On Some Stochastic Optimal Control Problems in Actuarial Mathematics

by

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I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

The event of ruin (bankruptcy) has long been a core concept of risk management interest in the literature of actuarial science. There are two major research lines. The first one focuses on distributional studies of some crucial ruin-related variables such as the deficit at ruin or the time to ruin. The second one focuses on dynamically controlling the probability that ruin occurs by imposing controls such as investment, reinsurance, or dividend payouts. The content of the thesis will be in line with the second research direction, but under a relaxed definition of ruin, for the reason that ruin is often too harsh a criteria to be implemented in practice.

Relaxation of the concept of ruin through the consideration of "exotic ruin" features, including for instance, ruin under discrete observations, Parisian ruin setup, two-sided exit framework, and drawdown setup, received considerable attention in recent years. While there has been a rich literature on the distributional studies of those new features in insurance surplus processes, comparably less contributions have been made to dynamically controlling the corresponding risk. The thesis proposes to analytically study stochastic control problems related to some "exotic ruin" features in the broad area of insurance and finance.

In particular, in Chapter 3, we study an optimal investment problem by minimizing the probability that a significant drawdown occurs. In Chapter 4, we take this analysis one step further by proposing a general drawdown-based penalty structure, which include for example, the probability of drawdown considered in Chapter 3 as a special case. Subsequently, we apply it in an optimal investment problem of maximizing a fund manager's expected cumulative income. Moreover, in Chapter 5 we study an optimal investment-reinsurance problem in a two-sided exit framework. All problems mentioned above are considered in a random time horizon. Although

the random time horizon is mainly determined by the nature of the problem, we point out that under suitable assumptions, a random time horizon is analytically more tractable in comparison to its finite deterministic counterpart.

For each problem considered in Chapters 3–5, we will adopt the dynamic programming principle (DPP) to derive a partial differential equation (PDE), commonly referred to as a Hamilton-Jacobi-Bellman (HJB) equation in the literature, and subsequently show that the value function of each problem is equivalent to a strong solution to the associated HJB equation via a verification argument. The remaining problem is then to solve the HJB equations explicitly. We will develop a new decomposition method in Chapter 3, which decomposes a nonlinear second-order ordinary differential equation (ODE) into two solvable nonlinear first-order ODEs. In Chapters 4 and 5, we use the Legendre transform to build respectively one-to-one correspondence between the original problem and its dual problem, with the latter being a linear free boundary problem that can be solved in explicit forms. It is worth mentioning that additional difficulties arise in the drawdown related problems of Chapters 3 and 4 for the reason that the underlying problems involve the maximum process as an additional dimension. We overcome this difficulty by utilizing a dimension reduction technique.

Chapter 6 will be devoted to the study of an optimal investment-reinsurance problem of maximizing the expected mean-variance utility function, which is a typical time-inconsistent problem in the sense that DPP fails. The problem is then formulated as a non-cooperative game, and a subgame perfect Nash equilibrium is subsequently solved. The thesis is finally ended with some concluding remarks and some future research directions in Chapter 7.

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Dedication

To my parents and grandparents.

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Chapter 1

Introduction

1.1 Background

A financial institution imposes controls stochastically on its wealth (assets) to achieve certain objectives. For instance, an integrated reinsurance and investment strategy is commonly employed by an insurance company for the purpose of increasing its underwriting capacity, stabilizing the underwriting results, protecting itself against catastrophic losses, and achieving financial growth. Failing to monitor the wealth process properly may lead to undesirable events such as bankruptcy, even for large institutions that are normally labelled as "too big to fall". A typical example is the collapse of the American Insurance Group and Lehman Brothers in the 2008 financial crisis, which were once respectively the largest insurance company and the fourth largest investment bank in the U.S..

The actuarial community has long been focusing on modelling the event of ruin (bankruptcy) and characterizing some crucial ruin-related variables such as the probability of ruin and the time to ruin. The reader is referred to Asmussen and Albrecher [9] or Landriault et al. [61] and references therein for a comprehensive review on ruin theory. In addition to knowing the distributional properties of ruin-related variables, it is equally important for the actuarial community to gain knowledge on how to reduce the risk of ruin by implementing controls. This motivates another research line that focuses on dynamically controlling ruin-related variables such as the ruin probability. For instance, Young [88] studied an optimal investing problem of minimizing the probability of ruin over a lifetime period under the Black-Scholes framework from the point view of individuals, which is commonly referred to as a *lifetime ruin problem*. Her work is followed by several variants including but not limited to adding borrowing constraints (e.g., Bayraktar and Young [25]), assuming various types of consumption (e.g., Bayraktar and Young [26]), under stochastic volatility (e.g., Bayraktar et al. [21]), under ambiguity aversion (see, Bayraktar and Zhang [28]), and allowing changes of model parameters subject to a shock (see, Moore and Young [75]). Another example is given by Schmidli [80] and Promislow and Young [78], which are two relatively early contributions on studying the problem of minimizing ruin probability by controlling investment and/or reinsurance strategies from the view point of insurance companies. Their works were later extended by Bai and Guo [14] to multiple assets with short-selling constraints, Luo et al. [69] with borrowing constraints, and Bai et al. [13] to a bivariate reserve process. The content of this thesis is mostly in line with the work on stochastically controlling the probability of ruin, but with the concept of ruin relaxed (as is introduced in the next paragraph). Also, note that there are other types of actuarial related stochastic control problems which have been studied over the years, such as the expected utility maximization of insurers (e.g., Zou and Cadenillas [96] [97]), the cumulative dividend payout maximization (e.g., Asmussen and Taksar [10], Marciniak and Palmowski [71]), and various problems in life insurance (e.g., Iwaki and Osaki [54], Kronborg and Steffensen [57], Mousa et al. [76]). The reader is referred to the aforementioned references for a more detailed review of this body of literature.

Recently, the concept of ruin was relaxed through the consideration of some "exotic ruin" features, including for instance, ruin under discrete observations (e.g., Albrecher et al. [2]), Parisian ruin setup (e.g., Loeffen et al. [68]), two-sided exit framework (e.g., Li et al. [63]), and drawdown setup (e.g., Landriault et al. [59]). For example, drawdown, which measures the current level of a process to its historical running maximum, is of particular interest for risk management purposes. In practice, a financial institution is suggested to monitor its wealth process and take actions correspondingly based on drawdown-related events rather than the event of ruin, for the obvious reason that ruin is too harsh a criteria that usually leads to the termination of the business. Due to its importance, there has been a rich literature on distributional studies of drawdown-related quantities and the reader is referred to Landriault et al. [60] and references therein for the most recent developments and a complete literature review. However, on the other hand, not much attention has been paid to controlling drawdown-related quantities dynamically. We are thus motivated to model and subsequently study stochastic control problems that involve exotic ruin features such as drawdown in Chapter 3 and 4. In addition, another stochastic control problem under a two-sided exit framework will be studied in Chapter 5 for similar reasons. This serves as the first main objective of the thesis.

In general, a stochastic control problem can be classified into three categories based on the time horizon it fits in, i.e., *infinite time horizon*, *finite deterministic time horizon*, and *finite random time horizon*. The time horizon is usually determined by the problem's nature. For instance, the classical consumption and investment problems (e.g., Merton [73]) are set in a finite definite time horizon; the lifetime ruin problem (e.g., Young [88]) mentioned above is set in a finite random horizon for the reason that any individual investor is subject to a nonnegligible mortality rate; for studies on asymptotic investment strategies of funds (e.g., Guasoni and Obloj [52]), an infinite time horizon is usually assumed to reflect the long-term feature. For stochastic control problems sitting in an infinite time horizon, a fairly general result is given by Bäuerle and Bayraktar [20], which showed that for a controlled diffusion process, an optimal solution to problems of minimizing hitting probabilities such as the probability of ruin in an infinite time horizon, is the one that maximizes the ratio of drift to volatility squared. For problems in the other two categories, additional difficulties are introduced due to the loss of some nice properties such as time homogeneity, and the analysis for each problem is usually carried out separately. The thesis will embrace the challenge and focus on solving stochastic control problems with a finite random time horizon (Chapters 3–5) and a finite deterministic time horizon (Chapter 6).

A primary goal of solving a stochastic optimal control problem is to characterize the *value function*, i.e., the optimal value of the objective function, and an optimal *control* that leads to it. A classical and powerful way to attain this goal is by utilizing the *dynamic programming principle* (DPP). The method typically leads to an associated *partial differential equation* (PDE) or *ordinary differential equation* (ODE) satisfied by the value function, which is commonly referred to as the *Hamilton-Jacobi-Bellman* (HJB) equation or *dynamic programming equation* in the literature. However, the derivation of the HJB equation is heuristic in the sense that it relies on several assumptions of the unknown value function such as twice continuously differentiability, which in most cases are difficult to verify in advance. To be mathematically rigorous, the procedure of showing equivalence between the value function and a solution to the associated HJB equation is commonly reverted by stating and subsequently proving a verification theorem. In other words, instead of showing that the value function satisfies the HJB equation, one shows that a strong (sufficiently smooth) solution (if exists) to the HJB equation is equivalent to the value function. Though the verification procedure is straightforward, it requires an explicit characterization of a strong solution to the HJB equation.

The verification procedure is adopted repeatedly in all problems considered in this thesis. The drawback mentioned above should not be a source of concern for the reason that an explicit strong solution to the HJB equation associated with each problem can be obtained (as we will see in later chapters). The reader should be aware that the focus of this thesis is then on solving the HJB equations, which generally speaking can be mathematically challenging. The challenge arises from the nature of the HJB equation, i.e., a second order nonlinear parabolic PDE, and such equations in general should be solved on a case-by-case basis. We would like to take the challenge and contribute to expanding the class of solvable HJB equations by utilizing existing approaches and/or developing new approaches. This serves as the second main objective of the thesis.

Although beyond the scope of the thesis, it is important to note that the lack of smoothness of the value function is quite common, which for instance can be caused by restricting the value of the controls in a closed interval. In such cases, the theory of *viscosity solution*, which is a weak formulation of solutions, steps in naturally. One then shows directly (by circumventing Ito's formula) that the value function is the solution (in viscosity sense) to the HJB equation. The reader is referred to Crandall et al. [44] and references therein for an excellent survey on viscosity solutions.

Moreover, it is also crucial to note that the failure of DPP, which is commonly referred to as *time inconsistency* in the literature, may occur under some circumstances. The failure can be caused by various reasons such as the form of the objective function. A typical example is given when the objective is to maximize a mean variance utility function. In the literature, there are mainly two approaches to tackle the time inconsistency. The first approach solves for an optimal strategy by optimizing the objective function based on today's information only, which is known as *pre-commitment* (e.g., Bäuerle [19], Bai and Zhang [16]). The strategy will be applied even if it no longer optimizes the objective function at a later time. Even though the approach is economically meaningful, the issue of time inconsistency is not really addressed. The second approach formulates the problem as a non-cooperative game, and a subgame perfect Nash equilibrium is subsequently solved. In other words, at every time point, there exists a player who solves for an *equilibrium strategy* by treating the decision-making as a game against all future players. The equilibrium strategies are thus time-consistent. The approach can be traced back to Strotz [84], and has recently been further developed by Björk and Murgoci [29] for a general class of objective functions in a Markovian framework. There are other works along the research direction dealing with problems under specific model setups; see, e.g., Ekeland and Lazrak [48], Basak and Chabakauri [18], and Czichowsky [47]. The second approach will be adopted in Chapter 6 to solve a stochastic control problem under the objective of maximizing a mean variance utility function.

1.2 Structure of the Thesis

The thesis is a collection of four research projects and is organized as follows. Chapter 2 is devoted to introducing the mathematical preliminaries. Chapters 3–6 are respectively devoted to solving a specific stochastic optimal control problems of interest in the broad area of insurance and/or finance. Finally, in Chapter 7, we end the thesis with concluding remarks and a brief discussion on future research directions. The motivation, the methodology, and the main results of each problem as well as the connections among them are specified as follows.

In Chapter 3, we study an optimal investing problem of minimizing the probability that a significant drawdown occurs over a lifetime investment. In the fund management industry, drawdown is interpreted as a measurement of the decline of portfolio value from its historic high-water mark (running maximum) (see, e.g., Figure 2.1). It is a frequently quoted risk metric to evaluate the performance of portfolio managers via performance measures such as the Calmar ratio and the Sterling ratio; see, e.g., Schuhmacher and Eling [81] for a list of existing drawdown-based performance measures. Drawdown focuses primarily on extreme downward risks (as opposed to other standard risk measures such as volatility and Beta), making it particularly relevant for risk management purposes. Also, drawdown can easily be measured and interpreted by both portfolio managers and clients. A significant drawdown not only leads to large portfolio losses but may also trigger a long-term recession. Bailey and Prado [17] recently provided some justification to the so-called "triple penance rule", where the recovery period was shown to be on average three times as long as the time to produce a drawdown. Also, drawdown is considered a key determinant of sustainable investments as investors tend to overestimate their tolerance to risk. For instance, a sharp drop in portfolio's value is often accompanied by investors exercising their fund redemption options. Moreover, investors tend to assess their investment success by comparing their current portfolio value to the historical maximum value. This resulted in much hardship during the global financial crisis of 2008 when substantial drops in portfolio value were experienced across the board. Therefore, portfolio managers have strong incentives to adopt strategies with low drawdown risks (and more stable growth rate).

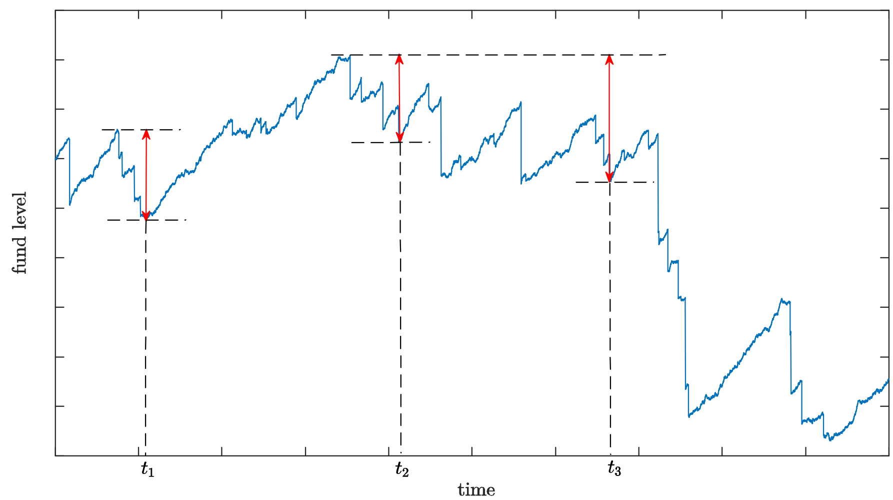


Figure 2.1. Examples of drawdown at different time points

Optimal investing problems related to drawdown risks have long focused on maximizing the long-term (asymptotic) growth rate of a portfolio subject to a strict drawdown constraint. Grossman and Zhou [50] pioneered the research topic by considering a market model with a risky asset and a risk-free asset in the Black-Scholes framework. This problem has been extended to a multi-asset framework and a general semimartingale framework by Cvitanic and Karatzas [46] and Cherny and Obloj [42], respectively. Klass and Nowicki [56] later showed that the strategy proposed by Grossman and Zhou [50] is not always optimal in a discrete-time setting. Moreover, the objective to maximize the long-term growth rate has been criticized because any strategy which coincides with the optimal strategy of Grossman and Zhou [50] after any fixed time is optimal. Roche [79] studied the infinite-horizon optimal consumption-investment problem for a power utility subject to the same drawdown constraint. Elie and Touzi [49] later extended Roche [79] to a general class of utility functions. Portfolio optimization problems with drawdown constraints are also considered in discrete-time settings (see, e.g., Chekhlov et al. [41] and Alexander and Baptista [3]).

The work appearing in Chapter 3 proposes to minimize the lifetime drawdown probability rather than impose a strict drawdown constraint, as is commonly done in the literature. This is because a strict drawdown constraint may not be attainable in some contexts (such as in the models developed in Chapter 3). In particular, under the Black-Scholes framework, we examine two financial market models: a market with two risky assets, and a market with a risk-free asset and a risky asset. Closed-form optimal trading strategies are derived under both models by utilizing a decomposition technique on the associated HJB equation. We show that it is optimal to minimize the portfolio variance when the fund value is at its historic high-water mark. Moreover, when the fund value drops, the proportion of wealth invested in the asset with a higher instantaneous rate of return should be increased. We also find that the instantaneous return rate of the minimum lifetime drawdown probability (MLDP) portfolio is never less than the return rate of the minimum variance (MV) portfolio. This supports the practical use of drawdown-based performance measures in which the role of volatility is replaced by drawdown.

In Chapter 4, we take the analysis one step further by proposing a general drawdown-based penalty structure. Under the proposed structure, investors' different degrees of aversion toward drawdown risks are captured by the embedded drawdown-based penalty functions. The general structure includes the drawdown probability considered in Chapter 3 and the expected time spent in drawdown considered in Angoshtari et al. [5] as special cases. Subsequently, we study an optimal investing problem of maximizing a fund manager's expected cumulative income over a lifetime investment period given that his/her income is determined by a particular penalty structure. To be more specific, a penalty (deduction in income rate) will be incurred once a predetermined drawdown level is reached, and the severity of the deduction is characterized by the related penalty function.

Under a market model consisting of a risk-free asset and a risky asset, we study in particular a constant penalty structure and a linear penalty structure. The two structures are of practical interest as they are consistent in spirit with the Calmar ratio (built on maximum drawdown) and the Sterling ratio (built on average drawdown), respectively, which are two frequently quoted performance measures in the financial industry. In both cases, closed-form expressions for the maximized cumulative income (MCI) and the MCI trading strategy are obtained by applying a dual approach. We find that in the nonpenalty region, the MCI strategy is equivalent to the MLDP strategy developed in Chapter 3, and as the fund level drops, the manager intends to invest more aggressively by increasing the proportion of wealth invested in the risky asset. However, in the penalty region, the manager can either be more or less aggressive depending on the particular choice of the penalty function.

In Chapter 5, we study a pair of optimal reinsurance-investment strategies under the two-sided exit framework which aims to (1) maximize the probability that the surplus reaches the target b before ruin occurs over the time horizon $[0, e_\lambda]$ (where e_λ is an independent exponentially distributed random time); (2) minimize the probability that ruin occurs before the surplus reaches the target b over the time horizon $[0, e_\lambda]$. A strong motivation to consider objectives (1) and (2) stems from the crucial role the two-sided exit probabilities $\mathbb{P}\left(\tau_b < \tau_0 \wedge e_\lambda \middle| X_0^u = x\right)$ and $\mathbb{P}\left(\tau_0 < \tau_b \wedge e_\lambda \middle| X_0^u = x\right)$ play in the analysis of insurance risk processes (e.g., spectrally negative Lévy processes and Markov-additive processes), and many recently proposed exotic ruin models. For the latter, we specifically mention the work on the discretely-observed ruin model (e.g., Albrecher et al. [2]), the loss-carry-forward tax model (e.g., Li et al. [63]), and the Parisian ruin model (e.g., Loeffen et al. [68]), to name a few.

The two-sided exit objectives (1) and (2) are closely related to earlier contributions made on the objective of minimizing a given ruin probability or reaching a bequest goal. Indeed, Promislow and Young [78] studied the optimal reinsurance-investment problem under the objective of minimizing the infinite-time ruin probability (i.e., a special case of objective (2) with $b = \infty$ and $\lambda = 0$). Under the same infinite-time horizon framework, Luo et al. [69] and Bai and Guo [14] further extended the work of Promislow and Young [78] by considering short-selling constraints and the presence of multiple risky assets, respectively. The objectives (1) and (2) are also in spirit related to the objective (3) of reaching a bequest goal, namely, $\sup_u \mathbb{P}\left(X^u_{e_\lambda \wedge \tau_0} \geq b \middle| X^u_0 = x\right)$, see, e.g., Bayraktar and Young [27] and references therein. A major difference between (3) and the two-sided exit objectives (1) and (2) is that the game in (3) ends at $e_\lambda \wedge \tau_0$, while the game in (1) and (2) ends at the earlier time $\tau_0 \wedge \tau_b \wedge e_\lambda$. Another difference is that Bayraktar and Young [27] considered the lifetime investment problem of an individual investor, while the focus of this paper is on the optimal reinsurance-investment problem of an insurer. See, for instance, Pestien and Sudderth [77], Karatzas [55], Browne [32][33][34], and Bayraktar et al. [24] for other related papers on the optimal control of an investor's wealth to reach a given level.

The work appearing in Chapter 5 assumes that the insurer can purchase proportional reinsurance and invest its wealth in a financial market consisting of a risk-free asset and a risky asset, where the dynamics of the latter is assumed to be correlated with the insurance surplus. By solving the associated HJB equation via a dual argument, an explicit expression for the optimal reinsurance-investment strategy is obtained. We find that the optimal strategy of objective (1) (objective (2) resp.) is always more aggressive (conservative resp.) than the strategy of minimizing the infinite-time ruin probability of Promislow and Young [78]. Due to the presence of the time factor e_λ, the optimal strategy under objective (1) or (2) may lead to more aggressive positions as the wealth level increases, a behavior which may be more consistent with industry practices.

Chapter 6 is devoted to the study of an optimal reinsurance-investment problem for an insurer under a mean-variance criterion in a dynamic setting. Two types of reinsurance policies are most commonly studied in the literature of optimal reinsurance-investment problems under a mean-variance criterion, i.e., proportional (quota-share) reinsurance (see, e.g., Zeng and Li [89], Shen and Zeng [82]) and excess-of-loss reinsurance (see, e.g., Gu et al. [51], Zhao et al. [94]). Given the extensive literature, two questions naturally arise which have not yet received much attention. 1) *Which reinsurance policy (of the two) is a better choice for an insurer under a mean-variance criterion*? 2) *Is there another form of reinsurance policy that is better than both under the same criterion*? We are thus motivated to take one step further by searching for the optimal reinsurance form within the class of combinations of proportional and excess-of-loss reinsurance policies.

In particular, we assume that the insurer purchases one of the two combinations (type I or type II) of proportional and excess-of-loss reinsurance policies. Under a mean-variance criterion, the reinsurance-investment problem is time-inconsistent in the sense that DPP fails. We then formulate the problem as a non-cooperative game and solve for a subgame perfect Nash equilibrium. Under a spectrally negative Lévy insurance model, we obtain the explicit equilibrium reinsurance-investment strategy by solving the extended HJB equation associated with type I (type II resp.) policy. The result shows that the excess-of-loss reinsurance is the optimal reinsurance form within the combined class of type I and type II reinsurance policies.

Finally, it is important to note that each of the Chapters 3–6 corresponds to a research project, which was written independently of each other. Although efforts have been made to keep the notation as consistent as possible, some inconsistencies may remain. The reader is therefore invited to treat each chapter separately from a notational standpoint.

Chapter 2

Mathematical Preliminaries

Consider a stochastic process $X = \{X_t\}_{t\geq 0}$ defined on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ satisfying the usual conditions. Let $\{\mathcal{F}_t^X\}_{t\geq 0}$ denote the natural filtration generated by X, defined by $\mathcal{F}_t^X = \sigma\{X_s : 0 \leq s \leq t\}$. Throughout the chapter, for convenience, we assume the state space of X is $\mathbb{R}$.

2.1 Lévy Process

Definition 2.1.1 *A real-valued stochastic process $X = \{X_t\}_{t\geq 0}$ with Càdlàg paths defined on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ is a Lévy process if*

(1) $\mathbb{P}(X_0 = 0) = 1$;

(2) for every $0 \leq s \leq t$, $X_t - X_s$ is independent of $\mathcal{F}_s$ and is identically distributed as X_{t-s}.

The famous Lévy-Khintchine formula is stated in the following theorem (Theorem 1.6 of Kyprianou [58]), which fully characterizes a Lévy process.

Theorem 2.1.1 *Suppose that $\mu \in \mathbb{R}$, $\sigma > 0$, and v is a measure concentrated on $\mathbb{R}\backslash\{0\}$ satisfying*

$$\int_{\mathbb{R}} (x^2 \wedge 1) v(\mathrm{d}x) < \infty.$$

From the triplet (μ, σ, v) define for each $s \in \mathbb{R}$,

$$\Psi(s) = i\mu s + \frac{1}{2}\sigma^2 s^2 + \int_{\mathbb{R}} (1 - e^{isx} + isx\mathbf{1}_{\{|x|\leq 1\}}) v(\mathrm{d}x).$$

Then, there exists a probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ on which a Lévy process is defined with characteristic exponent Ψ.

The measure v is referred to as Lévy measure in the literature. Lévy processes form a rich family of processes, including for instance, compound Poisson processes, Brownian motions, and stable processes, as special cases. The family itself is a subclass of Markov processes. Moreover, it is well known that a Lévy process exhibits strong Markov property.

We next introduce the definition the *Poisson random measure* as it is a convenient tool used to analyze the jump parts of a Lévy process.

Definition 2.1.2 *Let $(\mathbb{S}, \mathcal{S}, m)$ be an arbitrary σ-finite measure space. $N : \mathcal{S} \to \mathbb{Z}^+ \cup \{0, \infty\}$ is a Poisson random measure with intensity m if*

(1) for any subset B of $\mathcal{S}$, $N(B)$ is a Poisson random variable with parameter $m(B)$;

(2) for any disjoint subsets $B_1, B_2, \cdots, B_n$ of $\mathcal{S}$, $N(B_1), N(B_2), \cdots, N(B_n)$ are independent.

An important subclass of Lévy processes which draws tremendous attention from the actuarial community is the so-called spectrally negative Lévy processes

(SNLP), for the reason that a SNLP is particularly well suited to model a surplus process of an insurance company (as is done in Chapter 6). Spectrally negative Lévy processes are Lévy processes with v restricted to be a measure on $(-\infty, 0)$, namely, $v(0,\infty) = 0$. The SDE of a SNLP $X = \{X_t\}_{t\geq 0}$ can be written as

$$\mathrm{d}X_t = \mu \mathrm{d}t + \sigma \mathrm{d}B_t + \int_{-\infty}^{0} zN(\mathrm{d}z, \mathrm{d}t),\ X_0 = 0, \tag{2.1}$$

where $N(\mathrm{d}z, \mathrm{d}t)$ is a Poisson random measure with intensity $\mathrm{d}tv(\mathrm{d}z)$. Alternatively, SDE (2.1) can be written as

$$\mathrm{d}X_t = \left(\mu + \int_{-\infty}^{0} zv(\mathrm{d}z)\right)\mathrm{d}t + \sigma \mathrm{d}B_t + \int_{-\infty}^{0} z\tilde{N}(\mathrm{d}z, \mathrm{d}t),\ X_0 = 0, \tag{2.2}$$

where $\tilde{N}(\mathrm{d}z, \mathrm{d}t) := N(\mathrm{d}z, \mathrm{d}t) - v(\mathrm{d}z)\mathrm{d}t$ is the *compensated* Poisson random measure.

Note that the study on Lévy processes indeed forms a rich research field. The interested reader is referred to, e.g., Kyprianou [58] for a complete review on Lévy processes with its applications.

2.2 Controlled Lévy Diffusion Process

The Lévy process (2.2) is a special case of a *Lévy diffusion process*. A sufficient condition for the existence and uniqueness of a strong solution to the SDE of a Lévy diffusion is given in the following theorem (Theorem 1.19 of Øksendal and Sulem, Page 10).

Theorem 2.2.1 *Consider the following Lévy diffusion SDE in* $\mathbb{R}$*:* $X(0) = x_0 \in \mathbb{R}$ *and*

$$\mathrm{d}X_t = \mu(t, X_t)\mathrm{d}t + \sigma(t, X_t)\mathrm{d}B_t + \int_{\mathbb{R}} \gamma(t, X_{t-}, z)\tilde{N}(\mathrm{d}z, \mathrm{d}t), \tag{2.3}$$

where $\mu : [0,T] \times \mathbb{R} \to \mathbb{R}$*,* $\sigma : [0,T] \times \mathbb{R} \to \mathbb{R}$ *and* $\gamma : [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *satisfy the following conditions:*

(1) there exists a constant $C_1 < \infty$ *such that for all* $x \in \mathbb{R}$*,*

$$|\mu(t,x)|^2 + |\sigma(t,x)|^2 + \int_{\mathbb{R}} |\gamma(t,x,z)|^2 v(\mathrm{d}z) \leq C_1(1+|x|^2); \tag{2.4}$$

(2) there exists a constant $C_2 < \infty$ *such that for all* $x, y \in \mathbb{R}$*,*

$$|\mu(t,x)-\mu(t,y)|^2 + |\sigma(t,x)-\sigma(t,y)|^2 + \int_{\mathbb{R}} |\gamma(t,x,z)-\gamma(t,y,z)|^2 v(\mathrm{d}z) \leq C_2|x-y|^2. \tag{2.5}$$

Then, there exists a unique cádlág adapted solution X_t *such that* $\mathbb{E}\left[X_t^2\right] < \infty$ *for all* t*.*

A Lévy diffusion process $X = \{X_t\}_{t\geq 0}$ under a control $U = \{U_t\}_{t\geq 0}$ is described by the following SDE

$$\mathrm{d}X_t = \mu(t, X_t, U_t)\mathrm{d}t + \sigma(t, X_t, U_t)\mathrm{d}B_t + \int_{\mathbb{R}} \gamma(X_{t-}, u_{t-}, z)\tilde{N}(\mathrm{d}z, \mathrm{d}t),\ \ X_0 = x. \tag{2.6}$$

The control U is assumed to be cádlág and adapted, valued in $\mathbb{U} \subset \mathbb{R}^m$. All models developed in Chapters 3-6 will fall into the category of the processes described by (2.6). To be more specific, Chapters 3–5 deal with controlled diffusion processes (without jumps) and Chapter 6 is dealing with a controlled SNLP.

In most cases, we are interested in Markov controls. A $\mathbb{U}$-valued control U is called a *Markov control* if $U_t = u(t, X_t)$ for some measurable function $u : [0,\infty) \times \mathbb{R} \to \mathbb{U}$. In particular, if $U_t = u(X_t)$ for some measurable function $u : \mathbb{R} \to \mathbb{U}$, then U is called a *time homogeneous* Markov control. Under a Markov control, a controlled Lévy diffusion process stays within the class of Lévy diffusion processes.

2.3 Itô's Formula

Itô's formula is one of the most important mathematical tools in stochastic analysis. In this section, we state Itô's formula for semimartingales, which is its most general form.

Definition 2.3.1 *A stochastic process $X = \{X_t\}_{t\geq 0}$ with state space $\mathbb{R}$ is a semimartingale if it admits the decomposition*

$$X = X_0 + M + A,$$

where X_0 is finite and $\mathcal{F}_0$-measurable, M is a local martingale with $M_0 = 0$ and A is a finite variation process with $A_0 = 0$.

Theorem 2.3.1 *Let $X = \{X_t\}_{t\geq 0}$ be a semimartingale with state space $\mathbb{R}$ and f be a C^2 function on $\mathbb{R}$. Then $f(X)$ is a semimartingale and*

$$\begin{aligned} f(X_t) &= f(X_0) + \int_0^t f_x(X_{s-})\mathrm{d}X_s + \frac{1}{2}\int_0^t f_{xx}(X_{s-})\mathrm{d}\langle X^c\rangle_s \\ &\quad + \sum_{0\leq s\leq t}\left(f(X_s) - f(X_{s-}) - f_x(X_{s-})\Delta X_s\right), \end{aligned} \tag{2.7}$$

where f_x (f_{xx} resp.) is the first order (second order resp.) derivative of f with respect to X.

Note that formula (2.7) will be repeatedly used in the following chapters in the proof of verification theorems.

Chapter 3

Optimal Investment to Minimize the Probability of Lifetime Drawdown

3.1 Introduction

In this chapter, we consider the optimization problem of minimizing the probability that a significant drawdown occurs over a lifetime investment. Mathematically speaking, our problem is formulated as follows. On a filtered complete probability space $(\Omega, \mathcal{F}, \boldsymbol{F} = \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ satisfying the usual conditions, we consider a $\boldsymbol{F}$-progressively measurable trading strategy $\pi = \{\pi_t\}_{t\geq 0}$. The associated fund value process is denoted by $W^\pi = \{W_t^\pi\}_{t\geq 0}$ with initial value $W_0 = w > 0$. We define the (floored) running maximum of the fund value at time t by $M_t^\pi = \max\left\{\sup_{0\leq s\leq t} W_s^\pi, m\right\}$ with $m \geq w$. Note that the initial values w and m are fixed positive constants, and hence are independent of the trading strategy

π. The ratios $(M_t^\pi - W_t^\pi)/M_t^\pi$ and W_t^π/M_t^π are respectively called the *relative drawdown level* and the *relative fund level* at time t. To quantify and measure the drawdown risk, for a fixed significance level $\alpha \in (0,1)$, we define

$$\tau_\alpha^\pi = \inf\left\{t \geq 0 : M_t^\pi - W_t^\pi > \alpha M_t^\pi\right\},$$

to be the first time the relative drawdown of the fund value W^π exceeds the significance level $100\alpha\%$. Equivalently, the event $(\tau_\alpha^\pi > t)$ for some fixed $t > 0$ implies that the relative drawdown of the fund value in time period $[0,t]$ never exceeds α.

Our main objective is to solve for the optimal trading strategy $\pi^* = \{\pi_t^*\}_{t\geq 0}$ that minimizes the probability that a relative drawdown of size over α occurs before e_λ, the random time of death of a client with constant force of mortality $\lambda > 0$, i.e.,

$$\min_{\pi\in\Pi} \mathbb{P}\left\{\tau_\alpha^\pi < e_\lambda | W_0 = w, M_0 = m\right\}, \tag{3.1}$$

where Π is the set of admissible trading strategies defined as

$$\Pi = \left\{\pi : \pi \text{ is } \boldsymbol{F}\text{-progressively measurable and } \int_0^t \pi_s^2 \mathrm{d}s < \infty \text{ for any } t \geq 0\right\}. \tag{3.2}$$

Thus, e_λ is an $\mathcal{F}$-measurable exponentially distributed random variable with mean $1/\lambda > 0$, independent of the fund value process by assumption. For ease of notation, we denote the objective function in (3.1) as

$$\psi(w,m) = \min_{\pi\in\Pi} \mathbb{P}^{w,m}\left\{\tau_\alpha^\pi < e_\lambda\right\} = \min_{\pi\in\Pi} \mathbb{E}^{w,m}[e^{-\lambda\tau_\alpha^\pi}], \tag{3.3}$$

where the last equation is due to the independence of τ_α^π and e_λ. Here and henceforth, we write $\mathbb{E}^{w,m}[\cdot] = \mathbb{E}[\cdot|W_0 = w, M_0 = m]$.

As for other similar optimization problems (e.g., the minimum lifetime ruin probability (MLRP) of Young [88], Bayraktar and Young [25], Bayraktar and Zhang

[28] and references therein), we consider the drawdown probability over the lifetime of a client with a constant force of mortality. For the treatment of non-constant forces of mortality, one may adopt the approximative scheme of Moore and Young [74]. Finally, the solution of our resulting Hamilton-Jacobi-Bellman (HJB) equation does not possess a simple form, which makes its solution form difficult to guess. Instead, we decompose the HJB equation into two nonlinear equations of first order which are solved consecutively.

We point out that a recent paper by Angoshtari et al. [6] also studied the minimum drawdown probability problem but over an infinite-time horizon. By utilizing the results of Bäuerle and Bayraktar [20], the authors found that the minimum infinite-time drawdown probability (MIDP) strategy coincides with the minimum infinite-time ruin probability (MIRP) strategy which consists in maximizing the ratio of the drift of the value process to its volatility squared. However, we point out that such a relationship does not hold for a random (or finite) maturity setting such as in (3.3) as the time-change arguments in Bäuerle and Bayraktar [20] do not apply.

We will study the MLDP problem (3.3) by examining two different market models: a market with two risky assets and a market with a risk-free asset and a risky asset. It worth pointing out that several conclusions and implications of market model I are determinant to the subsequent analysis of market model II. Also, the following financial implications hold for both market models: (1) it is optimal to minimize the portfolio's variance when the fund value is at its historic high-water mark; (2) when the fund value drops, it is optimal to increase the proportion invested in the asset with a higher instantaneous rate of return (even though its volatility may also be higher). It follows that the instantaneous return rate of the MLDP strategy is never less than the return rate of the minimum variance

(MV) strategy, which supports the practical use of drawdown-based performance measures.

The rest of the chapter is organized as follows. The parallel Sections 3.2 and 3.3 are respectively devoted to the market models I and II. For each model, we provide a verification theorem, obtain closed-form expressions for the MLDP and its corresponding optimal trading strategy, as well as prove some properties of the optimal trading strategy. At the end of each section, we complement the analysis with some numerical examples.

3.2 Market model I

In this section, we study problem (3.3) under the market model consisting of two risky assets. We assume that the i-th risky asset ($i = 1, 2$) is governed by a geometric Brownian motion with dynamics

$$\mathrm{d}S_t^{(i)} = \mu_i S_t^{(i)} \mathrm{d}t + \sigma_i S_t^{(i)} \mathrm{d}B_t^{(i)}, \quad S_0^{(i)} > 0,$$

where $\mu_i \in \mathbb{R}$, $\sigma_i > 0$, and $\{B_t^{(i)}\}_{t\geq 0}$ is a standard Brownian motion on the filtered probability space $(\Omega, \mathcal{F}, \boldsymbol{F}, \mathbb{P})$. In addition, $\{B_t^{(1)}\}_{t\geq 0}$ and $\{B_t^{(2)}\}_{t\geq 0}$ are assumed to be dependent with

$$\mathrm{d}B_t^{(1)} \mathrm{d}B_t^{(2)} = \rho \mathrm{d}t,$$

where $\rho \in (-1, 1)$ is the correlation coefficient. To avoid triviality, we exclude cases where the two assets are either perfectly positively or negatively correlated. Given a trading strategy $\pi \in \Pi$ defined in (3.2), where π_t represents *the fraction of wealth invested in Asset* 1 at time t, the evolution of the fund value process W^π is governed by

$$
\begin{aligned}
\mathrm{d}W_t^\pi &= \pi_t W_t^\pi \frac{\mathrm{d}S_t^{(1)}}{S_t^{(1)}} + (1-\pi_t) W_t^\pi \frac{\mathrm{d}S_t^{(2)}}{S_t^{(2)}} \\
&= \left(\pi_t \mu_1 + (1-\pi_t)\mu_2\right) W_t^\pi \mathrm{d}t + \pi_t W_t^\pi \sigma_1 \mathrm{d}B_t^{(1)} + (1-\pi_t) W_t^\pi \sigma_2 \mathrm{d}B_t^{(2)} \qquad (3.4)
\end{aligned}
$$

with initial value $W_0 = w > 0$.

3.2.1 Verification theorem

We first prove a verification theorem for the MLDP. By a dimension reduction, the MLDP problem (3.3) will later be reduced to a one-dimensional stochastic control problem.

Let

$$D = \left\{(w, m) \in \mathbb{R}^2 : m(1-\alpha) \le w \le m \text{ and } m > 0\right\},$$

and define a differential operator $\mathcal{L}^\beta$ ($\beta \in \mathbb{R}$) as

$$\mathcal{L}^\beta f = (\beta\mu_1 + (1-\beta)\mu_2)\, x f_x + \frac{1}{2}\left(\beta^2\sigma_1^2 + (1-\beta)^2\sigma_2^2 + 2\rho\beta(1-\beta)\sigma_1\sigma_2\right) x^2 f_{xx} - \lambda f,$$

where f is a twice-differentiable function in x with $f_x := \frac{\partial f}{\partial x}$ and $f_{xx} := \frac{\partial^2 f}{\partial x^2}$.

Theorem 3.2.1 *Suppose that $f : D \to (0, 1]$ satisfies the following conditions:*

(1) For any fixed $m > 0$, $f(\cdot, m) \in C^2([m(1-\alpha), m])$ is strictly decreasing and strictly convex;

(2) For any fixed $w > 0$, $f(w, \cdot) \in C^1([w, w/(1-\alpha)])$ is strictly increasing;

(3) For any fixed $m > 0$ and $\beta \in \mathbb{R}$, $\mathcal{L}^\beta f(\cdot, m) \ge 0$ for $w \in [m(1-\alpha), m]$;

(4) For any fixed $m > 0$, there exists an admissible strategy $\pi^* : D \to \mathbb{R}$ such that $\mathcal{L}^{\pi^*} f(\cdot, m) = 0$ for $w \in [m(1-\alpha), m]$;

(5) For any $m > 0$, $f(m\,(1-\alpha)\,, m) = 1$;

(6) For any $m > 0$, $f_m(m, m) = 0$.

Then $f(w, m) = \psi(w, m)$ on D, where $\psi(w, m)$ is the MLDP defined in (3.3), and π^* is the corresponding optimal trading strategy.

Proof. For an admissible trading strategy π satisfying (3.2), we define a sequence of stopping time $\{\gamma_n^\pi\}_{n\in\mathbb{N}}$ with $\gamma_n^\pi = \inf\left\{t \geq 0 : \int_0^t \pi_s^2 \mathrm{d}s \geq n\right\}$. By applying Itô's formula to the process $e^{-\lambda t} f(W_t^\pi, M_t^\pi)$ for $t \in [0, \tau_{\alpha,n}^\pi]$, where $\tau_{\alpha,n}^\pi := \tau_\alpha^\pi \wedge \gamma_n^\pi$, and then using (3.4), we arrive at

$$\begin{aligned}
& e^{-\lambda\tau_{\alpha,n}^\pi} f(W_{\tau_{\alpha,n}^\pi}^\pi, M_{\tau_{\alpha,n}^\pi}^\pi) - f(w, m) \\
&= -\lambda \int_0^{\tau_{\alpha,n}^\pi} e^{-\lambda t} f(W_t^\pi, M_t^\pi)\mathrm{d}t + \int_0^{\tau_{\alpha,n}^\pi} e^{-\lambda t} f_w(W_t^\pi, M_t^\pi)\mathrm{d}W_t^\pi \\
&+ \frac{1}{2}\int_0^{\tau_{\alpha,n}^\pi} e^{-\lambda t} f_{ww}(W_t^\pi, M_t^\pi)(\mathrm{d}W_t^\pi)^2 + \int_0^{\tau_{\alpha,n}^\pi} e^{-\lambda t} f_m(W_t^\pi, M_t^\pi)\mathrm{d}M_t^\pi \\
&= \int_0^{\tau_{\alpha,n}^\pi} e^{-\lambda t} \mathcal{L}^\pi f(W_t^\pi, M_t^\pi)\mathrm{d}t + \int_0^{\tau_{\alpha,n}^\pi} e^{-\lambda t} f_w(W_t^\pi, M_t^\pi)\pi_t W_t^\pi \sigma_1 \mathrm{d}B_t^{(1)} \\
&+ \int_0^{\tau_{\alpha,n}^\pi} e^{-\lambda t} f_w(W_t^\pi, M_t^\pi)(1-\pi_t) W_t^\pi \sigma_2 \mathrm{d}B_t^{(2)},
\end{aligned} \tag{3.5}$$

Note that the operator $\mathcal{L}^\pi f(\cdot,\cdot)$ is applied on the argument w of f in (3.5). Also, the passage from the first to the second equality in (3.5) was made possible given that $f_m(W_t^\pi, M_t^\pi)\mathrm{d}M_t^\pi = 0$ a.s.. This is because either $\mathrm{d}M_t^\pi = 0$ when $W_t^\pi < M_t^\pi$ or $f_m(W_t^\pi, M_t^\pi) = 0$ when $W_t^\pi = M_t^\pi$ by condition (6). Taking the conditional expectation $\mathbb{E}^{w,m}[\cdot]$ on both sides of (3.5) and invoking condition (3), we obtain

$$\mathbb{E}^{w,m}\left[e^{-\lambda\tau_{\alpha,n}^\pi} f(W_{\tau_{\alpha,n}^\pi}^\pi, M_{\tau_{\alpha,n}^\pi}^\pi)\right] \geq f\,(w, m)\,, \tag{3.6}$$

for all $\pi \in \Pi$. Since f is assumed to be bounded, by the dominated convergence theorem and condition (5), it follows that

$$\mathbb{E}^{w,m}\left[e^{-\lambda\tau_\alpha^\pi}\right] \geq f\,(w, m)\,, \tag{3.7}$$

for all $\pi \in \Pi$. Further, by condition (4), there exists an admissible strategy $\pi^* : D \to \mathbb{R}$ such that the equality holds in (7.47). In other words, we deduce that

$$f(w,m) = \psi(w,m) = \inf_{\pi \in \Pi} \mathbb{E}^{w,m}\left[e^{-\lambda \tau_\alpha^\pi}\right] = \mathbb{E}^{w,m}[e^{-\lambda \tau_\alpha^{\pi^*}}],$$

which completes the proof. ■

Let f be the function satisfying all the conditions of Theorem 3.2.1. It is not difficult to see that $f(cw, cm) = f(w,m)$ for any constant $c > 0$. This scaling relation implies that we can reduce the dimension of f by considering

$$f(w,m) = f\left(\frac{w}{m}, 1\right) := g\left(\frac{w}{m}\right), \quad 1 - \alpha \leq \frac{w}{m} \leq 1, \tag{3.8}$$

where the ratio w/m is the relative fund level. Using the change of variable formulas $f_w = \frac{1}{m} g'$, $f_{ww} = \frac{1}{m^2} g''$, and $f_m = -\frac{w}{m^2} g'$, we immediately obtain the following corollary from Theorem 3.2.1.

Corollary 3.2.1 *Suppose that $g : [1-\alpha, 1] \to (0,1]$ satisfies the following conditions:*

(1) $g(\cdot) \in C^2([1-\alpha, 1])$ is strictly decreasing and strictly convex;

(2) $\mathcal{L}^\beta g(z) \geq 0$ for any $\beta \in \mathbb{R}$ and $z \in [1-\alpha, 1]$;

(3) There exists an admissible strategy $\pi^* : [1-\alpha, 1] \to \mathbb{R}$ such that $\mathcal{L}^{\pi^*} g(z) = 0$ for $z \in [1-\alpha, 1]$;

(4) $g(1-\alpha) = 1$;

(5) $g'(1) = 0$.

Then $g(z) = \phi(z) := \inf_{\pi \in \Pi} \mathbb{E}^{w,m}\left[e^{-\lambda \tau_\alpha^\pi}\right]$ for $z = \frac{w}{m} \in [1-\alpha, 1]$, and π^* is the corresponding optimal trading strategy.

3.2.2 MLDP and optimal trading strategy

In this section, we aim to solve for the MLDP $\phi(\cdot)$ and the corresponding optimal trading strategy π^*. By conditions (2) and (3) of Corollary 3.2.1, we have

$$\inf_{\beta\in\mathbb{R}}\left\{\mathcal{L}^{\beta}g(z)\right\}=0,\quad z\in[1-\alpha,1]. \tag{3.9}$$

By the first-order condition of Equation (3.9), the minimizer is given in the feedback form

$$\pi^*(z)=\frac{\sigma_2^2-\rho\sigma_1\sigma_2}{\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2}-\frac{(\mu_1-\mu_2)g'(z)}{(\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2)zg''(z)},\quad z\in[1-\alpha,1]. \tag{3.10}$$

Substituting (3.10) into (3.9) followed by algebraic manipulations, we obtain the nonlinear equation

$$\frac{A}{2}z^2g''-\frac{B}{2}\frac{(g')^2}{g''}-Czg'-\lambda g=0,\quad z\in[1-\alpha,1], \tag{3.11}$$

where $A:=\frac{\sigma_1^2\sigma_2^2(1-\rho^2)}{\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2}>0$, $B:=\frac{(\mu_2-\mu_1)^2}{\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2}\geq 0$, and $C:=\frac{(\mu_2-\mu_1)(\sigma_2^2-\rho\sigma_1\sigma_2)}{\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2}-\mu_2$.

Theorem 3.2.2 *Under market model I, the MLDP and its corresponding optimal trading strategy are respectively given by*

$$\phi(z)=\exp\left(-A\int_{h^{-1}(z)}^{h^{-1}(1-\alpha)}\frac{x}{k(x)}\mathrm{d}x\right), \tag{3.12}$$

and

$$\pi^*(z)=\frac{\sigma_2^2-\rho\sigma_1\sigma_2}{\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2}-\frac{\mu_1-\mu_2}{\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2}\frac{Ah^{-1}(z)}{A(h^{-1}(z))^2+k(h^{-1}(z))-Ah^{-1}(z)}, \tag{3.13}$$

for $z\in[1-\alpha,1]$, *where* $k(x):=\lambda+(A+C)x-Ax^2+\sqrt{(\lambda+Cx)^2+ABx^2}$ *and* $h(v):=\exp\left(-\int_v^0\frac{A}{k(x)}\mathrm{d}x\right)$ *for* $v\in(v_*,0]$ *with* $v_*:=\sup\{x<0:k(x)=0\}$. *Furthermore,* $\phi(\cdot),\pi^*(\cdot)\in C^\infty([1-\alpha,1])$.

Proof. In light of Equation (3.11) and Corollary 3.2.1, we consider the following non-linear equation

$$\begin{cases} \frac{A}{2}z^2G'' - \frac{B}{2}\frac{(G')^2}{G''} - CzG' - \lambda G = 0, \quad z \in (0,1], \\ G(1) = 1, \\ G'(1) = 0, \\ G''(z) > 0, \quad z \in (0,1]. \end{cases} \tag{3.14}$$

Next, we show that (3.14) admits a unique solution G and furthermore, $G \in C^\infty((0,1])$. The advantage to consider the function G is that it is independent of α.

Define two auxiliary functions

$$u(z) := \frac{zG'(z)}{z^2G''(z)} \text{ and } v(z) := \frac{zG'(z)}{G(z)}, \quad z \in (0,1]. \tag{3.15}$$

Since $G'(1) = 0$ and $G''(z) > 0$ for $z \in (0,1)$, we have $G'(z) < 0$ for $z \in (0,1)$, which further implies that both $u(z)$ and $v(z)$ are strictly negative functions on $(0,1)$. Dividing both sides of the first equation of (3.14) by $zG'(z)$, we obtain

$$\frac{A}{2u} - \frac{B}{2}u - C - \frac{\lambda}{v} = 0, \quad z \in (0,1). \tag{3.16}$$

Solving the algebraic equation (3.16) with $u(z) < 0$ and $v(z) < 0$, we have

$$\frac{1}{u} = \frac{\lambda + Cv + \sqrt{(\lambda + Cv)^2 + ABv^2}}{Av}. \tag{3.17}$$

Differentiating v in z from the second relation of (3.15) and subsequently using (3.17), it follows that

$$zv' = z\frac{(zG'' + G')G - z(G')^2}{G^2} = \frac{v}{u} + v - v^2 = \frac{1}{A}k(v), \tag{3.18}$$

where $k(x) := \lambda + (A + C)x - Ax^2 + \sqrt{(\lambda + Cx)^2 + ABx^2}$ for $x \in \mathbb{R}$. Since $k(\cdot) \in C^\infty(\mathbb{R})$, $k(0) = 2\lambda > 0$ and $\lim_{v\downarrow -\infty} k(v) = -\infty$, there exists some point v_*

such that

$$v_* := \sup\{x < 0 : k(x) = 0\} > -\infty.$$

Furthermore, by $v'(z)z'(v) = 1$, Equation (3.18) becomes

$$z'(v) = \frac{A}{k(v)} z(v),$$

which admits a unique solution

$$z(v) = h(v) := \exp\left(-\int_v^0 \frac{A}{k(x)} \mathrm{d}x\right), \quad v \in (v_*, 0], \tag{3.19}$$

under the boundary condition $z(0) = 1$. Moreover, by $v = h^{-1}(z)$, it can be shown that $v'(z) > 0$ for $z \in (0, 1]$, $v(1) = 0$, and $\lim_{z \downarrow 0} v(z) = v_*$. Now, letting $H(v) := G(h(v)) = G(z)$, it follows that

$$\begin{cases} \frac{\mathrm{d}H}{\mathrm{d}v} = \frac{\mathrm{d}G}{\mathrm{d}z}\frac{\mathrm{d}z}{\mathrm{d}v} = \frac{vG(z)}{z}\frac{Az}{k(v)} = \frac{Av}{k(v)} H(v), & v \in (v_*, 0], \\ H(0) = 1. \end{cases} \tag{3.20}$$

The solution to (3.20) is given by

$$H(v) = \exp\left(-A \int_v^0 \frac{x}{k(x)} \mathrm{d}x\right), \quad v \in (v_*, 0],$$

or equivalently, we have shown that (3.14) admits a unique solution

$$G(z) = \exp\left(-A \int_{h^{-1}(z)}^0 \frac{x}{k(x)} \mathrm{d}x\right) \in C^\infty((0, 1]).$$

Letting

$$g(z) := \frac{G(z)}{G(1-\alpha)} = \exp\left(-A \int_{h^{-1}(z)}^{h^{-1}(1-\alpha)} \frac{x}{k(x)} \mathrm{d}x\right), \quad z \in [1-\alpha, 1], \tag{3.21}$$

it is straightforward to verify that $g(\cdot)$ satisfies all the conditions of Corollary 3.2.1. Hence, we conclude that $g(z) = \phi(z)$ for $z \in [1-\alpha, 1]$ which proves (3.12). Finally, differentiating (3.21) yields

$$g'(z) = g(z) \frac{h^{-1}(z)}{z} \text{ and } g''(z) = g(z) \frac{A(h^{-1}(z))^2 + k(h^{-1}(z)) - Ah^{-1}(z)}{Az^2}. \tag{3.22}$$

Substituting (3.22) into (3.10) leads to the optimal strategy $\pi^*(\cdot) \in C^\infty([1-\alpha, 1])$ given in (3.13). $\pi^*(\cdot)$ is bounded in $[1-\alpha, 1]$ since it is countinuous in $[1-\alpha, 1]$ and $[1-\alpha, 1]$ is a compact set. Thus, $\pi^* \in \Pi$. This completes the proof. ■

We have some interesting observations to make of the MLDP strategy (3.13), which relate to the classical MV strategy.

1. Suppose that $\mu_1 = \mu_2$, the optimal strategy (3.13) reduces to a constant proportional strategy
$$\hat{\pi} = \frac{\sigma_2^2 - \rho\sigma_1\sigma_2}{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}. \tag{3.23}$$
It is easy to see from (3.4) that
$$\begin{aligned} & \min_{\pi \in \Pi} \mathrm{Var}\left[\log W_t^\pi\right] \\ = & \min_{\pi \in \Pi} \int_0^t \left(\pi_s^2\sigma_1^2 + (1-\pi_s)^2\sigma_2^2 + 2\pi_s(1-\pi_s)\sigma_1\sigma_2\rho\right) \mathrm{d}s = \mathrm{Var}\left[\log W_t^{\hat{\pi}}\right]. \end{aligned}$$
Hence, when $\mu_1 = \mu_2$, the MLDP strategy (3.13) coincides with the MV strategy (3.23).

2. Even if $\mu_1 \neq \mu_2$, we can see from (3.10) and condition (5) of Corollary 3.2.1 that
$$\pi^*(1) = \frac{\sigma_2^2 - \rho\sigma_1\sigma_2}{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2} = \hat{\pi}. \tag{3.24}$$
Relation (3.24) implies that, when the fund value is at its running maximum, the MLDP strategy is identical to the MV strategy.

3. By (3.4), we denote by $\mu^\pi := \mu_1\pi_t + \mu_2(1-\pi_t)$ the instantaneous return rate of the portfolio at time t under strategy π. By (3.10) and the fact that the MLDP ϕ is decreasing and convex, we have
$$\mu^{\pi^*} - \mu^{\hat{\pi}} = (\mu_2 - \mu_1)(\hat{\pi} - \pi^*(z)) = \frac{-(\mu_2-\mu_1)^2\phi'(z)}{(\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2)z\phi''(z)} \geq 0, \tag{3.25}$$

for all $z \in [1-\alpha, 1]$. In other words, the instantaneous return rate of the MLDP portfolio is never less than the return rate of the MV portfolio. This result supports the practical use of drawdown-based performance measures in which the role of volatility is replaced by drawdown. Intuitively speaking, this conclusion is consistent with the fact that volatility-based measures penalize for both upside and downside movements of the fund process while drawdown-based measures only penalize for downside movements.

This leads to a natural question: *How does the MLDP strategy behave when the fund value is away from a historic high-water mark?* We find that, as shown in the next proposition, it is optimal to increase the proportion invested in the asset with a higher instantaneous rate of return as the portfolio's relative drawdown level increases (even though this may increase the portfolio's variance).

Proposition 3.2.1 *Suppose that $\mu_1 \neq \mu_2$. We have*

$$(\mu_1 - \mu_2)\frac{\mathrm{d}\pi^*}{\mathrm{d}z} < 0, \quad z \in [1-\alpha, 1].$$

Proof. By (3.10) and the definitions of $u(\cdot)$ and $v(\cdot)$ in (3.15), it follows that the optimal strategy can be rewritten as

$$\pi^*(z) = \frac{\sigma_2^2 - \rho\sigma_1\sigma_2}{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2} + \frac{\mu_2 - \mu_1}{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2} u(z),$$

which implies that

$$(\mu_1 - \mu_2)\frac{\mathrm{d}\pi^*}{\mathrm{d}z} = -\frac{(\mu_2 - \mu_1)^2}{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}\frac{\mathrm{d}u}{\mathrm{d}z}.$$

By (3.18), we have

$$\frac{\mathrm{d}v}{\mathrm{d}z} = \frac{1}{Az}k(v) > 0. \tag{3.26}$$

On the other hand, solving v from (3.16), we obtain $v = \frac{2\lambda u}{A - Bu^2 - 2Cu}$ which yields

$$\frac{\mathrm{d}v}{\mathrm{d}u} = \frac{2\lambda A + 2\lambda B u^2}{(A - Bu^2 - 2Cu)^2} > 0. \tag{3.27}$$

Using (3.26), (3.27), and $\frac{\mathrm{d}v}{\mathrm{d}z} = \frac{\mathrm{d}v}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}z}$, we conclude that $\frac{\mathrm{d}u}{\mathrm{d}z} > 0$. This ends the proof. ■

Remark 3.2.1 *As for the market model II of Section 3.3, a proportional management fee of the fund with rate $\eta \in (0,1)$ can easily be incorporated into the above analysis. Then the dynamics of the fund value process (3.4) becomes*

$$\mathrm{d}W_t^\pi = (\pi_t \mu_1 + (1 - \pi_t)\,\mu_2 - \eta)\,W_t^\pi \mathrm{d}t + \pi_t W_t^\pi \sigma_1 \mathrm{d}B_t^{(1)} + (1 - \pi_t)\,W_t^\pi \sigma_2 \mathrm{d}B_t^{(2)}.$$

It is clear that the formulas of the MLDP (3.12) and the optimal trading strategy (3.13) still hold by simply replacing μ_1 and μ_2 by $\mu_1 - \eta$ and $\mu_2 - \eta$, respectively.

3.2.3 Numerical examples

In this section, we provide some numerical examples to illustrate the main results of Section 2. We consider a relative drawdown level of $\alpha = 0.2$ and an investor's expected future lifetime of 20 years (i.e. $\lambda = 0.05$).

In Figure 1, we set $\mu_1 = 0.1$, $\mu_2 = 0.15$, $\sigma_1 = 0.125$, $\sigma_2 = 0.15$ and $\rho = 0.2$. We first examine the diversification benefit by comparing in Figure 1 (left plot) the MLDP to the drawdown probability for investment in Asset 1 or 2 only. The drawdown probabilities for geometric Brownian motions were first derived by Taylor [86] and can also be found more recently in, e.g., Theorem 1 of Avram et. al. [11]. We recall this result here. For $S := \{S_t\}_{t \geq 0}$ a geometric Brownian motion with dynamics

$$\mathrm{d}S_t = \mu S_t \mathrm{d}t + \sigma S_t \mathrm{d}B_t, \quad S_0 := w > 0,$$

where $\mu \in \mathbb{R}$, $\sigma > 0$, and $\{B_t\}_{t \geq 0}$ is a standard Brownian motion, we define the first time the relative drawdown of S exceeds level α as $\tau_\alpha := \inf \{t \geq 0 : M_t - S_t > \alpha M_t\}$, where $M_t := \max \left\{ \sup_{0 \leq u \leq t} S_u, m \right\}$ and $m \geq w$. Then,

$$\mathbb{P}^z\{\tau_\alpha < e_\lambda\} := \mathbb{P}^{w,m}\{\tau_\alpha < e_\lambda\} = \frac{\beta^+ z^{\beta^-} - \beta^- z^{\beta^+}}{\beta^+ (1-\alpha)^{\beta^-} - \beta^- (1-\alpha)^{\beta^+}},$$

where $z := \frac{w}{m} \in [1-\alpha, 1]$ and $\beta^\pm = \frac{-\mu + \sigma^2/2 \pm \sqrt{(\mu - \sigma^2/2)^2 + 2\lambda\sigma^2}}{\sigma^2}$.

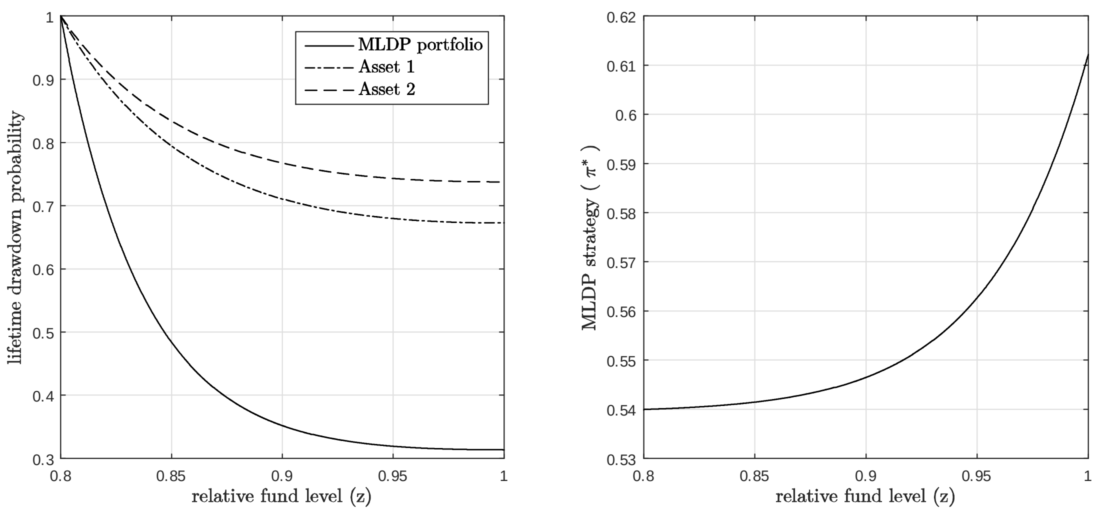


Figure 3.1. Lifetime drawdown probabilities (left) and the MLDP trading strategy (right)

We observe that the drawdown probabilities are considerably lower under the MLDP strategy (than investing in either Asset 1 or 2). In Figure 3.1 (right plot), we provide the curve of the corresponding MLDP strategy as a function of the relative fund level $z = w/m$. Notice that π is increasing in z, which is consistent with Proposition 3.2.1 as $\mu_1 = 0.1 < 0.15 = \mu_2$.

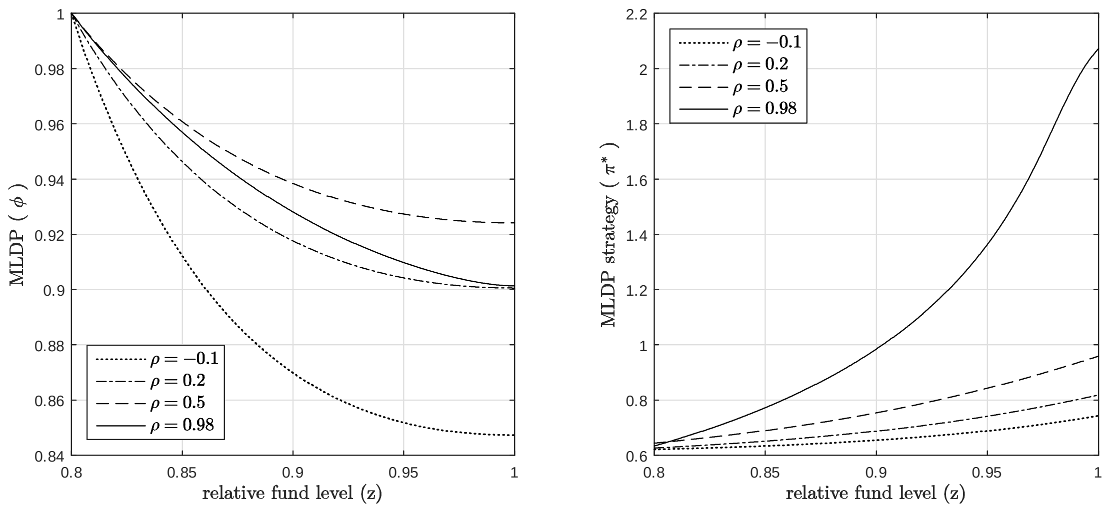


Figure 3.2. Impact of ρ on the MLDP (left) and the MLDP trading strategy (right)

Next, we are interested in studying the impact of the correlation coefficient ρ of the two risky assets on the MLDP and the corresponding optimal trading strategy. We set $\mu_1 = 0.05$, $\mu_2 = 0.3$, $\sigma_1 = 0.2$ and $\sigma_2 = 0.36$ to produce the numerical values of Figure 3.2. We find that neither of these two quantities is necessarily monotone in ρ. In the left plot, we observe that the MLDPs are first increasing and then decreasing in ρ for any $z \in [1-\alpha, 1]$. This shows that a selection of highly correlated assets (ρ close to -1 or 1 in this example) in a portfolio can help reduce the MLDP of the portfolio. In the right plot, we can see that the impact of ρ on the optimal strategy $\pi^*(z)$ is even more complex. However, when $z = 1$, we find that $\pi^*(1)$ is increasing in ρ. This observation can easily be verified from (3.24) as

$$(\sigma_2 - \sigma_1)\frac{\partial \pi^*(1)}{\partial \rho} = \frac{\sigma_1\sigma_2(\sigma_2-\sigma_1)^2(\sigma_2+\sigma_1)}{(\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2)^2} \geq 0.$$

Note that we choose $\sigma_2 = 0.36 > 0.2 = \sigma_1$.

3.3 Market model II

In this section, we examine the second market model consisting of a risk-free asset with constant interest rate $r > 0$ and a risky asset governed by a geometric Brownian motion with dynamics

$$\mathrm{d}S_t = \mu S_t \mathrm{d}t + \sigma S_t \mathrm{d}B_t, \quad S_0 > 0,$$

where $\mu \in \mathbb{R}$, $\sigma > 0$, and $\{B_t\}_{t\geq 0}$ is a standard Brownian motion defined on $(\Omega, \mathcal{F}, \boldsymbol{F}, \mathbb{P})$. To avoid triviality, a proportional management fee with rate $r < \eta < 1$ is continuously deducted from the fund. Therefore, for an admissible strategy $\pi \in \Pi$ representing *the fraction of wealth invested in the risky asset*, the dynamics of the fund value process W^π is then given by

$$\begin{aligned}\mathrm{d}W_t^\pi &= \pi_t W_t^\pi \frac{\mathrm{d}S_t}{S_t} + (1-\pi_t) r W_t^\pi \mathrm{d}t - \eta W_t^\pi \mathrm{d}t \\ &= (\pi_t(\mu - r) + r - \eta)\, W_t^\pi \mathrm{d}t + \pi_t W_t^\pi \sigma \mathrm{d}B_t, \end{aligned} \tag{3.28}$$

with initial value $W_0 = w > 0$.

At first glance, one may view market model II as a limiting case of market model I by letting $\sigma_2 \to 0$ and $\mu_2 = r$. However, as will be shown, the treatment of these two models and the associated HJB equations are structurally different. First, it is not obvious to find the limit of the MLDP (3.12) and the optimal strategy (3.13) by letting $\sigma_2 \to 0$ given that the form of h^{-1} is not fully explicit. Also, even if an explicit limit exists, the continuity of the MLDP and the optimal strategy w.r.t. σ_2 at $0+$ needs to be justified. Second, a major difference in the analysis of market model II is that we shall first narrow down the candidate pool of the optimal trading strategy. Interestingly, this intuition is based on some observations we made under market model I.

3.3.1 Verification theorem

We define a differential operator $\tilde{\mathcal{L}}^{\beta}$ ($\beta \in \mathbb{R}$) as

$$\tilde{\mathcal{L}}^{\beta} f = (\beta(\mu - r) + r - \eta)\, x f_x + \frac{1}{2}\beta^2\sigma^2 x^2 f_{xx} - \lambda f,$$

where f is a twice-differentiable function in x. Then we decompose the admissible set of trading strategies Π as

$$\Pi = \Pi_0 \cup \Pi_1,$$

where $\Pi_0 = \{\pi \in \Pi : \pi_t = 0 \text{ a.s. on } (M_t^{\pi} = W_t^{\pi})\}$ and $\Pi_1 = \Pi \backslash \Pi_0$. Therefore, Π_0 is the set of admissible strategies which has no risky investment whenever the associated fund value is at its running maximum. For any $\pi \in \Pi_0$, due to the absence of diffusion component when the fund value process reaches its running maximum and the negative drift $r - \eta$ of the value process at that moment, a new running maximum of the associated value process W^{π} will never occur, i.e.,

$$\mathrm{d}M_t^{\pi} = 0 \text{ a.s. for any } \pi \in \Pi_0 \text{ and } t > 0. \tag{3.29}$$

A verification theorem for the MLDP and the optimal trading strategy of market model II is given below.

Theorem 3.3.1 *Suppose that $f : D \to (0, 1]$ satisfies the following conditions:*

(1) For any fixed $m > 0$, $f(\cdot, m) \in C^2([m\,(1-\alpha)\,, m])$ is strictly decreasing and strictly convex;

(2) For any fixed $w > 0$, $f(w, \cdot) \in C^1\left([w, w/\,(1-\alpha)]\right)$ is strictly increasing;

(3) For any fixed $m > 0$ and $\beta \in \mathbb{R}$, $\mathcal{L}^{\beta} f(\cdot, m) \geq 0$ for $w \in [m\,(1-\alpha)\,, m]$;

(4) For any fixed $m > 0$, there exists an admissible strategy $\pi^* : D \to \mathbb{R}$ such that $\pi^* \in \Pi_0$ and $\mathcal{L}^{\pi^*} f(\cdot, m) = 0$ for $w \in [m\,(1-\alpha)\,, m]$;

(5) For any $m > 0$, $f(m(1-\alpha), m) = 1$;

Then $f(w,m) = \psi(w,m)$ on D, where $\psi(w,m)$ is the MLDP defined in (3.3), and π^* is the corresponding optimal trading strategy.

Proof. Suppose that $f : D \to (0,1]$ satisfies conditions (1)-(5) of Theorem 3.3.1 and $\pi^* \in \Pi_0$ is an admissible strategy satisfying condition (4). By condition (2) and the fact that M_t^π is a non-decreasing process, we know that $f_m(W_t^\pi, M_t^\pi)\mathrm{d}M_t^\pi \geq 0$ a.s.. Along the same lines as in the proof of Theorem 3.2.1, one can see that (7.2) still holds for all $\pi \in \Pi$. Moreover, by $\pi^* \in \Pi_0$ and (3.29), the equality holds in (7.2) for π^*. Using the same arguments as the rest of the proof of Theorem 3.2.1, we complete the proof of Theorem 3.3.1. ■

Similar as (3.8), the dimension of f in Theorem 3.3.1 can be reduced by considering

$$f(w,m) = f\left(\frac{w}{m}, 1\right) := g\left(\frac{w}{m}\right), \quad 1-\alpha \leq \frac{w}{m} \leq 1,$$

which immediately yields the following corollary.

Corollary 3.3.1 *Suppose that $g : [1-\alpha, 1] \to (0,1]$ satisfies the following conditions:*

(1) $g(\cdot) \in C^2([1-\alpha,1])$ is strictly decreasing and strictly convex;

(2) $\tilde{\mathcal{L}}^\beta g(z) \geq 0$ for any $\beta \in \mathbb{R}$ and $z \in [1-\alpha, 1]$;

(3) There exists an admissible strategy $\pi^* : [1-\alpha, 1] \to \mathbb{R}$ such that $\pi^* \in \Pi_0$ and $\tilde{\mathcal{L}}^{\pi^*} g(z) = 0$ for $z \in [1-\alpha, 1]$;

(4) $g(1-\alpha) = 1$;

(5) $\lim_{z\uparrow 1} g''(1) = \infty$ if $\mu \neq r$.

Then $g(z) = \phi(z) := \inf_{\pi\in\Pi} \mathbb{E}^{w,m}\left[e^{-\lambda\tau_\alpha^\pi}\right]$ for $z = \frac{w}{m} \in [1-\alpha, 1]$, and π^* is the corresponding optimal trading strategy.

In comparison to Corollary 3.2.1, the presence of the two new conditions $\pi^* \in \Pi_0$ and $\lim_{z\uparrow 1} g''(z) = \infty$ if $\mu \neq r$ may appear abrupt. However, both conditions are in agreement with conclusions reached under market model I. First, the condition $\pi^* \in \Pi_0$ is consistent with the conclusion that the MLDP strategy is identical to the MV strategy when the portfolio value is at its running maximum. On the other hand, one can argue $\pi^* \notin \Pi_1$. Otherwise, by (3.29), we should have $\mathbb{P}\{M_t^{\pi^*} > m$ for some $t > 0\} > 0$, which further implies that $g'(1) = 0$ from the proof of Theorem 3.2.2. Moreover, by the first-order condition, we have

$$\pi^*(z) = \begin{cases} -\frac{\mu-r}{\sigma^2}\frac{g'(z)}{zg''(z)}, & \text{if } \mu \neq r, \\ 0, & \text{if } \mu = r. \end{cases} \tag{3.30}$$

Substituting (3.30) into the equation $\tilde{\mathcal{L}}^{\pi^*} g(z) = 0$, we obtain the nonlinear equation

$$\frac{(\mu-r)^2}{2\sigma^2}\frac{(g')^2}{g''} + (\eta - r)zg' + \lambda g = 0, \quad z \in [1-\alpha, 1]. \tag{3.31}$$

However, by the conditions of Corollary 3.3.1, we have

$$\frac{(\mu-r)^2}{2\sigma^2}\frac{(g'(1))^2}{g''(1)} + (\eta - r)zg'(1) + \lambda g(1) \geq \lambda g(1) > 0,$$

which contradicts (3.31). Therefore, we deduce $\pi^* \in \Pi_0$ and $g'(1) \neq 0$, which further implies that $\lim_{z\uparrow 1} g''(z) = \infty$ if $\mu \neq r$ by (3.30).

Remark 3.3.1 *Under market model II, we have $\pi^* \in \Pi_0$, which implies that the fund value process will never reach a new running maximum by (3.29). Intuitively speaking, this conclusion is consistent with the fact that the objective function of the MLDP problem (3.3) only penalizes downside risk and does not offer incentives to* *reach a new running maximum. As shown in Proposition 3.3.1 and Figure 3.3 later, the MLDP strategy becomes more conservative as the fund value increases. As such, since $\eta > r$, when the fund value recovers its running maximum, it is preferable to invest all in the risk-free asset (even if the instantaneous return rate of the portfolio is negative) rather than "gamble" by investing a nonzero proportion of the portfolio in the risky asset and increase the exposure to substantial drawdowns.*

3.3.2 MLDP and optimal trading strategy

By (3.31) and Corollary 3.3.1, we only need to find a positive, strictly decreasing, strictly convex, and $C^2([1-\alpha,1])$ solution to the following nonlinear equation

$$\begin{cases} \frac{(\mu-r)^2}{2\sigma^2}\frac{(g')^2}{g''} + (\eta - r)zg' + \lambda g = 0, \quad z \in [1-\alpha, 1], \\ g(1-\alpha) = 1, \\ \lim_{z\uparrow 1} g''(z) = \infty, \text{ if } \mu \neq r. \end{cases} \tag{3.32}$$

Theorem 3.3.2 *Under market model II, the MLDP and its corresponding optimal trading strategy are respectively given by*

$$\phi(z) = \begin{cases} \exp\left(-\int_{\tilde{h}^{-1}(z)}^{\tilde{h}^{-1}(1-\alpha)} \frac{x}{\tilde{k}(x)}\mathrm{d}x\right), & \text{if } \mu \neq r, \\ \left(\frac{1-\alpha}{z}\right)^{\lambda/(\eta-r)}, & \text{if } \mu = r, \end{cases} \tag{3.33}$$

and

$$\pi^*(z) = \begin{cases} \frac{2}{\mu - r}\left(\eta - r + \frac{\lambda}{\tilde{h}^{-1}(z)}\right), & \text{if } \mu \neq r, \\ 0, & \text{if } \mu = r, \end{cases} \tag{3.34}$$

for $z \in [1-\alpha, 1]$, where

$$\tilde{k}(x) := -\frac{(\mu-r)^2}{2\sigma^2}\frac{x^2}{(\eta-r)x+\lambda} + x - x^2 \text{ and } \tilde{h}(v) := \exp\left(-\int_v^{-\lambda/(\eta-r)} \frac{1}{\tilde{k}(x)}\mathrm{d}x\right),$$

for $v \in (\tilde{v}_*, -\lambda/(\eta-r)]$ with $\tilde{v}_* = \frac{\eta-r-\lambda-\frac{(\mu-r)^2}{2\sigma^2}-\sqrt{\left(\eta-r-\lambda-\frac{(\mu-r)^2}{2\sigma^2}\right)^2+4\lambda(\eta-r)}}{2(\eta-r)}$. Furthermore, $\phi(\cdot), \pi^*(\cdot) \in C^\infty([1-\alpha,1])$.

Proof. For the simple case $\mu = r$, the solution to (3.32) is easily found to be $g(z) = \left(\frac{1-\alpha}{z}\right)^{\lambda/(\eta-r)}$ for $z \in [1-\alpha, 1]$. By Corollary 3.3.1, one concludes that $g(\cdot) = \phi(\cdot)$.

For the case $\mu \neq r$, similar to the proof of Theorem 3.2.2, we consider the following equation:

$$\begin{cases} \frac{(\mu-r)^2}{2\sigma^2}\frac{(G')^2}{G''} + (\eta - r)zG' + \lambda G = 0, \quad z \in (0,1], \\ G(1) = 1, \\ \lim_{z\uparrow 1} G''(z) = \infty, \\ G'(z) < 0, \quad z \in (0,1]. \\ G''(z) > 0, \quad z \in (0,1]. \end{cases} \tag{3.35}$$

We show that (3.35) admits a unique solution with $G \in C^\infty((0,1])$. First, substituting the auxiliary functions $u(\cdot)$ and $v(\cdot)$ defined in (3.15) into the first equation of (3.35) yields $\frac{(\mu-r)^2}{2\sigma^2}u = -\frac{\lambda}{v} - (\eta - r)$. This together with (3.18) leads to

$$zv' = \frac{v}{u} + v - v^2 = -\frac{(\mu-r)^2}{2\sigma^2}\frac{v^2}{(\eta-r)v+\lambda} + v - v^2 := \tilde{k}(v). \tag{3.36}$$

Note that $\tilde{k}(v) \in C^\infty((-\infty, -\lambda/(\eta-r)))$ with $\lim_{v\uparrow -\lambda/(\eta-r)} \tilde{k}(v) = \infty$ and $\lim_{v\downarrow -\infty} \tilde{k}(v) = -\infty$. Hence, we denote by

$$\begin{aligned} \tilde{v}_* &:= \sup\left\{x < -\lambda/(\eta-r) : \tilde{k}(x) = 0\right\} \\ &= \frac{\eta - r - \lambda - \frac{(\mu-r)^2}{2\sigma^2} - \sqrt{\left(\eta - r - \lambda - \frac{(\mu-r)^2}{2\sigma^2}\right)^2 + 4\lambda(\eta-r)}}{2(\eta-r)}. \end{aligned}$$

By (3.35), it is easy to see that $v(1) = G'(1) = -\lambda/(\eta-r)$. Moreover, by (3.36) and using the relation $z'(v)v'(z) = 1$, we obtain

$$z(v) = \tilde{h}(v) := \exp\left(-\int_v^{-\lambda/(\eta-r)} \frac{1}{\tilde{k}(x)}\mathrm{d}x\right), \quad v \in (\tilde{v}_*, -\lambda/(\eta-r)]. \tag{3.37}$$

Now, by (3.37), let $H(v) := G(\tilde{h}(v)) = G(z)$. It follows from the second relation of (3.15) and (3.36) that $H(v)$ is the solution to the following equation

$$\begin{cases} \frac{\mathrm{d}H}{\mathrm{d}v} = \frac{\mathrm{d}G}{\mathrm{d}z}\frac{\mathrm{d}z}{\mathrm{d}v} = \frac{vG(z)}{z}\frac{z}{\tilde{k}(v)} = \frac{v}{\tilde{k}(v)}H(v), \quad v \in (v_*, -\lambda/(\eta - r)], \\ H(-\lambda/(\eta - r)) = G(1) = 1. \end{cases}$$

Solving the above initial value problem, we have

$$G(z) = \exp\left(-\int_{v(z)}^{-\lambda/(\eta-r)} \frac{x}{\tilde{k}(x)}\mathrm{d}x\right) = \exp\left(-\int_{\tilde{h}^{-1}(z)}^{-\lambda/(\eta-r)} \frac{x}{\tilde{k}(x)}\mathrm{d}x\right) \in C^\infty((0,1]).$$

Finally, letting

$$g(z) := \frac{G(z)}{G(1-\alpha)} = \exp\left(-\int_{\tilde{h}^{-1}(z)}^{\tilde{h}^{-1}(1-\alpha)} \frac{x}{\tilde{k}(x)}\mathrm{d}x\right), \quad z \in [1-\alpha, 1], \tag{3.38}$$

it is straightforward to verify that $g(\cdot)$ satisfies all the conditions of Corollary 3.3.1, which ends the proof of (3.33). By differentiating (3.38) and further using (3.30), we obtain the optimal strategy π^* given in (3.34). π^* is bounded in $[1-\alpha, 1]$ since it is continuous in $[1-\alpha, 1]$ and $[1-\alpha, 1]$ is a compact set. Moreover, it is straightforward to see that $\pi^*(1) = 0$. Thus, $\pi^* \in \Pi_0$. This completes the proof. ■

The proof of the following proposition is similar to Proposition 3.2.1, and hence is omitted.

Proposition 3.3.1 *Under market model II, for $\mu \neq r$, we have*

$$(\mu - r)\frac{\mathrm{d}\pi^*}{\mathrm{d}z} < 0, \quad z \in [1-\alpha, 1].$$

By Theorem 3.3.2 and Proposition 3.3.1, the following implications of market model I also hold under market model II:

1. At high-water mark (i.e. $\pi^* \in \Pi_0$ or equivalently $\pi^*(1) = 0$), the MLDP strategy (3.34) is consistent with the MV strategy.

2. When the drawdown level increases, the MLDP strategy tends to increase the proportion invested in the asset with a higher return rate.

3. Similarly as in (3.25), it is easy to verify that the instantaneous return rate of the MLDP portfolio is never less than the return rate of the MV portfolio.

3.3.3 Numerical examples

We numerically implement the main results of Section 3 by first conducting a sensitivity analysis on the management fee rate η. For this purpose, we let $\alpha = 0.2$, $\lambda = 0.05$, $\mu = 0.12$, $\sigma = 0.12$ and $r = 0.05$. The numerical values of the MLDPs and the corresponding optimal trading strategies can be found in Figure 3.3 for various η values.

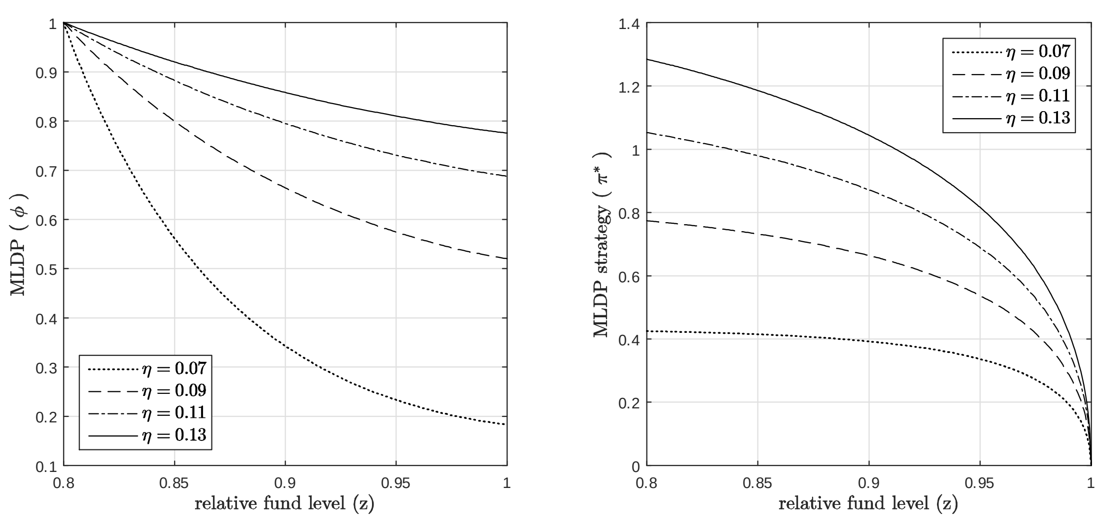


Figure 3.3. Impact of η on the MLDP (left) and the MLDP trading strategy (right)

For a fixed η, one can see that the MLDP satisfies all the conditions of Corollary 3.3.1. In particular, we see that $\phi'(1) < 0$, which is different from market model I (condition 5 of Corollary 3.2.1). For the optimal trading strategy, as $\mu > r$, we find π^* is decreasing in z which is consistent with Proposition 3.3.1. Moreover, we see that $\pi^*(1) = 0$ which satisfies condition (3) of Corollary 3.3.1. As for the impact of η, not surprisingly, we find that both the MLDP and the optimal trading strategy are increasing in η, i.e., a high management fee will incur a higher drawdown probability and result in a more aggressive investment strategy.

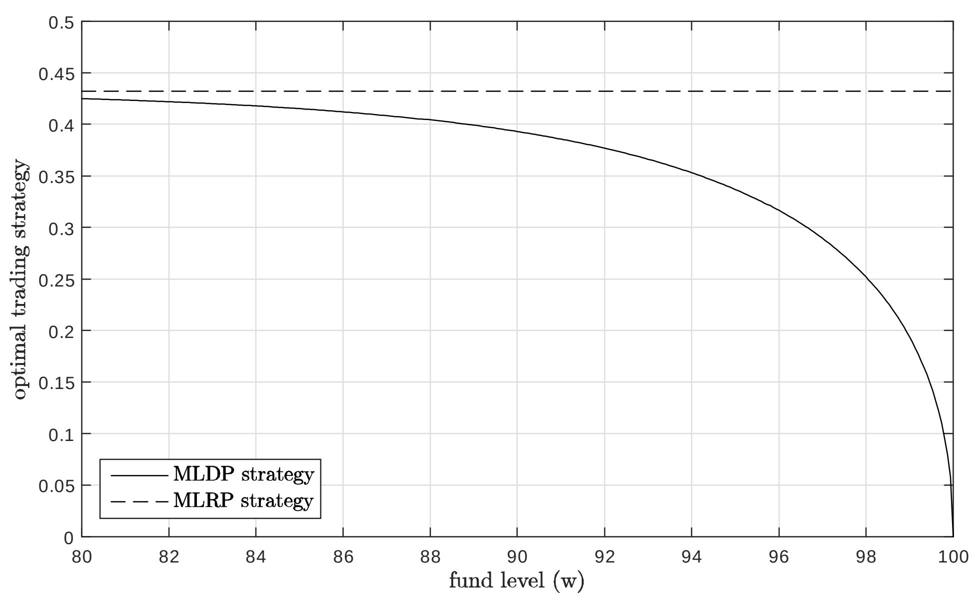


Figure 3.4. The MLDP and MLRP trading strategies

In Figure 3.4, we are interested in comparing the MLDP strategy with the MLRP strategy $\breve{\pi}$ of Young [88]. We recall that the MLRP strategy is a constant proportional strategy given by

$$\breve{\pi} = \frac{\mu - r}{\sigma^2(1 - \tilde{v}_*)}.$$

In Figure 3.4, we use the same parameter setting as in Figure 3.3 except we choose $\eta = 0.07$, the floored maximum $m = 100$, and the ruin level $w_r = 80$. We see that the MLDP strategy is always more conservative than the MLRP strategy. In fact, with some calculations, one can verify from (3.34) and Proposition 3.3.1 that

$$\pi^*(z) < \lim_{z \downarrow 0} \pi^*(z) = \frac{2}{\mu - r}\left(\eta - r + \frac{\lambda}{\tilde{v}_*}\right) = \breve{\pi}, \quad 0 < z \leq 1.$$

This relation is also proved in Theorem 3.2 of Angoshtari et al. [5]. Intuitively, this is because, for any admissible strategy, the first drawdown time of the associated wealth process always occurs before (or equal to) the ruin time. To prevent the occurrence of an earlier stopping time, an investor tends to adopt a more conservative strategy.

Chapter 4

Maximizing a Fund Manager's Income under Drawdown-based Penalties

4.1 Introduction

Fund managers implement investment strategies for investors and receive income (usually in the form of *financial compensation*) for providing the professional financial service. In most contracts, the fund manager's income structure is designed to be performance dependent and consistent with the main objectives of the fund under management (and hence, its investors' risk profile). This income structure varies quite significantly among investment products and can sometimes be customized for large investors. Subject to a given income structure, a natural question arises: *what (trading) strategy should a fund manager adopt to maximize his/her expected cumulative income over the investment period?* In this chapter, we propose to make use of the concept of *drawdown* to design a manager's income structure, and provide an answer to the above question. Indeed, drawdown is a natural risk metric to use in the design of a manager's income structure for similar reasons discussed in the introduction of the thesis. We are thus motivated to develop a dynamic drawdown-based income structure for a fund manager, and study the manager's optimal trading strategy under a cumulative income maximization objective.

We consider a financial market model consisting of a risk-free asset with constant interest rate $r > 0$ and a risky asset governed by a geometric Brownian motion with dynamics

$$\mathrm{d}S_t = \mu S_t \mathrm{d}t + \sigma S_t \mathrm{d}B_t, \quad S_0 > 0,$$

where $\mu \in \mathbb{R} > r$, $\sigma > 0$, and $\{B_t\}_{t\geq 0}$ is a standard Brownian motion defined on a filtered complete probability space $(\Omega, \mathcal{F}, \boldsymbol{F} = \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ satisfying usual conditions. A *proportional* consumption at rate $\eta \in (r, 1)$ is continuously deducted from the fund by the investor. We denote by $\pi = \{\pi_t\}_{t\geq 0}$ an $\boldsymbol{F}$-progressively measurable trading strategy, where π_t represents the *fraction* of fund invested in the risky asset at time t. The associated fund value process is denoted by $W^\pi = \{W_t^\pi\}_{t\geq 0}$, whose dynamics is given by

$$\begin{aligned} \mathrm{d}W_t^\pi &= (1-\pi_t) W_t^\pi r \mathrm{d}t + \pi_t W_t^\pi \frac{\mathrm{d}S_t}{S_t} - \eta W_t^\pi \mathrm{d}t \\ &= (r - \eta + (\mu - r)\pi_t) W_t^\pi \mathrm{d}t + \sigma \pi_t W_t^\pi \mathrm{d}B_t, \end{aligned} \tag{4.1}$$

with initial value $W_0 = w > 0$. We assumed that 0 is an absorbing state, i.e., $W_t^\pi = 0$, $\forall t \geq \inf\{s : W_s \leq 0\}$. We define the (floored) running maximum of the fund value at time t by

$$M_t^\pi = \max\left\{\sup_{0\leq s\leq t} W_s^\pi, m\right\}$$

with $m \geq w$. The ratios $(M_t^\pi - W_t^\pi)/M_t^\pi$ and W_t^π/M_t^π are respectively called the *relative drawdown level* and the *relative fund level* at time t.

The main objective of this chapter is to study the problem of maximizing a fund manager's cumulative income under drawdown based penalty schemes over a lifetime investment, i.e.,

$$v(w,m) = \sup_{\pi\in\Pi} \mathbb{E}^{w,m}\left[\int_0^{e_\lambda\wedge\tau_\alpha^\pi}\left(s_0 - \zeta\left(\frac{M_t^\pi - W_t^\pi}{M_t^\pi}\right)\right)\mathrm{d}t\right], \tag{4.2}$$

where s_0 is the ceiling rate at which the fund manager is paid; $\zeta : [0,1) \to \mathbb{R}$ is an increasing function representing the penalty on the relative drawdown level;

$$\tau_\alpha^\pi = \inf\left\{t \geq 0 : \frac{M_t^\pi - W_t^\pi}{M_t^\pi} > \alpha\right\}$$

is the first time the fund's relative drawdown level exceeds $\alpha \in (0,1]$; e_λ represents a random investment maturity (which can be triggered by exogenous factors, e.g., death) which is assumed to be an $\mathcal{F}$-measurable exponentially distributed random variable with mean $1/\lambda > 0$, independent of the fund value process; Π is the set of admissible trading strategies defined as

$$\Pi = \left\{\pi : \pi \text{ is } \boldsymbol{F}\text{-progressively measurable and } \int_0^t \pi_s^2 \mathrm{d}s < \infty \text{ for any } t \geq 0\right\}. \tag{4.3}$$

In particular, when $\alpha = 1$, since $W_t^\pi > 0$ for any $t \geq 0$, by (4.1) and (4.3), we have $\tau_\alpha^\pi = \inf\{\emptyset\} = \infty$ by convention.

Objective (4.2) generalizes two problems in the literature. First, by choosing $\zeta(\cdot) \equiv 0$, objective (4.2) is equivalent to minimizing the lifetime drawdown probability (MLDP), namely

$$\inf_{\pi\in\Pi} \mathbb{P}^{w,m}\left(\tau_\alpha^\pi < e_\lambda\right),$$

which was studied by Angoshtari et al. [4] and Chen et al. [39]. Second, by choosing $\zeta(\cdot) = \mathbf{1}_{\{\cdot\geq\beta\}}$ for some constant $\beta \in (0,1)$, and letting $\alpha = 1$ (or equivalently $\tau_\alpha^\pi = \infty$), objective (4.2) reduces to minimizing the cumulative time the fund's relative drawdown level exceeds β, namely

$$\inf_{\pi\in\Pi}\mathbb{E}^{w,m}\left[\int_0^{e_\lambda}\mathbf{1}_{\left\{\frac{M_t^\pi-W_t^\pi}{M_t^\pi}\geq\beta\right\}}\mathrm{d}t\right],$$

which was studied by Angoshtari et al. [5].

Since problem (4.2) does not admit a closed-form solution for a general penalty function ζ, we focus on the following two problems with specific drawdown penalty schemes:

$$v_1(w,m)=\sup_{\pi\in\Pi}\mathbb{E}^{w,m}\left[\int_0^{e_\lambda\wedge\tau_\alpha^\pi}\left(s_0-s_1\mathbf{1}_{\left\{\frac{M_t^\pi-W_t^\pi}{M_t^\pi}\geq\beta\right\}}\right)\mathrm{d}t\right],\quad 0<\beta<\alpha<1,\tag{4.4}$$

and

$$v_2(w,m)=\sup_{\pi\in\Pi}\mathbb{E}^{w,m}\left[\int_0^{e_\lambda}\left(s_0-s_1\left(\frac{M_t^\pi-W_t^\pi}{M_t^\pi}-\beta\right)_+\right)\mathrm{d}t\right],\quad 0<\beta<1,\tag{4.5}$$

where $s_1\in(0,s_0)$ is the penalty rate and $(x)_+=\max\{x,0\}$.

For objective (4.4), referred as the *constant penalty scheme*, the manager receives an income rate of s_0 whenever the relative drawdown level is less than β. When the fund's drawdown level exceeds β (but is less than α), the manager's income rate is reduced to s_0-s_1. If the fund's drawdown level ever reaches level α before the random maturity e_λ, the manager's operation of the fund stops (together with the income rate). For objective (4.5), referred as the *linear penalty scheme*, the manager's income rate is still s_0 whenever the relative drawdown level is below β. When the drawdown level exceeds β, the penalty is linearly proportional to the excess drawdown level over β, and hence the manager receives income at rate $s_0-s_1\left(\frac{M_t^\pi-W_t^\pi}{M_t^\pi}-\beta\right)$. The fund operation is only terminated at the random maturity e_λ. Roughly speaking, the constant penalty scheme penalizes the maximum drawdown, which is in the spirit of Calmar ratio, and the linear penalty scheme penalizes the average drawdown which is in the spirit of Sterling ratio.

The main implications of this chapter are summarized as follows. First, under both penalty schemes, the MCI strategy in the non-penalty region coincides with the minimum lifetime drawdown probability (MLDP) strategy derived in Chen et al. [39] under market model II, i.e., the only objective of the manager is to minimize the probability of triggering a penalty, independent of the magnitude of the income penalty rate. Second, under the constant penalty scheme, the MCI strategy in the penalty region could exhibit different behaviors as the fund level increases depending on which of the following two effects prevails: maintaining the fund risk at an acceptable level by investing conservatively to avoid large drawdowns and leaving the penalty region by investing more aggressively. Third, under the linear penalty scheme without the terminating drawdown level, we find that the manager always becomes more conservative as the fund level increases, as the effect of higher drawdown penalties dominates.

The rest of the chapter is organized as follows. In Section 4.2, we provide a verification theorem for the general drawdown-based penalty schemes. Sections 4.3 and 4.4 consider the two specific penalty schemes detailed above (the constant penalty scheme and linear penalty scheme, respectively) in great length. By applying a dual approach, we obtain closed-form expressions for the MCI and the MCI strategy in both cases. Numerical examples are provided at the end of each section to complement the analytic results. Some technical proofs are deferred to the Appendix.

4.2 Verification Theorem

We define a region

$$D = \left\{(w, m) \in \mathbb{R}^2 : \alpha m \leq w \leq m \text{ and } m > 0\right\},$$

and subsequently a differential operator $\mathcal{L}^{\pi}$ $(\pi \in \mathbb{R})$ as

$$\mathcal{L}^{\pi} f(w,m) = [r - \eta + \pi(\mu - r)]\, w f_w + \frac{1}{2}\sigma^2\pi^2 w^2 f_{ww} - \lambda f + s_0 - \zeta\left(\frac{m-w}{m}\right),$$

where f is a twice-differentiable function in w with $f_w := \frac{\partial f}{\partial w}$ and $f_{ww} := \frac{\partial^2 f}{\partial w^2}$.

We first give a verification theorem for the general objective (4.2). The proof of the verification theorem is postponed to the Appendix.

Theorem 4.2.1 *Suppose that $f : D \to \mathbb{R}$ is bounded and satisfies the following conditions:*

(1) $\forall m > 0$, $f(\cdot, m) \in C^2[(1-\alpha)m, m]$ ($f(\cdot, m) \in C^1$ at (finitely many) points where $\zeta(1 - \cdot/m)$ is discontinuous) is strictly increasing and strictly concave;

(2) $\forall w > 0$, $f(w, \cdot) \in C^1[w, \infty)$ is strictly decreasing;

(3) $\forall m > 0$ and $\pi \in \mathbb{R}$, $\mathcal{L}^{\pi} f(\cdot, m) \leq 0$ for $w \in [(1-\alpha)m, m]$;

(4) $\forall m > 0$, there exists an admissible strategy

$$\pi^* := \operatorname*{argmax}_{\pi \in \mathbb{R}} \{\mathcal{L}^{\pi} f(\cdot, m)\}$$

such that $\mathcal{L}^{\pi^*} f(\cdot, m) = 0$, $\forall w \in [(1-\alpha)m, m]$;

(5) $f(m(1-\alpha), m) = 0$ if $\alpha \in (0,1)$; $f(0,m) = \frac{s_0 - \zeta(1)}{\lambda}$ if $\alpha = 1$;

(6) $\lim_{w \to m^-} f_{ww}(w,m) = -\infty$;

Then $f = v$ on D, where v is the value function of objective (4.2), and π^* is the corresponding optimal trading strategy.

Let f be the function satisfying all the conditions of Theorem 4.2.1. It is not difficult to see that $f(cw, cm) = f(w,m)$ for any constant $c > 0$. This scaling relationship implies that we can reduce the dimension of f by considering

$$f(w,m) = f\left(\frac{w}{m}, 1\right) := g\left(\frac{w}{m}\right) = g(z), \quad 1 - \alpha \leq z := \frac{w}{m} \leq 1,$$

where the ratio w/m is the relative fund level. Using the change of variable formulas $f_w = \frac{1}{m} g_z$, $f_{ww} = \frac{1}{m^2} g_{zz}$, and $f_m = -\frac{w}{m^2} g_z$, the following corollary is a natural consequence of Theorem 4.2.1.

Corollary 4.2.1 *Suppose that $g : [1-\alpha, 1] \to \mathbb{R}$ is bounded and satisfies the following conditions:*

(1) $g(\cdot) \in C^2[1-\alpha, 1]$ ($g(\cdot) \in C^1$ at points where $\zeta(1-\cdot)$ is discontinuous) is strictly increasing and strictly concave;

(2) $g(\cdot)$ is a solution to

$$\begin{cases} (r-\eta) z g_z(z) - \delta \frac{g_z^2(z)}{g_{zz}(z)} - \lambda g(z) + s_0 - \zeta(1-z) = 0, \quad z \in (1-\alpha, 1), \\ \lim_{z\uparrow 1} g_{zz}(z) = -\infty, \\ g(1-\alpha) = 0, \ \textit{if } \alpha \in (0,1); \ g(0) = \frac{s_0 - \zeta(1)}{\lambda} \ \textit{if } \alpha = 1, \end{cases}$$

where $\delta = \frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2$. Then $g(z) = v(z) := \sup_{\pi\in\Pi} \mathbb{E}^{w,m}[\int_0^{e_\lambda \wedge \tau_\alpha^\pi} (s_0 - \zeta(\frac{M_t^\pi - W_t^\pi}{M_t^\pi}))\mathrm{d}t]$ and the optimal trading strategy is given by

$$\pi^*(z) = -\frac{\mu - r}{\sigma^2} \frac{g_z(z)}{z g_{zz}(z)},$$

for $z = \frac{w}{m} \in [1-\alpha, 1]$.

4.3 Constant Penalty Scheme

This section is devoted to the analysis of objective (4.4) under the constant penalty scheme. By Corollary 4.2.1 with $\zeta(x) = s_1 \mathbf{1}_{\{x \geq \beta\}}$, the associated HJB equation is given by

$$\begin{cases} (r-\eta) z g_z(z) - \delta \frac{g_z^2(z)}{g_{zz}(z)} - \lambda g(z) + s_0 - s_1 \mathbf{1}_{\{1-z\geq\beta\}} = 0, \quad z \in (1-\alpha, 1), \\ \lim_{z\uparrow 1} g_{zz}(z) = -\infty, \ g(1-\alpha) = 0. \end{cases} \tag{4.6}$$

4.3.1 MCI and Optimal Strategy

Rather than directly solving (4.6), we first consider the following free boundary problem (FBP):

$$\begin{cases} y^2\hat{\psi}_{yy} + \frac{\lambda+\eta-r}{\delta}y\hat{\psi}_y - \frac{\lambda}{\delta}\hat{\psi} + \frac{s_0}{\delta} - \frac{s_1}{\delta}\mathbf{1}_{\{y\geq y_\beta\}} = 0,\ 0<y_0<y<y_\alpha<\infty, \\ \hat{\psi}_y(y_0) = -1, \hat{\psi}_{yy}(y_0) = 0, \hat{\psi}_y(y_\beta) = \beta - 1, \hat{\psi}_y(y_\alpha) = \alpha - 1, \hat{\psi}(y_\alpha) = (\alpha-1)y_\alpha. \end{cases} \tag{4.7}$$

The solution to the FBP (4.7) is given in Lemma 4.3.1 below. The proof can be found in the Appendix. The main difficulty lies in proving the uniqueness of the solution to equation (4.12) and the convexity of $\hat{\psi}$.

Lemma 4.3.1 *For $y \in [y_0, y_\alpha]$, the solution to the FBP (4.7) is given by*

$$\hat{\psi}(y) = \begin{cases} \frac{(1-\gamma^+)y_0}{(\gamma^+-\gamma^-)\gamma^-}\left(\frac{y}{y_0}\right)^{\gamma^-} + \frac{(1-\gamma^-)y_0}{(\gamma^--\gamma^+)\gamma^+}\left(\frac{y}{y_0}\right)^{\gamma^+} + \frac{s_0}{\lambda},\ y\in[y_0,y_\beta], \\ \left[\frac{(\alpha-1)y_\alpha(\gamma^+-1)}{(\gamma^+-\gamma^-)} - \frac{(s_0-s_1)\gamma^+}{\lambda(\gamma^+-\gamma^-)}\right]\left(\frac{y}{y_\alpha}\right)^{\gamma^-} \\ +\left[\frac{(\alpha-1)y_\alpha(\gamma^--1)}{(\gamma^--\gamma^+)} - \frac{(s_0-s_1)\gamma^-}{\lambda(\gamma^--\gamma^+)}\right]\left(\frac{y}{y_\alpha}\right)^{\gamma^+} + \frac{s_0-s_1}{\lambda} \quad ,\ y\in(y_\beta,y_\alpha], \end{cases} \tag{4.8}$$

where

$$\begin{cases} \gamma^- = \frac{(r+\delta-\lambda-\eta)-\sqrt{(\lambda+\eta-r-\delta)^2+4\delta\lambda}}{2\delta} < 0, \\ \gamma^+ = \frac{(r+\delta-\lambda-\eta)+\sqrt{(\lambda+\eta-r-\delta)^2+4\delta\lambda}}{2\delta} \in (0,1). \end{cases} \tag{4.9}$$

The boundaries y_β, y_0 and y_α are respectively given by

$$y_\beta = \frac{(s_0-s_1)\gamma^+\gamma^-\left(\kappa_\alpha^{\gamma^-} - \kappa_\alpha^{\gamma^+}\right)}{\lambda(\alpha-1)(\gamma^+-1)\gamma^-\kappa_\alpha^{\gamma^--1} - \lambda(\alpha-1)(\gamma^--1)\gamma^+\kappa_\alpha^{\gamma^+-1} - \lambda(\beta-1)(\gamma^+-\gamma^-)}, \tag{4.10}$$

$y_0 = y_\beta/\kappa_0$ and $y_\alpha = y_\beta/\kappa_\alpha$, where κ_0 is the unique solution in $(1,\infty)$ to

$$\frac{1-\gamma^+}{\gamma^+-\gamma^-}x^{\gamma^--1} + \frac{1-\gamma^-}{\gamma^--\gamma^+}x^{\gamma^+-1} = \beta - 1, \tag{4.11}$$

and κ_α is the unique solution in $(0,1)$ to

$$\frac{(\alpha-1)\left[\left(\gamma^{+}-1\right)\gamma^{-}x^{\gamma^{-}-1}-\left(\gamma^{-}-1\right)\gamma^{+}x^{\gamma^{+}-1}\right]-(\beta-1)\left(\gamma^{+}-\gamma^{-}\right)}{\left(s_0-s_1\right)\gamma^{+}\gamma^{-}\left(x^{\gamma^{-}}-x^{\gamma^{+}}\right)}$$
$$=\frac{(\alpha-1)\left[\left(\gamma^{+}-1\right)x^{\gamma^{-}-1}-\left(\gamma^{-}-1\right)x^{\gamma^{+}-1}\right]-\left(\frac{1-\gamma^{+}}{\gamma^{-}}\kappa_0^{\gamma^{-}-1}-\frac{1-\gamma^{-}}{\gamma^{+}}\kappa_0^{\gamma^{+}-1}\right)}{s_1\left(\gamma^{+}-\gamma^{-}\right)+\left(s_0-s_1\right)\left(\gamma^{+}x^{\gamma^{-}}-\gamma^{-}x^{\gamma^{+}}\right)}. \tag{4.12}$$

Furthermore, $\hat{\psi}$ is strictly convex, strictly decreasing, and C^2 in $[y_0, y_\alpha]$ except at $y=y_\beta$, where it is C^1.

Lemma 4.3.2 *For $\hat{\psi}$ defined in (4.8), consider its Legendre transform defined as*

$$\psi(z)=\inf_{y\in[y_0,y_\alpha]}\left\{\hat{\psi}(y)+yz\right\},\ z\in[1-\alpha,1]. \tag{4.13}$$

Then $\psi(z)$ solves (4.6) and

$$\psi(z)=v_1(z):=\sup_{\pi\in\Pi}\mathbb{E}^{w,m}\left[\int_0^{e_\lambda\wedge\tau_\alpha^\pi}\left(s_0-s_1\mathbf{1}_{\left\{\beta<\frac{M_t^\pi-W_t^\pi}{M_t^\pi}\leq\alpha\right\}}\right)\mathrm{d}t\right],$$

for $z=\frac{w}{m}\in[1-\alpha,1]$. Furthermore, ψ is strictly concave, strictly increasing, and C^2 in $[1-\alpha,1]$ except at $z=1-\beta$, where it is C^1.

Proof. Lemma 4.3.1 shows that $\hat{\psi}$ is strictly convex, strictly decreasing, and C^2 in $[y_0, y_\alpha]$ except at $y=y_\beta$, where it is C^1. The property of Legendre transform immediately implies that ψ is strictly increasing, strictly concave, and C^2 except at $z=1-\beta$, where it is C^1. To show $\psi(\cdot)=v_1(\cdot)$, by Corollary 4.2.1, it is only left to verify that $\psi(\cdot)$ solves (4.6).

From the boundary conditions in (4.7) and Lemma 4.3.1, we recall that $\hat{\psi}_y(y_\alpha)=\alpha-1$ and $\hat{\psi}$ is strictly convex on $[y_0,y_\alpha]$. It implies that $\hat{\psi}(y)+y(1-\alpha)$ is decreasing on $[y_0,y_\alpha]$ and attains its infimum at $y=y_\alpha$. Thus

$$\psi(1-\alpha)=\hat{\psi}(y_\alpha)+y_\alpha(1-\alpha)=0, \tag{4.14}$$

and the "dual" of $z=1-\alpha$ is $y=y_\alpha$. Similarly, since $\hat{\psi}_y(y_0)=-1$ and $\hat{\psi}$ is strictly convex on $[y_0,y_a]$, we conclude that $\hat{\psi}(y)+y$ is increasing on $[y_0,y_\alpha]$ and attains its infimum at $y=y_0$. Thus, the dual of $z=1$ is $y=y_0$.

For $z\in(1-\alpha,1)$, by the first-order condition, the optimizer $y^*\in(y_0,y_\alpha)$ of (4.13) solves the equation

$$\hat{\psi}_y(y)=-z. \tag{4.15}$$

By Lemma 4.3.1, we deduce $y^*=I_1(-z)$, where $I_1:=(\hat{\psi})^{-1}$ is the inverse function of $\hat{\psi}_y$. It follows that

$$\psi(z)=\hat{\psi}(y^*)+y^*z=\hat{\psi}(I_1(-z))+I_1(-z)\,z,\quad z\in(1-\alpha,1). \tag{4.16}$$

Taking the first and second order derivatives with respect to z to (4.16) yields

$$\psi_z(z)=y^*=I_1(-z)\ \text{ and }\ \psi_{zz}(z)=-\frac{1}{\hat{\psi}_{yy}(y^*)}=-\frac{1}{\hat{\psi}_{yy}(I_1(-z))}. \tag{4.17}$$

Since the dual of $z=1$ is $y=y_0$, by the second relation of (4.17), we deduce

$$\lim_{z\uparrow 1}\psi_{zz}(z)=-\lim_{y^*\downarrow y_0}\frac{1}{\hat{\psi}_{yy}(y^*)}=-\infty. \tag{4.18}$$

Using (4.14)–(4.18), it is straightforward to verify that $\psi(z)$ solves (4.6). ■

Theorem 4.3.1 *For $z\in(1-\alpha,1]$,*

$$v_1(z)=\begin{cases} C^-\left(\frac{I_1(-z)}{y_\alpha}\right)^{\gamma^-}+C^+\left(\frac{I_1(-z)}{y_\alpha}\right)^{\gamma^+}+\frac{s_0-s_1}{\lambda}-I_1(-z)\,z,\,, & z\in(1-\alpha,1-\beta), \\ D^-\left(\frac{I_1(-z)}{y_0}\right)^{\gamma^-}+D^+\left(\frac{I_1(-z)}{y_0}\right)^{\gamma^+}+\frac{s_0}{\lambda}-I_1(-z)\,z\,, & z\in[1-\beta,1], \end{cases}$$

and the optimal trading strategy is given by

$$\pi_1^*(z)=\begin{cases} \begin{array}{l}\frac{\mu-r}{\sigma^2 z y_\alpha}\Big\{C^-\gamma^-(\gamma^--1)\left(\frac{I_1(-z)}{y_\alpha}\right)^{\gamma^--1} \\ \qquad +C^+\gamma^+(\gamma^+-1)\left(\frac{I_1(-z)}{y_\alpha}\right)^{\gamma^+-1}\Big\}\end{array}, & z\in(1-\alpha,1-\beta), \\ \frac{\mu-r}{\sigma^2 z}\frac{(1-\gamma^+)(\gamma^--1)}{(\gamma^+-\gamma^-)}\left(\left(\frac{I_1(-z)}{y_0}\right)^{\gamma^--1}-\left(\frac{I_1(-z)}{y_0}\right)^{\gamma^+-1}\right), & z\in[1-\beta,1], \end{cases} \tag{4.19}$$

where

$$
\begin{aligned}
C^{-} &= \left[\frac{(\alpha-1)\,y_{\alpha}\,(\gamma^{+}-1)}{(\gamma^{+}-\gamma^{-})}-\frac{(s_0-s_1)\,\gamma^{+}}{\lambda\,(\gamma^{+}-\gamma^{-})}\right],\\
C^{+} &= \left[\frac{(\alpha-1)\,y_{\alpha}\,(\gamma^{-}-1)}{(\gamma^{-}-\gamma^{+})}-\frac{(s_0-s_1)\,\gamma^{-}}{\lambda\,(\gamma^{-}-\gamma^{+})}\right],\\
D^{-} &= \frac{(1-\gamma^{+})\,y_0}{(\gamma^{+}-\gamma^{-})\,\gamma^{-}},\ D^{+}=\frac{(1-\gamma^{-})\,y_0}{(\gamma^{-}-\gamma^{+})\,\gamma^{+}},
\end{aligned}
$$

$I_1 = (\hat{\psi}_y)^{-1}$, *and the function* $\hat{\psi}$ *and the constants* y_0, y_α, y_β, $\gamma^{\pm}$ *are as given in Lemma 4.3.1.*

Proof. The result follows immediately from Corollary 3.2.1, Lemma 4.3.1, and Lemma 4.3.2. Moreover, π_1^* is bounded in $[1-\alpha, 1-\beta]$ and $[1-\beta, 1]$ given that it is continous in each interval and each interval is a compact set. Then, it is straightforward to verify that $\pi_1^* \in \Pi$. ■

Remark 4.3.1 *As we see from (4.19), the MCI strategy in the non-penalty region (i.e., for* $z \in [1-\beta, 1]$*) is independent of the penalty rate* s_1*. Moreover, one can verify that the MCI strategy in the non-penalty region coincides with the MLDP strategy in Theorem 3.2 of Chen et. al. [39]. This implies that, when the fund's relative drawdown level is small, the manager's goal is to minimize the probability that the relative drawdown level reaches level* β*, indifferently of the size of the penalty rate* s_1*.*

Next, we study the behavior of the optimal trading strategy π_1^*.

Proposition 4.3.1 *For* $z \in [1-\beta, 1]$, $\pi_1^*(z)$ *is decreasing. For* $z \in (1-\alpha, 1-\beta)$, $\pi_1^*(z)$ *is decreasing if*

$$(\alpha-1)\,\kappa_{\alpha}^{\gamma^{+}-1} \geq \beta - 1, \tag{4.20}$$

and increasing otherwise.

Proof. By the first relation of (4.17), for $y = I_1(-z) \in (y_0, y_\alpha)$,

$$\frac{\mathrm{d}\pi_1^*}{\mathrm{d}z} = \frac{\mathrm{d}\pi_1^*}{\mathrm{d}y}\frac{\mathrm{d}y}{\mathrm{d}z} = \frac{\mathrm{d}\pi_1^*}{\mathrm{d}y}\frac{\mathrm{d}^2 v_1}{\mathrm{d}z^2}.$$

By Lemma 4.3.2, v_1 is strictly concave on $[1-\alpha, 1]$, which implies $\frac{\mathrm{d}^2 v_1}{\mathrm{d}z^2} \leq 0$. Thus, it remains to determine the sign of $\frac{\mathrm{d}\pi_1^*}{\mathrm{d}y}$. For $y \in (y_0, y_\beta]$, i.e., $z \in [1-\beta, 1)$, differentiating (4.19) and using some algebraic manipulations, we obtain that

$$\frac{\mathrm{d}\pi_1^*}{\mathrm{d}y} \propto \gamma^+ - \gamma^- > 0,$$

where "$\propto$" means the relationship of positive proportional. Similarly for $y \in (y_\beta, y_\alpha)$, i.e., $z \in (1-\alpha, 1-\beta)$, we obtain that

$$\frac{\mathrm{d}\pi_1^*}{\mathrm{d}y} \propto -\left[(\alpha-1)\left(\gamma^+ - 1\right) - \frac{(s_0 - s_1)\,\gamma^+ \kappa_\alpha}{\lambda y_\beta}\right].$$

Clearly, $\frac{\mathrm{d}\pi_1^*}{\mathrm{d}y} \geq 0$ if

$$s_0 - s_1 \geq \frac{(\alpha-1)\left(\gamma^+ - 1\right)\lambda y_\beta}{\gamma^+ \kappa_\alpha}, \tag{4.21}$$

and $\frac{\mathrm{d}\pi_1^*}{\mathrm{d}y} < 0$, otherwise. Finally, replacing y_β on the right-hand side of (4.21) using (4.10), followed by some algebraic manipulations of the resulting inequality leads to (4.20). This completes the proof of Proposition 4.3.1. ■

4.3.2 Numerical Examples

In this subsection, we provide several numerical examples to the main results of Section 4.3.1, and study the sensitivity of the MCI strategy to some key model parameters. Throughout, we assume that $r = 0.03$, $\mu = 0.1$, $\sigma = 0.2$, and $s_0 = 1$. Also, the threshold drawdown levels of $\alpha = 0.5$ (fund termination), and $\beta = 0.2$ (income penalty) are considered.

Example 4.3.1 (MCI and MCI strategy) *In this example, by setting $\lambda = 0.05$, $\eta = 0.1$, and different levels of s_1, we obtain the corresponding MCI (left plot) and MCI strategies (right plot) in Figure 4.1.*

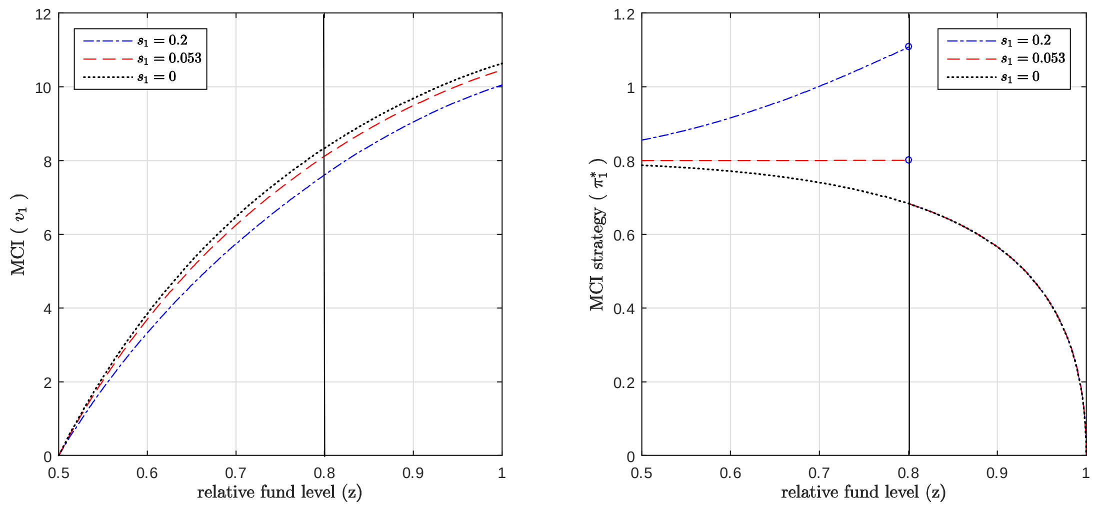


Figure 4.1. MCI (right) and MCI strategy (left)

The left plot shows that the MCI v_1 is increasing and concave. In the right plot, we first observe that

- *in the non-penalty region (i.e., $z \in [0.8, 1]$), the MCI strategy is independent of s_1. Indeed, from Remark 4.3.1, we know that the MCI strategy in the non-penalty region coincides with the MLDP strategy in Chen et al. [39]. As expected, we remark that the manager invests more conservatively as the fund level increases;*
- *in the penalty region (i.e., $z \in [0.5, 0.8]$), the MCI strategy can either be more aggressive or conservative as the fund level increases, depending on the*

penalty rate s_1. This can be explained by the dilemma the manager faces between maintaining the fund risk at an acceptable level by investing conservatively to avoid a drawdown of size α and (possibly) leaving the penalty region by investing more aggressively. In fact, there exists a critical level $s_1^* = 0.053$ such that (4.20) achieves equality. When the manager's income is penalized by s_1^*, the MCI strategy is indifferent to changes in fund level. When the penalty rate $s_1 > (< \text{resp.})\ s_1^*$, the manager has a strong (weak resp.) incentive to leave the penalty region and the investment strategy becomes more aggressive (conservative resp.) as the fund level increases. These implications are consistent with Proposition 4.3.1.

In particular, when $s_1 = 0$, the MCI strategy coincides with the MLDP strategy for any fund level $z \in (1-\alpha, 1]$. When $s_1 > 0$, the MCI strategy is always more aggressive than the MLDP strategy for any fund level $z \in (1-\alpha, 1]$, as the MCI strategy offers an incentive to leave the penalty region.

Example 4.3.2 (Impact of λ) *In this example, by setting $\eta = 0.1$ and different levels of λ, we obtain the MCI strategy in the penalty region with a small penalty* *rate* $s_1 = 0.05$ *(left plot) and a large penalty rate* $s_1 = 0.2$ *(right plot) in Figure 4.2.*

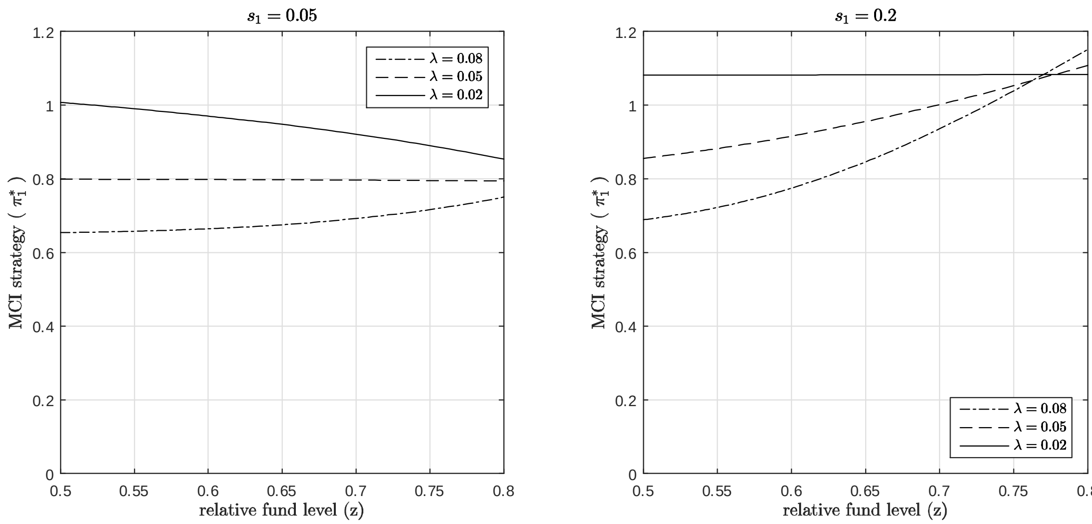


Figure 4.2. Impact of λ *on MCI strategy in the penalty region*

The left plot shows that, when the penalty rate s_1 *is relatively small (i.e.,* $s_1 = 0.05$*), the manager adopts a more aggressive strategy over a longer investment time horizon (i.e., for a smaller* λ*). Indeed, given that the manager's income penalty is rather small, the incentive to leave the penalty region (which is accomplished by adopting a more aggressive trading strategy) increases with the investment time horizon. However, the right plot shows that, for a large penalty rate* s_1*, a MCI strategy with a shorter time horizon may become more aggressive (than its counterpart with a longer time horizon). This is because the magnitude of the income penalty rate* s_1 *creates a strong incentive for the manager to leave the penalty region by investing more aggressively. For fund levels close to the non-penalty region, this incentive to leave the penalty region becomes more pressing for an investment with a shorter time horizon as the manager is also competing against time.*

Example 4.3.3 (Impact of η) *In this example, we study the impact of the consumption rate η on the manager's MCI strategy in the penalty region in Figure 4.3 by setting $\lambda = 0.05$, $\beta = 0.2$, and $s_1 = 0.1$.*

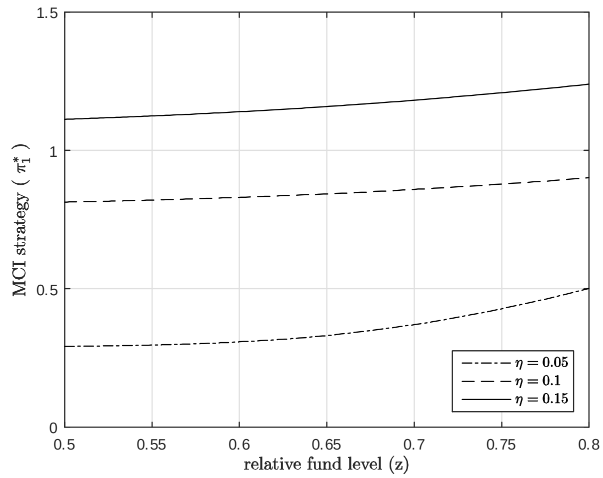


Figure 4.3. Impact of η in the penalty region

Not surprisingly, we see that a larger consumption rate η results in a more aggressive MCI strategy.

4.4 Linear Penalty Scheme

In this section, we parallel the work of Section 4.3 but for the objective function v_2 defined in (4.5). By Corollary 4.2.1 with a penalty of the form $\zeta(x) = s_1 (x - \beta)_+$, the associated HJB equation is given by

$$\begin{cases} (r-\eta)\, z g_z(z) - \delta \frac{g_z^2(z)}{g_{zz}(z)} - \lambda g(z) + s_0 - s_1 \mathbf{1}_{\{1-z>\beta\}} (1 - z - \beta) = 0, \\ \lim_{z \uparrow 1} g_{zz}(z) = -\infty, \ g(0) = \frac{s_0 - s_1(1-\beta)}{\lambda}. \end{cases} \tag{4.22}$$

4.4.1 MCI and Optimal Strategy

Here again, we propose to solve the HJB (4.22) by first considering the following FBP:

$$\begin{cases} y^2\hat{\Psi}_{yy} + \left[\frac{\eta-r+\lambda}{\delta}y - \frac{s_1}{\delta}\mathbf{1}_{\{y>\theta_\beta\}}\right]\hat{\Psi}_y - \frac{\lambda}{\delta}\hat{\Psi} + \frac{s_0-s_1(1-\beta)\mathbf{1}_{\{y>\theta_\beta\}}}{\delta} = 0, \ 0<\theta_0\le y<\infty \\ \hat{\Psi}_{yy}(\theta_0)=0, \ \hat{\Psi}_y(\theta_0)=-1, \ \hat{\Psi}_y(\theta_\beta)=\beta-1, \ \lim_{y\to\infty}\hat{\Psi}(y)=\frac{s_0-s_1(1-\beta)}{\lambda}. \end{cases} \tag{4.23}$$

As we will show, the solution to (4.23) can be expressed in terms of *confluent hypergeometric functions of the first kind* defined as

$$\Phi(j,k;x) = 1 + \sum_{i=1}^{\infty} \frac{(j)_i}{(k)_i}\frac{x^i}{i!}, \tag{4.24}$$

for $x\in\mathbb{R}$, $j\in\mathbb{R}$, and $k\in\mathbb{R}\backslash\mathbb{Z}^-$, where $(j)_i = j(j+1)\cdots(j+i-1)$. We state in Lemma 4.4.1 the properties of Φ which will be useful in the later analysis. Interested readers are referred to Abramowitz and Stegun [1] for more details.

Lemma 4.4.1 *For the confluent hypergeometric function of the first kind Φ defined in (4.24),*

(1) $k\Phi(j,k;x) + x\Phi(j+1,k+1;x) = k\Phi(j+1,k;x)$;

(2) $(1+j-k)\Phi(j,k;x) + (k-1)\Phi(j,k-1;x) = j\Phi(j+1,k;x)$;

(3) $\Phi(j,k;x) = \Phi(k-j,k;-x)e^x$;

(4) For $n=1,2,...,$ $\frac{\mathrm{d}^n\Phi(j,k;x)}{\mathrm{d}x^n} = \frac{(j)_n}{(k)_n}\Phi(j+n,k+n;x)$;

(5) If $k>j>0$, $\Phi(j,k;x) = \frac{\Gamma(k)}{\Gamma(k-j)\Gamma(j)}\int_0^1 e^{xt}t^{j-1}(1-t)^{k-j-1}\mathrm{d}t > 0$.

The solution to the FBP (4.23) is given in Lemma 4.4.2 and its proof can be found in the Appendix. The main difficulty lies in the analysis of the confluent hypergeometric function Φ.

Lemma 4.4.2 *The solution to the FBP (4.23) on $y \in [\theta_0, \infty)$ is given by*

$$\hat{\Psi}(y) = \begin{cases} \frac{\left(1-\gamma^+\right)\theta_0}{(\gamma^+-\gamma^-)\gamma^-}\left(\frac{y}{\theta_0}\right)^{\gamma^-} + \frac{\left(1-\gamma^-\right)\theta_0}{(\gamma^--\gamma^+)\gamma^+}\left(\frac{y}{\theta_0}\right)^{\gamma^+} + \frac{s_0}{\lambda}, \ y \in [\theta_0, \theta_\beta] \\ \frac{(1-\beta)\theta_\beta}{A\Phi\left(A+1,B;-s_1\left(\delta\theta_\beta\right)^{-1}\right)}\left(\frac{\theta_\beta}{y}\right)^A \Phi\left(A, B, -s_1\left(\delta y\right)^{-1}\right) + \frac{s_0-s_1(1-\beta)}{\lambda}, \ y \in (\theta_\beta, \infty), \end{cases} \tag{4.25}$$

where $A = -\gamma^-$, $B = \gamma^+ - \gamma^- + 1$, and $\gamma^\pm$ are as defined in (4.9). The boundaries θ_β and θ_0 are respectively given by $\theta_\beta = -s_1\left(\delta x_\beta\right)^{-1}$ and $\theta_0 = \theta_\beta/\kappa_0$, where x_β is the unique solution on $(-\infty, 0)$ to

$$\begin{aligned} &\left(\frac{1-\gamma^+}{\left(\gamma^+-\gamma^-\right)\gamma^-}\kappa_0^{\gamma^--1} + \frac{1-\gamma^-}{\left(\gamma^--\gamma^+\right)\gamma^+}\kappa_0^{\gamma^+-1}\right)\frac{A}{\delta}\Phi\left(A+1, B; x\right) \\ &= \frac{(1-\beta)}{\delta}\Phi\left(A, B; x\right) + \frac{(1-\beta)A}{\lambda}x\Phi\left(A+1, B; x\right), \end{aligned} \tag{4.26}$$

and κ_0 is defined in Lemma 4.3.1. Furthermore, $\hat{\Psi}$ is strictly convex, strictly decreasing and C^2 in $[\theta_0, \infty)$.

Lemma 4.4.3 *Let Ψ be the Legendre transform $\hat{\Psi}$, i.e.*

$$\Psi(z) = \inf_{y\in[\theta_0,\infty)}\left\{\hat{\Psi}(y) + yz\right\}, \ z \in (0, 1].$$

Then $\Psi(z)$ solves (4.22) and

$$\Psi(z) = v_2(z) := \sup_{\pi\in\Pi}\mathbb{E}^{w,m}\left[\int_0^{e_\lambda}\left(s_0 - s_1\left(\frac{M_t^\pi - W_t^\pi}{M_t^\pi} - \beta\right)_+\right)\mathrm{d}t\right],$$

for $z = \frac{w}{m} \in (0, 1]$. Furthermore, Ψ is strictly concave, strictly increasing, and C^2 in $(0, 1]$.

Theorem 4.4.1 *For $z \in (0, 1]$,*

$$v_2(z) = \begin{cases} \begin{array}{l} \frac{(1-\beta)\theta_\beta}{A\Phi\left(A+1,B;-s_1\left(\delta\theta_\beta\right)^{-1}\right)}\left(\frac{\theta_\beta}{I_2(-z)}\right)^A \Phi\left(A, B, -s_1\left(\delta I_2\left(-z\right)\right)^{-1}\right) \\ -I_2\left(-z\right)z + \frac{s_0-s_1(1-\beta)}{\lambda} \end{array}, \ z \in (0, 1-\beta), \\ \frac{\left(1-\gamma^+\right)\theta_0}{(\gamma^+-\gamma^-)\gamma^-}\left(\frac{I_2(-z)}{\theta_0}\right)^{\gamma^-} + \frac{\left(1-\gamma^-\right)\theta_0}{(\gamma^--\gamma^+)\gamma^+}\left(\frac{I_2(-z)}{\theta_0}\right)^{\gamma^+} - I_2\left(-z\right)z + \frac{s_0}{\lambda}, \ z \in [1-\beta, 1], \end{cases}$$

and the optimal trading strategy is given by

$$\pi_2^*(z)=\begin{cases}\frac{\mu-r}{\sigma^2}\frac{(A+1)\Phi\left(A+2,B;-s_1(\delta I_2(-z))^{-1}\right)}{\Phi\left(A+1,B;-s_1(\delta I_2(-z))^{-1}\right)}, & z\in(0,1-\beta),\\ \frac{\mu-r}{\sigma^2 z}\frac{\left(1-\gamma^+\right)\left(\gamma^--1\right)}{\left(\gamma^+-\gamma^-\right)}\left(\left(\frac{I_2(-z)}{\theta_0}\right)^{\gamma^--1}-\left(\frac{I_2(-z)}{\theta_0}\right)^{\gamma^+-1}\right), & z\in[1-\beta,1].\end{cases}$$

where $I_2:=(\hat{\Psi}_y)^{-1}$ is the inverse function of $\hat{\Psi}_y$. The constants A, B, $\gamma^\pm$, θ_0, θ_β are as given in Lemma 4.4.2.

The proof of Lemma 4.4.3 and Theorem 4.4.1 is similar to that of Lemma 4.3.2 and Theorem 4.3.1, respectively, and are therefore omitted. Next, we study the behavior of π_2^*.

Proposition 4.4.1 *For $z\in[1-\beta,1]$, $\pi_2^*(z)=\pi_1^*(z)$, where $\pi_1^*(z)$ is given in Theorem 4.3.1; for $z\in(0,1-\beta)$, $\pi_2^*(z)$ is strictly decreasing.*

Proof. We first show that $\pi_2^*(z)=\pi_1^*(z)$ on $[1-\beta,1]$. Clearly, $\pi_1^*(1)=\pi_2^*(1)=0$. By Theorems 4.3.1 and 4.4.1, it remains to show that $\frac{I_1(-z)}{y_0}=\frac{I_2(-z)}{\theta_0}$ for $z\in[1-\beta,1)$. Since $I_1=(\hat{\psi}_y)^{-1}$ and $I_2=(\hat{\Psi}_y)^{-1}$, we have

$$\hat{\psi}_y(I_1(-z))=\hat{\Psi}_y(I_2(-z))=-z. \tag{4.27}$$

Differentiating the first equation in both (4.8) and (4.25), we obtain

$$\begin{cases}\hat{\psi}_y(y)=\frac{1-\gamma^+}{\gamma^+-\gamma^-}\left(\frac{y}{y_0}\right)^{\gamma^--1}+\frac{1-\gamma^-}{\gamma^--\gamma^+}\left(\frac{y}{y_0}\right)^{\gamma^+-1}, & y\in[y_0,y_\beta],\\ \hat{\Psi}_y(y)=\frac{1-\gamma^+}{\gamma^+-\gamma^-}\left(\frac{y}{\theta_0}\right)^{\gamma^--1}+\frac{1-\gamma^-}{\gamma^--\gamma^+}\left(\frac{y}{\theta_0}\right)^{\gamma^+-1}, & y\in[\theta_0,\theta_\beta].\end{cases}$$

Using (7.3), (4.27) can be rewritten as

$$\phi\left(\frac{I_1(-z)}{y_0}\right)=\phi\left(\frac{I_2(-z)}{\theta_0}\right)=-z.$$

For a given $z \in [1-\beta, 1)$, both $\frac{I_1(-z)}{y_0}$ and $\frac{I_2(-z)}{\theta_0}$ solve for $\phi(x) = -z$ on $(1, \infty)$. Given that ϕ was shown to be a strictly increasing function on $(1, \infty)$ (see proof of Lemma 3.1 in Appendix), it follows that $\frac{I_1(-z)}{y_0} = \frac{I_2(-z)}{\theta_0}$ which further implies $\pi_2^*(z) = \pi_1^*(z)$ on $[1-\beta, 1)$.

Next, we show that π_2^* is strictly decreasing on $(0, 1-\beta)$. Recall that for $z \in (0,1)$,

$$\Psi(z) = \inf_{y\in[\theta_0,\infty)} \left\{ \hat{\Psi}(y) + yz \right\} = \hat{\Psi}(I_2(-z)) + I_2(-z)\, z.$$

We deduce that

$$\Psi_z(z) = I_2(-z) \text{ and } \Psi_{zz}(z) = -\frac{1}{\hat{\Psi}_{yy}(I_2(-z))}. \tag{4.28}$$

By the first relation in (4.28), for $y = I_2(-z) \in (\theta_\beta, \infty)$,

$$\frac{\mathrm{d}\pi_2^*}{\mathrm{d}z} = \frac{\mathrm{d}\pi_2^*}{\mathrm{d}y}\frac{\mathrm{d}y}{\mathrm{d}z} = \frac{\mathrm{d}\pi_2^*}{\mathrm{d}y}\frac{\mathrm{d}^2 v_2}{\mathrm{d}z^2}.$$

By Lemma 4.4.3, v_2 is strictly concave on $(0,1]$ which implies $\frac{\mathrm{d}^2 v_2}{\mathrm{d}z^2} < 0$. Thus, it remains to show that $\frac{\mathrm{d}\pi_2^*}{\mathrm{d}y} > 0$ for $y \in (\theta_\beta, \infty)$. For $y \in (\theta_\beta, \infty)$, i.e., $z \in (0, 1-\beta)$, by applying Properties (2) and (4) of Lemma 4.4.1 together with some algebraic manipulations, we obtain

$$\begin{aligned}
\frac{\mathrm{d}\pi_2^*}{\mathrm{d}y} &\propto \frac{\mathrm{d}}{\mathrm{d}y}\left[\frac{\Phi\left(A+2, B, -s_1(\delta y)^{-1}\right)}{\Phi\left(A+1, B, -s_1(\delta y)^{-1}\right)}\right] \\
&\propto (A+2-B)\,\Phi(A+2, B+1; -s_1(\delta y)^{-1})\Phi(A+1, B, -s_1(\delta y)^{-1}) \\
&- (A+1-B)\,\Phi(A+1, B+1; -s_1(\delta y)^{-1})\Phi(A+2, B, -s_1(\delta y)^{-1}).
\end{aligned}$$

Since $B \in (A+1, A+2)$, by Property (5) of Lemma 4.4.1, the functions $\Phi(A+2, B+1; -s_1(\delta y)^{-1})$, $\Phi(A+1, B, -s_1(\delta y)^{-1})$, and $\Phi(A+1, B+1; -s_1(\delta y)^{-1})$ are all positive for $y \in (\theta_\beta, \infty)$. Since $\hat{\Psi}$ is strictly convex on (θ_β, ∞) and

$$\hat{\Psi}_{yy} \propto \Phi\left(A+2, B, -s_1(\delta y)^{-1}\right),$$

we deduce that $\Phi\left(A+2, B, -s_1 (\delta y)^{-1}\right) > 0$ for $y \in (\theta_\beta, \infty)$. It follows that $\frac{\mathrm{d}\pi_2^*}{\mathrm{d}y} > 0$ for $y \in (\theta_\beta, \infty)$. The proof is therefore complete. ■

4.4.2 Numerical Examples

In this subsection, we provide several numerical examples in connection with the results of Section 4.4.1, and study the sensitivity of the MCI strategy to some key model parameters. For all the following examples, we set $r = 0.03$, $\mu = 0.1$, $\sigma = 0.2$, $s_0 = 1$, $\eta = 0.1$, and a threshold drawdown level of $\beta = 0.2$ to initialize the income penalty.

Example 4.4.1 (MCI and MCI strategy) *We consider $\lambda = 0.05$, and different levels for the penalty rate s_1. The MCI (left plot) and MCI strategies (right plot) are displayed in Figure 4.4.*

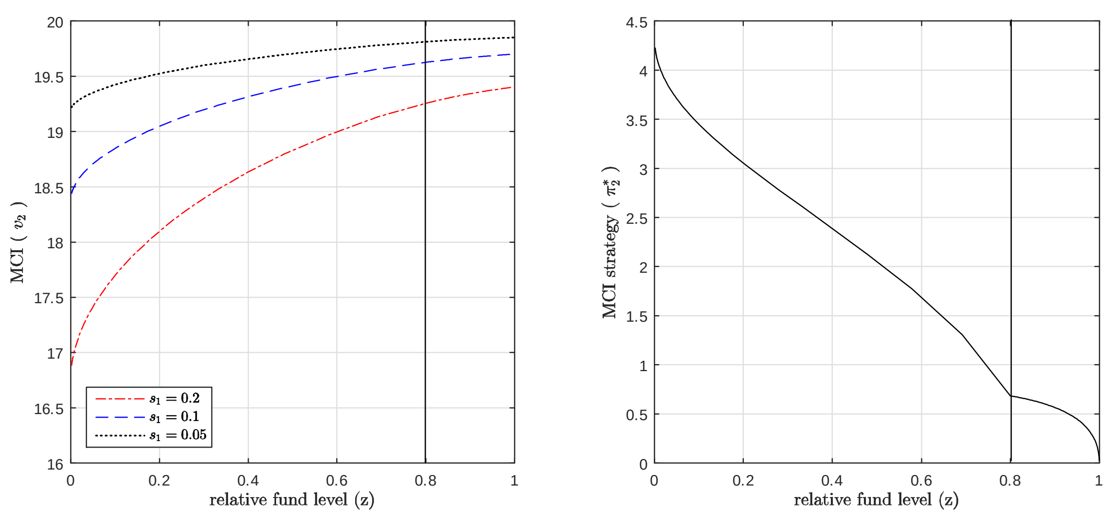


Figure 4.4. MCI (right) and MCI strategy (left) with different penalty rates

The left plot shows that the MCI v_2 is increasing and concave (as expected). From the right plot, we observe that

- *in the non-penalty region $z \in [0.8, 1]$, the MCI strategy again coincides with the MLDP strategy;*
- *in the penalty region $z \in (0, 0.8)$, the MCI strategy is independent of s_1 which is a consequence of removing the terminating drawdown level α.*

It immediately follows that the MCI v_2 is monotone decreasing in the penalty rate s_1 (as we can observe from the left plot). It is also interesting to note that as the fund level increases, the manager invests more conservatively as the fear of incurring a larger penalty dominates the appetite to take on more risk to possibly lower the penalty rate.

Example 4.4.2 (Impact of λ) *In this example, by setting different levels of λ, we obtain the corresponding MCI strategies in Figure 4.5. Note that a smaller λ implies a longer expected time horizon. Intuitively speaking, the manager intends to adopt a more aggressive strategy when the expected time horizon is longer, as we see in the non-penalty region $z \in [0.8, 1]$. However, the trend is for the most part reversed in the penalty region $z \in (0, 0.8)$. Indeed, given that the income penalty is high for low fund levels under the linear penalty scheme, a manager with a longer investment time horizon tends to adopt a more conservative strategy to avoid the risk of a heavy income penalty over a longer time period.*

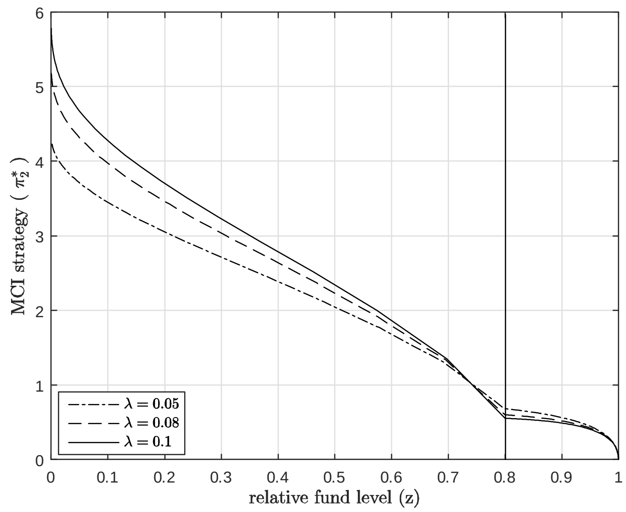


Figure 4.5. Impact of λ on MCI strategy

Chapter 5

A Pair of Optimal Reinsurance-Investment Strategies in the Two-sided Exit Framework

5.1 Introduction

Given that investment is an integral component of an insurer's risk management practices, risk models taking both insurance and investment risks into consideration have received a great deal of attention in the literature. In addition to investment, insurers frequently rely on reinsurance to control their risk exposure. Subject to a control on investment and reinsurance, optimization problems under various objective functions have become a popular research topic in the actuarial literature. Common objective functions include the *ruin probability minimization* (e.g., Young

[88], Promislow and Young [78], Chen et al. [40], Bayraktar and Zhang [28]), the *bequest goal optimization* (e.g., Bayraktar et al. [22][23], Bayraktar and Young [27]), the *expected utility maximization* (e.g., Liu and Ma [67], Bai and Guo [15], Liang and Bayraktar [64], Liang and Yuen [66]), as well as the traditional *mean-variance portfolio optimization criteria* (e.g., Bäuerle [19], Bai and Zhang [16], Bi et al. [30], Zeng et al. [90]).

In this chapter, for a controlled surplus process $\{X_t^u\}_{t\geq 0}$ under a reinsurance-investment strategy u, we study the optimal reinsurance-investment problem in the so-called *two-sided exit framework*, namely

$$\sup_u \mathbb{P}\left(\tau_b < \tau_0 \wedge e_\lambda \middle| X_0^u = x\right), \tag{5.1}$$

and

$$\inf_u \mathbb{P}\left(\tau_0 < \tau_b \wedge e_\lambda \middle| X_0^u = x\right), \tag{5.2}$$

where $x \in [0, b]$, $\tau_0 = \inf\{t \geq 0 : X_t^u < 0\}$, $\tau_b = \inf\{t \geq 0 : X_t^u > b\}$, and e_λ is a random time horizon modelled by an independent exponential random variable with mean $1/\lambda$. Objective (5.1) proposes to maximize the probability that the insurer's surplus reaches the target b before the time of ruin and the end of the time horizon e_λ, while objective (5.2) minimizes the probability that ruin occurs before the surplus reaches the target b and the end of the time horizon e_λ.

Under objectives (5.1) and (5.2), we consider an insurer whose surplus is modelled by a diffusion process. In addition, the insurer can purchase proportional reinsurance and invest its wealth in a financial market consisting of a risk-free asset and a risky asset, where the dynamics of the latter is assumed to be correlated with the insurance surplus. By applying the dual approach and solving the associated Hamilton–Jacobi–Bellman (HJB) equation, an explicit expression for the optimal reinsurance-investment strategy are obtained for both objectives (5.1) and (5.2). A closer examination of the resulting optimal reinsurance-investment strategies will reveal the following three main implications which we highlight here:

1. With the introduction of the time factor e_λ (measuring the insurer's tolerance to time) to the objectives (5.1) and (5.2), the corresponding optimal reinsurance-investment strategy may become more aggressive as the surplus level increases. This is in contrast to e.g., Promislow and Young [78] and Bai and Guo [14] where the optimal infinite-time reinsurance-investment strategy is always more conservative as the surplus level increases (which may not always be consistent with industry practices);

2. The optimal reinsurance-investment strategy under objective (5.1) (objective (5.2) resp.) is always more aggressive (conservative resp.) than the strategy to minimize the infinite-time ruin probability in Promislow and Young [78];

3. The optimal reinsurance-investment strategy under objective (5.1) is independent of the target level b, a result similar to Bayraktar and Young [27] under the objective of reaching a bequest goal.

It is worth pointing out that the reinsurance control plays an important role in the derivation of an explicit expression for the optimal strategy under objectives (5.1) and (5.2). In fact, if the insurer cannot manage the insurance risk by purchasing reinsurance, the corresponding HJB equation appears difficult to solve analytically (see Remark 5.3.1 for more details on this point). Also note that Luo et al. [70] considered objective (5.1) in the same context but with only the reinsurance control (without investment).

The remainder of this chapter is organized as follows. In Section 5.2, the mathematical framework under which the objectives (5.1) and (5.2) will be examined is formally defined. In Section 5.3, explicit expressions for the optimal reinsurance-investment strategies are given, and various implications are later discussed. In Section 5.4, some numerical examples are provided to support the theoretical findings of Section 5.3. Some tedious derivations are postponed to the Appendix.

5.2 Problem formulation

Consider a filtered complete probability space $(\Omega, \mathcal{F}, \mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ satisfying the usual conditions. Following Promislow and Young [78] and Bai and Guo [14], we assume the dynamics of the claim payment process is governed by

$$\mathrm{d}C_t = m\mathrm{d}t - \upsilon \mathrm{d}W_t^1, \tag{5.3}$$

where m and υ are positive constants and $\{W_t^1\}_{t\geq 0}$ is an $\mathbb{F}$-adapted standard Brownian motion. Suppose that the premium rate is given by $c = (1+\theta)m$, where the constant $\theta > 0$ is the insurer's safety loading. The dynamics of the primary surplus process without reinsurance and investment is given by

$$\mathrm{d}U_t = c\mathrm{d}t - \mathrm{d}C_t = \theta m\mathrm{d}t + \upsilon \mathrm{d}W_t^1. \tag{5.4}$$

The primary surplus model (5.4) is a commonly used approximation to the classical Cramér-Lundberg model.

Suppose that the insurer can manage the insurance liabilities by purchasing (proportional) reinsurance or acquiring new business (e.g., Bäuerle [19]). For $t \geq 0$, the reinsurance/new business level is denoted by the risk exposure rate $q_t \in [0, +\infty)$ where

- when $q_t \in [0, 1]$, it corresponds to situations where the insurer purchases a proportional reinsurance coverage. More specifically, the insurer diverts part of the premium income to the reinsurer at rate $(1 - q_t)(1 + \eta)m$ while the reinsurer is responsible for $(1 - q_t)100\%$ of the claim payment. Here, the reinsurer's safety loading η is assumed to satisfy the common assumption

$$\eta > \theta > 0. \tag{5.5}$$

- when $q_t \in [1, +\infty)$, it corresponds to situations where the insurer acquires new business by, e.g., acting as a reinsurer for other insurers whose risks are identically distributed. Hence, the insurer's safety loading on the new business (i.e., the portion of the risk exposure q_t over 1) is assumed to be η (see also Equation (5.6)). We continue to assume that the insurer's safety loading on the original insurance business is θ.

We exclude the strategies $q_t < 0$ because this implies the insurer over-reinsures the original underwritten business (by transferring more than 100% of the business to reinsurers), a practice which is not permitted under insurance regulation.

Remark 5.2.1 *Although there exists other forms of reinsurance contracts which may be more widely used in practice (e.g., stop-loss, excess-of-loss; see, e.g., Borch [31] and Arrow [7] for a more detailed discussion), we have chosen to limit the present analysis to the case involving proportional reinsurance only. This choice was made for reasons of mathematical tractability as it leads to a closed-form expression to the corresponding HJB equation. Nonetheless, it is worth pointing out that proportional reinsurance was shown to be the optimal form of reinsurance under certain setups (see, e.g., Theorem 3.1 of Cai et al. [35] and Theorem 13 in Centeno and Simões [38]).*

Remark 5.2.2 *(5.5) is a common assumption made in the literature, indicating that reinsurance business is usually more expensive. To better understand the as-*

sumption, we first notice that the case $\theta < \eta$ should be excluded as it leads to an arbitrage opportunity if the insurer reinsures the whole portfolio. In practice, the case $\theta = \eta$ seldomly happens for several reasons. First, undertaking the insured risk incurs additional costs for reinsurers. Second, to get rid of the undesirable part of the insured risk, in general insurers are willing to accept a higher loading. Third, $\eta > \theta$ also results from reinsurers' growing market power (see, e.g., Cummins and Weiss [45]).

Under the reinsurance strategy $\{q_t\}_{t\geq 0}$, the dynamics of the surplus process follows

$$\begin{aligned} \mathrm{d}R_t &= \mathrm{d}U_t - (1-q_t)(1+\eta)m\mathrm{d}t + (1-q_t)\mathrm{d}C_t \\ &= (\theta - \eta + \eta q_t)\, m\mathrm{d}t + q_t \upsilon \mathrm{d}W_t^1. \end{aligned} \tag{5.6}$$

We assume that the insurer can also invest in a financial market consisting of a risk-free bond with interest rate $r > 0$ and a risky stock whose price is governed by

$$\mathrm{d}S_t = \mu S_t \mathrm{d}t + \sigma S_t \left(\rho \mathrm{d}W_t^1 + \tilde{\rho}\mathrm{d}W_t^2\right),$$

where $\mu > r$, $\sigma > 0$, $\rho \in (-1, 1)$, $\tilde{\rho} := \sqrt{1-\rho^2}$, and $\{W_t^2\}_{t\geq 0}$ is another $\mathbb{F}$-adapted standard Brownian motion, independent of $\{W_t^1\}_{t\geq 0}$. We denote by π_t the *amount* of surplus invested in the stock at time t, and $\{X_t^u\}_{t\geq 0}$ the corresponding insurance surplus process under the reinsurance-investment strategy $u := (q_t, \pi_t)_{t\geq 0}$. Apart from the risky investment, the balance of the surplus is invested in the risk-free bond. Thus, by (5.6), the dynamics of the insurer's surplus follows

$$\begin{aligned} \mathrm{d}X_t^u &= (X_t^u - \pi_t)r\mathrm{d}t + \pi_t \frac{\mathrm{d}S_t}{S_t} + \mathrm{d}R_t \\ &= \left[rX_t^u + m(\theta - \eta + \eta q_t) + (\mu - r)\pi_t\right]\mathrm{d}t + (\upsilon q_t + \rho\sigma\pi_t)\mathrm{d}W_t^1 + \tilde{\rho}\sigma\pi_t \mathrm{d}W_t^2, \end{aligned}$$

or equivalently,

$$\mathrm{d}X_t^u = [rX_t^u + m(\theta - \eta + \eta q_t) + (\mu - r)\pi_t]\,\mathrm{d}t + \sqrt{\upsilon^2 q_t^2 + 2\rho\sigma q_t \pi_t + \sigma^2\pi_t^2}\mathrm{d}W_t, \tag{5.7}$$

with initial surplus $X_0^u = x > 0$ and $\{W_t\}_{t\geq 0}$ is defined as

$$W_t = \frac{\upsilon q_t + \rho\sigma\pi_t}{\sqrt{\upsilon^2 q_t^2 + 2\rho\sigma q_t\pi_t + \sigma^2\pi_t^2}} W_t^1 + \frac{\tilde{\rho}\sigma\pi_t}{\sqrt{\upsilon^2 q_t^2 + 2\rho\sigma q_t\pi_t + \sigma^2\pi_t^2}} W_t^2, \quad t \geq 0.$$

Note that it is straightforward to show that the quadratic variation $\langle W\rangle_t = t$ and thus $\{W_t\}_{t\geq 0}$ is a $\mathbb{F}$-adapted standard Brownian motion by Lévy's characterization of Brownian motion.

Throughout the chapter, we also assume that

$$\rho\frac{\mu - r}{\sigma} \leq \frac{\eta m}{\upsilon}. \tag{5.8}$$

Although the risk exposure rate q_t is assumed to be nonnegative, one will later see that the unconstrained optimal reinsurance strategies indeed will be nonnegative under condition (5.8), and hence correspond to the optimal reinsurance strategies with constraint. Note that condition (5.8) is clearly satisfied if $\rho \leq 0$, i.e. the insurance risk and financial risk are either negatively correlated or independent. Heuristically, when $\rho \leq 0$, both the optimal reinsurance strategy q_t and the optimal investment strategy π_t should be positive. Indeed, due to the safety loading condition (5.5) and the positiveness of the market price of risk $(\mu - r)/\sigma$, there is a natural hedge in holding a long position in both the insurance and financial markets when $\rho \leq 0$. For $\rho > 0$, condition (5.5) implies that the insurer will be discouraged from over reinsuring if the reinsurance premium rate η is high or the ratio m/υ is large, where the latter condition is consistent with Promislow and Young [78] which states that the diffusion approximation (5.3) is reasonable in actuarial practice when the ratio m/υ is large so that the probability of realizing negative claim payments is small.

Definition 5.2.1 (Admissible strategies) *The pair $u = (q_t, \pi_t)_{t\geq 0}$ is called an admissible reinsurance-investment strategy, i.e., $u \in \Pi$, if it satisfies the following conditions:*

(1) u is $\mathbb{F}$-progressively measurable;

(2) $q_t \in [0, +\infty)$ a.s. for any $t \geq 0$;

(3) $\int_0^t (q_s^2 + \pi_s^2)\mathrm{d}s < \infty$ a.s. for any $t \geq 0$.

For the threshold levels 0 and b with $0 \leq x \leq b$, we recall that $\tau_0 = \inf\{t \geq 0 : X_t^u < 0\}$ and $\tau_b = \inf\{t \geq 0 : X_t^u > b\}$ are two first passage times of the controlled wealth process $\{X_t^u\}_{t\geq 0}$. The main objective of this chapter is to study the optimal reinsurance-investment problems under the two-sided exit framework, i.e.,

$$\psi^+(x) = \sup_{u\in\Pi} \mathbb{P}_x(\tau_b < \tau_0 \wedge e_\lambda) = \sup_{u\in\Pi} \mathbb{E}_x\left[e^{-\lambda\tau_b} 1_{\{\tau_b<\tau_0\}}\right], \tag{5.9}$$

and

$$\psi^-(x) = \inf_{u\in\Pi} \mathbb{P}_x(\tau_0 < \tau_b \wedge e_\lambda) = \inf_{u\in\Pi} \mathbb{E}_x\left[e^{-\lambda\tau_0} 1_{\{\tau_0<\tau_b\}}\right]. \tag{5.10}$$

For ease of notation, we denote $\mathbb{P}_x$ the law of X^u given that $X_0^u = x \in [0, b]$.

Although the two objective functions (5.9) and (5.10) appear to be similar (in particular when $\lambda = 0$, (5.9) and (5.10) are equivalent), we will see that their corresponding reinsurance-investment strategies can be quite different. Interestingly, we also find that the associated optimal strategies are closely related to the strategy of minimizing the infinite-time ruin probability, namely

$$\inf_{u\in\Pi} \mathbb{P}_x(\tau_0 < \infty), \tag{5.11}$$

studied by Promislow and Young [78] and Bai and Guo [14]. Suppose the optimal reinsurance-investment strategies for objectives (5.9)–(5.11) are denoted by $u_+^* :=$

(q_+^*, π_+^*), $u_-^* := (q_-^*, \pi_-^*)$ and $u_0^* := (q_0^*, \pi_0^*)$, respectively. We find that

$$u_+^* \succ u_0^* \succ u_-^*,$$

where "$\succ$" is the relation of *aggressiveness* which is defined as follows.

Definition 5.2.2 *For $u_i = (q_{i,t}, \pi_{i,t})_{t\geq 0} \in \Pi$, $i = \{1, 2\}$, we say that u_1 is more aggressive than u_2, denoted by $u_1 \succ u_2$, if for any $t \geq 0$, $\mathbb{P}$-a.s.,*

$$q_{1,t} \geq q_{2,t} \geq 0, \quad \textit{and} \quad \pi_{1,t} \geq \pi_{2,t} \geq 0,$$

or

$$q_{1,t} \geq q_{2,t} \geq 0, \quad \textit{and} \quad \pi_{1,t} \leq \pi_{2,t} \leq 0.$$

In other words, u_1 is more aggressive than u_2, if the insurer buys less reinsurance and takes a larger position (either short or long) in the financial market.

We refer the reader to Theorem 5.3.5 and Remark 5.3.4 for more details on this assertion.

5.3 Main results

5.3.1 Objective function ψ^+

We first investigate objective (5.9). From (5.7), for any $u = (q, \pi) \in [0, \infty) \times \mathbb{R}$ and $\phi(x) \in C^2$, we define a differential operator $\mathcal{A}^u$ as

$$\mathcal{A}^u \phi = [rx + (\mu - r)\pi + m(\theta - \eta + \eta q)]\, \phi_x + \frac{1}{2}\left(\upsilon^2 q^2 + 2\rho\sigma\upsilon q\pi + \sigma^2\pi^2\right) \phi_{xx} - \lambda\phi, \tag{5.12}$$

where ϕ_x (ϕ_{xx} resp.) denotes the first (second resp.) order derivative of ϕ with respect to x.

By a standard argument, we first provide a verification theorem for the optimization problem. The proof of Theorem 5.3.1 is standard, and is thus omitted.

Theorem 5.3.1 (Verification theorem) *Suppose that a function $\phi(x) : [0, b] \to [0, 1]$ satisfies the following conditions:*

(1) $\phi(x) \in C^2[0, b]$ is strictly increasing and strictly concave;

(2) for any $u \in [0, \infty) \times \mathbb{R}$, $\mathcal{A}^u\phi(x) \le 0$ for $x \in (0, b)$;

(3) there exists an admissible feedback strategy $u_+^* : [0, b] \to [0, \infty) \times \mathbb{R}$ such that $\mathcal{A}^{u_+^*}\phi(x) = 0$ for $x \in (0, b)$;

(4) $\phi(0) = 0$ and $\phi(b) = 1$.

Then $\phi(x) = \psi^+(x)$ on $[0, b]$, u_+^* is an optimal reinsurance-investment strategy, and $\psi^+(x)$ is the associated objective function defined in (5.9).

From Theorem 5.3.1, we obtain the associated HJB equation for objective (5.9):

$$\begin{cases} \sup\limits_{u \in [0,\infty) \times \mathbb{R}} \{\mathcal{A}^u\phi(x)\} = 0, \quad x \in (0, b), \\ \phi(0) = 0 \text{ and } \phi(b) = 1. \end{cases} \tag{5.13}$$

Applying the first-order condition to (5.13) yields

$$\begin{cases} m\eta\phi_x + \phi_{xx}\upsilon^2 q_+^* + \phi_{xx}\rho\sigma\upsilon\pi_+^* = 0, \\ (\mu - r)\phi_x + \phi_{xx}\rho\sigma\upsilon q_+^* + \phi_{xx}\sigma^2\pi_+^* = 0. \end{cases}$$

Solving the above linear system, we obtain

$$q_+^* = \frac{[(\mu - r)\rho\upsilon - \sigma\eta m]\,\phi_x}{\sigma\upsilon^2\tilde{\rho}^2\phi_{xx}}, \quad \pi_+^* = \frac{[\rho\sigma m\eta - (\mu - r)\upsilon]\,\phi_x}{\sigma^2\upsilon\tilde{\rho}^2\phi_{xx}}. \tag{5.14}$$

By substituting (5.14) into the HJB equation (5.13), we obtain

$$\begin{cases} \left[rx - m(\eta - \theta)\right]\phi_x - \lambda\phi - \Lambda\frac{\phi_x^2}{\phi_{xx}} = 0, \quad x \in (0, b), \\ \phi(0) = 0 \text{ and } \phi(b) = 1, \end{cases} \tag{5.15}$$

where

$$\Lambda = \frac{(\mu - r)^2 \upsilon^2 - 2(\mu - r)\rho\sigma m\eta\upsilon + \sigma^2\eta^2 m^2}{2\sigma^2\upsilon^2\tilde{\rho}^2} > 0. \tag{5.16}$$

In order to solve (5.15), we first consider the corresponding free boundary problem (FBP):

$$\begin{cases} \Lambda y^2 \hat{\phi}_{yy}(y) + (\lambda - r)\, y\hat{\phi}_y(y) - \lambda\hat{\phi}(y) - m(\eta - \theta)\, y = 0, \quad 0 < y_b < y < y_0, \\ \hat{\phi}_y(y_0) = 0; \quad \hat{\phi}_y(y_b) = -b; \quad \hat{\phi}(y_0) = 0; \quad \hat{\phi}(y_b) + by_b = 1. \end{cases} \tag{5.17}$$

The solution to the FBP (5.17) is given in Lemma 5.3.1. The proof can be found in the Appendix.

Lemma 5.3.1 *The solution to the FBP (5.17) is given by*

$$\hat{\phi}(y) = \frac{(\eta - \theta)my}{r}\left[\frac{1 - \delta_-}{\delta_+ - \delta_-}\left(\frac{y}{y_0}\right)^{\delta_+ - 1} + \frac{\delta_+ - 1}{\delta_+ - \delta_-}\left(\frac{y}{y_0}\right)^{\delta_- - 1} - 1\right], \quad y \in [y_b, y_0], \tag{5.18}$$

where $\delta_\pm = \frac{-(\lambda - r - \Lambda) \pm \sqrt{(\lambda - r - \Lambda)^2 + 4\lambda\Lambda}}{2\Lambda}$, $y_0 = \kappa y_b$,

$$y_b = \frac{-r\delta_-}{(1 - \delta_-)\left[rb - (\eta - \theta)m\right] + (\eta - \theta)m(1 - \delta_-)\kappa^{1 - \delta_+}} > 0,$$

and κ *is the unique solution in* $(1, +\infty)$ *to*

$$\frac{1 - \delta_-}{\delta_-}x^{1 - \delta_+} + \frac{\delta_+ - 1}{\delta_+}x^{1 - \delta_-} - \frac{\delta_+ - \delta_-}{\delta_+\delta_-}\frac{(\eta - \theta)m - rb}{(\eta - \theta)m} = 0.$$

Furthermore, $\hat{\phi} \in C^2[y_b, y_0]$ *is strictly decreasing and strictly convex on* $[y_b, y_0]$.

Lemma 5.3.2 shows that the *Legendre transform* of the solution to the FBP (5.17) is the solution to (5.15), and corresponds to the value function ψ^+.

Lemma 5.3.2 *For $\hat{\phi}$ given in (5.18), consider its Legendre transform defined as*

$$\phi(x) = \inf_{y \in [y_b, y_0]} \left\{ \hat{\phi}(y) + xy \right\}, \quad x \in [0, b]. \tag{5.19}$$

Then, $\phi(x)$ solves (5.15) and furthermore $\phi(x) = \psi^+(x)$ on $[0, b]$, where $\psi^+(x)$ is the value function defined in (5.9).

Proof. From the boundary conditions in (5.17) and Lemma 5.3.1, we recall that $\hat{\phi}_y(y_b) = -b$ and $\hat{\phi}(y)$ is strictly convex on $[y_b, y_0]$. This implies $\hat{\phi}(y) + by$ is increasing on $[y_b, y_0]$ and attains its infimum at $y = y_b$ such that $\phi(b) = \hat{\phi}(y_b) + by_b = 1$. A similar argument yields $\phi(0) = \hat{\phi}(y_0) = 0$.

For $x \in (0, b)$, the optimizer $y^* \in (y_b, y_0)$ of (5.19) solves the equation

$$\hat{\phi}_y(y) = -x. \tag{5.20}$$

By Lemma 5.3.1, we deduce the optimizer $y^* = I(-x)$, where $I := (\hat{\phi}_y)^{-1}$ is the inverse function of $\hat{\phi}_y$. It follows that

$$\phi(x) = \hat{\phi}(y^*) + xy^* = \hat{\phi}(I(-x)) + xI(-x), \tag{5.21}$$

for $x \in (0, b)$. Taking the first and second order of derivatives to (5.21) with respect to x yields

$$\phi_x(x) = y^* = I(-x), \quad \phi_{xx}(x) = -\frac{1}{\hat{\phi}_{yy}(y^*)} = -\frac{1}{\hat{\phi}_{yy}(I(-x))}. \tag{5.22}$$

Using (5.20)–(5.22), it is straightforward to verify that $\phi(x)$ solves the HJB equation (5.15). Moreover, since $\hat{\phi}$ is strictly decreasing, strictly convex and C^2 on $[y_b, y_0]$, we deduce that ϕ is strictly increasing, strictly concave, and C^2 on $[0, b]$. Therefore, by Theorem 5.3.1, we conclude that $\phi(x) = \psi^+(x)$ on $[0, b]$. ■

The solution to objective (5.9) is given in the next theorem, and the proof can be found in the Appendix.

Theorem 5.3.2 *For $x \in (0, b)$, the optimal reinsurance-investment strategy $u_+^* = (q_+^*, \pi_+^*)$ for objective (5.9) is given by*

$$q_+^*(x) = A_+\delta_+ \left(\frac{I(-x)}{y_0}\right)^{\delta_+-1} - A_+\delta_- \left(\frac{I(-x)}{y_0}\right)^{\delta_--1}, \tag{5.23}$$

$$\pi_+^*(x) = B_+\delta_+ \left(\frac{I(-x)}{y_0}\right)^{\delta_+-1} - B_+\delta_- \left(\frac{I(-x)}{y_0}\right)^{\delta_--1}, \tag{5.24}$$

where

$$\begin{aligned} A_+ &= \frac{[\sigma\eta m - (\mu - r)\rho\upsilon]\,(\eta-\theta)m\,(\delta_+ - 1)\,(1-\delta_-)}{\sigma\upsilon^2\tilde{\rho}^2 r(\delta_+ - \delta_-)}, \\ B_+ &= \frac{[(\mu - r)\upsilon - \rho\sigma m\eta]\,(\eta-\theta)m\,(\delta_+ - 1)\,(1-\delta_-)}{\sigma^2\upsilon\tilde{\rho}^2 r(\delta_+ - \delta_-)}, \end{aligned}$$

and the associated value function is given by

$$\psi^+(x) = \frac{(\eta-\theta)m(1-\delta_-)(1-\delta_+)}{r(\delta_+ - \delta_-)} I(-x) \left[\left(\frac{I(-x)}{y_0}\right)^{\delta_+-1} - \left(\frac{I(-x)}{y_0}\right)^{\delta_--1}\right],$$

where $I = (\hat{\phi}_y)^{-1}$. The function $\hat{\phi}$ and the constants $\delta_\pm$, y_b, y_0, κ are given in Lemma 5.3.1.

Remark 5.3.1 *As pointed out earlier, the reinsurance control plays an important role in the derivation of an explicit solution to problem (5.9). If the insurer cannot manage the insurance risk by purchasing reinsurance (i.e., $q_t \equiv 1$ for all $t \geq 0$), then the corresponding problem has only one control variable π. The associated HJB equation becomes*

$$\sup_\pi \left\{ [rx + (\mu - r)\pi + m\theta]\,\phi_x + \frac{1}{2}\left(\upsilon^2 + 2\pi\upsilon\sigma\rho + \pi^2\sigma^2\right)\phi_{xx} - \lambda\phi \right\} = 0,$$

which further implies that

$$\left(rx + \theta m - \frac{(\mu - r)\upsilon\rho}{\sigma}\right)\phi_x - \lambda\phi - \frac{(\mu-r)^2}{2\sigma^2}\frac{\phi_x^2}{\phi_{xx}} + \frac{\upsilon^2\tilde{\rho}}{2}\phi_{xx} = 0, \tag{5.25}$$

with boundary conditions $\phi(0) = 0$ and $\phi(b) = 1$. In comparison to (5.13), equation (5.25) contains an additional term involving ϕ_{xx}, which makes the identification of a closed-form expression difficult.

As shown in the following proposition, the optimal reinsurance-investment strategy under objective (5.9) is independent of the target level b. This result largely simplifies the decision-making process as the selection of an appropriate target level might be difficult in general.

Proposition 5.3.1 *The optimal reinsurance-investment strategy $u_+^* = (q_+^*, \pi_+^*)$ is independent of b.*

Proof. From Theorem 5.3.2, it suffices to show that $I(-x)/y_0$ is independent of b. Recall from (5.20) that $\hat{\phi}_y(I(-x)) = -x$, where $\hat{\phi}$ is given in (5.18). Differentiating (5.18) yields that

$$\hat{\phi}_y(I(-x)) = -\frac{(\eta - \theta)\, m}{r} + g\left(\frac{I(-x)}{y_0}\right),$$

where the function

$$g(z) := \frac{(\eta-\theta)m}{r(\delta_+ - \delta_-)}\left[(1-\delta_-)\delta_+ z^{\delta_+ - 1} + (\delta_+ - 1)\delta_- z^{\delta_- - 1}\right], \quad z \in (0,1)$$

is increasing in z as $\delta_+ > 1 > 0 > \delta_-$. Moreover, $\lim_{z \downarrow 0} g(z) = -\infty$ and $\lim_{z \uparrow 1} g(z) = \frac{(\eta-\theta)m}{r} > 0$. In other words, $z = \frac{I(-x)}{y_0}$ is the unique solution to

$$g(z) = \frac{(\eta - \theta)\, m}{r} - x.$$

Since the function $g(\cdot)$ is independent of b, so is $I(-x)/y_0$. ■

5.3.2 Objective function ψ^-

In this subsection, we study objective (5.10) which, for convenience, is restated here:

$$\psi^-(x) = \inf_{u\in\Pi} \mathbb{P}_x\left(\tau_0 < \tau_b \wedge e_\lambda\right) = \inf_{u\in\Pi} \mathbb{E}_x\left[e^{-\lambda\tau_0} 1_{\{\tau_0<\tau_b\}}\right], \quad x \in [0, b].$$

The corresponding optimal reinsurance-investment strategy is denoted as $u_-^* := \left(q_-^*, \pi_-^*\right)$.

By letting $(q, \pi) \equiv (0, 0)$ in the dynamics (5.7) of the surplus process $\{X_t^u\}_{t\geq 0}$, we first observe that there exists a *safe level* $\frac{(\eta-\theta)m}{r}$ for this objective, i.e., $\psi^-(x) = 0$ for any $x \geq \frac{(\eta-\theta)m}{r}$.

Proposition 5.3.2 *We have $\psi^-(x) = 0$ for $x \geq \frac{(\eta-\theta)m}{r}$. A corresponding optimal strategy is given by $\left(q_-^*(x), \pi_-^*(x)\right) = (0, 0)$ for any $x \geq \frac{(\eta-\theta)m}{r}$.*

The existence of this safe level is a significant difference between objectives (5.9) and (5.10). Heuristically, objective (5.10) is easier to achieve as the game ends positively for the insurer when either the target level b is achieved or the exponential time horizon e_λ expires before ruin. However, an insurer with objective (5.9) shall reach the target level b before ruin occurs and the end of the time horizon e_λ. The optimal strategies under these two objectives will be formally compared in the next section. By Proposition 5.3.2, without loss of generality, we only consider objective (5.10) when

$$b \leq \frac{(\eta-\theta)\,m}{r}. \tag{5.26}$$

The proof of the following verification theorem is also standard, and is therefore omitted.

Theorem 5.3.3 (Verification theorem) *Suppose that a function $\phi(x) : [0,b] \to [0,1]$ satisfies the following conditions:*

(1) $\phi(x) \in C^2[0,b]$ is strictly decreasing and strictly convex;

(2) for any $u \in [0,\infty) \times \mathbb{R}$, $\mathcal{A}^u \phi(x) \geq 0$ for $x \in (0,b)$;

(3) there exists an admissible feedback strategy $u_-^* : [0,b] \to [0,\infty) \times \mathbb{R}$ such that $\mathcal{A}^{u_-^*} \phi(x) = 0$ for $x \in (0,b)$;

(4) $\phi(b) = 0$ and $\phi(0) = 1$.

Then $\phi(x) = \psi^-(x)$ on $[0,b]$, u_-^* is an optimal reinsurance-investment strategy, and $\psi^-(x)$ is the associated objective function defined in (5.10).

Theorem 5.3.3 implies that the associated HJB equation of objective (5.10) is given by

$$\begin{cases} \inf\limits_{u \in [0,\infty)\times\mathbb{R}} \{\mathcal{A}^u \phi(x)\} = 0, \quad x \in (0,b), \\ \phi(b) = 0 \text{ and } \phi(0) = 1. \end{cases} \tag{5.27}$$

By the first-order condition of (5.27), the feedback form of the optimal strategy is given by

$$q_-^* = \frac{[(\mu - r)\upsilon\rho - \sigma\eta m]\,\phi_x(x)}{\sigma\upsilon^2\tilde{\rho}^2\phi_{xx}(x)}, \quad \pi_-^* = \frac{[\sigma\rho\eta m - (\mu - r)\upsilon]\,\phi_x(x)}{\sigma^2\upsilon\tilde{\rho}^2\phi_{xx}(x)}. \tag{5.28}$$

Substituting (5.28) into the HJB equation (5.27) yields

$$\begin{cases} [rx - m(\eta - \theta)]\,\phi_x - \lambda\phi - \Lambda\frac{\phi_x^2}{\phi_{xx}} = 0, \quad x \in (0,b), \\ \phi(b) = 0 \text{ and } \phi(0) = 1, \end{cases} \tag{5.29}$$

where Λ is given by (5.16). Note that the equation in (5.29) is identical to the one in (5.15) but with different boundary conditions. To solve (5.29), we consider the

FBP:

$$\begin{cases} \Lambda y^2\hat{\phi}_{yy}(y) + (\lambda - r)y\hat{\phi}_y(y) - \lambda\hat{\phi}(y) + m(\eta-\theta)y = 0, \quad 0 \le \tilde{y}_b < y < \tilde{y}_0, \\ \hat{\phi}_y\left(\tilde{y}_0\right) = 0; \quad \hat{\phi}_y\left(\tilde{y}_b\right) = b; \quad \hat{\phi}\left(\tilde{y}_0\right) = 1; \quad \hat{\phi}\left(\tilde{y}_b\right) - b\tilde{y}_b = 0. \end{cases} \tag{5.30}$$

The solution to the FBP (5.30) is given in Lemma 5.3.3. The proof can be found in the Appendix.

Lemma 5.3.3 *(a) If $b < \frac{(\eta-\theta)m}{r}$, the solution to the FBP (5.30) is*

$$\hat{\phi}\left(y\right) = \frac{[rb-(\eta-\theta)m]y}{r}\left[\frac{1-\delta_-}{\delta_+-\delta_-}\left(\frac{y}{\tilde{y}_b}\right)^{\delta_+-1} + \frac{\delta_+-1}{\delta_+-\delta_-}\left(\frac{y}{\tilde{y}_b}\right)^{\delta_--1} - \frac{(\eta-\theta)\,m}{(\eta-\theta)m-rb}\right],$$

for $0 < \tilde{y}_b \le y \le \tilde{y}_0$, where $\delta_\pm = \frac{-(\lambda-r-\Lambda)\pm\sqrt{(\lambda-r-\Lambda)^2+4\lambda\Lambda}}{2\Lambda}$, $\tilde{y}_b = \tilde{y}_0/\tilde{\kappa}$,

$$\tilde{y}_0 = \frac{-r\delta_-}{(1-\delta_-)\left[rb-(\eta-\theta)m\right]\tilde{\kappa}^{\delta_+-1} + m(1-\delta_-)(\eta-\theta)} > 0,$$

and $\tilde{\kappa}$ is the unique solution in $(1,+\infty)$ to

$$\frac{1-\delta_-}{\delta_-}x^{\delta_+-1} + \frac{\delta_+-1}{\delta_+}x^{\delta_--1} + \frac{\delta_+-\delta_-}{\delta_+\delta_-}\frac{(\eta-\theta)m}{(\eta-\theta)m-rb} = 0.$$

Moreover, $\hat{\phi} \in C^2\left[\tilde{y}_b, \tilde{y}_0\right]$ is strictly increasing and strictly concave on $\left[\tilde{y}_b, \tilde{y}_0\right]$.

(b) If $b = \frac{(\eta-\theta)m}{r}$, the solution to the FBP (5.30) is

$$\hat{\phi}\left(y\right) = -\frac{(\eta-\theta)my}{r}\left[\frac{1}{\delta_+}\left(\frac{y}{\tilde{y}_0}\right)^{\delta_+-1} - 1\right], \quad y \in [0, \tilde{y}_0],$$

where $\delta_+ = \frac{-(\lambda-r-\Lambda)+\sqrt{(\lambda-r-\Lambda)^2+4\lambda\Lambda}}{2\Lambda}$ and $\tilde{y}_0 = \frac{r\delta_+}{(\delta_+-1)(\eta-\theta)m}$. Moreover, $\hat{\phi} \in C^2\left[0, \tilde{y}_0\right]$ is strictly increasing and strictly concave on $[0, \tilde{y}_0]$.

The proof of Lemma 5.3.4 is similar to that of Lemma 5.3.2, and is therefore omitted.

Lemma 5.3.4 *For $\hat{\phi}$ given in Lemma 5.3.3, consider its Legendre transform defined as*

$$\phi(x) = \sup_{y \in [\tilde{y}_b, \tilde{y}_0]} \left\{ \hat{\phi}(y) - xy \right\}, \quad x \in [0, b].$$

It follows that $\phi(x)$ solves (5.29) and further $\phi(x) = \psi^-(x)$ on $[0, b]$, where $\psi^-(x)$ is the value function defined in (5.10).

The proof of the following theorem can be found in the Appendix.

Theorem 5.3.4 *(a) For $x \in (0, b)$, if $b < \frac{(\eta-\theta)m}{r}$, the optimal reinsurance-investment strategy for objective (5.10) is given by*

$$q_-^*(x) = A_-\delta_+ \left(\frac{I(x)}{\tilde{y}_b} \right)^{\delta_+ - 1} - A_-\delta_- \left(\frac{I(x)}{\tilde{y}_b} \right)^{\delta_- - 1},$$

$$\pi_-^*(x) = B_-\delta_+ \left(\frac{I(x)}{\tilde{y}_b} \right)^{\delta_+ - 1} - B_-\delta_- \left(\frac{I(x)}{\tilde{y}_b} \right)^{\delta_- - 1},$$

where

$$A_- = \frac{[\sigma\eta m - (\mu - r)\upsilon\rho]\,[(\eta-\theta)m - rb]\,(\delta_+ - 1)(1-\delta_-)}{\sigma\upsilon^2\tilde{\rho}^2 r\,(\delta_+ - \delta_-)},$$

$$B_- = \frac{[(\mu - r)\upsilon - \sigma\rho\eta m]\,[(\eta-\theta)m - rb]\,(\delta_+ - 1)(1-\delta_-)}{\sigma^2\upsilon\tilde{\rho}^2 r\,(\delta_+ - \delta_-)},$$

and the associated value function is given by

$$\psi^-(x) = \frac{[(\eta-\theta)m - rb]\,(\delta_+ - 1)(1-\delta_-)}{r\,(\delta_+ - \delta_-)} I(x) \left[\left(\frac{I(x)}{\tilde{y}_b} \right)^{\delta_+ - 1} - \left(\frac{I(x)}{\tilde{y}_b} \right)^{\delta_- - 1} \right],$$

where $I(x) := \hat{\phi}_y^{-1}(x)$. The function $\hat{\phi}$ and the constants $\delta_\pm$, $\tilde{y}_b$, $\tilde{y}_0$, $\tilde{\kappa}$ are given in part (a) of Lemma 5.3.3.

(b) For $x \in (0, b)$, if $b = \frac{(\eta-\theta)m}{r}$, the optimal reinsurance-investment strategy for objective (5.10) is given by

$$q_-^*(x) = \frac{[\sigma\eta m - (\mu - r)\upsilon\rho]\,(\delta_+ - 1)[(\eta-\theta)m - rx]}{\sigma\upsilon^2\tilde{\rho}^2 r},$$

$$\pi_-^*(x) = \frac{[(\mu - r)\upsilon - \sigma\rho\eta m]\,(\delta_+ - 1)[(\eta-\theta)m - rx]}{\sigma^2\upsilon\tilde{\rho}^2 r},$$

and the associated value function is given by

$$\psi^{-}(x) = \frac{(\delta_{+}-1)\tilde{y}_0[(\eta-\theta)m-rx]}{r\delta_{+}}\left[\frac{(\eta-\theta)m-rx}{(\eta-\theta)m}\right]^{\frac{1}{\delta_{+}-1}},$$

where δ_{+} and $\tilde{y}_0$ are given in part (b) of Lemma 5.3.3.

Remark 5.3.2 *Note that if $b < \frac{(\eta-\theta)m}{r}$, the optimally controlled underlying process, denoted by $X^{u_{-}^{*}}$, can reach the upper level b with a positive probability as the volatility of $X^{u_{-}^{*}}$ is away from zero for all $x \in (0,b)$. On the contrary, if $b = (\eta-\theta)m/r$, $X^{u_{-}^{*}}$ can not reach the upper level b in any finite time horizon. In fact, by defining $Y_t := m(\eta-\theta) - rX_t^{u_{-}^{*}}$ for $t \geq 0$, one can verify that $\{Y_t\}_{t\geq 0}$ is a geometric Brownian motion, i.e., $Y_t > 0$ a.s., which further implies that $X_t^{u_{-}^{*}} < \frac{(\eta-\theta)m}{r}$ for any $t \geq 0$.*

5.3.3 A comparison of optimal strategies

In this section, we compare the optimal reinsurance-investment strategies u_{+}^{*} and u_{-}^{*} with the strategy $u_0^{*} = (q_0^{*}, \pi_0^{*})$ minimizing the infinite-time ruin probability, namely

$$\inf_{u\in\Pi} \mathbb{P}_x(\tau_0 < \infty). \tag{5.31}$$

Note that this comparison will be made only in the case where $0 < x < b \leq \frac{(\eta-\theta)m}{r}$ given that ψ^{-} was only analyzed under (5.26). We recall that Promislow and Young [78] have shown that for $0 < x < b \leq \frac{(\eta-\theta)m}{r}$,

$$\begin{cases} q_0^{*}(x) = \frac{[\sigma\eta m-(\mu-r)\upsilon\rho][(\eta-\theta)m-rx]}{\sigma\upsilon^2\tilde{\rho}^2\Lambda}, \\ \pi_0^{*}(x) = \frac{[(\mu-r)\upsilon-\sigma\rho\eta m][(\eta-\theta)m-rx]}{\sigma^2\upsilon\tilde{\rho}^2\Lambda}. \end{cases} \tag{5.32}$$

Theorem 5.3.5 *For $0 < x < b \leq \frac{(\eta-\theta)m}{r}$, we have the following relations for the optimal reinsurance-investment strategies u_+^*, u_-^* and u_0^* given in Theorems 5.3.2, 5.3.4, and equation (5.32), respectively:*

(1) $q_+^*(x) \geq q_0^*(x) \geq q_-^*(x) \geq 0$;

(2) $\pi_+^*(x) \geq \pi_0^*(x) \geq \pi_-^*(x) \geq 0$ if $\frac{\mu-r}{\sigma} \geq \frac{\rho\eta m}{\upsilon}$; $\pi_+^*(x) \leq \pi_0^*(x) \leq \pi_-^*(x) \leq 0$ if $\frac{\mu-r}{\sigma} \leq \frac{\rho\eta m}{\upsilon}$.

Proof. (1) By the condition $\rho\frac{\mu-r}{\sigma} \leq \frac{\eta m}{\upsilon}$ in (5.8), $q_+^*(x), q_0^*(x), q_-^*(x) \geq 0$ for any $x \in (0, b)$. Further, by (5.14), (5.15), Lemma 5.3.2 and (5.32), we deduce

$$\begin{aligned}
q_+^* - q_0^* &= \frac{[(\mu-r)\rho\upsilon - \sigma\eta m]\,\psi_x^+}{\sigma\upsilon^2\tilde{\rho}^2\psi_{xx}^+} - \frac{[\sigma\eta m - (\mu-r)\rho\upsilon]\,[(\eta-\theta)m - rx]}{\sigma\upsilon^2\tilde{\rho}^2\Lambda} \\
&= \frac{\sigma\eta m - (\mu-r)\rho\upsilon}{\sigma\upsilon^2\tilde{\rho}^2\Lambda\psi_x^+}\left(-\Lambda\frac{(\psi_x^+)^2}{\psi_{xx}^+} + [rx - (\eta-\theta)m]\,\psi_x^+\right) \\
&= \frac{\sigma\eta m - (\mu-r)\rho\upsilon}{\sigma\upsilon^2\tilde{\rho}^2\Lambda\psi_x^+}\lambda\psi^+ \\
&\geq 0,
\end{aligned}$$

where the last inequality is due to condition (5.8) and Theorem 5.3.1. Similarly, by (5.28), (5.29), (5.32) and Lemma 5.3.4, we can show that

$$\begin{aligned}
q_0^* - q_-^* &= \frac{[(\mu-r)\rho\upsilon - \sigma\eta m]\,[rx - (\eta-\theta)m]}{\sigma\upsilon^2\tilde{\rho}^2\Lambda} - \frac{[(\mu-r)\rho\upsilon - \sigma\eta m]\,\psi_x^-}{\sigma\upsilon^2\tilde{\rho}^2\psi_{xx}^-} \\
&= \frac{\sigma\eta m - (\mu-r)\rho\upsilon}{\sigma\upsilon^2\tilde{\rho}^2\Lambda\psi_x^-}\left(-[rx - (\eta-\theta)m]\,\psi_x^- + \Lambda\frac{(\psi_x^-)^2}{\psi_{xx}^-}\right) \\
&= -\frac{\sigma\eta m - (\mu-r)\rho\upsilon}{\sigma\upsilon^2\tilde{\rho}^2\Lambda\psi_x^-}\lambda\psi^- \\
&\geq 0.
\end{aligned}$$

(2) When $\frac{\mu-r}{\sigma} = \frac{\rho\eta m}{\upsilon}$, it is clear that $\pi_+^*(x) = \pi_0^*(x) = \pi_-^*(x) \equiv 0$. When $\frac{\mu-r}{\sigma} \neq \frac{\rho\eta m}{\upsilon}$, one can see that

$$q^*(x) = \frac{1}{\frac{\mu-r}{\sigma} - \frac{\rho m\eta}{\upsilon}}\frac{\sigma\eta m - (\mu-r)\rho\upsilon}{\upsilon^2}\pi^*(x), \tag{5.33}$$

where (q^*, π^*) stands for any of the three optimal strategies (q_+^*, π_+^*), (q_0^*, π_0^*), and (q_-^*, π_-^*). Since $\frac{\sigma\eta m-(\mu-r)\rho\upsilon}{\upsilon^2} \geq 0$ by (5.8), the result of part (2) then follows immediately from relation (5.33) and part (1). ■

The condition $\frac{\rho\eta m}{\upsilon} \leq (\geq)\frac{\mu-r}{\sigma}$ is used to determine the sign of π^*. Intuitively, part (2) of Theorem 5.3.5 indicates that the optimal strategy is to long (short resp.) the stock if its Sharpe ratio $\frac{\mu-r}{\sigma}$ is larger (less resp.) than the benchmark $\frac{\rho\eta m}{\upsilon}$. Further, note that when $\rho \leq 0$, both π^* and q^* (condition (5.8)) are positive because of the diversification benefit between the insurance and the financial risks together with the positiveness of η and $\frac{\mu-r}{\sigma}$.

Remark 5.3.3 *If* $0 < x < b < \frac{(\eta-\theta)m}{r}$, u_0^* *is also an optimal strategy for the objective functions (5.9) and (5.10) in the infinite-time horizon, i.e.,*

$$\sup_{u\in\Pi} \mathbb{P}_x\left(\tau_b < \tau_0\right), \tag{5.34}$$

and

$$\inf_{u\in\Pi} \mathbb{P}_x\left(\tau_0 < \tau_b\right). \tag{5.35}$$

In fact, (5.31), (5.34), and (5.35) fall into the more general setup of Bäuerle and Bayraktar [20] which have shown that the optimal strategy is the one that maximizes the ratio of the drift to the volatility squared of the underlying process (5.7). Note that the approach in Bäuerle and Bayraktar [20] does not hold for a random (or finite) maturity setting such as in objectives (5.9) and (5.10) as the time-change arguments do not apply then.

Note that the case $b = \frac{(\eta-\theta)m}{r}$ is specifically excluded from Remark 5.3.3 as u_0^* is clearly not an optimal strategy to (5.34). One can show (using the similar arguments as in Remark 5.3.2) that the controlled process $X^{u_0^*}$ can not reach the safe level $\frac{(\eta-\theta)m}{r}$ in a finite time horizon.

Remark 5.3.4 *In Theorem 5.3.5, it is shown that*

$$u_+^* \succ u_0^* \succ u_-^*,$$

where "$\succ$" is the relation of aggressiveness defined in Definition 5.2.2. Therefore, the optimal reinsurance-investment strategy u_+^* (u_-^* resp.) is more aggressive (conservative resp.) than the strategy u_0^*. This is consistent with the underlying objective of these strategies:

- *under objective (5.9) (which is the harshest of the three), the insurer takes on more risk as it shall not only avoid ruin but reach the target level within the time period;*
- *under objective (5.10) (which is the most moderate of the three), the insurer takes on less risk as it only requires the insurer to meet one of the following two conditions: no ruin before e_λ or reaching the target level b before ruin.*

5.4 Numerical examples

In this section, we provide some numerical examples to support the theoretic results of Section 5.3. Most notably, the effect of some model parameters on the optimal reinsurance-investment strategies will be examined. In the following examples, we consider the following joint set of exogenous parameters: $\mu = 0.08$, $r = 0.05$, $m = 3$, $v = 1$, $\sigma = 0.2$, $\theta = 0.1$ and $\eta = 0.2$. The other parameters x, b, λ and ρ may vary.

Example 5.4.1 (Effect of λ) *In this example, we examine the effect of the expected length of the time horizon on the optimal reinsurance-investment strategies. We let $\rho = 0$ and $b = 1$.*

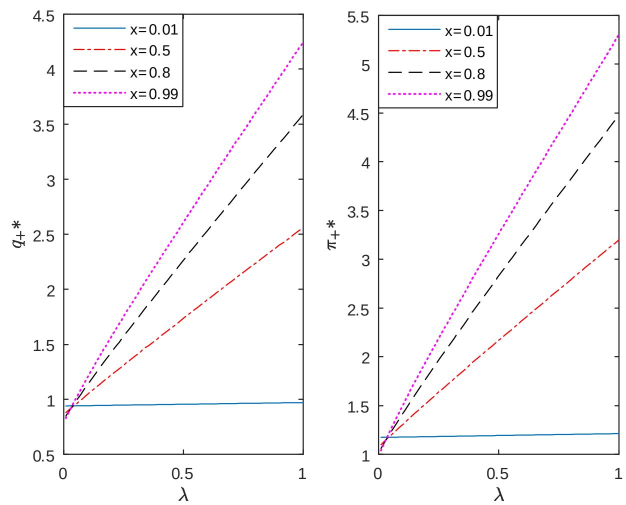


Figure 5.1. Effect of λ on (q_+^*, π_+^*)

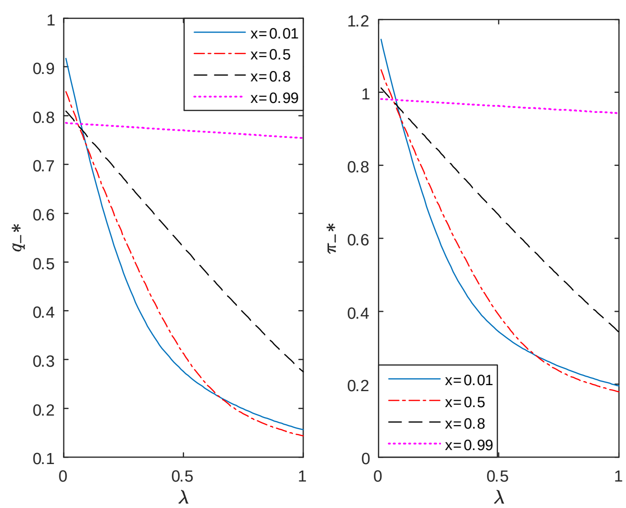


Figure 5.2. Effect of λ on (q_-^*, π_-^*)

Figure 5.1 (Figure 5.2 resp.) shows that the optimal reinsurance-investment *strategy under objective (5.9) (objective (5.10) resp.) increases (decreases resp.) in λ. The opposite trend result from the essential difference between the two objective functions. As λ increases (i.e. the expected time horizon becomes shorter), an insurer with objective (5.9) has to adopt a more aggressive strategy by undertaking more insurance and financial risks to reach the target level b over a shorter time period. On the contrary, Figure 5.2 shows that, as λ increases, the insurer with objective (5.10) will adopt a more conservative strategy as it only requires the insurer to avoid ruin over a relatively short time period.*

Example 5.4.2 (Effect of x) *In this example, we examine the effect of the surplus level on the optimal reinsurance-investment strategies. We consider $\rho = 0$ and $b = 1$.*

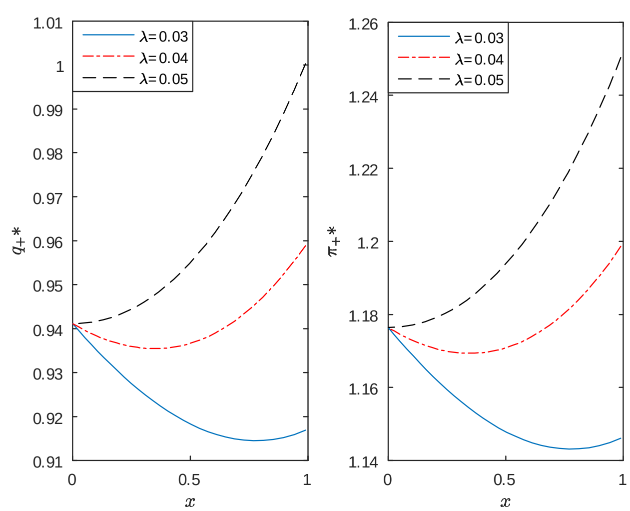


Figure 5.3. Effect of x on (q_+^*, π_+^*)

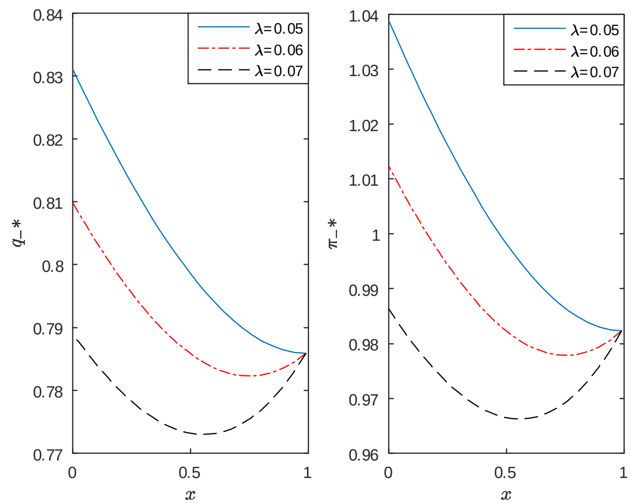


Figure 5.4. Effect of x on (q_-^*, π_-^*)

Figures 5.3 and 5.4 show that in general the strategies (q_+^*, π_+^*) and (q_-^*, π_-^*) are not monotone in the surplus level x. More specifically, when the surplus level x is small, both strategies decrease in x to avoid ruin (by reducing the risk position). For larger values of surplus x, an insurer will adopt a more aggressive strategy to win the game by achieving the target level b. Moreover, when λ is large (i.e. an expected shorter time horizon), the strategy u_+^* becomes monotone increasing in x as the time constraint factor for the insurer to reach the target level b dominates. Figures 5.3 and 5.4 also show that u_+^* is increasing in λ and u_-^* is decreasing in λ, a conclusion consistent with the last example.

Example 5.4.3 (Comparison of optimal strategies) *In this example, we compare the three optimal reinsurance-investment strategies u_+^*, u_0^*, and u_-^*. We let*

$b = 1$.

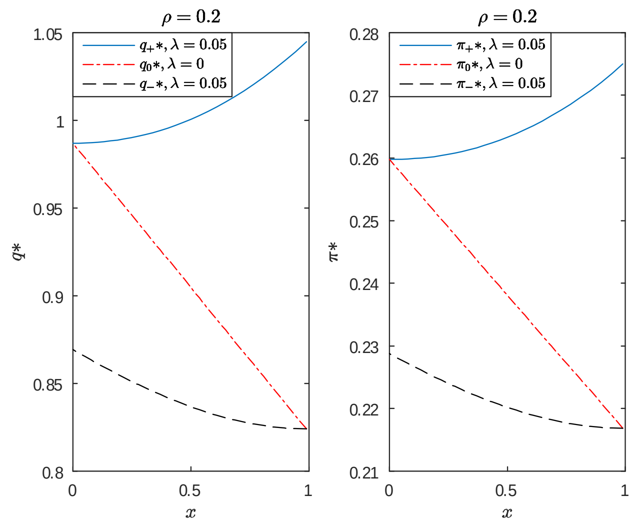


Figure 5.5. Comparison of u_+^*, u_0^*, *and* u_-^* *when* $\dfrac{\rho\eta m}{\upsilon} \leq \dfrac{\mu - r}{\sigma}$

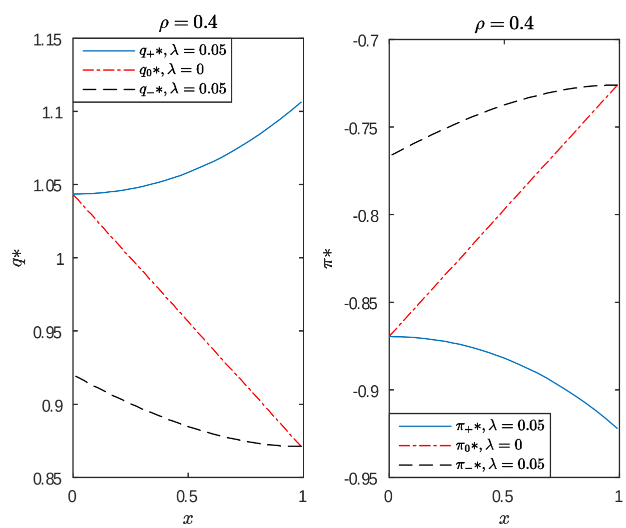


Figure 5.6. Comparison of u_+^*, u_0^*, *and* u_-^* *when* $\dfrac{\rho\eta m}{\upsilon} > \dfrac{\mu - r}{\sigma}$

The left panels of Figures 5.5 and 5.6 show that $q_+^* \geq q_0^* \geq q_-^* \geq 0$. *Furthermore, the right panel of Figure 5.5 shows that* $\pi_+^* \geq \pi_0^* \geq \pi_-^* \geq 0$ *if* $\frac{\rho\eta m}{\upsilon} \leq \frac{\mu - r}{\sigma}$, *and the right panel of Figure 5.6 shows that* $\pi_+^* \leq \pi_0^* \leq \pi_-^* < 0$ *if* $\frac{\rho\eta m}{\upsilon} > \frac{\mu - r}{\sigma}$. *All of them are consistent with Theorem 5.3.5 and the implications have been discussed in Remark 3.4.*

Example 5.4.4 (Effect of b on u_-^*) *Since* u_+^* *is independent of* b *as shown in Proposition 5.3.1, we only examine the effect of* b *on* u_-^*. *We let* $\rho = 0$ *and* $\lambda = 0.05$.

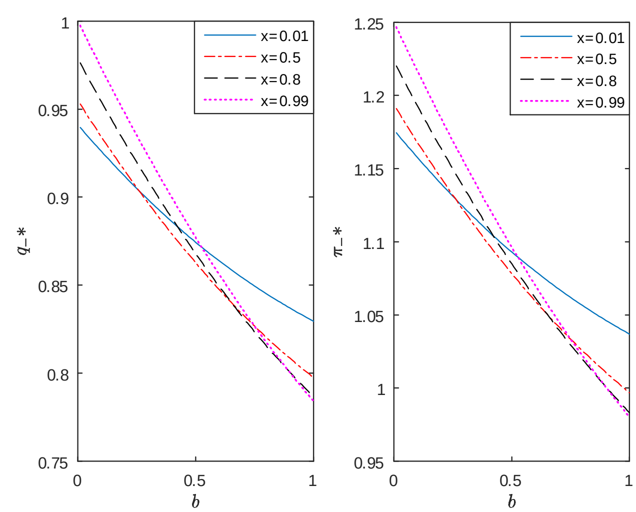


Figure 5.7. Effect of b *on* u_-^*

Figure 5.7 shows that u_-^* *decreases in* b, *i.e., for a high target level* b, *the insurer with objective (5.10) tends to adopt a more conservative strategy to avoid ruin because the chance of winning the game by reaching a high target level is small.*

Example 5.4.5 (Effect of ρ) *In this example, we examine the effect of* ρ *on the optimal reinsurance-investment strategies. We let* $\lambda = 0.05$ *and* $b = 2$.

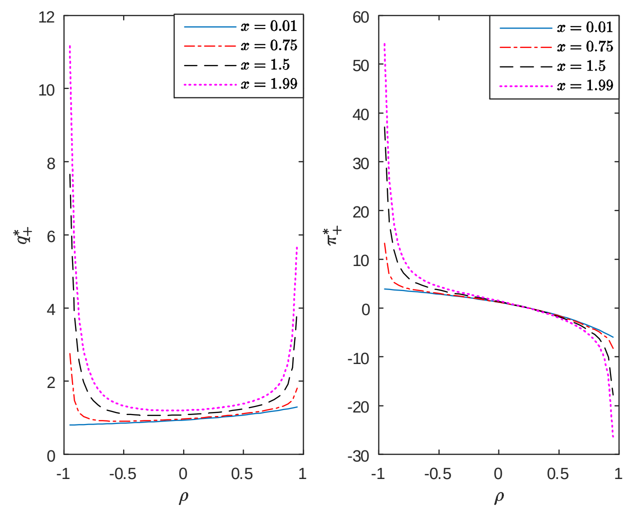


Figure 5.8. Effect of ρ on (q_+^*, π_+^*)

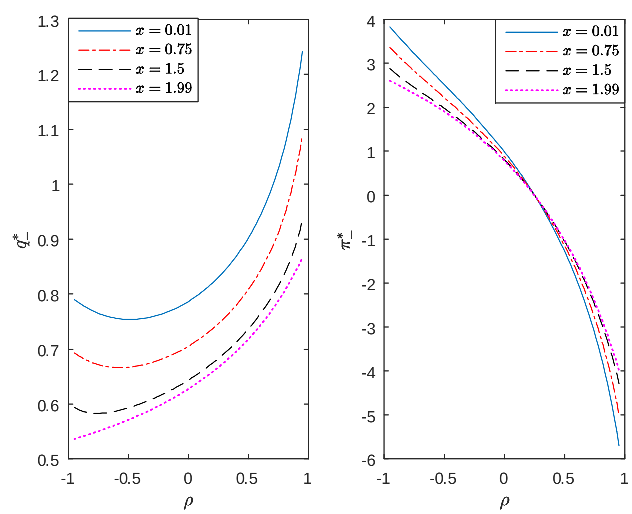


Figure 5.9. Effect of ρ on (q_-^*, π_-^*)

The right panels of Figures 5.8 and 5.9 show that the optimal investment strate-

gies π_+^* *and* π_-^* *are more aggressive when the correlation between the insurance and financial risks is strong. Correspondingly, to diversify the entire porfolio risk, the reinsurance strategies* q_+^* *and* q_-^* *also become more aggressive as* ρ *approaches* -1 *or* 1 *as shown in the left panels of Figures 5.8 and 5.9.*

Chapter 6

Equilibrium Investment-Reinsurance Strategy in a Combined Reinsurance Class

6.1 Introduction

An integrated reinsurance and investment strategy is commonly employed by an insurer (cedent) for the purpose of increasing its underwriting capacity, stabilizing the underwriting results, protecting itself against catastrophic losses, and achieving financial growth. In this chapter, we study an optimal reinsurance-investment problem for an insurer under a mean-variance criterion in a dynamic setting. We model the insurer's basic surplus process, that is, the surplus process without any reinsurance-investment strategy, by a *spectrally negative Lévy process*. The model is widely employed in the context of risk theory and ruin theory (see, e.g., Yang and Zhang [87], Chiu and Yin [43], Avram et al. [12], Landriault et al. [62]) in the actuarial science literature. It is a generalization of many insurance models studied in the context of reinsurance-investment problems, including the Brownian motion model (see, e.g., Promislow and Young [78]), the classical Cramér-Lundberg (C-L) model (see, e.g., Zeng et. al. [90]), and the jump diffusion model (e.g., Zeng et. al. [91]).

In particular, we consider two types of combinations, referred to as *type I reinsurance policy* (*type II reinsurance policy* resp.). Under type I policy, the reinsurer covers a proportion of the *excess loss* (the part exceeding a retention level) for each individual claim, while the insurer covers the remaining. This type of policy is seldom studied in the literature of finding optimal reinsurance policies, especially under dynamic settings. Under static settings, a similar reinsurance policy called *change-loss* policy is studied extensively (see, e.g., Cai et al. [35], Tan and Weng [85]). The difference between a type I policy and a change-loss policy is that the latter is applied to aggregate claims instead of each individual claim. Under type II policy, an insurer covers a proportion of each individual loss up to a retention level, while the remaining is ceded to a reinsurer. A rich literature has been contributed to investigating the optimal reinsurance under type II policy in both static settings (see, e.g., Centeno [36], Centeno and Simões [37]) and dynamic settings (see, e.g., Zhang et al. [93], Liang and Guo [65]).

By deriving the closed-form equilibrium reinsurance-investment strategy under type I (type II resp.) reinsurance policy, we find that under the expected value premium principle, it is optimal for the insurer to transfer all the excess losses to the reinsurer (to cover all the losses under the retention level resp.). In other words, we show that the **excess-of-loss** reinsurance is the optimal reinsurance form (within the combined class of type I and type II reinsurance policies we consider) for the time-consistent insurer under a mean-variance criterion. The result is consistent with several works in the literature, where under the expected value principle and various objective functions, the optimality of the excess-of-loss policy is shown, including maximizing the expected cumulative discounted dividend pay-outs (see, e.g., Asmussen et al. [8]), maximizing the expected utility of terminal wealth (see, e.g., Liang and Guo [65], Zeng and Luo [92]), and minimizing the ruin probability (see, e.g., Zhang et al. [93], Meng and Zhang [72], Bai et al. [13], Zhou and Cai [95]). Note that the conclusion might not hold under other premium principles (for instance, the variance principle), and the investigation is left for future research.

The remainder of this chapter is organized as follows. Section 6.2 describes the formulation of the model. Section 6.3 derives the explicit expressions of the equilibrium reinsurance-investment strategy and the corresponding equilibrium value function respectively under type I and type II reinsurance policies. Section 6.4 presents some numerical examples to illustrate our findings. Some technical proofs are postponed to the Appendix.

6.2 Model Formulation

Let $(\Omega, \mathcal{F}, \boldsymbol{F} = \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a filtered complete probability space satisfying the usual conditions and $T > 0$ be a finite time horizon. Consider an insurer's basic surplus process modeled by a spectrally negative Lévy process defined on $(\Omega, \mathcal{F}, \boldsymbol{F} = \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ with dynamics

$$\mathrm{d}U_t = c\mathrm{d}t + \sigma_1 \mathrm{d}B_t^{(1)} - \int_0^\infty zN(\mathrm{d}t, \mathrm{d}z), \; U_0 > 0,$$

where $c > 0$ is the premium rate, $\sigma_1 > 0$ is the volatility rate, $\left\{B_t^{(1)}\right\}_{t\geq 0}$ is an $\boldsymbol{F}$-adapted standard Brownian motion, and $N(\mathrm{d}t, \mathrm{d}z)$ is a Poisson point measure representing the number of insurance claims of size $(z, z + \mathrm{d}z)$ within the time period $(t, t + \mathrm{d}t)$. We denote the compensated measure of $N(\mathrm{d}t, \mathrm{d}z)$ by $\tilde{N}(\mathrm{d}t, \mathrm{d}z) = N(\mathrm{d}t, \mathrm{d}z) - v(z)\mathrm{d}t$, where v is a Lévy measure such that $\int_0^\infty zv\,(dz) < \infty$, representing the expected number of insurance claims of size $(z, z+\mathrm{d}z)$ within a unit time interval. The insurer's premium c is determined under the expected value principle, i.e., $c = (1 + \theta) \int_0^\infty zv\,(\mathrm{d}z)$, where $\theta > 0$ is the safety loading of the insurer.

The insurer manages the insurance liabilities by purchasing a combined reinsurance policy (strategy) $(m_t, p_t)_{t\in[0,T]}$, where $m_t \in [0, \infty)$ is a retention level and $p_t \in [0, 1]$[1] is a proportional reinsurance coverage. For an individual claim incurred at $t \in [0, T]$, denoted by Z_t, the retained loss of the insurer is represented by $l(Z_t) : [0, \infty) \to [0, \infty)$ with $0 \leq l(Z_t) \leq Z_t$, while the reinsurer covers the remaining loss $Z_t - l(Z_t)$. To be more specific, under type I and type II policy respectively, the retained loss function takes the form

$$l(Z_t) = Z_t \mathbf{1}_{\{Z_t \leq m_t\}} + (m_t + p_t (Z_t - m_t))\, \mathbf{1}_{\{Z_t > m_t\}}, \tag{6.1}$$

and

$$l(Z_t) = p_t Z_t \wedge m_t. \tag{6.2}$$

The premium rate of the reinsurance policy is given by

$$(1 + \eta) \int_0^\infty (z - l(z))v\,(\mathrm{d}z)\,,$$

determined again under the expected value principle, where η is the reinsurer's safety loading. It is commonly assumed in the literature that $\eta > \theta$, indicating that a reinsurance policy is usually more expensive. Under the reinsurance strategy, the dynamics of the surplus process is governed by

[1] We assume that the insurer is not allowed to acquire new reinsurance business on the same risk, i.e., the case $p_t > 1$ is excluded.

$$
\begin{aligned}
\mathrm{d}R_t &= \mathrm{d}U_t - (1+\eta)\int_0^\infty [z - l(z)]v\,(\mathrm{d}z)\,\mathrm{d}t + \int_0^\infty [z - l(z)]N\,(\mathrm{d}t, \mathrm{d}z) \\
&= (1+\theta)\int_0^\infty zv\,(\mathrm{d}z)\,\mathrm{d}t + \sigma_1 \mathrm{d}B_t^{(1)} - (1+\eta)\int_0^\infty [z - l(z)]v\,(\mathrm{d}z)\,\mathrm{d}t \\
&\quad - \int_0^\infty zN\,(\mathrm{d}t, \mathrm{d}z) + \int_0^\infty [z - l(z)]N\,(\mathrm{d}t, \mathrm{d}z) \\
&= \left[(\theta - \eta)\int_0^\infty zv\,(\mathrm{d}z) + \eta\int_0^\infty l(z)v\,(\mathrm{d}z)\right]\mathrm{d}t + \sigma_1 \mathrm{d}B_t^{(1)} - \int_0^\infty l(z)\tilde{N}\,(\mathrm{d}t, \mathrm{d}z)\,.
\end{aligned}
$$

Furthermore, suppose that the insurer invests in a financial market consisting of a risk-free asset with constant interest rate $r > 0$ and a risky asset governed by a geometric Brownian motion with dynamics

$$
\mathrm{d}S_t = \mu S_t \mathrm{d}t + \sigma_2 S_t \left(\rho \mathrm{d}B_t^{(1)} + \sqrt{1-\rho^2}\mathrm{d}B_t^{(2)}\right), \quad S_0 > 0,
$$

where $\mu > r$, $\sigma_2 > 0$, $\rho \in (-1, 1)$, and $\{B_t^{(2)}\}_{t\geq 0}$ is another $\boldsymbol{F}$-adapted standard Brownian motion, independent of $B^{(1)}$ and $N\,(\mathrm{d}t, \mathrm{d}z)$. We denote by π_t the amount of surplus invested in the risky asset at time t and $\{X_t^u\}_{t\geq 0}$ the corresponding insurance surplus process under a reinsurance-investment strategy $u := (m_t, p_t, \pi_t)_{t\in[0,T]}$. The dynamics of the surplus process $\{X_t^u\}_{t\in[0,T]}$ is then given by

$$
\begin{aligned}
\mathrm{d}X_t^u &= \pi_t \frac{\mathrm{d}S_t}{S_t} + (X_t^u - \pi_t)\,r\mathrm{d}t + \mathrm{d}R_t \\
&= \left[rX_t^u + (\mu - r)\,\pi_t + (\theta - \eta)\int_0^\infty zv\,(\mathrm{d}z) + \eta\int_0^\infty l(z)v\,(\mathrm{d}z)\right]\mathrm{d}t \\
&\quad + \sqrt{\sigma_1^2 + 2\rho\sigma_1\sigma_2\pi_t + \pi_t^2\sigma_2^2}\,\mathrm{d}B_t - \int_0^\infty l(z)\tilde{N}\,(\mathrm{d}t, \mathrm{d}z)\,, \qquad (6.3)
\end{aligned}
$$

where $\{B_t\}_{t\geq 0}$ is an $\boldsymbol{F}$-adapted standard Brownian motion, independent of $N(\mathrm{d}t, \mathrm{d}z)$.

Definition 6.2.1 *(Admissible strategy). Let $\mathbb{U}$ be the set of all admissible strategies. A strategy $u = (m_t, p_t, \pi_t)_{t\in[0,T]} \in \mathbb{U}$ if it satisfies the following conditions:*

(1) u is $\boldsymbol{F}$-progressively measurable;

(2) $\forall t \in [0, T]$, $m_t \in [0, \infty)$ and $p_t \in [0, 1]$;

(3) $\forall (t, x) \in [0, T] \times \mathbb{R}$, $\mathbb{E}_{t,x}[\int_0^T (p_s^2 + m_s^2 + \pi_s^2)\mathrm{d}s] < \infty$;

(4) $\forall (t, x) \in [0, T] \times \mathbb{R}$, the stochastic differential equation (6.3) has a unique strong solution.

The **main objective** of the chapter is to study the reinsurance-investment problem for a time-consistent insurer under a mean-variance criterion, i.e.,

$$\sup_{u \in \mathbb{U}} J^u(t, x), \tag{6.4}$$

where

$$J^u(t, x) = \mathbb{E}_{t,x}[X_T^u] - \frac{\gamma}{2}\mathrm{Var}_{t,x}[X_T^u], \ (t, x) \in [0, T] \times \mathbb{R}, \tag{6.5}$$

is the mean-variance criterion with $\gamma > 0$ reflecting the insurer's degree of risk aversion.

Problem (6.4) is a time-inconsistent problem in the sense that Bellman's optimality principle fails. We tackle the problem from a non-cooperative game point of view by defining an equilibrium strategy and its corresponding equilibrium value function.

Definition 6.2.2 *For an admissible strategy $u^* = (m_t^*, p_t^*, \pi_t^*)_{t \in [0,T]}$, we define the following strategy*

$$u_s^\varepsilon = \begin{cases} (\bar{m}, \bar{p}, \bar{\pi}), & t \le s < t + \varepsilon, \\ u_s^*, & t + \varepsilon \le s < T, \end{cases}$$

where $(\bar{m}, \bar{p}, \bar{\pi}) \in \mathcal{U} := [0, \infty) \times [0, 1] \times \mathbb{R}$ and $\varepsilon \in \mathbb{R}^+$. If for all $(t, x) \in [0, T] \times \mathbb{R}$,

$$\liminf_{\varepsilon \downarrow 0} \frac{J^{u^*}(t, x) - J^{u^\varepsilon}(t, x)}{\varepsilon} \ge 0,$$

then u^* is an equilibrium strategy and $J^{u^*}(t, x)$ is the corresponding equilibrium value function.

6.3 Equilibrium Reinsurance-Investment Strategy

6.3.1 General Framework

Throughout the chapter, we make the following additional integrability condition on the Lévy measure v :

$$\int_1^\infty z^2 v(\mathrm{d}z) < \infty.$$

For any $u = (m, p, \pi) \in \mathcal{U}$, we define an integral-differential operator $\mathcal{A}^u$ as

$$\begin{aligned}
\mathcal{A}^u \phi(t,x) &:= \lim_{\varepsilon \downarrow 0} \frac{\mathbb{E}_{t,x}\left[\phi\left(t+\varepsilon, X^u_{t+\varepsilon}\right)\right] - \phi(t,x)}{\varepsilon} \\
&= \left[rx + (\mu - r)\pi + (\theta - \eta)\int_0^\infty z v(\mathrm{d}z) + (1+\eta)\int_0^\infty l(z;t) v(\mathrm{d}z)\right]\phi_x(t,x) \\
&+ \frac{1}{2}\left(\sigma_1^2 + 2\rho\pi\sigma_1\sigma_2 + \pi^2\sigma_2^2\right)\phi_{xx}(t,x) + \phi_t(t,x) \\
&+ \int_0^\infty \left(\phi(t, x - l(z;t)) - \phi(t,x)\right) v(\mathrm{d}z), \qquad (6.6)
\end{aligned}$$

where $\phi(t,x) \in C^{1,2}([0,T]\times\mathbb{R})$, ϕ_t is the first order partial derivative with respect to t, and ϕ_x and ϕ_{xx} are respectively the first and second order partial derivative with respect to x.

We first provide a verification theorem. The proof of the theorem is postponed to the Appendix.

Theorem 6.3.1 *(Verification theorem). Suppose that there exist $V(t,x)$, $g(t,x) \in C^{1,2}([0,T]\times\mathbb{R})$ satisfying the following conditions*

(1) for all $(t,x) \in [0,T]\times\mathbb{R}$,

$$\sup_{u\in\mathcal{U}}\left\{\mathcal{A}^u V(t,x) - \mathcal{A}^u \frac{\gamma}{2} g^2(t,x) + \gamma g(t,x)\mathcal{A}^u g(t,x)\right\} = 0, \qquad (6.7)$$

and let u^* denote the optimal value to attain the supremum in (6.7);

(2) for all $(t,x) \in [0,T] \times \mathbb{R}$*,*

$$\mathcal{A}^{u^*} g(t,x) = 0; \tag{6.8}$$

(3) for $x \in \mathbb{R}$*,*

$$V(T,x) = x \text{ and } g(T,x) = x. \tag{6.9}$$

Then u^* *is an equilibrium reinsurance-investment strategy, and* $V(t,x) = J^{u^*}(t,x)$ *is the corresponding equilibrium value function. Furthermore,* $g(t,x) = \mathbb{E}_{t,x}[X_T^{u^*}]$*.*

Next, we study the equilibrium strategy and the corresponding equilibrium value function under the class of type I and type II policies, respectively.

6.3.2 Type I Reinsurance Policy

Under the class of type I reinsurance policies, consider a reinsurance-investment strategy $\hat{u} = (\hat{m}_t, \hat{p}_t, \hat{\pi}_t)_{t\in[0,T]}$. For an individual claim Z_t, the retained loss function of the insurer takes the form

$$\hat{l}\,(Z_t) = Z_t \mathbf{1}_{\{Z_t \le \hat{m}_t\}} + (\hat{m}_t + \hat{p}_t\,(Z_t - \hat{m}_t))\,\mathbf{1}_{\{Z_t > \hat{m}_t\}}, \tag{6.10}$$

while the reinsurer covers the remaining loss $(1-\hat{p}_t)\,(Z_t - \hat{m}_t)\,\mathbf{1}_{\{Z_t > \hat{m}_t\}}$. Note that type I reinsurance policy is a generalization of a proportional reinsurance (with $\hat{m}_t \equiv 0$) and an excess-of-loss reinsurance (with $\hat{p}_t \equiv 0$). The main results on the equilibrium strategy and the corresponding value function are given in Theorem 6.3.2. The proof of the theorem can be found in the Appendix.

Theorem 6.3.2 *Under the class of type I reinsurance policies, for* $(t,x) \in [0,T] \times \mathbb{R}$*, the equilibrium reinsurance-investment strategy* $\hat{u}^* = (\hat{m}_t^*, \hat{p}_t^*, \hat{\pi}_t^*)$ *for problem*

(6.4) is given by

$$\begin{cases} \hat{m}^*(t) = \frac{\eta}{\gamma}\mathrm{e}^{-r(T-t)}, \\ \hat{p}^*(t) = 0, \\ \hat{\pi}^*(t) = \frac{(\mu-r)}{\gamma\sigma_2^2}\mathrm{e}^{-r(T-t)} - \rho\frac{\sigma_1}{\sigma_2}, \end{cases} \tag{6.11}$$

and the corresponding equilibrium value function is

$$\hat{V}(t,x) = \mathrm{e}^{r(T-t)}x + \bar{A}(t), \tag{6.12}$$

where

$$\begin{aligned} \bar{A}(t) = & \frac{(\mu-r)^2}{2\gamma\sigma_2^2}(T-t) \\ & + \int_t^T \left\{ e^{r(T-s)} \left[-(\mu-r)\rho\frac{\sigma_1}{\sigma_2} + (\theta-\eta)\int_0^\infty zv(\mathrm{d}z) + \eta\int_0^\infty (z\wedge\hat{m}_s^*)v(\mathrm{d}z) \right] \right. \\ & \left. -\frac{\gamma}{2}e^{2r(T-s)} \left[\left(1-\rho^2\right)\sigma_1^2 + \int_0^\infty (z^2\wedge\hat{m}_s^{*2})v(\mathrm{d}z) \right] \right\} \mathrm{d}s. \end{aligned}$$

Furthermore,

$$\mathbb{E}_{t,x}\left[X_T^{\hat{u}^*}\right] = g(t,x) = \mathrm{e}^{r(T-t)}x + \bar{a}(t), \tag{6.13}$$

where

$$\bar{a}(t) = \bar{A}(t) + \frac{(\mu-r)^2}{2\gamma\sigma_2^2}(T-t) + \int_t^T \frac{\gamma}{2}e^{2r(T-s)} \left[\left(1-\rho^2\right)\sigma_1^2 + \int_0^\infty (z^2\wedge\hat{m}_s^{*2})v(\mathrm{d}z) \right] \mathrm{d}s.$$

Remark 6.3.1 *Theorem 6.3.2 implies that within the class of type I reinsurance policies, the excess-of-loss reinsurance policy is the optimal form for an insurer under a mean-variance criterion. Note that the equilibrium strategy is independent of the state variable x. The independence results from the constant risk aversion assumption; see Björk et al. [30] for a detailed discussion. Moreover, the equilibrium excess-of-loss strategy is independent of the parameters of the risky asset and the safety loading of the insurer, while the equilibrium investment strategy is independent of the safety loadings of the insurer and the reinsurer. In other words,* *the equilibrium reinsurance strategy is unaffected by the financial market, while the equilibrium investment strategy is unaffected by the price of the reinsurance market, and both strategies are unaffected by the price of the insurance market.*

The behavior of the equilibrium strategy is given in the following proposition. The proof is straightforward and hence omitted. The intuitions behind the behavior of the equilibrium strategy will be discussed in details in the numerical analysis.

Proposition 6.3.1 *For any $t \in [0,T]$, $\hat{m}_t^*$ is increasing in η and decreasing in r and γ; $\hat{\pi}_t^*$ is increasing in μ and decreasing in r, γ, σ_1, and σ_2.*

6.3.3 Type II Reinsurance Policy

Under the class of type II reinsurance policies, consider a reinsurance-investment strategy $\tilde{u} := (\tilde{m}_t, \tilde{p}_t, \tilde{\pi}_t)_{t\in[0,T]}$. For an individual claim Z_t, the retained loss function of the insurer takes the form

$$\begin{aligned}\tilde{l}\,(Z_t;t) &= \tilde{p}_t Z_t \wedge \tilde{m}_t \\ &= \tilde{p}_t Z_t \mathbf{1}_{\left\{Z_t \leq \frac{\tilde{m}_t}{\tilde{p}_t}\right\}} + \tilde{m}_t \mathbf{1}_{\left\{Z_t > \frac{\tilde{m}_t}{\tilde{p}_t}\right\}},\end{aligned} \tag{6.14}$$

while the reinsurer covers the remaining loss $Z_t - \tilde{p}_t Z_t \wedge \tilde{m}_t$. This case $\tilde{p}_t = 0$ is excluded as it is meaningless for the insurer to undertake the risk at a safety loading θ and cede the same risk at a higher safety loading η. A proportional type policy (with $\tilde{m}_t \equiv +\infty$) is also excluded as we have shown in Subsection 3.2 that it is not the optimal form. Note that type II policy includes an excess-of-loss policy (with $\tilde{p}_t \equiv 1$) as a special case. The main results on the equilibrium strategy and the corresponding value function are summarized in Theorem 6.3.3. The proof of the theorem can be found in the Appendix.

Theorem 6.3.3 *Under the class of type II reinsurance policies, for* $(t, x) \in [0, T] \times \mathbb{R}$*, the equilibrium reinsurance-investment strategy* $\tilde{u}^* = (\tilde{m}_t^*, \tilde{p}_t^*, \tilde{\pi}_t^*)$ *for problem (6.4) is given by*

$$\begin{cases} \tilde{m}^*(t) = \frac{\eta}{\gamma} \mathrm{e}^{-r(T-t)}, \\ \tilde{p}^*(t) = 1, \\ \tilde{\pi}^*(t) = \frac{(\mu - r)}{\gamma \sigma_2^2} \mathrm{e}^{-r(T-t)} - \rho \frac{\sigma_1}{\sigma_2}, \end{cases} \tag{6.15}$$

and the corresponding equilibrium value function is

$$\tilde{V}(t, x) = \hat{V}(t, x). \tag{6.16}$$

Furthermore,

$$\mathbb{E}_{t,x}\left[X_T^{\tilde{u}^*}\right] = \tilde{g}(t, x) = \hat{g}(t, x). \tag{6.17}$$

Remark 6.3.2 *Theorem 6.3.3 implies that within the class of type II reinsurance policies, the excess-of-loss reinsurance policy is also the optimal form for an insurer under a mean-variance criterion. Therefore, the excess-of-loss reinsurance policy is indeed optimal within the combined class of type I and type II reinsurance policies. However, whether the excess-of-loss reinsurance policy is the optimal form among all possible policies remains an open question to be investigated.*

Remark 6.3.3 *One sees from the proof of Theorem 6.3.2 (Theorem 6.3.3 resp.) that if the insurer is allowd to acquire new reinsurance business on the same risk, that is, if* $p_t > 1$ *is allowed, then the optimal reinsurance-investment strategy under the class of type I (type II resp.) policies is* $(\hat{m}_t^*, 0, \hat{\pi}_t^*)$ $((\tilde{m}_t^*, +\infty, \tilde{\pi}_t^*)$ *resp.), where* $\hat{m}_t^*$ *and* $\hat{\pi}_t^*$ *are given in (6.11) (*$\tilde{m}_t^*$ *and* $\tilde{\pi}_t^*$ *are given in (6.15) resp.). In other words, the optimal reinsurance form within the combined class of type I and type II policies is* $\tilde{l}\,(Z_t; t) = \tilde{m}_t^*$*, a reinsurance strategy independent of the claim size, if the restriction on acquiring new reinsurance business is released. However, whether it is reasonable to release the restriction remains debatable (at least to the author).*

6.4 Numerical Examples

Example 6.4.1 (Equilibrium strategies) *In this example, we examine the sensitivity of the equilibrium reinsurance-investment strategies given in (6.11) to different parameters. Unless otherwise stated, the parameters are given by* $r = 0.05$, $\mu = 0.1$, $\sigma_1 = 0.2$, $\sigma_2 = 0.3$, $\eta = 0.6$, $\rho = 0.5$, $\gamma = 1$, *and* $T = 9$. *The corresponding equilibrium strategy under the parameter setting is denoted by* (m^*, π^*).

In Figure 6.1, we plot the impact of r *on the reinsurance-investment strategy. Both* m^* *and* π^* *are decreasing (except for* m^* *at* $t = T$ *where it is a constant) as the risk-free rate increases. When large claims occur, the insurer might end up taking loans from banks to remain solvent. Thus, the insurer intends to undertake less insurance risk as borrowing money becomes more costly. As with the investment strategy, clearly a reasonable investment decision for the insurer is to decrease the amount invested in the financial market as the risk-free asset becomes more attractive.*

In Figure 6.2, we plot the impact of γ *on the reinsurance-investment strategy. We see clearly that as the insurer becomes more risk averse, it intends to take less insurance and financial risk.*

In Figure 6.3, we plot the impact of η *on the reinsurance strategy. We see that* m^* *increases as* η *increases. In other words, as the reinsurance policy becomes more expensive, the insurer intends to undertake more insurance risk by increasing the retention level.*

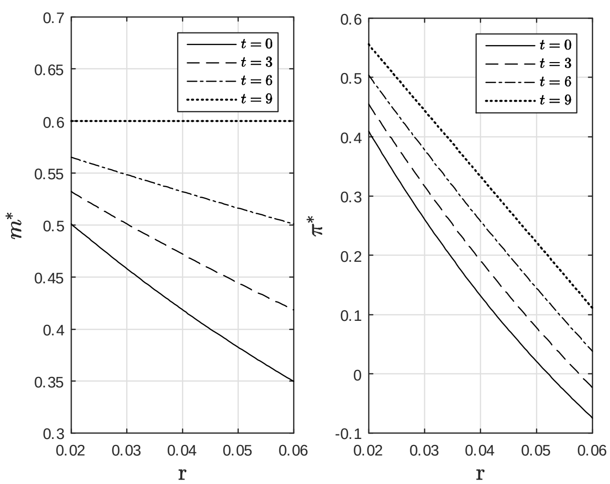


Figure 6.1. Impact of r.

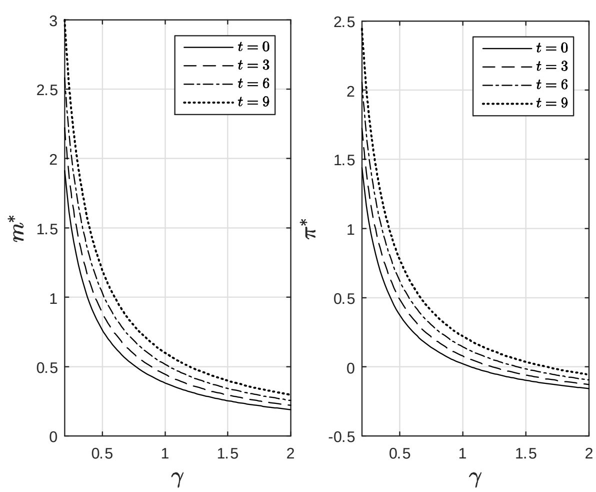


Figure 6.2. Impact of γ.

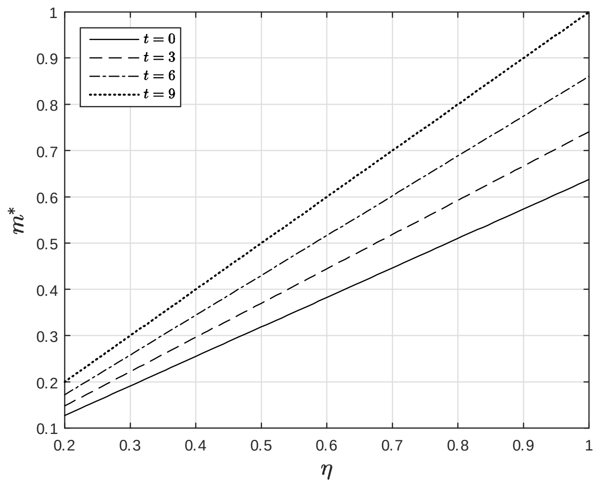


Figure 6.3. Impact of η on m^*.

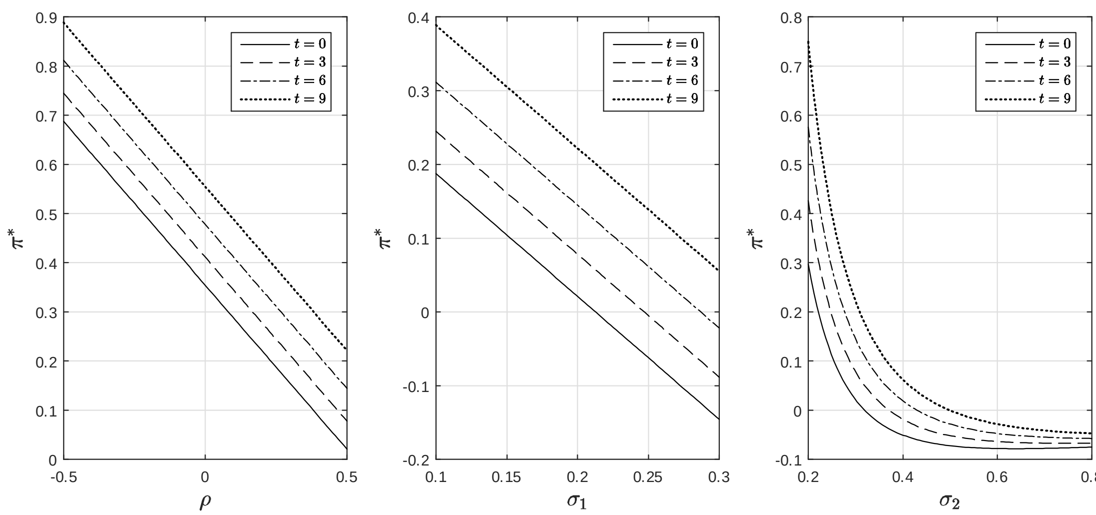


Figure 6.4. Impact of η (left), σ_1 (middle), and σ_2 (right) on π^*.

In Figure 6.4, we plot the impact of ρ, σ_1, and σ_2 on the investment strategy respectively. First, we see from the left panel that as ρ increases, π^* decreases. This *happens because the total volatility undertaken by the insurer increases. Second, we see from the middle panel that as the insurance market becomes more volatile, the amount invested in the financial market should be decreased as there is a positive correlation ($\rho = 0.5$) between the two markets. Finally, we see from the right panel that as the financial market becomes more volatile, the insurer intends to decrease the amount invested in it.*

Example 6.4.2 (Proportional vs excess-of-loss) *In this example, we assume that the basic surplus process follows a C-L model*

$$\mathrm{d}U_t = c\mathrm{d}t - \mathrm{d}\sum_{i=1}^{N(t)} Y_i,\ U_0 = u,$$

where $\{Y_i\}_{i=1}^{\infty}$ is a sequence of independent and identically distributed (i.i.d.) exponential random variables with common survival function $\overline{F}(y) := \mathrm{e}^{-\kappa y}$, $y > 0$, representing the amount of individual claims, and $\{N_t\}_{t\geq 0}$ is a Poisson process with intensity $\lambda > 0$ representing the number of claims, independent of $\{Y_i\}_{i=1}^{\infty}$. Applying equation (6.12) with $v(\mathrm{d}z) = \lambda F(\mathrm{d}z)$ and $\sigma_1 = 0$, the corresponding value function under the C-L model is given by

$$V_1(t,x) = \mathrm{e}^{r(T-t)}x + \bar{A}_1(t),\ (t,x) \in [0,T] \times \mathbb{R},$$

where

$$\bar{A}_1(t) = \int_t^T \left\{ \frac{(\mu - r)^2}{2\gamma\sigma^2} + \mathrm{e}^{r(T-s)} \left[(\theta - \eta)\lambda \mathbb{E}\left[Y\right] + \eta\lambda \int_0^{\frac{\eta}{\gamma}\mathrm{e}^{-r(T-s)}} \bar{F}(y)\mathrm{d}y \right] \right.$$
$$\left. -\gamma\lambda \mathrm{e}^{2r(T-s)} \int_0^{\frac{\eta}{\gamma}\mathrm{e}^{-r(T-s)}} y\bar{F}(y)\mathrm{d}y \right\} \mathrm{d}s.$$

Moreover, we have

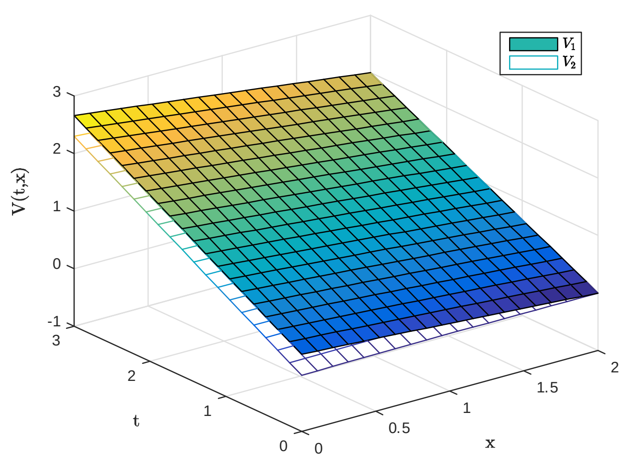
$$\mathbb{E}_1^{t,x}\left[X_T^{u^*}\right] = \mathrm{e}^{r(T-t)}x + \overline{a}_1(t),\ (t,x) \in [0,T] \times \mathbb{R},$$

where

$$\bar{a}_1(t) = \int_t^T \left\{ \frac{(\mu - r)^2}{\gamma\sigma^2} + \mathrm{e}^{r(T-s)} \left[(\theta - \eta)\lambda\mathbb{E}[Y] + \eta\lambda \int_0^{\frac{\eta}{\gamma}\mathrm{e}^{-r(T-s)}} \bar{F}(y)\mathrm{d}y \right] \right\} \mathrm{d}s,$$

and $\mathrm{Var}_1^{t,x}(X_T^{u^*}) = \frac{2}{\gamma}\left(\mathbb{E}_1^{t,x}\left[X_T^{u^*}\right] - V_1(t,x)\right)$. *Other parameters are given by* $r = 0.05$, $\mu = 0.1$, $\sigma_2 = 0.3$, $\gamma = 0.5$, $T = 1$, $\theta = 0.5$, $\eta = 0.6$, $\lambda = 1$ *and* $\kappa = 0.5$.

Under the model, we compare the value function V_1 *with the value function under the equilibrium proportional reinsurance* V_2 *in, e.g., Zeng et al. [91]. We see from the top panel in Figure 6.5 that* V_2 *is dominated by* V_1 *except at the boundary* $t = T$, *where* $V_1 = V_2$. *In other words, the equilibrium proportional reinsurance policy is not optimal when a larger class, i.e., the combined class of type I and type II reinsurance policies, is available, and the optimal form is an excess-of-loss type policy. We also see from the bottom panel in Figure 6.5 that when mean and variance are viewed separately, compared to the equilibrium proportional reinsurance, though the equilibrium excess-of-loss policy generates a higher terminal mean, the associated terminal risk is also higher.*



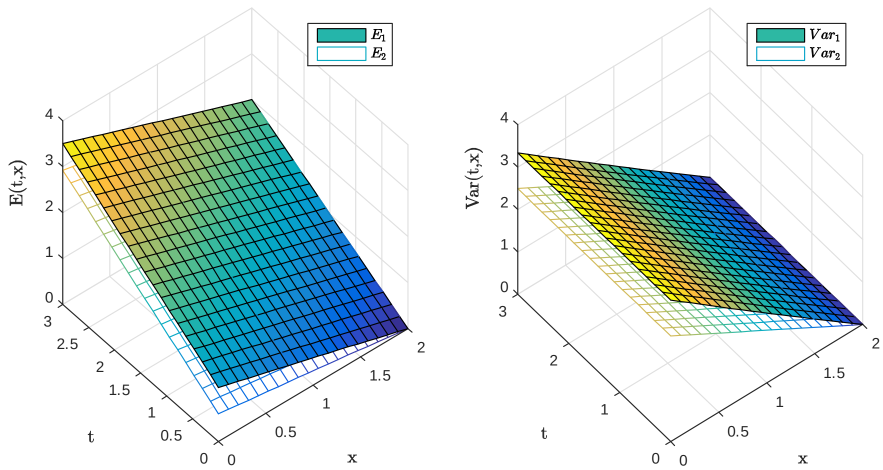


Figure 6.5. Excess-of-loss vs proportional.

Chapter 7

Concluding Remarks and Future Research

The main contribution of this thesis is that it has relaxed the concept of ruin in stochastic control problems by studying in drawdown setups (Chapter 3 and 4) and a two-sided exit framework (Chapter 5), and subsequently provided explicit solutions to those problems. Moreover, the in-length study provided deep insights into the optimal controls and the associated value functions. The thesis can be viewed as a first-step movement in stochastic control of "exotic" ruin features in the actuarial literature. It can be further extended but not limited to the following research directions.

The first direction is on relaxing the assumption on constant force of mortality. For the sake of mathematical tractability, the random time horizon is assumed to be exponentially distributed, i.e., the force of mortality is assumed to be a constant, for problems studied in Chapters 3–5. A relaxation of the constant force of mortality assumption usually comes with a sacrifice in mathematical tractability (e.g., Moore and Young [74]). My research goal is thus to push the study to the boundary on both directions, that is, finding a generalization as realistic as possible while maintaining mathematical tractability. A future attempt is to consider a state dependent force of mortality.

The second direction lies in decomposing a stochastic control problem in a drawdown setup into sub-problems that are easier to solve in principle. For example, by utilizing a perturbation approach, it is known that some drawdown related quantities can be bounded up and down by the product of a series of decomposed two-sided exit quantities (e.g., Landriault et al. [60]). In the limiting case, the upper and lower bounds converge to the corresponding drawdown quantity. Such decomposition is particularly appealing given that the drawdown problem is not as well-studied and understood as its related two-sided exit problems. When stochastic controls are imposed on both sides, two open questions are of primary concern: (1) Will the product of the optimized two-sided exit quantities converge in the limit to the optimized drawdown quantity? (2) If (1) is true, what is the relationship between the optimal controls of the drawdown problem and the two-sided exit problems? The research on this direction is meaningful as it potentially gives insights into drawdown problems and provides an alternative way to tackle or approximate drawdown problems by solving its related problems such as the two-sided exit problems.

The third direction is on measuring and controlling drawdown risks in insurance. While drawdowns have been substantially analyzed and applied in the financial industry with great success, the research of drawdowns in insurance have significantly lagged behind. Recently, developing drawdown-based insurance models (e.g., Landriault et. al [59]) has drawn some attention. However, to the best of my knowledge, few stochastic control problems in insurance have incorporated drawdowns as part of the models. Thus, introducing drawdown-based features into existing optimal control problems in insurance, for instance, optimal dividend/reinsurance/capital injection problems, is of both practical and theoretical interest, and will potentially create a rich research field.

Though being independent of the contribution of the other chapters, the findings in Chapter 6 are also very interesting as they raise the discussion on optimal reinsurance form of insurance companies. In contrast to the common procedure of studying a problem with a reinsurance control in a dynamic setting, i.e., assuming a specific form of a reinsurance policy and subsequently investigating its optimal value, Chapter 6 considered an optimal reinsurance form within a fairly general class of reinsurance policies. The study can be further generalized in at least two ways. The first one is to search for an optimal reinsurance form within a larger class of available reinsurance policies. The main challenge in this direction comes from the mathematical difficulties. The second generalization is to model the optimal reinsurance problem in a game theoretic framework. In reality, an insurance company cannot determine the optimal reinsurance without considering the objective of the reinsurance company and vice versa. As a result, it is natural to consider the bargain process as a noncooperative game and subsequently study the existence of a equilibrium.

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Appendix

Proof of Theorem 4.2.1

We first consider the case $\alpha \in (0,1)$. For an arbitrary $\pi \in \Pi$, we define a sequence of stopping time $\{\gamma_n^\pi\}_{n\in\mathbb{N}}$ with $\gamma_n^\pi = \inf\left\{t \geq 0 : \int_0^t \pi_s^2 \mathrm{d}s \geq n\right\}$. By applying Itô's formula to the process $e^{-\lambda t} f(W_t^\pi, M_t^\pi)$ for $t \in [0, \tau_{\alpha,n}^\pi]$, where $\tau_{\alpha,n}^\pi := \tau_\alpha^\pi \wedge \gamma_n^\pi$, and subsequently utilizing (4.1), we arrive at

$$\begin{aligned}
&e^{-\lambda\tau_{\alpha,n}^\pi} f(W_{\tau_{\alpha,n}^\pi}^\pi, M_{\tau_{\alpha,n}^\pi}^\pi) - f(w,m)\\
&= -\lambda \int_0^{\tau_{\alpha,n}^\pi} e^{-\lambda t} f(W_t^\pi, M_t^\pi)\mathrm{d}t + \int_0^{\tau_{\alpha,n}^\pi} e^{-\lambda t} f_w(W_t^\pi, M_t^\pi)\mathrm{d}W_t^\pi\\
&+ \frac{1}{2}\int_0^{\tau_{\alpha,n}^\pi} e^{-\lambda t} f_{ww}(W_t^\pi, M_t^\pi)(\mathrm{d}W_t^\pi)^2 + \int_0^{\tau_{\alpha,n}^\pi} e^{-\lambda t} f_m(W_t^\pi, M_t^\pi)\mathrm{d}M_t^\pi\\
&\leq \int_0^{\tau_{\alpha,n}^\pi} e^{-\lambda t}\left\{\mathcal{L}^\pi f(W_t^\pi, M_t^\pi) - [s_0 - \zeta\,(1 - W_t^\pi/M_t^\pi)]\right\}\mathrm{d}t\\
&+ \int_0^{\tau_{\alpha,n}^\pi} e^{-\lambda t} f_w(W_t^\pi, M_t^\pi)\pi_t W_t^\pi \sigma \mathrm{d}B_t.
\end{aligned} \tag{7.1}$$

Since $\{M_t^\pi\}_{t\geq 0}$ is nondecreasing and $f_m(W_t^\pi, M_t^\pi) < 0$ by considtion (2), we deduce that $f_m(W_t^\pi, M_t^\pi)\mathrm{d}M_t^\pi \leq 0$ a.s.. Thus, the inequality in (7.1) holds. Taking the conditional expectation $\mathbb{E}^{w,m}[\cdot]$ on both sides of (7.1) and invoking condition (3), we obtain

$$\mathbb{E}^{w,m}\left[e^{-\lambda\tau_{\alpha,n}^\pi} f(W_{\tau_{\alpha,n}^\pi}^\pi, M_{\tau_{\alpha,n}^\pi}^\pi) + \int_0^{\tau_{\alpha,n}^\pi} e^{-\lambda t}\left[s_0 - \zeta\,(1 - W_t^\pi/M_t^\pi)\right]\mathrm{d}t\right] \leq f\,(w,m)\,, \tag{7.2}$$

for all $\pi \in \Pi$. Since f is assumed to be bounded, by the dominated convergence theorem and condition (5), it follows that

$$\mathbb{E}^{w,m}\left[\int_0^{\tau^{\pi}_{\alpha,n}} e^{-\lambda t}\left[s_0-\zeta\left(1-W_t^{\pi}/M_t^{\pi}\right)\right]\mathrm{d}t\right]\le f\left(w,m\right),$$

for all $\pi \in \Pi$. Further, by conditions (4) and (6), there exists an admissible strategy π^* such that the equality holds in (7.1) because $\mathrm{d}M_t^{\pi^*}=0$ a.s.. Moreover, by condition (4), the equality in (7.2) also holds. In other words, we deduce that

$$\begin{aligned} f\left(w,m\right) &= \mathbb{E}^{w,m}\left[\int_0^{\tau^{\pi^*}_{\alpha,n}} e^{-\lambda t}\left[s_0-\zeta\left(1-W_t^{\pi^*}/M_t^{\pi^*}\right)\right]\mathrm{d}t\right] \\ &= \sup_{x\in\Pi}\mathbb{E}^{w,m}\left[\int_0^{\tau^{\pi}_{\alpha,n}} e^{-\lambda t}\left[s_0-\zeta\left(1-W_t^{\pi}/M_t^{\pi}\right)\right]\mathrm{d}t\right]=v(w,m). \end{aligned}$$

For the case $\alpha=1$. Consider a set $\mathcal{N}:=\left\{(w,m)\in D:\ w\neq 0\right\}$. For an arbitrary $\pi\in\Pi$, $\forall\left(w,m\right)\in\mathcal{N}$, $\tau^{\pi}_{\alpha}=+\infty$ by (4.1). By utilizing a similar set of arguments as in the case $\alpha\in(0,1)$, we deduce that $\forall\left(w,m\right)\in\mathcal{N}$,

$$\begin{aligned} f\left(w,m\right) &= \mathbb{E}^{w,m}\left[\int_0^{\infty} e^{-\lambda t}\left[s_0-\zeta\left(1-W_t^{\pi^*}/M_t^{\pi^*}\right)\right]\mathrm{d}t\right] \\ &= \sup_{x\in\Pi}\mathbb{E}^{w,m}\left[\int_0^{\infty} e^{-\lambda t}\left[s_0-\zeta\left(1-W_t^{\pi}/M_t^{\pi}\right)\right]\mathrm{d}t\right]=v(w,m). \end{aligned}$$

$\forall\left(w,m\right)\in D\backslash\mathcal{N}$, since we assume that 0 is an absorbing state, i.e., $W_t^{\pi}=0$, $\forall t\ge\inf\left\{s:W_s\le 0\right\}$,

$$f\left(0,m\right)=\mathbb{E}^{0,m}\left[\int_0^{\infty} e^{-\lambda t}\left[s_0-\zeta\left(1\right)\right]\mathrm{d}t\right]=\frac{s_0-\zeta(1)}{\lambda}.$$

Proof of Lemma 4.3.1

We first show that equation (4.11) admits a unique solution in $(1,\infty)$. Letting

$$\phi\left(x\right):=\frac{1-\gamma^{+}}{\gamma^{+}-\gamma^{-}}x^{\gamma^{-}-1}+\frac{1-\gamma^{-}}{\gamma^{-}-\gamma^{+}}x^{\gamma^{+}-1}, \tag{7.3}$$

it is straightforward to verify that

$$\phi_x(x) = \frac{(1-\gamma^+)(\gamma^- - 1)}{\gamma^+ - \gamma^-}(x^{\gamma^- - 2} - x^{\gamma^+ - 2}) > 0, \qquad x > 1.$$

Given that $\phi(1) = -1 < \beta - 1$ and $\lim_{x\to\infty}\phi(x) = 0$, it easily follows that (7.3) admits a unique solution in $(1, \infty)$.

Next, we show that equation (4.12) admits a unique solution in $(0, 1)$. Cross-multiplying (4.12) and subsequently multiplying each side of the resulting equation by $x^{1-\gamma^+-\gamma^-}$ yield

$$L(x) := (s_0 - s_1) H(x) + s_1 G(x) = 0, \tag{7.4}$$

where $H(x) := D_1 x^{1-\gamma^-} + D_4 x^{1-\gamma^+} + D_5$, $G(x) := D_2 x^{1-\gamma^+-\gamma^-} + D_3 x^{-\gamma^-} + D_6 x^{-\gamma^+}$, and

$$\begin{cases} D_1 = -(1-\gamma^+)\kappa_0^{\gamma^- - 1} < 0, \\ D_2 = (1-\beta)(\gamma^+ - \gamma^-) > 0, \\ D_3 = (1-\alpha)(\gamma^- - 1)\gamma^+ < 0, \\ D_4 = (1-\gamma^-)\kappa_0^{\gamma^+ - 1} > 0, \\ D_5 = -(1-\alpha)(\gamma^+ - \gamma^-) < 0, \\ D_6 = -(1-\alpha)(\gamma^+ - 1)\gamma^- < 0. \end{cases} \tag{7.5}$$

Note that the above multiplication by $x^{1-\gamma^+-\gamma^-}$ will make the function L monotone on $(0, 1)$ which simplifies the proof of the existence of a unique solution in $(0, 1)$ to (4.12). Since $1 > \gamma^+ > 0 > \gamma^-$ and $\beta \in (0, \alpha)$, it is easy to see that $\lim_{x\downarrow 0} L(x) = -\infty$ and $L(1) = (\alpha - \beta) s_0 (\gamma^+ - \gamma^-) > 0$. Then, it suffices to show that $L_x > 0$ on $(0, 1)$. Differentiating $H(x)$ yields

$$\begin{aligned} H_x(x) &= D_1\left(1-\gamma^-\right) x^{-\gamma^-} + D_4\left(1-\gamma^+\right) x^{-\gamma^+} \\ &= \left(1-\gamma^+\right)\left(1-\gamma^-\right)\kappa_0^{-1}\left(\left(\frac{\kappa_0}{x}\right)^{\gamma^+} - \left(\frac{\kappa_0}{x}\right)^{\gamma^-}\right). \end{aligned}$$

Since $1 > \gamma^+ > 0 > \gamma^-$ and $\kappa_0 > 1 > x > 0$, we deduce that $H_x(x) > 0$. It remains to show that $G_x(x) > 0$ on $(0,1)$. By $\beta \in (0, \alpha)$, we have

$$\begin{aligned}
G_x(x) = {} & D_2 \left(1 - \gamma^+ - \gamma^-\right) x^{-\gamma^+ - \gamma^-} + D_3 \left(-\gamma^-\right) x^{-1-\gamma^-} + D_6 \left(-\gamma^+\right) x^{-1-\gamma^+} \\
& > (1-\alpha) \left(\gamma^+ - \gamma^-\right) \left(1 - \gamma^+ - \gamma^-\right) x^{-\gamma^+ - \gamma^-} \\
& + (1-\alpha) \gamma^- \gamma^+ \left[\left(1 - \gamma^-\right) x^{-1-\gamma^-} - \left(1 - \gamma^+\right) x^{-1-\gamma^+}\right].
\end{aligned}$$

Further, since $1 - \gamma^+ - \gamma^- = \frac{\lambda + \eta - r}{\delta} > \frac{\lambda}{\delta} = -\gamma^- \gamma^+$, it follows that

$$G_x(x) > (1-\alpha) \left(-\gamma^+ \gamma^-\right) \left[\left(\gamma^+ - \gamma^-\right) x^{-\gamma^+ - \gamma^-} - \left(1 - \gamma^-\right) x^{-1-\gamma^-} + \left(1 - \gamma^+\right) x^{-1-\gamma^+}\right].$$

Hence, to show $G_x(x) > 0$, it suffices to show that, for any fixed $x \in (0,1)$,

$$\left(\gamma^+ - \gamma^-\right) x^{-\left(\gamma^+ + \gamma^-\right)} - \left(1 - \gamma^-\right) x^{-\left(1+\gamma^-\right)} + \left(1 - \gamma^+\right) x^{-\left(1+\gamma^+\right)} \geq 0,$$

which clearly holds by the convexity of $\chi(u) := x^u$ in $u \in \mathbb{R}$. Thus, $G_x(x) > 0$ on $(0,1)$. Therefore, (7.4) admits a unique solution $\kappa_\alpha \in (0,1)$.

Next, we show that y_β defined in (4.10) is positive. Rearranging (4.12) yields

$$\frac{k_1}{k_2} = \frac{(s_0 - s_1) \gamma^+ \gamma^- \left(\kappa_\alpha^{\gamma^-} - \kappa_\alpha^{\gamma^+}\right)}{s_1 (\gamma^+ - \gamma^-) + (s_0 - s_1) \gamma^+ \kappa_\alpha^{\gamma^-} - (s_0 - s_1) \gamma^- \kappa_\alpha^{\gamma^+}},$$

where

$$k_1 = (\kappa_\alpha)^{\gamma^+ + \gamma^- - 1} G(\kappa_\alpha),$$

and

$$k_2 = (\alpha - 1) \left[\left(\gamma^+ - 1\right) \kappa_\alpha^{\gamma^- - 1} - \left(\gamma^- - 1\right) \kappa_\alpha^{\gamma^+ - 1}\right] - \left[\frac{1 - \gamma^+}{\gamma^-} \kappa_0^{\gamma^- - 1} - \frac{1 - \gamma^-}{\gamma^+} \kappa_0^{\gamma^+ - 1}\right].$$

Since $1 > \gamma^+ > 0 > \gamma^-$ and $\kappa_\alpha \in (0,1)$, we deduce that $\frac{k_1}{k_2} < 0$. Furthermore, by (4.11) and $\kappa_\alpha \in (0,1)$, we can see that

$$k_1 - k_2 \propto (\alpha - 1) \left(\kappa_\alpha^{\gamma^- - 1} - \kappa_\alpha^{\gamma^+ - 1}\right) + \frac{\kappa_0^{\gamma^- - 1}}{\gamma^-} - \frac{\kappa_0^{\gamma^+ - 1}}{\gamma^+} < 0,$$

where "$\propto$" means the relationship of positive proportionality. Thus, we deduce $k_1 < 0$ and

$$y_\beta = \frac{(s_0 - s_1)\,\gamma^+\gamma^-\left(\kappa_\alpha^{\gamma^-} - \kappa_\alpha^{\gamma^+}\right)}{\lambda k_1} > 0.$$

It follows that y_β, $y_0 = y_\beta/\kappa_0$, and $y_\alpha = y_\beta/\kappa_\alpha$ are all well defined. Then, it is straightforward to verify that (4.8) is the solution to (4.7).

Finally, we show that $\hat{\psi}$ is strictly decreasing and strictly convex on $[y_0, y_\alpha]$. Since $\hat{\psi}_y(y_\alpha) = \alpha - 1 < 0$, it remains to show that $\hat{\psi}$ is strictly convex on $[y_0, y_\alpha]$. The second order derivative of $\hat{\psi}$ is given by

$$\hat{\psi}_{yy}(y) = \begin{cases} \frac{(1-\gamma^+)(\gamma^- -1)}{(\gamma^+ - \gamma^-)y_0}\left[\left(\frac{y}{y_0}\right)^{\gamma^- -2} - \left(\frac{y}{y_0}\right)^{\gamma^+ -2}\right], & y \in (y_0, y_\beta), \\ d_1\left(\frac{y}{y_\alpha}\right)^{\gamma^- -2} + d_2\left(\frac{y}{y_\alpha}\right)^{\gamma^+ -2}, & y \in (y_\beta, y_\alpha), \end{cases} \tag{7.6}$$

where

$$\begin{aligned} d_1 &= \left(\frac{(\alpha-1)(\gamma^+ - 1)}{y_\alpha} - \frac{(s_0 - s_1)\,\gamma^+}{\lambda y_\alpha^2}\right)\frac{\gamma^-(\gamma^- - 1)}{\gamma^+ - \gamma^-}, \\ d_2 &= \left(\frac{(\alpha-1)(\gamma^- - 1)}{y_\alpha} - \frac{(s_0 - s_1)\,\gamma^-}{\lambda y_\alpha^2}\right)\frac{\gamma^+(\gamma^+ - 1)}{\gamma^- - \gamma^+} > 0. \end{aligned}$$

Since $1 > \gamma^+ > 0 > \gamma^-$ and $y > y_0 > 0$, we deduce $\hat{\psi}_{yy}(y) > 0$ on (y_0, y_β). For $y \in (y_\beta, y_\alpha)$, we rewrite (7.6) as

$$\hat{\psi}_{yy}(y) = \left(\frac{y}{y_\alpha}\right)^{\gamma^- -2} J(y),$$

where

$$J(y) := d_1 + d_2\left(\frac{y}{y_\alpha}\right)^{\gamma^+ - \gamma^-}.$$

Since $J_y(y) = d_2(\gamma^+ - \gamma^-)\left(\frac{y}{y_\alpha}\right)^{\gamma^+ - \gamma^- -1} > 0$ and $y_\alpha = y_\beta/\kappa_\alpha$, it remains to show that

$$J(y_\beta) = d_1 + d_2\kappa_\alpha^{\gamma^+ - \gamma^-} > 0.$$

With some algebraic manipulations, one can check that

$$d_1 + d_2 \kappa_\alpha^{\gamma^+ - \gamma^-} > 0 \Longleftrightarrow D_1 \kappa_\alpha^{1-\gamma^-} \kappa_0^{1-\gamma^-} + D_4 \kappa_\alpha^{1-\gamma^+} \kappa_0^{1-\gamma^+} + \frac{D_5}{1-\beta} > 0,$$

where D_1, D_4, and D_5 are given in (7.5). Consider

$$Q\left(x\right) := D_1 x^{1-\gamma^-} \kappa_0^{1-\gamma^-} + D_4 x^{1-\gamma^+} \kappa_0^{1-\gamma^+} + \frac{D_5}{1-\beta}.$$

It remains to show that $Q\left(\kappa_\alpha\right) > 0$. By (7.5), it is easy to see that

$$\begin{aligned} Q\left(0\right) &= \frac{D_5}{1-\beta} < 0, \\ Q\left(1\right) &= \frac{\alpha - \beta}{1-\beta} > 0, \end{aligned}$$

and

$$Q_x\left(x\right) = \left(1-\gamma^+\right)\left(1-\gamma^-\right)(x^{-\gamma^+} - x^{-\gamma^-}) > 0, \quad x \in (0,1).$$

Thus, Q has a unique zero on $(0,1)$, denoted as x_0, such that

$$Q\left(x_0\right) = D_1 x_0^{1-\gamma^-} \kappa_0^{1-\gamma^-} + D_4 x_0^{1-\gamma^+} \kappa_0^{1-\gamma^+} + \frac{D_5}{1-\beta} = 0. \tag{7.7}$$

It remains to show that $\kappa_\alpha > x_0$, and we prove this by utilizing the monotonicity of H. Recall $G\left(x\right) = D_2 x^{1-\gamma^+ - \gamma^-} + D_3 x^{-\gamma^-} + D_6 x^{-\gamma^+}$. It is straightforward to verify that

$$G\left(\kappa_\alpha\right) = D_2 \kappa_\alpha^{1-\gamma^- - \gamma^+} + D_3 \kappa_\alpha^{-\gamma^-} + D_6 \kappa_\alpha^{-\gamma^+} \propto k_1 < 0.$$

Since $L\left(\kappa_\alpha\right) = \left(s_0 - s_1\right) H\left(\kappa_\alpha\right) + s_1 G\left(\kappa_\alpha\right) = 0$, it follows that

$$H\left(\kappa_\alpha\right) = D_1 \kappa_\alpha^{1-\gamma^-} + D_4 \kappa_\alpha^{1-\gamma^+} + D_5 > 0. \tag{7.8}$$

On the other hand, we deduce from (4.11) that

$$1 - \beta > \kappa_0^{\gamma^+ - 1} > \kappa_0^{\gamma^- - 1}. \tag{7.9}$$

By (7.7), (7.9), and $x_0 \in (0,1)$, we have

$$H(x_0) = D_1 x_0^{1-\gamma^-} + D_4 x_0^{1-\gamma^+} + D_5$$
$$= -\left(1-\gamma^-\right)\left(\kappa_0^{\gamma^+-1} - (1-\beta)\right)\left(x_0^{1-\gamma^-} - x_0^{1-\gamma^+}\right) < 0. \qquad (7.10)$$

It follows from (7.8), (7.10), and $H_x(x) > 0$ on $(0,1)$ that $\kappa_\alpha > x_0$. The proof is therefore complete.

Proof of Lemma 4.4.2

First, we show that equation (4.26) admits a unique solution on $(-\infty, 0)$, denoted as x_β. Rearranging (4.26) yields

$$\omega(x) := K_1 \Phi(A+1, B; x) + K_2 \Phi(A, B; x) + K_3 x \Phi(A+1, B; x) = 0, \qquad (7.11)$$

where $K_1 = -\left(\frac{1-\gamma^+}{(\gamma^+-\gamma^-)\gamma^-}\kappa_0^{\gamma^--1} + \frac{1-\gamma^-}{(\gamma^--\gamma^+)\gamma^+}\kappa_0^{\gamma^+-1}\right)\frac{A}{\delta} > 0$, $K_2 = \frac{(1-\beta)}{\delta} > 0$ and $K_3 = \frac{(1-\beta)}{\lambda} A > 0$. By (3) of Lemma 4.4.1, we have

$$\omega(x) = (K_1 + K_3 x) e^x \Phi(B-A-1, B; -x) + K_2 e^x \Phi(B-A, B; -x) := e^{-s}\varphi(s),$$

where $s = -x$ and

$$\varphi(s) = (K_1 - K_3 s)\Phi(B-A-1, B; s) + K_2 \Phi(B-A, B; s).$$

Since $\varphi(0) = K_1 + K_2 > 0$ and $\lim_{s\to\infty}\varphi(s) = -\infty$, it suffices to show that $\varphi(s)$ is decreasing on $(0,\infty)$. By (1) of Lemma 4.4.1, we rewrite φ as

$$\begin{aligned}\varphi(s) &= K_1\Phi(p-1, q; s) + K_2\Phi(p, q; s) \\ &\quad -K_3(q-1)\left[\Phi(p-1, q-1; s) - \Phi(p-2, q-1; s)\right], \qquad (7.12)\end{aligned}$$

where $p := B - A \in (1,2)$ and $q := B > 1$. By (2) of Lemma 4.4.1, we have

$$\begin{aligned}
\varphi_s(s) &= \frac{p-1}{q} K_1 \Phi(p, q+1; s) + K_2 \frac{p}{q} \Phi(p+1, q+1; s) \\
&\quad - K_3 \left[(p-1) \Phi(p, q; s) - (p-2) \Phi(p-1, q; s) \right] \\
&= \frac{p-1}{q} K_1 \Phi(p, q+1; s) + K_2 \left(\frac{p-q}{q} \Phi(p, q+1; s) + \Phi(p, q; s) \right) \\
&\quad - K_3 \left[(p-1) \Phi(p, q; s) - (p-2) \Phi(p-1, q; s) \right].
\end{aligned}$$

Therefore, showing $\varphi_s < 0$ is equivalent to showing $\forall s > 0$,

$$K_1 < \frac{(2-p)\,q}{p-1} K_3 \frac{\Phi(p-1, q; s)}{\Phi(p, q+1; s)} + \frac{(K_3(p-1) - K_2)\,q}{p-1} \frac{\Phi(p, q; s)}{\Phi(p, q+1; s)} + \frac{q-p}{p-1} K_2. \tag{7.13}$$

It is easy to verify that $K_3(p-1) = K_2$. Thus, (7.13) becomes

$$K_1 < K_3 \left[\frac{(2-p)\,q}{p-1} \frac{\Phi(p-1, q; s)}{\Phi(p, q+1; s)} + q - p \right], \ \forall s > 0.$$

With some tedious calculations, one can verify that $\frac{\Phi(p-1,q;s)}{\Phi(p,q+1;s)}$ is a decreasing function in s with

$$\min_{s>0} \frac{\Phi(p-1, q; s)}{\Phi(p, q+1; s)} = \frac{p-1}{q}.$$

Thus, it remains to show that

$$K_1 < K_3(2 - p + q - p). \tag{7.14}$$

Since $\kappa_0 \in (1, \infty)$ and $\gamma^- < 0$, we have $\kappa_0^{\gamma^- - 1} < 1$. With some further algebraic manipulations, we can verify that

$$\kappa_0^{\gamma^- - 1} < 1 \Longleftrightarrow K_1 < K_3 \left[2 - p + (q-p)(1-\beta) \right].$$

Thus, (7.14) holds as $\beta < 1$ and $q - p = A > 0$. Therefore, $\varphi(s)$ admits a unique zero on $(0, \infty)$ denoted as s_β, and equivalently, (7.11) admits a unique zero $x_\beta = -s_\beta$ on $(-\infty, 0)$.

It follows that $\theta_\beta = -s_1 (\delta x_\beta)^{-1}$ and $\theta_0 = \theta_\beta/\kappa_0$ are well defined, where κ_0 is defined in Lemma 4.3.1. Then, it is straightforward to verify that (4.25) is the solution to (4.23).

Finally, we show that $\hat{\Psi}$ is strictly decreasing and strictly convex on (θ_0, ∞). Note that $\hat{\Psi}(y)$ for $y \in [\theta_0, \theta_\beta]$ has the same form as $\hat{\psi}(y)$ in Lemma 4.3.1 for $y \in [y_0, y_\beta]$. Thus it suffices to show the monotonicity and convexity of $\hat{\Psi}$ on (θ_β, ∞). For $y \in (\theta_\beta, \infty)$,

$$\hat{\Psi}_y(y) = -C_3 A y^{-A-1} \Phi\left(A+1, B; -s_1 (\delta y)^{-1}\right),$$

where

$$C_3 = \frac{(\theta_\beta)^{A+1}(1-\beta)}{A\Phi\left(A+1, B; -s_1 (\delta\theta_\beta)^{-1}\right)} > 0.$$

Since $\lim_{y\to\infty} \hat{\Psi}_y(y) = 0$, it remains to show that $\hat{\Psi}$ is convex on (θ_β, ∞). The second order derivative of $\hat{\Psi}$ is given by

$$\hat{\Psi}_{yy}(y) = C_3 A (A+1) y^{-A-2} \Phi\left(A+2, B; -s_1 (\delta y)^{-1}\right) \propto \Phi\left(A+2, B; -s_1 (\delta y)^{-1}\right).$$

By (3) of Lemma 4.4.1 and let $s := s_1 (\delta y)^{-1}$,

$$\Phi\left(A+2, B; -s_1 (\delta y)^{-1}\right) = e^{-s}\Phi(B-A-2, B; s) = e^{-s}\Phi(p-2, q; s).$$

Since $p-2 \in (-1, 0)$, we deduce that

$$\frac{\mathrm{d}}{\mathrm{d}s}\Phi(p-2, q; s) = \frac{\mathrm{d}}{\mathrm{d}s}\sum_{i=0}^{\infty}\frac{(p-2)_i}{q_i}\frac{s^i}{i!} = \sum_{i=1}^{\infty}\frac{(p-2)_i}{q_i}\frac{s^{i-1}}{(i-1)!} < 0,$$

i.e., $\Phi(p-2, q; s)$ is decreasing in s with

$$\Phi(p-2, q; 0) = 1 \text{ and } \lim_{s\to\infty}\Phi(p-2, q; s) = -\infty.$$

Thus, $\Phi(p-2, q; s)$ admits a unique zero s^* on $(0, \infty)$. To show $\hat{\Psi}_{yy}(y) > 0$ for $y \in (\theta_\beta, \infty)$, it suffices to show that $s_1 (\delta\theta_\beta)^{-1} < s^*$. Recall $\varphi(s)$ defined in (7.12) is a decreasing function on $(0,\infty)$, and $\varphi(s)$ admits a unique zero $s_\beta = -x_\beta = s_1(\delta\theta_\beta)^{-1}$. Thus, it remains to show that $\varphi(s^*) < \varphi(s_\beta) = 0$. By (2) of Lemma 4.4.1, we have

$$(2-p)\Phi(p-1,q;s^*) + (q-1)\Phi(p-2,q-1;s^*) = (1-p+q)\Phi(p-2,q;s^*) = 0. \tag{7.15}$$

and

$$(p-q)\Phi(p-1,q;s^*) + (q-1)\Phi(p-1,q-1;s^*) = (p-1)\Phi(p,q;s^*). \tag{7.16}$$

By (7.14), the relation $K_2 = (p-1)K_3$, (7.15), and (7.16), one obtains

$$\begin{aligned}
\varphi(s^*) &= K_1\Phi(p-1,q;s^*) + K_2\Phi(p,q;s^*) \\
&\quad - K_3(q-1)\left[\Phi(p-1,q-1;s^*) - \Phi(p-2,q-1;s^*)\right] \\
&< K_3(2-p+q-p)\Phi(p-1,q;s^*) + K_3(p-1)\Phi(p,q;s^*) \\
&\quad - K_3(q-1)\Phi(p-1,q-1;s^*) + K_3(q-1)\Phi(p-2,q-1;s^*) \\
&= K_3\left[(q-p)\Phi(p-1,q;s^*) + (p-1)\Phi(p,q;s^*) - (q-1)\Phi(p-1,q-1;s^*)\right] \\
&= 0.
\end{aligned}$$

The proof is therefore complete.

Proof of Lemma 5.3.1

The general solution to the first equation in (5.17) is known to be of the form

$$\hat{\phi}(y) = c_1 y^{\delta_+} + c_2 y^{\delta_-} + c_3 y, \tag{7.17}$$

where $\delta_\pm = \frac{-(\lambda-r-\Lambda)\pm\sqrt{(\lambda-r-\Lambda)^2+4\lambda\Lambda}}{2\Lambda}$. Substituting (7.17) (and its first two derivatives) into the first equation of (5.17) yields

$$\begin{aligned}
0 &= y\left[(\theta-\eta)m - rc_3\right] + c_1 y^{\delta_+}\left[\Lambda\delta_+(\delta_+-1) + (\lambda-r)\delta_+ - \lambda\right] \\
&\quad + c_2 y^{\delta_-}\left[\Lambda\delta_-(\delta_--1) + (\lambda-r)\delta_- - \lambda\right] \\
&= y\left[(\theta-\eta)m - rc_3\right],
\end{aligned}$$

for any $y \in (y_b, y_0)$, from where we deduce

$$c_3 = \frac{(\theta - \eta)m}{r}. \tag{7.18}$$

Furthermore, by the boundary conditions in (5.17), we have

$$\begin{cases} c_1\delta_+ y_0^{\delta_+-1} + c_2\delta_- y_0^{\delta_--1} + \frac{(\theta-\eta)m}{r} = 0, \\ c_1 y_0^{\delta_+} + c_2 y_0^{\delta_-} + \frac{(\theta-\eta)m}{r} y_0 = 0, \\ c_1\delta_+ y_b^{\delta_+-1} + c_2\delta_- y_b^{\delta_--1} + \frac{(\theta-\eta)m}{r} = -b, \\ c_1 y_b^{\delta_+} + c_2 y_b^{\delta_-} + \frac{(\theta-\eta)m}{r} y_b + b y_b = 1. \end{cases} \tag{7.19}$$

Solving for c_1 and c_2 using the first two equations of (7.19) (and the last two equations of (7.19) for the second equality sign), we obtain

$$\begin{cases} c_1 = \frac{(\eta-\theta)m(1-\delta_-)}{r(\delta_+-\delta_-)} y_0^{1-\delta_+} = \frac{-r\delta_- - [(\theta-\eta)m+rb](1-\delta_-)y_b}{r(\delta_+-\delta_-)y_b^{\delta_+}} > 0, \\ c_2 = \frac{(\eta-\theta)m(\delta_+-1)}{r(\delta_+-\delta_-)} y_0^{1-\delta_-} = \frac{r\delta_+ - [(\theta-\eta)m+rb](\delta_+-1)y_b}{r(\delta_+-\delta_-)y_b^{\delta_-}} > 0. \end{cases} \tag{7.20}$$

Let $\kappa := \frac{y_0}{y_b} \in (1, +\infty)$. It follows from (7.20) that

$$\begin{cases} (\eta-\theta)m(1-\delta_-)\kappa^{1-\delta_+} = \frac{-r\delta_-}{y_b} - [(\theta-\eta)m + rb](1-\delta_-), \\ (\eta-\theta)m(\delta_+-1)\kappa^{1-\delta_-} = \frac{r\delta_+}{y_b} - [(\theta-\eta)m + rb](\delta_+-1). \end{cases} \tag{7.21}$$

Eliminating the term y_b from the above system of equations yields the following equation of κ:

$$f(\kappa) := \frac{1-\delta_-}{\delta_-}\kappa^{1-\delta_+} + \frac{\delta_+-1}{\delta_+}\kappa^{1-\delta_-} - \frac{\delta_+-\delta_-}{\delta_+\delta_-}\frac{(\eta-\theta)m - rb}{(\eta-\theta)m} = 0. \tag{7.22}$$

Since $\delta_+ > 1 > 0 > \delta_-$, it is straightforward to verify that f is a strictly increasing function on $[1, +\infty)$ with $f(1) = \frac{(\delta_+-\delta_-)rb}{\delta_+\delta_-(\eta-\theta)m} < 0$ and $\lim_{\kappa\uparrow+\infty} f(\kappa) = +\infty$. Therefore, (7.22) admits a unique solution $\kappa \in (1, +\infty)$. Substituting (7.18) and (7.20) into (7.17) yields

$$\hat{\phi}(y) = \frac{(\eta-\theta)my}{r}\left[\frac{1-\delta_-}{\delta_+-\delta_-}\left(\frac{y}{y_0}\right)^{\delta_+-1} - \frac{1-\delta_+}{\delta_+-\delta_-}\left(\frac{y}{y_0}\right)^{\delta_--1} - 1\right], \quad y_b \le y \le y_0, \tag{7.23}$$

where $y_0 = \kappa y_b$ and y_b can be determined from either equation in (7.21). By (7.22), it is easy to see that since $\delta_+ > 1 > 0 > \delta_-$ and $\kappa > 1$,

$$
\begin{aligned}
\frac{1}{y_b} &= \frac{(1-\delta_+)}{r\delta_+}\left[(\eta-\theta)m - rb - (\eta-\theta)m\kappa^{1-\delta_-}\right] \\
&> \frac{(1-\delta_+)}{r\delta_+}\left[(\eta-\theta)m - rb - (\eta-\theta)m\frac{(\eta-\theta)m - rb}{(\eta-\theta)m}\right] = 0.
\end{aligned}
$$

Thus, $y_b > 0$.

Moreover, from (7.23) it is easy to verify that $\hat{\phi}$ is strictly convex on $[y_b, y_0]$. In addition, by the boundary condition $\hat{\phi}_y(y_0) = 0$ and $\hat{\phi}_y(y_b) = -b$, we deduce that $\hat{\phi}$ is strictly decreasing on $[y_b, y_0]$.

Proof of Theorem 5.3.2

By (5.22) and Lemma 5.3.1, it follows that

$$
\begin{aligned}
\frac{\phi_x}{\phi_{xx}} &= -I(-x)\,\hat{\phi}_{yy}(I(-x)) \\
&= \frac{-(\eta-\theta)m(\delta_+-1)(1-\delta_-)}{r(\delta_+-\delta_-)}\left[\delta_+\left(\frac{I(-x)}{y_0}\right)^{\delta_+-1} - \delta_-\left(\frac{I(-x)}{y_0}\right)^{\delta_--1}\right].
\end{aligned}
\tag{7.24}
$$

Substituting (7.24) into (5.14) immediately leads to (5.23) and (5.24). As for the value function ψ^+, it follows from Lemma 5.3.2, (5.21) and (5.18) that

$$
\begin{aligned}
\psi^+(x) &= \hat{\phi}(I(-x)) + xI(-x) \\
&= \hat{\phi}(I(-x)) - \hat{\phi}_y(I(-x))\,I(-x) \\
&= \frac{(\eta-\theta)m(1-\delta_-)(1-\delta_+)}{r(\delta_+-\delta_-)}I(-x)\left[\left(\frac{I(-x)}{y_0}\right)^{\delta_+-1} - \left(\frac{I(-x)}{y_0}\right)^{\delta_--1}\right].
\end{aligned}
$$

Moreover, q_+^* and π_+^* are bounded in $[0, b]$ since both functions are continuous in the compact set $[0, b]$. Thus, (q_+^*, π_+^*) is admissible as conditions 1)–3) in Definition 5.2.1 are satisfied.

Proof of Lemma 5.3.3

The proof of Lemma 5.3.3 is similar to that of Lemma 5.3.1. Thus, we will skip some tedious calculations.

The general solution to the first equation in (5.30) is known to be given by (7.17). Substituting (7.17) (and its first two derivatives) into the first equation of (5.30) yields

$$c_3 = \frac{(\eta - \theta)\, m}{r}. \tag{7.25}$$

Furthermore, the boundary conditions of (5.30) implies

$$\begin{cases} c_1\delta_+\tilde{y}_0^{\delta_+-1} + c_2\delta_-\tilde{y}_0^{\delta_--1} + \frac{(\eta-\theta)m}{r} = 0, \\ c_1\tilde{y}_0^{\delta_+} + c_2\tilde{y}_0^{\delta_-} + \frac{(\eta-\theta)m}{r}\tilde{y}_0 = 1, \\ c_1\delta_+\tilde{y}_b^{\delta_+-1} + c_2\delta_-\tilde{y}_b^{\delta_--1} + \frac{(\eta-\theta)m}{r} = b, \\ c_1\tilde{y}_b^{\delta_+} + c_2\tilde{y}_b^{\delta_-} + \frac{(\eta-\theta)m}{r}\tilde{y}_b - b\tilde{y}_b = 0. \end{cases} \tag{7.26}$$

In what follows, we consider the two cases $b < \frac{(\eta-\theta)m}{r}$ and $b = \frac{(\eta-\theta)m}{r}$ separately.

(1) For $b < \frac{(\eta-\theta)m}{r}$, by (7.26), we obtain

$$\begin{cases} c_1 = \frac{-r\delta_- - m(1-\delta_-)(\eta-\theta)\tilde{y}_0}{r(\delta_+-\delta_-)\tilde{y}_0^{\delta_+}} = \frac{(1-\delta_-)[rb-(\eta-\theta)m]}{r(\delta_+-\delta_-)}\tilde{y}_b^{1-\delta_+}, \\ c_2 = \frac{r\delta_+ - m(\delta_+-1)(\eta-\theta)\tilde{y}_0}{r(\delta_+-\delta_-)\tilde{y}_0^{\delta_-}} = \frac{(\delta_+-1)[rb-(\eta-\theta)m]}{r(\delta_+-\delta_-)}\tilde{y}_b^{1-\delta_-}. \end{cases} \tag{7.27}$$

Note that $c_1, c_2 < 0$. Let $\tilde{\kappa} := \frac{\tilde{y}_0}{\tilde{y}_b} \in (1, +\infty)$. It follows from (7.27) that

$$\begin{cases} (1-\delta_-)\,[rb - (\eta-\theta)m]\,\tilde{\kappa}^{\delta_+-1}\tilde{y}_0 + m(1-\delta_-)(\eta-\theta)\tilde{y}_0 = -r\delta_-, \\ (\delta_+-1)\,[rb - (\eta-\theta)m]\,\tilde{\kappa}^{\delta_--1}\tilde{y}_0 + m(\delta_+-1)(\eta-\theta)\tilde{y}_0 = r\delta_+. \end{cases} \tag{7.28}$$

From (7.28), it follows that $\tilde{\kappa}$ shall satisfy

$$f(\tilde{\kappa}) := [(\eta-\theta)m - rb]\left[\delta_+(1-\delta_-)\tilde{\kappa}^{\delta_+-1} + \delta_-(\delta_+-1)\tilde{\kappa}^{\delta_--1}\right] + (\delta_+-\delta_-)(\eta-\theta)m = 0. \tag{7.29}$$

Since $b < \frac{(\eta-\theta)m}{r}$ and $\delta_+ > 1 > 0 > \delta_-$, it is not difficult to show that f is a strictly increasing function on $[1, +\infty)$ with $f(1) = -rb(\delta_+ - \delta_-) < 0$ and $\lim_{\tilde{\kappa}\uparrow+\infty} f(\tilde{\kappa}) = +\infty$. Thus, (7.29) admits a unique solution $\tilde{\kappa} \in (1, +\infty)$. Substituting (7.25) and (7.27) back into (7.17) yields

$$\hat{\phi}(y) = \frac{[rb - (\eta-\theta)m]y}{r}\left[\frac{1-\delta_-}{\delta_+ - \delta_-}\left(\frac{y}{\tilde{y}_b}\right)^{\delta_+ - 1} + \frac{\delta_+ - 1}{\delta_+ - \delta_-}\left(\frac{y}{\tilde{y}_b}\right)^{\delta_- - 1} - \frac{(\eta-\theta)m}{(\eta-\theta)m - rb}\right], \tag{7.30}$$

for $y \in [\tilde{y}_b, \tilde{y}_0]$, where $\tilde{y}_0$ can be determined from either equation in (7.28), and $\tilde{y}_b = \tilde{y}_0/\tilde{\kappa}$. By (7.29), it is easy to see that since $\delta_+ > 1 > 0 > \delta_-$, $\tilde{\kappa} > 1$ and $b < \frac{(\eta-\theta)m}{r}$,

$$\begin{aligned}\frac{1}{\tilde{y}_0} &= \frac{(\delta_+ - 1)}{r\delta_+}\left[(rb - (\eta-\theta)m)\,\tilde{\kappa}^{\delta_- - 1} + (\eta-\theta)\,m\right] \\ &> \frac{(\delta_+ - 1)}{r\delta_+}\left[(rb - (\eta-\theta)m)\,\frac{(\eta-\theta)\,m}{(rb - (\eta-\theta)m)} + (\eta-\theta)\,m\right] > 0.\end{aligned}$$

Hence, $\tilde{y}_0 > 0$.

Moreover, from (7.30) one can show that $\hat{\phi}$ is strictly concave on $[\tilde{y}_b, \tilde{y}_0]$. In addition, by the boundary conditions $\hat{\phi}_y(\tilde{y}_b) = b$ and $\hat{\phi}_y(\tilde{y}_0) = 0$, we deduce that $\hat{\phi}$ is strictly increasing on $[\tilde{y}_b, \tilde{y}_0]$.

(2) For $b = \frac{(\eta-\theta)m}{r}$, (7.26) reduces to

$$\begin{cases} c_1\delta_+\tilde{y}_0^{\delta_+ - 1} + c_2\delta_-\tilde{y}_0^{\delta_- - 1} + \frac{(\eta-\theta)m}{r} = 0, \\ c_1\tilde{y}_0^{\delta_+} + c_2\tilde{y}_0^{\delta_-} + \frac{(\eta-\theta)m}{r}\tilde{y}_0 = 1, \\ c_1\delta_+\tilde{y}_b^{\delta_+ - 1} + c_2\delta_-\tilde{y}_b^{\delta_- - 1} = 0, \\ c_1\tilde{y}_b^{\delta_+} + c_2\tilde{y}_b^{\delta_-} = 0. \end{cases} \tag{7.31}$$

From the last two equations, we deduce that $c_1\tilde{y}_b^{\delta_+} = c_2\tilde{y}_b^{\delta_-} = 0$. This further implies $c_2 = 0$ and $\tilde{y}_b = 0$. Moreover, the first two equations of (7.31) yield

$$\tilde{y}_0 = \frac{r\delta_+}{(\delta_+ - 1)(\eta-\theta)m} > 0, \quad c_1 = -\frac{(\eta-\theta)m}{r\delta_+\tilde{y}_0^{\delta_+ - 1}} < 0.$$

Substituting c_1, c_2, c_3 into (7.17), we obtain

$$\hat{\phi}\left(y\right) = -\frac{(\eta-\theta)my}{r}\left[\frac{1}{\delta_+}\left(\frac{y}{\tilde{y}_0}\right)^{\delta_+-1} - 1\right], \quad y \in [0, \tilde{y}_0]. \tag{7.32}$$

From (7.32) and the boundary conditions $\hat{\phi}_y\left(\tilde{y}_0\right) = 0$ and $\hat{\phi}_y\left(0\right) = b$, it is straightforward to verify that $\hat{\phi}$ is strictly concave and strictly increasing on $[0, \tilde{y}_0]$.

Proof of Theorem 5.3.4

Given that

$$\phi\left(x\right) = \sup_{y\in[\tilde{y}_b,\tilde{y}_0]}\left\{\hat{\phi}\left(y\right) - xy\right\} = \hat{\phi}\left(I(x)\right) - xI(x), \tag{7.33}$$

where $I(x) = \hat{\phi}_y^{-1}(x)$ for $x \in (0, b)$, we deduce that $\phi_x\left(x\right) = I(x)$ and $\phi_{xx}\left(x\right) = 1/\hat{\phi}_{yy}\left(I(x)\right)$. Using (7.17), it follows that

$$\begin{aligned}\frac{\phi_x(x)}{\phi_{xx}(x)} &= I(x)\hat{\phi}_{yy}\left(I(x)\right) \\ &= c_1\delta_+(\delta_+-1)I(x)^{\delta_+-1} + c_2\delta_-(\delta_--1)I(x)^{\delta_--1}. \end{aligned} \tag{7.34}$$

Note that for the case $b = \frac{(\eta-\theta)m}{r}$, $I(x)$ can be explicitly expressed as

$$I(x) = \hat{\phi}_y^{-1}(x) = \tilde{y}_0\left[\frac{(\eta-\theta)m - rx}{(\eta-\theta)m}\right]^{\frac{1}{\delta_+-1}}, \; x \in (0, b).$$

Substituting (7.34) into (5.28) together with the expressions for c_1 and c_2 in the proof of Lemma 5.3.3 yield the corresponding expressions of q_-^* and π_-^* for both cases: $b < \frac{(\eta-\theta)m}{r}$ and $b = \frac{(\eta-\theta)m}{r}$. Moreover, the expression of the value function $\phi\left(x\right)$ can be obtained immediately from the second equation of (7.33) and the expression of $\hat{\phi}\left(\cdot\right)$ in Lemma 5.3.3. Finally, q_-^* and π_-^* are bounded in $[0, b]$ since both functions are continuous in the compact set $[0, b]$. Thus, (q_-^*, π_-^*) is admissible as conditions 1)–3) in Definition 5.2.1 are satisfied.

Proof of Theorem 6.3.1

Let $u^* \in \mathbb{U}$ be a strategy that satisfies conditions (1) and (2) in Theorem 6.3.2. We first show that $g(t,x) = \mathbb{E}_{t,x}[X_T^{u^*}]$. By condition (2) of Theorem 6.3.2, i.e., $\mathcal{A}^{u^*} g(t,x) = 0$ for all $(t,x) \in [0,T] \times \mathbb{R}$, and Dynkin's formula, we obtain

$$\mathbb{E}_{t,x}[g(T, X_T^{u^*})] = g(t,x) + \mathbb{E}_{t,x}\left[\int_t^T \mathcal{A}^{u^*} g(s, X_s) \mathrm{d}s\right] = g(t,x),$$

where $\mathcal{A}^u$ is defined in equation (6.6). By the second equation in condition (3) of Theorem 6.3.2, i.e., $g(T,x) = x$ for all $x \in \mathbb{R}$, we further obtain

$$g(t,x) = \mathbb{E}_{t,x}[g(T, X_T^{u^*})] = \mathbb{E}_{t,x}[X_T^{u^*}].$$

Next, we show that $V(t,x) = J^{u^*}(t,x)$. Since (6.7) attains its supremum at $u^* \in \mathcal{U}$, and $\mathcal{A}^{u^*} g(t,x) = 0$, (6.7) can be rewritten as

$$0 = \mathcal{A}^{u^*} V(t,x) - \frac{\gamma}{2} \mathcal{A}^{u^*} g^2(t,x), \ (t,x) \in [0,T] \times \mathbb{R}. \tag{7.35}$$

It follows from the first equation in condition (3) of Theorem 6.3.2, i.e., $V(T,x) = x$ for all $x \in \mathbb{R}$, and Dynkin's formula that

$$\mathbb{E}_{t,x}[X_T^{u^*}] = \mathbb{E}_{t,x}[V(T, X_T^{u^*})] = V(t,x) + \int_t^T \mathcal{A}^{u^*} V(s, X_s) \mathrm{d}s. \tag{7.36}$$

Substituting (7.35) into (7.36) yields

$$V(t,x) = \mathbb{E}_{t,x}[X_T^{u^*}] - \frac{\gamma}{2} \int_t^T \mathcal{A}^{u^*} g^2(s, X_s) \mathrm{d}s. \tag{7.37}$$

Since $g(T,x) = x$ and again by Dynkin's formula, we have

$$\begin{aligned}\mathbb{E}_{t,x}[(X_T^{u^*})^2] = \mathbb{E}_{t,x}[g^2(T, X_T^{u^*})] &= g^2(t,x) + \int_t^T \mathcal{A}^{u^*} g^2(s, X_s) \mathrm{d}s \\ &= \left(\mathbb{E}_{t,x}[X_T^{u^*}]\right)^2 + \int_t^T \mathcal{A}^{u^*} g^2(s, X_s) \mathrm{d}s,\end{aligned}$$

or equivalently,

$$\mathrm{Var}_{t,x}[X_T^{u^*}] = \int_t^T \mathcal{A}^{u^*} g^2(s, X_s) \mathrm{d}s. \tag{7.38}$$

Finally, substituting (7.38) into (7.37) yields

$$V(t,x) = \mathbb{E}_{t,x}[X_T^{u^*}] - \frac{\gamma}{2}\mathrm{Var}_{t,x}[X_T^{u^*}] = J^{u^*}(t,x). \tag{7.39}$$

The remaining work is to show that $u^* \in \mathbb{U}$ is an equilibrium strategy defined in Definition 6.2.2. For any $\varepsilon > 0$ and $t \in [0,T]$, consider the strategy u^ε defined in Definition 6.2.2, i.e.,

$$u_s^\varepsilon = \begin{cases} \bar{u}, & t \le s < t+\varepsilon, \\ u_s^*, & t+\varepsilon \le s < T, \end{cases}$$

where $\bar{u} := (\bar{m}, \bar{p}, \bar{\pi})$ is an arbitrary constant strategy. By the definition of $J^u(t,x)$ in (6.5) and the fact that $g(T,x) = x$, we have

$$\begin{aligned}
&\mathbb{E}_{t,x}[J^{u^\varepsilon}(t+\varepsilon, X_{t+\varepsilon}^{\bar{u}})] - J^{u^\varepsilon}(t,x) \\
&= \mathbb{E}_{t,x}\left[\mathbb{E}_{t+\varepsilon, X_{t+\varepsilon}^{\bar{u}}}\left[X_T^{u^*} - \frac{\gamma}{2}(X_T^{u^*})^2\right] + \frac{\gamma}{2}\left(\mathbb{E}_{t+\varepsilon, X_{t+\varepsilon}^{\bar{u}}}[X_T^{u^*}]\right)^2\right] \\
&- \mathbb{E}_{t,x}\left[X_T^{u^\varepsilon} - \frac{\gamma}{2}(X_T^{u^\varepsilon})^2\right] - \frac{\gamma}{2}\left(\mathbb{E}_{t,x}[X_T^{u^\varepsilon}]\right)^2 \\
&= \frac{\gamma}{2}\mathbb{E}_{t,x}\left[\left(\mathbb{E}_{t+\varepsilon, X_{t+\varepsilon}^{\bar{u}}}[X_T^{u^*}]\right)^2\right] - \frac{\gamma}{2}\left(\mathbb{E}_{t,x}\left[\mathbb{E}_{t+\varepsilon, X_{t+\varepsilon}^{\bar{u}}}[X_T^{u^*}]\right]\right)^2 \\
&= \frac{\gamma}{2}\mathbb{E}_{t,x}\left[g^2(t+\varepsilon, X_{t+\varepsilon}^{\bar{u}})\right] - \frac{\gamma}{2}\left(\mathbb{E}_{t,x}\left[g(t+\varepsilon, X_{t+\varepsilon}^{\bar{u}})\right]\right)^2.
\end{aligned} \tag{7.40}$$

It follows from (7.39), (7.40) and the definition of u^ε that

$$\begin{aligned}
J^{u^\varepsilon}(t,x) &= \mathbb{E}_{t,x}[J^{u^\varepsilon}(t+\varepsilon, X_{t+\varepsilon}^{\bar{u}})] - \frac{\gamma}{2}\mathbb{E}_{t,x}\left[g^2(t+\varepsilon, X_{t+\varepsilon}^{\bar{u}})\right] + \frac{\gamma}{2}\left(\mathbb{E}_{t,x}\left[g(t+\varepsilon, X_{t+\varepsilon}^{\bar{u}})\right]\right)^2 \\
&= \mathbb{E}_{t,x}[V(t+\varepsilon, X_{t+\varepsilon}^{\bar{u}})] - \frac{\gamma}{2}\left\{\mathbb{E}_{t,x}\left[g^2(t+\varepsilon, X_{t+\varepsilon}^{\bar{u}})\right] - g^2(t,x)\right\} \\
&+ \frac{\gamma}{2}\left\{\left(\mathbb{E}_{t,x}\left[g(t+\varepsilon, X_{t+\varepsilon}^{\bar{u}})\right]\right)^2 - g^2(t,x)\right\}.
\end{aligned} \tag{7.41}$$

For any $u \in \mathcal{U}$, $\varepsilon > 0$ and $\phi \in C^{1,2}([0,T] \times \mathbb{R})$, define an operator

$$\mathcal{A}_\varepsilon^u \phi(t,x) = \mathbb{E}_{t,x}[\phi(t+\varepsilon, X_{t+e}^u)] - \phi(t,x). \tag{7.42}$$

By the definition of $\mathcal{A}^u$, we have

$$\lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \mathcal{A}^u_\varepsilon \phi(t,x) = \mathcal{A}^u \phi(t,x). \tag{7.43}$$

By (7.42), we can rewrite (7.41) as

$$J^{u^\varepsilon}(t,x) = V(t,x) + \mathcal{A}^{\bar{u}}_\varepsilon V(t,x) - \frac{\gamma}{2} \mathcal{A}^{\bar{u}}_\varepsilon g^2(t,x) + \frac{\gamma}{2} \left\{ \left(\mathbb{E}_{t,x} \left[g(t+\varepsilon, X^{\bar{u}}_{t+\varepsilon}) \right] \right)^2 - g^2(t,x) \right\}. \tag{7.44}$$

By Dynkin's formula, we conclude that

$$\mathbb{E}_{t,x}[g(t+\varepsilon, X^{\bar{u}}_{t+\varepsilon})] = g(t,x) + \mathbb{E}_{t,x} \left[\int_t^{t+\varepsilon} \mathcal{A}^{\bar{u}} g(s, X_s) \mathrm{d}s \right]. \tag{7.45}$$

Substituting (7.45) into (7.44) yields

$$J^{u^\varepsilon}(t,x) = V(t,x) + \mathcal{A}^{\bar{u}}_\varepsilon V(t,x) - \frac{\gamma}{2} \mathcal{A}^{\bar{u}}_\varepsilon g^2(t,x) + \gamma g(t,x) \mathbb{E}_{t,x} \left[\int_t^{t+\varepsilon} \mathcal{A}^{\bar{u}} g(s, X_s) \mathrm{d}s \right] + o(\varepsilon). \tag{7.46}$$

On the other hand, since $\bar{u}$ is an arbitrary strategy,

$$\mathcal{A}^{\bar{u}} V(t,x) - \frac{\gamma}{2} \mathcal{A}^{\bar{u}} g^2(t,x) + \gamma g(t,x) \mathcal{A}^{\bar{u}} g(t,x) \leq 0. \tag{7.47}$$

It follows from (7.47) and (7.43) that

$$\mathcal{A}^{\bar{u}}_\varepsilon V(t,x) - \frac{\gamma}{2} \mathcal{A}^{\bar{u}}_\varepsilon g^2(t,x) + \gamma g(t,x) \mathbb{E}_{t,x} \left[\int_t^{t+\varepsilon} \mathcal{A}^{\bar{u}} g(s, X_s) \mathrm{d}s \right] \leq o(\varepsilon). \tag{7.48}$$

Substituting (7.48) into (7.46) together with the fact that $V(t,x) = J^{u^*}(t,x)$, we obtain

$$J^{u^\varepsilon}(t,x) \leq J^{u^*}(t,x) + o(\varepsilon),$$

which further implies

$$\liminf_{\varepsilon \downarrow 0} \frac{J^{u^*}(t,x) - J^{u^\varepsilon}(t,x)}{\varepsilon} \geq 0.$$

Therefore, u^* is an equilibrium strategy.

Proof of Theorem 6.3.2

We verify that given $\hat{u}^*$, $\hat{V}$, and $\hat{g}$ defined respectively in (6.11), (6.12), and (6.13), condition (1)–(3) in Theorem 6.3.1 are satisfied. For ease of notations, we rewrite $\hat{V}$ and $\hat{g}$ in the following form, i.e.,

$$\begin{cases} \hat{V}\left(t,x\right) = A\left(t\right)x + \bar{A}\left(t\right), \ A\left(T\right) = 1, \ \bar{A}\left(T\right) = 0, \\ \hat{g}\left(t,x\right) = a\left(t\right)x + \bar{a}\left(t\right), \ a\left(T\right) = 1, \ \bar{a}\left(T\right) = 0, \end{cases} \tag{7.49}$$

and subsequently denote $A(t)$, $\bar{A}\left(t\right)$, $a(t)$, and $\bar{a}(t)$ by A, $\bar{A}$, a, and $\bar{a}$, and their first order derivative by A_t, $\bar{A}_t$, a_t and $\bar{a}_t$, respectively. By expanding $\mathcal{A}^u$ using (6.6), together with some simplifications, we rewrite the function on the right-hand side of (6.7) as

$$\begin{aligned} \hat{L}(\hat{m},\hat{p},\hat{\pi}) := {}& \hat{V}_t\left(t,x\right) + \left[\hat{C} + \int_0^\infty \hat{l}\left(z;t\right)v\left(\mathrm{d}z\right)\right]\hat{V}_x(t,x) \\ & + \frac{1}{2}\left(\sigma_1^2 + 2\rho\hat{\pi}\sigma_1\sigma_2 + \hat{\pi}^2\sigma_2^2\right)\hat{V}_{xx}(t,x) \\ & + \int_0^\infty \left(\hat{V}\left(t, x - \hat{l}\left(z;t\right)\right) - \hat{V}\left(t,x\right)\right)v\left(\mathrm{d}z\right) \\ & - \frac{\gamma}{2}\left(\sigma_1^2 + 2\rho\hat{\pi}\sigma_1\sigma_2 + \hat{\pi}^2\sigma_2^2\right)\hat{g}_x^2\left(t,x\right) \\ & - \frac{\gamma}{2}\int_0^\infty \left[\hat{g}\left(t, x - \hat{l}\left(z;t\right)\right) - \hat{g}\left(t,x\right)\right]^2 v\left(\mathrm{d}z\right), \end{aligned} \tag{7.50}$$

where $\hat{l}(t,z)$ is defined in (6.10) and

$$\hat{C} = rx + \left(\mu - r\right)\hat{\pi} + \left(\theta - \eta\right)\int_0^\infty zv\left(\mathrm{d}z\right) + \eta\int_0^\infty \hat{l}\left(z;t\right)v\left(\mathrm{d}z\right).$$

By utilizing (7.49), we further rewrite (7.50) as

$$\begin{aligned} \hat{L}(\hat{m},\hat{p},\hat{\pi}) \quad = \quad & -\frac{\gamma}{2}\left[\left(\sigma_1^2 + 2\rho\hat{\pi}\sigma_1\sigma_2 + \hat{\pi}^2\sigma_2^2\right)a^2 + \int_0^\infty a^2\hat{l}^2\left(z;t\right)v\left(\mathrm{d}z\right)\right] \\ & + A_t x + \bar{A}_t + \hat{C}v\left(\mathrm{d}z\right)A. \end{aligned} \tag{7.51}$$

By (6.10), we further have

$$\int_0^\infty \hat{l}\left(z;t\right)v\left(\mathrm{d}z\right) = \int_0^{\hat{m}} zv\left(\mathrm{d}z\right) + \hat{m}\left(1-\hat{p}\right)\int_{\hat{m}}^\infty v\left(\mathrm{d}z\right) + \hat{p}\int_{\hat{m}}^\infty zv\left(\mathrm{d}z\right), \tag{7.52}$$

and

$$
\begin{aligned}
&\int_0^{\infty} \hat{l}^2 (z;t)\, v\,(\mathrm{d}z) \\
&= \int_0^{\hat{m}} z^2 v\,(\mathrm{d}z) + [\hat{m}\,(1-\hat{p})]^2 \int_{\hat{m}}^{\infty} v\,(\mathrm{d}z) + \hat{p}^2 \int_{\hat{m}}^{\infty} z^2 v\,(\mathrm{d}z) + 2\hat{m}\,(1-\hat{p})\,\hat{p} \int_{\hat{m}}^{\infty} zv\,(dz)\,.
\end{aligned}
\tag{7.53}
$$

By substituting (7.52) and (7.53) into (7.51) and subsequently applying the first order condition respectively to $\hat{m}$, $\hat{p}$ and $\hat{\pi}$, we obtain the following system of equations, i.e.,

$$
\begin{cases}
\hat{m} = \frac{\eta A}{\gamma a^2} - \frac{\hat{p}\int_{\hat{m}}^{\infty}(z-\hat{m})v(\mathrm{d}z)}{\int_{\hat{m}}^{\infty}v(\mathrm{d}z)}, \\
\hat{p} = \left(\frac{\eta A}{\gamma a^2} - \hat{m}\right)\frac{\int_{\hat{m}}^{\infty}(z-\hat{m})v(\mathrm{d}z)}{\int_{\hat{m}}^{\infty}(z-\hat{m})^2 v(\mathrm{d}z)}, \\
\hat{\pi} = \frac{(\mu-r)A}{\gamma\sigma_2^2 a^2} - \rho\frac{\sigma_1}{\sigma_2}.
\end{cases}
\tag{7.54}
$$

We claim that $\hat{m}^* = \frac{\eta A}{\gamma a^2}$ and $\hat{p}^* = 0$ is the unique solution to the first two equations in (7.54). To see this, suppose that

$$
\hat{p} = \left(\frac{\eta A}{\gamma a^2} - \hat{m}\right)\frac{\int_{\hat{m}}^{\infty}(z-\hat{m})\,v\,(\mathrm{d}z)}{\int_{\hat{m}}^{\infty}(z-\hat{m})^2\,v\,(\mathrm{d}z)} \neq 0,
$$

then $\hat{m}^*$ is a solution to

$$
\int_{\hat{m}}^{\infty}(z-\hat{m})^2\,v\,(\mathrm{d}z)\int_{\hat{m}}^{\infty}v\,(\mathrm{d}z) = \left(\int_{\hat{m}}^{\infty}(z-\hat{m})\,v\,(\mathrm{d}z)\right)^2. \tag{7.55}
$$

We next prove that (7.55) does not admit a solution on $[0,\infty)$. Let

$$
\hat{f}\,(x) = \int_x^{\infty}(z-x)^2\,v\,(\mathrm{d}z)\int_x^{\infty}v\,(\mathrm{d}z) - \left[\int_x^{\infty}(z-x)\,v\,(\mathrm{d}z)\right]^2,\ x\in[0,\infty).
$$

Since $\hat{f}\,(0) > 0$, $\lim_{x\to\infty}\hat{f}\,(x) = 0$, and

$$
\begin{aligned}
\hat{f}_x\,(x) &= \int_x^{\infty} -2\,(z-x)\,v\,(\mathrm{d}z)\int_x^{\infty}v\,(\mathrm{d}z) - v\,(x)\int_x^{\infty}(z-x)^2\,v\,(\mathrm{d}z) \\
&\quad - 2\int_x^{\infty}(z-x)\,v\,(\mathrm{d}z)\int_x^{\infty}(-1)\,v\,(\mathrm{d}z) \\
&= -v\,(x)\int_x^{\infty}(z-x)^2\,v\,(\mathrm{d}z) < 0,\ x\in(0,\infty),
\end{aligned}
$$

$\hat{f}$ does not admit a zero on $[0,\infty)$, i.e., a solution to (7.55) does not exist on $[0,\infty)$. Thus,

$$\hat{u}^* = \left(\frac{\eta A}{\gamma a^2}, 0, \frac{(\mu - r) A}{\gamma \sigma_2^2 a^2} - \rho\frac{\sigma_1}{\sigma_2}\right) = \left(\frac{\eta}{\gamma}\mathrm{e}^{-r(T-t)}, 0, \frac{(\mu - r)}{\gamma \sigma_2^2}\mathrm{e}^{-r(T-t)} - \rho\frac{\sigma_1}{\sigma_2}\right)$$

is the unique solution to (7.54). By a second derivative test, it is straightforward to show that the supremum of $\hat{L}$ is attained at $\hat{u}^*$. By substituting $\hat{u}^*$ into (7.51) and using (6.12), it is straightforward to verify that the supremum is equal to 0, i.e., condition (1) in Theorem 6.3.1 is satisfied.

By expanding $\mathcal{A}^u$ using (6.6) and using $\hat{g}$ in (7.49), we have for any $(t,x) \in [0,T] \times \mathbb{R}$,

$$\mathcal{A}^u \hat{g}(t,x) = a_t x + \bar{a}_t + \left(rx + (\mu - r)\,\hat{\pi} + (\theta - \eta)\int_0^\infty z v\,(\mathrm{d}z) + \eta \int_0^\infty \hat{l}\,(z;t)\, v\,(\mathrm{d}z)\right) a. \tag{7.56}$$

Substituting $\hat{u}^*$ into (7.56) and utilizing the form of a and $\bar{a}$ yields $\mathcal{A}^{\hat{u}^*}\hat{g}(t,x) = 0$, i.e., condition (2) in Theorem 6.3.1 is satisfied.

Finally, one sees directly from (7.49) that $\hat{V}(T,x) = x$ and $\hat{g}\,(T,x) = x$, i.e., condition (3) in Theorem 6.3.1 is satisfied. Thus, by Theorem 6.3.1, $\hat{u}^*$ is the equilibrium reinsurance-investment strategy and $\hat{V}$ is the corresponding equilibrium value function. Furthermore, $\mathbb{E}_{t,x}[X_T^{\hat{u}^*}] = \hat{g}(t,x)$. The proof is therefore completed.

Proof of Theorem 6.3.3

We verify that given $\tilde{u}^*$, $\tilde{V}$, and $\tilde{g}$ defined respectively in (6.15), (6.16), and (6.17), condition (1)–(3) in Theorem 6.3.1 are satisfied. Since for any $(t,x) \in [0,T]\times\mathbb{R}$, $\tilde{V}(t,x) = \hat{V}(t,x)$ and $\tilde{g}(t,x) = \hat{g}(t,x)$, by using a similar set of arguments as in the proof of Theorem 6.3.2, we rewrite the function on the right-hand side of (6.7) as

$$\tilde{L}(\tilde{m},\tilde{p},\tilde{\pi}) = A_t x + \bar{A}_t + \tilde{C}A - \frac{\gamma}{2}\left(\sigma_1^2 + 2\rho\tilde{\pi}\sigma_1\sigma_2 + \tilde{\pi}^2\sigma_2^2\right) a^2 - \frac{\gamma}{2}\int_0^\infty a^2 \tilde{l}^2\,(z;t)\, v\,(\mathrm{d}z)\,, \tag{7.57}$$

where

$$\tilde{C} = rx + (\mu - r)\,\tilde{\pi} + (\theta - \eta)\int_0^\infty zv\,(\mathrm{d}z) + \eta\int_0^\infty \tilde{l}\,(z;t)\,v\,(\mathrm{d}z)\,.$$

Furthermore, by (6.14) we have

$$\int_0^\infty \tilde{l}\,(z;t)\,v\,(\mathrm{d}z) = \int_0^{\frac{\tilde{m}}{\tilde{p}}} \tilde{p}zv\,(\mathrm{d}z) + \tilde{m}\int_{\frac{\tilde{m}}{\tilde{p}}}^\infty v\,(\mathrm{d}z)\,, \tag{7.58}$$

and

$$\int_0^\infty \tilde{l}^2\,(z;t)\,v\,(\mathrm{d}z) = \int_0^{\frac{\tilde{m}}{\tilde{p}}} \tilde{p}^2z^2v\,(\mathrm{d}z) + \tilde{m}^2\int_{\frac{\tilde{m}}{\tilde{p}}}^\infty v\,(\mathrm{d}z)\,. \tag{7.59}$$

By substituting (7.58) and (7.59) into (7.57) and subsequently applying the first order condition respectively to $\tilde{m}$, $\tilde{p}$ and $\tilde{\pi}$, we obtain that

$$\begin{cases} \eta A\int_{\frac{\tilde{m}}{\tilde{p}}}^\infty v\,(\mathrm{d}z) - \gamma a^2\tilde{m}\int_{\frac{\tilde{m}}{\tilde{p}}}^\infty v\,(\mathrm{d}z) = 0, \\ -\gamma a^2\tilde{p}\int_0^{\frac{\tilde{m}}{\tilde{p}}} z^2v\,(\mathrm{d}z) + \eta A\int_0^{\frac{\tilde{m}}{\tilde{p}}} zv\,(\mathrm{d}z) = 0, \\ \tilde{\pi} - \left(\frac{(\mu-r)B}{\gamma\sigma_2^2b^2} - \rho\frac{\sigma_1}{\sigma_2}\right) = 0. \end{cases} \tag{7.60}$$

Solving the first equation in (7.60) yields $\tilde{m}^* = \frac{\eta A}{\gamma a^2}$. Let $\tilde{f}(\tilde{p}) := -\gamma a^2\tilde{p}\int_0^{\frac{\tilde{m}}{\tilde{p}}} z^2v\,(\mathrm{d}z) + \eta A\int_0^{\frac{\tilde{m}}{\tilde{p}}} zv\,(\mathrm{d}z)$. It is straightforward to verify that given $\tilde{m} = \tilde{m}^*$,

$$\begin{cases} \tilde{f}(1) = -\gamma a^2\int_0^{\tilde{m}^*} z(z-\tilde{m}^*)v(\mathrm{d}z) > 0, \\ \tilde{f}_{\tilde{p}}(\tilde{p}) = -\gamma a^2\int_0^{\frac{\tilde{m}^*}{\tilde{p}}} z^2v\,(\mathrm{d}z) < 0,\ \ \tilde{p}\in(0,1]. \end{cases}$$

Thus, $\tilde{f}(\tilde{p}) > 0$ for all $\tilde{p}\in(0,1]$. In other words, a critical point of $\tilde{L}$ does not exist on $[0,\infty)\times(0,1]\times\mathbb{R}$. The supremum $\tilde{L}$ is then attained at

$$\tilde{u}^* = \left(\frac{\eta A}{\gamma a^2}, 1, \frac{(\mu-r)\,A}{\gamma\sigma_2^2a^2} - \rho\frac{\sigma_1}{\sigma_2}\right) = \left(\frac{\eta}{\gamma}\mathrm{e}^{-r(T-t)}, 1, \frac{(\mu-r)}{\gamma\sigma_2^2}\mathrm{e}^{-r(T-t)} - \rho\frac{\sigma_1}{\sigma_2}\right).$$

Then it is straightforward to verify that

$$\sup_{\tilde{u}\in\mathcal{U}} \tilde{L}\,(\tilde{m},\tilde{p},\tilde{\pi}) = \tilde{L}\,(\tilde{m}^*,\tilde{p}^*,\tilde{\pi}^*) = \hat{L}\,(\hat{m}^*,\hat{p}^*,\hat{\pi}^*) = 0,$$

$$\mathcal{A}^{\tilde{u}^*}\tilde{g}(t,x) = \mathcal{A}^{\hat{u}^*}\hat{g}(t,x) = 0,$$

$\tilde{V}(T,x) = x$, and $\tilde{g}(T,x) = x$, i.e., conditions (1)–(3) in Theorem 6.3.1 are satisfied. Thus, by Theorem 6.3.1, $\tilde{u}^*$ is the equilibrium reinsurance-investment strategy and $\tilde{V}$ is the corresponding equilibrium value function. Furthermore, $\mathbb{E}_{t,x}\left[X_T^{\tilde{u}^*}\right] = \tilde{g}(t,x)$. The proof is therefore completed.