Classical Field Theory in the BV Formalism

by

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AUTHOR'S DECLARATION

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Abstract

This document is a review of the perspective on classical field theories presented in [2] and [3].

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1 Introduction

This document is essentially a review of the perspective on classical field theories presented in [2] and [3]. The only previously unwritten material is the presentation of AKSZ type theories in this language, but this is really only marginally original, and was explained to me by Kevin Costello.

The following is a summary of the contents of this document in the order in which they should be understood:

- Appendix A: the linear algebraic constructions underlying all the techniques in this paper.
- Section 2.1: the geometric interpretation of L_∞ algebras as formal derived stacks
- Appendix B: the functional analytic background necessary for doing the above constructions with sheaves of sections of vector bundles
- Section 2.2-2.3: the technology necessary for, and the definition of, classical field theories in the BV formalism
- Appendix C: the differential operators and D_M modules background necessary for a detailed description of the spaces of, and spaces of deformations of, classical field theories
- Section 2.4-2.5: Description of classical field theories by local action functionals and deformation theory of local L_∞ algebras classical field theories
- Section 3: a family of examples of classical field theories

2 Classical Field Theories

2.1 Formal Derived Stacks and Finite Dimensional L_∞ Algebras

Let k be the base field and $\mathfrak{g}$ an L_∞ algebra over k. Then $\mathfrak{g}$ can be understood as encoding the information of the infinitesimal neighbourhood of a point x in a space X. Precisely, $\mathfrak{g}$ can be interpretted as the -1 shifted tangent complex to a derived stack X at a geometric point $x \in X$, and this contains enough information to reconstruct the formal neighbourhood $\hat{X}_x$ of x. We can think of $\hat{X}_x$ as itself a space, with only one underlying geometric point but with a potentially interesting derived ring of functions, and we call such spaces formal derived stacks. We will not define derived stacks here, but will just define the category of formal derived stacks to be the opposite of the category of L_∞ algebras, and then explain how to understand these objects as describing the derived algebraic geometry of a space with a single underlying geometric point.

Recall from appendix 4 that for $\mathfrak{g}$ an L_∞ algebra we define the cdga of Chevalley-Eilenberg cochains by

$$C^\bullet(\mathfrak{g}; k) := \mathrm{Sym}_k^{\hat{\bullet}}(\mathfrak{g}^*[-1])$$

together with the differential d^{CE} determined on generators $\mathfrak{g}^*[-1]$ by the duals of the L_∞ structure maps. We define the formal derived stack $B\mathfrak{g}$ corresponding to $\mathfrak{g}$ by

$$B\mathfrak{g} := \mathrm{Spec}_k C^\bullet(\mathfrak{g}; k) \qquad \text{so that} \qquad \mathcal{O}(B\mathfrak{g}) = C^\bullet(\mathfrak{g}; k)$$

Here we define Spec_k to mean that $B\mathfrak{g}$ is the space with one underlying geometric point and ring of functions $C^\bullet(\mathfrak{g}; k)$. Note that this is a complete augmented cdga, with augmentation $\varepsilon : C^\bullet(\mathfrak{g}; k) \to k$ given by the projection to $\mathrm{Sym}^0_k = k$; the augmentation map is interpretted as the evaluation of a function at the underlying geometric point of $B\mathfrak{g}$, or dually as the inclusion of the geometric point $\mathrm{Spec}_k k$.

We define a vector bundle on $B\mathfrak{g}$ to be an L_∞ module N for $\mathfrak{g}$, and define its space of global sections by

$$\Gamma(B\mathfrak{g}, N) := C^\bullet(\mathfrak{g}; N) = \mathrm{Sym}^\bullet(\mathfrak{g}^*[-1]) \otimes N$$

Note that $\Gamma(B\mathfrak{g}, N)$ is a dg module for $\mathcal{O}(B\mathfrak{g})$, a derived version of the usual fact that the sheaf of sections of a vector bundle N over X is a sheaf of modules for the structure sheaf $\mathcal{O}_X$. In particular, we define the k-shifted tangent and cotangent bundles by

$$\Gamma(B\mathfrak{g}, T[k]B\mathfrak{g}) = C^\bullet(\mathfrak{g}; \mathfrak{g}[k+1]) \qquad \Gamma(B\mathfrak{g}, T^*[k]B\mathfrak{g}) = C^\bullet(\mathfrak{g}; \mathfrak{g}^*[k-1])$$

For $k = 0$ we drop the shift from the notation. The primary geometric features of these vector bundles are:

Proposition 2.1. *Let $\mathfrak{g}$ be a finite type L_∞ algebra.*

- *There is a natural universal derivation*

  $$d : \mathcal{O}(B\mathfrak{g}) \to \Gamma(B\mathfrak{g}, T^*B\mathfrak{g})$$

  defined on each degree $n \geq 1$ by a symmetrizing sum over the inclusions

  $$\mathrm{Sym}^n(\mathfrak{g}^*[-1]) \hookrightarrow \mathrm{Sym}^{n-1}(\mathfrak{g}^*[-1]) \otimes \mathfrak{g}^*[-1]$$

  and defined as 0 on Sym^0.

- *There is a natural non-degenerate pairing*

  $$\langle \cdot, \cdot \rangle_{B\mathfrak{g}} : \Gamma(B\mathfrak{g}, T[k]B\mathfrak{g}) \otimes \Gamma(B\mathfrak{g}, T^*[-k]B\mathfrak{g}) \to \mathcal{O}(B\mathfrak{g})$$

defined by contracting the additonal $\mathfrak{g}[1]$ *and* $\mathfrak{g}^*[-1]$ *tensor factors, which is a map of* $\mathcal{O}(B\mathfrak{g})$ *dg modules, and thus we can identify*

$$\Gamma(B\mathfrak{g}, T^*[-k]B\mathfrak{g}) = \Gamma(B\mathfrak{g}, (T[k]B\mathfrak{g})^*) := \mathrm{Hom}_{\mathcal{O}(B\mathfrak{g})}(\Gamma(B\mathfrak{g}, T[k]B\mathfrak{g}), \mathcal{O}(B\mathfrak{g}))$$

- *There is a natural map of* $\mathcal{O}(B\mathfrak{g})$ *dg modules*

$$\Gamma(B\mathfrak{g}, T[k]B\mathfrak{g}) \to \mathrm{Der}^k(\mathcal{O}(B\mathfrak{g})) \qquad \textit{defined by} \qquad X(f) = \langle X, df \rangle_{B\mathfrak{g}}$$

for each $X \in \Gamma(B\mathfrak{g}, T[k]B\mathfrak{g})$ *and* $f \in \mathcal{O}(B\mathfrak{g})$*, where* Der^k *denotes the algebra of cohomological degree* k *derivations.*

2.2 Local L_∞ Algebras and Modules

Throughout the remainder of this chapter, let M be a smooth manifold over $\mathbb{R}$ and C^∞_M denote the sheaf of smooth functions on M. Recall from Appendix 6.2 the definition:

Definition 2.2. *Let $E_1, ..., E_n, F$ be vector bundles on M and $\mathcal{E}_i, \mathcal{F}$ their sheaves of sections. We define the sheaf of polydifferential operators by:*

$$\mathrm{PolyDiff}(\mathcal{E}_1 \otimes ... \otimes \mathcal{E}_n, \mathcal{F}) = \mathrm{Diff}(\mathcal{E}_1, C^\infty_M) \otimes_{C^\infty_M} ... \otimes_{C^\infty_M} \mathrm{Diff}(\mathcal{E}_n, C^\infty_M) \otimes_{C^\infty_M} \mathcal{F}$$

There is a natural inclusion $\mathrm{PolyDiff}(\mathcal{E}_1 \otimes ... \otimes \mathcal{E}_n, \mathcal{F}) \hookrightarrow \mathrm{Hom}(\mathcal{E}_1 \hat{\otimes} ... \hat{\otimes} \mathcal{E}_n, \mathcal{F})$ defined by

$$(D_1 \otimes ... \otimes D_n \otimes f)(e_1 \otimes ... \otimes e_n) = (D_1 e_1)...(D_n e_n) f$$

for each $D_i \in \mathrm{Diff}(\mathcal{E}_i, C^\infty_M), e_i \in \mathcal{E}_i, f \in \mathcal{F}$. We let $\mathrm{Hom}_{\mathrm{loc}}(\mathcal{E}_1 \hat{\otimes} ... \hat{\otimes} \mathcal{E}_n, \mathcal{F})$ denote the image of this inclusion, and identify polydifferential operators with their corresponding maps of sheaves of sections throughout.

Definition 2.3. *A local L_∞ algebra on M is a graded vector bundle L on M, with sheaf of sections $\mathcal{L}$, together with a collection of polydifferential operators*

$$\{l_n : \mathcal{L}^{\hat{\otimes} n} \to \mathcal{L}[2-n]\}_{n \in \mathbb{N}^+}$$

making $\mathcal{L}$ into a sheaf on M of L_∞ algebras in the category NF*.*

A local L_∞ algebra L is called abelian if $l_n = 0$ for $n \geq 2$ and trivial if $l_n = 0$ for $n \geq 1$.

Given a local L_∞ algebra L, let $C^\bullet(\mathcal{L})$, the Chevalley-Eilenberg cochains on L, be the precosheaf of cdga's in the category NDF, defined by:

$$C^\bullet(\mathcal{L}(U)) \equiv \mathcal{O}(B\mathcal{L}(U)) := \mathrm{Sym}^{\hat{\bullet}}(\mathcal{L}(U)^*[-1]) = \prod_{n \in \mathbb{N}} (\mathcal{L}(U)[1]^{\hat{\otimes} n})^*_{S_n}$$

equipped with the Chevalley-Eilenberg differential; the notation $\mathcal{O}(B\mathcal{L}(U))$ indicates that we think of a local L_∞ algebra as determining a presheaf of formal derived stacks $B\mathcal{L}$ over the base manifold M.

Definition 2.4. *Let L be a local L_∞ algebra. A local L_∞ module for L is a graded vector bundle N on M, with sheaf of sections $\mathcal{N}$, and a differential operator $d : \mathcal{N} \to \mathcal{N}[1]$ satisfying $d^2 = 0$, such that*

- *$\mathcal{N}$ is a sheaf of L_∞ modules for the sheaf of L_∞ algebras $\mathcal{L}$ in the category* NF
- *The L_∞ module structure maps $\mathcal{L}^{\otimes n} \otimes \mathcal{N} \to \mathcal{N}[1-n]$ are given by polydifferential operators of each $n \in \mathbb{N}$*

Given a local L_∞ algebra L and a local L_∞ module N for L, we let $C^\bullet(\mathcal{L}, \mathcal{N})$, the Chevalley-Eilenberg cochains on L with coefficients in N, be the precosheaf of cdga's defined by

$$C^\bullet(\mathcal{L};\mathcal{N})(U) = C^\bullet(\mathcal{L}(U);\mathcal{N}(U)) = \prod_{n \in \mathbb{N}} \mathrm{Hom}(\mathcal{L}(U)[1]^{\hat{\otimes} n}, \mathcal{N}(U))_{S_n}$$

equipped with the Chevalley-Eilenberg differential.

Further, we define the subsheaves of local and/or reduced Chevalley-Eilenberg cochains by

$$\begin{aligned} C^\bullet_{\mathrm{loc}}(\mathcal{L};\mathcal{N})(U) &:= \prod_{n \in \mathbb{N}} \mathrm{Hom}_{\mathrm{loc}}(\mathcal{L}(U)[1]^{\hat{\otimes} n}, \mathcal{N}(U))_{S_n} \\ C^\bullet_{\mathrm{red}}(\mathcal{L};\mathcal{N})(U) &:= \prod_{n \in \mathbb{N}^+} \mathrm{Hom}(\mathcal{L}(U)[1]^{\hat{\otimes} n}, \mathcal{N}(U))_{S_n} \\ C^\bullet_{\mathrm{loc,red}}(\mathcal{L};\mathcal{N})(U) &:= \prod_{n \in \mathbb{N}^+} \mathrm{Hom}_{\mathrm{loc}}(\mathcal{L}(U)[1]^{\hat{\otimes} n}, \mathcal{N}(U))_{S_n} \end{aligned}$$

Each L_∞ module $\mathcal{N}(U)$ over $\mathcal{L}(U)$ defines a vector bundle over $B\mathcal{L}(U)$, and under this correspondence

$$\Gamma(B\mathcal{L}(U), \mathcal{N}(U)) = C^\bullet(\mathcal{L};\mathcal{N})(U) \cong \mathcal{O}(B\mathcal{L}(U)) \hat{\otimes} \mathcal{N}(U)$$

where $\cong$ here denotes only an isomorphism of vector spaces. Further, we let $\Gamma_{\mathrm{loc(red)}}(B\mathcal{L}(U), \mathcal{N}(U))$ denote the subspaces corresponding to the local (and/or reduced) cochains; note the geometric interpretation of the reduced cochains is as the space of sections of the vector bundle $\mathcal{N}$ vanishing to first order at the closed point of $B\mathcal{L}$.

In particular, for each $k \in \mathbb{Z}$ the vector bundle $L[k+1]$ on M together with the k^{th} shift of the first L_∞ bracket $l_1 : \mathcal{L}[k+1] \to \mathcal{L}[k+2]$ defines a local L_∞ module for L, and the corresponding vector bundle on $B\mathcal{L}$ is denoted $T[k]B\mathcal{L}$, so that

$$\Gamma(B\mathcal{L}, T[k]B\mathcal{L}) = C^\bullet(\mathcal{L}; \mathcal{L}[k+1])$$

There is a natural map

$$\Gamma(B\mathcal{L}, T[k]B\mathcal{L}) \to \mathrm{Der}^k(\mathcal{O}(B\mathcal{L}))$$

where $\mathrm{Der}^k(\mathcal{O}(B\mathcal{L}))$ denotes the cohomological degree k derivations of the precosheaf $\mathcal{O}(B\mathcal{L})$ of cdga's in the category NDF. For each open set $U \subset M$, and each vector field

$$X = \sum_{n\in\mathbb{N}} X_n \quad \in \quad C^\bullet(\mathcal{L}; \mathcal{L}[k+1])(U) = \prod_{n\in\mathbb{N}} \mathrm{Hom}(\mathcal{L}(U)[1]^{\hat{\otimes} n}, \mathcal{L}(U)[k+1])_{S_n}$$

define the action of the corresponding derivation $\tilde{X} \in \mathrm{Der}^k(\mathcal{O}(B\mathcal{L}(U)))$ on the subspace of linear functionals $l \in \mathcal{L}^*(U)[-1]$ by

$$\tilde{X}(l) = l \circ X = \sum_{n\in\mathbb{N}} l \circ X_n \quad \in \quad \prod_{n\in\mathbb{N}} (\mathcal{L}(U)^*[-1])^{\hat{\otimes} n}_{S_n}[k] = \mathcal{O}(B\mathcal{L}(U))[k]$$

and extend it to $\mathcal{O}(B\mathcal{L}(U))$ as a graded derivation. Let $\mathrm{Der}^k_{\mathrm{loc}}(\mathcal{O}(B\mathcal{L}))$ denote the image of $\Gamma_{\mathrm{loc}}(B\mathcal{L}, T[k]B\mathcal{L})$ under this map.

In particular, recalling $l_n \in \mathrm{Hom}_{\mathrm{loc}}(\mathcal{L}[1]^{\hat{\otimes} n}, \mathcal{L}[2])_{S_n}$, we can consider

$$Q_L := \sum_{n\in\mathbb{N}^+} l_n \quad \in \quad \prod_{n\in\mathbb{N}^+} \mathrm{Hom}_{\mathrm{loc}}(\mathcal{E}^{\hat{\otimes} n}, \mathcal{E}[1])_{S_n} = \Gamma_{\mathrm{loc,red}}(\mathcal{E}, T[1]\mathcal{E})$$

where $\mathcal{E}$ is the sheaf of sections of the vector bundle $E := L[1]$, interpreted as a sheaf of affine spaces with -1 shifted tangent complex given by the trivial local L_∞ algebra with underlying vector bundle L; in keeping with this interpretation, we abuse notation and use $\mathcal{E}$ for both the sheaf of sections $\mathcal{E}$ and the presheaf of linear spaces $B(\mathcal{E}[-1])$.

In fact, the data of $\mathcal{L}$ as a local L_∞ algebra is equivalent to that of $\mathcal{E}$ equipped with the vector field Q_L; the vector field and the maps underlying the L_∞ brackets are precisely the same data, the locality of the vector field is equivalent to the L_∞ brackets being polydifferential operators, and the L_∞ identities are equivalent to the derivation $\tilde{Q}_L \in \mathrm{Der}^1_{\mathrm{loc}}(\mathcal{O}(\mathcal{E}))$ satisfying $\tilde{Q}_L^2 = 0$. Note that the derivation $\tilde{Q}_L$ is precisely the Chevalley-Eilenberg differential on $\mathcal{O}(B\mathcal{L})$. The above equivalence and a generalization of it are stated precisely in propositions 2.10 and 2.11, respectively.

The cotangent bundle is a bit more subtle due to analytic issues in our infinite dimensional setting. We define

$$\Gamma(B\mathcal{L}, T^*[k]B\mathcal{L}) = C^\bullet(\mathcal{L}; \mathcal{L}^*[k-1])$$

equipped with the Chevalley-Eilenberg differential. However, $\mathcal{L}^*$ is not the sheaf of sections of a vector bundle and thus does not fit into the preceeding definition. We will be careful about this distinction whenever we discuss sections of the cotangent bundle in what follows.

There is a natural derivation

$$d : \mathcal{O}(B\mathcal{L}) = C^\bullet(\mathcal{L}) \to C^\bullet(\mathcal{L}; \mathcal{L}^*[-1]) = \Gamma(B\mathcal{L}, T^*B\mathcal{L})$$

given on each degree $n \geq 1$ by taking a symmetrizing sum over the inclusion maps

$$\mathrm{Sym}^n(\mathcal{L}^*[-1]) \hookrightarrow \mathrm{Sym}^{n-1}(\mathcal{L}^*[-1])\hat{\otimes}\mathcal{L}^*[-1]$$

and defined as 0 on $\mathrm{Sym}^0(\mathcal{L}^*[-1])$.

2.3 Invariant Symplectic Pairings and Hamiltonian Vector Fields

Definition 2.5. *Let L be a local L_∞ algebra and $E = L[1]$. A strictly local k-shifted symplectic structure on $B\mathcal{L}$ is a map of vector bundles*

$$\omega : E \otimes E \to \mathrm{Dens}_M[k]$$

which is graded-antisymmetric, fibrewise non-degenerate, and is invariant under the L_∞ structure, in the sense that the functionals

$$(Q_L \vee \omega)_n := I_M \circ \omega \circ (\mathbb{1}_{\mathcal{E}} \otimes l_n) \quad \in \mathrm{Hom}(\mathcal{E}_c^{\hat{\otimes} n}, \mathbb{C})$$

are S_n invariant for each $n \geq 2$, where $\{l_n : \wedge^n \mathcal{E} \to \mathcal{E}[1]\}_{n \in \mathbb{N}}$ are the poly-differential operators corresponding to the local vector field Q_L and I_M : $(\mathrm{Dens}_M)_c \to \mathbb{C}$ denotes the integration map.

We also define

$$\omega : E \to E^![k] \qquad \text{and} \qquad \Pi : E^! \to E[-k]$$

to be the corresponding vector bundle isomorphism and its inverse, respectively, where $E^! := E^\vee \otimes \mathrm{Dens}_M$ for $E^\vee$ the dual bundle to E and Dens_M the bundle of densities on M. Equivalently, we have $\omega : L \to L^![k-2]$ and $\Pi : L^! \to L[2-k]$.

Note that the invariance condition is trivial for a trivial L_∞ algebra, so that a pairing ω which satisfies the above hypotheses except for the invariance condition defines a shifted symplectic structure on the trivial local L_∞ algebra underlying L; moreover, in this situation the remaining invariance condition can be restated equivalently as the condition that $Q_L \in \Gamma_{\mathrm{loc,red}}(\mathcal{E}, T[1]\mathcal{E})$ be a symplectic vector field for the -1-shifted symplectic structure ω on $\mathcal{E}$, in the sense given in definition 2.12; see proposition 2.14 for the precise statement.

For a finite dimensional symplectic manifold (P, ω), each function $f \in$

$C^\infty(P)$ determines a vector field $X_f := \Pi(df)$ where $\Pi = \omega^{-1} : T^*M \to TM$ is the Poisson tensor corresponding to ω; this can equivalently be characterized by the condition $df = \iota_{X_f}\omega$. In our infinite dimensional setting most of the naive definitions fail to be well-defined in general, and we will have to be careful about various classes of functions for which we can make sense of these ideas.

A priori, a strictly local k-shifted symplectic structure on $B\mathcal{L}$ gives isomorphisms

$$\mathbb{1}_{\mathcal{O}(B\mathcal{L})}\hat{\otimes}\, \omega : C^\bullet(\mathcal{L}; \mathcal{L}[1-k]) \leftrightarrows C^\bullet(\mathcal{L}; \mathcal{L}^![-1]) : \mathbb{1}_{\mathcal{O}(B\mathcal{L})}\hat{\otimes}\, \Pi$$

of precosheaves of cochain complexes in the category Nuc. This gives an equivalence between the space of cohomological degree $-k$ vector fields on $B\mathcal{L}$ and something which is morally a space of 1-forms on $B\mathcal{L}$ of cohomological degree 0. Applying the map Π to 1-forms in this space which are in some sense the differential of a functional gives the Hamiltonian vector field corresponding to the given functional. In the next section, we explain one context in which this is made precise.

Finally, we are now in a position to just state the main definition of the chapter:

Definition 2.6. *A classical field theory on M is a local L_∞ algebra L equipped with a strictly local -1-shifted symplectic structure on $B\mathcal{L}$.*

2.4 Local Functionals and Local Hamiltonian Vector Fields

Let L be a local L_∞ algebra over M with sheaf of sections $\mathcal{L}$ and jet bundle $J(L)$; we denote the sheaf of sections of $J(L)$ by $\mathcal{J}(\mathcal{L})$. The following follows formally from the discussion in appendix 6.2:

Proposition 2.7. *$\mathcal{J}(\mathcal{L})$ is an L_∞ algebra object in the symmetric monoidal category $(D_M\text{-Mod}, \otimes_{C^\infty_M})$.*

We define the reduced Chevalley-Eilenberg cochains on $\mathcal{J}(\mathcal{L})$ by

$$C^\bullet_{\mathrm{red}}(\mathcal{J}(\mathcal{L})) = \prod_{n\in\mathbb{N}^+} \mathrm{Sym}^n_{C^\infty_M}(\mathcal{J}(\mathcal{L})^\vee[-1]) = \prod_{n\in\mathbb{N}^+} \mathrm{Hom}_{C^\infty_M}(\mathcal{J}(\mathcal{L})[1]^{\otimes n}, C^\infty_M)_{S_n}$$

which, equipped with the Chevalley-Eilenberg differential, defines a presheaf of reduced cdga's in the category D_M-Mod; here $\mathcal{J}(\mathcal{L})^{\otimes n}$ denotes the n^{th} tensor power over C^∞_M of the sheaf of sections of the jet bundle. Now

$$\mathrm{Hom}_{C^\infty_M}(\mathcal{J}(\mathcal{L})[1]^{\otimes n}, C^\infty_M) \cong \mathrm{Hom}_{C^\infty_M}(\mathcal{J}(\mathcal{L})[1], C^\infty_M)^{\otimes n} \cong \mathrm{PolyDiff}(\mathcal{L}[1]^{\otimes n}, C^\infty_M)$$

so that an element $F \in C^\bullet_{\mathrm{red}}(\mathcal{J}(\mathcal{L}))$ defines for each $n \geq 1$, a symmetric map $F_n : \mathcal{J}(\mathcal{L})[1]^{\otimes n} \to C^\infty(M)$ determined by

$$F_n(j(l_1) \otimes ... \otimes j(l_n)) = \sum_{i=1}^{k} D^i_1 l_1 ... D^i_n l_n$$

for all $l_i \in \mathcal{L}$, for some collection of differential operators $D^i_j \in \mathrm{Diff}(\mathcal{L}[1], C^\infty_M)$ defined for each $i = 1, ..., k,\ j = 1, ..., n$; here $j(l) \in \mathcal{J}(\mathcal{L})$ is the infinite jet extension of $l \in \mathcal{L}$.

Now, we define

$$\mathcal{O}_{\mathrm{loc}}(B\mathcal{L}) := \Gamma(M, \mathrm{Dens}_M) \otimes_{D_M} C^\bullet_{\mathrm{red}}(\mathcal{J}(\mathcal{L}))$$

which is the space of local action functionals on $B\mathcal{L}$. Evidently $C^\bullet_{\text{red}}(\mathcal{J}(\mathcal{L}))$ is the space of Lagrangians on the space of fields $\mathcal{L}[1]$, while $\Gamma(M, \text{Dens}_M)$ is the space of densities on M against which one can integrate the given Lagrangian; the tensor product is taken over D_M to account for the fact that some a priori different Lagrangian densities are equivalent via integration by parts, or equivalently to quotient by the space of Lagrangian densities which are total derivatives and thus define trivial action functionals.

Elements of $\mathcal{O}_{\text{loc}}(B\mathcal{L})$ can be thought of as functionals explicitly via the natural inclusion of presheaves valued in NDF: $\mathcal{O}_{\text{loc}}(B\mathcal{L}) \hookrightarrow C^\bullet(\mathcal{L}_c)$ defined on $U \subset M$ by

$$F|_U \otimes \eta \mapsto \prod_{n\in\mathbb{N}} \left[l \mapsto \int_M F_n(j(l)^{\otimes n})\ \eta \right]$$

where $\eta \in \text{Dens}_M(U)$ and $l \in \mathcal{L}_c(U)$, noting the integral is convergent since the $F_n \circ j^{\otimes n}$ maps $\mathcal{L}_c(U)$ to $C^\infty_c(U)$.

Now, we can restrict the universal derivation

$$d : C^\bullet(\mathcal{L}_c) \to C^\bullet(\mathcal{L}_c; \mathcal{L}_c^*) = \prod_{n\in\mathbb{N}} \text{Hom}(\mathcal{L}_c^{\hat{\otimes} n}, \mathcal{L}_c^*)_{S_n}$$

to the image of $\mathcal{O}_{\text{loc}}(B\mathcal{L})$. For $\tilde{F} = F \otimes \eta \in \mathcal{O}_{\text{loc}}(B\mathcal{L})$ we obtain $d\tilde{F} = \sum_{n\in\mathbb{N}} d\tilde{F}_n$ where $d\tilde{F}_n$ is defined by

$$d\tilde{F}_n(l_1 \otimes ... \otimes l_{n-1})(l_n) = \int_M F_n(j(l_1) \otimes ... \otimes j(l_n))\ \eta$$

Note that

$$d\tilde{F}_n(l_1 \otimes ... \otimes l_{n-1})(l_n) = \int_M \eta \sum_{i=1}^k D_1^i l_1 ... D_n^i l_n = \int_M \left(\sum_{i=1}^k \tilde{D}_n^i (\eta\ D_1^i l_1 ... D_{n-1}^i l_{n-1}) \right) l_n$$

by integration by parts, where $\tilde{D}_n^i$ is the formal adjoint of D_n^i, defined as the differential operator making the above integration by parts formula hold. This implies the more restrictive condition that $d\tilde{F}_n \in \text{Hom}_{\text{loc}}(\mathcal{L}^{\hat{\otimes}(n-1)}, \mathcal{L}^![-1]) \subset$

$\mathrm{Hom}_{\mathrm{loc}}(\mathcal{L}_c^{\hat{\otimes}(n-1)}, \mathcal{L}_c^*[-1])$ and thus we in fact obtain a map

$$d : \mathcal{O}_{\mathrm{loc}}(B\mathcal{L}) \to \prod_{n\in\mathbb{N}} \mathrm{Hom}_{\mathrm{loc}}(\mathcal{L}^{\hat{\otimes}n}, \mathcal{L}^![-1])_{S_n} = C^\bullet_{\mathrm{loc}}(\mathcal{L}; \mathcal{L}^![-1])$$

Now, given a strictly local k shifted symplectic structure on $B\mathcal{L}$ defined by $\omega : L \to L^![k-2]$, we obtain a map

$$\Pi_{\mathrm{loc}} : C^\bullet_{\mathrm{loc}}(\mathcal{L}; \mathcal{L}^![-1]) \to C^\bullet_{\mathrm{loc}}(\mathcal{L}; \mathcal{L}[1-k])$$

as discussed in the preceeding section. We can now formulate the definition of Hamiltonian vector field corresponding to a local actional functional:

Definition 2.8. *Let $F \in \mathcal{O}_{\mathrm{loc}}(B\mathcal{L})$ of cohomological degree 0, a local action funcitonal on $B\mathcal{L}$. The local Hamiltonian vector field corresponding to F is*

$$\Pi_{\mathrm{loc}}(dF) \in \Gamma_{\mathrm{loc}}(B\mathcal{L}; T[-k]B\mathcal{L}) = C^\bullet_{\mathrm{loc}}(\mathcal{L}; \mathcal{L}[1-k])$$

Further, a vector field $X \in C^\bullet_{\mathrm{loc}}(\mathcal{L}; \mathcal{L}[1-k])$ is called Hamiltonian if it is a local Hamiltonian vector field corresponding to some $F \in \mathcal{O}_{\mathrm{loc}}(B\mathcal{L})$.

Finally, we define the induced Poisson bracket of local functionals in $\mathcal{O}_{\mathrm{loc}}(B\mathcal{L})$:

Definition 2.9. *Let $F, G \in \mathcal{O}_{\mathrm{loc}}(B\mathcal{L})$. Define*

$$\{F, G\}_\omega = \mu \circ (\mathbb{1}_{C^\bullet(\mathcal{L})} \otimes \mathbb{1}_{C^\bullet(\mathcal{L})} \otimes \omega)(dF \hat{\otimes} dG) \quad \in C^\bullet(\mathcal{L})$$

where $\mu : C^\bullet(\mathcal{L})^{\hat{\otimes}2} \to C^\bullet(\mathcal{L})$ is the algebra multiplcation map.

2.5 Deformations of Local L_∞ Algebras and Local Action Functionals

Let L be a vector bundle with sheaf of sections $\mathcal{L}$ and $E = L[1]$. Recall that we think of the sheaf of sections $\mathcal{E}$ of E as the sheaf of affine spaces $B(\mathcal{E}[-1])$, where $B(\mathcal{E}[-1])(U)$ is the formal derived stack with -1 shifted tangent complex $\mathcal{E}(U)[-1]$ equal to the trivial L_∞ algebra with underlying vector space $\mathcal{L}(U)$.

Recall that in the discussion in section 2.2 we established the following:

Proposition 2.10. *Let $\mathcal{E}, \mathcal{L}$ as above. Then the following are equivalent:*

- *A collection of polydifferential operators*

$$\{l_n : \mathcal{L}^{\hat{\otimes} n} \to \mathcal{L}[2-n]\}_{n \in \mathbb{N}^+}$$

  which make L into a local L_∞ algebra.

- *A vector field*
$$Q_L \in \Gamma_{\mathrm{loc,red}}(\mathcal{E}, T[1]\mathcal{E})$$
  satisfying $Q_L^2 = 0$.

One can interpret the above proposition in a more general context: it shows that a deformation of a trivial local L_∞ algebra $\mathcal{E}[-1]$ into a general local L_∞ algebra $\mathcal{L}$ is equivalent to a square zero vector field on the sheaf of affine spaces modelled by $\mathcal{E}$. The analogous geometric description of the space of deformations of a general local L_∞ algebra is as follows:

Proposition 2.11. *Let L be a local L_∞ algebra with L_∞ brackets $\{l_n : \mathcal{L}^{\hat{\otimes} n} \to \mathcal{L}[2-n]\}_{n \in \mathbb{N}^+}$ and let $Q_L \in \Gamma_{\mathrm{loc,red}}(\mathcal{E}, T[1]\mathcal{E})$ be the corresponding local vector field. Then the following are equivalent:*

- *A collection of polydifferential operators*

$$\{\tilde{l}_n : \mathcal{L}^{\hat{\otimes} n} \to \mathcal{L}[2-n]\}_{n \in \mathbb{N}^+}$$

such that $\{l_n + \tilde{l}_n : \mathcal{L}^{\hat{\otimes} n} \to \mathcal{L}[2-n]\}_{n\in\mathbb{N}^+}$ *again make* L *into a local* L_∞ *algebra.*

- *A vector field*

$$V \in \Gamma_{\text{loc,red}}(B\mathcal{L}, T[1]B\mathcal{L})$$

such that $[Q_L, V] + \frac{1}{2}[V, V] = 0$*.*

Proof. Evidently the local vector field corresponding to the collection of poly-polydifferential operators $\{l_n + \tilde{l}_n\}_{n\in\mathbb{N}^+}$ is given by $\tilde{Q} = Q_L + V$. By the previous proposition, this defines an L_∞ structure if and only if $\tilde{Q}^2 = 0$, and

$$\tilde{Q}^2 = [Q_L, Q_L] + 2[Q_L, V] + [V, V] = 2[Q_L, V] + [V, V]$$

where the last equality follows again from the previous proposition since Q_L defines a local L_∞ algebra. □

Next, we wish to give the analogous results concerning the construction and deformation of classical field theories. To begin, we make the following definition:

Definition 2.12. *Let* $\mathcal{E}$ *as above and* ω *a strictly local* -1 *shifted symplectic form on* $\mathcal{E}$*. A local vector field* $X \in \Gamma_{\text{loc,red}}(\mathcal{E}, T[1]\mathcal{E})$ *is called symplectic if the functionals*

$$(X \vee \omega)_n := I_M \circ \omega \circ (\mathbb{1}_{\mathcal{E}} \otimes l_n) \quad \in \text{Hom}(\mathcal{E}_c^{\hat{\otimes} n}, \mathbb{C})$$

are invariant under S_n *for each* $n \geq 2$*; here* $\{l_n : \wedge^n \mathcal{E} \to \mathcal{E}[1]\}_{n\in\mathbb{N}}$ *are the polydifferential operators corresponding to the local vector field* X *and* $I_M : (\text{Dens}_M)_c \to \mathbb{C}$ *denotes the integration map.*

In particular, we have the following proposition, which can be interpreted as the vanishing of a certain Poisson cohomology module:

Proposition 2.13. *Let $\mathcal{E}$ as above, ω a strictly local -1 shifted symplectic form on $\mathcal{E}$ and $X \in \Gamma_{\mathrm{loc,red}}(\mathcal{E}, T[1]\mathcal{E})$ a symplectic vector field. Then X is Hamiltonian in the sense of definition 2.8, with corresponding local functional $F \in \mathcal{O}_{\mathrm{loc}}(\mathcal{E})$ defined by*

$$F = \sum_{n \in \mathbb{N}^+} I_M \circ \omega \circ (\mathbb{1}_{\mathcal{E}} \otimes l_n)$$

as an element of $C^\bullet(\mathcal{E}_c[-1])$, where $\{l_n : \wedge^n \mathcal{E} \to \mathcal{E}[1]\}_{n \in \mathbb{N}}$ are the polydifferential operators corresponding to the local vector field X and $I_M : (\mathrm{Dens}_M)_c \to \mathbb{C}$ denotes the integration map.

From this proposition, together with 2.10, we have:

Proposition 2.14. *Let E a vector bundle over M, consider $L = E[-1]$ as a trivial local L_∞ algebra, and let $\omega : E \to E^![-1]$ be a strictly local, -1 shifted symplectic pairing on $\mathcal{E}$. The following are equivalent:*

- *A collection of polydifferential operators $\{l_n : \mathcal{L}^{\hat{\otimes} n} \to \mathcal{L}[2-n]\}_{n \in \mathbb{N}^+}$ making $\mathcal{L}$ into a local L_∞ algebra for which ω is an invariant symplectic pairing.*
- *A symplectic vector field*

  $$Q_L \in \Gamma_{\mathrm{loc,red}}(\mathcal{E}, T[1]\mathcal{E})$$

  satisfying $Q_L^2 = 0$.
- *An element $S \in \mathcal{O}_{\mathrm{loc}}(\mathcal{E})$ of cohomological degree zero, which is at least quadratic as a polynomial on $\mathcal{E}$ and satisfies*

  $$\{S, S\}_\omega = 0$$

Motivated by this, we make the following definition:

Definition 2.15. *A functional $S \in \mathcal{O}_{\text{loc}}(B\mathcal{L})$ satisfies the classical master equation if*

$$\{S, S\}_\omega = 0$$

Thus, by the preceeding proposition, we have the equivalent definition of classical field theory:

Definition 2.16. *A classical field theory on M is a vector bundle E with a strictly local, -1 shifted symplectic structure ω on $\mathcal{E}$ and an action functional $S \in \mathcal{O}_{\text{loc}}(\mathcal{E})$ which is atleast quadratic as a polynomial and satisfies the classical master equation.*

Finally, we proceed to discuss deformations of classical field theories with a fixed symplectic form. Such a deformation is in particular a deformation of a local L_∞ algebra of the type described by the preceeding proposition; it simply must satisfy the additional condition that the symplectic form is still preserved by the vector field corresponding to the new L_∞ structure. Thus, we have:

Proposition 2.17. *Let L be a local L_∞ algebra with L_∞ brackets given by $\{l_n : \mathcal{L}^{\hat{\otimes} n} \to \mathcal{L}[2-n]\}_{n \in \mathbb{N}^+}$, ω a strictly local -1 shifted symplectic form on $B\mathcal{L}$, and $S \in \mathcal{O}_{\text{loc}}(B\mathcal{L})$ be the corresponding local action functional. Then the following are equivalent:*

- *A collection of polydifferential operators*

$$\{\tilde{l}_n : \mathcal{L}^{\hat{\otimes} n} \to \mathcal{L}[2-n]\}_{n \in \mathbb{N}^+}$$

  such that $\{l_n + \tilde{l}_n : \mathcal{L}^{\hat{\otimes} n} \to \mathcal{L}[2-n]\}_{n \in \mathbb{N}^+}$ again make L into a local L_∞ algebra $\tilde{L}$, such that ω again defines a strictly local symplectic form on $B\tilde{\mathcal{L}}$.

- *A symplectic vector field*

$$V \in \Gamma_{\text{loc,red}}(B\mathcal{L}, T[1]B\mathcal{L})$$

such that $[Q_L, V] + \frac{1}{2}[V, V] = 0$.

- *An element* $\tilde{S} \in \mathcal{O}_{\mathrm{loc}}(\mathcal{E})$ *of cohomological degree zero, which is at least quadratic as a polynomial on* $\mathcal{E}$ *and satisfies*

$$\{S, \tilde{S}\}_\omega + \frac{1}{2}\{\tilde{S}, \tilde{S}\}_\omega = 0$$

3 Formal AKSZ-Type Theories

In this section, we define a family of examples of classical field theories, first constructed in [1].

Let $\mathfrak{g}$ a finite dimensional L_∞ algebra over $\mathbb{R}$ with an invariant, skew-symmetric, non-degenerate pairing $\omega_{\mathfrak{g}} : \mathfrak{g}^{\otimes 2} \to k[n-2]$, so that $B\mathfrak{g}$ is n-shifted symplectic as a formal derived stack.

Then for each orientable manifold M of dimension $n+1$, we define a classical field theory on M with underlying local L_∞ algebra defined by

$$\mathcal{L} = T_0\mathrm{Maps}((\cdot)_{dR}, B\mathfrak{g})[-1] := \Omega_M^\bullet \otimes \mathfrak{g}$$

with local L_∞ algebra structure given on each $U \subset M$ as the tensor product of the cdga $\Omega_M^\bullet(U)$ and the L_∞ algebra $\mathfrak{g}$: explicitly, we have

$$l_1 : \mathcal{L} \to \mathcal{L}[1] \qquad \text{defined by} \qquad l_1 = d_M \otimes \mathbb{1}_{\mathfrak{g}} + \mathbb{1}_{\Omega_M^\bullet} \otimes l_1^{\mathfrak{g}}$$

and

$$l_n : \mathcal{L}^{\otimes n} \to \mathcal{L}[2-n] \qquad \text{defined by} \qquad l_n = \mu_M^n \otimes l_n^{\mathfrak{g}}$$

for $n \geq 2$, where we let $\mu_M^n : (\Omega_M^\bullet)^{\otimes n} \to \Omega_M^\bullet$ denote the (associative) multiplication of n elements, and $l_n^{\mathfrak{g}} : \mathfrak{g}^{\otimes n} \to \mathfrak{g}[2-n]$ denote the L_∞ brackets for $\mathfrak{g}$. Note that l_1 is differential operator, and the maps l_n for $n \geq 2$ are in fact maps of C_M^∞ modules, and thus in particular differential operators in a trivial way.

The initial insight of [1] is that local L_∞ algebras constructed as presheaves of formal mapping spaces in this way have a natural compatible, strictly local -1 shifted symplectic structure $\omega : \mathcal{L}^{\otimes 2}[1] \to \mathrm{Dens}_M[-1]$ defined on sections by:

$$\omega = \pi_{n+1} \circ (\mu_M \otimes \omega_{\mathfrak{g}}) : \Omega^\bullet(M)^{\otimes 2} \otimes \mathfrak{g}[1]^{\otimes 2} \to \mathrm{Dens}_M[-1]$$

where $\pi_{n+1} : \Omega_M^\bullet \to \Omega_M^{n+1}[n+1] \cong \mathrm{Dens}_M[n+1]$; note ω is in fact coming from an underlying map of vector bundles.

We now give two familiar examples of theories in this class:

3.1 Topological Classical Mechanics

Let V be a symplectic vector space, with symplectic form $\omega : V^{\otimes 2} \to \mathbb{C}$ and let $\mathfrak{g} = V[-1]$ be the -1 shifted tangent complex of V, viewed as a trivial L_∞ algebra. Then $\omega_g = \omega : \mathfrak{g}^{\otimes 2} \to \mathbb{C}[-2]$ defined a 0 shifted symplectic structure on $V = B\mathfrak{g}$. By the above construction, this yields a dimensional classical field theory on 1 manifolds called topological classical mechanics valued in V; this name comes from the fact that it is the classical field theory underlying topological quantum mechanics.

3.2 Classical Chern-Simons Theory

Let $\mathfrak{g}$ be a complex, semi-simple Lie algebra viewed as an L_∞ algebra concentrated in cohomological degree 0 with $l_n = 0$ for $n \neq 2$, and let $\omega_{\mathfrak{g}} := \kappa : \mathfrak{g}^{\otimes 2} \to \mathbb{C}$ denote the Killing form, which defines a 2 shifted symplectic structure on $B\mathfrak{g}$. By the above construction, this yields a classical field theory on 3 manifolds called classical Chern-Simons theory. Note that the action functional corresponding to this classical field theory, in the sense of definition 2.16 is given by

$$S(A) = \int_M \kappa(dA \wedge A + A \wedge [A \wedge A]) \qquad A \in \Omega^\bullet(M) \otimes \mathfrak{g}$$

4 Commutative Differential Graded Algebras and L_∞ Algebras

4.1 Graded Linear Algebra

Throughout the remainder of this paper fix a base field k of characteristic zero.

Definition 4.1. *A graded vector space V is a vector space along with a direct sum decomposition*

$$V = \bigoplus_{k \in \mathbb{Z}} V_k$$

For a graded vector space V, V_k is called homogenous degree k summand, and for $v \in V_k$ we let $|v| = k$.

Given a graded vector space V, we write $V[i]$ for the graded vector space given by shifting the degree of each summand by i, or explicitly by the decomposition

$$V[i] = \bigoplus_{k \in \mathbb{Z}} V_{k+i}$$

When we need to be more careful about degree, we include vector spaces into graded vector spaces concentrated in degree zero, and thus can write

$$V = \bigoplus_{k \in \mathbb{Z}} V_k[-k]$$

for the graded vector space with homogenous degree k summand V_k. Here the direct sum is of graded vector spaces (defined below).

Definition 4.2. *A graded algebra is an algebra A over k such that the underlying vector space is a graded vector space*

$$A = \bigoplus_{k \in \mathbb{Z}} A_k$$

and the multiplication satisfies

$$\mu(A_j \otimes A_k) \subset A_{j+k}$$

A graded algebra A is called graded-commutative if

$$a \cdot b = (-1)^{|a||b|} b \cdot a$$

Throughout let V, W be graded vector spaces.

Definition 4.3. *A (degree 0) morphism of graded vector spaces $f : V \to W$ is a collection of linear maps $(f_k : V_k \to W_k)_{k\in\mathbb{Z}}$.*

Definition 4.4. *A degree i morphism of graded vector spaces is a (degree 0) morphism of graded vector spaces $f : V \to W[i]$.*

Definition 4.5. *The dual of V is the graded vector space V^* given by the decomposition*

$$V^* = \bigoplus_{k\in\mathbb{Z}} V^*_{-k}$$

Definition 4.6. *The direct sum $V \oplus W$ is the graded vector space given by the decomposition*

$$V \oplus W = \bigoplus_{k\in\mathbb{Z}} (V \oplus W)_k \qquad (V \oplus W)_k = V_k \oplus W_k$$

Definition 4.7. *The tensor product $V \otimes W$ is the graded vector space given by the decomposition*

$$V \otimes W = \bigoplus_{k\in\mathbb{Z}} (V \otimes W)_k \qquad (V \otimes W)_k = \oplus_{i+j=k} V_i \otimes W_j$$

or more generally for graded vector spaces $V^1, ..., V^n$ with $V^i = \oplus_{k\in\mathbb{Z}} V^i_k$, we define

$$\bigotimes_{i=1}^{n} V^i = \bigoplus_{k\in\mathbb{Z}} \bigoplus_{i_1+...+i_n=k} V^1_{i_1} \otimes ... \otimes V^n_{i_n}$$

Definition 4.8. *The tensor algebra $\mathcal{T}(V)$ is a graded algebra, with underlying vector space given by*

$$\mathcal{T}(V) = \bigoplus_{n\in\mathbb{N}} \otimes^n V = \bigoplus_{n\in\mathbb{N}} \left(\bigoplus_{k\in\mathbb{Z}} \bigoplus_{i_1+...+i_n=k} V_{i_1} \otimes ... \otimes V_{i_n} \right)$$

where the grading is given by the direct sum decomposition in the final expression.

Definition 4.9. *The symmetric algebra* $\mathrm{Sym}^\bullet(V)$ *is the quotient algebra $\mathcal{T}(V)/I$ where I is the two-sided ideal generated by elements of the form*

$$v \otimes w - (-1)^{|v||w|} w \otimes v$$

Definition 4.10. *The alternating algebra $\wedge^\bullet(V)$ is the quotient algebra $\mathcal{T}(V)/I$ where I is the two-sided ideal generated by elements of the form*

$$v \otimes w + (-1)^{|v||w|} w \otimes v$$

The most trivial example of all these constructions is for $V = V_0$ a vector space V_0, so concentrated in degree zero, in which case they reduce to the original vector space operations, with resulting vector spaces all concentrated in degree zero.

If we take $V = V_0[-1]$ for a vector space V_0, so concentrated in degree 1, then the tensor algebra yield the usual tensor algebra with the usual graded algebra structure given by the degree of the tensor as a multilinear map. The symmetric algebra $\mathrm{Sym}^\bullet(V)$ yields the alternating algebra $\wedge^\bullet V$ equipped again with the usual graded algebra structure, and vice versa.

4.2 Commutative Differential Graded Algebras

In this section we recall the definition of the category of commutative differential graded algebras.

Definition 4.11. *A degree j derivation on a commutative graded algebra A over k with product μ is a $k--\text{Mod}$ map $d : A \to A[j]$ such that*

$$d \circ \mu = \mu \circ (d \otimes \mathbb{1}_A + (-1)^{j+1} \mathbb{1}_A \otimes d)$$

Let $\mathrm{Der}_k^j(A)$ denote the degree j derivations of A.

Definition 4.12. *A commutative differential graded algebra is a commutative graded algebra together with a degree 1 derivation $d : A \to A[1]$ such that $d^2 = 0$.*

A map of commutative differential graded algebras is a map of graded algebras over k which is also a map of cochain complexes.

Given a graded vector space V, let $A(V)$ denote the free graded commutative algebra. We have:

Proposition 4.13. *The free commutative algebra on a graded vector space V over k is given by*

$$A(V) = \mathrm{Sym}_k^\bullet(V) \qquad \textit{with} \qquad \mu((v_1 \cdot \ldots \cdot v_j) \otimes (w_1 \cdot \ldots \cdot w_l)) = v_1 \cdot \ldots \cdot v_j \cdot w_1 \cdot \ldots \cdot w_l$$

for each $v_1, ..., v_j, w_1, ..., w_l \in V$, where $\cdot$ denotes the symmetric algebra multiplication.

For $A(V)$ a free graded commutative algebra, a derivation is determined by its restriction to the generators:

Proposition 4.14. *There is a bijection*

$$\mathrm{Hom}_{k--\mathrm{Mod}}(V[-j], A(V)) \to \mathrm{Der}_k^j(A(V)) \qquad \textit{defined by} \qquad X \mapsto \mu \circ (\mathbb{1}_{A(V)} \otimes X) \circ \Delta$$

where $\Delta : A(V) \to A(V) \otimes A(V)$ is the symmetric algebra comultiplication. The inverse of this bijection is given by

$$\mathrm{Der}_k^j(C(V)) \to \mathrm{Hom}_{k--\mathrm{Mod}}(V[-j], A(V)) \qquad \textit{defined by} \qquad d \mapsto d \circ \pi$$

where $\pi : A(V) \to V$ *denotes the projection to* $\mathrm{Sym}^1(V) = V$*.*

This gives us a concrete way to construct commutative differential graded algebras. We make the following definition:

Definition 4.15. *A commutative differential graded algebra with underlying graded algebra given by a free graded commutative algebra is called quasi-free.*

4.3 L_∞ Algebras

Definition 4.16. *An L_∞ algebra over k is a graded vector space $\mathfrak{g}$ together with a family of k linear maps*

$$l_n : \wedge^n \mathfrak{g} \to \mathfrak{g}[2-n]$$

defined for $n \in \mathbb{N}$, satisfying the n-Jacobi identity

$$\sum_{\substack{\sigma \in S_n \\ j,k \geq 1 \\ j=n-k+1}} (-1)^? l_j(l_k(x_{\sigma(1)}, ..., x_{\sigma(k)}), x_{\sigma(k+1)}, ..., x_{\sigma(n)}) = 0$$

for each $n \in \mathbb{N}$.

In particular, $l_1 : \mathfrak{g} \to \mathfrak{g}[1]$ can be identified with a differential since the 1-Jacobi identity is just that $l_1 \circ l_1 = 0$. Thus an L_∞ algebra with $l_n = 0$ for $n \geq 2$ is just a cochain complex.

Further, an L_∞ algebra with $l_n = 0$ for $n \geq 3$ is just a differential graded Lie algebra: $l_2 : \wedge^2 \mathfrak{g} \to \mathfrak{g}$ can be identified with a Lie bracket since the 3 Jacobi identity reduces to the normal Jacobi identity and the 2 Jacobi identity gives the compatibility of the differential l_1 and the Lie bracket l_2. In particular, this implies that an L_∞ algebra with $l_n = 0$ for $n \neq 2$ is a graded Lie algebra. In general, l_2 need not satisfy the Jacobi identity, but will modulo terms involving l_3.

4.4 Commutative-Lie Koszul Duality

Let $\mathfrak{g}$ be a graded vector space which is degree-wise finite dimensional, and consider the free commutative algebra $A(\mathfrak{g}^*[-1])$. From proposition 4.14, a degree 1 differential on $A(\mathfrak{g}^*[-1])$ is is equivalent to a $k-$-Mod map

$l^* : \mathfrak{g}^*[-2] \to A(\mathfrak{g}^*[-1])$. Decomposing

$$\mathrm{Sym}^\bullet(\mathfrak{g}^*[-1]) = \bigoplus_{n\in\mathbb{N}} \mathrm{Sym}^n(\mathfrak{g}^*[-1]) = \bigoplus_{n\in\mathbb{N}} (\wedge^n \mathfrak{g}^*)[-n]$$

this is equivalent to a family of maps

$$l_n^* : \mathfrak{g}^*[n-2] \to \wedge^n \mathfrak{g}^* \qquad \text{or dually} \qquad l_n : \wedge^n \mathfrak{g} \to \mathfrak{g}[2-n]$$

defined for $n \in \mathbb{N}$. Now we have:

Proposition 4.17. *A degree 1 derivation d on $\mathrm{Sym}^\bullet(\mathfrak{g}^*[-1])$ satisfies $d^2 = 0$ if and only if the corresponding maps $\{l_n\}_{n\in\mathbb{N}}$ satisfy the n-Jacobi identity for each $n \in \mathbb{N}$.*

Proof. It suffices to check that $d^2 = 0$ on generators since then it follows on products by applying the graded Leibniz rule:

$$d^2(\mu(a \otimes b)) = \mu(d^2a \otimes b + da \otimes db - da \otimes db + a \otimes d^2b) = 0$$

To check it on generators, we need only verify that the direct summands of the map

$$d^2 \circ \pi : \mathfrak{g}^*[-1] \to \bigoplus_{n\in\mathbb{N}} \mathrm{Sym}^n(\mathfrak{g}^*[-1])$$

vanish for each $n \in \mathbb{N}$. By definition we have

$$d^2 \circ \pi = d \circ \bigoplus_{n\in\mathbb{N}} l_n^* = \mu \circ (\mathbb{1}_{A(\mathfrak{g}^*[-1])} \otimes \bigoplus_{n\in\mathbb{N}} l_n^*) \circ \Delta \circ \bigoplus_{n\in\mathbb{N}} l_n^*$$

and the component of this map corresponding to the n^{th} direct summand of the codomain is precisely the dual of the left hand side of the n Jacobi identity above. $\square$

The above has shown we have a bijection between quasi-free commutative differential graded algebras and L_∞ algebras with finitely many non-zero brackets (since we considered the non-completed symmetric algebra). We now make the following definition:

Definition 4.18. *Let $\mathfrak{g}$ be an L_∞ algebra. The Chevalley-Eilenberg complex of $\mathfrak{g}$ with coefficients in k is the commutative differential graded algebra*

$$C^\bullet(\mathfrak{g}; k) := \mathrm{Sym}^{\hat{\bullet}}(\mathfrak{g}^*[-1])$$

where $\mathrm{Sym}^{\hat{\bullet}}$ denotes the completion of $\mathrm{Sym}^\bullet$ with respect to the natural augmentation on the symmetric algebra given by projection to $\mathrm{Sym}^0 = k$.

Recall that an L_∞ algebra concentrated in degree 0 with $l_n = 0$ for $n \neq 2$ is just a Lie algebra. One can easily check that in this case the above definition agrees with the usual Chevalley-Eilenberg complex for Lie algebra cohomology with coefficients in the trivial module k.

4.5 L_∞ Modules and Chevalley-Eilenberg Cochains with Coefficients

In this section, we define L_∞ modules for L_∞ algebras, generalizing the definition of modules for strict Lie algebras. In analogy with the commutative-Lie Koszul duality discussed above, we will define an L_∞ module in such a way that it is equivalent data to determining a square zero differential defining the Chevalley-Eilenberg cochains with coefficients in this module, which will be a dg module for the cdga $C^\bullet(\mathfrak{g}; k)$.

Definition 4.19. *Let A be a cdga with differential d_A. A differential graded (dg) module V for A is a graded module V for the graded algebra underlying A, together with a differential $d_V : V \to V[1]$ making V into a cochain complex, such that*

$$d_V(av) = (d_A a)v + (-1)^{|a|} a(d_V v)$$

for each $a \in A$ of homogeneous degree and $v \in V$.

Now, let $\mathfrak{g}$ be an L_∞ algebra, N a cochain complex with differential $d_N : N \to N[1]$, and consider the graded vector space over k

$$V(\mathfrak{g}; N) := \mathrm{Sym}^{\hat{\bullet}}(\mathfrak{g}^*[-1]) \otimes_k N$$

We would like to define a map $d_N : V(\mathfrak{g}; N) \to V(\mathfrak{g}; N)[1]$ making this $V(\mathfrak{g}; N)$ into a dg module for the cdga $C^\bullet(\mathfrak{g}; k)$ with action map

$$\rho : C^\bullet(\mathfrak{g}; k) \otimes V(\mathfrak{g}; N) \to V(\mathfrak{g}; N) \qquad \text{defined by} \qquad f \otimes (g \otimes n) \mapsto fg \otimes n$$

for $f, g \in C^\bullet(\mathfrak{g}; k)$ and $n \in N$. In particular, this requires

$$d_N(f \otimes n) = d_N(f \cdot 1 \otimes n) = d^{\mathrm{CE}}(f) \otimes n + (-1)^{|f|} f \otimes d_N(1 \otimes n)$$

so that d_N is determined by its restriction to $N = 1 \otimes N \hookrightarrow V(\mathfrak{g}; k)$:

$$d_N|_N : N \to V(\mathfrak{g}; N)[1] = \prod_{n \in \mathbb{N}} \mathrm{Sym}^n(\mathfrak{g}^*[-1]) \otimes N[1]$$

This is equivalent to a family of maps

$$a_n : \wedge^n \mathfrak{g} \otimes N \to N[1-n]$$

for each $n \in \mathbb{N}$. The restriction that the resulting differential d_N satisfies $d_N^2 = 0$ imposes a family of equations on the $\{a_n\}$ analogous to the n-Jacobi identities in the definition of L_∞ algebra; we do not write them here.

Definition 4.20. *An L_∞ module for $\mathfrak{g}$ is a k module N together with a family of maps*

$$\{a_n : \wedge^n \mathfrak{g} \otimes N \to N[1-n]\}$$

such that the corresponding differential $d_N : V(\mathfrak{g}; N) \to V(\mathfrak{g}; N)[1]$ satisfies $d_N^2 = 0$.

Further, for N an L_∞ module, we define the Chevalley-Eilenberg cochains with coefficients in N to be the $C^\bullet(\mathfrak{g};k)$ dg module $C^\bullet(\mathfrak{g};N) := V(\mathfrak{g};N)$ together with this differential.

Let $\mathfrak{g}$ be an L_∞ algebra with structure maps given by $\{l_n : \wedge^n\mathfrak{g} \to \mathfrak{g}[2-n]\}_{n\in\mathbb{N}}$. Then for each $k \in \mathbb{N}$, the vector space $\mathfrak{g}[k]$ is an L_∞ module for $\mathfrak{g}$ with structure maps defined by

$$a_n := l_{n+1}[k] : \mathfrak{g}^{\otimes n} \otimes \mathfrak{g}[k] \to \mathfrak{g}[k][1-n]$$

Similarly, one defines an L_∞ module structure on $\mathfrak{g}^*[k]$ for each $k \in \mathbb{Z}$.

5 Nuclear Vector Spaces

Throughout this paper, we use many constructions from homological/homotopical algebra in situations where the underlying vector spaces are infinite dimensional. For example, given a local L_∞ algebra $\mathcal{L}$ on a manifold M, its evaluation on an open set $U \subset M$ yields an L_∞ algebra $\mathcal{L}(U)$, but with underlying vector space the infinite dimensional vector space of sections of the vector bundle L over U. In such situations, the correct mathematical statement is that we obtain an L_∞ algebra object in the appropriate category of infinite dimensional vector spaces. However, in order to make such statements, we need to ensure this category has all the of the neccessary structures to facilitate such definitions, such as duals, tensor products and enrichment over vector spaces. In this appendix, we summarize without proof the relevant facts to make sense of these notions in the contexts we use them.

5.1 Categories of Nuclear Vector Spaces

In this section, we record the relevant statements about the existence of appropriate categories of infinite dimensional vector spaces for the constructions of this paper. An early reference for these results is [4] and a complete treatment is given in [5].

Theorem 5.1. *There exists a full subcategory* Nuc *of the category of topological vector spaces, called the category of nuclear vector spaces, satisfying the following properties:*

- Nuc *is enriched over vector spaces*
- *There is a projective tensor product functor* $\hat{\otimes} : \mathrm{Nuc} \times \mathrm{Nuc} \to \mathrm{Nuc}$ *making* Nuc *a symmetric monoidal category*
- *Closed subspaces of nuclear vector spaces, and quotients of nuclear vector spaces by their closed subspaces, are nuclear*

- *Countable direct sums, and more generally countable colimits, of nuclear vector spaces are nuclear*

This is all of the information necessary to make sense of the linear algebraic constructions given in appendix 4 in the category Nuc, with the exception of the use of duals. These are described in the following:

Theorem 5.2. *There exist full subcategories* NF *and* NDF*, called the nuclear Fréchet spaces and nuclear dual Fréchet spaces, respectively, satisfying the following properties:*

- *Closed subspaces of* NF (NDF) *spaces, and quotients of* NF (NDF) *spaces by their closed subspaces, are* NF (NDF) *spaces*
- NF *is closed under countable limits and finite colimits*
- NDF *is closed under countable colimits and finite limits*
- NF *and* NDF *are symmetric monoidal subcategories of* Nuc
- *Taking the strong dual of a topological vector space gives equivalences of symmetric monoidal categories:*

$$(\cdot)^* : \mathrm{NF} \xrightarrow{\cong} \mathrm{NDF}^{op} \qquad\qquad (\cdot)^* : \mathrm{NDF} \xrightarrow{\cong} \mathrm{NF}^{op}$$

A further useful property of NF spaces is that they share the following descriptions of Hom spaces from finite dimensional linear algebra:

$$\mathrm{Hom}(E,F) = E^*\hat{\otimes}F \qquad \mathrm{Hom}(E^*,F) = E\hat{\otimes}F \qquad \mathrm{Hom}(E,F^*) = E^*\hat{\otimes}F^*$$

for each $E, F \in \mathrm{NF}$; these equalities are only claimed here at the level of vector spaces (without topology), which is the only enrichment we have referred to, but more precise statements are available.

5.2 Vector Spaces from Geometry

Throughout, let M be a smooth manifold with sheaf of smooth functions C_M^∞ and Dens_M the bundle of densities on M (by abuse of notation, we also use this notation for its sheaf of sections). Recall that for any vector bundle E over M, the sections of E form a sheaf $\mathcal{E}$ over M while the compactly supported sections form a cosheaf $\mathcal{E}_c$, with extension maps given by extension by zero. Further, define the sheaf of distributions $\mathcal{D}$ on M by

$$\mathcal{D}(U) = \mathrm{Dens}_c(U)^*$$

where the subscript c denotes compactly supported sections.

Now, for each vector bundle E over M define

$$E^! = E^\vee \otimes \mathrm{Dens}(M)$$

where $(\cdot)^\vee$ denotes the dual vector bundle, or equivalently dualizing the sheaf of sections in the category of C_M^∞ modules, and let $\mathcal{E}^!$ denote the sheaf of sections of $E^!$.

Moreover, let

- $\overline{\mathcal{E}} := \mathcal{E} \otimes_{C^\infty} \mathcal{D}$ the sheaf of distributional sections of E
- $\overline{\mathcal{E}_c} := \mathcal{E}_c \otimes_{C^\infty} \mathcal{D}$ the cosheaf of compactly supported distributional sections of E
- $\overline{\mathcal{E}^!} := \mathcal{E} \otimes_{C^\infty} \mathcal{D}$ the sheaf of distributional sections of $E^!$
- $\overline{\mathcal{E}^!_c} := \mathcal{E}_c \otimes_{C^\infty} \mathcal{D}$ the cosheaf of compactly supported distributional sections of $E^!$

We have:

Proposition 5.3. *Let E a vector bundle over M and $U \subset M$ open. Then we have:*

- *The vector spaces* $\mathcal{E}(U), \mathcal{E}_c(U)$ *are naturally objects in* NF*.*
- *The vector spaces* $\overline{\mathcal{E}}(U), \overline{\mathcal{E}_c}(U)$ *are naturally objects in* NDF*.*

Now, there is a natural pairing

$$\langle \cdot, \cdot \rangle : \mathcal{E} \otimes \mathcal{E}_c^! \to \mathbb{R} \qquad \text{defined by} \qquad \langle e, \xi \rangle = \int_M \langle e, \xi \rangle_E$$

where $\langle \cdot, \cdot \rangle_E : \mathcal{E} \otimes \mathcal{E}^* \to (\mathrm{Dens}_M)_c$ denotes the fibrewise duality pairing on E. There are equivalently pairings on

$$\mathcal{E}_c \otimes \mathcal{E}^! \qquad \mathcal{E}^! \otimes \mathcal{E}_c \qquad \mathcal{E}_c^! \otimes \mathcal{E}$$

each of which can be extended in the first arguement to distributional sections of the respective bundles maintaining finiteness of the pairing. Via this pairing, we have:

Proposition 5.4. *There are natural isomorphisms:*

- $\overline{\mathcal{E}} \cong (\mathcal{E}_c^!)^*$ *as presheaves.*
- $\overline{\mathcal{E}_c} \cong (\mathcal{E}^!)^*$ *as precosheaves.*
- $\overline{\mathcal{E}^!} \cong \mathcal{E}_c^*$ *as presheaves.*
- $\overline{\mathcal{E}_c^!} \cong \mathcal{E}^*$ *as precosheaves.*

where the target category is Nuc*.*

5.3 Integral Kernels

Note that for each $U \subset M$ there is a natural embedding

$$I : C^\infty \hookrightarrow \mathcal{D} \qquad \text{defined on } U \subset M \text{ by} \qquad I(f)(\xi) = \int_U f \cdot \xi$$

and more generally

$$I_E : \mathcal{E} \hookrightarrow \overline{\mathcal{E}} \qquad \text{defined by} \qquad e \mapsto e \otimes I(1)$$

Thus we have the commutative square

$$\mathcal{E}_c(M) \hookrightarrow (\mathcal{E}(M),\ \overline{\mathcal{E}_c}(M)) \hookrightarrow \overline{\mathcal{E}}(M)$$

in the category Nuc. Motivated by this, we make the following definition:

Definition 5.5. *Let E, F vector bundles over M, N with spaces of sections $\mathcal{E}, \mathcal{F}$, respectively. A general operator from $\mathcal{E}$ to $\mathcal{F}$ is a map*

$$P : \mathcal{E}_c(M) \to \overline{\mathcal{F}}(M)$$

in the category Nuc.

In particular, letting $\tilde{\mathcal{E}}(M), \tilde{\mathcal{F}}(M)$ denote any of the distributional and/or compactly supported versions of the spaces of global sections of $\mathcal{E}, \mathcal{F}$, respectively, we see that any map $\tilde{P} : \tilde{\mathcal{E}}(M) \to \tilde{\mathcal{F}}(M)$ determines a general operator $P : \mathcal{E}_c(M) \to \overline{\mathcal{F}}(M)$.

Moreover, letting $\mathrm{Hom}_{\mathrm{gen}}(\mathcal{E}, \mathcal{F})$ denotes the space of general operators P from $\mathcal{E}$ to $\mathcal{F}$, we have:

Theorem 5.6. *There is a natural isomorphism*

$$P_{(\cdot)} : \overline{\Gamma}(M \times N, E^! \boxtimes F) \cong \overline{\mathcal{E}^!} \hat{\otimes} \overline{\mathcal{F}} \to \mathrm{Hom}_{\mathrm{gen}}(\mathcal{E}, \mathcal{F}) \qquad P_{\xi \otimes f}(e) = \int_M f \cdot \langle \xi, \pi_M^*(e) \rangle_E$$

for $\xi \in \overline{\mathcal{E}}$ and $f \in \overline{\mathcal{F}}$, and where $\pi_M : M \times N \to M$ is the projection.

6 Differential Operators, Jet Bundles and Smooth D-Modules

6.1 Differential Operators and the Jet Bundle

Throughout let M a smooth manifold, and let C^∞_M denote the sheaf of smooth functions.

Definition 6.1. *Let E, F be vector bundles on M with sheaves of sections $\mathcal{E}, \mathcal{F}$. A (linear) differential operator is a map of sheaves of vector spaces $A : \mathcal{E} \to \mathcal{F}$ such that for any local coordinates and trivializations of E and F over $U \subset M$ we have*

$$A|_U = \sum_{|\alpha| \leq n} a_\alpha \circ \partial_\alpha$$

for some $n \in \mathbb{N}$ and $a_\alpha \in \mathrm{Hom}_{C^\infty(U)}(\mathcal{E}(U), \mathcal{F}(U))$ for each α, where α denotes a multi index and $|\alpha|$ denotes its total degree.

A differential operator $A : \mathcal{E} \to \mathcal{F}$ has order less than or equal to $k \in \mathbb{N}$ if for each point $x \in M$ there is a choice of coordinates and local trivializations in a neighbourhood of x such that the above description can be written as a sum over $|\alpha| \leq k$.

Let $\mathrm{Diff}(\mathcal{E}, \mathcal{F})$ denote the sheaf of differential operators from $\mathcal{E}$ to $\mathcal{F}$, considered as a sheaf of C^∞_M modules, and $\mathrm{Diff}^k(\mathcal{E}, \mathcal{F})$ the subsheaf of differential operators from $\mathcal{E}$ to $\mathcal{F}$ of order less than or equal to k.

For each $k \in \mathbb{N}$ and vector bundle E over M, there is a vector bundle $J^k(E)$ over M called the k^{th} jet bundle of E, with sheaf of sections $\mathcal{J}^k(\mathcal{E})$, and an injective differential operator $j^k \in \mathrm{Diff}^k(\mathcal{E}, \mathcal{J}^k(\mathcal{E}))$ called the k^{th} jet extension, defined by the fact that

$$\mathrm{Hom}_{C^\infty_M}(\mathcal{J}^k(\mathcal{E}), \mathcal{F}) \xrightarrow{\cong} \mathrm{Diff}^k(\mathcal{E}, \mathcal{F}) \qquad \text{via the map} \qquad \varphi \mapsto \varphi \circ j^k$$

That is, the functor $\mathrm{Diff}^k(\mathcal{E}, \cdot)$ is representable with representing object $\mathcal{J}^k(\mathcal{E})$

in the category of locally free, finitely generated sheaves of C^∞_M modules. The section $j^k(e) \in \mathcal{J}^k(\mathcal{E})$ thus contains all information about the section $e \in \mathcal{E}$ and all of its mixed partial derivatives in local coordinates of degree less than or equal to k.

For each $k \leq l$ there is a projection $J^l(E) \to J^k(E)$ dual to the natural inclusion $\mathrm{Diff}^k(\mathcal{E}, \mathcal{F}) \hookrightarrow \mathrm{Diff}^l(\mathcal{E}, \mathcal{F})$, and thus we have sequences of maps

$$\cdots \to J^{k+1}(E) \to J^k(E) \to \cdots \to J^1(E) \to E$$

and

$$\cdots \to \mathcal{J}^{k+1}(\mathcal{E}) \to \mathcal{J}^k(\mathcal{E}) \to \cdots \to \mathcal{J}^1(\mathcal{E}) \to \mathcal{E}$$

From this, we define a pro-vector bundle $J(E) := \lim_k J^k(E)$, the (infinite) jet bundle of E, which by definition has sheaf of sections given by the locally free, pro finitely generated sheaf of C^∞_M modules

$$\mathcal{J}(\mathcal{E}) = \lim_k \mathcal{J}^k(\mathcal{E})$$

Further, there is a pro differential operator $j := \lim_k j^k : \mathcal{E} \to \mathcal{J}(\mathcal{E})$ called the infinite jet extension, such that we have

$$\mathrm{Hom}_{C^\infty_M}(\mathcal{J}(\mathcal{E}), \mathcal{F}) \xrightarrow{\cong} \mathrm{Diff}(\mathcal{E}, \mathcal{F}) \qquad \text{via the map} \qquad \varphi \mapsto \varphi \circ j$$

6.2 D_M-Modules, the Jets Functor, and Polydifferential Operators

Define the sheaf of smooth differential operators on M by

$$D_M = \mathrm{Diff}(C^\infty_M, C^\infty_M)$$

viewed as a sheaf of C^∞_M modules, and additionally a sheaf of (non-commutative) algebras with product given by composition.

Definition 6.2. *The category* D_M-Mod *is the category of sheaves of left modules for the sheaf* D_M*.*

Note that we have an inclusion of sheaves of algebras $C^\infty_M \hookrightarrow D_M$ and thus have a pullback functor on categories of sheaves of modules D_M-Mod $\to$ C^∞_M-Mod. We have the following characterization of D_M modules with a locally free underlying C^∞_M module, or equivalently, given by the sheaf of sections of a vector bundle E:

Proposition 6.3. *Let E be a vector bundle with corresponding sheaf of sections $\mathcal{E}$, viewed as a C^∞_M module. A choice of extension of the* C^∞_M-Mod *structure on $\mathcal{E}$ to a* D_M-Mod *stucture is equivalent to a flat connection $\nabla : \mathcal{E} \to \Omega^1_M \otimes \mathcal{E}$.*

The simplest example of a D_M module is C^∞_M, via the inclusion $\mathrm{Diff}(C^\infty_M, C^\infty_M) \hookrightarrow \mathrm{Hom}(C^\infty_M, C^\infty_M)$ where Hom with corresponding flat connection given by the de Rham differential. Another important example for this paper is Dens_M, viewed as a left/right D_M module with flat connection given by

$$\nabla_X \omega = \pm \mathcal{L}_X \omega = \pm d\iota_X \omega$$

For $R, S \in D_M$-Mod we define a tensor product functor $\otimes_{C^\infty_M} : D_M\text{-Mod} \times D_M\text{-Mod} \to D_M\text{-Mod}$ by

$$R \otimes_{C^\infty_M} S \quad \in D_M\text{-Mod} \qquad \text{by} \qquad D(r \otimes s) = Dr \otimes s + r \otimes Ds$$

We have

Proposition 6.4. *The category* D_M-Mod *together with the tensor product* $\otimes_{C^\infty_M}$ *is symmetric monodial.*

Note that for any vector bundle E with sheaf of sections $\mathcal{E}$, the sheaf of sections of the jet bundle $\mathcal{J}(\mathcal{E})$ is naturally a D_M module, as it carries a flat connection defined by the criteria that a section $\varepsilon \in \mathcal{J}(\mathcal{E})$ is flat if and only if $\varepsilon = j(e)$ for $e \in \mathcal{E}$. Further, note that this in fact extends to a functor

$$J : C^\infty_M\text{-Mod}^{\mathrm{lf}} \cong \mathrm{VB}_M \to D_M\text{-Mod} \qquad \text{defined by} \qquad \mathcal{E} \cong E \mapsto \mathcal{J}(\mathcal{E})$$

where C^∞_M-Mod$^{\mathrm{lf}}$ denotes the full subcategory of C^∞_M-Mod on locally free sheaves of modules.

We make the following definition:

Definition 6.5. *Let* $E_1, ..., E_n, F$ *be vector bundles on* M *and* $\mathcal{E}_i, \mathcal{F}$ *their sheaves of sections. We define the sheaf of* C^∞_M *modules of polydifferential operators by:*

$$\mathrm{PolyDiff}(\mathcal{E}_1 \otimes ... \otimes \mathcal{E}_n, \mathcal{F}) = \mathrm{Diff}(\mathcal{E}_1, C^\infty_M) \otimes_{C^\infty_M} ... \otimes_{C^\infty_M} \mathrm{Diff}(\mathcal{E}_n, C^\infty_M) \otimes_{C^\infty_M} \mathcal{F}$$

Now, we have the following crucial proposition explaining the significance of polydifferential operators in the formulation of classical field theories given in definition 2.6:

Proposition 6.6. *Let* E_i, F *vector bundles over* M *with sheaves of sections* $\mathcal{E}, \mathcal{E}_i, F$ *for* $i = 1, ..., n$. *Then*

$$\mathrm{Hom}_{D_M}(\mathcal{J}(\mathcal{E}_1) \otimes_{C^\infty_M} ... \otimes_{C^\infty_M} \mathcal{J}(\mathcal{E}_n), \mathcal{J}(\mathcal{F})) = \mathrm{PolyDiff}(\mathcal{E}_1 \hat{\otimes} ... \hat{\otimes} \mathcal{E}_n, \mathcal{F})$$

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