\substack{u \in A \\ h(u)=d}} f(u). \tag{C.12}$$

On the other hand, we have

$$\inf_{u \in A} \hat{K}(\bar{\lambda}; u) \leqslant \sup_{\lambda \in \mathbb{R}} \inf_{u \in A} \hat{K}(\lambda; u), \tag{C.13}$$

together with (C.11), (C.12) and (C.13),

$$\sup_{\lambda \in \mathbb{R}} \inf_{u \in A} \hat{K}(\lambda; u) \leqslant \inf_{\substack{u \in A \\ h(u)=d}} f(u) = \inf_{u \in A} \hat{K}(\bar{\lambda}; u) \leqslant \sup_{\lambda \in \mathbb{R}} \inf_{u \in A} \hat{K}(\lambda; u). \tag{C.14}$$

Therefore, (C.9) follows from (C.14). □

**Remark C.0.7.** Observe that the hypotheses of Theorem C.0.6 *do not* include any assumption that the problem C.7 actually has a solution. The real message of Theorem C.0.6 is that the condition (4) on the constant $d$ is enough to ensure existence of $\bar{\lambda} \in \mathbb{R}$ with the properties stated at (C.9).

**Remark C.0.8.** Theorem 2.6.1 of Aubin [2], which is essential for the preceding argument (see (C.12)), is established by a direct argument based in the separating hyperplanes theorem. It may be of interest to note that Theorem 2.6.1 of Aubin [2] can also be established by a simple application of the Rockafellar variational method outlined in Section 3.1 although we shall not give the details here.

# Appendix D

# Technical Results of Rockafellar on Conjugates of Integral Functions

Integrals having certain convexity properties which can be analyzed in the light of the theory of conjugate convex functions are studied in this section. The results in this section are applicable in the study of problems in control theory, as well as for dealing with integrals of convex risk criterion functions. The arguments in this section are taken from R.T. Rockafellar [29].

Let $(S, \Sigma, \mu)$ be a measure space with a finite measure $\lambda$ in this section. Recalling Definition C.0.2 for *convex* functions, we also call $\phi : \mathbb{R}^n \to (\infty, +\infty]$ a *proper* convex function if $\phi$ is not identically $+\infty$, and *lower semi-continuous (l.s.c.)* at $x \in \mathbb{R}^n$ if for every sequence $\{x^{(m)}; m \in \mathbb{N}\} \subset \mathbb{R}^n$ which converges in the norm to $x$, we have

$$\phi(x) \leq \liminf_{m\to\infty} \phi(x^{(m)}). \tag{D.1}$$

**Definition D.0.1.** An integrand function $\phi : \mathbb{R}^n \times S \to (-\infty, +\infty]$ is called a *normal convex integrand* if $\phi(x, \omega)$ is proper convex and lower semi-continuous in $x \in \mathbb{R}^n$ for each $\omega \in S$, and if further there exists a *countable* collection $U$ of $\Sigma$-measurable functions $u : S \to \mathbb{R}^n$ having the following properties:

(1) for each $u \in U$, $\phi(u(\cdot), \cdot) : S \to (-\infty, +\infty]$ is $\Sigma$-measurable;

(2) for each $\omega \in S$, $U_\omega \cap \text{dom } \phi(\cdot, \omega)$ is dense in dom $\phi(\cdot, \omega)$, where

$$U_\omega := \{u(\omega) \mid u \in U\}. \tag{D.2}$$

We refer Lemma 2 of Rockafellar [29, p.528] for the proofs of the following lemma.

**Lemma D.0.2.** *A function $\phi : \mathbb{R}^n \times S \to (-\infty, +\infty]$ is a normal convex integrand if the following conditions hold*

*(a) for each fixed $x \in \mathbb{R}^n$, $\phi(x, \cdot) : S \to (-\infty, +\infty]$ is $\Sigma$-measurable;*

(b) *for each* $\omega \in S$, $\phi(\cdot, \omega) : \mathbb{R}^n \to (-\infty, +\infty]$ *is convex and lower semi-continuous with interior points in its effective domain* $\{x \in \mathbb{R}^n \mid \phi(x, \omega) < +\infty\}$.

**Remark D.0.3.** From Lemma D.0.2, both $J$ and $J^*$ given by (4.1.19) and (4.1.20) are normal convex integrands.

**Definition D.0.4.** Suppose $\mathcal{L}$ is real vector space of $\Sigma$-measurable functions from $S$ to $\mathbb{R}^n$ for some $n \in \mathbb{N}$. We say $\mathcal{L}$ is *decomposable* if it satisfies the following conditions:

(1) $\mathcal{L}$ contains every bounded $\Sigma$-measurable function from $S$ to $R^n$;

(2) if $u \in \mathcal{L}$ and $E \in \Sigma$, then $\mathcal{L}$ contains $\mathbb{1}_E \cdot u$, where $\mathbb{1}_E$ is the *indicator function* of set $E$ over $S$.

**Remark D.0.5.** As the most important cases of decomposable vector spaces, vector spaces $\mathcal{L}_p(S, \Sigma, \mu), p \in [1, +\infty]$ are all decomposable. However, even if $S$ is a compact Hausdorff space (e.g. the interval $[0, 1] \subset \mathbb{R}$) and $\Sigma$ is the Borel $\sigma$-algebra, the space $C(S)$ of all real-valued continuous functions on $S$ is definitely *not* decomposable by either (1) or (2) in Definition D.0.4.

From Theorem 2 of Rockafellar [29, p.532], we have the following.

**Theorem D.0.6.** *Let* $\mathbb{L}$ *and* $\mathbb{M}$ *be decomposable real vector spaces of* $\Sigma$*-measurable functions from* $S$ *to* $\mathbb{R}^n$ *such that* $\omega \to u'(\omega)y(\omega) : S \to \mathbb{R}$ *gives a* $\mu$*-integrable function for every* $u \in \mathbb{L}$ *and* $y \in \mathbb{M}$. *Let* $\phi : \mathbb{R}^n \times S \to (-\infty, +\infty]$ *be a normal convex integrand such that* $\phi(u(\cdot), \cdot) : S \to (-\infty, \infty]$ *is* $\mu$*-integrable for at least one* $u \in \mathbb{L}$, *and* $\phi^*(y(\cdot)) : S \to (-\infty, \infty]$ *is* $\mu$*-integrable for at least one* $y \in \mathbb{M}$. *Then*

$$\sup_{u \in \mathbb{L}} \left[ \int_S u'y \, d\mu - \int_S \phi(u) \, d\mu \right] = \int_S \phi^*(y) \, d\mu, \tag{D.3}$$

*for each* $y \in \mathbb{M}$, *where*

$$\phi^*(y) := \sup_{x \in \mathbb{R}^n} \{\langle x, y \rangle - \phi(x)\}, \qquad y \in \mathbb{R}^n. \tag{D.4}$$

**Remark D.0.7.** Theorem D.0.6 implies that the outside supremum over $u \in \mathbb{L}$ on the left can be shifted to *follow* the integral over $S$ and becomes a supremum over $x \in \mathbb{R}^n$, that is for each $y \in \mathbb{M}$ we have

$$\sup_{u \in \mathbb{L}} \left\{ \int_S [u'y - \phi(u)] \mathrm{d}\mu \right\} = \int_S \sup_{x \in \mathbb{R}^n} \{\langle x, y(\omega) \rangle - \phi(x)\} \mu(\mathrm{d}\omega) = \int_S \phi^*(y(\omega)) \mu(\mathrm{d}\omega). \tag{D.5}$$

Notice that decomposability of the spaces $\mathbb{L}$ and $\mathbb{M}$ is essential for this result.

We shall now establish the following simple consequence of Theorem D.0.6, which is frequently used in the calculation of Lagrangian functions in the thesis.

**Proposition D.0.8.** *Suppose that $\xi, \eta \in \mathcal{L}_2(\Omega, \mathcal{F}_T, P)$, that $J(\cdot)$ and its convex conjugate function $J^*(\cdot)$ are given by (4.1.19) and (4.1.20), and that Condition 2.2.1 holds. Then*

$$\inf_{u \in \mathcal{L}_2(\Omega, \mathcal{F}_T, P)} \{E[u\eta + J(\xi - u)]\} = E[\xi\eta] - E[J^*(\eta)], \tag{D.6}$$

**Proof.** Since $\xi \in \mathcal{L}_2(\Omega, \mathcal{F}_T, P)$, it follows

$$v := \xi - u \in \mathcal{L}_2(\Omega, \mathcal{F}_T, P), \quad \text{for all } u \in \mathcal{L}_2(\Omega, \mathcal{F}_T, P), \tag{D.7}$$

and then

$$\begin{aligned} & \inf_{u \in \mathcal{L}_2(\Omega, \mathcal{F}_T, P)} \{E[u\eta + J(\xi - u)]\} \\ = & \inf_{v \in \mathcal{L}_2(\Omega, \mathcal{F}_T, P)} \{E[(\xi - v)\eta + J(v)]\} \\ = & \; E[\xi\eta] - \sup_{v \in \mathcal{L}_2(\Omega, \mathcal{F}_T, P)} \{E[v\eta - J(v)]\}. \end{aligned} \tag{D.8}$$

We next evaluate the supremum on the right side of (D.8). For this, we shall use Theorem D.0.6 with

$$(S, \Sigma, \mu) = (\Omega, \mathcal{F}_T, P), \mathbb{L} = \mathbb{M} = \mathcal{L}_2(\Omega, \mathcal{F}_T, P), n = 1, \text{and } \phi = J. \tag{D.9}$$

We know that $\mathbb{L} = \mathbb{M} = \mathcal{L}_2(\Omega, \mathcal{F}_T, P)$ is a decomposable space (see Remark D.0.5). Since $\eta \in \mathcal{L}_2(\Omega, \mathcal{F}_T, P)$, it holds $v\eta$ is $P$-integrable for each $v \in \mathcal{L}_2(\Omega, \mathcal{F}_T, P)$. Then it follows from Theorem D.0.6 that

$$\sup_{v \in \mathcal{L}_2(\Omega, \mathcal{F}_T, P)} \{E[v\eta - J(v)]\} = E[J^*(\eta)]. \tag{D.10}$$

Therefore, (D.7) follows by combining (D.9) and (D.10). □

# Appendix E

# Functions of Bounded Variation

We denote by $\mathcal{C}[0,T]$ the vector space of all $\mathbb{R}$-valued *continuous functions* over $[0,T]$ with the usual uniform norm

$$\|f\|_u := \sup_{t\in[0,T]} |f(t)| \qquad \text{for all } f \in \mathcal{C}[0,T]. \tag{E.1}$$

Also denote by $(\mathcal{C}[0,T])^*$ the vector space of all linear functionals on $\mathcal{C}[0,T]$ which are continuous with the uniform norm at (E.1) and define the *functional norm* $\|\cdot\|_u^*$ on $(\mathcal{C}[0,T])^*$ in the usual way, namely

$$\|L\|_u^* := \sup_{\substack{f\in\mathcal{C}[0,T]\\ \|f\|_u=1}} |L(f)| \qquad \text{for all } L \in (\mathcal{C}[0,T])^*. \tag{E.2}$$

We now recall the basic facts about functions of bounded variation:

A **partition** of an interval $[0,t] \subset \mathbb{R}$ is a finite set of points $\{t_0, t_1, ..., t_n\}$ such that $0 = t_0 < t_1 < ... < t_n = t$, and denoted by $\mathcal{P}[0,t]$ the collection of all partitions of $[0,t]$. We define three extended real-valued increasing functions $T_\phi, P_\phi, N_\phi$ with respect to each real-valued function $\phi$ on $[0,t]$ with $t \in (0,\infty)$ as:

$$T_\phi(t) := \sup\left\{\sum_{i=1}^{n} |\phi(t_i) - \phi(t_{i-1})| \;\middle|\; \{t_0, t_1, ..., t_n\} \in \mathcal{P}[0,t], n \in \mathbb{N}\right\}, \tag{E.3}$$

$$P_\phi(t) := \sup\left\{\sum_{i=1}^{n} [\phi(t_i) - \phi(t_{i-1})]^+ \;\middle|\; \{t_0, t_1, ..., t_n\} \in \mathcal{P}[0,t], n \in \mathbb{N}\right\}, \tag{E.4}$$

$$N_\phi(t) := \sup\left\{\sum_{i=1}^{n} [\phi(t_i) - \phi(t_{i-1})]^- \;\middle|\; \{t_0, t_1, ..., t_n\} \in \mathcal{P}[0,t], n \in \mathbb{N}\right\}, \tag{E.5}$$

and a vector space of bounded variation functions as:

$$\mathcal{BV}_0[0,T] := \{\phi : [0,T] \to \mathbb{R} \mid \phi(0) = 0 \text{ and } \|\phi\|_T < \infty\}, \tag{E.6}$$

where the *total variation norm* $\|\cdot\|_T$ on $\mathcal{BV}_0[0,T]$ is given by:

$$\|\phi\|_T := T_\phi(T), \tag{E.7}$$

for all $\phi \in \mathcal{BV}_0[0,T]$. $(\mathcal{BV}_0[0,T], \|\cdot\|_T)$ is a normed space, and

$$\phi(\cdot) = P_\phi(\cdot) - N_\phi(\cdot) \quad \text{and} \quad T_\phi(\cdot) = P_\phi(\cdot) + N_\phi(\cdot) \quad \text{for all } \phi \in \mathcal{BV}_0[0,T]. \tag{E.8}$$

Next, we define a vector subspace of $\mathcal{BV}_0[0,T]$ as

$$\mathcal{BV}_0^r[0,T] := \{\phi \in \mathcal{BV}_0[0,T] \mid \phi \text{ is right continuous on } [0,T]\}, \tag{E.9}$$

with $\|\cdot\|_T$ on $\mathcal{BV}_0^r[0,T]$ again given by (E.7), that is

$$\|\phi\|_T := T_\phi(T), \qquad \text{for all } \phi \in \mathcal{BV}_0^r[0,T]. \tag{E.10}$$

It is well known that there is a norm preserving isomorphism between $(\mathcal{C}[0,T])^*$ and $\mathcal{BV}_0^r[0,T]$, (for example see Theorem 4.14.8 and Corollary 4.14.10 in Friedman [11, p. 182 - 184]) which asserts the following:

**Lemma E.0.1.** *(i) Let $\phi \in \mathcal{BV}_0^r[0,T]$, and define $l_\phi : \mathcal{C}[0,T] \to \mathbb{R}$ by*

$$l_\phi(f) = \int_0^T f(t)\phi(dt) \qquad \textit{all } f \in \mathcal{C}[0,T]. \tag{E.11}$$

*Then $l_\phi$ is a bounded linear functional on $\mathcal{C}[0,T]$ with norm given by*

$$\|l_\phi\|_u^* = \|\phi\|_T, \qquad \textit{all } \phi \in \mathcal{BV}_0^r[0,T]. \tag{E.12}$$

*(ii) For each $L \in (\mathcal{C}[0,T])^*$, there exists a unique $\phi \in \mathcal{BV}_0^r[0,T]$ such that*

$$L = l_\phi, \quad \textit{on } \mathcal{C}[0,T]. \tag{E.13}$$

**Remark E.0.2.** The integral in (E.11) is of course the *Lebesgue-Stieltjes* integral defined by

$$\int_0^T f(t)\phi(\mathrm{d}t) = \int_0^T f(t)P_\phi(\mathrm{d}t) - \int_0^T f(t)N_\phi(\mathrm{d}t), \tag{E.14}$$

(recall (E.4) and (E.5)). Recalling (E.3) and (E.7), we have

$$\begin{aligned} \left|\int_0^T f(t)\phi(\mathrm{d}t)\right| &\leqslant \left|\int_0^T f(t)P_\phi(\mathrm{d}t)\right| + \left|\int_0^T f(t)N_\phi(\mathrm{d}t)\right| \\ &\leqslant \int_0^T |f(t)|\, T_\phi(\mathrm{d}t) \leqslant \|f\|_u \cdot \|\phi\|_T . \end{aligned} \tag{E.15}$$

# Appendix F

# Miscellaneous Results

We shall need Proposition I-1 of Bismut [4, p.387]:

**Proposition F.0.1.** *Given any* $X \equiv (X_0, \dot{X}, \Lambda_X) \in \mathbb{B}$ *and* $Y \equiv (Y_0, \dot{Y}, \Lambda_Y) \in \mathbb{B}$ *(recall Lemma 4.1.1), define*

$$M(X,Y)(t) := X(t)Y(t) - X_0Y_0 - \int_0^t \{X(s)\dot{Y}(s) + \dot{X}(s)Y(s) + \Lambda'_X(s)\Lambda_Y(s)\}ds, \quad \text{(F.1)}$$

*for all* $t \in [0,T]$*. Then* $M(X,Y)$ *is a continuous* $\mathcal{F}_t$*-martingale with* $M(X,Y)(0) = 0$*.*

We shall also need the following result on linear integral equations (see Lemma 5.4 of Labbé and Heunis [18]):

**Lemma F.0.2.** *Suppose Condition 2.1.2 and Condition 2.1.5. For each* $\eta \in \Pi$ *(see (4.1.1)) there is a unique* $\xi \in \Pi$ *such that*

$$\eta(t) = \xi(t) + \theta(t) \int_0^t \xi'(s)dW(s) \qquad t \in [0,T] \text{ a.e.}, \quad \text{(F.2)}$$

*where* $\theta$ *is the* market price of risk *given by* (2.1.8).

Finally, we shall need Lemma 5.4.2 of [15, p.207]:

**Lemma F.0.3.** *For any given* $\mathcal{F}^*$*-measurable (recall Notation 2.1.4 (2)) process* $\eta : \Omega \times [0,T] \to \mathbb{R}^n$*, there exists an* $\mathbb{R}^N$*-valued* $\mathcal{F}^*$*-measurable process* $\nu$ *such that*

$$\begin{cases} \|\nu(t)\| \leqslant 1, \left|\delta^*_{\mathbb{R}^N}\left\{-\nu(t)\middle|A\right\}\right| \leqslant 1, & \text{a.e. on } \Omega \times [0,T], \\ \eta'(t)\nu(t) + \delta^*_{\mathbb{R}^N}\left\{-\nu(t)\middle|A\right\} = 0, & \text{a.e. on } O, \\ \eta'(t)\nu(t) + \delta^*_{\mathbb{R}^N}\left\{-\nu(t)\middle|A\right\} < 0, & \text{a.e. on } (\Omega \times [0,T]) \setminus O, \end{cases} \quad \text{(F.3)}$$

*where* $O := \{(\omega;t) \in \Omega \times [0,T] \mid \eta(\omega;t) \in A\}$*.*

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