# Packing and Covering Odd $(u,v)$-trails in a Graph

by

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I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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**Abstract**

In this thesis, we investigate the problem of packing and covering odd $(u, v)$-trails in a graph. A $(u, v)$-trail is a $(u, v)$-walk that is allowed to have repeated vertices but no repeated edges. We call a trail *odd* if the number of edges in the trail is odd. Given a graph $G$ and two specified vertices $u$ and $v$, the odd $(u, v)$-trail packing number, denoted by $\nu(u, v)$, is the maximum number of edge-disjoint odd $(u, v)$-trails in $G$. And, the odd $(u, v)$-trail covering number, denoted by $\tau(u, v)$, is the minimum size of an edge-set that intersects every odd $(u, v)$-trail in $G$. In 2016, Churchley, Mohar, and Wu, were the first ones to prove a constant factor bound on the covering-vs.-packing ratio, by showing that $\tau(u, v) \leq 8 \cdot \nu(u, v)$. Our main result in this thesis is an improved bound on the covering number: $\tau(u, v) \leq 5 \cdot \nu(u, v) + 2$. The proof leads to a polynomial-time algorithm to find, for any given $k \geq 1$, either $k$ edge-disjoint odd $(u, v)$-trails in $G$ or a set of at most $5k - 3$ edges intersecting all odd $(u, v)$-trails in $G$.

## Acknowledgements

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To the curiosity of the human mind.

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# Chapter 1

# Introduction

Min-max theorems are a classical theme in Combinatorics, with many such results arising from the study of packing and covering problems. For instance, Menger's theorem (see [8]) gives a tight min-max theorem between packing and covering edge-disjoint (internally vertex-disjoint) $(u, v)$-paths; the maximum number of edge-disjoint (respectively, internally vertex-disjoint) $(u, v)$-paths in $G$ is equal to the minimum number of edges (respectively, vertices) whose removal from the graph separates $u$ and $v$. The celebrated max-flow min-cut theorem generalizes Menger's theorem and shows that the total weight of $u \rightsquigarrow v$ paths that can be packed in an edge-capacitated directed graph is equal to the minimum capacity of arcs whose removal destroys every $u \rightsquigarrow v$ path. Another prominent example is the Lucchesi-Younger theorem (see [6]) which shows that the maximum number of edge-disjoint directed cuts equals the minimum size of an arc-set that intersects every directed cut. Motivated by the max-flow min-cut theorem, it is natural to investigate whether similar (tight or approximate) min-max theorems hold for other variants of path-packing and path-covering problems. Perhaps, the most prominent example of this type are Mader's min-max theorems [7] for packing and covering T-paths — a T-path is a path whose endpoints are two distinct vertices in $T$ and no internal vertex of the path is in $T$ — that generalize both the Tutte-Berge formula and Menger's theorem.

In this thesis, we investigate a min-max relationship between a different variant of the $(u, v)$-path packing and $(u, v)$-path covering problems, wherein we impose parity constraints on the paths. To motivate the definition of our packing and covering problems, first consider the packing problem of finding the maximum number of edge-disjoint odd $(u, v)$-paths, where a $(u, v)$-path is odd if it contains an odd number of edges. The corresponding covering problem is to find the minimum-size of an edge-set that intersects every odd $(u, v)$-path. As noted in [3], there are simple examples (see Figure A.1) where packing number is 1 (every pair of odd $(u,v)$-paths intersect), but the covering number can be arbitrarily large, showing that there is no nice bound on the ratio of the covering and packing numbers. In light of this, following [3], we study the problem of packing and covering odd $(u,v)$-trails. An odd $(u,v)$-trail is a $(u,v)$-walk with no repeated edges and an odd number of edges. The packing and covering number for odd $(u,v)$-trails are defined analogously. The graph in Figure 1.1 shows that we cannot hope to achieve a tight min-max theorem (like Menger's theorem) relating the covering and packing numbers of odd $(u,v)$-trails. Our main result establishes a bound of (roughly) 5 on the ratio between the covering and packing numbers for odd $(u,v)$-trails, which improves upon the bound of 8 proved in [3].

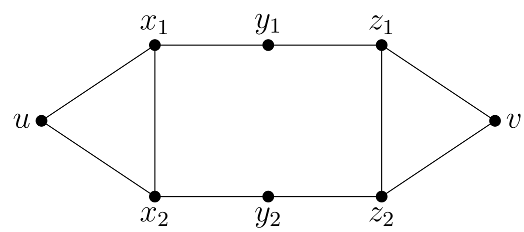



Figure 1.1: Graph with $\nu(u,v) = 1$ and $\tau(u,v) = 2$.

Another motivation for considering odd trails comes from the study of totally odd immersions. An *immersion* of a graph $H$ in another graph $G$ is a subgraph of $G$ consisting of a set $U \subseteq V(G)$ of vertices that are in bijective correspondence with the vertices of $H$, and edge-disjoint trails in $G$ representing the edges of $H$ that connect the corresponding vertices in $U$. An immersion is called *totally odd* if all the trails in $G$ corresponding to the edges of $H$ are of odd length. Hence, the problem of packing $k$ edge-disjoint odd $(u,v)$-trails in a graph is equivalent to finding a totally odd immersion of the two-vertex graph with $k$ parallel edges.

## 1.1 Our Results and Organization of the Thesis

Let $\tau(u,v)$ denote the minimum size of an edge-set that intersects every odd $(u,v)$-trail, and let $\nu(u,v)$ denote the maximum number of edge-disjoint odd $(u,v)$-trails in $G$. Our main result (see Theorem 6.1) is that $\tau(u,v) \leq 5 \cdot \nu(u,v) + 2$ for any graph $G$. This improves upon the result in [3], which shows that $\tau(u,v) \leq 8 \cdot \nu(u,v)$. We have recently learned that independent of our work, Churchley et al. [2] have also obtained an improved bound of 5 on the covering-vs.-packing ratio. We discuss briefly the ingredients building up to this result.

In Chapter 2, we discuss some preliminaries and simple reductions.

In Chapter 3, we prove that $\tau(u, v) \leq 6 \cdot \nu(u, v) + 4$ (see Theorem 3.5). Although this is weaker than our final result, the proof of this result introduces many of the key ingredients that we build upon in Chapters 4 and 5 to obtain our improved bound. We remark that the cover that we find is in fact identical to the one in [3], but we present a more refined analysis that yields our improved bound of roughly 6. Our analysis is also simpler and cleaner than the one in [3].

We conclude Chapter 3 with an outline for the rest of the thesis. We identify two bottlenecks involved in obtaining better results, which we deal with in Chapter 4 and 5. In Chapter 4, we improve upon the main technical building block (Theorem 3.4) used in Chapter 3 to obtain the factor-6 bound: we show that if $G$ does not have $k$ edge-disjoint trails of a certain type, then there exists a cover of size at most $5k - 5$ (see Theorem 4.1). Complementing this, we show in Chapter 5 how to remove the second bottleneck in the factor-6 result by weakening the conditions under which $k$ edge-disjoint trails of the type given by Theorem 4.1 can be converted to $k$ edge-disjoint odd $(u, v)$-trails (see Theorem 5.1). By combining Theorems 4.1 and 5.1, we prove the main result of this thesis i.e., Theorem 6.1 which implies that $\tau(u, v) \leq 5 \cdot \nu(u, v) + 2$. In Chapters 3 and 6, we also describe how one can design polynomial time algorithms from our proofs to obtain constant-factor approximations to $\tau(u, v)$ and $\nu(u, v)$.

In Chapter 7, we conclude the thesis by providing a family of graphs $\{G_k\}_{k\in\mathbb{N}}$ from [3] where $\nu(u, v; G_k) = k$ and $\tau(u, v, G_k) = 2k$, thereby showing that unlike the case of packing and covering $(u, v)$-paths, there is no tight min-max relationship between the covering and packing numbers. We then discuss a limitation of our techniques by providing a family of graphs $\{H_k\}_{k\in\mathbb{N}}$ where $\nu(u, v; H_k) = k$ and $\tau(u, v; H_k) = 2k$, but our techniques can at best yield a cover of size $3k$ for each $H_k$. This shows that our techniques can at best yield a factor-3 gap between $\tau(u, v)$ and $\nu(u, v)$.

## 1.2 Related Work

We have already mentioned the work of Churchley et al., who seem to be the first to consider the problem of finding bounds on the covering-vs.-packing ratio for odd $(u, v)$-trails. We now discuss some other related work. Since there is no nice bound on the ratio of the covering and packing numbers for odd $(u, v)$-paths, Schrijver and Seymour in [10] study the fractional packing of odd $(u, v)$-paths. They show that the minimum number of edges needed to intersect every odd $(u, v)$-path in a graph is at most twice the corresponding fractional packing number.

The notions of odd paths and trails can be generalized and abstracted in two ways. The first involves signed graphs (see the dynamic surveys of Zaslavsky [11, 12]), and there are various results on packing odd circuits in signed graphs, which are closely related to multicommodity flows (see [9], Chapter 75). The second involves group-labeled graphs. In this context, the work of Chudnovsky et al. [1] seems closely related to our work. They prove a min-max theorem for packing and covering vertex-disjoint non-zero $A$-paths in group-labeled graphs. Their result generalizes Mader's work on packing and covering $T$-paths.

# Chapter 2

# Preliminaries

In this chapter we give a brief introduction to the problem of packing and covering odd $(u, v)$-trails. We begin with some basic definitions and terminology.

**Definition 2.1** ($(x, y)$-trail) Given a multigraph $G = (V, E)$ and two vertices $x$ and $y$ in $V$, an $(x, y)$-trail is a sequence $T = (x_1y_1, x_2y_2, \ldots, x_ry_r)$ of *distinct* edges such that $x_1 = x$, $y_r = y$, and for all $1 < i \leq r$, $y_{i-1} = x_i$.

Informally, an $(x, y)$-trail is an $(x, y)$-walk that is allowed to have repeated vertices but no repeated edges. A trail having an odd (or even) number of edges in the sequence will be referred to as an odd (or even) trail.

**Definition 2.2** (Subtrail) Given a trail $T = (x_1y_1, \ldots, x_ry_r)$ of length $r$, a trail $S$ is said to be a *subtrail* of $T$ if there exist $1 \leq i \leq j \leq r$ such that $S = (x_iy_i, \ldots, x_jy_j)$ or $S = (y_jx_j, \ldots, y_ix_i)$.

We often overload notation, and use the same symbol to denote both the node-set and edge-set of an $(x, y)$-trail; the intended meaning will be clear from the context.

**Definition 2.3** (Packing number for odd $(u, v)$-trails) Given a multi-graph $G = (V, E)$ and two specified vertices $u$ and $v$, the *packing number for odd $(u, v)$-trails*, denoted by $\nu(u, v)$, is defined as the maximum number of edge-disjoint odd $(u, v)$-trails in $G$.

**Definition 2.4** (Odd $(u, v)$-trail Cover) Given a multi-graph $G = (V, E)$ and two specified vertices $u$ and $v$, an edge-set $C \subseteq E$ is said to be an *odd $(u, v)$-trail cover* if it intersects every odd $(u, v)$-trail in $G$.

**Definition 2.5** (Covering number for odd $(u, v)$-trails) Given a multi-graph $G = (V, E)$ and two specified vertices $u$ and $v$, the *covering number for odd $(u, v)$-trails*, denoted by $\tau(u, v)$, is the minimum size of an odd $(u, v)$-trail cover.

It is trivial to see that $\nu(u, v) \leq \tau(u, v)$. Our goal in this thesis is to show that $\tau(u, v)$ is upper bounded by a (small) constant times $\nu(u, v)$. Churchley et al. [3], seem to have been the first to investigate this problem; they show in [3] that $\tau(u, v) \leq 8 \cdot \nu(u, v)$. In this thesis, we build upon their ideas and prove an improved upper bound of (essentially) 5 on the ratio of the covering and packing numbers. The rest of this chapter deals with developing key ideas which will form the basis of our analysis. To avoid trivial settings, we make the following assumptions.

**Assumption 2.6**

1. $u$ and $v$ are not separated in $G$, as otherwise, $\nu(u, v) = \tau(u, v) = 0$.

2. $G$ is connected, as otherwise, we can focus on the component containing $u$ and $v$.

3. $G$ does not have a bridge $e = xy$ separating some vertex-set $S$ containing $\{u, v\}$ from $V \setminus S$, since otherwise, we can delete $V \setminus S$ and focus on $S$ as no odd $(u, v)$-trail uses $e$.

4. Assuming the above conditions hold, it is easy to detect if $\nu(u, v) = 0$, since $G$ has no odd $(u, v)$-trail if and only if $G$ is a bipartite graph with $u$ and $v$ on the same side of the bipartition. So we may assume that $\nu(u, v) > 0$, since otherwise, $\nu(u, v) = \tau(u, v) = 0$.

5. Assuming condition (4) holds, $G$ does not have a bridge $e$ separating $u$ and $v$, since otherwise, $\nu(u, v) = \tau(u, v) = 1$.

To sum up, we may assume that $G$ is 2-edge-connected. We will show later (see Lemma 2.9) that we may further assume that 2-edge-cuts of $G$ have a specific structure, but before delving into this, we build some intuition for the problem.

## 2.1 Candidates for Odd $(u, v)$-trail Cover

We will obtain upper bounds on the covering-vs.-packing ratio by proving results of the following form: for any positive integer $k$, we show that either there exist $k$ edge-disjoint odd $(u, v)$-trails in $G$ or there exists a cover of size $\mathcal{O}(k)$ (for a suitably small constant). To gain some insights, we discuss how one may obtain a cover for odd $(u, v)$-trails. Clearly, a trivial cover is a minimum $(u, v)$-cut. We will frequently use the notation $\lambda(x, y; H)$ to denote the minimum number of edges whose deletion from the graph $H$ separates vertices $x$ and $y$. Hence, if $\lambda(u, v; G) = \mathcal{O}(k)$, then we are done, but as Figure 2.1 shows, the size of this cover can be arbitrarily large even for graphs with packing number 1.

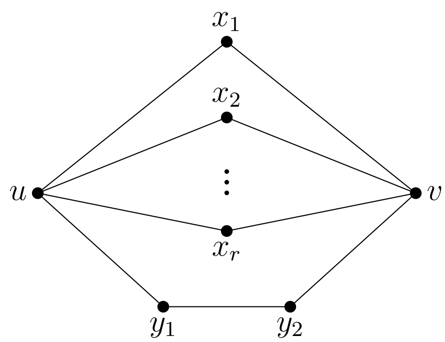



Figure 2.1: Graph with $\nu(u, v) = 1$ and $\lambda(u, v) = 1 + r$.

To do better, consider first the case where $G$ is bipartite, wherein the problem becomes trivial: if $u$ and $v$ are on different sides of the bipartition, then $\nu(u, v) = \tau(u, v) = \lambda(u, v; G)$, otherwise, there are no odd $(u, v)$-trails so $\nu(u, v) = \tau(u, v) = 0$. Motivated by these observations, our strategy will be to find a bipartite subgraph $G'$ of $G$ where $\lambda(u, v; G')$ is close to $\lambda(u, v; G)$. If $u$ and $v$ are on different sides of the bipartition of $G'$, then we are done since $\delta_G(X)$ where $X$ defines a min $(u, v)$-cut in $G'$, is close to the number of edge-disjoint $(u, v)$-paths in $G'$, all of which are odd. If none of these simple cases apply (see Figure 2.2), we will obtain a small cover by considering non-$(u, v)$-separating cuts. We will find a suitable set $X$ containing both $u$ and $v$, and take the cover to be the union of $\delta_G(X)$ and the nonbipartite edges internal to $X$. For example, we could consider $X = \{u, v, z_1, \ldots, z_q\}$ for the graph in Figure 2.2.

**Lemma 2.7** *Suppose $G'$ is a bipartite subgraph of $G$ with $u$ and $v$ on the same side of the bipartition. Let $F$ denote $E(G) \setminus E(G')$. Then for any vertex-set $X$ containing both $u$ and $v$, the edge-set $\delta(X) \cup (E(X) \cap F)$ is an odd $(u, v)$-trail cover.*

*Proof.* Let $H$ be the graph obtained from $G$ by removing the edges in $\delta(X) \cup (E(X) \cap F)$. The connected component in $H$ containing $u$ and $v$ is a subgraph of $(X, E(X) \setminus F)$, which is (by definition) bipartite with $u$ and $v$ on the same side of the bipartition. Thus, $H$ has no odd $(u, v)$-trails and hence $\delta(X) \cup (E(X) \cap F)$ is an odd $(u, v)$-trail cover of $G$. □

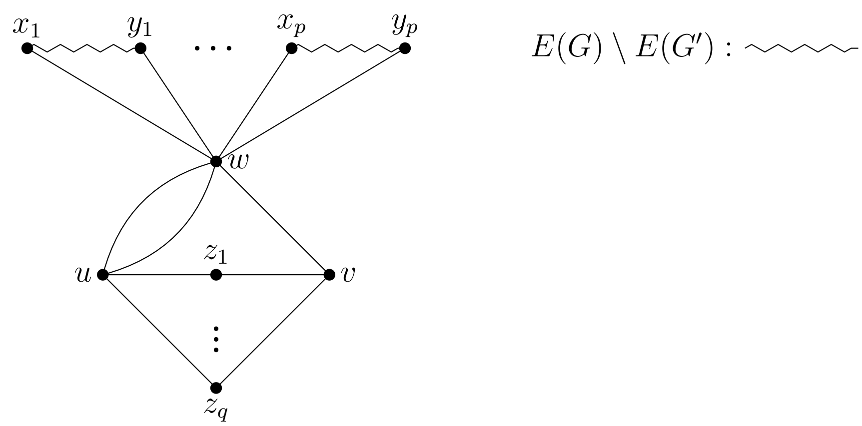



Figure 2.2: Graph with $\nu(u, v) = 1$ and $\lambda(u, v; G) = \lambda(u, v; G') = 1 + q$.

## 2.2 Reducing 2-edge-cuts

To aid us in finding a suitable set $X$, it will be convenient to consider another reduction that allows us to assume that 2-edge-cuts in $G$ (and their counterparts in $G'$) can be assumed to have a special structure.

**Reduction 2.8** Suppose that $\delta(C) = \{e_1, e_2\}$ is a 2-edge-cut in $G$ which separates vertex-set $C$ from $\{u, v\}$. Let $x$ and $y$ (which could be the same node) be endpoints of $e_1$ and $e_2$ in $C$, respectively. We may replace $C$ with an appropriate gadget, without changing the packing or covering number, by applying one of the following transformations, as illustrated in Figure 2.3:

1. If all $(x, y)$-trails in the induced subgraph $G[C]$ have even length, then replace $C$ with a single vertex of degree two obtained by identifying $x$ and $y$ into a new vertex $x'$. We refer to the edges incident with $x'$ as $f_1$ and $f_2$.

2. If $x \neq y$ and all $(x, y)$-trails in $G[C]$ have odd length, then replace $C$ with an edge $x'y'$.

3. If $x \neq y$ and $G[C]$ has at least one $(x, y)$-trail of each parity, then replace $C$ with a triangle $x'y'z'$ where $z'$ is a new vertex.

4. If $x = y$ and $G[C]$ has at least one $(x, y)$-trail of each parity, then replace $C$ with a vertex $x'$ and add a loop at $x'$.

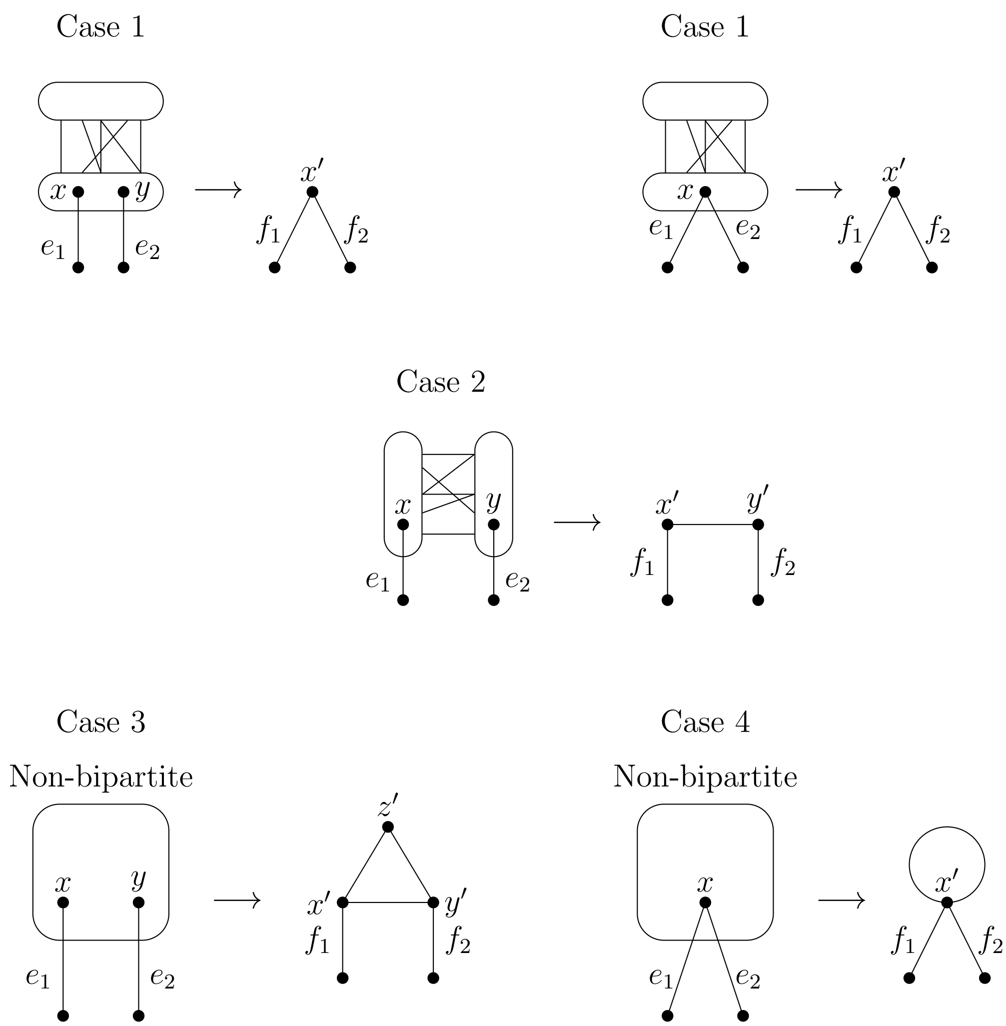



Figure 2.3: Replacing 2-edge-cuts with gadgets.

We can show that the packing number does not change by the above reduction as follows. Let $\tilde{G}$ be the graph obtained by replacing every such component $C$ with the corresponding gadget. Note that there can be at most one odd $(u, v)$-trail $G$ using nodes of $C$, call it $T$, and at most one odd $(u, v)$-trail of $\tilde{G}$ using nodes of the gadget, call it $\tilde{T}$. In each case above, it is easy to see that $T$ can be mapped to an odd $(u, v)$-trail in $\tilde{G}$ and conversely, $\tilde{T}$ can be mapped to an odd $(u, v)$-trail in $G$.

To show that the covering number also remains unchanged, let $A$ be a cover in $G$, and $\tilde{A}$ be a cover in $\tilde{G}$. If $A \cap (\delta(C) \cup E(C)) = \emptyset$, then $A$ remains a cover in $\tilde{G}$, otherwise

$(A \setminus (\delta(C) \cup E(C))) \cup \{e_1\}$ is a cover in $\tilde{G}$; in both cases the size of the cover does not increase. Similarly, from $\tilde{A}$ we can obtain a cover of $G$ of size at most $|\tilde{A}|$, which is either $\tilde{A}$, if $\tilde{A}$ does not use edges incident to the nodes of the gadget replacing $C$, or $\{e \in \tilde{A} : e \text{ is not incident to a gadget-node}\} \cup \{e_1\}$ otherwise.

To apply the above reduction, we consider each inclusion-wise maximal vertex-set $C \subset V(G)$ with $u, v \notin C$ and $|\delta(C)| = 2$. For brevity, and slightly abusing notation, we refer to such a $C$ as a *maximal 2-edge-cut*. Note that since $G$ is 2-edge-connected, maximal 2-edge-cuts are disjoint; also there are no edges crossing between two maximal 2-edge-cuts (see Figure 2.4). We apply Reduction 2.8 by replacing each maximal 2-edge-cut $C$ with the corresponding gadget to obtain the graph $\tilde{G} = (\tilde{V}, \tilde{E})$.

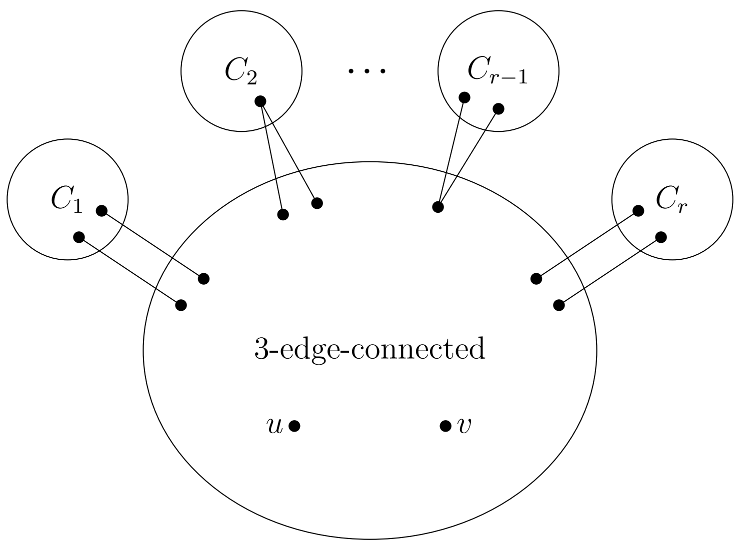



Figure 2.4: Identifying maximal 2-edge-cuts.

**Lemma 2.9** Given a bipartite subgraph $\tilde{G}_1 = (\tilde{V}, \tilde{E}_1)$ of $\tilde{G} = (\tilde{V}, \tilde{E})$, we can obtain a bipartite subgraph $\tilde{G}_2 = (\tilde{V}, \tilde{E}_2)$ satisfying the following conditions:

1. $|\tilde{E}_2| \geq |\tilde{E}_1|$.
2. For any maximal 2-edge-cut $C$ (which is one of the four cases as discussed in Reduction 2.8) of the input graph $G$ with $\delta(C) = \{e_1, e_2\}$, let $f_1, f_2$ be the corresponding edges in the gadget (as shown in Figure 2.3). Then, at least one of $f_1, f_2$ is in $\tilde{E}_2$, and in cases 2 and 3 of Reduction 2.8, both $f_1$ and $f_2$ are in $\tilde{E}_2$.

*Proof.* We construct the required bipartite subgraph $\tilde{G}_2$ by starting with the bipartition given by $\tilde{G}_1$ on vertices not in any of the gadgets and extending the bipartition to the vertices in a gadget as follows:

1. In cases 1 and 4, we can put the single vertex $x'$ to the partition which maximizes $\tilde{E}_2$. Let $a, b$ be the neighbors of $x'$ not in the gadget. If $a$ and $b$ are on the same side of the bipartition in $\tilde{G}_1$, then we get $f_1, f_2 \in \tilde{E}_2$. Otherwise, exactly one of them belongs to $\tilde{E}_2$. Note that the loop in case 4 is never a part of any bipartition.

2. In cases 2 and 3, we have two distinct vertices $x'$ and $y'$ in the gadget and hence we can (independently) add them to the partition opposite to their neighbor not in the gadget.

□

**Lemma 2.10** Let $G'$ be a bipartite subgraph of $\tilde{G}$ with $u$ and $v$ on the same side of the bipartition and let $F$ denote $E(G) \setminus E(G')$. Suppose that $G'$ satisfies condition (2) of Lemma 2.9. Let

$$J = \{f : f \text{ is a loop}, \exists \text{ maximal 2-edge-cut } C \text{ in } G \text{ such that } f \in E(C),\ |\delta(C) \setminus F| = 1\} .$$

Then, for any vertex-set $X$ containing both $u$ and $v$, the edge-set $\delta(X) \cup (E(X) \cap (F \setminus J))$ is an odd $(u, v)$-trail cover.

*Proof.* Since $G'$ satisfies condition (2) of Lemma 2.9, the 2-edge-cuts which have exactly one $F$-edge in the cut correspond to cases 1 and 4 in Reduction 2.8. Observe that self loops arise only in case 4. By Lemma 2.7, we know that $\delta(X) \cup (E(X) \cap F)$ is an odd $(u, v)$-trail cover. Hence, it is sufficient to show that adding back edges in $E(X) \cap J$ to $H = G - (\delta(X) \cup (E(X) \cap F))$ does not create any odd $(u, v)$-trails. Suppose, for the sake of contradiction, there is an odd $(u, v)$-trail in $H + (E(X) \cap J)$, call it $T$. Let $e = zz \in E(X) \cap J$ be a loop used in $T$. Let $x$ and $y$ be the neighbors of $z$ outside the gadget. Without loss of generality assume that $yz \in F$. Since $z \in X$, the edge $yz$ is either in $E(X)$ or $\delta(X)$. In both cases the edge $yz$ is part of $\delta(X) \cup (E(X) \cap (F \setminus J))$ contradicting the existence of $T$. □

# Chapter 3

# An Upper Bound of 6 on the Covering-vs.-Packing Ratio

Recall from Chapter 2 that the hard case in packing and covering odd $(u,v)$-trails is when $\lambda(u,v;G)$ is large and $u,v$ are on the same side of the bipartition in a maximum[1] bipartite subgraph $G'$ of $G$. The following lemma states a property of a maximum bipartite subgraph and is frequently used in our analysis. Its proof follows directly from Lemma 3.19, which is stated and proved in Section 3.2.

**Lemma 3.1** A maximum bipartite subgraph $G'$ of $G$ satisfies the following property: for any $X \subseteq V$, we have $|\delta(X) \setminus (E(G) \setminus E(G'))| \geq |\delta(X) \cap (E(G) \setminus E(G'))|$.

For the rest of this chapter, unless otherwise stated, we assume that $G = (V, E)$ is the graph obtained after applying Reduction 2.8 to the input graph and that $G$ satisfies Assumption 2.6. As noted earlier, Assumption 2.6 simply rules out some trivial cases, and Reduction 2.8 does not change the packing or covering numbers. We also assume that $G'$ satisfies Lemma 2.9. Further, since we seek covers of the form $\delta(X) \cup (E(X) \cap F)$ with $u,v \in X$, we may assume that $G$ does not have certain loops as discussed in Lemma 2.10. We now state this reduction, which depends on the choice of $G'$, and assume that this too has been applied to arrive at the graph $G$. Throughout, $F$ denotes $E(G) \setminus E(G')$.

**Reduction 3.2** Let $J = \{f : f \text{ is a loop at } z,\ \exists \text{ a maximal 2-edge-cut } C \text{ of the input graph s.t. } z \in C,\ |\delta(C) \setminus F| = 1\}$. Remove all edges in $J$ from $G$.

[1]While computing a maximum bipartite subgraph is NP-hard [5], we will show later (see Section 3.2) how we can instead work with a *maximal* bipartite subgraph.

Since $G$ is *reduced*, the 2-edge-cuts in $G$ consist of the gadget vertices. Since $G'$ is a maximum bipartite graph, any cut $\delta(C)$ having exactly one bipartite edge is a 2-edge-cut as a consequence of Lemma 3.1. Therefore, such cuts correspond to cases 1 and 4 of Reduction 2.8. The loops arising in Reduction 3.2 are from case 4 of Reduction 2.8 and since they are removed we get the following property.

**Property 3.3** For any vertex-set $C \subset V$ not containing $u, v$ and satisfying $|\delta(C) \setminus F| = 1$, there are no $F$-edges with both endpoints in $C$. Equivalently, for any vertex-set $X$ not containing $u$ and $v$ either $|\delta(X) \setminus F| > 1$ or $F \cap E(X) = \emptyset$ holds.

In Chapter 2 we discussed ideas for constructing an odd $(u, v)$-trail cover and the difficulties associated with some trivial covers. Since our aim in this thesis is to obtain good upper bounds on the covering number in terms of the packing number, it is important to understand the challenges associated with packing (many) odd $(u, v)$-trails. An odd $(u, v)$-trail has two aspects: a) parity (i.e., odd length) and b) connectivity from $u$ to $v$. Since $u$ and $v$ are on the same side of the bipartition in $G'$, every odd $(u, v)$-trail in $G$ uses an odd number of edges from $F$. In this thesis, we only deal with odd trails that use exactly one edge from $F$. As $\lambda(u, v; G')$ is large, there exist many edge-disjoint paths (of even length) from $u$ to $v$. So instead of just searching for odd $(u, v)$-trails, we will also look for odd $(u, u)$-trails and odd $(v, v)$-trails with the hope that the large $(u, v)$-edge-connectivity in $G'$ can be used to modify odd $(u, u)$-trails and odd $(v, v)$-trails into odd $(u, v)$-trails. To make the search process easier, we first unify all three kinds of odd trails, odd $(u, u)$-trails, odd $(v, v)$-trails, and odd $(u, v)$-trails, into one class by using a symbol $s$ to denote $\{u, v\}$. Since $s$ is not an actual vertex in $G$, we introduce the following notation to clarify what we mean when the symbol $s$ shows up in mathematical statements.

1. For a vertex-set $X$, $s \in X$ denotes $u, v \in X$.
2. For a vertex-set $X$, $s \notin X$ denotes $u, v \notin X$.
3. An $(s, t)$-path refers to a path having one of its end in $\{u, v\}$ and the other at $t$.
4. If $T$ is an odd $(s, s)$-trail, then $T$ is one of the following: an odd $(u, u)$-trail, an odd $(v, v)$-trail, or an odd $(u, v)$-trail.
5. $\lambda(s, t; H)$ is equivalent to $\lambda(\{u, v\}, t; H)$ i.e., the maximum number of edge-disjoint $(s, t)$-paths in a graph $H$. Or equivalently, the minimum number of edges in $H$ whose deletion separates $t$ from both $u$ and $v$.

Thus, all three kinds of odd trails are encompassed by the class of odd $(s,s)$-trails.

Our approach to solve the hard case involves finding $k$ edge-disjoint odd $(s,s)$-trails; if such a collection does not exist, then we exhibit a small odd $(s,s)$-trail cover (which in turn covers all odd $(u,v)$-trails). Since an odd $(s,s)$-trail is not necessarily an odd $(u,v)$-trail, we use the large edge connectivity between $u$ and $v$ in $G'$ to bridge the trail from $u$ to $v$.

The main technical result of this chapter is Theorem 3.4, which easily yields a bound of (essentially) 6 on the covering-vs.-packing ratio.

**Theorem 3.4** *For every nonnegative integer $k$, the graph $G$ has either:*

*i. $k$ edge-disjoint odd $(s,s)$-trails, each containing exactly one edge of $F$; or*

*ii. A vertex-set $S$ containing $s$ with $|\delta(S)| + |E(S) \cap F| \leq 6k - 6$.*

The above theorem is a variant of Lemma 2.1 in [3], wherein the second conclusion states that there exists a vertex-set $S$ containing $s$ such that $|\delta(S) \setminus F| + 2|E(S) \cap F| \leq 4k - 4$, which in turn is used to obtain a cover, $\delta(S) \cup (F \cap E(S))$, of size at most $8k - 8$. In fact, the set $S$ (and hence the cover) that we find is *identical* to the one in [3], but we give an improved analysis yielding the $6k - 6$ bound stated above. Our proof is also somewhat simpler and cleaner than the one in [3].

Before delving into the proof of Theorem 3.4, we show how this readily yields a bound of 6 on the covering-vs.-packing ratio.

**Theorem 3.5** *Let $\tilde{G}$ be a graph with two specified vertices $u$ and $v$. Let $k$ be a positive integer. Then, $\tilde{G}$ has either $k$ edge-disjoint odd $(u,v)$-trails or a set of at most $6k-2$ edges intersecting all odd $(u,v)$-trails. Hence, we have $\tau(u,v) \leq 6 \cdot \nu(u,v) + 4$ for every graph $\tilde{G}$.*

*Proof.* First of all, we may assume that $\lambda(u,v;\tilde{G}) \geq 6k - 1$, since otherwise a min $(u,v)$-cut is an odd $(u,v)$-trail cover of size at most $6k - 2$. To avoid trivial settings we may also assume that Assumption 2.6 holds. Hence $\tilde{G}$ is 2-edge-connected. Now we apply Reduction 2.8 from Chapter 2 to replace maximal 2-edge-cuts in $\tilde{G}$ with the corresponding gadgets. As mentioned earlier, the packing and covering numbers remain unchanged by these reductions. We reuse the symbol $\tilde{G}$ to refer to the resulting graph. Now, consider a maximum bipartite subgraph $G_1$ of $\tilde{G}$. Using Lemma 3.1, at least half the edges in every cut of $\tilde{G}$ are bipartite edges. Hence, we have $\lambda(u,v;G_1) \geq \lceil \lambda(u,v;\tilde{G})/2 \rceil \geq \lceil (6k-1)/2 \rceil = 3k$.

If $u$ and $v$ are on the opposite sides of the bipartition in $G_1$, then we can obtain at least $3k$ edge-disjoint odd $(u,v)$-paths in $G_1$ (which is more than what we need). Therefore, we may assume that $u$ and $v$ are on the same side of the bipartition in $G_1$. Let $G'$ be the bipartite subgraph of $\tilde{G}$ obtained by using Lemma 2.9 on $G_1$. Thus, $G'$ has at least one edge, and in some cases both edges, of a maximal 2-edge-cut of the input graph. Let $G$ denote the graph obtained from $\tilde{G}$ by applying Reduction 3.2 with respect to $G'$. Thus, $G, G', F := E(G) \setminus E(G')$ satisfy the hypothesis of Theorem 3.4. Hence, for the given $k$, one of the two conclusions of the theorem holds.

If there is a vertex-set $S$ containing $u$ and $v$ which satisfies $|\delta(S)| + |E(S) \cap F| \leq 6k-6$, then we are done since $\delta(S) \cup (E(S) \cap F)$ is an odd $(u, v)$-trail cover. Note that adding back the loops deleted during Reduction 3.2 does not create an odd $(u, v)$-trail, which is not covered by $\delta(S) \cup (E(S) \cap F)$, as shown in Lemma 2.10.

Otherwise, we get $k$ edge-disjoint odd $(s, s)$-trails each containing exactly one edge from $F$. Let $F'$ denote the subset of $F$-edges used in these trails. As discussed in the beginning of this chapter, we show how to obtain the desired $k$ edge-disjoint odd $(u, v)$-trails from the $k$ (edge-disjoint) odd $(s, s)$-trails using $\lambda(u, v; G') \geq 3k$. To this end, we construct a flow network $H$ as follows. Begin with the graph $\hat{G}(F')$ and assign unit capacity to all the edges. Next, add two new vertices $s^*$ and $t^*$. Then, add three new edges; the first one between $s^*$ and $u$ with capacity $3k$, the second one between $v$ and $t^*$ with capacity $k$ and the third one between $t$ and $t^*$ with capacity $2k$. If we can argue that $\lambda(s^*, t^*; H) = 3k$ holds, then we can obtain $3k$ edge-disjoint paths in $H$ such that $2k$ of them join $u$ with $t$ and $k$ of them join $u$ with $v$. The $2k$ edge-disjoint $(u, t)$-paths can be combined with edges in $F'$ to obtain $k$ edge-disjoint odd $(u, u)$-trails. Each of the remaining $k$ paths (which have even length since they only use bipartite edges) connecting $u$ with $v$ can be matched to an odd $(u, u)$-trail to obtain the desired $k$ edge-disjoint odd $(u, v)$-trails.

We now show that $\lambda(s^*, t^*; H) = 3k$ by analyzing cuts $\delta(X)$ in $H$ where $s^* \in X$ and $t^* \notin X$. We use $c(e)$ to denote the capacity of an edge $e$ and $c(\delta_H(X))$ to denote the sum of capacities of the edges in $\delta_H(X)$. If $u \notin X$, then $c(\delta_H(X)) \geq c(s^*u) = 3k$. So suppose $u \in X$ and $v \notin X$, then $c(\delta_H(X)) \geq \lambda(u, v; G') \geq 3k$. Next, if $u, v \in X$ and $t \notin X$, then $c(\delta_H(X)) \geq \lambda(s, t; \hat{G}(F')) + c(vt^*) = 3k$. Finally, when $u, v, t \in X$, we have $c(\delta_H(X)) \geq c(vt^*) + c(tt^*) = 3k$. This completes the proof of the theorem. □

While Theorem 3.5 does not consider polytime computation, we show in Section 3.2 that the underlying construction(s) can be suitably adapted to obtain in polynomial time, either $k$ edge-disjoint odd $(u, v)$-trails or a cover of size at most $6k - 2$.

## 3.1 Proof of Theorem 3.4

The structure of odd $(s, s)$-trails in the first conclusion of Theorem 3.4 motivates the following idea. For a fixed subset $F_0 \subseteq F$, let $\hat{G}(F_0)$ denote the graph obtained from $G'$ by adding a new vertex $t$ and for each $e = xy \in F_0$, adding a new edge between $x$ and $t$, and another edge between $y$ and $t$. If $\lambda(s, t; \hat{G}(F_0)) = 2 \cdot |F_0|$, then we can obtain a collection $\mathcal{P}$ of $2 \cdot |F_0|$ edge-disjoint $(s, t)$-paths in $\hat{G}(F_0)$. For each edge $f = xy \in F_0$, let $X_f$ and $Y_f$ denote the $(s, t)$-paths in $\mathcal{P}$ which use the edge $xt$ and $yt$, respectively. It is easy to see that we can obtain $|F_0|$ edge-disjoint odd $(s, s)$-trails (each using exactly one edge of $F$) by combining the paths $X_f$ and $Y_f$ with the edge $f$ for each $f \in F_0$.

Consider a *maximum-size* subset $F'$ of $F$ satisfying $\lambda(s, t; \hat{G}(F')) = 2 \cdot |F'|$. If $k' := |F'| \geq k$ then the first conclusion of the lemma holds trivially. So, we may assume that $k' < k$. We reserve the notation $\hat{G}$ to denote $\hat{G}(F')$. For any vertex-set $X \subseteq V$, we use $\hat{d}(X)$ as a shorthand for $|\delta_{\hat{G}}(X)|$. The following equation is another way of expressing $\hat{d}(X)$, where $\deg(X, F')$ counts the incidences between the edges of $F'$ and the vertices in $X$.

$$\hat{d}(X) = |\delta(X) \setminus F| + \deg(X, F')$$

Note that for any vertex-set $X \subseteq V$ containing $s$, we can bound $|\delta(X) \setminus F| + |E(X) \cap F'| + |\delta(X) \cap F'|$ by $\hat{d}(X)$. Since $G'$ is a maximum bipartite subgraph of $G$, we have $|\delta(X) \cap F| \leq |\delta(X) \setminus F| \leq \hat{d}(X)$. Hence, $2 \cdot \hat{d}(X)$ serves as a useful upper bound for $\delta(X) \cup (E(X) \cap F')$. Thus, a good upper bound on $|E(X) \cap (F \setminus F')|$ combined with a small $\hat{d}(X)$ gives us a small odd $(s, s)$-trail cover $\delta(X) \cup (E(X) \cap F)$. We now discuss the construction in [3], which yields a set $X$ containing $s$ with $\hat{d}(X) \leq 4k'$ and $E(X) \cap (F \setminus F') = \emptyset$, thereby yielding a cover of size at most $8k'$, which is the bound obtained in [3]. We improve the analysis to obtain a better bound of $6k'$.

**Claim 3.6** If $s \in X \subseteq V$, then $\hat{d}(X) \geq 2k'$.

*Proof.* This follows from the max-flow min-cut theorem since $\lambda(s, t; \hat{G}) = 2k'$. □

**Lemma 3.7** The function $\hat{d}$ is submodular i.e., for any $X, Y \subseteq V$, we have the following inequality.

$$\hat{d}(X) + \hat{d}(Y) \geq \hat{d}(X \cap Y) + \hat{d}(X \cup Y)$$

Moreover, equality holds if and only if no edge of $\mathbf{G}'$ has one endpoint in $X \setminus Y$ and the other in $Y \setminus X$.

*Proof.* By submodularity inequality for the cuts in $G'$ we get,

$$|\delta(X) \setminus F| + |\delta(Y) \setminus F| \geq |\delta(X \cap Y) \setminus F| + |\delta(X \cup Y) \setminus F| \ , \tag{3.1}$$

where equality holds if and only if no edge of $G'$ has one endpoint in $X \setminus Y$ and the other in $Y \setminus X$. Using inclusion-exclusion principle we get the following equation.

$$\deg(X, F') + \deg(Y, F') = \deg(X \cap Y, F') + \deg(X \cup Y, F') \tag{3.2}$$

Combining Inequality 3.1 and Equation 3.2, we get the desired result. □

Let $S_0 \subseteq V$ be the minimal min $(s,t)$-cut in $\hat{G}$. Observe that $\hat{d}(S_0) = 2k'$. It is well known (see [4], Chapter 3) that (a) the minimal min $(s,t)$-cut is unique, and (b) every min $(s,t)$-cut contains the minimal min $(s,t)$-cut. One simple proof of this follows from the submodularity of the cut function. Given this, we get the following claim.

**Claim 3.8** If $X$ is a vertex-set containing $s$ with $\hat{d}(X) = 2k'$, then $X \supseteq S_0$.

Although $\hat{d}(S_0)$ is small, $E(S_0)$ could contain many edges from $F \setminus F'$, so we need to devise a way of getting rid of such edges. This leads us to the notion of an $e$-inhibiting set.

**Definition 3.9** ($e$-inhibiting set) A vertex-set $S_e$ containing $s$ is said to be $e$-inhibiting for some $e \in F \setminus F'$ if $S_e$ is a min $(s,t)$-cut in $\hat{G}(F' \cup \{e\})$.

Since $F'$ is a maximum-size [2] subset satisfying $\lambda(s,t;\hat{G}(F')) = 2 \cdot |F'|$, the size of a min $(s,t)$-cut in $\hat{G}(F' \cup \{e\})$ is strictly less than $2k' + 2$.

**Lemma 3.10** If $S_e$ is an $e$-inhibiting set, then $\hat{d}(S_e) + \deg(S_e, e) < 2k' + 2$, and hence, $e \notin E(S_e)$. Moreover, if $e \in E(S_0) \cap (F \setminus F')$, then $\hat{d}(S_e) = 2k' + 1$, and hence, $e \notin E(S_e) \cup \delta(S_e)$.

*Proof.* By definition of an $e$-inhibiting set, we have $|\delta_{\hat{G}(F' \cup \{e\})}(S_e)| < 2k' + 2$, which is equivalent to $\hat{d}(S_e) + \deg(S_e, e) < 2k' + 2$. Since $\hat{d}(S_e) \geq 2k'$, we have $\deg(S_e, e) < 2$, which implies $e \notin E(S_e)$. Further, if $e \in E(S_0) \cap (F \setminus F')$, then $\hat{d}(S_e) = 2k' + 1$ since otherwise $\hat{d}(S_e) = 2k'$ implies $S_e \supseteq S_0$, contradicting $e \notin E(S_e)$. Hence $\deg(S_e, e) = 0$, which implies $e \notin E(S_e) \cup \delta(S_e)$. □

[2]Note that maximality of $F'$ is sufficient to get the same conclusion.

Next, for each $e \in E(S_0) \cap (F \setminus F')$, define $I_e$ to be a minimal $e$-inhibiting set i.e., a minimal min $(s,t)$-cut in $\hat{G}(F' \cup \{e\})$. As mentioned earlier, such minimal min $(s,t)$-cuts are unique.

**Lemma 3.11** For $e \in E(S_0) \cap (F \setminus F')$, we have $I_e \subsetneq S_0$.

*Proof.* By Lemma 3.10, there exists an $e$-inhibiting set $S_e$ with $\hat{d}(S_e) = 2k' + 1$. Using submodularity of $\hat{d}$ we get the following tight inequality.

$$2k' < \hat{d}(S_e \cap S_0) \leq \hat{d}(S_e) + \hat{d}(S_0) - \hat{d}(S_e \cup S_0) \leq 2k' + 1 + 2k' - 2k' = 2k' + 1$$

Thus, $S_e \cap S_0 \subsetneq S_0$ is also an $e$-inhibiting set. Using arguments similar to the ones used in deriving Claim 3.8, every $e$-inhibiting set contains $I_e$. Hence the result follows. $\square$

For $e \in E(S_0) \cap (F \setminus F')$, let $A_e := S_0 \setminus I_e$ denote the complement of $I_e$ in $S_0$.

**Lemma 3.12** For any pair of edges $e_1, e_2 \in E(S_0) \cap (F \setminus F')$, either $I_{e_1} = I_{e_2}$ or $A_{e_1} \cap A_{e_2} = \emptyset$ (i.e., $I_{e_1} \cup I_{e_2} = S_0$). Moreover, in the latter case, there are no edges of $G'$ crossing from $A_{e_1}$ to $A_{e_2}$.

*Proof.* Suppose that $I_{e_1} \neq I_{e_2}$. We may assume that $I_{e_1}$ is not an $e_2$-inhibiting set, since otherwise $I_{e_1} \cap I_{e_2}$ is a smaller $e_2$-inhibiting set, contradicting the minimality of $I_{e_2}$. Similarly, $I_{e_2}$ is not an $e_1$-inhibiting set. Using submodularity of $\hat{d}$, we get the following tight inequality.

$$2k' + 1 < \hat{d}(I_{e_1} \cap I_{e_2}) \leq \hat{d}(I_{e_1}) + \hat{d}(I_{e_2}) - \hat{d}(I_{e_1} \cup I_{e_2}) \leq 2k' + 1 + 2k' + 1 - 2k' = 2k' + 2$$

Thus, we have $\hat{d}(I_{e_1} \cap I_{e_2}) = 2k' + 2$ and $\hat{d}(I_{e_1} \cup I_{e_2}) = 2k'$. By Lemma 3.11, we have $I_{e_1}, I_{e_2} \subsetneq S_0$, and therefore $I_{e_1} \cup I_{e_2} = S_0$ (or equivalently $A_{e_1} \cap A_{e_2} = \emptyset$). Also, the tightness of submodular inequality implies that none of the edges in $G'$ cross from $I_{e_1} \setminus I_{e_2}$ $(= A_{e_2})$ to $I_{e_2} \setminus I_{e_1}$ $(= A_{e_1})$. $\square$

Observe that the intersection of all such minimal $e$-inhibiting sets has no $(F \setminus F')$-edge within it. Hence, our goal is to bound the value of $\hat{d}$ for such a vertex-set. Let $\mathcal{I} := \{I_e : e \in E(S_0) \cap (F \setminus F')\}$ be the collection of all minimal inhibiting sets corresponding to $(F \setminus F')$-edges having both endpoints in $S_0$. Let $N := |\mathcal{I}|$. Arbitrarily label the elements of $\mathcal{I}$ as $I_1, \ldots, I_N$. For each $1 \leq j \leq N$, let $A_j$ denote $S_0 \setminus I_j$.

**Lemma 3.13** For any edge $e \in E(S_0) \cap (F \setminus F')$ and $1 \leq j \leq N$, either $e \in E(I_j)$ or $e \in E(A_j)$.

*Proof.* Suppose, for the sake of contradiction, there exists an edge $e = xy \in (F \setminus F') \cap E(S_0)$ with $x \in I_j$ and $y \in A_j$. Observe that $I_j \neq I_e$, since $e$ has one of its endpoints in $I_j$. By definition, $e \in E(A_e)$, therefore, $A_e \cap A_j \neq \emptyset$, which contradicts Lemma 3.12. Thus, the conclusion of the lemma holds. $\square$

Combining Lemmas 3.12 and 3.13, we get the following corollary.

**Corollary 3.14** For any pair of distinct indices $1 \leq j, j' \leq N$, we have $E(A_j, A_{j'}) \setminus F' = \emptyset$ where $E(A, B)$ denotes the set of edges crossing from $A$ to $B$.

We will often refer to the vertex-sets in $\{A_1, \ldots, A_N\}$ as blobs. For any subset of indices $J \subseteq \{1, \ldots, N\}$, let $S_J := S_0 \cap \left(\bigcap_{j \in J} I_j\right) = S_0 \setminus \left(\bigcup_{j \in J} A_j\right)$ denote the intersection of $S_0$ and the minimal inhibiting sets indexed by the elements in $J$. We reserve the notation $A_0$ for the vertex-set $S_{\{1,\ldots,N\}} = S_0 \setminus (A_1 \cup \cdots \cup A_N)$. By definition, there are no edges in $F \setminus F'$ having both endpoints in $A_0$. We summarize key observations until now through Figure 3.1.

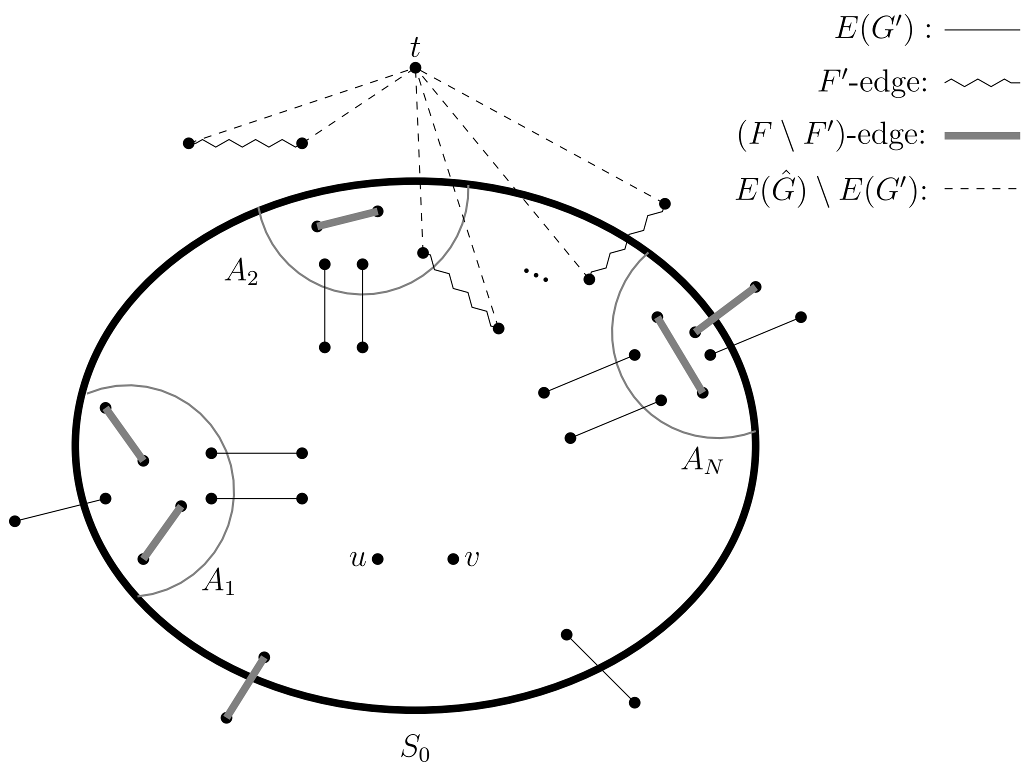



Figure 3.1: Structure of $S_0$.

The following lemma shows that for each blob removed from $S_0$, the value of $\hat{d}$ goes up by one.

**Lemma 3.15** For any $J \subseteq \{1, \ldots, N\}$, we have $\hat{d}(S_J) = 2k' + |J|$.

*Proof.* We will prove the lemma via induction on $|J|$. For $|J| = 0$, the statement is trivially true since $\hat{d}(S_\emptyset) = \hat{d}(S_0) = 2k'$. For $|J| = 1$, the statement is true since $\hat{d}(S_{\{j\}}) = \hat{d}(I_j) = 2k' + 1$ for any $1 \leq j \leq N$. Suppose that the statement is true for all index-sets having fewer than $p$ indices for some $2 \leq p \leq N$. Consider any index-set $J$ containing $p$ indices and let $j$ be one such index. We use the submodularity of $\hat{d}$ to get the following equation.

$$\hat{d}(S_J) = \hat{d}(S_{J \setminus \{j\}}) + \hat{d}(I_j) - \hat{d}(S_{J \setminus \{j\}} \cup I_j) = (2k' + p - 1) + (2k' + 1) - 2k' = 2k' + p$$

We used the tight version of submodular inequality since there are no $G'$ edges having one endpoint in $S_{J \setminus \{j\}} \setminus I_j = A_j$ and the other in $I_j \setminus S_{J \setminus \{j\}} = \bigcup_{i \in J \setminus \{j\}} A_i$, as shown in Lemma 3.12. Since the statement is true for $|J| = p$, the statement is true for all $0 \leq |J| \leq N$ by induction. $\square$

We now attribute the increase in $\hat{d}$ to the bipartite edges crossing from $A_0$ to $A_j$.

**Lemma 3.16** For each $1 \leq j \leq N$,

$$|E(A_j, A_0) \setminus F| = 1 + |E(A_j, V \setminus S_0) \setminus F| + \deg(A_j, F')$$

*Proof.* Comparing $\hat{d}(I_j)$ and $\hat{d}(S_0)$, we get the desired result.

$$\begin{aligned}\hat{d}(I_j) - \hat{d}(S_0) &= |\delta(I_j) \setminus F| + \deg(I_j, F') - |\delta(S_0) \setminus F| - \deg(S_0, F') \\ &= |E(A_j, A_0) \setminus F| - |E(A_j, V \setminus S_0) \setminus F| - \deg(A_j, F')\end{aligned}$$

$\square$

For each $0 \leq j \leq N$, let $r_j := |E(A_j, V \setminus S_0) \setminus F| + \deg(A_j, F')$ denote the number of edges crossing from $A_j$ to $(V \cup \{t\}) \setminus S_0$ in $\hat{G}$. The above lemma shows that the number of bipartite edges crossing from $A_0$ to $A_j$ is $1 + r_j$. Enumerating the edges in $\hat{G}$ on the boundary of $S_0$ gives us the following equation.

$$\hat{d}(S_0) = \sum_{0 \leq j \leq N} r_j \tag{3.3}$$

This brings us to the final lemma in the proof.

**Lemma 3.17** For any $1 \le j \le N$, we have $r_j > 0$.

*Proof.* Suppose, for the sake of contradiction, that $r_j = 0$ for some $1 \le j \le N$ i.e., there are no incidences of $F'$ edges in the blob $A_j$ and there are no bipartite edges crossing from $A_j$ to $V \setminus S_0$. By Corollary 3.14, there are no edges in $G - F'$ crossing from one blob to another. Hence, $|\delta(A_j) \setminus F| = |E(A_j, A_0) \setminus F| = 1$. But then by Property 3.3, there are no $F$-edges with both endpoints in $A_j$, which contradicts the assumption that $I_j$ is a minimal $e$-inhibiting set for some $e \in E(S_0) \cap (F \setminus F')$. Hence $r_j > 0$ for every $1 \le j \le N$. □

Combining Lemma 3.17 and Equation 3.3, we get the following corollary.

**Corollary 3.18** $N \le 2k'$.

The above corollary gives us an upper bound of $8k'$ on the odd $(s, s)$-trail cover $\delta(A_0) \cup (E(A_0) \cap F)$. But we can do better. Using Lemma 3.13, we know that there are no $(F \setminus F')$-edges crossing from $A_0$ to any of the blobs $A_j$. Thus, we get the following bound.

$$|\delta(A_0) \cap (F \setminus F')| \le |\delta(S_0) \cap (F \setminus F')| \le |\delta(S_0) \setminus F| \le \hat{d}(S_0)$$

By Lemma 3.15 and Corollary 3.18, we have $\hat{d}(A_0) = 2k' + N \le 4k'$. Hence, $A_0$ satisfies the second conclusion of the lemma as shown below.

$$\begin{aligned}|\delta(A_0)| + |E(A_0) \cap F| &= |\delta(A_0) \setminus F| + |\delta(A_0) \cap (F \setminus F')| + |\delta(A_0) \cap F'| + |E(A_0) \cap F'| \\ &\le |\delta(A_0) \setminus F| + \deg(A_0, F') + |\delta(A_0) \cap (F \setminus F')| \\ &\le \hat{d}(A_0) + \hat{d}(S_0) \le 6k' \le 6k - 6\end{aligned}$$

## 3.2 Algorithm for Theorem 3.5

In this section we discuss how an efficient algorithm can be designed for Theorem 3.5. While we do not get into the details of the exact running time, we will show that the running time is bounded above by a polynomial in $n := |V(G)|$, $m := |E(G)|$, and $k$. Since the proofs of Theorem 3.4 and 3.5 are constructive, they can easily be turned into algorithms. But, designing an algorithm running in polynomial time is nontrivial since the proof for Theorem 3.5 uses a maximum bipartite subgraph $G'$ of $G$, which is NP-hard to find (see [5]), and the proof for Theorem 3.4 uses a maximum edge-set $F'$ satisfying $\lambda(s, t; \hat{G}(F')) = 2 \cdot |F'|$, which we do not know how to find. Many of the results in our proofs were derived by inferring properties of a suitably chosen flow network. Such aspects can easily be turned into efficient subroutines by using algorithms for the well studied maximum flow problem.

The only property of a maximum bipartite subgraph that we use in the proofs is that for every cut, at least half the edges are bipartite. So, we overcome the problem of finding a maximum bipartite subgraph of $G$ by working with an (inclusion-wise) edge-maximal bipartite subgraph $G'$, and whenever we encounter a *bad* cut that shares fewer than half of its edges with the bipartite subgraph, we swap the bipartite edges in the cut with the nonbipartite edges in the cut to obtain a strictly larger bipartite subgraph of $G$, as shown in the following lemma.

**Lemma 3.19** Let $G = (V, E)$ be a graph and $G_1 = (V, E_1)$ a bipartite subgraph of $G$. Let $F_1 = E \setminus E_1$. Suppose there exists a cut $\delta(X)$ in $G$ such that $|\delta(X) \cap F| > |\delta(X) \setminus F|$. Then we can obtain a bipartite subgraph $G_2 = (V, E_2)$ of $G$ such that $|E_2| > |E_1|$.

*Proof.* If $G_1$ is not edge-maximal, then there exists an edge $e \notin G_1$, which has exactly one endpoint in each partition of $G_1$. We can add this edge to $G_1$ to obtain a larger bipartite subgraph of $G$. So, we may assume that $G_1$ is edge-maximal. Let $(Z_1, V \setminus Z_1)$ be the vertex partition of $G_1$. Let $Z_2 = Z_1 \Delta X$ be the symmetric difference of $Z_1$ and $X$. We can obtain a larger bipartite subgraph $G_2 = (V, E_2)$ of $G$ by considering the vertex partition $(Z_2, V \setminus Z_2)$, and adding all edges in $G$ that have exactly one endpoint in $Z_2$. It is easy to verify that $|E_2| = |E_1| + |\delta(X) \cap F| - |\delta(X) \setminus F| > |E_1|$. □

In the proof of Theorem 3.5, we use the fact that the edge-connectivity between $u$ and $v$ is at least $3k$ in the bipartite subgraph. If it's not the case with the bipartite subgraph that we are currently working with, then a minimum $(u, v)$-cut is a *bad* cut. Next, in Lemma 3.17, we assume that the cut defined by the blob $A_j$ satisfies $|\delta(A_j) \setminus F| \geq |\delta(A_j) \cap F|$. If it's not the case, then $\delta(A_j)$ is a *bad* cut. Finally, to finish the proof of Theorem 3.4, we use $|\delta(S_0) \setminus F| \geq |\delta(S_0) \cap F|$. Again, if it's not the case, then $\delta(S_0)$ is a *bad* cut. In each of these cases we can obtain a larger bipartite subgraph and restart the algorithm with the new larger bipartite subgraph. Hence, the number of calls to the subroutine for Theorem 3.4 is $\mathcal{O}(m)$.

The other difficulty to overcome is finding a maximum-size edge-set $F' \subseteq F$ (see Theorem 3.4) satisfying $\lambda(s, t; \hat{G}(F')) = 2 \cdot |F'|$. Recall from Definition 3.9 and Lemma 3.10, that we used maximality of $F'$ to derive the existence of inhibiting sets. So, we may as well use a maximal-size $F'$ and the proof still holds. Note that a maximal $F'$ can be obtained as follows: Start with $F' = \emptyset$ and as long as there exists an edge $e \in F \setminus F'$ satisfying

$\lambda(s,t;\hat{G}(F' \cup \{e\})) = 2 \cdot |F'| + 2$, add $e$ to $F'$. Hence, we can design an algorithm for Theorem 3.5 which runs in polynomial time.

## 3.3 Concluding Remarks: Outline for the remainder of the Thesis

The remainder of the thesis is devoted to improving the bound on the covering-vs.-packing ratio from 6 to 5. Examining the proof of Theorem 3.5, there are two bottlenecks that contribute to the bound of 6 that we need to circumvent to obtain an improved bound.

First, (the second conclusion in) Theorem 3.4 yields a cover of size at most $6k-6$. In Chapter 4, we show how to improve this bound to $5k-5$. Second, in order to modify the collection $\mathcal{T}$ of $k$ edge-disjoint odd $(s,s)$-trails each containing exactly one edge of $F$ (given by the first conclusion of Theorem 3.4) to obtain $k$ edge-disjoint odd $(u,v)$-trails, we require that:

there are $k$ edge-disjoint $(u,v)$-paths in the bipartite graph $G'$ that are edge-disjoint from $\mathcal{T}$. (*)

This requires that the $(u,v)$-edge-connectivity in the bipartite graph, $\lambda(u,v;G')$, be at least $3k$, and hence, that the $(u,v)$-edge-connectivity in the original graph be at least $6k-1$. One can show that $3k$ is the tight threshold needed for $\lambda(u,v,G')$ in order to achieve (*) (see the example in Figure 5.10). However, in Chapter 5, we show that a weaker requirement suffices to modify odd $(s,s)$-trails to obtain odd $(u,v)$-trails: given a collection of $2k$ edge-disjoint $(u,v)$-paths in $G'$, *that need not however be edge-disjoint from* $\mathcal{T}$, we show that we can obtain $k$ edge-disjoint odd $(u,v)$-trails. Consequently, we only need the $(u,v)$-edge-connectivity in the original graph to be (at least) $4k-1$.

Chapter 6 combines these ingredients and proves an improved bound of (essentially) 5 on the covering-vs.-packing ratio.

# Chapter 4

# Deficient Sets

The main result of this chapter is a strengthening of Theorem 3.4 to obtain Theorem 4.1 (stated below). We reuse the notation from Chapter 3 i.e., $G$ is a *reduced* graph, $G'$ is a maximum bipartite subgraph of $G$ with $u$ and $v$ on the same side of the bipartition and also satisfying the conclusion of Lemma 2.9, the loops arising in Reduction 3.2 have been removed, and $s$ refers to $\{u, v\}$.

**Theorem 4.1** *For every nonnegative integer $k$, the graph $G$ has either:*

*i. $k$ edge-disjoint odd $(s, s)$-trails each containing exactly one edge of $F$; or*

*ii. A vertex-set $S$ containing $s$ with $|\delta(S)| + |E(S) \cap F| \leq 5k - 5$.*

This chapter is devoted to proving the above theorem. Since the hypotheses of both Theorem 3.4 and Theorem 4.1 are the same, we will invoke Theorem 3.4 for the given $k$. If we obtain $k$ edge-disjoint odd $(s, s)$-trails each containing exactly one edge of $F$, then we are trivially done. Otherwise, we are in the premise of second conclusion of Theorem 3.4. So, reusing the terminology and notation from Chapter 3, we have a maximum-size edge-set $F' \subseteq F$ with $k' = |F'| < k$ that satisfies $\lambda(s, t; \hat{G}(F')) = 2k'$. Recall that $S_0$ is a minimal min $(s, t)$-cut in $\hat{G} = \hat{G}(F')$, which is partitioned into $A_0$ and the blobs $A_1, \ldots, A_N$, with $N \leq 2k'$. Also, recall that $\hat{d}(S)$ denotes the size of boundary of $S$ in $\hat{G}$, and we have $\hat{d}(A_0) = 2k' + N \leq 4k'$, and hence $|\delta(A_0)| + |E(A_0) \cap F| \leq 6k'$.

To obtain the improved bound in Theorem 4.1, we will show that when $N > k'$, there exists a subset $A \subseteq A_0$ containing $s$ with $\hat{d}(A) \leq 3k'$. This yields the second conclusion of Theorem 4.1, as shown below.

$$
\begin{aligned}
|\delta(A)| + |E(A) \cap F| &= |\delta(A) \setminus F| + |\delta(A) \cap (F \setminus F')| + |(E(A) \cup \delta(A)) \cap F'| \\
&\leq |\delta(A) \setminus F| + \deg(A, F') + |\delta(S_0) \cap (F \setminus F')| \\
&\leq \hat{d}(A) + \hat{d}(S_0) \leq 5k'
\end{aligned}
$$

The key idea that we develop in this chapter is generalizing the notion of inhibiting sets. Recall that for any edge $e \in E(S_0) \cap (F \setminus F')$, an $e$-inhibiting set $S_e$ is a min $(s,t)$-cut in $\hat{G}(F' \cup \{e\})$, and we have $|\delta_{\hat{G}(F' \cup \{e\})}(S_e)| < 2 \cdot |F'| + 2$. Thus, an $e$-inhibiting set shows that $\lambda(s, t; \hat{G}(F' \cup \{e\})) < 2k' + 2$. We generalize the idea as follows. Consider a nonempty collection $X = \{f_1, \ldots, f_l\}$ of $l$ edges in $F'$ and another collection $Y = \{e_1, \ldots, e_{l+1}\}$ of $l+1$ edges in $F \setminus F'$. Let $\tilde{F}$ denote $(F' \cup Y) \setminus X$. Since $F'$ is a maximum-size subset, we must have $\lambda(s, t; \hat{G}(\tilde{F})) < 2 \cdot |\tilde{F}|$ i.e., there exists a min $(s,t)$-cut in $\hat{G}(\tilde{F})$ having fewer than $2k' + 2$ edges in the cut. As we will see later (see Lemma 4.5), we provide sufficient conditions under which $\lambda(s, t; \hat{G}(\tilde{F})) = 2 \cdot |\tilde{F}|$ holds; since this condition cannot hold, this implies that the sufficient conditions are not met.

## 4.1 Internal Structure of $S_0$

In this section, we derive results related to the internal connectivity of the blobs $A_1, \ldots, A_N$. Recall that for any $0 \leq j \leq N$, we use $r_j$ to denote $|E(A_j, V \setminus S_0) \setminus F| + \deg(A_j, F')$. The following lemma shows that every blob $A_j$ is internally connected.

**Lemma 4.2** For any $1 \leq j \leq N$, the induced bipartite subgraph $G'[A_j]$ is connected.

*Proof.* Suppose, for the sake of contradiction, that the induced bipartite subgraph $G'[A_j]$ is disconnected. Let $X$ be one of the components in $G'[A_j]$. Using Lemma 3.16, we have $|E(A_j, A_0) \setminus F| = 1 + |E(A_j, V \setminus S_0) \setminus F| + \deg(A_j, F')$, which can be rewritten as follows.

$$
\begin{aligned}
&|E(X, A_0) \setminus F| + |E(A_j \setminus X, A_0) \setminus F| \\
&= 1 + |E(X, V \setminus S_0) \setminus F| + |E(A_j \setminus X, V \setminus S_0) \setminus F| + \deg(X, F') + \deg(A_j \setminus X, F')
\end{aligned}
$$

The above equation implies that, either $|E(X, A_0) \setminus F| \leq |E(X, V \setminus S_0) \setminus F| + \deg(X, F')$ or $|E(A_j \setminus X, A_0) \setminus F| \leq |E(A_j \setminus X, V \setminus S_0) \setminus F| + \deg(A_j \setminus X, F')$. The first case implies that $\hat{d}(S_0 \setminus X) \leq 2k'$ and the second case implies that $\hat{d}(S_0 \setminus (A_j \setminus X)) \leq 2k'$. In both cases, we contradict the minimality of $S_0$. $\square$

**Lemma 4.3** For a fixed $1 \leq j \leq N$ and a fixed edge $e = xy \in (F \setminus F') \cap E(A_j)$, let $H$ be the graph obtained from $G'[A_0 \cup A_j]$ by contracting $A_0$ into a single vertex $\alpha$, adding a new vertex $\beta$ and adding two edges $x\beta$ and $y\beta$. Then, $\lambda(\alpha, \beta; H) = 2$.

*Proof.* First of all, $H$ is connected since $G'[A_j]$ is connected. Suppose, for the sake of contradiction, that $e'$ is a bridge separating $\alpha$ and $\beta$ in $H$. Let the component containing $\alpha$ in $H - e'$ be $X$ i.e., $E_H(X, V(H) \setminus X) = \{e'\}$. Since $\deg_H(\alpha) = 1 + r_j \geq 2$ (see Lemma 3.17) and $\deg_H(\beta) = 2$, there are at least two vertices in both $X$ and $V(H) \setminus X$. It follows that $A_j \cap X \neq \emptyset$ and $A_j \setminus X \neq \emptyset$. Again, since $G'[A_j]$ is connected, there is a bipartite edge crossing from $A_j \cap X$ to $A_j \setminus X$, which is precisely $e'$. Combining all these observations, we get that all the neighbors of $\alpha$ in $H$ are in $A_j \cap X$ and both neighbors of $\beta$ (i.e., $x$ and $y$) are in $A_j \setminus X$.

Consider the vertex-set $A_j \setminus X$ in $\hat{G}$. If the set of edges in $\hat{G}$ crossing from $A_j \setminus X$ to $(V \cup \{t\}) \setminus S_0$ is nonempty, then we arrive at a contradiction since we get $\hat{d}(S_0 \setminus (A_j \setminus X)) \leq 2k'$. Otherwise, $|\delta(A_j \setminus X) \setminus F| = 1$, which contradicts Property 3.3 since $e = xy$ is an edge in $F \setminus F'$ with both endpoints in $A_j \setminus X$. Hence, $\lambda(\alpha, \beta; H) = 2$.

□

## 4.2 Interplay between $\{A_j\}_{j=0}^N$ and $F'$

Let the $k'$ edges in $F'$ be $x_1y_1, \ldots, x_{k'}y_{k'}$. Since $\lambda(s, t; \hat{G}) = 2k'$, we can obtain $2k'$ edge-disjoint $(s,t)$-paths in $\hat{G}$. Let the $2k'$ paths be labeled $X_1, \ldots, X_{k'}, Y_1, \ldots, Y_{k'}$ depending on which edge among $x_1t, \ldots, x_{k'}t, y_1t, \ldots, y_{k'}t$ is used in the path. Since $\hat{d}(S_0) = 2k'$, each path in $\mathcal{P} = \{X_1, \ldots, X_{k'}, Y_1, \ldots, Y_{k'}\}$ uses exactly one edge crossing from $S_0$ to $(V \cup \{t\}) \setminus S_0$ in $\hat{G}$. Also, using $\hat{d}(S_0) = \sum_{0 \leq j \leq N} r_j$, we can associate each path $P \in \mathcal{P}$ with a unique vertex-set $A_j$ (for some $0 \leq j \leq N$) satisfying $E_{\hat{G}}(A_j, V(\hat{G}) \setminus S_0) \cap P \neq \emptyset$ i.e., $P$ crosses from $S_0$ to $(V \cup \{t\}) \setminus S_0$ via the vertex-set $A_j$. Therefore, each $A_j$ is associated with exactly $r_j$ distinct paths in $\mathcal{P}$.

We now construct an auxiliary graph $\hat{H}$ as follows. Consider $N+1$ vertices $b_0, \ldots, b_N$ corresponding to the vertex-sets $A_0, \ldots, A_N$. For each $1 \leq i \leq k'$, do the following: let $A_j, A_{j'}$ (for some $0 \leq j, j' \leq N$) be the vertex-sets associated with $X_i$ and $Y_i$, then add an edge between the vertices $b_j$ and $b_{j'}$. Note that whenever $j = j'$, the edge is a loop at $b_j$. It is trivial to see that for any $0 \leq j \leq N$, the degree of $b_j$ (where loops contribute 2 to the degree) is $r_j$. We now investigate the (connected) components in $\hat{H}$. The following lemma provides a lower bound on the number of tree components which do not contain the vertex $b_0$.

**Lemma 4.4** Let $Q := \{C : C \text{ is a component in } \hat{H} \text{ with } b_0 \notin C \text{ and } |E(C)| = |V(C)| - 1\}$. Then, $|Q| \geq N - k'$.

*Proof.* For any component $C$, we have $|E(C)| \geq |V(C)| - 1$. For any $C \notin Q$ and $b_0 \notin C$, we have $|E(C)| \geq |V(C)|$. Using a simple counting argument over the edges in each of the components in $\hat{H}$ we get $E(\hat{H}) \geq V(\hat{H}) - |Q| - 1$, which gives $|Q| \geq N - k'$, as desired. $\square$

Lemma 4.4 shows that there are at least $N - k'$ tree components in $\hat{H}$ which do not contain the vertex $b_0$. Note that such a tree component has at least two vertices since $r_j > 0$ for all $1 \leq j \leq N$. We now label the components in $Q$ as $C_1, \ldots, C_q$ where $q := |Q|$. For any $1 \leq i \leq q$, let $D_i := \{j : 1 \leq j \leq N \text{ and } b_j \in C_i\}$ be the subset of indices corresponding to the vertices in $C_i$. We call each $D_i$ a *deficient* set. The reason for calling such sets deficient is due to the fact that a subset of $|D_i|$ blobs are associated with $2 \cdot (|D_i| - 1)$ paths in $\mathcal{P}$, which correspond to $|D_i| - 1$ edges (call the subcollection as $F'_i$) in $F'$.

We have now identified potential subsets ($F'_i$ corresponding to each $D_i$) of $|D_i| - 1$ edges in $F'$ which can be removed. What remains now is to identify a suitable subset of $|D_i|$ edges in $F \setminus F'$ which can be substituted for the removed edges to obtain a strictly larger $F'$. In fact, we will show that under suitable cut assumptions, one edge (of our choice) in $(F \setminus F') \cap E(A_j)$ for each $j \in D_i$ can be picked to be added to $F' \setminus F'_i$.

## 4.3 Saturated Deficient Sets

Recall that Lemma 4.3 provides us with a sufficient condition to route two units of flow (via edges in $G'$) from $A_0$ to the endpoints of an edge (of our choice) in $(F \setminus F') \cap E(A_j)$ for any $A_j$. The following lemma states a sufficient condition to route $2 \cdot |D_i|$ units of flow (via edges in $G'$) from $A_0$ to the $2 \cdot |D_i|$ endpoints of suitably chosen $|D_i|$ edges in $F \setminus F'$.

**Lemma 4.5** For a fixed $1 \leq i \leq q$, let $S := S_0 \cap \left(\bigcap_{j \in D_i} I_j\right) = S_0 \setminus \left(\bigcup_{j \in D_i} A_j\right)$ denote the intersection of $S_0$ and the minimal inhibiting sets indexed by elements in $D_i$. Suppose that for every $S' \subseteq S$ containing $s$, we have $\hat{d}(S') \geq \hat{d}(S) = 2k' + |D_i|$. Then, there exists an edge-set $\tilde{F} \subseteq F$ of cardinality $|F'| + 1$ and satisfying $\lambda(s, t; \hat{G}(\tilde{F})) = 2 \cdot |\tilde{F}|$.

*Proof.* Let $T$ be the node obtained by identifying $(V \cup \{t\}) \setminus S$ in $\hat{G}$ into a single vertex. Call the resulting graph $\bar{G}$. Since $\hat{d}(S') \geq 2k' + |D_i|$ for every subset $S' \subseteq S$ containing $s$, there exists an $(s,T)$-flow of value $2k' + |D_i|$ in $\bar{G}$. Since $\hat{d}(S) = 2k' + |D_i|$, the $(s,T)$-flow saturates all edges crossing from $S$ to $T$. Observe that for every $0 \leq j \leq N$ and $j \notin D_i$, all edges crossing from $A_j$ to $(V \cup \{t\}) \setminus S_0$ are saturated in this $(s,T)$-flow. Also, for every $j \in D$, all $1 + r_j$ bipartite edges, crossing from $A_0$ to $A_j$, are saturated. For each $j \in D_i$, we fix an edge $e^j = x^j y^j \in (F \setminus F') \cap E(A_j)$. Let $\tilde{F} := (F' \setminus F'_i) \cup \{e^j\}_{j \in D_i}$. In the graph $\hat{G}(\tilde{F})$, we can now independently (without disturbing the paths supplying flow to the endpoints of edges in $(F' \setminus F'_i)$ use Lemma 4.3 for each $j \in D_i$ to obtain two $(s,t)$-paths for each $e^j$ by using some two edges among the $1 + r_j$ bipartite edges crossing from $A_0$ to $A_j$. Hence, $\lambda(s,t; \hat{G}(\tilde{F})) = 2 \cdot |\tilde{F}|$. □

Since we assume that $F'$ is a maximum-size edge-set, the assumptions in the lemma do not hold for any of the deficient sets i.e., for each $1 \leq i \leq q$, there exists a subset $B_i \subsetneq S_0 \setminus \left( \bigcup_{j \in D_i} A_j \right)$ containing $s$ that satisfies $\hat{d}(B_i) \leq 2k' + |D_i| - 1$. We now have all the tools to finish the proof of Theorem 4.1.

## 4.4 Finishing up the Proof of Theorem 4.1

Let $R := \{1, \ldots, N\} \setminus \bigcup_{1 \leq i \leq q} D_i$ be the collection of indices for which the corresponding blob is not in any of the deficient sets. Let $S_R$ denote $S_0 \setminus \left( \bigcup_{j \in R} A_j \right)$. Observe that $A := S_R \cap B_1 \cap \cdots \cap B_q$ is a proper subset of $A_0$ since every index $1 \leq j \leq N$ appears in one of the index-sets in $\{R, D_1, \ldots, D_q\}$. We now use submodular inequality to show that $\hat{d}(A) \leq 3k'$, which will conclude the proof. Recall from Claim 3.6 that $\hat{d}(X) \geq 2k'$ for every vertex-set $X$ containing $s$.

$$
\begin{aligned}
& \hat{d}(B_1 \cap B_2) && \leq \hat{d}(B_1) + \hat{d}(B_2) - \hat{d}(B_1 \cup B_2) \\
\Longrightarrow\ & \hat{d}(B_1 \cap B_2) && \leq (2k' + |D_1| - 1) + (2k' + |D_2| - 1) - 2k' = 2k' + |D_1| + |D_2| - 2
\end{aligned}
$$

$$
\begin{aligned}
& \hat{d}(B_1 \cap B_2 \cap B_3) && \leq \hat{d}(B_1 \cap B_2) + \hat{d}(B_3) - \hat{d}((B_1 \cap B_2) \cup B_3) \\
\Longrightarrow\ & \hat{d}(B_1 \cap B_2 \cap B_3) && \leq (2k' + |D_1| + |D_2| - 2) + (2k' + |D_3| - 1) - 2k' \\
\Longrightarrow\ & \hat{d}(B_1 \cap B_2 \cap B_3) && \leq 2k' + |D_1| + |D_2| + |D_3| - 3 \\
& \qquad \vdots && \qquad \vdots \\
& \hat{d}(B_1 \cap \cdots \cap B_q) && \leq 2k' + \sum_{1 \leq i \leq q} |D_i| - q
\end{aligned}
$$

Intersecting $S_R$ with $B_1 \cap \cdots \cap B_q$ gives,

$$
\begin{aligned}
& \hat{d}(A) && \leq \hat{d}(B_1 \cap \cdots \cap B_q) + \hat{d}(S_R) - \hat{d}((B_1 \cap \cdots \cap B_q) \cup S_R) \\
\Longrightarrow\ & \hat{d}(S) && \leq (2k' + \sum_{1 \leq i \leq q} |D_i| - q) + (2k' + |R|) - 2k' \\
\Longrightarrow\ & \hat{d}(S) && \leq 2k' + N - q \leq 2k' + N - (N - k') \leq 3k'
\end{aligned}
$$

# Chapter 5

# Odd $(s, s)$-trails to Odd $(u, v)$-trails

Recall from the proof of Theorem 3.5 in Chapter 3, wherein for some $k \geq 1$, the $k$ edge-disjoint odd $(s, s)$-trails from Theorem 3.4 can be used to obtain $k$ edge-disjoint odd $(u, v)$-trails provided $\lambda(u, v; G') \geq 3k$. As the example in Figure 5.1 indicates, the requirement that $\lambda(u, v; G') \geq 3k$ is tight if we want to obtain $k$ edge-disjoint $(u, v)$-paths that are also edge-disjoint with the $k$ (edge-disjoint) odd $(s, s)$-trails.

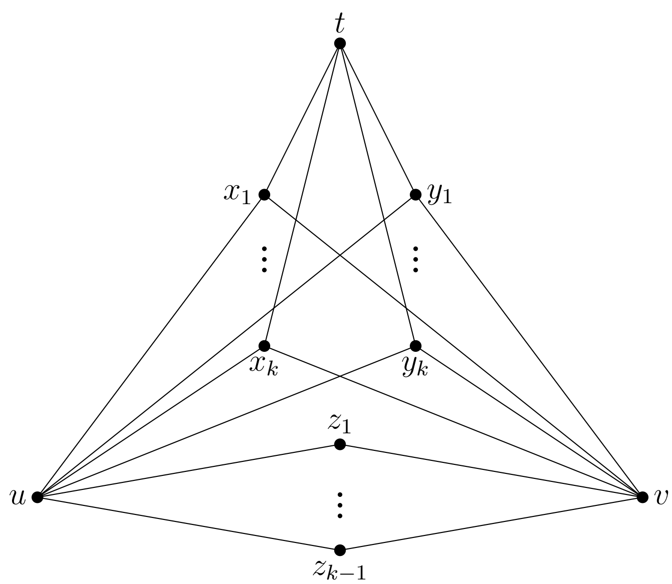



Figure 5.1: Graph $H$ with $\lambda(u, v; H) = 3k - 1$ and $\lambda(t, \{u, v\}; H) = 2k$.

To get $\lambda(u,v;G') \geq 3k$, we have to assume that $\lambda(u,v;G) \geq 6k-1$. In this chapter, we devise a way to obtain the desired $k$ edge-disjoint odd $(u,v)$-trails from $k$ edge-disjoint odd $(s,s)$-trails provided $\lambda(u,v;G') \geq 2k$. Hence, we may start with the assumption that $\lambda(u,v;G) \geq 4k-1$ and still obtain the desired trails. For the rest of this chapter, $G'$ is a maximum bipartite subgraph with $u$ and $v$ on the same side of the bipartition and $F$ denotes $E(G) \setminus E(G')$. For brevity, we call a collection of trails $\mathcal{T}$ in $G$ as *nice* if all the following conditions hold:

i The trails in $\mathcal{T}$ are edge-disjoint.

ii Each trail $T \in \mathcal{T}$ uses exactly one edge in $F$.

iii Each trail $T \in \mathcal{T}$ is one among the following: an odd $(u,u)$-trail, an odd $(v,v)$-trail, or an odd $(u,v)$-trail.

For a given positive integer $k$, if the first conclusion of Theorem 4.1 holds, then we get a *n*ice collection of trails $\mathcal{T}$ with cardinality $k$. We now state the main result of this chapter and devote the rest of this chapter to proving it.

**Theorem 5.1** *Let $\mathcal{T}$ be a nice collection of trails in $G$. If $\lambda(u,v;G') \geq 2 \cdot |\mathcal{T}|$, then we can obtain $|\mathcal{T}|$ edge-disjoint odd $(u,v)$-trails in $G$, in polynomial time.*

The main idea in the proof of this theorem is as follows. We have a nice collection of trails $\mathcal{T}$ and a collection $\mathcal{P}$ of $2 \cdot |\mathcal{T}|$ edge-disjoint even $(u,v)$-paths in $G'$. While the trails in $\mathcal{T}$ meet the parity requirement, the paths in $\mathcal{P}$ connect $u$ to $v$, so we replace portions (subtrails) of trails in $\mathcal{T}$ with suitably chosen portions (subpaths) from paths in $\mathcal{P}$ to obtain the desired outcome. We defer the technical details to Section 5.2. In the following section we introduce the notion of a *contact*.

## 5.1 Contacts

Given a *nice* collection of trails $\mathcal{T}$, we denote the subset of all odd $(u,u)$-trails, odd $(v,v)$-trails, and odd $(u,v)$-trails in $\mathcal{T}$ by $\mathcal{T}_{uu}$, $\mathcal{T}_{vv}$, and $\mathcal{T}_{uv}$, respectively. Let $k_{uu}(\mathcal{T})$, $k_{vv}(\mathcal{T})$, and $k_{uv}(\mathcal{T})$ denote $|\mathcal{T}_{uu}|$, $|\mathcal{T}_{vv}|$, and $|\mathcal{T}_{uv}|$, respectively. The edge-set $F'(\mathcal{T})$ will denote the subset of edges in $F$ used by the trails in $\mathcal{T}$. To keep notation simple, we will drop the argument $\mathcal{T}$ when its clear from the context. Note that $|F'| = |\mathcal{T}|$. The techniques developed in this chapter are elementary but involve some case analyses. We will use figures wherever necessary to make the proof more accessible. For the rest of this chapter, we use solid arcs (straight or curved) to represent trails (not necessarily edges) in $G'$ and zigzagged arcs are used to represent edges in $F'$.

Let $e = xy$ be an edge in $F'$. Consider the trail $T_e \in \mathcal{T}$ that uses the edge $e$. We can partition $T_e$, as shown in Figure 5.2, into three disjoint subtrails (possibly of zero length) as a trail from $a$ to $x$, the edge $xy$, and the trail from $b$ to $y$ such that $a, b \in \{u, v\}$. We call the $(a, x)$-trail and the $(b, y)$-trail the legs of $T_e$. We will refer to the endpoints $a$ and $b$ as the heads and the endpoints $x$ and $y$ as the tails of corresponding legs. Both legs of an odd $(u, u)$-trail are called $uu$-legs and both legs of an odd $(v, v)$-trail are called $vv$-legs. For an odd $(u, v)$-trail, the leg with the vertex $u$ as its head is called a $u$-leg and the other leg (with $v$ as its head) is called a $v$-leg. We denote by $\mathcal{L}_u(\mathcal{T})$, $\mathcal{L}_{uu}(\mathcal{T})$, $\mathcal{L}_v(\mathcal{T})$, and $\mathcal{L}_{vv}(\mathcal{T})$, the set of all $u$-legs, $uu$-legs, $v$-legs, and $vv$-legs of trails in $\mathcal{T}$, respectively. Again, we will drop the argument $\mathcal{T}$ when its clear from the context. Let $\mathcal{L} = \mathcal{L}_u \cup \mathcal{L}_{uu} \cup \mathcal{L}_v \cup \mathcal{L}_{vv}$ be the collection of all legs in $\mathcal{T}$.

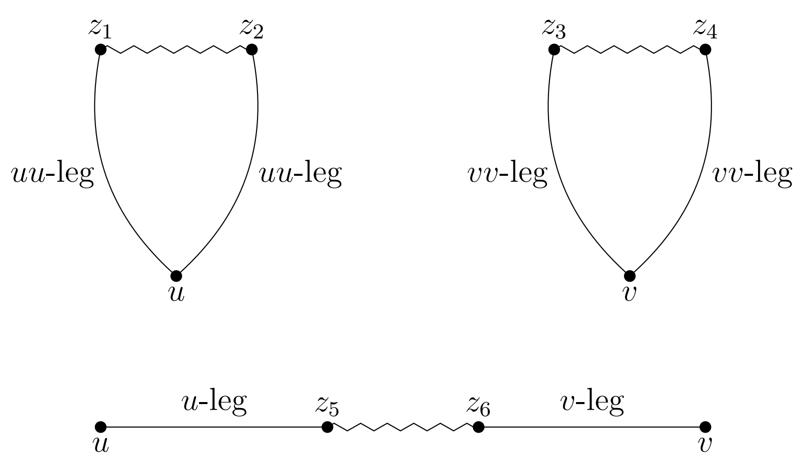



Figure 5.2: Legs of an odd $(u, u)$-trail, an odd $(v, v)$-trail, and an odd $(u, v)$-trail.

**Definition 5.2** (Contact) Let $L$ be a leg in $\mathcal{L}$ and $P = x_0x_1 \dots x_r$ be an $(u, v)$-path in $G'$. A *contact* between $P$ and $L$ is a *maximal subpath* $S$ of $P$ containing at least one edge (refer to Figure 5.3) such that $S$ is also a subtrail of $L$ i.e., for any pair of integers $0 \le i < j \le r$, $x_i \dots x_j$ is a contact between $P$ and $L$ if $x_i \dots x_j$ is a subtrail of $L$, but neither of $x_{i-1} \dots x_j$ (if $i > 0$) or $x_i \dots x_{j+1}$ (if $j < r$) is a subtrail of $L$.

For example, in Figure 5.3, the segments between $x_1$ and $y_1$ and between $x_2$ and $y_2$ are contacts between $P$ and the legs of $T$. Note that $w$ is not a contact.

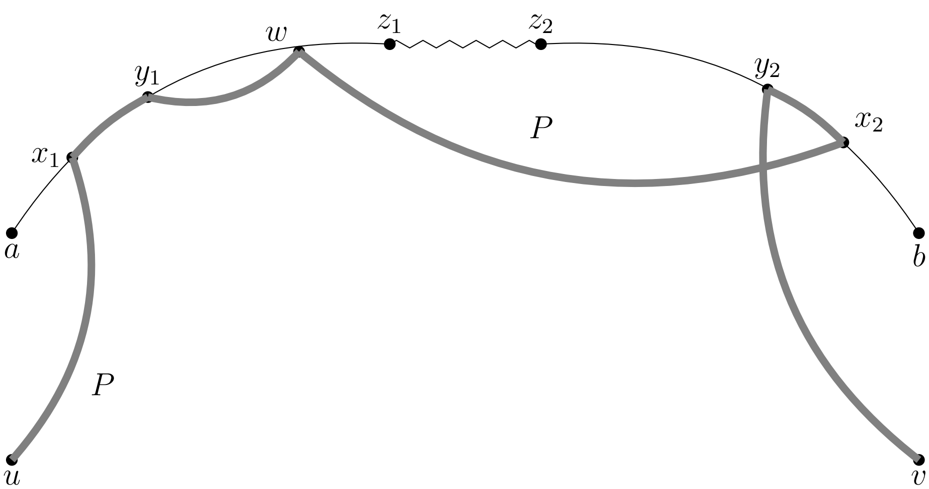



Figure 5.3: Contacts between a $(u, v)$-path $P$ in $G'$ and legs of an odd $(a, b)$-trial $T$ in $\mathcal{T}$ where $a, b \in \{u, v\}$.

For a leg $L \in \mathcal{L}$ and a $(u, v)$-path $P = x_0 x_1 \ldots x_r$ in $G'$, we define the function $\mathcal{C}$ which counts the number of contacts between $P$ and $L$ as follows.

$$\mathcal{C}(P, L) = |\{(i, j) : 0 \leq i < j \leq r \text{ such that } x_i \ldots x_j \text{ is a contact between P and L }\}|$$

By definition, contacts between $P$ and $L$ are edge-disjoint. With a slight abuse of notation, we will use $\mathcal{C}(P, \mathcal{L})$ to denote $\sum_{L \in \mathcal{L}} \mathcal{C}(P, L)$. If $\mathcal{C}(P, \mathcal{L}) = 0$, then we know that $P$ is edge-disjoint from every trail in $\mathcal{T}$, otherwise we use the term *first contact* of $P$ to refer to the first contact $P$ makes with some leg in $\mathcal{L}$ as we traverse from $u$ to $v$. Similarly, *last contact* of $P$ refers to the last contact made by $P$ with some leg in $\mathcal{L}$ as we traverse from $u$ to $v$. Whenever $\mathcal{C}(P, \mathcal{L}) = 1$, the *first contact* and the *last contact* are the same. If $\mathcal{P}$ is a collection of edge-disjoint $(u, v)$-paths in $G'$, then we use $\mathcal{C}(\mathcal{P}, \mathcal{L})$ to denote $\sum_{P \in \mathcal{P}} \mathcal{C}(P, \mathcal{L}) = \sum_{P \in \mathcal{P}} \sum_{L \in \mathcal{L}} \mathcal{C}(P, L)$. For convenience, we overload the notation $\mathcal{C}(\mathcal{P}, \mathcal{T})$ to denote $\mathcal{C}(\mathcal{P}, \mathcal{L}(\mathcal{T}))$.

## 5.2 Proof of Theorem 5.1

We give an overview of the proof before delving into the details. First, in Lemma 5.3, we show that having $2 \cdot |\mathcal{T}|$ edge-disjoint $(u, v)$-paths in $G'$ leads to at least one of the five conclusions of the lemma being satisfied. Next, we argue that, in each of these cases we can obtain another *nice* collection of trails $\mathcal{T}'$ which is better than $\mathcal{T}$ in terms of $\mathcal{C}(\mathcal{P}, \mathcal{T}')$ being lesser or $k_{uv}(\mathcal{T}')$ being closer to $|\mathcal{T}|$. This is the key component of the proof and is discussed in Lemma 5.4. Finally, to finish up the proof, we show, using a simple potential function, that after a polynomial number of invocations of Lemma 5.4 we get the desired trails. We now discuss each of the steps in detail.

**Lemma 5.3** Let $\mathcal{T}$ be a *nice* collection of trails in $G$ and let $\mathcal{L}$ denote $\mathcal{L}(\mathcal{T})$. Suppose $\mathcal{P}$ is a collection of at least $2 \cdot |\mathcal{T}|$ edge-disjoint $(u, v)$-paths in $G'$. Then, at least one of the following conclusions holds:

i. There exist at least $k_{uu} + k_{vv}$ distinct paths $P_1, \ldots, P_l \in \mathcal{P}$ such that $\mathcal{C}(P_i, \mathcal{L}) = 0$ for each $1 \leq i \leq l$.

ii. There exists $P \in \mathcal{P}$ such that $P$ makes its first contact with a $vv$-leg.

iii. There exists $P \in \mathcal{P}$ such that $P$ makes its last contact with a $uu$-leg.

iv. There exist two distinct paths $P, P' \in \mathcal{P}$ and a leg $L \in \mathcal{L}_u \cup \mathcal{L}_{uu} \cup \mathcal{L}_v$ such that both $P$ and $P'$ make their first contact with $L$.

v. There exist two distinct paths $P, P' \in \mathcal{P}$ and a leg $L \in \mathcal{L}_u \cup \mathcal{L}_v \cup \mathcal{L}_{vv}$ such that both $P$ and $P'$ make their last contact with $L$.

*Proof.* If the first conclusion holds, then we are done. So, we may assume that there are at most $k_{uu} + k_{vv} - 1$ paths in $\mathcal{P}$ which make no contact with any leg in $\mathcal{L}$. Hence, there are at least $2 \cdot |\mathcal{T}| - (k_{uu} + k_{vv} - 1) = 2k_{uv} + k_{uu} + k_{vv} + 1$ paths in $\mathcal{P}$ which make at least one contact with some leg in $\mathcal{L}$. Let $\mathcal{P}'$ be the collection of such paths. We will be concerned with the first and the last contact of paths in $\mathcal{P}'$. Suppose there exists a path $P \in \mathcal{P}'$ which makes its first contact with a $vv$-leg or makes its last contact with a $uu$-leg, then the second or the third conclusion holds. So, we may assume that every path $P \in \mathcal{P}'$ makes its first contact with a leg in $\mathcal{L}_u \cup \mathcal{L}_{uu} \cup \mathcal{L}_v$ and makes its last contact with a leg in $\mathcal{L}_u \cup \mathcal{L}_v \cup \mathcal{L}_{vv}$. By definition of legs, we know that $|\mathcal{L}_u|, |\mathcal{L}_v| \leq k_{uv}(\mathcal{T})$, $|\mathcal{L}_{uu}| \leq 2k_{uu}(\mathcal{T})$, and $|\mathcal{L}_{vv}| \leq 2k_{vv}(\mathcal{T})$. Hence, we get the following inequalities.

$$|\mathcal{L}_u \cup \mathcal{L}_{uu} \cup \mathcal{L}_v| \leq 2k_{uu} + 2k_{uv}$$
$$|\mathcal{L}_u \cup \mathcal{L}_v \cup \mathcal{L}_{vv}| \leq 2k_{vv} + 2k_{uv}$$

Now, $|\mathcal{P}'| \geq 2k_{uv} + k_{uu} + k_{vv} + 1 > \min(2k_{uu} + 2k_{uv}, 2k_{vv} + 2k_{uv})$, and hence, by pigeonhole principle there exist two distinct paths $P, P' \in \mathcal{P}' \subseteq \mathcal{P}$, satisfying at least one of the last two conclusions. $\square$

We now state and prove the key lemma of this chapter, which lets us modify $\mathcal{T}$ so that we can obtain another *nice* collection of trails $\mathcal{T}'$ which is *better* than $\mathcal{T}$.

**Lemma 5.4** Let $\mathcal{T}$ be a *nice* collection of trails in $G$ and let $\mathcal{P}$ be a collection of at least $2 \cdot |\mathcal{T}|$ edge-disjoint $(u, v)$-paths in $G'$. Then, we can find a *nice* collection of trails $\mathcal{T}'$ in $G$ satisfying one of the following conditions:

i. $k_{uv}(\mathcal{T}') = |\mathcal{T}|$.

ii. $\mathcal{C}(\mathcal{P}, \mathcal{T}') \leq \mathcal{C}(\mathcal{P}, \mathcal{T})$ and $k_{uv}(\mathcal{T}') = k_{uv}(\mathcal{T}) + 1$.

iii. $\mathcal{C}(\mathcal{P}, \mathcal{T}') < \mathcal{C}(\mathcal{P}, \mathcal{T})$ and $k_{uv}(\mathcal{T}') = k_{uv}(\mathcal{T})$.

iv. $\mathcal{C}(\mathcal{P}, \mathcal{T}') < \mathcal{C}(\mathcal{P}, \mathcal{T})$ and $k_{uv}(\mathcal{T}') = k_{uv}(\mathcal{T}) - 1$.

*Proof.* If $k_{uv}(\mathcal{T}) = |\mathcal{T}|$, then the first conclusion holds trivially by taking $\mathcal{T}' = \mathcal{T}$. So, we may assume that $k_{uv}(\mathcal{T}) < |\mathcal{T}|$ i.e, $\mathcal{T}$ has at least one odd $(u, u)$-trail or at least one odd $(v, v)$-trail. Observe that $\mathcal{T}, \mathcal{P}$ satisfy the assumptions stated in Lemma 5.3 and hence, at least one of the conclusions of the lemma holds. We handle each of the cases separately. In the accompanying figures, paths in $\mathcal{P}$ are shown using thick gray lines and dashed lines; the dashed portion of the paths is not relevant to the proof and can possibly make contacts with arbitrary legs in $\mathcal{L}$.

1. There are at least $k_{uu} + k_{vv}$ paths in $P$ which make no contact with any of the legs in $\mathcal{L}$. We can combine such paths with $k_{uu}$ odd $(u, u)$-trails and $k_{vv}$ odd $(v, v)$-trails in $\mathcal{T}$, to get $|\mathcal{T}|$ odd $(u, v)$-trails in total. Such a collection of trails satisfies the first conclusion of this lemma.

2. There exists $P \in \mathcal{P}$ that makes its first contact with a leg $L_1 \in \mathcal{L}_{vv}$ (see Figure 5.4). Let the first vertex in the first contact between $P$ and $L_1$ be $x$. Let the odd $(v, v)$-trail in $\mathcal{T}$ corresponding to $L_1$ be $T$. Let the other leg of $T$ be $L_2$ (possibly empty). We can now obtain an odd $(u, v)$-trail $T'$ as follows: Starting from $u$ traverse $P$ until we reach the vertex $x$, then traverse $L_1$ from $x$ to the tail of $L_1$, then traverse the $F'$-edge used in $T$, followed by traversing the leg $L_2$ from its tail to its head $(= v)$. Since $\mathcal{C}(P, T') \leq \mathcal{C}(P, T)$ and $\mathcal{C}(Q, T') \leq \mathcal{C}(Q, T)$ for any other path $Q \in \mathcal{P}$, we have $\mathcal{C}(\mathcal{P}, \mathcal{T}') \leq \mathcal{C}(\mathcal{P}, \mathcal{T})$ where $\mathcal{T}' = (\mathcal{T} \cup \{T'\}) \setminus \{T\}$. Hence, $\mathcal{T}'$ satisfies the second conclusion.

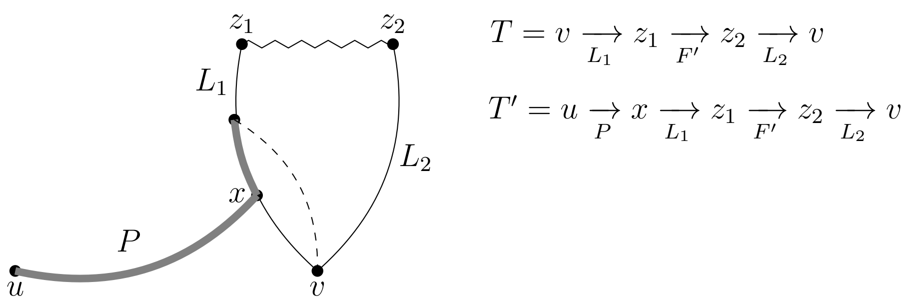



Figure 5.4: Path $P$ makes its first contact at a $vv$-leg $L_1$.

3. The case when a path $P \in \mathcal{P}$ makes its last contact with a leg $L_1 \in \mathcal{L}_{uu}$ (see Figure 5.5) is similar to Case 2 and is handled similarly.

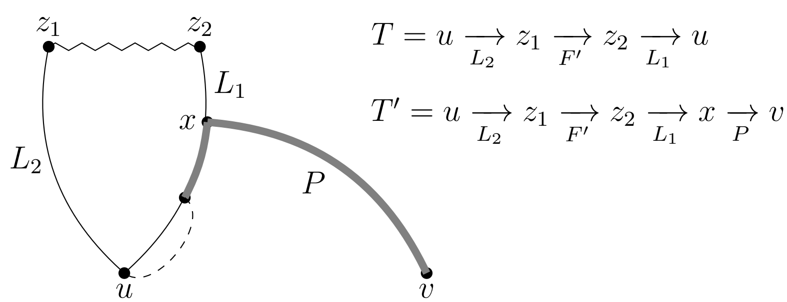
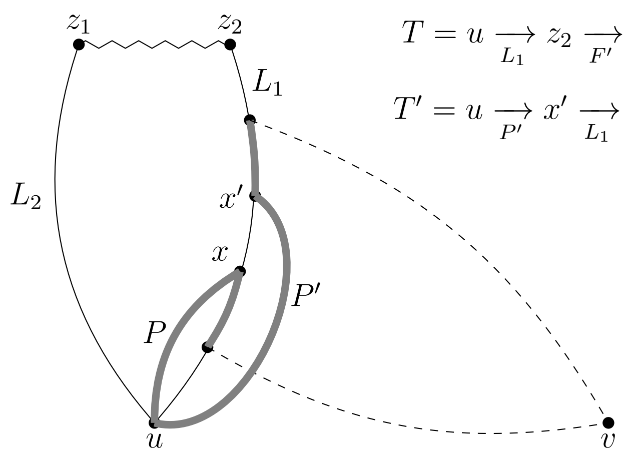



Figure 5.5: Path $P$ makes its last contact at a $uu$-leg $L_1$.

4. There exist two paths $P, P' \in \mathcal{P}$ and a leg $L_1 \in \mathcal{L}_u \cup \mathcal{L}_{uu} \cup \mathcal{L}_v$ such that both $P, P'$ make their first contact with $L_1$. We handle this case using two subcases:

   (a) Suppose that $L_1 \in \mathcal{L}_u \cup \mathcal{L}_{uu}$ (see Figure 5.6 which corresponds to $L_1 \in \mathcal{L}_{uu}$). Let the trail in $\mathcal{T}$ corresponding to $L_1$ be $T$. Let the other leg of $T$ be $L_2$ (possibly empty). Let the first vertex in the first contact between $P$ and $L_1$ be $x$ and the first vertex in the first contact between $P'$ and $L_1$ be $x'$. Without loss of generality, we may assume that $x$ is closer, along $L_1$, to the head of $L_1$ (i.e., $u$) than $x'$. We now construct an odd trail $T'$ as follows: We start from $u$ and traverse $P'$ until we reach $x'$, then traverse $L_1$ from $x'$ to the tail of $L_1$, then traverse the $F'$-edge used in $T$, followed by traversing the leg $L_2$ completely from its tail to its head. Let $\mathcal{T}' = (\mathcal{T} \cup \{T'\}) \setminus \{T\}$. Observe that $T$ and $T'$ are both either odd $(u, u)$-trails or odd $(u, v)$-trails. Hence, $k_{uv}(\mathcal{T}) = k_{uv}(\mathcal{T}')$.

Since the first contact made by $P$ with $L_1$ is no longer a contact between $P$ and the leg $L_1'$ (corresponding to $L_1$) of $T'$, we have $\mathcal{C}(P,T') < \mathcal{C}(P,T)$. For any other path $Q \in \mathcal{P} \setminus \{P\}$, we have $\mathcal{C}(Q,T') \leq \mathcal{C}(Q,T)$. Hence, $\mathcal{T}'$ satisfies the third conclusion of this lemma.

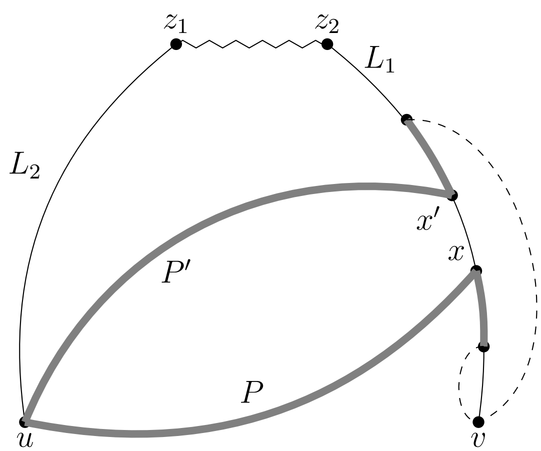



$$T = u \underset{L_1}{\longrightarrow} z_2 \underset{F'}{\longrightarrow} z_1 \underset{L_2}{\longrightarrow} u$$

$$T' = u \underset{P'}{\longrightarrow} x' \underset{L_1}{\longrightarrow} z_2 \underset{F'}{\longrightarrow} z_1 \underset{L_2}{\longrightarrow} u$$

Figure 5.6: Paths $P, P'$ make their first contact on a $uu$-leg $L_1$.

(b) Suppose that $L_1 \in \mathcal{L}_v$ (see Figure 5.7). Let the odd $(u,v)$-trail in $\mathcal{T}$ corresponding to $L_1$ be $T$. Let the other leg of $T$ be $L_2 \in \mathcal{L}_u(\mathcal{T})$ (possibly empty). Let the first vertex in the first contact between $P$ and $L_1$ be $x$ and the first vertex in the first contact between $P'$ and $L_1$ be $x'$. Without loss of generality, we may assume that $x$ is closer, along $L_1$, to the head of $L_1$ (i.e., $v$) than $x'$. We now construct an odd $(u,u)$-trail $T'$ as follows: We start from $u$ and traverse $L_2$ completely from its head to its tail, then traverse the $F'$-edge in $T$, then traverse $L_1$ from its tail towards its head until we reach the vertex $x'$, then traverse $P'$ (in the direction from $v$ to $u$) from $x'$ to $u$. Let $\mathcal{T}' = (\mathcal{T} \cup \{T'\}) \setminus \{T\}$. Since $T$ is an odd $(u,v)$-trail and $T'$ is an odd $(u,u)$-trail, we have $k_{uv}(\mathcal{T}') = k_{uv}(\mathcal{T}) - 1$. Observe that the first contact made by $P$ with $L_1$ is no longer a contact between $P$ and the leg $L_1' \in L_{uu}(\mathcal{T}') \setminus \{L_2\}$ of $T'$, hence $\mathcal{C}(P,T') < \mathcal{C}(P,T)$. For any other path $Q \in \mathcal{P} \setminus \{P\}$, we have $\mathcal{C}(Q,T') \leq \mathcal{C}(Q,T)$. Therefore, $\mathcal{T}'$ satisfies the fourth conclusion of this lemma.



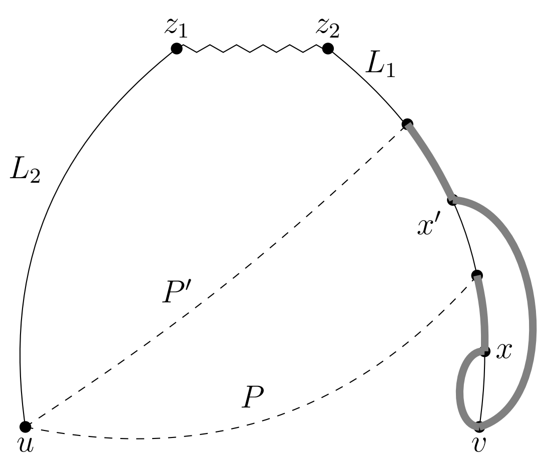

$$T = u \underset{L_2}{\longrightarrow} z_1 \underset{F'}{\longrightarrow} z_2 \underset{L_1}{\longrightarrow} v$$

$$T' = u \underset{L_2}{\longrightarrow} z_1 \underset{F'}{\longrightarrow} z_2 \underset{L_1}{\longrightarrow} x' \underset{P'}{\longrightarrow} u$$

Figure 5.7: Paths $P, P'$ make their first contact on a $v$-leg $L_1$.

5. There exist two paths $P, P' \in \mathcal{P}$ and a leg $L_1 \in \mathcal{L}_u \cup \mathcal{L}_v \cup \mathcal{L}_{vv}$ such that both $P, P'$ make their last contact with $L_1$. We handle this case using two subcases:

   (a) The case when $L_1 \in \mathcal{L}_v \cup \mathcal{L}_{vv}$ (see Figure 5.8 which corresponds to $L_1 \in \mathcal{L}_v$) is similar to subcase (a) of case 4, and is handled similarly.

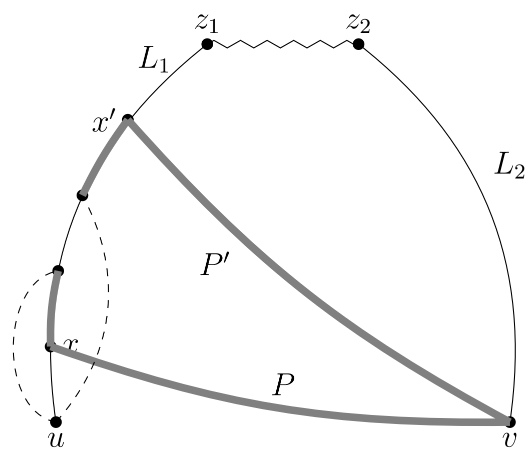



$$T = u \underset{L_2}{\longrightarrow} z_1 \underset{F'}{\longrightarrow} z_2 \underset{L_1}{\longrightarrow} v$$

$$T' = u \underset{L_2}{\longrightarrow} z_1 \underset{F'}{\longrightarrow} z_2 \underset{L_1}{\longrightarrow} x' \underset{P'}{\longrightarrow} v$$

Figure 5.8: Paths $P, P'$ make their last contact on a $v$-leg $L_1$.

   (b) The case when $L_1 \in \mathcal{L}_u$ (see Figure 5.9) is similar to subcase (b) of case 4, and is handled similarly.




$$T = u \xrightarrow[L_1]{} z_1 \xrightarrow[F']{} z_2 \xrightarrow[L_2]{} v$$

$$T' = v \xrightarrow[L_2]{} z_2 \xrightarrow[F']{} z_1 \xrightarrow[L_1]{} x' \xrightarrow[P']{} v$$

Figure 5.9: Paths $P, P'$ make their last contact on a $u$-leg $L_1$.

This concludes the proof of the lemma. □

### 5.2.1 Finishing up the Proof of Theorem 5.1

Since $\lambda(u, v; G') \geq 2 \cdot |\mathcal{T}|$, we can get a collection $\mathcal{P}$ of $2 \cdot |\mathcal{T}|$ edge-disjoint $(u, v)$-paths in $G'$. Let $\mathcal{T}^0$ denote the $\mathcal{T}$ in the statement of the theorem. We begin with $\mathcal{T}^0$ and repeatedly apply Lemma 5.4 until we obtain a collection $\mathcal{T}^*$ such that $k_{uv}(\mathcal{T}^*) = |\mathcal{T}^*| = |\mathcal{T}^0|$. We now argue that this process terminates in a polynomial number of iterations, which will conclude the proof. Consider the potential function $h$ defined for a *nice* collection of trails $\mathcal{A}$ as

$$h(\mathcal{A}) = 2 \cdot \mathcal{C}(\mathcal{P}, \mathcal{A}) - k_{uv}(\mathcal{A}) \ .$$

Consider any iteration where we invoke Lemma 5.4 and move from a *nice* collection of trails $\mathcal{T}$ to another *nice* collection of trails $\mathcal{T}'$ with $k_{uv}(\mathcal{T}') < |\mathcal{T}'|$. Then, one of the last three conclusions of the lemma must apply to $\mathcal{T}'$. It is easy to see that in each case, we have $h(\mathcal{T}') \leq h(\mathcal{T}) - 1$. Observe that $0 \leq \mathcal{C}(\mathcal{P}, \mathcal{A}) \leq |E(G')|$ for any *nice* collection of trails $\mathcal{A}$ since the paths in $\mathcal{P}$ are edge-disjoint and for a fixed $P \in \mathcal{P}$, the contacts made by $P$ with legs in $\mathcal{L}(\mathcal{A})$ are edge-disjoint. Hence, $-|\mathcal{A}| \leq h(\mathcal{A}) \leq 2 \cdot |E(G')|$. Therefore, the above process terminates in at most $2 \cdot |E(G')| + |\mathcal{T}^0|$ iterations. Thus, the running time of the algorithm is bounded above by a polynomial in $m$ and $k$.

### 5.2.2 Tightness of Assumptions

We now provide an example showing that the assumptions in Theorem 5.1 are tight for $|\mathcal{T}| \geq 2$. Consider the graph $G' + F'$ as shown in Figure 5.10. Observe that $\lambda(u, v; G') = 2k + 1$. Consider a *nice* collection of $k + 1$ trails $\mathcal{T} = \{ux_iy_iv\}_{i=1}^{k-1} \cup \{uz_1z_2u\} \cup \{vz_5z_6v\}$ in $G$. Since $\lambda(u, v; G') = 2 \cdot |\mathcal{T}| - 1$, and the conclusion of Theorem 5.1 is not satisfied, we can conclude that the assumptions in the theorem are tight.

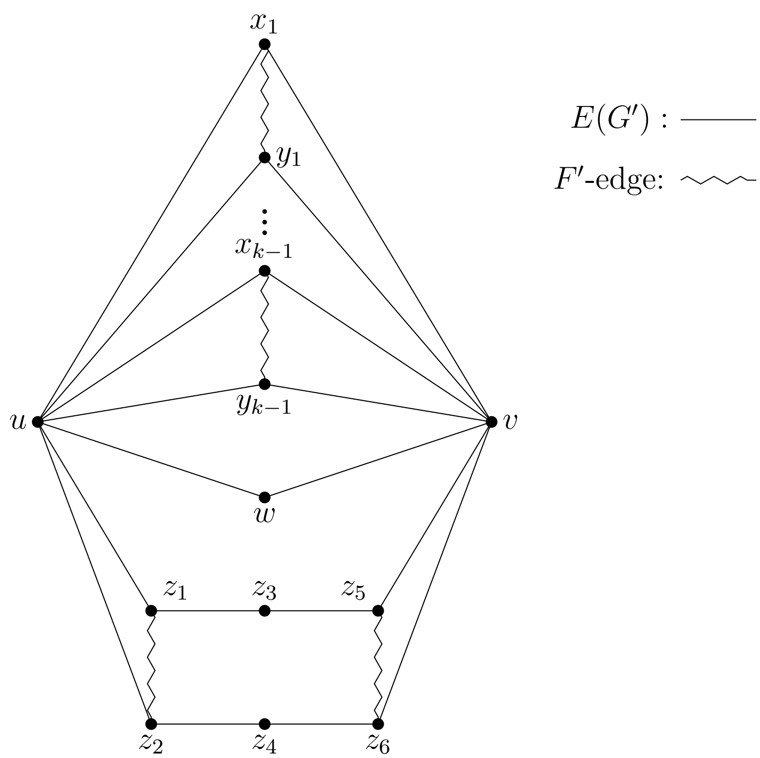



Figure 5.10: Example showing that the assumptions in Theorem 5.1 are tight for $|\mathcal{T}| \geq 2$.

# Chapter 6

# An Improved Bound: $\tau(u,v) \leq 5 \cdot \nu(u,v) + 2$

In this chapter, we combine results from Chapters 4 and 5 to prove the following theorem.

**Theorem 6.1** *Let $\tilde{G}$ be a graph with two specified vertices $u$ and $v$. Let $k$ be a positive integer. Then, $G$ has either $k$ edge-disjoint odd $(u,v)$-trails or a set of at most $5k-3$ edges intersecting all odd $(u,v)$-trails. Hence, we have $\tau(u,v) \leq 5 \cdot \nu(u,v) + 2$ for every graph $\tilde{G}$. Moreover, we can find these $k$ trails or a cover of size at most $5k-3$, in polynomial time.*

*Proof.* The proof is divided into two parts. First, we show that either $\tilde{G}$ has $k$ edge-disjoint odd $(u,v)$-trails or there is an odd $(u,v)$-trail cover of size at most $5k-3$. Then, we show how to achieve these outcomes in polynomial time. The first part is similar to the proof of Theorem 3.5. To begin with, we may assume that $\lambda(u,v;\tilde{G}) \geq 4k-1$, since otherwise a min $(u,v)$-cut is an odd $(u,v)$-trail cover of size at most $4k-2$, which is at most $5k-3$. To avoid trivial settings, we may also assume that Assumption 2.6 holds. Hence, $\tilde{G}$ is 2-edge-connected. Now we apply Reduction 2.8 from Chapter 2, to replace maximal 2-edge-cuts in $\tilde{G}$ with the corresponding gadgets. As mentioned earlier, the packing and covering numbers remain unchanged by these reductions. We reuse the symbol $\tilde{G}$ to refer to the resulting graph. Now, consider a maximum bipartite subgraph $G_1$ of $\tilde{G}$. Since at least half the edges in every cut of $\tilde{G}$ are bipartite edges, we have $\lambda(u,v;G_1) \geq \lceil \lambda(u,v;\tilde{G})/2 \rceil \geq \lceil (4k-1)/2 \rceil = 2k$.

If $u$ and $v$ are on the opposite sides of the bipartition in $G_1$, then we can obtain at least $2k$ edge-disjoint odd $(u,v)$-paths in $G_1$ (which is more than what we need). Therefore, we may assume that $u$ and $v$ are on the same side of the bipartition in $G_1$. Let $G'$

be the bipartite subgraph of $\tilde{G}$ obtained by using Lemma 2.9 on $G_1$. Thus, $G'$ has at least one edge, and in some cases both edges, of a maximal 2-edge-cut of $\tilde{G}$. Let $G$ denote the graph obtained from $\tilde{G}$ by applying Reduction 3.2 with respect to $G'$. Thus, $G, G', F := E(G) \setminus E(G')$ satisfy the hypothesis of Theorem 4.1. Hence, for the given $k$, one of the two conclusions of the theorem holds.

If the second conclusion holds, then we get a vertex-set $S$ containing $u$ and $v$ that satisfies $|\delta(S)| + |E(S) \cap F| \leq 5k - 5$. We are done, in this case, since $\delta(S) \cup (E(S) \cap F)$ is an odd $(u, v)$-trail cover. Note that adding back the loops deleted during Reduction 3.2 does not create an odd $(u, v)$-trail which is not covered by $\delta(S) \cup (E(S) \cap F)$, as shown in Lemma 2.10. Otherwise, we get $k$ edge-disjoint odd $(s, s)$-trails each of which uses exactly one edge from $F$, call the collection $\mathcal{T}$. In fact, $\mathcal{T}$ is a *nice* (defined in Chapter 5) collection of trails in $G$. Since $\lambda(u, v; G') \geq 2k = 2 \cdot |\mathcal{T}|$, we can use Theorem 5.1 to modify the trails in $\mathcal{T}$ to obtain the desired $k$ edge-disjoint odd $(u, v)$-trails. This concludes the first part of the proof.

Next, to find the desired $k$ trails or a small cover in polynomial time, we have to overcome two difficulties: a) Obtaining a maximum bipartite subgraph which is used in the proof of Theorem 4.1 and in the first part of this proof; and b) Finding a maximum edge-set $F'$ satisfying $\lambda(s, t; \hat{G}(F')) = 2 \cdot |F'|$, as required in the proof of Theorem 4.1. We deal with the first difficulty by using the same strategy as used in Section 3.2 while designing a polytime algorithm for Theorem 3.5. We start with an edge-maximal bipartite subgraph $G'$, and if any of the underlying arguments fail because of a *bad* cut $\delta(X)$ satisfying $|\delta(X) \setminus F| < |\delta(X) \cap F|$, then we improve $G'$ by using Lemma 3.19 and restart with the new larger bipartite subgraph.

We circumvent the difficulty in finding a maximum-size subset $F' \subseteq F$ satisfying $\lambda(s, t; \hat{G}(F')) = 2 \cdot |F'|$, by working with an inclusion-wise maximal subset. This can be done by simply adding edges to $F'$ (initially empty) as long as we can send $2 \cdot |F'|$ units of flow from $s$ to $t$ in the graph $\hat{G}(F')$. If we proceed through the proof of Theorem 4.1 with a maximal subset $F'$, then the proof shows that we will either get a vertex-set $A \subseteq A_0$ satisfying $\hat{d}(A) \leq 3k - 3$ or a deficient saturated set. In the latter case, we can move to a larger set $F'$ and restart the algorithm with this new $F'$. Since every time we restart (either because of $G'$ or $F'$), we increase the size of the corresponding structure, the entire process takes polynomial time. $\square$

Let $\mathcal{A}$ denote the polynomial time algorithm for Theorem 6.1 which takes as input a graph $\tilde{G} = (\tilde{V}, \tilde{E})$, two vertices $u$ and $v$, and a positive integer $k$ and either returns $k$ edge-disjoint odd $(u, v)$-trails or an odd $(u, v)$-trail cover of size at most $5k - 3$. We can use $\mathcal{A}$ to get approximate packing and covering as shown in the following corollaries.

**Corollary 6.2** Given a graph $\tilde{G} = (\tilde{V}, \tilde{E})$ with two specified vertices $u$ and $v$, we can find an odd $(u, v)$-trail cover $C \subseteq \tilde{E}$ of $\tilde{G}$ satisfying $|C| \leq 5 \cdot \tau(u, v) + 2$, in polynomial time.

*Proof.* Recall from Chapter 2 that we can efficiently detect whether $\tilde{G}$ has an odd $(u, v)$-trail. If $\nu(u, v) = 0$, then the empty edge-set is a cover of the required size. Otherwise, we iterate over positive integers starting from 1 until we reach an integer $k$ for which $\mathcal{A}$ gives $k$ edge-disjoint odd $(u, v)$-trails, and for $k + 1$, the algorithm gives an odd $(u, v)$-trail cover $C$ of size at most $5k + 2$. It follows that

$$k \leq \nu(u, v) \leq \tau(u, v) \leq |C| \leq 5k + 2 \leq 5 \cdot \tau(u, v) + 2 \ .$$

□

**Corollary 6.3** Given a graph $\tilde{G} = (\tilde{V}, \tilde{E})$ with two specified vertices $u$ and $v$, we can find $k$ edge-disjoint odd $(u, v)$-trails in $\tilde{G}$ such that $\nu(u, v) \leq 4k + 2$, in polynomial time.

*Proof.* The case when there are no odd $(u, v)$-trails in $\tilde{G}$ is easily handled so we may assume that $\nu(u, v) > 0$. As with the previous corollary, we iterate over positive integers starting from 1 until we reach an integer $k$ for which $\mathcal{A}$ gives $k$ edge-disjoint odd $(u, v)$-trails, and for $k + 1$, the algorithm gives an odd $(u, v)$-trail cover $C$ of size at most $5k + 2$. If the cover provided by $\mathcal{A}$ is a min $(u, v)$-cut in $\tilde{G}$, then in fact we know that $|C| \leq 4k + 2$ (see the proof of Theorem 6.1). Thus, $\nu(u, v) \leq 4k + 2$. On the other hand, if the cover provided is of the form $\delta(X) \cup (F \cap E(X))$, then we have $|C| \leq 5k$. A better bound on $\nu(u, v)$ is $\lfloor |\delta(X)|/2 \rfloor + |F \cap E(X)|$. In this case, we can show that the packing number is bounded above by $3k$ as follows,

$$\begin{aligned}
\left\lfloor \frac{|\delta(X)|}{2} \right\rfloor + |F \cap E(X)| &\leq \frac{|\delta(X)| + 2 \cdot |F \cap E(X)|}{2} \\
&\leq \frac{(|\delta(X)| + |F \cap E(X)|) + (|F \cap E(X)|)}{2} \\
&\leq \frac{5k + k}{2} = 3k \text{ since } |F \cap E(X)| = |F' \cap E(X)| < k + 1 \ .
\end{aligned}$$

The result follows since we show in both cases that $k \leq \nu(u, v) \leq 4k + 2$. □

# Chapter 7

# Conclusions

In this thesis, we showed an approximate duality between packing and covering odd $(u,v)$-trails by proving that for any graph $G$ we have $\tau(u,v) \leq 5 \cdot \nu(u,v) + 2$. As mentioned earlier, Churchley et al. [2], have also obtained this improved bound of 5. We say approximate duality since the covering number can be twice as large as the packing number. Such a family of graphs (from [3]) is shown in Figure 7.1. Before concluding this chapter, we construct a family of graphs $\{H_k\}_{k\in\mathbb{N}}$ where for each $H_k$, the packing number is $k$, the covering number is $2k$, and additionally, any cover of the type $\delta(X) \cup (E(X) \cap F)$ has cardinality at least $3k$. Thereby showing that covers of the type $\delta(X) \cup (E(X) \cap F)$ (where $u, v \in X$) are not necessarily optimal covers.

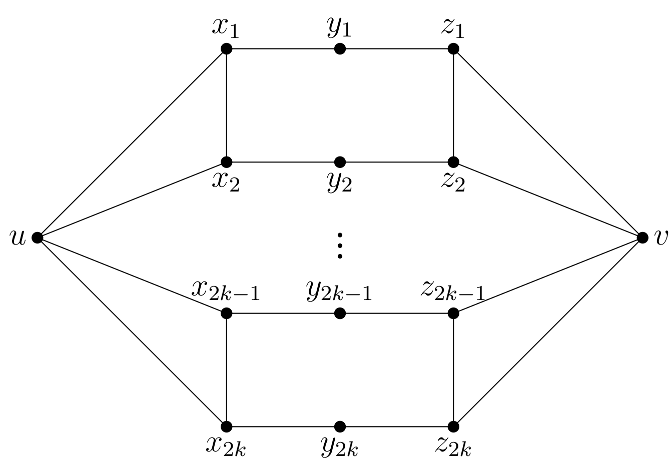



Figure 7.1: Graph with $\nu(u,v) = k$ and $\tau(u,v) = 2k$.

Let us call the graph $H_k$ (for some fixed $k \geq 1$) in Figure 7.2 as $G$. Observe that every odd trail $T$ in $G$ uses at least one of the odd cycles in $\{x_i a_i b_i x_i\}_{i=1}^k \cup \{y_i c_i d_i y_i\}_{i=1}^k \cup \{z_i e_i f_i z_i\}_{i=1}^k$. For each $1 \leq i \leq k$, the two parallel edges between $u$ and $x_i$ along with the edge $vz_i$ form a (nonseparating) cut. Hence, we cannot find two edge-disjoint trails $T_1, T_2$ such that both use at least one odd cycle among $\{x_i a_i b_i x_i, y_i c_i d_i y_i, z_i e_i f_i z_i\}$. It follows that the packing number is at most $k$. It is trivial to find $k$ edge-disjoint odd $(u,v)$-trails in $G$ and hence $\nu(u,v) = k$. Next, the covering number is at most $2k$ since the edge-set consisting of all the edges (including parallel copies) between $u$ and $x_i$ for all $1 \leq i \leq k$ is an odd $(u,v)$-trail cover. Let $C_i$ denote the vertex-set $\{u, v, x_i, y_i, z_i, a_i, b_i, c_i, d_i, e_i, f_i\}$. Deleting at most one edge from the induced subgraph $G[C_i]$ does not destroy all the odd $(u,v)$-trails using at least one odd cycle from $G[C_i]$ hence we need at least $2k$ edges in any cover. Thus, $\tau(u,v) = 2k$.

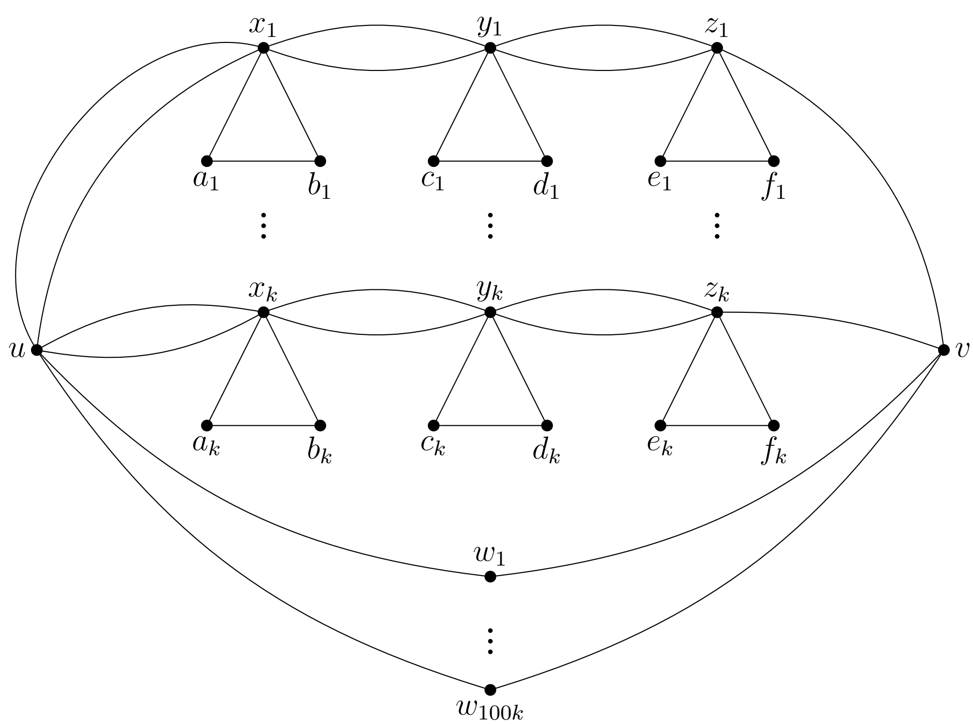



Figure 7.2: Graph with $\nu(u,v) = k$, $\tau(u,v) = 2k$, and $|\delta(X)| \cup |E(X) \cap F| \geq 3k$ for any vertex-set $X$ containing $u$ and $v$.

Since $\lambda(u, v; G) \geq 100k$, a minimum $(u,v)$-cut is not a good candidate for an odd $(u,v)$-trail cover. Deleting one edge from every odd cycle in $G$ gives us a cover of size $3k$

since there are $3k$ edge-disjoint odd cycles in $G$. But observe that we can easily construct examples with more number of odd cycles for the same $k$. Let us now analyze how our algorithm performs on this example. Since this example is a hard instance of the problem, the covers given by our algorithm will be of the form $\delta(X) \cup (F \cap E(X))$ for some $X$ containing $u$ and $v$. We now analyze the steps in Theorems 3.4 and 4.1 by redrawing $G$ (where $s$ denotes $\{u, v\}$) as shown in Figure 7.3. The zigzagged edges are $F'$-edges and the thick gray edges are $(F \setminus F')$-edges. Observe that $S_0 = V(G_s) \setminus \{a_1, \dots, a_k, b_1, \dots, b_k\}$. For each $1 \leq i \leq k$, the smallest $e$-inhibiting set corresponding to $(F \setminus F')$-edges $c_i d_i$ and $e_i f_i$ is $S_0 \setminus A_i$ where $A_i = \{x_i, y_i, z_i, c_i, d_i, e_i, f_i\}$. Hence $A_0 = \{s, w_1, \dots, w_{100k}\}$. It is easy to check that $|\delta(A_0) \setminus F| = 3k$. Since there are no $F$-edges within $A_0$ and on the boundary of $A_0$, $|\delta(A_0)| + |E(A_0) \cap F| = 3k$. In fact, such a result holds for every vertex-set $X$ containing $s$.

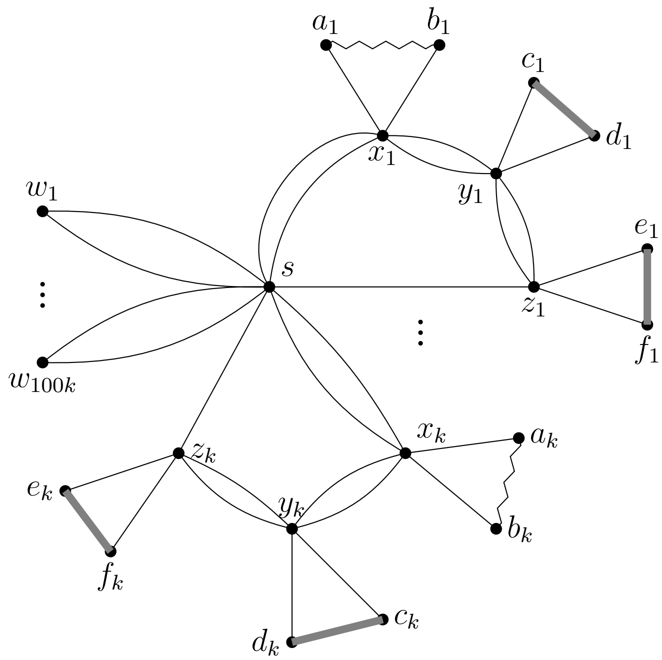



Figure 7.3: Graph from Figure 7.2 with vertices $u$ and $v$ identified into a vertex $s$.

Hence, for the general case, we cannot achieve an upper bound better than $3 \cdot \nu(u, v)$ on the size of covers of type $\delta(X) \cup (E(X) \cap F)$. It would be interesting to know if we can push the lower bound of $3 \cdot \nu(u, v)$ by using other kinds of cover.

Lastly, while we have produced approximate packing of odd $(u, v)$-trails, we do not know whether we can find $\nu(u, v)$ edge-disjoint odd $(u, v)$-trails in polynomial time. Similarly, while we know how to obtain approximate covers, we do not know whether we can find an optimal cover in polynomial time. Also, without any considerations of designing polytime algorithms, we do not have any arguments to show a better bound on the covering-vs.-packing ratio.

# References


[1] Maria Chudnovsky, Jim Geelen, Bert Gerards, Luis A. Goddyn, Michael Lohman, and Paul D. Seymour. Packing non-zero a-paths in group-labelled graphs. *Combinatorica*, 26(5):521–532, 2006.

[2] Ross Churchley. Personal communication. 2016.

[3] Ross Churchley, Bojan Mohar, and Hehui Wu. Weak duality for packing edge-disjoint odd ($u$, $v$)-trails. In *Proceedings of the Twenty-Seventh Annual ACM-SIAM Symposium on Discrete Algorithms, SODA 2016, Arlington, VA, USA, January 10-12, 2016*, pages 2086–2094, 2016.

[4] William J. Cook, William H. Cunningham, William R. Pulleyblank, and Alexander Schrijver. *Combinatorial Optimization*. John Wiley & Sons, Inc., New York, NY, USA, 1998.

[5] Richard M. Karp. Reducibility among combinatorial problems. In *Proceedings of a symposium on the Complexity of Computer Computations, held March 20-22, 1972, at the IBM Thomas J. Watson Research Center, Yorktown Heights, New York.*, pages 85–103, 1972.

[6] C. L. Lucchesi and D. H. Younger. A minimax theorem for directed graphs. *J. London Math. Soc*, 17:369–374, 1978.

[7] W. Mader. Über die maximalzahl kreuzungsfreierh-wege. *Archiv der Mathematik*, 31(1):387–402, 1978.

[8] Karl Menger. Zur allgemeinen kurventheorie. *Fund. Math.*, 10:96–115, 1927.

[9] A. Schrijver. *Combinatorial Optimization: Polyhedra and Efficiency*. Algorithms and Combinatorics. Springer Berlin Heidelberg, 2002.

[10] Alexander Schrijver and Paul D. Seymour. Packing odd paths. *J. Comb. Theory, Ser. B*, 62(2):280–288, 1994.

[11] Thomas Zaslavsky. Glossary of signed and gain graphs and allied areas. *Electron. J. Combin.*, 5:Dynamic Surveys 9, 41 pp. (electronic), 1998.

[12] Thomas Zaslavsky. A mathematical bibliography of signed and gain graphs and allied areas. *Electron. J. Combin.*, 5, 1998. Manuscript prepared with Marge Pratt.

# Appendix

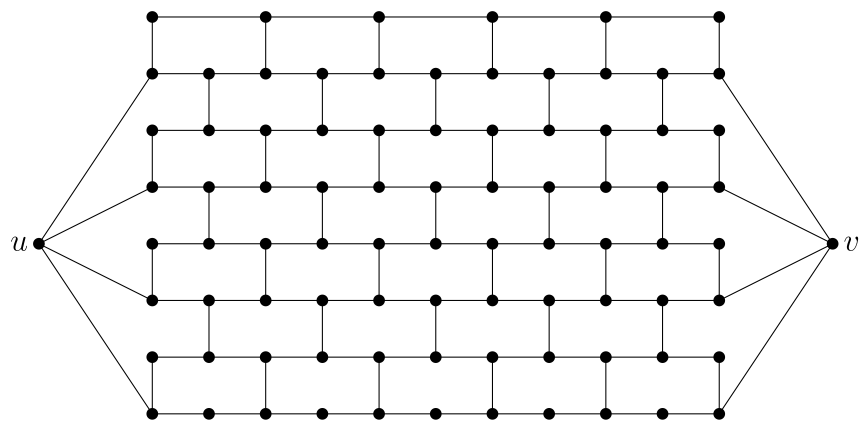



Figure A.1: A graph where no two odd $(u, v)$-paths are edge-disjoint.