Matrix Formulations of Matching Problems

by

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I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my readers.

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Abstract

Finding the maximum size of a matching in an undirected graph and finding the maximum size of branching in a directed graph can be formulated as matrix rank problems. The Tutte matrix, introduced by Tutte as a representation of an undirected graph, has rank equal to the maximum number of vertices covered by a matching in the associated graph. The branching matrix, a representation of a directed graph, has rank equal to the maximum number of vertices covered by a branching in the associated graph. A mixed graph has both undirected and directed edges, and the matching forest problem for mixed graphs, introduced by Giles, is a generalization of the matching problem and the branching problem. A mixed graph can be represented by the matching forest matrix, and the rank of the matching forest matrix is related to the size of a matching forest in the associated mixed graph. The Tutte matrix and the branching matrix have indeterminate entries, and we describe algorithms that evaluate the indeterminates as rationals in such a way that the rank of the evaluated matrix is equal to the rank of the indeterminate matrix. Matroids in the context of graphs are discussed, and matroid formulations for the matching, branching, and matching forest problems are given.

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Contents

Chapter 1

Introduction

The problem of finding a matching or a related structure in a graph can be formulated as a matrix problem. We focus on three formulations, each specific to a particular type of graph.

The *Tutte matrix* was introduced by Tutte as a representation of a graph with undirected edges. The number of vertices covered by a maximum matching in an undirected graph is equal to the rank of the corresponding Tutte matrix. The *branching matrix* for directed graphs has associations with Maxwell's rule in electrical engineering, and with Cayley's formula for counting trees (Rényi [27]). The rank of the branching matrix is equal to the number of vertices covered by a maximum branching in the associated directed graph. A *mixed graph* has both undirected and directed edges, and a matching forest in a mixed graph was introduced by Giles [14] as a generalization of matchings and branchings. The matching forest matrix is the sum of the Tutte matrix and the branching matrix, and it has rank equal to the maximum number of vertices covered by a matching forest in the associated mixed graph.

These three matrix representations and their relation to matching structures are described in Chapter 3, after the necessary linear algebra tools are developed in Chapter 2.

The entries in the Tutte matrix and the branching matrix are indeterminates, and general methods for evaluating indeterminates as rationals in such a way that an evaluated matrix has the same rank as the corresponding matrix of indeterminates are shown in Chapter 4. Specific algorithms for evaluations of the Tutte matrix, the branching matrix, and the matching forest matrix are given in Chapter 5.

Determining if a graph has a matching of a particular weight, rather than finding the usual maximum or minimum weight matching, is an example of an *exact* problem. The possibility of using matrix formulations to find solutions to exact matching and exact branching problems is discussed in Chapter 6.

The final chapter, Chapter 7, gives matroid formulations of the matching, branching, and matching forest problem in graphs.

Chapter 2

Matrix rank

The *rank* of a matrix can be defined in several ways. We use a definition which explicitly states the equivalence between rank and linear independence of vectors: the rank of a matrix A is the maximum number of linearly independent columns in A. Equivalently, the rank of a matrix is the maximum number of linearly independent rows. A matrix is *nonsingular* if it has both full row rank and full column rank, and therefore, determining if a matrix is nonsingular when the rank is known is trivial, as is calculating the rank of a nonsingular matrix. We show that the problems of calculating the rank of a matrix and determining if a matrix is nonsingular are equivalent. That is, if the rank of a matrix can be determined efficiently, then so can nonsingularity, and if nonsingularity can be efficiently computed, then so can matrix rank. Properties of matrix rank needed in Chapter 3 are proven in this chapter. These are standard properties that can be found in most linear algebra texts.

2.1 Rank and nonsingularity

Let $A = (a_{ij})$ be a matrix with rows indexed by R and columns indexed by C. If $X \subseteq R$ and $Y \subseteq C$, then $A[X;Y]$ denotes the submatrix of A which uses rows X and columns Y. The complement of X, $R \backslash X$ is denoted $\overline{X}$, and similarly, $\overline{Y} = C \backslash Y$. The nonsingular submatrix $A[X;Y]$ is a *maximal* nonsingular submatrix if for all $x \in \overline{R}$ and $y \in \overline{Y}$, the larger submatrix $A[X \cup x; Y \cup y]$ is singular. A maximal nonsingular submatrix in A can be constructed as follows:

Choose $X \subseteq R$ and $Y \subseteq C$ such that $A[X;Y]$ is nonsingular. For example, choose $X = Y = \emptyset$. While there exists $x \in \overline{X}$ and $y \in \overline{Y}$ such that $A[X \cup x; Y \cup y]$ is nonsingular, replace X with $X \cup \{x\}$ and replace Y with $Y \cup \{y\}$.

Theorem 2.1. *The size of a maximal nonsingular submatrix of A is equal to the rank of A.*

Proof. Let $A[X;Y]$ be a maximal nonsingular submatrix of A. If $X = R$ or $Y = C$ then the theorem is clearly true, so assume $X \subset R$ and $Y \subset C$. Let $x \in \overline{X}$ and $y \in \overline{Y}$. Since $A[X \cup x; Y \cup y]$ is singular, row x is in the row space of $A[X; Y \cup y]$. This is true for all $x \in \overline{X}$, and so by taking suitable multiples of rows in X, all entries in $A[\overline{X}; Y \cup y]$ can be eliminated. If $\widetilde{A} = (\tilde{a}_{ij})$ denotes the matrix A after the Gaussian elimination of $A[\overline{X}; Y \cup y]$, then since Gaussian elimination does not affect rank, rank $\widetilde{A} =$ rank A.

Suppose there exists $i \in \overline{X}$ and $j \in \overline{Y}$ such that $\tilde{a}_{ij} \neq 0$. Then

$$\det \widetilde{A}[X \cup i; Y \cup j] = \pm \tilde{a}_{ij} \det A[X;Y] \neq 0.$$

This implies that $A[X \cup i; Y \cup j]$ is nonsingular, which is a contradiction, and therefore $\tilde{a}_{ij} = 0$ for every $i \in \overline{X}, j \in \overline{Y}$, and rank $A =$ rank $A[X;Y]$. □

2.2 Submodularity

The set of all subsets of the finite set $\mathcal{X}$ is denoted $2^{\mathcal{X}}$. A function $f : 2^{\mathcal{X}} \to \mathbb{Z}$ is *submodular* if the inequality

$$f(A) + f(B) \geq f(A \cap B) + f(A \cup B)$$

holds for all $A, B \subseteq \mathcal{X}$.

Theorem 2.2. *The rank function on the set of columns of a matrix is submodular.*

Proof. Let M be a matrix with rows and columns indexed by X and Y respectively, and assume $A, B \subseteq Y$. Let Z be a maximal set of independent columns in $M[X; A \cap B]$, and extend Z to $\widetilde{Z}$, where $\widetilde{Z}$ is a maximal set of independent columns in $M[X; A \cup B]$. Then

$$\begin{aligned} \text{rank } (A \cup B) &= |\widetilde{Z}| \\ &= |\widetilde{Z} \cap A| + |\widetilde{Z} \cap B| - |\widetilde{Z} \cap (A \cap B)| \\ &= |\widetilde{Z} \cap A| + |\widetilde{Z} \cap B| - |Z| \\ &= |\widetilde{Z} \cap A| + |\widetilde{Z} \cap B| - \text{rank } (A \cap B) \end{aligned} \tag{2.1}$$

Both $\widetilde{Z} \cap A$ and $\widetilde{Z} \cap B$ are independent, and therefore

$$|\widetilde{Z} \cap A| + |\widetilde{Z} \cap B| \leq \text{rank } A + \text{rank } B. \tag{2.2}$$

Submodularity follows from (2.1) and (2.2). □

The set of all matrices over the field $\mathcal{F}$ is denoted by $M_{\mathcal{F}}$. A function $f : M_{\mathcal{F}} \to \mathbb{Z}$ is *submodular* if the inequality

$$f(A[X_1; Y_1]) + f(A[X_2; Y_2]) \geq f(A[X_1 \cap X_2; Y_1 \cup Y_2]) + f(A[X_1 \cup X_2; Y_1 \cap Y_2])$$

holds for all $A \in M_{\mathcal{F}}$ and all subsets of rows X_1, X_2 of A and all subsets of columns Y_1, Y_2 of A.

Theorem 2.3. *The rank function is submodular.*

Proof. Let $A = (a_{ij}) \in M_{\mathcal{F}}$, where $\mathcal{F}$ is any field. Let X and Y index the rows and columns respectively of A, and assume $X_1, X_2 \subseteq X$ and $Y_1, Y_2 \subseteq Y$. Consider

$$B = \begin{pmatrix} I & | & A \end{pmatrix},$$

where I is the identity matrix, and X indexes the rows and columns of I. For any $X' \subseteq X$ and $Y' \subseteq Y$, the following holds:

$$\text{rank } A[X'; Y'] = \text{rank } B[X; Y' \cup \overline{X'}] - |\overline{X'}|.$$

In particular,

$$\begin{aligned} \text{rank } A[X_1; Y_1] &= \text{rank } B[X; Y_1 \cup \overline{X_1}] - |\overline{X_1}|, \\ \text{rank } A[X_2; Y_2] &= \text{rank } B[X; Y_2 \cup \overline{X_2}] - |\overline{X_2}|, \\ \text{rank } A[X_1 \cap X_2; Y_1 \cup Y_2] &= \text{rank } B[X; (Y_1 \cup Y_2) \cup (\overline{X_1} \cup \overline{X_2})] - |\overline{X_1} \cup \overline{X_2}|, \\ \text{rank } A[X_1 \cup X_2; Y_1 \cap Y_2] &= \text{rank } B[X; (Y_1 \cap Y_2) \cup (\overline{X_1} \cap \overline{X_2})] - |\overline{X_1} \cap \overline{X_2}|. \end{aligned} \tag{2.3}$$

From Theorem 2.2, it follows that

$$\begin{aligned} &\text{rank } B[X; Y_1 \cup \overline{X_1}] + \text{rank } B[X; Y_2 \cup \overline{X_2}] \geq \\ &\qquad \text{rank } B[X; (Y_1 \cup \overline{X_1}) \cup (Y_2 \cup \overline{X_2})] + \text{rank } B[X; (Y_1 \cup \overline{X_1}) \cap (Y_2 \cup \overline{X_2})]. \end{aligned} \tag{2.4}$$

Using that $|\overline{X_1}| + |\overline{X_2}| = |\overline{X_1} \cap \overline{X_2}| + |\overline{X_1} \cup \overline{X_2}|$, and $(Y_1 \cap Y_2) \cup (\overline{X_1} \cap \overline{X_2}) \subseteq (Y_1 \cup \overline{X_1}) \cap (Y_2 \cup \overline{X_2})$, the Theorem follows from (2.3) and (2.4). □

With Theorem 2.3 we can prove that a submatrix formed by the intersection of a maximal set of independent rows with a maximal set of independent columns is nonsingular.

Corollary 2.4. *If X is a maximal set of independent rows in the matrix A, and Y is a maximal set of independent columns in A, then $A[X;Y]$ is nonsingular.*

Proof. Let R be the index set for the rows of A, and let C be the index set for the columns. By the submodularity of the rank function,

$$\text{rank } A[X;Y] + \text{rank } A \geq \text{rank } A[X;C] + \text{rank } A[R;Y].$$

Since X and Y are maximal independent sets,

$$\text{rank } A[R;Y] = \text{rank } A[X;C] = \text{rank } A,$$

and therefore $A[X;Y]$ has full rank. □

2.3 Symmetric matrices

An $n \times n$ matrix A is *symmetric* if it is equal to its transpose: $A = A^\top$. A *skew-symmetric* matrix is equal to the negative of its transpose: $A = -A^\top$. (Unless otherwise specified, we assume the field we are working over is the rationals.)

If A is a matrix with rows and columns indexed by V, and $X \subseteq V$, then $A[X;X]$ is a *principal submatrix* of A. The principal submatrix $A[X;X]$ is denoted $A[X]$.

Theorem 2.5. *If A is a symmetric or skew-symmetric matrix, and X indexes a maximal set of independent rows of A, then $A[X]$ is nonsingular.*

Proof. By symmetry, X is also a maximal set of independent columns of A. The theorem then follows from Corollary 2.4. □

We note that Theorem 2.5 does not hold when A is not symmetric, as shown by the first row in $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 2 \end{smallmatrix}\right)$.

Theorem 2.1 equated the rank of a matrix to the size of its largest nonsingular submatrix. With a symmetric or skew-symmetric matrix, this can be strengthened to the size of a largest nonsingular principal submatrix.

Corollary 2.6. *The size of a largest nonsingular principal submatrix in a symmetric or skew-symmetric matrix is equal to the rank of the matrix.*

Proof. Let A be a symmetric or skew-symmetric matrix. The rank of A is an upper bound on the size of a nonsingular principal submatrix, and, from Theorem 2.5, there is a principal submatrix whose size is equal to the rank of A. □

Two properties of the determinant function are that $\det A[X] = \det A[X]^\top$ and $\det(-A[X]) = (-1)^{|X|} \det A[X]$.

Corollary 2.7. *Skew symmetric matrices have even rank.*

Proof. If A is skew-symmetric, then $A[X] = -A[X]^\top$. From the above properties of the determinant function it follows that a nonsingular principal submatrix of A must have even size. The result then follows from Corollary 2.6. □

2.4 Nonsingularity of the sum of two matrices

Let A and B be $n \times n$ matrices. Suppose every column of A except for one is the same as the corresponding column in B. Let C be the $n \times n$ matrix with columns equal to those of A and B, and on the one column that A and B differ, the corresponding column of C is equal to the sum of the column in A and the column in B. For example, if $A = \left(v_1 \ v_2 \ a_3 \ v_4 \right)$ and $B = \left(v_1 \ v_2 \ b_3 \ v_4 \right)$, then $C = \left(v_1 \ v_2 \ a_3{+}b_3 \ v_4 \right)$. The linearity of the determinant function states that

$$\det A + \det B = \det C.$$

Repeated use of this linearity property yields an equation for the determinant of the sum of two arbitrary matrices. The determinant of the empty matrix appears in the equation, which by convention is 1. Assume the rows and columns of A and B are indexed by $V \subset \mathbb{Z}$, and for $X = \{x_1, \ldots, x_k\} \subseteq V$ and $Y = \{y_1, \ldots, y_k\} \subseteq V$, define $\operatorname{sign}(X, Y)$ to be $(-1)^{\sum_{i=1}^{k}(x_i + y_i)}$. The following is then standard.

Theorem 2.8. *If $A = (a_{ij})$ and $B = (b_{ij})$, where $i, j \in V$, then*

$$\det(A + B) = \sum_{X \subseteq V} \sum_{\substack{Y \subseteq V \\ |Y| = |X|}} \operatorname{sign}(X, Y) \det A[X; Y] \det B[\overline{X}; \overline{Y}].$$

We prove a weaker version of the theorem, where $\operatorname{sign}(X, Y)$ is replaced with ± 1.

Proof. For $X \subseteq V$, define $C^X = (c_{ij})$ to be the $V \times V$ matrix where

$$c_{ij} = \begin{cases} a_{ij}, & \text{if } i \in X; \\ b_{ij}, & \text{if } i \in \overline{X}. \end{cases}$$

That is, $C^X[X; V] = A[X; V]$, and $C^X[\overline{X}; V] = B[\overline{X}; V]$. Repeated use of the linearity of the determinant function on the rows of $A + B$ gives

$$\det(A + B) = \sum_{X \subseteq V} \det C^X \tag{2.5}$$

For $X, Y \subseteq V$, define $D^{X,Y} = (d_{ij})$ to be the $V \times V$ matrix where

$$d_{ij} = \begin{cases} a_{ij}, & \text{if } i \in X \text{ and } j \in Y; \\ b_{ij}, & \text{if } i \in \overline{X} \text{ and } j \in \overline{Y}; \\ 0, & \text{otherwise.} \end{cases}$$

Then $D^{X,Y}[X;Y] = A[X;Y]$ and $D^{X,Y}[\overline{X};\overline{Y}] = B[\overline{X};\overline{Y}]$. All other entries of $D^{X,Y}$ are zero, and therefore $D^{X,Y}$ is singular whenever $|X| \neq |Y|$.

Repeated use of the linearity of the determinant function on the columns of C^X gives

$$\det C^X = \sum_{\substack{Y \subseteq V \\ |X|=|Y|}} \det D^{X,Y}. \tag{2.6}$$

When $|X| = |Y|$, then

$$\det D^{X,Y} = \pm \det A[X;Y] \det B[\overline{X};\overline{Y}]. \tag{2.7}$$

Combining equations 2.5 to 2.7, the theorem follows. □

2.5 Pfaffians

Let $\mathcal{X}$ be a finite set, and suppose the subsets $X_1, \ldots, X_k \subseteq \mathcal{X}$ are disjoint and nonempty. If $\mathcal{X}$ is the union of the sets X_i, then $\Pi = X_1, \ldots, X_k$ is a *partition* of $\mathcal{X}$. For $\mathcal{X} = \{1, \ldots, 2n\}$, let $\mathcal{P}(2n)$ be the set of partitions of $\mathcal{X}$ into pairs. For example,

$$\mathcal{P}(4) = \big\{\{(1,2),(3,4)\}, \quad \{(1,3),(2,4)\}, \quad \{(1,4),(2,3)\}\big\}.$$

For $\Pi = \{(i_1, j_1), \ldots, (i_n, j_n)\} \in \mathcal{P}(2n)$, define σ_Π to be the following permutation:

$$\sigma_\Pi = \begin{pmatrix} 1 & 2 & \cdots & 2n-1 & 2n \\ i_1 & j_1 & \cdots & i_n & j_n \end{pmatrix}$$

The sign of σ_Π is denoted $\text{sign}(\Pi)$. The sign function is invariant on the order of the pairs in Π: permutations $\left(\begin{smallmatrix} 1 & 2 & 3 & 4 \\ i_1 & j_1 & i_2 & j_2 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 & 2 & 3 & 4 \\ i_2 & j_2 & i_1 & j_1 \end{smallmatrix}\right)$ have the same sign. Changing the order within a pair, however, affects the sign function by a factor of -1: $\text{sign}\left(\begin{smallmatrix} 1 & 2 \\ i_1 & j_1 \end{smallmatrix}\right) = -\text{sign}\left(\begin{smallmatrix} 1 & 2 \\ j_1 & i_1 \end{smallmatrix}\right)$.

Let $A = (a_{ij})$ be a $2n \times 2n$ skew symmetric matrix, and let $\Pi \in \mathcal{P}(2n)$. Define

$$a_\Pi = \text{sign}(\Pi) a_{i_1 j_1} \cdots a_{i_n j_n}.$$

The -1 factor that happens when the order within a pair (i_k, j_k) of Π changes, is cancelled with the -1 that comes from $a_{j_k i_k} = -a_{i_k j_k}$, and therefore a_Π is well defined. The *Pfaffian* of A is defined as

$$\text{pf } A = \sum_{\Pi \in \mathcal{P}(2n)} a_\Pi$$

For example,

$$\text{pf} \begin{bmatrix} 0 & a_{12} & a_{13} & a_{14} \\ -a_{12} & 0 & a_{23} & a_{24} \\ -a_{13} & -a_{23} & 0 & a_{34} \\ -a_{14} & -a_{24} & -a_{34} & 0 \end{bmatrix} = a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23}.$$

If A is $m \times m$ and m is odd, then $\mathcal{P}(m)$ is empty and pf A is identically zero. The following two theorems relate the Pfaffian of A to the determinant of A, and give a row expansion formula for computing the Pfaffian. For their proof, see Godsil [16].

Theorem 2.9 (Cayley). *If A is a skew-symmetric matrix, then*

$$\det A = (\text{pf } A)^2.$$

Theorem 2.10. *If $A = (a_{ij})$ is a skew-symmetric matrix with rows and columns indexed by X, then*

$$\text{pf } A = \sum_{i=2}^{n} (-1)^{1+i} a_{1i} \text{pf } A[X \backslash (1 \cup i)].$$

Chapter 3

Matrix formulations

The *adjacency matrix* $A = (a_{ij})$, and the *incidence matrix* $B = (b_{ij})$, are two ways to represent the graph $G = (V, E)$ by a matrix. In the adjacency matrix, the rows and columns are indexed by the vertices of G, with $a_{ij} = 1$ if vertex i is adjacent to vertex j, and $a_{ij} = 0$ otherwise. The incidence matrix has rows indexed by V and columns by E, and is defined by $b_{ij} = 1$ if vertex i is incident with edge j, and $b_{ij} = 0$ otherwise. Graph theory problems can often be formulated in terms of an appropriate matrix representation of the graph: the ij^{th} entry of A^k is the number of walks of length k between vertex i and vertex j; a set of edges in G does not contain a circuit if and only if the corresponding columns of B are independent over $\mathbb{F}_2$.

Matrix representations for undirected graphs, directed graphs, and graphs with both directed and undirected edges are given in this chapter. We explain how these representations determine the existence of specific structures, namely *matchings*, *branchings*, and *matching forests*, in the associated graphs.

3.1 The Tutte matrix

A *matching* in a graph $G = (V, E)$ is a subset M of E such that every $v \in V$ is incident with at most one edge in M. If M is a matching and $v \in V$ is incident to an edge in M, then v is *M-covered.* A vertex which is not M-covered is *M-exposed.* A *maximum matching* in G covers a maximum number of vertices, and the number of vertices missed by a maximum matching is the *deficiency* of G, denoted $def(G)$. The number of vertices covered by a maximum matching is then $|V| - def(G)$. A matching that covers every vertex is *perfect.*

Let $G = (V, E)$ be a graph, and let $\{z_e : e \in E\}$ be algebraically independent indeterminates. Form the skew-symmetric *Tutte matrix* $T = (t_{ij})$, with rows and columns indexed by V, and

$$t_{ij} = \begin{cases} \pm z_{ij}, & \text{if } ij \in E; \\ 0, & \text{otherwise.} \end{cases}$$

Suppose G has an odd number of vertices. Then G can not have a perfect matching, and we know from Corollary 2.7 that the skew-symmetric matrix T is singular when n is odd. When G has an even number of vertices, consider the Pfaffian of T:

$$\text{pf } T = \sum_{\Pi \in \mathcal{P}(n)} b_\Pi.$$

Each $\Pi \in \mathcal{P}(n)$ partitions V into pairs, and $b_\Pi \neq 0$ if and only if there is a perfect matching in G which corresponds to Π. Furthermore, $b_{\Pi_1} = \pm b_{\Pi_2}$ only when $\Pi_1 = \Pi_2$, and hence there is no cancellation of nonzero terms in the Pfaffian. Therefore G has a perfect matching if and only if the Pfaffian of T is nonzero. Using Theorem 2.9 to equate the square of the Pfaffian with the determinant, the following theorem is immediate.

Theorem 3.1 (Tutte). *If $G = (V, E)$ is a graph with Tutte matrix T, then G has a perfect matching if and only if T is nonsingular.*

Suppose $A \subseteq E$. The subgraph of G whose vertex set is the ends of A and whose edge set is A is denoted $G[A]$. The subgraph $G[E \backslash A]$ can be denoted as $G \backslash A$, and when $A = \{e\}$, we write $G \backslash a$ for $G \backslash \{a\}$.

Similarly, if $X \subseteq V$, then $G[X]$ denotes the subgraph with vertex set X and edge set all $e \in E$ such that both ends of e are in X. The subgraph $G[V \backslash X]$ can be denoted as $G \backslash X$, and when $X = \{v\}$, we write $G \backslash v$ for $G \backslash \{v\}$.

Corollary 3.2. *If T is the Tutte matrix for the graph $G = (V, E)$, and $X \subseteq V$, then $G[X]$ has a perfect matching if and only if $T[X]$ is nonsingular.*

Proof. This follows from Theorem 3.1. □

Corollary 3.3. *If T is the Tutte matrix for the graph G, then*

$$\text{rank } T = |V| - def(G).$$

Proof. By Theorem 2.6, the rank of T is the maximum size of $X \subseteq V$ such that $T[X]$ is nonsingular. By Corollary 3.2, $T[X]$ is nonsingular if and only if $G[X]$ has a perfect matching. □

Let $G = (V, E)$ be a bipartite graph with vertex partition $V = (V_1, V_2)$. If T is the Tutte matrix for G, and $A = T[V_1; V_2]$, then

$$T = \begin{array}{c} \\ V_1 \\ V_2 \end{array} \!\!\begin{array}{c} \begin{array}{cc} V_1 & V_2 \end{array} \\ \begin{pmatrix} 0 & A \\ -A^{\top} & 0 \end{pmatrix} \end{array}$$

Therefore, in the bipartite case, the Tutte matrix T can be restricted to $T[V_1; V_2]$. We refer to $T[V_1; V_2]$ as the *bipartite Tutte matrix* for G.

Corollary 3.4. *If $G = (V, E)$ is a bipartite graph with bipartite Tutte matrix T, then the rank of T is the number of edges in a maximum matching of G.*

Proof. The rank of the bipartite Tutte matrix is half the rank of the corresponding Tutte matrix. By Corollary 3.3, this is the same as half the number of vertices covered by a maximum matching in G, which is equal to the number of edges in a maximum matching. □

3.2 The branching matrix

Let $G = (V, \vec{E})$ be a directed graph. The directed edges $\vec{E}$ in G are called *arcs*, and if an arc is directed from vertex v to u, then v is the *tail* of the arc, and u is the *head*. A *branching* in G is a subset $F \subseteq \vec{E}$ where every vertex in V is the head of at most one arc in F, and F contains no cycles. (See Figure 3.1.) A vertex which is not the head of any arc in F is called a *root* of F, and if $v \in V$ is not a root of F, then F *covers* v.

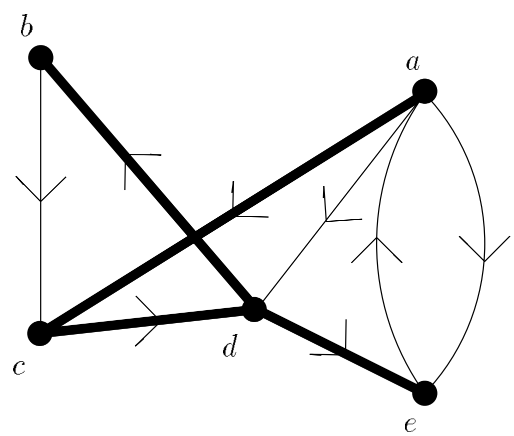

Figure 3.1: A branching with root a.

Let $\{x_{ij} : (i,j) \in \vec{E}\}$ be algebraically independent indeterminates. The *branching matrix* $B = (b_{ij})$ for G has rows and columns indexed by V, and

$$b_{ij} = \begin{cases} -x_{ij}, & \text{if } i \neq j \text{ and } (i,j) \in \vec{E}; \\ 0, & \text{if } i \neq j \text{ and } (i,j) \notin \vec{E}; \\ \sum_{k \neq i} b_{i,k} & \text{if } i = j. \end{cases}$$

The entries in any row of B sum to zero, and therefore the branching matrix is singular. A branching with exactly one root, such as the one indicated in Figure 3.1, is an *arborescence*. The matrix in 3.1 is the branching matrix for the graph in Figure 3.1. (Note that $x_{ae} \neq x_{ea}$.)

$$\begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array} \begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{pmatrix} x_{ea} & 0 & 0 & 0 & -x_{ea} \\ 0 & x_{db} & 0 & -x_{db} & 0 \\ -x_{ac} & -x_{bc} & (x_{ac} + x_{bc}) & 0 & 0 \\ -x_{ad} & 0 & 0 & x_{ad} & 0 \\ -x_{ae} & 0 & 0 & -x_{de} & (x_{de} + x_{ae}) \end{pmatrix} \end{array} \qquad (3.1)$$

Theorem 3.5 (Chaiken and Kleitman). *If B is the branching matrix for the directed graph $G = (V, \vec{E})$, and $v \in V$, then there is a one-to-one correspondence between arborescences in G, rooted at v, and terms in the determinant of $B[V \backslash v]$.*

Proof. Assume $V \backslash v = \{1, \ldots, n\}$ and for $i \in V \backslash v$, define $\vec{E}_i$ to be all $a \in \vec{E}$ such that i is the head of a. Let $\Pi = \{\vec{E}_i\}_{i \in V \backslash v}$, and let Π^* be the *transversals* of Π. That is, Π^* consists of all arc sets $a_1, a_2, \ldots, a_n$ such that $a_i \in \vec{E}_i$. For all $\mu \in \Pi^*$, define B_μ to be the branching matrix corresponding to $G_\mu = (V, \mu)$. Note that B_μ has one different indeterminate per row. Using the linearity of the determinant function, we can write

$$\det B[V\backslash v] = \sum_{\mu \in \Pi^*} \det B_\mu. \tag{3.2}$$

If $\mu \in \Pi^*$ is not a branching, then $G_\mu[U]$ has a circuit, for some $U \in V\backslash v$. It follows that B_μ decomposes into block diagonal $B_\mu[U]$ and $B_\mu[V\backslash(v \cup U)]$. The columns of $B_\mu[U]$ sum to zero, and therefore B_μ is singular.

Suppose $\mu \in \Pi^*$ is a branching. Up to simultaneous row and column permutations, B_μ is lower triangular. For example, if the order of the rows in B_μ corresponds to the order of a depth first search in the branching μ, then B_u will be lower triangular. Since each diagonal entry of B_μ is nonzero, it follows that B_μ is nonsingular. Since simultaneous row and column permutations do not affect the sign of the determinant and each diagonal term in B_μ is positive, the determinant of B_μ is positive.

Only transversals $\mu \in \Pi^*$ that are branchings contribute to (3.2), and the matrices corresponding to the $\mu \in \Pi^*$ that are branchings each have a unique set of indeterminates, and therefore there is no cancellation among the branching terms. □

Corollary 3.6. *The arc set A is an arborescence rooted at v if and only if $\prod_{ij \in A} x_{ji}$ is a term* $\det B[V\backslash v]$.

Corollary 3.7. *If B is the branching matrix for the directed graph $G = (V, \vec{E})$, and $v \in V$, then G has an arborescence rooted at v if and only if $B[V\backslash v]$ is nonsingular.*

Theorem 3.5 can be generalized.

Theorem 3.8 (Chaiken and Kleitman). *If B is the branching matrix for the directed graph $G = (V, \vec{E})$, and $U \subseteq V$, then there is a one-to-one correspondence between branchings of G that cover U and terms in $B[U]$. In particular, there is a branching that covers U if and only if $B[U]$ is nonsingular.*

Theorem 3.8 can be proved by the same method as Theorem 3.5.

A *maximum branching* in a directed graph is a branching with a maximum number of heads, or equivalently, a minimum number of roots. If B is the branching matrix for $G = (V, \vec{E})$, and $B[U]$ is nonsingular for some $U \subseteq V$, then by Theorem 3.8, there is a branching in G with $|U|$ heads. Since the rank of B is an upper bound on the size of a nonsingular principal submatrix of B, the rank of B is also an upper bound on the number of heads in a branching of G. This limit is attained; that is, G has a branching with exactly rank B heads. To prove this we use an analogy to Theorem 2.5. (If $U \subseteq V$ is a maximal set of independent rows in a skew symmetric matrix A, then $A[U]$ is nonsingular.) For the branching matrix, the assumption that the set of independent rows is maximal is not needed.

Lemma 3.9. *If B is the branching matrix for $G = (V, \vec{E})$ and the rows U are independent in B, then $B[U]$ is nonsingular.*

Proof. The proof is by induction on $|U|$, and is clearly true when U is empty. Let $U \subseteq V$ be k independent rows in B, and assume $B[U \backslash u]$ is nonsingular for all $u \in U$. Choose a minimal set X where $U \subseteq X \subseteq V$ and rank $B[U; X] =$ rank $B[U; V]$. If $X \neq U$, then choose $x \in X \backslash U$, and find $u \in U$ such that $b_{u,x} \neq 0$. (Such a u exists since $B[U; V]$ does not have any zero columns.) Then $b_{uu} \neq 0$, and b_{uu} has an indeterminate z that occurs exactly once in $B[U \backslash u]$. If $B[U](z \leftarrow 0)$ denotes the matrix $B[U]$ with 0 substituted for z, then by the linearity of the determinant function,

$$\det B[U] = \pm z \det B[U \backslash u] + \det B[U](z \leftarrow 0).$$

By the induction hypothesis, $B[U \backslash u]$ is nonsingular, and therefore $B[U]$ is nonsingular.

□

It can now be shown that the rank of a branching matrix is equal to the maximum number of heads in a branching of the associated graph.

Theorem 3.10 (Chaiken and Kleitman). *If $G = (V, \vec{E})$ is a directed graph with branching matrix B, then* rank B *is the maximum number of heads in a branching of G.*

Proof. From Theorem 3.8, the rank of B is an upper bound on the number of vertices covered by a branching. If $U \subseteq V$ is a maximal set of independent rows in B, then by Theorem 3.9 $B[U]$ is nonsingular, and by Theorem 3.8 there is a branching in G that covers U. □

3.3 The matching forest matrix

A *mixed graph* has both arcs and undirected edges. If $G = (V, E, \vec{E})$ is a mixed graph, then (V, E) is an undirected graph, and $(V, \vec{E})$ is a directed graph. The *underlying undirected graph* for G is the graph with all arcs in G replaced with undirected edges. In a mixed graph, undirected edges have two heads: if $e \in E$ is incident to vertices u and v, then both u and v are a head of e. The set $F \subseteq E \cup \vec{E}$ is a *matching forest* (Giles [14]) in G if

- each $v \in V$ is the head of at most one element of F, and
- F contains no circuits in the underlying undirected graph for G.

So if F is a matching forest for G, then $F \backslash \vec{E}$ is a matching in (V, E), $F \backslash E$ is a branching in $(V, \vec{E})$, and F is a forest in the underlying undirected graph for G. A *maximum matching forest* is a matching forest with a maximum number of heads. A vertex which is the head of an arc or edge in a matching forest is said to be *covered* by the matching forest, and a *perfect matching forest* covers all of V. Vertices not covered by a matching forest are *roots* of that matching forest and the deficiency of G is the number of roots of a maximum matching forest.

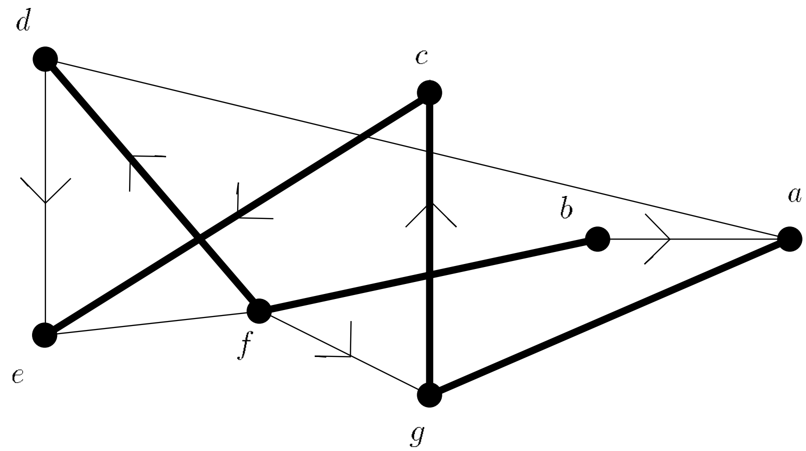

Figure 3.2: A perfect matching forest in a mixed graph

If T is the Tutte matrix for (V, E) and B is the branching matrix for $(V, \vec{E})$, then $T + B$ is a formulation for finding matching forests in $(V, E, \vec{E})$.

Theorem 3.11. *If $G = (V, E, \vec{E})$ is a mixed graph with Tutte matrix T for (V, E) and branching matrix B for $(V, \vec{E})$, then G has a perfect matching forest if and only if $T + B$ is nonsingular.*

Proof. Suppose $\det(T + B) \neq 0$. From Theorem 2.8 there exist $X, Y \subseteq V$ such that $T[X, Y]$ is nonsingular and $B[\overline{X}, \overline{Y}]$ is nonsingular. Let $Z \subseteq V$ be such that $X \cup Z$ is a maximal set of independent rows in T. By Theorem 2.5, $T[X \cup Z]$ is nonsingular, and by Theorem 3.1 there is a perfect matching M of $G[X \cup Z]$. The rows $\overline{X}$ are independent in B, and therefore so are $\overline{X \cup Z}$. By Lemma 3.9, $B[\overline{X \cup Z}]$ is nonsingular, and by Theorem 3.5 there is a branching F of $(V, \vec{E})$ with heads $\overline{X \cup Z}$. Letting $J = M \cup F$, J is a perfect matching forest of G.

To prove the converse direction, suppose J is a perfect matching forest of G. Then for some $X \subseteq V$, $M = J \cap E$ is a matching which covers X, and $F = J \cap \vec{E}$ is a branching which covers $\overline{X}$. By Theorems 3.1 and 3.5 respectively, both $T[X]$ and $B[\overline{X}]$ are nonsingular, and therefore $\det T[X] \det B[\overline{X}] \neq 0$. If $Y \subseteq V$ and $Y \neq X$, then any term in the determinant of $B[\overline{Y}]$ has a different set of indeterminates than any term in the determinant of $B[\overline{X}]$, and therefore $\det T[Y] \det B[\overline{Y}] \neq \det T[X] \det B[\overline{X}]$. By Theorem 2.8, $T + B$ is nonsingular. □

If all edges in G are undirected, then the branching matrix for G is zero, and Theorem 3.11 reduces to Theorem 3.1. As with the corresponding theorems for undirected graphs and directed graphs, Theorem 3.11 generalizes for a matching forest covering $U \subseteq V$. The proof is similar to Theorem 3.11.

Theorem 3.12. *If $G = (V, E, \vec{E})$ is a mixed graph with Tutte matrix T for (V, E) and branching matrix B for $(V, \vec{E})$, and $U \subseteq V$, then $B[U]$ is nonsingular if and only if $G[U]$ has a perfect matching forest.*

It was previously shown that the rank of the Tutte matrix is the number of vertices covered by a maximum matching in the associated undirected graph (Corollary 3.3), and that the rank of the branching matrix is the number of heads in a maximum branching in the associated directed graph (Theorem 3.10). Similarly, the rank of the matching forest matrix is the number of heads in a maximum matching forest in the associated mixed graph. We first prove that similar to the Tutte matrix and the branching matrix, the matching forest matrix has a maximal nonsingular submatrix which is principal.

Lemma 3.13. *Let $G = (V, E, \vec{E})$ be a mixed graph with Tutte matrix T and branching matrix B. If X is a maximal set of independent rows in $T + B$, then $(T + B)[X]$ is nonsingular, and therefore there is a nonsingular principal submatrix in $T + B$ with size equal to the rank of $T + B$.*

Proof. Let $X \subseteq V$ be a maximal set of independent rows of $T + B$. Then there exists $Y \subseteq V$ such that $(T+B)[X, Y]$ is nonsingular, and from Theorem 2.8 for the determinant of the sum of two matrices,

$$\sum_{R \subseteq X} \sum_{\substack{C \subseteq Y \\ |C|=|R|}} \pm \det T[R, C] \det B[X \backslash R, Y \backslash C] \neq 0.$$

Therefore there exists $R \subseteq X$ and $C \subseteq Y$ such that both $T[R, C]$ and $B[X \backslash R, Y \backslash C]$ are nonsingular. Choose a maximal set of rows Z such that $R \subseteq Z \subseteq V$ and $T[Z; V]$ is nonsingular, and by Lemma 3.9, $B[X \backslash (R \cup Z)]$ is nonsingular. Therefore, again by the formula in Theorem 2.8 for the sum of two matrices, $(T + B)[X \cup Z]$ is nonsingular, and since X is a maximal set of independent rows of $T + B$, $Z \subseteq X$. □

Theorem 3.14. *If $T + B$ is the matching forest matrix for the graph G, then the rank of $T + B$ is the number of heads in a maximum matching forest of G.*

Proof. From Theorem 3.12, the number of heads in a maximum matching forest in G is equal to the size of the largest nonsingular principal submatrix of $T + B$, and from Lemma 3.13, $T + B$ has a nonsingular principal submatrix with size equal to the rank of $T + B$. □

When G is an undirected graph, Theorem 3.14 reduces to Corollary 3.3. When G is directed, Theorem 3.14 reduces to Theorem 3.10.

Chapter 4

Optimal Evaluations

If the entries of an $n \times n$ matrix M are rational, then the determinant of M can be computed in $O(n^3)$ arithmetic steps. When M has indeterminate entries, the determinant of M has up to $n!$ terms, and therefore cannot be computed in polynomial time.

Let $\widetilde{M}$ be a matrix obtained by substituting rational values for the indeterminate entries in M. We call $\widetilde{M}$ an *evaluation* of M. If z is an indeterminate in M then $\tilde{z}$ denotes the value of z in $\widetilde{M}$, and $\widetilde{M}(z \leftarrow x)$ denotes the evaluation with x substituted everywhere for z.

Suppose the determinant of M is zero. Then for every value of $\tilde{z}$, the determinant of $\widetilde{M}$ is also zero, and therefore an evaluation of a singular matrix is also singular. However, if M is nonsingular, it is possible to choose a singular evaluation $\widetilde{M}$. For example, $M = \left(\begin{smallmatrix} z_1 & z_1 \\ z_3 & z_4 \end{smallmatrix}\right)$ is nonsingular, but any evaluation with $\tilde{z}_3 = \tilde{z}_4$ is singular. If $\widetilde{M}$ is an evaluation of M and rank $\widetilde{M}$ = rank M, then $\widetilde{M}$ is *optimal.* The methods for finding optimal evaluations given in this chapter are used in the algorithms of Chapter 5.

4.1 A Matrix Decomposition

A row or column in the matrix $M = (m_{ij})$ is *avoidable* if it can be removed from M without changing the rank of M. Equivalently, an avoidable row is a linear combination of other rows in M.

Suppose the rows and columns of M are indexed by X and Y respectively, where X and Y are disjoint. If $U \subseteq X$ and $V \subseteq Y$, then $M\backslash(U \cup V)$ denotes the matrix M with rows U removed, and columns V removed; that is, $M\backslash(U \cup V) = M[X\backslash U; Y\backslash V]$. When $U = \{u\}$, $M\backslash u$ is used instead of $M\backslash\{u\}$. (We note that in Chapter 5 where the results of this chapter are applied, the row and column indices are not disjoint. However, for convenience here we will use this notation, and the full form $M[X\backslash U; Y\backslash V]$ will be used only when necessary.) If $y \in U \cup V$ and y is not avoidable, then y is *unavoidable*, and rank $M\backslash y =$ rank $M - 1$. There are two possibilities with respect to the avoidable set of $M\backslash y$ compared to the avoidable set of M: a row or column that was avoidable before y was removed will still be avoidable after the removal of y, and hence the avoidable set does not decrease, but a row or column that was unavoidable in M may become avoidable in $M\backslash y$.

The following decomposition of a matrix M is from Geelen [12]:

$$D(M) = \{x \in X \cup Y : \text{rank } M\backslash x = \text{rank } M\}$$

$$A(M) = \{x \in X \cup Y : D(M\backslash x) = D(M)\},$$

$$C(M) = (X \cup Y)\backslash(D(M) \cup A(M)).$$

The avoidable rows of M are denoted by $D^R(M)$, and $D^C(M)$ denotes the avoidable columns. Similarly,

$$A^R(M) = A(M) \cap X, \qquad A^C(M) = A(M) \cap Y,$$
$$C^R(M) = C(M) \cap X, \qquad C^C(M) = C(M) \cap Y.$$

When the context is clear, D, C, and A are used for $D(M), C(M)$, and $A(M)$ respectively. The rank 3 matrix (4.1)

$$M = \begin{array}{c} \\ e \\ f \\ g \\ h \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 2 & 3 \\ 1 & 5 & 0 & -1 \end{pmatrix} \end{array} \tag{4.1}$$

has the following decomposition:

$$D^R = \{e, f\}, \qquad C^R = \{g\}, \qquad A^R = \{h\},$$
$$D^C = \{a, b\}, \qquad C^C = \{c, d\}, \qquad A^C = \emptyset.$$

When a row or column is removed from a matrix, the rank decreases by at most one.

Theorem 4.1. *If W is a set of rows and columns in the matrix M, then*

$$\text{rank } M \leq \text{rank } M \backslash W + |W|.$$

We will prove that Theorem 4.1 is met with equality when $W = A(M) \cup C^R(M)$.

Lemma 4.2. *If x is unavoidable in M, then $D(M) \subseteq D(M \backslash x)$. Specifically,* *(i)* *If* $x \in A(M)$*, then* $D(M \backslash x) = D(M)$*.*

(ii) *If* $x \in C^R(M)$*, then* $D^R(M) = D^R(M \backslash x)$*, and* $D^C(M) \subset D^C(M \backslash x)$*.*

(iii) *For each* $x \in C^R(M)$*, there exists* $z \in C^C(M)$ *such that* $z \in D^C(M \backslash x)$*, and* $x \in D^R(M \backslash z)$*.*

Proof.

(i) This is a restatement of the definition of A.

(ii) Let $x \in C^R(M)$. Since x is unavoidable, removing x does not decrease the avoidable set. Further, since x is not in A, the avoidable set must actually increase. Removing an unavoidable row does not affect avoidable rows, so the new avoidable element must be a column.

(iii) Let $x \in C^R(M)$, and let $z \in D^C(M \backslash x) \backslash D^C(M)$. (Such a z exists, by part (ii).) If $z \in A^C(M)$ then x would be unavoidable in $M \backslash z$, and rank $M \backslash \{x, z\} =$ rank $M - 2$. This is a contradiction, since $z \in D(M \backslash x)$ implies rank $M \backslash \{x, z\} =$ rank $M \backslash x =$ rank $M - 1$, and therefore $z \in C^C(M)$.

□

Parts (ii) and (iii) could equivalently have been expressed in terms of $x \in C^C(M)$. Part (i) describes the effect on the avoidable set when an element of A is removed from the matrix; the effect on the sets A and C is given by the following theorem.

Theorem 4.3 (Geelen). *If* $x \in A(M)$ *then*

$$D(M) = D(M \backslash x),$$

$$C(M) = C(M \backslash x), \text{ and}$$

$$A(M) \backslash x = A(M \backslash x).$$

Proof. Again, the avoidable set doesn't change by definition. Let $x \in A(M)$ and let $y \in C(M)$. By Lemma 4.2(iii), there exists $z \in C(M)$ such that $z \in D(M\backslash y)$, and hence $z \in D(M\backslash(y \cup x))$. By Lemma 4.2($i$), $z \notin D(M\backslash x)$ and therefore the avoidable set of $M\backslash x$ does not equal the avoidable set of $M\backslash\{x, y\}$. Using 4.2(ii), $y \in C(M\backslash x)$, and therefore, when $x \in A(M)$, $C(M) \subseteq C(M\backslash x)$.

Suppose there exists $u \in A(M)\backslash x$ such that $u \notin A(M\backslash x)$. By Lemma 4.2 ($i$), u is unavoidable in $M\backslash x$, and therefore $u \in C(M\backslash x)$, and

$$\text{rank } M\backslash\{x, u\} = \text{rank } M\backslash x - 1 = \text{rank } M - 2. \tag{4.2}$$

By Lemma 4.2 (iii), $u \in C(M\backslash x)$ implies there exists $v \in C(M\backslash x)$ such that v is in the avoidable set of $M\backslash\{x, y\}$. This gives

$$\text{rank } M\backslash\{x, v\} = \text{rank } M\backslash x - 1 = \text{rank } M - 2 \tag{4.3}$$

$$\text{rank } M\backslash\{x, v, u\} = \text{rank } M\backslash\{x, u\} = \text{rank } M - 2 \tag{4.4}$$

Further, $v \in C(M\backslash x)$ means $v \notin D(M)$ and since $D(M\backslash u) = D(M)$, it follows that $v \notin D(M\backslash u)$. Therefore

$$\text{rank } M\backslash\{u, v\} = \text{rank } M\backslash u - 1 = \text{rank } M - 2. \tag{4.5}$$

Two of x, u, v must be both columns or both rows, but all choices for pairs to be in the same row or in the same column lead to a contradiction. For example, suppose x and v are both rows. From equation (4.5), v is unavoidable in $M\backslash u$, and from equation (4.4), v is avoidable in $M\backslash\{u, x\}$. This contradicts Lemma 4.2, and therefore $A(M)\backslash x \subseteq A(M\backslash x)$.

By definition, $D(M) = D(M\backslash x)$ for all $x \in A(M)$. Therefore, if $C(M) \subseteq C(M\backslash x)$ and $A(M)\backslash x \subseteq A(M\backslash x)$, then $C(M) = C(M\backslash x)$ and $A(M)\backslash x = A(M\backslash x)$.

□

With Theorem 4.3, the decomposition of a matrix can be related to its rank.

Theorem 4.4 (Geelen). *If M is a matrix with decomposition D, C, A, then*

$$\text{rank } M = |A| + |C^R| + \text{rank } M[D^R; D^C \cup C^C].$$

Proof. From Lemma 4.3, each time an element from A is removed from M, the decomposition stays the same. Hence when all elements from A are removed, the rank decreases by the size of A:

$$\text{rank } M = |A| + \text{rank } M[D^R \cup C^R; D^C \cup C^C]. \tag{4.6}$$

The sets C and D for $M[D^R \cup C^R; D^C \cup C^C]$ are the same as the C and D for M, and the rank of $M[D^R \cup C^R; D^C \cup C^C]$ decreases by one each time an unavoidable row is removed. By Lemma 4.2(ii), removing a row from C does not affect row dependencies, and therefore

$$\text{rank } M[D^R \cup C^R; D^C \cup C^C] = |C^R| + \text{rank } M[D^R; D^C \cup C^C]. \tag{4.7}$$

Combining equations (4.6) and (4.7) gives Theorem 4.4. □

Corollary 4.5. *Every row and column of $M[D^R; D^C \cup C^C]$ is avoidable.*

Proof. Suppose $M[D^R; D^C \cup C^C]$ has an unavoidable column y. Since all columns from A have been removed, $y \in C(M[D^R; D^C \cup C^C])$. From Lemma 4.2, there must also be a row in $C(M[D^R; D^C \cup C^C])$. However, by Theorem 4.3, all the rows in $M[D^R; D^C \cup C^C]$ are avoidable, and therefore $M[D^R; D^C \cup C^C]$ does not have an unavoidable column. □

4.2 Series Classes

Two avoidable columns x and y of a matrix M are said to be *in series*, or *codependent*, if rank $M\backslash\{x,y\} = \text{rank } M - 1$. If M is skew symmetric with row and column index V, and x and y are avoidable, then x and y being in series is equivalent to rank $M[V\backslash\{x,y\}] = \text{rank } M - 2$. The following is standard.

Theorem 4.6. *Codependence is transitive.*

Proof. Let x, y and z be columns in a matrix M, and assume that x and y are codependent, and that x and z are codependent. Since y is avoidable,

$$\text{rank } M\backslash y = \text{rank } M, \tag{4.8}$$

and by the definition of codependent,

$$\text{rank } M\backslash\{x,y\} = \text{rank } M\backslash\{y,z\} = \text{rank } M - 1. \tag{4.9}$$

Using the submodularity of the rank function,

$$\text{rank } M\backslash\{x,y\} + \text{rank } M\backslash\{y,z\} \geq \text{rank } M\backslash\{x,y,z\} + \text{rank } M\backslash y. \tag{4.10}$$

Substituting (4.8) and (4.9) into (4.10) gives

$$\text{rank } M\backslash x - 2 \geq \text{rank } M\backslash\{x,y,z\}. \tag{4.11}$$

Since rank decreases by at most one each time a row or column is removed, the inequality of (4.11) is met with equality. Therefore the rank of $M\backslash\{x\}$ decreases by one when z is removed, and rank $M\backslash\{x,z\} = \text{rank } M - 1$. It follows that x and z are codependent. □

Using the transitivity of codependence, the avoidable columns of M can be partitioned into sets $\mathcal{D}_1, \ldots, \mathcal{D}_k$ for some integer k, such that two columns x and y are in the same set $\mathcal{D}_i$, if and only if rank $M \backslash \{x, y\} =$ rank $M - 1$. The sets $\mathcal{D}_1, \ldots, \mathcal{D}_k$ are called the *series classes* of M. Series classes in the context of matroids are discussed in Chapter 7. Series classes determine which columns become unavoidable once an avoidable column is removed. If $\mathcal{D}$ is a series class of M, the rank of $\mathcal{D}$ refers to the rank of the submatrix of M of the columns in $\mathcal{D}$.

Theorem 4.7. *If there are k series classes on the columns of M, and M has n columns, then* rank $M \geq n - k$.

Proof. When one column is removed from each series class, the remaining $n - k$ columns are unavoidable. □

Corollary 4.8. *If D is a series class on the columns of M, then the rank of D is at least $|D| - 1$.*

4.3 Improving Evaluations

One method for finding an optimal evaluation is to use a random evaluation, where the indeterminates are chosen from a large set. This is discussed in Section 6.4. The algorithms in Chapter 5 use a different approach: start with an arbitrary evaluation, and change the value of an indeterminate if doing so improves the evaluation. If an evaluation is not optimal, then a change which increases the rank is an improvement. Such a change is not always possible. For example, the bipartite graph G in Figure 4.1 has a perfect matching. (Edges af, be, ch, dg is one example.) From Corollary 3.4, the bipartite Tutte matrix for G is nonsingular, but the evaluation in (4.12) is singular, and there is no single change to an indeterminate evaluation which results in an optimal evaluation. A less

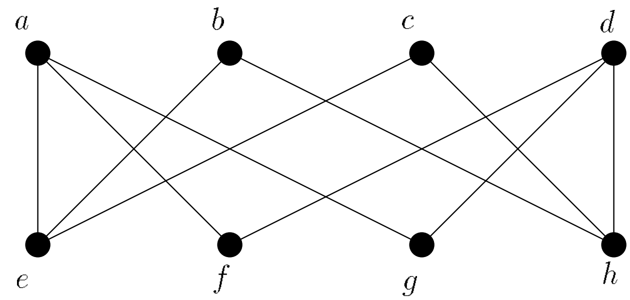

Figure 4.1: The bipartite graph G

restrictive definition for an improved evaluation than one with higher rank is needed.

$$T = \begin{pmatrix} z_{ea} & z_{eb} & z_{ec} & 0 \\ z_{fa} & 0 & 0 & z_{fd} \\ z_{ga} & 0 & 0 & z_{gd} \\ 0 & z_{hb} & z_{hc} & z_{hd} \end{pmatrix}, \quad \widetilde{T} = \begin{pmatrix} 1 & -1 & -1 & 0 \\ 2 & 0 & 0 & 1 \\ 2 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix} \tag{4.12}$$

Improvement in an evaluation is denoted by $\succeq$. If $\widetilde{M_1}$ and $\widetilde{M_2}$ are two evaluations of the matrix M, then $\widetilde{M_2} \succeq \widetilde{M_1}$ means either:

(i) rank $\widetilde{M_2} >$ rank $\widetilde{M_1}$; or

(ii) rank $\widetilde{M_2} =$ rank $\widetilde{M_1}$ and $D(\widetilde{M_2}) \supseteq D(\widetilde{M_1})$; or

(iii) rank $\widetilde{M_2} =$ rank $\widetilde{M_1}$, $D(\widetilde{M_2}) = D(\widetilde{M_1})$, and $C(\widetilde{M_2}) \supseteq C(\widetilde{M_1})$.

Strict improvement is denoted by $\succ$. Conditions (ii) and (iii) can alternatively be stated as:

(ii') rank $\widetilde{M_2} =$ rank $\widetilde{M_1}$, and for any row or column x, rank $\widetilde{M_2}\backslash x \geq$ rank $\widetilde{M_1}\backslash x$;

(iii') rank $\widetilde{M_2} =$ rank $\widetilde{M_1}$, and for any row x there exists a column y such that rank $\widetilde{M_2}\backslash\{x, y\} \geq$ rank $\widetilde{M_1}\backslash\{x, y\}$.

Let D, C, A be the decomposition of $\widetilde{M}$, where M is a matrix with indeterminate entries. If an indeterminate occurs exactly once in $M[D^R; D^C]$, then the rank of $\widetilde{M}$ can be increased with a single change to the value of the indeterminate.

Theorem 4.9 (Cunningham and Geelen). *If z occurs exactly once in $M[D^R; D^C]$ and m times in M, then for any $S \subset \mathbb{Z}$ with $|S| > m$, there exists $x \in S$ such that* $\operatorname{rank} \widetilde{M}(z \leftarrow x) > \operatorname{rank} \widetilde{M}$.

Proof. Assume that z occurs only once in $M[D^R; D^C]$, in row i and column j, and that z occurs m times in M. Let $a_1, \ldots, a_m$ be indeterminates, and let $\widetilde{M}_a$ be the evaluation $\widetilde{M}$ with the m entries of $\tilde{z}$ replaced with $\tilde{z} + a_1, \ldots, \tilde{z} + a_m$, such that $\tilde{z} + a_1$ is in $\widetilde{M}_a[i; j]$. Choose rows X and columns Y such that $i \notin X$, $j \notin Y$, and $\widetilde{M}[X; Y]$ is a maximal nonsingular submatrix of $\widetilde{M}$. Then

$$\det \widetilde{M}_a[X \cup i; Y \cup j] = \sum_{k=1}^{m} c_k a_k + p(a_1, \ldots, a_m) + c, \tag{4.13}$$

where $c_1, \ldots, c_m$ and c are constants, and each term in the polynomial $p(a_1, \ldots, a_m)$ has degree at least 2 and at most m.

If $\tilde{a}_k = 0$ for $k = 1, \ldots, m$, then $c = \det \widetilde{M}[X \cup i; Y \cup j]$, and since $\widetilde{M}[X \cup i; Y \cup j]$ is singular, $c = 0$. If $\tilde{a_k} = 0$ for $k = 2, \ldots, m$, then $c_1 = \det \widetilde{M}[X; Y]$, and therefore $c_1 \neq 0$.

Let $l \in \{2, \ldots, m\}$, and assume a_l is in row i' and column j' of $\widetilde{M}$. If $i' \notin X \cup i$ or $j' \notin Y \cup j$, then a_l is not in $\widetilde{M}_a[X \cup i; Y \cup j]$, and hence $c_l = 0$. Suppose $i' \in X \cup i$ and $j' \in Y \cup j$. If $\tilde{a_k} = 0$ for $k \in \{2, \ldots, m\} \backslash l$, then $c_l = \pm \det \widetilde{M}[(X \cup i) \backslash i'; (Y \cup j) \backslash j']$. From the assumption that z appears only once in $M[D^R; D^C]$, either i' or j' is unavoidable, hence $\widetilde{M}[(X \cup i) \backslash i'; (Y \cup j) \backslash j']$ is singular. Therefore

$$c_l = 0 \qquad \text{for all } l \in \{2, \ldots, m\}. \tag{4.14}$$

Combining $c = 0, c_1 \neq 0$ and (4.14), equation (4.13) simplifies to

$$\det \widetilde{M}_a[X \cup i; Y \cup j] = c_1 a_1 + p(a_1, \ldots, a_m). \tag{4.15}$$

Since the linear term $c_1 a_1$ is nonzero, it follows that $\widetilde{M}_a[X \cup i; Y \cup j]$ is nonsingular.

Let a be an indeterminate, and set $a_k = a - \tilde{z}$ for all $k \in \{1, \ldots, m\}$, so that $\widetilde{M}_a$ replaces each $\tilde{z}$ in $\widetilde{M}$ with a. The determinant of $\widetilde{M}(z \leftarrow a)[X \cup i; Y \cup j]$ is a non-zero polynomial of degree at most m in a, and therefore has at most m integer roots. If S is any set of more than m integers, then there exists $x \in S$ such that $\widetilde{M}(z \leftarrow x)[X \cup i; Y \cup j]$ is nonsingular, and rank $\widetilde{M}(z \leftarrow x) >$ rank $\widetilde{M}$. □

When z does not occur exactly once in the avoidable set, an improvement can still be guaranteed, but the required size of the set S may increase. Assume n is either the number of columns in M or the number of rows in M, whichever is greater.

Lemma 4.10. *If z occurs m times in M, and $|S| = 2mn + k$ where $S \subset \mathbb{Z}$, then there exist at least k integers $x \in S$ such that $\widetilde{M}(z \leftarrow x) \succeq \widetilde{M}$.*

Proof. Let D, C, A be the partition of $\widetilde{M}$, and assume $S \subset \mathbb{Z}$, with $|S| = 2mn + k$. Let $i \in D^R(\widetilde{M})$ and $j \in D^C(\widetilde{M})$, and let$\widetilde{M}[X; Y]$ be a maximal nonsingular submatrix of $\widetilde{M}$ with $i \notin X$ and $j \notin Y$. The determinant of $\widetilde{M}[X; Y]$ is a polynomial of degree at most m in $\tilde{z}$, and if x is not a root of the polynomial, then either rank $\widetilde{M}(z \leftarrow x) >$ rank $\widetilde{M}$, or rank $\widetilde{M}(z \leftarrow x) =$ rank $\widetilde{M}$ and i and j remain avoidable. There are at most n pairs i, j to consider, and hence at most mn possible x for which either rank $\widetilde{M}(z \leftarrow x) <$ rank $\widetilde{M}$ or rank $\widetilde{M}(z \leftarrow x) =$ rank $\widetilde{M}$ and $D(\widetilde{M}(z \leftarrow x)) \not\supseteq D(\widetilde{M})$. Therefore, there exists $S' \subseteq S$ such that $|S'| = mn + k$, and for all $x \in S'$, either rank $\widetilde{M}(z \leftarrow x) >$ rank $\widetilde{M}$, or rank $\widetilde{M}(z \leftarrow x) =$ rank $\widetilde{M}$ and $D(\widetilde{M}(z \leftarrow x)) \supseteq D(\widetilde{M})$.

If $x \in S'$ is such that $\widetilde{M}(z \leftarrow x) \not\succeq \widetilde{M}$, then $C(\widetilde{M})$ is nonempty. Let $i \in C^R(\widetilde{M})$. From

Lemma 4.2 (*iii*), there exists $j \in C^C(\widetilde{M})$ such that rank $\widetilde{M}\backslash\{i,j\}$ = rank $\widetilde{M} - 1$. Let $\widetilde{M}[X;Y]$ be a maximal nonsingular submatrix of $\widetilde{M}\backslash\{i,j\}$. The determinant of $\widetilde{M}[X;Y]$ is a polynomial of degree at most m in $\tilde{z}$, and therefore, at most m values of $\tilde{z}$ can make i or j unavoidable in $\widetilde{M}(z \leftarrow x)$. There are at most n sets of rows and columns to consider, so there are at most mn possible such x. Since $|S'| = mn + k$, this leaves at least k different $x \in S'$ such that $\widetilde{M}(z \leftarrow x) \succeq \widetilde{M}$. □

Lemma 4.11. *Suppose $\widetilde{M}_1$ and $\widetilde{M}_2$ are matrix evaluations, and D, C, A is the partition of $\widetilde{M}_1$. If $\widetilde{M}_2 \succeq \widetilde{M}_1$ and* rank $\widetilde{M}_2[D^R; D^C \cup C^C] >$ rank $\widetilde{M}_1[D^R; D^C \cup C^C]$, *then $\widetilde{M}_2 \succ \widetilde{M}_1$.*

Proof. Suppose rank $\widetilde{M}_2$ = rank $\widetilde{M}_1$. Since rank $\widetilde{M}_1 = |A| + |C^R| +$ rank $\widetilde{M}_1[D^R; D^C \cup C^C]$ and rank $\widetilde{M}_2[D^R; D^C \cup C^C] >$ rank $\widetilde{M}_1[D^R; D^C \cup C^C]$, the partition of M_2 is different than the partition of M_1. By assumption, $D(\widetilde{M}_2) \supseteq D$, and if $D(\widetilde{M}_2) = D$ then $C(\widetilde{M}_2) \supset C$. It follows that either $D(\widetilde{M}_2) \supset D$, or the avoidable sets of $\widetilde{M}_1$ and $\widetilde{M}_2$ are the same, and $C(\widetilde{M}_2) \supset C$. □

Statements similar to Theorem 4.9 can be made about improvements when an indeterminate occurs once or twice in $M[D^R; D^C \cup C^C]$.

Theorem 4.12 (Geelen).

(i) If z occurs exactly once in $M[D^R; D^C \cup C^C]$ and m times in M, then there exists $x \in \{1, \ldots, 2mn + 2\}$ such that $\widetilde{M}(z \leftarrow x) \succ \widetilde{M}$.

(ii) If z occurs exactly twice in $M[D^R; D^C \cup C^C]$, m times in M, and is in two different series classes with respect to the columns of $\widetilde{M}[D^R; D^C \cup C^C]$, then there exists $x \in \{1, \ldots, 2mn + 2\}$ such that $\widetilde{M}(z \leftarrow x) \succ \widetilde{M}$.

Proof.

(i) From Theorem 4.9, we can assume z occurs in $M[D^R; C^C]$. If $S = \{1, \ldots, 2mn+2\}$, then from Lemma 4.10, there are at least 2 different $x \in S$ such that $\widetilde{M}(z \leftarrow x) \succeq$

$\widetilde{M}$. From Corollary 4.5, every row and column of $\widetilde{M}[D^R; D^C \cup C^C]$ is avoidable, so applying Theorem 4.9 to $\widetilde{M}[D^R; D^C \cup C^C]$, it follows that in any set with at least 2 integers, there is an integer x in the set such that

$$\text{rank } \widetilde{M}[D^R; D^C \cup C^C](\tilde{z} \leftarrow x) > \text{rank } \widetilde{M}[D^R; D^C \cup C^C].$$

The theorem follows from Lemma 4.11.

(ii) Suppose $m_{ij} = z$ and $m_{hk} = z$, where columns j and k are in two different series classes of $\widetilde{M}[D^R; D^C \cup C^C]$. Then there exist $X' \subseteq D^R \backslash \{i\}$ and $Y' \subseteq (D^C \cup C^C) \backslash \{j, k\}$ such that $\widetilde{M}[X'; Y']$ is a maximum nonsingular submatrix of $\widetilde{M}[D^R; D^C \cup C^C]$. The determinant of $\widetilde{M}[X' \cup \{i\}; Y' \cup \{j\}]$ is linear in $\tilde{z}$, and therefore any change in $\tilde{z}$ will increase the rank of $\widetilde{M}[D^R; D^C \cup C^C]$. By Theorem 4.4, either the rank of $\widetilde{M}$ increases, or the decomposition changes. There are at most mn values for $\tilde{z}$ which make an avoidable row or column unavoidable, and at most mn values for $\tilde{z}$ which remove a row or column from C. If x is not one of these $2mn$ values and x is not the present value of z, then $\widetilde{M}(z \leftarrow x) \succ \widetilde{M}$.

□

Chapter 5

Rank completion algorithms

The formulations of chapter 3 involve calculating the rank of a matrix with indeterminate entries. From the discussion in Section 6.4, the formulations are not directly useful computationally. For example, when G is a bipartite graph with bipartite Tutte matrix T, there are as many terms in the determinant of T as there are perfect matchings in G, and therefore computing the determinant of T has the same order as counting all perfect matchings. Counting the number of perfect matchings in general graphs, even in the bipartite case, is $\mathcal{NP}$-hard (Lovász and Plummer [22], pg. 307).

In the maximum branching problem for a directed graph, the substitution of any positive rational for each indeterminate yields an optimal evaluation. This is because such a substitution ensures each term in the determinate is positive, and therefore one term in the permutation expansion cannot cancel another term. A similar approach is not always possible for undirected and mixed graphs; there are Tutte matrices and matching forest matrices for which every evaluation has both positive and negative terms in the determinant.(See Chapter 6 for examples of such matrices.) Algorithms that use the matrix decomposition to find an optimal evaluation of the bipartite Tutte matrix, the Tutte matrix, and the matching forest matrix are given here. Each algorithm follows the same general method: given an evaluation $\widetilde{M}$ of M and a set of integers S, if there exists an indeterminate z of M and $a \in S$ such that $\widetilde{M}(z \leftarrow a) \succ \widetilde{M}$, then replace $\widetilde{M}$ with $\widetilde{M}(z \leftarrow a)$.

5.1 An optimal evaluation of the bipartite Tutte matrix

An arbitrary evaluation of the bipartite Tutte matrix of a graph is either optimal, or can be improved by changing the value of a single indeterminate. A stronger version of the following theorem is in Geelen [13].

Theorem 5.1. *If T is the bipartite Tutte matrix for the bipartite graph $G = (V, E)$, and $\widetilde{T}$ is an evaluation of T, then either $\widetilde{T}$ is optimal, or there exists $e \in E$ and $a \in \{1, \ldots, 2n+2\}$ such that $\widetilde{T}(z_e \leftarrow a) \succ \widetilde{T}$.*

Proof. Let D, C, A be the partition of $\widetilde{T}$. If z is in $T[D^R; D^C]$, then since every indeterminate occurs only once in the bipartite Tutte matrix, it follows from Theorem 4.9 that rank $\widetilde{T}(\tilde{z} \leftarrow x) >$ rank $\widetilde{T}$ for any $x \neq \tilde{z}$.

Similarly, if z is in $T[D^R; C^C]$, then by Theorem 4.12(i), there exists an integer x in $\{1, \ldots, 2n+2\}$ such that $\widetilde{T}(\tilde{z} \leftarrow x) \succ \widetilde{T}$.

Both the rank and the size of the avoidable set are bounded, so improvements can be made until the partition D, C, A of the evaluation $\widetilde{T}$ is such that all entries in $T[D^R; D^C \cup C^C]$ are zero. For such an evaluation,

$$\text{rank } \widetilde{T}[D^R; D^C \cup C^C] = \text{rank } T[D^R; D^C \cup C^C] = 0. \tag{5.1}$$

With (5.1) and Theorem 4.4, it follows that rank $\widetilde{T} = |A| + |C^R|$. Theorem 4.1 states that rank $T \leq |A| + |C^R|$, and since the rank of T is an upper bound on the rank of any evaluation of T, $\widetilde{T}$ is optimal. $\square$

Geelen proves that when an evaluation $\widetilde{T}$ is not optimal, there exist $e \in E$ and $a \in \{1, \ldots, n\}$ such that either rank $\widetilde{T}(z_e \leftarrow a) >$ rank $\widetilde{T}$ or $D(\widetilde{T}(z_e \leftarrow a)) \supset D(\widetilde{T})$.

5.2 Matchings

Suppose $G = (V, E)$ does not have a perfect matching, but for any $v \in V$, the subgraph $G[V \backslash v]$ has a perfect matching. Such a graph is called *hypomatchable.*

5.2.1 Hypomatchable graphs and odd components

Suppose the Tutte matrix T for $G = (V, E)$ is such that every row and column of T is avoidable, and rank $T = |V| - 1$. Then $T[V \backslash v]$ is nonsingular for any $v \in V$, and by Corollary 3.2, the subgraph $G[V \backslash v]$ has a perfect matching for every $v \in V$. Thus the graph G is hypomatchable. Clearly a hypomatchable graph must have an odd number of vertices. Two examples of hypomatchable graphs are given in Figure 5.1. The Gallai-Edmonds decomposition of a graph, discussed next, finds hypomatchable components in a graph.

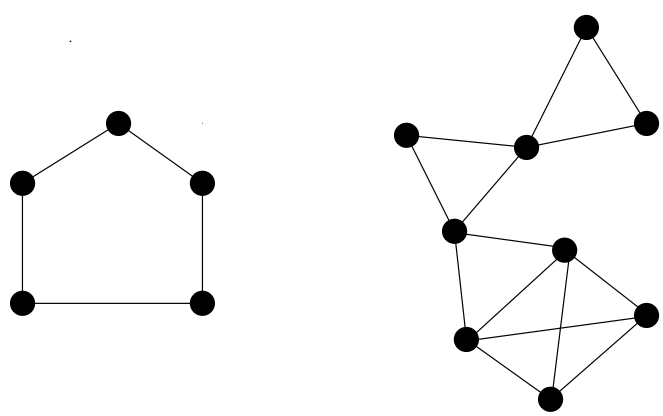

Figure 5.1: Hypomatchable graphs

Assume $G = (V, E)$ is a graph and let $A \subset V$. An *odd component* of $G[V \backslash A]$ is a maximal connected subgraph of $G[V\backslash A]$ with an odd number of vertices. The number of odd components in $G[V\backslash A]$ is denoted by $odd(G\backslash A)$.

If M is a matching of G, then unless M contains an edge uw where $u \in A$ and $w \notin A$, then M cannot cover all of A. Consider the case that A is the single vertex $v \in V$. Suppose $G[V\backslash\{v\}]$ has two odd components. The vertex v cannot be used to cover both odd components simultaneously, and therefore G does not have a perfect matching. Similarly, if $A \subset V$ is such that the number of odd components of $G[V\backslash A]$ is greater than the size of A, then G can not be perfectly matched. Tutte's matching theorem relates the existence of a perfect matching to the nonexistence of an odd component that can not be perfectly matched.

Theorem 5.2 (Tutte). *The graph $G = (V, E)$ has a perfect matching if and only if $odd(G\backslash A) \leq |A|$ for all $A \subseteq V$.*

Closely related to this is the *Tutte-Berge Formula*, which relates the deficiency of a graph (number of vertices not covered in a maximum matching) to the maximum difference between the number of odd components in $G\backslash A$ and the size of A. The deficiency of G is denoted by $def(G)$.

Theorem 5.3 (Tutte-Berge Formula). *For any graph $G = (V, E)$,*

$$def(G) = \max\{odd(G\backslash A) - |A| : A \subseteq V\}.$$

5.2.2 The Gallai-Edmonds decomposition

Let $G = (V, E)$ be a graph, and let D be the vertices not covered by at least one matching in G. Let $A \subseteq V\backslash D$ be the vertices incident to a vertex in D, and let $C = V\backslash(D \cup A)$. Note that D, A and C are well defined.

Theorem 5.4 (Gallai-Edmonds Structure Theorem). *If G is a graph with D, A and C defined as above, then the following hold:*

(i) every odd component of $G \backslash A$ is in $G[D]$, and every component of $G[D]$ is hypo-matchable;

(ii) $G[C]$ has a perfect matching;

(iii) if M is a maximum matching of G, then M perfectly matches $G[C]$, and every vertex in A is matched in M to a distinct component in $G[D]$;

(iv) the number of vertices covered by a maximum matching is $|V| - (odd(G \backslash A) - |A|)$.

It can be shown that (iv) implies (i), (ii), and (iii). For a proof of the Gallai-Edmonds Structure Theorem, see Lovász and Plummer [22] pg. 93-98.

For an example of the Gallai-Edmonds structure for a graph, see Figure 5.2.

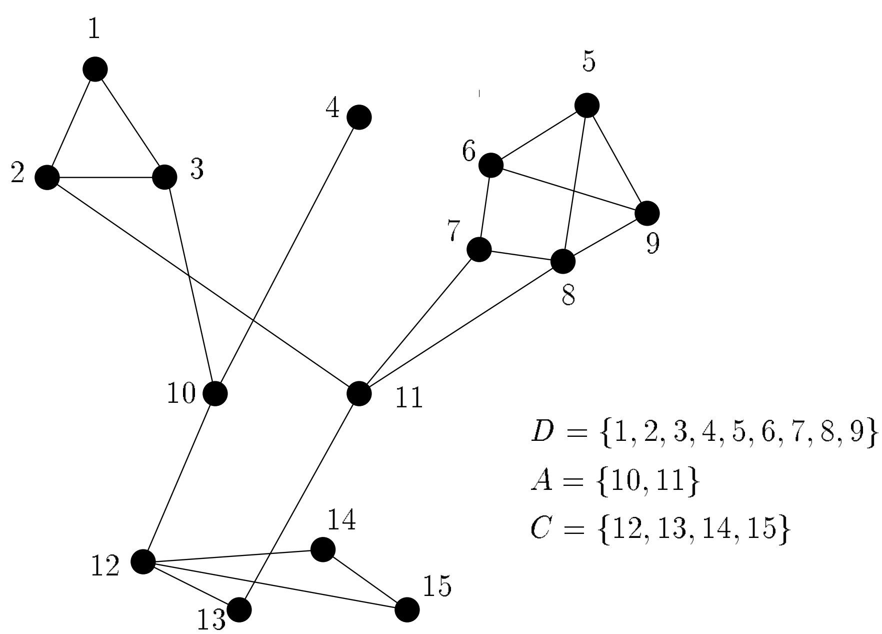

Figure 5.2: Example of a Gallai-Edmonds decomposition

The polynomial time Edmonds' Matching Algorithm for the undirected graph $G = (V, E)$ takes as input any matching M of G, and either finds a new matching M' where $|M'| > |M|$, or finds $A \subseteq V$ such that $odd(G \backslash A) - |A| > 0$, and $odd(G \backslash A) - |A|$ is the number of vertices not covered by M. From the Tutte-Berge Formula, M is a maximum matching.

The partition D, C, A of an optimal evaluation of T is the same as the sets D, C, and A from the Gallai-Edmonds decomposition of a graph: just as there are no non-zero entries in $T[D^R; C^C]$, there are no edges between vertices in D and C; the hypomatchable components in $G[D]$ correspond to the blocks from the series classes of $T[D]$; the submatrix $T[C]$ has full rank and the subgraph $G[C]$ is perfectly matchable; and finally, the size of the maximum matching, $|V| - (odd(G \backslash A) - |A|$ can be rewritten as $|A| + |C| + |D| - (odd(G[D]) - |A|)$, and this is equal to $2|A| + |C| + \text{rank } T[D^R; D^C]$.

5.2.3 An optimal evaluation of the Tutte matrix

Similar to an evaluation of the bipartite Tutte matrix, an evaluation of the Tutte matrix of a graph is either optimal, or can be improved by changing the value of a single indeterminate. A stronger version of the following theorem is in Geelen [12].

Theorem 5.5. *If T is the Tutte matrix for the graph $G = (V, E)$, and $\widetilde{T}$ is an evaluation of T, then either $\widetilde{T}$ is optimal, or there exists $e \in E$ and $a \in \{1, \ldots, 4n + 2\}$ such that $\widetilde{T}(z_e \leftarrow a) \succ \widetilde{T}$.*

Proof. Assume $\widetilde{T}$ is an evaluation with partition D, C, A. Since the Tutte matrix is symmetric, the partition D, C, A of $\widetilde{T}$ is symmetric, and in particular, $D^R = D^C$. Any indeterminate in $T[D^R; C^C]$ occurs exactly once in $T[D^R; C^C]$, and therefore, from Theorem 4.12(i), improvements in the evaluation can be made until all entries of $T[D^R; C^C]$ are zero.

Consider the series classes on the columns of $\widetilde{T}[D]$. By definition, the columns x and y belong to different series classes if and only if rank $\widetilde{T}\backslash\{x,y\}$ = rank $\widetilde{T}$. Suppose that the indeterminate z is in column i and j, and that i and j are in different series classes. By symmetry, z is also in row j, and row j can be removed without affecting column dependencies. Therefore rank $\widetilde{T}[V\backslash j; V\backslash\{i,j\}]$ = rank $\widetilde{T}$, and there exist $X \subseteq V\backslash j$ and $Y \subseteq V\backslash\{i,j\}$ such that $\widetilde{T}[X;Y]$ is a maximal nonsingular submatrix of $\widetilde{T}$. The indeterminate z is not in $T[X;Y]$, and although $\widetilde{T}[X\cup j; Y\cup i]$ is singular, its determinant is linear in $\tilde{z}$. Hence $\widetilde{T}[X\cup j; Y\cup i](\tilde{z} \leftarrow x)$ is nonsingular for any $x \neq \tilde{z}$, and therefore rank $\widetilde{T}(\tilde{z} \leftarrow x) >$ rank $\widetilde{T}$.

Suppose an evaluation $\widetilde{T}$ with partition D, C, A is such that every entry of $T[D^R; C^C]$ is zero, and each indeterminate pair z in $T[D]$ occurs in the same series class on the columns of $\widetilde{T}[D]$. By Theorem 4.4, it follows that rank $\widetilde{T}$ = rank $\widetilde{T}[D] + |A| + |C^R|$. From Theorem 4.1, rank $T \leq$ rank $T[D] + |A| + |C^R|$.

Let $\mathcal{D}_1, \ldots, \mathcal{D}_n$ be the series classes of $\widetilde{T}[D]$, and consider the submatrix $\widetilde{T}[D;\mathcal{D}_k]$ for an arbitrary series class $\mathcal{D}_k \subseteq D$. Suppose z occurs in row i and column j of $T[D]$, where $j \in \mathcal{D}_k$. By the skew-symmetry of the Tutte matrix, z is also in row j and column i. By the assumption that z occurs in two columns of $\mathcal{D}_k$, $i \in \mathcal{D}_k$, and every entry of $\widetilde{T}[D\backslash\mathcal{D}_k;\mathcal{D}_k]$ is zero.

The submatrix $\widetilde{T}[D]$ therefore consists of diagonal blocks $\widetilde{T}[\mathcal{D}_1], \ldots, \widetilde{T}[\mathcal{D}_n]$, and

$$\text{rank } \widetilde{T}[D] = \sum_{k=1}^{n} \text{rank } \widetilde{T}[\mathcal{D}_k].$$

Since all rows and columns of $\widetilde{T}[D]$ are avoidable, all rows and columns of each blocks $\widetilde{T}[\mathcal{D}_k]$ must be avoidable, for $k \in \{1, \ldots, n\}$. This same block structure must occur in

$T[D]$, hence

$$\operatorname{rank} T[D] = \sum_{k=1}^{n} \operatorname{rank} T[\mathcal{D}_k].$$

By Corollary 4.8, rank $\widetilde{T}[\mathcal{D}_k] \geq \mathcal{D}_k| - 1$ for all $k \in \{1, \ldots, n\}$, and since each row and column is avoidable, rank $\widetilde{T}[\mathcal{D}_k] = |\mathcal{D}_k| - 1$.

Each block $\widetilde{T}[\mathcal{D}_k]$ is skew symmetric, so having rank one less than its size, means it is an optimal evaluation of $T[\mathcal{D}_i]$. Hence rank $\widetilde{T}[D]$ = rank $T[D]$, and $\widetilde{T}$ is an optimal evaluation of T. □

Geelen proves that if an evaluation $\widetilde{T}$ is not optimal, then there exist $e \in E$ and $a \in \{1, \ldots, n\}$ such that either rank $\widetilde{T}(z_e \leftarrow a) >$ rank $\widetilde{T}$, or $D(\widetilde{T}(z_e \leftarrow a)) \supset D(\widetilde{T})$.

5.3 Branchings

Any evaluation which substitutes positive integers for the indeterminates in the branching matrix for a directed graph is optimal. This follows from Section 3.2.

Theorem 5.6 (Barahona and Pulleybank). *If B is the branching matrix for the directed graph $G = (V, \vec{E})$, and $\widetilde{B}$ is an evaluation of B with $\tilde{z} > 0$ for all indeterminates z in B, then $\widetilde{B}$ is optimal.*

If all the indeterminates in the branching matrix are evaluated as $+1$, then the determinant counts arborescences in G.

Theorem 5.7 (Barahona and Pulleybank). *If B is the branching matrix for the directed graph $G = (V, \vec{E})$, and $\widetilde{B}$ is an evaluation of B with $\tilde{x} = 1$ for all indeterminates x in B, then* $\det \widetilde{B}[V \backslash v]$ *is the number of arborescences in G with root v.*

Proof. Let $\mathcal{A}$ be the arc sets corresponding to arborescences of G with root v. From Corollary 3.6,

$$\det B = \sum_{A \in \mathcal{A}} \prod_{ij \in A} x_{ji}.$$

Hence if $\tilde{x}_a = 1$ for all $a \in \vec{E}$, then $\det \widetilde{B} = |\mathcal{A}|$. □

We note that determining if a directed graph has an arborescence is easy.

Theorem 5.8. *The directed graph $G = (V, \vec{E})$ with $r \in V$ has an arborescence rooted at r if and only if there is a directed path in G from r to v for all $v \in V$.*

In a *strongly connected* directed graph $G = (V, \vec{E})$, there is a directed path from v to u for each vertex pair $v, u \in V$.

Corollary 5.9. *A directed graph $G = (V, E)$ has an arborescence rooted at every $v \in V$ if and only if G is strongly connected.*

5.4 Matching forests

Hypomatchable undirected graphs (Section 5.2.1) have the property that for any vertex v in the graph, there is a matching that covers every vertex except v. We will call a mixed graph $G = (V, E, \vec{E})$ hypomatchable if G does not have a perfect matching forest, but for every $v \in V$, there is a matching forest in G that covers $V \backslash v$.

A vertex $v \in V$ is a *cut vertex* in a graph $G = (V, E)$ if the edge set can be partitioned into nonempty subsets E_1 and E_2 such that $G[E_1]$ and $G[E_2]$ have only the vertex v in common. A *block* in a graph $G = (V, E)$ is a maximal subgraph $G' = (V', E')$ such G' does not have a cut-vertex.

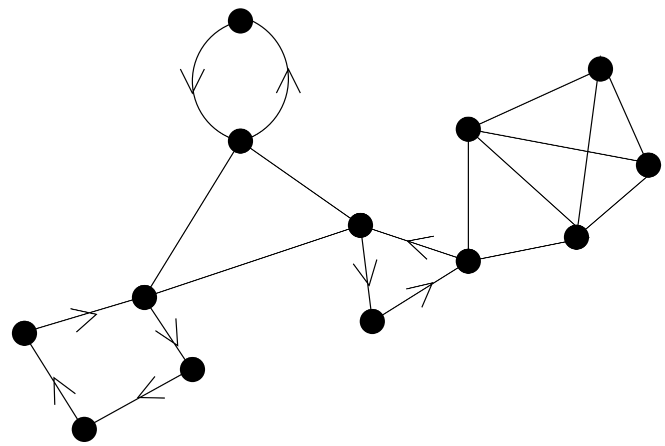

Figure 5.3: A hypomatchable tree

Lemma 5.10 (Giles). *If each block in a mixed graph G is either a directed graph or an undirected graph with an odd number of vertices, then G does not have a perfect matching forest. Moreover, if G is hypomatchable, then each undirected block is hypomatchable, and each directed block is strongly connected.*

Lemma 5.10 can be proved by induction on the number of blocks.

If each block in a mixed graph G is either a hypomatchable undirected subgraph of G, or a strongly connected directed subgraph, then we call G a *hypomatchable tree.* (See Figure 5.3.) The algorithm presented in the following section finds an optimal evaluation of a matching forest matrix, and, in the process, finds hypomatchable trees in the corresponding mixed graph.

5.4.1 An optimal evaluation of the matching forest matrix

Similar to the algorithms for an optimal evaluation of the Tutte matrix and the bipartite Tutte matrix, an optimal evaluation of the matching forest matrix is found by starting with an arbitrary evaluation, and improving the evaluation by changing the value of an indeterminate.

Theorem 5.11. *If $T+B$ is the matching forest matrix for the mixed graph $G = (V, E, \vec{E})$, and $\widetilde{T} + \widetilde{B}$ is an evaluation of $T + B$, then either $\widetilde{T} + \widetilde{B}$ is optimal, or there exist $e \in E$, $\vec{e} \in \vec{E}$, and $a, b \in \{1, \ldots, 4|E| + 2\}$ such that $(\widetilde{T} + \widetilde{B})(z_e \leftarrow a, x_{\vec{e}} \leftarrow b) \succ \widetilde{T} + \widetilde{B}$.*

Let $\widetilde{T} + \widetilde{B}$ be an evaluation of $T + B$, and let D, C, A be the partition of $\widetilde{T} + \widetilde{B}$. From Theorems 4.9 and 4.12(i), a single perturbation can improve the evaluation if any of the following three initial conditions are not satisfied by $\widetilde{T} + \widetilde{B}$:

1. The indeterminates in $(T + B)[D^R; D^C]$ appear in pairs.
2. The indeterminates in $(T + B)[D^R; C^C]$ appear in pairs.
3. For any series class X on the columns of $(\widetilde{T} + \widetilde{B})[D^R; D^C \cup C^C]$, the indeterminates in $(\widetilde{T} + \widetilde{B})[D^R; X]$ appear in pairs.

Assume the evaluation $\widetilde{T} + \widetilde{B}$ satisfies these three initial conditions, and let $X \subseteq D^C \cup C^C$ be a series class on the columns of $(\widetilde{T} + \widetilde{B})[D^R; D^C \cup C^C]$. Consider the submatrix $(\widetilde{T} + \widetilde{B})[D^R; X]$. If there is a nonzero entry in row i of $(\widetilde{T} + \widetilde{B})[D^R; X]$, then by condition (3) and the structure of the matching forest matrix, column $i \in X$. Therefore all entries of $(\widetilde{T} + \widetilde{B})[D^R \backslash X; X]$ are zero and $(\widetilde{T} + \widetilde{B})[D^R; D^C \cup C^C]$ decomposes into blocks corresponding to each of the series classes on its columns. Determining if $\widetilde{T} + \widetilde{B}$ is optimal is equivalent to determining if $(\widetilde{T} + \widetilde{B})[X]$ is optimal for each series class X.

Let $X \subseteq D^C \cup C^C$ be a series class. Define $M = (m_{ij})$ and $\widetilde{M} = (\tilde{m}_{ij})$ to be $(T+B)[X]$ and $(\widetilde{T} + \widetilde{B})[X]$ respectively.

Lemma 5.12. *Every row and column of $\widetilde{M}$ is avoidable, and* rank $\widetilde{M} = |X| - 1$.

Proof. From the previous observation that every entry in $(\widetilde{T} + \widetilde{B})[D^R \backslash X; X]$ is zero, rank $\widetilde{M}$ = rank $(\widetilde{T} + \widetilde{B})[D^R \backslash X; X] \geq |X| - 1$ (Corollary 4.8). Furthermore, since all rows and columns of $(\widetilde{T} + \widetilde{B})[D^R \backslash X; X]$ are avoidable, every row and column of $\widetilde{M}$ is also avoidable and the rank must be strictly less than the number of columns. □

Lemma 5.13. *If y is a vector in the nullspace of $\widetilde{M}$, and no change in the evaluation of an indeterminate will make $\widetilde{M}$ nonsingular, then $y_i = y_j$ for all arcs ij in $G[X]$.*

Proof. Suppose the vector y is in the nullspace of $\widetilde{M}$ and let ij be an arc in $G[X]$. Then

$$\widetilde{M}[X; X\backslash i]y\backslash y_i = -y_i\widetilde{M}[X; i],$$

and in particular,

$$\widetilde{M}[X\backslash i]y\backslash y_i = -y_i\widetilde{M}[X\backslash i; i]. \tag{5.2}$$

From Lemma 5.12, $\widetilde{M}[X\backslash i]$ is nonsingular, and since $\widetilde{M}(z \leftarrow a)$ does not have full rank for any $a \in \mathbb{Z}$,

$$\det \widetilde{M}[X\backslash i] = (-1)^{i+j} \det \widetilde{M}[X\backslash i; X\backslash j]. \tag{5.3}$$

The lemma follows from using Cramer's rule together with (5.3) to solve the system in (5.2) for y_j. □

The next step is to combine some columns of $\widetilde{M}$ and form a new matrix, denoted $\widehat{M}$. The columns are combined in such a way that there is a bijection between vectors in the nullspace of $\widetilde{M}$ and vectors in the nullspace $\widehat{M}$. For every $U \subseteq X$ such that the directed component of $G[U]$ is a maximal connected subgraph of $G[X]$, combine the columns U of $\widetilde{M}$ into one column in $\widehat{M}$, where the new column is equal to the sum of the individual columns. An example of this operation is Figure 5.4, where x, y, z are indeterminates from the Tutte matrix, and a, b, c, d, e are indeterminates from the branching matrix. All the entries of $\widehat{M}$ come from the Tutte matrix, since the entries in a connected component of the branching matrix sum to zero. Let $\widehat{X}$ index the columns of $\widehat{M}$.

$$M = \begin{pmatrix} a & -a & 0 & 0 & 0 & 0 \\ -b & b & -x & -y & 0 & 0 \\ 0 & x & 0 & -z & 0 & 0 \\ 0 & y & z & c & 0 & -c \\ 0 & 0 & 0 & -d & d & 0 \\ 0 & 0 & 0 & 0 & -e & e \end{pmatrix} \quad \widetilde{M} = \begin{pmatrix} 1 & -1 & 2 & 0 & 0 & 0 \\ -1 & 1 & -5 & 3 & 0 & 0 \\ 0 & 5 & 0 & -7 & 0 & 0 \\ 0 & -3 & 7 & 6 & 0 & -6 \\ 0 & 0 & 0 & -4 & 4 & 0 \\ 0 & 0 & 0 & 0 & 3 & -3 \end{pmatrix} \quad \widehat{M} = \begin{pmatrix} 0 & 2 & 0 \\ 0 & -5 & 3 \\ 5 & 0 & -7 \\ -3 & 7 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Figure 5.4: Constructing $\widehat{M}$

Lemma 5.14. *Every row and column of $\widehat{M}$ is avoidable, and* rank $\widehat{M} = |\widehat{X}| - 1$.

Proof. The rows of $\widetilde{M}$ are avoidable, and since combining columns in a linear way does not affect row dependency, the rows of $\widehat{M}$ are also avoidable.

Suppose y is in the nullspace of $\widetilde{M}$, and assume column x in $\widehat{M}$ is the sum of columns $i_1, \ldots, i_j$ in $\widetilde{M}$. From Theorem 5.13, $y_{i_1} = y_{i_2} = \cdots = y_{i_j}$. If $\hat{y}$ is the vector formed by replacing the entries $y_{i_1}, \ldots, y_{i_j}$ from y with the single entry y_{i_1}, for all entries whose corresponding columns in $\widetilde{M}$ have been combined into one column in $\widehat{M}$, then $\widehat{M}\hat{y} = 0$. Similarly, a vector $\hat{y}$ in the nullspace of $\widehat{M}$ can be expanded into a vector in the nullspace of $\widetilde{M}$, and therefore there is a bijection between the nullspace of $\widetilde{M}$ and the nullspace of $\widehat{M}$. Since the nullspace of $\widetilde{M}$ has dimension 1, so does the nullspace of $\widehat{M}$, and therefore the rank of $\widehat{M}$ is one less than the number of columns. □

If a single change to the entries of $\widehat{M}$ can increase the rank of $\widehat{M}$, then either the rank of $\widetilde{M}$ also increases, or the bijection between the two nullspaces no longer holds. If the bijection does not hold, then the property described in Lemma 5.13 no longer holds, and there exist i, j such that ij is an arc in $G[X]$, and $\det \widetilde{M}[X\backslash i] \neq (-1)^{i+j} \det \widetilde{M}[X\backslash i; X\backslash j]$. A single change to the value of the entry corresponding to the arc ij will make $\widetilde{M}$ have full rank. Next we consider series classes on the rows of $\widehat{M}$.

Lemma 5.15. *If the rank of $\widehat{M}$ cannot be increased by a single change to any entry, and Y is a nontrivial series class on the rows of $\widehat{M}$, then $\widehat{M}[Y; \widehat{X}]$ has $|Y|$ nonzero columns.*

Proof. From Corollary 4.8, rank $\widehat{M}[Y;\widehat{X}] \geq |Y|-1$, and since every column in $\widetilde{M}[Y;\widehat{X}]$ is avoidable, there must be at least $|Y|$ nonzero columns. From Theorem 4.9(ii), both entries of an indeterminate pair occur in the same series class, and at most once in a single row of the Tutte matrix. Therefore the number of nonzero columns cannot be more than the number of nonzero rows, and hence $\widehat{M}[Y;\widehat{X}]$ has exactly $|Y|$ nonzero columns. □

Lemma 5.15 implies some structure of $G[X]$.

Corollary 5.16. *If the rank of $\widehat{M}$ can not be increased by a single change in any entry, and Y is a series class on the columns of $\widehat{M}$, then $G[Y]$ has only undirected edges. Moreover, if $\overline{G}[X]$ denotes the undirected components of $G[X]$, then $G[Y]$ is a maximal hypomatchable component of $\overline{G}[X]$.*

Proof. If $i,j \in Y$ are such that $ij \in \vec{E}$, then by the construction of $\widehat{M}$, columns i and j would be added together. This contradicts Lemma 5.15, and therefore there are no arcs in $G[Y]$.

Since $\widehat{M}[Y;\widehat{X}]$ has exactly $|Y|$ nonzero columns, there is a Tutte matrix $\widetilde{T}[Y]$ such each nonzero column in $\widehat{M}[Y;\widehat{X}]$ corresponds to a column in $\widetilde{T}[Y]$. Since $\widehat{M}[Y\backslash y;\widehat{X}\backslash x]$ has full row rank for any $y \in Y, x \in \widehat{X}$, $\widetilde{T}[Y\backslash y]$ is nonsingular. Therefore, there is a perfect matching in $G[Y]$ which covers $Y\backslash y$ for any $y \in Y$, and $G[Y]$ is hypomatchable.

Any indeterminate in $M[Y;\widehat{X}]$ occurs twice in $M[Y;\widehat{X}]$, and therefore if $ij \in E$ and $i \in Y$, then $j \in Y$. Hence $G[Y]$ is a maximal hypomatchable component of $\overline{G}[X]$. □

Further structure in $G[X]$ is obtained by considering more than one series class on the rows of $\widehat{M}$.

Lemma 5.17. *If the rank of $\widehat{M}$ can not be increased by a single change to any entry, and $Y_1, \ldots Y_k$ are nontrivial series classes on the rows of $\widehat{M}$ with $|Y_1 \cup \cdots \cup Y_k| = m$, then $\widehat{M}[Y_1 \cup \cdots \cup Y_k;\widehat{X}]$ has at least $m-k+1$ nonzero columns.*

Proof. From Theorem 4.7, rank $\widehat{M}[Y_1 \cup \cdots \cup Y_k; \widehat{X}] \geq m - k$. Hence there are at least $m - k$ nonzero columns, and since each column is avoidable, there are at least $m - k + 1$ nonzero columns. □

Corollary 5.18. *If the rank of $\widehat{M}$ can not be increased by a single change in any entry, then each block of $G[X]$ has either only undirected edges, or only directed edges.*

Proof. Suppose $G[X]$ has a block with both undirected and directed edges, and let $B \subseteq X$ be the vertices in the block. Consider the undirected components of $G[B]$. From Corollary 5.16, the undirected blocks are hypomatchable subgraphs, each contained in some $G[Y_i]$, where Y_i is a series class on the rows of $\widehat{M}$. Assume $G[B]$ has k undirected components, corresponding to k series classes $Y_1, \ldots Y_k$. Also from Corollary 5.16, if a is an arc in $G[B]$ then the head of a is in a different series class than the tail of a. Since $G[B]$ is a block, there must be at least k arcs connecting the k undirected components. But then $\widehat{M}[Y_1 \cup \cdots \cup Y_k; \widehat{X}]$ has at most $|Y_1 \cup \cdots \cup Y_k| - k$ columns, which contradicts Lemma 5.17. Therefore, any block of $G[X]$ has either only undirected edges, or only directed edges. □

Corollary 5.19. *If the rank of $\widehat{M}$ can not be increased by a single change in any entry, then the undirected blocks of $G[X]$ are hypomatchable.*

Proof. This follows from Corollary 5.18 together with Corollary 5.16. □

We can now prove that the evaluation of M is optimal.

Theorem 5.20. *If the rank of $\widehat{M}$ cannot be increased by a single change to any entry, then the evaluation $\widetilde{M}$ is optimal, and $G[X]$ is a hypomatchable tree.*

Proof. From Corollary 5.19, if the rank of $\widehat{M}$ cannot be increased by a single change to any entry, then the blocks of the corresponding subgraph $G[X]$ are either hypomatchable undirected blocks, or blocks with only directed edges. From Lemma 5.10, $G[X]$ does not have a perfect matching matching forest. Since rank $\widetilde{M} = |X| - 1$, the evaluation is optimal, and since every row and column is avoidable, there is a maximum matching forest in $G[X]$ which misses any vertex in X, and $G[X]$ is a hypomatchable tree. □

It follows from Theorem 5.20 that if an evaluation $\widetilde{T} + \widetilde{B}$ satisfies the 3 initial conditions regarding the location of indeterminate pairs in $(\widetilde{T} + \widetilde{B})[D^R; D^C \cup C^C]$ and if for each series class $X \subseteq D^C \cup C^C$, the submatrix $(\widetilde{T} + \widetilde{B})[X]$ is optimal, then $\widetilde{T} + \widetilde{B}$ is optimal.

5.4.2 A minmax theorem

From the Tutte-Berge Formula (5.3) for undirected graphs, a matching in the undirected graph $G = (V, E)$ can not cover more than $|V| - (odd(G \backslash A) - |A|)$ vertices, where A is any subset of V. Both the Gallai-Edmonds decomposition and an optimal Tutte matrix evaluation find a set A for which this inequality is met with equality. An inequality similar to the Tutte-Berge formula can be given for the size of a maximum matching forest in a mixed graph (Giles [15]).

Let $G = (V, E, \vec{E})$ be a mixed graph, and assume $A \subseteq V$ is such that $G \backslash A$ has k hypomatchable trees. Let $D \subseteq V$ be all vertices in such a tree, and define C to be $V \backslash (A \cup D)$. Suppose that there are no arcs directed from A or C to D, and suppose also that there are no edges incident to both a vertex in D and a vertex in C, as in Figure 5.5, where $k = 2$.

If $|A| < k$, then at least $|A| - k$ vertices can not be covered by a matching forest in G, and if $tree(G \backslash A)$ is defined to be the number of hypomatchable trees in $G \backslash A$, then the following theorem is immediate.

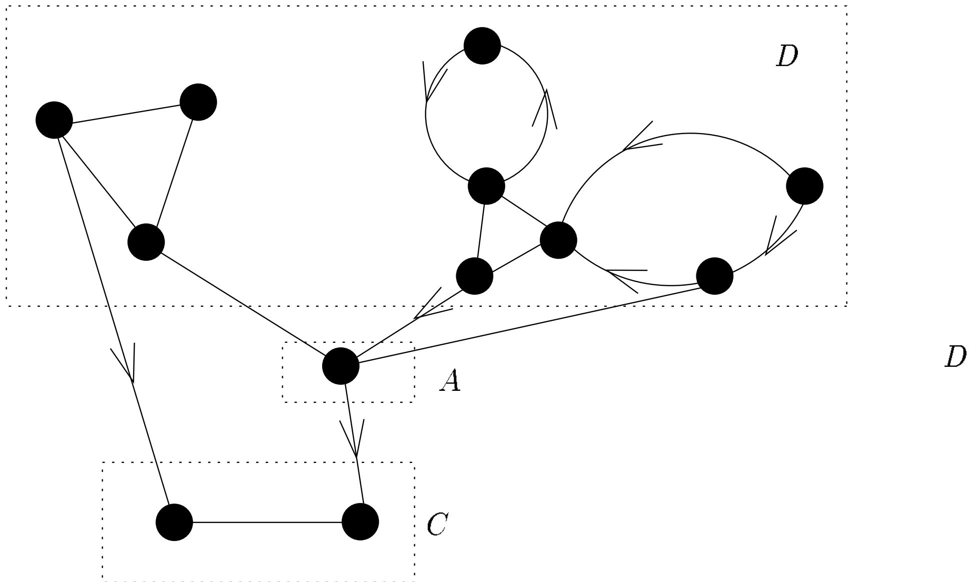

Figure 5.5: Decomposition of a mixed graph

Theorem 5.21. *If $G = (V, E, \vec{E})$ is a mixed graph, then*

$$def(G) \geq \max\{tree(G \backslash A) - |A| : A \subseteq V\}.$$

With an optimal evaluation of the matching forest matrix, Theorem 5.21 can be strengthened to a minmax formula.

Lemma 5.22. *If $\widetilde{T} + \widetilde{B}$ is an optimal evaluation of the matching forest matrix for the mixed graph $G = (V, E, \vec{E})$, and D, C, A is the partition of $\widetilde{T} + \widetilde{B}$, then all entries in $(\widetilde{T} + \widetilde{B})[D^R; C^C]$ are zero, and there are no arcs in G directed from a vertex in A^C to a vertex in D^R.*

Proof. From Lemma 3.13, $(T + B)[X]$ is nonsingular whenever X is a maximal set of independent rows in $T + B$. Let x be any avoidable row of $T + B$. Since there exists a maximal set of independent rows which does not include x, there is a maximal nonsingular submatrix which does not include column x and therefore column x is also avoidable. Since $\widetilde{T}+\widetilde{B}$ is optimal, if i is an avoidable row in $\widetilde{T}+\widetilde{B}$, then it is also an avoidable column, and $D^R \subseteq D^C$.

Suppose there is an entry in row i of $(\widetilde{T}+\widetilde{B})[D^R; C^C]$. Since column i is in D^C, the indeterminate occurs only once in $(T+B)[D^R; C^C]$, and hence an improvement in the evaluation can be made. This contradicts the assumption that $\widetilde{T}+\widetilde{B}$ is optimal, and therefore all entries of $(\widetilde{T}+\widetilde{B})[D^R; C^C]$ are zero.

Similarly, suppose there is an entry in row i of $(\widetilde{T}+\widetilde{B})[D^R; A^C]$. If the entry corresponds to the head of an arc, then the indeterminate occurs only once in $(T+B)[D^R; D^C]$, and an improvement in the evaluation can be made. This is a contradiction, and therefore there are no arcs directed from A^C to D^R. □

For an alternate version of the next theorem, see Giles [14].

Theorem 5.23. *If $G = (V, E, \vec{E})$ is a mixed graph, then*

$$def(G) = \max\{tree(G \backslash A) - |A| : A \subseteq V\}.$$

Proof. Inequality was shown in Theorem 5.21. Let $\widetilde{T}+\widetilde{B}$ be an optimal evaluation of the matching forest matrix for G, and assume D, C, A is the partition of $\widetilde{T}+\widetilde{B}$. Then

$$|V| - def(G) = \text{rank } \widetilde{T}+\widetilde{B} = |A| + |C^R| + \text{rank } (\widetilde{T}+\widetilde{B})[D^R; D^C \cup C^C]. \qquad (5.4)$$

Assume there are k series classes on the columns of $(\widetilde{T}+\widetilde{B})[D^R; D^C \cup C^C]$. From Lemma 5.22 and the block structure of $(\widetilde{T}+\widetilde{B})[D^R; D^C \cup C^C]$, it follows that

$$\text{rank } (\widetilde{T}+\widetilde{B})[D^R; D^C \cup C^C] = \text{rank } (\widetilde{T}+\widetilde{B})[D^R; D^C] = |D^R| - k.$$

Each series class in $(\widetilde{T}+\widetilde{B})[D]$ corresponds to a hypomatchable tree in $G\backslash A$, and therefore (5.4) can be rewritten as

$$\begin{aligned} |V| - def(G) &= |A^R| + |C^R| + |D^R| - (trees(G\backslash A^C) - |A^C|) \\ &= |V| - (trees(G\backslash A^C) - |A^C|). \end{aligned}$$

Thus the inequality in Theorem 5.21 is met with equality. □

5.5 Finding a matching from an optimal evaluation

Once an optimal evaluation of the bipartite Tutte matrix, Tutte matrix, or matching forest matrix is found, the size of a maximum matching or maximum matching forest can be determined from the rank of the evaluation. An optimal evaluation can also be used to explicitly find the maximum matching or maximum matching forest.

Suppose the Tutte matrix T for $G = (V, E)$ has rank r, and let $\widetilde{T}$ be an optimal evaluation of T. By Corollary 2.6, there exists $U \subseteq V$ such that $|U| = r$, and $\widetilde{T}[U]$ is nonsingular. Then $T[U]$ is nonsingular, and there is a maximum matching of G which covers $U = \{u_1, \ldots u_r\} \subseteq V$. Using the row expansion form of the Pfaffian (Theorem 2.10) and the fact that the Pfaffian of $\widetilde{T}[U]$ is non-zero, there exists $i \in \{2, \ldots, r\}$ such that $\tilde{t}_{u_1,u_i} \neq 0$ and $\widetilde{T}[U\backslash\{u_1, u_i\}]$ is nonsingular. Therefore, there exists $e = u_1u_i \in E$ such that $G[U\backslash\{u_1, u_i\}]$ has a perfect matching M, and then $M \cup \{e\}$ is a maximum matching of G. By repeatedly applying the process, a maximum matching can be found.Using the ideas of Cheriyan [5], this method can be improved to obtain the matching in $\mathcal{O}(|V|^3)$.

Similar arguments applied to an optimal evaluation of the matching forest matrix can be used to explicitly find a matching forest in the corresponding mixed graph.

Chapter 6

Exact problems

Problems in graph theory often involve a maximum or a minimum, such as finding a matching with a maximum number of edges. If weights are assigned to each edge in an undirected graph, then the *maximum weight matching problem* is to find a perfect matching such that the total weight of all the edges in the matching is a maximum.

There exist efficient algorithms for finding a maximum or minimum weight matching in an undirected graph (Edmonds [7]). Once the maximum and minimum weight of a perfect matching in a graph are known, the *exact* matching problem becomes interesting: does there exist a perfect matching with a particular weight?

6.1 The weighted branching matrix

Given a directed graph $G = (V, \vec{E})$ and weights w_a for each $a \in \vec{E}$, the *exact weighted arborescence problem* is to determine if G has an arborescence of a specific weight. The *weighted branching matrix*, $C = (c_{ij})$, is a formulation for this problem, where

$$c_{ij} = \begin{cases} -t^{w_{ij}}, & \text{if } i \neq j \text{ and } (i,j) \in \vec{E}; \\ 0, & \text{if } i \neq j \text{ and } (i,j) \notin \vec{E}; \\ \sum_{k \neq i} c_{i,k}, & \text{if } i = j. \end{cases}$$

The weighted branching matrix solves the exact weighted arborescence problem.

Theorem 6.1 (Barahona and Pulleybank). *If $G = (V, \vec{E})$ is a directed graph with weights w_a for each $a \in \vec{E}$, C is the weighted branching matrix for G, and $v \in V$, then the number of arborescences in G with weight w and root v is equal to the coefficient of t^w in the determinant of $C[V \backslash v]$.*

Proof. The weighted branching matrix is an evaluation of the branching matrix with $x_a = t^{w_a}$ for all $a \in \vec{E}$. From Corollary 3.6, for each arborescence of G with root v and weight w, t^w is a term in the determinant of $C[V \backslash v]$. □

6.2 The weighted Tutte matrix

Suppose $G = (V, E)$ is an undirected graph with weights w_e for each $e \in E$. If $\{z_e : e \in E\}$ are algebraically independent indeterminates, then the skew-symmetric *weighted Tutte matrix* $A = (a_{ij})$, with rows and columns indexed by V, is defined as follows:

$$a_{ij} = \begin{cases} \pm z_{ij} t^{w_{ij}}, & \text{if } ij \in E; \\ 0, & \text{otherwise.} \end{cases}$$

The next theorem follows immediately from the one-to-one correspondence between perfect matchings of G and terms in the Pfaffian of T.

Theorem 6.2 (Lovász). *If $G = (V, E)$ is a graph with weights w_e for each $e \in E$, and weighted Tutte matrix A, then G has a perfect matching of weight w if and only if the coefficient of t^w in the Pfaffian of A is non-zero.*

If $G = (V, E)$ is a weighted bipartite graph, with bipartition V_1, V_2, and A is the weighted Tutte matrix for G, then the *weighted bipartite Tutte matrix* is $A[V_1; V_2]$. For bipartite weighted graphs, Theorem 6.2 can be stated with respect to the determinant of the bipartite Tutte matrix.

Theorem 6.3 (Lovász). *If $G = (V, E)$ is a bipartite graph with weights w_e for each $e \in E$, and A is the weighted bipartite Tutte matrix for G, then G has a perfect matching of weight w if and only if the coefficient of t^w in the determinant of A is non-zero.*

The weighted Tutte matrix formulates a solution to the exact weight perfect matching problem. In the given form, this formulation is not computationally useful, since it requires computing the determinant of a matrix with indeterminate entries. If the indeterminates are evaluated as rationals, then, because of the sign factor in the determinant function, two non-zero terms may cancel each other. The *permanent* of a matrix has a similar permutation expansion to the determinant, except the sign function is absent: if A is an $n \times n$ matrix, the permanent of A is defined as

$$\text{per } A = \sum_{\sigma \in \mathcal{S}_n} a_{1,\sigma(1)} \cdots a_{n,\sigma(n)}.$$

If A is the weighted bipartite Tutte matrix for the bipartite graph G, and each entry a_{ij} is evaluated as $t^{w_{ij}}$, then the coefficient of t^w in the permanent of $\widetilde{A}$ is equal to the number of perfect matchings in G with weight w. This is not a computationally feasible solution to the exact matching problem, however, since calculating the permanent of a matrix is $\mathcal{NP}$-hard (L.Valiant (1979)).

6.3 Pfaffian orientations

Another possible solution for the exact weight matching problem is to choose signs for the entries in the weighted bipartite Tutte matrix, $a_{ij} = \pm t^{w_{ij}}$, such that

$$\operatorname{sign}(\sigma) a_{1,\sigma(1)} \cdots a_{n,\sigma(n)} \geq 0$$

for all permutations σ. This problem was first posed by Polya in 1913. For example, in the matrix (6.1), if a non-negative value is assigned to every $+$ and a non-positive value to every $-$, then every term in the determinant is non-negative. Such a matrix is called *sign-nonsingular*.

$$\begin{bmatrix} + & + & - \\ 0 & + & + \\ + & - & + \end{bmatrix} \tag{6.1}$$

When the matrix is the Tutte matrix T for a general graph G, then giving a sign to each indeterminate in T corresponds to assigning a direction, or orientation, to each edge in G. A *Pfaffian orientation* of G is an orientation such that all the terms in the Pfaffian of T have the same sign. A graph is *Pfaffian* if it has a Pfaffian orientation.

Suppose G is Pfaffian and bipartite. If each indeterminate a_{ij} in the bipartite weighted Tutte matrix A is evaluated as $\pm t^{w_{ij}}$, according to the orientation of G, then the coefficient of t^w in $\det \widetilde{A}$ is the number of perfect matchings in G with weight w.

Not all graphs are Pfaffian. For example, the 3×3 matrix that has a non-zero entry in each term is not sign-nonsingular, and therefore the complete bipartite graph $K_{3,3}$ does not have a Pfaffian orientation.

Theorem 6.4 (Kasteleyn). *Every planar graph is Pfaffian, and a Pfaffian orientation* *can be constructed in polynomial time.*

Kasteleyn's Theorem does not classify all Pfaffian graphs, as there are non-planar graphs that are Pfaffian. A theorem of Little [20] proves that a bipartite graph is Pfaffian if and only if it does not "contain" a $K_{3,3}$.

6.4 Complexity and random evaluations

The idea of a random evaluation is presented in Lovász [22]; if the entries of an evaluation are chosen from a large enough set, then the probability that a matrix is nonsingular when a random evaluation of it is singular can be made arbitrarily small. We first consider the probability that a random integer vector is a root of a polynomial.

Theorem 6.5 (Zippel). *If $p(x_1, \ldots, x_m)$ is a nonzero polynomial, the degree of x_i in p is at most d for all $i \in \{1, \ldots, m\}$, and a_i are selected with uniform probability from $\{1, \ldots, M\}$, then the probability that $p(a_1, \ldots, a_m) = 0$ is no more than $\frac{dm}{M}$.*

Proof. The proof is by induction on the number of variables in the polynomial.

If $m = 1$, then $p = p(a_1)$ is a polynomial of degree at most d, and therefore has at most d roots. The probability of selecting a_1 from $\{1, \ldots, M\}$ such that a_1 is a root of p is at most $\frac{d}{M}$.

Suppose the probability that $(a_1, \ldots a_m)$ is a root of a polynomial with m variables is at most $\frac{dm}{M}$ when a_i is selected with uniform probability from $\{1, \ldots, M\}$ for $i \in \{1, \ldots, m\}$, and the degree of any variable in the polynomial is at most d. Let p be a polynomial with $m + 1$ variables, each of degree at most d. Then p can be written as

$$p(x_1, x_2, \ldots, x_{m+1}) = x_1^d y_d + x_1^{d-1} y_{d-1} + \ldots + x_1 y_1 + y_0,$$

where y_i is a polynomial with at most m indeterminates, and each indeterminate in y_i

has degree at most d, for $i \in \{0, \ldots, d\}$.

Suppose $p(a_1, a_2, \ldots, a_{m+1}) = 0$. Either $\tilde{y}_i = 0$ for all $i \in \{0, \ldots, d\}$, or some $\tilde{y}_i \neq 0$, and a_1 is a root of $f(x) = x^d \tilde{y}_d + x^{d-1}\tilde{y}_{d-1} + \ldots + x\tilde{y}_1 + \tilde{y}_0$. The probability that $\tilde{y}_i = 0$ for all i is bounded above by the probability that $\tilde{y}_i = 0$ for some i, which is at most $\frac{dm}{M}$ by the induction hypothesis. If $\tilde{y}_i \neq 0$ for some $1 \leq i \leq d$, then $f(x)$ has at most d roots, and the probability that $a_1 \in \{1, \ldots, M\}$ is a root of f is at most $\frac{d}{M}$. Combining the two cases gives that the probability of selecting a root of $p(x_1, \ldots, x_{m_1})$ when each a_i is selected at random from $\{1, \ldots, M\}$, is at most $\frac{dm}{M} + \frac{d}{M} = \frac{d(m+1)}{M}$, and the theorem is true by induction. □

Theorem 6.5 can be applied to the determinant of a matrix with randomly selected integer entries.

Corollary 6.6 (Lovász). *If $A = (a_{ij})$ is a nonsingular matrix with m different indeterminate entries, each occurring at most d times in A, and $\tilde{a}_{ij}$ are selected with uniform probability from $\{1, \ldots, M\}$, then the probability that $\widetilde{A} = (\tilde{a}_{ij})$ is singular is no more than $\frac{dm}{M}$.*

Random evaluations provide a probabilistic solution to the exact matching problem in bipartite graphs.

Corollary 6.7 (Lovász). *Suppose $A = (a_{ij}t^{w_{ij}})$ is the weighted bipartite Tutte matrix for the bipartite graph $G = (V, E)$, where G has a perfect matching of weight w. If each $\tilde{a}_{ij}$ is selected with uniform probability from $\{1, \ldots, M\}$, then the probability that the coefficient of t^w in the determinant of $\widetilde{M}$ is zero is no more than $\frac{|V|}{2M}$.*

Proof. Since the coefficient of t^w in $\det A$ is a polynomial with $\frac{|V|}{2}$ indeterminates, each of degree 1, the corollary follows from Theorem 6.5. □

Chapter 7

Matroids

In Chapter 2, the three concepts of matrix rank, matrix singularity, and linear independence of vectors were shown to be equivalent, and in Chapters 3 and 5, these concepts were used to prove the existence of certain structures in graphs. *Matroids* are an abstraction of rank, singularity, and linear independence, and are discussed here.

7.1 Definitions and examples

Matroid theory began in the 1930's with Whitney [32], and was expanded by Tutte [30] and by Edmonds [9]. A matroid can be defined by a finite set $\mathcal{S}$ and a set of axioms, together with either subsets of $\mathcal{S}$ or a function defined on subsets of $\mathcal{S}$. We give three definitions for a matroid.

The Independence Axioms

Let $\mathcal{S}$ be a finite set, and let $\mathcal{I} \subseteq 2^{\mathcal{S}}$. The pair $(\mathcal{S}, \mathcal{I})$ is a matroid if it satisfies the following *independence axioms*:

(I1) The null set is in $\mathcal{I}$.

(I2) Every subset of a set in $\mathcal{I}$ is also in $\mathcal{I}$.

(I3) For any subset X of $\mathcal{S}$, the maximal subsets of X that are in $\mathcal{I}$ have the same cardinality.

The set $\mathcal{S}$ is called the *ground set* of the matroid. If $X \in \mathcal{I}$, then X is *independent*, otherwise X is *dependent*. A *circuit* is a minimal dependent set, and a maximal independent set is a *base*. From axiom (I2), only the bases are needed when listing the independent elements of a matroid.

The Base Axioms

The *base axioms* restrict the matroid $(\mathcal{S}, \mathcal{I})$ to $(\mathcal{S}, \mathcal{B})$ where $\mathcal{B} = \{X \in \mathcal{I} : X \text{ is a base}\}$. The pair $(\mathcal{S}, \mathcal{B})$ is a matroid if the following hold:

(B1) $\mathcal{B}$ is nonempty.

(B2) If $B_1, B_2 \in B$ and $x \in B_1$, then there exists $y \in B_2$ such that $(B_1 \backslash x) \cup \{y\} \in \mathcal{B}$.

Axiom (B2) is called the *exchange axiom* for matroids.

Both the independence and base axioms define a matroid by giving restrictions on a family of subsets of a finite set $\mathcal{S}$. Alternatively, a matroid can be given by the set $\mathcal{S}$ and a function defined on $2^{\mathcal{S}}$. One such function is the *rank function* r, where $r : 2^{\mathcal{S}} \to \mathbb{Z}^+$.

The Rank Axioms

If $\mathcal{S}$ is a finite set and $r : 2^{\mathcal{S}} \to \mathbb{Z}^+$, then $(\mathcal{S}, r)$ is a matroid if, for all $X, Y \subseteq \mathcal{S}$, the following *rank axioms* are satisfied:

(R1) $r(X) \leq |X|$.

(R2) If $X \subseteq Y$, then $r(X) \leq r(Y)$.

(R3) Submodular inequality: $r(X) + r(Y) \geq r(X \cap Y) + r(X \cup Y)$.

The independence, base, and rank axioms are not an exhaustive set of matroid definitions. Other axioms use either different subsets of $\mathcal{S}$ than those described here, or a different function or operator on $2^{\mathcal{S}}$. See Fujishige [11] or von Randow [31] for further axiomatic definitions of matroids.

The different axiom sets are equivalent; for every matroid $(\mathcal{S}, \mathcal{I})$ satisfying the independence axioms, there are corresponding pairs $(\mathcal{S}, \mathcal{B})$ and $(\mathcal{S}, r)$ satisfying the base and rank axioms respectively. For a proof of the equivalence of these axioms, together with the *span* and *circuit* axioms, see von Randow [31].

The following examples are from Cook et al. [10], Fujishige [11], Recski [26], Truemper [28], and von Randow [31].

Partition Matroids

Let $\Pi = X_1, ..., X_k$ be a partition of the finite set $\mathcal{S}$, and let $d_1, \ldots, d_k$ be nonnegative integers. Consider $(\mathcal{S}, \mathcal{I})$, where $\mathcal{I} = \{I \in \mathcal{S} : |I \cap X_i| \leq d_i \text{ for all } i = 1, \ldots, k\}$. Clearly (I1) is satisfied, and if $X \subseteq Y$ then $(X \cap X_i) \subseteq (Y \cap X_i)$ and (I2) is satisfied. Assume $X \subseteq \mathcal{S}$, and let $Y \in \mathcal{I}$ be a maximal subset of X. Since $|Y| = \sum_{i=1}^{k} |Y \cap X_i|$, and $|Y|$ is maximal, $|Y \cap X_i| = \min\{d_i, |X \cap X_i|\}$. Therefore the size of Y depends only on X, Π, and d_i, and (I3) is satisfied. Thus $(\mathcal{S}, \mathcal{I})$ satisfies the independence axioms, and is a *partition matroid.*

Forest Matroids

For a graph $G = (V, E)$, define $\mathcal{I}$ to be all edge sets of forests in G. That is,

$$\mathcal{I} = \{F \subseteq E : F \text{ does not contain a circuit}\}.$$

Then $(E, \mathcal{I})$ satisfies the three independence axioms for a matroid, and is called the *forest matroid.* Similarly, if $G = (V, E)$ is connected and $\mathcal{B}$ is the set of spanning trees of G, then $(E, \mathcal{B})$ satisfies the base axioms.

Linear Matroids

For a matrix M with rows indexed by X and columns indexed by Y, let

$$\mathcal{I} = \{Y' \subseteq Y : M[X;Y'] \text{ has full column rank}\}.$$

Consider $(Y, \mathcal{I})$. Since the empty set is independent, $(Y, \mathcal{I})$ satisfies (I1). Any subset of independent columns is independent so (I2) is satisfied. Axiom (I3) is also satisfied; if $A \subseteq Y$, then a maximal independent set of columns from A has size equal to the rank of $M[X; A]$. Therefore $(Y, \mathcal{I})$ is a matroid, called a *linear matroid.*

For $Y' \subseteq Y$, define $r(Y')$ to be the rank of $M[X;Y']$. Axioms (R1) and (R2) hold for (Y, r), by properties of linear algebra, and Theorem 2.2 showed that submodularity holds. Therefore the linear matroid also satisfies the rank axioms.

Two examples of linear matroids are the *matching matroid* and the *branching matroid.* If $G = (V, E)$ is a graph, let

$$\mathcal{I} = \{W \subseteq V : \text{ there is a matching in } G \text{ which covers } W\}.$$

Consider $(V, \mathcal{I})$. Axiom (I1) is satisfied, and if there is a matching M that covers $W \subseteq V$, then M also covers any subset of W, hence (I2) is satisfied. If T is the $V \times V$ Tutte matrix for G, then by Theorem 3.1, there is a matching covering $W \subseteq V$ if and only if $T[U]$ is nonsingular for some $U \supseteq W$. Let $X \subseteq V$. If $W \in \mathcal{I}$ is a maximal subset of X, then $|W| = \text{rank } T[V; X]$, which is independent of W. Therefore (I3) is satisfied and $(V, \mathcal{I})$ is a matroid. (For a proof which uses augmenting paths to show that (I3) is satisfied, see Cook et al. [10].) The branching matroid for a directed graph graph $G = (V, A)$ is defined by

$$\mathcal{I} = \{W \subseteq V : \text{ there is a branching in } G \text{ which covers } W\}.$$

With similar arguments to those for the matching matroid, it can be shown that $(V, \mathcal{I})$ satisfies the independence axioms for a matroid.

7.2 Matroid duals

If $M = (\mathcal{S}, \mathcal{I})$ is a matroid then the *dual* of M is $M^* = (\mathcal{S}, \mathcal{I}^*)$, where

$$\mathcal{I}^* = \{X \in \mathcal{S} : X \subseteq \overline{Y} \text{ for some base } Y \in \mathcal{I}\}.$$

Equivalently, $M^* = (\mathcal{S}, \mathcal{B}^*)$, where

$$\mathcal{B}^* = \{\overline{B} : B \text{ is a base for } M\}.$$

Theorem 7.1. *The dual of a matroid is a matroid.*

A proof of Theorem 7.1 is given in Cook et al. [10]. Note that $\mathcal{I}^{**} = \mathcal{I}$ and therefore if M is a matroid, $M^{**} = M$.

If $M = (\mathcal{S}, \mathcal{I})$ is a matroid, then $X \in \mathcal{S}$ is *coindependent* if X is independent in M^*. This terminology is generalized; if Y is a matroidal term with respect to the matroid M, then it is referred to as co-Y with respect to the dual M^*. For example, X is a *cocircuit* in M if it is a circuit in M^*, and the complement of a base in M is a *cobase* (Fujishige [11].)

We construct the dual of the linear matroid defined by the independence axioms on the columns of the matrix M. First, $\widetilde{M}$ is formed by Gaussian elimination on the rows of M. Since we are only interested in independent columns, and column dependencies are not affected by Gaussian elimination on rows, $\widetilde{M}$ can be used in place of M. A zero row does not affect column dependencies, so this row is dropped, leaving the matrix $\mathbf{M}$.

$$M = \begin{pmatrix} 1&0&0&1&1 \\ 0&2&0&2&0 \\ 0&0&1&1&1 \\ 1&2&1&3&2 \end{pmatrix}, \quad \widetilde{M} = \begin{pmatrix} 1&0&0&1&1 \\ 0&1&0&1&0 \\ 0&0&1&1&1 \\ 0&0&0&0&0 \end{pmatrix}, \quad \mathbf{M} = \begin{pmatrix} 1&0&0&1&1 \\ 0&1&0&1&0 \\ 0&0&1&1&1 \end{pmatrix}. \tag{7.1}$$

Note that the first three columns of $\mathbf{M}$ are a base of the column space of $\mathbf{M}$. To construct $\mathbf{M}^*$, columns 4 and 5 are transposed, and the identity matrix is appended:

$$\mathbf{M}^* = \begin{pmatrix} 1&1&1&1&0 \\ 1&0&1&0&1 \end{pmatrix}. \tag{7.2}$$

Columns 4 and 5 are a base of $\mathbf{M}^*$.

Suppose M is a matrix representation of a linear matroid with ground set Y. If $X \subseteq Y$ is a base of M, then by Gaussian elimination on the rows, the columns X can be made into the identity matrix (dropping zero rows if necessary.) Permute columns and relabel the rows to form $\mathbf{M}$, and construct $\mathbf{M}^*$ as in (7.2) of the example:

$$\mathbf{M} = \begin{array}{c} \\ X \end{array} \!\!\begin{array}{c} \begin{matrix} X & \overline{X} \end{matrix} \\ \begin{pmatrix} I & C \end{pmatrix} \end{array}, \quad \mathbf{M}^* = \begin{array}{c} \\ \overline{X} \end{array} \!\!\begin{array}{c} \begin{matrix} X & \overline{X} \end{matrix} \\ \begin{pmatrix} C^\top & I \end{pmatrix} \end{array}.$$

We show that $\mathbf{M}^*$ is the dual of M. Consider the column sets Z_1 and Z_2 of $\mathbf{M}$, where $Z_1 \subseteq X$, $Z_2 \subseteq \overline{X}$, and $|Z_1| + |Z_2| = |X|$. By the construction of $\mathbf{M}$,

$$\begin{aligned} \det \mathbf{M}[X; Z_1 \cup Z_2] &= \det \mathbf{M}[\overline{Z}_1; Z_2] \\ &= \det C[\overline{Z}_1; Z_2] \\ &= \det C^\top[Z_2; \overline{Z}_1] \\ &= \det \mathbf{M}^*[\overline{X}; X \backslash (Z_1 \cup Z_2)]. \end{aligned} \tag{7.3}$$

From (7.3), $Z_1 \cup Z_2$ is a base of M if and only if $X \backslash (Z_1 \cup Z_2)$ is a base of $\mathbf{M}^*$, and therefore $\mathbf{M}^*$ is the dual of M.

7.3 Matroid intersection and partition

Suppose $M_1 = (\mathcal{S}, \mathcal{I}_1)$ and $M_2 = (\mathcal{S}, \mathcal{I}_2)$ are matroids on the same ground set $\mathcal{S}$, with rank functions r_1 and r_2 respectively. The *matroid intersection problem* is to determine if there exists $B \subseteq \mathcal{S}$ such that B is a base in both M_1 and M_2. Suppose $X \subseteq \mathcal{S}$ and $J \in \mathcal{I}_1 \cap \mathcal{I}_2$. Then

$$|J| = |J \cap X| + |J \cap (\overline{X})| \leq r_1(X) + r_2(\overline{X}),$$

and therefore the minimum of $r_1(X) + r_2(\overline{X})$ over all $X \subseteq \mathcal{S}$ is an upper bound on the size of any set that is independent in both M_1 and M_2. This upper bound is met with equality.

Theorem 7.2 (Edmonds). *If $M_1 = (\mathcal{S}, \mathcal{I}_1)$ and $M_2 = (\mathcal{S}, \mathcal{I}_2)$ are matroids with rank functions r_1 and r_2, then*

$$\max\{|J| : J \in \mathcal{I}_1 \cap \mathcal{I}_2\} = \min\{r_1(X) + r_2(X) : X \in \mathcal{S}\}$$

An algorithm of Edmonds solves the matroid intersection problem.

The *matroid partition problem* is to find a partition S_1, S_2 of the ground set $\mathcal{S}$ such that S_1 is a base of $M_1 = (\mathcal{S}, \mathcal{I}_1)$ and S_2 is a base of $M_2 = (\mathcal{S}, \mathcal{I}_2)$. Suppose that $J_1 \subseteq \mathcal{S}$ and $J_2 \subseteq \mathcal{S}$ are disjoint, and that $J_1 \in \mathcal{I}_1$ and $J_2 \in \mathcal{I}_2$. If $X \subseteq \mathcal{S}$ and we let $J = J_1 \cup J_2$, then the following holds:

$$\begin{aligned} |J| &= |J \backslash X| + |J \cap X| \\ &\leq |S \backslash X| + |J_1 \cap X| + |J_2 \cap X| \\ &\leq |\overline{X}| + r_1(X) + r_2(X). \end{aligned}$$

This upper bound is met with equality.

Theorem 7.3 (Edmonds). *If $M_1 = (\mathcal{S}, \mathcal{I}_1)$ and $M_2 = (\mathcal{S}, \mathcal{I}_2)$ are matroids with rank functions r_1 and r_2, then*

$$\max\{|J_1 \cup J_2| : J_1 \in \mathcal{I}_1 \text{ and } J_2 \in \mathcal{I}_2)\} = \min\{r_1(X) + r_2(X) + |\overline{X}| : X \in \mathcal{S}\}$$

The intersection and partition problems are equivalent, as is shown in Recski [26].

Theorem 7.4. *The base pair (B_1, B_2) solves the partition problem for the matroids M_1 and M_2 if and only if $(B_1, \overline{B}_2)$ solves the matroid intersection problem for M_1 and M_2^*.*

Proof. Assume B_1 is a base for the matroid M_1 and B_2 is a base for M_2, and assume M_1 and M_2 have the same ground set $\mathcal{S}$. Suppose (B_1, B_2) solves the matroid partition problem for M_1 and M_2. Then $B_1 \cup B_2 = \mathcal{S}$ and $B_1 \cap B_2 = \emptyset$, and therefore $\overline{B}_2 = B_1$. Since $\overline{B}_2$ is a base for the dual of M_2, $(B_1, \overline{B}_2)$ solves the matroid intersection problem for M_1 and M_2^*. Similarly, if B_1 is a solution to the matroid intersection problem for M_1 and M_2^*, then $B_2 = \overline{B}_1$ is a basis for M_2, and the pair (B_1, B_2) solves the matroid partition problem for M_1 and M_2. □

The intersection and partition problems can be generalized for more than two matroids. Applications of the intersection and partition problem are given in the next section.

7.4 Matroid formulations

The three problems of finding a maximum matching in an undirected bipartite graph, a maximum branching in a directed graph, and a maximum matching forest in a mixed graph, all of which were formulated as matrix rank problems in Chapter 3, can be formulated as either matroid intersection or matroid partition problems.

7.4.1 Bipartite matchings

Let $G = (V, E)$ be a bipartite graph, with bipartition $V = V_1 \cup V_2$. For each $i \in \{1, \ldots, |V_1|\}$, define $E_i \subseteq E$ to be the set of edges incident with vertex v_i. Then $\Pi_1 = E_1, \ldots, E_{|V_1|}$ is a partition of E. (See Figure 7.1.)

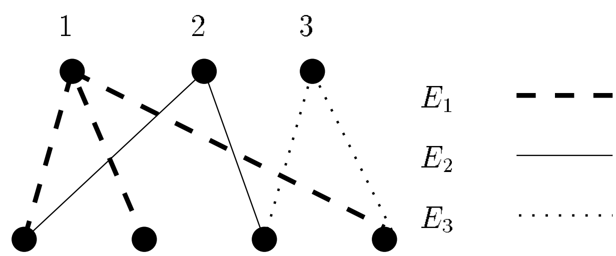

Figure 7.1: the edge partition Π_1

Let M_1 be the partition matroid with ground set E, partition Π_1, and intersection sizes $d_i = 1$ for all i. (Cook et al. [10].) Define the partition matroid M_2 similarly, with respect to V_2. Suppose $F \subseteq E$ is independent in both M_1 and M_2. Then every $v \in V$ is incident with at most one edge in F, and therefore F is a matching in G, and a solution to the matroid intersection problem for M_1 and M_2 solves the maximum matching problem in G.

7.4.2 Arborescences

Let $G = (V, \vec{E})$ be a connected, directed graph. We describe matroids M_1 and M_2 such that a set that is independent in both M_1 and M_2 corresponds to a branching in G. (Cook et al. [10].)

Let $M_1 = (\vec{E}, \mathcal{F})$ be the forest matroid (Section 7.1) for the underlying undirected graph for G. For each $v \in V$, let $X_v = \{a \in \vec{E} : v \text{ is the head of } a\}$. A partition of $\vec{E}$ is then given by $\Pi = \{X_v : v \in V\}$. Let $M_2 = (\vec{E}, \mathcal{I})$ be the partition matroid with respect to Π, with $d_v = 1$ for all $v \in V$. The bold edges in the graph in Figure 7.2 are an example of an independent set in M_2. The common intersection sets for M_1 and M_2

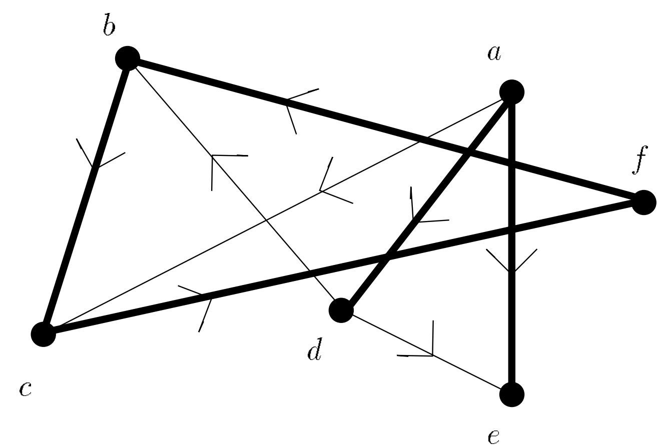

Figure 7.2: An independent set in M_2

are branchings of G, and a solution to the intersection problem for M_1 and M_2 is an arborescence in G.

7.4.3 Matching forests

For a mixed graph $G = (V, E, A)$, let M_1 be the matching matroid for the undirected graph $G_1 = (V, E)$ and let M_2 be the branching matroid for the directed graph $G_2 = (V, A)$. (The matching matroid and the branching matroid are described in Section 7.1.) If

$I_1 \subset V$ is independent in M_1, $I_2 \subseteq V$ is independent in M_2, and I_1 and I_2 are disjoint, then there is a matching M covering I_1 and a branching F covering I_2 such that the union of M and F is a matching forest that covers $I_1 \cup I_2$. Therefore, if $I_1 \cup I_2$ is a partition of V, then there is a matching forest in G that covers the vertices I_1 with edges and the vertices I_2 with arcs, and a solution to the matroid partition problem for M_1 and M_2 solves the maximum matching forest problem.

We remark that an algorithm for finding a bipartite matching, branching, or matching forest by the given matroid formulations requires an oracle for testing the rank in matroids M_1 and M_2.

7.4.4 Solving the intersection problem by rank completion

The maximum bipartite matching, maximum branching, and maximum matching forest problems can be solved by a maximum rank completion of the bipartite Tutte matrix, the branching matrix, or the matching forest matrix. For linear matroids, the matroid intersection problem can also be considered as an optimal evaluation problem.

Suppose M_1 and M_2 are linear matroids on the matrices A_1 and A_2 respectively, and assume A_1 is $n_1 \times m$ and A_2 is $n_2 \times m$. From section 7.2, we can assume n_1 is the rank of M_1, and n_2 is the rank of M_2. Construct the $(n_1 + m) \times (n_2 + m)$ matrices A and Z, where

$$\mathbf{A} = \left[\begin{array}{c|c} 0 & A_1 \\ \hline A_2^\top & 0 \end{array}\right], \quad \text{and} \quad \mathbf{Z} = \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & \begin{matrix} z_1 & & \\ & \ddots & \\ & & z_m \end{matrix} \end{array}\right]. \qquad (7.4)$$

We will show that the rank of $A + Z$ is determined by the size of the largest set of columns independent in both A_1 and A_2, and therefore show that the matroid intersection problem can be solved by finding an optimal evaluation of $A + Z$ (Murota [24]).

Theorem 7.5. *Let M_1 and M_2 be representable matroids with matrices A_1 and A_2 respectively, and suppose M_1 and M_2 have the same ground set V. If A and Z are the matrices from equation 7.4, then*

$$\max\{|J| : J \in \mathcal{I}_1 \cap \mathcal{I}_2\} = \text{rank } (A + Z) - |V|.$$

Proof. Suppose the columns $J \subseteq V$ are independent in both A_1 and A_2, and assume $|J| = l$. There is a set of rows, R_1, in A_1 and a set of rows, R_2, in A_2 such that both $A_1[R_1; J]$ and $A_2[R_2; J]$ are nonsingular. Let $\widetilde{A_1} = A_1[R_1; V]$ and $\widetilde{A_2} = A_2[R_2; V]$, and relabel R_1 and R_2 as R. Consider the matrix $\widetilde{A} + \widetilde{Z}$, where

$$\widetilde{A} = \begin{array}{c|c|c|} & R & V \\ \hline R & 0 & \widetilde{A_1} \\ \hline V & \widetilde{A_2}^{\top} & 0 \\ \hline \end{array}, \quad \text{and} \quad \widetilde{Z} = \begin{array}{c|c|c|} & R & V \\ \hline R & 0 & 0 \\ \hline V & 0 & \begin{matrix} z_1 & & \\ & \ddots & \\ & & z_m \end{matrix} \\ \hline \end{array}. \tag{7.5}$$

Applying Theorem 2.8 to $\widetilde{A} + \widetilde{Z}$, and using that a nonsingular submatrix of $\widetilde{Z}$ must be a principal submatrix of $\widetilde{Z}[V]$, we get

$$\det(\widetilde{A} + \widetilde{Z}) = \sum_{X \subseteq V} \det \widetilde{Z}[X] \det \widetilde{A}[(R \cup V) \backslash X]. \tag{7.6}$$

When $X \subseteq V$, $(R \cup V) \backslash X = R \cup (V \backslash X)$, and $\widetilde{A}[R \cup (V \backslash X)]$ is nonsingular only if

$|V \backslash X| = l$, or equivalently, $|X| = m - l$. Equation 7.6 can then be rewritten as

$$\det(\widetilde{A} + \widetilde{Z}) = \sum_{\substack{X \subseteq V \\ |X| = m-l}} \det \widetilde{Z}[X] \det \widetilde{A}[R \cup (V \backslash X)]. \tag{7.7}$$

When $X_1, X_2 \subseteq V$ are distinct, the determinant of $\widetilde{Z}[X_1]$ and the determinant of $\widetilde{Z}[X_2]$ are distinct, and there is no cancellation of terms in (7.7). By the assumption that $J \subseteq V$ was a common independent set, $\widetilde{A}[R \cup J]$ is nonsingular and the term $\det \widetilde{Z}[V \backslash J] \det \widetilde{A}[R \cup J]$ is nonzero. Therefore, from (7.7), the rank of $A + Z$ is at least $m + l$.

To finish the proof, we note that we can choose a nonsingular submatrix $A + Z$ with size equal to the rank of $(A + Z)$ by starting with the independent rows V in $A + Z$ and extending to a row base $V \cup R_1$, and extending the independent columns V in $A + Z$ to a column base $V \cup R_2$. Letting $\widetilde{A} = A[V \cup R_1; V \cup R_2]$ and $\widetilde{Z} = Z[V \cup R_1; V \cup R_2]$, the matrix sum $\widetilde{A} + \widetilde{Z}$ is nonsingular, and using equation (7.7), there exists a common independent set of columns J in A_1 and A_2, where $|J| = \text{rank } (A + Z) - m$. □

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