Risk Measures and Capital Allocation Principles for Risk Management

by

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I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

Risk measures (or premium principles) and capital allocation principles play a significant role in risk management. Regulators and companies in the financial markets usually adopt an appropriate risk measure, for example, Value-at-Risk (VaR) or Tail Value-at-Risk (TVaR), to determine the benchmarks. However, these risk measures are determined from the loss functions with constant weights, not random weight functions.

This thesis proposes new approaches to determine risk measures from two perspectives. Firstly, we will generalize the definition of the tail subadditivity for distortion risk measures; we define the generalized GlueVaR (a linear combination of VaR and TVaRs) to approach any coherent distortion risk measure. Secondly, we will research the risk measures (or premium principles) and capital allocation principles based on the loss functions with random weight functions.

The new reinsurance premium principles are derived similarly to the new risk measures. The two thresholds for the weight in the loss function can be employed by reinsurance companies as benchmarks when pricing the reinsurance products. The capital allocation principles derived based on the weighted loss functions are both mathematically and economically reasonable. Many of the risk measures and allocation principles, including the new risk measures, can be covered by this model.

The results of this thesis have not only unified many of the risk measures and capital allocation principles, but also provided new and practical models.

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Dedication

This is dedicated to my dear parents, Youfa Wang and Mingxing Wang, and dear grandparents, Zhi Wang and Zhiqing Liu.

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$\mathfrak{Q}_{\alpha,X,Y}$	common tail region for X and Y at a confidence level $\alpha \in (0,1)$	10
$\Omega_{X,Y}$	common tail region of X and Y	11
$\Omega_{X_1,\ldots,X_n}$	common tail region of a random vector $(X_1,\ldots,X_n)$	11
$\Omega_{\alpha,X}$	tail region of a risk X defined by VaR_α	11
Ω^e_X	tail region of a risk X defined by expectation	11
Ω_{α,S_n}	tail region for the aggregate risk S_n defined by VaR_α	12
$\Omega^e_{S_n}$	tail region for the aggregate risk S_n defined by expectation	12
$\Omega_{\alpha,X_1,\ldots,X_n}$	common tail region as union of tail regions for a portfolio of risks $(X_1,\ldots,X_n)$ defined by VaR_α	12
$\Omega^e_{X_1,\ldots,X_n}$	common tail region as union of tail regions for a portfolio of risks $(X_1,\ldots,X_n)$ defined by expectation	12
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$\Omega^{e*}_{X_1,\ldots,X_n}$	common tail region as intersection of tail regions for a portfolio of risks $(X_1,\ldots,X_n)$ defined by expectation	32
$\leq_{st}$	(multivariate) usual stochastic order	59

Chapter 1

Introduction

1.1 Premium Principles and Risk Measures

In Chapter 2, Chapter 3 and Chapter 4, we will propose two risk measures and a new premium principle to price reinsurance contracts. One of the two risk measures is a risk measure satisfying the property of tail-subadditivity, and the other is derived based on the weighted loss functions. In the literature, Young (2004) concluded three methods to attain premium principles, the *"ad hoc method"*, the *"characterization method"* and the *"economic method"*. If an actuary checks the desirable properties for a new premium principle, it is called the *"ad hoc method"*. Sometimes, they might firstly list the axioms for the potential premium principle and then derive the principle following these axioms by the *"characterization method"*. Moreover, particular economic theory could be applied to determine premium principles and it is defined as the *"economic method"*. In fact, most of the premium principles are not derived by only one method, and we can combine these three methods together to find new premium principles.

A risk measure H is a mapping from $\mathcal{X}$ to $\mathbb{R} = (-\infty, +\infty)$, where $\mathcal{X}$ is the set of loss random variables, namely, for any $X \in \mathcal{X}$, $H(X) \in (-\infty, +\infty)$. If $\mathcal{X}$ is the set of insured (covered) loss random variables faced by an insurer (a reinsurer), a risk measure $H : \mathcal{X} \to \mathbb{R}$ is also called an insurance (a reinsurance) premium principle. Some desired properties or axioms for the premium principles or risk measures are as follows.

1. *Risk loading*: $H[X] \geq \mathbb{E}[X]$ for all $X \in \mathcal{X}$.
2. *No unjustified risk loading*: If a risk $X \in \mathcal{X}$ is equal to a constant $c \geq 0$ (almost everywhere), then $H[X]$=c.
3. *Maximal loss (no rip-off)* : $H[X] \leq$ ess-sup$X$ for all $X \in \mathcal{X}$.
4. *Translation equivalence (translation invariance)*: $H[X + a] = H[X] + a$ for all $X \in \mathcal{X}$ and all $a \geq 0$.
5. *Scale equivalence (scale invariance, positive homogeneity)*: $H[bX] = bH[X]$ for all $X \in \mathcal{X}$ and all $b \geq 0$.
6. *Subadditivity*: $H[X + Y] \leq H[X] + H[Y]$ for all $X, Y \in \mathcal{X}$.
7. *Monotonicity*: If $X \leq Y$ a.s., then $H[X] \leq H[Y]$.
8. *Preserves first stochastic dominance (FSD) ordering*: If $S_X(t) \leq S_Y(t)$ for all $t \geq 0$, then $H[X] \leq H[Y]$, where S_X and S_Y are the survival functions of X and Y, respectively.
9. *Preserves stop-loss (SL) ordering*: If $\mathbb{E}[(X-d)_+] \leq \mathbb{E}[(Y-d)_+]$ for all $d \geq 0$, then $H[X] \leq H[Y]$.
10. *Law invariance*: $H(X) = H(Y)$ for all $X, Y \in \mathcal{X}$ such that X and Y have the same probability law.

In Chapter 2, we generalize the concept (Belles et al., 2014a and 2014b) of tail subadditivity for distortion risk measures and give sufficient and necessary conditions for a distortion risk measure to be tail subadditive. As applications of the derived results, we rectify Theorem 6.1 (Belles et al., 2014a) and Theorem 5.1 (Yin and Zhu, 2016) regarding the sufficient conditions for a distortion risk measure to be tail subadditive. We also introduce the generalized tail subadditive GlueVaR risk measures, which can be used to approach any coherent risk measure. To further illustrate the applications of the tail subadditivity, we propose a tail distortion principle for decision makers to determine the required solvency capitals or for insurers to calculate insurance premiums for a portfolio of risks. The tail distortion principle depends both on extreme tail events and on the dependence of the risks in a portfolio.

In Chapter 3, the weights in the model (Bellini et al., 2014) are generalized as functions of loss random variables, and the corresponding risk measures are defined as the weighted quantiles. Then, the quadratic and identity functions are adopted as two special functions to quantify the positive and negative parts in the objective function. Furthermore, the properties of the weighted quantiles are proposed. In the quadratic case, we define the risk measure as the weighted expectile and investigate the properties when the weight functions are segmented by a special risk measure. In fact, the expectiles and the weighted premium principle can be treated as two special cases of the weighted expectile. Based on a specially defined weight function, the weighted expectile can be a coherent risk measure. In the identic case, the new risk measure is defined as weighted VaR. Moreover, we compare the weighted VaR with the classical VaR, the median shortfall (Kuo, 2013), and the averaged VaR (Hera et al., 2012). In addition, numerical examples are provided following the theories.

In Chapter 4, given ceded functions, two classes of new premium principles for pricing reinsurance contracts are proposed by minimizing the objective function with both the insurer's and reinsurer's risks considered. Quadratic and identity functions are applied to quantify the risks, and two classes of reinsurance premium principles are derived. In fact, the weight functions in the objective function can be either constants or functions of the loss random variables. Also, the properties of the new principles are studied in two cases. In the quadratic case, the minimum weight factor acceptable and the preferred threshold for the reinsurer are suggested. Moreover, numerical examples with loss random variables following exponential and Pareto distributions are provided. In the identic case, the premium principle can be lower than the premium based on the classical VaR (Heilmann, 1989) when the weight functions are both constants.

1.2 Capital Allocation Principles

If there are n business lines in a company, the capital allocation problem is usually described as a decomposition of the given total capital based on individual risks for n business lines. The total capital is assumed to be K. The individual losses are X_1, X_2, …, X_n, and so the aggregated loss is $S = \sum_{i=1}^{n} X_i$. We use K_i to denote the allocated capital for the ith business line for $i = 1, 2, \ldots, n$.

Common Capital Allocations are the following:

(a) *Haircut*: for certain $\alpha \in (0, 1)$ and $i = 1, 2, \ldots, n$,

$$K_i = \frac{F_{X_i}^{-1}(\alpha)}{\sum_{i=1}^{n} F_{X_i}^{-1}(\alpha)} K,$$

where $F_{X_i}^{-1}(\alpha)$ is the VaR at confidence level α for X_i.

(b) *Quantile*: for $i = 1, 2, \ldots, n$,

$$K_i = \frac{F_{X_i}^{-1}(F_{S^c}(K))}{\sum_{i=1}^{n} F_{X_i}^{-1}(F_{S^c}(K))} K,$$

where $S^c = \sum_{i=1}^{n} F_{X_i}^{-1}(U)$ with U being a uniform random variable on $(0, 1)$.

(c) *Covariance*: for $i = 1, 2, \ldots, n$,

$$K_i = \frac{\mathrm{Cov}(X_i, \sum_{i=1}^{n} X_i)}{\mathrm{Var}(\sum_{i=1}^{n} X_i)} K.$$

(d) *Conditional-Tail-Expectation (CTE)*: for certain $\alpha \in (0,1)$ and $i = 1, 2, \ldots, n$,

$$K_i = \frac{\mathbb{E}[X_i|S > F_S^{-1}(\alpha)]}{\mathbb{E}[S|S > F_S^{-1}(\alpha)]}K,$$

where $S = \sum_{i=1}^n X_i$.

(e) *Tail covariance premium adjusted (Wang, 2014)*:

$$K_i = \frac{\mathrm{TVaR}_\alpha(X_i|S) + a\frac{TCov_\alpha(X_i|S)}{\sqrt{TV_\alpha(S)}}}{\mathrm{TVaR}_\alpha(S) + a\sqrt{TV_\alpha(S)}}K$$

where a is a non-negative constant and $TV_\alpha(X) = \mathrm{Var}[X|X > \mathrm{VaR}_\alpha(X)]$, $\mathrm{TVaR}_\alpha(X_i) = \mathbb{E}[X|X > \mathrm{VaR}_\alpha(X)]$, $\mathrm{TVaR}_\alpha(X_i|S) = \mathbb{E}[X_i|S > \mathrm{VaR}_\alpha(S)]$, and $TCov_\alpha(X_i|S) = \mathrm{Cov}[X_i, S|S > \mathrm{VaR}_\alpha(S)]$.

In Furman and Zitikis (2008b), several axioms are introduced for the weighted capital allocation principle. They consider the business units $X_1, X_2, \ldots, X_n$ in a portfolio and want to allocate total initial capital K into the n business lines, namely, the allocated capital should be $K_1, K_2, \ldots, K_n$. We know that the aggregated loss is $S = \sum_{i=1}^n X_i$. In this problem, $K_i = A(X_i, S)$ for $i = 1, 2, \ldots, n$. For $I \subseteq N = \{1, 2, \ldots, n\}$ and $S_I = \sum_{i \in I} X_i$, the axioms are as following:

1. *No undercut*: $\sum_{i \in I} A(X_i, S) \le A(S_I, S_I)$.
2. *Consistent no-undercut*: $A(S_I, S) \le A(S_I, S_I)$.
3. *No negative loading*: $A(X_i, S) \ge \mathbb{E}[X_i]$.
4. *No unjustified loading*: $A(X_i, S) = c$, if $X_i = c$.
5. *Full additivity*: $\sum_{i \in N} A(X_i, S) = A(S, S)$.
6. *Consistency*: $\sum_{i \in I} A(X_i, S) = A(S_I, S)$.
7. *Sub-translation invariance*: $A(X_i + a, S + a) \le A(X_i, S) + a$ for any constant $a \ge 0$.

8. *Translation invariance*: $A(X_i + a, S + a) = A(X_i, S) + a$ for any constant $a \geq 0$.

9. *Super-translation invariance*: $A(X_i+a, S+a) \geq A(X_i, S)+a$ for any constant $a \geq 0$.

10. *Sub-scale invariance*: $A(bX_i, \sum_{j\neq i} X_j + bX_i) \leq bA(X_i, S)$ for any constant $b \geq 0$.

11. *Scale invariance*: $A(bX_i, \sum_{j\neq i} X_j + bX_i) = bA(X_i, S)$ for any constant $b \geq 0$.

12. *Super-scale invariance*: $A(bX_i, \sum_{j\neq i} X_j + bX_i) \geq bA(X_i, S)$ for any constant $b \geq 0$.

In Chapter 5, we establish a general model for determining optimal capital allocation principles based on the unified model (Dhaene et al., 2012 and Belles et al., 2014b). The weighted loss functions with the capital deficit risk and capital surplus risk are considered, and so the model is both mathematically and economically reasonable. Following the generalized principles, the capital allocation principles for the business driven and aggregate portfolio driven types are derived with the quadratic and identity quantifying functions. In addition, the add on and off capital allocation principles are proposed. Then, we investigate the properties of these principles. In fact, the capital allocation principles in Dhaene et al. (2012) and Belles et al. (2014b) are special cases of the generalized allocation principle in our revised model.

In Chapter 6, we provide concluding remarks.

Throughout this paper, "increasing" means "non-decreasing" while "decreasing" means "non-increasing".

Chapter 2

Tail Subadditivity of Distortion Risk Measures with Applications in Portfolio Risk Management

2.1 Introduction

Risk measures have been used extensively in insurance and finance as a tool of risk management. One of the important functions of risk measures is to determine the required regulatory capitals for investment and insurance portfolios and to price insurance and reinsurance products. Mathematically, a risk measure is a mapping $\rho : \mathcal{X} \to \mathbb{R} = (-\infty, \infty)$, where $\mathcal{X}$ is a set of loss random variables or risks defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. When $\mathcal{X}$ is the set of the losses faced by a bank or an insurer, a risk measure ρ can be used to determine the required regulatory capitals for the bank or insurer. When $\mathcal{X}$ is the set of the insured risks faced by an insurer, a risk measure can represent a premium calculation principle. In this case, for an insured risk $X \in \mathcal{X}$, $\rho(X)$ is the premium assigned to the insured risk X. In this chapter, we denote $L^p = L^p(\Omega, \mathcal{F}, \mathbb{P})$ for $p \in [0, \infty]$, which is the set of all random variables, defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with finite p-th moment for $0 \le p \le \infty$. In particular, L^0 means the set of all random variables while L^∞ represents the set of all bounded random variables. In determining the regulatory capitals, the VaR and the TVaR are two popular risk measures used by regulators in finance and insurance. In insurance risk management, many premium principles, such as net premium principle, expected value principle, Dutch principle, Wang's principle, and many others, have been employed in insurance pricing.

For any loss random variable $X \in L^0$, the VaR of X at a given confidence level $\alpha \in (0,1)$ is defined as

$$\mathrm{VaR}_\alpha(X) = \inf\{x \in \mathbb{R} : \mathbb{P}(X \leq x) \geq \alpha\} = F_X^{-1}(\alpha),$$

which is the left continuous inverse of the distribution $F_X(x) = P(X \leq x) = 1 - S_X(x)$. When the regulatory capitals of a bank or an insurance company are determined by VaR_α, the solvency probability of the company is at least the confidence level α. However, the risk measure of VaR does not satisfy the subadditivity[1], which is one property desired by many regulators in determining the capital reserves. For any loss random variable $X \in L^1$, the TVaR of X at a given confidence level $\alpha \in (0,1)$ is defined as

$$\mathrm{TVaR}_\alpha(X) = \frac{1}{1-\alpha}\int_\alpha^1 \mathrm{VaR}_q(X)\,\mathrm{d}q,$$

which has the following equivalent expression:

$$\mathrm{TVaR}_\alpha(X) = \mathrm{VaR}_\alpha(X) + \frac{1}{1-\alpha}\int_{\mathrm{VaR}_\alpha(X)}^\infty S_X(x)dx.$$

Both the VaR and TVaR are the special cases of distortion risk measures. A distortion risk measure $\rho_g : \mathcal{X} \to \mathbb{R}$, with a distortion function[2] g, is defined as

$$\rho_g(X) = \int_{-\infty}^0 [g(S_X(x)) - 1]dx + \int_0^{+\infty} g(S_X(x))dx. \tag{2.1}$$

[1]A risk measure $\rho : \mathcal{X} \to \mathbb{R}$ is siad to satisfy the subadditive property if $\rho(X+Y) \leq \rho(X) + \rho(Y)$ for all $X, Y \in \mathcal{X}$.

[2]A function $g : [0,1] \to [0,1]$ is said to be a distortion function if g is increasing in $[0,1]$ with $g(0) = 0$ and $g(1) = 1$.

Note that, for any random variable X,

$$\mathbb{E}[X] = \int_{-\infty}^{0} [S_X(x) - 1]dx + \int_{0}^{+\infty} S_X(x)dx. \tag{2.2}$$

The idea behind a distortion risk measure is to distort the survival function of a random variable using a distortion function such that the 'distorted expectation' can provide more flexible and reasonable measures for the risk. Mathematically, a distortion risk measure $\rho_g(X)$ is a Choquet integral $\int_\Omega X\mathrm{d}\mu\,(= \int X\mathrm{d}\mu)$ over the sample space Ω with respect to the set function $\mu = g \circ P$, which is defined on the σ-algebra $\mathcal{F}$ as $\mu(A) = g(P(A))$ for any $A \in \mathcal{F}$. A brief review of Choquet integrals and their properties is given in Section 2.2. Roughly speaking, a distortion risk measure of a risk is 'its expectation' over a sample space under a distorted measure.

It is well known that $\mathrm{TVaR}_\alpha(X)$ is a distortion risk measure with the distortion function

$$g_{\alpha,\mathrm{TVaR}}(x) = \frac{x}{1-\alpha}\mathbb{I}_{\{0\le x\le 1-\alpha\}} + \mathbb{I}_{\{1-\alpha<x\le 1\}} = \min\left\{\frac{x}{1-\alpha},\, 1\right\},$$

where and throughout the chapter, $\mathbb{I}_A$ is the indicator function of an event A or a Bernoulli random variable, and it equals to 1 if A holds true and 0 otherwise. In addition, $\mathrm{VaR}_\alpha(X)$ or the left continuous inverse $F_X^{-1}(\alpha)$ is a distortion risk measure with the following left continuous distortion function:

$$g_{\alpha,\mathrm{VaR}}(x) = \mathbb{I}_{\{1-\alpha<x\le 1\}}.$$

Furthermore, the right continuous inverse $F_X^{-1+}(\alpha) = \sup\{x \in \mathbb{R} : F_X(x) \le \alpha\}$ of the distribution function $F_X(x) = P(X \le x)$ is a distortion risk measure with the following right continuous distortion function:

$$g^{+}_{\alpha,\mathrm{VaR}}(x) = \mathbb{I}_{\{1-\alpha\le x\le 1\}}.$$

In terms of Choquet integrals, a distortion risk measure $\rho_g : \mathcal{X} \to \mathbb{R}$ is said to be subadditive in $\mathcal{X}$ if

$$\int_\Omega (X+Y)\,\mathrm{d}g \circ P \leq \int_\Omega X \mathrm{d}g \circ P + \int_\Omega Y \mathrm{d}g \circ P \tag{2.3}$$

holds for any two random variables $X, Y \in \mathcal{X}$. Any distortion risk measure including VaR satisfies positive homogeneity, translation-invariance, and monotonicity. For any risk X and a confidence level $\alpha \in (0,1)$, it holds that $\mathrm{TVaR}_\alpha(X) \geq \mathrm{VaR}_\alpha(X)$. Moreover, TVaR_α is the smallest subadditive distortion risk measure (indeed the smallest coherent risk measure[3]) satisfying $\mathrm{TVaR}_\alpha(X) \geq \mathrm{VaR}_\alpha$ for all $X \in \mathcal{X}$ under a given confidence level α. However, as pointed out by Belles et al. (2014a), subadditivity or coherence might be an expensive (even unrealistic) requirement on the derminations of premiums and regulatory capitals. Belles et al. (2014a) introduced a subclass of distortion risk measures, called GlueVaR risk measures, which satisfy the tail subadditive property, a weaker requirement than subadditivity. In introducing the concept of tail subadditivity, they defined a tail region of a random variable X at a confidence level $\alpha \in (0,1)$ as

$$\mathfrak{Q}_{\alpha,X} = \{\omega \in \Omega : X(\omega) > F_X^{-1}(\alpha)\} = \{X > F_X^{-1}(\alpha)\}. \tag{2.4}$$

For any two random variables $X, Y \in \mathcal{X}$, the common tail region for the two random variables at a confidence level $\alpha \in (0,1)$ is defined as

$$\begin{aligned} \mathfrak{Q}_{\alpha,X,Y} &= \mathfrak{Q}_{\alpha,X} \cap \mathfrak{Q}_{\alpha,Y} \cap \mathfrak{Q}_{\alpha,X+Y} \\ &= \{X > F_X^{-1}(\alpha),\, Y > F_Y^{-1}(\alpha),\, X+Y > F_{X+Y}^{-1}(\alpha)\}. \end{aligned} \tag{2.5}$$

Instead of calculating the expectations of risks under distorted measures over the sample space Ω in (2.3), Belles et al. (2014a) calculated the expectations of risks under distorted measures over the the common tail region. By their definition, a distortion risk measure ρ_g is said to be tail-subadditive for a pair of random

[3] See Kusuoka (2001) for details.

variables $X, Y \in \mathcal{X}$ at a confidence level $\alpha \in (0,1)$ if

$$\int_{\mathfrak{Q}_{\alpha,X,Y}} (X+Y) dg \circ P \leq \int_{\mathfrak{Q}_{\alpha,X,Y}} X \mathrm{d}g \circ P + \int_{\mathfrak{Q}_{\alpha,X,Y}} Y dg \circ P \tag{2.6}$$

holds, where the integrals are Choquet integrals over the the common tail region $\mathfrak{Q}_{\alpha,X,Y}$. Theorem 6.1 of Belles et al. (2014a) shows that if the distortion function of a GlueVaR is concave on $[0, 1-\alpha)$, then the GlueVaR risk measure is tail-subadditive for any pair of random variables $X, Y \in X$. We will point out in Section 2.3, the condition that the distortion function of a GlueVaR is concave on $[0, 1-\alpha)$ in Theorem 6.1 of Belles et al. (2014a) should be that the distortion function of a GlueVaR is concave on $[0, 1-\alpha]$. Interesting applications of the GlueVaR risk measures in insurance, finance, and other fields have been discussed in Belles et al. (2014a, 2014b).

In this chapter, we first give more general definitions of common tail regions and the tail subadditivity.

Definition 2.1. We call a set $\Omega_X \subset \Omega$ a tail region of a random variable X if $\Omega_X \in \mathcal{F}$. Furthermore, we call a set $\Omega_{X,Y} \subset \Omega$ a common tail region of X and Y if $\Omega_{X,Y} \in \mathcal{F}$ and $\Omega_{X,Y} = \Omega_{Y,X}$. More general, we call $\Omega_{X_1,\ldots,X_n} \subset \Omega$ a common tail region of a random vector $(X_1, ..., X_n)$ if $\Omega_{X_1,\ldots,X_n} \in \mathcal{F}$ and $\Omega_{X_1,\ldots,X_n} = \Omega_{X_{\pi(1)},\ldots,X_{\pi(n)}}$ for any permutation $(\pi(1), ..., \pi(n))$ of $(1, ..., n)$.

There are different ways to define common tail regions for risks in a portfolio. We will focus on the following common tail regions that are often concerned in risk management. First, the tail regions of a risk X often arise in solvency risk management are

$$\Omega_{\alpha,X} = \{\omega \in \Omega : X \geq \mathrm{VaR}_\alpha(X)\} = \{X \geq \mathrm{VaR}_\alpha(X)\} \tag{2.7}$$

and $\Omega^-_{\alpha,X} = \{X < \mathrm{VaR}_\alpha(X)\}$. The former describes a right tail and the latter is a left tail. Moreover, the right tail

$$\Omega^e_X = \{\omega \in \Omega : X \geq \mathbb{E}(X)\} = \{X \geq \mathbb{E}[X]\} \tag{2.8}$$

is an important tail region in pricing premiums. Indeed, $\mathrm{VaR}_\alpha(X)$ and $\mathbb{E}[X]$ are respectively the important benchmarks in determining the required solvency capital and in calculating insurance premiums. Correspondingly, the following two common tail regions

$$\Omega_{\alpha,S_n} = \{S_n \geq \mathrm{VaR}_\alpha(S_n)\} \tag{2.9}$$

and

$$\Omega^e_{S_n} = \{S_n \geq \mathbb{E}[S_n]\} \tag{2.10}$$

are often concerned in portfolio risk management, where $S_n = X_1 + \cdots + X_n$ is the aggregate risk of a portfolio consisting of risks $X_1, ..., X_n$. Other interesting common tail regions include

$$\Omega_{\alpha,X_1,...,X_n} = \{X_1 \geq \mathrm{VaR}_\alpha(X_1)\} \cup \cdots \cup \{X_n \geq \mathrm{VaR}_\alpha(X_n)\} \tag{2.11}$$

and

$$\Omega^e_{X_1,...,X_n} = \{X_1 \geq \mathbb{E}[X_1]\} \cup \cdots \cup \{X_n \geq \mathbb{E}[X_n]\}. \tag{2.12}$$

All the above tail regions and the common tail regions describe the extreme tail events concerned by decision makers in portfolio risk management. Furthermore, the common tail region of a risk portfolio $(X_1, ..., X_n)$ is defined as $\Omega_{\alpha,X_1+\cdots+X_n} = \Omega_{\alpha,S_n}$, where $S_n = X_1 + \cdots + X_n$ is the aggregate risk of the portfolio.

In this chapter, we define tail subadditivity for distortion risk measures as follows.

Definition 2.2. Let $\Omega_{X,Y}$ be a common tail region for $X, Y \in \mathcal{X}$. For a distortion function g, the distortion risk measure $\rho_g : \mathcal{X} \to \mathbb{R}$ is said to be tail subadditive for a pair of random variables $X, Y \in \mathcal{X}$ if

$$\int_{\Omega_{X,Y}} (X+Y)\,\mathrm{d}g \circ P \leq \int_{\Omega_{X,Y}} X\mathrm{d}g \circ P + \int_{\Omega_{X,Y}} Y\mathrm{d}g \circ P, \tag{2.13}$$

where the integrals are Choquet integrals over the common tail region $\Omega_{X,Y}$. In general, a distortion risk measure $\rho_g : \mathcal{X} \to \mathbb{R}$ is said to be tail subadditive for $X_1, ..., X_n \in \mathcal{X}$ if

$$\int_{\Omega_{X_1,\ldots,X_n}} \sum_{i=1}^{n} X_i \,\mathrm{d}g \circ P \leq \sum_{i=1}^{n} \int_{\Omega_{X_1,\cdots,X_n}} X_i \,\mathrm{d}g \circ P. \tag{2.14}$$

In addition, the distortion risk measure $\rho_g : \mathcal{X} \to \mathbb{R}$ is said to be tail subadditive in $\mathcal{X}$ if (2.13) holds for any pair of random variables $X, Y \in \mathcal{X}$.

The rest of the chapter is organized as follows. In Section 2.2, we recall some preliminaries about Choquet integrals and convex functions. In Section 2.3, we study sufficient and necessary conditions for a distortion risk measure to be tail subadditive. We rectify Theorem 6.1 of Belles et al. (2014a) and Theorem 5.1 of Yin and Zhu (2016) about the sufficient conditions for a distortion risk measure to be tail subadditive. We also discuss the generalized tail subadditive GlueVaR risk measures, which generalizes the tail subadditive GlueVaR risk measures introduced by Belles et al. (2014a). Moreover, the generalized tail subadditive GlueVaR risk measures can be used to approach any coherent risk measures. To further illustrate the applications of the tail subadditivity, in Section 2.4, we propose a tail distortion principle for decision makers to determine the capital reserves or for insurers to calculate insurance premiums for risks in a portfolio. The properties of the proposed principle are discussed as well. In Section 2.5, we illustrate the applications of the tail distortion principle by multivariate Pareto distributions.

2.2 Preliminaries about Choquet integrals and convex functions

In this section, we review some concepts and results about Choquet integrals and recall some results about convex functions, which will be used in this chapter.

Let Ω be a basic set and 2^Ω be the family of all subsets of Ω. Let $\mathcal{S} \subset 2^\Omega$ be a σ-algebra, which is also called a set system of Ω. In addition, $(\Omega, \mathcal{S})$ is called a measurable space.

Definition 2.3. A set function μ on a set system $\mathcal{S}$ is said to be monotone if $A \subset B$ implies $\mu(A) \leq \mu(B)$ for any $A,\ B \in \mathcal{S}$, and it is said to be submodular if $A,\ B \in \mathcal{S}$ such that $A \cup B \in \mathcal{S},\ A \cap B \in \mathcal{S}$ implies

$$\mu(A \cup B) + \mu(A \cap B) \leq \mu(A) + \mu(B). \tag{2.15}$$

Definition 2.4. Let $\mu : \mathcal{S} \to \mathbb{R}_+ = [0, \infty)$ a monotone set function with $\mu(\Omega) < \infty$ and $X : \Omega \to \mathbb{R}$ be a measurable function on the measurable space $(\Omega, \mathcal{S})$. The (asymmetric) Choquet integral of the function X with respect to the set function μ is denoted by $\int_\Omega X d\mu$ or $\int X d\mu$, which is defined as

$$\int X d\mu = \int_\Omega X d\mu = \int_{-\infty}^{0} [S_{\mu,X}(x) - \mu(\Omega)]dx + \int_0^\infty S_{\mu,X}(x)dx, \tag{2.16}$$

where $S_{\mu,X}(x) = \mu(\{X > x\})$ denotes the survival function of X with respect to μ. Moreover, for $A \in \mathcal{S}$, the Choquet integral of the function X on the set A with respect to the set function μ is denoted by $\int_A X d\mu$, which is defined as

$$\int_A X d\mu = \int_{-\infty}^{0} [\mu(A \cap \{X > x\}) - \mu(A)]dx + \int_0^\infty \mu(A \cap \{X > x\})dx. \tag{2.17}$$

Note that if we define a set function $\mu_A(B) = \mu(A \cap B)$ for $B \in \mathcal{S}$, then (2.17) can be rewritten as

$$\int_A X d\mu = \int_\Omega X d\mu_A. \tag{2.18}$$

Furthermore, if $\mu(A) = 0$, then $\int_A X d\mu = 0$. Roughly speaking, the Choquet integral of a risk over a set is the 'expectation' of the risk under a distorted measure. We recall the following properties of Choquet integrals from Denneberg (1994).

Lemma 2.5. *Let $(\Omega, \mathcal{S})$ be a measurable space. If $\mu : \mathcal{S} \to \mathbb{R}_+$ is a monotone set function and $X, Y : \Omega \to \mathbb{R}$ are measurable functions, then*

(a) $\int \mathbb{I}_A \, d\mu = \mu(A)$ *for any* $A \in \mathcal{S}$.

(b) $\int (aX + b) d\mu = a \int X d\mu + b\, \mu(\Omega)$ *for any* $a \geq 0$ *and* $b \in \mathbb{R}$.

(c) $\int (X + Y) d\mu = \int X d\mu + \int Y d\mu$ *if* X *and* Y *are comonotonic.*

(d) $\int X d\mu \leq \int Y d\mu$ *if* $X \leq Y$.

(e) $\int (X+Y) d\mu \leq \int X d\mu + \int Y d\mu$ *if* μ *is submodular and* X, Y *are* μ*-essentially* $> -\infty$.

(f) *A measurable function* X *is* μ*-essentially* $> -\infty$ *if and only if* $\lim_{x \to \infty} \mu((X > x)) = 0$.

Remark 2.6. *In this chapter, we will focus on distortion risk measures. If* X *is a random variable on the probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *and let* $\mu = g \circ P : \mathcal{F} \to [0, 1]$ *be a distorted measure with a distortion function* g*, then we have the following assertions:*

(i) *The distorted measure* $g \circ P$ *is a monotone set function on* $\mathcal{F}$.

(ii) *The random variable* X *is* $g \circ P$*-essentially* $> -\infty$ *if the distortion function* g *is right continuous at* 0*, namely,* $g(0+) = g(0) = 0$.

Throughout the chapter, $f(a+)$ *and* $f(a-)$ *mean the right and left limits of a function* f *at* a*, respectively.* $\square$

Furthermore, in the following lemma, we recall the known results about convex/-concave functions from Niculescu and Persson (2006).

Lemma 2.7. *(a) Let* $f : [a, b] \to \mathbb{R}$ *be a function bounded from below on every compact subinterval of* $[a, b]$*. Then* f *is concave if and only if* f *is a midpoint concave, that is,*

$$f\Big(\frac{x + y}{2}\Big) \geq \frac{f(x) + f(y)}{2}$$

for any $x, y \in [a, b]$.

(b) Let $f : [a, b] \to \mathbb{R}$ ba a concave function, then

$$\frac{f(a) + f(b)}{2} - f\Big(\frac{a+b}{2}\Big) \leq \frac{f(c) + f(d)}{2} - f\Big(\frac{c+d}{2}\Big)$$

for all $a \leq c \leq d \leq b$.

The following lemma reviews the properties (Dhaene et al., 2012; McNeil et al., 2005) of the left and right continuous inverses of a distribution function, which will be used in this chapter.

Lemma 2.8. *Let X be a random variable with the distribution function $F_X(x) = P(X \leq x)$. Then, for any $\alpha \in (0, 1)$, it holds that*

$$F_X(F_X^{-1}(\alpha)-) = P(X < F_X^{-1}(\alpha)) < \alpha \leq P(X \leq F_X^{-1}(\alpha)) = F_X(F_X^{-1}(\alpha)). \quad (2.19)$$

2.3 Tail subadditivity of distortion risk measures

In this section, we discuss tail subadditivity of distortion risk measures with applications.

Theorem 2.9. *Let g be a distortion function satisfying $g(0+) = g(0) = 0$, $\alpha \in (0, 1)$ be a confidence level, and $\Omega_{X,Y}$ be a common tail region of a pair of random variables X and Y.*

(a) For a pair of random variables $X, Y \in L^p$, if the distortion function g is concave on $[0,\ P(\Omega_{X,Y})]$, then the distortion risk measure $\rho_g : L^p \to \mathbb{R}$ is tail subadditive for the pair of random variables X, Y.

(b) Assume that $(\Omega, \mathcal{F}, \mathbb{P})$ is atomless[4]. If the distortion risk measure $\rho_g : L^p \to \mathbb{R}$ is tail subadditive in L^p, then g is concave on $[0,\ p]$, where $p = \max\{P(\Omega_{\mathbb{I}_A, \mathbb{I}_B}) : A, B \in \mathcal{F}\}$.

[4] A probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is atomless if for any $A \in \mathcal{F}$ with $P(A) > 0$, there exists $B \in \mathcal{F}$ such that $B \subset A$ and $0 < P(B) < P(A)$.

Proof. (a) Define the set function $(g \circ P)_{\Omega_{X,Y}} : \mathcal{F} \to \mathbb{R}$ as

$$(g \circ P)_{\Omega_{X,Y}}(A) = (g \circ P)(A \cap \Omega_{X,Y}) \tag{2.20}$$

for $A \in \mathcal{F}$. Obviously, $(g \circ P)_{\Omega_{X,Y}}$ is a monotone set function on $\mathcal{F}$ since g is an increasing function on $[0, 1]$.

Assume that g is concave on $[0, P(\Omega_{X,Y})]$. For any subsets $A, B \in \mathcal{F}$, denote $u = P((A \cap B) \cap \Omega_{X,Y})$, $v = P((A \cup B) \cap \Omega_{X,Y})$, $a = P(A \cap \Omega_{X,Y})$, and $b = P(B \cap \Omega_{X,Y})$. Note that

$$\begin{aligned} u + v &= P((A \cap B) \cap \Omega_{X,Y}) + P((A \cup B) \cap \Omega_{X,Y}) \\ &= P(A \cap \Omega_{X,Y}) + P(B \cap \Omega_{X,Y}) = a + b. \end{aligned}$$

Furthermore, we have

$$0 \le P((A \cap B) \cap \Omega_{X,Y}) \le P(A \cap \Omega_{X,Y}) \le P((A \cup B) \cap \Omega_{X,Y}) \le P(\Omega_{X,Y})$$

and

$$0 \le P((A \cap B) \cap \Omega_{X,Y}) \le P(B \cap \Omega_{X,Y}) \le P((A \cup B) \cap \Omega_{X,Y}) \le P(\Omega_{X,Y}),$$

thus, $0 \le u \le a, b \le v \le P(\Omega_{X,Y})$ satisfying $a + b = u + v$. Hence, by Lemma 2.7 (b), we have $g(a) + g(b) \ge g(u) + g(v)$, which means that

$$g(P((A \cup B) \cap \Omega_{X,Y})) + g(P((A \cap B) \cap \Omega_{X,Y})) \le g(P(A \cap \Omega_{X,Y})) + g(P(B \cap \Omega_{X,Y})),$$

namely, $(g \circ P)_{\Omega_{X,Y}}$ is a submodular set function on $\mathcal{F}$.

For any random variable $Z \in L^p$, we have that $(g \circ P)_{\Omega_{X,Y}}(Z > x) = g(P(\Omega_{X,Y} \cap (Z > x)) \to g(0)$ as $x \to \infty$ since $P(\Omega_{X,Y} \cap (Z > x)) \searrow P(\emptyset) = 0$ as $x \to \infty$ and $g(0+) = g(0) = 0$. Hence, Z is $(g \circ P)_{\Omega_{X,Y}}$-essentially $> -\infty$. Thus, by Lemma

2.5 (e), we have that

$$\int (X+Y)d(g\circ P)_{\Omega_{X,Y}} \leq \int Xd(g\circ P)_{\Omega_{X,Y}} + \int Yd(g\circ P)_{\Omega_{X,Y}}, \tag{2.21}$$

which is equivalent to

$$\int_{\Omega_{X,Y}} (X+Y)dg\circ P \leq \int_{\Omega_{X,Y}} Xdg\circ P + \int_{\Omega_{X,Y}} Ydg\circ P. \tag{2.22}$$

Hence, (2.13) holds and thus the distortion risk measure ρ_g is tail subadditive for the pair of random variables $X, Y \in L^p$.

(b) If the distortion risk measure ρ_g is tail subadditive in L^p, then (2.13) holds for any pair of random variables $X, Y \in L^p$. Thus, for any subsets $A,\ B \subset \mathcal{F}$, by (2.13), we have that

$$\int_{\Omega_{\mathbb{I}_A,\mathbb{I}_B}} (\mathbb{I}_A+\mathbb{I}_B)\mathrm{d}g\circ P \leq \int_{\Omega_{\mathbb{I}_A,\mathbb{I}_B}} \mathbb{I}_A\,\mathrm{d}\,g\circ P + \int_{\Omega_{\mathbb{I}_A,\mathbb{I}_B}} \mathbb{I}_B\,\mathrm{d}g\circ P, \tag{2.23}$$

which is equivalent to

$$\int (\mathbb{I}_A+\mathbb{I}_B)\mathrm{d}(g\circ P)_{\Omega_{\mathbb{I}_A,\mathbb{I}_B}} \leq \int \mathbb{I}_A\,\mathrm{d}\,(g\circ P)_{\Omega_{\mathbb{I}_A,\mathbb{I}_B}} + \int \mathbb{I}_B\,\mathrm{d}(g\circ P)_{\Omega_{\mathbb{I}_A,\mathbb{I}_B}}. \tag{2.24}$$

By Denneberg (1994, page 69), we have that for any monotone set function $\mu : \mathcal{F} \to \mathbb{R}_+$ and any $A, B \in \mathcal{F}$, it holds

$$\int (\mathbb{I}_A+\mathbb{I}_B)d\mu = \mu(A\cup B) + \mu(A\cap B),$$

which, together with (2.24) and Lemma 2.5(a), implies that

$$\begin{aligned} & (g\circ P)((A\cup B)\cap\Omega_{\mathbb{I}_A,\mathbb{I}_B}) + (g\circ P)((A\cap B)\cap\Omega_{\mathbb{I}_A,\mathbb{I}_B}) \\ \leq\ & (g\circ P)(A\cap\Omega_{\mathbb{I}_A,\mathbb{I}_B}) + (g\circ P)(B\cap\Omega_{\mathbb{I}_A,\mathbb{I}_B}), \end{aligned} \tag{2.25}$$

which means that $(g\circ P)_{\Omega_{\mathbb{I}_A,\mathbb{I}_B}}$ is a submodular set function on $\mathcal{F}$. Thus, if $P(\Omega_{\mathbb{I}_A,\mathbb{I}_B}) = 0$, then g is concave on $[0, P(\Omega_{\mathbb{I}_A,\mathbb{I}_B})]$. If $P(\Omega_{\mathbb{I}_A,\mathbb{I}_B}) > 0$, then for any $0 \leq u < v \leq P(\Omega_{\mathbb{I}_A,\mathbb{I}_B})$, there exists $C, D \in \mathcal{F}$ so that $C \subset D \subset \Omega_{\mathbb{I}_A,\mathbb{I}_B}$, $P(C) = u$, and $P(D) = v$ since L^p is atomless. Denote by $E = D \setminus C$, thus $P(E) = v - u$. Note that $\frac{u-v}{2} < u - v$, thus, there exists $F \subset E$ such that $P(F) = \frac{v-u}{2}$. Now, define $A' = C \cup F$ and $B' = D \setminus F$, and thus $P(A') = P(B') = \frac{u+v}{2}$, $P(A' \cap B') = P(C) = u$, and $P(A' \cup B') = P(D) = v$. Note that $A', B' \subset \Omega_{\mathbb{I}_A, \mathbb{I}_B}$ and that $(g \circ P)_{\Omega_{\mathbb{I}_A, \mathbb{I}_B}}$ is a submodular set function on $\mathcal{F}$, hence, we have

$$\begin{aligned} & g(P((A' \cup B') \cap \Omega_{\mathbb{I}_A, \mathbb{I}_B})) + g(P((A' \cap B') \cap \Omega_{\mathbb{I}_A, \mathbb{I}_B})) \\ \leq \; & g(P(A' \cap \Omega_{\mathbb{I}_A, \mathbb{I}_B})) + g(P(B' \cap \Omega_{\mathbb{I}_A, \mathbb{I}_B})), \end{aligned}$$

which implies that

$$g\Big(\frac{u+v}{2}\Big) \geq \frac{g(u) + g(v)}{2}.$$

Hence, the distortion function g is concave on $[0,\, P(\Omega_{\mathbb{I}_A, \mathbb{I}_B})]$ for any $A, B \in \mathcal{F}$. Thus, g is concave on $[0,\, p]$. ☐

Corollary 2.10. *Let g be a distortion function satisfying $g(0+) = g(0) = 0$ and $\alpha \in (0,1)$ be a confidence level. Assume that the common tail region $\Omega_{X,Y}$ is defined as $\Omega_{X,Y} = \mathfrak{Q}_{\alpha,X,Y}$ given by (2.5) or $\Omega_{X,Y} = \mathfrak{Q}_{\alpha,X+Y} = \{X + Y > F^{-1}_{X+Y}(\alpha)\}$ for $X, Y \in L^p$. Then, the following assertions hold.*

(a) If g is concave on $[0, 1-\alpha]$, then the distortion risk measure $\rho_g : L^p \to \mathbb{R}$ is tail subadditive in L^p.

(b) If the distortion risk measure $\rho_g : L^p \to \mathbb{R}$ is tail subadditive in $L^p = L^p(\Omega, \mathcal{F}, \mathbb{P})$ and $(\Omega, \mathcal{F}, \mathbb{P})$ is atomless, then g is concave on $[0, 1-\alpha]$.

Proof. (a) For any $\alpha \in (0,1)$, by Lemma 2.8, we have that

$$P(\mathfrak{Q}_{\alpha,X,Y}) \leq P(X + Y > F^{-1}_{X+Y}(\alpha)) = 1 - F_{X+Y}(F^{-1}_{X+Y}(\alpha)) \leq 1 - \alpha$$

and

$$P(\mathfrak{Q}_{\alpha,X+Y}) = P(X + Y > F^{-1}_{X+Y}(\alpha)) = 1 - F_{X+Y}(F^{-1}_{X+Y}(\alpha)) \leq 1 - \alpha.$$

Thus, if g is concave on $[0, 1-\alpha]$, then g is concave on $[0, P(\Omega_{X,Y})]$. Hence, by Theorem 2.9, we have that ρ_g is tail subadditive for any $X, Y \in L^p$, which means that ρ_g is tail subadditive in L^p.

(b) If $(\Omega, \mathcal{F}, \mathbb{P})$ is atomless, then there exists a set $A_\alpha \in \mathcal{F}$ such that $P(A_\alpha) = \alpha$. Let $A = B = \Omega \setminus A_\alpha = A_\alpha^c$. Thus, if $\Omega_{X,Y}$ is defined as $\mathfrak{Q}_{\alpha,X,Y}$ for $X, Y \in L^p$, we have

$$\begin{aligned} \Omega_{\mathbb{I}_{A_\alpha^c},\mathbb{I}_{A_\alpha^c}} &= \{\omega \in \Omega : \mathbb{I}_{A_\alpha^c}(\omega) > F^{-1}_{\mathbb{I}_{A_\alpha^c}}(\alpha),\, 2\,\mathbb{I}_{A_\alpha^c}(\omega) > F^{-1}_{2\,\mathbb{I}_{A_\alpha^c}}(\alpha)\} \\ &= \{\omega \in \Omega : \mathbb{I}_{A_\alpha^c}(\omega) > F^{-1}_{\mathbb{I}_{A_\alpha^c}}(\alpha)\} \end{aligned}$$

since VaR satisfies positive homogeneity. On the other hand, if $\Omega_{X,Y}$ is defined as $\mathfrak{Q}_{\alpha,X+Y}$ for $X, Y \in L^p$, we have

$$\Omega_{\mathbb{I}_{A_\alpha^c},\mathbb{I}_{A_\alpha^c}} = \{\omega \in \Omega : 2\,\mathbb{I}_{A_\alpha^c}(\omega) > F^{-1}_{2\,\mathbb{I}_{A_\alpha^c}}(\alpha)\} = \{\omega \in \Omega : \mathbb{I}_{A_\alpha^c}(\omega) > F^{-1}_{\mathbb{I}_{A_\alpha^c}}(\alpha)\}.$$

It is easy to see that $F^{-1}_{\mathbb{I}_{A_\alpha^c}}(\alpha) = 0$. Thus, $P(\Omega_{\mathbb{I}_{A_\alpha^c},\mathbb{I}_{A_\alpha^c}}) = P(\mathbb{I}_{A_\alpha^c} > 0) = P(\mathbb{I}_{A_\alpha^c} = 1) = P(A_\alpha^c) = 1-\alpha$, which implies $p = \max\{P(\Omega_{\mathbb{I}_A,\mathbb{I}_B}) : A, B \in \mathcal{F}\} = 1-\alpha$ since $P(\Omega_{X,Y}) \le 1-\alpha$ for any $X, Y \in L^p$. Hence, if ρ_g is tail subadditive in L^p, then by Theorem 2.9, we have that g is concave on $[0, p] = [0, 1-\alpha]$. It completes the proof. □

Remark 2.11. *We point out that if $X = Y$ and X has a distribution $F_X(x)$, then $\mathfrak{Q}_{\alpha,X} = \mathfrak{Q}_{\alpha,Y} = \{X > F_X^{-1}(\alpha)\}$ and $\mathfrak{Q}_{\alpha,X,Y} = \{X > F_X^{-1}(\alpha),\, 2X > F_{2X}^{-1}(\alpha)\} = \{X > F_X^{-1}(\alpha)\}$. Hence, in this case, $\mathfrak{Q}_{\alpha,X,Y} = \{X > F_X^{-1}(\alpha)\}$ and $P(\mathfrak{Q}_{\alpha,X,Y}) = P(X > F_X^{-1}(\alpha)) = 1 - F(F_X^{-1}(\alpha)) \le 1-\alpha$. In particular, if $F_X(x)$ is continuous at $F_X^{-1}(\alpha)$, then $F(F_X^{-1}(\alpha)) = \alpha$. Hence, it is obvious that the statement "because $P(\mathfrak{Q}_{\alpha,X,Y}) < 1-\alpha$" in the proof of Theorem 6.1 of Belles et al. (2014a) should be modified to the statement "because $P(\mathfrak{Q}_{\alpha,X,Y}) \le 1-\alpha$" and the conclusion "concave in $[0, 1-\alpha)$" in Theorem 6.1 of Belles et al. (2014a) should be corrected to "concave in $[0, 1-\alpha]$". The same comments apply to Theorem 5.1 of Yin and Zhu (2016). Corollary 2.10 not only gives the corrected version of Theorem 6.1 of Belles et al. (2014a) and Theorem 5.1 of Yin and Zhu (2016) but also obtains the* *necessary condition for a distortion risk measure to be tail subadditive when the common tail region is defined by $\mathfrak{Q}_{\alpha,X,Y}$ and $\mathfrak{Q}_{\alpha,X+Y}$.* $\square$

Furthermore, using Theorem 2.9, we can recover the well-known result about the subadditivity of distortion risk measures as follows.

Corollary 2.12. *Let g be a distortion function satisfying $g(0+) = g(0) = 0$. The following assertions hold.*

(a) If g is concave on $[0,1]$, then the distortion risk measure $\rho_g : L^p \to \mathbb{R}$ is subadditive in L^p.

(b) If the distortion risk measure $\rho_g : L^p \to \mathbb{R}$ is subadditive in $L^p = L^p(\Omega, \mathcal{F}, \mathbb{P})$ and $(\Omega, \mathcal{F}, \mathbb{P})$ is atomless, then g is concave on $[0,1]$.

Proof. The results of (a) and (b) follow immediately from Theorem 2.9 by letting $\Omega_{X,Y} = \Omega$ for any $X, Y \in L^p$. $\square$

Next, we generalize Belles et al. (2014a)'s GlueVaR risk measure and discuss the properties of the generalized GlueVaR.

Definition 2.13. (Generalized GlueVaR) Define a distortion function $g^{h_1,\ldots,h_n}_{\alpha_1,\ldots,\alpha_n}$ as

$$g^{h_1,\ldots,h_n}_{\alpha_1,\ldots,\alpha_n}(u) = \begin{cases} \frac{h_1}{1-\alpha_1}u, & 0 \le u \le 1-\alpha_1, \\ h_1 + \frac{h_2-h_1}{\alpha_1-\alpha_2}[u-(1-\alpha_1)], & 1-\alpha_1 < u \le 1-\alpha_2, \\ \cdots, & \cdots, \\ h_k + \frac{h_{k+1}-h_k}{\alpha_k-\alpha_{k+1}}[u-(1-\alpha_k)], & 1-\alpha_k < u \le 1-\alpha_{k+1}, \\ \cdots, & \cdots, \\ h_{n-1} + \frac{h_n-h_{n-1}}{\alpha_{n-1}-\alpha_n}[u-(1-\alpha_{n-1})], & 1-\alpha_{n-1} < u \le 1-\alpha_n, \\ 1, & 1-\alpha_n < u \le 1, \end{cases}$$

where

$$1 > \alpha_1 > \alpha_2 > \cdots > \alpha_k > \cdots > \alpha_n > 0$$

and

$$0 \leq h_1 \leq h_2 \leq \cdots \leq h_k \leq \cdots \leq h_n \leq 1.$$

The distortion risk measure under the distortion function $g_{\alpha_1,\ldots,\alpha_n}^{h_1,\ldots,h_n}$ is called a generalized GlueVaR (GGlueVaR), denoted by $\mathrm{GGlueVaR}_{\alpha_1,\ldots,\alpha_n}^{h_1,\ldots,h_n}$.

Geometrically, the distortion function $g_{\alpha_1,\ldots,\alpha_n}^{h_1,\ldots,h_n}$ is a piecewise linear function over the interval $[0,1]$. However, we can show that a $\mathrm{GGlueVaR}_{\alpha_1,\ldots,\alpha_n}^{h_1,\ldots,h_n}$ risk measure under this distortion function $g_{\alpha_1,\ldots,\alpha_n}^{h_1,\ldots,h_n}$ is a linear combination of n TVaRs and one VaR. Furthermore, any coherent distortion risk measure can be approached by a series of generalized tail subadditive GlueVaR risk measures. These results, together with other properties of the generalized GlueVaR, are presented below.

Proposition 2.14. *The generalized GlueVaR risk measure* $\mathrm{GGlueVaR}_{\alpha_1,\ldots,\alpha_n}^{h_1,\ldots,h_n}(X)$ *of a risk* X *can be expressed as*

$$\mathrm{GGlueVaR}_{\alpha_1,\ldots,\alpha_n}^{h_1,\ldots,h_n}(X) = \sum_{i=1}^{n} \omega_i \,\mathrm{TVaR}_{\alpha_i}(X) + \omega_{n+1}\,\mathrm{VaR}_{\alpha_n}(X), \tag{2.26}$$

where

$$\omega_1 = h_1 - \frac{h_2 - h_1}{\alpha_1 - \alpha_2}(1-\alpha_1), \tag{2.27}$$

$$\omega_k = \Big(\frac{h_k - h_{k-1}}{\alpha_{k-1} - \alpha_k} - \frac{h_{k+1} - h_k}{\alpha_k - \alpha_{k+1}}\Big)(1-\alpha_k) \tag{2.28}$$

for $k = 2, \ldots, n-1$,

$$\omega_n = \frac{h_n - h_{n-1}}{\alpha_{n-1} - \alpha_n}(1-\alpha_n), \tag{2.29}$$

and

$$\omega_{n+1} = 1 - \sum_{k=1}^{n} \omega_k = 1 - h_n. \tag{2.30}$$

Moreover,

$$\mathrm{VaR}_{\alpha_n}(X) \leq \mathrm{GGlueVaR}_{\alpha_1,\ldots,\alpha_n}^{h_1,\ldots,h_n}(X) \leq \mathrm{TVaR}_{\alpha_1}(X) \tag{2.31}$$

if

$$\frac{1}{1-\alpha_1} \geq \frac{h_1}{1-\alpha_1} \geq \frac{h_2 - h_1}{\alpha_1 - \alpha_2} \geq \frac{h_3 - h_2}{\alpha_2 - \alpha_3} \geq \cdots \geq \frac{h_n - h_{n-1}}{\alpha_{n-1} - \alpha_n}. \tag{2.32}$$

Proof. Let $g_{\alpha_k,\mathrm{TVaR}}$ and $g_{\alpha_n,\mathrm{VaR}}$ be the distortion function of $\mathrm{TVaR}_{\alpha_k}(X)$ and $\mathrm{VaR}_{\alpha_n}(X)$, respectively, then

$$\begin{aligned} & g_{\alpha_k,\mathrm{TVaR}}(u) = \frac{u}{1-\alpha_k}\mathbb{I}_{\{0\le u\le 1-\alpha_k\}} + \mathbb{I}_{\{1-\alpha_k<u\le 1\}} \\ = & \frac{u}{1-\alpha_k}\left(\mathbb{I}_{\{0\le u\le 1-\alpha_1\}} + \sum_{i=2}^{k}\mathbb{I}_{\{1-\alpha_{i-1}<u\le 1-\alpha_i\}}\right) + \sum_{i=k}^{n-1}\mathbb{I}_{\{1-\alpha_i<u\le 1-\alpha_{i+1}\}} + \mathbb{I}_{\{1-\alpha_n<u\le 1\}}, \end{aligned}$$

and $\mathrm{VaR}_{\alpha_n}(X) = \mathbb{I}_{\{1-\alpha_n<u\le 1\}}$. Note that the GGlueVaR distortion function can reexpressed as the following form:

$$\begin{aligned} & g_{\alpha_1,\ldots,\alpha_n}^{h_1,\ldots,h_n}(u) \\ = & \frac{h_1}{1-\alpha_1}\, u\,\mathbb{I}_{\{0\le u\le 1-\alpha_1\}} + \sum_{k=1}^{n-1}\left(h_k + \frac{h_{k+1}-h_k}{\alpha_k-\alpha_{k+1}}(u-(1-\alpha_k))\right)\mathbb{I}_{\{1-\alpha_k<u\le 1-\alpha_{k+1}\}} \\ & +\mathbb{I}_{\{1-\alpha_n<u\le 1\}}. \end{aligned}$$

Hence, to show (2.26), it suffices to prove for any $0\le u\le 1$, the following equation holds:

$$\begin{aligned} & \frac{h_1}{1-\alpha_1}\, u\,\mathbb{I}_{\{0\le u\le 1-\alpha_1\}} + \sum_{k=1}^{n-1}\left(h_k + \frac{h_{k+1}-h_k}{\alpha_k-\alpha_{k+1}}(u-(1-\alpha_k))\right)\mathbb{I}_{\{1-\alpha_k<u\le 1-\alpha_{k+1}\}} \\ & +\mathbb{I}_{\{1-\alpha_n<u\le 1\}} \\ = & \sum_{k=1}^{n}\omega_k\, g_{\alpha_k,\mathrm{TVaR}}(u) + \omega_{n+1}\, g_{\alpha_n,\mathrm{VaR}}(u). \end{aligned} \tag{2.33}$$

In doing so, equating the coefficients for $\mathbb{I}_{\{0\le u\le 1-\alpha_1\}}, \mathbb{I}_{\{1-\alpha_k<u\le 1-\alpha_{k+1}\}}$, $k=1,2,\ldots,n-1$, and $\mathbb{I}_{\{1-\alpha_n<u\le 1\}}$ on both sides of (2.33), we see that ω_k, $k=1,2,...,n,n+1$ must satisfy the following equations:

$$\frac{h_1}{1-\alpha_1}u = \frac{\omega_1}{1-\alpha_1}u + \frac{\omega_2}{1-\alpha_2}u + \ldots + \frac{\omega_n}{1-\alpha_n}u + \omega_{n+1}, \tag{2.34}$$

$$h_1 + \frac{h_2-h_1}{\alpha_1-\alpha_2}\big(u-(1-\alpha_1)\big) = \omega_1 + \frac{\omega_2}{1-\alpha_2}u + \ldots + \frac{\omega_n}{1-\alpha_n}u + \omega_{n+1}, \tag{2.35}$$

$$h_2 + \frac{h_3-h_2}{\alpha_2-\alpha_3}[u-(1-\alpha_2)] = \omega_1 + \omega_2 + \frac{\omega_3}{1-\alpha_3}u + \ldots + \frac{\omega_n}{1-\alpha_n}u + \omega_{n+1}, \tag{2.36}$$

$$\cdots$$

$$h_k+\frac{h_{k+1}-h_k}{\alpha_k-\alpha_{k+1}}[u-(1-\alpha_k)]=\sum_{i=1}^{k}\omega_i+\frac{\omega_{k+1}}{1-\alpha_{k+1}}u+\ldots+\frac{\omega_n}{1-\alpha_n}u+\omega_{n+1}, \tag{2.37}$$

$$\cdots$$

$$h_{n-2}+\frac{h_{n-1}-h_{n-2}}{\alpha_{n-2}-\alpha_{n-1}}[u-(1-\alpha_{n-2})]=\sum_{i=1}^{n-2}\omega_i+\frac{\omega_{n-1}}{1-\alpha_{n-1}}u+\frac{\omega_n}{1-\alpha_n}u+\omega_{n+1}, \tag{2.38}$$

$$h_{n-1}+\frac{h_n-h_{n-1}}{\alpha_{n-1}-\alpha_n}[u-(1-\alpha_{n-1})]=\sum_{i=1}^{n-1}\omega_i+\frac{\omega_n}{1-\alpha_n}u+\omega_{n+1}, \tag{2.39}$$

$$1=\sum_{i=1}^{n+1}\omega_i. \tag{2.40}$$

Hence, by subtracting (2.35) from (2.34) and subtracting (2.36) from (2.35) , we get

$$\omega_1=h_1-\frac{h_2-h_1}{\alpha_1-\alpha_2}(1-\alpha_1) \quad \text{and} \quad \omega_2=\Big(\frac{h_2-h_1}{\alpha_1-\alpha_2}-\frac{h_3-h_2}{\alpha_2-\alpha_3}\Big)(1-\alpha_2).$$

Similarly, for $k=3,\ldots,n-1$, we have

$$\omega_k=\Big(\frac{h_k-h_{k-1}}{\alpha_{k-1}-\alpha_k}-\frac{h_{k+1}-h_k}{\alpha_k-\alpha_{k+1}}\Big)(1-\alpha_k).$$

Moreover, subtracting (2.39) from (2.38), we obtain

$$h_{n-1}+\frac{h_n-h_{n-1}}{\alpha_{n-1}-\alpha_n}\big(u-(1-\alpha_{n-1})\big)-1=\frac{\omega_n}{1-\alpha_n}u-\omega_n,$$

which implies that

$$\omega_n=\frac{h_n-h_{n-1}}{\alpha_{n-1}-\alpha_n}(1-\alpha_n).$$

Hence, $\omega_k, k=1,2,...,n+1$ have the expressions given in (2.27)-(2.29). Moreover, it is easy to verify that

$$\sum_{j=1}^{k}\omega_j=h_k-\frac{h_{k+1}-h_k}{\alpha_k-\alpha_{k+1}}(1-\alpha_k) \tag{2.41}$$

holds for $k = 1, 2, \ldots, n-1$. Thus, $\sum_{j=1}^{n} \omega_j = \sum_{j=1}^{n-1} \omega_j + \omega_n = h_n$, and hence, by (2.40), we obtain $\omega_{n+1} = 1 - h_n$. Therefore, (2.26) holds when $\omega_k, k = 1, ..., n+1$, are given by (2.27)-(2.30). In addition, (2.31) holds since $g_{\alpha_n,\mathrm{VaR}}(u) \leq g_{\alpha_1,\ldots,\alpha_n}^{h_1,\ldots,h_n}(u) \leq g_{\alpha_1,\mathrm{TVaR}}(u)$ for $0 \leq u \leq 1$ if (2.32) holds. $\square$

Corollary 2.15. *For a confidence level* $0 < \alpha < 1$ *and the risk measure* $\mathrm{GGlueVaR}_{\alpha_1,\ldots,\alpha_n}^{h_1,\ldots,h_n}$, *if* α_k *and* h_k, $k = 1, 2, \ldots, n$, *satisfy*

$$\frac{h_1}{1-\alpha_1} \geq \frac{h_2 - h_1}{\alpha_1 - \alpha_2} \geq \frac{h_3 - h_2}{\alpha_2 - \alpha_3} \geq \cdots \geq \frac{h_n - h_{n-1}}{\alpha_{n-1} - \alpha_n}, \tag{2.42}$$

and the common tail region $\Omega_{X,Y}$ *is defined as* $\mathfrak{Q}_{\alpha_n,X,Y}$ *or* $\mathfrak{Q}_{\alpha_n,X+Y}$ *for any* $X, Y \in L^p$, *then the risk measure* $\mathrm{GGlueVaR}_{\alpha_1,\ldots,\alpha_n}^{h_1,\ldots,h_n}$ *is tail subadditive in* L^p.

Proof. It is obvious that the distortion function $g_{\alpha_1,\ldots,\alpha_n}^{h_1,\ldots,h_n}$ of $\mathrm{GGlueVaR}_{\alpha_1,\ldots,\alpha_n}^{h_1,\ldots,h_n}$ is concave on $[0, 1-\alpha_n]$ if the gradients of the distortion function $g_{\alpha_1,\ldots,\alpha_n}^{h_1,\ldots,h_n}$ on $[0, 1-\alpha_n]$ is decreasing, namely, if (2.42) holds. Thus, the GGlueVaR risk measure is tail subadditive by Corollary 2.10. $\square$

Proposition 2.16. *Assume that a series of* $\{(\alpha_1^n, ..., \alpha_n^n), n = 1, 2, ...\}$ *satisfy* $1 > \alpha_1^n > \cdots > \alpha_k^n > \cdots > \alpha_n^n > 0$, $\lim_{n\to\infty} \alpha_1^n = 1$, $\lim_{n\to\infty} \alpha_n^n = 0$, *and* $\lim_{n\to\infty} \max\{\alpha_i^n - \alpha_{i+1}^n, i = 1, ..., n-1\} = 0$. *For any coherent distortion risk measure* $\rho_g : L^1 \to \mathbb{R}$ *with a concave distortion function* g, *it holds that for any* $X \in L^1$,

$$\rho_g(X) = \lim_{n\to\infty} \Big(\sum_{i=1}^{n} \omega_i^n \ \mathrm{TVaR}_{\alpha_i^n}(X) + \omega_{n+1}^n \ \mathrm{VaR}_{\alpha_n^n}(X) \Big), \tag{2.43}$$

where

$$\begin{aligned}
\omega_1^n &= h_1^n - \frac{h_2^n - h_1^n}{\alpha_1^n - \alpha_2^n}(1-\alpha_1^n), \\
\omega_k^n &= \Big(\frac{h_k^n - h_{k-1}^n}{\alpha_{k-1}^n - \alpha_k^n} - \frac{h_{k+1}^n - h_k^n}{\alpha_k^n - \alpha_{k+1}^n} \Big)(1-\alpha_k^n) \ \text{ for } \ k = 2, \ldots, n-1, \\
\omega_n^n &= \frac{h_n^n - h_{n-1}^n}{\alpha_{n-1}^n - \alpha_n^n}(1-\alpha_n^n), \\
\omega_{n+1}^n &= 1 - \sum_{k=1}^{n} \omega_k^n = 1 - h_n^n,
\end{aligned}$$

and $h_k^n = g(1-\alpha_k^n)$ *for any* $k = 1, ..., n$*;* $n = 1, 2, ...$

Proof. Note that the distortion function g is concave on $[0,1]$ and thus g is also continuous on $[0,1]$. Hence, it is obvious that the function g can be approached by a series of the distortion functions g_n, where $g_n(u) = g_{\alpha_1^n,...,\alpha_n^n}^{h_1^n,...,h_n^n}(u)$, $n = 1, 2, ...$, are defined as in Definition 2.13, and the coefficients α_k^n, h_k^n, $k = 1, ..., n$, $n = 1, 2, ...$, satisfy the conditions of Proposition 2.16. Thus, for any $u \in [0,1]$, $g(u) = \lim_{n\to\infty} g_n(u) = \lim_{n\to\infty} g_{\alpha_1^n,...,\alpha_n^n}^{h_1^n,...,h_n^n}(u)$, which, together with the definitions of the distortion risk measures and the dominated convergence theorem, implies that $\rho_g(X) = \lim_{n\to\infty} \rho_{g_n}(X)$. Hence, by (2.26), we obtain (2.43). □

Remark 2.17. In particular, we can take

$$\alpha_k^n = 1 - \frac{k}{n+1}$$

for any $k = 1, ..., n$.

Then

$$\rho_g(X) = \lim_{n\to\infty} \Big(\sum_{i=1}^{n} \omega_i^n \ \mathrm{TVaR}_{\alpha_i^n}(X) + \omega_{n+1}^n \ \mathrm{VaR}_{\alpha_n^n}(X) \Big),$$

where

$$\begin{aligned}
\omega_1^n &= 2h_1^n - h_2^n, \\
\omega_k^n &= k(2h_k^n - h_{k-1}^n - h_{k+1}^n), \ \text{ for } k = 2, \ldots, n-1, \\
\omega_n^n &= n(h_n^n - h_{n-1}^n), \\
\omega_{n+1}^n &= 1 - h_n^n,
\end{aligned}$$

and $h_k^n = g(\frac{k}{n+1})$ for any $k = 1, ..., n$; $n = 1, 2, ...$

2.4 Applications of tail subadditivity in portfolio risk management

To consider more applications of tail subadditivity, we fist give the following lemma.

Lemma 2.18. *Let g be a distortion function and let $A \in \mathcal{F}$ with $g(P(A)) > 0$. If $g(x) = \theta x$ for $0 < x \leq P(A)$ and $0 < \theta \leq 1/P(A)$, then*

$$\int_A X dg \circ P = \theta P(A)\, \mathbb{E}[X|A]. \tag{2.44}$$

Proof. Let $\mu = g \circ P$. Thus, by the definition of the Choquet integral, we have

$$\int_A X dg \circ P = \int_A X d\mu$$

$$= \mu(A)\Big[\int_{-\infty}^{0} \Big(\frac{\mu(A \cap (X > x))}{\mu(A)} - 1\Big)dx + \int_0^{\infty} \frac{\mu(A \cap (X > x))}{\mu(A)} dx\Big]$$

$$= \mu(A)\Big[\int_{-\infty}^{0} (S_{X|A}(x) - 1)dx + \int_0^{\infty} S_{X|A}(x)dx\Big],$$

where $\mu(A) = g(P(A))$ and

$$S_{X|A}(x) = \frac{\mu(A \cap \{X > x\})}{\mu(A)} = \frac{g(P(A \cap \{X > x\}))}{g(P(A))}$$

for $x \in \mathbb{R}$. If $g(x) = \theta x$ for $0 < x \leq P(A)$ and $0 < \theta \leq 1/P(A)$, then

$$S_{X|A}(x) = \frac{P(A \cap \{X > x\})}{P(A)} = P(X > x|A).$$

Note that the survival function of $X|A$ is $P(X > x|A) = \frac{P(A \cap \{X > x\})}{P(A)}$, hence,

$$\mathbb{E}[X|A] = \int_{-\infty}^{0} \Big(\frac{P(A \cap \{X > x\})}{P(A)} - 1\Big)dx + \int_0^{\infty} \frac{P(A \cap \{X > x\})}{P(A)} dx.$$

Thus, $\int_A X d\mu = \mathbb{E}[X|A]\mu(A) = \theta P(A)\mathbb{E}[X|A]$. □

Using Theorem 2.9 and Lemma 2.18, we obtain the following proposition about the tail subadditivity of distortion risk measures.

Proposition 2.19. *Let $(X_1, ..., X_n)$ be a portfolio of risks and $S_n = X_1 + \cdots + X_n$ be the aggregate risk of the portfolio. For a common tail region $\Omega_{X_1,...,X_n}$ with $p_n = P(\Omega_{X_1,...,X_n}) > 0$, if the distortion function g is concave on $[0, p_n]$, then,*

$$\int_{\Omega_{X_1,\dots,X_n}} S_n \, dg \circ P \leq \sum_{i=1}^{n} \int_{\Omega_{X_1,\dots,X_n}} X_i \, dg \circ P. \tag{2.45}$$

Moreover, if the distortion function $g(x) = \theta x$ for $0 < x \leq p_n$ and $0 < \theta \leq 1/p_n$, then,

$$\int_{\Omega_{X_1,\dots,X_n}} S_n \, dg \circ P = \sum_{i=1}^{n} \int_{\Omega_{X_1,\dots,X_n}} X_i \, dg \circ P \tag{2.46}$$

with

$$\int_{\Omega_{X_1,\dots,X_n}} S_n \, dg \circ P = \theta \, p_n \, \mathbb{E}[S_n \,|\, \Omega_{X_1,\dots,X_n}] \tag{2.47}$$

and

$$\int_{\Omega_{X_1,\dots,X_n}} X_i \, dg \circ P = \theta \, p_n \, \mathbb{E}[X_i \,|\, \Omega_{X_1,\dots,X_n}] \tag{2.48}$$

for $i = 1, ..., n$.

Proof. By Theorem 2.9, we have

$$\begin{aligned}
&\int_{\Omega_{X_1,\dots,X_n}} S_n \, dg \circ P = \int_{\Omega_{X_1,\dots,X_n}} (S_{n-1} + X_n) \, dg \circ P \\
\leq \; &\int_{\Omega_{X_1,\dots,X_n}} S_{n-1} \, dg \circ P + \int_{\Omega_{X_1,\dots,X_n}} X_n \, dg \circ P \\
\leq \; &\cdots \leq \int_{\Omega_{X_1,\dots,X_n}} X_1 \, dg \circ P + \cdots + \int_{\Omega_{X_1,\dots,X_n}} X_n \, dg \circ P.
\end{aligned}$$

Moreover, by Lemma 2.18, we see that (2.47) and (2.48) hold. Thus,

$$\int_{\Omega_{X_1,\dots,X_n}} S_n \, dg \circ P = \theta \, p_n \, \mathbb{E}[S_n|\Omega_{X_1,\dots,X_n}] = \theta \, p_n \sum_{i=1}^{n} \mathbb{E}[X_i|\Omega_{X_1,\dots,X_n}],$$

which, together with (2.48), implies that (2.46) holds. □

Proposition 2.19 motivates us to develop a new approach for decision makers to determine the capital reserves or for insurers to calculate insurance premiums of risks in a portfolio. In doing so, consider a portfolio of insurance policies with n policies, denote X_i by the risk of policy i for $i = 1, 2, \dots, n$, the decision maker of the portfolio has to determine premiums for different insured risks. For example, if X_1 is the risk of a policyholder, she/he may buy multiple insurances on X_1, say λX_1 for $\lambda > 0$, then the decision maker needs to determine the premium of λX_1. Moreover, if both X_1 and X_2 are the risks of a policyholder, she/he may buy one insurance on $X_1 + X_2$ under an umbrella insurance policy. Thus, the decision maker needs to determine the premium of $X_1 + X_2$. In general, for a portfolio of risks $(X_1, \dots, X_n)$, the decision maker needs to determine the premium of $h(X_1, \dots, X_n)$, where $h : \mathbb{R}^n \to \mathbb{R}$ is a function that is called an operation function in this chapter. The similar situations arise for the regulator to determine the required capitals for a company with several sub-companies. In this case, if a company has n sub-companies, the loss/risk of sub-company i is X_i for $i = 1, 2, \dots, n$. Assume that ρ is the risk measure used by the regulator to determine the required capital of a risk. The company may request the regulator to determine the regulatory capitals on individual sub-companiesnies, say that $\rho(X_i)$ is the regulatory capital for sub-company i for $i = 1, 2, \dots, n$, if $\sum_{i=1}^{n} \rho(X_i) < \rho(S_n)$. Here $\rho(S_n)$ is the regulatory capital for the company when the company requests the regulator to determine the regulatory capital on its aggregate losses. This is one of the major reasons why subadditivity is an important property of risk measures. Now, we propose a new approach for decision makers to determine required capitals or to calculate premiums for risks in a portfolio.

Definition 2.20. Consider a portfolio of risks $(X_1, ..., X_n)$, a common tail region $\Omega_{X_1,...,X_n}$ with $p_n = P(\Omega_{X_1,...,X_n}) > 0$, and a distortion function g. For any operation function $h : \mathbb{R}^n \to \mathbb{R}$, denote the risk measure (capital reserve/premium) of risk $h(X_1, ..., X_n)$ by $H_g(h(X_1, ..., X_n))$, which is defined by

$$H_g\big(h(X_1, ..., X_n)\big) = \frac{1}{p_n} \int_{\Omega_{X_1,...,X_n}} h(X_1, ..., X_n)\, dg \circ P. \tag{2.49}$$

We call (2.49) the tail distortion principle for a portfolio of risks $(X_1, ..., X_n)$.

This tail distortion principle satisfies the following properties.

Proposition 2.21. *For a portfolio of risks $(X_1, ..., X_n)$, a common tail region $\Omega_{X_1,...,X_n}$ with $p_n = P(\Omega_{X_1,...,X_n}) > 0$, a distortion function g, the tail distortion principle H_g defined by (2.49) satisfies the following properties.*

(a) For any $\lambda \geq 0$ and $i = 1, ..., n$, $H_g(\lambda X_i) = \lambda H_g(X_i)$.

(b) For any $c \in \mathbb{R}$ and $i = 1, ..., n$, $H_g(X_i + c) = H_g(X_i) + c$.

(c) For a pair of X_i and X_j, $i, j \in \{1, ..., n\}$, if X_i and X_j are comonotonic, then $H_g(X_i + X_j) = H_g(X_i) + H_g(X_j)$.

(d) For a pair of X_i and X_j, $i, j \in \{1, ..., n\}$, if $X_i \leq X_j$, then $H_g(X_i) \leq H_g(X_j)$.

(e) If the distortion function g is concave on $[0, p_n]$, then, for any $i, j \in \{1, ..., n\}$, it holds that $H_g(X_i + X_j) \leq H_g(X_i) + H_g(X_j)$. Moreover, $H_g(\sum_{i=1}^n X_i) \leq \sum_{i=1}^n H_g(X_i)$.

(f) If the distortion function $g(x) = \theta x$ for $0 < x \leq p_n$ and $0 < \theta \leq 1/p_n$, then, $H_g(S_n) = \sum_{i=1}^n H_g(X_i)$ with $H_g(X_i) = \theta\, \mathbb{E}[X_i \,|\, \Omega_{X_1,...,X_n}]$ for $i = 1, ..., n$.

Proof. The proofs of (a)-(d) follow from the definition of H_g, (2.17), and Lemma 2.5, while the proofs of (e) and (f) follows from Proposition 2.19. $\square$

We point out that the risk measure H_g defined (2.49) can be viewed as a 'conditional expectation' of a risk in the portfolio, conditioning on a common tail region of the portfolio, with respect to the distorted measure $g \circ P$. This type of conditional expectations was discussed by Denneberg (1994b) and Young (1998). In particular, Proportion 2.21 (a)-(c) can be also obtained by using Proposition 3.3 of Young (1998).

To apply the tail distortion principle, we consider the four common tail regions defined in (2.9)-(2.12). First, for any $\alpha \in (0,1)$ and any random variable X, by Lemma 2.8, we have

$$P(\Omega_{\alpha,X}) = P(X \geq F_X^{-1}(\alpha)) \geq 1-\alpha. \tag{2.50}$$

Hence,

$$P(\Omega_{\alpha,S_n}) = P(S_n \geq \mathrm{VaR}_\alpha(S_n)) \geq 1-\alpha \tag{2.51}$$

and

$$P(\Omega_{\alpha,X_1,\dots,X_n}) \geq P(X_i \geq \mathrm{VaR}_\alpha(X_i)) \geq 1-\alpha \tag{2.52}$$

for $i = 1, \dots, n$.

Next, we show the following lemma.

Lemma 2.22. *For any random variable X,*

$$P(\Omega_X^e) = P(X \geq \mathbb{E}[X]) > 0. \tag{2.53}$$

Proof. It can be reduced to proof that $P(Y \geq 0) > 0$ if $\mathbb{E}[Y] = 0$, where Y is denoted as $Y = X - \mathbb{E}[X]$. If X is a degenerated random variable, $P(Y \geq 0) \geq P(Y = 0) = 1 > 0$. If X is not a degenerated random variable, we suppose $P(Y \geq 0) = 0$, then $P(Y < 0) = 1$. Then there exists $y_0 < 0$ and $\delta > 0$, such that $0 < P(Y \leq y) < 1$ for $y \in (y_0 - \delta, y_0)$. In addition,

$$\mathbb{E}[Y] = \int_{-\infty}^{0} (S_Y(y) - 1)dy + \int_{0}^{\infty} S_Y(y)dy.$$

For $y \geq 0$,

$$S_Y(y) = P(Y > y) \leq P(Y \geq 0) = 0.$$

For $y < 0$,

$$\begin{aligned} S_Y(y) - 1 &= P(Y > y) - 1 = P(Y \geq 0) + P(y < Y < 0) - 1 \\ &= P(Y < 0) - P(Y \leq y) - 1 = -P(Y \leq y) \end{aligned}$$

Hence,

$$\mathbb{E}[Y] = -\int_{-\infty}^{0} P(Y \leq y)dy \leq -\int_{y_0-\delta}^{y_0} P(Y \leq y)dy < 0,$$

which contradicts with $\mathbb{E}[Y] = 0$. Thus, $P(Y \geq 0) > 0$, namely, (2.53) holds. □

We point out that if X has a continuous distribution with a symmetric density function about its mean, then $P(X \geq \mathbb{E}[X]) = 1/2$. By (2.53), we have

$$P(\Omega^e_{S_n}) = P(S_n \geq \mathbb{E}[S_n]) > 0 \tag{2.54}$$

and

$$P(\Omega^e_{X_1,\dots,X_n}) \geq P(X_i \geq \mathbb{E}[X_i]) > 0 \tag{2.55}$$

for $i = 1, \dots, n$. Hence, all the four common tail regions Ω_{α,S_n}, $\Omega_{\alpha,X_1,\dots,X_n}$, $\Omega^e_{S_n}$, and $\Omega^e_{X_1,\dots,X_n}$ satisfy the conditions of Propositions 2.19 and 2.21 on a common tail region. In addition,

$$\begin{aligned} \Omega^*_{\alpha,X_1,\dots,X_n} &= \{X_1 \geq \mathrm{VaR}_\alpha(X_1), \cdots, X_n \geq \mathrm{VaR}_\alpha(X_n)\}, & (2.56) \\ \Omega^{e*}_{X_1,\dots,X_n} &= \{X_1 \geq \mathbb{E}[X_1], \cdots, X_n \geq \mathbb{E}[X_n]\}, & (2.57) \end{aligned}$$

are also interesting common tail regions. However,

$$P(\Omega^*_{\alpha,X_1,\dots,X_n}) > 0$$

and

$$P(\Omega^{e*}_{X_1,\dots,X_n}) > 0$$

may not hold for some random vectors. All these common tail regions describe the important extreme events concerned by decision makers in risk management. For instance, if reserves are determined by VaR, then Ω_{α,S_n} represents the event that aggregate risks of the portfolio will exceed the reserve of the aggregate risk, while $\Omega_{\alpha,X_1,...,X_n}$ means that at least one sub-portfolio will be in the state of technical insolvency. In addition, $\Omega^*_{\alpha,X_1,...,X_n}$ implies that all the sub-portfolios will be in the state of technical insolvency.

Indeed, VaR is a key benchmark for the regulator to determine the required capital. In calculating insurance premiums, the expected risk $\mathbb{E}[X]$ is called the net premium principle and it is an important benchmark for insurance pricing in the sense that a premium on a risk is often required to be bigger than the expectation of the risk, which is one desirable property called non-negative loading.

The CTE principle plays an important rule in determining the required capitals. In the CTE principle, the reserve of the portfolio risks $(X_1, ..., X_n)$ is first determined by $\mathbb{E}[S_n|S_n \geq \mathrm{VaR}_\alpha(S_n)]$ and then $\mathbb{E}[X_i|S_n \geq \mathrm{VaR}_\alpha(S_n)]$ is allocated to sub-portfolio i for $i = 1, ..., n$. The CTE allocation principle has been extensively studied by Cai and Li (2005), Chiragiev and Landsman (2007), Dhaene, et al. (2008), Landsman and Valdez (2003), and many others. The CTE principle is additive in the sense that

$$\mathbb{E}[S_n|S_n \geq \mathrm{VaR}_\alpha(S_n)] = \sum_{i=1}^{n} \mathbb{E}[X_i|S_n \geq \mathrm{VaR}_\alpha(S_n)]. \tag{2.58}$$

This principle can be viewed as a special case of the tail distortion principle.

By using Proposition 2.19, we propose allocation principles based on common tail regions. For a common tail region $\Omega_{X_1,...,X_n}$ with $P(\Omega_{X_1,...,X_n}) = p_n > 0$. The total reserves of the portfolio risks $(X_1, ..., X_n)$ is first determined by $\theta\,\mathbb{E}[S_n|\Omega_{X_1,...,X_n}]$ and then $\theta\,\mathbb{E}[X_i|\Omega_{X_1,...,X_n}]$ is allocated to sub-portfolio i for $i = 1, ..., n$, where $0 < \theta \leq 1/p_n$. The principle is called the conditional tail principle. The parameter $\theta \in (0,\, 1/p_n]$ can be viewed as an adjustment coefficient of the conditional tail principle.

When $\theta = 1/p_n$, we have a very conservative principle for the portfolio with

$$\frac{1}{p_n}\,\mathbb{E}[S_n|\Omega_{X_1,\dots,X_n}] = \sum_{i=1}^{n} \frac{1}{p_n}\mathbb{E}[X_i|\Omega_{X_1,\dots,X_n}]. \tag{2.59}$$

When $\theta = 1$, we obtain a more general CTE principle with

$$\mathbb{E}[S_n|\Omega_{X_1,\dots,X_n}] = \sum_{i=1}^{n} \mathbb{E}[X_i|\Omega_{X_1,\dots,X_n}]. \tag{2.60}$$

When $\theta = p_n$, we have a more optimistic principle for the portfolio with

$$\begin{aligned} p_n\,\mathbb{E}[S_n|\Omega_{X_1,\dots,X_n}] &= \mathbb{E}[S_n\,\mathbb{I}_{\{\Omega_{X_1,\dots,X_n}\}}] \\ &= \sum_{i=1}^{n} \mathbb{E}[X_i\,\mathbb{I}_{\{\Omega_{X_1,\dots,X_n}\}}] = \sum_{i=1}^{n} p_n\,\mathbb{E}[X_i|\Omega_{X_1,\dots,X_n}]. \end{aligned} \tag{2.61}$$

Thus, by using the six common tail regions Ω_{α,S_n}, $\Omega_{\alpha,X_1,\dots,X_n}$, $\Omega^e_{S_n}$, $\Omega^e_{X_1,\dots,X_n}$, $\Omega^*_{\alpha,X_1,\dots,X_n}$, and $\Omega^{e*}_{X_1,\dots,X_n}$, we obtain different allocation principles. The allocation principles conditioning on the tail of the aggregate risk:

$$\theta\,\mathbb{E}[S_n|S_n \geq \mathrm{VaR}_\alpha(S_n)] = \sum_{i=1}^{n} \theta\,\mathbb{E}[X_i|S_n \geq \mathrm{VaR}_\alpha(S_n)] \tag{2.62}$$

and

$$\theta\,\mathbb{E}[S_n|S_n \geq \mathbb{E}[S_n]] = \sum_{i=1}^{n} \theta\,\mathbb{E}[X_i|S_n \geq \mathbb{E}[S_n]]. \tag{2.63}$$

The allocation principles conditioning on that at least one tail of individual risks will occur:

$$\begin{aligned} &\theta\,\mathbb{E}[S_n|X_1 \geq \mathrm{VaR}_\alpha(X_1)\,\text{or}\,\cdots\,\text{or}\,X_n \geq \mathrm{VaR}_\alpha(X_n)] \\ = &\sum_{i=1}^{n} \theta\,\mathbb{E}[X_i|X_1 \geq \mathrm{VaR}_\alpha(X_1)\,\text{or}\,\cdots\,\text{or}\,X_n \geq \mathrm{VaR}_\alpha(X_n)] \end{aligned} \tag{2.64}$$

and

$$
\begin{aligned}
& \theta\, \mathbb{E}[S_n | X_1 \geq \mathbb{E}[X_1] \,\mathrm{or}\, \cdots \mathrm{or}\, X_n \geq \mathbb{E}[X_n]] \\
= & \sum_{i=1}^{n} \theta\, \mathbb{E}[X_i | X_1 \geq \mathbb{E}[X_1] \,\mathrm{or}\, \cdots \mathrm{or}\, X_n \geq \mathbb{E}[X_n]]. \qquad (2.65)
\end{aligned}
$$

The allocation principles conditioning on that all the tails of individual risks will occur:

$$
\begin{aligned}
& \theta\, \mathbb{E}[S_n | X_1 \geq \mathrm{VaR}_\alpha(X_1), \ldots, X_n \geq \mathrm{VaR}_\alpha(X_n)] \\
= & \sum_{i=1}^{n} \theta\, \mathbb{E}[X_i | X_1 \geq \mathrm{VaR}(X_1), \ldots, X_n \geq \mathrm{VaR}(X_n)] \qquad (2.66)
\end{aligned}
$$

and

$$
\begin{aligned}
& \theta\, \mathbb{E}[S_n | X_1 \geq \mathbb{E}[X_1], \ldots, X_n \geq \mathbb{E}[X_n]] \\
= & \sum_{i=1}^{n} \theta\, \mathbb{E}[X_i | X_1 \geq \mathbb{E}[X_1], \ldots, X_n \geq \mathbb{E}[X_n]]. \qquad (2.67)
\end{aligned}
$$

More importantly, when we uses the tail distortion principle H_g to determine the capital reserves or premiums for a portfolio risks, the premiums/reserves $H_g(X_i)$ depends both on the extreme tail events and on the dependence of risks in the portfolio.

Let $(X_1, ..., X_n)$ be risks in a portfolio. Assume that the joint distribution $(X_1, ..., X_n)$ is a multivariate elliptical distribution (Xu and Mao, 2012) or a multivariate Pareto type II distribution, calculate the right sides of (2.62)-(2.65) for $\theta = 1/p_n, 1$, and p_n, respectively, consider the influence of the correlation coefficients between X_i and X_j on these values.

Proposition 2.23. *Let g be a distortion function and*

$$
\Omega^{or}_{X_1,\ldots,X_n} = \{X_1 \geq t_1 \, or \cdots or \, X_n \geq t_n\}
$$

be a common tail region of a random vector $(X_1, ..., X_n)$ *with*

$$p_n(t_1, ..., t_n) = P(X_1 \geq t_1 \, or \cdots or \, X_n \geq t_n) > 0,$$

Then,

$$\begin{aligned} H_g(X_i) &= t_i \cdot \frac{g(p_n(t_1, ..., t_n))}{p_n(t_1, ..., t_n)} + \int_{-\infty}^{t_i} \Big[\frac{g\big(S_{X_i}(x_i) - D_i(x_i|t_1, \ldots, t_n)\big)}{p_n(t_1, ..., t_n)} \\ & \quad - \frac{g(p_n(t_1, ..., t_n))}{p_n(t_1, ..., t_n)} \Big] dx_i + \frac{1}{p_n(t_1, ..., t_n)} \int_{t_i}^{\infty} g\big(S_{X_i}(x_i)\big) dx_i \end{aligned} \quad (2.68)$$

for $i = 1, ..., n$*, where* S_{X_i} *is the survival function of* X_i *and*

$$\begin{aligned} & D_i(x_i|t_1, \ldots, t_n) \\ = \; & P(X_1 < t_1, \ldots, X_i < t_i, \ldots, X_n < t_n) - P(X_1 < t_1, \ldots, X_i \leq x_i, \ldots, X_n < t_n) \\ = \; & P(X_1 \geq t_1 \; or \; \ldots \; or \; X_i > x_i \; or \; \ldots \; or \; X_n \geq t_n) - p_n(t_1, ..., t_n). \end{aligned} \quad (2.69)$$

Proof. Note that

$$\begin{aligned} & P((X_i > x_i) \cap (X_1 \geq t_1 \, \text{or} \cdots \text{or} \, X_n \geq t_n)) \\ = \; & P(X_i > x_i) - P((X_i > x_i) \cap (X_1 < t_1, \cdots X_n < t_n)). \end{aligned} \quad (2.70)$$

It is easy to see that if $x_i \geq t_i$, then

$$P((X_i > x_i) \cap (X_1 < t_1, \ldots, x_n < t_n)) = 0,$$

and if $x_i < t_i$, then

$$\begin{aligned} & P((X_i > x_i) \cap (X_1 < t_1, \ldots, X_n < t_n)) \\ = \; & P(X_1 < t_1, \ldots, x_i < X_i < t_i, \ldots, X_n < t_n) \\ = \; & P(X_1 < t_1, \ldots, X_i < t_i, \ldots, X_n < t_n) - P(X_1 < t_1, \ldots, X_i \leq x_i, \ldots, X_n < t_n) \\ = \; & D_i(x_i|t_1, \ldots, t_n). \end{aligned}$$

Thus,

$$P((X_i > x_i) \cap (X_1 \geq t_1 \,\text{or}\, \cdots \,\text{or}\, X_n \geq t_n)) = \begin{cases} S_{X_i}(x_i) - D_i(x_i|t_1, \ldots, t_n), & x_i < t_i, \\ S_{X_i}(x_i), & x_i \geq t_i. \end{cases}$$

Hence,

$$\begin{aligned} & \int_{\Omega^{or}_{X_1,\ldots,X_n}} X_i \,\mathrm{d}g \circ P \\ = & \int_{-\infty}^{0} \big[g(P((X_i > x_i) \cap (X_1 \geq t_1 \,\text{or}\, \cdots \,\text{or}\, X_n \geq t_n))) - g(p_n(t_1, ..., t_n))\big] dx_i \\ & + \int_0^{\infty} g(P((X_i > x_i) \cap (X_1 \geq t_1 \,\text{or}\, \cdots \,\text{or}\, X_n \geq t_n)) dx_i. \end{aligned}$$

Thus, if $t_i \geq 0$, we obtain

$$\begin{aligned} & \int_{\Omega^{or}_{X_1,\ldots,X_n}} X_i \,\mathrm{d}g \circ P \\ = & \int_{-\infty}^{0} \Big[g\big(S_{X_i}(x_i) - D_i(x_i|t_1, \ldots, t_n)\big) - g(p_n(t_1, ..., t_n))\Big] dx_i \\ & + \int_0^{t_i} g\big(S_{X_i}(x_i) - D_i(x_i|t_1, \ldots, t_n)\big) dx_i + \int_{t_i}^{\infty} g\big(S_{X_i}(x_i)\big) dx_i \\ = & \; t_i \, g(p_n(t_1, ..., t_n)) + \int_{-\infty}^{t_i} \Big[g\big(S_{X_i}(x_i) - D_i(x_i|t_1, \ldots, t_n)\big) - g(p_n(t_1, ..., t_n))\Big] dx_i \\ & + \int_{t_i}^{\infty} g\big(S_{X_i}(x_i)\big) dx_i. \end{aligned} \tag{2.71}$$

If $t_i < 0$, we obtain

$$\begin{aligned} & \int_{\Omega^{or}_{X_1,\ldots,X_n}} X_i \,\mathrm{d}g \circ P \\ = & \int_{-\infty}^{t_i} \Big[g\big(S_{X_i}(x_i) - D_i(x_i|t_1, \ldots, t_n)\big) - g(p_n(t_1, ..., t_n))\Big] dx_i \\ & + \int_{t_i}^{0} \Big[g(S_{X_i}(x_i)) - g(p_n(t_1, ..., t_n))\Big] dx_i + \int_0^{\infty} g(S_{X_i}(x_i)) dx_i \\ = & \; t_i \, g(p_n(t_1, ..., t_n)) + \int_{-\infty}^{t_i} \Big[g\big(S_{X_i}(x_i) - D_i(x_i|t_1, \ldots, t_n)\big) - g(p_n(t_1, ..., t_n))\Big] dx_i \\ & + \int_{t_i}^{\infty} g\big(S_{X_i}(x_i)\big) dx_i, \end{aligned} \tag{2.72}$$

which, together with (2.71) and

$$H_g(X_i) = \frac{1}{p_n(t_1, ..., t_n)} \int_{\Omega^{or}_{X_1,...,X_n}} X_i \, \mathrm{d}g \circ P,$$

means that (2.68) holds for any t_i. $\square$

Remark 2.24. *(i) By letting $g(u) = u$ for $u \in [0,1]$, we have*

$$\begin{aligned} & \mathbb{E}[X_i | X_1 \geq t_1 \, or \cdots or \, X_n \geq t_n] \\ = \; & t_i + \int_{-\infty}^{t_i} \Big[\frac{S_{X_i}(x_i) - D_i(x_i | t_1, \dots, t_n)}{p_n(t_1, ..., t_n)} - 1 \Big] dx_i \\ & + \frac{1}{p_n(t_1, ..., t_n)} \int_{t_i}^{\infty} S_{X_i}(x_i) dx_i. \end{aligned}$$

(ii) If $(X_1, ..., X_n)$ is a non-negative random vector and $t_i \geq 0$, then

$$\begin{aligned} & \mathbb{E}[X_i | X_1 \geq t_1 \, or \cdots or \, X_n \geq t_n] \\ = \; & \frac{\mathbb{E}[X_i]}{p_n(t_1, ..., t_n)} - \frac{1}{p_n(t_1, ..., t_n)} \int_0^{t_i} D_i(x_i | t_1, \dots, t_n) dx_i. \end{aligned}$$

(iii) If $(X_1, ..., X_n)$ has a continuous joint distribution $F(x_1, ..., x_n) = P(X_1 \leq x_1, ..., X_n \leq x_n)$, then

$$D_i(x_i | t_1, ..., t_n) = F(t_1, \dots, t_i, \dots, t_n) - F(t_1, \dots, x_i, \dots, t_n).$$

$\square$

Proposition 2.25. *Let g be a distortion function and $\Omega^*_{X_1,...,X_n} = \{X_1 \geq t_1, ..., X_n \geq t_n\}$ be a common tail region of a random vector $(X_1, ..., X_n)$ with*

$$p^*_n(t_1, ..., t_n) = P(\Omega^*_{X_1,...,X_n}) = P(X_1 \geq t_1, \dots, X_n \geq t_n) > 0.$$

Then,

$$H_g(X_i) = t_i \cdot \frac{g(p^*_n(t_1, ..., t_n))}{p^*_n(t_1, ..., t_n)} + \frac{1}{p^*_n(t_1, ..., t_n)} \int_{t_i}^{\infty} g(G_i(x_i | t_1, ..., t_n)) dx_i \quad (2.73)$$

for $i = 1, ..., n,$ *where*

$$G_i(x_i|t_1, \ldots, t_n) = P(X_1 \geq t_1, \ldots, X_i > x_i, \ldots, X_n \geq t_n). \tag{2.74}$$

Proof. Note that if $x_i < t_i$, then

$$P((X_i > x_i) \cap (X_1 \geq t_1, \ldots, X_n \geq t_n)) = p_n^*(t_1, ..., t_n),$$

and if $x_i \geq t_i$, then

$$\begin{aligned} P((X_i > x_i) \cap (X_1 \geq t_1, \ldots, X_n \geq t_n)) &= P(X_1 \geq t_1, \ldots, X_i > x_i, \ldots, X_n \geq t_n) \\ &= G_i(x_i|t_1, ..., t_n). \end{aligned}$$

Thus,

$$\begin{aligned} \int_{\Omega^*_{X_1,\ldots,X_n}} X_i \,\mathrm{d}g \circ P &= \int_0^\infty g(P((X_i > x_i) \cap (X_1 \geq t_1, \ldots, X_n \geq t_n)))dx_i \\ &+ \int_{-\infty}^0 \big[g(P((X_i > x_i) \cap (X_1 \geq t_1, \ldots, X_n \geq t_n))) \\ &- g(p_n^*(t_1, ..., t_n))\big] dx_i. \end{aligned}$$

Thus, if $t_i \geq 0$, we obtain

$$\int_{\Omega^*_{X_1,\ldots,X_n}} X_i \,\mathrm{d}g \circ P = t_i \, g(p_n^*(t_1, ..., t_n)) + \int_{t_i}^\infty g(G_i(x_i|t_1, ..., t_n))dx_i. \tag{2.75}$$

If $t_i < 0$, we obtain

$$\begin{aligned} &\int_{\Omega^*_{X_1,\ldots,X_n}} X_i \,\mathrm{d}g \circ P \\ &= \int_{t_i}^0 \big[g(G_i(x_i|t_1, ..., t_n)) - g(p_n^*(t_1, ..., t_n))\big] dx_i + \int_0^\infty g(G_i(x_i|t_1, ..., t_n))dx_i \\ &= t_i \, g(p_n^*(t_1, ..., t_n)) + \int_{t_i}^\infty g(G_i(x_i|t_1, ..., t_n))dx_i, \end{aligned} \tag{2.76}$$

which, together with (2.75) and $H_g(X_i) = \frac{1}{p_n(t_1,...,t_n)} \int_{\Omega^{or}_{X_1,\ldots,X_n}} X_i \,\mathrm{d}g \circ P$, means that (2.73) holds for any t_i. □

Remark 2.26. *(i) By letting $g(u) = u$ for $u \in [0, 1]$, we have*

$$\mathbb{E}[X_i|X_1 \geq t_1, \ldots, X_n \geq t_n] = t_i + \frac{1}{p_n^*(t_1, ..., t_n)} \int_{t_i}^{\infty} G_i(x_i|t_1, ..., t_n) dx_i.$$

(ii) If $X_1, ..., X_n$ are independent, then

$$\mathbb{E}[X_i|X_1 \geq t_1, \ldots, X_n \geq t_n] = \mathbb{E}[X_i|X_i \geq t_i].$$

(iii) If $(X_1, ..., X_n)$ has a continuous joint survival function $S(x_1, ..., x_n) = P(X_1 > x_1, ..., X_n > x_n)$, then

$$G_i(x_i|t_1, ..., t_n) = S(t_1, \ldots, x_i, \ldots, t_n).$$

Note that $S(x_1, ..., x_n) \neq 1 - F(x_1, ..., x_n)$. □

2.5 Numerical Examples

In this section, we will consider the multivariate Pareto type II model, which is also investigated in Chiragiev and Landsman (2007), with joint survival function

$$S_{\boldsymbol{X}}(\boldsymbol{x}) = (1 + \sum_{i=1}^{n} \frac{x_i - \mu_i}{\sigma_i})^{-\beta}, \; x_i > \mu_i, \; i = 1, 2, 3,$$

where $n = 3$, $\beta > 1$, $\mu_1 = \mu_2 = \mu_3 = 0$, $\sigma_1 > 0$, $\sigma_2 > 0$ and $\sigma_3 > 0$. In this model, the coefficient of X_i and X_j is $\mathrm{corr}(X_i, X_j) = 1/\beta$ for any $i \neq j$ and $1 \leq i, j \leq n$. We will consider three parameter assumptions: when $\beta = 1.5$, $\sigma_1 = 0.32$, $\sigma_2 = 0.94$ and $\sigma_3 = 0.16$; when $\beta = 2.5$, $\sigma_1 = 0.96$, $\sigma_2 = 2.82$ and $\sigma_3 = 0.48$; when $\beta = 4.5$, $\sigma_1 = 2.24$, $\sigma_2 = 6.58$ and $\sigma_3 = 1.12$. The expectation of each capital line calculated by $\mathbb{E}[X_i] = \frac{\sigma_i}{\beta - 1}$ are same following these three parameter assumptions. Note that

	X_1	X_2	X_3
$\mathrm{VaR}_{0.95}(X_i)(\beta = 1.5)$	2.0378	5.9860	1.0189
$\mathrm{VaR}_{0.95}(X_i)(\beta = 2.5)$	2.2219	6.5268	1.1109
$\mathrm{VaR}_{0.95}(X_i)(\beta = 4.5)$	2.1188	6.2239	1.0594
$\mathrm{TVaR}_{0.95}(X_i)(\beta = 1.5)$	6.7534	19.8380	3.3767
$\mathrm{TVaR}_{0.95}(X_i)(\beta = 2.5)$	4.3432	12.7580	2.1201
$\mathrm{TVaR}_{0.95}(X_i)(\beta = 4.5)$	3.3641	9.8821	1.6821
$\mathbb{E}[X_i](\beta = 1.5,\ 2.5,\ 4.5)$	0.64	1.88	0.32

TABLE 2.1: Values of $\mathrm{VaR}_{0.95}(X_i)$ and $\mathbb{E}[X_i]$

$\mathrm{TVaR}_{0.95}(X_i)$ is calculated by

$$\begin{aligned}
\mathrm{TVaR}_{0.95}(X_i) &= \mathrm{VaR}_{0.95}(X_i) + \frac{\int_{\mathrm{VaR}_{0.95}(X_i)}^{\infty}(1+\frac{x}{\sigma_i})^{-\beta}dx}{1-0.95} \\
&= \mathrm{VaR}_{0.95}(X_i) + \frac{\sigma_i}{0.05(\beta-1)}(1+\frac{\mathrm{VaR}_{0.95}(X_i)}{\sigma_i})^{-\beta+1} \\
&= \mathrm{VaR}_{0.95}(X_i) + \frac{\mathbb{E}[X_i]}{0.05}(1+\frac{\mathrm{VaR}_{0.95}(X_i)}{\sigma_i})^{-\beta+1}
\end{aligned}$$

The values of $\mathrm{VaR}_{0.95}(X_i)$, $\mathrm{TVaR}_{0.95}(X_i)$ and $\mathbb{E}[X_i]$ are shown in Table 2.1.

Note that for any $t_i \in \mathbb{R}$, $i = 1, \dots, n$,

$$Pr(X_1 \geq t_1, \dots, X_n \geq t_n) \leq Pr(S_n \geq \sum_{i=1}^{n} t_i) \leq Pr(X_1 \geq t_1 \,\text{or}\, \cdots \,\text{or}\, X_n \geq t_n).$$

If $t_i = \mathrm{VaR}_\alpha(X_i)$ and $\mathrm{VaR}_\alpha(S_n) \leq \sum_{i=1}^{n} \mathrm{VaR}_\alpha(X_i)$, then

$$Pr(X_1 \geq \mathrm{VaR}_\alpha(X_1), \dots, X_n \geq \mathrm{VaR}_\alpha(X_n)) \leq Pr(S_n \geq \sum_{i=1}^{n} \mathrm{VaR}_\alpha(X_i)) \leq Pr(S_n \geq \mathrm{VaR}_\alpha(S_n)).$$

Example 2.1. *According to the formula for multivariate Pareto portfolios in Chiragiev and Landsman (2007), we will provide the results of* (2.62) *with* $\alpha = 0.95$ *when* $\beta = 1.5,\ 2.5,\ 4.5$ *in Table* 2.2*. Firstly, we can get the survival function for the aggregate risk* S*:*

$$\overline{F}_S(s) = \sum_{i=1}^{3} \frac{\sigma_i^2(1+\frac{s}{\sigma_i})^{-\beta}}{\prod_{j=1, j\neq i}^{3}(\sigma_i - \sigma_j)}$$

by adopting (2.2) and (4.2) in Chiragiev and Landsman (2007). Then, we have $\mathrm{VaR}_{0.95}(S_n) = 8.6005$ *when* $\beta = 1.5$, $\mathrm{VaR}_{0.95}(S_n) = 8.8005$ *when* $\beta = 2.5$,

	$\beta = 1.5$	$\beta = 2.5$	$\beta = 4.5$
$\mathbb{E}[X_1 \mid \Omega_{0.95,S_n}]$	5.7746	3.0389	1.9178
$\mathbb{E}[X_2 \mid \Omega_{0.95,S_n}]$	19.3110	12.2184	9.4468
$\mathbb{E}[X_3 \mid \Omega_{0.95,S_n}]$	2.7541	1.3640	0.8138
$\mathbb{E}[S_n \mid \Omega_{0.95,S_n}]$	27.8396	16.6213	12.1784

TABLE 2.2: On the tail $\Omega_{0.95,S_n}$

	$\beta = 1.5$	$\beta = 2.5$	$\beta = 4.5$
$\mathbb{E}[X_1 \mid \Omega^e_{S_n}]$	2.2496	1.3686	1.1019
$\mathbb{E}[X_2 \mid \Omega^e_{S_n}]$	7.2580	4.7418	4.0235
$\mathbb{E}[X_3 \mid \Omega^e_{S_n}]$	1.0828	0.6364	0.4977
$\mathbb{E}[S_n \mid \Omega^e_{S_n}]$	10.5905	6.7468	5.6232

TABLE 2.3: On the tail $\Omega^e_{S_n}$

$\mathrm{VaR}_{0.95}(S_n) = 7.9888$ *when* $\beta = 4.5$, *and* $\mathbb{E}[S_n] = 2.84$ *when* $\beta = 1.5,\ 2.5,\ 4.5$. *For the allocation principle conditioned on the tail of aggregate risk with* $\Omega_{0.95,S_n} = \{S_n \geq \mathrm{VaR}_{0.95}(S_n)\}$,

$$p_n = P(\Omega_{0.95,S_n}) = 0.05$$

when $\beta = 1.5,\ 2.5,\ 4.5$. *The values of* $\mathbb{E}[X_i|\Omega]$, $i = 1, 2, 3$, *and* $\mathbb{E}[S_n|\Omega]$, *where* Ω *represents the corresponding common tail region, are calculated based on the formulas concluded in Theorem 2 and Theorem 3 of Chiragiev and Landsman (2007).*

The results for (2.63) *are illustrated in Table* 2.3. *For the allocation principle conditioned on the tail of aggregate risk with* $\Omega^e_{S_n} = \{S_n \geq \mathbb{E}[S_n]\}$, $p_n = P(\Omega^e_{S_n}) = 0.1958$ *when* $\beta = 1.5$; $p_n = P(\Omega^e_{S_n}) = 0.2887$ *when* $\beta = 2.5$; *and* $p_n = P(\Omega^e_{S_n}) = 0.6205$ *when* $\beta = 4.5$.

Next, we will calculate the principles based on (2.68) and (2.73) in Examples 2.2 and 2.3.

Example 2.2. *For the allocation principles conditioned on that at least one tail of individual risks will occur by* (2.64) *with* $\alpha = 0.95$ *and* (2.65),

	$\beta = 1.5$	$\beta = 2.5$	$\beta = 4.5$
$\mathbb{E}[X_1\|\Omega_{0.95,X_1,X_2,X_3}]$	3.6836	2.3118	1.7402
$\mathbb{E}[X_2\|\Omega_{0.95,X_1,X_2,X_3}]$	10.8206	6.7910	5.1120
$\mathbb{E}[X_3\|\Omega_{0.95,X_1,X_2,X_3}]$	1.8418	1.1559	0.8701
$\mathbb{E}[S_n\|\Omega_{0.95,X_1,X_2,X_3}]$	16.3460	10.2587	7.7223

TABLE 2.4: On the tail $\Omega_{0.95,X_1,X_2,X_3}$

we will calculate them by (2.68) *in Table* 2.4. *By* (2.69), *without loss of generality, for* $i = 1$, *we get*

$$\begin{aligned}
& D_1(x_1|t_1,t_2,t_3) \\
= \; & P(X_1 > x_1 \text{ or } X_2 \geq t_2 \text{ or } X_3 \geq t_3) - p_3(t_1,t_2,t_3) \\
= \; & P(X_1 > x_1 \text{ or } X_2 \geq t_2 \text{ or } X_3 \geq t_3) - P(X_1 \geq t_1 \text{ or } X_2 \geq t_2 \text{ or } X_3 \geq t_3) \\
= \; & (1+\frac{x_1}{\sigma_1})^{-\beta} - (1+\frac{t_1}{\sigma_1})^{-\beta} - (1+\frac{x_1}{\sigma_1}+\frac{t_2}{\sigma_2})^{-\beta} + (1+\frac{t_1}{\sigma_1}+\frac{t_2}{\sigma_2})^{-\beta} - (1+\frac{x_1}{\sigma_1}+\frac{t_3}{\sigma_3})^{-\beta} \\
& +(1+\frac{t_1}{\sigma_1}+\frac{t_3}{\sigma_3})^{-\beta} + (1+\frac{x_1}{\sigma_1}+\frac{t_2}{\sigma_2}+\frac{t_3}{\sigma_3})^{-\beta} - (1+\frac{t_1}{\sigma_1}+\frac{t_2}{\sigma_2}+\frac{t_3}{\sigma_3})^{-\beta}.
\end{aligned}$$

For the principle conditioned on the tail with at least one tail with $X_i \geq \mathrm{VaR}_{0.95}(X_i)$, $i = 1, 2, 3$, *when* $\beta = 1.5$, $p_n = P(\Omega_{0.95,X_1,X_2,X_3}) = 0.1022$, *where*

$$\Omega_{0.95,X_1,X_2,X_3} = \{X_1 \geq \mathrm{VaR}_{0.95}(X_1) \, or \, X_2 \geq \mathrm{VaR}_{0.95}(X_2) or \, X_3 \geq \mathrm{VaR}_{0.95}(X_3)\}.$$

When $\beta = 2.5$, $p_n = 0.1157$; *and when* $\beta = 4.5$, $p_n = 0.1271$.

For the allocation principles conditioned on that at least one tail of individual risks will occur by (2.65), *the results are in Table* 2.5. *For the principle conditioned on the tail with at least one tail with* $X_i \geq \mathbb{E}(X_i)$, $i = 1, 2, 3$, *when* $\beta = 1.5$, $p_n = P(\Omega^e_{X_1,X_2,X_3}) = 0.3630$, *where*

$$\Omega^e_{X_1,X_2,X_3} = \{X_1 \geq \mathbb{E}[X_1] \, or \, X_2 \geq \mathbb{E}[X_2] or \, X_3 \geq \mathbb{E}[X_3]\}.$$

When $\beta = 2.5$, $p_n = 0.5400$; *and when* $\beta = 4.5$, $p_n = 0.6374$.

Example 2.3. *For the allocation principles conditioned on that all the tails of individual risks will occur by* (2.66) *with* $\alpha = 0.95$, *we will calculate them by*

	$\beta = 1.5$	$\beta = 2.5$	$\beta = 4.5$
$\mathbb{E}[X_1 \mid \Omega^e_{X_1,X_2,X_3}]$	1.4738	1.0045	0.8690
$\mathbb{E}[X_2 \mid \Omega^e_{X_1,X_2,X_3}]$	4.3294	2.9507	2.5527
$\mathbb{E}[X_3 \mid \Omega^e_{X_1,X_2,X_3}]$	0.7369	0.5022	0.4345
$\mathbb{E}[S_n \mid \Omega^e_{X_1,X_2,X_3}]$	6.5401	4.4574	3.8562

TABLE 2.5: On the tail $\Omega^e_{X_1,X_2,X_3}$

	$\beta = 1.5$	$\beta = 2.5$	$\beta = 4.5$
$\mathbb{E}[X_1 \mid \Omega^*_{0.95,X_1,X_2,X_3}]$	14.8969	7.3268	4.5272
$\mathbb{E}[X_2 \mid \Omega^*_{0.95,X_1,X_2,X_3}]$	43.7597	21.5225	13.2986
$\mathbb{E}[X_3 \mid \Omega^*_{0.95,X_1,X_2,X_3}]$	7.4485	3.6634	2.2636
$\mathbb{E}[S_n \mid \Omega^*_{0.95,X_1,X_2,X_3}]$	66.1051	32.5127	20.0894

TABLE 2.6: On the tail $\Omega^*_{0.95,X_1,X_2,X_3}$

(2.73) *in Table 2.6. By* (2.74)*, for* $i = 1$*, we have*

$$G_1(x_1|t_1, t_2, t_3) = P(X_1 > x_1, X_2 \geq t_1, X_3 \geq t_3).$$

For the principle conditioned on that all the tails of individual risks will occur with $\{X_i \geq \mathrm{VaR}_{0.95}(X_i)\}$*,* $i = 1, 2, 3$*, when* $\beta = 1.5$*,* $p_n = P(\Omega^*_{0.95,X_1,X_2,X_3}) = 0.0111$*, where*

$$\Omega^*_{0.95,X_1,X_2,X_3} = \{X_1 \geq \mathrm{VaR}_{0.95}(X_1), X_2 \geq \mathrm{VaR}_{0.95}(X_2), X_3 \geq \mathrm{VaR}_{0.95}(X_3)\}.$$

When $\beta = 2.5$*,* $p_n = 0.0056$*; and when* $\beta = 4.5$*,* $p_n = 0.0024$*.*

Table 2.7 illustrates the allocation principles conditioned on that all the tails of individual risks will occur by (2.67)*. For the principle conditioned on that all the tails of individual risks will occur with* $\{X_i \geq \mathbb{E}(X_i)\}$*,* $i = 1, 2, 3$*, when* $\beta = 1.5$*,*

$$p_n = P(\Omega^{e*}_{X_1,X_2,X_3}) = 0.0540,$$

where

$$\Omega^{e*}_{X_1,X_2,X_3} = \{X_1 \geq \mathbb{E}[X_1], X_2 \geq \mathbb{E}[X_2], X_3 \geq \mathbb{E}[X_3]\}.$$

When $\beta = 2.5$*,* $p_n = 0.0642$*; and when* $\beta = 4.5$*,* $p_n = 0.0617$*.*

	$\beta = 1.5$	$\beta = 2.5$	$\beta = 4.5$
$\mathbb{E}[X_1 \vert \Omega^{e*}_{X_1,X_2,X_3}]$	5.1196	2.5585	1.8283
$\mathbb{E}[X_2 \vert \Omega^{e*}_{X_1,X_2,X_3}]$	15.0388	7.5156	5.3707
$\mathbb{E}[X_3 \vert \Omega^{e*}_{X_1,X_2,X_3}]$	2.5598	1.2793	0.9142
$\mathbb{E}[S_n \vert \Omega^{e*}_{X_1,X_2,X_3}]$	22.7182	11.3534	8.1132

TABLE 2.7: On the tail $\Omega^{e*}_{X_1,X_2,X_3}$

According Tables 2.1-2.7, we have the following conclusions of the formulas derived in this chapter.

(i) If the expected loss remains same for different β, the capital allocated to each capital line decreases as β increases for the six different tail regions $\Omega_{0.95,S_n}$, $\Omega^e_{S_n}$, $\Omega_{0.95,X_1,X_2,X_3}$, $\Omega^e_{X_1,X_2,X_3}$, $\Omega^*_{0.95,X_1,X_2,X_3}$ and $\Omega^{e*}_{X_1,X_2,X_3}$, as illustrated in Tables 2.2-2.7. As β increases, the coefficient of X_i and X_j decreases, hence, the corresponding loss of each capital line will decrease, and it is reasonable to keep a lower reserve capital instead.

(ii) In the examples, if the multivariate distribution of the individual risks are same, the capital allocated to the corresponding capital line based on the three kinds of common tail regions defined by $\mathrm{VaR}_{0.95}$ is larger than the capital based on the regions defined by the expectations by comparing Table 2.2 and Table 2.3, Table 2.4 and Table 2.5, or Table 2.6 and Table 2.7.

(iii) Moreover, as illustrated in Table 2.4 and Table 2.6, or Table 2.5 and Table 2.7, if the common tail regions are unions of the corresponding individual tail regions, the amounts of capital are less than those with the common tail regions as the intersections of the corresponding individual tail regions for either $\mathrm{VaR}_{0.95}$ or expectation as the benchmarks of the same multivariately distributed risks. Since the intersections of the tail regions are more conservative than the unions, these results are reasonable.

Chapter 3

Risk Measures Based on Weighted Loss Functions

3.1 Introduction

In the literature, there are many axioms defined for risk measures. Artzner et al. (1999) proposed the four axioms for a coherent risk measure. Additionally, Kusuoka (2001) investigated the law invariant coherent risk measures. Moreover, the convex risk measures were analyzed in Föllmer and Schied (2002), and they argued that if a risk measure satisfies the axioms of monotonicity and translation invariance, it belongs to the set of monetary risk measures. Jouini et al. (2006) further talked about the constraints based on which the law invariant risk measures can attain the Fatou property. Then, Pichler (2015) researched on the premiums and reserves adjusted by distortions. In addition, Cai and Mao (2016) derived a class of risk measures for the required regulatory capital from a regulator's perspective. For convex analysis, the properties of convex functions are provided in Niculescu and Persson (2006) and Rockafellar (2011).

Bellini et al. (2014) defined the generalized quantiles and categorized them as risk measures. Moreover, Mao and Cai (2016) generalized their model based on the rank-dependent expected utility theory. Firstly, we will introduce the model of Bellini et al. (2014). For a random variable X, the generalized quantiles of X in their model is defined as:

$$c_\alpha(X) = \arg\min_{c\in\mathbb{R}} f_\alpha(c, X), \tag{3.1}$$

where

$$f_\alpha(c, X) = \mathbb{E}[\alpha\Phi_1((X-c)_+) + (1-\alpha)\Phi_2((X-c)_-)],$$

in which Φ_1 and Φ_2 are convex functions, and $\alpha \in (0, 1)$.

Obviously, the objective function in their model is a linear combination of the expectations of $(X-c)_+$ and $(X-c)_-$. In this chapter, we will generalize the weight factors in the objective function from constants to random factors. Namely, the weight factor α for the positive part is modified to be $g(X)$, and $1-\alpha$ for the negative part is set to be $h(X)$. The objective function in our model is a weighted combination of the expectations of the risks $(X-c)_+$ and $(X-c)_-$. Thus, the objective loss function is generalized to be

$$f_{g,h}(c, X) = \mathbb{E}[g(X)\Phi_1((X-c)_+)] + \mathbb{E}[h(X)\Phi_2((X-c)_-)], \tag{3.2}$$

in which $g(X)$, $h(X)$ are non-negative functions of X, and the optimization problem is generalized to be

$$c^*_{g,h}(X) = \arg\min_{c\in\mathbb{R}} f_{g,h}(c, X), \tag{3.3}$$

where the weight functions g and h may depend on X. In this case, we write $g_X(x)$ and $h_X(x)$ instead, for example, $g_X(x) = \gamma\mathbb{I}_{\{x\le\mathbb{E}[X]\}} + \beta\mathbb{I}_{\{x>\mathbb{E}[X]\}}$. In this chapter, we call $c^*_{g,h}(X)$, the minimizers of (3.3), the *weighted quantiles* of X.

This generalized model can be applied to different optimization situations. Firstly, it can minimize the insurer's potential risk. If X represents the loss covered by an insurer and c is the insurance premium, then $(X-c)_+$ describes the deficit risk for the insurer. If the insurance premium is not enough to cover the insured loss, insurance companies will incur a loss $(X-c)_+$. We suppose that Φ_1 is the function used to quantify this kind of risk, and $\mathbb{E}[g(X)\Phi_1((X-c)_+)]$ is the quantitative deficit risk for insurance companies by employing the weight function $g(x)$. In addition, $(X-c)_- = (c-X)_+$ can be treated as the insurer's risk or cost when the premium is overpriced. Additional cost for insurance companies might be produced due to tax payments. Moreover, an overpriced insurance contract might not be competitive. If the insured individuals feel that they have been overcharged for the insurance contract, they might switch providers, or surrender from current contracts. Hence, $\mathbb{E}[h(X)\Phi_2((X-c)_-)]$ is the quantitative risk for the insurer on the overcharged premium.

In fact, this model can be illustrated from the insurer's perspective by considering the loss of the insured. Note that $(X-c)_- = (c-X)_+$ can be treated as the overpaid risk of the insured. Some insurance customers may face a situation in which the insurance premium is overcharged since their actual loss could be much less than the amount of the paid premium. In this case, $\mathbb{E}[h(X)\Phi_2((X-c)_-)]$ is the quantitative risk for the insured due to the overcharged premium. Then the optimization problem is to minimize both the insurer's deficit risk and the insured's overcharged risk.

Moreover, the regulators can also employ this model to determine the appropriate amount of capital as solvency benchmarks. In this case, if c is the required solvency capital or reserve, then $(X-c)_+$ is the shortfall risk for a company and $(X-c)_- = (c-x)_+$ is the over-required capital. In model (3.1), α plays a role as the sensitive factor in the regulator's decision to quantify these two kinds of risks. If $\alpha > \frac{1}{2}$, the regulator will be more conservative about the current financial situation since they may anticipate that the potential shortfall risk grow due to certain extreme events in the near future. If $\alpha < \frac{1}{2}$, the regulator will be positive about the financial situation and so companies can benefit from a more flexible cash flow. Our model (3.3) will generalize the sensitive factor from constants to functions, which is more reasonable to model the variation of the risk.

Furthermore, if X in (3.2) is assumed to be the loss random variable of one business line, and c is assumed to be the capital put in the corresponding business line as in Dhaene et al. (2012), or Chapter 5 in this paper, $f_{g,h}(c, X)$ can be treated as the loss risk measure, or loss function for the business line by choosing appropriate functions for g, h, Φ_1 and Φ_2. Hence, this model can be applied to determine the appropriate capital for a single business line.

Since $(X - c)_+$ could be unlimited, it is reasonable to assume $g(x) \geq h(x)$ from the insurer's and the regulator's perspective. However, from the insured or the companies' perspective, a relative lower weight on the unlimited loss might be preferred, and so they would rather model the weight factors as $g(x) \leq h(x)$. The minimizers will be reduced to the generalized quantiles in Bellini et al. (2014) when $g(x) = \alpha$ and $h(x) = 1 - \alpha$, where $\alpha \in (0, 1)$. In model (3.3), the weights for the two parts of risks can depend on the loss X. For instance, we may modify $g_X(x)$ with higher values for large X if the right-tail needs more consideration.

In addition, we will see that the weighted expectiles defined in this generalized model (3.3) can cover the weighted premium principles defined in Heilmann (1989) and Kamps (1998). Heilmann (1989) proposed the class of weighted premium principles induced by the Esscher transform, which is defined as

$$H(X) = \frac{\mathbb{E}[Xh(X)]}{\mathbb{E}[h(X)]},$$

where $h(x) \geq 0$, $0 < \mathbb{E}[h(X)] < +\infty$ and $\mathbb{E}[Xh(X)] < +\infty$ for a loss random variable $X \geq 0$. In fact, $H(X)$ is the solution to the optimization problem:

$$H(X) = \arg\min_{c \in \mathbb{R}} \mathbb{E}[(X - c)^2 h(X)] \tag{3.4}$$

as in Kamps (1998). Furman and Zitikis (2008a) further investigated the properties of the weighted premium principle, like ordering, invariance and explicit formulas for some specially distributed loss. When $g(x) = h(x)$, and $\Phi_1 = \Phi_2 = x^2$, the objective function (3.2) is reduced to

$$f_{g,h}(c, X) = \mathbb{E}[h(X)((X-c)_+^2 + (X-c)_-^2)].$$

In this case, the optimization problem (3.3) is reduced to (3.4) since

$$(X-c)_+^2 + (X-c)_-^2 = (X-c)^2$$

always holds.

3.2 Preliminaries

The following lemma is similar to Lemma 2.1 of Mao and Cai (2016).

Lemma 3.1. *Let $g(c) = \mathbb{E}[\xi D((X-c)_+)]$ and $h(c) = \mathbb{E}[\xi D((X-c)_-)]$, where ξ is a non-negative random variable. D is a convex and increasing function defined on $\mathbb{R}^+$. Assume $g(c) < +\infty$ and $h(c) < +\infty$ for any $c \in \mathbb{R}$, then*

$$g'_+(c) = -\mathbb{E}[\xi D'_-((X-c)_+)\mathbb{I}_{\{X>c\}}],$$

$$g'_-(c) = -\mathbb{E}[\xi D'_+((X-c)_+)\mathbb{I}_{\{X\geq c\}}],$$

$$h'_+(c) = \mathbb{E}[\xi D'_+((X-c)_-)\mathbb{I}_{\{X\leq c\}}]$$

and

$$h'_-(c) = \mathbb{E}[\xi D'_-((X-c)_-)\mathbb{I}_{\{X<c\}}],$$

which are all finite. If D is differentiable with $D'(0) = 0$, then

$$g'(c) = -\mathbb{E}[\xi D'((X-c)_+)]$$

and

$$h'(c) = \mathbb{E}[\xi D'((X-c)_-)].$$

Proof. For any $c, x \in \mathbb{R}$ and $\delta \neq 0$, denote

$$w_\delta(c,x) = \frac{D((x-(c+\delta))_+) - D((x-c)_+)}{\delta}.$$

Note that $D((x)_+)$ is increasing in $x \in \mathbb{R}$ since both $D(x)$ and $(x)_+$ are increasing in $x \in \mathbb{R}$. Thus, for $\delta \in (-\infty, 0) \cup (0, \infty)$, it holds that $w_\delta(c,x) \leq 0$ for any $c, x \in \mathbb{R}$. Considering that both $D(x)$ and $(x)_+$ are convex, $D((x)_+)$ is also convex. Recall that for a convex function $u : \mathbb{R} \to \mathbb{R}$ and any given $y \in \mathbb{R}$, it holds that $(u(x) - u(y))/(x-y)$ is increasing in $x \in \mathbb{R}$. Hence,

$$w_\delta(c,x) = \frac{D((x-(c+\delta))_+) - D((x-c)_+)}{\delta} = (-1) \times \frac{D((x-(c+\delta))_+) - D((x-c)_+)}{x-(c+\delta)-(x-c)}$$

is increasing in $\delta \in (-\infty, 0) \cup (0, \infty)$. Thus, for $\delta > -1$ and $\delta \neq 0$,

$$w_{-1}(c,x) \leq w_\delta(c,x) \leq 0 \quad \textit{for all } c, x \in \mathbb{R}.$$

Noting that $\lim_{\delta \to 0^+} (x-(c+\delta))_+ = (x-c)_+$, we have that

$$\lim_{\delta \to 0^+} \frac{D((x-(c+\delta))_+) - D((x-c)_+)}{(x-(c+\delta))_+ - (x-c)_+} = D'_-((x-c)_+).$$

Also,

$$\begin{aligned}
\lim_{\delta \to 0^+} \frac{(x-(c+\delta))_+ - (x-c)_+}{\delta} &= \lim_{\delta \to 0^+} \frac{X-(c+\delta)-(X-c)}{\delta} \mathbb{I}_{\{X \geq c+\delta\}} \\
&= -\mathbb{I}_{\{X > c\}}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\lim_{\delta \to 0^+} w_\delta(c,x) &= \lim_{\delta \to 0^+} \frac{D((x-(c+\delta))_+) - D((x-c)_+)}{\delta} \\
&= \lim_{\delta \to 0^+} \frac{D((x-(c+\delta))_+) - D((x-c)_+)}{(x-(c+\delta))_+ - (x-c)_+} \times \lim_{\delta \to 0^+} \frac{(x-(c+\delta))_+ - (x-c)_+}{\delta} \\
&= -D'_-((x-c)_+)\mathbb{I}_{\{x > c\}}.
\end{aligned}$$

Since $\mathbb{E}[\xi w_{-1}(c,x)] = \mathbb{E}[\xi D((x-c)_+)] - \mathbb{E}[\xi D((x-(c-1))_+)]$ does exist and is finite,

$$\mathbb{E}[|\xi w_\delta(c,x)|] \leq -\mathbb{E}[\xi w_{-1}(c,x)] < \infty$$

for $\delta > -1$ and $\delta \neq 0$. Hence, by applying the dominated convergence theorem, we conclude that $g'_+(c)$ exists and yields

$$\begin{aligned}
g'_+(c) &= \lim_{\delta \to 0^+} \frac{g(c+\delta) - g(c)}{\delta} \\
&= \lim_{\delta \to 0^+} \mathbb{E}[\xi w_\delta(c,x)] \\
&= \mathbb{E}[\xi \lim_{\delta \to 0^+} w_\delta(c,x)] \\
&= -\mathbb{E}[\xi D'_-((x-c)_+)\mathbb{I}_{\{x>c\}}],
\end{aligned}$$

which is finite. On the other hand,

$$\lim_{\delta \to 0^-} \frac{D((x-(c+\delta))_+) - D((x-c)_+)}{(x-(c+\delta))_+ - (x-c)_+} = D'_+((x-c)_+)$$

and

$$\begin{aligned}
&\lim_{\delta \to 0^-} \frac{(x-(c+\delta))_+ - (x-c)_+}{\delta} \\
&= \lim_{\delta \to 0^-} \frac{X-(c+\delta) - (X-c)}{\delta} \mathbb{I}_{\{X \geq c\}} \\
&= -\mathbb{I}_{\{x \geq c\}}.
\end{aligned}$$

Thus,

$$\lim_{\delta \to 0^-} w_\delta(c,x) = -D'_+((x-c)_+)\mathbb{I}_{\{x \geq c\}}.$$

Similarly, $g'_-(c)$ exists, and

$$\begin{aligned}
g'_-(c) &= \lim_{\delta \to 0^-} \frac{g(c+\delta) - g(c)}{\delta} \\
&= \lim_{\delta \to 0^-} \mathbb{E}[\xi w_\delta(c,x)] \\
&= \mathbb{E}[\xi \lim_{\delta \to 0^-} w_\delta(c,x)] \\
&= -\mathbb{E}[\xi D'_+((x-c)_+)\mathbb{I}_{\{x \geq c\}}],
\end{aligned}$$

which is finite. For $h(c)$, note that $(x)_-$ is convex, and thus $D((x)_-)$ is convex as well. Hence, using the arguments similar to those for $g(c)$, it is easy to verify that

$$\begin{aligned}
h'_+(c) &= \lim_{\delta \to 0^+} \frac{f(c+\delta) - f(c)}{\delta} \\
&= \mathbb{E}[\lim_{\delta \to 0^+} \xi \times \frac{D((x-(c+\delta))_-) - D((x-c)_-)}{(x-(c+\delta))_- - (x-c)_-} \times \lim_{\delta \to 0^+} \frac{(x-(c+\delta))_- - (x-c)_-}{\delta}] \\
&= \mathbb{E}[\xi D'_+((x-c)_-)\mathbb{I}_{\{x \leq c)\}}]
\end{aligned}$$

and

$$\begin{aligned}
h'_-(c) &= \lim_{\delta \to 0^-} \frac{f(c+\delta) - f(c)}{\delta} \\
&= \mathbb{E}[\xi D'_-((x-c)_-)\mathbb{I}_{\{x<c\}}],
\end{aligned}$$

which are both finite. If D is differentiable with $D'(0) = 0$, then

$$g'_+(c) = g'_-(c) = -\mathbb{E}[\xi D'((x-c)_+)\mathbb{I}_{\{x>c\}}] = -\mathbb{E}[\xi D'((x-c)_+)]$$

since $(x-c)_+ = 0$ for $x \leq c$ and $D'(0) = 0$. Similarly, we have $h'_+(c) = h'_-(c) = \mathbb{E}[\xi D'((x-c)_-)]$. □

We recall a well-known result (Rockafellar, 2011) about convex optimization problems without constraints in the following lemma.

Lemma 3.2. *Let $f(x)$ be a convex function on R. Then, c is a minimizer of $\min_{x\in\mathbb{R}} f(x)$ if and only if $0 \in [f'_-(c), f'_+(c)]$, where $f'_-(c)$ and $f'_+(c)$ are the left and right derivatives of f at c.*

Define $G_X(x) = Pr(X < x)$ and $G_X^{-1+}(q) = \sup\{x \in \mathbb{R} : G_X(x) \leq q\}$. According to Wichura (2001), $G_X(x)$ is increasing and left-continuous, and $F_X(x) = Pr(X \leq x)$ is increasing and right-continuous. Also, $G_X^{-1+}(q)$ is increasing and right-continuous, and $F_X^{-1}(q)$ is increasing and left-continuous. Moreover,

$$G^{-1+}(q) = \lim_{u \searrow q} F^{-1}(u) = F^{-1}(q+). \tag{3.5}$$

Since $F^{-1}(q)$ is increasing, we have $F^{-1}(q+) \geq F^{-1}(q)$, or $G^{-1+}(q) \geq F^{-1}(q)$. Further,

$$G_X(x) \leq q \leq F_X(x) \tag{3.6}$$

if and only if

$$G^{-1+}(q) \leq x \leq F^{-1}(q). \tag{3.7}$$

Dhaene et al. (2012) referred the p-mixed inverse function $F_X^{-1(p)}(q)$ of the distribution function $F_X(x) = Pr(X \leq x)$ for any random variable X and $q \in (0,1)$ as

$$F_X^{-1(p)}(q) = pF_X^{-1}(q) + (1-p)F_X^{-1+}(q), \tag{3.8}$$

where $0 \leq p \leq 1$,

$$F_X^{-1}(q) = \inf\{x \in \mathbb{R} : F_X(x) \geq q\}, \tag{3.9}$$

$$F_X^{-1+}(q) = \sup\{x \in \mathbb{R} : F_X(x) \leq q\}, \tag{3.10}$$

with $\inf\{\emptyset\} = +\infty$, $\sup\{\emptyset\} = -\infty$. Then, for any random variable X and for all x with $0 < F_X(x) < 1$, there exists $p_x \in [0,1]$ such that $F_X^{-1(p_x)}(F_X(x)) = x$ since

$$F_X^{-1}(F_X(x)) \leq x \leq F_X^{-1+}(F_X(x)) = F^{-1}(F_X(x)+)$$

and by the definition of $F_X^{-1(p)}(q)$ according to (3.8).

Usually, the right-continuous reverse of the distribution function $F_X(x)$ is defined as (3.10), or (3.11) for $q \in (0,1)$. Lemma 3.3 will prove that the definition of right-continuous inverse $G_{X,g,h}^{-1+}$ for $G_{X,g,h}$ in Proposition 3.18 based on x_B in Lemma 3.3 is equivalent to x_A by (3.10), or (3.11).

Lemma 3.3. *Let X be a random variable, for $0 < q < 1$,*

$$x_A = \sup\{x \in \mathbb{R} : \ Pr(X \leq x) \leq q\} \tag{3.11}$$

and

$$x_B = \sup\{x \in \mathbb{R} : \ Pr(X < x) \leq q\}. \tag{3.12}$$

Then $x_A = x_B$.

Proof. Denote $A_q = \{x \in \mathbb{R} : \ Pr(X \leq x) \leq q\}$ and $B_q = \{x \in \mathbb{R} : \ Pr(X < x) \leq q\}$. Clearly, $A_q \subseteq B_q$ since $Pr(X < x) \leq Pr(X \leq x)$. Thus, $x_A \leq x_B$. Since $Pr(X < x) \nearrow 1$, $x_B < +\infty$. For x_B, there exist $x_n \in B_q$ such that $x_1 \leq x_2 \leq \ldots \leq x_n \leq \ldots$ and $x_n \nearrow x_B$ as $n \to \infty$ and $Pr(X < x_n) \leq q$. Thus, $\lim_{n\to\infty} Pr(X < x_n) = Pr(X < x_B)$ since $Pr(X < x)$ is left-continuous. Hence, $Pr(X < x_B) \leq q$, or $\sup B_q$ is attainable. If $F(x_B) = F(x_B^-)$, then $F(x_B) \leq q$. Thus, $x_B \in A_q$, and so $x_B \leq x_A$. Hence, $x_A = x_B$. If $F(x_B) > F(x_B^-)$, then we can conclude that $A_q = (-\infty, x_B)$. Indeed, for any $x < x_B$, $Pr(X < x) \leq q$. There exist $x_n \in (x, x_B)$ such that $x_n \searrow x$ and $Pr(X < x_n) \leq q$. Thus, $\lim_{n\to\infty} Pr(X \leq x_n - \frac{1}{n}) \leq q$, which is equivalent to $Pr(X \leq x) \leq q$. Hence, $x \in A_q$. Note that $Pr(X < x)$ is left-continuous with $Pr(X \leq x)$ as the right limit of $Pr(X < x)$ at x. $\square$

Now $G_X^{-1+}(q) = \sup\{x \in \mathbb{R} : G_X(x) \leq q\} = \sup\{x \in \mathbb{R} : F_X(x) \leq q\} = F_X^{-1+}(q)$, so the above results with regard to G_X^{-1+} can be substituted by F_X^{-1+}.

3.3 Properties of Weighted Quantiles

In this chapter, we assume that $\Phi_1, \Phi_2 : \mathbb{R}_+ \to \mathbb{R}_+$ are two non-degenerated, convex and increasing functions. Now we consider the minimization problem (3.3) with $g \geq 0$, $h \geq 0$, $0 < \mathbb{E}[g(X)] < +\infty$ and $0 < \mathbb{E}[h(X)] < +\infty$. The minimizer $c_{g,h}(X)$ of minimization problem (3.3) is called a weighted quantile of X. Moreover, we suppose both $\mathbb{E}[g(X)\Phi_1((X - x)_+)] < +\infty$ and $\mathbb{E}[h(X)\Phi_2((X - x)_-)] < +\infty$ for all $x \in \mathbb{R}$. Firstly, we will analyze the properties of the weighted quantiles in Proposition 3.4 and 3.6.

Proposition 3.4. *Let* $\Phi_1, \Phi_2 : \mathbb{R}_+ \to \mathbb{R}_+$ *be two non-degenerated, convex and increasing functions, and* $f_{g,h}(c, X)$ *be the objective function defined by* (3.2)*. Let* $c^*_{g,h} = c^*_{g,h}(X)$ *be the weighted quantile of* X*. Then, the following results hold.*

(a) *The set of minimizers of the minimization problem* (3.3) *is a closed interval, namely,*

$$\arg\min_{c\in\mathbb{R}} \ f_{g,h}(c,X) = [c^{*-}_{g,h}, \ c^{*+}_{g,h}].$$

(b) $c^*_{g,h} \in \arg\min_{c\in\mathbb{R}} \ f_{g,h}(c,X)$ *if and only if*

$$\begin{cases} \mathbb{E}[g(X)\Phi^{'}_{1-}((X-c^*_{g,h})_+)\mathbb{I}_{\{X>c^*_{g,h}\}}] \le \mathbb{E}[h(X)\Phi^{'}_{2+}((X-c^*_{g,h})_-)\mathbb{I}_{\{X\le c^*_{g,h}\}}], \\ \mathbb{E}[g(X)\Phi^{'}_{1+}((X-c^*_{g,h})_+)\mathbb{I}_{\{X\ge c^*_{g,h}\}}] \ge \mathbb{E}[h(X)\Phi^{'}_{2-}((X-c^*_{g,h})_-)\mathbb{I}_{\{X< c^*_{g,h}\}}], \end{cases}$$

where $\Phi^{'}_{i-}$ *and* $\Phi^{'}_{i+}$ *represent the corresponding left and right derivatives of* Φ_i*,* $i = 1, 2$.

(c) $c^{*+}_{g,h} = c^{*-}_{g,h}$ *if both* Φ_1 *and* Φ_2 *are strictly convex.*

(d) *If* Φ_1 *and* Φ_2 *are differentiable,* $\Phi^{'}_{1+}(0) = \Phi^{'}_{2+}(0) = 0$*, or* X *follows a continuous distribution, then the minimizers of the minimization problem* (3.3) *are the solutions to the following equation:*

$$\mathbb{E}[g(X)\Phi^{'}_{1}(X-c)_+] = \mathbb{E}[h(X)\Phi^{'}_{2}(X-c)_-]. \tag{3.13}$$

Proof. (a) Firstly, we have $f_{g,h}(c,X) < +\infty$ since $\mathbb{E}[g(X)\Phi_1((X-c)_+)]$ and $\mathbb{E}[h(X)\Phi_2((X-c)_-)]$ are finite. Obviously, $f_{g,h}(c,X)$ is non-negative and convex. For non-degenerated functions $\Phi_1(x)$ and $\Phi_2(x)$, there exists $x_0 \in \mathbb{R}$ such that $\Phi^{'}_{1+}(x_0) > 0$, which illustrates that $\Phi_1(x) \ge \Phi_1(x_0) + \Phi^{'}_{1+}(x_0)(x - x_0)$ for $x > x_0$. Moreover, as $c \to +\infty$, we have $(x-c)_+ \to 0$ and $(x-c)_- \to +\infty$. As $c \to -\infty$, we have $(x-c)_+ \to +\infty$ and $(x-c)_- \to 0$. Then, by the Monotone Convergence Theorem,

$$\lim_{c\to+\infty} f_{g,h}(c,X) = \lim_{c\to+\infty} \mathbb{E}[h_X(X)\Phi_2(X-c)_-] = +\infty$$

and

$$\lim_{c\to-\infty} f_{g,h}(c,X) = \lim_{c\to-\infty} \mathbb{E}[g_X(X)\Phi_1(X-c)_+] = +\infty$$

since both Φ_1 and Φ_2 are increasing functions. Thus, $f_{g,h}(c,X)$ is finite, non-negative and convex with

$$\lim_{c\to-\infty} f_{g,h}(c,X) = \lim_{c\to+\infty} f_{g,h}(c,X) = +\infty.$$

Hence, the set of the minimizers should be a closed interval, denoted as $[c^{*-}_{g,h},\ c^{*+}_{g,h}]$, where $c^{*-}_{g,h}$ is the lower bound and $c^{*+}_{g,h}$ is the upper bound.

(b) Note that $f_{g,h}(c,X)$ is a convex function. Hence, $c^*_{g,h}$ is the minimizer if and only if

$$0 \in \Big[\frac{\partial^- f_{g,h}}{\partial c}, \frac{\partial^+ f_{g,h}}{\partial c}\Big].$$

By Lemma 3.1, we can get

$$\begin{aligned}\frac{\partial^+ f_{g,h}}{\partial c}(c,X) = &-\mathbb{E}[g_X(X)\Phi'_{1-}((X-c)_+)\mathbb{I}_{\{X>c\}}] \\ &+\mathbb{E}[h_X(X)\Phi'_{2+}((X-c)_-)\mathbb{I}_{\{X\le c\}}]\end{aligned} \tag{3.14}$$

and

$$\begin{aligned}\frac{\partial^- f_{g,h}}{\partial c}(c,X) = &-\mathbb{E}[g_X(X)\Phi'_{1+}((X-c)_+)\mathbb{I}_{\{X\ge c\}}] \\ &+\mathbb{E}[h_X(X)\Phi'_{2-}((X-c)_-)\mathbb{I}_{\{X<c\}}].\end{aligned} \tag{3.15}$$

Then, the inequalities in (b) can be arrived at.

(c) If both Φ_1 and Φ_2 are strictly convex, then

$$g(x)\Phi_1((x-c)_+) + h(x)\Phi_2((x-c)_-)$$

is strictly convex at c. Thus, $f_{g,h}(c,X)$ is strictly convex at c. Hence, the minimizer is unique, i.e. $c^{*+}_{g,h} = c^{*-}_{g,h}$.

(d) If Φ_1 and Φ_2 are differentiable with $\Phi_{i+}^{'} = \Phi_{i-}^{'}$, $i = 1, 2$, then

$$\begin{cases} \quad \mathbb{E}[g(X)\Phi_1^{'}((X - c_{g,h}^{*})_{+})\mathbb{I}_{\{X > c_{g,h}^{*}\}}] \\ \leq \mathbb{E}[h(X)\Phi_2^{'}((X - c_{g,h}^{*})_{-})\mathbb{I}_{\{X \leq c_{g,h}^{*}\}}], \\ \quad E[g(X)\Phi_1^{'}((X - c_{g,h}^{*})_{+})\mathbb{I}_{\{X \geq c_{g,h}^{*}\}}] \\ \geq \mathbb{E}[h(X)\Phi_2^{'}((X - c_{g,h}^{*})_{-})\mathbb{I}_{\{X < c_{g,h}^{*}\}}]. \end{cases}$$

If $\Phi_{1+}^{'}(0) = \Phi_{2+}^{'}(0) = 0$, or X follows a continuous distribution ($P(X = c_{g,h}^{*}) = 0$), then the minimizers are the solutions to

$$\mathbb{E}[g(X)\Phi_1^{'}(X - c)_{+}] = \mathbb{E}[h(X)\Phi_2^{'}(X - c)_{-}].$$

□

Remark 3.5. *According to Proposition 3.4 (c), if both Φ_1 and Φ_2 are strictly convex, then $c_{g,h}^{*-} = c_{g,h}^{*+}$, which means that the minimizer is unique. For example, when $\Phi_1(x) = \Phi_2(x) = x^2$, the minimizer is unique.* □

The following proposition will provide the properties of the weighted quantile as risk measures when g and h depend on X.

Proposition 3.6. *Let $\Phi_1, \Phi_2 : \mathbb{R}_{+} \to \mathbb{R}_{+}$ be two non-degenerated, convex and increasing functions, and $f_{g,h}(c, X)$ be the objective function defined by (3.2). Let $c_{g,h}^{*-}(X)$ and $c_{g,h}^{*+}(X)$ be the lower and upper weighted quantiles of X as denoted in Proposition 3.4 (a). Then, the following properties hold.*

(a) Translation invariance: $[c_{g,h}^{*-}(X+m),\ c_{g,h}^{*+}(X+m)] = [c_{g,h}^{*-}(X)+m,\ c_{g,h}^{*+}(X)+m]$ if $g_{X+m}(X+m) = g_X(X)$ and $h_{X+m}(X+m) = h_X(X)$ for any $m \in \mathbb{R}$.

(b) Positive homogeneity: $[c_{g,h}^{*-}(\lambda X),\ c_{g,h}^{*+}(\lambda X)] = [\lambda c_{g,h}^{*-}(X),\ \lambda c_{g,h}^{*+}(X)]$ if $\Phi_1(x) = \Phi_2(x) = x^{\beta}$ with $\beta \geq 1$, $g_{\lambda X}(\lambda x) = \eta(\lambda) g_X(x)$ and $h_{\lambda X}(\lambda x) = \eta(\lambda) h_X(x)$ for any $\lambda \in \mathbb{R}_{+}$, where η is any nonnegative functional.

(c) Constancy: $c_{g,h}^{*-}(m) = c_{g,h}^{*+}(m) = m$ for any constant $m \in \mathbb{R}$.

Proof.

(a) If $g_{X+m}(X+m) = g_X(X)$ and $h_{X+m}(X+m) = h_X(X)$, then $f_{g,h}(c, X+m) = f_{g,h}(c-m, X)$. Thus, $c^{*-}_{g,h}(X+m) = c^{*-}_{g,h}(X)+m$ and $c^{*+}_{g,h}(X+m) = c^{*+}_{g,h}(X)+m$.

(b) If $\Phi_1(x) = \Phi_2(x) = x^\beta$ with $\beta \geq 1$, $\beta \geq 1$, $g_{\lambda X}(\lambda x) = \eta(\lambda) g_X(x)$ and $h_{\lambda X}(\lambda x) = \eta(\lambda) h_X(x)$, then

$$f_{g,h}(c, \lambda X) = \lambda^\beta \eta(\lambda) f_{g,h}(\frac{c}{\lambda}, X)$$

for $\lambda \in \mathbb{R}_+$.

(c) If $X = m$, $f_{g,h}$ attains its minimum 0 if and only if $c^{*-}_{g,h} = c^{*+}_{g,h} = m$, namely, $c^*_{g,h} = m$. $\square$

Remark 3.7. *Let $g_X(x) = \beta_1 \mathbb{I}_{\{x>\rho(X)\}} + \gamma_1 \mathbb{I}_{\{x\leq\rho(X)\}}$ and $h_X(x) = \beta_2 \mathbb{I}_{\{x>\rho(X)\}} + \gamma_2 \mathbb{I}_{\{x\leq\rho(X)\}}$, where ρ is a distortion risk measure. Then, the weighed quantiles satisfy the property of translation invariance since $g_{X+m}(X+m) = g_X(X)$ and $h_{X+m}(X+m) = h_X(X)$ hold when ρ is a distortion risk measure. Also, the property of positive homogeneity can be satisfied since $g_{\lambda X}(\lambda x) = g_X(x)$ and $h_{\lambda X}(\lambda x) = h_X(x)$ in this case. Therefore, the weighted quantiles based on the weight functions $g_X(x)$ and $h_X(x)$ defined by a distortion risk measures ρ will satisfy the first two properties in Proposition 3.6.* $\square$

We will investigate the property of monotonicity when g and h are not dependent on X in Proposition 3.8.

Proposition 3.8. *Let $\Phi_1, \Phi_2 : \mathbb{R}_+ \to \mathbb{R}_+$ be two non-degenerated, convex, and increasing functions, and $f_{g,h}(c, X)$ be the objective function defined by (3.2). Let $c^{*-}_{g,h}(X)$, $c^{*+}_{g,h}(X)$ be the lower and upper weighted quantiles of X as denoted in Proposition 3.4 (a). Then*

(a) Monotonicity of X: if $X \leq_{st} Y$, then $c^{*-}_{g,h}(X) \geq c^{*-}_{g,h}(Y)$ and $c^{*+}_{g,h}(X) \leq c^{*+}_{g,h}(Y)$ when $g(x)$ is increasing and $h(x)$ is decreasing.

(b) Monotonicity of g and h: $c^{*-}_{g,h}(X)$ and $c^{*+}_{g,h}(X)$ are increasing in g and decreasing in h, namely, $c^{*-}_{g_1,h}(X) \leq c^{*-}_{g_2,h}(X)$ if $g_1(x) \leq g_2(x)$ and $c^{*-}_{g,h_1}(X) \geq c^{*-}_{g,h_2}(X)$ if $h_1(x) \leq h_2(x)$.

Proof.

(a) If $g(x)$ is increasing and $h(x)$ is decreasing, then

$$-g(x)\Phi'_{1-}((x-c)_+)\mathbb{I}_{\{x>c\}} + h(x)\Phi'_{2+}((x-c)_-)\mathbb{I}_{\{x\leq c\}}$$

and

$$-g(x)\Phi'_{1+}((x-c)_+)\mathbb{I}_{\{x\geq c\}} + h(x)\Phi'_{2-}((x-c)_-)\mathbb{I}_{\{x< c\}}$$

are decreasing functions of x. Hence, if $X \leq_{st} Y$, then

$$\frac{\partial^+ f_{g,h}}{\partial c}(c,X) \geq \frac{\partial^+ f_{g,h}}{\partial c}(c,Y)$$

and

$$\frac{\partial^- f_{g,h}}{\partial c}(c,X) \geq \frac{\partial^- f_{g,h}}{\partial c}(c,Y)$$

by (3.14) and (3.15), respectively. Thus,

$$\{c\in\mathbb{R}: \frac{\partial^+ f_{g,h}}{\partial c}(c,Y)\geq 0\} \subseteq \{c\in\mathbb{R}: \frac{\partial^+ f_{g,h}}{\partial c}(c,X)\geq 0\}$$

and

$$\{c\in\mathbb{R}: \frac{\partial^- f_{g,h}}{\partial c}(c,X)\leq 0\} \subseteq \{c\in\mathbb{R}: \frac{\partial^- f_{g,h}}{\partial c}(c,Y)\leq 0\}$$

Also, we know that

$$c^{*-}_{g,h}(X) = \inf\{c\in\mathbb{R}: \frac{\partial^+ f_{g,h}}{\partial c}(c,X)\geq 0\} \tag{3.16}$$

and

$$c^{*+}_{g,h}(X) = \sup\{c\in\mathbb{R}: \frac{\partial^- f_{g,h}}{\partial c}(c,X)\leq 0\} \tag{3.17}$$

by Rockafellar (2011). Thus, $c^{*-}_{g,h}(X) \leq c^{*-}_{g,h}(Y)$ and $c^{*+}_{g,h}(X) \leq c^{*+}_{g,h}(Y)$.

(b) If $g_1(x) \le g_2(x)$, then

$$\frac{\partial^+ f_{g_1,h}}{\partial c}(c, X) \ge \frac{\partial^+ f_{g_2,h}}{\partial c}(c, X)$$

and

$$\frac{\partial^- f_{g_1,h}}{\partial c}(c, X) \ge \frac{\partial^- f_{g_2,h}}{\partial c}(c, X).$$

Similar as (a), we can get

$$\{c \in \mathbb{R} : \frac{\partial^+ f_{g_2,h}}{\partial c}(c, X) \ge 0\} \subseteq \{c \in \mathbb{R} : \frac{\partial^+ f_{g_1,h}}{\partial c}(c, X) \ge 0\}$$

and

$$\{c \in \mathbb{R} : \frac{\partial^- f_{g_1,h}}{\partial c}(c, X) \le 0\} \subseteq \{c \in \mathbb{R} : \frac{\partial^- f_{g_2,h}}{\partial c}(c, X) \le 0\}.$$

By adopting (3.16) and (3.17) again, $c^{*+}_{g_1,h} \le c^{*+}_{g_2,h}$ and $c^{*-}_{g_1,h} \le c^{*-}_{g_2,h}$. Similarly, $c^{*+}_{g,h_2} \ge c^{*+}_{g,h_1}$ and $c^{*-}_{g,h_2} \ge c^{*-}_{g,h_1}$ if $h_1(x) \le h_2(x)$. □

In the rest of this chapter, we will consider the applications of the optimizers of (3.3).

3.4 Weighted Expectiles

In this section, we attain the weighted expectiles as the minimizers when $\Phi_1(x) = \Phi_2(x) = x^2$.

Proposition 3.9. *(Minimizers with Quadratic Functions) If $\Phi_1(x) = \Phi_2(x) = x^2$, the minimizer $c^*_{g,h}(X)$ of (3.3) is the unique solution to*

$$\mathbb{E}[h_X(X)(X - c)] + \mathbb{E}[(g_X(X) - h_X(X))(X - c)_+] = 0, \tag{3.18}$$

or

$$\mathbb{E}[g_X(X)(X - c)_+] = \mathbb{E}[h_X(X)(X - c)_-], \tag{3.19}$$

or

$$c = \frac{\mathbb{E}[Xh_X(X)] + \mathbb{E}[(g_X(X) - h_X(X))(X-c)_+]}{\mathbb{E}[h_X(X)]}. \tag{3.20}$$

Proof. This result directly follows Proposition 3.4 (c) and (d). $\square$

Remark 3.10. *If $g_X(x) = h_X(x)$, then the risk measure in Proposition 3.9 is reduced to the weighted premium principle in Kamps (1989) by* (3.20). $\square$

Remark 3.11. *If $g(x) = \alpha$ and $h(x) = 1-\alpha$, where $\alpha \in (0,1)$ is a constant, the minimizer in Proposition 3.9 is reduced to be an expectile denoted by e_α, introduced by Newey and Powell (1987). As is known, e_α is the unique solution to*

$$\alpha\mathbb{E}[(X-c)_+] = (1-\alpha)\mathbb{E}[(X-c)_-].$$

In addition, e_α is a coherent risk measure if $\alpha \geq \frac{1}{2}$ and converges in weak Wasserstein distance, see Bellini et al. (2014). Obviously, the minimizer in Proposition 3.9 is a generalization of the expectile risk measure. $\square$

Definition 3.12. (Weighted Expectiles) For a random variable X, the unique solution $c^*_{g,h}(X)$ to

$$\mathbb{E}[g_X(X)(X-c)_+] = \mathbb{E}[h_X(X)(X-c)_-],$$

where $g_X(x) \geq h_X(x) \geq 0$, is called the *weighted expectile* of X.

Proposition 3.13. *(Properties of the Weighted Expectile) Let $c_X = c^*_{g,h}(X)$ be the weighted expectile of X.*

(a) Risk Loading: If $g_X(x) \geq h_X(x)$, then $c_X \geq \frac{\mathbb{E}[Xh_X(X)]}{\mathbb{E}[h_X(X)]}$. If $g_X(x) \geq h_X(x)$ and $\mathrm{Cov}(X, h_X(X)) \geq 0$, then $c_X \geq \mathbb{E}[X]$.

(b) Maximal Loss: $c_X \leq \sup X$ if $\mathrm{Cov}(X, h_X(X)) \leq 0$.

(c) *Subadditivity:* $c_{X+Y} \leq c_X + c_Y$ *if* g *and* h *satisfy* $g \geq h$,

$$\frac{\mathbb{E}[(g_{X+Y}(X+Y) - h_{X+Y}(X+Y))(X-c_X)_+]}{\mathbb{E}[h_{X+Y}(X+Y)]} \leq \frac{\mathbb{E}[(g_X(X) - h_X(X))(X-c_X)_+]}{\mathbb{E}[h_X(X)]}, \tag{3.21}$$

$$\frac{\mathbb{E}[(g_{X+Y}(X+Y) - h_{X+Y}(X+Y))(Y-c_Y)_+]}{\mathbb{E}[h_{X+Y}(X+Y)]} \leq \frac{\mathbb{E}[(g_Y(Y) - h_Y(Y))(Y-c_Y)_+]}{\mathbb{E}[h_Y(Y)]} \tag{3.22}$$

and

$$\frac{\mathbb{E}[(X+Y)h_{X+Y}(X+Y)]}{\mathbb{E}[h_{X+Y}(X+Y)]} \leq \frac{\mathbb{E}[Xh_X(X)]}{\mathbb{E}[h_X(X)]} + \frac{\mathbb{E}[Yh_Y(Y)]}{\mathbb{E}[h_Y(Y)]}. \tag{3.23}$$

Proof. (a) Since $g_X(x) \geq h_X(x)$ for any $x \in \mathbb{R}$ and $(x - c_X)_+ \geq 0$, $\mathbb{E}[(g_X(X) - h_X(X))(X - c_X)_+] \geq 0$, the conclusion is obvious. If $\mathrm{Cov}(X, h_X(X)) \geq 0$, $\frac{\mathbb{E}[Xh_X(X)]}{\mathbb{E}[h_X(X)]} \geq \mathbb{E}[X]$. Hence, $c_X \geq \mathbb{E}[X]$.

(b) If $\sup X = \infty$, then $c_X \leq \infty$ holds. If $\sup X = M < \infty$, then $X \leq M$ and $E[X] \leq M$, and it must be $c_X \leq M$. Otherwise, assume $c_X > M$, we have $E[(g_X(X) - h_X(X))(X - c_X)_+ = 0$. So $c_X = \frac{\mathbb{E}[Xh_X(X)]}{\mathbb{E}[h_X(X)]}$. And if $\mathrm{Cov}(X, h_X(X)) \leq 0$, $\mathbb{E}[h_X(X)X] \leq \mathbb{E}[h_X(X)]E[X]$. Hence, $c_X \leq \mathbb{E}[X] \leq M$, which contradicts the prior assumption. In all, $c_X \leq \sup\ X$.

(c) According to (3.20), we have

$$c_X = \frac{\mathbb{E}[Xh_X(X)]}{\mathbb{E}[h_X(X)]} + \frac{\mathbb{E}[(g_X(X) - h_X(X))(X-c_X)_+]}{\mathbb{E}[h_X(X)]},$$

$$c_Y = \frac{\mathbb{E}[Yh_Y(Y)]}{\mathbb{E}[h_Y(Y)]} + \frac{\mathbb{E}[(g_Y(Y) - h_Y(Y))(Y-c_Y)_+]}{\mathbb{E}[h_Y(Y)]}$$

and

$$\begin{aligned} c_{X+Y} = & \frac{\mathbb{E}[(g_{X+Y}(X+Y) - h_{X+Y}(X+Y))(X+Y-c_{X+Y})_+]}{\mathbb{E}[h_{X+Y}(X+Y)]} \\ & + \frac{\mathbb{E}[(X+Y)h_{X+Y}(X+Y)]}{\mathbb{E}[h_{X+Y}(X+Y)]}. \end{aligned}$$

Suppose $c_{X+Y} > c_X + c_Y$, then

$$(X+Y-c_{X+Y})_+ \leq (X+Y-c_X-c_Y)_+ \leq (X-c_X)_+ + (Y-c_Y)_+.$$

Hence,

$$\begin{aligned}
&\frac{\mathbb{E}[(g_{X+Y}(X+Y)-h_{X+Y}(X+Y))(X+Y-c_{X+Y})_+]}{\mathbb{E}[h_{X+Y}(X+Y)]}\\
\leq\ &\frac{\mathbb{E}[(g_{X+Y}(X+Y)-h_{X+Y}(X+Y))(X-c_X)_+]}{\mathbb{E}[h_{X+Y}(X+Y)]}\\
&+\frac{\mathbb{E}[(g_{X+Y}(X+Y)-h_{X+Y}(X+Y))(Y-c_Y)_+]}{\mathbb{E}[h_{X+Y}(X+Y)]}\\
\leq\ &\frac{\mathbb{E}[(g_X(X)-h_X(X))(X-c_X)_+]}{\mathbb{E}[h_X(X)]}+\frac{\mathbb{E}[(g_Y(Y)-h_Y(Y))(Y-c_Y)_+]}{\mathbb{E}[h_Y(Y)]}
\end{aligned}$$

by (3.21) and (3.22). Now we have

$$\begin{aligned}
c_{X+Y}=\ &\frac{\mathbb{E}[(g_{X+Y}(X+Y)-h_{X+Y}(X+Y))(X+Y-c_{X+Y})_+]}{\mathbb{E}[h_{X+Y}(X+Y)]}\\
&+\frac{\mathbb{E}[(X+Y)h_{X+Y}(X+Y)]}{\mathbb{E}[h_{X+Y}(X+Y)]}\\
\leq\ &\frac{\mathbb{E}[Xh_X(X)]}{\mathbb{E}[h_X(X)]}+\frac{\mathbb{E}[Yh_Y(Y)]}{\mathbb{E}[h_Y(Y)]}+\frac{\mathbb{E}[(g_X(X)-h_X(X))(X-c_X)_+]}{\mathbb{E}[h_X(X)]}\\
&+\frac{\mathbb{E}[(g_Y(Y)-h_Y(Y))(Y-c_Y)_+]}{\mathbb{E}[h_Y(Y)]}\\
=\ &c_X+c_Y,
\end{aligned}$$

which contradicts the assumption $c_{X+Y}>c_X+c_Y$. Thus, $c_{X+Y}\leq c_X+c_Y$.

$\square$

In the following corollaries, we will consider the weight functions defined by $g_X(x)=\gamma\mathbb{I}_{\{x\leq\rho(X)\}}+\beta\mathbb{I}_{\{x>\rho(X)\}}$, where ρ is a risk measure. Note that $\rho(X)$ represents a benchmark, and it can be chosen as $\mathbb{E}[X]$, $\mathrm{VaR}_{\alpha^*}(X)$, where $\alpha^*\in[0,1]$, or other appropriate risk measures. In fact, we prefer to choose the distortion risk measures and consequently, the weighted quantiles can attain good properties as risk measures. Usually, we assume $\beta\geq\gamma$ to illustrate that a higher weight will be put on the region where X is no less than the benchmark $\rho(X)$.

Corollary 3.14. *Let $h_X(x)=\lambda$ and $g_X(x)=\gamma\mathbb{I}_{\{x\leq\mathbb{E}[X]\}}+\beta\mathbb{I}_{\{x>\mathbb{E}[X]\}}$ with $\beta\geq\gamma\geq\lambda>0$, $x\in\mathbb{R}$. Assume $\mathbb{E}[X]\geq0$ and $\mathbb{E}[Y]\geq0$. Then, $c_{X+Y}\leq c_X+c_Y$. We call such a weighted expectile c_X a 3-parameter expectile denoted by $e_{\lambda,\beta,\gamma}(X)$.*

Proof. Obviously, $g_X(x)$ is increasing in x. Also, for any $x \in \mathbb{R}$, $g_{X+Y}(x) \leq g_X(x)$ and $g_{X+Y}(x) \leq g_Y(x)$ since $\mathbb{E}[X+Y] \geq \mathbb{E}[X]$, $\mathbb{E}[Y]$. Note that $c_X \geq \mathbb{E}[X]$, $c_Y \geq \mathbb{E}[Y]$ by (3.20), $h_X(x) = \lambda$, and $g_X(x) \geq h_X(x)$. Hence,

$$
\begin{aligned}
&\frac{\mathbb{E}[(g_{X+Y}(X+Y) - h_{X+Y}(X+Y))(X+Y-c_{X+Y})_+]}{\mathbb{E}[h(X+Y)]} \\
\leq &\frac{\mathbb{E}[(g_{X+Y}(X+Y) - h_{X+Y}(X+Y))(X-c_X)_+]}{\mathbb{E}[h(X+Y)]} \\
&+ \frac{\mathbb{E}[(g_{X+Y}(X+Y) - h_{X+Y}(X+Y))(Y-c_Y)_+]}{\mathbb{E}[h(X+Y)]} \\
= &\frac{1}{\lambda}\mathbb{E}[g_{X+Y}(X+Y)(X-c_X)_+] + \frac{1}{\lambda}\mathbb{E}[g_{X+Y}(X+Y)(Y-c_Y)_+] \\
&- \mathbb{E}[(X-c_X)_+ + (Y-c_Y)_+].
\end{aligned}
$$

Note that

$$
g_X(X+Y) = \begin{cases} \gamma, & X+Y \leq \mathbb{E}[X], \\ \beta, & X+Y > \mathbb{E}[X]. \end{cases}
$$

and $\mathbb{E}[X] \leq c_X$, $\mathbb{E}[Y] \leq c_Y$. Also,

$$
g_X(X) = \begin{cases} \gamma, & X \leq \mathbb{E}[X], \\ \beta, & X > \mathbb{E}[X]. \end{cases}
$$

If $X+Y > \mathbb{E}[X]$ but $X \leq c_X$, then

$$
g_{X+Y}(X+Y)(X-c_X)_+ = 0 = g_X(X)(X-c_X)_+;
$$

If $X+Y > \mathbb{E}[X]$ but $X > c_X$, then

$$
g_{X+Y}(X+Y)(X-c_X)_+ = \beta(X-c_X) = g_X(X)(X-c_X)_+.
$$

Hence, if $X+Y > \mathbb{E}[X]$, it holds that $g_X(X+Y)(X-c_X)_+ = g_X(X)(X-c_X)_+$. If $X+Y \leq \mathbb{E}[X]$ but $X \leq c_X$, then

$$
g_{X+Y}(X+Y)(X-c_X)_+ = 0 = g_X(X)(X-c_X)_+;
$$

If $X+Y \le \mathbb{E}[X]$ but $X > c_X$, then

$$g_{X+Y}(X+Y)(X-c_X)_+ = \gamma(X-c(X)) \le g_X(X)(X-c_X)_+$$

since $\gamma \le \beta$. Thus,

$$g_{X+Y}(X+Y)(X-c_X)_+ \le g_X(X)(X-c_X)_+.$$

Similarly,

$$g_{X+Y}(X+Y)(Y-c_Y)_+ \le g_Y(Y)(Y-c_Y)_+.$$

Thus, (3.21) and (3.22) can be satisfied. Also, it is trivial that (3.23) holds. Hence, $c_{X+Y} \le c_X + c_Y$ by Proposition 3.13 (c). $\square$

Corollary 3.15. *Let $e_{\lambda,\beta,\gamma}(X)$ be the 3-parameter expectile defined in Corollary 3.14. Then, $e_{\lambda,\beta,\gamma}(X) \le e_{\lambda,\beta,\gamma}(Y)$ if $X \le Y$.*

Proof. According to (3.20), $c_X \ge \mathbb{E}[X]$, $c_Y \ge \mathbb{E}[Y]$, and c_X satisfies

$$e_{\lambda,\beta,\gamma}(X) = \mathbb{E}[X] + \frac{\mathbb{E}[(g_X(X)-\lambda)(X-e_{\lambda,\beta,\gamma}(X))_+]}{\lambda}, \tag{3.24}$$

and c_Y satisfies

$$e_{\lambda,\beta,\gamma}(Y) = \mathbb{E}[Y] + \frac{\mathbb{E}[(g_Y(Y)-\lambda)(Y-e_{\lambda,\beta,\gamma}(Y))_+]}{\lambda}. \tag{3.25}$$

Also, we have

$$\begin{aligned}
&(g_X(X)-\lambda)(X-e_{\lambda,\beta,\gamma}(X))_+ \\
&= ((\gamma-\lambda)\mathbb{I}_{\{X\le\mathbb{E}[X]\}} + (\beta-\lambda)\mathbb{I}_{\{X>\mathbb{E}[X]\}})(X-e_{\lambda,\beta,\gamma}(X))_+ \\
&= (\beta-\lambda)(X-e_{\lambda,\beta,\gamma}(X))_+ \\
&\ge 0
\end{aligned}$$

and

$$\begin{aligned}
&(g_Y(Y)-\lambda)(Y-e_{\lambda,\beta,\gamma}(Y))_+ \\
&=((\gamma-\lambda)\mathbb{I}_{\{Y\le\mathbb{E}[Y]\}}+(\beta-\lambda)\mathbb{I}_{\{Y>\mathbb{E}[Y]\}})(Y-e_{\lambda,\beta,\gamma}(Y))_+ \\
&=(\beta-\lambda)(Y-e_{\lambda,\beta,\gamma}(Y))_+ \\
&\ge 0
\end{aligned}$$

since $e_{\lambda,\beta,\gamma}(X)\ge\mathbb{E}[X]$ and $e_{\lambda,\beta,\gamma}(Y)\ge\mathbb{E}[Y]$. Suppose $e_{\lambda,\beta,\gamma}(X)>e_{\lambda,\beta,\gamma}(Y)$ if $X\le Y$. Then, $(X-e_{\lambda,\beta,\gamma}(X))_+\le(Y-e_{\lambda,\beta,\gamma}(Y))_+$. Thus, the RHS of (3.24) is equal or less than the RHS of (3.25), namely, $e_{\lambda,\beta,\gamma}(X)\le e_{\lambda,\beta,\gamma}(Y)$, which contradicts with what we have supposed. Hence, $e_{\lambda,\beta,\gamma}(X)\le e_{\lambda,\beta,\gamma}(Y)$ if $X\le Y$. □

The relationship between the weighted expectile in Corollary 3.14 and the classical expectile is investigated in Corollary 3.16.

Corollary 3.16. *Let $e_{\lambda,\beta,\gamma}(X)$ be the 3-parameter expectile defined in Corollary 3.14 and $e_\alpha(X)$ be the classical expectile at confidence level α. Then, the following results hold.*

(a) $e_{\lambda,\beta,\gamma}(X)\ge e_\alpha(X)$ *for any* $\beta\ge\gamma\ge\lambda>0$ *and* $\frac{\beta}{\lambda}\ge\frac{\alpha}{1-\alpha}$.

(b) $e_{\lambda,\beta_1,\gamma}(X)\le e_{\lambda,\beta_2,\gamma}(X)$ *for any* $\beta_1\le\beta_2$, $\beta_i\ge\gamma\ge\lambda>0$, $i=1,2$.

Furthermore, if $\gamma=\alpha$ and $\lambda=1-\alpha$, where $\alpha\in[\frac{1}{2},1)$, we call such a weighted expectile c_X a 2-parameter expectile denoted by $e_{\alpha,\beta}$. Then, the following results hold.

(c) $e_{\alpha_1,\beta}(X)\ge e_{\alpha_2,\beta}(X)$ *for any* $\beta\ge\alpha_1\ge\alpha_2$.

(d) $e_{\alpha,\beta}(X)\ge e_\alpha(X)$ *for any* $\beta\ge\alpha$, *and* $e_{\alpha,\alpha}(X)=e_\alpha(X)$.

(e) $e_{\alpha,\beta_1}(X)\le e_{\alpha,\beta_2}(X)$ *for any* $\beta_1\le\beta_2$, $\beta_i\ge\alpha$, $i=1,2$.

Proof. (a) According to (3.20), $e_{\lambda,\beta,\gamma}(X)\ge\mathbb{E}[X]$ since $g_X(x)\ge h_X(x)$. Then, $e_{\alpha,\beta,\gamma}(X)$ satisfies

$$\mathbb{E}[(\beta\mathbb{I}_{\{X>\mathbb{E}[X]\}}+\gamma\mathbb{I}_{\{x\le\mathbb{E}[X]\}})(X-e_{\lambda,\beta,\gamma}(X))_+]=\lambda\mathbb{E}[(X-e_{\lambda,\beta,\gamma}(X))_-],$$

which is equivalent to

$$\frac{\beta}{\lambda}\mathbb{E}[(X - e_{\lambda,\beta,\gamma}(X))_+] = \mathbb{E}[(X - e_{\lambda,\beta,\gamma}(X))_-] \tag{3.26}$$

since

$$\begin{aligned}
\mathbb{E}[\beta\mathbb{I}_{\{X>\mathbb{E}[X]\}}(X - e_{\lambda,\beta,\gamma}(X))_+] &= \beta\mathbb{E}[(X - e_{\lambda,\beta,\gamma}(X)) \cdot \mathbb{I}_{\{X>\mathbb{E}[X]\}} \cdot \mathbb{I}_{\{X\geq e_{\lambda,\beta,\gamma}(X)\}}] \\
&= \beta\mathbb{E}[(X - e_{\lambda,\beta,\gamma}(X)) \cdot \mathbb{I}_{\{X\geq e_{\lambda,\beta,\gamma}(X)\}}] \\
&= \beta\mathbb{E}[(X - e_{\lambda,\beta,\gamma}(X))_+]
\end{aligned}$$

and

$$\mathbb{E}[\gamma\mathbb{I}_{\{x\leq\mathbb{E}[X]\}}(X-e_{\lambda,\beta,\gamma}(X))_+] = \gamma\mathbb{E}[(X-e_{\lambda,\beta,\gamma}(X))\cdot\mathbb{I}_{\{X\leq\mathbb{E}[X]\}}\cdot\mathbb{I}_{\{X>e_{\lambda,\beta,\gamma}(X)\}}] = 0.$$

And $e_\alpha(X)$ satisfies

$$\frac{\alpha}{1-\alpha}\mathbb{E}[(X - e_\alpha(X))_+] = \mathbb{E}[(X - e_\alpha(X))_-]. \tag{3.27}$$

If we suppose $e_{\lambda,\beta,\gamma}(X) < e_\alpha(X)$, the RHS of (3.26) would be equal or less than the RHS of (3.27) and the LHS of (3.26) would be no less than the RHS of (3.27). The equivalence of both sides of (3.26) and (3.27) hold if and only if $\mathbb{E}[(X - e_{\lambda,\beta,\gamma}(X))_+] = \mathbb{E}[(X - e_\alpha(X))_+] = 0$ and $\mathbb{E}[(X - e_{\lambda,\beta,\gamma}(X))_-] = \mathbb{E}[(X - e_\alpha(X))_-]$, which illustrates that $e_{\lambda,\beta,\gamma}(X) = e_\alpha(X) = \mathbb{E}[X]$, and it contradicts with $e_{\lambda,\beta,\gamma}(X) < e_\alpha(X)$. Hence, we can get either (i) RHS of (3.26) $\leq$ RHS of (3.27) and LHS of (3.26) $>$ RHS of (3.27), or (ii) RHS of (3.26) $<$ RHS of (3.27) and LHS of (3.26) $\geq$ RHS of (3.27), which is not true obviously. Hence, $e_{\lambda,\beta,\gamma}(X) \geq e_\alpha(X)$.

(b) Note that $e_{\lambda,\beta_1,\gamma}(X)$ satisfies

$$\beta_1\mathbb{E}[(X - e_{\lambda,\beta_1,\gamma}(X))_+] = \lambda\mathbb{E}[(X - e_{\lambda,\beta_1,\gamma}(X))_-],$$

β	$e_{\alpha,\beta}/\mathbb{E}[X]$	$e_{\alpha,\beta}/\mathrm{VaR}_{\alpha^*}(X)$
0.90	204.01	204.01
0.92	205.27	204.95
0.94	206.51	205.92
0.96	207.73	206.87
0.98	208.93	207.81

TABLE 3.1: $X \sim Exp(100)$ with $\alpha = 0.9$

and $e_{\lambda,\beta_2,\gamma}(X)$ satisfies

$$\beta_2\mathbb{E}[(X - e_{\lambda,\beta_2,\gamma}(X))_+] = \lambda\mathbb{E}[(X - e_{\lambda,\beta_2,\gamma}(X))_-].$$

Similar to the proof in (a), we get $e_{\lambda,\beta_1,\gamma}(X) \le e_{\lambda,\beta_2,\gamma}(X)$ for $\beta_1 \le \beta_2$.

(c) By (a).

(d) By (c).

(e) By (b).

□

Remark 3.17. *The weighed expectiles $e_{\lambda,\beta,\gamma}$ defined in Corollary 3.14 and $e_{\alpha,\beta}$ defined in Corollary 3.16 are coherent for $X \in L^{p+}$.* □

Example 3.1 and Example 3.2 illustrate the values of $e_{\alpha,\beta}$ when $\rho(X) = \mathbb{E}[X]$ and $\rho(X) = \mathrm{VaR}_\alpha(X)$. Note that $e_{\alpha,\beta}/\rho(X)$ is denoted as the 2-parameter expectile with risk measure ρ in Tables 3.1-3.6.

Example 3.1. *For $X \sim Exp(100)$ with $F_X(x) = 1 - e^{-x/100}$, $x > 0$. We consider two cases: (i) $g_X(x) = \beta\mathbb{I}_{\{x>\mathbb{E}[X]\}} + \alpha\mathbb{I}_{\{x\le\mathbb{E}[X]\}}$, $h_X(x) = 1-\alpha$. (ii) $g_X(x) = \beta\mathbb{I}_{\{x>\mathrm{VaR}_{\alpha^*}(X)\}} + \alpha\mathbb{I}_{\{x\le\mathrm{VaR}_{\alpha^*}(X)\}}$, where $\alpha^* = 0.95$, $h_X(x) = 1-\alpha$. The results are in second and third columns of Tables 3.1, 3.2 and 3.3 when $\alpha = 0.9,\ 0.8,\ 0.7$.*

Example 3.2. *For $X \sim Pareto(3, 200)$ with density function $f(x) = \frac{3\cdot 200^3}{(x+200)^4}$, $x > 0$. We consider two cases: (i) $g_X(x) = \beta\mathbb{I}_{\{x>\mathbb{E}[X]\}} + \alpha\mathbb{I}_{\{x\le\mathbb{E}[X]\}}$, $h_X(x) = 1-\alpha$. (ii) $g_X(x) = \beta\mathbb{I}_{\{x>\mathrm{VaR}_{\alpha^*}(X)\}} + \alpha\mathbb{I}_{\{x\le\mathrm{VaR}_{\alpha^*}(X)\}}$, where $\alpha^* = 0.95$, $h_X(x) = 1-\alpha$. The results are in the second and third columns of Tables 3.4, 3.5 and 3.6 when $\alpha = 0.9,\ 0.8,\ 0.7$.*

β	$e_{\alpha,\beta}/\mathbb{E}[X]$	$e_{\alpha,\beta}/\mathrm{VaR}_{\alpha^*}(X)$
0.80	160.35	160.35
0.85	163.41	162.21
0.90	166.33	164.05
0.95	169.12	165.88

TABLE 3.2: $X \sim Exp(100)$ with $\alpha = 0.8$

β	$e_{\alpha,\beta}/\mathbb{E}[X]$	$e_{\alpha,\beta}/\mathrm{VaR}_{\alpha^*}(X)$
0.70	134.68	134.68
0.75	137.81	136.31
0.80	140.78	137.93
0.85	143.61	139.53
0.90	146.31	141.13
0.95	148.89	142.71

TABLE 3.3: $X \sim Exp(100)$ with $\alpha = 0.7$

β	$e_{\alpha,\beta}/\mathbb{E}[X]$	$e_{\alpha,\beta}/\mathrm{VaR}_{\alpha^*}(X)$
0.90	254.74	254.74
0.92	257.03	256.88
0.94	259.29	259.00
0.96	261.51	261.10
0.98	263.70	263.19

TABLE 3.4: $X \sim Pareto(3, 200)$ with $\alpha = 0.9$

β	$e_{\alpha,\beta}/\mathbb{E}[X]$	$e_{\alpha,\beta}/\mathrm{VaR}_{\alpha^*}(X)$
0.80	182.16	182.16
0.85	186.86	185.93
0.90	191.39	189.65
0.95	195.77	193.35

TABLE 3.5: $X \sim Pareto(3, 200)$ with $\alpha = 0.8$

β	$e_{\alpha,\beta}/\mathbb{E}[X]$	$e_{\alpha,\beta}/\mathrm{VaR}_{\alpha^*}(X)$
0.70	144.85	144.85
0.75	149.20	147.94
0.80	153.38	151.01
0.85	157.41	154.05
0.90	161.29	157.07
0.95	165.04	160.06

TABLE 3.6: $X \sim Pareto(3, 200)$ with $\alpha = 0.7$

According to Tables 3.1-3.6, we can arrive at the following conclusions, which correspond with Corollary 3.16 (d)-(f). Note that $e^{exp}_{\alpha,\beta}/\rho(X)$ is denoted as the 2-parameter expectile when $X \sim Exp(100)$, and $e^{pareto}_{\alpha,\beta}/\rho(X)$ is denoted as the 2-parameter expectile when $X \sim Pareto(3, 200)$.

(i) Tables 3.1-3.3 and Tables 3.4-3.6 illustrate that the 2-parameter expectile increases as α increases for any fixed $\beta \geq \alpha$, ρ and X. For example,

$$e^{exp}_{0.8,0.9}/\mathbb{E}[X] = 166.33 \leq 204.01 = e^{exp}_{0.9,0.9}/\mathbb{E}[X]$$

and

$$e^{pareto}_{0.7,0.9}/\mathrm{VaR}_{0.95}(X) = 157.07 \leq 189.65 = e^{pareto}_{0.8,0.9}/\mathrm{VaR}_{0.95}(X).$$

In addition, the first rows in Tables 3.1-3.6 are equal since $e_{\alpha,\alpha}(X) = e_\alpha(X)$ whatever the risk measure ρ is. Also, in the same column, the values of 2-parameter expectiles in other rows are always larger than the values in the first row since $e_{\alpha,\beta}(X) \geq e_\alpha(X)$ for any $\beta \geq \alpha$.

(ii) In each column of the Tables 3.1-3.6, we can see that the 2-parameter expectile increases as β increases for any fixed $\alpha \leq \beta$, ρ and X. For instance, in the third column of Table 3.4,

$$e^{pareto}_{0.9,0.92}/\mathrm{VaR}_{0.95}(X) = 256.88 \leq 263.19 = e^{pareto}_{0.9,0.98}/\mathrm{VaR}_{0.95}(X).$$

(iii) By comparing the values in the second column and the third column of Tables 3.1-3.6, we find that the 2-parameter expectiles with $\rho(X) = \mathrm{VaR}_{0.95}(X)$ are always less than those with $\rho(X) = \mathbb{E}[X]$ for any fixed $\alpha \leq \beta$ and X since $\mathrm{VaR}_{0.95}(X) \geq \mathbb{E}[X]$ for $X \sim Exp(100)$ or $X \sim Pareto(3, 200)$. Note that $\rho(X)$ plays as a benchmark to determine the threshold of X with a larger value. If the benchmark increases and other assumptions in the objective function are fixed, the minimizer in our model might decrease.

(iv) Moreover, $e^{exp}_{\alpha,\beta}/\rho(X) \leq e^{pareto}_{\alpha,\beta}/\rho(X)$ is true for any fixed $\alpha \leq \beta$ when $\rho(X) = \mathrm{VaR}_{0.95}(X)$ *or* $\mathbb{E}[X]$ by checking the corresponding values for the 2-parameter expectile. For example,

$$e^{exp}_{0.9,0.96}/\mathbb{E}[X] = 207.73 \leq 261.51 = e^{pareto}_{0.9,0.96}/\mathbb{E}[X]$$

and

$$e^{exp}_{0.9,0.98}/\mathrm{VaR}_{0.95}(X) = 207.81 \leq 263.19 = e^{pareto}_{0.9,0.98}/\mathrm{VaR}_{0.95}(X),$$

see Table 3.1 and Table 3.4. Conclusions are same when we compare the values in Table 3.2 and Table 3.5, or Table 3.3 and Table 3.6.

3.5 Weighted VaRs

In this section, we consider the case when $\Phi_1(x) = \Phi_2(x) = x$, and investigate the properties of the corresponding minimizers.

Proposition 3.18. *(Minimizers with Identity Functions) Let $\Phi_1(x) = \Phi_2(x) = x$, the minimizers $c^*_{g,h}(X)$ of (3.3) are expressed as*

$$c^*_{g,h}(X) = pH^{-1}_{X,g,h}(\alpha_{X,g,h}) + (1-p)G^{-1+}_{X,g,h}(\alpha_{X,g,h}), \tag{3.28}$$

where

$$\alpha_{X,g,h} = \frac{\mathbb{E}[g(X)]}{\mathbb{E}[g(X)+h(X)]},$$

$$H^{-1}_{X,g,h}(\alpha) = \inf\{x \in \mathbb{R} : H_{X,g,h}(x) \geq \alpha\}$$

and

$$G^{-1+}_{X,g,h}(\alpha) = \sup\{x \in \mathbb{R} : G_{X,g,h}(x) \leq \alpha\},$$

with

$$H_{X,g,h}(x) = \frac{\mathbb{E}[(g(X)+h(X))\mathbb{I}_{\{X \leq x\}}]}{\mathbb{E}[g(X)+h(X)]}$$

and

$$G_{X,g,h}(x) = \frac{\mathbb{E}[(g(X)+h(X))\mathbb{I}_{\{X<x\}}]}{\mathbb{E}[g(X)+h(X)]}.$$

In particular,

$$c^*_{g,h}(X) = H^{-1}_{X,g,h}(\alpha_{X,g,h})$$

if $H^{-1}_{X,g,h}(x)$ is continuous at $\alpha_{X,g,h}$.

Proof. Following Proposition 3.4 (b), we can get that $c^*_{g,h} = c^*_{g,h}(X)$ are the solutions to

$$\begin{cases} \mathbb{E}[g(X)\mathbb{I}_{\{X>c^*_{g,h}\}}] \leq \mathbb{E}[h(X)\mathbb{I}_{\{X\leq c^*_{g,h}\}}], \\ \mathbb{E}[g(X)\mathbb{I}_{\{X\geq c^*_{g,h}\}}] \geq \mathbb{E}[h(X)\mathbb{I}_{\{X<c^*_{g,h}\}}]. \end{cases} \tag{3.29}$$

Note that (3.29) can be rewritten as

$$\frac{\mathbb{E}[g(X)+h(X))\mathbb{I}_{\{X<c^*_{g,h}\}}]}{\mathbb{E}[g(X)+h(X)]} \leq \frac{\mathbb{E}[g(X)]}{\mathbb{E}[g(X)+h(X)]} \leq \frac{\mathbb{E}[g(X)+h(X))\mathbb{I}_{\{X\leq c^*_{g,h}\}}]}{\mathbb{E}[g(X)+h(X)]}, \tag{3.30}$$

which is equivalent to

$$G_{X,g,h}(c^*_{g,h}) \leq \frac{\mathbb{E}[g(X)]}{\mathbb{E}[g(X)+h(X)]} \leq H_{X,g,h}(c^*_{g,h}). \tag{3.31}$$

It is easy to verify that $G_{X,g,h}(x)$ is increasing and left-continuous, while $H_{X,g,h}(x)$ is increasing and right-continuous. Moreover, $G_{X,g,h}(x) = \lim_{y\to x^-} H_{X,g,h}(y)$ and $G_{X,g,h}(x) \leq H_{X,g,h}(x)$. In addition, $H_{X,g,h}(x) \to 1$ as $x \to \infty$ and $H_{X,g,h}(x) \to 0$ as $x \to -\infty$ under the assumption that $\mathbb{E}[g(X)+h(X)] < \infty$. Hence, $H_{X,g,h}(x)$ is a distribution function. Let

$$H^{-1}_{X,g,h}(\alpha) = \inf\{x \in \mathbb{R} : H_{X,g,h}(x) \geq \alpha\} \tag{3.32}$$

be the left-continuous inverse of H. For $\alpha = 0$ and $\alpha = 1$, use convention $\inf\emptyset = \infty$ and $\sup\emptyset = -\infty$. For a distribution function F, we have $F^{-1}(0) = -\infty$ and $F^{-1+}(1) = \infty$. Denote

$$G^{-1+}_{X,g,h}(\alpha) = \sup\{x \in \mathbb{R} : G_{X,g,h}(x) \leq \alpha\} \tag{3.33}$$

By (3.5), we have

$$G^{-1+}_{X,g,h}(\alpha) = \lim_{u \searrow \alpha} H^{-1}_{X,g,h}(u) = H^{-1}_{X,g,h}(\alpha+).$$

Since $H^{-1}_{X,g,h}(\alpha)$ is increasing, we have

$$H^{-1}_{X,g,h}(\alpha+) \geq H^{-1}_{X,g,h}(\alpha).$$

or

$$G^{-1+}_{X,g,h}(\alpha) \geq H^{-1}_{X,g,h}(\alpha).$$

Thus, (3.31) is equivalent to

$$H^{-1}_{X,g,h}(\frac{\mathbb{E}[g(X)]}{\mathbb{E}[g(X)+h(X)]}) \leq c^*_{g,h} \leq G^{-1+}_{X,g,h}(\frac{\mathbb{E}[g(X)]}{\mathbb{E}[g(X)+h(X)]}) \tag{3.34}$$

by the equivalence of (3.6) and (3.7). Hence, (3.28) holds. □

We call the risk measures defined by (3.28) as the *weighted VaRs*. When g and h are defined as in Remark 3.7, the weighted quantiles satisfy positive homogeneity and translation invariance. We firstly provide the formulas to calculate the weighted quantiles when $\Phi_1(x) = \Phi_2(x) = x$ in Corollary 3.19 and 3.20. The corresponding weighted VaR in Corollary 3.19 is defined as a multi-parameter VaR.

Corollary 3.19. *Let $\Phi_1(x) = \Phi_2(x) = x$, $c^* = c^*_{\beta_1,\beta_2,\gamma_1,\gamma_2,\rho}$ be the solutions to (3.28) when*

$$g_X(x) = \beta_1 \mathbb{I}_{\{x>\rho(X)\}} + \gamma_1 \mathbb{I}_{\{x\leq\rho(X)\}}$$

and

$$h_X(x) = \beta_2 \mathbb{I}_{\{x>\rho(X)\}} + \gamma_2 \mathbb{I}_{\{x\leq\rho(X)\}},$$

where ρ is a risk measure, $\beta_1,\ \gamma_1,\ \beta_2,\ \gamma_2 \in \mathbb{R}_+$. Let $q_1^* = F_X(\rho(X))$ and $q_2^* = G_X(\rho(X)) = F_X(\rho(X)-)$, where $G_X(x) = Pr(X < x)$. Denote

$$\alpha_{X,g,h} = \frac{\mathbb{E}[g(X)]}{\mathbb{E}[g(X)+h(X)]} = \frac{\beta_1(1-q_1^*)+\gamma_1 q_1^*}{d^*}$$

as defined in Proposition 3.18, where $d^* = \mathbb{E}[g(X) + h(X)]$. *Then, the following results hold:*

(i) If $\alpha_{X,g,h} < \frac{(\gamma_1+\gamma_2)q_2^*}{d^*}$, $c^* = F_X^{-1(p)}(\frac{\beta_1+(\gamma_1-\beta_1)q_1^*}{\gamma_1+\gamma_2})$;

(ii) If $\alpha_{X,g,h} = \frac{(\gamma_1+\gamma_2)q_2^*}{d^*}$, $c^* = p\rho(X) + (1-p)F_X^{-1+}(\frac{\beta_1+(\gamma_1-\beta_1)q_1^*}{\gamma_1+\gamma_2})$;

(iii) If $\frac{(\gamma_1+\gamma_2)q_2^*}{d^*} < \alpha_{X,g,h} < \frac{(\gamma_1+\gamma_2)q_1^*}{d^*}$, $c^* = \rho(X)$;

(iv) If $\alpha_{X,g,h} = \frac{(\gamma_1+\gamma_2)q_1^*}{d^*}$, $c^* = pF_X^{-1}(\frac{\beta_1+(\beta_2-\gamma_2)q_1^*}{\beta_1+\beta_2}) + (1-p)\rho(X)$;

(v) If $\alpha_{X,g,h} > \frac{(\gamma_1+\gamma_2)q_1^*}{d^*}$, $c^* = F_X^{-1(p)}(\frac{\beta_1+(\beta_2-\gamma_2)q_1^*}{\beta_1+\beta_2})$;

where $p \in [0,1]$.

Proof. According to Proposition 3.18, we have

$$H_{X,g,h}(x) = \frac{\mathbb{E}[((\beta_1+\beta_2)\mathbb{I}_{\{X>\rho(X)\}} + (\gamma_1+\gamma_2)\mathbb{I}_{\{X\le\rho(X)\}})I_{\{X\le x\}}]}{d^*}$$

and

$$G_{X,g,h}(x) = \frac{\mathbb{E}[((\beta_1+\beta_2)\mathbb{I}_{\{X>\rho(X)\}} + (\gamma_1+\gamma_2)\mathbb{I}_{\{X\le\rho(X)\}})\mathbb{I}_{\{X<x\}}]}{d^*}.$$

For $H_{X,g,h}(x)$, if $x \ge \rho(X)$, then

$$H_{X,g,h}(x) = \frac{(\beta_1+\beta_2)(F_X(x) - q_1^*) + (\gamma_1+\gamma_2)q_1^*}{d^*};$$

If $x < \rho(X)$, then

$$H_{X,g,h}(x) = \frac{(\gamma_1+\gamma_2)F_X(x)}{d^*}.$$

Note that $H_{X,g,h}(x)$ is increasing and right-continuous with

$$H = H_{X,g,h}(\rho(X)) = \frac{(\gamma_1+\gamma_2)q_1^*}{d^*} \tag{3.35}$$

and $G_{X,g,h}(x)$ is increasing and left-continuous with

$$G = G_{X,g,h}(\rho(X)) = \frac{(\gamma_1+\gamma_2)q_2^*}{d^*}. \tag{3.36}$$

In addition,

$$H_{X,g,h}(\rho(X)-) = G_{X,g,h}(\rho(X)) = G$$

and

$$G_{X,g,h}(\rho(X)+) = H_{X,g,h}(\rho(X)) = H.$$

Thus, if $\alpha_{X,g,h} \geq H$, let

$$H_{X,g,h}(x) = \frac{(\beta_1+\beta_2)(F_X(x)-q_1^*)+(\gamma_1+\gamma_2)q_1^*}{d^*} = \alpha_{X,g,h},$$

then

$$H^{-1}_{X,g,h}(\alpha_{X,g,h}) = F_X^{-1}(\frac{\beta_1+(\beta_2-\gamma_2)q_1^*}{\beta_1+\beta_2});$$

If $G \leq \alpha_{X,g,h} < H$, $H^{-1}_{X,g,h}(\alpha_{X,g,h}) = \rho(X)$;

If $\alpha_{X,g,h} < G$, let

$$H_{X,g,h}(x) = \frac{(\gamma_1+\gamma_2)F_X(x)}{d^*} = \alpha_{X,g,h},$$

then

$$H^{-1}_{X,g,h}(\alpha_{X,g,h}) = F_X^{-1}(\frac{\beta_1+(\gamma_1-\beta_1)q_1^*}{\gamma_1+\gamma_2}).$$

For $G_{X,g,h}(x)$, if $x > \rho(X)$, then

$$G_{X,g,h}(x) = \frac{(\beta_1+\beta_2)(G_X(x)-q_1^*)+(\gamma_1+\gamma_2)q_1^*}{d^*};$$

If $x \leq \rho(X)$, then

$$G_{X,g,h}(x) = \frac{(\gamma_1+\gamma_2)G_X(x)}{d^*}.$$

Similarly as $H^{-1}_{X,g,h}$, if $\alpha_{X,g,h} > H$,

$$G^{-1+}_{X,g,h}(\alpha_{X,g,h}) = F_X^{-1+}(\frac{\beta_1+(\beta_2-\gamma_2)q_1^*}{\beta_1+\beta_2});$$

If $G < \alpha_{X,g,h} \leq H$, $G^{-1+}_{X,g,h}(\alpha_{X,g,h}) = \rho(X)$;

If $\alpha_{X,g,h} \leq G$,

$$G^{-1+}_{X,g,h}(\alpha_{X,g,h}) = F_X^{-1+}(\frac{\beta_1+(\gamma_1-\beta_1)q_1^*}{\gamma_1+\gamma_2}).$$

$\square$

The corresponding weighted VaRs in Corollary 3.20 is defined as 4-parameter VaRs.

Corollary 3.20. *Under the same assumptions and notations as in Corollary 3.19. Assume F_X is continuous. Let $q^* = F_X(\rho(X))$. Then, the following results hold:*

(i) If $q^* < \frac{\beta_1}{\beta_1+\gamma_2}$, $c^* = F_X^{-1(p)}(\frac{\beta_1+(\beta_2-\gamma_2)q^*}{\beta_1+\beta_2})$;

(ii) If $q^* = \frac{\beta_1}{\beta_1+\gamma_2}$, $c^* = \rho(X)$;

(iii) If $q^* > \frac{\beta_1}{\beta_1+\gamma_2}$, $c^* = F_X^{-1(p)}(\frac{\beta_1+(\gamma_1-\beta_1)q^*}{\gamma_1+\gamma_2})$;

where $p \in [0,1]$. Assume F_X is continuous and strictly increasing. Then, the following results hold:

(i) If $q^* \leq \frac{\beta_1}{\beta_1+\gamma_2}$, $c^* = F_X^{-1}(\frac{\beta_1+(\beta_2-\gamma_2)q^*}{\beta_1+\beta_2})$;

(ii) If $q^* > \frac{\beta_1}{\beta_1+\gamma_2}$, $c^* = F_X^{-1}(\frac{\beta_1+(\gamma_1-\beta_1)q^*}{\gamma_1+\gamma_2})$.

Proof. If F_X is continuous, $q^* = q_1^* = q_2^* = F_X(\rho(X))$, where q_1^* and q_2^* are as defined in Corollary 3.19. In addition,

$$G = H = \frac{(\gamma_1+\gamma_2)q^*}{d^*}$$

by (3.35) and (3.36). Hence, the results in Corollary 3.19 are reduced to be

(i) If $\alpha_{X,g,h} > \frac{(\gamma_1+\gamma_2)q^*}{d^*}$, $c^* = F_X^{-1(p)}(\frac{\beta_1+(\beta_2-\gamma_2)q^*}{\beta_1+\beta_2})$;

(ii) If $\alpha_{X,g,h} = \frac{(\gamma_1+\gamma_2)q^*}{d^*}$, $c^* = \rho(X)$;

(iii) If $\alpha_{X,g,h} < \frac{(\gamma_1+\gamma_2)q^*}{d^*}$, $c^* = F_X^{-1(p)}(\frac{\beta_1+(\gamma_1-\beta_1)q^*}{\gamma_1+\gamma_2})$;

where $p \in [0,1]$. Moreover, we have

$$\alpha_{X,g,h} > \frac{(\gamma_1+\gamma_2)q^*}{d^*} \iff q^* < \frac{\beta_1}{\beta_1+\gamma_2},$$

$$\alpha_{X,g,h} = \frac{(\gamma_1+\gamma_2)q^*}{d^*} \iff q^* = \frac{\beta_1}{\beta_1+\gamma_2}$$

and

$$\alpha_{X,g,h} < \frac{(\gamma_1+\gamma_2)q^*}{d^*} \iff q^* > \frac{\beta_1}{\beta_1+\gamma_2}.$$

If F_X is continuous and strictly increasing, then F_X^{-1} is also continuous and strictly increasing. Also,

$$F_X^{-1}(\frac{\beta_1+(\beta_2-\gamma_2)q^*}{\beta_1+\beta_2}) = F_X^{-1}(\frac{\beta_1+(\gamma_1-\beta_1)q^*}{\gamma_1+\gamma_2}) = F_X^{-1}(\frac{\beta_1}{\beta_1+\gamma_2}) = \rho(X)$$

if $q^* = \frac{\beta_1}{\beta_1+\gamma_2}$. □

Remark 3.21. *In Corollary 3.20, for $q^* < \frac{\beta_1}{\beta_1+\gamma_2}$, $c^* > F_X^{-1}(q^*)$, and if $\beta_2 \geq \gamma_2$, c^* is increasing in q^*; if $\beta_2 < \gamma_2$, c^* is decreasing in q^*. For $q^* > \frac{\beta_1}{\beta_1+\gamma_2}$, $c^* < F_X^{-1}(q^*)$, and if $\gamma_1 \geq \beta_1$, c^* is increasing in q^*; if $\gamma_1 < \beta_1$, c^* is decreasing in q^*. Hence, for any $\hat{q}_1^* = Pr(X \leq \rho_1(X))$ and $\hat{q}_2^* = Pr(X \leq \rho_2(X))$, without loss of generality, we suppose $\hat{q}_1^* \geq \hat{q}_2^*$. Denote c_1^* and c_2^* being the optimal risk measure determined by ρ_1 and ρ_1 correspondingly. For $\hat{q}_2^* > \frac{\beta_1}{\beta_1+\gamma_2}$, if $\gamma_1 \geq \beta_1$, $F_X^{-1}(\frac{\beta_1}{\beta_1+\gamma_2}) > c_1^* \geq c_2^*$; if $\gamma_1 < \beta_1$, $c_1^* \leq c_2^* < F_X^{-1}(\frac{\beta_1}{\beta_1+\gamma_2})$. For $\hat{q}_1^* < \frac{\beta_1}{\beta_1+\gamma_2}$, if $\beta_2 \geq \gamma_2$, $c_1^* \geq c_2^* > F_X^{-1}(\frac{\beta_1}{\beta_1+\gamma_2})$; if $\beta_2 < \gamma_2$, $F_X^{-1}(\frac{\beta_1}{\beta_1+\gamma_2}) < c_1^* \leq c_2^*$. If $\hat{q}_2^* \leq \frac{\beta_1}{\beta_1+\gamma_2} \leq \hat{q}_1^*$, then $c_2^* \leq F_X^{-1}(\frac{\beta_1}{\beta_1+\gamma_2}) \leq c_1^*$.* □

The following corollary illustrates the relationships between the weighted VaR and the classical VaR, and the monotonicity of the weighted VaR.

Corollary 3.22. *Let $g_X(x) = \beta\mathbb{I}_{\{x>\rho(X)\}} + \alpha\mathbb{I}_{\{x\leq\rho(X)\}}$ and $h_X(x) = 1-\alpha$, where $\beta \geq \alpha$ and $\beta \geq 1-\alpha$ with $\alpha \in (0,1)$. We call such a weighted VaR a 2-parameter VaR denoted by $\mathrm{VaR}_{\alpha,\beta}(X)$ as the solution to (3.28). Assume F_X is strictly increasing and continuous. Then, the following results hold.*

(a) $\mathrm{VaR}_{\alpha,\beta}(X) \geq \mathrm{VaR}_\alpha(X)$ *for any $\beta \geq \alpha$ and $\beta \geq 1-\alpha$, and* $\mathrm{VaR}_{\alpha,\alpha}(X) = \mathrm{VaR}_\alpha(X)$.

(b) $\mathrm{VaR}_{\alpha,\beta_1}(X) \leq \mathrm{VaR}_{\alpha,\beta_2}(X)$ *for any $\beta_1 \leq \beta_2$, $\beta_i \geq \alpha$, $\beta_i \geq 1-\alpha$, $i = 1, 2$.*

(c) $\mathrm{VaR}_{\alpha,\beta}(X) \le \mathrm{VaR}_{\alpha,\beta}(Y)$ *if* $\rho(X) = \mathrm{VaR}_{\alpha^*}(X)$ *and* $X \le Y$*, where* $\alpha^* \in (0,1)$.

Proof. (a) Note that

$$\mathrm{VaR}_{\alpha,\beta}(X) = \begin{cases} F_X^{-1}(\frac{\beta}{\beta+1-\alpha}), & q^* \le \frac{\beta}{\beta+1-\alpha}, \\ F_X^{-1}(\beta(1-q^*)+\alpha q^*), & q^* > \frac{\beta}{\beta+1-\alpha}, \end{cases}$$

where q^* is defined as in Corollary 3.20. If $q^* \le \frac{\beta}{\beta+1-\alpha}$, $\frac{\beta}{\beta+1-\alpha} \ge \frac{\alpha}{\alpha+1-\alpha} = \alpha$ since $f(x) = \frac{x}{x+1-\alpha}$ is an increasing function of x. And when $q^* > \frac{\beta}{\beta+1-\alpha}$, $\beta(1-q^*)+\alpha q^* \ge \alpha(1-q^*)+\alpha q^* = \alpha$. Hence, $\mathrm{VaR}_{\alpha,\beta}(X) \ge \mathrm{VaR}_{\alpha}(X)$ for any $\beta \ge \alpha$ and $\beta \ge 1-\alpha$.

(b) In fact, we firstly have $\frac{\beta_1}{\beta_1+1-\alpha} \le \frac{\beta_2}{\beta_2+1-\alpha}$. In addition,

$$\mathrm{VaR}_{\alpha,\beta_1}(X) = \begin{cases} F_X^{-1}(\frac{\beta_1}{\beta_1+1-\alpha}), & q^* \le \frac{\beta_1}{\beta_1+1-\alpha}, \\ F_X^{-1}(\beta_1(1-q^*)+\alpha q^*), & \frac{\beta_1}{\beta_1+1-\alpha} < q^* \le \frac{\beta_2}{\beta_2+1-\alpha}, \\ F_X^{-1}(\beta_1(1-q^*)+\alpha q^*), & q^* > \frac{\beta_2}{\beta_2+1-\alpha}. \end{cases}$$

and

$$\mathrm{VaR}_{\alpha,\beta_2}(X) = \begin{cases} F_X^{-1}(\frac{\beta_2}{\beta_2+1-\alpha}), & q^* \le \frac{\beta_1}{\beta_1+1-\alpha}, \\ F_X^{-1}(\frac{\beta_2}{\beta_2+1-\alpha}), & \frac{\beta_1}{\beta_1+1-\alpha} < q^* \le \frac{\beta_2}{\beta_2+1-\alpha}, \\ F_X^{-1}(\beta_2(1-q^*)+\alpha q^*), & q^* > \frac{\beta_2}{\beta_2+1-\alpha}. \end{cases}$$

by Corollary 3.20. If $q^* \le \frac{\beta_1}{\beta_1+1-\alpha}$, then $\frac{\beta_1}{\beta_1+1-\alpha} \le \frac{\beta_2}{\beta_2+1-\alpha}$ since $f(x) = \frac{x}{x+1-\alpha}$ is an increasing function of x. If $\frac{\beta_1}{\beta_1+1-\alpha} < q^* \le \frac{\beta_2}{\beta_2+1-\alpha}$, then $\beta_1(1-q^*) + \alpha q^* = \beta_1 + (\alpha-\beta_1)q^* \le \beta_1 + (\alpha-\beta_1)\frac{\beta_1}{\beta_1+1-\alpha} = \frac{\beta_1}{\beta_1+1-\alpha} \le \frac{\beta_2}{\beta_2+1-\alpha}$. If $q^* > \frac{\beta_2}{\beta_2+1-\alpha}$, $\beta_1(1-q^*)+\alpha q^* \le \beta_2(1-q^*)+\alpha q^*$. Thus, $\mathrm{VaR}_{\alpha,\beta_1}(X) \le \mathrm{VaR}_{\alpha,\beta_2}(X)$.

(c) It is trivial, since $q^* = \alpha^*$, and $\mathrm{VaR}_{q^*}(X) \le \mathrm{VaR}_{q^*}(Y)$ if $X \le Y$.

$\square$

β	$\frac{\beta}{\beta+1-\alpha}$	$\mathrm{VaR}_{\alpha,\beta}/\mathbb{E}[X]$	$\mathrm{VaR}_{\alpha,\beta}/\mathrm{VaR}_{\alpha^*}(X)$	c^{**}	$MS_\alpha(X)$
0.90	0.90	230.26	230.26	152.35	299.57
0.92	0.9020	232.23	231.26		
0.94	0.9038	234.18	232.28		
0.96	0.9056	236.09	233.30		
0.98	0.9074	237.95	234.34		

TABLE 3.7: $X \sim Exp(100)$ with $\alpha = 0.9$

β	$\frac{\beta}{\beta+1-\alpha}$	$\mathrm{VaR}_{\alpha,\beta}/\mathbb{E}[X]$	$\mathrm{VaR}_{\alpha,\beta}/\mathrm{VaR}_{\alpha^*}(X)$	c^{**}	$MS_\alpha(X)$
0.80	0.8000	160.94	160.94	120.40	230.26
0.85	0.8095	165.82	162.20		
0.90	0.8182	170.47	163.48		
0.95	0.8261	174.92	164.77		

TABLE 3.8: $X \sim Exp(100)$ with $\alpha = 0.8$

β	$\frac{\beta}{\beta+1-\alpha}$	$\mathrm{VaR}_{\alpha,\beta}/\mathbb{E}[X]$	$\mathrm{VaR}_{\alpha,\beta}/\mathrm{VaR}_{\alpha^*}(X)$	c^{**}	$MS_\alpha(X)$
0.7	0.7000	120.40	120.40	102.98	189.71
0.75	0.7143	125.28	121.23		
0.80	0.7273	129.93	122.08		
0.85	0.7391	134.37	122.93		
0.90	0.7500	138.63	123.79		
0.95	0.7600	142.71	124.65		

TABLE 3.9: $X \sim Exp(100)$ with $\alpha = 0.7$

We now employ Example 3.3 and Example 3.4 to illustrate the results in Corollary 3.22 for Exponential and Pareto distributed random variables with the same mean. Note that $\mathrm{VaR}_{\alpha,\beta}/\rho(X)$ is denoted as the 2-parameter VaR with risk measure ρ.

Example 3.3. *For $X \sim Exp(100)$ with $F_X(x) = 1 - e^{-x/100}$, $x > 0$. We consider two cases: (i) $g_X(x) = \beta\mathbb{I}_{\{x>\mathbb{E}[X]\}} + \alpha\mathbb{I}_{\{x\le\mathbb{E}[X]\}}$, $h_X(x) = 1-\alpha$, and $q^* = 1 - e^{-1} = 0.6321$. (ii) $g_X(x) = \beta\mathbb{I}_{\{x>\mathrm{VaR}_{\alpha^*}(X)\}} + \alpha\mathbb{I}_{\{x\le\mathrm{VaR}_{\alpha^*}(X)\}}$, $h_X(x) = 1-\alpha$, where $\alpha^* = 0.95$, and $q^* = \alpha^*$. The results are in the third and fourth columns of Tables 3.7, 3.8 and 3.9 when $\alpha = 0.9,\ 0.8,\ 0.7$.*

Example 3.4. *For $X \sim Pareto(3, 200)$ with density function $f(x) = \frac{3\cdot 200^3}{(x+200)^4}$, $x > 0$ and $F_X^{-1}(p) = 200((1-p)^{-\frac{1}{3}} - 1)$. We consider two cases: (i) $g_X(x) = \beta\mathbb{I}_{\{x>\mathbb{E}[X]\}} + \alpha\mathbb{I}_{\{x\le\mathbb{E}[X]\}}$, $h_X(x) = 1-\alpha$ and $q^* = 1 - (\frac{200}{100+200})^3 = 0.7037$. (ii) $g_X(x) = \beta\mathbb{I}_{\{x>\mathrm{VaR}_{\alpha^*}(X)\}} + \alpha\mathbb{I}_{\{x\le\mathrm{VaR}_{\alpha^*}(X)\}}$, $h_X(x) = 1-\alpha$, where $\alpha^* = 0.95$ and*

β	$\frac{\beta}{\beta+1-\alpha}$	$\mathrm{VaR}_{\alpha,\beta}/\mathbb{E}[X]$	$\mathrm{VaR}_{\alpha,\beta}/\mathrm{VaR}_{\alpha^*}(X)$	c^{**}	$MS_\alpha(X)$
0.90	0.9000	230.89	230.89	173.12	342.8
0.92	0.9020	233.74	232.33		
0.94	0.9038	236.56	233.80		
0.96	0.9056	239.34	235.28		
0.98	0.9074	242.08	236.79		

TABLE 3.10: $X \sim Pareto(3, 200)$ with $\alpha = 0.9$

β	$\frac{\beta}{\beta+1-\alpha}$	$\mathrm{VaR}_{\alpha,\beta}/\mathbb{E}[X]$	$\mathrm{VaR}_{\alpha,\beta}/\mathrm{VaR}_{\alpha^*}(X)$	c^{**}	$MS_\alpha(X)$
0.80	0.8000	142.00	142.00	119.01	230.88
0.85	0.8095	147.60	143.43		
0.90	0.8182	153.03	144.89		
0.95	0.8261	158.30	146.39		

TABLE 3.11: $X \sim Pareto(3, 200)$ with $\alpha = 0.8$

β	$\frac{\beta}{\beta+1-\alpha}$	$\mathrm{VaR}_{\alpha,\beta}/\mathbb{E}[X]$	$\mathrm{VaR}_{\alpha,\beta}/\mathrm{VaR}_{\alpha^*}(X)$	c^{**}	$MS_\alpha(X)$
0.7	0.7000	98.76	98.76	93.77	176.41
0.75	0.7143	103.66	99.59		
0.80	0.7273	108.40	100.44		
0.85	0.7391	113.01	101.29		
0.90	0.7500	117.48	102.16		
0.95	0.7600	121.83	103.03		

TABLE 3.12: $X \sim Pareto(3, 200)$ with $\alpha = 0.7$

$q^* = \alpha^*$. The results are in the third and fourth columns of Tables 3.10, 3.11 and 3.12 when $\alpha = 0.9,\ 0.8,\ 0.7$.

Now we will compare the risk measure determined by Corollary 3.20 with other risk measures in literature.

Remark 3.23. *Let $c_q^* = pF_X^{-1}(q) + (1-p)F_X^{-1+}(q)$ and $\rho(X) = F_X^{-1}(q)$, where $q \in (0,1)$, then $q^* = q$. In Corollary 3.20, for any given $p \in [0,1]$, we can get $c^* > c_q^*$ if $q < \frac{\beta_1}{\beta_1+\gamma_2}$; $c^* < c_q^*$ if $q > \frac{\beta_1}{\beta_1+\gamma_2}$; $c^* = c_q^*$ if $q = \frac{\beta_1}{\beta_1+\gamma_2}$.* □

Remark 3.24. *Theorem 1 in Heras et al. (2012) derived a class of risk measure from the pairwise minimizer $(c^{**}, \tau^{**}) = \arg\min_{(c,\tau)\in\mathbb{R}^2} V(c,\tau)$, where $V(c,\tau) = \tau + \frac{1}{1-q}\int_0^\infty (|c-x| - \tau)_+ f(x)dx$. If $F_X(x)$ is continuous, $c^{**} = \frac{1}{2}[F_X^{-1}(\frac{1+q}{2}) + F_X^{-1}(\frac{1-q}{2})]$. If $q \geq \frac{1}{3}$, $\frac{1-q}{2} \leq q \leq \frac{1+q}{2}$. Let $\overline{q}_2 = F_X(c^{**})$. Since $F_X(x)$ is continuous, $F_X^{-1}(\overline{q}_2) = \frac{1}{2}[F_X^{-1}(\frac{1+q}{2}) + F_X^{-1}(\frac{1-q}{2})]$. Let $\rho_1(X) = \rho_2(X) = \mathrm{VaR}_q(X) = F_X^{-1}(q)$. Now $q^* = q$ and $c^* = F_X^{-1}(\overline{q}_1)$, where $\overline{q}_1 = \frac{\beta_1+(\beta_2-\gamma_2)q^*}{\beta_1+\beta_2}$ if $q \leq \frac{\beta_1}{\beta_1+\gamma_2}$ and $\overline{q}_1 =$*

$\frac{\beta_1+(\gamma_1-\beta_1)q^*}{\gamma_1+\gamma_2}$ *if* $q > \frac{\beta_1}{\beta_1+\gamma_2}$. *The relationship between* c^{**} *and* c^* *depends on the values of the parameters in the model assumption.* $\square$

Example 3.5. *For* $X \sim Exp(100)$, $c^{**} = \frac{1}{2}[F_X^{-1}(\frac{1+\alpha}{2})+F_X^{-1}(\frac{1-\alpha}{2})] = -50\ln(\frac{1-\alpha^2}{4})$. *For* $X \sim Pareto(3,200)$, $c^{**} = \frac{1}{2}[F_X^{-1}(\frac{1+\alpha}{2})+F_X^{-1}(\frac{1-\alpha}{2})] = 100((\frac{1-\alpha}{2})^{-\frac{1}{3}}+(\frac{1+\alpha}{2})^{-\frac{1}{3}}-2)$. *We will calculate* c^{**} *when* $\alpha = 0.9,\ 0.8,\ 0.7$ *in the fifth column in Table 3.7-3.12.*

Remark 3.25. *Kuo et al. (2013) proposed the median shortfall of a risk* X *as* $MS_q(X) = \mathrm{VaR}_{\frac{1+q}{2}}(X)$ *at the confidence level* $q \in (0,1)$. *In Corollary 3.20 (i), if* $\rho(X) = \mathrm{VaR}_q(X)$ *and* F_X *is continuous, then* $c^* = F_X^{-1}(\frac{\beta_1(1-q)-\gamma_2 q}{\beta_1+\beta_2}+q)$. *Let* $\beta_1+\beta_2 = 2$, *where* $\beta_1 > 1$, *and* $\frac{\beta_1-1}{\gamma_2} = \frac{q}{1-q}$. *Then,* $\frac{\beta_1(1-q)-\gamma_2 q}{\beta_1+\beta_2}+q = \frac{\beta_1(1-q)-\gamma_2 q}{2}+q = \frac{\beta_1(1-q)-(1-q)(\beta_1-1)}{2}+q = \frac{1-q}{2}+q = \frac{1+q}{2}$. *Hence,* $MS_q(X)$ *is a special case of the risk measure proposed in Corollary 3.20, which provides another interpretation of* $MS_q(X)$. $\square$

Example 3.6. *For* $X \sim Exp(100)$, $MS_\alpha(X) = F_X^{-1}(\frac{1+\alpha}{2}) = 100\ln(\frac{1-\alpha}{2})$. *For* $X \sim Pareto(3,200)$, $MS_\alpha(X) = 200((\frac{1-\alpha}{2})^{-\frac{1}{3}}-1)$. *We will calculate* $MS_\alpha(X)$ *when* $\alpha = 0.9,\ 0.8,\ 0.7$ *in the sixth column in Table 3.7-3.12.*

Remark 3.26. *If* $\beta_1 = \gamma_1 = \alpha$ *and* $\beta_2 = \gamma_2 = 1-\alpha$, *then* $g(x) = \alpha$ *and* $h(x) = 1-\alpha$ *in Corollary 3.20. For (i) and (iii),* $\frac{\beta_1(1-q)-\gamma_2 q}{\beta_1+\beta_2}+q = \frac{\beta_1(1-q)+\gamma_1 q}{\gamma_1+\gamma_2} = \alpha$. *For (ii), the condition* $\beta_1(1-q) = \gamma_2 q$ *illustrates that* $q = \alpha$. *Hence, we can conclude* $c^* = F_X^{-1(p)}(\alpha)$ *in this case.* $\square$

According to Tables 3.7-3.12, we can obtain the following conclusions. Note that $\mathrm{VaR}^{exp}_{\alpha,\beta}/\rho(X)$ is denoted as the 2-parameter VaR when $X \sim Exp(100)$ and $\mathrm{VaR}^{pareto}_{\alpha,\beta}/\rho(X)$ is denoted as the 2-parameter VaR when $X \sim Pareto(3,200)$.

(i) Tables 3.7-3.9 and Tables 3.10-3.12 illustrate that the 2-parameter VaR increases as α increases for any fixed $\beta \geq \alpha$, ρ and X. For example,

$$\mathrm{VaR}^{exp}_{0.8,0.9}/\mathbb{E}[X] = 170.47 \leq 230.26 = \mathrm{VaR}^{exp}_{0.9,0.9}/\mathbb{E}[X]$$

and

$$\mathrm{VaR}^{pareto}_{0.7,0.9}/\mathrm{VaR}_{0.95}(X) = 102.16 \leq 144.89 = \mathrm{VaR}^{pareto}_{0.8,0.9}/\mathrm{VaR}_{0.95}(X).$$

In addition, the first rows in Tables 3.7-3.12 are equal since $\mathrm{VaR}_{\alpha,\alpha}(X) = \mathrm{VaR}_{\alpha}(X)$ whatever the risk measure ρ is. Also, in the same column, the values of 2-parameter VaR in other rows are always larger than the values in the first row since $\mathrm{VaR}_{\alpha,\beta}(X) \geq \mathrm{VaR}_{\alpha}(X)$ for any $\beta \geq \alpha$.

(ii) In each column of the Tables 3.7-3.12, we can see that the 2-parameter VaR increases as β increases for any fixed $\alpha \leq \beta$, ρ and X. For instance, in the third column of Table 3.10,

$$\mathrm{VaR}^{pareto}_{0.9,0.92}/\mathrm{VaR}_{0.95}(X) = 233.74 \leq 242.08 = \mathrm{VaR}^{pareto}_{0.9,0.98}/\mathrm{VaR}_{0.95}(X).$$

(iii) By comparing the values in the second column and the third column of Tables 3.7-3.12, we find that the 2-parameter VaRs with $\rho(X) = \mathrm{VaR}_{0.95}(X)$ are always less than those with $\rho(X) = \mathbb{E}[X]$ for any fixed $\alpha \leq \beta$ and X due to the same reason for the 2-parameter expectiles.

(iv) Also, $\mathrm{VaR}^{exp}_{\alpha,\beta}/\rho(X) \leq \mathrm{VaR}^{pareto}_{\alpha,\beta}/\rho(X)$ for any fixed $\beta \geq \alpha$ when $\alpha = 0.9$ and $\rho(X) = \mathrm{VaR}_{0.95}(X)$ *or* $\mathbb{E}[X]$. However, when $\alpha = 0.8$ *or* 0.7, $\mathrm{VaR}^{exp}_{\alpha,\beta}/\rho(X) \geq \mathrm{VaR}^{pareto}_{\alpha,\beta}/\rho(X)$ when $\rho(X) = \mathrm{VaR}_{0.95}(X)$ *or* $\mathbb{E}[X]$. In this case, this type of monotonicity is different from the 2-parameter expectiles.

(v) In the fifth column of the tables, we find that c^{**} is lower than the 2-parameter VaR and $MS_\alpha(X)$ with $\alpha = 0.7,\ 0.8,\ 0.9$. Meanwhile, $MS_\alpha(X)$ is always higher than the 2-parameter VaR and c^{**} with the same $\alpha = 0.7,\ 0.8,\ 0.9$. Hence, we can conclude that the risk measure proposed in Kuo et al. (2013) is more conservative, though it satisfies the property of ellipticity. Moreover, the 2-parameter VaR lies between c^{**} and $MS_\alpha(X)$ when $\alpha \geq 0.7$ in our examples.

Chapter 4

New Premium Principles for Pricing Reinsurance Contracts

4.1 Introduction

It is well known that an optimal reinsurance problem usually has four components: the support set of the ceded functions, the insurance premium principle, the reinsurance premium principle, and the objective function under certain risk measures. In the literature, scholars usually investigate the optimization problem by deriving optimal ceded loss functions when the reinsurance premium principles are given. The contributions of Cai and Tan (2007), Cai et al. (2008), and Tan et al. (2009) are notable since they firstly derived the optimal ceded functions with the stop-loss or quota-share type under VaR and TVaR. In fact, the partial hedging methods under VaR and TVaR in Cong et al. (2013, 2014) are developed based on these ideas. Following their work, Chi and Tan (2011) simplified the proof, and Cheung (2010) provided a geometric approach. Then, Chi and Weng (2013) studied the problem with ceded functions subject to the Vajda condition. Cai et al. (2013) further did research on the optimal reciprocal reinsurance under the joint survival probability and the joint profitable probability. By analyzing the optimal reinsurance problem considering the initial capital and default risk, two layer stop-loss ceded functions are established as the optimal reinsurance strategies in Cai et al. (2014). Moreover, the optimal reinsurance problem based on the expectiles is researched in Cai and Weng (2014). After that, Cai et al. (2015) determined an optimal reinsurance design with two types of constraints under the convex combination of the VaR risk measures.

Whereas most of the former results are based on the expected value premium principle, Kaluszka (2001) studied the optimal reinsurance problem under the mean-variance premium principle. Then, the VaR and CTE constrained reinsurance premium principles are adopted by Zhou et al. (2011) to find the optimal reinsurance with a combination of the quota-share and stop-loss ceded functions. Building on their work, Chi and Tan (2013) considered the optimization problem under a general premium principle. A new method is presented by Cui et al. (2013) to illustrate the optimal reinsurance strategy on a tradeoff between the insurer's risk under distortion risk measures and a general reinsurance premium principle. Additionally, Zheng and Cui (2014), and Zheng et al. (2014) further discussed the optimal reinsurance with premium constraint and expected reinsurance premium under distortion risk measures. To date, the dependence structure is considered by Cai and Wei (2012), and they analyzed the optimal reinsurance problem with positively dependent risks.

In most studies on optimal reinsurance, as we have concluded, the reinsurance premium principles are assumed to be given, and the problem is to find the optimal ceded loss function under certain constraints. However, in practice, an insurer may negotiate the type of the ceded loss functions with the reinsurers, and then the reinsurers determine the premium of the contracts. Finally, the insurer will decide which contract to purchase. Hence, we have to consider the following problem: given a general ceded function in the objective function, what should the optimal reinsurance premium principle be?

Let $X \geq 0$ be the initial total loss faced by the insurance company, $I(X)$ be the ceded loss function, or the loss covered by the reinsurer, and c be the reinsurance premium. $I(X)$ can be stop-loss, quota-share or other version of functionals, but it lies within the function set $\mathfrak{I}$ below,

$$\mathfrak{I} = \{I : 0 \leq I(X) \leq X, \ for \ any \ X \geq 0\}.$$

In this problem, the insurer's loss is defined as $X - I(X) + c$ and reinsurer's loss is defined as $(I(X) - c)_+$. Let $\varphi_1, \varphi_2 : [0, +\infty) \to [0, +\infty)$ be the functions adopted to quantify the risks for the reinsurer and insurer, respectively.

For the insurance company, X is the initial total loss. By purchasing the reinsurance contract with the reinsurance premium rate at c, $I(X)$ is ceded to the reinsurance company. Thus, the total loss for the insurer is $X - I(X) + c$. The expectation of the total quantitative loss for the insurer is $\mathbb{E}[h(X)\varphi_2(X - I(X) + c)]$ by adopting the quantifying function φ_2 and weight function $h(x)$. Therefore, $\mathbb{E}[h(X)\varphi_2(X - I(X) + c)]$ is the risk measure of the insurer in this contract. Moreover, in this reinsurance contract, the reinsurer receives the reinsurance premium c and covers loss $I(X)$, and hence, the residual loss is $(I(X) - c)_+$, which is a decreasing function of c. Namely, $(I(X) - c)_+$ decreases as c increases. Consequently, the expectation of the total quantitative loss faced by the reinsurer is $\mathbb{E}[g(X)\varphi_1((I(X) - c)_+)]$ by employing the quantifying function φ_1 and weight function $g(x)$. Note that the weight functions $g(x)$ and $h(x)$ are non-negative functions with $0 < \mathbb{E}[g(X)] < \infty$ and $0 < \mathbb{E}[h(X)] < \infty$ for random variable X. Our goal is to find the optimal premium c such that the weighted risk measure or weighted expected quantitative loss of both the insurer and reinsurer can be minimized. Therefore, the objective function is

$$f_1(c) = \mathbb{E}[g(X)\varphi_1((I(X) - c)_+)] + \mathbb{E}[h(X)\varphi_2(X - I(X) + c)]. \tag{4.1}$$

The optimal reinsurance premium should be the minimizer for this objective function. Namely, this optimal reinsurance premium is

$$c^*_{X,I,g,h} = \arg\min_{c \in R} f_1(c). \tag{4.2}$$

Additionally, if the weight functions $g(x)$ and $h(x)$ are non-negative constants, the objective function is reduced to

$$f_2(c) = \alpha \mathbb{E}[\varphi_1((I(X) - c)_+)] + (1 - \alpha)\mathbb{E}[\varphi_2(X - I(X) + c)], \tag{4.3}$$

where $\alpha = \frac{g}{g+h} \in (0, 1)$ is the weight on the reinsurer's loss while $1 - \alpha$ is the weight on the insurer's loss. Now $c^*_{X,I,g,h}$ can be rewritten as $c^*_{X,I,\alpha}$ with confidence level α. In this case, the optimal reinsurance premium is denoted as $c^*_\alpha(I_1(X))$ for risk X with ceded function I_1 and $c^*_\alpha(I_2(Y))$ for risk Y with ceded function I_2. Then, the premium is denoted as $c^*_\alpha(I_1(X) + I_2(Y))$ for the combined ceded loss $I_1(X) + I_2(Y)$ in Proposition 4.8 (a).

4.2 Preliminaries

Lemma 3.1 in Chapter 3 has provided a formula to determine the first derivative of the function $g(c) = E[D((X - c)_+)]$, where D is a convex and nondecreasing function defined on $\mathbb{R}^+$.

Lemma 4.1. *Let $g(c) = \mathbb{E}[\xi D((X - c)_+)]$, where ξ is a non-negative random variable. D is a convex and increasing function defined on $\mathbb{R}^+$. Assume $g(c) < +\infty$ for any $c \in \mathbb{R}$, then*

$$g'_+(c) = -\mathbb{E}[\xi D'_-((X - c)_+)\mathbb{I}_{\{X>c\}}],$$

$$g'_-(c) = -\mathbb{E}[\xi D'_+((X - c)_+)\mathbb{I}_{\{X\geq c\}}],$$

and both of them are finite. If D is differentiable with $D'(0) = 0$, then

$$g'(c) = -\mathbb{E}[\xi D'((X-c)_+)].$$

Lemma 3.2 in Chapter 3 introduced a result about the well-known conclusion for the minimizers of a convex program with no constraints according to Rockafellar (2011).

Lemma 4.2. *Let $f(x)$ be a convex function on R. Then c is a minimizer of $\min_{x\in\mathbb{R}} f(x)$ if and only if $0 \in [f'_-(c), f'_+(c)]$, where $f'_-(c)$ and $f'_+(c)$ are the left and right derivatives of f at c.*

4.3 Optimal Reinsurance Premiums and Their Properties

In this chapter, we assume φ_1, φ_2: $[0,+\infty) \to [0,+\infty)$ are non-degenerated, convex and increasing functions. In addition, the objective function defined in (4.1) satisfies $g \geq 0$, $h \geq 0$, $0 < \mathbb{E}[g(X)] < +\infty$, $0 < \mathbb{E}[h(X)] < +\infty$, $\mathbb{E}[g(X)\varphi_1((I(X)-x)_+)] < \infty$, and $\mathbb{E}[h(X)\varphi_2(X - I(X) + x)] < \infty$ for any $x \in R$.

Theorem 4.3. *Let $\varphi_1, \varphi_2 : \mathbb{R}_+ \to \mathbb{R}_+$ be two non-degenerated, convex and increasing functions, and $f_1(c)$ be the objective function defined by (4.1). The optimal reinsurance premium principles $c^*_{X,I,g,h}$ do exist and are the solutions to*

$$\begin{cases} \mathbb{E}[h(X)\varphi'_{2+}(X - I(X) + c)] \geq \mathbb{E}[g(X)\varphi'_{1-}((I(X)-c)_+)\mathbb{I}_{\{I(X)>c\}}], \\ \mathbb{E}[g(X)\varphi'_{1+}((I(X)-c)_+)\mathbb{I}_{\{I(X)\geq c\}}] \geq \mathbb{E}[h(X)\varphi'_{2-}(X - I(X) + c)], \end{cases} \tag{4.4}$$

where φ'_{i-} and φ'_{i+} represent the corresponding left and right derivatives of φ_i, $i = 1, 2$. If φ_1 and φ_2 are strictly convex, the solution is unique. If φ_1 and φ_2 are differentiable and $\varphi'_1(0) = \varphi'_2(0) = 0$, the optimal reinsurance premium principles

$c^*_{X,I,g,h}$ *are solutions to*

$$\mathbb{E}[h(X)\varphi_2'(X - I(X) + c)] = \mathbb{E}[g(X)\varphi_1'((I(X) - c)_+)]. \tag{4.5}$$

Proof. Similar to Proposition 3.4 (a) in Chapter 3, we can get that $f_1(c)$ is finite, non-negative, convex, and satisfies

$$\lim_{c\to-\infty} f_1(c) = \lim_{c\to+\infty} f_1(c) = +\infty.$$

And the remaining results can be concluded by Lemma 4.1 and Lemma 4.2. □

Remark 4.4. *If g and h are non-negative constants,* (4.4) *is reduced to*

$$\begin{cases} (1-\alpha)\mathbb{E}[\varphi_{2+}'(X - I(X) + c)] \geq \alpha\mathbb{E}[\varphi_{1-}'((I(X) - c)_+)\mathbb{I}_{\{I(X)>c\}}], \\ \alpha\mathbb{E}[\varphi_{1+}'(I(X) - c)_+\mathbb{I}_{\{I(X)\geq c\}}] \geq (1-\alpha)\mathbb{E}[\varphi_{2-}'(X - I(X) + c)], \end{cases} \tag{4.6}$$

where $\alpha = \frac{g}{g+h}$*. And* (4.5) *is reduced to*

$$(1-\alpha)\mathbb{E}[\varphi_2'(X - I(X) + c)] = \alpha\mathbb{E}[\varphi_1'((I(X) - c)_+)], \tag{4.7}$$

where $\alpha = \frac{g}{g+h}$*. Here,* $c^*_{X,I,g,h}$ *is rewritten as* $c^*_{X,I,\alpha}$*.* □

4.4 Reinsurance Premium Principles with Quadratic Functions

4.4.1 Formula and Properties for Reinsurance Premium Principles with Quadratic Functions

When the quantifying functions are quadratic functions, the formula to derive the new reinsurance premium principle is provided in Proposition 4.5. Proposition 4.6 will study the properties when the weight functions are functions of X, and Proposition 4.8 is for the case with constant weights.

Proposition 4.5. *(Reinsurance Premium Principles with Quadratic Functions) If $\varphi_1(x) = \varphi_2(x) = x^2$, the optimal reinsurance premium $c^*_{X,I,g,h}$ is the unique solution to*

$$\mathbb{E}[h(X)]c = \mathbb{E}[g(X)(I(X) - c)_+] - \mathbb{E}[h(X)(X - I(X))], \tag{4.8}$$

or

$$c = \frac{\mathbb{E}[g(X)(I(X) - c)_+] - \mathbb{E}[h(X)(X - I(X))]}{\mathbb{E}[h(X)]}. \tag{4.9}$$

If g and h are non-negative constants, (4.8) is reduced to

$$(1 - \alpha)c = \alpha\mathbb{E}[(I(X) - c)_+] - (1 - \alpha)\mathbb{E}[X - I(X)], \tag{4.10}$$

where $\alpha = \frac{g}{g+h}$.

Proof. If $\varphi_1(x) = \varphi_2(x) = x^2$, $\varphi_1'(0) = \varphi_2'(0) = 0$. Also, φ_1 and φ_2 are strictly convex. Hence, the results follow Theorem 4.3. □

Note that $c^*_{X,I,g,h}$ has three random components: X, the non-negative underlying risk; I, the ceded loss function; and g and h, the weight functions, or the constant weight α in (4.10). Firstly, we will introduce the properties of $c^*_{X,I,g,h}$ with respect to X and I in Proposition 4.6 and Proposition 4.8.

Proposition 4.6. *(Properties of the Reinsurance Premium Principles with Quadratic Functions-1) Let $c^* = c^*_{X,I,g,h}$ be the unique solution to (4.8).*

(a) Monotonicity of I: $c^*_{X,I_1,g,h} \leq c^*_{X,I_2,g,h}$ if $I_1(X) \leq I_2(X)$.

(b) Translation invariance: $c^*_{X,I+d,g,h} = c^*_{X,I,g,h} + d$ for any $d \in R$, where $(I + d)(X) = I(X) + d$.

(c) Maximal loss: $c^*_{X,I,g,h} \leq \sup\ I(X)$.

(d) Monotonicity of g and h:

(i) For fixed $h(x)$, $c^*_{X,I,g_1,h} \leq c^*_{X,I,g_2,h}$ if $g_1(x) \leq g_2(x)$;

(ii) For fixed $g(x)$, $c^*_{X,I,g,h_1} \geq c^*_{X,I,g,h_2}$ if $h_1(x) \leq h_2(x)$;

(iii) If the weight functions $g_i(x)$ and $h_i(x)$ satisfy $g_i(x) + h_i(x) = 1$ for $i = 1, 2$, then $c^*_{X,I,g_1,h_1} \leq c^*_{X,I,g_2,h_2}$ if $g_1(x) \leq g_2(x)$.

Proof. (a) Firstly write $c_1 = c^*_{X,I_1,g,h}$ and $c_2 = c^*_{X,I_2,g,h}$. If $I_1(X) \leq I_2(X)$, assume $c_1 > c_2$, then $I_1(X) - c_1 < I_2(X) - c_2$. Thus,

$$(I_1(X) - c_1)_+ \leq (I_2(X) - c_2)_+.$$

In addition, by Proposition 4.5, we have

$$\mathbb{E}[h(X)]c_1 = \mathbb{E}[g(X)(I_1(X) - c_1)_+] - \mathbb{E}[h(X)(X - I_1(X))]$$

and

$$\mathbb{E}[h(X)]c_2 = \mathbb{E}[g(X)(I_2(X) - c_2)_+] - \mathbb{E}[h(X)(X - I_2(X))].$$

Hence,

$$\begin{aligned}\mathbb{E}[h(X)](c_1 - c_2) = \mathbb{E}[g(X)((I_1(X) - c_1)_+ - (I_2(X) - c_2)_+)] \\ + \mathbb{E}[h(X)(I_1(X) - I_2(X))] \leq 0.\end{aligned}$$

Since both g and h are nonnegative functions, $c_1 - c_2 \leq 0$, thus contradicting what has been assumed. Hence, $c_1 \leq c_2$.

(b) For the ceded loss $I_d(X) = I(X) + d$, denote the corresponding premium as c^*_d, which is the unique solution to

$$\mathbb{E}[h(X)]c_d = \mathbb{E}[g(X)(I_d(X) - c_d)_+] - \mathbb{E}[h(X)(X - I_d(X))],$$

which is equivalent to

$$\mathbb{E}[h(X)](c_d - d) = \mathbb{E}[g(X)(I(X) - (c_d - d))_+] - \mathbb{E}[h(X)(X - I(X))].$$

Since c^*, the premium with the ceded function I, is the unique solution to

$$\mathbb{E}[h(X)]c = \mathbb{E}[g(X)(I(X) - c)_+] - \mathbb{E}[h(X)(X - I(X))],$$

$c^*_d - d$ must be equal to c^*. Hence, $c^*_d = c^* + d$ for any $d \in R$.

(c) If $\sup\ I(X) = \infty$, then $c^* \leq \sup\ I(X)$. And if $\sup\ I(X) = M < \infty$, then $I(X) \leq M$ and $\mathbb{E}[I(X)] \leq M$. Note that $c \leq M$ must hold now. Otherwise, assume $c > M$, then $\mathbb{E}[g(X)(I(X) - c)_+] = 0$. Thus, c^* will satisfy

$$\mathbb{E}[h(X)]c = \mathbb{E}[h(X)(I(X) - X)]. \tag{4.11}$$

Now, the LHS of (4.11) is strictly greater than $M\mathbb{E}[h(X)]$ since $\mathbb{E}[h(X)] > 0$, while $\mathbb{E}[h(X)(I(X) - X)] \leq M\mathbb{E}[h(X)] - \mathbb{E}[Xh(X)] \leq M\mathbb{E}[h(X)]$ since $h(x)$ is non-negative and $X \geq 0$, which yields a contradiction.

(d) (i) Denote $c_1 = c^*_{X,I,g_1,h}$ and $c_2 = c^*_{X,I,g_2,h}$. By (4.9), c_i, is the solution to

$$c = \frac{\mathbb{E}[g_i(X)(I(X) - c)_+] - \mathbb{E}[h(X)(X - I(X))]}{\mathbb{E}[h(X)]}, \tag{4.12}$$

$i = 1, 2$. Suppose $c_1 > c_2$, then

$$\mathbb{E}[g_1(X)(I(X) - c_1)_+] > \mathbb{E}[g_2(X)(I(X) - c_2)_+] \tag{4.13}$$

by (4.12). Also, if $g_1(x) \leq g_2(x)$, then

$$\mathbb{E}[g_1(X)(I(X) - c_1)_+] \leq \mathbb{E}[g_2(X)(I(X) - c_2)_+],$$

which contradicts (4.13). Thus, $c_1 > c_2$ if $g_1(x) \leq g_2(x)$.

(ii) Denote $c_1 = c^*_{X,I,g,h_1}$ and $c_2 = c^*_{X,I,g,h_2}$. By (4.9), c_i is solution to

$$c = \frac{\mathbb{E}[g(X)(I(X) - c)_+] - \mathbb{E}[h_i(X)(X - I(X))]}{\mathbb{E}[h_i(X)]},$$

$i = 1, 2$. Suppose $c_1 < c_2$, then

$$\begin{aligned}&\frac{\mathbb{E}[g(X)(I(X)-c_1)_+] - \mathbb{E}[h_1(X)(X - I(X))]}{\mathbb{E}[h_1(X)]} \\ <\ &\frac{\mathbb{E}[g(X)(I(X)-c_1)_+] - \mathbb{E}[h_2(X)(X - I(X))]}{\mathbb{E}[h_2(X)]}.\end{aligned}$$

However,

$$\begin{aligned}&\frac{\mathbb{E}[g(X)(I(X)-c_1)_+] - \mathbb{E}[h_1(X)(X - I(X))]}{\mathbb{E}[h_1(X)]} \\ \geq\ &\frac{\mathbb{E}[g(X)(I(X)-c_2)_+] - \mathbb{E}[h_2(X)(X - I(X))]}{\mathbb{E}[h_2(X)]}\end{aligned}$$

since $(I(X) - c_1)_+ \geq (I(X) - c_2)_+$, $X - I(X) \geq 0$, $0 \leq h_1(x) \leq h_2(x)$ and $0 < \mathbb{E}[h_1(X)] \leq \mathbb{E}[h_2(X)]$. Therefore, $c_1 \geq c_2$ if $h_1 \leq h_2$.

(iii) If $g_i(x) + h_i(x) = 1$, $i = 1, 2$, then denote $c_1 = c^*_{X,I,g_1,h_1}$ and $c_2 = c^*_{X,I,g_2,h_2}$. By (4.9), c_i is the solution to

$$c = \frac{\mathbb{E}[g_i(X)(I(X)-c)_+] - \mathbb{E}[(1-g_i(X))(X - I(X))]}{\mathbb{E}[1-g_i(X)]}. \tag{4.14}$$

Suppose $c_1 > c_2$, which is equivalent to

$$\begin{aligned}&\frac{\mathbb{E}[g_1(X)(I(X)-c_1)_+] - \mathbb{E}[(1-g_1(X))(X - I(X))]}{\mathbb{E}[1-g_1(X)]} \\ >\ &\frac{\mathbb{E}[g_2(X)(I(X)-c_2)_+] - \mathbb{E}[(1-g_2(X))(X - I(X))]}{\mathbb{E}[1-g_2(X)]}\end{aligned}$$

by (4.14). However,

$$\begin{aligned}&\frac{\mathbb{E}[g_1(X)(I(X)-c)_+] - \mathbb{E}[(1-g_1(X))(X - I(X))]}{\mathbb{E}[1-g_1(X)]} \\ \leq\ &\frac{\mathbb{E}[g_2(X)(I(X)-c)_+] - \mathbb{E}[(1-g_2(X))(X - I(X))]}{\mathbb{E}[1-g_2(X)]}\end{aligned}$$

since $(I(X)-c_1)_+ \leq (I(X)-c_2)_+$, $0 \leq g_1 \leq g_2 \leq 1$ and $0 < \mathbb{E}[g_i(X)] < 1$, $i = 1, 2$. Hence, $c_1 \leq c_2$ in this case.

$\square$

Remark 4.7. *Note that Proposition 4.6 (d) illustrates that $c_{X,I,g,h}$ is a monotone function of g and h. For the unique solution to (4.10), we can conclude that $c^*_{X,I,\alpha_1} \le c^*_{X,I,\alpha_2}$ for any fixed X and I if $\alpha_1 \le \alpha_2$.* □

Proposition 4.8. *(Properties of the Reinsurance Premium Principles with Quadratic Functions-2) Let $c^*_\alpha(I(X)) = c^*_{X,I,\alpha}$ be the unique solution to (4.10).*

(a) Subadditivity: $c^*_\alpha(I_1(X) + I_2(Y)) \le c^*_\alpha(I_1(X)) + c^*_\alpha(I_2(Y))$.

(b) Risk Loading: $c^*_{X,I,\alpha} \ge 0$ if $\alpha \ge \alpha_1$; And $c^*_{X,I,\alpha} \ge E[I(X)]$ if $\alpha \ge \alpha_2$, where

$$\alpha_1 = 1 - \frac{\mathbb{E}[I(X)]}{\mathbb{E}[X]}, \tag{4.15}$$

and

$$\alpha_2 = 1 - \frac{\mathbb{E}[(I(X) - \mathbb{E}[I(X)])_+]}{\mathbb{E}[(I(X) - E[I(X)])_+] + \mathbb{E}[X]}. \tag{4.16}$$

In addition, $\alpha_2 \ge \alpha_1$ holds.

Proof. (a) Write c_s as the optimal reinsurance premium for $I_s(X+Y) = I_1(X) + I_2(Y)$ and the total loss $X+Y$, c_1 and c_2 as the optimal reinsurance premiums for $I_1(X)$ and $I_2(Y)$. Then

$$(1-\alpha)c_s = \alpha\mathbb{E}[(I_s(X+Y) - c_s)_+] - (1-\alpha)\mathbb{E}[X+Y-I_s(X+Y)],$$

$$(1-\alpha)c_1 = \alpha\mathbb{E}[(I_1(X) - c_1)_+] - (1-\alpha)\mathbb{E}[X - I_1(X)],$$

and

$$(1-\alpha)c_2 = \alpha\mathbb{E}[(I_2(Y) - c_2)_+] - (1-\alpha)\mathbb{E}[Y - I_2(Y)].$$

Thus,

$$\begin{aligned}(1-\alpha)(c_s - c_1 - c_2) = {} & \alpha\mathbb{E}[(I_1(X) + I_2(Y) - c_s)_+] \\ & - \alpha\mathbb{E}[(I_1(X) - c_1)_+] - \alpha\mathbb{E}[(I_2(Y) - c_2)_+].\end{aligned} \tag{4.17}$$

If

$$c_s > c_1 + c_2$$

and

$$\begin{aligned}\mathbb{E}[(I_1(X)+I_2(Y)-c_s)_+] &\leq \mathbb{E}[(I_1(X)+I_2(Y)-c_1-c_2)_+] \\ &\leq \mathbb{E}[(I_1(X)-c_1)_+]+\mathbb{E}[(I_2(Y)-c_2)_+],\end{aligned}$$

the LHS of (4.17) is positive; however, the RHS of (4.17) is non-positive. Hence, $c_s \leq c_1+c_2$.

(b) If

$$\alpha \geq 1-\frac{\mathbb{E}[I(X)]}{\mathbb{E}[X]},$$

$f_2'(0)<0$, and then $c \geq 0$ since $f_2'(c)=0$, and $f_2(c)$ is convex with $f_2''(c) \geq 0$. Similarly, if

$$\alpha \geq 1-\frac{\mathbb{E}[(I(X)-\mathbb{E}[I(X)])_+]}{\mathbb{E}[(I(X)-\mathbb{E}[I(X)])_+]+\mathbb{E}[X]},$$

$f_2'(\mathbb{E}(I(X)))<0$, and so $c \geq \mathbb{E}[I(X)]$. Since $\mathbb{E}[I(X)] \geq 0$ and $\mathbb{E}[(I(X)-\mathbb{E}[I(X)])_+] \geq 0$,

$$\frac{\mathbb{E}[I(X)]}{\mathbb{E}[X]} \geq \frac{\mathbb{E}[(I(X)-\mathbb{E}[I(X)])_+]}{\mathbb{E}[(I(X)-\mathbb{E}[I(X)])_+]+\mathbb{E}[X]}.$$

Given that

$$\mathbb{E}[I(X)] \geq \mathbb{E}[(I(X)-\mathbb{E}[I(X)])_+]$$

and

$$\mathbb{E}[(I(X)-\mathbb{E}[I(X)])_+]+\mathbb{E}[X] \geq \mathbb{E}[X],$$

then

$$\frac{\mathbb{E}[I(X)]}{\mathbb{E}[X]} \geq \frac{\mathbb{E}[(I(X)-\mathbb{E}[I(X)])_+]}{\mathbb{E}[(I(X)-\mathbb{E}[I(X)])_+]+\mathbb{E}[X]}.$$

Hence, $\alpha_2 \geq \alpha_1$.

□

Remark 4.9. In practice, the underlying loss random variables may follow different distributions since the insurance companies insure various types of risks. For example, the risks insured by the casualty insurance contracts might vary from the risks insured by the life or health insurance contracts. As a result, the ceded *functions adopted by the reinsurance companies are different for these various risks. To describe the distinction, we denote two types of the underlying risks as X and Y in Proposition 4.8 (a). And the corresponding ceded function are defined as $I_1(x)$ and $I_2(x)$. For example, we can choose $I_1(x) = (x-d)_+$ and $I_2(x) = kx$, where $d \in R_+$, $k \in [0,1]$. Then, $I_1(X) + I_2(Y)$ is the combined loss ceded to the reinsurer. Proposition 4.8 (a) illustrates that the premium will be lower if the insurance companies buy one reinsurance contract for different risks together by the pricing formula (4.10). Ceding $I_1(X) + I_2(Y)$ to the reinsurer is a win-win situation for both the insurer and reinsurer. The insurer will pay a lower reinsurance premium and so save on the main costs. Adopting the premium based on the combined ceded loss allows the quantity of purchases for this kind of reinsurance product to grow, and thus, the reinsurer can gain more profits. Also, this combination is quite flexible since one can combine n different losses together by choosing n corresponding ceded functions. Furthermore, if we choose $I_1(X) = (X-d_1)_+$, $I_2(Y) = (Y-d_2)_+$ and $I_3(X+Y) = (X+Y-d_3)_+$. If $d_3 \geq d_1+d_2$, $(X+Y-d_3)_+ \leq (X+Y-d_1-d_2)_+ \leq (X-d_1)_+ + (Y-d_2)_+$, which illustrates $I_3(X+Y) \leq I_1(X)+I_2(Y)$, then $c^*_\alpha(I_3(X+Y)) \leq c^*_\alpha(I_1(X))+c(I_2(Y))$.* □

Remark 4.10. *According to Proposition 4.8 (b), the premium is non-negative (if $\alpha \geq \alpha_1$) and no-less than $\mathbb{E}[I(X)]$ (if $\alpha \geq \alpha_2$). Hence, α_1 can be viewed as the minimum weight factor acceptable for the reinsurer, and α_2 is the preferred threshold of the weight factor for the reinsurer. Cai et al. (2015) suggested that one possible general method for optimal reinsurance designs is to maximize or minimize objective functions under certain constraints based on another party's goal. Thus, these two thresholds for α are just the constraints based on the reinsurer's various objects, and the optimal reinsurance premium is attainable by (4.10).* □

4.4.2 Relationship with the Reinsurance Premium Principle Based on the Expectiles

For the non-negative random variable X, the relationship between the new reinsurance premium with quadratic functions and the reinsurance premium based on the expectiles will be proposed in Proposition 4.11 and Corollary 4.12. The expectile, denoted as $e_\alpha(X)$, is defined as

$$e_\alpha(X) = \arg\min_{c\in\mathbb{R}} \ \alpha\mathbb{E}[(X-c)_+^2] + (1-\alpha)\mathbb{E}[(X-c)_-^2], \tag{4.18}$$

where $\alpha \in (0,1)$.

As is known, the expectile $e_\alpha(X)$ is the unique solution to

$$e_\alpha(X) = \mathbb{E}[X] + \frac{2\alpha-1}{1-\alpha}\mathbb{E}[(X-e_\alpha(X))_+].$$

For the ceded loss $I(X)$, the reinsurance premium based on the expectiles is denoted as $e_\alpha(I(X))$. The reinsurance premium based on the expectiles is always non-negative and is no less than $\mathbb{E}[I(X)]$ if $\alpha \geq \frac{1}{2}$. Moreover, the premium equals $\mathbb{E}[I(X)]$ if $\alpha = \frac{1}{2}$.

Additionally, we can always arrive at the following proposition about the relationship between the reinsurance premium with quadratic functions and the reinsurance premium based on the expectiles for $X \geq 0$.

Proposition 4.11. *Let $\varphi_1(x) = \varphi_2(x) = x^2$ and $c^* = c^*_{X,I,\beta_1,\gamma_1,\beta_2,\gamma_2,q}$ be the reinsurance premium by (4.9) when*

$$g(x) = \beta_1\mathbb{I}_{\{I(x)>\rho(I(X))\}} + \gamma_1\mathbb{I}_{\{I(x)\leq\rho(I(X))\}}$$

and

$$h(x) = \beta_2\mathbb{I}_{\{I(x)>\rho(I(X))\}} + \gamma_2\mathbb{I}_{\{I(x)\leq\rho(I(X)))\}},$$

where ρ is a risk measure. Let $e_\alpha(I(X))$ be the reinsurance premium based on the expectiles. Then,

$$c^* \leq e_\alpha(I(X)) \tag{4.19}$$

when $\beta_2 = \gamma_2 = \beta$ and $\beta_1,\ \gamma_1 \leq \frac{\alpha}{1-\alpha}\beta$.

Proof. By rewriting the equations, we can see that $e_\alpha = e_\alpha(I(X))$ is the solution to

$$e_\alpha = \mathbb{E}[I(X)] - \mathbb{E}[(I(X) - e_\alpha)_+] + \frac{\alpha}{1-\alpha}\mathbb{E}[(I(X) - e_\alpha)_+].$$

Note that c^* is the solution to

$$c = \frac{\mathbb{E}[g(X)(I(X) - c)_+]}{\beta} - \mathbb{E}[X] + \mathbb{E}[I(X)]$$

when $\beta_2 = \gamma_2 = \beta$. Then, if $\beta_1,\ \gamma_1 \leq \frac{\alpha}{1-\alpha}\beta$, we have

$$c^* \leq \frac{\alpha}{1-\alpha}\mathbb{E}[(I(X) - c^*)_+] - \mathbb{E}[X] + \mathbb{E}[I(X)].$$

If $c^* > e_\alpha(I(X))$, then

$$\frac{\alpha}{1-\alpha}\mathbb{E}[(I(X) - c^*)_+] \leq \frac{\alpha}{1-\alpha}\mathbb{E}[(I(X) - e_\alpha(I(X)))_+].$$

Also, we have $I(X) - e_\alpha(I(X)) \leq I(X) \leq X$ for $X \geq 0$. Hence, $\mathbb{E}[(I(X) - e_\alpha(I(X)))_+] \leq \mathbb{E}[X]$. Thus, $c^* \leq e_\alpha(I(X))$, which contradicts $c^* > e_\alpha(I(X))$ as assumed. Therefore, $c^* \leq e_\alpha(I(X))$. □

Corollary 4.12. *Let $c^*_{X,I,\alpha}$ be the reinsurance premium by (4.10) and $e_\alpha(I(X))$ be the reinsurance premium based on the expectiles. Then,*

$$c^*_{X,I,\alpha} \leq e_\alpha(I(X)). \tag{4.20}$$

Proof. Let $\beta_1 = \gamma_1 = \frac{\alpha}{1-\alpha}\beta$ in Proposition 4.11. □

Remark 4.13. *Note that (4.20) means that the new reinsurance premium principle with quadratic quantifying functions and constant weights is always lower than the reinsurance premium based on the expectiles at the same confidence level α. The reinsurance premium principle based on the expectiles according to (4.18) considers only the potential risks from the reinsurance contract, namely, the ceded loss. In (4.18), $(I(X) - c)_+$ is the residual loss faced by the reinsurer, and $(c - I(X))_+$ is the overcharged loss for the insurer. Hence, the reinsurance premiums based on the expectiles are optimizers to minimize the combination of these risks. However, in* *the new reinsurance premium formula, $X - I(X) + c = (c - I(X))_+ + X - (c - I(X))_-$ is the loss faced by the insurer as a combination of the overpaid loss $(c - I(X))_+$ and $X - (c - I(X))_- = X - (I(X) - c)_+ \geq 0$. In this case, more information is included in the optimization problem (4.2). Hence, the minimizer based on the new model might be lower than the reinsurance premium based on the expectiles. Here, $c^*_{X,I,\alpha}$, the minimizer of (4.3), is suggested as a reasonable reinsurance premium principle for two reasons. Firstly, (4.3) covers more risk from the insurer's point of view. Secondly, the new premium formula provides a more competitive price.* $\square$

4.5 Reinsurance Premium Principles with Identity Functions

4.5.1 Formula and Properties for Reinsurance Premium Principles with Identify Functions

In this section, we will talk about the class of reinsurance premium principles when the quantifying functions are identity functions if $g(x) \geq h(x)$, see Proposition 4.14. Furthermore, we express a special class of the principles as the mixed inverses of the distributions of $I(X)$ at different confidence levels in Corollary 4.16.

Proposition 4.14. *(Reinsurance Premium Principles with Identity Functions) If $\varphi_1(x) = \varphi_2(x) = x$, the reinsurance premium $c^*_{X,I,g,h}$ are the solutions to*

$$\mathbb{E}[g(X)\mathbb{I}_{\{I(X)>c\}}] \leq \mathbb{E}[h(X)] \leq \mathbb{E}[g(X)\mathbb{I}_{\{I(X)\geq c\}}]. \tag{4.21}$$

If $g(x) \geq h(x)$, then (4.21) is equivalent to

$$c^*_{X,I,g,h} = F^{-1(p)}_{X,I,g}\left(\frac{\mathbb{E}[g(X)] - \mathbb{E}[h(X)]}{\mathbb{E}[g(X)]}\right), \tag{4.22}$$

where

$$F^{-1(p)}_{X,I,g}(\alpha) = pH^{-1}_{X,I,g}(\alpha) + (1-p)G^{-1+}_{X,I,g}(\alpha)$$

with $p \in [0,1]$,

$$H^{-1}_{X,I,g}(\alpha) = \inf\{x \in R : H_{X,I,g} \geq \alpha\}$$

and

$$G^{-1+}_{X,I,g}(\alpha) = \sup\{x \in R : G_{X,I,g} \leq \alpha\},$$

where

$$H_{X,I,g}(x) = \frac{\mathbb{E}[g(X)\mathbb{I}_{\{I(X)\leq x\}}]}{\mathbb{E}[g(X)]}$$

and

$$G_{X,I,g}(x) = \frac{\mathbb{E}[g(X)\mathbb{I}_{\{I(X)< x\}}]}{\mathbb{E}[g(X)]}.$$

In addition,

$$c^*_{X,I,g,h} = H^{-1}_{X,I,g}(\frac{\mathbb{E}[g(X)] - \mathbb{E}[h(X)]}{\mathbb{E}[g(X)]}),$$

if $H^{-1}_{X,I,g}(x)$ *is continuous at* $\frac{\mathbb{E}[g(X)]-\mathbb{E}[h(X)]}{\mathbb{E}[g(X)]}$.

Proof. The equation (4.4) in Theorem 4.3 can be rewritten as (4.21), which is equivalent to

$$G_{X,I,g}(c) \leq \frac{\mathbb{E}[g(X)] - \mathbb{E}[h(X)]}{\mathbb{E}[g(X)]} \leq H_{X,I,g}(c). \tag{4.23}$$

It is easy to verify that $G_{X,I,g}(x)$ is increasing and left-continuous, while $H_{X,I,g}(x)$ is increasing and right-continuous. Moreover, $G_{X,I,g}(x) = \lim_{y\to x^-} H_{X,I,g}(y)$ and $G_{X,I,g}(x) \leq H_{X,I,g}(x)$ In addition, $H_{X,I,g}(x) \to 1$ as $x \to \infty$ and $H_{X,I,g}(x) \to 0$ as $x \to -\infty$ under the assumption that $\mathbb{E}[g(X)] < \infty$. Hence, $H_{X,I,g}(x)$ is a distribution function. Let $H^{-1}_{X,I,g}(\alpha) = \inf\{x \in R : H_{X,I,g}(x) \geq \alpha\}$ be the left-continuous inverse of $H_{X,I,g}$. For $\alpha = 0$ and $\alpha = 1$, use convention $\inf \emptyset = \infty$, $\sup \emptyset = -\infty$. For a distribution function F, we have $F^{-1}(0) = -\infty$ and $F^{-1+}(1) = \infty$. Denote

$$G^{-1+}_{X,I,g}(\alpha) = \sup\{x \in R : G_{X,I,g}(x) \leq \alpha\}.$$

Similar to Exercise 8 on page $1 - 12$ in Wichura (2001), we have

$$G^{-1+}_{X,I,g}(\alpha) = \lim_{u \searrow \alpha} H^{-1}_{X,I,g}(u) = H^{-1}_{X,I,g}(\alpha+).$$

Since $H^{-1}_{X,I,g}(\alpha)$ is increasing, we have

$$H^{-1}_{X,I,g}(\alpha+) \geq H^{-1}_{X,I,g}(\alpha).$$

or

$$G^{-1+}_{X,I,g}(\alpha) \geq H^{-1}_{X,I,g}(\alpha).$$

The relationship between (5) and (18) in Wichura (2001) indicates that (4.21) is equivalent to

$$H^{-1}_{X,I,g}\left(\frac{\mathbb{E}[g(X)] - \mathbb{E}[h(X)]}{\mathbb{E}[g(X)]}\right) \leq c \leq G^{-1+}_{X,I,g}\left(\frac{\mathbb{E}[g(X)] - \mathbb{E}[h(X)]}{\mathbb{E}[g(X)]}\right).$$

If $g(x) \geq h(x)$, then

$$0 \leq \frac{\mathbb{E}[g(X)] - \mathbb{E}[h(X)]}{\mathbb{E}[g(X)]} \leq 1.$$

Let

$$\begin{aligned}
&F^{-1(p)}_{X,I,g}\left(\frac{\mathbb{E}[g(X)] - \mathbb{E}[h(X)]}{\mathbb{E}[g(X)]}\right) \\
&= pH^{-1}_{X,I,g}\left(\frac{\mathbb{E}[g(X)] - \mathbb{E}[h(X)]}{\mathbb{E}[g(X)]}\right) + (1-p)G^{-1+}_{X,I,g}\left(\frac{\mathbb{E}[g(X)] - \mathbb{E}[h(X)]}{\mathbb{E}[g(X)]}\right),
\end{aligned}$$

where $p \in [0, 1]$. Then we can conclude that (4.21) is reduced to (4.22). □

Remark 4.15. *Since $(X - c)_+$ is unlimited, it is reasonable for the reinsurer to assume $g(x) \geq h(x)$. Similarly, it is also reasonable to suppose that $\beta_1 \geq \beta_2 \geq 0$, $\gamma_1 \geq \gamma_2 \geq 0$ in Corollary 4.16, Corollary 4.17, Corollary 4.18, and Corollary 4.20, and that $\alpha \geq \frac{1}{2}$ in Corollary 4.21 and Corollary 4.22.* □

Corollary 4.16. *Let $\varphi_1(x) = \varphi_2(x) = x$ and $c^* = c^*_{X,I,\beta_1,\gamma_1,\beta_2,\gamma_2,g}$ be the reinsurance premium principle by (4.22) when*

$$g(x) = \beta_1 \mathbb{I}_{\{I(x) > \rho(I(X))\}} + \gamma_1 \mathbb{I}_{\{I(x) \leq \rho(I(X))\}}$$

and

$$h(x) = \beta_2 \mathbb{I}_{\{I(x) > \rho(I(X))\}} + \gamma_2 \mathbb{I}_{\{I(x) \leq \rho(I(X))\}},$$

where ρ is a risk measure, $\beta_1 \geq \beta_2 \geq 0$, $\gamma_1 \geq \gamma_2 \geq 0$. Let

$$q_1^* = F_{I(X)}(\rho(I(X)))$$

and

$$q_2^* = G_{I(X)}(\rho(I(X))) = F_{I(X)}(\rho(I(X))-),$$

where $G_{I(X)}(x) = Pr(I(X) < x)$. Denote

$$\alpha_{X,I,g,h} = \frac{\mathbb{E}[g(I(X)) - h(I(X))]}{\mathbb{E}[g(I(X))]} = \frac{(\beta_1 - \beta_2)(1 - q_1^*) + (\gamma_1 - \gamma_2)q_1^*}{d^*}$$

as defined in Proposition 4.14, where $d^* = \mathbb{E}[g(X)]$. Then, the following results hold:

(i) if $\alpha_{X,I,g,h} < \frac{\gamma_1 q_2^*}{d^*}$, $c^* = F_{I(X)}^{-1(p)}(\frac{(\beta_1-\beta_2)(1-q_1^*)+(\gamma_1-\gamma_2)q_1^*}{\gamma_1})$;

(ii) if $\alpha_{X,I,g,h} = \frac{\gamma_1 q_2^*}{d^*}$, $c^* = p\rho(I(X)) + (1-p)F_{I(X)}^{-1+}(\frac{(\beta_1-\beta_2)(1-q_1^*)+(\gamma_1-\gamma_2)q_1^*}{\gamma_1})$;

(iii) if $\frac{\gamma_1 q_2^*}{d^*} < \alpha_{X,I,g,h} < \frac{\gamma_1 q_1^*}{d^*}$, $c^* = \rho(I(X))$;

(iv) if $\alpha_{X,I,g,h} = \frac{\gamma_1 q_1^*}{d^*}$, $c^* = pF_{I(X)}^{-1}(\frac{(\beta_1-\beta_2)(1-q_1^*)+(\beta_1-\gamma_2)q_1^*}{\beta_1}) + (1-p)\rho(I(X))$;

(v) if $\alpha_{X,I,g,h} > \frac{\gamma_1 q_1^*}{d^*}$, $c^* = F_{I(X)}^{-1(p)}(\frac{(\beta_1-\beta_2)(1-q_1^*)+(\beta_1-\gamma_2)q_1^*}{\beta_1})$;

where $F_X^{-(p)}(x) = pF_X^{-1}(x) + (1-p)F_X^{-1+}(x)$ with $F_X^{-1}(\alpha) = \inf\{x \in R : Pr(X \leq x) \geq \alpha\}$, $F_X^{-1+}(\alpha) = \sup\{x \in R : Pr(X < x) \leq \alpha\}$ and $p \in [0,1]$.

Proof. Now we have

$$H_{X,I,g,h}(x) = \frac{\mathbb{E}[(\beta_1 \mathbb{I}_{\{I(X)>\rho(I(X))\}} + \gamma_1 \mathbb{I}_{\{I(X)\leq\rho(I(X))\}})\mathbb{I}_{\{I(X)\leq x\}}]}{d^*}$$

and

$$G_{X,I,g,h}(x) = \frac{\mathbb{E}[(\beta_1 \mathbb{I}_{\{I(X)>\rho(I(X))\}} + \gamma_1 \mathbb{I}_{\{I(X)\leq\rho(I(X))\}})\mathbb{I}_{\{I(X)<x\}}]}{d^*}.$$

For $H_{X,I,g,h}(x)$, if $x \geq \rho(I(X))$, then

$$H_{X,I,g,h}(x) = \frac{\beta_1 F_{I(X)}(x) + (\gamma_1 - \beta_1)q}{d^*};$$

if $x < \rho(I(X))$, then

$$H_{X,I,g,h}(x) = \frac{\gamma_1 F_{I(X)}(x)}{d^*}.$$

Note that $H_{X,I,g,h}(x)$ is increasing and right-continuous with

$$H^* = H_{X,I,g,h}(\rho(I(X))) = \frac{\gamma_1 q_1^*}{d^*},$$

and $G_{X,I,g,h}(x)$ is increasing and left-continuous with

$$G^* = G_{X,I,g,h}(\rho(I(X))) = \frac{\gamma_1 q_2^*}{d^*}.$$

Moreover,

$$H_{X,I,g,h}(\rho(I(X))-) = G_{X,I,g,h}(\rho(I(X))) = G^*$$

and

$$G_{X,I,g,h}(\rho(I(X))+) = H_{X,I,g,h}(\rho(I(X))) = H^*.$$

Therefore, if $\alpha_{X,I,g,h} \geq H^*$, let

$$H_{X,I,g,h}(x) = \frac{\beta_1 F_{I(X)}(x) + (\gamma_1 - \beta_1)q}{d^*} = \alpha_{X,I,g,h},$$

then

$$H^{-1}_{X,I,g,h}(\alpha_{X,I,g,h}) = F^{-1}_{I(X)}(\frac{(\beta_1 - \beta_2)(1 - q_1^*) + (\beta_1 - \gamma_2)q_1^*}{\beta_1});$$

if $G^* \leq \alpha_{X,I,g,h} < H^*$, $H^{-1}_{X,I,g,h}(\alpha_{X,I,g,h}) = \rho(I(X))$;
if $\alpha_{X,I,g,h} < G^*$, let

$$H_{X,I,g,h}(x) = \frac{\gamma_1 F_{I(X)}(x)}{d^*} = \alpha_{X,I,g,h},$$

then

$$H^{-1}_{X,I,g,h}(\alpha_{X,I,g,h}) = F^{-1}_{I(X)}(\frac{(\beta_1 - \beta_2)(1 - q_1^*) + (\gamma_1 - \gamma_2)q_1^*}{\gamma_1}).$$

For $G_{X,I,g,h}(x)$, if $x > \rho(I(X))$, then

$$G_{X,I,g,h}(x) = \frac{\beta_1 G_{I(X)}(x) + (\gamma_1 - \beta_1)q}{d^*};$$

if $x \le \rho(I(X))$, then

$$G_{X,I,g,h}(x) = \frac{\gamma_1 G_{I(X)}(x)}{d^*}.$$

Similarly, for $H^{-1}_{X,I,g,h}$, if $\alpha_{X,I,g,h} > H^*$, then

$$G^{-1+}_{X,I,g,h}(\alpha_{X,I,g,h}) = F^{-1+}_{I(X)}(\frac{(\beta_1-\beta_2)(1-q_1^*)+(\beta_1-\gamma_2)q_1^*}{\beta_1});$$

if $G^* < \alpha_{X,I,g,h} \le H^*$, $G^{-1+}_{X,I,g,h}(\alpha_{X,I,g,h}) = \rho(I(X))$;
if $\alpha_{X,I,g,h} \le G^*$, then

$$G^{-1+}_{X,I,g,h}(\alpha_{X,I,g,h}) = F^{-1+}_{I(X)}(\frac{(\beta_1-\beta_2)(1-q_1^*)+(\gamma_1-\gamma_2)q_1^*}{\gamma_1}).$$

□

Corollary 4.17. *Apply the same assumptions and notations as in Corollary 4.16. Assume $F_{I(X)}$ is continuous. Let $q^* = F_{I(X)}(\rho(I(X)))$. Then, the following results hold.*

(i) if $q^* < \frac{\beta_1-\beta_2}{\beta_1-\beta_2+\gamma_2}$, $c^* = F^{-1(p)}_{I(X)}(\frac{(\beta_1-\beta_2)(1-q^*)+(\beta_1-\gamma_2)q^*}{\beta_1})$;

(ii) if $q^* = \frac{\beta_1-\beta_2}{\beta_1-\beta_2+\gamma_2}$, $c^* = \rho(I(X))$;

(iii) if $q^* > \frac{\beta_1-\beta_2}{\beta_1-\beta_2+\gamma_2}$, $c^* = F^{-1(p)}_{I(X)}(\frac{(\beta_1-\beta_2)(1-q^*)+(\gamma_1-\gamma_2)q^*}{\gamma_1})$;

where $p \in [0,1]$. Assume F_X is continuous and strictly increasing. Then, the following results hold:

(iv) if $q^* \le \frac{\beta_1-\beta_2}{\beta_1-\beta_2+\gamma_2}$, $c^* = F^{-1}_{I(X)}(\frac{(\beta_1-\beta_2)(1-q^*)+(\beta_1-\gamma_2)q^*}{\beta_1})$;

(v) if $q^* > \frac{\beta_1-\beta_2}{\beta_1-\beta_2+\gamma_2}$, $c^* = F^{-1}_{I(X)}(\frac{(\beta_1-\beta_2)(1-q^*)+(\gamma_1-\gamma_2)q^*}{\gamma_1})$.

Proof. Similar to the proof of Corollary 3.19 in Chapter 3. □

The premium principle with identity functions defined in Corollary 4.16 when $F_{I(X)}$ is continuous will satisfy the properties in Corollary 4.18.

Corollary 4.18. *Let $c^* = c^*_{X,I,\beta_1,\gamma_1,\beta_2,\gamma_2,q^*}$, where $q^* = F_{I(X)}(\rho(I(X)))$, be the reinsurance premium in Corollary 4.17. Then, the following results hold.*

(a) c^* is decreasing in β_2 and γ_2.

(b) $c^*_{X,mI,\beta_1,\gamma_1,\beta_2,\gamma_2,q^*_{mI(X)}} = mc^*_{X,I,\beta_1,\gamma_1,\beta_2,\gamma_2,q^*_{I(X)}}$ if $\rho(mX) = m\rho(X)$ for any $m \in R_+$.

(c) $c^*_{X,I_1,\beta_1,\gamma_1,\beta_2,\gamma_2,q^*_{I_1(X)}} \leq c^*_{Y,I_2,\beta_1,\gamma_1,\beta_2,\gamma_2,q^*_{I_2(Y)}}$ if $I_1(X) \leq I_2(Y)$ and $\rho(I(X)) = \mathrm{VaR}_{\alpha^*}(I(X))$, where $\alpha^* \in (0,1)$.

(d) Monotonicity of q^*:

(i) if $0 \leq q^* < \frac{\beta_1-\beta_2}{\beta_1-\beta_2+\gamma_2}$, then c^* is increasing (decreasing) in q^* when $\beta_2 \geq (\leq)\gamma_2$;

(ii) if $q^* = \frac{\beta_1-\beta_2}{\beta_1-\beta_2+\gamma_2}$, c^* is increasing in q^*;

(iii) if $\frac{\beta_1-\beta_2}{\beta_1-\beta_2+\gamma_2} < q^* \leq 1$, then c^* is increasing (decreasing) in q^* when $\beta_2 + \gamma_1 \geq (\leq)\beta_1 + \gamma_2$.

Proof. (a) $F_X^{-1+}(x)$ and $F_X^{-1}(x)$ are increasing in x.

(b) If $\rho(mX) = m\rho(X)$ for any $m \in R_+$, then

$$\begin{aligned} q^*_{mI(X)} &= P(mI(X) \leq \rho(mI(X))) = P(mI(X) \leq m\rho(I(X))) \\ &= P(I(X) \leq \rho(I(X))) = q^*_{I(X)}. \end{aligned}$$

Also, $F_X^{-1+}(x)$ and $F_X^{-1}(x)$ satisfy the property of translation invariance with respect to X.

(c) Firstly, $q^*_{I_1(X)} = q^*_{I_2(Y)} = \alpha^*$ if $\rho(I(X)) = \mathrm{VaR}_{\alpha^*}(I(X))$. In addition, $F_X^{-1+}(x) \leq F_Y^{-1+}(x)$ and $F_X^{-1}(x) \leq F_Y^{-1}(x)$ if $X \leq Y$.

(d) The same as (a).

□

Remark 4.19. *Note that the property of subadditivity cannot be satisfied by c^* defined in Corollary 4.18 since both $F_X^{-1+}(x)$ and $F_X^{-1}(x)$ do not satisfy it. One special case of Corollary 4.18 (d) is that c^* is increasing in $q^* \in [0,1]$ when $\gamma_1 \geq \beta_1 \geq \beta_2 \geq \gamma_2$.* □

4.5.2 Relationship between the Reinsurance Premium Principles with Identity Functions and the Classical Reinsurance Premium

According to Heilmann (1989), the optimal premium c are the minimizers of the expectation of loss function defined by

$$L(X,c) = \alpha(X-c)_+ + (1-\alpha)(X-c)_-,$$

where $\alpha \in (0,1)$. Here we denote the minimizers as $F_X^{-1(p)}(\alpha)$, where $p \in [0,1]$, and we call Heilmann (1989)'s model the classical model and $F_X^{-1(p)}(\alpha)$ the classical premium. The classical reinsurance premium by Heilmann (1989)'s model is denoted as

$$c_\alpha^* = F_{I(X)}^{-1(p)}(\alpha) \tag{4.24}$$

and $c_\alpha^* = F_{I(X)}^{-1}(\alpha)$ if $F_{I(X)}^{-1}$ is continuous at α.

Corollary 4.20. *Let $c^* = c^*_{X,I,\beta_1,\gamma_1,\beta_2,\gamma_2,q^*}$, where $q^* = F_{I(X)}(\rho(I(X)))$, be the reinsurance premium in Corollary 4.17 and $c_{q^*} = F_{I(X)}^{-1(p)}(q^*)$. Then, the following results hold:*

(i) if $q^* < \frac{\beta_1-\beta_2}{\beta_1-\beta_2+\gamma_2}$, then $c^* > c_{q^*}$;

(ii) if $q^* = \frac{\beta_1-\beta_2}{\beta_1-\beta_2+\gamma_2}$, then $c^* = c_{q^*}$;

(iii) if $q^* > \frac{\beta_1-\beta_2}{\beta_1-\beta_2+\gamma_2}$, then $c^* < c_{q^*}$;

for any given $p \in [0,1]$.

Proof. (i) If $q^* < \frac{\beta_1-\beta_2}{\beta_1-\beta_2+\gamma_2}$, then $\frac{(\beta_1-\beta_2)(1-q^*)+(\beta_1-\gamma_2)q^*}{\beta_1} > q^*$. Thus, $c^* > c_{q^*}$ since both $F^{-1}(x)$ and $F^{-1+}(x)$ are non-decreasing in x.

(ii) It is trivial according to Corollary 4.17.

(iii) If $q^* > \frac{\beta_1-\beta_2}{\beta_1-\beta_2+\gamma_2}$, then $\frac{(\beta_1-\beta_2)(1-q^*)+(\gamma_1-\gamma_2)q^*}{\gamma_1} < q^*$. Thus, $c^* < c_{q^*}$ since both $F^{-1}(x)$ and $F^{-1+}(x)$ are increasing in x.

□

If $\beta_1 = \gamma_1$ and $\beta_2 = \gamma_2$ hold for g and h in Corollary 4.16, we can arrive at the results for the identity functions in Corollary 4.21. Without loss of generality, we assume $g = \alpha$, $h = 1-\alpha$, where $\alpha \geq \frac{1}{2}$ since $\beta_1 \geq \beta_2$.

Corollary 4.21. *If $\varphi_1(x) = \varphi_2(x) = x$, $g = \alpha$, $h = 1-\alpha$, where $\alpha \in [\frac{1}{2}, 1]$, the reinsurance premium by (4.22) is*

$$c^* = F_{I(X)}^{-1(p)}(2 - \frac{1}{\alpha}). \tag{4.25}$$

Also, $c^* \geq F_{I(X)}^{-1}(2-\frac{1}{\alpha}) = \mathrm{VaR}_{2-\frac{1}{\alpha}}(I(X))$. Moreover, $c^* = \mathrm{VaR}_{2-\frac{1}{\alpha}}(I(X))$ if $F_{I(X)}^{-1}(x)$ is continuous at $2-\frac{1}{\alpha}$.

Proof. Since $\beta_1 = \gamma_1 = \alpha$ and $\beta_2 = \gamma_2 = 1-\alpha$, $\frac{(\beta_1-\beta_2)(1-q^*)+(\beta_1-\gamma_2)q^*}{\beta_1}$ in (i) of Corollary 4.17 is reduced to $2-\frac{1}{\alpha}$, as well as $\frac{(\beta_1-\beta_2)(1-q^*)+(\gamma_1-\gamma_2)q^*}{\gamma_1}$ in (iii). In addition, $(\beta_1-\beta_2)(1-q^*) = \gamma_2 q^*$ illustrates that $q^* = 2-\frac{1}{\alpha}$. Hence,

$$H_{X,I,g,h}^{-1}(\frac{(\beta_1-\beta_2)(1-q^*)+(\gamma_1-\gamma_2)q^*}{\beta_1(1-q^*)+\gamma_1 q^*}) = H_{X,I,g,h}^{-1}(2-\frac{1}{\alpha}) = F_{I(X)}^{-1}(2-\frac{1}{\alpha})$$

and

$$G_{X,I,g,h}^{-1+}(\frac{(\beta_1-\beta_2)(1-q^*)+(\gamma_1-\gamma_2)q^*}{\beta_1(1-q^*)+\gamma_1 q^*}) = G_{X,I,g,h}^{-1+}(2-\frac{1}{\alpha}) = F_{I(X)}^{-1+}(2-\frac{1}{\alpha}).$$

□

Corollary 4.22. *Let c_α^* be the classical reinsurance premium defined by (4.24) and c^* be the premium defined by (4.25). Then*

$$c^* \leq c_\alpha^*$$

for any given $p \in [0,1]$ and $\alpha \in [\frac{1}{2}, 1]$.

Proof. Since

$$\alpha - (2 - \frac{1}{\alpha}) = \alpha + \frac{1}{\alpha} - 2 \geq 2\sqrt{\alpha \times \frac{1}{\alpha}} - 2 = 0$$

holds for any $\alpha \in (0,1]$,

$$\alpha \geq 2 - \frac{1}{\alpha}.$$

In addition, $2 - \frac{1}{\alpha} \in [0,1]$ if and only if $\alpha \in [\frac{1}{2}, 1]$. Thus,

$$F_{I(X)}^{-1(p)}(\alpha) \geq F_{I(X)}^{-1(p)}(2 - \frac{1}{\alpha})$$

for any given $p \in [0,1]$ since both $F^{-1}(x)$ and $F^{-1+}(x)$ are non-decreasing in x. □

Remark 4.23. *We should emphasize two points by comparing c_α^* and c^* inspired by Corollary 4.22. One is that the new optimal premium is always competitive in the reinsurance market since it is lower than the classical reinsurance premium at the same confidence level. The other is that the new principal is derived based on the total expected loss of the insurer and the reinsurer; however, the classical one covers losses only for the reinsurer. Hence, this new reinsurance premium principle proposed in Corollary 4.22 is more reasonable than the classical one.* □

4.6 Numerical Examples

In this section, we will apply the conclusions to the exponential and two-parameter Pareto distributions with the same mean. In the following examples, the mean of the underlying loss is 100. Note that there are two thresholds for the weight α in the objective function for the new reinsurance premiums as in Proposition 4.8

α	c^*
0.4	0.7759
0.5	13.5958
0.6	28.8393
0.7	48.1203
0.8	75.1189
0.9	121.9283

TABLE 4.1: $I(x) = (x - 50)_+$

(b). One is $\alpha_1 = 1 - \frac{E[I(X)]}{E[X]}$. If $\alpha \geq \alpha_1$, the premium is non-negative. The other one is $\alpha_2 = 1 - \frac{\mathbb{E}[(I(X)-\mathbb{E}[I(X)])_+]}{\mathbb{E}[(I(X)-\mathbb{E}[I(X)])_+]+\mathbb{E}[X]}$. If $\alpha \geq \alpha_2$, the premium is no less than the expectation of the ceded loss.

Example 4.1. *If $X \sim Exp(\theta)$, the optimal reinsurance premium c^* with the quadratic functions is the solution to*

$$\alpha\theta e^{-\frac{c+d}{\theta}} - (1-\alpha)(\theta(1-e^{-\frac{d}{\theta}}) + c) = 0$$

for the stop-loss ceded function, $I(x) = (x-d)_+$. Moreover, the optimal reinsurance premium c^* is the solution to

$$\alpha k\theta e^{-\frac{c}{k\theta}} - (1-\alpha)((1-d)\theta + c) = 0$$

for the quota-share ceded function $I(x) = kx$.

If $X \sim Exp(100)$, for $I(x) = (x-d)_+$ with $d = 50$, according to Proposition 4.8 (b), the threshold for a non-negative premium is $\alpha_1 = 0.3935$. And for the case where a premium is no less than $E[I(X)] = 60.6531$, the threshold for the weight has to be no less than $\alpha_2 = 0.7515$. We calculated the reinsurance premium c^* for $\alpha = 0.4,\ 0.5,\ 0.6,\ 0.7,\ 0.8,\ 0.9$ in Table 4.1.

Now we calculate the premiums for the quota-share type reinsurance contract under the condition $\mathbb{E}[(X-d)_+] = \mathbb{E}[kX]$ to guarantee that the expectations of the covered losses are equal for the two different types of ceded functions. In this way,

α	c^*
0.4	0.6545
0.5	11.1342
0.6	22.9605
0.7	37.2420
0.8	56.3955
0.9	88.1893

TABLE 4.2: $I(x) = 0.606531x$

we get $k = 0.606531$ *by* $\mathbb{E}[(X-50)_+] = \mathbb{E}[kX] = 60.6531$*, and the thresholds are* $\alpha_1 = 0.3935$ *and* $\alpha_2 = 0.8176$*, correspondingly. Table 4.2 calculates the premiums for* $I(x) = 0.606531x$ *under the same confidence levels as for the stop-loss type ceded loss function.*

We point out that Table 4.1 and Table 4.2 demonstrate that, for the same weight α*, the reinsurance premium for the stop-loss reinsurance is higher than the reinsurance premium for the quota-share reinsurance. This is a reasonable result since the stop-loss reinsurance will cover more loss than the quota-share reinsurance when the underlying loss becomes larger.*

Example 4.2. *If* $X \sim$ *Pareto*(τ, θ)*, the optimal reinsurance premium with the quadratic functions* c^* *is the solution to*

$$\alpha\frac{\theta}{\tau-1}(\frac{\theta}{\theta+d+c})^{\tau-1} - (1-\alpha)\frac{\theta}{\tau-1}(1-(\frac{\theta}{\theta+d})^{\tau-1}+c) = 0$$

for the stop-loss ceded function $I(x) = (x-d)_+$*; and is the solution to*

$$\alpha k\frac{\theta}{\tau-1}(\frac{\theta}{\theta+\frac{c}{k}})^{\tau-1} - (1-\alpha)((1-k)\frac{\theta}{\tau-1}+c) = 0$$

for the quota-share ceded function $I(x) = kx$*.*

Let $X \sim$ *Pareto*$(3, 200)$ *and* $I(x) = (x-d)_+$ *with* $d = 50$*. Similarly, according to Proposition 4.8* (b)*, the thresholds for a positive premium and a premium no less than* $\mathbb{E}[I(X)] = 64$ *are* $\alpha_1 = 0.3600$ *and* $\alpha_2 = 0.7114$*, respectively. Table*

α	c^*
0.4	5.0075
0.5	19.1974
0.6	36.8958
0.7	60.6901
0.8	96.9323
0.9	169.0288

TABLE 4.3: $I(x) = (x - 50)_+$

α	c^*
0.4	4.0746
0.5	15.1618
0.6	28.3457
0.7	45.3859
0.8	70.4751
0.9	118.8603

TABLE 4.4: $I(x) = 0.64x$

4.3 provides the results under the assumption of step-wise increasing confidential level as in the example for the Exponentially distributed random variables.

For $I(x) = kx$ with $k = 0.64$, it holds that $\mathbb{E}[I(X)] = 64$. Then the thresholds are $\alpha_1 = 0.36$ and $\alpha_2 = 0.7785$. Table 4.4 provides the reinsurance premiums at the corresponding confidence level α. The same comments can be applied to the Pareto distributed loss random variables.

Remark 4.24. *In fact, satisfying the second threshold $\alpha \geq \alpha_2$ requires a higher weight on the reinsurer's loss in (4.3), which implies that more of the reinsurer's loss should be considered. If we take $\alpha \to 1$, the objective function includes only the reinsurer's total loss and it is beneficial for the reinsurance company to minimize this objective function. However, the insurer might not accept the reinsurance premium priced under this condition since it is high but covers less of their loss. Hence, the modified objective function with a percentage of the initial risk transferred from the insurer to the reinsurer is more reasonable. Considering the insurer's preferences, a relatively smaller quota of the insurer's loss should also be considered in the objective function.* □

α	e_α
0.4	48.1631
0.5	60.6531
0.6	74.9810
0.7	92.6673
0.8	117.0813
0.9	159.3032

TABLE 4.5: $I(x) = (x - 50)_+$

α	e_α
0.4	52.0872
0.5	60.6531
0.6	70.1869
0.7	81.6859
0.8	97.2600
0.9	123.7392

TABLE 4.6: $I(x) = 0.606531x$

α	e_α
0.4	49.0954
0.5	64.0000
0.6	82.1306
0.7	106.0665
0.8	142.0661
0.9	213.1674

TABLE 4.7: $I(x) = (x - 50)_+$

Now we will give an example for the reinsurance premiums calculated based on the expectile risk measure.

Example 4.3. *The premiums for $X \sim Exp(100)$ with $I(x) = (x - 50)_+$ and $I(x) = 0.606531x$, and $X \sim Pareto(3, 200)$ with $I(x) = (x-50)_+$ and $I(x) = 0.64x$ are calculated in Table 4.5-4.8. The results are consistent with those results in Corollary 4.12. Moreover, the premiums based on the expectiles are always greater than our new premium principle, if the confidence level is the same as the constant weight α as in (4.3).*

Next, the corresponding c_α^* and c^* in Corollary 4.22 will be calculated in Example 4.4 and Example 4.5.

α	e_α
0.4	53.3751
0.5	64
0.6	76.5327
0.7	92.7027
0.8	116.5850
0.9	163.0365

TABLE 4.8: $I(x) = 0.64x$

α	c^*_α	c^*
0.8	110.9438	88.6294
0.85	139.7120	123.4434
0.9	180.2585	169.7325
0.95	249.5732	244.5039

TABLE 4.9: $I(x) = (x - 50)_+$

α	c^*_α	c^*
0.8	97.6174	84.0830
0.85	115.0662	105.1988
0.9	139.6589	133.2746
0.95	181.7004	178.6257

TABLE 4.10: $I(x) = 0.606531x$

α	c^*_α	c^*
0.8	91.9952	67.4802
0.85	126.4144	106.5456
0.9	180.8869	166.0306
0.95	292.8835	283.7871

TABLE 4.11: $I(x) = (x - 50)_+$

Example 4.4. *If $X \sim Exp(100)$, the values of c^*_α and c^* with $\alpha = 0.8,\ 0.85,\ 0.9,\ 0.95$ for the stop-loss ceded function $I(x) = (x - 50)_+$ are shown in Table 4.9. And the corresponding premiums for $I(x) = 0.606531x$ are illustrated in Table 4.10.*

By comparing the examples with the same loss random variable and ceded function, we can see that, when $\alpha > \alpha_2$, the optimal reinsurance premiums with identity functions are always higher than the premiums with quadratic functions. For instance, the premium with quadratic functions is 75.1189, and the premium with identify functions is 88.6294, when $X \sim Exp(100)$, $I(x) = (x - 50)_+$ and $\alpha = 0.8 > \alpha_2 = 0.7515$.

α	c^*_α	c^*
0.8	90.8769	75.1873
0.85	112.9052	100.1892
0.9	147.7676	138.2595
0.95	219.4454	213.6237

TABLE 4.12: $I(x) = 0.64x$

Example 4.5. *If* $X \sim Pareto(3, 200)$*,* c^*_α *and* c^* *with* $\alpha = 0.8,\ 0.85,\ 0.9,\ 0.95$ *are illustrated in Table 4.11 and Table 4.12. By comparing Table 4.3 and Table 4.11, we find that the values of* c^* *are less than the values of* c^*_α *for the stop-loss type ceded function.*

Chapter 5

Capital Allocation Principles Based on Weighted Loss Functions

5.1 Literature Review

In Merton and Perold (1993), *"risk capital"* is defined as "the smallest amount that can be invested to insure the value of the firm's net assets against a loss in value relative to the risk-free investment of those net assets." Also, they proposed that "as defined, risk capital differs from both the *regulatory capital*, which attempts to measure risk capital according to a particular accounting standard, and from the *cash capital*, which represents the up-front cash required to execute a transaction. *Cash capital* is a component of *working capital* that includes financing of operating expenses like salaries and rent." Buch et al. (2011) argued that there are mainly three trends of research on the risk capital allocation problems: Denault (2001), Kalkbrener (2005), Tasche (2004), Buch and Dorfleitner (2008) contributed to mathematical finance field; Dhaene et al. (2003), Furman and Zitikis (2008b), Gatzert and Schmeiser (2008) investigated the benefits for insurance companies by applying proper capital allocation principles; Merton and Perold (1993), and

Stoughton and Zechner (2007) identified whether the principles are sensible from an economic perspective.

Dhaene et al. (2012) unified the capital allocation principles as the optimizers of an optimization problem. Proportional capital allocation principles, such as quantile, covariance and CTE, can be classified into this optimization problem under certain model assumptions. However, the haircut principle cannot be covered by the optimization problem since VaR does not satisfy sub-additivity. To solve this problem, Belles et al. (2014b) assumed the weight functions in Dhaene et al. (2012) to be integrable and concluded the generalized principles under the GlueVaR risk measure as defined in Belles et al. (2014a). The haircut principle is just a special case of the optimizers under GlueVaR, so it can be included in the generalized capital allocation problem.

The standard deviation type principle by Buhlmann (1970) and the Tail Covariance Premium Adjusted (TCPA) principle by Wang (2014) are based on the property of additivity for covariance or tail covariance. Note that the principle in Wang (2014) can be unified by the generalized model in Belles et al. (2014b) if we define an appropriate weight function. In addition, Laeven and Goovaerts (2004) investigated the dynamic capital allocation problems under the distortion risk measures by adopting the stochastic model and updating the distortion functions.

Moreover, the formulas for some specially distributed loss random variables are derived by the following researchers. Barges et al. (2009) calculated the capital allocation principles for the multivariate distribution constructed by the exponential marginals and the mixed exponential marginals with the Farlie-Gumbel-Morgenstern (FGM) copulas under the TVaR risk measure. They arrived at the close formula for the capital allocation principle and further suggest an approximation method. Additionally, the capital allocation principles derived in Cossette et al. (2013) provide close formulas under TVaR and covariance for the multivariate distribution constructed by the mixed erlang marginals with the FGM copulas.

Note that the work by Barges et al. (2009) and Cossette et al. (2013) can be applied to the case when the number of business lines is no less than two. However, Wang (2014) only calculated the TCPA principle for two business lines, and the principle is determined for the multivariate distribution constructed by the exponential marginals with the FGM copulas.

Cai and Wei (2014) defined new notions of dependence structure for the optimal capital allocation problems. You and Li (2014) further studied the capital allocation concerning mutually interdependent random risks, and they proved that more capital should be allocated to the risk with a larger reversed hazard rate when risks are coupled by an Archimedean copula for risk-averse insurers with decreasing convex loss functions. In addition, they developed sufficient conditions to exclude the worst capital allocations for random risks with Archimedean copulas. Inspired by the works of generalized quantiles in Bellini et al. (2014), we will investigate the optimization problem considering both the positive and negative parts of the potential loss for each business line.

5.1.1 Dhaene's Unified Capital Allocation Model

On the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, $(X_1, X_2, \ldots, X_n)$ is a random vector, and it is treated as a portfolio of n individual losses. Moreover, the aggregate loss S is defined by

$$S = \sum_{j=1}^{n} X_j. \tag{5.1}$$

In addition, the initial aggregate capital K is fixed and given. As in most of the capital allocation problems, the object is to allocate the risk capital K among n business lines based on an optimization criteria, namely, to determine the values of $K_1, \ldots, K_n$ satisfying

$$\sum_{j=1}^{n} K_j = K, \tag{5.2}$$

the full allocation requirement.

In Dhaene et al. (2012), the values of K_j, $j = 1, \ldots, n$ are assumed to be non-negative real numbers, and the value of the initial aggregate capital K is positive. The capital allocation principles are determined by the optimizers $(K_1, \ldots, K_n)$ for the unified optimization problem:

$$\min_{(K_1,\ldots,K_n)\in\mathbb{R}^n} \sum_{j=1}^{n} \nu_j \mathbb{E}[\xi_j D(\frac{X_j - K_j}{\nu_j})], \ s.t. \sum_{j=1}^{n} K_j = K,$$

where ν_j, $j = 1, \ldots, n$, are non-negative real numbers such that $\sum_{j=1}^{n} \nu_j = 1$, ξ_j, $j = 1, \ldots, n$, are non-negative random variables such that $\mathbb{E}[\xi_j] = 1$, and D is a non-negative function.

In this model, ξ_j can be a function of either X_j or S. If it is a function of X_j, the allocation principle is called *the business unit driven capital allocation*; If it is a function of S, the principle is called *the aggregate portfolio driven capital allocation*. Furthermore, ξ_j, $j = 1, \ldots, n$, are assumed to satisfy $\mathbb{E}[|\xi_j|] < \infty$ to cover the haircut allocation principle defined by equation (3) in Belles et al. (2014b).

The optimal allocation principles with quadratic and identity quantifying functions, i.e. $D(x) = x^2$ and $D(x) = x$, are determined by Theorem 1 and Theorem 2 in Dhaene et al. (2012). In their paper, Theorem 1 is proved through a geometric approach. Belles et al. (2014b) further arrived at a similar conclusion under the generalized assumption.

5.2 Background for the Generalized Capital Allocation Problem

In the models of Dhaene et al. (2012) and Belles et al. (2014b), they considered the differences between X_j, the potential loss in the jth business line, and K_j, the capital that will be allocated into the jth business line. In detail, they denoted the distance between X_j and K_j as $|X_j - K_j|$, and added appropriate weight functions and parameters to model the total risk. Their models can cover most of the capital allocation and premium principles by choosing different weight functions and parameters, see Table 1 and Table 2 in Dhaene et al. (2012).

Mathematically, the positive part of a function f is defined by

$$f^+ = \max(f, 0),$$

and the negative part of f is defined by

$$f^- = -\min(f, 0).$$

Note that f^+ and f^- are both non-negative functions. The functions f and $|f|$ can be expressed as $f = f^+ - f^-$ and $|f| = f^+ + f^-$ in terms of f^+ and f^-. In addition, $f^+ = \frac{|f|+f}{2}$ and $f^- = \frac{|f|-f}{2}$. The capital allocation problem in Dhaene et al. (2012) only considers the absolute value of the difference $X_j - K_j$, or the distance between X_j and K_j. However, it might be more reasonable to determine new capital allocation principles based on the objective functions with both $(X_j - K_j)_+$ and $(X_j - K_j)_-$ included.

Economically, if we allocate K_j on the jth business line, then $(X_j - K_j)_+$ illustrates the possible loss over the attained capital. Therefore, the positive part of $X_j - K_j$ represents the actual risk faced by the jth business line. Here, we defined it as *the capital deficit risk* for the jth business line. For each business line, for example, the manager in one branch of a multinational enterprise would like to reduce this kind of risk becasue the more capital was to be allocated in their business, the less capital deficit risk they might face. Additionally, the probability of going bankrupt (or business failure) for their branch could be reduced. Considering this preference for each business line, we build *Capital Allocation Problem I*:

$$\min_{(K_1,K_2,\cdots,K_n)\in\mathbb{R}^n} \sum_{j=1}^{n} \nu_j \mathbb{E}[\xi_j D_1(\frac{(X_j - K_j)_+}{\nu_j})] \tag{5.3}$$

such that $\sum_{j=1}^{n} K_j = K$.

The objective function in (5.3) represents the total capital deficit risk faced by all business lines, and our goal can be to determine the capital allocation principles by minimizing the total deficit risk. In (5.3), D_1 is the function adopted to quantify the capital deficit risk for the jth business line. And ν_j, $j = 1, \ldots, n$, are non-negative constants to model the business volume or risk exposure for the jth business line. In the term $D_1(\frac{(X_j - K_j)_+}{\nu_j})$, ν_j can make the capital deficit risk faced by the corresponding business line more comparable with other business lines by normalizing these risks. Moreover, ν_j also represents the weight put on the term $\mathbb{E}[\xi_j D_1(\frac{(X_j - K_j)_+}{\nu_j})]$, to illustrate the significance of each business line. Also, the weight ξ_j can be functions of the aggregated loss $S = \sum_j^n X_j$, or any reasonable function of X_j, $j = 1, 2, \ldots, n$. Based on Dhaene et al. (2012), if ξ_j is function of X_j, it is called *the business unit driven capital allocation*; If it is a function of $S = \sum_j^n X_j$, it is then called *the aggregate portfolio driven capital allocation*. Note that ξ_j might play the similar role as ν_j, however, it might vary with the loss random variables X_j, $j = 1, \ldots, n$.

Meanwhile, $(X_j - K_j)_- = (K_j - X_j)_+$ could illustrate the future capital received by the jth business line over the possible loss, and we define it as the *capital surplus risk* for the jth business line. Obviously, a larger $(K_j - X_j)_+$ means that the allocation on the jth business is over budgeted. It is known that money does have an opportunity cost, therefore, the over budgeted amount in the jth business line could be invested in other business lines with a higher profit rate. Considering that it might be beneficial from the perspective of the whole company, a lower risk of $(K_j - X_j)_+$ is preferred as well.

Therefore, a trade-off of the benefits between each business line and the whole company (or the whole country) should be considered when we allocate capital or make capital budgeting. In this sense, we should minimize the total capital deficit risk for business lines and the total capital surplus risk at the same time from a global perspective. Now, we have *Capital Allocation Problem II*:

$$\min_{(K_1,K_2,\cdots,K_n)\in\mathbb{R}^n}\sum_{j=1}^{n}\{\nu_j\mathbb{E}[\xi_j D_1(\frac{(X_j-K_j)_+}{\nu_j})]+\omega_j\mathbb{E}[\psi_j D_2(\frac{(X_j-K_j)_-}{\omega_j})]\}, \quad (5.4)$$

such that $\sum_{j=1}^{n} K_j = K$. For the capital surplus risk, the weights ψ_j and parameters ω_j share the similar meaning as ξ_j and ν_j for the capital deficit risk. Moreover, ξ_j and ψ_j satisfy $\mathbb{E}[|\xi_j|] < \infty$ and $\mathbb{E}[|\psi_j|] < \infty$ to include the VaR based capital allocation. Additionally, D_2 is the function quantifying the capital surplus risk. If we let ξ_j, $j = 1, \ldots, n$ or D_2 in the second part of the objective function in (5.4) be equal to zero, this model can be reduced to the *Capital Allocation Problem I.*

5.3 Preliminaries

In Chapter 3, the formula is derived to determine the first derivatives of the two types of functions in Lemma 3.1.

Lemma 5.1. *Let $g(c) = \mathbb{E}[\xi D((X-c)_+)]$ and $h(c) = \mathbb{E}[\xi D((X-c)_-)]$, where ξ is a non-negative random variable. D is a convex and increasing function defined on $\mathbb{R}^+$. Assume $g(c) < +\infty$ and $h(c) < +\infty$ for any $c \in \mathbb{R}$, then*

$$g'_+(c) = -\mathbb{E}[\xi D'_-((X-c)_+)I_{\{X>c\}}],$$

$$g'_-(c) = -\mathbb{E}[\xi D'_+((X-c)_+)I_{\{X\geq c\}}],$$

$$h'_+(c) = \mathbb{E}[\xi D'_+((X-c)_-)I_{\{X\leq c\}}]$$

and

$$h'_-(c) = \mathbb{E}[\xi D'_-((X-c)_-)I_{\{X<c\}}],$$

which are all finite. If D is differentiable with $D'(0) = 0$, then

$$g'(c) = -\mathbb{E}[\xi D'((X-c)_+)]$$

and

$$h'(c) = \mathbb{E}[\xi D'((X-c)_-)].$$

In Rockafellars (2011), the ordinary convex program (P) is defined as:

$$\min_{\boldsymbol{x} \in C} f_0(\overline{\boldsymbol{x}})$$

subject to constraints:

$$f_i(\overline{\boldsymbol{x}}) \leq 0, \ i = 1, \dots, r,$$

$$f_i(\overline{\boldsymbol{x}}) = 0, \ i = r+1, \dots, m,$$

where C is a non-empty convex subset in R_n, f_i, $i = 0, \dots, r$ are finite convex functions on C, and f_i, $i = r+1, \dots, m$ are affine functions on C. In addition, Theorem 28.3 provides a sufficient and necessary condition for the program (P). The conditions (a)-(c) in Lemma 5.2 are called the Karush-Kuhn-Tucker (KKT) conditions.

Lemma 5.2. *(Theorem 28.3 Rockafellars (2011)) Let (P) be an ordinary convex program. Let* $\overline{\boldsymbol{u}}^*$ *and* $\overline{\boldsymbol{x}}$ *be random vectors in* R^m *and* R^n*, respectively. In order for* $\overline{\boldsymbol{u}}^*$ *to be a Kuhn-Tucker vector for (P), it is necessary and sufficient that* $(\overline{\boldsymbol{u}}^*, \overline{\boldsymbol{x}})$ *be a saddle-point of the Lagrangian* $L = f_0(\boldsymbol{x}) + \lambda_1 f_1(\boldsymbol{x}) + \dots, +\lambda_m f_m(\boldsymbol{x})$ *of (P). Moreover, this condition holds if and only if* $\overline{\boldsymbol{x}}$ *and the components* λ_i *of* $\overline{\boldsymbol{u}}^*$ *satisfy*

(a) $\lambda_i \geq 0$, $f_i(\overline{\boldsymbol{x}}) \leq 0$ *and* $\lambda_i f_i(\overline{\boldsymbol{x}}) = 0$, $i = 1, \dots, r$,

(b) $f_i(\overline{\boldsymbol{x}}) = 0$ *for* $i = r+1, \dots, m$,

(c) $0 \in [\partial f_0(\overline{\boldsymbol{x}}) + \lambda_1 \partial f_1(\overline{\boldsymbol{x}}) + \dots, +\lambda_m \partial f_m(\overline{\boldsymbol{x}})]$.

Here, the set of all sub-gradients of a function f at $\boldsymbol{x}$ is called the sub-differential and is denoted as $\partial f(\boldsymbol{x})$, which equals $\nabla f(\boldsymbol{x})$ when $f(\boldsymbol{x})$ is differentiable. Grasmair (2015) concluded Theorem 10 for the optimization problems with non-empty, closed and unbounded constraints.

Lemma 5.3. *Assume that $S \subseteq R^n$ is non-empty and closed, and that $f : S \to \overline{R}$ satisfies lower semi-continuous and*

$$\lim_{\|\boldsymbol{x}\| \to \infty} f(\boldsymbol{x}) = \infty,$$

then the optimization problem $\min_{\boldsymbol{x} \in S} f(\boldsymbol{x})$ admits at least one global minimizer $\boldsymbol{x}^*$.

Also, Theorem 3.4.2 in Mokhtar et al. (1993) illustrates the relationship between a local optimal solution and a global optimal solution for convex optimization programs.

Lemma 5.4. *Let S be a nonempty convex set in E_n, and let $f : S \to E_1$ be convex on S. Consider the problem to minimize $f(\boldsymbol{x})$ subject to $\boldsymbol{x} \in S$. Assume that $\overline{\boldsymbol{x}} \in S$ is a local optimal solution to the problem.*

1. *Then, $\overline{\boldsymbol{x}}$ is a global optimal solution.*
2. *If either $\overline{\boldsymbol{x}}$ is a strict local minimum, or if π is strictly convex, then $\overline{\boldsymbol{x}}$ is the unique global optimal solution, and it is also a strong local minimum.*

Lemma 5.5. *If $f'_+(x) \leq (<)0$ and $f'_-(x) \leq (<)0$, then $f(x)$ is (strictly) decreasing in x.*

5.4 Capital Allocation Problem II

Given the aggregate capital K, determine the allocation vector $\boldsymbol{k} = (K_1, \ldots, K_n)$ from the optimization problem:

$$\min_{(K_1, K_2, \cdots, K_n) \in \mathbb{R}^n} \sum_{j=1}^{n} \{\nu_j \mathbb{E}[\xi_j D_1(\frac{(X_j - K_j)_+}{\nu_j})] + \omega_j \mathbb{E}[\psi_j D_2(\frac{(X_j - K_j)_-}{\omega_j})]\}$$

such that $\sum_{j=1}^{n} K_j = K$, where ν_j, ω_j are non-negative real numbers, ξ_j, ψ_j are non-negative random variables such that $0 < \mathbb{E}[|\xi_j|] < \infty$, $0 < \mathbb{E}[|\psi_j|] < \infty$,

$\mathbb{E}[\xi_j D_1(\frac{(X_j-x)_+}{\nu_j})] < \infty$, $\mathbb{E}[\psi_j D_2(\frac{(X_j-x)_-}{\omega_j})] < \infty$ for any $x \in R$, $i = 1, \ldots, n$, and $D_1, D_2 : \mathbb{R}_+ \to \mathbb{R}_+$ are two non-degenerated, convex and increasing functions.

In the literature, as in the unified model of Dhaene et al. (2012), the initial capital K is assumed to be positive, but it can definitely be negative, such as when the company has a debt to be paid in the next year. In this case, the problem is generalized to allocate the debt into the business lines. Then, we calculate the allocated capital for each business line based on the principles. If the allocated capital is positive, that amount of capital will be allocated to the business line; if the capital is negative, we will withdraw that amount of capital from that business line. Thus, we assume $K \in \mathbb{R}$ instead.

In this chapter, we will consider some convex quantifying functions, for example, the quadratic and identity functions. Since strong duality holds for convex optimization problems, the KKT conditions are sufficient and necessary for the existence of the solutions to the convex optimization problems. Moreover, it is known that $g(x) + f(x)$ is a convex function if both $g(x)$ and $f(x)$ are convex functions, and $h(x) = g(f(x))$ is also a convex function if $g(x)$ is a convex and increasing function and f is convex. In model (5.4), $(X_j - K_j)_+$ and $(X_j - K_j)_-$ are convex functions of K_j. In addition, $D_1(\frac{(X_j-K_j)_+}{\nu_j})$ and $D_2(\frac{(X_j-K_j)_-}{\omega_j})$ are convex functions of K_j if both $D_1(x)$ and $D_2(x)$ are convex and increasing functions. Hence, the objective function in model (5.4) is a convex function.

Hence, for *Capital Allocation Problem II*, since there are no constraints like

$$f_i(\overline{\boldsymbol{x}}) \leq 0, \ \ i = 1, \ldots, r,$$

the KKT conditions in Lemma 5.2 can be reduced to

(b) $f_i(\overline{\boldsymbol{x}}) = 0$ for $i = r+1, \ldots, m$,

(c) $0 \in [\partial f_0(\overline{\boldsymbol{x}}) + \lambda_{r+1}\partial f_{r+1}(\overline{\boldsymbol{x}}) + \ldots, +\lambda_m \partial f_m(\overline{\boldsymbol{x}})]$.

When D_1 and D_2 are convex and increasing functions, the objective function is a convex function with a closed convex support set $\sum_{j=1}^n K_j = K$, and so the optimization problem (5.4) is a convex optimization problem. The local minimizers determined by the reduced conditions (b) and (c) are also the global minimizers by Lemma 5.4.

Theorem 5.6. *Let $D_1, D_2 : \mathbb{R}_+ \to \mathbb{R}_+$ be non-degenerated, convex and increasing functions. For Capital Allocation Problem II, there exists at least one global minimizer, and K_j, $j = 1, 2, \ldots, n$, the global optimal allocated capital on the jth business line, and λ, an auxiliary real-valued variable, are the solutions to*

$$\begin{cases} \mathbb{E}[\psi_j D'_{2+}(\frac{(X_j - K_j)_-}{\omega_j})\mathbb{I}_{\{X_j \le K_j\}}] \ge \mathbb{E}[\xi_j D'_{1-}(\frac{(X_j - K_j)_+}{\nu_j})\mathbb{I}_{\{X_j > K_j\}}] - \lambda, \; j = 1, 2, \ldots, n, \\ \mathbb{E}[\xi_j D'_{1+}(\frac{(X_j - K_j)_+}{\nu_j})\mathbb{I}_{\{X_j \ge K_j\}}] \ge \mathbb{E}[\psi_j D'_{2-}(\frac{(X_j - K_j)_-}{\omega_j})\mathbb{I}_{\{X_j < K_j\}}] + \lambda, \; j = 1, 2, \ldots, n, \\ \sum_{j=1}^n K_j = K. \end{cases} \tag{5.5}$$

If D_1 and D_2 are differentiable with $D'_1(0) = D'_2(0) = 0$, then K_j, $j = 1, 2, \ldots, n$ and λ, an auxiliary real-valued variable are the solutions to

$$\begin{cases} \mathbb{E}[\psi_j D'_2(\frac{(X_j - K_j)_-}{\omega_j})] = \mathbb{E}[\xi_j D'_1(\frac{(X_j - K_j)_+}{\nu_j})] - \lambda, \; j = 1, 2, \ldots, n, \\ \sum_{j=1}^n K_j = K. \end{cases} \tag{5.6}$$

Proof. In model (5.4), the objective function is

$$\pi(\boldsymbol{k}) = \sum_{j=1}^n \{\nu_j \mathbb{E}[\xi_j D_1(\frac{(X_j - K_j)_+}{\nu_j})] + \omega_j \mathbb{E}[\psi_j D_2(\frac{(X_j - K_j)_-}{\omega_j})]\}, \tag{5.7}$$

and $D_1(x)$ and $D_2(x)$ are continuous on $(0, \infty)$ with

$$\lim_{x \to +\infty} D_1(x) = \lim_{x \to +\infty} D_2(x) = +\infty$$

since D_1 and D_2 are convex and increasing functions on $(0, \infty)$. In addition, $\pi(\boldsymbol{k})$ is continuous. Considering that D_1 and D_2 are also non-degenerated, we have

$$\lim_{\|\boldsymbol{k}\| \to +\infty} \pi(\boldsymbol{k}) = +\infty.$$

Moreover, $\{\boldsymbol{k} \in R^n | \boldsymbol{k} : \sum_{j=1}^n K_j = K\}$, the constraint of model (5.4), is a non-empty, closed and unbounded set. Hence, $\pi(\boldsymbol{k})$ can attain a global minimizer within $\{\boldsymbol{k} \in R^n | \boldsymbol{k} : \sum_{j=1}^n K_j = K\}$ by Lemma 5.3. For the optimization problem (5.4), the Lagrangian objective function is

$$L(\boldsymbol{k}, \lambda) = \sum_{j=1}^n \{\nu_j \mathbb{E}[\xi_j D_1(\frac{(X_j - K_j)_+}{\nu_j})] + \omega_j \mathbb{E}[\psi_j D_2(\frac{(X_j - K_j)_-}{\omega_j})]\} + \lambda(\sum_{j=1}^n K_j - K),$$

and $\boldsymbol{k} = (K_1, \ldots, K_n)$ can be the optimizer if and only if $\sum_{j=1}^n K_j = K$ and $0 \in \left[\frac{\partial^- L}{\partial K_j}, \frac{\partial^+ L}{\partial K_j}\right]$, for $j = 1, \ldots, n$. Similar to Proposition 3.4 in Chapter 3, the optimizers are the solutions to (5.5) by Lemma 5.1 and Lemma 5.2. In addition, these optimizers are global optimizers according to Lemma 5.4. $\square$

Remark 5.7. *As proposed, Capital Allocation Problem I is actually a special case of Capital Allocation Problem II if the part of capital surplus risk in the objective function of model* (5.4) *equals zero. Therefore, we can obtain similar results for Capital Allocation Problem I by setting $\psi_j = 0$, $j = 1, \ldots, n$, or $D_2(x) = 0$.* $\square$

In the following sections, the capital allocation principles are defined as *the quadratic capital allocation* if $D_1(x) = D_2(x) = x^2$ and *the identic capital allocation* if $D_1(x) = D_2(x) = x$.

5.5 Quadratic Capital Allocations

Theorem 5.8. *(Quadratic Capital Allocation) For Capital Allocation Problem II, if $D_1(x) = D_2(x) = x^2$, then K_j, $j = 1, 2, \ldots, n$, the global optimal allocated capital on the jth business line, and λ, an auxiliary real-valued variable, are the* *solutions to*

$$\begin{cases} \mathbb{E}[\frac{\xi_j}{\nu_j}(X_j - K_j)_+] = \mathbb{E}[\frac{\psi_j}{\omega_j}(X_j - K_j)_-] + \frac{\lambda}{2},\ j = 1, 2, \dots, n, \\ \sum_{j=1}^{n} K_j = K. \end{cases} \tag{5.8}$$

Or equivalently, K_j*,* $j = 1, 2, \dots, n$*, the global optimal allocated capital on the* j*th business line, are the solutions to*

$$\begin{cases} \mathbb{E}[\frac{\xi_i}{\nu_i}(X_i - K_i)_+] - \mathbb{E}[\frac{\psi_i}{\omega_i}(X_i - K_i)_-] = \mathbb{E}[\frac{\xi_j}{\nu_j}(X_j - K_j)_+] - \mathbb{E}[\frac{\psi_j}{\omega_j}(X_j - K_j)_-] \\ \sum_{j=1}^{n} K_j = K. \end{cases} \tag{5.9}$$

for $i \neq j$ *and* $(i, j) \in \{1, 2, \dots, n\}^2$*. Hence, the optimizers are equivalently the solutions to these* $n - 1$ *equations with* $\sum_{j=1}^{n} K_j = K$*.*

Proof. Note that $D_1'(0) = D_2'(0) = 0$ if $D_1(x) = D_2(x) = x^2$. In addition, the condition that $D_1, D_2 : \mathbb{R}_+ \to \mathbb{R}_+$ are non-degenerated, convex and increasing functions can be satisfied when $D_1(x) = D_2(x) = x^2$. Then, the result is trivial by Theorem 5.6. □

Note that (5.8) can be rewritten as

$$\begin{cases} h_{X_j,\xi_j,\psi_j}(K_j) = \frac{\lambda}{2} - \frac{\mathbb{E}[\psi_j X_j]}{\omega_j},\ j = 1, \dots, n \\ \sum_{j=1}^{n} K_j = K, \end{cases} \tag{5.10}$$

where

$$\begin{aligned} h_{X_j,\xi_j,\psi_j}(x) &= \mathbb{E}[(\frac{\xi_j}{\nu_j} - \frac{\psi_j}{\omega_j})(X_j - x)_+] - \frac{\mathbb{E}[\psi_j]}{\omega_j}x \\ &= \mathbb{E}[(\frac{\xi_j}{\nu_j} - \frac{\psi_j}{\omega_j})(X_j - x)_-] - \frac{\mathbb{E}[\xi_j]}{\nu_j}x + \mathbb{E}[(\frac{\xi_j}{\nu_j} - \frac{\psi_j}{\omega_j})X_j] \end{aligned} \tag{5.11}$$

and $h_{X_j,\xi_j,\psi_j}(x)$ is denoted as h_j for simplicity. Firstly, h_j is a continuous function of x. By Lemma 5.1, the right-derivative of h_j is

$$\begin{aligned}
h'_{j+} &= -\mathbb{E}[(\frac{\xi_j}{\nu_j} - \frac{\psi_j}{\omega_j})\mathbb{I}_{\{X_j > x\}}] - \frac{\mathbb{E}[\psi_j]}{\omega_j} \\
&= -\mathbb{E}[\frac{\xi_j}{\nu_j}\mathbb{I}_{\{X_j > x\}}] - \mathbb{E}[\frac{\psi_j}{\omega_j}\mathbb{I}_{\{X_j \leq x\}}] \\
&\leq 0,
\end{aligned}$$

and the left-derivative is

$$\begin{aligned}
h'_{j-} &= -\mathbb{E}[(\frac{\xi_j}{\nu_j} - \frac{\psi_j}{\omega_j})\mathbb{I}_{\{X_j \geq x\}}] - \frac{\mathbb{E}[\psi_j]}{\omega_j} \\
&= -\mathbb{E}[\frac{\xi_j}{\nu_j}\mathbb{I}_{\{X_j \geq x\}}] - \mathbb{E}[\frac{\psi_j}{\omega_j}\mathbb{I}_{\{X_j < x\}}] \\
&\leq 0.
\end{aligned}$$

Thus, h_j is a continuous and decreasing function of x. If h_j is not strictly decreasing, the solutions to (5.10) are not unique. Now we will provide one sufficient condition for the existence of a unique solution.

Proposition 5.9. *If $\mathbb{E}[\xi_j] > 0$ and $\mathbb{E}[\psi_j] > 0$, $j = 1, \ldots, n$, then the capital allocation in Theorem 5.8 is unique.*

Proof. Let h_j be defined by (5.10). Note that $\nu_j > 0$ and $\omega_j > 0$. If $\mathbb{E}[\psi_j] > 0$, then $h_j \to -\infty$ as $x \to \infty$, and $h_j \to \infty$ as $x \to -\infty$. Also, if $\mathbb{E}[\xi_j] > 0$ and $\mathbb{E}[\psi_j] > 0$, then $h'_{j+} < 0$ and $h'_{j-} < 0$. In fact, for h'_{j+}, if $\frac{\xi_j}{\nu_j} \leq \frac{\psi_j}{\omega_j}$, then

$$h'_{j+} = -\mathbb{E}[\frac{\xi_j}{\nu_j}] + \mathbb{E}[(\frac{\xi_j}{\nu_j} - \frac{\psi_j}{\omega_j})\mathbb{I}_{\{X_j \leq x\}}] < 0;$$

else if $\frac{\xi_j}{\nu_j} > \frac{\psi_j}{\omega_j}$, then

$$h'_{j+} = -\mathbb{E}[(\frac{\xi_j}{\nu_j} - \frac{\psi_j}{\omega_j})\mathbb{I}_{\{X_j > x\}}] - \mathbb{E}[\frac{\psi_j}{\omega_j}] < 0.$$

Hence, h_j is a strictly decreasing function of x by Lemma 5.5. Thus, $h_j^{-1}(x)$ does exist and is also a continuous and strictly decreasing function of x. Let

$$h_j^{-1}(x) = \inf\{y \in R: \ h_j(y) \leq x\}, \ x \in R$$

be the inverse of h_j. The range of h_j is R and the domain of h_j^{-1} is the range of h_j. Thus, (5.10) is reduced to

$$\begin{cases} K_j = h^{-1}_{X_j,\xi_j,\psi_j}(\dfrac{\lambda}{2} - \dfrac{\mathbb{E}[\psi_j X_j]}{\omega_j}), \ j = 1, \ldots, n \\ K = \displaystyle\sum_{j=1}^{n} K_j. \end{cases} \tag{5.12}$$

Based on (5.12), λ is determined by

$$\sum_{j=1}^{n} h^{-1}_{X_j,\xi_j,\psi_j}(\frac{\lambda}{2} - \frac{\mathbb{E}[\psi_j X_j]}{\omega_j}) = K. \tag{5.13}$$

Considering that $h_j^{-1}(x)$, $j = 1, \ldots, n$ are all continuous and strictly decreasing functions of x, $\sum_{j=1}^{n} h^{-1}_{X_j,\xi_j,\psi_j}(\frac{\lambda}{2} - \frac{\mathbb{E}[\psi_j X_j]}{\omega_j})$ is a continuous and strictly decreasing function of λ. Thus, Equation (5.13) has a unique solution for λ. Hence, the capital allocation principle can be determined by

$$K_j = h^{-1}_{X_j,\xi_j,\psi_j}(\frac{\lambda}{2} - \frac{\mathbb{E}[\psi_j X_j]}{\omega_j})$$

and is also unique. $\square$

Remark 5.10. *If $\frac{\xi_j}{\nu_j} = \frac{\psi_j}{\omega_j}$ and $D_1(x) = D_2(x) = x^2$, model (5.4) is reduced to be*

$$min_{(K_1,\ldots,K_n)\in\mathbb{R}^n} \sum_{j=1}^{n} \nu_j E[\xi_j(\frac{X_j - K_j}{\nu_j})^2], \ s.t. \sum_{j=1}^{n} K_j = K.$$

The auxiliary variable λ in Theorem 5.8 is expressed as

$$\lambda = \frac{2(\sum_{j=1}^{n} \frac{\mathbb{E}[\xi_j X_j]}{\mathbb{E}[\xi_j]} - K)}{\sum_{j=1}^{n} \frac{1}{\mathbb{E}[\frac{\xi_j}{\nu_j}]}}.$$

Hence,

$$K_j = \frac{\mathbb{E}[\xi_j X_j]}{\mathbb{E}[\xi_j]} + \frac{\nu_j}{\mathbb{E}[\xi_j]} \cdot \frac{K - \sum_{i=1}^n \frac{\mathbb{E}[\xi_i X_i]}{\mathbb{E}[\xi_i]}}{\sum_{i=1}^n \frac{\nu_i}{\mathbb{E}[\xi_i]}}. \tag{5.14}$$

Further, if $\sum_{j=1}^n \nu_j = 1$ *and* $\mathbb{E}[\xi_j] = 1$, *the results in Theorem 5.8 can be reduced as in Theorem* 1 *of Dhaene et al. (2012). Hence, the standard deviation principle, CTE, distortion risk measure, exponential principle, and Esscher principle are special cases of the business unit driven allocation principles by assuming*

$$\nu_j = \frac{\mathbb{E}[\xi_j X_j]}{\sum_{j=1}^n \mathbb{E}[\xi_j X_j]}.$$

Also, the aggregate portfolio driven allocation principles in Table 2 *of Dhaene et al. (2012) can be included in our revised model. In fact, the objective functions of Capital Allocation Problem II and Dhaene's Unified Capital Allocation Model are equal since*

$$\mathbb{E}[\frac{\xi_j}{\nu_j}(X_j - K_j)_+(X_j - K_j)_-] = 0$$

in the quadratic case. □

According to Theorem 5.8, we can easily get the corresponding capital allocation principles for the business unit driven and aggregate portfolio driven types by letting $\xi_j(\cdot) = g_j(\cdot) = g_j(X_j)$ or $g_j(S)$, $\psi_j(\cdot) = h_j(\cdot) = h_j(X_j)$ or $h_j(S)$.

5.5.1 The Add On and Off Capital Allocation Principle

Note that (5.14) can be rewritten as

$$K_j = \frac{\frac{\nu_j}{\mathbb{E}[\xi_j]}}{\sum_{i=1}^n \frac{\nu_i}{\mathbb{E}[\xi_i]}} K + \frac{\frac{\nu_j}{\mathbb{E}[\xi_j]}}{\sum_{i=1}^n \frac{\nu_i}{\mathbb{E}[\xi_i]}} \Big(\sum_{i=1}^n (\frac{\nu_i \mathbb{E}[\xi_j X_j]}{\nu_j \mathbb{E}[\xi_i]} - \frac{\mathbb{E}[\xi_i X_i]}{\mathbb{E}[\xi_i]}) \Big). \tag{5.15}$$

We call the capital allocation principle defined by (5.15) *the add on and off capital allocation principle*. Here, $\frac{\nu_j}{\mathbb{E}[\xi_j]}$ represents the weight function normalized business volume (or risk exposure) for the jth business line. Both the initial aggregate capital K and $\sum_{i=1}^n (\frac{\mathbb{E}[\xi_j X_j]}{\mathbb{E}[\xi_j]} - \frac{\mathbb{E}[\xi_i X_i]}{\mathbb{E}[\xi_i]})$ will be multiplied by the ratio of the jth normalized volume to the total normalized volume, respectively. If $\mathbb{E}[\xi_j]$ and ν_j

are equal for each j, then (5.15) can be reduced to

$$K_j = \frac{K}{n} + \frac{\sum_{i=1}^{n}(\frac{\mathbb{E}[\xi_j X_j]}{\mathbb{E}[\xi_j]} - \frac{\mathbb{E}[\xi_i X_i]}{\mathbb{E}[\xi_i]})}{n}. \tag{5.16}$$

Furthermore, we can determine new capital allocation principles by setting different functionals for ξ_j. For example, if ξ_j, $j = 1, \ldots, n$ are constants, then

$$K_j = \frac{K}{n} + \frac{\sum_{i=1}^{n}(\mathbb{E}[X_j] - \mathbb{E}[X_i])}{n}. \tag{5.17}$$

Now the generalized model (5.4) can be reduced to the classical quadratic capital allocation problem:

$$\min_{(K_1,K_2,\cdots,K_n)\in\mathbb{R}^n} \sum_{j=1}^{n} \mathbb{E}[(X_j - K_j)^2].$$

If we further assume that the expectations of X_j, $j = 1, \ldots, n$ are equal, then $K_j = \frac{K}{n}$ for all $j = 1, \ldots, n$. If the expectations are not equal, the allocated capital for the jth business line is the sum of $\frac{K}{n}$ and $\frac{\sum_{i=1}^{n}(\mathbb{E}[X_j]-\mathbb{E}[X_i])}{n}$, the averaged sum of $\mathbb{E}[X_j - \mathbb{E}[X_i]$, $i = 1, 2, \ldots, n$. Moreover, the result in (5.17) does not depend on the distributions of X_j, $j = 1, \ldots, n$ but the expectations of them. Also, the principle is not a proportional capital allocation principle as concluded in Dhaene et al. (2012).

We call the allocation principle

$$K_j = \frac{K}{n} + \frac{\sum_{i=1}^{n}(\frac{\mathbb{E}[\xi_j X_j]}{\mathbb{E}[\xi_j]} - \frac{\mathbb{E}[\xi_i X_i]}{\mathbb{E}[\xi_i]})}{n}, \ j = 1, \ldots, n$$

the *averaged add on and off capital allocation principle*. In this case, $\frac{\nu_j}{\mathbb{E}[\xi_j]}$ are equal for each j, which means that the weight functions normalized business volume are equal for each business line. In this principle, the amount of $\frac{K}{n}$ is the base capital allocated into each business line. Then, if

$$\frac{\sum_{i=1}^{n}(\frac{\mathbb{E}[\xi_j X_j]}{\mathbb{E}[\xi_j]} - \frac{\mathbb{E}[\xi_i X_i]}{\mathbb{E}[\xi_i]})}{n} \geq 0,$$

this amount of capital will be added to the jth business line. But if the figure is negative,

$$\frac{\sum_{i=1}^{n}(\frac{\mathbb{E}[\xi_j X_j]}{\mathbb{E}[\xi_j]} - \frac{\mathbb{E}[\xi_i X_i]}{\mathbb{E}[\xi_i]})}{n} < 0,$$

the absolute value of that figure will be subtracted or withdrawn from the jth business line.

In Dhaene et al. (2012), the proportional principles in Table 1 and Table 2 are unified as

$$K_j = K \frac{\mathbb{E}[\xi_j X_j]}{\sum_{i=1}^{n} \mathbb{E}[\xi_i X_i]},$$

$j = 1, \ldots, n$. We will follow the proportional principles and introduce the various types of the *averaged add on and off capital allocation principles* as below. The *add on and off capital allocation principle* can be achieved by substituting $\frac{1}{n}$ by $\frac{\frac{\nu_j}{\mathbb{E}[\xi_j]}}{\sum_{i=1}^{n} \frac{\nu_i}{\mathbb{E}[\xi_i]}}$. For the business unit driven type,

(a) *The standard deviation type* is the case when $\xi_j = 1 + a_j \frac{X_j - \mathbb{E}[X_j]}{\sigma_{X_j}}$, where σ_{X_j} is the standard deviation of X_j and $a_j \geq 0$. Then, $\frac{\mathbb{E}[\xi_j X_j]}{\mathbb{E}[\xi_j]} = \mathbb{E}[X_j] + a_j \sigma_{X_j}$. The allocated capital to the jth business line is

$$K_j = \frac{K}{n} + \frac{\sum_{i=1}^{n}(\mathbb{E}[X_j] + a_j \sigma_{X_j} - \mathbb{E}[X_i] - a_i \sigma_{X_i})}{n}. \tag{5.18}$$

(b) *The CTE type* is available if $\xi_j = \frac{1}{1-p_j} \mathbb{I}_{\{X_j > F_{X_j}^{-1}(p_j)\}}$, where $p_j \in (0,1)$. Now, $\frac{\mathbb{E}[\xi_j X_j]}{\mathbb{E}[\xi_j]} = CTE_{p_j}(X_j) = \mathbb{E}[X_j | X_j > F_{X_j}^{-1}(p_j)]$. The allocated capital to the jth business line is

$$K_j = \frac{K}{n} + \frac{\sum_{i=1}^{n}(CTE_{p_j}(X_j) - CTE_{p_i}(X_i))}{n}. \tag{5.19}$$

(c) *The distortion risk measure type* is attained if $\xi_j = \frac{g_j^{'}(\overline{F}_{X_j}(X_j))}{\mathbb{E}[g_j^{'}(\overline{F}_{X_j}(X_j))]}$, where $\overline{F}_{X_j}(\cdot)$ is the survival function of X_j, and g_j is a distortion function defined as $g_j : [0,1] \mapsto [0,1]$ with $g_j^{'} > 0$ and $g_j^{''} < 0$. Then, $\frac{\mathbb{E}[\xi_j X_j]}{\mathbb{E}[\xi_j]} = \mathbb{E}[X_j g_j^{'}(\overline{F}_{X_j}(X_j))]$, which is the spectral risk measure with respect to g. The allocated capital to the jth business line is

$$K_j = \frac{K}{n} + \frac{\sum_{i=1}^{n}(\mathbb{E}[X_j g_j'(\overline{F}_{X_j}(X_j))] - \mathbb{E}[X_i g_i'(\overline{F}_{X_i}(X_i))])}{n}. \tag{5.20}$$

The CTE type is a special case of this principle.

(d) *The exponential type* is the principle by assuming $\xi_j = \int_0^1 \frac{e^{\gamma a_j X_j}}{\mathbb{E}[e^{\gamma a_j X_j}]} d\gamma$, where $a_j > 0$. Thus, $\frac{\mathbb{E}[\xi_j X_j]}{\mathbb{E}[\xi_j]} = \frac{1}{a_j} \ln \mathbb{E}[e^{a_j X_j}]$. The allocated capital to the jth business line is

$$K_j = \frac{K}{n} + \frac{\sum_{i=1}^{n}(\frac{1}{a_j} \ln \mathbb{E}[e^{a_j X_j}] - \frac{1}{a_i} \ln \mathbb{E}[e^{a_i X_i}])}{n}. \tag{5.21}$$

(e) *The Esscher transform type* is derived when $\xi_j = \frac{e^{a_j X_j}}{\mathbb{E}[e^{a_j X_j}]}$, where $a_j > 0$. Note that $\frac{\mathbb{E}[\xi_j X_j]}{\mathbb{E}[\xi_j]}$ is the Esscher transform of X_j. The allocated capital to the jth business line is

$$K_j = \frac{K}{n} + \frac{\sum_{i=1}^{n}(\frac{\mathbb{E}[X_j e^{a_j X_j}]}{\mathbb{E}[e^{a_j X_j}]} - \frac{\mathbb{E}[X_i e^{a_i X_i}]}{\mathbb{E}[e^{a_i X_i}]})}{n}. \tag{5.22}$$

Compared with Table 1 in Dhaene et al. (2012), these allocation principles are not proportional. For the aggregate portfolio driven type, we can attain the following parallel results illustrated by Table 2 in Dhaene et al. (2012).

(a) *The standard covarience type* is the case when $\xi_j = 1 + a_j \frac{S - \mathbb{E}[S]}{\sigma_S}$, where σ_S is the standard deviation of S and $a_j \geq 0$, $\frac{\mathbb{E}[\xi_j X_j]}{\mathbb{E}[\xi_j]} = \mathbb{E}[X_j] + a_j \frac{\mathrm{Cov}[X_j, S]}{\sigma_S}$. The allocated capital to the jth business line is

$$K_j = \frac{K}{n} + \frac{\sum_{i=1}^{n}(\mathbb{E}[X_j] + a_j \frac{\mathrm{Cov}[X_j,S]}{\sigma_S} - \mathbb{E}[X_i] - a_i \frac{\mathrm{Cov}[X_i,S]}{\sigma_S})}{n}. \tag{5.23}$$

(b) *The CTE type* is available if $\xi_j = \frac{1}{1-p_j} \mathbb{I}_{\{S > F_S^{-1}(p_j)\}}$, where $p_j \in (0,1)$, $\frac{\mathbb{E}[\xi_j X_j]}{\mathbb{E}[\xi_j]} = \mathbb{E}[X_j | S > F_S^{-1}(p_j)]$. The allocated capital to the jth business line is

$$K_j = \frac{K}{n} + \frac{\sum_{i=1}^{n}(\mathbb{E}[X_j | S > F_S^{-1}(p_j)] - \mathbb{E}[X_i | S > F_S^{-1}(p_i)])}{n}. \tag{5.24}$$

(c) *The distortion risk measure type* is if $\xi_j = \frac{g_j^{'}(\overline{F}_S(S))}{E[g_j^{'}(\overline{F}_S(S))]}$, where $\overline{F}_S(\cdot)$ is the survival function of S, $g_j : [0,1] \mapsto [0,1]$, $g_j^{'} > 0$ and $g_j^{''} < 0$. Then, $\frac{\mathbb{E}[\xi_j X_j]}{\mathbb{E}[\xi_j]} = \mathbb{E}[X_j g_j^{'}(\overline{F}_S(S))]$. The allocated capital to the jth business line is

$$K_j = \frac{K}{n} + \frac{\sum_{i=1}^{n}(\mathbb{E}[X_j g_j^{'}(\overline{F}_S(S))] - \mathbb{E}[X_i g_i^{'}(\overline{F}_S(S))])}{n}. \tag{5.25}$$

(d) *The exponential type* is the principle by assuming $\xi_j = \int_0^1 \frac{e^{\gamma a_j S}}{\mathbb{E}[e^{\gamma a_j S}]} d\gamma$, where $a_j > 0$. Thus, $\frac{\mathbb{E}[\xi_j X_j]}{E[\xi_j]} = \mathbb{E}[X_j \int_0^1 \frac{e^{\gamma a_j S}}{\mathbb{E}[e^{\gamma a_j S}]} d\gamma]$. The allocated capital to the jth business line is

$$K_j = \frac{K}{n} + \frac{\sum_{i=1}^{n}(\mathbb{E}[X_j \int_0^1 \frac{e^{\gamma a_j S}}{\mathbb{E}[e^{\gamma a_j S}]} d\gamma] - \mathbb{E}[X_i \int_0^1 \frac{e^{\gamma a_i S}}{\mathbb{E}[e^{\gamma a_i S}]} d\gamma])}{n}. \tag{5.26}$$

(e) *The Esscher transform type* is derived when $\xi_j = \frac{e^{a_j S}}{\mathbb{E}[e^{a_j S}]}$, where $a_j > 0$, and we get $\frac{\mathbb{E}[\xi_j X_j]}{\mathbb{E}[\xi_j]} = \frac{\mathbb{E}[X_j e^{a_j S}]}{\mathbb{E}[e^{a_j S}]}$. The allocated capital to the jth business line is

$$K_j = \frac{K}{n} + \frac{\sum_{i=1}^{n}(\frac{\mathbb{E}[X_j e^{a_j S}]}{\mathbb{E}[e^{a_j S}]} - \frac{\mathbb{E}[X_i e^{a_i S}]}{\mathbb{E}[e^{a_i S}]})}{n}. \tag{5.27}$$

5.5.2 Properties of the Averaged Add On and Off Capital Allocation Principles

The averaged add on and off capital allocation principles have the following properties, but some of them cannot be satisfied by the general quadratic capital allocation principle in Theorem 5.8 since even the principle defined by (5.16) cannot satisfy them as a special case.

5.5.2.1 No Negative Loading, Riskless Allocation and No Unjustified Loading

No negative loading is defined as $K_j \geq E[X_j]$, $j = 1, 2, \ldots, n$, and the principle defined by (5.16) can satisfy it under some conditions as provided in Proposition

5.11. Our general allocation may satisfy this property under certain conditions illustrated by the following proposition.

Proposition 5.11. *Let K_j, $j = 1, \ldots, n$ be the capital allocation principle defined by (5.16). Then, the following results hold:*

(a) $K_j \geq \mathbb{E}[X_j]$ *if* $K \geq \sum_{j=1}^{n} \frac{\mathbb{E}[\xi_j X_j]}{\mathbb{E}[\xi_j]}$ *and* $\mathrm{Cov}(\xi_j, X_j) \geq 0$*;*

(b) $K_j \leq \mathbb{E}[X_j]$ *if* $K \leq \sum_{j=1}^{n} \frac{\mathbb{E}[\xi_j X_j]}{\mathbb{E}[\xi_j]}$ *and* $\mathrm{Cov}(\xi_j, X_j) \leq 0$*;*

(c) $K_j = \frac{\mathbb{E}[\xi_j X_j]}{\mathbb{E}[\xi_j]}$ *if* $K = \sum_{j=1}^{n} \frac{\mathbb{E}[\xi_j X_j]}{\mathbb{E}[\xi_j]}$.

Proof. It is trivial by (5.16). □

The axioms of *riskless allocation* in Maume-Deschamps et al. (2015) and *no unjustified loading* in Furman and Zitikis (2008b) are axioms considering the business line without risk, namely, the corresponding risk random variable is assumed as a constant c. Both of their papers argue that the allocated capital to the riskless business line should be c, and the capital allocation problem is revised to be the problem of allocating $K - c$ among other risky business lines. The allocation principles in Maume-Deschamps et al. (2015) satisfy this property, however, our general principles in Theorem 5.8 fail to satisfy it since, as a special case of our proposed model, the principles derived by (5.16) do not satisfy it. In fact, $K_j = c$ when $X_j = c$ holds only if $K = \sum_{j=1}^{n} \frac{\mathbb{E}[\xi_j X_j]}{\mathbb{E}[\xi_j]}$, see Proposition 5.11 (c). Obviously, this property might highly depend on the value of the initial capital K.

In most of the models in the literature, the initial capital K is usually assumed as fixed and exogenously given. For instance, the initial capital may be estimated by some risk measures. Based on Proposition 5.11, we find that the initial capital plays a significant role in determining whether the underlying principle is an effective one. Hence, determining the amount of K is still a potential problem for future research, although we may apply risk measures to calculate it endogenously.

5.5.2.2 Translation Invariance

Now we will check the axiom of *translation invariance* for the allocation by (5.16). Let

$$(K_1^*, \dots, K_n^*) = A_{X_1+a_1,\dots,X_n+a_n}(K)$$

and

$$(K_1^{**}, \dots, K_n^{**}) = A_{X_1,\dots,X_n}(K - \sum_{j=1}^{n} a_j),$$

where

$$A_{X_1,\dots,X_n}(K) = (K_1, \dots, K_n)$$

is as defined in Maume-Deschamps et al. (2015). Moreover, we assume the weight functions as $\xi_j = g_j(X_1, \dots, X_n)$, $j = 1, \dots, n$ in Proposition 5.12 and Proposition 5.15.

Proposition 5.12. *The allocation by* (5.16) *satisfies the axiom of translation invariance if*

$$\frac{\mathbb{E}[g_j(X_1 + a_1, \dots, X_n + a_n)X_j]}{\mathbb{E}[g_j(X_1 + a_1, \dots, X_n + a_n)]} = \frac{\mathbb{E}[g_j(X_1, \dots, X_n)X_j]}{\mathbb{E}[g_j(X_1, \dots, X_n)]} \tag{5.28}$$

holds for $j = 1, \dots, n$.

Proof. If (5.28) holds for $j = 1, \dots, n$, then

$$\frac{\mathbb{E}[g_j(X_1 + a_1, \dots, X_n + a_n)(X_j + a_j)]}{\mathbb{E}[g_j(X_1 + a_1, X_n + a_n)]} = \frac{\mathbb{E}[g_j(X_1, \dots, X_n)(X_j + a_j)]}{\mathbb{E}[g_j(X_1, \dots, X_n)]}$$

holds for $j = 1, \dots, n$. Thus,

$$K_j^* = K_j + a_j - \frac{\sum_{j=1}^{n} a_j}{n}.$$

In addition,

$$K_j^{**} = K_j - \frac{\sum_{j=1}^{n} a_j}{n}$$

holds for the capital allocation by (5.16). Hence, $K_j^* = K_j^{**} + a_j$, $j = 1, \dots, n$. $\square$

Remark 5.13. *Note that $g_j(x_1,\ldots,x_n)$ may depend on the random variables X_j, $j=1,\ldots,n$, and we denote them as $g_j^{X_1,\ldots,X_n}(x_1,\ldots,x_n)$. If we assume*

$$g_j^{X_1,\ldots,X_n}(x_1,\ldots,x_n)=\beta_j\mathbb{I}_{\{x_1>\rho_1(X_1),\ldots,x_n>\rho_n(X_n)\}}+\gamma_j\mathbb{I}_{\{x_1\leq\rho_1(X_1),\ldots,x_n\leq\rho_n(X_n)\}},$$

where ρ_j are distortion risk measures, $\beta_j,\ \gamma_j\in\mathbb{R}_+$, $j=1,\ldots,n$, then

$$g_j^{X_1+a_1,\ldots,X_n+a_1}(X_1+a_1,\ldots,X_n+a_n)=g_j^{X_1,\ldots,X_n}(X_1,\ldots,X_n).$$

Obviously, the condition (5.28) *in Proposition 5.12 can be satisfied.* □

5.5.2.3 Scale Invariance and Continuity

Furman and Zitikis (2008b) defined the axioms of *scale invariance, sub-scale invariance, and super-scale invariance*, which are only defined for the jth allocated capital as $A(X_j,S)=K_j$. In addition, Maume-Deschamps et al. (2015) defined the axioms of *positive homogeneity* and *continuity* and applied the notation $A_{X_1,\ldots,X_n}(K)=(K_1,\ldots,K_n)$ to these two axioms.

Definition 5.14. For any $b\in R^+$,

(a) *scale invariance*: $A(bX_j,\sum_{i\neq j}X_i+bX_j)=bA(X_j,S)$;

(b) *sub-scale invariance*: $A(bX_j,\sum_{i\neq j}X_i+bX_j)\leq bA(X_j,S)$;

(c) *super-scale invariance*: $A(bX_j,\sum_{i\neq j}X_i+bX_j)\geq bA(X_j,S)$;

(d) *positive homogeneity*: $A_{bX_1,\ldots,bX_n}(bK)=bA_{X_1,\ldots,X_n}(K)$;

(e) *continuity*: $lim_{\epsilon\to 0}A_{X_1,\ldots,(1+\epsilon)X_j,\ldots,X_n}(K)=A_{X_1,\ldots,X_n}(K)$.

If we suppose $(K_1^*,\ldots,K_j^*,\ldots,K_n^*)=A_{X_1,\ldots,b_jX_j,\ldots,X_n}(K)$, where $b_j\in R^+$, and

$$(K_1^{**},\ldots,K_j^{**},\ldots,K_n^{**})=A_{bX_1,\ldots,bX_j,\ldots,bX_n}(K),$$

where $b\in R^+$.

Proposition 5.15. *Let K_j, $j = 1, \dots, n$ be the capital allocation principle defined by (5.16). Assume (5.28) holds for $j = 1, \dots, n$. Then,*

(a) scale invariance holds if $b_j = 1$ or $K = \sum_{i=1, i\neq j}^{n} \frac{\mathbb{E}[\xi_i X_i]}{E[\xi_i]}$;

(b) sub-scale invariance holds if $(b_j - 1)(K - \sum_{i=1, i\neq j}^{n} \frac{\mathbb{E}[\xi_i X_i]}{E[\xi_i]}) \leq 0$;

(c) super-scale invariance holds if $(b_j - 1)(K - \sum_{i=1, i\neq j}^{n} \frac{\mathbb{E}[\xi_i X_i]}{E[\xi_i]}) \geq 0$;

(d) continuity holds.

Assume

$$\frac{\mathbb{E}[g_j(bX_1, \dots, bX_j, \dots, bX_n)X_j]}{\mathbb{E}[g_j(bX_1, \dots, bX_j, \dots, bX_n)]} = \frac{\mathbb{E}[g_j(X_1, \dots, X_j, \dots, X_n)X_j]}{\mathbb{E}[g_j(X_1, \dots, X_j, \dots, X_n)]} \tag{5.29}$$

holds for $j = 1, \dots, n$. Then,

(e) positive homogeneity holds.

Proof. Note that (a), (b), and (c) can be verified since K_j^* and K_i^* $(i \neq j)$, $j = 1, \dots, n$ have the following expressions.

$$\begin{aligned}
K_j^* &= b_j \frac{\mathbb{E}[g_j(X_1, \dots, b_j X_j, \dots, X_n)X_j]}{\mathbb{E}[g_j(X_1, \dots, b_j X_j, \dots, X_n)]} + \frac{1}{n}(K - \sum_{i=1, i\neq j}^{n} \frac{\mathbb{E}[\xi_i X_i]}{\mathbb{E}[\xi_i]} \\
&\quad - b_j \frac{\mathbb{E}[g_j(X_1, \dots, b_j X_j, \dots, X_n)X_j]}{\mathbb{E}[g_j(X_1, \dots, b_j X_j, \dots, X_n)]}) \\
&= b_j (\frac{\mathbb{E}[g_j(X_1, \dots, b_j X_j, \dots, X_n)X_j]}{\mathbb{E}[g_j(X_1, \dots, b_j X_j, \dots, X_n)]} + \frac{1}{n}(K - \sum_{i=1, i\neq j}^{n} \frac{\mathbb{E}[\xi_i X_i]}{\mathbb{E}[\xi_i]} - \\
&\quad \frac{\mathbb{E}[g_j(X_1, \dots, b_j X_j, \dots, X_n)X_j]}{\mathbb{E}[g_j(X_1, \dots, b_j X_j, \dots, X_n)]})) + (1 - b_j)\frac{1}{n}(K - \sum_{i=1, i\neq j}^{n} \frac{\mathbb{E}[\xi_i X_i]}{\mathbb{E}[\xi_i]}) \\
&= b_j (\frac{\mathbb{E}[g_j(X_1, \dots, X_j, \dots, X_n)X_j]}{\mathbb{E}[g_j(X_1, \dots, X_j, \dots, X_n)]} + \frac{1}{n}(K - \sum_{i=1, i\neq j}^{n} \frac{\mathbb{E}[\xi_i X_i]}{\mathbb{E}[\xi_i]} \\
&\quad - b_j \frac{\mathbb{E}[g_j(X_1, \dots, X_j, \dots, X_n)X_j]}{\mathbb{E}[g_j(X_1, \dots, X_j, \dots, X_n)]})) + (1 - b_j)\frac{1}{n}(K - \sum_{i=1, i\neq j}^{n} \frac{\mathbb{E}[\xi_i X_i]}{\mathbb{E}[\xi_i]}) \\
&= b_j K_j + \frac{1}{n}(1 - b_j)(K - \sum_{i=1, i\neq j}^{n} \frac{\mathbb{E}[\xi_i X_i]}{E[\xi_i]}),
\end{aligned}$$

where the third equation holds if (5.28) holds for $j = 1, \ldots, n$. In addition,

$$\begin{aligned}
K_i^* &= \frac{\mathbb{E}[\xi_i X_i]}{\mathbb{E}[\xi_i]} + \frac{1}{n}(K - \sum_{l=1, l\neq i, l\neq j}^{n} \frac{\mathbb{E}[\xi_l X_l]}{E[\xi_l]} - \frac{\mathbb{E}[\xi_i X_i]}{\mathbb{E}[\xi_i]} \\
&\quad - b_j \frac{\mathbb{E}[g_j(X_1, \ldots, b_j X_j, \ldots, X_n) X_j]}{\mathbb{E}[g_j(X_1, \ldots, b_j X_j, \ldots, X_n)]}) \\
&= \frac{\mathbb{E}[\xi_i X_i]}{\mathbb{E}[\xi_i]} + \frac{1}{n}(K - \sum_{l=1, l\neq i, l\neq j}^{n} \frac{\mathbb{E}[\xi_l X_l]}{\mathbb{E}[\xi_l]} - \frac{\mathbb{E}[\xi_i X_i]}{\mathbb{E}[\xi_i]} \\
&\quad - b_j \frac{\mathbb{E}[g_j(X_1, \ldots, X_j, \ldots, X_n) X_j]}{\mathbb{E}[g_j(X_1, \ldots, X_j, \ldots, X_n)]}) \\
&= K_i + \frac{1}{n}(1 - b_j) \frac{\mathbb{E}[\xi_j X_j]}{\mathbb{E}[\xi_j]},
\end{aligned}$$

where the second equation holds if (5.28) holds again.

For (d), to prove

$$lim_{\epsilon \to 0} A_{X_1, \ldots, (1+\epsilon) X_j, \ldots, X_n}(K) = A_{X_1, \ldots, X_j, \ldots, X_n}(K)$$

is equally to prove

$$lim_{b_j \to 1} A_{X_1, \ldots, b_j X_j, \ldots, X_n}(K) = A_{X_1, \ldots, X_j, \ldots, X_n}(K),$$

which holds since

$$lim_{b_j \to 1} K_j^* = K_j$$

and

$$lim_{b_j \to 1} K_i^* = K_i$$

for $i \neq j$.

For (e), we have

$$\begin{aligned}
K_j^{**} &= b \frac{\mathbb{E}[g_j(bX_1, \ldots, bX_j, \ldots, bX_n) X_j]}{\mathbb{E}[g_j(bX_1, \ldots, bX_j, \ldots, bX_n)]} + \frac{1}{n}(bK - b \sum_{i=1}^{n} \frac{\mathbb{E}[g_i(bX_1, \ldots, bX_i, \ldots, bX_n) X_i]}{\mathbb{E}[g_i(bX_1, \ldots, bX_i, \ldots, bX_n)]}) \\
&= b(\frac{\mathbb{E}[g_j(bX_1, \ldots, bX_j, \ldots, bX_n) X_j]}{\mathbb{E}[g_j(bX_1, \ldots, bX_j, \ldots, bX_n)]} + \frac{1}{n}(K - \sum_{i=1}^{n} \frac{\mathbb{E}[g_i(bX_1, \ldots, bX_i, \ldots, bX_n) X_i]}{\mathbb{E}[g_i(bX_1, \ldots, bX_i, \ldots, bX_n)]})) \\
&= bK_j,
\end{aligned}$$

if (5.29) holds for $j = 1, \ldots, n$. $\square$

5.5.2.4 Consistency and Additivity

In Furman and Zitikis (2008b), the allocated capital to the jth line is defined as $A(X_j, S)$, and they defined $S_I = \sum_{i \in I} X_i$, where $I \subseteq N = \{1, 2, \ldots, n\}$. The axiom of *consistency* is satisfied if

$$\sum_{i \in I} A(X_i, S) = A(S_I, S)$$

holds. Maume-Deschamps et al. (2015) defined the *sub-additivity* for capital allocation principles as

$$\sum_{i \in I} A(X_i, S) \leq A(S_I, S).$$

Now we will check this axiom for the allocation by (5.17) in this section by adopting the definition in Furman and Zitikis (2008b).

Proposition 5.16. *For the capital allocation by* (5.17)*, the following results hold.*

(a) $\sum_{i \in I} A(X_i, S) \geq A(S_I, S)$ *if* $K \geq \sum_{i=1}^{n} \mathbb{E}[X_i]$.

(b) $\sum_{i \in I} A(X_i, S) \leq A(S_I, S)$ *if* $K \leq \sum_{i=1}^{n} \mathbb{E}[X_i]$.

(c) $\sum_{i \in I} A(X_i, S) > A(S_I, S)$ *if* $K > \sum_{i=1}^{n} \mathbb{E}[X_i]$ *and* $1 < r < n$.

(d) $\sum_{i \in I} A(X_i, S) < A(S_I, S)$ *if* $K < \sum_{i=1}^{n} \mathbb{E}[X_i]$ *and* $1 < r < n$.

(e) $\sum_{i \in I} A(X_i, S) = A(S_I, S)$ *if* $K = \sum_{i=1}^{n} \mathbb{E}[X_i]$*, or* $r = 1$ *or* n.

Proof. Firstly, we denote $card(S_I) = r$, and we have

$$A(S_I, S) = \frac{K}{n - r + 1} + \mathbb{E}[S_I] - \frac{\sum_{i=1}^{n} \mathbb{E}[X_i]}{n - r + 1}$$

and

$$\sum_{i\in I} A(X_i,S) = \sum_{i\in I} \Big(\frac{K}{n} + \mathbb{E}[X_i] - \frac{\sum_{i=1}^n \mathbb{E}[X_i]}{n}\Big)$$
$$= \frac{r}{n}K + \mathbb{E}[S_I] - \frac{r}{n}\sum_{i=1}^n \mathbb{E}[X_i].$$

Thus,

$$\sum_{i\in I} A(X_i,S) - A(S_I,S) = \Big(\frac{r}{n} - \frac{1}{n-r+1}\Big)\Big(K - \sum_{i=1}^n \mathbb{E}[X_i]\Big).$$

Also, $\frac{r}{n} - \frac{1}{n-r+1} \geq 0$ since $1 \leq r \leq n$. Then the results can be concluded. □

Remark 5.17. *If the allocation by* (5.15) *or* (5.16) *satisfy these properties, they will converge to the allocation derived by* (5.17) *since the properties only hold when the loading factors ξ_i are constants. Meanwhile, we can conclude that the property for the capital allocation depends on the value of the initial capital K and the total expected loss $\sum_{i=1}^n \mathbb{E}[X_i]$. By Proposition 5.16 (a), $\sum_{i=1}^n \mathbb{E}[X_i]$ can be treated as a threshold of the initial capital for a subadditivitive allocation. For the allocation principle by* (5.17)*, we can find that the axioms of no negative loading and sub-additivity will be satisfied at the same time if the initial capital is no less than this threshold.* □

According to the Proposition 5.16, the axioms of *super-additivity*, *strictly sub-additivity* and *strictly super-additivity* can be defined.

Definition 5.18. (Axioms for Additivity)

(a) Super-additivity: $\sum_{i\in I} A(X_i,S) \leq A(S_I,S)$.

(b) Strictly sub-additivity: $\sum_{i\in I} A(X_i,S) > A(S_I,S)$.

(c) Strictly super-additivity: $\sum_{i\in I} A(X_i,S) < A(S_I,S)$.

The corresponding dynamic *averaged add on and off* capital allocation under distortion risk measures can be realized by adopting the updated distortion functions proposed in Laeven and Goovaerts (2004).

			$K=1$			$K=3$		
α_1	α_2	α_3	K_1	K_2	K_3	K_1	K_2	K_3
0.05	0.05	1	0.0279	0.0279	0.9442	0.8230	0.8230	1.3540
0.05	0.5	1	-0.6554	0.7557	0.8997	0.6744	1.0901	1.2355
0.5	0.5	1	0.2615	0.2615	0.4770	0.9530	0.9530	1.0939
1	1	1	0.3333	0.3333	0.3333	1	1	1
0.05	1	1	-0.7829	0.8915	0.8915	0.6147	1.1927	1.1927
0.05	2	1	-0.8842	0.9993	0.8850	0.5475	1.3046	1.1478
2	2	1	0.3753	0.3753	0.2493	1.0395	1.0395	0.9210
0.05	10	1	-1.1996	1.3349	0.8647	0.2962	1.6891	1.0148
10	10	1	0.4041	0.4041	0.1918	1.1223	1.1223	0.7554
0.05	100	1	-1.5409	1.6979	0.8429	-0.0765	2.1396	0.9369
100	100	1	0.4168	0.4168	0.1663	1.1676	1.1676	0.6648

TABLE 5.1: Example 5.1-1

Now we will provide numerical examples for the quadratic capital allocation.

Example 5.1. *If $D_1(x) = D_2(x) = x^2$, $\nu_j = \omega_j = 1$, $\xi_j = 1 - e^{-\alpha_j X_j}$, $\psi_j = e^{-\alpha_j X_j}$, $X_j \sim Exp(\beta_j)$, $j = 1, \dots, n$. The optimal capital allocations should be solutions to*

$$\begin{cases} \left(\frac{1}{\beta_j} e^{-\beta_j K_j} - \frac{2\beta_j}{(\alpha_j+\beta_j)^2} e^{-(\alpha_j+\beta_j)K_j}\right)\mathbb{I}_{\{K_j \geq 0\}} \\ \quad + \left(\frac{1}{\beta_j} - \frac{2\beta_j}{(\alpha_j+\beta_j)^2} + K_j \frac{\beta_j - \alpha_j}{\alpha_j + \beta_j}\right)\mathbb{I}_{\{K_j < 0\}} \\ \qquad\qquad + \frac{\beta_j}{(\alpha_j+\beta_j)^2} - K_j \frac{\beta_j}{\alpha_j+\beta_j} = \frac{\lambda}{2}, \ j = 1, 2, \dots, n, \\ \qquad\qquad\qquad \sum_{j=1}^{n} K_j = K. \end{cases} \tag{5.30}$$

Suppose $\beta_1 = \beta_2 = \beta_3 = 1$. The results are in Table 5.1. Then, if we suppose $\alpha_1 = \alpha_2 = \alpha_3 = 1$, we get Table 5.2.

5.6 Identic Capital Allocations

When $D_1(x) = D_2(x) = x$, we can obtain the capital allocation principles in Theorem 5.19.

			$K=1$			$K=3$		
β_1	β_2	β_3	K_1	K_2	K_3	K_1	K_2	K_3
1	1	1	0.3333	0.3333	0.3333	1.0000	1.0000	1.0000
1	5	10	1.0298	0.0922	-0.1220	1.7168	0.7028	0.5804
1	1	5	0.8345	0.8345	-0.6690	1.3120	1.3120	0.3760
1	1	10	0,9367	0.9367	-0.8734	1.3467	1.3467	0.3066
1	5	5	0.9868	0.0066	0.0066	1.6688	0.6656	0.6656
1	10	10	1.0423	-0.0212	-0.0212	1.7675	0.6163	0.6163
5	5	10	0.3652	0.3652	0.2697	1.0500	1.0500	0.9000
5	10	10	0.3989	0.3005	0.3005	1.1028	0.9486	0.9486

TABLE 5.2: Example 5.1-2

Theorem 5.19. *(Identic Capital Allocation) For Capital Allocation Problem II with* $\mathbb{E}[|\xi_j|] < \infty$ *and* $\mathbb{E}[|\psi_j|] < \infty$*, if* $D_1(x) = D_2(x) = x$*, then* K_j*,* $j = 1, 2, \ldots, n$*, the global optimal allocated capital on the* j*th business line, and* λ*, an auxiliary real-valued variable, are the solutions to*

$$\begin{cases} \mathbb{E}[\psi_j \mathbb{I}_{\{X_j \le K_j\}}] \ge \mathbb{E}[\xi_j \mathbb{I}_{\{X_j > K_j\}}] - \lambda, \; j = 1, 2, \ldots, n, \\ \mathbb{E}[\xi_j \mathbb{I}_{\{X_j \ge K_j\}}] \ge \mathbb{E}[\psi_j \mathbb{I}_{\{X_j < K_j\}}] + \lambda, \; j = 1, 2, \ldots, n, \\ \displaystyle\sum_{j=1}^{n} K_j = K. \end{cases} \tag{5.31}$$

This theorem can be easily proved by Theorem 5.6. The equation (5.31) in Theorem 5.19 can be rewritten as

$$\begin{cases} G_{X_j,\xi_j,\psi_j}(K_j) \le \dfrac{\mathbb{E}[\xi_j] - \lambda}{\mathbb{E}[\xi_j + \psi_j]} \le H_{X_j,\xi_j,\psi_j}(K_j), \; j = 1, 2, \ldots, n, \\ \displaystyle\sum_{j=1}^{n} K_j = K, \end{cases} \tag{5.32}$$

where

$$G_{X_j,\xi_j,\psi_j}(x) = \frac{\mathbb{E}[(\xi_j + \psi_j)\mathbb{I}_{\{X_j < x\}}]}{\mathbb{E}[\xi_j + \psi_j]}$$

and

$$H_{X_j,\xi_j,\psi_j}(x) = \frac{\mathbb{E}[(\xi_j + \psi_j)\mathbb{I}_{\{X_j \le x\}}]}{\mathbb{E}[\xi_j + \psi_j]}.$$

It is easy to verify that $G_{X_j,\xi_j,\psi_j}(x)$ is increasing and left-continuous, while $H_{X_j,\xi_j,\psi_j}(x)$ is increasing and right-continuous. Moreover,

$$G_{X_j,\xi_j,\psi_j}(x) = \lim_{y \to x^-} H_{X_j,\xi_j,\psi_j}(y)$$

and

$$G_{X_j,\xi_j,\psi_j}(x) \leq H_{X_j,\xi_j,\psi_j}(x).$$

In addition, $H_{X_j,\xi_j,\psi_j}(x) \to 1$ as $x \to \infty$ and $H_{X_j,\xi_j,\psi_j}(x) \to 0$ as $x \to -\infty$ if $\mathbb{E}[|\xi_j|] < \infty$ and $\mathbb{E}[|\psi_j|] < \infty$. Hence, $H_{X_j,\xi_j,\psi_j}(x)$ is a distribution function. Let

$$H^{-1}_{X_j,\xi_j,\psi_j}(\alpha) = \inf\{x \in R : \; H_{X_j,\xi_j,\psi_j}(x) \geq \alpha\}, \; for \; 1 \leq \alpha \leq 1$$

be the left-continuous inverse of H_{X_j,ξ_j,ψ_j}. For $\alpha = 0$ and $\alpha = 1$, using convention $\inf \emptyset = \infty$, $\sup \emptyset = -\infty$. For a distribution function F, we have $F^{-1}(0) = -\infty$ and $F^{-1+}(1) = \infty$. Denote

$$G^{-1+}_{X_j,\xi_j,\psi_j}(\alpha) = \sup\{x \in R : \; G_{X_j,\xi_j,\psi_j}(x) \leq \alpha\} \; for \; 1 \leq \alpha \leq 1.$$

Similar to Exercise 8, page $1-12$ in Wichuta (2001), we have

$$G^{-1+}_{X_j,\xi_j,\psi_j}(\alpha) = \lim_{u \searrow \alpha} H^{-1}_{X_j,\xi_j,\psi_j}(u) = H^{-1}_{X_j,\xi_j,\psi_j}(\alpha+).$$

Since $H^{-1}_{X_j,\xi_j,\psi_j}$ is increasing, we have

$$H^{-1}_{X_j,\xi_j,\psi_j}(\alpha+) \geq H^{-1}_{X_j,\xi_j,\psi_j}(\alpha)$$

or

$$G^{-1+}_{X_j,\xi_j,\psi_j}(\alpha) \geq H^{-1}_{X_j,\xi_j,\psi_j}(\alpha).$$

By the relation between (5) and (18) of Wichuta (2001), the conditions in (5.32) are equivalent to

$$\begin{cases} H^{-1}_{X_j,\xi_j,\psi_j}(\dfrac{\mathbb{E}[\xi_j]-\lambda}{\mathbb{E}[\xi_j+\psi_j]}) \leq K_j \leq G^{-1+}_{X_j,\xi_j,\psi_j}(\dfrac{\mathbb{E}[\xi_j]-\lambda}{\mathbb{E}[\xi_j+\psi_j]}),\ j=1,2,\ldots,n, \\ \displaystyle\sum_{j=1}^{n} K_j = K, \end{cases} \tag{5.33}$$

Note that for each j, it must hold that $0 \leq \frac{\mathbb{E}[\xi_j]-\lambda}{\mathbb{E}[\xi_j+\psi_j]} \leq 1$. Thus,

$$\begin{aligned} & -\mathbb{E}[\psi_j] \leq \lambda \leq \mathbb{E}[\xi_j],\ j=1,\ldots,n, \\ \iff & \max\{-\mathbb{E}[\psi_j]\} \leq \lambda \leq \min\{\mathbb{E}[\xi_j]\}, \\ \iff & -\min\{\mathbb{E}[\psi_j]\} \leq \lambda \leq \min\{\mathbb{E}[\xi_j]\}, \end{aligned}$$

Let $F^{-1(p_j)}_{X_j,\xi_j,\psi_j}(\alpha) = p_j H^{-1}_{X_j,\xi_j,\psi_j}(\alpha) + (1-p_j) G^{-1+}_{X_j,\xi_j,\psi_j}(\alpha)$, $p_j \in [0,1]$. Then, (5.33) is reduced to

$$\begin{cases} K_j = F^{-1(p_j)}_{X_j,\xi_j,\psi_j}(\dfrac{\mathbb{E}[\xi_j]-\lambda}{\mathbb{E}[\xi_j+\psi_j]}),\ j=1,2,\ldots,n, \\ K = \displaystyle\sum_{j=1}^{n} K_j. \end{cases} \tag{5.34}$$

Hence, λ will be the solution to

$$\sum_{j=1}^{n} F^{-1(p_j)}_{X_j,\xi_j,\psi_j}(\frac{\mathbb{E}[\xi_j]-\lambda}{\mathbb{E}[\xi_j+\psi_j]}) = K.$$

In particular, if we take $p_j = p$ for $j=1,\ldots,n$, λ will be the solution to

$$\sum_{j=1}^{n} F^{-1(p)}_{X_j,\xi_j,\psi_j}(\frac{\mathbb{E}[\xi_j]-\lambda}{\mathbb{E}[\xi_j+\psi_j]}) = K. \tag{5.35}$$

Note that $\underline{\lambda} \leq \lambda \leq \overline{\lambda}$, where $\underline{\lambda} = -\min\{\mathbb{E}[\psi_j]\}$, $\overline{\lambda} = \min\{\mathbb{E}[\xi_j]\}$. Hence, if $K \in (\underline{K}, \overline{K})$, where

$$\underline{K} = \sum_{j=1}^{n} F^{-1(p)}_{X_j,\xi_j,\psi_j}(\frac{\mathbb{E}[\xi_j]-\overline{\lambda}}{\mathbb{E}[\xi_j+\psi_j]}) = -\infty$$

since some $\mathbb{E}[\xi_j] = \overline{\lambda}$, and

$$\overline{K} = \sum_{j=1}^{n} F^{-1(p)}_{X_j,\xi_j,\psi_j}(\frac{\mathbb{E}[\xi_j] - \underline{\lambda}}{\mathbb{E}[\xi_j + \psi_j]}) = \infty$$

since some $-\mathbb{E}[\psi_j] = \underline{\lambda}$, then the solutions to (5.35) do exist.

Remark 5.20. *If $\xi_j = \psi_j$ for $j = 1, \ldots, n$, the optimization problem is*

$$min_{K_1,K_2,\cdots,K_n} \sum_{j=1}^{n} \mathbb{E}[\xi_j|X_j - K_j|] \quad s.t. \sum_{j=1}^{n} K_j = K,$$

and it is solved by Theorem 4 in Dhaene et al. (2012). In fact, $\sum_{j=1}^{n} \nu_j = 1$ can be removed from the model in Dhaene et al. (2012) since the conclusion still holds without this condition. Suppose $\mathbb{E}[\xi_j] = \mathbb{E}[\psi_j] = \beta$ in Theorem 5.19, (5.35) is reduced to

$$\sum_{j=1}^{n} F^{-1(p)}_{X_j,\xi_j,\psi_j}(\frac{\beta - \lambda}{2\beta}) = K.$$

Let S^c be the comonotonic sum with $S^c = \sum_{j=1}^{n} F^{-1}_{X_j,\xi_j,\psi_j}(U)$, then take $K = F^{-1(p)}_{S^c}(\frac{\beta-\lambda}{2\beta})$, where $F^{-1(p)}_{S^c}$ is the p-mixed inverse of S^c. Then, $F_{S^c}(K) = \frac{\beta-\lambda}{2\beta}$. Thus, $\lambda = \beta(1 - 2F_{S^c}(K))$. Therefore, p is the solution to

$$\sum_{j=1}^{n} F^{-1(p)}_{X_j,\xi_j,\psi_j}(F_{S^c}(K)) = K.$$

Hence, $K_j = F^{-1(p)}_{X_j,\xi_j,\psi_j}(F_{S^c}(K))$, which generalizes Theorem 4 of Dhaene et al. (2012). □

Formulas for business unit driven and aggregate portfolio driven capital allocations can be derived by letting $\xi_j = g_j(X_j)$, $\psi_j = h_j(X_j)$, or $\xi_j = g_j(S)$, $\psi_j = h_j(S)$.

Example 5.2. *If $D_1(x) = D_2(x) = x$, $\nu_j = \omega_j = 1$, $\xi_j = 1 - e^{-\alpha_j X_j}$, $\psi_j = e^{-\alpha_j X_j}$, $X_j \sim Exp(\beta_j)$, $j = 1, \ldots, n$. The optimal capital allocations should be the solutions to*

$$\begin{cases} e^{-\beta_j K_j}\mathbb{I}_{\{K_j \geq 0\}} + \mathbb{I}_{\{K_j < 0\}} = \dfrac{\beta_j}{\alpha_j + \beta_j} + \lambda, \ j = 1, 2, \ldots, n, \\ \displaystyle\sum_{j=1}^{n} K_j = K. \end{cases} \tag{5.36}$$

			$K = 1$			$K = 3$		
r_1	r_2	r_3	K_1	K_2	K_3	K_1	K_2	K_3
0.200	0.500	0.500	0.6316	0.1842	0.1842	1.6118	0.6941	0.6941
0.250	0.500	0.500	0.5793	0.2104	0.2104	1.5017	0.7492	0.7492
0.333	0.500	0.500	0.4944	0.2528	0.2528	1.3243	0.8378	0.8378
0.500	0.500	0.500	0.3333	0.3333	0.3333	1.0000	1.0000	1.0000
0.890	0.500	0.500	0.0027	0.4987	0.4987	0.4097	1.2952	1.2952
0.894	0.500	0.500	-0.0018	0.5009	0.5009	0.4047	1.2926	1.2926
0.900	0.500	0.500	-0.0217	0.5108	0.5108	0.3973	1.3014	1.3014
1.000	0.500	0.500	-0.3863	0.6931	0.6931	0.2790	1.3605	1.3605

TABLE 5.3: Example 5.2-1

			$K = 1$			$K = 3$		
β_1	β_2	β_3	K_1	K_2	K_3	K_1	K_2	K_3
1	1	1	0.3333	0.3333	0.3333	1.0000	1.0000	1.0000
1	5	10	0.9167	0.0621	0.0212	2.7409	0.1843	0.0747
1	1	5	0.4942	0.4942	0.0117	1.4437	1.4437	0.1126
1	1	10	0.5261	0.5261	-0.0522	1.4775	1.4775	0.0451
1	5	5	0.8832	0.0584	0.0584	2.6383	0.1809	0.1809
1	10	10	0.9539	0.0230	0.0230	2.8477	0.0761	0.0761
5	5	10	0.4192	0.4192	0.1616	1.3717	1.3717	0.2566
5	10	10	0.5893	0.2054	0.2054	2.4840	0.2580	0.2580

TABLE 5.4: Example 5.2-2

Now we suppose $\beta_1 = \beta_2 = \beta_3 = 1$ *and* $r_j = \frac{\beta_j}{\alpha_j+\beta_j}$, $j = 1, 2, 3$, *the capital allocated in this example is shown in Table 5.3. Also, we can get Table 5.4 if* $\alpha_1 = \alpha_2 = \alpha_3 = 1$.

Chapter 6

Conclusion

In Chapter 2, the property of tail subadditivity was generalized from for a pair of random variables to a distortion risk measure. Moreover, by applying the sufficient conditions for a tail subadditive distortion risk measure, we proved that any coherent risk measure can be approached by the GlueVaR risk measures, which is expressed as the linear combinations of VaR and TVaR. As is known, for many popularly adopted distributions, VaR and TVaR have closed-form expressions. Hence, the value of the benchmark for loss random variables under certain coherent risk measures can be approached by the corresponding combinations of the values of VaR and TVaR for the random variables. Also, this property can be applied to portfolio risk management; in particular, the property of tail additivity can be satisfied under certain conditions for a flexible common tail region as illustrated. Finally, the tail distortion principle defined in this paper is coherent and easy to calculate in practice.

In Chapter 3, we proposed the new risk measure of weighted quantiles with the weighted expectiles and weighted VaRs as two special cases. In addition, we proved that the 3-parameter expectile, as a special case of the weighted expectile, is a coherent risk measure when ρ is the expectation risk measure. The 3-parameter expectile not only generalizes the expectiles, but also can be more flexibly defined since the weight for the region where the loss random variable is greater than the benchmark defined by ρ can be modified. With these properties, it is more reasonable to apply this risk measure in industry. For instance, the regulators or insurance companies would pay more attention to the larger loss and may put a relatively higher weight on the region where the loss random variable is greater than a chosen benchmark. Moreover, the multi-parameter VaRs derived with the identity functions can be lower or higher than the classical VaR, which is determined by the relationship between the q^* and the ratios derived in Corollary 3.19 and 3.20. But the 2-paramter VaR is greater than the risk measure introduced by Heras et al. (2012) and less than the median shortfall under the assumptions in our numerical examples. Finally, we can see that the numerical examples clearly illustrate the properties that the corresponding risk measures satisfy.

In Chapter 4, we derived two classes of new reinsurance premium principles that satisfy most of the axioms for risk measures. For example, the premium principles with quadratic functions are monotonic functions of the ceded loss functions and weight functions. Especially, if the weight factors are constants, the premium with quadratic functions satisfies the property of subadditivity. Additionally, we suggest that the reinsurer calculate benchmarks for reinsurance premiums based on the two thresholds for the constant weights as defined in Proposition 4.8. Also, the reinsurance premium based on the expectiles can be no less than the new premium with quadratic functions under the condition in Proposition 4.11. Moreover, this inequality for the two premium principles always holds when the weight factors are constants. With the identity functions and specially defined weight functions, the premium principle is a mixed inverse of a distribution function at a certain confidence level in five cases. If we further assume that the distribution function of the ceded loss is continuous and strictly increasing, the results are reduced to two cases. Moreover, the premium can satisfy translation invariance and monotonicity under the conditions in Corollary 4.18. In this case, when the weight factors are constants, the premium is always less than the premium calculated based on the classical risk measure by Heilmann (1989). Hence, we can conclude that the two classes of reinsurance premium principles determined with constant weights are more reasonable and competitive since they are lower, satisfy most of the axioms for risk measures, and include more of the insurer's risks in the pricing model.

In Chapter 5, capital allocation principles for the quadratic and identity quantifying functions were derived. For the model assumption, we proposed only the capital deficit risk and the capital surplus risk. However, other potential risks can be included in this model by supposing various frameworks of the weight functions ξ_j and ψ_j, $j = 1, \ldots, n$. The theories from Enterprize Risk Management (ERM) will guide us to integrate more risks in the model, such as financial, operational and strategic risks, if necessary. These allocation principles are also applicable to the multivariate distributions constructed by any marginal distribution and any dependence structure, though most of the examples are provided as "i.i.d.". It is a general model, so many allocation principles can be generated.

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