Cyclically 5-Connected Graphs

by

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I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

Tutte's Four-Flow Conjecture states that every bridgeless, Petersen-free graph admits a nowhere-zero 4-flow. This hard conjecture has been open for over half a century with no significant progress in the first forty years. In the recent decades, Robertson, Thomas, Sanders and Seymour [8, 9, 10, 11] has proved the cubic version of this conjecture. Their strategy involved the study of the class of cyclically 5-connected cubic graphs. It turns out a minimum counterexample to the general Four-Flow Conjecture is also cyclically 5-connected. Motivated by this fact, we wish to find structural properties of this class in hopes of producing a list of minor-minimal cyclically 5-connected graphs.

Acknowledgements

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Table of Contents

List of Tables

List of Figures

Chapter 1

Introduction

1.1 History and Motivation

In graph theory, a well studied area is colouring.

Definition A graph G is k-colourable if there exists an assignment of k colours to the vertices of the graph such that no adjacent vertices have the same colour.

A well-known difficult problem is the Four Colour Theorem.

Theorem 1.1 (Four Colour Theorem). *Any planar graph G is four colourable.*

This theorem was first proposed by Francis Guthrie in 1852 (see the history survey [13] for more information). However, it was not until 1976 that Kenneth Appel and Wolfgang Haken [1, 2, 3] produced a full correct proof of the theorem. Their method involved checking over 1900 possible counterexamples with a computer. In 1997 Robertson, Sanders, Seymour and Thomas [6] gave a simpler proof of the theorem while still requiring the aid of a computer.

A generalization of this theorem is the Four-Flow Conjecture, first proposed by William Tutte in 1953 [14].

Conjecture 1.2 (Four-Flow Conjecture). *If G is a bridgeless graph and G does not contain the Petersen graph as a minor, then G admits a nowhere-zero four-flow.*

Definition A graph G admits a k-flow if there exists an assignment of orientation and values $1, 2, ..., k-1$ to every edge such that for every vertex v, the sum of the values of edges going into v equals the sum of the values of the edges going out of v.

The Four-Flow Conjecture (1.2) remains open. The cubic version (every vertex has degree 3) however was recently proved by Robertson, Sanders, Seymour and Thomas [7, 10, 11, 8] in a series of papers.

Theorem 1.3. *Let G be a cubic bridgeless graph. If G does not contain the Petersen graph as a minor, then G admits a nowhere-zero four-flow.*

Robertson et al.'s approach involves the study of the class of cyclically 5-connected graphs, whose definition is as follows.

Definition: Suppose A, B are subgraphs of G and $X \subseteq V(G)$ where $V(A) \cup V(B) = V(G), V(A) \cap V(B) = X, E(A) \cup E(B) = E(G), E(A) \cap E(B) = \emptyset$. If both A, B contain a cycle, then we say A, B forms a *cyclic k-separation* and X is a *cyclic k-cut* where $k = |X|$. Then G is *cyclically k-connected* if G contains a cyclic m-cut X where $m \geq k$ and does not contain a cyclic n-cut where $n < k$.

The key observation is that a minimum counterexample to Theorem 1.3 is cyclically 5-connected (See [9]). This is why it is rather important to understand the structure of cubic cyclically 5-connected graphs. Robertson et al.'s first step was to identify the list of minor-minimal cubic cyclically 5-connected graphs. This was also independently proved by McCuaig in [5]. The list of graphs is as follows: Petersen, Triplex, Box, Ruby and Dodecahedron (see 1.1)

The next step in the papers by Robertson et al. is to understand how to construct larger cubic cyclically 5-connected graphs from the list of minor-minimal graphs. This eventually led to reducing Theorem 1.3 to apex and doublecross graphs, two classes of graphs that are almost planar, whose definitions are as follow.

Definition: A graph G is *apex* if there exists a vertex $v \in V(G)$ such that $G \backslash v$ is planar. A graph G is *doublecross* if there exists an embedding of G on the plane such that there are at most two crossings and all crossings, if any, are incident to the infinite face.

Note that the class of apex and doublecross graphs are both minor-closed. They both contain the class of planar graphs and are Petersen-free. Robertson et al. [7] showed that a minimum counterexample to Theorem 1.3 is either apex or doublecross, thus reducing Theorem 1.3 to the following.

Theorem 1.4. *Let G be a cubic bridgeless graph that does not contain the Petersen graph as a minor. If G is apex or doublecross, then G admits a nowhere-zero 4-flow.*

The second part to Robertson et al.'s proof involves tackling Theorem 1.4. This equally difficult task was completed by modifying their proof for the Four-Colour Theorem [10, 11], thus proving Theorem 1.3.

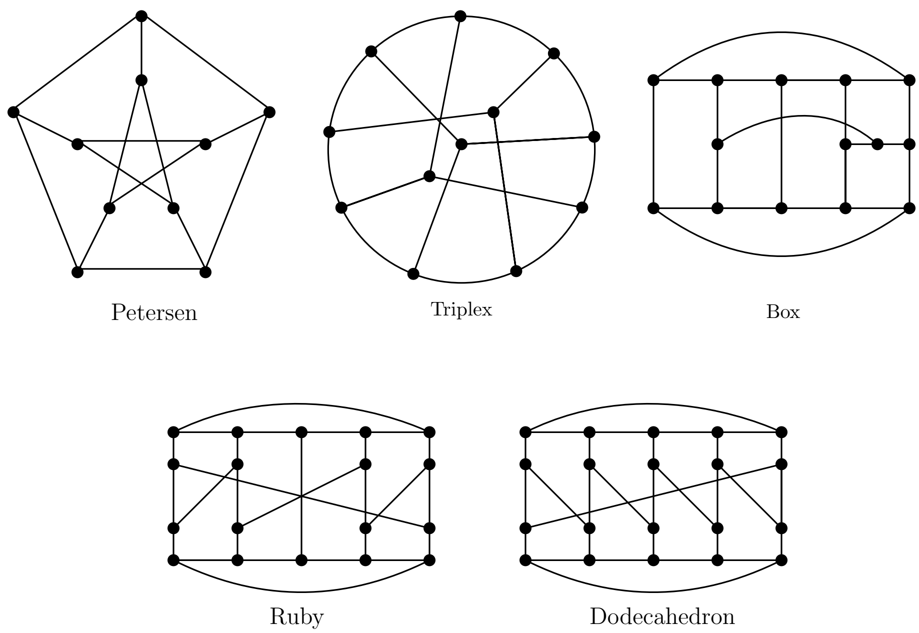


Figure 1.1: Minor Minimal Cubic Cyclically 5-Connected Graphs

We like to point out that the general non-cubic version of Theorem 1.4 remains open. We believe the doublecross part follows from the cubic version but the apex portion might require more in-depth analysis, similar to ones used in solving the Four-Colour Theorem.

For the general Four-Flow Conjecture, it turns out that a minimum counterexample is also cyclically 5-connected and has minimum degree 3. To the best of our knowledge, this is not written in any literature but it might be a known accepted fact by researchers in this field. Nevertheless, we will provide a proof for this in Chapter 2. For the purpose of brevity, we will use CkC to denote a graph that is cyclically k-connected, has girth (length of shortest cycle) k and contains minimum degree 3. Then, it is conceivable that studying the general class of C5C graphs is the correct approach in solving Tutte's Four-Flow Conjecture. The first step, analogous to the cubic case, is to find a list of minor-minimal C5C graphs. However, due to the various degrees of the vertices in G, this becomes a much harder task than the cubic version. It is not enough to just delete an edge and observe the effect locally. Thus, we developed the following strategies based on the cuts of the graphs.

1.2 Outline of Strategy

Definition: If A, B is a k-separation of G with $V(A) \cap V(B) = X$, then we say X is a cut that *separates* G into subgraphs A, B.

Definitions: Let X be a cyclic 5-cut with respect to a cyclic 5-separation A_1, A_2. We say A_i is an *acyclic side* of X if either $A_i \backslash X$ does not contain a cycle or $A_i \backslash X$ is a 5-cycle. We say A_i is a *cyclic side* of X if $A_i \backslash X$ contains a cycle and $A_i \backslash X$ is not the 5-cycle. Then we say the cyclic 5-cut X with respect to A_1, A_2 is

- a *doubly-acyclic* cut if both A_1, A_2 are acyclic sides,
- a *doubly-cyclic* cut if both A_1, A_2 are cyclic sides,
- a *mixed* cut if one of A_1, A_2 is an acyclic side and the other one is a cyclic side.

Now, we will divide the class of C5C graphs into three subclasses, based on the types of cyclic 5-cuts they contain.

Definitions: Let G be a C5C graph. If G contains a doubly-cyclic cut, then we say that G is a *doubly-cyclic* graph. If G contains a doubly-acyclic cut, then we say that G is a *doubly-acyclic* graph. If for all cyclic 5-separations A, B and their induced cyclic 5-cut X, X is a mixed cut with respect to A, B, then G is a *mixed* C5C graph.

Our belief is that doubly-cyclic and doubly-acyclic graphs either contain certain restricted structures or admit a reduction to smaller C5C graphs. Mixed C5C graphs, on the other hand, do not seem to have a natural way of obtaining any structure nor admitting a reduction. Thus our main focus for the thesis is to find partial structural properties about mixed C5C graphs by analyzing the acyclic side of mixed cuts. In the following subsections, we will briefly explain the potential strategies for analyzing doubly-cyclic and doubly-acyclic graphs and why we believe they are essentially "finite" problems. The analysis of doubly-cyclic and doubly-acyclic graphs will be left to another work.

1.2.1 Doubly-Cyclic Graphs

Suppose G is a doubly-cyclic C5C graph with a doubly-cyclic cut X and two cyclic sides A, B. Note that both sides contain a cycle disjoint from the cut X. Since the graph is cyclically 5-connected, there exists 5 edge disjoint paths from each of the cycles to X. Let C be the cycle in B and P_i for $i = 1, 2, 3, 4, 5$ be the five paths connecting C to X. Then,

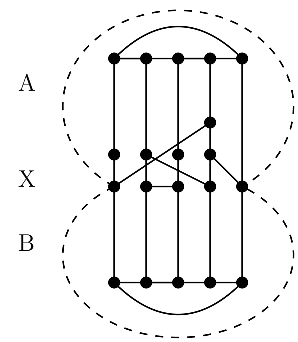


Figure 1.2: Example of Doubly-Cyclic Graph and Jellyfishing

consider the graph G' obtained by deleting all edges in B except for those in C and P_i for $i = 1, 2, 3, 4, 5$ and then suppressing all degree-2 vertices. This is equivalent to replacing B with a 5-cycle and 5 edges. The hope is that G' remains a C5C graph.

If not, G' might have a 4-cycle. The 4-cycle must be created by two vertices in X and two vertices in C. This implies that there exists an edge x_1x_2 where $x_1, x_2 \in X$. If $d(x_1), d(x_2) \geq 4$ in G, then we can attempt to delete the edge x_1x_2 to obtain a smaller C5C graph. Suppose one of x_1, x_2 has degree 3. Without loss of generality, assume $d(x_1) = 3$. Then x_1 has one neighbour $x_A \in V(A)$ and another one $x_B \in V(H)$ other than x_2. Then, note that $X' = (X \backslash x_1) \cup \{x_A\}$ is also a cyclic 5-cut in G.

Now, we can attempt to repeat the same process with respect to X'. We will once again, either find a smaller C5C graph or find a degree-3 vertex in X' and push the cut further into A (See Figure 1.2). This will produce a chain of degree-3 vertices in A. This structure will either stop after hitting the cycle in A or, by Ramsey-type arguments, it can be shown that the chain exhibits certain repeated patterns. If the chain stops, then it provides us an explicit finite structure of what A is. If the chain repeats, then we can remove a portion of the repeated structure, shortening the chain, producing a C5C minor of G. We call this method of studying the chain of degree-3 vertices the "jellyfish" method. We believe that there should not be many, if any at all, minor-minimal C5C graphs that are doubly-cyclic. We wish to point out that none of the graphs in the list of minor-minimal cubic cyclically 5-connected graphs are doubly-cyclic. In particular, Petersen, Triplex, Box and Ruby are doubly-acyclic and Dodecahedron is mixed.

1.2.2 Doubly-Acyclic Graphs

Now suppose G is a doubly-acyclic C5C graph with a doubly-acyclic cut X and two acyclic sides A, B. Consider the subgraph $A \backslash X$. Note that $A \backslash X$ is either a 5-cycle or a forest.

If $A \backslash X$ is a 5-cycle, we will show that A has a very specific structure. Since G has minimum degree 3, every vertex in the 5-cycle has at least one other neighbour in X. Since G has girth 5, no two vertices in C have a common neighbour in X. This implies that the subgraph A is a 5-cycle with five edges that matches the vertices in the cycle to X. We call this subgraph C_J (See Figure 4.1).

Now, suppose $A \backslash X$ is a forest. Since every vertex has degree at least 3 in G, the leaves in $A \backslash X$ has at least two neighbours in X. Then, by the Pigeonhole Principle, $A \backslash X$ contains at most ten leaves; otherwise, two of the leaves have two common neighbours in X, forming a 4-cycle, contradicting the girth of G. This implies that the total number of leaves in $A \backslash x$ and $B \backslash X$ cannot exceed 10, and $A \backslash x, B \backslash X$ are subdivisions of a finite list of forests. The question is could there be a very long subdivision.

Suppose $A \backslash X$ contains a very long path P of degree-2 vertices. Since G has minimum degree 3, each vertex in P has at least one neighbour in X. Since the size of X is limited, again, by using Ramsey-type arguments, it can be shown that the path contains subpaths $u_1u_2u_3$ and $v_1v_2v_3$ where the neighbours of $N(u_i) \cap X = N(v_i) \cap X$. Then, we can shorten the path by removing everything between u_2 and v_2, creating a smaller C5C graph. If such long chains do not exist, it is conceivable that one can list out all the possibilities of such acyclic side. Then, by checking all combinations of two such acyclic sides, one can verify and list out the graphs that are doubly-acyclic and minor-minimal C5C. Similar techniques will be used in Chapter 3. We will comment on these similarities at the end of Chapter 4.

1.2.3 Mixeded Graphs

Now, if G is a mixed C5C graph, the strategy is not as obvious. Suppose we replace the cyclic side with a 5-cycle and five edges to produce a minor G', similar to the jellyfish method, it is unclear whether G' is still cyclically 5-connected. The jellyfish method depends on the existence of the five paths from each cycle to the cut on both sides so it might not give us much structure about G. Thus, we will first attempt to study the acyclic side. This method should be very similar to the one for doubly-acyclic graphs.

Our initial goal is to find a list of minor-minimal acyclic sides in the sense that If an acyclic side A is not in the list, then it can be replaced by A', a minor of A that is in the list and the resulting graph remains C5C. Since mixed C5C graphs seem to be the trickiest to handle, it will be the main focus of this thesis.

Given a mixed cut X and an acyclic side A and cyclic side B. Suppose there exists a vertex $x \in X$ such that $d_B(x) = 1$ where $N_B(x) = \{x'\}$. Then, consider the cut $X' = X \cup \{x'\} \backslash \{x\}$, $A' = A \cup xx', B' = B \backslash xx'$. Note that A' still contains a cycle. Since B contains a cycle disjoint from x, the same cycle still exists in B'. Then, A', B' is a cyclic 5-separation and X' is a cyclic 5-cut. Since G is a mixed graph, X' is a mixed cut. Now we have two cases, whether A' or B' is the acyclic side.

If B' becomes the acyclic side and A' becomes the cyclic side, then G must be very similar to a doubly-acyclic graph. G is essentially two acyclic sides joined together with an extra edge. Thus, the analysis of such graphs should be very similar to the analysis of doubly-acyclic graphs. We will comment on this similarity at the end of Chapter 3 but will leave the analysis to another work.

If A' remains as the acyclic side, we say X can be *pushed* along the edge xx'. Note that this is a very natural notion when analyzing cuts. This will allow us, in some sense, to study the largest acyclic side of a mixed graph G and try to reduce it by replacing the acyclic side with something smaller. We call a mixed graph *push-consistent* if for all mixed cuts X, the acyclic side stays acyclic after pushing along any pushable edges. Push-consistent graphs are the main focus of this thesis.

1.3 Main Results

The following are the main results of this thesis. First, as promised, we provide a proof that the minimum counterexample to te general 4-flow conjecture is C5C.

Theorem 1.5 (Theorem 2.1)**.** *If G is a minimum counterexample to the Four-Flow Conjecture, then G is cyclically 5-connected with minimum degree 3 and girth 5.*

Next, we consider when it is possible to reduce the acyclic side of a mixed cut.

Theorem 1.6. *Let G be a mixed C5C graph. Let X be a mixed cut in G with acyclic side A and cyclic side B. If $d_B(x) \geq 2$ for all $x \in X$ and X is an independent set, then A is:*

- *either isomorphic to one of the 43 graphs in $\mathcal{L}$ (See Figures 3.5) or,*
- *A can be replaced by A', a minor of A where $A' \in \mathcal{L}$ and the resulting graph is C5C.*

Then, we focus specifically on push-consistent graphs. Since we can freely push a mixed cut X towards the cyclic side, we can essentially push all mixed cuts until it becomes "non-pushable". This creates a large acyclic side to work with, allowing us to reduce the list of acyclic sides to 12 graphs. In this theorem, we also allow X to not be an independent set.

Theorem 1.7. *Let G be a minor-minimal C5C graph. Suppose G is push-consistent and X is a non-pushable cut with acyclic side A. Then A is isomorphic to a graph in $\mathcal{L}'$ (see Figure 4.1).*

Lastly, we use local reduction tools such as vertex/edge deletion to reduce the list of acyclic sides even further to merely 4 possible subgraphs. At the same time, we also gain certain local structural properties about these graphs.

Theorem 1.8 (Theorem 5.1)**.** *Let G be a push-consistent mixed C5C graph. Let X be a non-pushable mixed cut with acyclic side A. If G is minor-minimal C5C, then all of the following holds.*

1. *A is isomorphic to one of $C_J, A_0, D_{3,4}, D_{4,4}$ (See Figure 4.1) and,*
2. *every edge e in G is incident to a degree 3 vertex v such that there exists a 5-cycle containing v but not e and,*
3. *every vertex v is adjacent to a degree-3 vertex u such that there exists a 5-cycle containing u but not v and,*
4. *every degree-3 component has size at least 4.*

Our hope is that, given more time, this theory can be further developed until we can prove dodecahedron is the only minor-minimal C5C graph that is also push-consistent.

1.4 Outline of Thesis

In Chapter 2, we will provide some basic definitions and necessary background on cyclically 5-connected graphs. We will also provide a proof of Theorem 2.1: a minimum counterexample to the Four-Flow Conjecture is a C5C graph.

Chapter 3 is dedicated to studying the acyclic sides of a mixed C5C graph. The goal is to produce a list of minor-minimal acyclic sides. In other words, given an acyclic side

A of a C5C graph G, when is it possible to find A', a minor of A such that the graph G' obtained from G by replacing A with A' remains C5C. The main result is Theorem 3.5, which is equivalent to Theorem 1.6.

In Chapter 4, we formally define the idea of *pushing* a cut and the concept of *push-consistent* mixed C5C graphs. Then, we will show that the acyclic side of push-consistent graphs is only one of 19 graphs, a subset of the list obtained in Chapter 3. The 12 graphs are those found in $\mathcal{L}'$ (See Figure 4.1). The main result is Theorem 4.2, which implies Theorem 1.7. We will also include a brief discussion of how to adapt our current strategy when analyzing doubly-acyclic and non-push-consistent graphs at the end of chapter 4.

The goal of Chapter 5 is to further refine the list of minor-minimal acyclic sides of a push-consistent graph. This chapter will use local reduction techniques such as deleting edges, vertices or small components and study the remaining graph. By the end of chapter 5, we can show that if G is a minor-minimal of C5C graphs that is also push-consistent, then it must contain certain structural properties and its acyclic side is isomorphic to only one of four possibilities. We are also able to gain certain structural information and thus proving Theorem 5.1.

The eventual goal is to extend the results in Chapter 5 and obtain more structural properties by performing larger local reductions. Our hope is that one can eventually show that if G does not admit any local reductions, then the size of a degree-3 component of G is larger than 15, at which point we can prove that G is none other than the Dodecahedron.

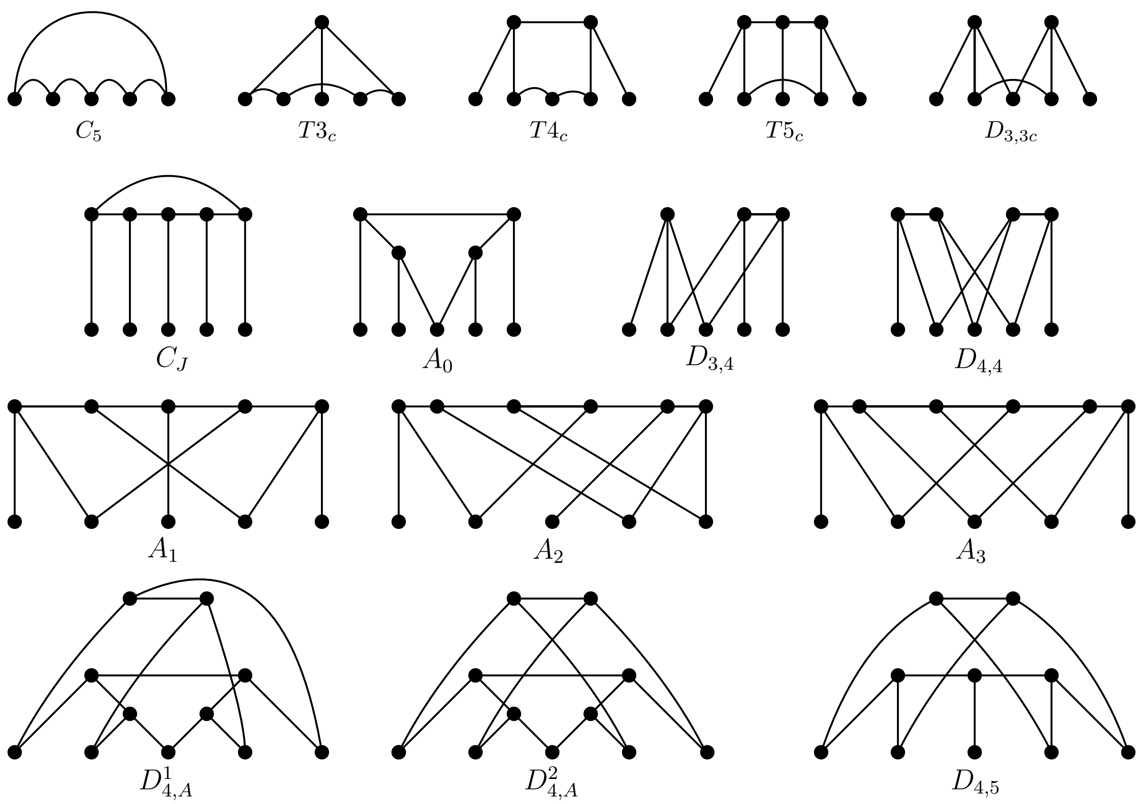


Figure 1.3: The List $\mathcal{L}'$

Chapter 2

Preliminary Background

2.1 Definitions

First, we will repeat some of the terminology mentioned in Chapter 1.

Let X be a cut in a graph G. Let A, B be two subgraphs of G such taht $E(A), E(B)$ is a bipartition of $E(G)$ and $V(A) \cap V(B) = X$. Then, we say X *separates* A, B.

For our purposes, a graph, G, is *cyclically k-connected (CkC)* if it has minimum degree 3, girth k, contains a cyclic n-cut where $n \geq k$ and for all cyclic m-cuts, $m \geq k$.

Let $S3, S4, T4, S5, T5, T5'$ be the graphs illustrated in Figure 2.1.

Let X be a cut that separates A, B. We say a side A is:

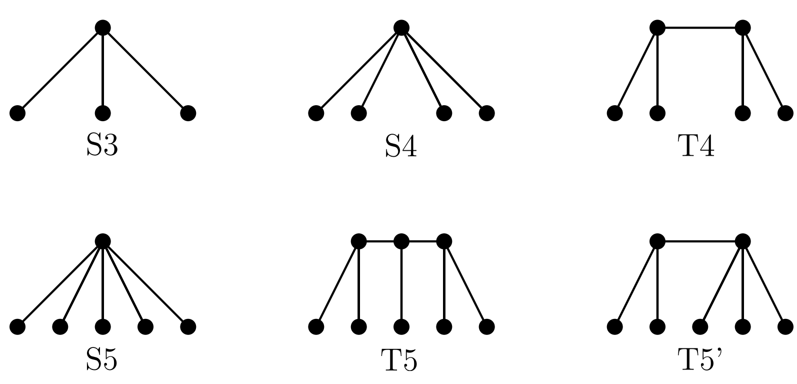


Figure 2.1: Trivial Sides and Trivial Cuts

- *trivial* if A is isomorphic to one of $S3, S4, T4, S5, T5, T'$,
- an *acyclic side* A contains a cycle and $A \backslash X$ is either isomorphic to a 5-cycle or contains no cycles,
- a *cyclic side* if both $A, A \backslash X$ contain a cycle and $A \backslash X$ is not isomorphic to a 5-cycle.

We say a cut X is a *cyclic k-cut* if X separates G into two subgraphs, A, B and both A, B contains a cycle.

The cut X with respect to sides A, B is:

- *trivial* if one of the sides is trivial,
- *doubly-acyclic* if A, B are both acyclic sides,
- *doubly-cyclic* if A, B are both cyclic sides,
- *mixed* if one of A, B is an acyclic side and the other is a cyclic side.

Let G be a C5C graph. The graph G is:

- *doubly-acyclic* if G contains a doubly-acyclic cut,
- *doubly-cyclic* if G contains a doubly-cyclic cut,
- *mixed* if all cyclic 5-cuts in G are mixed cuts.

2.2 Minimum Counterexample is C5C

In this section, we will allude to the importance of cyclically 5-connected graphs by proving.

Theorem 2.1. *If G is a minimum counterexample to the 4-flow conjecture, then G is cyclically 5-connected.*

The following are two well known theorems about flows.

Theorem 2.2. *A graph G admits a nowhere-zero k-flow if and only if G admits a nowhere-zero $\mathbb{Z}_k$-flow.*

Theorem 2.3. *Let $\mathbb{G}_1, \mathbb{G}_2$ be two finite abelian groups of the same size. If a graph G admits a nowhere-zero $\mathbb{G}_1$-flow, then it also admits a nowhere-zero $\mathbb{G}_1$-flow.*

These are two theorems proven by Tutte in 1953. We will omit their proofs. One can refer to the textbook [4] for more information. The two theorems imply that to prove a graph G admits a nowhere-zero 4-flow, it suffices to show that G admits a nowhere-zero $\mathbb{Z}_2 \times \mathbb{Z}_2$-flow.

Note that in a $\mathbb{Z}_2 \times \mathbb{Z}_2$-flow, the orientation of the edges does not matter. Consider the following functions. Let $\phi : E(G) \to \{1, 2, 3\}$. Given ϕ, for $i = 1, 2, 3$, let $\phi_i : V(G) \to \mathbb{Z}_2$ such that $\phi_i(v) \equiv |\{e : e \in \delta(v), \phi(e) = i\}| (\bmod 2)$. We say ϕ *flows properly* at a vertex v if $\phi_1(v) = \phi_2(v) = \phi_3(v)$. We say G has a *proper flow* ϕ if ϕ flows properly at all vertices in G. Then, note that G admits a nowhere-zero $\mathbb{Z}_2 \times \mathbb{Z}_2$-flow if and only if G has a proper flow ϕ.

We also require the following well-known theorems about connectivity and planarity [4, 12]. Their proofs will be omitted in this paper.

Theorem 2.4 (Menger's Theorem)**.** *Let s, t be two non-adjacent vertices in G. The size of the smallest cut that disconnects s from t equals the largest number of vertex disjoint paths from s to t.*

Theorem 2.5 (Two-Path Theorem)**.** *Let s_1, t_1, s_2, t_2 be vertices in a graph G. There does not exist two vertex disjoint paths P_1, P_2 where P_i denote a path from s_i to t_i for $i = 1, 2$ if and only if there exist pairwise disjoint vertex sets $A_1, ..., A_k \subset V(G) - \{s_1, t_1, s_2, t_2\}$ such that for all $1 \leq m, n \leq k$ where $m \neq n$:*

- $|N(A_m)| \leq 3$,
- $N(A_m) \cap A_n = \emptyset$,
- *if G' is a graph obtained from G by deleting all vertices in $A_1 \cup ... \cup A_k$ and for each $1 \leq p \leq k$, add edges so $N(A_p)$ becomes a clique and add edges s_1t_1, s_2t_2, then there exists an embedding of G' on the plane such that s_1t_1, s_2t_2 forms the only crossing.*

The Two-Path Theorem was first proved by Seymour in 1978. We will use the following corollary for the purpose of our proof.

Corollary 2.6. Let G be a cyclically 4-connected graph with at least four vertices x_1, x_2, x_3, x_4. If there does not exist two vertex disjoint paths P_{13}, P_{24} where P_{ij} is a path from x_i to x_j, then the graph G' obtained by adding the cycle $C = x_1x_2x_3x_4$ has a planar embedding with C on the outer face.

Proof: It follows from the Two-Path Theorem that there exist pairwise disjoint vertex sets $A_1, ..., A_k$ such that for all $1 \leq i, j \leq k$ where $i \neq j$:

- $|N(A_i)| \leq 3$,
- $N(A_i) \cap A_j = \emptyset$,
- if H is a graph obtained from G by deleting all vertices in $A_1 \cup ... \cup A_k$ and for each $1 \leq i \leq k$, add edges so $N(A_i)$ becomes a clique and add edges x_1x_3, x_2x_4, then there exists an embedding of G' on the plane such that x_1x_3, x_2x_4 forms the only crossing.

Without loss of generality, we may assume the crossing is in the outer face. Consider the graph $H' = H \cup C \backslash \{x_1x_3, x_2x_4\}$. It is evident that H' is planar and has an embedding where C is in the outer face. Note that $N(A_i)$ separates A_i from the rest of the graph. Since $|N(A_i)| \leq 3$ and G is cyclically 4-connected, A_i is a single vertex and $|N(A_i)| = 3$ for all $1 \leq i \leq k$. Consider now adding back the deleted vertices of A_i and their incident edges for $1 \leq i \leq k$. It follows that this resulting graph remains planar. Then, by deleting the extra edges amongst $N(A_i)$ for $1l3i \leq k$, we obtain G'. It follows that G' is planar with C on the outer face, as required. $\square$

We will also prove the following lemma.

Lemma 2.7. If G is planar and bridgeless, then G admits a nowhere-zero 4-flow.

Proof:

Given a planar embedding of G, let G' be the planar dual of G. Since G is bridgeless, G' is loopless and G' is 4-colourable. Let h be a homomorphism that maps G' to K_4. Suppose the edges and vertices of K_4 are labeled. Let $E_1, E_2, E_3 \subset E(K_4)$ be three distinct perfect matchings of K_4. Let ϕ be a flow such that $\phi(e) = i$ if $h(e) \in E_i$. We will show that ϕ is a proper flow on G.

Given a vertex $v \in V(G)$, note that the edges incident to v in G forms a boundary walk in G' which corresponds to a closed walk $W(v)$ in K_4. Given a closed walk W, let $w_i = |E(W) \cap E_i| (\text{mod } 2)$ for $i = 1, 2, 3$. Note that $w_i = \phi_i(v)$ for $i = 1, 2, 3$, thus it suffices to show that $w_1 = w_2 = w_3$ with respect to $Wv)$ for all $v \in V(G)$.

We will prove the stronger statement that for any closed walk W on K_4, $w_1 = w_2 = w_3$. First, if $E(W) = \emptyset$, it is trivially true and there does not exist any closed walks where $|E(W)| = 1$. If $|E(W)| = 2$, it corresponds to doubling back on the same edge and $|E(w)| = 3$ implies W is a triangle. In both cases, $w_1 = w_2 = w_3$. If W is a 4-cycle, note that $w_1 = w_2 = w_3$ as well. If $|E(W)| \geq 4$, then W can be partitioned into $W^1, W^2, ..., W^k$ where W^j is a closed walk of length 2 or a triangle or a 4-cycle for all $1 \leq j \leq k$. Since $w_1^j = w_2^j = w_3^j$ for all $1 \leq j \leq k$, it follows that $w_1 = w_2 = w_3$, as required. □

To facilitate the proof of Theorem 2.1, we will first prove some properties about cyclically 4-connected graphs.

Lemma 2.8. Let G be a cyclically 4-connected graph with minimum degree 3 and girth 5. Let $X = \{x_1, x_2, x_3, x_4\} \subseteq V(G)$ be a cut that separates G into two subgraphs A, B where both A, B contain a cycle. Let C be a cycle in A. Then, there exist four vertex disjoint paths from X to four vertices of C in A.

Proof: Let G' be a graph obtained by adding vertices c, x and edges that connects c to every vertex in C and x to every vertex of X. Let S be a smallest cut that disconnects c from x. We will prove that $|S| \geq 4$.

Suppose for the sake of contradiction, $|S| \leq 3$. Since G is connected, $|S| \neq \emptyset$. Let M', N' be two subgraphs separated by S in G'. Without loss of generality, we may assume $c \in M', x \in N'$. Let $M = M' \backslash c, N = N' \backslash x$. Note that S separates M, N in G. Since G has girth 5, $|V(C)| \geq 5, d(c) \geq 5$. This implies that $|V(M)| = |V(M')| - 1 \geq |\{c\} \cup V(C)| - 1 \geq 6 - 1 = 5$. Note also that x has four neighbours and each vertex in X is adjacent to vertices in A and B. There exists at least one vertex of X not in S which implies at least one of the vertices in $A \backslash X$ is in N. Then, $|V(N)| = |V(N')| - 1 \geq |\{x\} \cup X| + 1 - 1 \geq 5$. This implies that $V(M), V(N) \neq S$ and S is an actual cut in G. Since G is cyclically 4-connected, at least one of M, N is a forest. Assume M is a forest. Note that $|V(M)| \geq 5$ so $M \backslash S \neq \emptyset$. Since every vertex in $M \backslash S$ has degree at least 3, M has at least three leaves. Since S are the only vertices in M that can have degree less than 3, S are the leaves in M and $|S| = 3$. This implies that S is a trivial cut and M is isomorphic to a $S3$, contradicting $|V(M)| \geq 5$. Note that the same arguments can be made about N. This implies that neither M, N are forests, a contradiction.

Since $|S| \geq 4$, by Menger's Theorem (2.4), there exist paths P_i' from x to c for $i = 1, 2, 3, 4$ such that the only common vertices between any two paths are c and x. Since x has exactly four neighours, without loss of generality, we may assume $x_i \in V(P_i')$ for $i = 1, 2, 3, 4$. Since each path contains exactly one vertex of X, it follows that no paths contain any vertices in $B\backslash X$. Then, for $i = 1, 2, 3, 4$, let P_i be a path that starts at x_i, follows P'_i towards c and stops at the first instance of a vertex in C. Note that P_i for $i = 1, 2, 3, 4$ are the four desired vertex-disjoint paths. $\square$

Given $X = \{x_1, x_2, x_3, x_4\}$, let $T4_{ij,kl}$ be a graph with vertices $X \cup \{u, v\}$ and edges $\{uv, ux_i, ux_j, vx_k, vx_l\}$. Note that these are graphs isomorphic to $T4$.

Lemma 2.9. Let G be a cyclically 4-connected graph with minimum degree 3 and girth 5. Let $X = \{x_1, x_2, x_3, x_4\} \subseteq V(G)$ be a cut that separates G into two subgraphs A, B where both A, B contain a cycle. Then, each subgraph A, B contains at least two of the following $T4_{12,34}, T4_{13,24}, T4_{14,23}$ as minors.

Proof: Note that it suffices to prove the lemma is true for A only.

Suppose there exists a cycle C in A such that $V(C) \cap X = \emptyset$. By Lemma 2.8, there exist four vertex disjoint paths P_i for $i = 1, 2, 3, 4$ from C to X. Without loss of generality, we may assume for $i = 1, 2, 3, 4$, P_i is a path from x_i to c_i, $c_i \in V(C)$ and vertices c_1, c_2, c_3, c_4 appears in this cyclic order in C. Then, by deleting the path from c_1 to c_4 in C, it is clear that A contains a $T4_{12,34}$ minor. Similarly, by deleting the path from c_1 to c_2, it is clear that A contains a $T4_{14,23}$ minor, as required.

Suppose there exist a cycle C where $|V(C) \cap X| = 1$. We can once again apply Lemma 2.8. Note that this time, one of the paths will be a single vertex $V(C) \cap X$. Without loss of generality, assume that $x_1 = V(C) \cap X$ and for $i = 1, 2, 3, 4$, P_i are paths from x_i to c_i where $c_i \in V(C)$ and c_1, c_2, c_3, c_4 appear in this cyclic order in C. Then, once again, by deleting the path from x_1 to x_4 or the path from x_1 to x_2, we get two different $T4$ minors of A, as required.

Lastly, suppose for the sake of contradiction that for all cycles C in A, $|V(C) \cap X| \geq 2$. This implies that $A\backslash X$ is a forest. Let F be a component of $A\backslash X$. Note that no two vertices of F have a common neighbour in X, otherwise, it forms a cycle C such that $|V(C) \cap X| = 1$. Note that since all vertices of G has degree 3, each leaf of F is adjacent to at least two vertices of X. Then, by pigeonhole principle, F has at most two leaves. If F has two leafs, it follows that F has only two vertices, each adjacent to two distinct vertices of X. Now, we claim that $A\backslash X$ contains at least two components. Suppose for the sake of contradiction it has at most one component. Since A has a cycle of length at least 5, $|V(A)| \geq 5$, $A\backslash X$ contains at least one component. Let F be that component. It follows from previous arguments that F is either a single vertex of degree at least 3 or a path of two vertices. However, in both cases, it follows that either A does not contain a cycle or A contains a cycle of length less than 5, a contradiction. Now we can assume $A\backslash X$ has at least two components F_1, F_2. We will show that no component is a single vertex. Suppose for the sake of contradiction that $F_1 = v$. Note that v has at least three neighbours in X. However, this implies that any leaf in any other component has at least two common neighbours with v, forming a 4-cycle, a contradiction. Thus we may assume that F_1, F_2 both have two leafs. Since they do not form any 4-cycles, each component produces a distinct $T4$ minor of A, as required. $\square$

Corollary 2.10. Each subgraph A, B contains at least two distinct perfect matchings of X as minors. Also, $|V(A)\backslash X|, |V(B)\backslash X| \geq 2$.

Now, we will proceed to prove our main theorem of this chapter.

Lemma 2.11. If G is a minimum counterexample to the Four-Flow Conjecture, then G is simple.

Proof: Let G be a bridgeless graph that does not contain the Petersen graph as a minor and all bridgeless proper minors of G has a proper flow ϕ but not G itself. Note that any minor of G automatically does not contain the Petersen graph as a minor.

Suppose for the sake of contradiction that e, f are parallel edges with ends u, v. Consider $G' = G\backslash f$.

First, assume e is not a bridge in G'. Then, it follows that G' is bridgeless. Since G' also does not contain the Petersen graph as a minor, let ϕ' be a proper flow for G'. Without loss of generality, we may assume $\phi'(e) = 1$. Let $\phi(e) = 2, \phi(f) = 3$ and $\phi(g) = \phi'(g)$ for all other edges $g \in E(G)$. Note that $\phi_1(u) \equiv \phi'_1(u) - 1 \equiv \phi'_2(u) + 1 \equiv \phi'_3(u) + 1 \equiv \phi_2(u) \equiv \phi_3(u) (\text{mod } 2)$. Then, by symmetry, $\phi_1(v) = \phi_2(v) = \phi_3(v)$. Since $\phi(g) = \phi'(g)$ for all edges $g \neq e \neq f$, it follows that ϕ is a proper flow for G, a contradiction.

If e is a bridge in G', consider $G'' = G'\backslash e$. Note that all components of G'' are bridgeless and does not contain the Petersen graph as a minor. Let ϕ'' be a proper flow on G''. Let $\phi(e) = \phi(f) = 1$ and $\phi(g) = \phi''(g)$ for all other edges $g \in E(G)$. Since, $\phi_1(u) = \phi''_1(u) + 2$, $\phi_1(u) = \phi_2(u) = \phi_3(u)$. Then, it follows that ϕ is a proper flow on G, a contradiction. $\square$

Lemma 2.12. If G is a minimum counterexample to the Four-Flow Conjecture, then G is 2-connected.

Proof: For the sake of contradiction, assume G is not 2-connected. It is obvious that G is connected. Since G is bridgeless, G has a cut vertex v. Let A, B be the two subgraphs separated by v. Note that since G is bridgeless, $d_A(v), d_B(v) \geq 2$. Then it follows that A, B are both bridgeless minors of G, do not have the Petersen graph as a minor and they both have a proper flow. Then, by assigning the same flow values to the edges of G, we also obtain a proper flow for G, a contradiction. $\square$

Lemma 2.13. If G is a minimum counterexample to the Four-Flow Conjecture, then G is 3-connected.

Proof: First, note that a degree one vertex does not exist since G is 2-connected. If G has a degree two vertex v incident to edges e, f. Then consider $G' = G/f$. Let ϕ' be a proper flow for G'. Then, extend ϕ' to ϕ by assigning $\phi(e) = \phi'(f)$ and $\phi(g) = \phi'(g)$ for all other edges $g \neq e$. Note that $\phi_1(w) = \phi_2(w) = \phi_3(w)$ for all $w \in V(G)$, thus G has a proper flow, a contradiction.

Now, for the sake of contradiction, assume G has a cut set $\{u, v\}$. Let A, B be the subgraphs separated by u, v. Note that there exists a path from u to v in both A and B, otherwise u is a cut vertex in G. Let A' be the graph A with an extra edge $e = uv$. Similarly, let B' be the graph B with an extra edge $f = uv$. Note that A', B' are minors of G. Note that e is not a bridge in A' since there already exist a path from u to v in A. No other edges are bridges, otherwise, they are also a bridge in G. This implies that A' is bridgeless. By symmetry, B is also bridgeless. By our assumption, A', B' has a proper flow ϕ^A, ϕ^B respectively. Without loss of generality, we may assume that $\phi^A(e) = \phi^B(f) = 1$. Let ϕ be a flow such that $\phi(g) = \phi^A(g)$ if $g \in A$ and $\phi(g) = \phi^B(g)$ if $g \in B$. Note that for all $w \neq u, v$, it is clear that $\phi_1(w) = \phi_2(w) = \phi_3(w)$. If $w = u, v$, then $\phi_1(w) = \phi_1^A(w) - 1 + \phi_1^B(w) - 1($ mod 2$) = \phi_1^A(w) + \phi_1^B(w) = \phi_2^A(w) + \phi_2^B(w) = \phi_3^A(w) + \phi_3^B(w) = \phi_2(w) = \phi_3(w)$. This proves that ϕ is a proper flow for G, a contradiction. $\square$

Lemma 2.14. If G is a minimum counterexample to the Four-Flow Conjecture, then G is triangle-free.

Proof: Suppose for the sake of contradiction, G has a triangle with vertices u, v, w. Consider the graph $G' = G \backslash vw$. Since G is 3-connected, G' is bridgeless. Let ϕ' be a proper flow for G'.

If $\phi'(uv) = \phi'(uw)$, without loss of generality, we may assume $\phi'(uv) = 1 = \phi'(uw)$. Then, let $\phi(uv) = \phi(uw) = 2, \phi(vw) = 3$ and $\phi(e) = \phi'(e)$ for all other edges e. Note that $\phi_1(v) \equiv \phi_1'(v) - 1 \equiv \phi_2'(v) + 1 \equiv \phi_3'(v) + 1 \equiv \phi_2(v) \equiv \phi_3(v)(\text{mod } 2)$. By symmetry, $\phi_1(w) = \phi_2(w) = \phi_3(w)$. Also, $\phi_1(u) \equiv \phi_1'(u) - 2 \equiv \phi_2'(u) + 2 \equiv \phi_2(u) \equiv \phi_3'(u) \equiv \phi_3(u)($

mod 2). Note that all other vertices x, $\phi_i(x) = \phi'_i(x)$ for $i = 1, 2, 3$. Then, ϕ is a proper flow for G, a contradiction.

If $\phi'(uv) \neq \phi'(uw)$, without loss of generality, we may assume that $\phi'(uv) = 1, \phi'(uw) = 2$. Then, let $\phi(uv) = 2, \phi(uw) = 1, \phi(vw) = 3$ and $\phi(e) = \phi'(e)$ for all other edges e. Note that $\phi_i(u) = \phi'_i(u)$ for $i = 1, 2, 3$. Also, $\phi_1(v) \equiv \phi'_1(v) - 1 \equiv \phi'_2(v) + 1 \equiv \phi'_3(v) + 1 \equiv \phi_2(v) \equiv \phi_3(v) (\text{mod } 2)$. By symmetry, $\phi_1(w) = \phi_2(w) = \phi_3(w)$. Since $\phi_i(x) = \phi'_i(x)$ for $i = 1, 2, 3$ and all other vertices x, ϕ is a proper flow for G, a contradiction. $\square$

Lemma 2.15. If G is a minimum counterexample to the Four-Flow Conjecture, then G is cyclically 4-connected.

Proof: For the sake of contradiction, suppose there exists a cut $X = \{x_1, x_2, x_3\}$ that separates the graph into two subgraphs A, B each containing a cycle. Since G does not contain a triangle, all vertices has degree at least 3 and A, B both contain a cycle, $|V(A)\backslash X|, |V(B)\backslash X| \geq 2$. Let $a \in V(A)\backslash X, b \in V(B)\backslash X$. Let G^A be the graph obtained from A by adding a vertex b and edges bx_1, bx_2, bx_3. Similarly, let G^B be the graph obtained from B by adding a vertex a and edges ax_1, ax_2, ax_3. Note that since G is 3-connected, by Menger's Theorem (2.4), there exist three internally vertex disjoint paths from any vertex in $V(A)\backslash X$, to any vertex in $V(B)\backslash X$. Note that these paths must go through X. It follows that G^A, G^B are proper minors of G. It is also clear that G^A, G^B are bridgeless and does not contain the Petersen graph as a minor. Let ϕ^A, ϕ^B be two proper flows on G^A, G^B respectively. Note that $\phi^A(bx_1) \neq \phi^A(bx_2) \neq \phi^A(bx_3)$ and $\phi^B(ax_1) \neq \phi^B(ax_2) \neq \phi^B(ax_3)$. Without loss of generality, assume that $\phi^A(bx_i) = \phi^B(ax_i) = i$ for $i = 1, 2, 3$. Then, let $\phi(e)$ retain the same value as $\phi^A(e)$ and $\phi^B(e)$ for all edges $e \in E(G)$. Note that $\phi_1(x_1) \equiv \phi^A_1(x_1) + \phi^B_1(x_1) - 2 \equiv \phi^A_2(x_1) + \phi^B_2(x_1) \equiv \phi^A_3(x_1) + \phi^B_3(x_1) \equiv \phi_2(x_1) \equiv \phi_3(x_1) ($ mod 2). Then, $\phi_1(x_1) = \phi_2(x_1) = \phi_3(x_1)$. By symmetry, ϕ is a proper flow on x_2, x_3 as well. Since the flow values did not change for all other edges, ϕ is a proper flow on G, a contradiction. $\square$

Lemma 2.16. If G is a minimum counterexample to the Four-Flow Conjecture, then G has girth 5.

Proof: Suppose for the sake of contradiction, G contains a cycle C of length 4 or less. It follows from previous lemmas that $C = uvwx$ has length 4. Let $G' = G\backslash\{uv, wx\}$. First, we will show that G' is bridgeless.

Suppose there exist a bridge yz. Let A, B be two components of $G'\backslash yz$. Without loss of generality, we may assume that $u, x, y \in V(A)$. Note that u, x, y forms a cut in G. Since G

is cyclically 4-connected, u, x, y forms a trivial cut and they all have a common neighbour. However, since ux is an edge, G contains a triangle, a contradiction.

Since G' is bridgeless and does not contain the Petersen graph as a minor, it has a proper flow ϕ'. Without loss of generality, we may assume that $\phi'(ux) = 1$.

If $\phi'(ux) = \phi(vw)$, let $\phi(ux) = \phi(vw) = 2, \phi(uv) = \phi(wx) = 3$ and $\phi(e) = \phi'(e)$ for all other edges e. Note that $\phi_1(u) \equiv \phi'_1(u) - 1 \equiv \phi'_2(u) + 1 \equiv \phi'_3(u) + 1 \equiv \phi_2(u) \equiv \phi_3(u) (\text{mod } 2)$. Then, by symmetry, it follows that ϕ is a proper flow on G, a contradiction.

If $\phi'(ux) \neq \phi'(vw)$, without loss of generality, we may assume $\phi'(vw) = 2$. Let $\phi(ux) = 2, \phi(vw) = 1, \phi(uv) = \phi(wx) = 3$ and $\phi(e) = \phi'(e)$ for all other edges e. Note that $\phi_1(u) \equiv \phi'_1(u) - 1 \equiv \phi'_2(u) + 1 \equiv \phi'_3(u) + 1 \equiv \phi_2(u) \equiv \phi_3(u) (\text{mod } 2)$. It follows from similar arguments that ϕ is a proper flow on G, a contradiction. $\square$

Lemma 2.17. Let G be a minimum counterexample to the Four-Flow conjecture and G contains a cut $X = \{x_1, x_2, x_3, x_4\}$ that separates G into two subgraphs A, B where both sides contains a cycle. Then, for all $i = 1, 2, 3, 4, d_A(x_i)$ or $d_B(x_i) = 1$.

Proof: Suppose for the sake of contradiction, there exist x_i such that $d_A(x_i), d_B(x_i) \geq 2$. Without loss of generality, we may assume $i = 1$. Let G' be the graph obtained from G by splitting the vertex x_1 into x_A, x_B such that e is incident to x_A, x_B if and only if e is incident to x_1 and $e \in E(A), E(B)$ respectively.

First, note that since G is 3-connected, G' has no cut vertex and thus G' is bridgeless. We claim that G' also does not contain the Petersen graph as a minor. Suppose for the sake of contradiction that G' does contain the Petersen graph as a minor. Since G does not, it follows that the minor uses both vertices x_A, x_B. This implies that x_2, x_3, x_4 is a cut of the Petersen minor. Since the only vertex cut of size less than or equals to 3 in the Petersen graph is a trivial cut, it implies that one of A, B has $S3$ as a minor with x_2, x_3, x_4 as leaves. Without loss of generality, we may assume A is the subgraph with the $S3$ minor. Note that by Lemma 2.9, A contains a minor isomorphic to a $T4$. It follows that A also contains a minor that is isomorphic to $S3$ with x_2, x_3, x_4 as leaves and does not use x_1. Then, it follows that G contains the Petersen graph as a minor, a contradiction.

Since G' is bridgeless and does not contain the Petersen graph as a minor, it has a proper flow ϕ'. Note that ϕ' is also a proper flow on G, a contradiction. $\square$

Note that the above lemma implies that X induces an edge cut of size 4. Note that for each edge in the cut, by choosing either ends, we can form cyclic 4-cuts. Thus, given a cyclic 4-cut X, we can always find another cyclic 4-cut X' such that $d_A(x_i') = 1$ for all $i = 1, 2, 3, 4$.

Now we are ready to prove our main lemma.

Proof of Theorem 2.1: Let G be a minimum counterexample to the Four-Flow Conjecture. It follows from previous lemmas that G is cyclically 4-connected with girth 5. For the sake of contradiction, assume G is not cyclically 5-connected. Then, G contains a cut X of size 4 that separates G into A, B where both contain a cycle. Recall that by Corollary 2.10, each subgraph, A, B, contains at least two distinct perfect matchings of X as minors. We will break it into two cases.

Case 1: One of A, B contains 3 minors that are distinct perfect matchings of X.

Without loss of generality, assume A is the subgraph that contains 3 minors that are distinct perfect matchings of X. As shown at the end of Lemma 2.17, we can also assume that $d_A(x_i) = 1$ for $i = 1, 2, 3, 4$ and A still contains three minors that are distinct perfect matchings of X. Note that $A \backslash X$ is connected, otherwise a proper subset of X forms a smaller cut, contradicting G being cyclically 4-connected. Then, each perfect matching minor induces two disjoint path of A that matches the vertices of X and each path has length at least 2. Since $A \backslash X$ is connected, there exist a path connecting the two induced paths. This implies that A also contains $T4_{12,34}, T4_{13,24}, T4_{14,23}$ as minors. Let $A_{12,34}, A_{13,24}, A_{14,23}$ be subgraphs obtained from G by replacing the subgraph A with $T4_{12,34}, T4_{13,24}, T4_{14,23}$ respectively. By Lemma 2.9, B also contains at least two distinct $T4$ minors. Without loss of generality, we may assume that B contains $T4_{12,34}$ and $T4_{13,24}$ as minors. Let $B_{12,34}, B_{13,24}$ be subgraphs obtained from G by replacing the subgraph B with $T4_{12,34}, T4_{13,24}$ respectively. It is clear that $A_{12,34}, A_{13,24}, A_{14,23}, B_{12,34}, B_{13,24}$ are all bridgeless minors of G and does not contain the Petersen graph as a minor.

For ease of notation, we define the following. Let G' be a graph obtained from G by replacing one of the subgraphs A, B with a $T4$. Let ϕ be a proper flow on G'. Let e_i be the edge of the $T4$ incident to x_i for $i = 1, 2, 3, 4$ and e be the other edge in the $T4$ not incident to a vertex in X. Without loss of generality, we may always assume $\phi(e) = 3$. If $\phi(e_i) = t_i$ for $i = 1, 2, 3, 4$ where $t_i = 1, 2$, we say that ϕ induces $t_1t_2t_3t_4$ on X. Note that suppose G^A, G^B are two graphs obtained from G by replacing A, B respectively with a $T4$ and each has a proper flow ϕ^A, ϕ^B that induces the same value $t_1t_2t_3t_4$ on X. Then, let $\phi(e) = \phi^A(e)$ for all $e \in E(A)$ and $\phi(e) = \phi^B(e)$ for all $e \in E(B)$. Then $\phi_i^A(x_j) + \phi_i^B(x_j) - \phi_i(x_j)$ is either 0 or 2 for $i = 1, 2, 3$ and $j = 1, 2, 3, 4$. This implies that ϕ is a proper flow on G. So our goal is to find flows ϕ^A, ϕ^B for the minors G^A, G^B respectively that induces the same values on X.

Consider a proper flow on $A_{12,34}$. Without loss of generality, we may assume it induces 1212 on X. Then, consider a proper flow on $B_{12,34}$. without loss of generality, it either induces 1212 or 1221 on X. If it's the first case, then G also has a proper flow, a contradiction. Then we assume $B_{12,34}$ has a flow that induces 1221 on X. By a similar argument, a flow on $A_{13,24}$ induces either 1221 a contradiction, or 1122. Thus, we may assume that $A_{12,34}, A_{13,24}$ induces $1212, 1122$ respectively on X. Consider a flow on $B_{13,24}$. Without loss of generality, it induces either 1221 or 1122 on X. However, both implies G has a proper flow, a contradiction.

Case 2: Both subgraphs A, B contains exactly two distinct minors that are perfect matchings of X.

Note that this implies A, B each contains only two distinct $T4$ minors as well. Without loss of generality, we may assume A has $T4_{12,34}$ and $T4_{14,23}$ as minors and B has $T4_{12,34}$ as a minor.

First, suppose that B also has $T4_{13,24}$ as a minor. Similarly as before, we induce minors: $A_{12,34}, A_{14,23}, B_{12,34}, B_{13,24}$. Without loss of generality, we may assume a proper flow of $B_{12,34}$ induces 1212 on X. Following similar argument as before, a proper flow on $A_{12,34}$ must induce 1221 on X. Then, a proper flow of $B_{13,24}$ must induce 1122 on X. Without loss of generality, we may assume a proper flow on $A_{14,23}$ induces either 1122 or 1212 on X. However, both implies G has a proper flow, a contradiction.

Now, assume that B has $T4_{14,23}$ as a minor. By the Two-Path Theorem (2.6), each subgraph A, B can be embedded in a plane such that the vertices of X is on the outer face. Since A, B has the same two $T4_{12,34}$ and $T4_{14,23}$ as minors, they can both be embedded in the plane such that x_1, x_2, x_3, x_4 appears in the same cyclic order in the outer face. This implies that G has a planar embedding. However, by Lemma 2.7, G has a proper flow, a contradiction. $\square$

Chapter 3

Acyclic Sides of Mixed C5C Graphs

The goal of this chapter is to determine when can we replace the acyclic side of a mixed graph with a minor so that the resulting graph is still C5C. This is essentially the function of Lemma 3.1 which we ;rove in the first section. Then, in the remaining sections of the chapter, we explicitly determine what are the structures that cannot be replaced with, producing in some sense a minor-minimal of acyclic side of mixed C5C graphs.

In the Doubly-Acyclic section of Chapter 1, we have briefly discussed what an acyclic side can be. We know that if A is an acyclic side, then A is either isomorphic to C_J or $A \backslash X$ is a forest. So we know C_J is a "minor-minimal" acyclic side. The rest of the chapter is focused on acyclic sides where $A \backslash X$ is a forest.

3.1 Subpattern Operation

Definition: In a mixed C5C graph G with mixed cut X and acyclic side A, let $u, v \in A \backslash X$. Denote $X(u)$ as the set of edges going from u to X. A *subpattern operation* is either:

- deleting an edge of A, or
- contracting the edge uv and deleting either $X(u)$ or $X(v)$.

We say A' is a *subpattern* of A if A' can be reached through a series of subpattern operations from A. A' is a *proper* subpattern if A' is a subpattern of A and $A' \neq A$.

In this section, we will discuss given a mixed C5C graph G with mixed cut X and acyclic side A, when is it possible to replace A with a subpattern A' of A so the new resulting graph remains C5C.

Lemma 3.1. Let G be a C5C graph with minimum degree 3 and a mixed cyclic 5-cut X. Let A and B be the acyclic and cyclic side respectively where $B\backslash X$ is connected and $d_{B\backslash X}(x) \geq 2$ for all $x \in X$. If A' is a subpattern of A such that:

1. $E(A') \cap E(X) = \emptyset$
2. A' is not the following graph: a 5-cycle C and four edges e, f, g, h, where $|V(C) \cap V(X)| = 1$ and e, f, g, h form a matching from the vertices in $X\backslash V(C)$ to the vertices in $V(C)\backslash X$,
3. A' is not the following graph: a 5-cycle C and three edges e, f, g, where $|V(C) \cap V(X)| = 2$ and e, f, g form a matching from the vertices in $X\backslash V(C)$ to the vertices in $V(C)\backslash X$,
4. A' is not the following graph: a path P of nine vertices and two edges e, f where $X \subset V(P)$, edges e and f form a matching amongst the four vertices in $V(P)\backslash X$ such that $P \cup e \cup f$ does not contain any cycles of length three,
5. for every $S \subseteq V(A')$ where $|S| \leq 4$ and for every collection of components F in $A'\backslash S$ where $V(F) \cap X = \emptyset$, $G(V(F) \cup S)$ does not contain a cycle,
6. every $v \in V(A')\backslash X$ has degree ≥ 3 in A'
7. every $x \in X$ is adjacent to some vertex $v \in V(A')\backslash X$,
8. A' contains a cycle
9. the girth of A' is at least five,

then, $G' = A' \cup B\backslash E(X)$ is a C5C graph with minimum degree three and girth 5.

Here are some brief explanation to why we have these nine conditions. Condition 1 suggests that X should be an independent set. We will discuss in more detail what happens if X is not an independent set in later sections. Conditions 2-4 are obstructions where if A is replaced with those minors, the resulting G' might contain a cyclic 4-cut. We will characterize this more formally in Corollary 3.2. Condition 5 prevents the existence of any cyclic k-cuts in A' where $k < 5$. Condition 6-7 guarantee tat G' has minimum degree 3. Condition 8 makes sure that a cyclic k-cut still exists; in particular, X is still a cyclic 5-cut. Condition 9 keeps the girth of G' at 5. Now we will proceed on proving the lemma.

Proof of Lemma 3.1: First, note that G' has minimum degree three and both A' and B contains a cycle. So G' contains a cyclic k-cut for some k. Next, we claim G' has girth five. From Condition 1 and 9, it follows that every 4-cycle, C in G' contains two vertices of X, a vertex in $A'\backslash X$ and a vertex in $B\backslash X'$. However, by the definition of subpattern operations, it follows that $C \in A$, contradicting the fact that A is a C5C graph. So now, it suffices to show that G' does not contain a cyclic k-cut where $k \leq 4$.

Assume for the purpose of contradiction that A' is a minimal counterexample. Choose Y, C, D where Y is a cut of size at most 4 that separates G' into subgraphs C, D, both containing a cycle, such that:

- Y has minimum size,
- $|V(C) \cap X| \leq |V(D) \cap X|$ and
- $|V(C) \cap X\backslash Y|$ has minimum size.

Denote the graph induced by the vertices $V(A') \cap V(D)\backslash(X \cup Y), V(A') \cap V(C)\backslash(X \cup Y), V(B) \cap V(C)\backslash(X \cup Y), V(B) \cap V(D)\backslash(X \cup Y)$ as $Q_{A',D}, Q_{A',C}, Q_{B,C}, Q_{B,D}$ respectively. Let $m = |Y|$. By condition 5, it follows that G' is connected and $m \neq 0$. Thus $1 \leq |Y| = m \leq 4$.

Case 1: $X \cap Y = \emptyset$.

Subcase i: $X \cap V(C) = \emptyset$. If $Y \subseteq V(B)$ then Y is a cyclic m-cut in G, a contradiction. If $m - 2 \leq |Y \cap V(B)| \leq m - 1$, then it follows that $V(Q_{A',C}) = \emptyset$ and hence there exists a cycle in $B \cap C$. This implies that $Y \cap V(B)$ is a cyclic n-cut in G' where $n < m$, contradicting the choice of Y. If $1 \leq |Y \cap V(B)| \leq m-2$, then it follows that $V(Q_{B,C}) = \emptyset$ and there is a cycle in $A' \cap C$. This implies $Y \cap V(A')$ is a cyclic n-cut in G' where $n < m$, contradicting the minimal size assumption of Y. If $Y \cap V(B) = \emptyset$, then A' violates condition 5 with $S = Y$ and $F = Q_{A',C}$, a contradiction.

Subcase ii: $|X \cap V(C)| = 1$. Let $x = X \cap V(C)$. If $Y \subseteq V(B)$, then $Q_{A',C} = \emptyset$. This implies that $d_{A'}(x) = 0$ violating condition 7, a contradiction. If $1 \leq |Y \cap V(B)| \leq m-1$, then x is a cut vertex in C. It follows that the cycle in C is either in $A \cap C$ or in $B \cap C$. If the cycle is in $A \cap C$, then $x \cup (Y \cap V(A'))$ is a cyclic n-cut in G' where $n \leq 4$ contradicting that $|C \cap X\backslash Y|$ is minimum. Similarly, if the cycle is in $B \cap C$, then $x \cup (Y \cap V(B))$ is a cyclic n-cut where $n \leq 4$, contradicting that $|C \cap X \backslash Y|$ has minimum size. Lastly, if $Y \subseteq V(A')$, then $V(Q_{B,C}) = \emptyset$. It follows then $d_B(x) = 0 \leq 2$, a contradiction .

Subcase iii: $|X \cap V(C)| = 2$. Let $\{u, v\} = X \cap V(C)$. If $Y \subseteq V(B)$, then $V(Q_{A',C}) = \emptyset$. This implies that A' violates condition 6, a contradiction. If $|Y \cap V(B)| = m - 1$, then A' contains a cut vertex. Then, the cycle in A' is either in $A' \cap C$ or $A' \cap D$ because A'. So either $V(A') \cap V(C) \cap (X \cup Y)$ or $V(A') \cap V(D) \cap (X \cup Y)$ forms a smaller cyclic n-cut in G', contradicting the choice of Y. Lastly, assume $0 \leq |Y \cap V(B)| \leq m - 2$. Note that since $d_B(u), d_B(v) \geq 2$ and B has girth 5, it follows that $B \cap C$ contains a cycle. Then, $\{u, v\} \cup (Y \cap V(B))$ is a cyclic n-cut in G where $n \leq 4$, a contradiction.

Case 2: $|X \cap Y| = 1$. Let $v = X \cap Y$.

Subcase i: $X \cap V(C) = v$. If $2 \leq |Y \cap V(B)| \leq 4$, then, v is a cut vertex in C. This implies that the cycle in C is either in $A' \cap C$ or $B \cap C$. In the first instance, if we let $S = Y \cap V(A'), F = Q_{A',C}$, we contradict condition 5. In the second instance, $Y \cap V(B)$ forms a cyclic n-cut in G where $n \leq 4$, a contradiction. If $|Y \cap V(B)| = 1$, then $Y \subseteq V(A')$. Then, if we let $S = Y, F = Q_{A',C}$, we once again contradict condition 5. Note that $v \in X \cap Y$ so $|Y \cap V(B)| \geq 1$.

Subcase ii: $|X \cap V(C)| = 2$. Let $\{v, x\} = X \cap V(C)$. If $Y \subseteq V(B)$, then it follows that $V(Q_{A',C}) = \emptyset$. Then A' violates condition 7, a contradiction. Suppose $|Y \cap V(B)| = m - 1$. We will show that at least one of $A' \cap C$ and $B \cap C$ contain a cycle. Suppose for the sake of contradiction that both $A' \cap C$ and $B \cap C$ are acyclic, it follows that $|V(Q_{A',C})| \leq 1$. From condition 7, it follows that v and x have a common neighbour in A'. This implies that v and x have no common neighbours in $B \cap C$. It then follows that at least one of v and x will have less than two neighbours in B, a contradiction. So we may assume either $A' \cap C$ or $B \cap C$ contains a cycle. Then either $(Y \cup X) \cap (V(A') \cap V(C))$ or $(Y \cup X) \cap (V(B) \cap V(C))$ forms a cyclic n-cut in G' where $n \leq m$ contradicts the minimum size of Y. Lastly, assume $|Y \cap V(B)| \leq 2$, then either $B \cap C$ is acyclic or $B \cap C$ contains a cycle. The first instance implies that $d_B(v) < 2$ while the second instance implies that $Y \cap V()$ is a cyclic n-cut in G where $n \leq 4$, both forming contradictions.

Subcase iii: Suppose $|X \cap V(C)| = 3$. If $Y \subseteq V(B)$, then the cycle in A' is either in $A' \cap C$ or in $A' \cap D$. This implies that either $X \cap V(C)$ or $X \cap V(D)$ is a cyclic 3-cut in G', contradicting the choice of Y. Now suppose $|Y \cap V(B)| = m - 1$, let $y = V(A') \cap Y \backslash X$. For similar reasons as before, $A' \cap C$ and $A' \cap D$ are acyclic. Furthermore, this implies that $A' \cap C$ and $A' \cap D$ is one of the following: a collection of edges amongst X and Y, or S3, or S3 and an edge, or S4, or T4. By checking the possible combinations that satisfies condition 6-9, it follows that A' is the graph described in condition 2, a contradiction. Now suppose $|Y \cap V(B)| \leq 2$, then $B \cap C$ and $B \cap D$ are acyclic; otherwise, one of $X \cup Y \cap V(C)$ and $X \cup Y \cap V(D)$ forms a cyclic n-cut in G' where $n < 5$, contradicting the choice of Y and C. So we may assume both $B \cap C$ and $B \cap D$ contains no cycles. However, it then follows that at least one of the vertices in $X \cap V(C) \backslash Y$ has less than two neighbours in $B \backslash X$, a contradiction.

Case 3: $|X \cap Y| = 2$. Note that $|Y| \geq 2$.

Subcase i: $|X \cap V(C)| = 2$. If $Y \subseteq V(B)$, then $V(Q_{A',C}) = \emptyset$. This implies that $B \cap C$ contains a cycle and Y is a cyclic m-cut in G, a contradiction. If $|Y \cap V(B)| = 3$, then $Y \cap V(A')$ and $Y \cap V(B)$ are trivial cuts; otherwise they form cyclic n-cuts where $n < 5$ that contradict the choice of Y and C. However, any combination of trivial cuts of size three or less and edges creates a cycle of length four or less, contradicting the girth assumption. If $Y \subseteq V(A')$, then A' violates condition 5 with $S = Y, F = Q_{A',C}$, a contradiction.

Subcase ii: $|X \cap V(C)| = 3$. Let $v = X \cap V(C) \backslash Y$. If $Y \subseteq V(B)$, then both $A' \cap C$ and $A' \cap D$ are both acyclic. By condition 9, it follows that $A' \cap C$ is S3 and $A' \cap D$ is T4. Then, A' is isomorphic to the graph described in condition 3, a contradiction. If $2 \leq |Y \cap V(B)| \leq 3$, with similar arguments as before, $A' \cap C$ and $B \cap C$ are acyclic. However once again, every combination of trivial cuts forms either a cycle of length less than 5 or a vertex in X has less than two neighbours in B, a contradiction.

Case 4: $|X \cap Y| = 3$. Note then $|Y| \geq 3$.

Subcase i: $|X \cap V(C)| = 3$. Note that by condition 5, $A' \cap C$ is acyclic. Also, $B \cap C$ is acyclic or else $Y \cap V(B)$ is a cyclic n-cut in G with $n < 5$, a contradiction. Then, $A' \cap C$ and $B \cap C$ are unions of collections of edges, S3, S4 and T4. By checking all combinations, it follows that all cycles in C has length at most four, a contradiction.

Subcase ii: $|X \cap C| = 4$. If $Y \subseteq V(A')$, then either $B \cap C$ or $B \cap D$ contains a cycle. This is because $B \backslash X$. However, this implies $X \cap V(C)$ or $X \cap V(D)$ is a cyclic 4-cut in G, a contradiction. If $Y \subseteq V(B)$, then $A' \cap C$ and $A' \cap D$ are acyclic. Then, by checking different combinations of trivial 4-cuts, A' is the graph described in condition 4, a contradiction.

Case 5: $Y \subseteq X$.

Since B is connected, $V(Q_{B,C}) = \emptyset$. Since C contains a cycle, $A' \cap C$ also contains a cycle. This implies that Y is a cyclic 4-cut in A', contradicting condition 5. $\square$

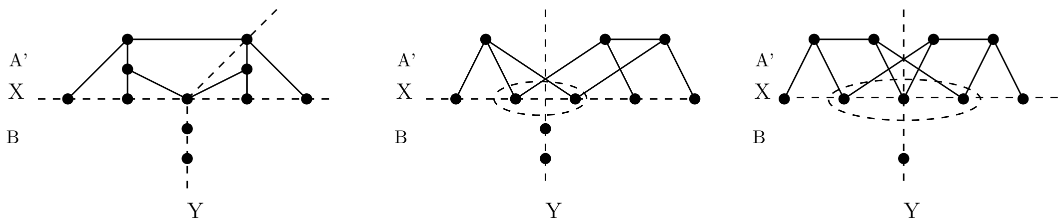


Figure 3.1: Problematic Cuts in Lemma 3.1

Corollary 3.2. If A' is isomorphic to one of the graphs mentioned in Conditions 2-4 of Lemma 3.1 and the resulting G' is not a C5C graph, then G' contains a cut Y of size 4. More specifically, G' resembles one of the following (See Figure 3.1).

Proof: Following the proof of Lemma 3.1, if we assume A satisfies conditions 5-9 of Lemma 3.1, G' is not a C5C graph if and only if we are in the following three subcases: Case 2 subcase iii, Case 3 subcase ii and Case 4 subcase ii. Note that in these three subcases, we get a specific cyclic 4-cut where A' is isomorphic to one of the graphs mentioned in Conditions 2-4. This proves our corollary. $\square$

3.2 Patterns

In this section, we will discuss how to apply Lemma 3.1 and Corollary 3.2 to find minors of an acyclic side. Our goal in this section is to determine, given an acyclic side A, whether A have a minor A' such that the graph G' obtained by replacing the acyclic side A with A' remains C5C. Our main result is Lemma 3.5.

Definition: A *pattern* is a forest G and a labeling X of the vertices of G, where for each vertex v in G, its label $X(v)$ is a subset of $\{1, 2, 3, 4, 5\}$ and the following conditions are satisfied:

- *Min-degree condition*: for all $v \in V(G), d(v) + |X(v)| \geq 3$
- *Cover condition*: $| \cup_{v \in V(G)} X(v)| = 5$

- *Girth condition*: if $u, v \in V(G)$ are neighbours or have a common neighbour, then $X(u) \cap X(v) = \emptyset$ and for all $u, v \in V(G)$, $|X(u) \cap X(v)| \leq 1$

- *Cycle condition*: either there exist two vertices $u, v \in V(G)$ in the same component in G such that $|X(u) \cap X(v)| \geq 1$ or there exists components F_1, F_2 of G such that $|(\cup_{u \in F_1} X(u)) \cap (\cup_{v \in F_2} X(v))| \geq 2$

Definition: A pattern G is a *C5C pattern* if it satisfies the following condition.

(C5C condition): Let $S \subseteq V(G)$ with $|S| = m \leq 4$, F be the union of some components of $G \backslash S$ such that $|X'| \leq m - 4$ where $X' = \cup_{v \in V(F)} X(v)$. Let $F_1, F_2, ..., F_k$ be components of the graph induced by $V(F) \cup S$. Then, for all such S, F, i) for any $u, v \in F_i$ for $1 \leq i \leq k$, $X(u) \cap X(v) \cap X' = \emptyset$ and ii) for $1 \leq i, j, \leq k$, $|(\cup_{u \in V(F_i)} X(u)) \cap (\cup_{v \in V(F_j)} X(v)) \cap X'| \leq 1$.

Patterns that do not satisfy the C5C condition will be called *non-C5C patterns*.

Note that a C5C pattern is a representations of an acyclic side in a mixed C5C graph.

Definition: Let F be a pattern. Let $Gr(F)$ denote the graph where $V(Gr(F)) = V(F) \cup \{x_1, x_2, x_3, x_4, x_5\}, E(Gr(F)) = E(F) \cup \{vx_i | v \in V(F), i \in X(v)\}$.

Proposition 3.3. Let F be a pattern. Then F is a C5C pattern if and only if $Gr(F)$ satisfies conditions 5 - 9 for A' in Lemma 3.1.

Proof: In the forward direction, note that the min-degree and the cover condition respectively implies condition 6 and 7 in Lemma 3.1. Since X is an independent set, the girth condition implies condition 9. It is also easy to check that the cycle condition implies the condition 8.

Now, we claim that condition 5 follows from the C5C condition. Suppose for the sake of contradiction that there exist $S \subseteq V(Gr(F))$ where $|S| \leq 4$ and T, a collection of components of $Gr(F) \backslash S$ where $V(T) \cap X = \emptyset$ and $G(T \cup S)$ contains a cycle, C. Note that S might contain vertices of X. Let $S' = S \backslash X$, $X' = S \backslash S'$. Let $F_1, ..., F_k$ be components of $G(V(F) \cup S')$. We will show that $S', T, F_1, ..., F_k$ violates the C5C condition. Note that $\cup_{v \in V(T)} X(v) = S \backslash S'$. Then $| \cup_{v \in V(T)} X(v)| = |S \backslash S'| \leq 4 - |S'|$. We will break into cases based on the size of $|V(C) \cap X'|$.

First note that since $F \backslash X$ does not contain a cycle, $V(C) \cap X' \neq \emptyset$. If $|V(C) \cap X'| = 1$, then there exists a component F_i that contains the path $C \backslash X$ for some $1 \leq i \leq k$. Note that the endpoints of this path have a common neighbour in X; so there exists $u, v \in V(F_i)$ such that $X(u) \cap X(v) \cap X' \neq \emptyset$, contradicting the C5C condition.

If $|V(C) \cap X'| = 2$, then there exists two components F_i, F_j that contains the two paths $C \backslash X$ for some $1 \leq i, j \leq k$. Similarly, it follows that $|(\cup_{v \in V(F_i)} X(v)) \cap (\cup_{v \in V(F_j)} X(v)) \cap X'| \geq 2$, contradicting the C5C condition.

Lastly, suppose $|V(C) \cap X'| \geq 3$. Let $F'_1, F'_2, ... F'_l$ be components that contain $C \backslash X$. If $l \leq 2$, the analysis is the same as before, where we can find components that contradicts the C5C condition. Suppose $l \geq 3$. Let $1 \leq i \leq l$. Note that each component F'_i is a tree, since F is a forest. By the min-degree condition, each leaf v in F'_i has at least two neighbours in X. For all leaves $v \notin S'$, all of its neighbours are in X'. Note that $|X'| \geq |V(C) \cap X'| \geq 3$. This implies that $|S'| \leq 1$. Then, there exists at most one component F'_j that contains the vertices in S' for some $1 \leq j \leq k$. Then it follows that $| \cup_{v \in V(F'_i)} X(v) \cap X'| \geq 3$ for all $i \neq j$ and $| \cup_{v \in V(F'_i)} X(v) \cap X'| \geq 2$. Since $|X'| \leq 4$ and $l \geq 3$, by the pigeonhole principle, it follows that there exists two components, F'_p, F'_q for some $1 \leq p, q \leq l$ such that $| \cup_{v \in V(F'_p)} X(v) \cap \cup_{v \in V(F'_q} X(v) \cap X'| \geq 2$, contradicting the C5C condition. This completes the proof of the forward direction.

For the backward direction, it is clear that the min-degree and the cover conditions implies condition 6 and 7 of Lemma 3.1 respectively. Since F is a forest, every cycle in $Gr(F)$ contains some vertex in X. Then, condition 9 also follows from the girth condition.

Now we claim that condition 8 implies the cycle condition. Let C be a cycle in A'. Since $A' \backslash X$ is a forest, $V(C) \cap X \neq \emptyset$. Suppose $|V(C) \cap X| = 1$. Then, there exists a component of F that contains $C \backslash X$. Furthermore, this component contains two vertices u, v such that $X(u) \cap X(v) \neq \emptyset$, satisfying the cycle condition. If $|V(C) \cap X| = 2$, we can similarly find two components F_1, F_2 of F that contains $C \backslash X$. It is clear that $|(\cap_{v \in V(F_1)} X(v)) \cap (\cap_{v \in V(F_2)} X(v))| \geq 2$, satisfying the cycle condition. Lastly, if $|V(C) \cap X| \geq 3$, let $F_1, ..., F_k$ be the components of F that contains the components of $C \backslash X$. Since every vertex in $A' \backslash X$ has degree 3 or more, it follows that F_i are tree with at least 3 leaves for $1 \leq i \leq k$. Then, $| \cup_{v \in V(F_i)} X(v)| \geq 3$ for all $1 \leq i \leq k$. By the pigeonhole principle there exists two components F_i, F_j where $1 \leq i, j \leq k$ such that $|(\cap_{v \in V(F_i)} X(v)) \cap (\cap_{v \in V(F_j)} X(v))| \geq 2$, satisfying the cycle condition.

Lastly, we will show that condition 5 implies the C5C condition. Suppose for the sake of contradiction that there exists a set $S \in V(F)$, F', a union of components of $F \backslash S$ and F_1, F_2 where $|S| + | \cup_{v \in V(F)} X(v)| \leq 4$ and F_1, F_2 are components of $F \cup S$ such that either i) $X(u) \cap X(v) \neq \emptyset$ for some $u, v \in V(F_1)$ or ii) $|(\Cup_{v \in F_1}) \cap (\Cup_{v \in F_2})| \geq 2$. In both cases, $Gr(F_1 \cup F_2)$ contains a cycle. Thus, let $S' = S \cup \{x_i : i \in X(v), v \in V(F)\}$, then $S', Gr(F_1), Gr(F_2)$ violates condition 5, a contradiction. $\square$

Two patterns F and H are *isomorphic* if the two forests F and H are isomorphic and there is a bijection between the labeling of the vertices of the graphs.

Note that subpattern operations can be defined in the similar fashion.

Definition: A *subpattern operation* is either:

- deleting an edge of G
- deleting an element of $X(v)$ for some $v \in G$
- contracting the edge uv and let u' be the new vertex labelled with either $X(u)$ or $X(v)$.

Then we define G' to be a *subpattern* of G if G is a pattern that can be obtained from G through a series of subpattern operations. We say G' is a proper subpattern of G if G' is a subpattern of G and G' is not isomorphic to G.

Below are the graphs in conditions 2-4 of Lemma 3.1 described as patterns:

1. A path of four vertices v_1, v_2, v_3, v_4 where $|X(v_1)| = |X(v_4)| = 2, |X(v_2)| = |X(v_3)| = 1, |X(v_1) \cap X(v_4)| = 1, |X(v_i) \cap X(v_j)| = 0$ for all other pairs of i, j. Let A_0 denote this pattern.
2. A vertex v and an edge uw where $|X(v)| = 3, |X(u)| = |X(w)| = 2, |X(v) \cap X(u)| = |X(v) \cap X(w)| = 1, |X(u) \cap X(w)| = 0$. Let $D_{3,4}$ denote this pattern.
3. Two edges v_1v_2, v_3v_4 where $|X(v_i)| = 2$ for all $1 \leq i \leq 4, |X(v_1) \cap X(v_3)| = |X(v_2) \cap X(v_3)| = |X(v_2) \cap X(v_4)| = 1, |X(v_i) \cap X(v_j)| = 0$ for all other pairs of i, j. Let $D_{4,4}$ denote this pattern.

Definition: A pattern G is *basic* if G is $A_0, D_{3,4}$ or $D_{4,4}$. A pattern G is *pseudobasic* if G is a C5C pattern that is not basic and only has basic proper subpatterns. Note that basic patterns satisfy the C5C condition so they are C5C patterns. Also, note that if G is basic, then G is the only C5C subpattern of itself.

Proposition 3.4. A C5C pattern is either:

- basic,

- pseudobasic, or
- has a pseudobasic subpattern

The above proposition follows from the definition of basic and pseudobasic patterns. The main result is the following lemma where we determine the actual list of pseudobasic patterns. Now, we can rewrite our main Lemma 1.6 in terms of patterns in the following form.

Theorem 3.5. *G is a basic or a pseudobasic pattern if and only if G is in $\mathcal{L}$.*

Please see the end of the chapter (Figure 3.5) for a complete list $\mathcal{L}$.

To prove the above lemma, we will break patterns into different classes.

Definition: A pattern G is a *path pattern* if G is a path. A pattern G is a *tree pattern* if G is a tree that is not a path. G is a *forest pattern* if G is a forest that is not a tree.

The next four subsections are devoted to finding the list of pseudobasic patterns.

3.2.1 Non-C5C Patterns

To facilitate the process of finding the list for pseudobasic path and tree patterns, we will need the following definition and lemmas about non-C5C patterns.

Definition: Let (G, X) be a pattern and T be a subgraph of G. Then we call $(T, X|T)$ a *patch* of (G, X). We will abuse the notation and say T is a patch of G. Note that a patch is not necessarily a pattern.

Definition: Let N_{C5C} be a patch of a path with five vertices v_1, v_2, v_3, v_4, v_5 such that $X(v_1) = 12, X(v_2) = 3, X(v_4) = 1, X(v_5) = 23$.

Definition: A pattern G is N_{C5C}*-embedded* if G is isomorphic to a pattern that contains $N_{C5C} = v_1v_2v_3v_4v_5$ as a patch such that

- v_1, v_5 are leaves of G, and
- v_2, v_4 have degree 2 in G.

Lemma 3.6. Let T be a path or tree pattern. Then T is a non-C5C pattern if and only if T is N_{C5C}-embedded.

First we re-introduce the following operation for patches:

Definition: For a patch P, define $Gr(P)$ as a graph where $V(Gr(P)) = V(P) \cup \{x_i | i \in \cup_{v \in V(P)} X(v)\}$ and $E(Gr(P)) = E(P) \cup \{vx_i | v \in V(P), i \in X(v)\}$.

Proof of Lemma 3.6: For the backward direction, if T is N_{C5C}-embedded, then, the C5C condition is violated with $S = v_3, F = \{v_1, v_2, v_4, v_5\}, k = 1 F_1 = F \cup S$ where v_1, v_2, v_3, v_4, v_5 are the five vertices of N_{C5C}. Thus T is a non-C5C pattern.

For the forward direction, assume T violates the C5C condition with S, F as stated in the definition. Assume T is a minimal counterexample and choose S, F such that $|V(F)|$ is as small as possible. Let $H_1, ..., H_k$ be components of F. Note that for each $s \in S$ and $1 \leq i \leq k$, there exists at most one edge $sv \in E(T)$ where $v \in V(H_i$. Otherwise, T contains a cycle. Let $H_i' = Gr(H_i \cup S)/backslash\{sx_j : 1 \leq j \leq 5\}$.

We claim that H_i' for $1 \leq i \leq k$ is acyclic. Suppose for the sake of contradiction that, without loss of generality, H_1' contains a cycle. Let $s = V(H_1) \cap S$. As shown before, s has exactly one neighbour $s' \in V(H_1)$. Then, consider the set $S' = S \cup s' \backslash s$. Let $F' = H_1 \backslash s', F_1' = H_1$. Then note that S', F', F_1 violates the C5C condition and this contradicts the minimality of F.

This along with the min-degree condition implies that for $a \leq i \leq k$, H_i' is S3, or S4 or T4 where the leaves are either vertices in S or vertices in X. Next we claim that H_i is not isomorphic to S3 or S4. Suppose for the sake of contradiction, without loss of generality, H_1 is isomorphic to S3 or S4. Then, there does not exist $H_2, ..., H_k$ or the girth condition is violated. Since $Gr(F \cup S)$ contains a cycle, $|X(S) \cap (\cup_{v \in V(H_1)} X(v))| \geq 2$. Since H_1 has only one vertex, this violates the girth condition.

It follows that H_i is isomorphic to T4 for all $1 \leq i \leq k$. Next we claim that $X(S) \cap (\cup_{v \in V(F)} X(v)) = \emptyset$. Suppose for the sake of contradiction that there exists an edge sx_i for some $1 \leq i \leq 5$ and $x_i \in V(H_j')$. If $k = 0$, then $Gr(S)$ contains a cycle by assumption. However, this cycle violates the girth condition, a contradiction. If $k \geq 1$, then the edge sx_i and the graph H_1' forms a 4-cycle, forming a contradiction as well.

It then follows that H_i is isomorphic to T4 for $1 \leq i \leq k$ and $Gr(F \cup S) \backslash \{x_j s : 1 \leq j \leq 5, s \in S\}$ contains a cycle. By minimality of F, $k = 2$. Then, by observing the graph and by the girth condition $|S| = 1$. Without loss of generality, assume H_i is an edge $u_i v_i$ for $i = 1, 2$, $X(u_1) = 12, X(v_1) = 3, X(u_2) = 23, X(v_2) = 1$ and v_1, v_2 are neighbours of $s = S$. Note then the graph contains N_{C5C} , a contradiction. $\square$

Lemma 3.7. Let T be a path or tree pattern. Let T' be a subpattern of T. If T' is a non-C5C pattern, then T is either a non-C5C pattern or contains a pseudobasic subpattern.

Proof: If T is a non-C5C pattern, then the lemma follows. So me may assume T is a C5C pattern and also that T is minimal under subpattern operations. Hence T does not contain any pseudobasic subpatterns. Since T' is a non-C5C pattern, it follows from Lemma 3.6 that T' is N_{C5C}-embedded. Without loss of generality, suppose T' contains the path $P' = (v_1', v_2', v_3', v_4', v_5')$ where $X(v_1') = 12, X(v_2') = 3, X(v_4') = 1, X(v_5') = 23$.

Consider the following operation. Let P be a path in T. For $u \in V(P)$, define $X_P(u) = \cup_{v \in V(T_u)}(X(v)$ where T_u is the connected component of $T/backslash(V(P) - u)$ that contains u.

Since T' is a subpattern of T, there exists a path P that contains a set of five vertices $v_1, v_2, ..., v_5$ where $X(v_i') \subseteq X_P(v_i)$ for $i = 1, 2..., 5$. In particular, $45 \in X_P(v_3)$. Since subpattern operations do not create leaves, we can assume that v_1, v_5 are leaves in T. Since T is a C5C pattern, there exists vertex u such that $u \in P, u \neq v_3, X_P(u) \cap 45 \neq \emptyset$. Suppose $u \neq v_1, v_5$. Without loss of generality, let u be between v_1 and v_3 in P and $4 \in X_P(u)$. Then, T contains the subpattern $(12, 4, 5, 1, 23)$ isomorphic to the pseudobasic pattern $(12, 3, 4, 2, 15) = A_4$ a contradiction. Now, without loss of generality, assume $u = v_1$. Then $X_P(u) = X(u)$ and $4 \in X(u)$. This implies that T contains the subpattern $(14, 3, 5, 1, 23)$ isomorphic to the pseudobasic pattern $(12, 3, 4, 1, 35) = A_1$, a contradiction. $\square$

Lemma 3.8. Let G be a path or tree pattern where there exists a vertex $v \in V(G)$ such that $d_G(v) + |X(v)| > 3$. Then, G is either a non-C5C pattern or G is a C5C pattern that contains a pseudobasic subpattern.

Proof: Suppose for the purpose of contradiction that G is a C5C pattern with a minimum number of vertices where there exists a vertex $v \in V(G)$ such that $d_G(v) + |X(v)| > 3$ and G does not contain any pseudobasic subpatterns. For $u \in N(v)$ in G, let G_u represent the patch obtained from G by removing all vertices in the set $\{w \in V(G) :$ there exists a path from w to u that does not contain $v\}$. For $i \in X(v)$, let G_i be the patch obtained from removing i from $X(v)$.

First assume $d_G(v) \geq 4$. Let G_u' be a patch obtained from G_u by removing all labels at v so $X(v) = \emptyset$ in G_u'. Note that for all $u \in N(v)$, G_u' does not violate the min-degree nor the girth condition. Since $d_G(v) \geq 3$, there exist at least three leaves in G'_u. By pigeonhole principle, there exist two leaves, x, y such that $X(x) \cap X(y) \neq \emptyset$; therefore for all $u \in G(v)$, G'_u does not violate the cycle condition. For $u \in N(v)$, let $M(u) = \{1,2,3,4,5\} \backslash \cup_{w \in V(G'_u)} X(w)$. Note that for all $u \in N(v)$, $|M(u)| \leq 1$, otherwise, G violates the C5C condition with $S = v, F \cup S = G'_u$. We claim there exists $u \in N(v)$ such that G'_u does not violate the cover condition and hence is a subpattern. If for the sake of contradiction, G'_u violates the cover condition; hence $|M(u)| = 1$. Then let G' be the pattern obtained by adding $M(u)$ to $X(v)$ in G'_u. Note that G' does not violate the min-degree, cycle and girth conditions for the same reasons as G_u. Note also that G' does not violate the cover condition. Therefore, G' is a subpattern of G. By Lemma 3.7, G' does not violate the C5C condition, implying that G either contains a pseudobasic subpattern or is not minimal, a contradiction. Now, we may assume G'_u is a subpattern of G. Note that G_u does not violate the C5C condition, otherwise, by Lemma 3.7, G either violates the C5C condition or is not a pseudobasic subpattern, a contradiction in both cases. Therefore G'_u satisfies the C5C condition and G_u is a subpattern of G, a contradiction.

Now, assume $d_G(v) = 3, |X(v)| \geq 1$. Then as above, G_i satisfies the girth, min-degree and cycle conditions. If there exists $i \in X(v)$ such that G_i satisfies the cover condition, then G_i is a pattern. By Lemma 3.7, G_i is a C5C subpattern of G, a contradiction. Now assume that i G_i violates the cover condition for all $i \in X(v)$. This implies that $|X(v)| = 1$, otherwise G violates the C5C condition with $S = v, F = G \backslash S$. By pigeonhole principle, there exist two leaves x, y such that $X(x) \cap X(y) \neq \emptyset$. Therefore, there exists $u \in N(v)$ such that G_u does not violate the cycle condition. Note that G_u also satisfies the girth and min-degree conditions. We claim that G_u does not violate the cover condition and thus is a pattern. For the sake of contradiction, assume G_u violates the cover condition. Consider G'_u, a patch of G_u by removing the labels of v, so $X(v) = \emptyset$. Note that G'_u contains a cycle and $| \cup_{w \in V(G'_u)} X(w)| \leq 3$. This implies that G violates the C5C condition with $S = v, F = G \backslash v$, a contradiction. Therefore G_u is a pattern. As before, by Lemma 3.7, G_u does not violate the C5C condition; thus G_u is a subpattern of G, a contradiction.

Now assume $d_G(v) = 2, |X(v)| \geq 2$. Then as above, for all $i \in X(v)$, G_i satisfies the girth and min-degree conditions. We claim that G_i does not violate the cycle condition for all $i \in X(v)$. Suppose for the sake of contradiction that there exists $i \in X(v)$ such that G_i violates the cycle condition. So $Gr(G_i)$ is acyclic. Note that G_i contains at least two leaves and a vertex v that is not a leaf. This implies that $Gr(G_i)$ is isomorphic to T5. However, it follows that for one of the leaves x, $X(x) \cap X(v) = i$. Since x and v are neighbours, it implies G violates the girth condition, a contradiction. Now, we claim that there exists $i \in X(v)$ such that G_i does not violate the cover condition. Suppose for the sake of contradiction, for all $i \in X(v)$, G_i violates the cover condition. Since, Consider the patch G' obtained by removing all labels of v from G. Note that G' contains at least two leaves. Let x, y be two of its leaves. Since G_i violates the cover condition for all $i \in X(v)$, then $X(v) \cap X(x) = X(v) \cap X(y) = \emptyset$. This implies that $X(x) \cup X(y) \leq 3$ and thus $X(x) \cap X(y) \neq \emptyset$. Then, $Gr(G')$ is not acyclic, which implies that G violates the C5C condition with $S = v, F = G \backslash v$, a contradiction. Therefore we may assume that there exists $i \in X(v)$ such that G_i satisfies the cover condition and thus G_i is a pattern. Then, as before, it follows from Lemma 3.7, G_i satisfies the C5C condition, which implies G_i is a subpattern of G, a contradiction.

Lastly, assume that $d_G(v) = 1, X(v) \geq 3$. Note that this implies v is a leaf. As above, for all $i \in X(v), G_i$ does not violate the girth nor the min-degree conditions. We claim that G_i does not violate the cycle condition for $i \in X(v)$. For the sake of contradiction, assume G_i violates the cycle condition for some $i \in X(v)$. This implies that for all $j \in X(v), u \in V(G)$ where $j \neq iandu \neq v$, $j \notin X(u)$. Since $|X(v) \backslash i| \geq 3$, G violates the C5C condition with $S = v, F = G \backslash v$, a contradiction. Note that there exists $i \in X(v)$ such that G_i does not violate the cover condition, otherwise, G violates the C5C condition with $S = v, F = G \backslash v$. This implies that there exists $i \in X(v)$ such that G_i is a pattern. Then, as before, it follows from Lemma 3.7, G_i satisfies the C5C condition, which implies G_i is a subpattern of G, a contradiction. □

3.2.2 Path Patterns

Let P be a path pattern with the path $v_1 v_2 ... v_p$. Then let $(X(v_1), X(v_2), ..., X(v_p))$ be the notation we use to represent P. Also abusing the notation, instead of separating the elements inside a set with commas, $X(v)$ is represented by a string of its elements.

For example, $(12, 3, 4, 15)$ is a representation of the first basic pattern A_0 (See Figure 3.2).

Let $\mathcal{A}$ be a list with the following path patterns (See Figure 3.2).

- $A_1 = (12, 3, 4, 1, 35)$
- $A_2 = (12, 3, 4, 1, 5, 34)$
- $A_3 = (12, 3, 4, 1, 3, 45)$
- $A_4 = (12, 3, 4, 5, 13)$

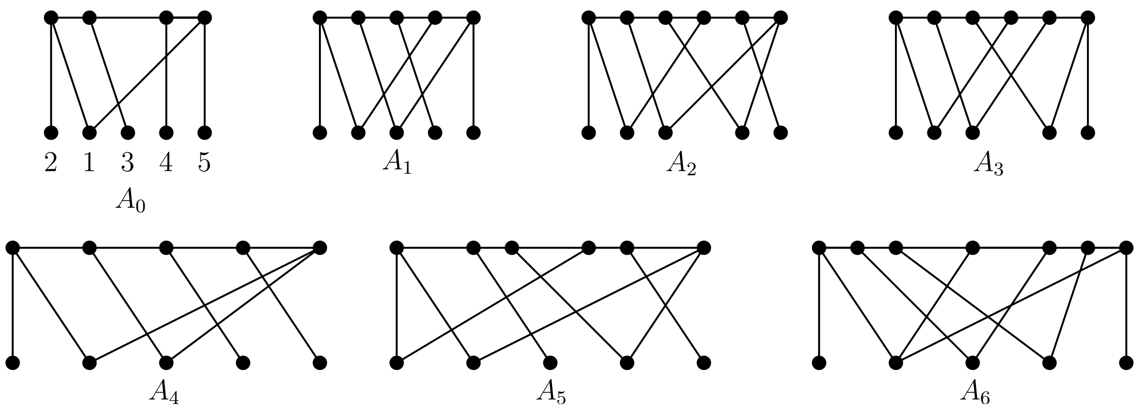


Figure 3.2: Basic (A_0) and Pseudobasic ($\mathcal{A}$) Path Patterns

- $A_5 = (12, 3, 4, 2, 5, 14)$
- $A_6 = (12, 3, 4, 1, 3, 4, 15)$

Lemma 3.9. A path pattern P is a pseudobasic pattern if and only if P is isomorphic to a pattern in $\mathcal{A}$.

Proof: For the backward direction, the reader can inspect that all path patterns in $\mathcal{A}$ are C5C path patterns, are not basic, and contain only basic proper subpatterns, so they are pseudobasic path patterns.

For the forward direction, assume by contradiction that $P = (X(v_1), X(v_2), ..., X(v_p))$ is a minimal counterexample path pattern that is not basic and only has basic proper subpatterns but is not in $\mathcal{A}$.

By Lemma 3.8, it follows that $|X(v_1)| = |X(v_p)| = 2, |X(v_i)| = 1$ for all $1 < i < p$. Without loss of generality, let $X(v_1) = 12$. By the cover condition, $p \geq 3$. However, if $p = 3$, by the cover condition, P is isomorphic to $(12, 3, 45)$, violating the cycle condition, a contradiction. Thus, we may assume that $p \geq 4$. By the girth condition, $X(v_1) \cap X(v_2) = X(v_1) \cap X(v_3) = X(v_2) \cap X(v_3) = \emptyset$,. Without loss of generality, let $X(v_2) = 3, X(v_3) = 4$. Hence $P = (12, 3, 4, ...)$. We will now case on possible values of $X(v_p)$.

Case 1: $X(v_p) = 13$. By the cover condition, there exists $4 \leq i \leq p - 1$ such that $5 \in X(v_i)$. Then, P contains the pseudobasic subpattern, $(12, 3, 4, 5, 13) = A_4$, a contradiction.

Case 2: $X(v_p) = 14$. By the cover condition, there exists $4 \leq i \leq p-1$ such that $5 \in X(v_i)$. Note that by the girth condition, since $4 \in X(v_3), X(v_p)$, $p \geq 6$. First, we claim that for all $4 \leq j \leq p-1$, $2, 3 \notin X(v_j)$ Suppose not and there exists $4 \leq j < i$ such that $2 \in X(v_j)$, then P contains the pseudobasic subpattern $(12, 3, 4, 2, 5, 14) = A_5$, a contradiction. . If there exists $4 \leq j < i$ such that $3 \in X(v_j)$, then P contains the subpattern $(12, 4, 3, 5, 14)$ which is isomorphic to the pseudobasic pattern $(12, 3, 4, 5, 13) = A_4$, a contradiction. If there exists $i < j < p$ such that $2 \in X(v_j)$, then P contains the subpattern $(14, 2, 5, 3, 12)$, isomorphic to the pseudobasic pattern $(12, 3, 4, 5, 13) = A_4$, a contradiction. If there exists $i < j < p$ such that $3 \in X(v_j)$, then P contains the subpattern $(12, 4, 5, 3, 14)$ which is isomorphic to the pseudobasic pattern $(12, 3, 4, 5, 13) = A_4$, a contradiction. It follows from the claim and the girth condition that $X(v_{p-1}) = 5$ and $12345 \cap X_(v_{p-2}) = \emptyset$, a contradiction.

Case 3: $X(v_p) = 15$. Note that $v_p \neq v_5$ otherwise P is the basic pattern A_0, contradicting our assumption. First, we claim that for all $3 < i < p, 2 \notin X(v_i)$. Suppose for the purpose of contradiction, there exists $3 < i < p$ such that $2 \in X(v_i)$, then $(15, 2, 4, 3, 12)$ is a subpattern of P isomorphic to the pseudobasic pattern $(12, 3, 4, 5, 13) = A_4$, a contradiction. By the girth condition and our claim, it follows that $X(v_{p-1}) = 3$ or $X(v_{p-1}) = 4$.

Subcase i: $X(v_{p-1}) = 3$. Then by our claim and the girth condition, $X(v_{p-2}) = 4$. Note that by the girth condition $v_3 \neq v_{p-2}, v_{p-3}, v_{p-4}$. If there exists $4 \leq i \leq p-3$ such that $5 \in X(v_i)$, then $(12, 5, 4, 3, 15)$ is a subpattern of P isomorphic to the pseudobasic pattern $(12, 3, 4, 5, 13) = A_4$, a contradiction. If for all $4 \leq i \leq p-3, 5 \notin X(v_i)$, then by our claim and the girth condition,$X(v_{p-3}) = 1$. However, by the girth condition, it follows that $X(v_{p-4}) \neq 1, 2, 3, 4, 5$, a contradiction.

Subcase ii: $X(v_{p-1}) = 4$. Then by the girth condition and our claim, $X(v_{p-2}) = 3, v_{p-3} \neq v_3$ and $234 \cap X(v_{p-3}) = \emptyset$. If $1 = X(v_{p-3})$, then P contains the pseudobasic pattern $(12, 3, 4, 1, 3, 4, 15) = A_6$, a contradiction. If $5 \in X(v_i)$, then $(12, 5, 3, 4, 15)$ is a subpattern of P isomorphic to the pseudobasic pattern $(12, 3, 4, 5, 13) = A_4$, a contradiction. This implies that $12345 \cap X(v_{p-3}) = \emptyset$, a contradiction.

Note that by symmetry from Case 1-3, we may assume that $2 \notin X(v_p)$.

Case 4: $X(v_p) = 34$. By the girth condition, $X(v_{p-1}) = 2$ or 3 or 5. First, we claim that $X(v_{p-1}) \neq 1, 2$. Suppose for the sake of contradiction that $X(v_{p-1}) = 1, 2$; without loss of generality, assume $X(v_{p-1}) = 1$. By the cover condition, there exists $4 \leq i \leq p-2$ such that $5 \in X(v_i)$. Then P contains the subpattern $(12, 3, 5, 1, 34)$, isomorphic to the pseudobasic pattern $(12, 3, 4, 1, 35) = A_1$, a contradiction. Now, we may assume $X(v_{p-1}) = 5$. By the girth condition, $v_{p-2} \neq v_3, 345 \cap X(v_{p-2}) = \emptyset$. Without loss of generality, assume $X(v_{p-2}) = 1$. Then P contains the pseudobasic subpattern $(12, 3, 4, 1, 5, 34) = A_2$, a contradiction.

Case 5: $X(v_p) = 35$. By the girth condition, $v_p \neq v_4, 35 \cap X(v_{p-1}) = \emptyset$. We claim that for all $4 \leq i \leq p-1$, $12 \cap X(v_i) = \emptyset$. Suppose for the purpose of contradiction that there exists $4 \leq i \leq p-1$ such that $\{1,2\} \cap X(v_i) \neq \emptyset$. Without loss of generality, assume $X(v_i) = 1$. Then P contains the pseudobasic subpattern $(12, 3, 4, 1, 35) = A_1$, a contradiction. From our claim and the girth condition, it follows that $X(v_{p-1}) = 4, v_3 \neq v_{p-1}, v_{p-2}$. However, from our claim and the girth condition, it follows that $12345 \cap X(v_{p-2}) = \emptyset$, a contradiction.

Case 6: $X(v_p) = 45$. By the girth condition, $v_3 \neq v_{p-1}, 45 \cap X(v_{p-1}) = \emptyset$. First we claim that $X(v_{p-1}) \neq 3$. Otherwise, by the girth condition, $X(v_{p-2})$ is either 1 or 2 and P contains a subpattern isomorphic to the pseudobasic pattern $(12, 3, 4, 1, 3, 45) = A_3$, a contradiction. Thus we may assume $12 \cap X(v_{p-1}) \neq \emptyset$. Without loss of generality, assume $1 \in X(v_{p-1})$. Then it follows from the girth condition that $v_3 \neq v_{p-2}$ and $X(v_{p-2})$ is either 2 or 3. If $X(v_{p-2}) = 2$, then P contains $(45, 1, 2, 4, 3, 12)$ isomorphic to the pseudobasic pattern $(12, 3, 4, 1, 5, 34) = A_2$, a contradiction. If $X(v_{p-2}) = 3$, then P contains $(12, 4, 3, 1, 45)$ as a subpattern which is isomorphic to the pseudobasic pattern $(12, 3, 4, 1, 35) = A_1$, a contradiction.

This completes the proof and shows that we have a complete pseudobasic patterns for paths. $\square$

3.2.3 Tree Patterns

This subsection is dedicated to show that we have a complete list of pseudobasic tree patterns. First, we introduce the following notation for tree patterns with three leaves.

Let T be a tree pattern with three leaves. The tree can be separated into three edge disjoint paths, $P_1 = v - 1v_{1,2}...v, P_2 = v_{2,1}v2, 2...v, P_3 = v_{3,1}v_{3,2}...v$ all starting with a leaf and ending with a common vertex, v. Then, the following notation is used to represent this pattern $((X(v_1), X(v_{1,2}), ..., vX(v)), (X(v_{2,1}), X(v_{2,2}), ..., vX(v)), (X(v_{3,1}), X(v_{3,2}), ..., vX(v)))$.

Let $\mathcal{B}$ be the following tree patterns (See Figure 3.3).

- $B_1 = ((12, 4, v), (13, v), (23, 5, v))$
- $B_2 = ((12, v), (23, 5, v), (34, v))$
- $B_3 = ((12, 5, v), (23, v), (34, 1, v))$

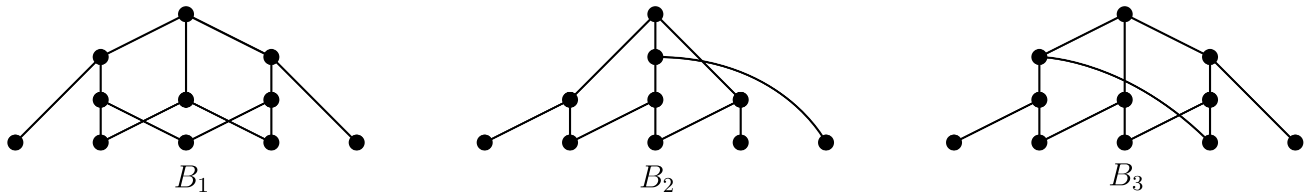


Figure 3.3: Pseudobasic Tree Patterns ($\mathcal{B}$)

Lemma 3.10. T is a pseudobasic tree pattern with three leaves if and only if T is isomorphic to a pattern in $\mathcal{B}$.

Proof: For the backward direction, it is easy to verify that all patterns in $\mathcal{B}$ are C5C tree patterns, are not basic patterns and contains only basic subpatterns. For the forward direction, assume by contradiction that T is a pseudobasic tree pattern not in $\mathcal{L}$. By Lemma 3.8, it follows that $X(v) = \emptyset, |X(v_{i,1})| = 2, |X(v_{i,j}| = 1$, for $i = 1, 2, 3$ and $j > 1$. We will now case on the possible values of $X(v_{i,1})$ for $i = 1, 2, 3$.

Case 1: $|X(v_{1,1}) \cup X(v_{2,1}) \cup X(v_{3,1})| = 3$

Without loss of generality,let $X(v_{1,1}) = 12, X(v_{2,1}) = 13, X(v_{3,1}) = 23$. By the cover condition, there exist vertices u, w where $X(u) = 4, X(w) = 5$. Note that $u \neq v \neq w$. Without loss of generality, assume that $u \in V(P_1)$. If $w \in V(P_1)$, then T contains the subpattern $(23, 1, 5, 4, 12)$, isomorphic to the pseudobasic pattern $(12, 3, 4, 5, 13) = A_4$, a contradiction. If $w \notin V(P_1)$, without loss of generality, assume $w \in V(P_3)$. Then T contains the pseudobasic subpattern $((12, 4, v), (13, v), (23, 5, v)) = B_1$, a contradiction.

Now we analyze when $|X(v_{1,1}) \cup X(v_{2,1}) \cup X(v_{3,1})| = 4$. There are two possible choices for the leaves, either they all share a common element, or they do not all share a common element.

Case 2: $|X(v_{1,1}) \cup X(v_{2,1}) \cup X(v_{3,1})| = 4$ and$X(v_{1,1}) \cap X(v_{2,1}) \cap X(v_{3,1}) \neq \emptyset$

Without loss of generality, let $X(v_{1,1}) = 14, X(v_{2,1}) = 24, X(v_{3,1}) = 34$.

Claim 1: there does not exist k, m, n, where $k = 1, 2, 3, m < n$ such that $5 \in X(v_{k,n}), (123\backslash k) \cap X(v_{k,m}) \neq \emptyset$. Suppose for the sake of contradiction that there exist k, m, n, where $k = 1, 2, 3, m < n$ such that $5 \in X(v_{k,n}), (123\backslash k) \cap X(v_{k,m}) \neq \emptyset$. Without loss of generality, assume $k = 1$ and $2 \in X(v_{1,m})$. Then T contains the subpattern $(14, 2, 5, 3, 12)$ isomorphic to the pseudobasic pattern $(12, 3, 4, 5, 13) = A_4$, a contradiction.

By the cover condition, there exists a vertex u such that $X(u) = 5$. Without loss of generality, assume $u = v_{1,j} \in V(P_1)$ and assume for all $1 < k < j, 5 \notin X(V_{1,k})$.

Claim 2: $j = 2$. Suppose for the sake of contradiction that $j \neq 2$. By our assumption, $5 \neq X(v_{1,2})$. By the girth condition, $1, 4 \neq X(v_{1,2})$. This implies that $X(v_{1,2}) = 2$ or 3, contradicting claim 1.

Now, we assume $v_{1,2} \neq v$ and $X(v_{1,2}) = 5$.

Claim 3: that there does not exist a vertex $w \in P_2 \cup P_3$ such that $1 \in X(w)$. Suppose not and without loss of generality, assume $w \in P_2$. Note that $w \neq v$. Then T contains the subpattern $(24, 1, 3, 5, 14)$ isomorphic to $(12, 3, 4, 5, 13) = A_4$, a contradiction.

Now, by the girth condition, we know that $|P_2| \geq 2$ or $|P_3| \geq 2$. Without loss of generality, suppose there exist $v_{2,2} \neq v$. By claim 3 and the girth condition, $X(v_{2,2}) = 5$ or $X(v_{2,2}) = 3$.

Subcase i: $X(v_{2,2}) = 5$. By the girth condition, there exists another vertex between u, v or $v_{2,2}, v$. Without loss of generality, assume there exist $v_{1,3} \neq v$. By symmetry from Claim 3, $2 \notin X(P_1), X(P_3)$. Then by the girth condition, it follows that $X(v_{1,3}) = 3$. Since $3 \in X(v_{1,3})$, by symmetry to Claim 3, $5 \notin X(v_{3,2})$. By the girth condition and symmetry to Claim 3, $1, 2, 3, 4 \notin X(v_{3,2})$ and thus $v_{3,2} = v$. By the girth condition, there exists $v_{1,4} \neq v$. Note that by the girth condition and symmetry to Claim 3, $235 \cap X(v_{1,4}) = \emptyset$. Note that $X(v_{1,4}) \neq 1$, otherwise T contains the subpattern $(24, 1, 3, 5, 14)$ which is isomorphic to the pseudobasic pattern $(12, 3, 4, 5, 13) = A_4$, a contradiction. So we may assume $X(v_{1,4}) = 4$. By the girth condition, there exists $v_{1,5} \neq v$. However, $1 \notin X(v_{1,5})$ for similar reason as before, $2 \notin X(v_{1,5})$ from a symmetric argument of Claim 3. $3, 4 \notin X(v_{1,5})$ by the girth condition, and $5 \notin X(v_{1,5})$ from Claim 1. This results in a contradiction.

Subcase ii: $X(v_{2,2}) = 3$. Note that by symmetry to the previous subcase, we may assume that $X(v_{3,2}) \neq 5$. It follows from Claim 1 that $5 \notin X(P_2), X(P_3)$. It follows from Claim 3 that $1 \notin X(P_2), X(P_3)$. Then, T violates the C5C condition with $S = v, F = T \backslash V(P_1)$, a contradiction.

Case 3: $|X(v_{1,1}) \cup X(v_{2,1}) \cup X(v_{3,1})| = 4$ and $X(v_{1,1}) \cap X(v_{2,1}) \cap X(v_{3,1}) = \emptyset$

Without loss of generality, let $X(v_{1,1}) = 12, X(v_{2,1}) = 23, X(v_{3,1}) = 34$. By the cover condition, there exists a vertex, u, where $X(u) = 5$. Note that $5 \notin X(P_2)$ so $u \notin V(P_2)$. Otherwise T contains the pseudobasic subpattern $((12, v), (23, 5, v), (34, v)) = B_2$, a contradiction. Without loss of generality, assume that $u = v_{1,i} \in V(P_1)$ and for all $1 < j < i, 5 \notin X(v_{1,j})$.

Claim 1: $u = v_{1,2}$. Suppose for the purpose of contradiction that $v_{1,2} \neq u$. By the girth condition, $X(v_{1,2}) = 3$ or 4. However, if $X(v_{1,2}) = 3$, then T contains the subpattern $(12, 3, 5, 4, 23)$, isomorphic to $(12, 3, 4, 5, 13) = A_4$ and if $X(v_{1,2}) = 4$, then T contains the subpattern $(12, 4, 5, 2, 34)$ isomorphic to $(12, 3, 4, 1, 35) = A_1$, both result in a contradiction.

Now we may assume that $5 = X(v_{1,2})$. Claim 2: $1 \notin X(P_2)$. Otherwise T contains the subpattern $(23, 1, 4, 5, 12)$ isomorphic to $(12, 3, 4, 5, 13) = A_4$, a contradiction.

Claim 3: $v \neq v_{2,2}$. Suppose for the purpose of contradiction that $v = v_{2,2}$. By the girth condition, $v \neq v_{3,2}$. Note that $X(v_{3,2}) \neq 1, X(v_{3,2}) \neq 2$. Otherwise, T contains the subpattern $(12, 5, 4, 1, 34)$ or $(12, 5, 4, 2, 34)$, both isomorphic to $(12, 3, 4, 1, 45) = A_1$, a contradiction. It follows then $X(v_{3,2}) = 5$. By the girth condition, $v \neq v_{1,3}$ or $v \neq v_{3,3}$. However, for similar reasons as before, $12345 \cap X(v_{3,3}) = \emptyset$ and by symmetry, $12345 \cap X(v_{1,3}) = \emptyset$, a contradiction.

Then, it follows from Claim 2, Claim 3, the girth condition and our assumption of $5 \notin X(P_2)$ that $X(v_{2,2}) = 4$. By the girth condition, our assumption and Claim 2, it follows that $v_{2,3} = v$. By the girth condition, there exists $v_{3,2} \neq v$ and $X(v_{3,2}) \neq 3, 4$. Note that $X(v_{3,2}) \neq 5$ otherwise T contains $(23, 4, 1, 5, 34)$ isomorphic to $(12, 3, 4, 5, 13) = A_4$, a contradiction. Note that $X(v_{3,2}) \neq 1$ otherwise T contains $(23, 4, 5, 1, 34)$ isomorphic to $(12, 3, 4, 5, 13) = A_4$, a contradiction. Then $X(v_{3,2}) = 3$.. For similar reasons as before, $12345 \cap X(v_{3,3}) = \emptyset$. This implies that $v = v_{3,3}$. However, this implies that T violates the C5C condition with $S = v, F = T \backslash V(P_1)$, a contradiction.

Case 4: $|X(v_{1,1}) \cup X(v_{2,1}) \cup X(v_{3,1})| = 5$

Without loss of generality, let $X(v_{1,1}) = 12, X(v_{2,1}) = 23, X(v_{3,1}) = 45$. By the C5C condition, there exists $u \in V(P_1) \cup V(P_2)$ such that $45 \cap X(u) \neq \emptyset$. Without loss of generality, assume $u = v_{1,i} \in P_1$, $X(u) = 4$ and for all $1 < j < i, 45 \cap X(v_{1,j}) = \emptyset$.

Claim 1: $u = v_{1,2}$. Suppose for the sake of contradiction that $v_{1,2} \neq u$ so $X(v_{1,2}) \neq 4, 5$. By the girth condition, $X(v_{1,2}) \neq 1, 2$. Then $X(v_{1,2}) = 3$ and T contains the subpattern $(12, 3, 4, 5, 23)$ isomorphic to $(12, 3, 4, 5, 13) = A_4$, a contradiction.

Claim 2: $v = v_{3,2}$. Suppose for the purpose of contradiction that $v_{3,2} \neq v$. By the girth condition, $X(v_{3,2}) \neq 4, 5$. Note that $X(v_{3,2}) \neq 1, 2$ or else T contains the subpattern $(45, 1, 3, 4, 12)$ or $(45, 2, 3, 4, 12)$, both isomorphic to the pseudobasic pattern $(12, 3, 4, 1, 35) = A_1$, a contradiction. Then it follows that $X(v_{3,2}) = 3$. For similar reasons, $12345 \cap X(v_{3,3}) = \emptyset$ and thus $v = v_{3,3}$. By the girth condition, it follows that there exists

$v_{2,2} \neq v$ and $X(v_{2,2}) \neq 2, 3$. Note that if $X(v_{2,2}) = 4$ or $X(v_{2,2}) = 5$, then T contains the subpattern $(23, 4, 1, 3, 45)$ or $(23, 5, 1, 3, 45)$, both isomorphic to the pseudobasic pattern $(12, 3, 4, 1, 35) = A_1$, a contradiction. Then, it follows that $X(v_{2,2}) = 1$. However, then T contains the subpattern $(23, 1, 5, 4, 12)$, isomorphic to the pseudobasic subpattern $(12, 3, 4, 5, 13) = A_4$, a contradiction.

Now, it follows from the girth condition and Claim 2 that there exists $v_{1,3} \neq v$. By the girth condition, $X(v_{1,3}) \neq 1, 2, 4$. If $X(v_{1,3}) = 3$, then T contains the subpattern $(12, 4, 3, 2, 45)$ isomorphic to the pseudobasic pattern $(12, 3, 4, 1, 35) = A_1$, a contradiction. So it follows that $X(v_{1,3}) = 5$. By the girth condition and Claim 2, there exists $v_{1,4} \neq v$ and $X(v_{1,4}) \neq 4, 5$. For the same reason as before, $X(v_{1,4}) \neq 3$. This implies $X(v_{1,4}) = 1$ or $X(v_{1,4}) = 2$. Then T contains either the subpattern $(12, 4, 5, 1, 3, 45)$ or the subpattern $(12, 4, 5, 2, 3, 45)$, both isomorphic to $(12, 3, 4, 1, 5, 34) = A_2$, a contradiction.

This completes the proof and shows that B_1, B_2, B_3 are the only pseudobasic tree patterns with three leaves. $\square$

Lemma 3.11. Let T be a C5C tree pattern with more than four leaves. Then T is not a pseudobasic pattern.

Proof: Assume by contradiction that T is a C5C tree pattern with more than four leaves and T is a pseudobasic pattern. From Lemma 3.8, it follows that T only has vertices of degree 3 or less and $X(v) = \emptyset$ if $d(v) = 3$. Let $u, w, u_1, u_2, w_1, w_2 \in V(T)$ where u_1, u_2, w_1, w_2 are leaves and $d(u) = d(w) = 3$, for $i = 1, 2$, let P_i denote the path from u_i to u and Q_i denote the path from w_i to w, such that for $i = 1, 2$, all vertices in P_i, Q_i that are not endpoints have degree 2 in T. Then consider the patch T' obtained by removing all vertices in $T \backslash (V(P_1) \cup V(P_2) \cup V(Q_1) \cup V(Q_2))$ and adding an edge between u and w. Note that T' satisfies the girth condition and the min-degree condition. Since T' has four leaves, by pigeonhole principle, it follows that there exists two leaves $x, y \in V(T')$ such that $X(x) \cap X(y) \neq \emptyset$ and thus T' satisfies the cycle condition.

If T' satisfies the cover condition, then T' is a subpattern of T. Then, by Lemma 3.7, T' satisfies the C5C condition and thus is a pattern. Since T' has four leaves, T' is not a basic pattern which implies T contains a pseudobasic subpattern, a contradiction.

If T' violates the cover condition, then $X(T') \neq 12345$. Since T' has four leaves, by pigeonhole principle, $5 - |X(T[])| = 1$. Without loss of generality, assume $1 \notin X(T')$. Denote T^* as the patch obtained by deleting all vertices in $V(P_1) \backslash u$ and adding 1 to $X(u)$. Note that T^* is a patch of T and T^* does not violate the min-degree conditions. Since 1 is only in $X(u), T^*$ also satisfies the girth condition. Since T^* has three leaves, by pigeonhole principle, T^* satisfies the cycle condition. We claim that T^* satisfies the cover condition. For the sake of contradiction, assume T^* violates the cover condition. Since T^* has three leaves and $1 \notin X(v)$ for all $v \neq u$, then $|\cup_{v \in V(T^*)} X(v)| = 4$. Without loss of generality, assume $2 \notin \cup_{v \in V(T^*)} X(v)$. Note that this implies $X(w_1) \cap X(w_2) \neq \emptyset$. Then, T violates the C5C condition with $S = w, F = T|(V(Q_1) \cup V(Q_2) \backslash w)$, a contradiction. Then, it follows that T^* is a pattern and by Lemma 3.7, T^* does not violate the C5C condition. Since T^* has three leaves, it is not a basic pattern. This implies that T has a pseudobasic subpattern, a contradiction. $\square$

Remark 3.12. Let T be a tree pattern with four leaves. Let u, w be the vertices of degree 3 in T. Let P be the path from u to w. If P is not an edge, then the same arguments can be made to show that T is not a pseudobasic subpattern.

Lemma 3.13. Let T be a tree pattern with at least four leaves. Then T is not a pseudobasic pattern.

Proof: Assume for the purpose of contradiction that T is a pseudobasic pattern. from the previous lemma and remark, we may assume that T has four leaves and the two degree three vertices are adjacent to each other.

For $i \in \{1, 2\}$ let P_i, Q_i be paths from u_i to u and w_i to w respectively where u_i, w_i are leaves and u, w are the two degree 3 vertices of T and $V(P_i) \cap V(Q_i) = \emptyset$. . We will now case on the possible values of $X(u_i), X(w_i)$ for $i \in \{1, 2\}$. Note that by Lemma 3.8, $|X(u_i)|, |X(w_i)| = 2$.

Case 1: $X(u_1) \cap X(u_2) \neq \emptyset$ or $X(v_1) \cap X(v_2) \neq \emptyset$

Without loss of generality, assume that $X(u_1) \cap X(u_2) \neq \emptyset, X(u_1) = 12, X(u_2) = 23$.

Subcase 1.1: there exists $i \in \{1, 2\}$ such that $X(w_i) \subseteq X(u_1) \cup X(u_2)$

Without loss of generality, assume that $X(w_1) = 13$. It follows from Lemma 3.7 that there exists a vertex $v \in P_i$ for some $i = 1, 2$ such that $X(v) \cap 45 \neq \emptyset$. Without loss of generality, assume $X(v) = 4, v \in P_1$. By the cover condition, there exists a vertex z such that $5 \in X(z)$. If $z \in P_1$, then T contains subpattern $(23, 1, 5, 4, 12)$ or $(23, 1, 4, 5, 12)$, isomorphic to the pseudobasic pattern $(12, 3, 4, 5, 13) = A_4$, a contradiction. If $z \in P_2$, then T contains the subpattern $((12, 4, v), (13, v), (23, 5, v)) = B_1$, a contradiction. If $z \in Q_i$ for $i = 1, 2$, then T contains the subpattern $((12, 4, vv), (23, v), (13, 5, v))$ isomorphic to $((12, 4, v), (13, v), (23, 5, v)) = B_1$, a contradiction.

Subcase 1.2: For $i = 1, 2, |X(w_i) \cap (X(u_1) \cup X(u_2))| \leq 1$ and there exists $i = 1, 2$ such that $X(u_1) \cap X(u_2) \cap X(w_i) = \emptyset$,.

Subcase 1.2.1: $X(w_2) = 34$.

By the cover condition, there exists a vertex z such that $5 \in X(z)$. Note that $z \notin P_2$. Otherwise T contains the pseudobasic subpattern $((12, v), (23, 5, v), (34, v)) = B_2$, a contradiction.

We claim that $z \notin P_1$. Suppose for the sake of contradiction that $z \in P_1$. We will show that all possible values for $X(w_2)$ will result in a contradiction. By the girth condition, $X(w_2) \neq 12, 23, 34$. By subcase 1.1, $X(w_2) \neq 13$. By symmetry of subcase 1.1, $X(w_2) \neq 24$. If $X(w_2) = 14$, then T contains the subpattern $((14, v), (12, 5, v), (23, v))$ isomorphic to $((12, v), (23, 5, v), (34, v)) = B_2$, a contradiction. If $X(w_2) = 15$ or $X(w_2) = 25$, then T contains the subpattern $(12, 5, 3, 4, 15)$ or $(12, 5, 3, 4, 25)$ isomorphic to $(12, 3, 4, 5, 13) = A_4$, a contradiction. If $X(w_2) = 35$, it follows from Lemma 3.7, there exists a vertex $z' \in Q_i$ for some $i \in \{1, 2\}$ such that $X(z') \cap 12 \neq \emptyset$. If $z' \in Q_1, 1 \in X(z')$, then T contains the pseudobasic subpattern $((12, 5, v), (23, v), (34, 1, v)) = B_3$, a contradiction. If $z' \in Q_1, 2 \in X(z')$, then T contains $(34, 2, 5, 1, 23)$ isomorphic to $(12, 3, 4, 5, 13) = A_3$, a contradiction. If $z' \in Q_2, 1 \in X(z')$ or $2 \in X(z')$, then T contains $(35, 1, 4, 5, 12)$ or $(35, 2, 4, 5, 12)$, isomorphic to the pseudobasic subpattern $(12, 3, 4, 1, 35) = A_1$, a contradiction. Now, lastly, if $X(w_2) = 45$, it follows from Lemma 3.7, there exists $z' \in Q_i$ for some $i \in \{1, 2\}$ such that $X(z') \cap 12 \neq \emptyset$. For similar reasons as before, $z' \notin Q_1$. If $z' \in Q_2, 1 \in X(z')$ or $2 \in X(z')$, then T contains $(45, 1, 3, 5, 12)$ or $(45, 2, 3, 5, 12)$, isomorphic to the pseudobasic subpattern $(12, 3, 4, 1, 35) = A_1$, a contradiction. So this proves the claim and $z \notin P_i$ for $i = 1, 2$.

Now, we may assume that $z \in Q_i$ for some $i = 1, 2$. By Lemma 3.7, there exists a vertex $z' \in P_j$ for some $j \in \{1, 2\}$ such that $4 \in X(z')$. Whether $z \in Q_1$ or $z \in Q_2$, if $z' \in P_1$, then T contains the subpattern $((34, 5, v), (23, v), (12, 4, v))$ isomorphic to $((12, 5, v), (23, v), (34, 1, v)) = B_3$, a contradiction. Whether $z \in Q_1$ or $z \in Q_2$, if $z' \in P_2$, then T contains the subpattern $(23, 4, 1, 5, 34)$ isomorphic to $(12, 3, 4, 5, 13) = A_4$, a contradiction. This completes all possible cases for subcase 1.2.1.

Note by subcase 1.1, for $i = 1, 2$, $X(w_{i,2}) \neq 13$. Then, by symmetry to subcase 1.2.1, we may assume that for $i = 1, 2$, $13 \cap X(w_i) = \emptyset$.

Subcase 1.2.2: for $i = 1, 2$, $13 \cap X(w_i) = \emptyset$.

By assumption, there exists some $i \in \{1, 2\}$ such that $123 \cap X(w_i) = \emptyset$; it follows there exists $i = 1, 2$ such that $X(w_i) = 45$. Without loss of generality, assume $X(w_1) = 45$.

By the girth condition, $X(w_2) \neq 45$. Since $1, 3 \notin X(w_2)$, without loss of generality, we may assume $X(w_2) = 24$. Note that $X(u_1) = 12, X(u_2) = 23, X(w_1) = 45, X(w_2) = 24$ is symmetric to $X(w_1) = 34, X(w_2) = 35, X(u_1) = 12, X(u_2) = 23$ which is another instance of subcase 1.2.1.

Subcase 1.3: $X(u_1) \cap X(u_2) \cap X(w_1) \cap X(w_2) \neq \emptyset$

Without loss of generality, let $X(w_1) = 24, X(w_2) = 25$. It follows from Lemma 3.7 that there exists a vertex $z \in P_i$ for some $i \in \{1, 2\}$ such that $X(z) \cap 45 \neq \emptyset$. Without loss of generality, assume $z \in P_1, X(z) = 4$. Then T contains the subpattern $(12, 4, 3, 5, 24)$ isomorphic to the pseudobasic pattern $(12, 3, 4, 5, 13) = A_4$ a contradiction.

By symmetry, we can now assume that $X(u_1) \cap X(u_2) = \emptyset, X(w_1) \cap X(w_2) = \emptyset$.

Case 2: $X(u_1) \cap X(u_2) = \emptyset, X(w_1) \cap X(w_2) = \emptyset$

Subcase 2.1: $|X(u_1) \cup X(u_2) \cup X(w_1) \cup X(w_2)| = 4$

Without loss of generality, we may assume that $X(u_1) = 12, X(u_2) = 34, X(w_1) = 23, X(w_2) = 14$. By the cover condition, there exists a vertex z such that $X(z) = 5$. Without loss of generality, assume $z \in P_1$. Then, T contains the subpattern $((23, v), (12, 5, v), (14, v))$ isomorphic to $((12, v), (23, 5, v), (34, v)) = B_2$, a contradiction.

Subcase 2.2: $|X(u_1) \cup X(u_2) \cup X(w_1) \cup X(w_2)| = 5$

Without loss of generality, $X(u_1) = 12, X(u_2) = 34, X(w_1) = 23, X(w_2) = 45$. Then T contains the pseudobasic subpattern $((12, v), (23, 5, v), (34, v)) = B_2$, a contradiction. □

Remark 3.14. Note that Lemma 3.10 - 3.13 implies that B_1, B_2 and B_3 are the only pseudobasic tree patterns.

3.2.4 Forest Patterns

First, we define the following as *trivial* patches (see Figure 2.1).

- $S3$ = a single vertex v where $|X(v)| = 3$,

- $S4$ = a single vertex v where $|X(v)| = 4$,
- $S5$ = a single vertex v where $|X(v)| = 5$,
- $T4$ = an edge u, v where $|X(u)| = |X(v)| = 2, X(u) \cap X(v) = \emptyset$,
- $T5$ = a path uvw where $|X(u)| = |X(w)| = 2, |X(v)| = 1, X(u) \cap X(v), X(u) \cap X(w), X(v) \cap X(w) = \emptyset$,
- $T5'$ = an edge u, v where $|X(u)| = 2, |X(v)| = 3, X(u) \cap X(v) = \emptyset$,

Proposition 3.15. Let T be a connected patch that satisfies the min-degree and girth conditions. Then T is a trivial patch if and only if T does not satisfy the cycle condition.

Proof: Note that the forward direction is trivial. For the backward direction, by pigeonhole principle, T has at most two leaves. This implies T is a path. By pigeonhole, T has at most three vertices. Then, by inspection, one can check that T is a trivial patch. □

Let F be a forest pattern with k components $T_1, ..., T_k$. Then we write $F = T_1 + T_2 + ... + T_k$.

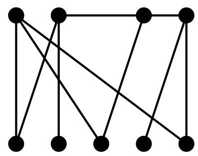
Proposition 3.16. If $F = T_1 + T_2 + ... + T_k$ is a forest C5C pattern, then for $1 \leq i \leq k$, either T_i is a trivial patch or is a C5C pattern.

Proof: Suppose for the purpose of contradiction that there exists $1 \leq i \leq k$ such that T_i is not a trivial patch nor a C5C pattern. It follows that T_i satisfies the cycle condition. By definition, T_i satisfies the min-degree and girth conditions. Note that T_i also satisfies the C5C condition, otherwise, F is not a C5C pattern. Since T_i is not a C5C pattern, T_i violates the cover condition. However, this implies that F violates the C5C condition with $S = \emptyset, F_1 = T_i, F_2 = \emptyset$, a contradiction. □

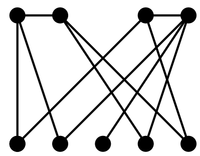
Definition: Let T_i for $1 \leq i \leq k$ be patches where T_i is a path pattern, or a tree pattern or a trivial patch. We define $T_1 \times T_2 \times ... \times T_k = \{F : F$ is a forest pattern and $F = T_1 + T_2 + ... + T_k\}$.

Define:

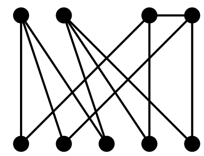
- $D_{3,A} = S3 \times A_0$,



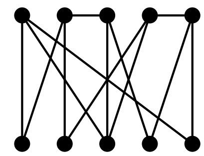
$D_{3,5}$



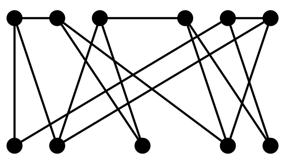
$D_{4,5'}$



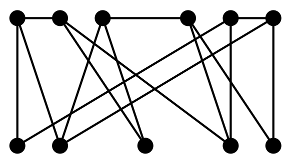
$D_{3,3,4}$



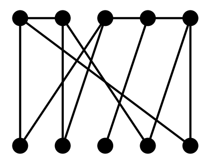
$D_{3,4,4}$



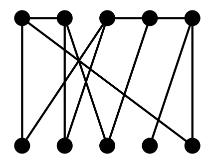
$D^1_{4,4,4}$



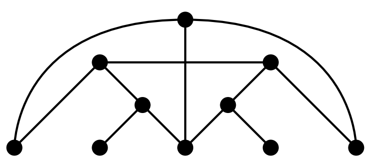
$D^2_{4,4,4}$



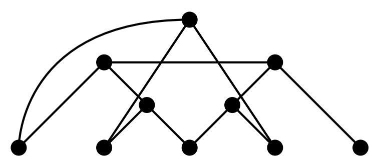
$D^1_{4,5}$



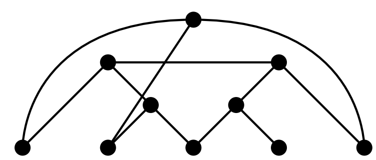
$D^2_{4,5}$



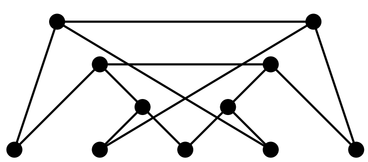
$D^1_{3,A}$



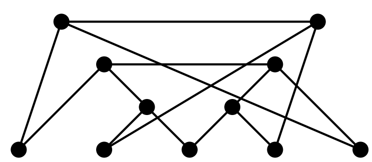
$D^2_{3,A}$



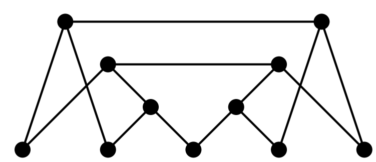
$D^3_{3,A}$



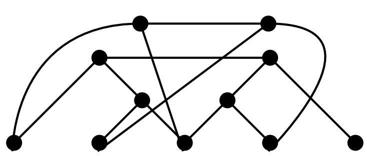
$D^1_{4,A}$



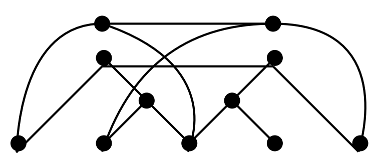
$D^2_{4,A}$



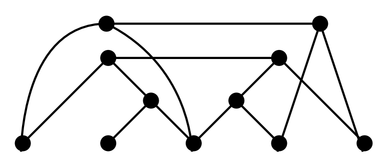
$D^3_{4,A}$



$D^4_{4,A}$



$D^5_{4,A}$



$D^6_{4,A}$

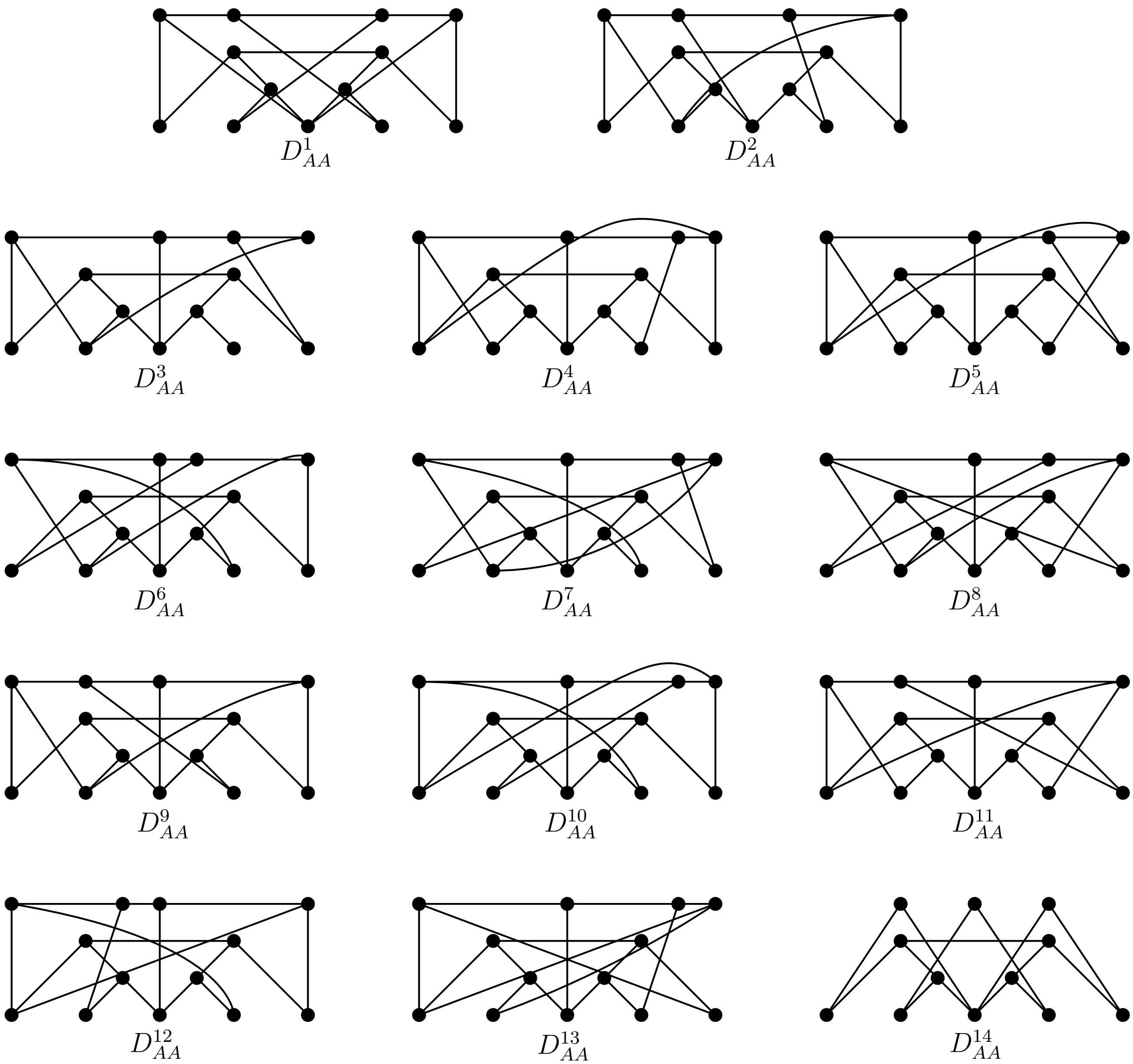


Figure 3.4: Basic and Pseudobasic Forest Patterns ($\mathcal{D}$)

- $D_{4,A} = T4 \times A_0$,
- $D_{A,A} = A_0 \times A_0$,
- $D_{3,5} = S3 \times T5$,
- $D_{4,5} = T4 \times T5$,
- $D_{4,5'} = T4 \times T5'$,
- $D_{3,3,4} = S3 \times S3 \times T4$,
- $D_{3,4,4} = S3 \times T4 \times T4$,
- $D_{4,4,4} = T4 \times T4 \times T4$.

Define $\mathcal{D}$ to be the union of the above sets (See Figure 3.3).

Note that $|D_{3,5}|, |D_{4,5'}|, |D_{3,3,4}|, |D_{3,4,4}| = 1$. Abusing the notation, we will also use them to represent the actual forest. Namely, up to isomorphism,

- $D_{3,5} = (135) + (12, 3, 45)$,
- $D_{4,5'} = (12, 45) + (15, 234)$.
- $D_{3,3,4} = (123) + (345) + (14, 25)$,
- $D_{3,4,4} = (135) + (12, 34) + (23, 45)$,

Also note that $|D_{3,A}| = 3, |D_{4,A}| = 6, |D_{A,A}| = 14, |D_{4,5}| = 2, |D_{4,4,4}| = 2$.

Lemma 3.17. Let F be a forest C5C pattern. Then F is a pseudobasic pattern if and only if $F \in \mathcal{D}$.

Proof: For the backward direction, first note that all patterns in $\mathcal{D}$ satisfy the min-degree, cycle, girth, cover and C5C conditions. Let $F \in \mathcal{D} \backslash D_{4,5'}$. Note that performing subpattern operations on F either violates the min-degree condition or results in complete removal of a component of F. However, if we remove any of the components of F, F will violate the cycle condition or violate the cover condition or result in a basic pattern. If $F = D_{4,5'}$, then any subpattern operation will result in a trivial patch or a basic pattern Note also that no patterns in $\mathcal{D}$ is a basic pattern. Therefore all patterns in $\mathcal{D}$ are pseudobasic patterns.

For the forward direction, let $F = F_1 + F_2 + ... + F_k$ be a forest C5C pseudobasic pattern. First, note that for all $1 \leq i \leq k$, if F_i is a C5C pattern, then F_i is A_0. Otherwise, F_i is a subpattern of F, contradicting the fact that F only has basic subpatterns.

Then, it follows from the previous proposition that all components, F_i are either trivial patches or A_0. For the sake of contradiction, assume $F \notin \mathcal{D}$. We will check all possible combinations. Note that since F is a pseudobasic pattern, it follows that $k \geq 2$. Note that no component of F is isomorphic to $S5$. Otherwise, any leaf in another component along with the vertex in $S5$ violates the girth condition.

Case 1: There exists $1 \leq i \leq k$ such that F_i is isomorphic to A_0

If there exists $1 \leq j \leq k, j \neq i$ such that F_j is isomorphic to A_0, then $F_i + F_j \in D_{A,A}$ is a pseudobasic subpattern of F, a contradiction. If there exists $1 \leq j \leq k, j \neq i$ such that F_j is isomorphic to $S3$, or $S4$ or $T5'$, by deleting vertices and labels, F contains a subpattern isomorphic to an element of $S3 \cup A_0 = D_{3,A}$, a contradiction. If there exists $1 \leq j \leq k, j \neq i$ such that F_j is isomorphic to $T4$, then $F_i + F_j \in D_{4,A}$ is a pseudobasic subpattern of F, a contradiction. If there exists $1 \leq j \leq k, j \neq i$ such that F_j is isomorphic to $T5$, then $F_i + F_j \in D_{5,A}$ is a pseudobasic subpattern of F, a contradiction.

Case 2: No components of F is isomorphic to A_0 and there exists $1 \leq i \leq k$ such that F_i is isomorphic to $T5$.

If there exists $1 \leq j \leq k, j \neq i$ such that F_j is isomorphic to $S3$ or $S4$ or $T5'$, then F contains a subpattern isomorphic to $S3 \cup T5 = D_{3,5}$, a contradiction. If there exists $1 \leq j \leq k, j \neq i$ such that F_j is isomorphic to $T4$, then F contains a subpattern isomorphic to $T4 \cup T5 = D_{4,5}$, a contradiction. If there exists $1 \leq j \leq k, j \neq i$ such that F_j is isomorphic to $T5$, then consider the trivial patch F'_j obtained by contracting an edge in F_j and keeping the labels of the leaves. Then $F'_j + F_i = T4 + T5 = D_{4,5}$ is a pseudobasic subpattern of F, a contradiction.

Case 3: No component of F is isomorphic to A_0 or $T5$ and there exists $1 \leq i \leq k$ such that F_i is isomorphic to $T5'$.

Note that there does not exist $1 \leq j \leq k$ where $j \neq i$ such that $F_j = S3, S4, T5'$. Otherwise, a leaf in F_i and a leaf in F_j violates the girth condition. If there exists $1 \leq j \leq k$ such that $F_j = T4$, then $F_i + F_j = T4 + T5' = D_{4,5'}$ is a pseudobasic subpattern of F, a contradiction.

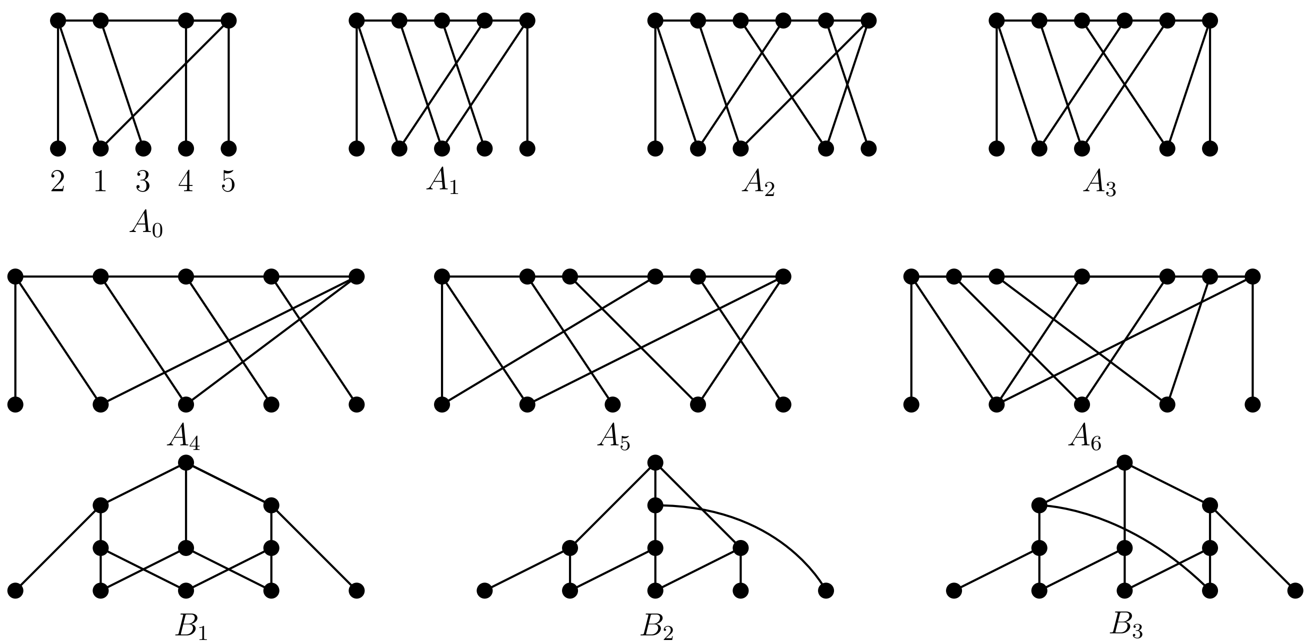
Case 4: For all $1 \le i \le k$, F_i is isomorphic $S3$ or $S4$ or $T4$.

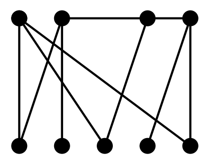
As before, by the girth condition, for all $1 \le i \le k$, F_i not isomorphic to $S4$. If $k = 2$, by the cover, girth and cycle condition, one can check that F is either $(123) + (14, 25)$ or $(12, 34) + (23, 45)$ which are both basic patterns, a contradiction. Now, assume $k \ge 3$, then by girth condition, there exist at most two components of F isomorphic to $S3$. If there exist two components isomorphic to $S3$, then F contains a pseudobasic subgraph isomorphic to $D_{3,3,4}$, a contradiction. If there exists exactly one component isomorphic to $S3$, then F contains a pseudobasic subgraph isomorphic to an element of $D_{3,4,4}$. If no components of F is isomorphic to $S3$, then F contains a subgraph isomorphic to an element of $D_{4,4,4}$, a contradiction.

Therefore, $\mathcal{D}$ contains the entire list of pseudobasic tree patterns. $\square$

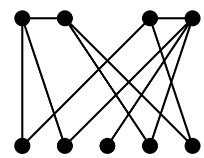
Note that the list $\mathcal{L}$ is simply the union of C_J, the basic patterns and pseudobasic patterns. Thus Lemma 3.5 follows from Lemma 3.9-3.17.

We also like to point out that Lemma 3.1 requires the $d_B(x) \ge 2$ for all $x \in X$ where X is a mixed cut with cyclic side B. However, this might not always be the case. In the next chapter, we will discuss exactly when we are allowed to apply Lemma 3.1 and what happens if we are not.

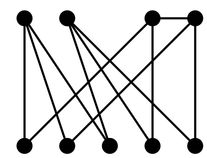

2 1 3 4 5
A_0
A_1
A_2
A_3
A_4
A_5
A_6
B_1
B_2
B_3



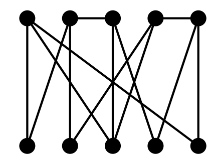
$D_{3,5}$



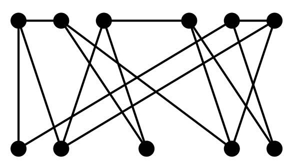
$D_{4,5'}$



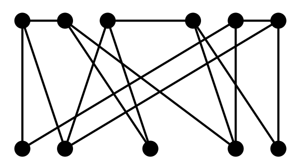
$D_{3,3,4}$



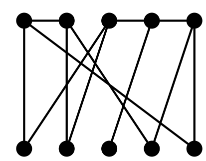
$D_{3,4,4}$



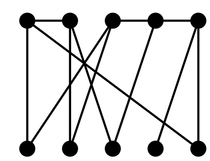
$D^1_{4,4,4}$



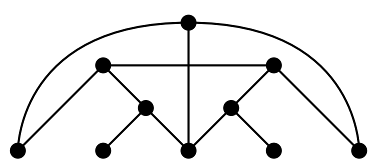
$D^2_{4,4,4}$



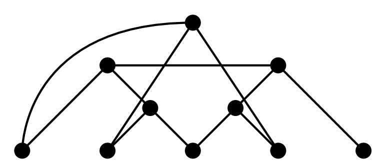
$D^1_{4,5}$



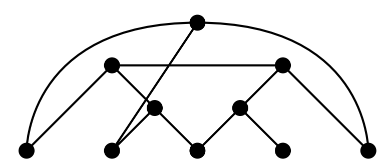
$D^2_{4,5}$



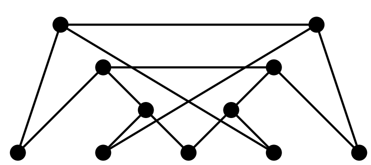
$D^1_{3,A}$



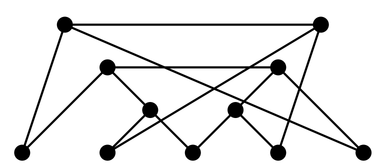
$D^2_{3,A}$



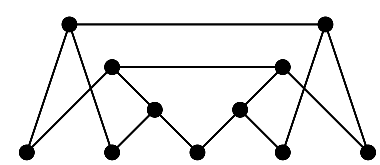
$D^3_{3,A}$



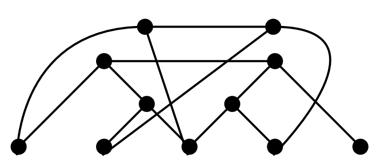
$D^1_{4,A}$



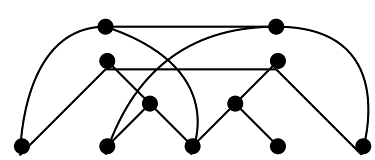
$D^2_{4,A}$



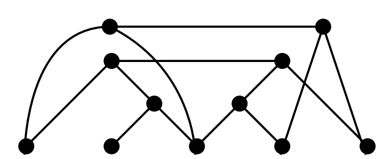
$D^3_{4,A}$



$D^4_{4,A}$



$D^5_{4,A}$



$D^6_{4,A}$

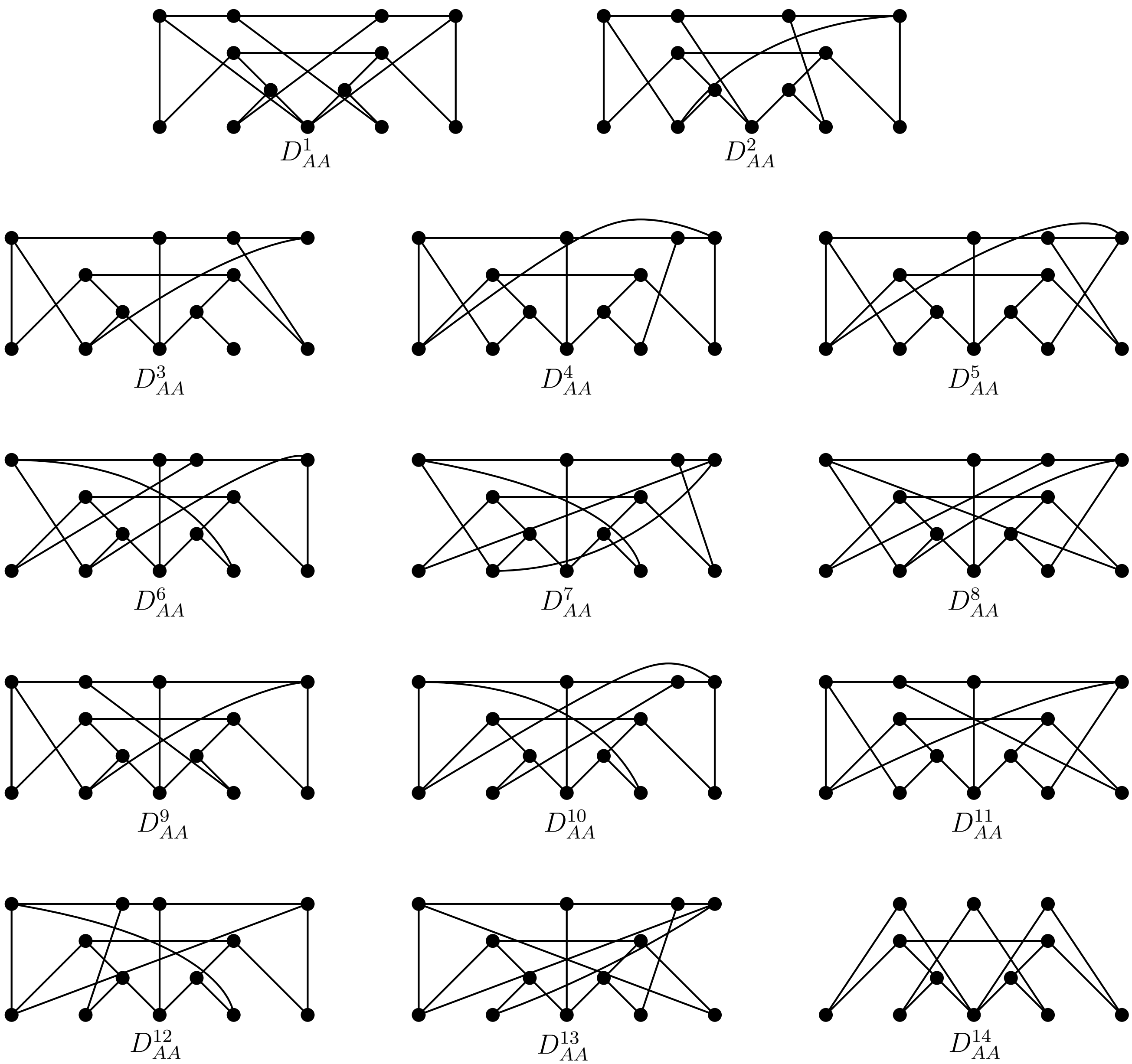


Figure 3.5: The Complete List $\mathcal{L}$

Chapter 4

Push-Consistent Graphs

As a reminder, we will once again define the notion of pushing and push-consistent.

Definition: Let G be a mixed C5C graph with mixed cut X, acyclic and cyclic sides A, B respectively. Suppose for some $x \in X, d_B(x) = 1$. Let x' be the neighbour of x in B. Note that since G is a mixed C5C graph, it follows that $(X \backslash x) \cup x'$ is also a mixed cyclic 5-cut. Let A', B' be subgraphs of G where $V(A') = V(A) \cup x', E(A') = E(A) \cup xx', V(B') = V(B) \backslash x', E(B') = E(B) \backslash xx'$. If $A' \backslash X$ contains no cycles and $B' \backslash X$ still contains a cycle, then we say the mixed cut X is *pushable* along the edge xx' with respect to A, B, X. If in a mixed C5C graph G, for all A, B, X, x and x', the cut X is pushable along xx' with respect to A, B, X, then we call G a *push-consistent* C5C graph. We say X is *non-pushable* if there does not exist an edge xx' for X to push along.

The point of defining push-consistent graphs is so that we can apply Lemma 3.1. Consider a mixed C5C graph G with mixed cut X, cyclic and cyclic sides A, B. If there exists $x \in X$ such that $d_B(x) = 1$, then we cannot apply Lemma 3.1. However, if the edge incident to x in B is pushable, then, we can push along that edge and attempt to apply Lemma 3.1 to the newly obtained cut. Thus, assuming all cuts are pushable gives us more structure to work with. We will briefly discuss the case when a graph is not push-consistent at the end of this chapter.

First, observe the following.

Proposition 4.1. In a push-consistent graph, a mixed cut X is not pushable if and only if $d_B(x) \geq 2$ for all $x \in X$ and all cyclic side B with respect to X.

Proof: This follows from the definition of pushable and push-consistent. $\square$

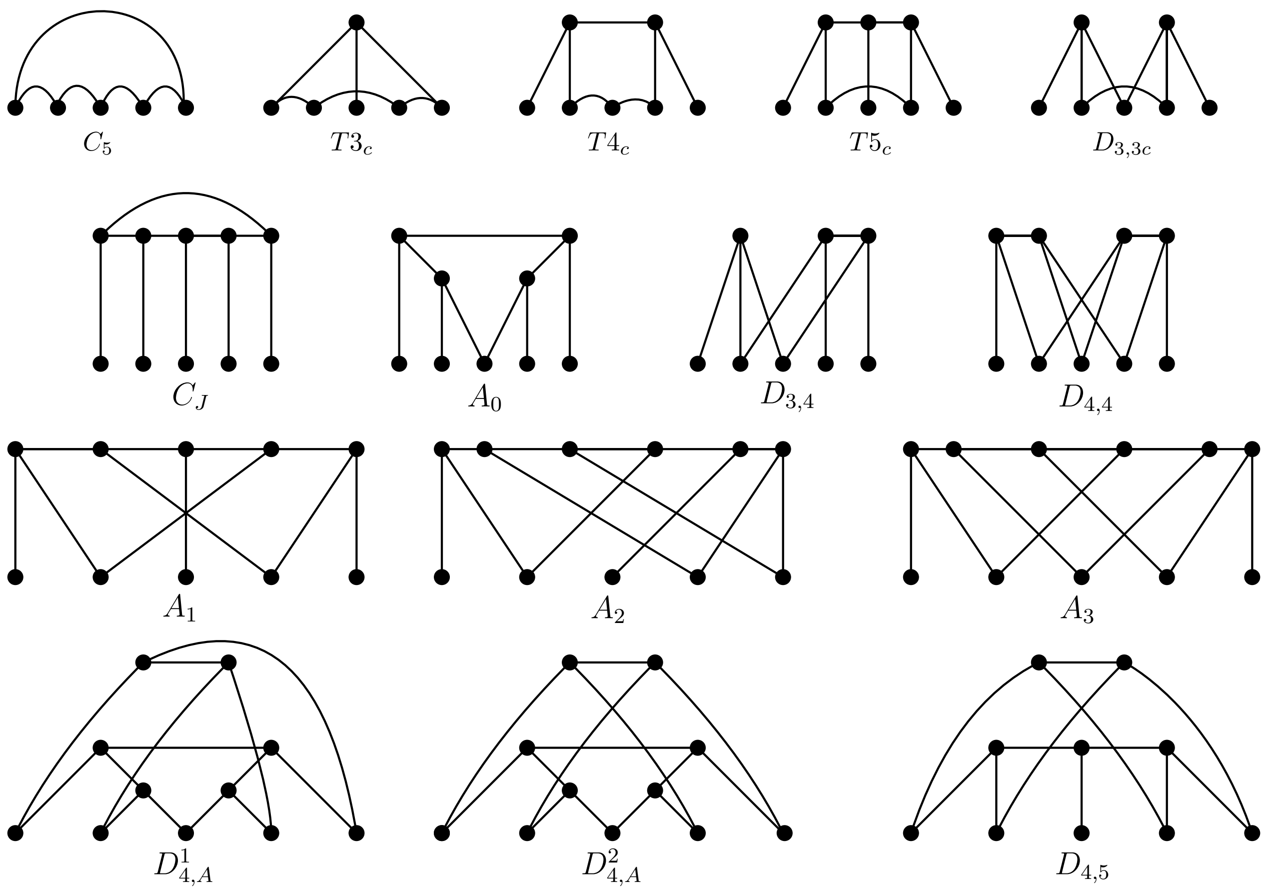


Figure 4.1: The List $\mathcal{L}'$

Definition: Let G be a mixed C5C graph. If for all G', X, A, B where X is a mixed cut with acyclic and cyclic side A, B respectively and G' is a proper minor of G obtained by performing subpattern operations on A, G' is not C5C, then we say G is *acyclically minimal*.

Our goal of this section is to prove the following lemma about push-consistent graphs.

Theorem 4.2. *If G be an acyclically minimal mixed C5C graph and X is a non-pushable mixed cut, then for all acyclic sides A with respect to X, A is isomorphic to $Gr(F)$ where F is a pattern in $\mathcal{L}'$ (See Figure 4.1).*

Note that Theorem 4.2 implies Theorem 1.7.

4.1 Non-Independent Mixed Cuts

Another problem Lemma 3.1 runs into is when X is not an independent set. Let e be an edge with both ends in X. Since the cyclic side contains a cycle disjoint from X, the cycle is also disjoint from e. Then, we may assume that e can be thrown into the acyclic side. Then, the idea is to push the cut X, if possible, otherwise apply Lemma 3.1 to remove the edge e. However, we run into problems if the resulting acyclic side after removing e becomes a basic pattern. This section deals with this problem.

First, note the following.

Proposition 4.3. Let G be an acyclically minimal push-consistent mixed C5C graph. Let X be a non-pushable mixed cut with an acyclic side A. If X is independent, then A is isomorphic to $Gr(F)$ where $F \in \mathcal{L}$.

Proof: From Proposition 4.1, we know that $d_B(x) \geq 2$ for all $x \in X$. For the sake of contradiction, assume $F \notin \mathcal{L}$. It follows from Lemma 3.5 and Proposition 3.4 that there exists F', a pseudobasic subpattern of F. By Lemma 3.1, we can use subpattern operations and reduce $Gr(F)$ to $Gr(F')$ and construct a minor G' of G that is C5C, contradicting G being acyclically minimal. $\square$

Now, consider the following graphs. Let $\mathcal{L}_c$ be the graphs in Figure 4.2.

Then, we claim the following is true.

Proposition 4.4. Let G be an acyclically minimal push-consistent mixed C5C graph. Let X be a non-pushable mixed cut with an acyclic side A. If X is not an independent set, then A is isomorphic to one of the graphs in $\mathcal{L}_c$.

Proof: Let E' be the set of edges that have both ends in X. Since $B \backslash X$ contains a cycle, we may assume that $E' \subset E(A)$. Let $G' = G \backslash E', A' = A \backslash E'$. We have two possibilities: either A' no longer contains a cycle, or A' still contains a cycle.

First, let us assume that A' no longer contains a cycle. This implies that either A' is empty or it is isomorphic to a trivial tree or combinations of trivial trees that does not yield any cycles. If A' is empty, then E' forms a C_5. By the girth constraint, there does not exist any other edges, implying that $A = C_5$. If A' is a trivial tree, it is isomorphic to

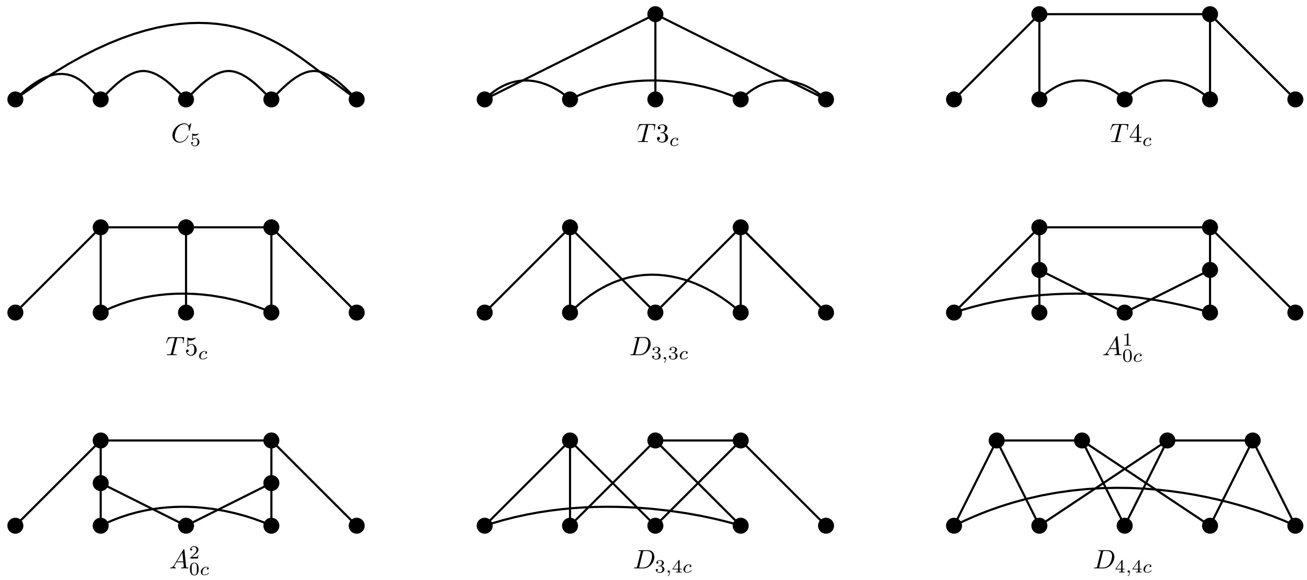


Figure 4.2: The List $\mathcal{L}_c$

$S3, T4, T5$, or $T5'$. Since A has a cycle and has girth at least 5, if $A' = S3$, then $A = S3_c$. For similar reasons, it follows that if $A' = T4, T5$, A' is $T4_c, T5_c$ respectively. If $A' = T5'$, any edges amongst X will violate the girth constraint and thus $A' \neq T5'$. Lastly, if A' is a combination of trivial trees, the only possible case that does not yield in any cycles is $D_{3,3}$ which implies that $A = D_{3,3c}$.

Now, let us assume that A' still contains a cycle. Note that A' satisfies Conditions 1, 5, 6, 8, 9 of Lemma 3.1. It also satisfies Condition 6, otherwise, a proper subset of X forms a cyclic k-cut in G where $k < 5$, contradicting G being C5C. Note that by Proposition 4.1, $d_B(v) \geq 2$. Then it follows from Lemma 3.1 and the fact that G' is not C5C, A' does not satisfy one of the conditions 2-4. Thus, we may assume A' is isomorphic to $Gr(F)$ where F is a basic pattern.

By Corollary 3.2, there exists a specific cut Y of size 4 in G'. The fact that Y is not a cut in G implies that there exists an edge $e \in E'$ that goes across the cut Y. Now we claim that $|E'| = 1$. Otherwise, by the same corollary, adding the edge e back to G' forms a C5C graph, contradicting the fact that G is acyclically minimal. Then, by the girth constraint, one can check that A can only be one of the following: $A^1_{0c}, A^2_{0c}, D_{3,4c}, D_{4,4c}$. $\square$

Thus, it follows from the previous two propositions that to prove Theorem 4.2, it suffices to show that if A is an acyclic side of a non-pushable cut X in a push-consistent acyclically minimal graph G, then A is not isomorphic to a graph in $\mathcal{L} \cup \mathcal{L}_c \backslash \mathcal{L}'$. We will proceed by first proving an acyclic side must have certain properties. Next, we will show if A is one of the extra graphs, then G induces another mixed cut with an acyclic side A' that violates those properties.

4.2 Acyclically Minimal Mixed Graphs

The goal of the next Lemma is to prove that an acyclic side of a non-pushable cut X in an acyclically minimal graph G must have certain properties. At the same time, we will also show why $Gr(F)$ is not in $\mathcal{L}'$ where F is isomorphic to $D_{4,5'}, B_1, B_2, B_3$.

Lemma 4.5. Let G be an acyclically minimal push-consistent mixed C5C graph and let X be a mixed cut with acyclic side A. Then, the following is true:

1. If there exists a vertex $v \in V(A)\backslash X$ where $d(v) > 3$, then $d(v) = 4$ and $|N(v) \cap X| = 3$.
2. All vertices in $V(A)\backslash X$ have degree 3.
3. If there exists $v \in V(A)\backslash X$ such that $N(v) \cap X = \emptyset$, then there exists $u \in N(v)$ such that $N(u)\backslash\{v\} \subset X$.
4. $N(v) \cap X \neq \emptyset$ for all $v \in V(A)\backslash X$.

For a summary of the proof, please see Table 4.1.

Proof: We will prove the Claims in the given order. Latter claims may require the validity assumption of previous claims.

Let X' be a non-pushable mixed cut obtained by pushing the cut X as far as possible. Let A' be an acyclic side with respect to X'. Note that by the previous two propositions, A' is isomorphic to either a graph in $\mathcal{L}_c$ or the graph of a pattern in $\mathcal{L}$.

Claim 1

Let us assume that there exists a vertex $v \in V(A)\backslash X$ such taht $d(v) \geq 4$. By inspection of the patterns in $\mathcal{L}$ and te graphs in $\mathcal{L}_c$, $D_{4,5'}$ is the only pattern that contains a vertex of degree 4. This implies that A' is isomorphic to $D_{4,5'}$. Now, we can attempt to reverse the pushing process to determine what X may be. However, since $v \notin X$, $X = X'$ and $A = A'$. Then, Claim 1 follows.

Claim 2

For the sake of contradiction, assume Claim 2 is false where there exists a vertex v with degree larger than 3. It follows that $v \in V(A')$ and still has degree larger than 3. Then, A' is isomorphic to $Gr(D_{4,5'}) = Gr((12, 34)+(23, 145)) = Gr((X(v_1), X(v_2))+(X(u_1), X(u_2)))$. Consider $G' = G\backslash u_2x_1$. Since G is minimally acyclic, G' is not a C5C graph. Then it follows from Corollary 3.2 that G' contains a C4C cut $Y = \{x_2, x_3, x_4, y\}$ where $y \in V(B)\backslash X$.This implies that $Y' = Y \cup u_2$ is a mixed cyclic 5-cut in G. Let C be an acyclic side with respect to Y' and D be the resulting cyclic side. Then we get the following two scenarios, either $x_1 \in V(C)$ or $x_5 \in V(C)$.

If $x_1 \in C$, then $d_D(u_2) = 1$. Consider pushing the cut Y' along the edge u_2x_5 and obtain a new cut Y^* and new acyclic and cyclic sides C^*, D^*. Note that $u_2 \in C^*$ and u_1, x_5 are neighbours of u_2 that are not in Y^*, contradicting Claim 1.

If $x_5 \in C$, then $d_D(u_2) = 1$. Similarly, we can push the cut Y' along the edge u_2x_1 so u_2 is now in the acyclic side of the cut. Then, as before, the acyclic side contains u_2 and u_1, x_1 are neighbours of u_2 that are not in Y^*, contradicting Claim 1. This proves Claim 2.

Claim 3

For Claim 3, if there exists $v \in V(A)\backslash X$ such that $N(v) \cap X = \emptyset$, then $N(v) \cap X' = \emptyset$. It follows that the component containing v in $A'\backslash X'$ has at least three leaves. Thus F is a pseudobasic true pattern where $Gr(F) = A'$. We can again attempt to reverse the pushing process to determine the possible cases of X while keeping $v \notin X$ and $N(v) \cap X = \emptyset$. If A' is isomorphic to B_1, B_2, then $X = X'$. If A' is isomorphic $B_3 = ((12, 4, v), (23, v), (34, 5, v))$, either $X = X'$ or $X = X' \cup \{v_{1,1}\}\backslash x_1$, implying A is isomorphic to B_2. In all possibilities, A is isomorphic to B_1, B_2, B_3. For all of the pseudobasic tree patterns, there exists only one such v and v is always adjacent to at least one leaf u of F. Thus by inspection, Claim 3 is true.

Claim 4

For the sake of contradiction, assume Claim 4 is false. By inspection, F is a pseudobasic tree pattern where $A' = Gr(F)$. We will follow the standard tree pattern notation.

If $F = B_1 = ((23, 1, v), (24, v), (34, 5, v))$, consider G' obtained by removing v and $v_{2,1}$ from G and adding an edge $v_{1,2}v_{3,2}$. Note that this is the same as performing subpattern operations to A to reach $A' = Gr((23, 1, 5, 34)) = Gr(A_0)$. By Proposition 3.2, there exists a cut Y in G' and without loss of generality, we may assume $x_3, v_{3,1} \in Y$. Let $Y' = Y \cup v_{2,1}$. Note that Y' is a mixed cut in G. Let C be an acyclic side with respect to Y'. Note that either $x_2 \in C$ or $x_4 \in C$. If $x_2 \in C$, we can keep pushing the cut Y' and reach a non-pushable cut Y^* where x_2 is in an acyclic side with respect to Y^*. However, note that $d(x_2) \geq 4$, contradicting Claim 2. Similarly, if $x_4 \in C$, $d(x_4) \geq 4$, also contradicting Claim 2.

If $F = B_2 = ((23, v), (34, 5, v), (14, v))$, obtain G' by deleting $v_{3,1}$ and adding the edge x_1v. Note that this is equivalent to performing subpattern operations on A to achieve $A' = Gr((23, 1, 5, 34)) = Gr(A_0)$. By Proposition 3.2, there exists a cut Y in G' and without loss of generality, we may assume $x_3, v_{2,2} \in Y$. Let $Y' = Y \cup v_{3,1}$. Note that Y' is a mixed cut in G. Let C be an acyclic side with respect to Y'. We have two cases, either $x_1 \in C$ or $x_5 \in C$.

If $x_1 \in C$, then we can first push the cut along $v_{3,1}x_4$. This implies that $v_{3,1}$ is now in the acyclic side. We can push the cut again along e edge $v_{3,2}x_5$ to obtain a new cut Y^*. Note that v is still in the acyclic side and for all $u \in N(v), N(u) \cap Y^* \leq 1$, contradicting Claim 3.

If $x_5 \in C$, then we may assume $x_4 \in C$. However, $d(x_4) \geq 4$, contradicting Claim 2.

Lastly, if $F = B_3 = ((23, 1, v), (34, v), (14, 5, v))$, obtain G' by deleting $v_{3,1}$ and $v_{3,2}$ and adding an edge vx_5. Note that this turns A' into $Gr((23, 1, 5, 34)) = Gr(A_0)$. Then, once again we may assume there exists a cut Y where $x_3, v \in Y$. Then, $Y' = Y \cup v_{3,1}$ is a mixed cut in G creating subgraphs C, D where $C \backslash Y'$ is acyclic. We have two cases, either $x_1, x_2 \in C$ or $x_4, x_5 \in C$. However, note that $d(x_1), d(x_4) \geq 4$ so in either case, it will contradict Claim 2. □

Now, we will use the above properties to eliminate more graphs as potential acyclic sides. The proof is very similar to the ones above where we will first obtain a a minor G' with a basic acyclic side, induce a 5-cut in the original graph, push the cut until we reach one that contradicts one of the above claims. Refer to the table at the end of this lemma for a condensed short-hand version of the proof.

Lemma 4.6. Let G be an acyclically minimal push-consistent mixed C5C graph and let X be a non-pushable cut in G with an acyclic side $A = Gr(F)$. Then A is not isomorphic to $A^1_{0c}, A^2_{0c}, D_{3,4c}, D_{4,4c}$ and F is not isomorphic to the following: $A_4, A_5, A_6, D^1_{A,A}4, D_{3,3,4}, D_{3,4,4}, D_{4,4,4}, D_{3,A}$, four of $D_{4,A}, D_{3,5}$ and $D_{4,5}$.

For a summary of this proof, please see Table 4.2.

Proof: First, for the sake of contradiction, we will assume X is not independent.

Suppose $A = A^1_{0c}$ is $Gr(F) \cup e$ where $F = (23, 1, 5, 34) = (X(v_1), X(v_2), X(v_3), X(v_4))$ and $e = x_1x_4$. Let $G' = G \backslash e$. By Corollary 3.2, there exists a C4C cut Y in $G'$where $v_3, x_3 \in Y$. Note that $Y' = Y \cup \{x_4\}$ is a 5-cut in G. Let C be an acyclic side with respect to Y'. Then, either $x_1 \in C$ or $x_5 \in C$. If $x_1 \in C$, note that $d(x_1) = 4$, contradicting Claim 2. If $x_5 \in C$, we can push Y' along the edge x_4x_1 so that x_4 is now in the acyclic side. Note that $d(x_4) = 4$, contradicting Claim 2.

Suppose $A = A^2_{0c} = Gr(F) \cup e$ where $F = (23, 1, 5, 34) = (X(v_1), X(v_2), X(v_3), X(v_4))$ and $e = x_2x_4$. Let $G' = G \backslash e$. By Corollary 3.2, there exists a C4C cut Y in $G'$where $v_3, x_3 \in Y$. Note that $Y' = Y \cup \{x_4\}$ is a cyclic 5-cut in G. Let C be an acyclic side with respect to Y'. Then, either $x_2 \in C$ or $x_5 \in C$. If $x_2 \in C$, note that $d(x_2) = 4$, contradicting Claim 2. If $x_5 \in C$, we can push Y' along the edge x_4x_2 so x_4 is now in the acyclic side. Note that $d(x_4) = 4$, contradicting Claim 2.

Suppose $A = D_{3,4c} = Gr(F) \cup e$ where $F = (123) + (24, 35)$ and $e = x_1x_4$. Let $G' = G \backslash e$. By Corollary 3.2, there exists a C4C cut Y in $G'$where $x_2, x_3 \in Y$. Note that $Y' = Y \cup \{x_4\}$ is a cyclic 5-cut in G. Let C be an acyclic side with respect to Y'. Then, either $x_1 \in C$ or $x_5 \in C$. If $x_1 \in C$, note that $d(x_1) = 4$, contradicting Claim 2. If $x_5 \in C$, we can push Y' along the edge x_4x_1 so x_4 is now in the acyclic side. Note that $d(x_4) = 4$, contradicting Claim 2.

If $A = D_{4,4c}$ is $Gr(F) \cup e$ where $F = (12, 34) + (23, 45)$ and $e = x_1x_5$. Let $G' = G \backslash e$. By Corollary 3.2, there exists a C4C cut Y in $G'$where $x_2, x_3, x_4 \in Y$. Note that $Y' = Y \cup \{x_4\}$ is a cyclic 5-cut in G. Let C be an acyclic side with respect to Y'. Then, either $x_1 \in C$ or $x_1 \notin C$. If $x_1 \in C$, note that $d(x_1) = 4$, contradicting Claim 2. If $x_1 \notin C$, we can push Y' along the edge x_5x_1 so that x_5 is now in the acyclic side. Note that $d(x_5) = 4$, contradicting Claim 2.

Now, for the sake of contradiction, we assume that $A = Gr(F)$ where $F = A_4, A_5, A_6$. We will be using standard path pattern notation.

If $F = A_4 = (23, 4, 1, 5, 34)$, obtain G' by deleting v_2 and adding the edge v_1v_3. Note that this turns A' into $Gr((23, 1, 5, 34)) = Gr(A_0)$. Then, once again we may assume there exists a cut Y where $x_3, v_4 \in Y$. Then, $Y' = Y \cup v_2$ is a mixed cut in G with some acyclic side C. We have two cases, either $x_1 \in C$ or $x_4 \in C$. If $x_4 \in C$, note that $d(x_4) \geq 4$, contradicting Claim 2. If $x_1 \in C$, we can first push along the following sequence of edges: v_2x_4, v_4x_5 to obtain a new cut X' so that v_2, v_4 is now in the acyclic side. Then, note that $N(v_3) \cap X' = \emptyset$, contradicting Claim 4.

If $F = A_5 = (23, 1, 4, 2, 5, 34)$, obtain G' by deleting v_3, v_4 and adding the edge v_2v_5. Note that this turns A' into $Gr((23, 1, 5, 34)) = Gr(A_0)$. Then, once again we may assume there exists a cut Y where $x_3, v_5 \in Y$. Then, $Y' = Y \cup v_3$ is a mixed cut in G with some acyclic side C. We have two cases, either $x_1, x_2 \in C$ or $x_4, x_5 \in C$. Note that $d(x_2), d(x_4) \geq 4$. Then, in either cases, it contradicts Claim 2.

If $F = A_6 = (23, 1, 5, 3, 1, 5, 34)$, obtain G' by deleting v_3, v_4, v_5 and adding the edge v_2v_6. Note that this turns A into $Gr((23, 1, 5, 34)) = Gr(A_0)$. Then, once again we may assume there exists a cut Y where $x_3, v_6 \in Y$. Then, $Y' = Y \cup v_3$ is a mixed cut in G with some acyclic side C. We have two cases, either $x_1 \in C$ or $x_5 \in C$. However, note that $d(x_1), d(x_5) \geq 4$ so in either cases, it contradicts Claim 2.

We will now eliminate some of the forest patterns. We will use standard forest pattern notations. Let $A = Gr(F)$. Note that F has at most three components, $F = F_1 + F_2 + F_3$ where f_3 may be empty. The vertices in F_1, F_2, F_3 will have labels v_i, u_j, w_k respectively. If a component contains only one vertex, the subscript indices label of the vertex will be dropped.

If A is isomorphic to $Gr(D^1_{A,A}4)$ where $D^1_{A,A}4 = F_1 + F_2, F_1 = (23, 1, 5, 34)$ and $F_2 = (13, 2, 4, 35)$ Obtain G' by deleting F_2. Note that this turns A into A_0. By Proposition 3.2, we may assume there exists a cut Y in G' with $x_3, v_2 \in Y$ and $x_1, x_2, x_4, x_5 \notin Y$. Then, note that $Y' = Y \cup u_2$ is a mixed cut in G with some acyclic side C. Then, we have either $x_1 \in C$ or $x_5 \in C$. Note that $d(x_1), d(x_5) \geq 4$. Then, in either cases, it contradicts Claim 2.

If $F = D_{3,3,4} = (123) + (24, 35) + (145)$, obtain G' by deleting w. Note that this turns A into $D_{3,4}$. By Proposition 3.2, there exists a cut Y in G' with $x_2, x_3 \in Y$. Then, note that $Y' = Y \cup w$ is a mixed cut in G with some acyclic side C. Then, either $x_1 \in C$ or $x_4 \in C$. However, note that $d(x_1), d(x_5) \geq 4$ so in either cases, it contradicts Claim 2.

If $F = D_{3,4,4} = (123) + (24, 35) + (14, 25)$, obtain G' by deleting w_1, w_2. Note that this turns A into $D_{3,4}$. By Proposition 3.2, there exists a cut Y in G' with $x_2, x_3 \in Y$. Then, note that $Y' = Y \cup w_1$ is a mixed cut in G with some acyclic side C. Then, either $x_1 \in C$ or $x_4 \in C$. However, note that $d(x_1), d(x_4) \geq 4$ so in either cases,it contradicts Claim 2.

Now we assume F is isomorphic to one of $D_{4,4,4}$.

If $F = (12, 34) + (23, 45) + (15, 24)$, obtain G' by deleting w_1, w_2. Note that this turns A into $D_{4,4}$. By Proposition 3.2, there exists a cut Y in G' with $x_2, x_3, x_4 \in Y$. Then, note that $Y' = Y \cup w_1$ is a mixed cut in G with some acyclic side C. Then, either $x_1 \in C$ or $x_5 \in C$. However, note that $d(x_1), d(x_5) \geq 4$ so in either cases, it contradicts Claim 2.

If $F = (12, 34) + (23, 45) + (14, 25)$, obtain G' by deleting w_1, w_2. Note that this turns A into $D_{4,4}$. By Proposition 3.2, there exists a cut Y in G' with $x_2, x_3, x_4 \in Y$. Then, note that $Y' = Y \cup w_1$ is a mixed cut in G with some acyclic side C. Then, either $x_1 \in C$ or $x_5 \in C$. However, note that $d(x_1), d(x_5) \geq 4$ so in either cases, it contradicts Claim 2.

We will now assume $F \in D_{A,3}$. Let $F = (23, 1, 5, 34) + F_2$ where F_2 is (135) or (124) or (125).

Obtain G' by deleting v. Note that this turns A into A_0. By Proposition 3.2, there exists a cut Y in G' with $x_3, v_2 \in Y$ and $x_1, x_2, x_4, x_5 \notin Y$. Then, note that $Y' = Y \cup u$ is a mixed cut in G with some acyclic side C. Then, either $x_1, x_2 \in C$ or $x_4, x_5 \in C$. However, note that $d(x_1) \geq 4$ so by Claim 2, $x_1 \notin C$. Then, $x_4, x_5 \in C$. Note also that regardless of what F_2 is, one of x_4, x_5 has degree at least four, once again contradicting Claim 2.

Now we will eliminate four of the $D_{4,A}$ patterns. Let $F = (23, 1, 5, 34) + F_2$ where F_2 is one of the following $(12, 45), (13, 24), (13, 25), (13, 45)$

Obtain G' by deleting F_2. Note that this turns A into A_0. By Proposition 3.2, there exists a cut Y in G' with $x_3, v_2 \in Y$ and $x_1, x_2, x_4, x_5 \notin Y$. Then, note that $Y' = Y \cup u_2$ is a mixed cut in G with some acyclic side C. Then, either $x_1, x_2 \in C$ or $x_4, x_5 \in C$. However, note that $d(x_1) \geq 4$ so by Claim 2, $x_1 \notin C$. Then, $x_4, x_5 \in C$. Note that regardless which F_2 it is, one of x_4 or x_5 has degree at least four, once again contradicting Claim 2.

We will make two additional claims about non-pushable cuts X.

Claim 5: If there exists a vertex $v \in V(A) \backslash x$ such that $N(v) \subseteq X$, then $|V(A) \backslash X| \leq 5$.

Claim 6: If A is a forest pattern with one component F_1 where $Gr(F_1)$ is isomorphic to $T4$, then $|V(A) \backslash X| \leq 6$.

The proof for Claim 5 is if there exists a vertex $v \in V(A) \backslash X$ such that $N(v) \in X$, by inspection, it implies that A is isomorphic to $Gr(D_{3,4}), Gr(D_{3,5}), Gr(D_{A,3})$. Then, by inspection, Claim 5 is true.

The proof for Claim 6 is if A contains a component isomorphic to $T4$, by inspection, it implies that A is isomorphic to one of the following: $Gr(D_{3,4}), Gr(D_{4,5}), Gr(D_{A,4})$. Then, Claim 6 follows.

Now we will eliminate $D_{3,5}$ and $D_{4,5}$.

If $F = D_{3,5} = (123) + (24, 1, 35)$, obtain G' by deleting u_2 and adding the edge u_1u_3. Note that this turns A into $D_{3,4}$. By Proposition 3.2, there exists a cut Y in G' with $x_2, x_3 \in Y$ and $x_1, x_4, x_5 \notin Y$. Then, note that $Y' = Y \cup vu2$ is a mixed cut in G with some acyclic side C. Then, either $x_1 \in C$ or $x_4, x_5 \in C$. If $x_1 \in C$, then note that $d(x_1) \geq 4$, contradicting Claim 2. If $x_4, x_5 \in C$, then we can push the cut Y' along the edge u_2x_1 to obtain a new mixed cut Y^* and new acyclic side C^*. Note that $N(v) \subseteq Y^*$ so we may assume v is in the acyclic side. Then, $v, u_1, u_2, u_3, x_4, x_5 \in C^* \backslash Y^*$, contradicting Claim 5.

If $F = (12, 34) + (23, 1, 45)$, obtain G' by deleting u_2 and adding the edge u_1u_3. Note that this turns A into $D_{4,4}$. By Proposition 3.2, there exists a cut Y in G' with $x_2, x_3, x_4 \in Y$ and $x_1, x_5 \notin Y$. Then, note that $Y' = Y \cup u_2$ is a mixed cut in G with some acyclic side C. Then, either $x_1 \in C$ or $x_5 \in C$. If $x_1 \in C$, then note that $d(x_1) \geq 4$, contradicting Claim 2. If $x_5 \in C$, then we can push the cut Y' along the edge u_2x_1 to obtain a new mixed cut Y^* and new acyclic side C^*. Note that $Gr(u_1u_2)$ with respect to $Y^*$is isomorphic to $T4$ so we may assume u_1, u_2 in the acyclic side. Note that $d(x_5) \geq 3$ and since x_5 is not adjacent to x_1, x_2, x_3, x_4, there exists a vertex $z \neq u_3$ where $z \notin Y^*$. Then, note that $v_1, v_2, u_1, u_2, u_3, x_5, w \in V(C^*) \backslash Y^*$, contradicting Claim 6.

This concludes our proof. $\square$

Below, we summarize the proofs of the previous two lemmas. For the first table, the first column indicates the claim that we assume is false. The second column indicates all the possibilities for F where $Gr(F) = A$. The third column indicates how to obtain G' with a basic side A'. The fourth column indicates the intersection of the cyclic 5-cut Y' in G with $V(A)$. The fifth column (PEoC), indicates the possible elements of an acyclic side C with respect to Y'. The sixth column indicates the first few edges we should push the cut Y' along. The last column indicates which vertex will eventually contradict which

Claim	F	G'	$Y' \cap V(A)$	PEoC	Pushed Edges	Contradiction
2	$D_{4,5'} = (12,34) + (23,145)$	$G \backslash u_2x_1$	x_2, x_3, x_4, u_2	x_1	u_2x_5	u_2, Claim 1
				x_5	u_2x_1	u_2, Claim 1
4	$B_1 = ((23,1,v),(24,v),(34,5,v))$	$G \backslash v_{2,1}/vv_{1,2}$	$x_3, v_{2,1}, v_{3,1}$	x_2		x_2, Claim 2
				x_4		x_4, Claim 2
	$B_2 = ((23,v),(34,5,v),(14,v))$	$G \backslash v_{3,1}x_4/vv_{1,1}$	$x_3, v_{2,2}, v_{3,1}$	x_1	$v_{3,1}x_4, v_{2,2}x_5$	v, Claim 3
				x_5		x_4, Claim 2
	$B_3 = ((23,1,v),(34,v),(14,5,v))$	$G \backslash v_{3,1}/v_{3,2}v$	$x_3, v, v_{3,1}$	x_1		x_1, Claim 2
				x_4		x_4, Claim 2

Table 4.1: Summary of Proof of Lemma 4.5

previously proven claim. The second table is similar, except we remove the first column and start directly by assuming for the sake of contradiction what A might be. Note that Claim 1, 3, 5 ,6 are proven by inspection. We will use standard path, tree and forest pattern notations.

Remark 4.7. Let G be an acyclically minimal push-consistent mixed C5C graph with a non-pushable mixed cut X and an acyclic side $A = Gr(F)$. If A is isomorphic to:

- $D_{4,5}$, then F is isomorphic to $D^1_{4,5} = (12,3,45) + (15,24)$,
- $D_{4,A}$, then F is isomorphic to either $D^1_{4,A} = (23,1,5,34) + (14,25)$ or $D^2_{4,A} = (23,1,5,34) + (15,24)$,
- $D_{A,A}$, then F is isomorphic to one of the other thirteen patterns that is not $(23,1,5,34)+(13,2,4,35)$.

Our next step is to show that A is not isomorphic to any of the graphs in $D_{A,A}$ as well.

4.2.1 Patterns in $D_{A,A}$

Lemma 4.8. Let G be an acyclically minimal push-consistent mixed C5C graph. Let X be a non-pushable cut and A be an acyclic side with respect to X. Then A is not isomorphic to an element of $D_{A,A}$.

Note that if A is isomorphic to $D_{A,A}$, we can attempt to remove one of the A_0 and by Proposition 3.2, it induces a cut of size three in the cyclic part of the graph. Similarly, if

G	G'	$Y' \cap V(A)$	PEoC	Pushed Edges	Contradiction
$A^1_{0c}=$ $(23,1,5,34) \cup x_1x_4$	$G\backslash x_1x_4$	x_3, v_3x_4	x_1		x_1, Claim 2
			x_5	x_4x_1	x_4, Claim 2
$A^2_{0c} =$ $Gr((23,1,5,34)) \cup x_2x_4$	$G\backslash x_2x_4$	x_3, v_3x_4	x_2		x_2, Claim 2
			x_5	x_4x_2	x_4, Claim 2
$D_{3,4c} =$ $Gr((123)+(23,45)) \cup x_1x_4$	$G\backslash x_1x_4$	x_2, x_3x_4	x_1		x_1, Claim 2
			x_5	x_4x_1	x_4, Claim 2
$D_{4,4c} =$ $Gr((12,34)+(23,45)) \cup x_1x_5$	$G\backslash x_1x_5$	x_2, x_3x_4, x_5	x_1		x_1, Claim 2
			Not x_1	x_5x_1	x_5, Claim 2
$Gr(A_4) =$ $Gr((23,4,1,5,34))$	$G\backslash v_1x_4/v_2v_1$	x_3, v_1, v_3	x_1	v_1x_4, v_4x_5	v_2, Claim 4
			x_4		x_4, Claim 2
$Gr(A_5) =$ $Gr((23,1,4,2,5,34))$	$G\backslash v_1x_4/v_2v_1$	x_3, v_3, v_5	x_2		x_2, Claim 2
			x_4		x_4, Claim 2
$Gr(A_6) =$ $Gr((23,1,5,3,1,5,34))$	$G\backslash\{v_3x_5, v_4x_3, v_5x_1\}$ $\{v_2v_3, v_3v_4, v_4v_5, v_5v_6\}$	x_3, v_3, v_5	x_1		x_1, Claim 2
			x_5		x_5, Claim 2
$Gr(D^1_{A,A}4) =$ $Gr((23,1,5,34)+(13,2,4,35))$	$G\backslash$ one of the $Gr(A_0)$	x_3, v_2, u_2	x_1		x_1, Claim 2
			x_5		x_5, Claim 2
$Gr(D_{3,3,4}) =$ $Gr((123)+(24,35)+(145))$	$G\backslash w$	x_2, x_3, w	x_1		x_1, Claim 2
			x_5		x_5, Claim 2
$Gr(D_{3,4,4}) =$ $Gr((123)+(24,35)+(14,25))$	$G\backslash\{w_1, w_2\}$	x_2, x_3, w_2	x_1		x_1, Claim 2
			x_4		x_4, Claim 2
$Gr(D^1_{4,4,4}) =$ $Gr((12,34)+(23,45)+(15,24))$	$G\backslash\{w_1, w_2\}$	x_2, x_3, x_4, w_1	x_1		x_1, Claim 2
			x_5		x_5, Claim 2
$Gr(D^2_{4,4,4}) =$ $Gr((12,34)+(23,45)+(14,25))$	$G\backslash\{w_1, w_2\}$	x_2, x_3, x_4, w_1	x_1		x_1, Claim 2
			x_5		x_5, Claim 2
$Gr(D^1_{A,3}) =$ $Gr((23,1,5,34)+(135))$	$G\backslash u$	x_2, v_2, u	x_1		x_1, Claim 2
			x_5		x_5, Claim 2
$Gr(D^2_{A,3}) =$ $Gr((23,1,5,34)+(124))$	$G\backslash u$	x_2, v_2, u	x_1		x_1, Claim 2
			x_4		x_4, Claim 2
$Gr(D^3_{A,3}) =$ $Gr((23,1,5,34)+(125))$	$G\backslash u$	x_2, v_2, u	x_1		x_1, Claim 2
			x_5		x_5, Claim 2
$Gr(D^3_{A,4}) =$ $Gr((23,1,5,34)+(12,45))$	$G\backslash\{u_1, u_2\}$	x_2, v_2, u_2	x_1		x_1, Claim 2
			x_5		x_5, Claim 2
$Gr(D^4_{A,4}) =$ $Gr((23,1,5,34)+(13,24))$	$G\backslash\{u_1, u_2\}$	x_2, v_2, u_2	x_1		x_1, Claim 2
			x_4		x_4, Claim 2
$Gr(D^5_{A,4}) =$ $Gr((23,1,5,34)+(13,25))$	$G\backslash\{u_1, u_2\}$	x_2, v_2, u_2	x_1		x_1, Claim 2
			x_5		x_5, Claim 2
$Gr(D^6_{A,4}) =$ $Gr((23,1,5,34)+(13,45))$	$G\backslash\{u_1, u_2\}$	x_2, v_2, u_2	x_1		x_1, Claim 2
			x_5		x_5, Claim 2
$Gr(D_{3,5}) =$ $Gr((123)+(24,1,35))$	$G\backslash u_2x_1/u_2u_3$	x_2, x_3, u_2	x_1		x_1, Claim 2
			x_5	u_2x_1	Claim 5
$Gr(D^2_{4,5}) =$ $Gr((12,34)+(23,1,45))$	$G\backslash u_2x_1/u_2u_3$	x_2, x_3, u_2	x_1		x_1, Claim 2
			x_5	u_2x_1	Claim 6

Table 4.2: Summary of Proof of Lemma 4.6

we remove the other A_0, we obtain another cut of size three in the cyclic side. The idea is to analyze all possible combinations of how these two cuts interact and deduce that no such two cuts can exist, providing a contradiction. Thus, to prove the above lemma, we will first prove the following.

Lemma 4.9. Let G be a mixed C5C graph with a non-pushable cut independent set X and cyclic side B such that there does not exist $v \in V(B)$ where $N(V) \subseteq X$. Let $Y = \{x, y, z\}, Y' = \{x', y', z'\}$ be vertex cut sets of B where $X \cap Y = x, X \cap Y' = x'$. Let $(D_1, D_2), (D'_1, D'_2)$ be subgraphs of B separated by Y and Y' respectively. For $i, j = 1, 2$, let $E_{i,j} = D_i \cap D'_j$ be a quadrant of B. Then, the following situations does not exist:

1. $x = x'$ and each quadrant contains x and exactly one other vertex of X.
2. $x \neq x'$, the quadrant that contains both x and x' contains one other vertex of X and the quadrant that does not contain x and x' contains the remaining two vertices of X.
3. $x \neq x'$, the quadrant that contains both x, x' does not contain any other vertices of X and every other quadrant contains exactly one other vertex of X.

Proof: For the sake of contradiction, we assume that we are in one of the three cases mentioned above. For the rest of the proof, we will only analyze the subgraph B. Note that $d(u) \geq 2$ for all $u \in X$ and $d(v) \geq 3$ for all $v \in V(B)\backslash X$.

Let $S_{i,j} = V(E_{i,j}) \cap (Y \cup Y')$ be the quadrant cut of a quadrant $E_{i,j}$ since $S_{i,j}$ separates $E_{i,j}$ from the rest of B. Let $S'_{i,j} = S_{i,j} \cup (X \cap V(E_{i,j}))$. Note that $S'_{i,j}$ is a cut in G. Since G is a C5C graph, if $|S'_{i,j}| \leq 4$, then $E_{i,j}$ is acyclic.

We will first prove the following Claims:

Claim 1: For all $i, j = 1, 2$, $|S_{i,j}| \geq 2$.

Assume for the sake of contradiction that $|S_{i,j}| = 1$ for some $i, j = 1, 2$. Note that for all three cases, there does not exist a quadrant that contains more than three vertices of X. Then, $|S'_{i,j}| \leq 4$ and $E_{i,j}$ is a forest. Since every vertex in B has degree 2 or more, every leaf of $E_{i,j}$ is in its quadrant cut. Since a forest has at least two leafs,this contradicts our assumption that $|S_{i,j}| \leq 1$.

Claim 2: If there exist some quadrant $E_{i,j}$ where $|X \cap S_{i,j}| = 1$ and $|X \cap V(E_{i,j})| = 2$, then $|S_{i,j}| \geq 3$.

Suppose for the sake of contradiction, there exists a quadrant $E_{i,j}$ such that $|S_{i,j}| \leq 2$. It follows from Claim 1 that $|S_{i,j}| = 2$. Then, from our assumption, $|S_{i,j} \cup (X \cap V(E_{i,j}))| = 3$ so $E_{i,j}$ is a forest. More specifically, since $|S_{i,j}| = 2$, $E_{i,j}$ is a path. Since only vertices in X can have degree less than 3, $V(E_{i,j}) \backslash S_{i,j} \subseteq X$. It follows that $E_{i,j}$ is a path of length two and there is an edge between the two vertices in $X \cap V(E_{i,j})$, contradicting X being an independent set.

Claim 3: Suppose for some quadrant $E_{i,j}$, $S_{i,j} = \{x_1, u, v\}$ and $V(E_{i,j}) \cap X = \{x_1, x_2\}$. Then, $E_{i,j}$ is either a tree with five vertices and $S_{i,j}$ as leaves, or a length three path with x_1 as one of the ends.

Note that $|S_{i,j} \cup (X \cap V(E_{i,j}))| = 4 \leq 5$, it follows that $E_{i,j|}$ is acyclic. Since, all vertices in $V(E_{i,j}) \backslash S_{i,j}$ have degree at least 2, $E_{i,j}$ is either a path or a tree with three leaves. If $E_{i,j}$ is a path, since X is an independent set, $E_{i,j}$ is either x_1uxv or x_1vxu. If $E_{i,j}$ is a tree with three leaves, it has at most one vertex of degree 3. Note that that vertex cannot be x_2, otherwise x_1x_2 is an edge, contradicting X being independent. Therefore, $E_{i,j}$ has another vertex of degree 3 w. Then, $E_{i,j}$ has either edges wx_1, wx_2, wu, x_2v or edges wx_1, wx_2, wv, x_2u.

Now, we will proceed by assuming we are in one of the three cases.

Case 1:

For the sake of contradiction, we assume that $x = x'$ and each quadrant contains x and exactly one other vertex of X. Let $x_{i,j} = X \cap V(E_{i,j}) \backslash \{x\}$.

Subcase 1.1: $y = y', z = z'$

Note that for all quadrants, $S_{i,j} = \{x, y, z\}$. Since every quadrant contains exactly one vertex of X that is not x, it follows from Claim 3 that every quadrant $E_{i,j}$ is either a length three path or a tree with three leaves. It also follows that in each quadrant, there exists a pair of vertices in $S_{i,j}$ that has a common neighbour. Then, by pigeonhole principle, there exists a pair of vertices in $S_{i,j}$ that contains two common neighbours in B, contradicting the girth condition of G.

Subcase 1.2: $y = y', z \neq z'$.

Without loss of generality, we may assume that $z, z' \in S_{1,1}$. Then, note that $z, z' \notin S_{2,2}$. However, this implies that $S_{2,2} = \{x, y\}$. Note that $E_{2,2}$ also contains one other vertex of X that is not x, contradicting Claim 2.

Subcase 1.3: $y \neq y', z \neq z'$.

First, note that $|Y \cap S_{i,j}| = 2$ for all $i, j = 1, 2$. Suppose for the sake of contradiction that $|Y \cap S_{i,j}| \neq 2$. Then, $Y \subseteq V(D_1')$ or $Y \subseteq V(D_2')$. Without loss of generality, assume that $Y \subseteq V(D_1')$. However, this implies at least one of $|S_{1,2}|$ or $|S_{2,2}|$ is less than 3, contradicting Claim 2. Therefore we may assume that $|Y \cap S_{i,j}| = 2$.

By symmetry, we may assume that $|Y' \cap S_{i,j}| = 2$ for all $i, j = 1, 2$ as well. Then, let $y \in V(D_1'), z \in V(D_2'), y' \in V(D_1), z' \in V(D_2')$. Note that $|S_{i,j} \cup (X \cap E_{i,j})| = 4$ so by Claim 2, every quadrant is isomorphic to either a path or a tree with three leaves. We will first prove that no quadrant is isomorphic to a path. Suppose for the sake of contradiction, that there exists a quadrant isomorphic to a path. Without loss of generality, let $E_{1,1}$be the path $y'x_{1,1}yx$. It then follows that $d_{E1,2(y')} \geq 2$. By Claim 2, it follows that $E_{1,2}$ cannot be a tree with three leaves. Then, $E_{1,2}$ is the path $xy'x_{1,2}z$. However, note that $y'x_{1,1}yx$ is a cycle of length 4, contradicting G being C5C. Therefore, all quadrants are isomorphic to a tree with three leaves. However, this implies that the degree for the vertices y, z, y', z' is 2, contradicting the degree condition.

Case 2:

For the sake of contradiction, suppose $x \neq x'$, there exists a quadrant that contains x, x' and exactly one other vertex of X and the quadrant that does not contain x and x' contains the remaining two vertices of X. Without loss of generality, let $x \in V(D_1'), x' \in V(D_1), V(E_{1,1}) \cap X = \{x, x', x_{1,1}\}$ and $V(E_{2,2}) \cap X = \{x_{2,2}, x'_{2,2}\}$.

We will first prove that $|S_{1,1}| \geq 4$. Suppose for the sake of contradiction that $|S_{1,1}| \leq 3$. Since $|S_{1,1} \cup (X \cap V(E_{1,1}))| \leq 4$, $E_{1,1}$ is a forest. First, assume $S_{1,1} = \{x, x'\}$. This implies that $E_{1,1}$ is a path since it has at most two leafs. Then, $X_{1,1}$ must be adjacent to both x, x', contradicting X being an independent set. Now, assume that $S_{1,1} = \{x, x', u\}$ for some vertex $u \in Y \cup Y' \backslash X$. This implies that $E_{1,1}$ is either a path or a tree with three leaves. Note that for similar reason as before, if $E_{1,1}$ is isomorphic to a path, it will contradict X being an independent set. Thus, we may assume that $E_{1,1}$ is a tree with three leaves. Note that $x_{1,1}$ has at least two neighbours and since X is an independent set $x_{1,1}$ is adjacent to u and another vertex $v \in V(S_{1,1}) \backslash S_{1,1}$. Since $E_{1,1}$ is a tree with three leaves, it has at most one vertex of degree 3. This implies that $V(E_{1,1}) = \{x, x', u, v, x_{1,1}\}$. Since v has degree at least three and the girth of G is at least 5, $N(v) = \{x, x', x_{1,1}\}$, contradicting our initial assumption that there does not exist a vertex in B where all of its neighbours are in X. Therefore, $|S_{1,1}| \geq 4$.

Next, we will prove that $|S_{2,2}| \geq 3$. Suppose for the sake of contradiction that $|S_{2,2}| \leq 2$. By Claim 1, $|S_{2,2}| = 2$. Let $S_{2,2} = \{u, v\}$. Since $|S_{2,2} \cup (X \cap V(E_{2,2}))| = 4$, $E_{2,2}$ is acyclic. Since u, v are the only vertices that can have degree one, there does not exist any vertices of degree 3 in $E_{2,2}$ so $V(E_{2,2}) = \{u, v, x_{2,2}, x'_{2,2}\}$. Since X is an independent set, $N(x_{2,2}) = N(x'_{2,2}) = \{u, v\}$, creating a cycle of length 4, a contradiction. Therefore, $|S_{2,2}| \geq 3$.

Now, note that $7 \leq |S_{1,1}| + |S_{2,2}| = (|S_{1,1} \cap Y| + |S_{1,1} \cap Y' \backslash Y|) + (|S_{2,2} \cap Y'| + |S_{2,2} \cap Y \backslash Y'|) = |Y| + |Y'| = 6$, a contradiction.

Case 3:

Suppose for the sake of contradiction that $x \neq x'$, the quadrant that contains both x, x' does not contain any other vertices of X and every other quadrant contains exactly one other vertex of X.

Without loss of generality, we may assume $x_{1,2} \in V(E_{1,2}) \backslash (Y \cup Y')$, $x_{2,1} \in V(E_{2,1}) \backslash (Y \cup Y')$, $x_{2,2} \in V(E_{2,2}) \backslash (Y \cup Y')$.

First, we will prove that $|S_{1,2}| = |S_{2,1}| = 3$ and $|S_{1,1}| \geq 5$.

Note that $|S_{1,2}| + |S_{2,1}| = |Y| + |Y'| = 6$. It follows from Claim 2 that $|S_{1,2}| = |S_{2,1}| = 3$. This implies that $d_{E_{1,2}}(x') = d_{E_{2,1}}(x) = 1$ and $d_{E_{1,1}}(x), d_{E_{1,1}}(x') \geq 2$. Now, suppose for the sake of contradiction, $|S_{1,1}| \leq 4$. Then, $|S_{1,1} \cup (X \cap V(E_{1,1}))| = 4$. This implies that $E_{1,1}$ is a forest and contains at least two leaves. Note that $S_{1,1} \backslash X$ are the only vertices that can be leaves so $|S_{1,1} \backslash X| \geq 2$. Then, it follows that $S_{1,1}| = 4$, $E_{1,1}$ has two leaves and is a path. However, this implies that xx' is an edge, contradicting X being an independent set. Therefore, $|S_{1,1}| \geq 4$.

By Claim 1, $|S_{2,2}| \geq 2$. Then, note that $6 = |Y| + |Y'| = |S_{1,1}| + |S_{2,2}| \geq 5 + +2 = 7$, a contradiction.

This completes the proof of this lemma. $\square$

Proof of Lemma 4.8: Let F be a pattern such that $Gr(F)$ is isomorphic to A. It follows from Lemma 4.6 that F is not $D^1_{A,A}4$. Now, we will use the previous lemma to eliminate the other 13 possible cases of F.

Suppose for the sake of contradiction that F is isomorphic to one of the other $D_{A,A}$ patterns. Since G is acyclically minimal, it follows from Proposition 4.4 that X is an independent set. By Corollary 3.2, each component of F that is isomorphic to A_0 induces a cut of size 3 in B. Then, the subgraph B contains two cuts of size three Y, Y' where $|Y \cap X| = |Y' \cap X| = 1$. Note that this fits our criteria for Lemma 4.9. We will use same notation and terminology mentioned in Lemma 4.9.

If $F = (23, 1, 5, 34) + (13, 4, 2, 35)$, by Lemma 4.9, $Y \cap Y' = x_3$, and each quadrant contains exactly one other vertex of $X \backslash \{x_3\}$, contradicting Case 1 of Lemma 4.9.

If $F = (23, 1, 5, 34) + (12, 3, 4, 25)$, by Corollary 3.2, $Y \cap Y' \cap X = \emptyset$. Without loss of generality, $x_3 \in Y, x_2 \in Y', \{x_1, x_2, x_3\} = X \cap V(E_{1,1})$ and $\{x_4, x_5\} = X \cap V(E_{2,2})$. However, this contradicts Case 2 of Lemma 4.9. Note that similarly, if F is one of the following patterns:

- $(23, 1, 5, 34) + (12, 3, 5, 24)$
- $(23, 1, 5, 34) + (12, 3, 4, 15)$
- $(23, 1, 5, 34) + (12, 3, 5, 14)$,

it also contradicts Case 2 of Lemma 4.9.

Lastly, if F is one of the following:

- $(23, 1, 5, 34) + (24, 3, 1, 25)$
- $(23, 1, 5, 34) + (24, 3, 5, 12)$
- $(23, 1, 5, 34) + (25, 3, 1, 24)$
- $(23, 1, 5, 34) + (25, 3, 4, 12)$
- $(23, 1, 5, 34) + (14, 3, 2, 15)$
- $(23, 1, 5, 34) + (14, 3, 5, 12)$
- $(23, 1, 5, 34) + (15, 3, 2, 14)$
- $(23, 1, 5, 34) + (15, 3, 4, 12)$

for similar reasons, one can check that it will contradict Case 3 of Lemma 4.9. This proves that F is not any of the patterns in $D_{A,A}$. $\square$

Note, the list is now reduced to $A_0 - A_3, D_{3,4}, D_{4,4}, (12, 3, 45) + (15, 24), (23, 1, 5, 34) + (14, 25), (23, 1, 5, 34) + (15, 24)$. Combined with Proposition 4.1, this completes the proof of Theorem 4.2.

4.3 A Comment on Doubly-Acyclic and Non-Push-Consistent Graphs

Consider a C5C graph with a cyclic 5-cut X that separates G into A, B. Suppose A is an acyclic side. Our plan this far is to replace A with a proper minor of A' and hope that the resulting graph is still C5C. However, as stated before, the biggest challenge in applying Lemma 3.1 is that we cannot always guarantee $d_B(x) \geq 2$ for all $x \in X$. This is why we have introduced the idea of pushing. So that we can deal with these troublesome edges. However, the new problem is after we push along an edge, we cannot always guarantee the new side A' remains an acyclic side. This is the case with doubly-acyclic and non-push-consistent graphs.

However, note that we can modify Lemma 3.1 to deal with this issue. We just need to modify condition 7 in the lemma to "for all $x \in X$, x is adjacent to some vertex in $A' \backslash X$ and if x is adjacent to at least two vertices in $A \backslash X$, then x is still adjacent to at least two vertices in $A' \backslash X$". Note that this does not change the proof of Lemma 3.1.

With this modification, however, it affects the list of pseudobasic patterns since the cover condition is essentially changed. This means we will have a slightly larger list of pseudobasic subpatterns. Then, to find graphs that are minor-minimal C5C and also are doubly-acyclic, we simply need to check all combinations of the newly obtained list of acyclic sides. For non-push-consistent graphs, we will also check all possible combinations of the new list of acyclic sides but with an extra edge between them. Both should not be a difficult task after the modification.

This also implies that by checking combinations of the current list of basic and pseudobasic patterns, we can obtain a partial list of minor-minimal C5C graphs that are either doubly-acyclic or non-push-consistent. After some attempt, the only minor-minimal graphs we obtained are Petersen, Triplex and Box.

Chapter 5

Local Reduction

In this chapter, our goal is to determine the characteristics of minor-minimal C5C graphs that are also mixed and push consistent.

Let $S \subseteq V(G)$ be a set of degree 3 vertices. We say S is a *degree-3 component* if the graph induced by S is connected and $d(v) \geq 4$ for every $v \in N(S)$.

Our main theorem is the following.

Theorem 5.1. *Let G be a push-consistent mixed C5C graph. Let X be a non-pushable mixed cut with acyclic side A. If G is minor-minimal C5C, then all of the following holds.*

1. *A is isomorphic to one of $C_J, A_0, D_{3,4}, D_{4,4}$ (See Figure 4.1) and,*
2. *every edge e in G is incident to a degree 3 vertex v such that there exists a 5-cycle containing v but not e and,*
3. *every vertex v is adjacent to a degree-3 vertex u such that there exists a 5-cycle containing u but not v and,*
4. *every degree-3 component has size at least 4.*

5.1 CkC Graphs with Minimum Degree 3

In this chapter, we would like to delete vertices, edges or even components. However, it is not always clear the resulting graph still contains a cyclic k-cut. Tis section provides some insight to this matter.

Lemma 5.2. Let G be a graph with minimum degree 3. Then, G:

- is K_4,
- is $K_{3,3}$, or
- contains two edge-disjoint cycles.

Proof: For the sake of contradiction, assume G is not $K_4, K_{3,3}$ and does not contain two edge-disjoint cycles. We will first show that G is 3-connected. For the sake of contradiction, suppose there exists a cut of G, S, where $|S| \leq 2$. Let A, B be two subgraphs separated by S. Since A, B do not contain cycles, they are forests and contain at least two leaves. Note that the vertices in S are the only ones that can have degree one in A, B. This implies $|S| = 2$ and each vertex in S has degree one in each of the subgraphs A, B. Then, the vertices in S has degree 2, contradicting the minimum degree of G. Thus G is 3-connected.

Let u, v be two vertices of G. Let P_1, P_2, P_3 be three internally vertex-disjoint paths from u to v. We will break it into two cases.

Case 1: Suppose there exists a vertex w such that $w \notin V(P_i)$ for $i = 1, 2, 3$.

Since G is 3-connected, there exist three internally vertex-disjoint paths from w to the vertices in $V(P_1) \cup V(P_2) \cup V(P_3)$. For $i = 1, 2, 3$, let Q_i be paths from w to w_i such that w_1, w_2, w_3 are the only vertices in $Q_1 \cup Q_2 \cup Q_3$ that are also vertices of $P_1 \cup P_2 \cup P_3$. Suppose there exists $i, j, k = 1, 2, 3, i \neq j$ such that $w_i, w_j \in V(P_k)$. Without loss generality, we may assume that $w_1, w_2 \in V(P_1)$. Note that P_2, P_3 forms a cycle that is edge-disjoint from the cycle induced by W_1, W_2, P_1, a contradiction. Thus for all $i = 1, 2, 3, V(P_i) \cap \{w_1, w_2, w_3\} < 2$. Then, by pigeonhole, for all $i = 1, 2, 3, V(P_i) \cap \{w_1, w_2, w_3\} = 1$ and $w_i \neq u, v$. Without loss of generality, we may assume that $w_i \in V(P_i)$ for $i = 1, 2, 3$. Let $G' = \cup_{i=1,2,3}(P_i \cup Q_i)$. Note that G' is a subdivision of $K_{3,3}$. For convenience, we will relabel u, v, w as u_1, u_2, u_3 respectively and we will relabel the paths for $i, j = 1, 2, 3$, where $P_{i,j}$ is a path from u_i to w_j.

We will now prove the following.

Claim: there does not exist a path R with ends r, r' in G such that $r, r' \in V(G')$ and $V(R) \cap V(G') = \{r, r'\}$.

Suppose for the sake of contradiction that there exists such path R. First, suppose there exist $i, j = 1, 2, 3$ such that $r, r' \in V(P_{i,j})$. Without loss of generality, assume $r, r' \in V(P_{1,1})$. Then, $R, P_{1,1}$ induces a cycle that is edge-disjoint from the cycle formed by $P_{2,2}, P_{3,2}, P_{3,3}, P_{2,3}$, a contradiction. Now, suppose there exist $i, j, k = 1, 2, 3, j \neq k$

such that $r \in V(P_{i,j}), r' \in V(P_{i,k})$. Without loss of generality, assume $r \in V(P_{1,1}), r' \in V(P_{1,2})$. Then $R, P_{1,1}, P_{1,2}$ induces a cycle that is edge-disjoint from the cycle formed by $P_{2,2}, P_{3,2}, P_{3,3}, P_{2,3}$, a contradiction. Then, it implies that there exist $i, j, k, l = 1, 2, 3, i \neq k, j \neq l$ such that $r \in V(P_{i,j}), r' \in V(P_{k,l})$. Without loss of generality, we may assume $r \in V(P_{1,1}), r' \in V(P_{2,2})$. Then, $R, P_{1,1}, P_{1,2}, P_{2,2}$ induces a cycle that is edge-disjoint from $P_{2,1}, P_{3,1}, P_{3,3}, P_{2,3}$, a contradiction. This proves our claim.

Now, note that since $G \neq K_{3,3}$, $G \neq G'$. Since the graph is 3-connected, G contains a path R with ends r, r' such that $V(R) \cap V(G') = \{r, r'\}$ contradicting our claim. Note that this implies G does not have a subgraph that is a subdivision of $K_{3,3}$.

Case 2: Given x, y, P_1, P_2, P_3, $V(G) = V(P_1 \cup P_2 \cup P_3)$.

Since G does not have any parallel edges, $V(G) \neq \{x, y\}$. Let w be another vertex of G and without loss of generality, assume $w \in V(P_1)$. Since G is 3-connected, thre exists an edge ww' such that $ww' \notin E(P)$. If $w' \in V(P_1)$, then ww', P_1 induces a cycle that is edge-disjoint from P_2, P_3, a contradiction. Then, by symmetry, without loss of generality, we may assume $w' \in V(P_2)$ and $w, w' \neq u, v$. Let $G' = P_1 \cup P_2 \cup P_3 \cup ww'$. Note that G' is a subdivision of K_4. For convenience, we will relabel the vertices u, v, w, w' as u_1, u_2, u_3, u_4 and relabel the paths as $P_{i,j}$ for $i, j = 1, 2, 3, 4, i \neq j$ where $P_{i,j}$ denotes a path from u_i to u_j. Note that $V(G) = V(G')$.

If $V(G) = \{u_1, u_2, u_3, u_4\}$, since there are no parallel edges and all vertices have degree at least 3, $G = G' = K_4$, a contradiction. Then, there exists a vertex $x \neq u_1, u_2, u_3, u_4$. Without loss of generality, assume $x \in V(P_{1,2})$ Since G is 3connected and $V(G) = V(G')$, there exists an edge xx' such that $xx' \notin E(P_{1,2})$. If $x' \in V(P_{1,2})$, then $xx', P_{1,2}$ induces a cycle that is edge disjoint from $P_{2,3}, P_{3,4}, P_{2,4}$, a contradiction. If $x' \in V(P_{1,3})$, then $xx', P_{1,2}, P_{1,3}$ induces a cycle that is edge-disjoint from $P_{2,3}, P_{3,4}, P_{2,4}$, a contradiction. Then, by symmetry, $x' \notin V(P_{1,4}), V(P_{2,3}), V(P_{2,4})$. This also implies that $x, x' \neq u_1, u_2, u_3, u_4$ and $x' \in V(P_{3,4})$. Note that $G' \cup xx'$ is a subdivision of $K_{3,3}$. Then, by similar arguments as Case 1, this is a contradiction. □

The above lemma implies the following corollary:

Corollary 5.3. If G has minimum degree 3 and has girth 5, then G contains a cut X that separates G into A, B where both subgraphs contains a cycle.

Proof: It follows from the previous lemma that G contains two edge-disjoint cycles, C, C'. Let $X = V(C), A = C, B = G \backslash C$. Note that X separates G into A, B and $C \in A, C' \in B$. □

5.2 Structural Properties of Minor-Minimal Mixed Graphs

Proposition 5.4. If A is an acyclic side of a cut X in a minor-minimal C5C graph that is also mixed and push-consistent, then every vertices in $A \backslash X$ has degree 3 and is adjacent to a vertex of X.

Proof: Note that A is isomorphic to a graph in $\mathcal{L}'$. Then, by checking all graphs in the list, one can verify that the proposition is true.

Lemma 5.5. Let G be a minor-minimal C5C graph. If G is a push-consistent mixed C5C graph, then the vertices of degree 4 or more forms an independent set.

Proof: For the sake of contradiction, suppose there exists an edge $e = uv$ where $d(u), d(v) \geq 4$. Let $G' = G \backslash e$. By assumption, G' is not C5C. Note that G' has girth 5 and minimum degree 3 so by Corollary 5.3, there exists a cut X' where $|X'| \leq 4$ that separates G' into subgraphs A', B' and both contains a cycle. Note that X' is not a cut in G, otherwise, it contradicts G being C5C. Let $X = X' \cup \{u\}$. Without loss of generality, assume $v \in B'$. Let $A = A', B = B' \cup \{e\}$. Note that X separates G into A, B and both contains a cycle. This implies that $|X| = 5$ and one of A, B is the acyclic side. Note that $v \in B$ and $d_B(v) \geq 4$. It follows from Proposition 5.4 that B is not the acyclic side. Note that $d_B(u) = 1$ so the cut can X be pushed along the edge e from u to v. However then, $u \in A, d_A(u) \geq 4$ contradicting Proposition 5.4. □

This implies the following:

Corollary 5.6. Let G be a minor-minimal C5C graph that is also mixed and push-consistent with a mixed cut X and acyclic side A. Then $A \notin \mathcal{L}_{\mathcal{C}}$. In other words, $A \in \mathcal{L}'$.

Lemma 5.7. Let G be a minor minimal C5C graph that is also push-consistent. Then, for every degree-3 component S, $|S| > 1$.

Proof: Suppose for the sake of contradiction that there exists such vertex u. Consider $G' = G \backslash u$. Note that G' still has minimum degree 3 and girth 5. Since G is minor-minimal C5C, G' contains a cut X' of size at most 4 that separates the graph into A', B', both containing a cycle. Note that $N(u) \not\subseteq V(A')$, otherwise, V is a cut in G, contradicting G being C5C. By symmetry, $N(u) \not\subseteq V(B')$. Then, without loss of generality, we may assume $v_1 \in V(A') \backslash X', v_2 \in V(B') \backslash X'$. Note that $X = X' \cup \{u\}$ is a cut in G. Since G is C5C, it follows that $|X| = 5$ and either v_1 or v_2 is in the acyclic side with respect to X. However, $d(v_1), d(v_2) \geq 4$, contradicting Proposition 5.4. □

5.2.1 T5 Substitution

Now, we like to replace certain acyclic sides that contain T5 as a subpattern with a T5. By Corollary 5.3, we know the resulting graph should still contain a cyclic k-cut as long as we guarantee the minimum degree is still 3 and girth is 5. The problem is what if the resulting graph has a cyclic k-cut where $k \leq 4$. Thus, we would like to perform pushing on cuts of size less than 5.

The following proposition extends the idea of pushing a cut to one with size at most 4. Note that in this context, we no longer care about whether a side is acyclic or cyclic sides. All we want is for the sides, after pushing, to still contain a cycle.

Proposition 5.8. Let G be a graph with minimum degree 3 and girth 5. Let X be a cut of size at most 4 that separates G into two subgraphs A, B where both contains a cycle. The following is true:

1. If a vertex $x \in X$ has only one neighbor $x' \notin V(A)$, then the cut X can be pushed along the edge xx' to obtain a new cut that still separates G into two subgraphs that both contains a cycle.
2. If xx' is an edge where $x, x' \in X$ and $xx' \in E(A)$, then the subgraphs $A' = A \backslash xx', B' = B \cup \{xx'\}$ still contains a cycle.
3. If $v \in V(A) \backslash X$ and $N(v) \subseteq X$, then the subgraphs $A' = A \backslash v, B' = B \cup \{vu : u \in N(v)\}$ both contains a cycle.

Proof: Let C_A, C_B be cycles in A, B respectively. For the first statement, note that $xx' \notin C_A, C_B$ thus the new subgraphs created by the new cut will still contain the two respective cycles.

For the second statement, note that C_B is still in B' so B' contains a cycle. Suppose for the sake of contradiction that A' does not contain a cycle. This implies that A' is and edge, or $S3$, or $S4$ or $T4$. However, this implies A contains a cycle of length 4 or less, contradicting the girth of G. Therefore, A' still contains a cycle.

For the last statement, it is easy to see that B' still contains a cycle. Suppose for the sake of contradiction that A' is acyclic. Then A' is $S3$, or $S4$, or $T4$. However, both implies that A contains a cycle of length 4, contradicting the girth of G. □

The following lemma gives us some structure when replacing an acyclic side of a cut with a $T5$.

Lemma 5.9. Let G be a mixed C5C graph with a non-pushable cut X, acyclic side A and cyclic side B. Suppose A has $T5 = (X(v_1), X(v_2), X(v_3)) = (12, 3, 45)$ as a subpattern. Let G' be a graph obtained by replacing A with $T5$. Then, either G' is a C5C graph, or for every cut Y' of size at most 4 where $|Y'|$ is minimum, that separates G' into two subgraphs A', B', both containing a cycle, the following is true:

1. $Y' \cap \{v_1, v_2, v_3\} \neq \emptyset$,
2. $|X \cap (V(A') \backslash V(B'))|, |X \cap (V(B') \backslash V(A')| \geq 2$,
3. $|Y' \cap X| \leq 1$,
4. Y' can be pushed to a cut Y'' such that $Y'' \cap \{v_1, v_2, v_3\} = v_3$.

Proof: Suppose G' is not C5C. Let X' be a cut of size at most 4 that separates G' into two subgraphs A', B' both containing a cycle. Note that since $T5$ is a subpattern of A, G' has girth 5. Since X is non-pushable, $d_B(x) \geq 2$ for all $x \in x$ and thus G has minimum degree 3. This means we can apply Proposition 5.8 and push the cut X' along edges to obtain different cuts. Let $V = \{v_1, v_2, v_3\}$.

Claim 1: $X' \cap V \neq \emptyset$

If $X' \cap V = \emptyset$, then X' is also a cut in G, contradicting G being C5C.

Claim 2: $|X \cap (V(A') \backslash V(B'))|, |X \cap (V(B') \backslash V(A')| \geq 2$.

For the sake of contradiction, assume $x \in V(A') \backslash V(B'), X \backslash \{x\} \subseteq V(B')$. First, assume that $x = x_3$. Since $X \backslash \{x_3\} \subseteq V(B')$ v_2 should be the only vertiex in $V \cap X'$. This implies we can push the cut along the edge v_2x_3, creating a cut that is disjoint from V, contradicting Claim 1. If $v \neq x_3$, using similar arguments, X' can always be pushed to create a cut that is disjoint from V, contradicting Claim 1.

Claim 3: $|X' \cap X| \leq 1$

Suppose for the sake of contradiction, assume $|X' \cap X| \geq 2$. Then it follows that $|X \backslash X'| \leq 3$. Then, by pigeonhole principle, there exists a vertex $x \in X$ such that either $X \cap (V(A') \backslash V(B')) = x$ or $X \cap (V(B') \backslash V(A')) = x$, contradicting Claim 2.

Claim 4: X' can be pushed to a cut Y' such that $Y' \cap V = v_3$.

Suppose $v_1 \in X'$. Note that $d_{A'}(v_1), d_{B'}(v_1) \neq 0$, otherwise v_1 is not needed in the cut. Since $d(v_1) = 3$, there exists an edge incident to v_1 that we can push on. By symmetry, if $v_3 \in X'$, there also exists an edge that we can push the cut on. Since v_1 is not adjacent to

v_3, it follows that we can always push the cut X' away from those two vertices to obtain a cut Y' such that $Y' \cap \{v_1, v_3\} = \emptyset$. Then, it follows from Claim 1 that $Y' \cap V = v_2$. $\square$

Before proceeding on replacing an acyclic side with a T5, we need to first prove the following lemma. This will allow us to determine whether a small subgraph contains a cycle or not.

Lemma 5.10. Let G be a graph with girth 5 and no isolated vertices. Let $S = \{x_1, x_2, y_1, y_2, y_3\} \subseteq V(G)$ such that $d(x_1), d(x_2) \geq 2, d(v) \geq 3$ for all $v \in V(G)\backslash S$. Then, either G contains a cycle or for every vertex $v \in V(G)$, there exists $i = 1, 2$ such that u is at distance at most 2 from x_i and $x_i \neq u$.

Proof: Suppose G does not contain a cycle. Note that y_1, y_2, y_3 are the only vertices that can be leaves in G, so G is either a path or a tree with 3 leaves. If G is a path, there does not exist any vertices of degree 3 or more so $V(G) = S$. Since x_1, x_2 are not leaves, it is easy to check that they are at distance at most two from each other and any other vertices are at distance at most 2 from one of them. Now assume G is a tree with three leaves. Then, there exists only one vertex of degree 3, so $|V(G)\backslash S| \leq 1$. If $V(G) = S$, it is clear that x_1, x_2 are adjacent to each other and all other vertices are adjacent to one of them. If $V(G)\backslash S = v$, then x_1, x_2 are at most 2 away from each other and all other vertices are either adjacent to one of them or adjacent to v. Since v must be adjacent to at least one of x_1, x_2, our claim is still true. $\square$

Lemma 5.11. Let A be a graph isomorphic to $Gr(F)$ where $F \in \mathcal{L}'$. Let $e = xv \in E(A)$ where $x \in X, v \notin X$. Let A' be the graph obtained by deleting e and suppressing any degree 2 edges. Then, one of the following is true:

1. A' contains a 4-cycle,
2. A' does not contain a cut of size at most 4 that separates A' into A'', B'' such that A'' contains a cycle,
3. the neighbors of v that is not x are two degree 3 vertices.

Proof: For this lemma, we just have to check it is true for every $F \in \mathcal{L}'$. Note that if G' is acyclic, then it is trivially true. If G' is isomorphic to a graph in $\mathcal{L}_C$ or to F' where F' is a C5C pattern, then by definition of patterns, our claim is also true. Thus, we just need to show that if G' is not a C5C pattern, it either contains a 4-cycle or in the original graph, v is adjacent to two other vertices of degree 3.

Assume $F = A_0 = (23, 1, 5, 34)$. If $e = x_1v_2$, then G' contains a 4-cycle $v_1v_3v_4x_3$. This is similarly true if $e \in \{v_1x_2, v_3x_5, v_4x_4\}$. If $e \in \{x_3v_1, v_4x_3\}$, then G' is isomorphic to $T5$ which is acyclic, so our claim is true.

The rest of the cases use similar analogy. We summarize the result in the following (Table 5.1).

Note that the only instance we get a cut of size at most 4 is when $F = D^1_{4,5}$ and $e = u_2x_3$. In this case, u_2 is adjacent to two other vertices u_1, u_3 that are degree 3. This completes the proof for our lemma. $\square$

Lemma 5.12. Let G be a minor-minimal C5C graph that is also push-consistent. Let $e \in E(G)$. If $e = st$ where $d(s) = 3, d(t) \geq 4$, then either:

1. s is in a 5-cycle that does not contain e, or
2. s is adjacent to two other vertices s_1, s_2 of degree 3.

Proof: Let G' be the graph obtained from G by deleting the edge e and suppressing any degree 2 vertices. Since G is minor-minimal C5C, G' is not C5C. Let s_1, s_2 be the neighbors of s.

Assume that all 5-cycle of G that contains s also contains e. This implies that G' has girth 5. Since G' has minimum degree 3, it follows that there exists a cut X' in G' of size at most 4 that separates the graph into A', B' both containing a cycle. Note that we may assume $\{s_1, s_2\}$ and t are not in the same subgraph A', B', otherwise X' is also a cut in G. Without loss of generality, let $s_1, s_2 \in V(A'), t \in V(B')$. Let $X = X' \cup \{s\}, A =$ the graph induced by the $V(A') \cup \{s\}, B =$ the graph induced by $E(G)\backslash E(A)$. Note that A', B' are minors of A, B respectively. Then, X is a cut in G that separates A, B both containing a cycle. Since G is a mixed C5C graph, it implies that $|X'| = 4, |X| = 5$ and either A or B is the acyclic side. Since $d(t) \geq 4$, by Proposition 5.4, A is the acyclic side. Then, we can first push the cut along the edge e and keep pushing along other edges if needed until we reach a non-pushable cut Y with acyclic side A_Y. Note that by Corollary 5.4, $t \notin V(A_Y)\backslash V(B_Y)$. Thus $t \in Y$. Note that $X \in V(A_Y), A' \subseteq A_Y$. In particular, X' forms a cut that separates A' from the rest of the graph in $A_Y\backslash e$ and A' contains a cycle. Then, it follows from Lemma 5.11, s is adjacent to two other vertices of degree 3, proving our lemma. $\square$

Now, we are ready to replace acyclic sides with a T5 structure.

$F =$	$e =$	G'
$A_0 =$ $(23, 1, 4, 35)$	$x_1v_2, x_2v_1, x_4v_4, x_5v_3$	contains a 5-cycle
	x_3v_1, v_4x_3	is acyclic
$D_{3,4} =$ $((123) + (24, 35))$	v_1x_1, u_1x_4, u_2x_5	contains a 4-cycle
	$v_1x_2, v_1x_3, x_2u_1, u_2x_3$	is acyclic
$D_{4,4}$ $= ((12, 34) + (23, 45))$	$v_1x_1, v_2x_4, x_1u_1, u_2x_5$	contains a 5-cycle
	$v_1x_2, v_2x_3, u_1x_3, u_2x_4$	is isomorphic to $D_{3,4}$
$A_1 =$ $(12, 4, 3, 2, 45)$	v_1x_1, v_2x_4 v_3x_3, v_4x_2, v_5x_5	contains a 4-cycle
	v_1x_2, v_5x_4	is isomorphic to A_0
$A_2 =$ $(12, 4, 5, 2, 3, 45)$	v_1x_1, v_2x_4, v_3x_5 v_4x_2, v_5x_3, v_6x_4	contains a 4-cycle
	v_1x_2	is isomorphic to $(12, 3, 4, 5, 23)$
	v_6x_5	is isomorphic to A_1
$A_3 =$ $(12, 3, 4, 2, 3, 45)$	v_1x_1, v_2x_3, v_3x_4 v_4x_2, v_5x_3, v_6x_5	contains a 4-cycle
	v_1x_2, v_6x_4	is isomorphic to A_1
$D^1_{A,4} =$ $((23, 1, 5, 34) + (15, 24))$	$v_1x_2, v_2x_1, v_3x_5, v_4x_4$	contains a 4-cycle
	v_1x_3, v_4x_3	is isomorphic to $D^1_{4,5}$
	$u_1x_1, u_1x_5, u_2x_2, u_2x_4$	is isomorphic to a graph in $D_{A,3}$
$D^2_{A,4} =$ $((23, 1, 5, 34) + (14, 25))$	$v_1x_2, v_2x_1, v_3x_5, v_4x_4$	contains a 4-cycle
	v_1x_3, v_4x_3	is isomorphic to $D^1_{4,5}$
	$u_1x_1, u_1x_4, u_2x_2, u_2x_5$	is isomorphic to a graph in $D_{A,3}$
$D^1_{4,5} =$ $((15, 24) + (12, 3, 45))$	$v_1x_1, v_1x_5, v_2x_2, v_2x_4$	contains a 4-cycle
	$u_1x_1, u_1x_2, u_3x_4, u_3x_5$	is isomorphic to $D_{4,4}$
	u_2x_3	contains a cut of size 4

Table 5.1: Summary of Proof of Lemma 5.11

5.2.2 Reducing $D^1_{4,5}, A_1, A_2, A_3$

Lemma 5.13. Let G be a minor-minimal C5C graph that is also push-consistent. If X is a non-pushable cut with acyclic side A and cyclic side B, then A is not isomorphic to $D^1_{4,5}$.

Proof: For the sake of contradiction, assume $A = D^1_{4,5} = (15, 24) + (12, 3, 45) = (X(u_1), X(u_2) + (X(v_1), X(v_2), x(v_3)$. Let $G' = G\backslash\{u_1, u_2\}$. Since G is minor-minimal C5C, G' is not a C5C graph. By Lemma 5.9, there exists a cut Y' that separates G' into subgraphs C', D' such that $Y' \cap \{v_1, v_2, v_3\} = v_2, |Y' \cap X| \leq 1, |X \cap (V(C')\backslash V(D')| \geq 2$ and $|X \cap (V(D')\backslash V(C'))| \geq 2$. Since $Y' \cup \{u_1, u_2\}$ is a cut in G and G is C5C, $Y'| = 4$. We will break it into three cases.

Case 1: $|Y' \cap X| = 1, Y \cap X \neq x_3$.

Without loss of generality, assume $Y \cap X = x_1$. Note that x_4, x_5 are in the same subgraph, either C' or D'. Without loss of generality, assume $x_4, x_5 \in V(D')$. Since $|X \cap (V(C')\backslash V(D'))| \geq 2$, it follows that $x_2, x_3 \in V(C')$. Let $Y = Y' \cup \{u_2\}, C = C' \cup \{x_2u_2\}, D = D' \cup \{x_1u_1, u_1u_2, x_5u_1, x_4u_2\}$. Note that Y is a cut that separates G into C, D, both containing a cycle. Then either C or D is acyclic. Since $Y' \cap X = x_1$, we know that $x_2, x_5 \notin Y$. Note that $d(x_2), d(x_5) \geq 4$, contradicting Proposition 5.4.

Before proceeding on case 2, we will prove the following:

Claim 1: There exists distinct degree 3 vertices w_1, w_2, w_4, w_5 where for each $i = 1, 2, 4, 5$, w_i is adjacent to both x_i and x_3.

Consider the edge x_1v_1. Note that v_1 is not adjacent to two vertices of degree 3. Then, by Lemma 5.12, there exists a 5-cycle in G that contains the edges x_2v_1, v_1v_2. Since X is an independent set, it follows that x_1, x_3 has a common neighbor. By symmetry, we can obtain the vertices w_1, w_2, w_4, w_5. By Lemma 5.5, $d(w_i) = 3$ for $i = 1, 2, 4, 5$. Note that $w_1 \neq w_2$, otherwise, $N(w_1) = \{x_1, x_2, x_3\}$, contradicting Lemma 5.7. Thus, they are all distinct vertices, proving our claim.

Case 2: $Y' \cap X = \emptyset$ and Y' cannot be pushed to another cut such that contains vertices of X.

This implies that x_1, x_2 are in the same subgraph and x_4, x_5 are in the same subgraph. Without loss of generality, assume $x_1, x_2 \in V(C'), x_3, x_5, x_5 \in V(D')$. We will push as far as we can towards D' until we reach an non-pushable cut Y^* with subgraphs C^*, D^*. By our assumption in this case, $x_3 \notin Y^*$. Note that $x_1, x_2 \in V(C^*), x_3 \in V(D^*)$ and $x_1, x_2, x_3 \notin Y^*$. By our claim, it follows that the common neighbors $w_1, w_2 \in Y^*$. Let

$Y = Y^*\backslash\{v_3\} \cup \{x_1, x_2\}, C = C^* \cap B, D =$ the graph induced by all the edges not in C. Note that C has girth 5 and $d_C(x_1), d_C(x_2) \geq 2$. Since G has girth 5 and x_1, x_2 already has a common neighbor v_1, they are not adjacent to each other and do not share another common neighbor. Then, by Lemma 5.10, C contains a cycle. Since $D^* \subseteq D$, D also contains a cycle. Then, Y is a cut that separates G into C, D, both containing a cycle. Note that $d_D(x_1), d_D(x_2) \geq 2$ and the other vertices in Y has already been pushed as far towards D^* as possible. This implies that Y is also non-pushable. However, since $w_1, w_2 \in Y$, Y is not an independent set, contradicting Corollary 5.6.

Case 3: $Y' \cap X = x_3$

Before proceeding, we make the following claim.

Claim 2: There exists four paths $P_{i,j} = x_i z_{i,j} z'_{i,j} x_j$ for $i = 1, 2$ and $j = 4, 5$.

By Case 1, we may assume that G' does not contain a C4C cut that contains any of the vertices x_1, x_2, x_4, x_5. Thus we may assume all such cuts separates x_1 and x_4. Consider the graph $G' \cup x_1x_4$. Note that this graph is a minor of G. It then follows that this graph contains a 4-cycle and thus there exists such path $P_{1,4} = x_1 z_{1,4} z'_{1,4} x_4$ in B. By symmetry, Claim 2 is true.

Note that $z_{1,i} \neq z_{2,j}$ for $i, j = 4, 5$, otherwise G contains a 4-cycle $x_1 z_{1,i} x_2 v_1$. By symmetry, $z'_{i,4}$ is distinct from $z'_{j,5}$ for $i, j = 1, 2$. Since X is an independent set $z_{i,j}, z'_{i,j}$ are also not v_3 for $i = 1, 2, j = 4, 5$. Note that $|Y'\backslash\{v_2, x_3\}| = 2$. Without loss of generality, assume $z_{1,4} \in Y'$. Note that $z_{1,4} \neq z_{1,5}$, otherwise, Y' can be pushed along the edge $z_{1,4}x_1$, creating a cut that contains two vertices of X, contradicting Lemma 5.9. This implies that $z_{i,j}$ are distinct vertices for $i = 1, 2, j = 4, 5$. Note that X is an independent set and by Lemma 5.5, the vertices $z_{i,j}, z'_{i,j}$ are vertices of degree 3 for $i = 1, 2, j = 4, 5$. Then, it follows that the four paths $P_{i,j}$ are internally disjoint paths for $i = 1, 2, j = 4, 5$. However, this contradicts $|Y'\backslash\{v_2, x_3\}| = 2$. $\square$

The above two lemmas imply the following:

Corollary 5.14. Let G be a minor-minimal C5C graph that is also mixed and push-consistent. If there exists an edge $e = uv$ where $d(v) \geq 4$, then there exists a 5-cycle C such that $u \in V(C), e \notin E(C)$.

The next three lemmas eliminate A_1, A_2, A_3 as potential acyclic sides.

Lemma 5.15. Let G be a minor-minimal C5C graph that is also mixed and push-consistent. If X is a non-pushable cut with acyclic side A and cyclic side B, then A is not isomorphic to A_1.

Proof: Suppose for the sake of contradiction, G contains a cut X that separates the graph into A, B where A is isomorphic to $A_1 = (X(v_1), X(v_2), X(v_3), X(v_4), X(v_5)) = (12, 4, 3, 2, 45)$. Consider G', the graph obtained by deleting the edges v_2x_4, v_4x_2 and suppressing the degree 2 vertices. This is equivalent to replacing A with $T5 = (12, 3, 45)$. Note that G' has minimum degree 3 and girth 5. Then by Lemma 5.9, there exists a cut Y' of size at most 4 that separates the G' into two subgraphs C', D' both containing a cycle. We may also assume that $|Y' \cap X| \leq 1, Y' \cap (V(A) \backslash X) = v_3$.

Suppose $x_3 \in Y'$. By Lemma 5.9, we may assume $x_1, x_2 \in V(C'), x_4, x_5 \in V(D')$. Consider pushing the cut Y' along the edge v_3v_5. This produces a new cut Y'' separating two subgraphs C'', D'' both containing a cycle. Let $Y = Y'' \cup \{v_2\}$. Note that this is a cut that still separates C'', D'' in G'. It is also a cut that separates G into two similar subgraphs $C = C'' \cup \{x_2v_4\}, D = D'' \cup \{x_4v_2\}$ where both contains a cycle. Since G is mixed C5C, either C or D is acyclic. However, $d(x_2), d(x_4) \geq 4, x_2, x_4 \notin Y$, contradicting Proposition 5.4.

If $X \cap Y' = \emptyset$, the argument is very similar to the one made above where $y = (Y' \cup \{v_2, v_4\}) \backslash \{v_3\}$ is a C5C cut in G and $x_2, x_4 \notin Y$, a contradiction. If $x_1, x_5 \in Y'$, the argument is similar as well.

Now, suppose $|Y' \cap X| = 1, x_1, x_3, x_5 \notin Y'$. Without loss of generality, assume $x_2 \in Y'$. By Lemma 5.9, we may also assume $x_1, x_3 \in V(C'), x_4, x_5 \in V(D')$. Consider $Y = (Y'' \cup \{v_2, v_4\}) \backslash v_3$. For similar reasons as before, Y is a C5C cut in G with an acyclic side C and cyclic side D. Since $d(x_4) \geq 4, x_4, x_5 \in V(D)$. This implies that $x_1, x_3 \in V(C)$. Consider pushing the cut along the edges v_2x_4, v_4v_5 to obtain a new C5C cut Z. Note that v_3 is still in the acyclic side with respect to Z. However, no neighbors of v_3 is in Z, contradicting Proposition 5.4. $\square$

Lemma 5.16. Let G be a minor-minimal C5C graph that is also mixed and push-consistent. If X is a non-pushable cut with acyclic side A and cyclic side B, then A is not isomorphic to A_2.

Proof: Before proceeding, we need the following claim:

There exists vertices $z_{1,4}, z_{3,5} \in V(B)$ where $z_{i,j}$ is adjacent to both z_i, z_j.

Consider the edge x_2v_1. By Lemma 5.14, G contains a cycle of length 5 that contains the edges x_1v_1, v_1v_2. Since X is an independent set, it follows that x_1, x_3 has a common neighbor, $z_{1,3}$. By applying the same analysis to the edge x_4v_6, we get the vertex $z_{2,5}$. Note that these two vertices are not in A, thus they are in B. By Proposition 5.5, $z_{1,4}, z_{3,5}$ are degree 3 vertices, implying that they are also distinct.

If $x_1 \in Y'$, then by Lemma 5.9, we may assume $x_2, x_3 \in V(C'), x_4, x_5 \in V(D')$. Then, by our claim, $z_{3,5} \in Y'$. By Lemma 5.5, $z_{3,5}$ has one other neighbor $z \neq x_3, x_5$. Note that $z \in V(C')$ or $z \in V(D')$. This implies Y' can be pushed to either x_3 or x_5, creating a new cut Y''. However, this implies that $|Y'' \cap X| = 2$, contradicting Lemma 5.9.

Note that by the same argument, if one of x_2, x_3, x_4 or $x_5 \in Y'$, then at least one of $z_{1,4}, z_{3,5}$ is also in Y'. Then, it also implies that Y' can be pushed to another cut that contains two vertices of X, contradicting Lemma 5.9. Thus we may assume $Y' \cap X = \emptyset$.

Then, by Lemma 5.9, we may assume without loss of generality $x_1, x_2 \in V(C'), x_4, x_5 \in V(D')$. Then, we have two cases; either $x_3 \in V(C')$ or $x_3 \in V(D')$. If $x_3 \in V(C')$, then both $z_{1,4}, z_{3,5} \in Y'$. This implies that Y' can be pushed away from both vertices towards two other vertices in X, contradicting Lemma 5.9. If $x_3 \in V(D')$, then first push Y' along the edge v_5v_1 to obtain the cut Y''. Note that $Y'' \cup \{v_4\}$ is still a cut that separates G into two subgraphs C, D. Note that C, D still contains a cycle. Since G is C5C, $|Y| = 5$ and either C or D is the acyclic side. However, $d(x_2), d(x_4) \geq 4$, contradicting Proposition 5.4.
□

Lemma 5.17. Let G be a minor-minimal C5C graph that is also mixed and push-consistent. If X is a non-pushable cut with acyclic side A and cyclic side B, then A is not isomorphic to A_3.

Proof: Assume for the sake of contradiction that $A = (12, 3, 4, 2, 3, 45) = (X(v_1), X(v_2), X(v_3), X(v_4)$
Consider the graph G' obtained by deleting the edges v_2x_3, v_3x_4, v_4x_2 and suppressing the degree 2 vertices. This is equivalent to replacing A with $T5 = (12, 3, 45)$. By Lemma 5.9, there exists a cut Y' of size at most 4 such that $v_5 \in Y'$ and $|Y' \cap X| \leq 1$. Let C', D' be the two graphs separated by Y' in G'.

Suppose $x_1 \in Y'$. By Lemma 5.9, we may assume that $x_2, x_3 \in V(C'), x_4, x_5 \in V(D')$. First, push Y' along the edge v_5v_6 to obtain a new cut Y'' with subgraphs C'', D''. Consider $Y = Y' \cup \{v_3\}$. Note that Y is a cut in G that separates the grpah into two subgraphs C, D both still contains a cycle. Since G is C5C, $|Y| = 5$ and either C or D is the acyclic side. However, $d(v_2), d(v_4) \geq 4$, contradicting Lemma 5.4. Thus symmetry, $x_1, x_5 \notin Y'$.

If $x_2 \in Y'$, we can also first push the cut along v_5v_6 and add v_3 to get a cut Y that separates G into two subgraphs both containing a cycle. Once again, either x_3 or x_4 is in the acyclic side. However, $d(x_3), d(x_4) \geq 2$, a contradiction.

By symmetry, $x_2, x_4 \notin Y'$. If $x_3 \in Y'$, the exact same argument can be made. This implies that $Y' \cap X = \emptyset$. Then, without loss of generality and by Lemma 5.9, we may assume that $x_1, x_2, x_3 \in V(C'), x_4, x_5 \in V(D')$. Once again, we can push the cut along v_5v_6 and add v_3 to create a C5C mixed cut Y in G. Then, by the same argument, $d(x_2), d(x_4) \geq 4$, a contradiction. $\square$

5.2.3 Proving Theorem 5.1

Note that by this point, we have proven the first statement in Theorem 5.1. The rest of this section is to prove the other statements in the theorem.

Lemma 5.18. Let G be a minor-minimal C5C graph that is also mixed and push-consistent. Then, for every degree-3 component $S, |S| > 2$.

Suppose for the sake of contradiction that there exists two degree 3 vertices v_1, v_2 that are adjacent to each other and all other neighbors have degree 4 or higher. Let $X = \{a_1, a_2, b_1, b_2\}$ be the vertices where a_i, b_i be the other two neighbors of v_i for $i = 1, 2$. Since G has girth 5, all vertices in X are distinct. Let $G' = G \backslash \{v_1, v_2\}$. It follows from Corollary 5.3 that there exists a cut Y' of size at most 4 that separates G' into A', B' both containing a cycle. Assume $|Y'|$ is minimum.

Claim 1: $Y' \cap X = \emptyset$.

Suppose for the sake of contradiction, $Y' \cap X \neq \mathit{emptyset}$. Note that if $|Y' \cap X| \geq 3$, then Y' is still a cut in G and both sides still contains a cycle, contradicting G being C5C. Thus we may assume there exists at least one vertex $x \in X \cap (V(A') \backslash V(B'))$ and another vertex $x' \in X \cap (V(B') \backslash V(A'))$. Without loss of generality, assume $x = a_1$. If a_1 is the only vertex in $X \cap (V(A') \backslash V(B')$, then $Y = Y' \cup \{u_1\}$ is a C5C cut in G. This implies that either a_1 or x' is in the acyclic side. However, since both a_1, x' have degree 4 or more, it contradicts Proposition 5.4. Thus, by symmetry and pigeonhole principle, $|X \cap (V(A') \backslash V(B'))|, |X \cap (V(B') \backslash V(A'))| = 2$. Thus $Y' \cap X = \emptyset$.

Furthermore, note that if $a_1, b_1 \in V(A')$ and $a_2, b_2 \in V(B')$, then $Y = Y' \cup \{u_1\}$ is also a C5C cut in G, contradicting Proposition 5.4. Thus, without loss of generality, we may assume that $a_1, a_2 \in V(A')$ and $b_1, b_2 \in V(B')$.

Claim 2: for $i, j = 1, 2$, the vertices a_i, b_j do not have a common neighbor in G'.

Suppose for the sake of contradiction that for some $i, j = 1, 2$, a_i, b_j has a common neighbor in G'. Note that $i \neq j$, otherwise a_i, b_i has two common neighbors in G, forming a 4-cycle, a contradiction. Without loss of generality, assume a_1, b_2 has a common neighbor u. By Lemma 5.5, u has degree 3. Since $a_1 \in V(A'), b_2 \in V(B')$ and $a_1, b_2 \notin Y'$, it follows that $u \in Y'$. However, since u has only one other neighbor, the cut Y' can be pushed either along the edge ua_1 or the edge ub_2, contradicting Claim 1. This proves Claim 2.

Now, consider the edge v_1b_1 in G. By Lemma 5.14, there exists a cycle containing the edges a_1v_1, v_1v_2. It follows from Claim 2 that a_1, a_2 has a common neighbor a. By symmetry, b_1, b_2 also has a common neighbor b. Note that by Lemma 5.5, a, b are degree 3 vertices

Note that a, b are not adjacent to each other. Suppose for the sake of contradiction, that $ab \in E(G)$. Let F be the subgraph of G induced by the vertices u_1, u_2, a, b and X. Note that X is a cut that separates G into F, F' where F' is the graph induced by all edges not in F. Sice G is C5C, $|X| = 4$ and F contains a cycle, it follows that F' is acyclic and either $F' = S4$ or $T4$. If $F' = S4$, then G contains a 4-cycle, a contradiction. If $F' = T4$, it follows that G is the Petersen graph. However, the Petersen graph is doubly-acyclic, not mixed, a contradiction.

Claim 3: $a, b \notin Y'$.

We will prove by way of contradiction. Without loss of generality, assume $a \in Y'$. Let a' be the third neighbor of a. Note that $a' \notin X$, otherwise G contains a cycle of length less than 5. Also, from previous argument, $a' \neq b$. Since $a_1, a_2 \in V(A')$ and $a \in Y'$, we may assume $a' \in V(B')$. Consider the edge aa_2 in G. By Lemma 5.14, there exists a 5-cycle containing a_1a, aa'. Thus, there exists a path $a_1zz'a''$ in G. We will break into two cases, either $z' = b_1$ or $z' \neq b_1$.

Case 1: $z' = b_1$

This implies that $z = v_1$. By Lemma 5.5, a' has degree 3. Let a'' be the third neighbor of a'. Note that $a'' \neq b_2, b, a, a_1, a_2$, otherwise, G contains a cycle of length less than 5, a contradiction. Note that $a'' \notin Y'$, otherwise, the cut Y' can be pushed along the edges, $aa', a'b_1$, contradicting Claim 1. Since $a'' \notin Y'$ and $a' \in V(B')\backslash V(A')$, $b' \in V(B')\backslash V(A')$ as well. Now, consider the edge aa_2. By Lemma 5.14, there exists a 5-cycle containing a_2a, aa'. It then follows form Claim 2 that a'' and a_2 has a common neighbor, y. Since $a''' \notin Y'$, it follows that $y \in Y'$. By Proposition 5.5, y has degree 3, a neighbor other than a_2, a'''. Let y' be the third neighbor of y. If $y' \in V(B')$, then Y' can be pushed along the edge ya_2, contradicting Claim 1. If $y' \in V(A')$, then Y' can be pushed along the edges $y'a''', aa', a'a'', a''b_1$, contradicting Claim 1. This completes our proof of Case 1.

Case 2: $z' \neq b_1$.

Note that by Claim 2, $z' \neq b_2$. Since G has girth 5, it follows from Case 1 that $z, z' \neq a, a_1, a_2, b_1, b_2, b$. Note that since $a_1 \in V(A')$ and $a' \in V(B')$, at least one of $a, a' \in Y'$. By Lemma 5.5, z has degree 3. If $z \in Y'$, then Y' can be pushed either along the edge za_1 or zz'. By Claim, 1, Y' cannot be pushed along za_1. Thus without loss of generality, we may assume that $z' \in Y'$ and $z \in V(A')\backslash V(B')$.

Now consider applying Lemma 5.14 to the edge a_2a. By symmetry, there exists a path $a_2yy'a$ where $y, y' \neq a_1, a_2, b_1, b_1, a, b$. We will first prove that $z' = y'$.

Suppose for the sake of contradiction, $z' \neq y'$. Without loss of generality, we may assume $y' \in Y'$. Note that $N(a') = \{a, z', y'\} \subseteq Y'$. It follows from Proposition 5.8 that a is not needed in the cut, contradicting the minimality assumption of $|Y'|$. Thus, we may assume $z' = y'$. Note that $z' \in Y', z, y \in V(A')\backslash V(B')$.

Note that $z \neq y$, otherwise, za_1aa_2 forms a 4-cycle in G. Now, we will prove that $d(z') \geq 4$. For the sake of contradiction, if $d(z') = 3$, $N(z') = \{y, z, a'\}$. Then, Y' can be pushed along the edge za' and a is once again no longer needed in the cut, contradicting the minimality assumption of $|Y'|$. Thus, $d(z') \geq 4$.

Then, consider applying Lemma 5.14 on the edge $z'a'$. This implies there exists a path $a'w'w$ where $w' \neq z', b_1, b_2$ and w is adjacent to either a_1 or a_2. By Lemma 5.5, $d(w) = 3$. Note that one of w or w' is in Y'. By similar argument as before, the cut can always be pushed onto w', thus we may assume without loss of generality that $w' \in Y'$. However, this implies that $N(a') = \{z', w', a\} \subseteq Y'$. By Proposition 5.8, a is not needed in the cut, contradicting the minimality of $|Y'|$. This completes the proof for Case 2. Thus we may assume there does not exist a cut Y' where $a \in Y'$. By symmetry, we may also assume $b \notin Y'$.

Claim 4: for $i, j = 1, 2$, there exists 4 disjoint paths $a_ia_{i,j}b_{i,j}b_j$ where $a_{i,j}, b_{i,j}$ are degree 3 vertices and $|Y \cap \{a_{i,j}, b_{i,j}\}| = 1$.

For $i, j = 1, 2$, consider the graph $G_{i,j}$ obtained from G' by adding the edge a_ib_j. Since every CkC cut, where $k \leq 4$ in G' separates a_i and b_j, it follows that $G_{i,j}$ does not contain any CkC cuts. Note that $G_{i,j}$ is a minor of G. This implies that $G_{i,j}$ contains a cycle of length 4 or less. Thus, there exists a path of length $a_ia_{i,j}b_{i,j}b_j$ in G. Since Y' separates a_i from b_j, at least one of $a_{i,j}$ or b_Pi, j is in Y'. Note that by Lemma 5.5, $a_{i,j}, b_{i,j}$ have degree 3.If $a_{i,j} \in Y'$, by Claim 1, the third neighbor of $a_{i,j} \notin V(B')$ and the cut can be pushed along the edge $a_{i,j}b_{i,j}$. Then, by similar argument, it follows that the third neighbor of $b_{i,j} \notin V(A')$. By symmetry, it implies that only one of $a_{i,j}, b_{i,j} \in Y'$ and the cut can be easily pushed from one to the other.

Now, we will show that $a_{i,j}, b_{k,l}$ are distinct vertices for $i, j, k, l = 1, 2$. First, note that $a_{i,j} \neq a$; otherwise, the cut can be pushed towards a contradicting Claim 3. Note that $a_{i,j} \neq a_{k,j}$ if $i \neq k$ for $i, j, k = 1, 2$; otherwise, $a_{i,j}, a_i, a, a_k$ forms a 4-cycle in G. Note that $a_{i,j} \neq a_{i,k}$ if $j \neq k$ for $i, j, k = 1, 2$; otherwise, the cut can be pushed from $a_{i,j}$ to a_i, contradicting Claim 1. Since $d(a_{i,j}) \neq 4$, $a_{i,j} \neq b_{k,l}$ for any $i, j, k, l = 1, 2$. Then, by symmetry, all of the vertices $a_{i,j}, b_{k,l}$ are distinct, proving our claim.

Let a' the third neighbor of $a_{1,1}$. without loss of generality, we may assume $Y' = \{a_{1,1}, a_{1,2}, a_{2,1}, a_{2,2}\}$. Note that $a' \neq a_{i,j}$ for any $i, j = 1, 2$; otherwise, the cut can be pushed from $a_{1,1}$ to a_1, contradicting Claim 1. Now, apply Lemma 5.14 to the edge $a_1 a_{1,1}$. This implies there exists a path $a' z_1 z_2 b_{1,1}$ disjoint from $a_{1,1}$. Note that $a', b_{1,1} \notin Y'$. Since $b_{1,1} \not b_{i,j}$ where $(i, j) \neq (1, 1)$, then $z' \neq a_{i,j}$. Then, it follows that $z' \notin Y'$ and thus $z \in Y'$. Let $z = a_{i,j}$ for some $(i, j) \neq (1, 1)$. Note that $z' \in V(B') \notin V(A')$ so $z' = b_{i,j}$. Then, $N(b_{1,1}) = \{a_{1,1}, b_1, b_{i,j}\}$. Then, the cut Y' can be pushed along the edges $a_{1,1}b_{1,1}, a_{i,j}b_{i,j}, b_{1,1}b_1$, contradicting Claim 1.

This completes our proof. $\square$

It follows from the previous lemma that no acyclic side of a minor-minimal C5C graph that is also push-consistent is isomorphic to $Gr(F)$ where F is an element of $D_{4,A}$. Thus, we have the following corollary:

Corollary 5.19. Let G be a minor-minimal C5C graph that is also mixed and push-consistent with a mixed cut X and acyclic side A. Then, A is isomorphic to $A_0, D_{3,4}, D_{4,4}, C_5$.

Note that we also have the following:

Lemma 5.20. If G is a minor-minimal mixed C5C graph that is also mixed and push-consistent and S is a degree-3 componenet, then $|S| \geq 4$.

Proof: Suppose for the sake of contradiction, there exist degree 3 vertices v_1, v_2, v_3, vertices $u_1, u_1', u_2, u_3, u_3'$ with degree more than 3 and edges $v_1u_1, v_1u_1', v_1v_2, v_2v_3, v_2v_2', v_3u_3, v_3u_3'$. By applying Lemma 5.14 on v_2u_2, one of the following $u_1u_3, u_1u_3', u_1'u_3, u_1'u_3'$ is an edge in G. However, this contradicts Lemma 5.5. $\square$

Lemma 5.21. Let G be a minor-minimal C5C graph that is also mixed and push-consistent . Then, for every edge e, there exists a 5-cycle C and a degree 3 vertex v such that $V(C) \cap V(e) = \{v\}$.

Proof: Note that by Lemma 5.14, the lemma is true if e is incident to a degree 4 vertex. Then, it follows from Lemma 5.5 that we only need to consider edges e that are incident to two degree 3 vertices. Assume for the sake of contradiction that there exists an edge $e = uv$ such that every 5-cycles that contains u also contains v. It follows from Lemma 5.14 that $d(u) = d(v) = 3$. For $i = 1, 2$ let u_i, v_i be the respective neighbours of u, v. Let G' be the graph obtained by deleting e and suppressing u, v. Note that since G has girth 5, u_i is not adjacent to v_j for $i, j = 1, 2$. Then, it follows that G' also has girth at least 5. Since G is minor-minimal C5C, G' contains a cut Y' with $|Y'| \leq 4$ that separates G' into subgraphs A', B' both containing a cycle. Note that u_1u_2 and v_1v_2 are not in the same subgrpah, otherwise, Y' is also a CkC cut in G where $k \leq 4$. Without loss of generality, let $u_1u_2 \in E(A'), v_vv_2 \in E(B')$. Let $Y = Y' \cup \{v\}, A = (A' \backslash u_1u_2) \cup \{u_1u, uu_2, uv\}, B = (B' \backslash v_1v_2) \cup \{v_1v, vv_2\}$. Note that Y separates G into A, B both containing a cycle. Thus, $|Y| = 5$ and either A or B is the acyclic side. Note that $u_1, u_2 \in V(A), v_1, v_2 \in V(B)$. Then, the cut can be pushed back and forth along the edge uv. Thus, without loss of generality, we may assume that A is the acyclic side. Consider pushing the cut Y as far as we can towards B to obtain a non-pushable cut Y^* with acyclic side A^*. Note that $Y \subseteq V(A^*)$. By Corollary 5.19, A^* is isomorphic to $A_0, D_{2,4}, D_{4,4}$ or C_5. We will break into two cases, either $v \in Y^*$ or $v \notin Y^*$.

If $v \in Y^*$, then $N_{A^*}(v) = u$. Suppose $A^* = A_0 = (23, 1, 5, 34) = (X(w_1), X(w_2), X(w_3), X(w_4,))$. Note that $v \in X$. Since $d_{A^*}(x_3) = 2, v \neq x_3$. If $v = x_1, x_2, x_4, x_5$, then u is in a 5-cycle that does not contain v, contradicting our original assumption. We can apply similar analysis to check that A^* is also not isomorphic to $D_{3,4}, D_{4,4}, C_5$, contradicting Corollary 5.19.

Suppose $v \notin Y^*$. Note that A' is a minor of A^*. First, suppose A^* is isomorphic to $A)9$. Note that by deleting any edge in A_0 that is not incident to a vertex of X, the graph becomes acyclic, contradicting A' containing a cycle. By the same analysis, A^* is also not isomorphic to $D_{3,4}, D_{4,4}, C_5$, contradicting Corollary 5.19. $\square$

Lemma 5.22. Let G be a minor-minimal C5C graph that is also mixed and push-consistent. Then, for every vertex v, there exists a 5-cycle C and a vertex $u \in N(v)$ such that $u \in V(C)$ but $v \notin V(C)$.

Proof: Suppose for the sake of contradiction that there exists a vertex v such that for any 5-cycle C that contains a degree-3 vertex $u \in N(V)$, C also contains v. It follows from Lemma 5.14 that $d(v) = 3$. Let u_1, u_2, u_3 be the neighbours of v. Consider $G' = G \backslash v$ and suppressing any degree-2 vertices.

First, we claim that G' has girth 5. For the sake of contradiction, suppose G' has a cycle C' of length less than 5. Let C be the cycle in G that becomes C' after deleting v

and suppressing any degree 2 vertices. Since G has girth 5, C contains at least one of the suppressed vertices.

Suppose C contains exactly one of the suppressed vertices. Without loss of generality, assume u_1 is the suppressed vertex. Let W_1, w_1' be the two neighbours of u_1 other than v. Since C only contains one suppressed vertices, it follows that $|C| - |C'| = 1$ and $|C| = 5$. Then, by our assumption, $v \in V(C)$. Note that $w_1, u, w_1', v \in V(C)$ and $w_1 u, uw_1' \in E(C)$. Then, either w_1 or w_1' is adjacent to v, implying G contains a 3-cycle, a contradiction.

Suppose C contains two suppressed vertices. Without loss of generality, assume they are u_1, u_2 and let w_i, w_i' be the other neighbours of u_i for $i = 1, 2$. Then, $u_1, w_1, w_1', u_2, w_2, w_2' \in V(C)$. The, it follows $|C| = 6, |C'| = 4$ and there exists a perfect matching amongst w_1, w_1' to w_2, w_2'. Without loss of generality, assume $w_1 w_2, w_1' w_2'$ are edges in G. By Lemma 5.5, at least one of w_1, w_2 has degree 3. Without loss of generality, assume $d(w_1) = 3$ and let z_1 be the third neighbour of w_1. Similarly, one of w_1', w_2' has degree 3. We will break into two cases. In the first case, suppose $d(w_1') = 3$. Let z_1' be the third neighbour of w_1'. Consider the cut $X = \{v, u_1, w_1, w_2, u_2\}$ that clearly separates a 5-cycle from the rest of the graph. Consider pushing the cut along the edges $vu_3, u_1 w_1', w_1' z_1', w_1 z_1$ to obtain a new cut Y with acyclic side A. Note that $u_1 \in V(A)$ but $N(u) \cap Y = \emptyset$, contradicting Proposition 5.4. In the second case, suppose $d(w_2') = 3$. Let z_2' be the third neighbour of w_2'. Once again, consider the same cut X and now we push along the edges $vu_3, u_1 w_1' w_1 z_1, u_2 w_2', w_2' z_2'$ to obtain a new cut Y and acyclic side A. Note that A is isomorphic to $Gr(A_1)$. Let Y^* be the non-pushable cut we reach by pushing Y along other edges. Let A^* be the newly obtained acyclic side. Note that $|V(A^*)| \geq |V(A)| = |V(C_5)| > |V(Gr(A_0))|, |V(Gr(D_{3,4}))|, |V(Gr(D_{4,4}))|$. Since A is not isomorphic to C_5, it follows that A^* is not isomorphic to $A_0, D_{3,4}, D_{4,4}, C_5$, contradiction Corollary 5.19.

Lastly, if C contains three suppressed vertices, then $4 \geq |C'| = |C| - 3 \geq 9 - 3 = 6$, a contradiction. Thus, there does not exist such cycle C, proving our claim.

Note that G' has minimum degree 3. Since G is C5C minimal, it follows that G' contains a CkC cut Y' where $k \leq 4$. Consider $Y = Y' \cup \{v\}$. It follows that Y is a C5C cut in G. Consider pushing the cut Y towards the cyclic side until we reach a non-pushable cut Y^* with acyclic side A^* and cyclic side B^*. We will break into two cases, either $v \in Y^*$ or $v \notin Y^*$.

Suppose $v \in Y^*$. Since Y^* is non-pushable and $d(v) = 3$, $d_{B^*}(v) = 2, d_{A^*}(v) = 1$. By Corollary 5.19, A^* is isomorphic to one of $A_0, D_{3,4}, D_{4,4}, C_5$. However, by checking every possible vertex in the cut where $d_{A^*}(v) = 1$, u_1 is always in a 5-cycle that does not contain v, a contradiction.

Now, suppose $v \notin Y^*$. Note that $Y' \subseteq V(A^*)$. Since Y' is a CkC cut, it follows that

A^* contains a cycle C where $v \notin V(C)$. By Corollary 5.19, A^* is isomorphic to one of $A_0, D_{3,4}, D_{4,4}, C_5$. However, note that by deleting any vertex in $V(A^*) \backslash Y^*$, these graphs becomes acyclic, a contradiction. $\square$

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