The Existence of Balanced Tournament Designs and Partitioned Balanced Tournament Designs

by

Shane Bauman

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I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

A balanced tournament design of order n, BTD(n), defined on a $2n$-set V, is an arrangement of the $\binom{2n}{2}$ distinct unordered pairs of elements of V into an $n \times (2n-1)$ array such that (1) every element of V occurs exactly once in each column and (2) every element of V occurs at most twice in each row. We will show that there exists a BTD(n) for n a positive integer, $n \neq 2$. For $n = 2$, a BTD(n) does not exist. If the BTD(n) has the additional property that it is possible to permute the columns of the array such that for every row, all the elements of V appear exactly once in the first n pairs of that row and exactly once in the last n pairs of that row then we call the design a partitioned balanced tournament design, PBTD(n). We will show that there exists a PBTD(n) for n a positive integer, $n \geq 5$, except possibly for $n \in \{9, 11, 15\}$. For $n \leq 4$ a PBTD(n) does not exist.

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Chapter 1

Introduction

The purpose of this paper is to investigate the existence of two related types of combinatorial designs. The first design we investigate is a balanced tournament design.

Definition 1.1 *A* balanced tournament design *of order n, BTD(n), defined on a $2n$-set V, is an arrangement of the $\binom{2n}{2}$ distinct unordered pairs of elements of V into an $n \times (2n-1)$ array such that*

1. *every element of V occurs exactly once in each column, that is, every column is* Latin*, and*
2. *every element of V occurs at most twice in each row*

Example 1.1 *An example of a BTD(3)*

16	35	23	45	24
25	46	14	13	36
34	12	56	26	15

We will show that there exists a BTD(n) for n a positive integer, $n \neq 2$. For $n = 2$ a BTD(n) does not exist.

The second related design we investigate is the partitioned balanced tournament design. A partitioned balanced tournament design is a BTD with an additional property.

Definition 1.2 *Let V be a set of cardinality $2n$. A* partitioned balanced tournament design *of order n, PBTD(n), is a BTD defined on V, for which it is possible to permute the columns so that for every row of the array, all the elements of V appear exactly once in the first n pairs of that row and exactly once in the last n pairs of that row.*

Example 1.2 *A PBTD(5) due to D.R. Stinson [16].*

90	58	46	12	37	28	59	40	16
24	79	13	80	56	10	47	29	38
36	14	89	57	49	20	67	18	35
15	23	70	69	50	48	39	26	17
78	60	25	34	27	19	45	30	68

We will show that there exists a PBTD(n) for n a positive integer, $n \geq 5$, except possibly for $n \in \{9, 11, 15\}$. For $n \leq 4$ a PBTD(n) does not exist.

Chapter 2

Balanced Tournament Designs

A sports league with $2n$ teams is setting up a tournament with n rounds in which every team plays in every round and throughout the course of the tournament each team will play every other team. The league has access to n different playing fields and these fields are of unequal quality. Therefore, in the spirit of fairness, the league wants to ensure that in the $2n-1$ games that each team plays at most 2 of these games are played on any particular field. We see that these requirements correspond to the conditions of a balanced tournament design. In this context the columns refer to the rounds of the tournament and the rows correspond to the playing fields.

We will prove the existence of BTDs for all values of n except $n=2$. This result was originally proved in 1977 by Schellenberg et al [12]. The proof we will present here is a simpler proof presented by Lamken and Vanstone in 1985 [8]. This proof uses the idea of a factored BTD.

Definition 2.1 *A* factored balanced tournament design *of order n, FBTD(n), is a BTD(n) with the additional condition that in each row there exist n cells, called a* factor*, which contain all* $2n$ *elements of* V.

Example 2.1 *An example of a FBTD(4). The pairs in each factor are underlined.*

$\underline{12}$	67	03	$\underline{70}$	$\underline{34}$	45	$\underline{56}$
35	$\underline{13}$	$\underline{57}$	$\underline{46}$	16	$\underline{20}$	24
47	$\underline{40}$	26	$\underline{15}$	$\underline{27}$	$\underline{36}$	10
$\underline{60}$	$\underline{25}$	$\underline{14}$	23	50	17	$\underline{37}$

Example 2.2 *An example of a FBTD(6) due to Lamken and Vanstone [8]. In this example A stands for 10 and B stands for 11. As before, the pairs in each factor are underlined.*

$B8$	$\underline{46}$	62	$\underline{A0}$	03	$\underline{B3}$	$\underline{91}$	17	$A5$	$\underline{58}$	$\underline{27}$
$\underline{69}$	$B9$	$\underline{57}$	73	$\underline{A1}$	14	$\underline{B4}$	$\underline{02}$	28	$A6$	$\underline{38}$
$A7$	$\underline{70}$	$B0$	$\underline{68}$	84	$\underline{A2}$	25	$\underline{B5}$	$\underline{13}$	39	$\underline{49}$
40	$A8$	$\underline{81}$	$B1$	$\underline{79}$	95	$\underline{A3}$	36	$\underline{B6}$	$\underline{42}$	$\underline{50}$
$\underline{53}$	51	$A9$	$\underline{92}$	$B2$	$\underline{80}$	06	$\underline{A4}$	47	$\underline{B7}$	$\underline{61}$
$\underline{12}$	23	$\underline{34}$	45	$\underline{56}$	67	$\underline{78}$	89	$\underline{90}$	01	$\underline{AB}$

Lamken and Vanstone prove the stronger result that an FBTD(n) exists for all values of $n \neq 2$. We begin by showing that an FBTD(n) exists for all odd n.

Theorem 2.1 *If n is odd, then an FBTD(n) exists.*

Proof: Let $n = 2k+1$ and define $V = \mathbb{Z}_n \times \{1,2\}$. For notational convenience we denote the element $(x,i) \in V$ by x_i. Thus $V = \{0_1, 0_2, 1_1, 1_2, \ldots, (2k)_1 (2k)_2\}$. We begin by constructing a $(2k+1) \times (4k+1)$ array. We will index the rows of the array by $0, \ldots, 2k$, the elements of $\mathbb{Z}_{2k+1}$, and the columns by $0, \ldots, 4k$, the elements of $\mathbb{Z}_{4k+1}$. We will assign pairs to each of the cells in row 0 and then we will obtain row i from row 0 by replacing each ordered pair (x_s, y_t) by $((x+i)_s, (y+i)_t)$. The array we obtain is displayed below, except that due to space limitations we display the transpose of the array instead.

	0	1	...	$2k$
0	$0_1, 0_2$	$1_1, 1_2$	...	$(2k)_1, (2k)_2$
1	$1_1, 2k_1$	$2_1, 0_1$	...	$0_1, (2k-1)_1$
2	$2_1, (2k-1)_1$	$3_1, (2k)_1$	...	$1_1, (2k-2)_1$
$\vdots$	$\vdots$	$\vdots$		$\vdots$
k	$k_1, (k+1)_1$	$(k+1)_1, (k+2)_1$	...	$(k-1)_1, (k)_1$
$k+1$	$1_2, 2k_2$	$2_2, 0_2$	...	$0_2, (2k-1)_2$
$\vdots$	$\vdots$	$\vdots$		$\vdots$
$2k$	$k_2, (k+1)_2$	$(k+1)_2, (k+2)_2$	...	$(k-1)_2, (k)_2$
$2k+1$	$1_1, 2k_2$	$2_1, 0_2$	...	$0_1, (2k-1)_2$
$\vdots$	$\vdots$	$\vdots$		$\vdots$
$3k$	$k_1, (k+1)_2$	$(k+1)_1, (k+2)_2$	...	$(k-1)_1, (k)_2$
$3k+1$	$1_2, 2k_1$	$2_2, 0_1$	...	$0_2, (2k-1)_1$
$\vdots$	$\vdots$	$\vdots$		$\vdots$
$4k$	$k_2, (k+1)_1$	$(k+1)_2, (k+2)_1$	...	$(k-1)_2, (k)_1$

First, we will check to ensure that every unordered pair of distinct elements of V appears in the array. With any unordered pair of distinct elements $\{x_i, y_j\} \subseteq V$, we associate two 3-tuples $(x - y \pmod{n}, i, j)$ and $(y - x \pmod{n}, j, i)$ which we call the *differences* of x_i and y_j. For notational convenience, we will denote these differences by $(x-y)_{ij}$ and $(y-x)_{ji}$, respectively, where it is to be understood from the context that $x - y$ and $y - x$ are elements of $\mathbb{Z}_n$. Since n is odd, the two differences of $\{x_i, y_j\} \subseteq V$ are distinct.

The following result is well known.

Lemma 2.1 *The pairs* $\{a_i, b_j\} \subset \mathbb{Z}_n \times \{i, j\}$ *and* $\{c_i, d_j\} \subset \mathbb{Z}_n \times \{i, j\}$ *have the same differences if and only if there exists an element* $x \in \mathbb{Z}_n$ *such that*

$$\begin{aligned} \{a+x, b+x\} &= \{c, d\} \quad \text{if} \quad i = j \\ (a+x, b+x) &= (c, d) \quad \text{if} \quad i \neq j; \end{aligned}$$

furthermore, this element x *is unique if and only if* $(a-b)_{ij} \neq (b-a)_{ji}$.

Observe that, for $i = 1, 2$, the 2k differences associated with the pairs

$$\{1_i, 2k_i\}, \{2_i, 2k-1_i\}, \ldots, \{k_i, k+1_i\}$$

from row 0 are all distinct and are precisely the elements of $(\mathbb{Z}_{2k+1} \setminus \{0\}) \times \{i\} \times \{i\}$. Observe also that the $4k+2$ differences associated with the pairs

$$\{0_1, 0_2\}, \{1_1, 2k_2\}, \{2_1, 2k-1_2\}, \ldots, \{2k_1, 1_2\}$$

from row 0 are all distinct and are precisely the elements of the set $(\mathbb{Z}_{2k+1} \times \{1\} \times \{2\}) \cup (\mathbb{Z}_{2k+1} \times \{2\} \times \{1\})$. Hence, by the above lemma and since row i is a translate of row 0, all the unordered pairs of distinct elements of V are contained in the array precisely once.

In the first row, every element of V appears twice except 0_1 and 0_2. Therefore, since every row is a translate of the first row, each row contains every element of V at most twice. Therefore the array satisfies the row condition of BTDs. Furthermore the pairs in columns $0, \ldots, 2k$ give a factor for each row.

Columns $0, 2k+1, 2k+2, \ldots, 4k$, each contain all the elements of V exactly once. However columns $1, \ldots, 2k$ contain some elements of V twice and therefore these columns violate the column condition on BTDs. However if we can switch pairs within their row to alter these columns so that they satisfy the column condition then we will not have affected the row condition or the fact that each row has a factor and we will have a FBTD.

Consider column i, where $1 \leq i \leq k$. For each pair $\{x_1, y_1\}$ in this column, the differences are $(-2i)_{11}$ and $(2i)_{11}$. Therefore this column contains the pairs

$$\{x_1, (x-2i)_1\}, \{(x-2i)_1, (x-4i)_1\}, \ldots, \{(x-(t-1)2i)_1, x_1\},$$

where t is the period of $-2i \pmod{2k+1}$. The period must divide $2k+1$ and therefore t must be odd. The column can be partitioned into $(2k+1)/t$ cosets each containing t pairs. We can similarly partition column $k+i$ and column $2k+i$ from sets two and three respectively. Therefore the coset

$$\{x, x-2i\}, \{x-2i, x-4i\}, \{x-4i, x-6i\} \ldots, \{x-2(t-1)i, x\}$$

appears in each set with its corresponding subscripts.

We will make the following exchanges. For each odd s, interchange

$$\{(x-2si)_1, (x-2(s+1)i)_1\} \text{ and } \{(x-2si)_2, (x-2(s+1)i)_2\}.$$

Also interchange

$$\{x_2, (x-2i)_2\} \text{ and } \{x_1, (x-2i)_2\}$$

as well as

$$\{(x-2(t-1)i)_1, x_1\} \text{ and } \{(x-2(t-1)i)_1, x_2\}$$

If we make these exchanges for every coset in column i, for all $1 \leq i \leq k$ then every column will contain every element of V exactly once. Since all our interchanges were within the same row we have then created a FBTD. □

We will illustrate the proof of the theorem by an example.

Example 2.3 *Let $n = 5$, thus $k = 2$. The original array constructed in the proof.*

	0	1	2	3	4	5	6	7	8
0	$0_1, 0_2$	$1_1, 4_1$	$2_1, 3_1$	$1_2, 4_2$	$2_2, 3_2$	$1_1, 4_2$	$2_1, 3_2,$	$1_2, 4_1$	$2_2, 3_1$
1	$1_1, 1_2$	$2_1, 0_1$	$3_1, 4_1$	$2_2, 0_2$	$3_2, 4_2$	$2_1, 0_2$	$3_1, 4_2,$	$2_2, 0_1$	$3_2, 4_1$
2	$2_1, 2_2$	$3_1, 1_1$	$4_1, 0_1$	$3_2, 1_2$	$4_2, 0_2$	$3_1, 1_2$	$4_1, 0_2,$	$3_2, 1_1$	$4_2, 0_1$
3	$3_1, 3_2$	$4_1, 2_1$	$0_1, 1_1$	$4_2, 2_2$	$0_2, 1_2$	$4_1, 2_2$	$0_1, 1_2,$	$4_2, 2_1$	$0_2, 1_1$
4	$4_1, 4_2$	$0_1, 3_1$	$1_1, 2_1$	$0_2, 3_2$	$1_2, 2_2$	$0_1, 3_2$	$1_1, 2_2,$	$0_2, 3_1$	$1_2, 2_1$

For $i = 1$ the period of $-2i$ is 5. Therefore the entire column forms the coset: {1,4}, {4,2}, {2,0}, {0,3}, {3,1}. Similarly the entire column forms the coset for $i = 2$: {2,3}, {3,4}, {4,0}, {0,1}, {1,2}. Therefore after all the interchanges we obtain the following array, which is our desired FBTD.

	0	1	2	3	4	5	6	7	8
0	$0_1, 0_2$	$1_1, 4_1$	$2_1, 3_1$	$1_1, 4_2$	$2_1, 3_2$	$1_2, 4_2$	$2_2, 3_2,$	$1_2, 4_1$	$2_2, 3_1$
1	$1_1, 1_2$	$2_1, 0_1$	$3_2, 4_2$	$2_2, 0_2$	$3_1, 4_1$	$2_1, 0_2$	$3_1, 4_2,$	$2_2, 0_1$	$3_2, 4_1$
2	$2_1, 2_2$	$3_1, 1_2$	$4_1, 0_1$	$3_2, 1_2$	$4_2, 0_2$	$3_1, 1_1$	$4_1, 0_2,$	$3_2, 1_1$	$4_2, 0_1$
3	$3_1, 3_2$	$4_2, 2_2$	$0_2, 1_2$	$4_1, 2_1$	$0_1, 1_1$	$4_1, 2_2$	$0_1, 1_2,$	$4_2, 2_1$	$0_2, 1_1$
4	$4_1, 4_2$	$0_2, 3_2$	$1_1, 2_2$	$0_1, 3_1$	$1_2, 2_2$	$0_1, 3_2$	$1_1, 2_1,$	$0_2, 3_1$	$1_2, 2_1$

The previous theorem proved the existence of FBTD(n) for odd n. Now we will prove the existence for almost all even n using a method of doubling. The proof of this theorem uses the idea of mutually orthogonal Latin squares.

Definition 2.2 *A* Latin square of side n *is an* $n \times n$ *array in which each cell contains a single element from an* n*-set* S*, such that each element of* S *occurs exactly once in each row and exactly once in each column.*

Definition 2.3 *Let* L_1 *and* L_2 *be two Latin squares of side* n*. Let* $L_i(j,k)$ *be the element in the* j*th row and* k*th column of* L_i*.* L_1 *and* L_2 *are* orthogonal *if* $L_1(a,b) = L_1(c,d)$ *and* $L_2(a,b) = L_2(c,d)$ *implies that* $a = c$ *and* $b = d$*. We refer to a set of Latin squares which are pairwise orthogonal as a set of* mutually orthogonal Latin squares*, MOLS.*

We will need the following result proved by Bose, Shrikhande and Parker in 1960 [1].

Lemma 2.2 *There exist two orthogonal Latin squares of side* n *for all* $n \neq 2$ *or 6.*

Theorem 2.2 *If an FBTD(n) exists,* $n > 3$*, then an FBTD(2n) also exists.*

Proof Let A be an FBTD(n) on the set $U = \{1, \ldots, 2n\}$.We obtain A_1 from A in the following way. If a is an element of the factor of its row then leave it as a, otherwise replace a by $\overline{a}$. We obtain A_2 from A in a similar way. If a is an element of the factor of its row then this time replace a by $\overline{a}$, otherwise leave it as a.

Let $B = \begin{array}{|c|}\hline A_1 \\ \hline A_2 \\ \hline \end{array}$ and let $V = U \cup \overline{U}$, where $\overline{U} = \{\overline{1}, \ldots, \overline{2n}\}$. B is a $2n \times (2n-1)$ array of pairs of elements from V. Since every element of U appears in each column of A, every element of V appears in every column of B. Also if an element of U appeared twice in a row of A then it must appear once in the factor for that row and once not in that factor. Therefore every element of V appears at most once in a row of B.

Since every pair of distinct elements of U appears in A, every pair of elements of the form $\{x, y\}$ or $\{\overline{x}, \overline{y}\}$, where $x \neq y$ must appear in B. To extend B to an FBTD(n) we need to add the pairs of the form $\{x, \overline{y}\}$.

Let C_1 and C_2 be two orthogonal Latin squares of side $2n$ on the elements of U and $\overline{U}$ respectively. Since $n > 3$ we know that two orthogonal Latin squares of side $2n$ exist by the theorem of Bose, Shrikhande and Parker mentioned previously. Let C be the array obtained by superimposing C_1 on C_2. C is the $2n \times 2n$ array for which cell (i, j) will contain the ordered pair (x, y), where x is the element in cell (i, j) of C_1 and y is the element in cell (i, j) of C_2. We will use the notation $C = C_1 \circ C_2$ to indicate that C is the superposition of C_1 and C_2. Every pair of the form $\{x, \overline{y}\}$ will appear in C and every element of V will appear exactly once in each row of C and exactly once in each column of C.

Now we let $D = \boxed{B \mid C}$ Thus D is an $2n \times (4n-1)$ array where every pair of distinct elements of V appears in a cell of D. Every element of V appears exactly once in each column and at most twice in any row. In each row, the pairs that originally belonged to C provide a factor for that row. Therefore D is an FBTD(n). □

Example 2.4 *We will construct an FBTD(8) from the FBTD(4) given in Example 2.1. We will use the following matrix C constructed from two orthogonal Latin squares of side 8.*

03	30	21	56	47	74	65	12
35	06	17	60	71	42	53	24
27	14	05	72	63	50	41	36
52	61	70	07	16	25	34	43
40	73	62	15	04	37	26	51
76	45	54	23	32	01	10	67
64	57	46	31	20	13	02	75
11	22	33	44	55	66	77	00

We obtain the following FBTD(8):

12	$\overline{67}$	$\overline{03}$	70	34	$\overline{45}$	56	$0\overline{3}$	$3\overline{0}$	$2\overline{1}$	$5\overline{6}$	$4\overline{7}$	$7\overline{4}$	$6\overline{5}$	$1\overline{2}$
$\overline{35}$	13	57	46	$\overline{16}$	20	$\overline{24}$	$3\overline{5}$	$0\overline{6}$	$1\overline{7}$	$6\overline{0}$	$7\overline{1}$	$4\overline{2}$	$5\overline{3}$	$2\overline{4}$
$\overline{47}$	40	$\overline{26}$	15	27	36	$\overline{10}$	$2\overline{7}$	$1\overline{4}$	$0\overline{5}$	$7\overline{2}$	$6\overline{3}$	$5\overline{0}$	$4\overline{1}$	$3\overline{6}$
60	25	14	$\overline{23}$	$\overline{50}$	$\overline{17}$	37	$5\overline{2}$	$6\overline{1}$	$7\overline{0}$	$0\overline{7}$	$1\overline{6}$	$2\overline{5}$	$3\overline{4}$	$4\overline{3}$
$\overline{12}$	67	03	$\overline{70}$	$\overline{34}$	45	$\overline{56}$	$4\overline{0}$	$7\overline{3}$	$6\overline{2}$	$1\overline{5}$	$0\overline{4}$	$3\overline{7}$	$2\overline{6}$	$5\overline{1}$
35	$\overline{13}$	$\overline{57}$	$\overline{46}$	16	$\overline{20}$	24	$7\overline{6}$	$4\overline{5}$	$5\overline{4}$	$2\overline{3}$	$3\overline{2}$	$0\overline{1}$	$1\overline{0}$	$6\overline{7}$
47	$\overline{40}$	26	$\overline{15}$	$\overline{27}$	$\overline{36}$	10	$6\overline{4}$	$5\overline{7}$	$4\overline{6}$	$3\overline{1}$	$2\overline{0}$	$1\overline{3}$	$0\overline{2}$	$7\overline{5}$
$\overline{60}$	$\overline{25}$	$\overline{14}$	23	50	17	$\overline{37}$	$1\overline{1}$	$2\overline{2}$	$3\overline{3}$	$4\overline{4}$	$5\overline{5}$	$6\overline{6}$	$7\overline{7}$	$0\overline{0}$

We are now ready to prove the existence of an FBTD(n) for all n except $n \neq 2$.

Theorem 2.3 *An FBTD(n) exists for all n except $n = 2$.*

Proof We begin by showing that a BTD(2) does not exist. A BTD(2) would have 2 rows and 3 columns. Let V be the set $\{1, 2, 3, 4\}$. Without loss of generality assume that 1 occurs twice in the first row. Since there are only two cells in each column, the element that does not share a cell with 1 in the first row, call this element a, must occur in the cell in the second row for these two columns. However in the remaining column, 1 must share a cell with a in the second row. Therefore a occurs three times in the second row and thus this array is not a BTD. Therefore a BTD(2) does not exist and therefore an FBTD(2) does not exist.

Now consider the factorization of n. If n is odd, then Theorem 2.1 implies the existence. If n is a power of 2 other than $n = 2$, then Example 2.1 and Theorem 2.2 imply the existence. If $n = 2^r m$ where $r \geq 1$, m is odd and $m > 3$ then Theorem 2.1 and Theorem 2.2 imply the existence. If $n = 2^r \cdot 3$, then Example 1.1 gives the existence for $r = 0$, Example 2.2 gives the existence for $r = 1$ and then Theorem 2.2 implies the existence for $r \geq 2$. Therefore an FBTD(n)exists for all $n \neq 2$. $\Box$

Chapter 3

Partitioned Balanced Tournament Design

3.1 Definitions and Connections to Other Designs

In every row of a BTD there are exactly two elements which do not appear twice in that row. We call these elements the *deficient* elements of the BTD. If for every row of the BTD, these two elements appear together in the same cell then we say that the BTD is *linked*. For some linked BTDs the columns can be permuted to obtain a partitioned BTD. We remind the reader that a *partitioned balanced tournament design* of order n, PBTD(n) is a BTD for which it is possible to permute the columns such that for every row the first n pairs form a factor and the last n pairs also form a factor.

The definition of a PBTD can also be explained using Howell designs.

Definition 3.1 *Let V be a set of $2n$ elements. A* Howell design of side s and order $2n$, H(s, $2n$) *is an $s \times s$ array in which each cell is either empty or contains an unordered pair of elements of V such that*

1. *each row and column is Latin (that is every element of V occurs exactly once in each column and row) and*

2. *every unordered pair of V occurs in at most one cell of the array*

It follows from the definition that $n \leq s \leq 2n - 1$.

If we can partition the columns of a BTD(n) into three sets C_1, C_2 and C_3 of sizes $n-1$, $n-1$, 1 respectively so that the columns of $C_1 \cup C_3$ form an H($n, 2n$) and the columns of $C_2 \cup C_3$ form an H($n, 2n$) then we have a PBTD(n).

Example 3.1 *We repeat the PBTD(5) given in Example 1.2. The partition of the columns is indicated.*

90	58	46	12	37	28	59	40	16
24	79	13	80	56	10	47	29	38
36	14	89	57	49	20	67	18	35
15	23	70	69	40	48	39	26	17
78	60	25	34	27	19	45	30	68

C_1 C_3 C_2

Since we can construct two H(n, $2n$) from a PBTD(n), we can also consider this construction in reverse and construct a PBTD(n) from two H(n, $2n$). However there is one condition we must impose on the Howell designs. They must be almost disjoint.

Definition 3.2 *Two Howell designs H($n, 2n$), H_1 and H_2 are* almost disjoint *if there exists one row/column of H_1 such that all the pairs of that row/column appear in a row/column of H_2 and there are no other pairs that are common to both H_1 and H_2.*

Therefore the row/column that is common to both designs becomes C_3 and the remaining rows/columns from the two designs become C_1 and C_2 respectively in the construction of the PBTD(n).

Example 3.2 *[13] An example of two almost disjoint Howell designs H(7,14) which can be used to construct a PBTD(7).*

$A_1 =$

$2\overline{3}$	$\alpha 4$	$3\overline{6}$	$5\overline{4}$	$\infty\overline{5}$	$6\overline{2}$	$1\overline{1}$
$\infty\overline{4}$	$3\overline{1}$	$5\overline{3}$	$1\overline{5}$	$\alpha 6$	$4\overline{6}$	$2\overline{2}$
$\alpha 5$	$\infty\overline{6}$	$6\overline{5}$	$4\overline{1}$	$1\overline{2}$	$2\overline{4}$	$3\overline{3}$
$3\overline{2}$	$2\overline{5}$	$\alpha\overline{1}$	$6\overline{3}$	$5\overline{6}$	$\infty 1$	$4\overline{4}$
$1\overline{6}$	$6\overline{4}$	$4\overline{2}$	$\infty 3$	$2\overline{1}$	$\alpha\overline{3}$	$5\overline{5}$
$4\overline{5}$	$1\overline{3}$	$\infty 2$	$\alpha\overline{2}$	$3\overline{4}$	$5\overline{1}$	$6\overline{6}$
$6\overline{1}$	$5\overline{2}$	$1\overline{4}$	$2\overline{6}$	$4\overline{3}$	$3\overline{5}$	$\alpha\infty$

$A_2 =$

$\overline{34}$	34	$\infty\overline{2}$	$\alpha 2$	56	$\overline{56}$	$1\overline{1}$
$\alpha 3$	$\infty\overline{3}$	$\overline{46}$	46	$\overline{15}$	15	$2\overline{2}$
45	$\overline{45}$	26	$\overline{26}$	$\alpha 1$	$\infty\overline{1}$	$3\overline{3}$
$\overline{16}$	16	$\alpha\overline{5}$	$\infty 5$	23	$\overline{23}$	$4\overline{4}$
$\infty 6$	$\alpha\overline{6}$	$\overline{13}$	13	$\overline{24}$	24	$\overline{5}5$
12	$\overline{12}$	35	$\overline{35}$	$\infty 4$	$\alpha\overline{4}$	$6\overline{6}$
$\overline{25}$	25	14	$\overline{14}$	$\overline{36}$	36	$\alpha\infty$

In addition to Howell designs, PBTDs are also connected to room squares

Definition 3.3 *Let V be a set of size $2n+2$. A* Room square *of side $2n+1$, RS($2n+1$), is a $(2n+1)\times(2n+1)$ array R in which each cell is either empty or contains an unordered* *pair of elements chosen from V, such that each column and row of R is Latin and every pair of elements of V occurs in exactly one cell of R.*

Every row or column of R contains n empty cells and $n+1$ filled cells. Therefore a $t \times t$ empty subarray of R must have $t \leq n$.

Definition 3.4 *A Room square of side $2n+2$ which contains an empty $n \times n$ subarray of cells is called a maximum empty subarray Room square, MESRS($2n+2$).*

Theorem 3.1 *[16] A PBTD($n+1$) exists if and only if there exists a MESRS($2n+1$).*

Proof Let R be a MESRS($2n+1$) defined on V, where $|V| = 2n+2$. We can permute rows and columns of R such that the $n \times n$ empty subarray T appears in the lower right corner of R. We can continue to permute the rows and columns of R such that it has the form $R = \begin{array}{|c|c|}\hline D & C_1 \\ \hline C_2 & T \\ \hline\end{array}$ where D is a diagonal subarray and C_1 and C_2 are completely filled subarrays.

Every element of V must appear $2n+1$ times in R. Each element appears once in each column of C_1 and once in each row of C_2. Thus each element of V appears once on the diagonal of D. Let C_3 be $(n+1) \times 1$ array which is the projection of the diagonal of D into a single column. The $(n+1) \times (n+1)$ array $\begin{array}{|c|c|}\hline C_3 & C_1 \\ \hline\end{array}$ is an H($n+1$,$2n+2$) and the $(n+1) \times (n+1)$ array $\begin{array}{|c|}\hline C_3^T \\ \hline C_2 \\ \hline\end{array}$ is also an H($n+1$,$2n+2$). The only pairs that these two Howell designs have in common are those of C_3 and thus $\begin{array}{|c|c|c|}\hline C_1 & C_2^T & C_3 \\ \hline\end{array}$ is a PBTD($n+1$).

The construction that we used to transform the MESRS($2n+1$) into a PBTD($n+1$) is reversible and thus a MESRS($2n+1$) exists if and only if a PBTD($n+1$) exists. □

12	34	56	X	X
35				
46				
X			36	X
X			X	45

12	34	56	X	X
35	X	X	14	
46	X	X		
X	15	24	36	X
X	26	13	X	45

Figure 3.1: An X indicates an empty cell

3.2 The Nonexistence of PBTD(n) for $n = 2, 3, 4$

Since a PBTD(n) is a special case of a BTD(n), a PBTD(2) does not exist since a BTD(2) does not exist. In the previous section we pointed out the connection between PBTDs and MESRSs. A PBTD(3) exists if and only if a MESRS(5) exists. A MESRS(5) is a special case of an RS(5). We will show that a PBTD(3) does not exist by showing that an RS(5) does not exist.

Theorem 3.2 *[14] An RS(5) does not exist.*

Proof Let $V = \{1, \ldots, 6\}$. We can permute columns of a Room square and thus we can assume that the first three cells of the first row are filled with the pairs 12, 34 and 56. Similarly we can permute the rows of a Room square such that the second and third cells of the first column are 35 and 46. This is equivalent to 36 and 45 by permuting 5 and 6. See the first array in Figure 3.1.

We now consider the placement of the pair 36. It must appear in the 2×2 subarray in the lower right hand corner of the array. By permuting rows and columns we can assume that it appears in cell (4,4). The pair 45 must also appear in the 2×2 subarray in the lower right hand corner but it cannot appear in cells (4,5) or (5,4) since this would require 12 to appear in row 4 or column 4. Therefore 45 must appear in cell (5,5).

Cell (4,2) must be filled with a pair and 15 and 25 are the only possibilities. These pairs are equivalent under the permutation of 1 and 2. Thus assume that (4,2) contains

15, and this implies that 24, 26, and 13 appear in cells (4,3), (5,2) and (5,3) respectively as shown in the second array in Figure 3.1. Therefore cells (2,2), (2,3), (3,2) and (3,3) must be empty.

The only place left for the pair 14 is cell (2,4) but this placement implies 26 must appear in cell (2,5) but 26 already appears in cell (5,2). Therefore an RS(5) does not exist and thus a PBTD(3) does not exist. □.

Theorem 3.3 *[16] A PBTD(4) does not exist*

Proof If a PBTD(4) exists then there exists two H(4,8), H_1 and H_2 such that there is one row/column of H_1 which contains the same pairs as one row/column of H_2 but no other pair occurs in both H_1 and H_2.

The graph $G(H)$ of a Howell design H is the graph on vertex set S, the symbol set of the design, whose edges are the pairs occurring in the cells of H. Thus we need to find two 4-regular graphs on eight vertices, G_1 and G_2, corresponding to H_1 and H_2 respectively, such that $G_1 \cup G_2 = K_8$ and $G_1 \cap G_2 = M$ where M is a perfect matching of both graphs.

Figure 3.2 contains the six, 4-regular graphs on eight vertices. First we need to check if any of these correspond to an H(4,8). Let us begin with the first graph. By rearranging the rows and columns of the design we can assume that the pairs 12, 13, 15, 17 appear in the cells along the diagonal. Now consider placing the pair 35. It can be placed in cell (1,4) or cell (4,1). Since the transpose of a Howell design is also a Howell design and so far the only pairs we have placed are on the diagonal we can assume that 35 is placed in

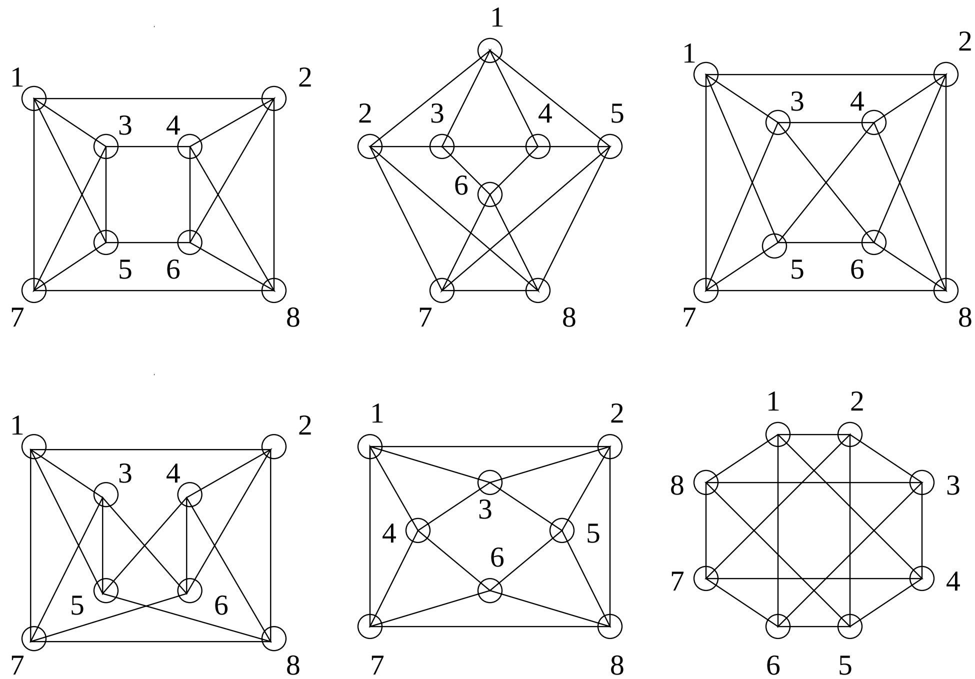

Figure 3.2: The six non-isomorphic 4-regular graphs on eight vertices

cell (1,4). This placement implies that the pair 57 must appear in cell (1,2).

12	57		
	13		
		15	
35			17

However this placement implies that the pair 37 must appear in cell (3,3) which is already filled with the pair 15. Therefore this graph does not correspond to a Howell design.

With similar arguments, it is possible to show that the only one of the six graphs that corresponds to an H(4,8) is the last one which is $K_{4,4}$. A Howell design corresponding to

$K_{4,4}$ is given below.

12	56	78	34
58	14	23	67
47	38	16	25
36	27	45	18

The complement of $K_{4,4}$ is two copies of K_4. K_4 has K_3 as a subgraph and $K_{4,4}$ has no odd cycles. Therefore it is impossible to find a graph which is isomorphic to $K_{4,4}$ which has K_4 as a subgraph. Therefore it is impossible to find two 4-regular graphs on eight vertices, G_1 and G_2, corresponding to H_1 and H_2 respectively, such that $G_1 \cup G_2 = K_8$ and $G_1 \cap G_2 = M$ where M is a perfect matching of both graphs. Therefore a PBTD(4) does not exist. □

3.3 Existence of PBTD(n) for $n \equiv 1 \pmod 4$, $n \geq 5$

We will consider the existence of PBTD(n) in three cases, $n \equiv 1 \pmod 4$, $n \equiv 3 \pmod 4$ and finally $n \equiv 0 \pmod 2$. Many of the constructions given make use of other combinatorial designs. In most cases we will simply state the relevant existence results and refer the reader to the literature for the proofs. We begin with a recursive construction that makes use of MOLS which were defined in Definition 2.2.

Theorem 3.4 *[16] Suppose there exist PBTD(m) and PBTD(n) and a pair of MOLS of side n. Then there exists a PBTD(mn).*

Proof Let A be a PBTD(m) defined on the set $M = \{1, 2, \ldots, 2m\}$. We order the columns of A so that the first m columns are an H($m, 2m$) and the last m columns are an H($m, 2m$). Therefore the mth column is the column of deficient pairs. We also order each pair of A arbitrarily.

Let L_1 and L_2 be a pair of MOLS of order n both defined on the set $N = \{1, 2, \ldots, n\}$. Let L be the array obtained by superimposing L_1 on L_2, $L = L_1 \circ L_2$. We define $L_{u,v}$ to be the array obtained from L by replacing each ordered pair (a, b) in L by the ordered pair (a_u, b_v).

Let $B_{u,v}$ be a PBTD(n) defined on the set $\{1_u, 2_u, \ldots, n_u, 1_v, 2_v, \ldots, n_v\}$. Again we order the columns of $B_{u,v}$ so that the first n columns are an H$(n, 2n)$ and the last n columns are an H$(n, 2n)$. Therefore the nth column is the column of deficient pairs.

Replace every pair (u, v) of A which is not a deficient pair by the array $L_{u,v}$ and replace every pair of A which is deficient by the array $B_{u,v}$. Call the resulting array C. We claim that C is a PBTD(mn).

Clearly C is defined on the set $\{1_i, 2_i, \ldots, n_i | 1 \leq i \leq 2m\}$, and this is a set of cardinality $2mn$. C has mn rows and $(2m - 2)n + 2n - 1 = 2mn - 1$ columns. Thus C has the correct dimensions.

Consider the unordered pair of distinct elements $\{x_u, y_v\}$. If $u = v$ then $x \neq y$ since we want a pair of distinct elements. Every element of M appears in the mth column of A since every column of A is Latin. Let w be the element that appears with u in the mth columns of A. Therefore there is a subarray of C, namely $B_{u,w}$ which contains all pairs of distinct elements from the set $\{1_u, \ldots, n_u, 1_w, \ldots, n_w\}$. Thus the unordered pair $\{x_u, y_u\}$ appears in C.

If $u \neq v$ and (u, v) is a deficient pair then there is a subarray of C, namely $B_{u,v}$ which contains all pairs of distinct elements from the set $\{1_u, \ldots, n_u,\ 1_v, \ldots, n_v\}$. Thus the unordered pair $\{x_u, y_v\}$ appears in C.

If $u \neq v$ and (u, v) is not a deficient pair then the unordered pair $\{x_u, y_v\}$ appears in the subarray of C, $L_{u,v}$. Thus every pair of distinct elements from

$$\big\{x_i | x \in \{1, 2, \dots , n\}, i \in \{1, 2, \dots , 2m\}\big\}$$

appears in C.

We will verify that the columns of C are Latin. Consider the element x_u. Since the columns of A are Latin, the element u appears in every column. Thus for every column of C there is a part of this column which is either a column of the subarray $L_{u,v}$ (or $L_{v,u}$) or a column of the subarray $B_{u,v}$ (or $B_{v,u}$). Every column of $L_{u,v}$ and $B_{u,v}$ contains the element x_u. Therefore x_u appears in every column of C and thus the columns of C are Latin.

Now we will verify that the first mn columns of C are an H(mn, $2mn$) and that the last mn columns are an H(mn, $2mn$) and thus the mnth column of C is the column of deficient pairs.

We have already verified that the columns of C are Latin. Now for each row we need to verify that every element appears once in the first mn columns and once in the last mn columns.

Consider any row r of C and any element $x_u \in \{a_i \mid a \in \{1, 2, \dots , n\}, i \in \{1, 2, \dots 2m\}\}$. In the corresponding row of A, there is one pair, say $\{u, v\}$, containing u in the first m columns because A is a PBTD(m). If $\{u, v\}$ is the deficient pair of this row of A, then element x_u occurs once in the first n columns of each row of $B_{u,v}$ because B is a PBTD(n). Hence x_u is in a cell of the first mn columns of row r of C. If $\{u, v\}$ is not the deficient pair of this row of A, then element x_u occurs once in each row of $L_{u,v}$, and hence x_u is in a cell of the first mn columns of row r of C. Since this is true for every element $x_u \in \{a_i \mid a \in \{1, 2, \dots , n\}, i \in \{1, 2, \dots 2m\}\}$, every element occurs in precisely one cell in the first mn columns of row r.

Similarly, every element occurs in precisely one cell in the last mn columns of row r. Thus C satisfies the row property of a PBTD(mn). Therefore C is a PBTD(mn). □

For the next construction we need some more definitions.

Definition 3.5 *Let V be a set of v elements. Let $G_1, G_2, \dots, G_m$ be a partition of V into m sets. A $\{G_1, G_2, \dots, G_m\}$-*frame *F with subset size k, index λ, and Latinicity μ is a square array of side v which satisfies the properties indicated below. We index the rows and columns of F by the elements of V.*

1. *Each cell is either empty or contains a k-subset of V.*

2. *Let F_i be the subsquare of F indexed by the elements of G_i. F_i is empty for $i = 1, 2, \dots, m$.*

3. *Let $x \in G_i$. Row x of F contains each element of $V - G_i$ μ times and column x of F contains each element of $V - G_i$ μ times.*

4. *Let $x \in G_i$ and $y \in G_j$. If $i = j$ then x and y never appear together in the same cell. If $i \neq j$ then x and y appear together in λ cells.*

Definition 3.6 *Let V be a set of v elements. Let $G_1, G_2, \dots, G_m$ be a partition of V into m sets. Let F be a $\{G_1, G_2, \dots, G_m\}$-frame. If there is a $\{G_1, G_2, \dots, G_m\}$-frame H with subset size k, index λ and latinicity μ such that if cell (a, b) of H is non-empty then cell (a, b) of F is empty, then H is called a* complement *of F and denoted $\overline{F}$. If a complement of F exists, we call F a* complementary *$\{G_1, G_2, \dots, G_m\}$-*frame. *A complementary $\{G_1, G_2, \dots, G_m\}$-frame is said to be* skew *if at most one of the cells (i, j) and (j, i) is nonempty, where $i \neq j$.*

The *type* of a $\{G_1, G_2, \ldots, G_m\}$-frame is the multiset $\{|G_1|, |G_2|, \ldots, |G_m|\}$. We will say that a frame has type $t_1^{u_1} t_2^{u_2} \ldots t_s^{u_s}$ if there are u_i G_js of cardinality t_i where $1 \leq i \leq s$. Since all the frames we deal with have $\mu = \lambda = 1$ and $k = 2$, we will simply denote a frame by its type.

Many of the constructions that we will describe involve creating an array, F, by superimposing a complementary $\{G_1, G_2, \ldots, G_m\}$-frame, say F_1 and its complement, F_2. Let F be the superposition of F_1 and F_2, $F = F_1 \circ F_2$. Let $x \in G_i$. Row x of F_1 contains each element of $V - G_i$ exactly once. Since each filled cell of F_1 contains a pair of elements, thus exactly half of the cells indexed by $V - G_i$ are filled. Similarly exactly half of the cells indexed by $V - G_i$ in column x are filled. Since F_2 is the complement of F_1, every cell of F contains a pair of elements, except for those cells indexed by (x, y), where $x, y \in G_i$ for some i.

This construction also uses mutually orthogonal partitioned incomplete Latin squares.

Definition 3.7 *Let $P = \{S_1, S_2, \ldots, S_m\}$ be a partition of a set S where $m \geq 2$. A* partitioned incomplete Latin square *having partition P is an $|S| \times |S|$ array L, indexed by the elements of S that satisfies the following properties.*

1. *A cell of L either contains an element of S or is empty.*

2. *The subarrays indexed by $S_i \times S_i$ are empty for $1 \leq i \leq m$.*

3. *Let $j \in S_i$. Row j of L contains every element of $S - S_i$ exactly once and column j of L contains every element of $S - S_i$ exactly once.*

The type *of L is the multiset $\{|S_i|, |S_i|, \ldots, |S_m|\}$. If there are u_i $S_j s$ of cardinality t_i, $1 \leq i \leq k$, we say that L has type $t_1^{u_1} t_2^{u_2} \ldots t_k^{u_k}$.*

Definition 3.8 *Let L and M be a pair of partitioned incomplete Latin squares with partition P. We say that L and M are* orthogonal partitioned incomplete Latin squares

(OPILS) if the array formed by the superposition of L and M contains every ordered pair in $S \times S - \bigcup_{i=1}^{m}(S_i \times S_i)$ exactly once. A set of m partitioned incomplete Latin squares with partition P is called a set of m mutually orthogonal partitioned incomplete Latin squares of type $\{|S_i|, |S_i|, \ldots, |S_m|\}$, *if each pair of distinct squares is orthogonal.*

We now have all the necessary ingredients for the next construction

Theorem 3.5 *[9] If there exists a complementary $\{G_1, G_2, \ldots, G_m\}$-frame ($m \geq 2$), a pair of OPILS with partition $\{G_1, G_2, \ldots, G_m\}$ and PBTD($|G_i| + 1$) for $1 \leq i \leq m$ then there is a PBTD($(\sum_{i=1}^{m} |G_i|) + 1$).*

Proof Let $V = \{v_1, \ldots, v_n\}$ and $\overline{V} = \{\overline{v}_1, \ldots, \overline{v}_n\}$ with the following bijection, $f : V \to \overline{V}$, between the two sets: $f(v_i) = \overline{v}_i$. Let $\{G_1, G_2, \ldots, G_m\}$ be a partition of V. Let $\{H_1, H_2, \ldots, H_m\}$ be the partition of $\overline{V}$ obtained by applying the bijection to the partition of V. Thus $H_i = \{\overline{v}_j \mid v_j \in G_i\}$ for $1 \leq i \leq m$.

Let F_1 be a complementary $\{G_1, G_2, \ldots, G_m\}$-frame defined on V. Let F_2 be the complement of F_1. Let F_3 be the array obtained by applying the bijection f to all the elements in the cells of F_2. Let F be the array obtained by superimposing F_1 on F_3, $F = F_1 \circ F_3$.

Let L_1 and L_2 be a pair of OPILS with partition $\{G_1, G_2, \ldots, G_m\}$. Let L_3 be the array obtained by applying the bijection f to every element in the cells of L_2. Let L be the superposition of L_1 on L_3, $L = L_1 \circ L_3$. Like F, every cell of L contains a pair of elements, except for those cells indexed by (x, y), where $x, y \in G_i$ for some i.

Let B_i be a PBTD($|G_i| + 1$) defined on $V_i' = G_i \bigcup H_i \bigcup \{\alpha, \infty\}$. We write B_i in the following form.

$B_i =$

D_i	E_i	C_i
R_{i1}	R_{i2}	$\alpha\infty$

Therefore the subarray $\begin{array}{|c|c|}\hline D_i & C_i \\ \hline R_{i1} & \alpha\infty \\ \hline \end{array}$ is an H($|G_i|+1, 2|G_i|+2$) and the subarray $\begin{array}{|c|c|}\hline E_i & C_i \\ \hline R_{i2} & \alpha\infty \\ \hline \end{array}$ is an H($|G_i| + 1, 2|G_i| + 2$).

Now we will construct a PBTD($(\sum_{i=1}^{m} |G_i|) + 1$) from F, L and $B_1, B_2, \ldots, B_m$. Essentially we will fill in the empty subarrays in F and L with subarrays from the B_is and then add an additional row and column.

$$B = \begin{array}{|cccc|cccc|c|}\hline D_1 & & & & E_1 & & & & C_1 \\ & D_2 & F & & & E_2 & L & & C_2 \\ & & \ddots & & & & \ddots & & \vdots \\ & & & D_m & & & & E_m & C_m \\ \hline R_{11} & R_{21} & \ldots & R_{m1} & R_{12} & R_{22} & \ldots & R_{m2} & \alpha\infty \\ \hline \end{array}$$

The array B has dimensions $(n + 1) \times (2n + 1)$ and it contains elements from the set $V' = V \bigcup \overline{V} \bigcup \{\alpha, \infty\}$ which is a set of cardinality $2v + 2$.

Consider any unordered pair of elements $\{x, y\}$, $x \neq y$ from V'. If $x, y \in V_i'$ for some i then the pair $\{x, y\}$ occurs in one of the subarrays that were part of B_i. If x and y do not both belong to the same V_i', but both x and y are in V or both x and y are in $\overline{W}$ then the pair $\{x, y\}$ appears in F. If x and y do not both occur in the same V_i', but one of them is in V and the other is in $\overline{V}$ then the pair $\{x, y\}$ appears in L. Thus every unordered pair of distinct elements from V' appears in B.

Consider any of the first n columns of B. Let $x \in V'$. This column passes through the subarray D_i for some i and if $x \in V_i'$ then x appears in the part of the column that belongs to D_i or in the pair R_{i1}. If $x \notin V_i'$ then x appears in the part of the column that belongs to F since F is the superposition of a complementary frame and its complement. Thus this column is Latin.

Consider any of the last $n+1$ columns of B except for the very last column. Again let $x \in V'$. This column passes through the subarray E_i for some i and if $x \in V_i'$ then x appears in the part of the column that belongs to E_i or in the pair R_{i2}. If $x \notin V_i'$ then x appears in the part of the column that belongs to L since L is the superposition of two OPILSs. Thus this column is Latin.

The union of the pairs of the last column is $(\bigcup_{i=1}^{m} C_i) \bigcup \{\alpha, \infty\} = V'$ and thus the last column is Latin.

Consider any of the rows of B other than the last row. This row passes through the subarray D_i and the subarray E_i for some i. Let $x \in V'$. If $x \notin V_i'$ then x appears in this row in a pair that belongs to F and in a pair that belongs to L. If $x \in V_i'$ then x appears in this row in a pair that belongs to D_i and in a pair that belongs to E_i unless it was a deficient element for the row of B_i that this row corresponds to and in this case x appears only once in this row, in the pair of C_i. Also $(\bigcup_{i=1}^{m} R_{i1}) \bigcup \{\alpha, \infty\} = (\bigcup_{i=1}^{m} R_{i2}) \bigcup \{\alpha, \infty\} = V'$. Therefore the first n columns of B along with the last column are an $\mathrm{H}(n+1, 2n+2)$ and the last $n+1$ columns of B are an $\mathrm{H}(n+1, 2n+2)$.

Therefore B is a PBTD$(\sum_{i=1}^{m} |G_i| + 1)$. $\square$

We need one more construction to complete the case for PBTD(n) where $n \equiv 1 \pmod 4$. It is used to prove that a PBTD(13) exists.

Theorem 3.6 *[11] Let* $n \equiv 0 \pmod 2$*. If there exists a complementary* 2^n *frame and a pair of OPILS of type* 2^n*, then there is a PBTD(*$2n+1$*).*

Proof Let $V = \{u_i, v_i \mid i = 1, 2, \ldots, n\}$ and let $\overline{V} = \{\overline{u}_i, \overline{v}_i, \mid i = 1, 2, \ldots, n\}$. Let $f : V \to \overline{V}$ be the following bijection: $f(v_i) = \overline{v}_i, f(u_i) = \overline{u}_i$. Let $G_i = \{u_i, v_i\}$..

Let F_1 be a complementary $\{G_1, G_2, \ldots, G_n\}$-frame defined on V. Thus F_1 is of type 2^n. Let F_2 be a complement of F_1. Let F_3 be array obtained by applying the bijection f to every element in the cells of F_2. Thus F_3 is also of type 2^n. Let F be the array obtained by superimposing F_1 on F_3, $F = F_1 \circ F_3$.

Let L_1 and L_2 be OPILS of type 2^n defined on V with partition

$$\big\{\{u_1, u_2\}, \{v_1, v_2\}, \dots, \{u_{n-1}, u_n\}, \{v_{n-1}, v_n\}\big\}.$$

Let L_3 be the array obtained by applying the bijection f to every element in the cells of L_2. Therefore L_3 is partitioned incomplete Latin square defined on $\overline{V}$ with partition

$$\big\{\{\overline{u}_1, \overline{u}_2\}, \{\overline{v}_1, \overline{v}_2\}, \dots, \{\overline{u}_{n-1}, \overline{u}_n\}, \{\overline{v}_{n-1}, \overline{v}_n\}\big\}.$$

Let L be the array obtained by superimposing L_1 on L_3, $L = L_1 \circ L_3$.

We are going to fill the empty subarrays in F and L with small arrays. Let $A_i = \begin{array}{|c|c|}\hline \alpha\overline{u}_i & \infty u_i \\ \hline \infty\overline{v}_i & \alpha v_i \\ \hline\end{array}$, $C_i = \begin{array}{|c|}\hline \overline{v}_i v_i \\ \hline \overline{u}_i u_i \\ \hline\end{array}$ and $R_i = \begin{array}{|c|c|}\hline u_i v_i & \overline{u}_i\overline{v}_i \\ \hline\end{array}$ for $i = 1, 2, \dots, n$.

For $i = 1, 3, \dots, n-1$, $i \equiv 1 \pmod 2$.

Let $B_i = \begin{array}{|c|c|}\hline \alpha u_i & \infty\overline{u}_i \\ \hline \infty\overline{u}_{i+1} & \alpha u_{i+1} \\ \hline\end{array}$, $C'_i = \begin{array}{|c|}\hline \overline{u}_{i+1} u_{i+1} \\ \hline u_i\overline{u}_i \\ \hline\end{array}$ and $R'_i = \begin{array}{|c|c|}\hline \overline{u}_i u_{i+1} & u_i\overline{u}_{i+1} \\ \hline\end{array}$.

For $i = 2, 4, \dots, n$, $i \equiv 0 \pmod 2$.

Let $B_i = \begin{array}{|c|c|}\hline \alpha\overline{v}_{i-1} & \infty v_{i-1} \\ \hline \infty v_i & \alpha\overline{v}_i \\ \hline\end{array}$, $C'_i = \begin{array}{|c|}\hline v_i\overline{v}_i \\ \hline v_{i-1}\overline{v}_{i-1} \\ \hline\end{array}$ and $R'_i = \begin{array}{|c|c|}\hline \overline{v}_i v_{i-1} & \overline{v}_{i-1} v_i \\ \hline\end{array}$.

We fill in the empty subarrays in F with the A_i. Using the C_i and R_i, we add an additional row and column to obtain H_1, a $(2n+1) \times (2n+1)$ array:

$$H_1 = \begin{array}{|cccc|c|}\hline A_1 & & & & C_1 \\ & A_2 & & & C_2 \\ & F & \ddots & & \vdots \\ & & & A_n & C_n \\ \hline R_1 & R_2 & \dots & R_n & \alpha\infty \\ \hline\end{array}$$

We fill in the empty subarrays of L with the B_i. Using the C_i' and R_i' we add an additional row and column to obtain H_2, a $(2n+1) \times (2n+1)$ array:

$$H_2 = \begin{array}{|cccc|c|}\hline B_1 & & & & C_1' \\ & B_2 & & & C_2' \\ & L & \ddots & & \vdots \\ & & & B_n & C_n' \\ \hline R_1' & R_2' & \dots & R_n' & \alpha\infty \\ \hline \end{array}$$

It is clear that both H_1 and H_2 are $\text{H}(2n+1, 4n+2)$ defined on the set $V' = V \bigcup \overline{V} \bigcup \{\alpha, \infty\}$. The final column of H_2 is filled with the same pairs as the final column of H_1 so we can permute the rows of H_2 so that its final column is identical to the final column of H_1. Let this final column be $\begin{array}{|c|}\hline C \\ \hline \alpha\infty \\ \hline \end{array}$. Therefore we can write H_1 and H_2 in the following form:

$$H_1 = \begin{array}{|c|c|}\hline K_1 & C \\ \hline R & \alpha\infty \\ \hline \end{array} \qquad H_1 = \begin{array}{|c|c|}\hline K_2 & C \\ \hline R' & \alpha\infty \\ \hline \end{array}$$

Therefore if we can show that every distinct pair of element of V' appears in either H_1 or H_2, then the following array, B, is a PBTD$(2n+1)$.

$$B = \begin{array}{|c|c|c|}\hline K_1 & K_2 & C \\ \hline R & R' & \alpha\infty \\ \hline \end{array}$$

Clearly the pair $\{\alpha, \infty\}$ appears in B. The elements α and ∞ appear in a pair with every element of $V \bigcup \overline{V}$ in the A_i and B_i subarrays.

All the unordered pairs of the form $\{u_i, v_j\}$, $\{\overline{u}_i, \overline{v}_j\}$, $\{u_i, u_j\}$, $\{\overline{u}_i, \overline{u}_j\}$, $\{v_i, v_j\}$ and $\{\overline{v}_i, \overline{v}_j\}$, where $i \neq j$ appear in F. All pairs of the form $\{u_i, v_i\}$, and $\{\overline{u}_i, \overline{v}_i\}$ appear in the R_i subarrays.

All the pairs of the form $\{u_i, \overline{u}_i\}$, and $\{v_i, \overline{v}_i\}$ appear in the C_i subarrays. All pairs of the form $\{u_i, \overline{v}_i\}$, $\{\overline{u}_i, v_i\}$, $\{u_i, \overline{v}_j\}$ and $\{\overline{u}_i, v_j\}$ where $i \neq j$ appear in L. The remaining pairs of the form $\{u_i, \overline{u}_j\}$ and $\{v_i, \overline{v}_j\}$ where $i \neq j$ appear in L and the R'_i subarrays.

Thus every distinct pair of elements of V' appears in B and thus B is a PBTD($2n+1$).

□

We now have all the necessary constructions to prove the existence of PBTD(n) for $n \equiv 1 \pmod 4$, $n \neq 9$. However, to use these constructions, we need the existence of some complementary frames and some pairs of OPILS. We state here the necessary existence results and refer the reader to the literature for their proofs.

We start with the necessary results for complementary frames.

Lemma 3.1 *[9] For n a positive integer, $n \geq 4$, there exists a complementary 4^n frame.*

Lemma 3.2 *[9] Let q be a prime power, $q \equiv 1 \pmod 4$. There exists a complementary 2^{q+1} frame.*

We also need the following result for OPILS.

Lemma 3.3 *[17] For any $h \geq 2$, there exists a pair of OPILS of type h^n if and only if $n \geq 4$.*

We now have all the necessary pieces to prove the existence of PBTD($4n+1$), with one possible exception.

Theorem 3.7 *[9, 11] Let n be a positive integer. Then there exists a PBTD($4n+1$), except possibly for $n = 2$.*

Proof Since $5 \equiv 1 \pmod 4$ is a prime power, thus by Lemma 3.2 there is a complementary 2^6 frame. By Lemma 3.3 there exists a pair of OPILS of type 2^6. Therefore, by Theorem 3.6, a PBTD(13) exists.

By Lemma 3.1 and Lemma 3.3 there exist complementary 4^n frames for $n \geq 4$ and a pair of OPILS of type 4^n for $n \geq 4$. Example 3.1 is a PBTD(5). Therefore we can use Theorem 3.5 to construct a PBTD($4n+1$) for $n \geq 4$. □

3.4 Existence of PBTD(n) for $n \equiv 3 \pmod 4$, $n \geq 7$

The next case we will consider is the existence of PBTD(n) where $n \equiv 3 \pmod 4$, $n \geq 7$. To handle this case we will need three more constructions.

Theorem 3.8 *[11] If there exists a complementary 2^n frame, a pair of OPILS of type $t_1^{w_1} t_2^{w_2} \ldots t_k^{w_k}$ where $\sum_{i=1}^{k} w_i t_i = 2n$ and $w_i \equiv 0 \pmod 2$ for all i, and a pair of orthogonal Latin squares of order $t_i + 1$ for $i = 1, 2, \ldots, k$, then there is a PBTD($2n+1$).*

Proof For $1 \leq s \leq k$ and $\sum_{i=1}^{s-1} \frac{w_i}{2} + 1 \leq j \leq \sum_{i=1}^{s} \frac{w_i}{2}$, let $U^j = \{u_1^j, u_2^j, \ldots, u_{t_s}^j\}$, $V^j = \{v_1^j, v_2^j, \ldots, v_{t_s}^j\}$, $\overline{U}^j = \{\overline{u}_1^j, \overline{u}_2^j, \ldots, \overline{u}_{t_s}^j\}$ and $\overline{V}^j = \{\overline{v}_1^j, \overline{v}_2^j, \ldots, \overline{v}_{t_s}^j\}$.

Let $q = \sum_{i=1}^{k} \frac{w_i}{2}$. Let $V = \bigcup_{i=1}^{q} (U^i \cup V^i)$ and let $\overline{V} = \bigcup_{i=1}^{q} (\overline{U}^i \cup \overline{V}^i)$. Let $f : V \to \overline{V}$ be the bijection defined by $f(x) = \overline{x}$. For $1 \leq j \leq q$, and $1 \leq i \leq t_s$, where $t_s = |U^j|$, let $G_i^j = \{u_i^j, v_i^j\}$ and let $\overline{G}_i^j = \{\overline{u}_i^j, \overline{v}_i^j\}$.

Let F_1 be a complementary $\{G_1^1, G_2^1, \ldots, G_{t_1}^1, \ldots, G_1^q, G_2^q, \ldots, G_{t_k}^q\}$-frame defined on V. F_1 is a frame of type 2^n. Let F_2 be a complement of F_1. Let F_3 be the array obtained by applying the bijection f to every element in the cells of F_2. Thus F_3 is a $\{\overline{G}_1^1, \overline{G}_2^1, \ldots, \overline{G}_{t_1}^1, \ldots, \overline{G}_1^q, \overline{G}_2^q, \ldots, \overline{G}_{t_k}^q\}$-frame of type 2^n. Let F be the array obtained by superimposing F_1 on F_3, $F = F_1 \circ F_3$.

Let L_1 and L_2 be a pair of OPILS of type $t_1^{w_1} t_2^{w_2} \ldots t_k^{w_k}$ defined on V with partition $\{U^1, U^2, \ldots, U^q, V^1, V^2, \ldots, V^q\}$. Let L_3 be the array obtained by applying the bijection f to every element in the cells of L_2. Thus L_3 in a partitioned Latin square defined on

$\overline{V}$ with partition $\{\overline{U}^1, \overline{U}^2, \dots, \overline{U}^q, \overline{V}^1, \overline{V}^2, \dots, \overline{V}^q\}$. Let L be the array obtained by the superimposing L_1 on L_3, $L = L_1 \circ L_3$.

Consider the set U^j, $1 \leq j \leq q$. Let $|U^j| = t_i$. Let M^1 and M^2 be a pair of orthogonal Latin squares of side $t_i + 1$. Thus let M_j be the array obtained by superimposing M^1 on M^2 where M^1 is defined on the set $U^j \cup \{\alpha\}$ and M^2 is defined on the set $\overline{U}^j \cup \{\infty\}$. Similarly consider the set V^j, $1 \leq j \leq q$. Let $|V^j| = t_i$. Thus let N_j be the array obtained by superimposing M^1 on M^2 where M^1 is defined on the set $V^j \cup \{\infty\}$ and M^2 is defined on the set $\overline{V}^j \cup \{\alpha\}$. In addition we assume that M_j and N_j are partitioned as outlined below.

$$M_j = \begin{array}{|c|c|}\hline & u_1^j\overline{u}_1^j \\ A_j & u_2^j\overline{u}_2^j \\ & \vdots \\ & u_{t_i}^j\overline{u}_{t_i}^j \\ \hline D_j & \alpha\infty \\ \hline \end{array} \qquad N_j = \begin{array}{|c|c|}\hline & v_1^j\overline{v}_1^j \\ B_j & v_2^j\overline{v}_2^j \\ & \vdots \\ & v_{t_i}^j\overline{v}_{t_i}^j \\ \hline E_j & \alpha\infty \\ \hline \end{array}$$

Let $J_j = \begin{array}{|c|}\hline u_1^j\overline{u}_1^j \\ \hline u_2^j\overline{u}_2^j \\ \hline \vdots \\ \hline u_{t_i}^j\overline{u}_{t_i}^j \\ \hline \end{array}$ and let $K_j = \begin{array}{|c|}\hline v_1^j\overline{v}_1^j \\ \hline v_2^j\overline{v}_2^j \\ \hline \vdots \\ \hline v_{t_i}^j\overline{v}_{t_i}^j \\ \hline \end{array}$

For $1 \leq j \leq q$, and $1 \leq i \leq t_s$, where $t_s = |U^j|$, let $F_i^j = \begin{array}{|c|c|}\hline \alpha u_i^j & \infty\overline{u}_i^j \\ \hline \infty v_i^j & \alpha\overline{v}_i^j \\ \hline \end{array}$, $C_i^j = \begin{array}{|c|}\hline v_i^j\overline{v}_i^j \\ \hline u_i^j\overline{u}_i^j \\ \hline \end{array}$ and $R_i^j = \begin{array}{|c|c|}\hline \overline{u}_i^j\overline{v}_i^j & u_i^j v_i^j \\ \hline \end{array}$.

Now we construct two $(2n+1) \times (2n+1)$ arrays as follows.

$$H_1 = \begin{array}{|ccccccc|c|}\hline
F_1^1 & & & & & & & C_1^1 \\
 & \ddots & & & F & & & \vdots \\
 & & F_{t_1}^1 & & & & & C_{t_1}^1 \\
 & & & \ddots & & & & \vdots \\
 & & & & F_1^q & & & C_1^q \\
 & & & & & \ddots & & \vdots \\
 & & & & & & F_{t_k}^q & C_{t_k}^q \\ \hline
R_1^1 & \dots & R_{t_1}^1 & \dots & R_1^q & \dots & R_{t_k}^q & \alpha\infty \\ \hline
\end{array}$$

$$H_2 = \begin{array}{|cccccc|c|}\hline
A_1 & & & & & & J_1 \\
 & \ddots & & L & & & \vdots \\
 & & A_q & & & & J_q \\
 & & & B_1 & & & K_1 \\
 & & & & \ddots & & \vdots \\
 & & & & & B_q & K_q \\ \hline
D_1 & \dots & D_q & E_1 & \dots & E_q & \alpha\infty \\ \hline
\end{array}$$

We need to show that H_1 and H_2 are both H$(2n+1{,}4n+2)$ defined on $V' = V \cup \overline{V} \cup \{\alpha, \infty\}$. First we note that both H_1 and H_2 are $(2n+1) \times (2n+1)$ arrays and that $|V'| = 4n+2$.

Any element of V' which does not appear in a row of F appears in the F_i^j array that is used to fill in the hole of that row or it appears in the C_i^j array that is appended to that row. Similarly, any element of V' which does not appear in a column of F appears in the F_i^j array that is used to fill in the hole of that column, or it appears in the R_i^j

array that is appended to that column. Thus the columns and rows of H_1 are Latin and H_1 is an H($2n+1$,$4n+2$).

Any element of V' which does not appear in a row of L appears in the A_j or B_j array that is used to fill in the hole of that row or it appears in the J_j or K_j array that is appended to that row. Similarly any element of V' which does not appear in a column of L appears in the A_j or B_j array that is used to fill in the hole of that column or it appears in the D_j or E_j array that is appended to that column. Thus the columns and rows of H_1 are Latin and H_2 is an H($2n+1$,$4n+2$).

The final column of H_1 contains the same set of pairs as the final column of H_2. Therefore we can permute the rows of H_1, which does not affect the fact that it is an H($2n+1$,$4n+2$), so that the last column of H_1 is identical to the last column of H_2. Call this last column C. Thus we can write H_1 in the form $H_1 = \begin{array}{|c|c|}\hline S_1 & C \\ \hline\end{array}$ and we can write H_2 in the form $H_2 = \begin{array}{|c|c|}\hline S_2 & C \\ \hline\end{array}$. Thus we let $B = \begin{array}{|c|c|c|}\hline S_1 & S_2 & C \\ \hline\end{array}$. If we can verify that every possible pair of distinct elements of V' appears in B, then B is a PBTD($2n+1$) with partitioning given by H_1 and H_2.

Clearly the pair $\{\alpha, \infty\}$ appears in the array. The element α appears with the u_i^j and the $\overline{v}_i^j$ in the F_i^j subarrays, with the $\overline{u}_i^j$ in the A_j subarrays and with the v_i^j in the B_j subarrays. The element ∞ appears with the v_i^j and the $\overline{u}_i^j$ in the F_i^j subarrays and with the $\overline{v}_i^j$ in the B_j subarrays and with the u_i^j in the A_j subarrays. Thus α and ∞ appear with every element of V'.

All the unordered pairs of the form $\{u_i^j, v_x^y\}$, $\{\overline{u}_i^j, \overline{v}_x^y\}$, $\{u_i^j, u_x^y\}$, $\{\overline{u}_i^j, \overline{u}_x^y\}$, $\{v_i^j, v_x^y\}$ and $\{\overline{v}_i^j, \overline{v}_x^j\}$, where $i \neq x$ and $j \neq y$ appear in F. All pairs of the form $\{u_i^j, v_i^j\}$, and $\{\overline{u}_i^j, \overline{v}_i^j\}$ appear in the R_i^j subarrays.

All the unordered pairs of the form $\{u_i^j, \overline{u}_i^j\}$, and $\{v_i^j, \overline{v}_i^j\}$ appear in the C_i^j subarrays. All pairs of the form $\{u_i^j, \overline{v}_i^j\}$, $\{\overline{u}_i^j, v_i^j\}$, $\{u_i^j, \overline{v}_x^y\}$ and $\{\overline{u}_i^j, v_x^y\}$ where $i \neq x$ and $j \neq y$ appear in L.

The unordered pairs of the form $\{u_i^j, \overline{u}_x^y\}$, where $i \neq x$, appear in L if $j \neq y$, otherwise they appear in the A_j or D_j subarrays. The pairs of the form $\{v_i^j, \overline{v}_x^y\}$, where $i \neq x$, appear in L if $j \neq y$, otherwise they appear in the B_j or E_j subarrays.

Thus every unordered pair of distinct elements from V' appears in a cell of B. Thus B is a PBTD$(2n+1)$. $\square$

The following construction is almost the same as the previous construction.

Theorem 3.9 *[11] Let* $2n = tw + 6$ *where* $w \equiv 0 \pmod 2$*. If there exists a complementary* 2^n *frame, a pair of OPILS of type* $t^w 6$ *and a pair of orthogonal Latin squares of side* $t+1$*, then there is a PBTD(*$2n+1$*).*

Proof The construction for this PBTD is nearly identical to the previous construction. We have the same partitioning of the sets V and $\overline{V}$ as in the previous construction, except that the sets U^j, V^j, $\overline{U}^j$ and$\overline{V}^j$ sets have cardinality t for $1 \leq j \leq w$ and $|U^{w+1}| = |V^{w+1}| = |\overline{U}^{w+1}| = |\overline{V}^{w+1}| = 3$. Let $U^{w+1} = \{1,2,3\}$, $V^{w+1} = \{4,5,6\}$, $\overline{U}^{w+1} = \{\overline{1},\overline{2},\overline{3}\}$ and $\overline{V}^{w+1} = \{\overline{4},\overline{5},\overline{6}\}$. Let $f : V \to \overline{V}$ be the bijection defined by $f(x) = \overline{x}$.

The rest of the construction is the same as before, except that instead of using the superposition of two orthogonal Latin squares of side 7 to fill in the empty subarray of size 6 in L which is of type $t^w 6$, we will use the first of the two H(7,14) given in Example 3.2.

The rows and columns of a Howell design are Latin so this does not affect the Latinicty of the rows and columns of the PBTD. The final column of A contains pairs of the same form as those of the C_i^j so we can partition the array in the same manner as we partitioned the superimposed orthogonal Latin squares and fill in the hole in L in the same manner as before.

We only need to check that all the necessary pairs appear in A. The element α appears with $1, 2, 3, \overline{4}, \overline{5}$ and $\overline{6}$ in the F_i^{w+1} arrays and it appears with $4, 5, 6, \overline{1}, \overline{2}$ and $\overline{3}$

in A. The element ∞ appears with $4, 5, 6, \overline{1}, \overline{2}$ and $\overline{3}$ in the F_i^{w+1} arrays and it appears with $1, 2, 3, \overline{4}, \overline{5}$ and $\overline{6}$ in A.

The pairs of the form (x, y) and $(\overline{x}, \overline{y})$, where $x \neq y$ appear in F, therefore we need the pairs of the form $(x, \overline{y})$ to appear in A and they do. Therefore A contains all the necessary pairs and therefore this construction produces a PBTD$(2n + 1)$ where $n = tw + 6$ and $w \equiv 0 \pmod 2$. □

Our last construction for this case requires a starter-adder construction for PBTDs.

Definition 3.9 *For n a positive, odd integer, a* starter *for a PBTD(n+1) is a partition*

$$S = \{\{x_1, y_1\}, \{x_2, y_2\}, \ldots, \{x_{n-2}, y_{n-2}\}, \{x_{n-1} = \alpha, y_{n-1}\}, \{x_n = \infty, y_n\}, \{w_1, w_2\}\}$$

of $\mathbb{Z}_{2n} \cup \{\alpha, \infty\}$ *such that*

$$\left(\bigcup_{i=1}^{n-2} \{\pm(x_i - y_i)\} \right) \bigcup \{\pm(w_1 - w_2)\} = \mathbb{Z}_{2n} \setminus \{0, n\},$$

and $w_1 - w_2 \in \{1, 3, 5, \ldots, 2n-1\}$. *Let* H *be the subgroup* $\{0, 2, 4, \ldots, 2n-2\} \subset \mathbb{Z}_{2n}$. *An* adder *for* S *is a bijection* $a : S \setminus \{w_1, w_2\} \to H$ *such that*

$$\left(\bigcup_{i=1}^{n-2} \{x_i + a(x_i, y_i), y_i + a(x_i, y_i)\} \right) \bigcup \{y_{n-1} + a(\alpha, y_{n-1}), y_n + a(\infty, y_n)\} = \mathbb{Z}_{2n} \setminus \{u, v\}$$

where $u - v = n \in \mathbb{Z}_{2n}$. *(Observe that the adder defines a bijection from* $\mathbb{Z}_{2n} \setminus \{w_1, w_2\}$ *to* $\mathbb{Z}_{2n} \setminus \{u, v\}$.*)*

Theorem 3.10 *[9] Let n be a positive odd integer. Let*

$$S = \{\{x_1, y_1\}, \{x_2, y_2\}, \ldots, \{x_{n-2}, y_{n-2}\}, \{x_{n-1} = \alpha, y_{n-1}\}, \{x_n = \infty, y_n\}, \{w_1, w_2\}\}$$

be a starter for a PBTD($n+1$). Let H be the subgroup $\{0, 2, 4, \ldots, 2n-2\} \subset \mathbb{Z}_{2n}$. Let the bijection $a : S \setminus \{w_1, w_2\} \to H$ be an adder for S. Let $u, v \in \mathbb{Z}_{2n}$ be the pair of elements in the definition of the adder whose difference is n. There exists a PBTD($n+1$).

Proof Let B be an $(n+1) \times (2n+1)$ array constructed as follows. Label the rows of B, 0,1, ... , n and the columns of B, 0,1, ... , $2n$. For any $g \in \mathbb{Z}_{2n}$, define $\alpha + g = \alpha$ and $\infty + g = \infty$. For $1 \le i \le n$ and $0 \le j \le n-1$ place $\{x_i + a(x_i, y_i) + j, y_i + a(x_i, y_i) + j\}$ in the cell $(j, a(x_i, y_i) + j)$ and place $\{x_i + a(x_i, y_i) + n + j, y_i + a(x_i, y_i) + n + j\}$ in the cell $(j, a(x_i, y_i) + n + j)$. All of these calculations are done modulo $2n$. In row n, the final row, and column j place the pair $\{w_1 + j, w_2 + j\}$ for $0 \le j \le 2n-1$. In row j and column $2n$, the final column, place the pair $\{u+j, v+j\}$ for $0 \le j \le n-1$. In cell $(n, 2n)$ place the pair $\{\alpha, \infty\}$.

We claim that B is a PBTD($n+1$) defined on the set $V = \mathbb{Z}_{2n} \cup \{\alpha, \infty\}$. Since the range of a is H and n is odd, every cell in the array is filled. Clearly the pair $\{\alpha, \infty\}$ appears in B. Since $x_{n-1} = \alpha$ and $x_n = \infty$ all the pairs of the form $\{\infty, g\}$ and $\{\alpha, g\}$, $g \in \mathbb{Z}_{2n}$, are contained in B. Since

$$\left(\bigcup_{i=1}^{n-2} \{\pm(x_i - y_i)\} \right) \bigcup \{\pm(w_1 - w_2)\} = \mathbb{Z}_{2n} \setminus \{0, n\},$$

all the differences are distinct. In addition, $u - v = n$, and thus every pair $\{a, b\} \subset \mathbb{Z}_{2n}$ is contained in a cell of B. Therefore all unordered pairs of elements from V appear in B.

The pairs in column 0 are precisely all the pairs of S. Therefore, column 0 is Latin. Since the pairs in column k, $1 \le k \le 2n-1$, are the pairs of

$$\{\{x_i + k, y_i + k\} | 1 \le i \le n\} \cup \{w_1 + k, w_2 + k\},$$

the Latinicity of column 0 implies the Latinicity of column k. Column $2n$ contains the pairs $\{\alpha,\infty\},\{0,n\},\{1,n+1\},\ldots,\{n-1,2n-1\}$ and thus this column is Latin as well.

By construction, the pairs in cells $(0,h)$, $h\in H$, together with cell $(0,2n)$ are those of

$$\{\{x_i+a(x_i,y_i),y_i+a(x_i,y_i)\}|1\le i\le n\}\cup(\{u,v\}).$$

These pairs partition $\mathbb{Z}_{2n}\cup\{\infty,\alpha\}$. By construction, the pairs in cells $(0,k)$, $k\in K=\{1,3,5,\ldots,2n-1\}$, together with cell $(0,2n)$ are those of

$$\{\{x_i+a(x_i,y_i)+n,y_i+a(x_i,y_i)+n\}|1\le i\le n\}\cup(\{u,v\}).$$

These pairs partition $\mathbb{Z}_{2n}\cup\{\infty,\alpha\}$.

Every row j, $1\le j\le n-1$, inherits this property from row 0; that is, cells (j,h), $h\in H$, and cell $(j,2n)$ partition $\mathbb{Z}_{2n}\cup\{\infty,\alpha\}$, as do cells (j,k), $k\in K$, and cell $(j,2n)$.

Row n also has this property because $w_1-w_2\in K$. Therefore, columns $0,2,4,\ldots,2n-2,2n$ are an $\mathrm{H}(n+1,2n+2)$, as are columns $1,3,5,\ldots,2n-1,2n$.

Therefore B is a PBTD$(n+1)$. □

Below are the starter-adder pairs for $n=5$ given in [9] and the PBTD(6) generated by the construction.

$n = 5$

$S\ \{x_i, y_i\}$:	7 9	0 6	1 2	α 3	∞ 4
$a(x_i, y_i)$:	6	2	8	4	0
$\{w_1, w_2\}$:	5 8				
$\{u, v\}$:	1 6				

∞4	80	28	45	α7	∞9	35	73	90	α2	16
α3	∞5	91	39	56	α8	∞0	46	84	01	27
12	α4	∞6	02	40	67	α9	∞1	57	95	38
06	23	α5	∞7	13	51	78	α0	∞2	68	49
79	17	34	α6	∞8	24	62	89	α1	∞3	56
58	69	70	81	92	03	14	25	36	47	$\alpha\infty$

Example 3.3 *We give starter-adder pairs for PBTD(n + 1) where* $n = 7, 9, 11, 13, 15, 17, 19, 21$ *found in [9] and [10].*

$n = 7$

$S\ \{x_i, y_i\}$:	*5 7*	*1 4*	*10 0*	*8 13*	*6 12*	α *9*	∞ *11*
$a(x_i, y_i)$:	*12*	*6*	*2*	*10*	*8*	*4*	*0*
$\{w_1, w_2\}$:	*2 3*						
$\{u, v\}$:	*1 8*						

$n = 9$

$S\ \{x_i, y_i\}$:	*0 1*	*2 4*	*3 6*	*7 11*	*10 16*	*5 12*	*9 17*	α *14*	∞ *15*
$a(x_i, y_i)$:	*14*	*0*	*2*	*6*	*8*	*4*	*12*	*16*	*10*
$\{w_1, w_2\}$:	*8 13*								
$\{u, v\}$:	*1 10*								

$n = 11$

$S\ \{x_i, y_i\}$:	2 4	3 6	5 9	8 13	12 18	14 21	11 19	7 16
	10 20	α 15	∞ 17					
$a(x_i, y_i)$:	0	14	2	6	4	16	12	18
	8	20	10					
$\{w_1, w_2\}$:	0 1							
$\{u, v\}$:	10 21							

$n = 13$

$S\ \{x_i, y_i\}$:	2 4	3 6	5 9	7 12	10 16	14 21	17 25	15 24
	13 23	11 22	8 20	α 18	∞ 19			
$a(x_i, y_i)$:	0	12	18	4	24	8	14	2
	22	10	16	20	6			
$\{w_1, w_2\}$:	0 1							
$\{u, v\}$:	7 20							

$n = 15$

$S\ \{x_i, y_i\}$:	2 4	3 6	5 9	7 12	8 14	15 22	17 25	20 29
	18 28	16 27	11 23	13 26	10 24	α 19	∞ 21	
$a(x_i, y_i)$:	0	2	10	16	12	26	22	4
	24	28	20	14	6	18	8	
$\{w_1, w_2\}$:	0 1							
$\{u, v\}$:	6 21							

$n = 17$

$S\ \{x_i, y_i\}$:	2 4	3 6	5 9	7 12	8 14	11 18	17 25	19 28
	21 31	22 33	20 32	16 29	13 27	15 30	10 26	α 23
	∞ 24							
$a(x_i, y_i)$:	0	2	16	22	12	6	32	18
	14	10	30	24	20	26	4	8
	28							
$\{w_1, w_2\}$:	0 1							
$\{u, v\}$:	10 27							

$n = 19$

$S\ \{x_i, y_i\}$:	2 4	3 6	5 9	7 12	8 14	10 17	15 23	18 27
	21 31	24 35	25 37	20 33	22 36	19 34	16 32	13 30
	11 29	α 26	∞ 28					
$a(x_i, y_i)$:	0	2	4	14	22	12	10	26
	6	34	24	8	16	36	18	32
	28	30	20					
$\{x_t, y_t\}$:	0 1							
$\{u, v\}$:	16 35							

$n = 21$

$S\ \{x_i, y_i\}$:	*2 4*	*3 6*	*5 9*	*7 12*	*8 14*	*10 17*	*11 19*	*20 29*
	21 31	*23 34*	*25 37*	*28 41*	*26 40*	*24 39*	*22 38*	*15 32*
	18 36	*16 35*	*13 33*	α *27*	∞ *30*			
$a(x_i, y_i)$:	*0*	*2*	*4*	*14*	*24*	*10*	*20*	*8*
	12	*30*	*40*	*26*	*16*	*6*	*38*	*34*
	22	*36*	*28*	*32*	*18*			
$\{w_1, w_2\}$:	*0 1*							
$\{u, v\}$:	*15 36*							

We will use the PBTDs constructed by the starter-adder construction as well as a combinatorial design called an incomplete orthogonal array to construct other PBTDs. We begin with the definition of an incomplete orthogonal array.

Definition 3.10 *Let V be a finite set of cardinality n. Let K be a subset of V of size k. An* incomplete orthogonal array *$IA(n, k, s)$ is an $(n^2 - k^2) \times s$ array defined on the set V such that any two columns give, in their horizontal pairs, every ordered pair of elements of $(V \times V) - (K \times K)$ exactly once.*

An IA(n, k, s) is equivalent to a set of $s - 2$ mutually orthogonal Latin squares of side n which are missing a subsquare of side k. We do not have to be able to fill in the $k \times k$ missing subsquares with orthogonal Latin squares of side k.

We are now ready for the construction.

Theorem 3.11 *[10] Let $n \equiv 1 \pmod 2$. If there exists a PBTD$(n + 1)$ generated by a starter-adder pair on $\mathbb{Z}_{2n} \cup \{\alpha, \infty\}$, a PBTD$(m)$, a PBTD$(m + k)$, a pair of orthogonal Latin squares of order m and an IA$(m + k, k, 4)$, then there is a PBTD$((n + 1)m + k)$.*

Proof Let B be a PBTD$(n + 1)$ generated by a starter-adder pair on $\mathbb{Z}_{2n} \cup \{\alpha, \infty\}$ as in the proof of Theorem 3.10. Unlike that proof, we will label the columns of B,

$C_1, \ldots, C_{2n+1}$, and the rows of B, $R_1, \ldots, R_{n+1}$. Thus in C_{2n+1}, R_i contains the pair $\{u_i, v_i\}$, where $|u_i - v_i| = n$ for $i = 1, \ldots, n$ and R_{n+1} contains the pair $\{\alpha, \infty\}$. Also $C_1 \cup C_3 \cup \ldots, \cup C_{2n+1}$ is an $\mathrm{H}(n+1, 2n+2)$ and $C_2 \cup C_4 \cup \ldots \cup C_{2n} \cup C_{2n+1}$ is an $\mathrm{H}(n+1, 2n+2)$.

Let $\{s_1, t_1\}$ be a pair in the first column and the jth row, where $1 \leq j \leq n$, such that $|s_1 - t_1|$ is odd. Let $s_i = s_1 + i - 1 \pmod{2n}$ and $t_i = t_1 + i - 1 \pmod{2n}$ for $1 \leq i \leq 2n$. Therefore by the way B was constructed, $\{s_i, t_i\}$ occurs in C_i and R_{j+i-1} (mod n). We will permute the rows of B so that $\{s_i, t_i\}$ appears in cell (i, i) if $1 \leq i \leq n$ or in cell $(i-n, i)$ if $n+1 \leq i \leq 2n$. This permutation of columns does not change the fact that $C_1 \cup C_3 \cup \ldots, \cup C_{2n+1}$ is an $\mathrm{H}(n+1, 2n+2)$ and $C_2 \cup C_4 \cup \ldots \cup C_{2n} \cup C_{2n+1}$ is an $\mathrm{H}(n+1, 2n+2)$.

Let $M = \{1, 2, \ldots, m\}$. Let L_1 and L_2 be a pair of orthogonal Latin squares of side m defined on M. Let L be the array obtained by superimposing L_1 and L_2, $L = L_1 \circ L_2$. Let L_{xy} be the array obtained by replacing each pair (a, b) in L by the pair (a_x, b_y).

Let $\beta = \{\beta_1, \beta_2, \ldots, \beta_k\}$ and let $\gamma = \{\gamma_1, \gamma_2, \ldots, \gamma_k\}$. We have an $\mathrm{IA}(m+k, k, 4)$ and this is equivalent to a pair of orthogonal Latin squares of side $m+k$ which are missing a subsquare of side k. Let N_{s_i} be one of these two arrays defined on $(M \times \{s_i\}) \cup \beta$ with its missing subsquare defined on β. Let N_{t_i} be the other array of this pair of arrays and let it be defined on $(M \times \{t_i\}) \cup \gamma$ with its missing subsquare defined on γ. Let $N_{s_i t_i}$ be the array obtained by superimposing N_{s_i} on N_{t_i}, $N_{s_i t_i} = N_{s_i} \circ N_{t_i}$. We can write $N_{s_i t_i}$ in the following way. $N_{s_i t_i} = \begin{array}{|c|c|} \hline A_i & E_i \\ \hline D_i & O \\ \hline \end{array}$, where A_i is an $m \times m$ array, E_i is an $m \times k$ array, D_i is a $k \times m$ array and O is an empty array of size $k \times k$.

Let B_i be a $\mathrm{PBTD}(m)$ defined on $M \times \{u_i, v_i\}$ for $i = 1, 2, \ldots, n$. We will write the B_i in the following way, $B_i = \begin{array}{|c|c|c|} \hline F_i & G_i & H_i \\ \hline \end{array}$, where F_i and G_i are $m \times (m-1)$ arrays and H_i is an $m \times 1$ array. The B_i are written such that $F_i \cup H_i$ and $G_i \cup H_i$ are $\mathrm{H}(m, 2m)$.

Let B_{n+1} be a PBTD$(m+k)$ defined on $(M \times \{\alpha, \infty\}) \cup (\beta \cup \gamma)$. We will write B_{n+1} in the following way, $B_{n+1} = \boxed{K_1 \mid K_2 \mid K_3 \mid K_4 \mid K_5}$, where K_1 and K_2 are $(m+k) \times (m-1)$ arrays, K_4 and K_5 are $(m+k) \times k$ arrays and K_3 is an $(m+k) \times 1$ array. B_{n+1} is written such that $K_1 \cup K_4 \cup K_3$ and $K_2 \cup K_5 \cup K_3$ are H$(m+k, 2m+2k)$.

Let $V = \mathbb{Z}_{2n} \cup \{\alpha, \infty\}$ and let $W = \beta \cup \gamma$. We now construct the array P defined on $(M \times V) \cup W$.

Replace each pair $\{s_i, t_i\}$ with the $m \times m$ array, A_i, for $i = 1, \ldots, 2n$. Replace each pair $\{u_i, v_i\}$, which occurs in the final column, by the $m \times (2m-1)$ array, B_i, for $i = 1, \ldots, n$. Replace the pair $\{\alpha, \infty\}$ by O_{2m-1}, which is an empty array of size $m \times (2m-1)$. Replace all the remaining pairs of B, $\{x, y\}$, by the $m \times m$ array, L_{xy}. The resulting array is of size $m(n+1) \times (2mn + 2m - 1)$. To this array we add k new rows in the form $\boxed{D_1 \mid D_2 \mid \ldots \mid D_{2n} \mid O_{2m-1+k}}$, where O_{2m-1+k} is an empty array of size $k \times (2m-1+k)$. We also add $2k$ new columns to the array. (These columns will overlap with the new rows added, but in the cells where they overlap both the new rows and the new columns will be empty.) The array of the $2k$ new columns is of the following form.

E_1	E_{n+1}
E_{n+2}	E_2
E_3	E_{n+3}
$\vdots$	$\vdots$
E_n	E_{2n}
O_{2k}	

O_{2k} is an empty array of size $(m+k) \times 2k$.

Therefore the resulting array is of size $(m(n+1)+k) \times (2mn + 2m - 1 + 2k)$ with an empty subarray of size $(m+k) \times (2m-1+2k)$ in the lower right hand corner. Fill in this empty array with the array B_{n+1}. We call this array P.

A_1				A_{n+1}				F_1	G_1	H_1	E_1	E_{n+1}
	A_2				A_{n+2}			F_2	G_2	H_2	E_{n+2}	E_2
		$\ddots$				$\ddots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
			A_n				A_{2n}	F_n	G_n	H_n	E_n	E_{2n}
$L_{x_t y_t}$	$\dots$						$L_{x_{t-1} y_{t-1}}$					
D_1	D_2	$\dots$	D_n	D_{n+1}	$\dots$		D_{2n}	K_1	K_2	K_3	K_4	K_5
C'_1	C'_2	$\dots$	C'_n	C'_{n+1}	$\dots$		C'_{2n}	D_1	D_2	D_3	D_4	D_5

We claim that P is a PBTD$((n+1)m+k)$. P is an $(m(n+1)+k)\times(2mn+2m-1+2k)$ array defined on the set $V' = (M \times V) \cup W$ which is a set of cardinality $2nm + 2m + 2k$. We must ensure that any unordered pair from V' occurs in a cell of P.

Unordered pairs of the form $\{\beta_i, \beta_j\}$, and $\{\gamma_i, \gamma_j\}$ where $1 \le i < j \le k$, and $\{\beta_i, \gamma_j\}$ where $1 \le i, j \le k$ occur in the subarray B_{n+1}. Also unordered pairs of the form $\{a_x, \beta_i\}$, $\{a_x, \gamma_i\}$ $\{a_x, b_y\}$ where $a, b \in M$, $a_x \ne b_y$, $x, y \in \{\alpha, \infty\}$ and $1 \le i \le k$, occur in B_{n+1}.

The subarray $N_{s_i t_i}$ contains every ordered pair of $((M\times\{s_i\})\cup\beta)\times((M\times\{t_i\})\times\gamma) - (\beta\times\gamma)$. Since $s_i = s_1 + i - 1 \pmod{2n}$ and $t_i = t_1 + i - 1 \pmod{2n}$ for $1 \le i \le 2n$, thus $\{s_1, s_2, \dots, s_{2n}\} = \{t_1, t_2, \dots, t_{2n}\} = \mathbb{Z}_{2n}$. Therefore the unordered pairs of the form $\{a_x, \beta_i\}$ and $\{a_x, \gamma_i\}$ where $a \in M$, $x \in \mathbb{Z}_{2n}$, and $1 \le i \le k$ occur in the $N_{s_i t_i}$ subarrays. Also unordered pairs of the form $\{a_x, b_y\}$ where $a, b \in M$, $x, y \in \mathbb{Z}_{2n}$ and $|x-y| \equiv |s_1 - t_1| \pmod{2n}$ (that is (x, y) is an (s_i, t_i) pair) occur in the $N_{s_i t_i}$ subarrays.

Unordered pairs of the form $\{a_x, b_y\}$, where $a, b \in M$, $x, y \in \mathbb{Z}_{2n}$ and $|x - y| \equiv n \pmod{2n}$ occur in the B_i subarrays. The remaining unordered pairs of the form $\{a_x, b_y\}$, where $a, b \in M$, $x, y \in \mathbb{Z}_{2n} \cup \{\alpha, \infty\}$ with at most one of x and y in $\{\alpha, \infty\}$, $|x - y| \not\equiv n \pmod{2n}$ and $|x - y| \not\equiv |s_1 - t_1| \pmod{2n}$ appear in the L_{xy} subarrays. Therefore every unordered pair of V' appears in P.

Next we want to show that every element of V' appears exactly once in each column. Consider any of the first $2mn$ columns. Each of these columns intersects an A_i and a D_i subarray, for $1 \leq i \leq 2n$. The part of the column that intersects these subarrays contains the elements $(M \times \{s_i, t_i\}) \cup \beta \cup \gamma$. The rest of this column is made up of columns of L_{xy} subarrays. Since the columns of B, the PBTD$(n+1)$ P was created from, are Latin, thus the rest of the elements in this column are $M \times (\mathbb{Z}_{2n} \cup \{\alpha, \infty\} - \{s_i, t_i\})$. Therefore this column contains every element of V'.

Now consider the next $2m - 1$ columns. These columns intersect every B_i for $1 \leq i \leq n+1$. Each of the columns of the B_i are Latin and thus this column consists of the elements of

$$\bigcup_{i=1}^{n}(M \times \{u_i, v_i\}) \bigcup (M \times \{\alpha, \infty\} \bigcup \beta \bigcup \gamma$$

However $\bigcup_{i=1}^{n}\{u_i, v_i\} = \mathbb{Z}_{2n}$ and thus these columns contain every element of V'.

Finally consider the last $2k$ columns. The first k columns of these columns intersect each of the E_i subarrays, where $i \equiv 1 \pmod 2$, as well the K_4 subarray. Therefore each of these columns contain the elements

$$\bigcup_{i\ odd} \big((M \times \{s_i\}) \cup (M \times \{t_i\})\big) \bigcup (M \times \{\alpha, \infty\}) \bigcup \beta \bigcup \gamma$$

Since $|s_1 - t_1| \equiv 1 \pmod 2$, thus $\bigcup_{j=0}^{n-1}\{s_{2j+1}, t_{2j+1}\} = \mathbb{Z}_{2n}$. Therefore each of these columns contains every element of V'.

The next k columns intersect each of the E_i subarrays, where $i \equiv 0 \pmod 2$, as well the K_5 subarray. Therefore each of these columns contain the elements

$$\bigcup_{i\ even} ((M \times \{s_i\}) \cup (M \times \{t_i\})) \bigcup (M \times \{\alpha, \infty\}) \bigcup \beta \bigcup \gamma$$

Since $|s_1 - t_1| \equiv 1 \pmod 2$, thus $\bigcup_{j=1}^{n} \{s_{2j}, t_{2j}\} = \mathbb{Z}_{2n}$. Therefore each of these columns contains every element of V'.

We also need to show that every element of V' appears at most twice in each row. We have indicated a partitioning of the columns of P in the diagram outlining the structure of P. We group the columns of P in the following way $S_1 = C'_1 \cup C'_3 \cup \ldots \cup C'_{2n-1} \cup D_1 \cup D_4 \cup D_3$ and $S_2 = C'_2 \cup C'_4 \cup \ldots \cup C'_{2n} \cup D_2 \cup D_5 \cup D_3$. We will show that each element of V' appears once in S_1 and once in S_2. Note that $S_1 \cap S_2 = D_3$, which is a single column.

For our argument we need to note some properties of B. Consider row j of B for $1 \leq j \leq n$. If $x \in \mathbb{Z}_{2n} \cup \{\alpha, \infty\} - \{u_j, v_j\}$ then x appears once in the even numbered columns of B and once in the odd numbered columns of B. We also note that if $\{s_i, t_i\}$ appears in this row then $\{s_{i+n}, t_{i+n}\}$ also appears in this row. The parity of i is opposite to that of $i + n$.

Now consider the set of rows of P that were constructed from row j of B where $1 \leq j \leq n$, that is the first mn rows. Consider the element a_x where $a \in M$ and $x \in \mathbb{Z}_{2n} \cup \{\alpha, \infty\} - \{s_i, t_i, s_{i+n}, t_{i+n}, u_j, v_j\}$. The element a_x appears twice in this row, because the element x appeared twice in row j of B. It appears in the subarrays L_{xy} and L_{xz} for some y and z. The columns C'_i were created from column i of B. The element a_x will appear once in a C'_i where i is even and once in a C'_i where i is odd. Thus a_x appears once in S_1 and once in S_2.

If $x \in \{s_i, t_i\}$ then a_x appears once in the subarray L_{xy} for some y and either once in the subarray A_i or once in the subarray E_i. Thus a_x appears once in S_1 and once in S_2. If $x \in \{s_{i+n}, t_{i+n}\}$ then a_x appears appears once in the subarray L_{xy} for some y and either once in the subarray A_{i+n} or once in the subarray E_{i+n}. Thus a_x appears once in S_1 and once in S_2.

If $x \in \{u_j, v_j\}$ then a_x appears once in F_j and once in G_j, except for one pair which appears only once in H_j. Thus a_x appears once in S_1 and once in S_2. Note that H_j is a single column and belongs to both S_1 and S_2.

The elements in $\beta \cup \gamma$ appear once in A_i or E_i and once in A_{i+n} or E_{i+n}. Thus these elements appear once in S_1 and once in S_2.

Now we consider the next m rows of P. These were created from the last row of B. Every element of $\mathbb{Z}_{2n}$ appears twice in this final row, once in an odd numbered column and once in an even numbered column. The elements α and ∞ appear once in the final column. Thus the element a_x where $a \in M$ and $x \in \mathbb{Z}_{2n}$ appears in the subarrays L_{xy} and L_{xz} for some y and z. The element a_x appears once in S_1 and once in S_2. The elements of $(M \times \{\alpha, \infty\}) \cup \beta \cup \gamma$ appear once in $K_1 \cup K_4$ and once in $K_2 \cup K_5$ except for one pair of elements which appears exactly once in K_3. Thus every element of $(M \times \{\alpha, \infty\}) \cup \beta \cup \gamma$ appears once in S_1 and once in S_2. Note that K_3 is a single column and belongs to both S_1 and S_2.

Finally consider the last k rows of P. These rows are constructed by the union of the D_i arrays where each D_i array corresponds to a (s_i, t_i) pair, for $1 \leq i \leq 2n$ and the last k rows of the B_{n+1} array. Thus the element a_x where $a \in M$ and $x \in \mathbb{Z}_{2n}$ appears twice in each row. Once in a D_i subarray where i is odd and once in a D_i subarray where i is even. Thus a_x appears once in S_1 and once in S_2. The elements of $(M \times \{\alpha, \infty\}) \cup \beta \cup \gamma$ appear once in $K_1 \cup K_4$ and once in $K_2 \cup K_5$ except for one pair of elements which appears exactly once in K_3. Thus every element of $(M \times \{\alpha, \infty\}) \cup \beta \cup \gamma$ appears once in S_1 and once in S_2.

Therefore every element of V' appears at most twice in each row of P. Also every element appears exactly once in the columns of S_1 and exactly once in the columns of S_2. Therefore both S_1 and S_2 are $\mathrm{H}((n+1)m+k, 2(n+1)m+k)$. Thus P is a PBTD$((n+1)m+k)$. $\square$

We now have all the necessary constructions to examine the existence of PBTD(n), for $n \equiv 3 \pmod 4$. To use these constructions we need some results about the existence of complementary frames, pairs of OPILS and incomplete orthogonal arrays. We give the necessary existence results here and refer the reader to the literature for their proofs.

Lemma 3.4 *[9] If $q \equiv 1 \pmod 2$, $q \geq 7$ then there exists a complementary frame of type 2^q.*

Lemma 3.5 *[15] There exists a complementary frame of type $4^4 6$.*

Lemma 3.6 *[11] There exists a pair of OPILS of types $4^4 6$, $4^5 6$, $4^8 6$, $4^{11} 6$, $8^5 6$, $7^6 8^4$ and $19^8 7^2$.*

Lemma 3.7 *[3] For $k \geq 1$, an IA$(n,k,4)$ exists if and only if $n \geq 3k$, $(n,k) \neq (6,1)$.*

Using these lemmas and the constructions we have outlined we can construct the necessary base cases for the recursive construction for PBTD(n) where $n \equiv 3 \pmod 4$.

Theorem 3.12 *[11] Let $n \equiv 1 \pmod 2$, $n \geq 3$. There is a PBTD$(6n+1)$.*

Proof Let $n \equiv 1 \pmod 2$, $n \geq 3$. By Lemma 3.4 there exists a complementary 2^{3n} frame. By Lemma 3.3 there exist a pair of OPILS of type 3^{2n}. By Lemma 2.2 there exists a pair of orthogonal Latin squares of side 4. Thus using Theorem 3.8, there exists a PBTD$(6n+1)$. $\square$

Theorem 3.13 *[11] There exists PBTD(n) for*

$$n \in \{71, 99, 111, 131, 171, 183, 191\} \cup \{75, 167\} \cup \{27, 39, 47, 51\}.$$

Proof Lemma 3.4 gives the existence of the complementary frames listed in the tables below. Lemma 3.3 and Lemma 3.6 give the existence of OPILS listed in the tables below. Lemma 2.2 gives the existence of the pairs of orthogonal Latin squares given in the tables below.

We use Theorem 3.8 to construct PBTDs for the values of n, t_1, t_2, w_1 and w_2 given in the tables below.

n	t_1	w_1	Complementary Frame	OPILS	OLS	PBTD
35	7	10	2^{35}	7^{10}	8	71
49	7	14	2^{49}	7^{14}	8	99
55	11	10	2^{55}	11^{10}	12	111
65	13	10	2^{65}	13^{10}	14	131
85	17	10	2^{85}	17^{10}	18	171
91	7	26	2^{91}	7^{26}	8	183
95	19	10	2^{95}	19^{10}	20	191

n	t_1	w_1	t_2	w_2	Complementary Frame	OPILS	OLS	PBTD
37	7	6	8	4	2^{37}	$7^6 8^4$	$8, 9$	75
83	19	8	7	2	2^{83}	$19^8 7^2$	$8, 20$	167

We use Theorem 3.9 to construct PBTDs for the values of n, t and w given in the table below.

n	t	w	Complementary Frame	OPILS	Orthogonal Latin Squares	PBTD
13	4	5	2^{13}	$4^{5}6$	5	27
19	4	8	2^{19}	$4^{8}6$	5	39
23	8	5	2^{23}	$8^{5}6$	9	47
25	4	11	2^{25}	$4^{11}6$	5	51

□

Theorem 3.14 *[11] Let* $n \in \{59, 63, 83, 87, 107, 123, 143, 159, 179\}$. *Then there exists a PBTD(n).*

Proof These constructions use already constructed PBTDS. By Theorem 3.10, a PBTD(q) exists for $q \in \{6, 8, 10, 12, 14, 16, 22\}$. For $q \in \{13, 17, 25\}$, a PBTD(q) exists by Theorem 3.7. PBTD(7) is given in Example 3.2.

Lemma 2.2 gives the existence of the necessary pairs of orthogonal Latin squares and Lemma 3.7 gives the existence of the necessary incomplete orthogonal arrays.

Therefore Theorem 3.11 gives the existence of the PBTDs for the values of $n+1, m$ and k given in the table below.

k	PBTD$(n+1)$	PBTD(m)	PBTD$(m+k)$	OLS	IA	PBTD
3	8	7	10	7	$(10,3,4)$	59
3	6	10	13	10	$(13,3,4)$	63
3	8	10	13	10	$(13,3,4)$	83
3	6	14	17	14	$(17,3,4)$	87
5	6	17	22	17	$(22,5,21)$	107
3	12	10	13	10	$(13,3,4)$	123
3	14	10	13	10	$(13,3,4)$	143
3	12	13	16	13	$(16,3,4)$	159
3	8	22	25	22	$(25,3,4)$	179

□

Theorem 3.15 *[11] There exists a PBTD(23).*

Proof By Lemma 3.5, there exists a complementary frame of type $4^4 6$. By Lemma 3.6, there exists a pair of OPILS of type $4^4 6$. Examples 3.1 and 3.2 are a PBTD(5) and a PBTD(7) respectively. Therefore by Theorem 3.5 there exists a PBTD(23). □

Theorem 3.16 *[11] Let $n \equiv 3 \pmod 4$. There exists a PBTD(n) for $7 \leq n \leq 191$, except possibly for $n = 11$ or 15.*

Proof The following table lists the constructions used for PBTD(n) for $n \equiv 3 \pmod 4$ for $7 \leq n \leq 191$, and $n \neq 11, 15$.

n	Construction		n	Construction	
7	Example 3.2		107	$6 \cdot 17 + 5$	3.14
19	$6 \cdot 3 + 1$	3.12	111	$11 \cdot 10 + 1$	3.13
23	$4 \cdot 4 + 6 + 1$	3.15	115	$5 \cdot 23$	3.4
27	$4 \cdot 5 + 6 + 1$	3.13	119	$7 \cdot 17$	3.4
31	$6 \cdot 5 + 1$	3.12	123	$12 \cdot 10 + 3$	3.14
35	$5 \cdot 7$	3.4	127	$6 \cdot 21 + 1$	3.12
39	$4 \cdot 8 + 6 + 1$	3.13	131	$13 \cdot 10 + 1$	3.13
43	$6 \cdot 7 + 1$	3.12	135	$5 \cdot 27$	3.4
47	$8 \cdot 5 + 6 + 1$	3.13	139	$6 \cdot 23 + 1$	3.12
51	$4 \cdot 11 + 6 + 1$	3.13	143	$14 \cdot 10 + 3$	3.14
55	$6 \cdot 9 + 1$	3.12	147	$7 \cdot 21$	3.4
59	$8 \cdot 7 + 3$	3.14	151	$6 \cdot 25 + 1$	3.12
63	$6 \cdot 10 + 3$	3.14	155	$5 \cdot 31$	3.4
67	$6 \cdot 11 + 1$	3.12	159	$12 \cdot 13 + 3$	3.14
71	$7 \cdot 10 + 1$	3.13	163	$6 \cdot 27 + 1$	3.12
75	$7 \cdot 6 + 8 \cdot 4 + 1$	3.13	167	$19 \cdot 8 + 7 \cdot 2 + 1$	3.13
79	$6 \cdot 13 + 1$	3.12	171	$17 \cdot 10 + 1$	3.13
83	$8 \cdot 10 + 3$	3.14	175	$5 \cdot 35$	3.4
87	$6 \cdot 14 + 3$	3.14	179	$8 \cdot 22 + 3$	3.14
91	$7 \cdot 13$	3.4	183	$7 \cdot 26 + 1$	3.13
95	$5 \cdot 19$	3.4	187	$6 \cdot 31 + 1$	3.12
99	$7 \cdot 14 + 1$	3.13	191	$19 \cdot 10 + 1$	3.13
103	$6 \cdot 17 + 1$	3.12			

□

The recursive construction requires two more existence results for complementary frames and OPILS.

Lemma 3.8 *[11] Let m be a positive integer, $m \geq 4$, $m \neq 6, 10$ and let t be a non-negative integer such that $0 \leq t \leq 3m$. There there is a complementary frame of type $(4m)^4(2t)^1$.*

Lemma 3.9 *[15] Let m be a positive integer, $m \geq 4$, $m \neq 6, 10$ and let t be a non-negative integer such that $0 \leq t \leq 3m$. There there is a pair of OPILS of type $(4m)^4(2t)^1$.*

Theorem 3.17 *[11] Let $n \equiv 3 \pmod 4$, $n \geq 7$. There exists a PBTD(n), except possibly for $n = 11, 15$.*

Proof Let m positive integer such that $m \geq 11$. Let t be a non-negative integer such that $0 \leq t \leq 3m$. Thus by Lemma 3.8 and Lemma 3.9 there exists a complementary frame of type $(4m)^4(2t)^1$ and a pair of OPILS of type $(4m)^4(2t)^1$, respectively. Since $m > 2$, by Theorem 3.7 there exists a PBTD$(4m+1)$. If there exists a PBTD$(2t+1)$, then by Theorem 3.5 there exists a PBTD$(16m+2t+1)$.

Let $t = 9, 3, 13, 15$. For each of these values $t \leq 33 \leq 3m$. By Theorem 3.16 there exists a PBTD$(2t+1)$ for each of these values of t. Therefore there exists PBTD(n) for the following four cases:

1. $n \equiv 3 \pmod{16}$, $n \geq 195$,
2. $n \equiv 7 \pmod{16}$, $n \geq 183$,
3. $n \equiv 11 \pmod{16}$, $n \geq 203$,
4. $n \equiv 15 \pmod{16}$, $n \geq 207$.

We combine these results with Theorem 3.16 and we obtain for $n \equiv 3 \pmod 4$, $n \neq 11, 15$ that there exists a PBTD(n). $\square$

3.5 Existence of PBTD(n) for $n \equiv 0 \pmod 2$, $n \geq 6$

The final case we consider is the existence of PBTD(n) where $n \equiv 0 \pmod 2$, $n \geq 6$. For this case we need four additional constructions.

The first construction is quite similar to the construction of Theorem 3.4

Theorem 3.18 *[16] If there exists a PBTD($2m$) and a pair of orthogonal Latin squares of side m then there exists a PBTD($6m$).*

Proof Let A be the following array defined on $\mathbb{Z}_{12}$.

$A =$

<table>
<tr><td>7 8</td><td>1 5</td><td>10 11</td><td>2 4</td><td rowspan="2">0 3
6 9</td><td>1 11</td><td>5 10</td><td>4 8</td><td>2 7</td></tr>
<tr><td>1 2</td><td>7 11</td><td>4 5</td><td>8 10</td><td>5 7</td><td>1 8</td><td>2 10</td><td>4 11</td></tr>
<tr><td>9 11</td><td>6 8</td><td>2 3</td><td>0 5</td><td rowspan="2">1 4
7 10</td><td>0 8</td><td>2 9</td><td>3 11</td><td>5 6</td></tr>
<tr><td>3 5</td><td>0 2</td><td>8 9</td><td>6 11</td><td>2 6</td><td>0 11</td><td>5 9</td><td>3 8</td></tr>
<tr><td>0 4</td><td>9 10</td><td>6 7</td><td>1 3</td><td rowspan="2">2 5
8 11</td><td>3 10</td><td>4 6</td><td>0 7</td><td>1 9</td></tr>
<tr><td>6 10</td><td>3 4</td><td>0 1</td><td>7 9</td><td>4 9</td><td>3 7</td><td>1 6</td><td>0 10</td></tr>
</table>

The cells which contain four elements are considered as belonging to both rows they intersect. Using this assumption we make some important observations about A. Each column of A contains every element of $\mathbb{Z}_{12}$. Every row of A contains every element of $\mathbb{Z}_{12}$ twice, once in the first five columns and once in the last five columns. Also every unordered pair of $\mathbb{Z}_{12}$ appears as a subset of the elements of a cell in A.

Let L_1 and L_2 be a pair of MOLS of order m both defined on the set $N = \{1, 2, \ldots, m\}$. Let L be the array obtained by superimposing L_1 on L_2, $L = L_1 \circ L_2$. We define $L_{u,v}$ to be the array obtained from L by replacing each ordered pair (a, b) in L by the ordered pair (a_u, b_v).

Let $B_{s,t,u,v}$ be a PBTD($2m$) defined on the set $\{x_i : x \in \{1, 2, \ldots m\}, i \in \{s, t, u, v\}\}$. We order the columns of $B_{s,t,u,v}$ so that the first $2m$ columns are an H($2m, 4m$) and the last $2m$ columns are an H$(2m, 4m)$. Therefore the $2m$th column of the array is the column of deficient pairs.

Replace every cell of A which contains a pair, say (u, v), by the array $L_{u,v}$ and replace every cell of A which contains a quadruple, say (s, t, u, v), by the array $B_{s,t,u,v}$. Call the resulting array C. We claim that C is PBTD$(6m)$.

C is defined on the set $V = \{1_i, 2_i, \ldots, m_i | 0 \leq i \leq 11\}$, and this is a set of cardinality $12m$. C has $6m$ rows and $8m + 4m - 1 = 12m - 1$ columns. Thus C has the correct dimensions.

Consider the unordered pair of distinct elements $\{x_u, y_v\}$. If $u = v$ then $x \neq y$ since we want a pair of distinct elements. Every element of $\mathbb{Z}_{12}$ appears in the fifth column of A. Let Q be the quadruple that contains u in the fifth column of A. Therefore, there is a subarray of C, namely B_Q which contains all pairs of distinct elements from the set $\{x_i : x \in \{1, 2, \ldots m\}, i \in Q\}$. Thus the unordered pair $\{x_u, y_u\}$ appears in C.

If $u \neq v$ and $\{u, v\}$ is a subset of Q, the quadruple of A which contains u, then there is a subarray of C, namely B_Q which contains all pairs of distinct elements from the set $\{x_i : x \in \{1, 2, \ldots m\}, i \in Q\}$. Thus the unordered pair $\{x_u, y_v\}$ appears in C.

If $u \neq v$ and $\{u, v\}$ is not a subset of Q, then the unordered pair $\{x_u, y_v\}$ appears in the subarray, $L_{u,v}$ of C. Thus every unordered pair of distinct elements from V appears in C.

We will verify that the columns of C are Latin. Consider the element x_u. Since the columns of A are Latin, the element u appears in every column. Thus for every column of C there is a part of this column which is either a column of the subarray $L_{u,v}$ (or $L_{v,u}$) or a column of the subarray B_Q where Q is the quadruple of A that contains u. Every column of $L_{u,v}$ and B_Q contains the element x_u. Therefore x_u appears in every column of C and thus the columns of C are Latin.

Now we will verify that the first $6m$ columns of C are an H$(6m, 12m)$ and that the last $6m$ columns are an H($6m$, $12m$) and thus the $6m$th column of C is the column of deficient pairs.

We have already verified that the columns of C are Latin. Now for each row we need to verify that every element appears twice, once in the first $6m-1$ columns and once in the last $6m-1$ columns except for the row where the pair appears only once in the $6m$th column.

Consider the element x_u. For every row of A the element u appears twice except for the rows where it appears in the quadruple. For the rows where it appears twice, it appears once in the first four columns and once in the last four columns. When it appears in the quadruple it only appears in the fifth column. First let us consider the rows where u is not in the quadruple. Let us say that in this row u appears with v in the first four columns and with w in the last four columns. Thus in the rows of C corresponding to this row of A, x_u will appear in the rows of $L_{u,v}$ (or $L_{v,u}$) which appears in the first $4m$ columns of C and in $L_{u,w}$ (or $L_{w,u}$) which appears in the last $4m$ columns of C.

Now in the rows of C corresponding to the row of A where u was in the quadruple, x_u will appear in the subarray B_Q. Since B_Q is a PBTD($2m$), x_u will appear twice in each row, except for one row where x_u is in the deficient pair and appears in $2m$th column. When x_u appears twice in a row of B_Q, it will appear once in the first $2m-1$ columns of B and once in the last $2m-1$ columns of B_Q. Therefore we have x_u appearing in the first $6m-1$ columns of C and once in the last $6m-1$ columns, except for when x_u appears in the deficient pair of B_Q and then x_u appears in the $6m$th column of C.

Therefore C is a PBTD($6m$). □

The second construction is similar to the construction of Theorem 3.5.

Theorem 3.19 *[9] If there exists a complementary $\{G_1, G_2, \ldots, G_m\}$-frame where $m \geq 2$, a pair of OPILS with partition $\{G_1, G_2, \ldots, G_m\}$, a pair of orthogonal Latin squares* *of side n and PBTD($n|G_i|+1$) for $1 \leq i \leq m$ then there is a PBTD($n\sum_{i=1}^{m}|G_i|+1$).*

Proof Let $V = \{v_1, \ldots, v_p\}$ and $\overline{V} = \{\overline{v}_1, \ldots, \overline{v}_p\}$ with the following bijection, $f : V \rightarrow \overline{V}$, between the two sets: $f(v_i) = \overline{v}_i$. Let $\{G_1, G_2, \ldots, G_m\}$ be a partition of V. Let $\{H_1, H_2, \ldots, H_m\}$ be the partition of $\overline{V}$ obtained by applying the bijection to the partition of V. Thus $H_i = \{\overline{v}_j \mid v_j \in G_i\}$ for $1 \leq i \leq m$.

Let F_1 be a complementary $\{G_1, G_2, \ldots, G_m\}$-frame defined on V. Let F_2 be the complement of F_1. Let F_3 be the array obtained by applying the bijection f to all the elements in the cells of F_2. Let F be the array obtained by superimposing F_1 on F_3, $F = F_1 \circ F_3$.

Let L_1 and L_2 be a pair of OPILS with partition $\{G_1, G_2, \ldots, G_m\}$. Let L_3 be the array obtained by applying the bijection f to every element in the cells of L_2. Let L be the superposition of L_1 on L_3, $L = L_1 \circ L_3$. Like F, every cell of L contains a pair of elements except for those cells indexed by (x, y), where $x, y \in G_i$ for some i.

Let $N = \{1, 2, \ldots, n\}$. Let M_1 and M_2 be a pair of orthogonal Latin squares of side n defined on N. Let M be the array obtained by superimposing M_1 and M_2, $M = M_1 \circ M_2$. Let $M_{x,y}$ be the array obtained by replacing each pair (a, b) in M by the pair (a_x, b_y).

Let A' be the array $A' = [FL]$. Replace a cell in A' which contains the pair (x, y) by the array $M_{x,y}$. Replace each empty cell of A' by an $n \times n$ empty array. Let A be the resulting array. Thus A is an $np \times 2np$ array.

Let B_i be a PBTD($n|G_i|+1$) defined on $V'_i = (N \times (G_i \cup H_i)) \cup \{\alpha, \infty\}$. We will write B_i in the following form.

$B_i =$

D_i	E_i	C_i
R_{i1}	R_{i2}	$\alpha\infty$

so that the array $\begin{array}{|c|c|}\hline D_i & C_i \\ \hline R_{i1} & \alpha\infty \\ \hline\end{array}$ is an $\mathrm{H}(n|G_i|+1, 2n|G_i|+2)$ and the array $\begin{array}{|c|c|}\hline E_i & C_i \\ \hline R_{i2} & \alpha\infty \\ \hline\end{array}$ is an $\mathrm{H}(n|G_i|+1, 2n|G_i|+2)$.

Now we will construct a PBTD$(n\sum_{i=1}^{m}|G_i|+1)$ from F, L and $B_1, B_2, \ldots, B_m$. Essentially we will fill in the empty $n|G_i| \times n|G_i|$ subarrays in A with subarrays from the B_is and then add an additional row and column.

$$B = \begin{array}{|cccccccc|c|}\hline D_1 & & & & E_1 & & & & C_1 \\ & D_2 & & A & & E_2 & & & C_2 \\ & & \ddots & & & & \ddots & & \vdots \\ & & & D_m & & & & E_m & C_m \\ \hline R_{11} & R_{21} & \ldots & R_{m1} & R_{12} & R_{22} & \ldots & R_{m2} & \alpha\infty \\ \hline\end{array}$$

The array B has dimensions $(np+1) \times (2np+1)$ and it contains elements from the set $V' = N \times (V \cup \overline{V}) \cup \{\alpha, \infty\}$ which is a set of cardinality $2np+2$.

We want to show that every unordered pair of V' appears in B. Clearly the unordered pair $\{\alpha, \infty\}$ appears in B. Now consider the unordered pairs $\{\alpha, a_x\}$ and $\{\infty, a_x\}$ where $a \in N$ and $x \in V \cup \overline{V}$. The element a_x belongs to a unique V_i' and thus these pairs appear in the subarray B_i.

Consider the unordered pair $\{a_x, b_y\}$ where a_x and b_y are distinct elements, $a, b \in N$ and $x, y \in V \cup \overline{V}$. If $a_x, b_y \in V_i'$ for some i then the unordered pair $\{a_x, b_y\}$ appears in the subarray B_i. If a_x and b_y do not both belong to V_i' for some i and $x, y \in V$ or $x, y \in \overline{V}$ then the unordered pair $\{x, y\}$ appears in F and thus the unordered pair $\{a_x, b_y\}$ appears in $M_{x,y}$. If a_x and b_y do not both belong to V_i' for some i and one of x and y belongs to V and the other belongs to $\overline{V}$ then assume without loss of generality that $x \in V$ and $y \in \overline{V}$. Thus the pair $\{x, y\}$ appears in L and therefore the unordered pair $\{a_x, b_y\}$ appears in

$M_{x,y}$. Therefore every pair of distinct elements from V' appears in B.

Consider any of the first np columns of B. This column passes through the subarray D_i for some i. The elements α and ∞ appear in every column of D_i. Therefore they appear in the column we are considering. Now consider the element a_x where $a \in N$ and $x \in V \cup \overline{V}$. If $x \in V_i'$ then a_x appears in the part of the column that belongs to D_i or in the pair R_{i1}. If $x \notin V_i'$ then we will consider the column of A' from which this column was created. The element x appears in the part of the column that belongs to F, since F is the superposition of a complementary frame defined on V and its complement defined on $\overline{V}$ and since $x \notin V_i'$. Assume that in A', x appears with the element y. Thus in B, a_x appears in the part of the column that belongs to the array $M_{x,y}$, since $M_{x,y}$ is the superposition of two orthogonal Latin squares. Thus the first np columns of B are Latin.

Consider any of the last $np+1$ columns of B except the very last column. This column passes through the subarray E_i for some i. The elements α and ∞ appear in every column of E_i. Therefore they appear in the column we are considering. Now consider the element a_x where $a \in N$ and $x \in V \cup \overline{V}$. If $x \in V_i'$ then a_x appears in the part of the column that belongs to E_i or in the pair R_{i2}. If $x \notin V_i'$ then we will consider the column of A' from which this column was created. The element x appears in the part of the column that belongs to L, since L is the superposition of two OPILS where one is defined on V and the other is defined on $\overline{W}$ and since $x \notin V_i'$. Assume that in A', x appears with the element y. Thus in B, a_x appears in the part of the column that belongs to the array $M_{x,y}$. Thus this column is Latin. The union of the pairs of the last column is

$$\bigcup_{i=1}^{m} C_i \bigcup \{\alpha, \infty\} = \bigcup_{i=1}^{m} N \times (G_i \cup H_i) \bigcup \{\alpha, \infty\} = V'$$

Thus the last column is Latin.

Consider any of the rows of B other than the last row. This row passes through the subarray D_i and the subarray E_i for some i. The elements α and ∞ appear once in each row of D_i and once in each row of E_i. Thus α and ∞ appear twice in this row, once in the first np columns and once in the last $np+1$ columns. Now consider the element a_x where $a \in N$ and $x \in V \cup \overline{V}$. If $a_x \in V_i'$ then a_x appears in this row in a pair that belongs to D_i and in a pair that belongs to E_i, unless it was a deficient element for the row of B_i that this row corresponds to and in this case x appears only once in this row, in the pair of C_i. Thus a_x appears once in the first np columns and once in the last $np+1$ columns, unless it is a deficient element and in that case it appears only in the last column. If $a_x \notin V_i'$ then consider the row of A' from which this row was created. The element x appears in a pair, say with element y, that belongs to F and in a pair, say with element z that belongs to L. Therefore in B, a_x appears in the first np columns in the part of the row that belongs to $M_{x,y}$ and in the last np columns in the part of the row that belongs to $M_{x,z}$. Also

$$\bigcup_{i=1}^{m} R_{i1} \bigcup \{\alpha, \infty\} = \bigcup_{i=1}^{m} R_{i2} \bigcup \{\alpha, \infty\} = V'$$

Therefore the first np columns of B along with the last column are an $\mathrm{H}(np+1, 2np+2)$ and the last $np+1$ columns of B are an $\mathrm{H}(np+1, 2np+2)$.

Therefore B is a PBTD$(n\sum_{i=1}^{m} |G_i| + 1)$. $\square$

For the third construction we need some more definitions.

Definition 3.11 *Let A be a square array. A* transversal *of A is a set of cells such that there is exactly one cell from each row and each column in the set. A transversal is* skew *if it has the property that if the cell (x, y) is in the transversal then so is the cell (y, x).*

Definition 3.12 *Let $V = \bigcup_{i=1}^{n} V_i$ and let $\overline{V} = \bigcup_{i=1}^{n} \overline{V}_i$ where $|V_i| = |\overline{V}_i| = t$ for all i. Thus $|V| = |\overline{V}| = tn$. Let $f : V \to \overline{V}$ be the bijection defined as follows: $f(v_i) = \overline{v}_i$. Let F*

be a complementary $\{V_1, V_2, \ldots, V_n\}$*-frame of type* t^n*. Let* F' *be the complement of* F*. Let* $\overline{F}$ *be the array obtained by applying the bijection* f *to every element in the cells of* F'*. Thus* $\overline{F}$ *is a* $\{\overline{V}_1, \overline{V}_2, \ldots, \overline{V}_n\}$*-frame of type* t^n*. Let* A *be the array of pairs formed by superimposing* F *on* $\overline{F}$*,* $A = F \circ \overline{F}$*. Suppose* A *has a transversal* T *with the following properties:*

1. *Every element of* $(V - V_i) \cup (\overline{V} - \overline{V}_i)$ *occurs exactly once in* T *for some* i*.*

2. T *contains* t *empty cells from the hole* F_i*.*

Let $L = \{L_1, L_2\}$ *be a pair of OPILS of type* t^n *defined on* V *with the partition* $\{V_1, V_2, \ldots, V_n\}$*. Let* L_3 *be the array obtained by applying the bijection* f *to every element in the cells of* L_2*. Thus* L_3 *is a partitioned incomplete Latin square defined on* $\overline{V}$ *with partition* $\{\overline{V}_1, \overline{V}_2, \ldots, \overline{V}_n\}$*. Let* N *be the array of pairs formed by superimposing* L_1 *and* L_3*,* $N = L_1 \circ L_3$*. Let* S *be a transversal of* N *with the following properties:*

1. *Every element of* $(V - V_i) \cup (\overline{V} - \overline{V}_i)$ *occurs precisely once in* S *for the same* i *as* T*.*

2. S *contains* t *empty cells from the* i*th hole of* N*.*

If we can order the pairs of T *and* S *so that every element of* $(V - V_i) \cup (\overline{V} - \overline{V}_i)$ *occurs exactly once as a first coordinate and exactly once as a second coordinate, then we say that the complementary frame* F *and the pair of OPILS* $L = \{L_1, L_2\}$ share an ordered transversal $T \cup S$.

Example 3.4 *A skew frame of type* 2^5*.*

		21 41			40 31	10 20			30 11
			20 40	41 30			11 21	31 10	
	40 21			31 01			00 41	20 30	
41 20					30 00	01 40			21 31
30 40			00 31			41 11			10 01
	31 41	01 30					40 10	11 00	
	20 11	40 00			10 41			01 21	
21 10			41 01	11 40					00 20
11 31			30 21	00 10			20 01		
	10 30	31 20			01 11	21 00			

We give a skew transversal for the skew frame above. The transversal is indicated by the pairs that appear in their cells except for the empty cells where the actual cell is given. Since the transversal is skew we only list half of the pairs, the others are implied.

$$T: \quad 10,20 \quad 11,21 \quad 31,01 \quad 30,00 \quad (9,9) \quad (10,10)$$

The following array is the result of superimposing two OPILS of type 2^5*, where the second array has had the bijection applied to all its elements. This array shares a transversal with the skew frame of the previous example. To see this fact notice that the skew transversal for the skew frame will contain the pairs* (x, y) *and* $(\overline{x}, \overline{y})$ *and the corresponding transversal (which in this case is the same set of cells) for the pair of OPILS contains* *the pairs* $(y, \overline{x})$ *and* $(\overline{y}, x)$.

		$\overline{21}$ 41	$\overline{41}$ 20	$\overline{30}$ 40	$\overline{40}$ 31	10 $\overline{20}$	21 $\overline{10}$	11 $\overline{31}$	30 $\overline{11}$
		$\overline{40}$ 21	$\overline{20}$ 40	$\overline{41}$ 30	$\overline{31}$ 41	20 $\overline{11}$	11 $\overline{21}$	31 $\overline{10}$	10 $\overline{30}$
21 $\overline{41}$	40 $\overline{21}$			$\overline{31}$ 01	$\overline{01}$ 30	$\overline{40}$ 00	$\overline{00}$ 41	20 $\overline{30}$	31 $\overline{20}$
41 $\overline{20}$	20 $\overline{40}$			$\overline{00}$ 31	$\overline{30}$ 00	$\overline{01}$ 40	$\overline{41}$ 01	30 $\overline{21}$	21 $\overline{31}$
30 $\overline{40}$	41 $\overline{30}$	31 $\overline{01}$	00 $\overline{31}$			$\overline{41}$ 11	$\overline{11}$ 40	$\overline{00}$ 10	$\overline{10}$ 01
40 $\overline{31}$	31 $\overline{41}$	01 $\overline{30}$	30 $\overline{00}$			$\overline{10}$ 41	$\overline{40}$ 10	$\overline{11}$ 00	$\overline{01}$ 11
$\overline{10}$ 20	$\overline{20}$ 11	40 $\overline{00}$	01 $\overline{40}$	41 $\overline{11}$	10 $\overline{41}$			$\overline{01}$ 21	$\overline{21}$ 00
$\overline{21}$ 10	$\overline{11}$ 21	00 $\overline{41}$	41 $\overline{01}$	11 $\overline{40}$	40 $\overline{10}$			$\overline{20}$ 01	$\overline{00}$ 20
$\overline{11}$ 31	$\overline{31}$ 10	$\overline{20}$ 30	$\overline{30}$ 21	00 $\overline{10}$	11 $\overline{00}$	01 $\overline{21}$	20 $\overline{01}$		
$\overline{30}$ 11	$\overline{10}$ 30	$\overline{31}$ 20	$\overline{21}$ 31	10 $\overline{01}$	01 $\overline{11}$	21 $\overline{00}$	00 $\overline{20}$		

We are now ready to describe the next construction.

Theorem 3.20 *[7] Let* m *be a positive integer such that* $m \neq 2, 6$. *Suppose that there exists*

1. *a complementary frame* F_1 *of type* t^n *and a pair of OPILS of type* t^n *with a shared ordered transversal,*
2. *an* $IA(m + k, k, 4)$,
3. *a* $PBTD(tm + 1)$ *and*
4. *a* $PBTD(tm + k + 1)$.

Then there is a PBTD$(tmn + k + 1)$.

Proof Let $V = \bigcup_{i=1}^{n} V_i$ and let $\overline{V} = \bigcup_{i=1}^{n} \overline{V}_i$ where $|V_i| = |\overline{V}_i| = t$ for all i. Thus $|V| = |\overline{V}| = tn$. Let $f : V \to \overline{V}$ be the bijection defined as follows: $f(v_i) = \overline{v}_i$. Let $M = \{1, 2, \ldots, m\}$. Let $U = \beta \cup \gamma$ where $\beta = \{\alpha_1, \alpha_2, \ldots, \alpha_k\}$ and $\gamma = \infty_1, \infty_2, \ldots, \infty_k\}$.

Let F_1 be a complementary $\{V_1, V_2, \ldots, V_n\}$-frame of type t^n. Let F_2 be the complement of F_1. Let F_3 be the array obtained by applying the bijection f to every element in the cells of F_2. Let F be the array obtained by superimposing F_1 on F_3, $F = F_1 \circ F_3$. Let $L' = \{L_1, L_2\}$ be a pair of OPILS of type t^n defined on V with partition $\{V_1, V_2, \ldots, V_n\}$. Let L_3 be the array obtained by applying the bijection f to every element in the cells of L_2. Thus L_2 is a partitioned incomplete Latin square defined on $\overline{V}$ with partition $\{\overline{V}_1, \overline{V}_2, \ldots, \overline{V}_n\}$. Let L be the array of pairs formed by superimposing L_1 and L_3, $L = L_1 \circ L_3$.

F_1 and L' share an ordered transversal. Let T be the transversal of F such that every element of $(V - V_n) \cup (\overline{V} - \overline{V}_n)$ occurs precisely once in T and also T contains t empty cells from hole F_n. Let S be the transversal of L such that every element of $(V - V_n) \cup (\overline{V} - \overline{V}_n)$ occurs precisely once in S and also S contains t empty cells from the last hole of L defined on $V_n \cup \overline{V}_n$. The pairs in $T \cup S$ are ordered so that every element of $(V - V_n) \cup (\overline{V} - \overline{V}_n)$ occurs once as a first coordinate and once as a second coordinate. Let $a(i)$ denote the pair in T which occurs in the ith row of F and let $b(i)$ be the pair in T which occurs in the ith column of F for $i = 1, 2, \ldots, t(n-1)$. Similarly let $c(i)$ denote the pair in S which occurs in the ith row of L and let $d(i)$ be the pair in S which occurs in the ith column of L for $i = 1, 2, \ldots, t(n-1)$.

Since $m \neq 2, 6$ then by Lemma 2.2, there exists a pair of orthogonal Latin squares of side m defined on M. Let this pair be N_1 and N_2 and let N be the superposition of N_1 on N_2, $N = N_1 \circ N_2$. Let N_{xy} be the array obtained by replacing every pair (a, b) in N

by the pair $((a,x),(b,y))$.

Since there exist an IA$(m+k,k,4)$, there exist a pair of mutually orthogonal Latin squares of side $m+k$ which are missing a subsquare of side k. Let this pair be I_1 and I_2. Let I be the array of pairs obtained by superimposing I_1 and I_2, $I = I_1 \circ I_2$. Let I_{xy} be the array obtained from I in the following way. Replace each element of I_1 that belongs to the missing subsquare by an element of β and replace every other element a of I_1 by pair (a,x). Replace each element of I_2 that belongs to the missing subsquare by an element of γ and replace every other element b of I_2 by pair (b,y). Thus I_{xy} is defined on $(M \times \{x,y\}) \cup U$ where the missing subsquare is defined on U. I_{xy} can be written in the form, $I_{xy} = \begin{array}{|c|c|}\hline A_{xy} & C_{xy} \\ \hline R_{xy} & 0 \\ \hline \end{array}$ where 0 is an empty array of side k.

Let $B_1 = [F,L]$. Replace each pair (x,y) in $F-T$ and $L-S$ by the array N_{xy}. Replace each ordered pair (x,y) in $T \cup S$ with A_{xy}. Let B_2 be the resulting array. Thus B_2 has dimensions $mnt \times 2mnt$.

To B_2 we add k new rows. The array of new rows is given below. Let $w = (n-1)t$. These rows have $2tmn + 2k$ columns because we are going to add $2k$ new columns to B_2 as well. The subarrays labelled E_z are empty arrays of dimensions $k \times z$.

$R_{b(1)}$	$\ldots$	$R_{b(w)}$	E_{mt}	$R_{d(1)}$	$\ldots$	$R_{d(w)}$	E_{mt}	E_{2k}

The array of new columns is given below. The array E is an empty array of dimensions $(mt+k) \times 2k$.

$C_{a(1)}$	$C_{c(1)}$
$\vdots$	$\vdots$
$C_{a(w)}$	$C_{c(w)}$
E	

The new rows and columns intersect but they intersect only in empty cells. We will fill the empty subarrays in the new rows and columns as well as the empty subarrays in F and L with PBTDs. Let D_i be a PBTD$(mt+1)$ defined on $(M \times (V_i \cup \overline{V}_i)) \cup \{\alpha, \infty\}$ for $i = 1, 2, \ldots, n-1$. D_i can be written in the following form where the first mt columns along with the last column is an H$(mt+1, 2mt+2)$ and the last $mt+1$ columns is an H$(mt+1, 2mt+2)$.

$D_i =$

D_i^1	D_i^2	D_i^3
D_i^4	D_i^5	$\alpha\infty$

Let G be a PBTD$(mt+k+1)$ defined on $(M \times (V_n \cup \overline{V}_n)) \cup U \cup \{\alpha, \infty\}$. G can be written in the following form where $G_1 \cup G_3 \cup G_5$ and $G_2 \cup G_4 \cup G_5$ are H$(mt+k+1, 2mt+2k+2)$.

$G =$

G_1	G_2	G_3	G_4	G_5
				$\alpha\infty$
mt	mt	k	k	1

Therefore filling in the empty subarrays we obtain the following array which we will call B.

$B =$

D_1^1					D_1^2							D_1^3
	D_2^1					D_2^2						D_2^3
		$\ddots$					$\ddots$					$\vdots$
			D_{n-1}^1					D_{n-1}^2				D_{n-1}^3
				G_1					G_2	G_3	G_4	G_5
D_1^4	D_2^4	$\ldots$	D_{n-1}^4		D_1^5	D_2^5	$\ldots$	D_{n-1}^5				$\alpha\infty$
C_1					C_2					C_3	C_4	C_5

We claim that B is a PBTD$(mnt+k+1)$ with both $C_1 \cup C_3 \cup C_5$ and $C_2 \cup C_4 \cup C_5$ being H$(mnt+k+1, 2mnt+2k+2)$.

Let $V' = (M \times (V \cup \overline{V})) \cup U \cup \{\alpha, \infty\}$. Thus $|V'| = 2tmn + 2k + 2$. B is defined on V' and has dimensions $(tmn+k+1) \times (2tmn+2k+1)$.

We have to show that any unordered pair of distinct elements from V' occurs in a cell of B. G contains all the unordered pairs of distinct elements of $(M \times (V_n \cup \overline{V}_n)) \cup U \cup \{\alpha, \infty\}$. The array D_i contains all the unordered pairs of distinct elements of $(M \times (V_i \cup \overline{V}_i)) \cup \{\alpha, \infty\}$ for $1 \leq i \leq n-1$.

Every element of $(V - V_n) \cup (\overline{V} - \overline{V}_n)$ occurs exactly once as a first co-ordinate and exactly once as a second co-ordinate in the ordered pairs of the cells of $T \cup S$. For every pair (x, y) which occurs in a cell of $T \cup S$, the pairs of the array I_{xy} appear in B. The array I_{xy} contains all the ordered pairs of $(((M \times \{x\}) \cup \beta) \times ((M \times \{y\}) \cup \gamma)) - (\beta \times \gamma)$. Therefore all the unordered pairs where one element is from U and one element is from $M \times ((V \cup \overline{V}) - (V_n \cup \overline{V}_n))$ appear in B. We also have the ordered pairs of $(M \times \{x\}) \times (M \times \{y\})$ where the pair (x, y) occurs in a cell of $T \cup S$ occurring in B.

The array N_{xy} contains all the ordered pairs of $(M \times \{x\}) \times (M \times \{y\})$. The array N_{xy} occurs in B for all pairs (x, y) which occur in the cells of $F - T$ and $L - S$.

The unordered pair $\{x, y\}$ occurs in B_1 as long as x and y are not both in $V_i \cup \overline{V}_i$ for some i. Therefore B contains all the unordered pairs where one element is from $M \times \{x\}$ and the other is from $M \times \{y\}$ where $x \neq y$ and x and y are not both in $V_i \cup \overline{V}_i$ for some i. The pair either appears in an N_{xy} or I_{xy} subarray. Therefore B contains all the unordered pairs of distinct elements from V'.

Now we need to show that every column of B is Latin. Every column of B_1 intersects a hole. Assume that the column of B_1 we are considering intersects the ith hole of F or L. Thus this column contains all the elements of $(V \cup \overline{V}) - (V_i \cup \overline{V}_i)$. Exactly one cell in this column belongs to $T \cup S$. Every pair (x, y) of F and L is replaced either by the array N_{xy} or the array A_{xy}. If a pair of F or L is replaced by the array A_{xy} then the array R_{xy} will be appended to the columns which intersect A_{xy}. The new set of columns created from the one we started with has an $mt \times mt$ hole in it which we fill with part of the columns of D_i. The remaining part of these columns of D_i we append to the bottom of these columns. However for the nth hole we fill it with columns from G and these columns extend beyond the hole and make the new columns as long as the other columns that had arrays appended to them.

Every column of N_{xy} contains every element of $M \times \{x, y\}$. The columns of I_{xy} which intersect A_{xy} contain all the elements of $(M \times \{x, y\}) \cup U$. Thus by replacing each pair (x, y) by the array N_{xy} or by the array A_{xy}, we obtain a column that contains all the elements of $(M \times ((V \cup \overline{V}) - (V_i \cup \overline{V}_i))) \cup U$. Every column of D contains all the elements of $(M \times (V_i \cup \overline{V}_i)) \cup \{\alpha, \infty\}$. Therefore a column of B that filled in its hole with part of the columns of D is Latin.

The columns which filled in the hole with part of the columns of G correspond to a column in B_1 which intersected the nth hole. Therefore the cells of $T \cup S$, in these columns would have been empty. Therefore every pair (x, y) in these columns would be replaced by the array N_{xy}. The columns of G contain every element of $(M \times (V_n \cup \overline{V}_n)) \cup U \cup \{\alpha, \infty\}$ and therefore these columns of B are Latin as well.

Now we consider the last $2k+1$ columns of B. The first k of these columns are made of a column from each of $C_{a(1)}, \ldots C_{a(w)}$ and then a column of G. The next k columns are made of a column from each of $C_{c(1)}, \ldots C_{c(w)}$ and then a column of G. Every column of C_{xy} contains all the elements of $M \times \{x, y\}$. Since $a(1), \ldots a(w)$ are all the pairs of T and $c(1), \ldots c(w)$ are all the pairs of S, thus $\bigcup_{i=1}^{w} a(i) = \bigcup_{i=1}^{w} c(i) = (V - V_n) \cup (\overline{V} - \overline{V}_n)$. Therefore each of these columns contains all the elements of $M \times (V - V_n) \cup (\overline{V} - \overline{V}_n)$ and with the column from G we obtain the rest of the elements of V'.

We now consider the last column. It is constructed from the last column of each D_i where $1 \leq i \leq n-1$ and a column from G. Thus it is Latin.

To finish the proof we need to show that for each row of B, every element of V' appears once in the columns of $C_1 \cup C_3 \cup C_5$ and once in the columns of $C_2 \cup C_4 \cup C_5$.

Every row of B_1 intersects two holes. Let these holes be the ith hole of F and the ith hole of L. Thus this row contains all the elements of $(V \cup \overline{V}) - (V_i \cup \overline{V}_i)$ in the first tn columns of B_1 and also again in the last tn columns. As before every pair (x, y) of B_1 is replaced either by the array N_{xy} or the array A_{xy}. If a pair of F is replaced by the array A_{xy} then the array C_{xy} will be appended to the rows which intersect A_{xy}. The columns of this array will fall in the set C_3. Similarly if a pair of L is replaced by the array A_{xy} then the array C_{xy} will be appended to the rows which intersect A_{xy} and the columns of this array will fall in the set C_4. The first hole in these new rows will be filled with D_i^1 and the second hole will be filled with D_i^2. D_i^3 will be appended to the rows and its column will fall in the set C_5. However we fill the nth holes of F and L with rows from G_1 and G_2 respectively.

Every row of N_{xy} contains every element of $M \times \{x, y\}$. The rows of I_{xy} which intersect A_{xy} contain all the elements of $(M \times \{x, y\}) \cup U$. Thus by replacing each pair (x, y) by the array N_{xy} or by the array A_{xy} we obtain a row that contains all the elements of $(M \times ((V \cup \overline{V}) - (V_i \cup \overline{V}_i))) \cup U$ in the columns of $C_1 \cup C_3$ and also again in the columns of $C_2 \cup C_4$. Every row of D contains all the elements of $(M \times (V_i \cup \overline{V}_i)) \cup \{\alpha, \infty\}$ once in the columns of D_i^1 and D_i^3 and then also once in the columns of D_i^2 and D_i^3. Therefore a row of B that filled in the holes with rows of D_i has every element of V' appearing once in the columns of $C_1 \cup C_3 \cup C_5$ and once in the columns of $C_2 \cup C_4 \cup C_5$.

The rows which filled in the holes with rows of G_1 and G_2 correspond to rows which in B_1 intersected the nth hole of F and L. Therefore the cells of $T \cup S$ in these rows would have been empty. Therefore every pair (x, y) in these rows would be replaced by the array N_{xy}. The rows of G contain every element of $(M \times (V_n \cup \overline{V}_n)) \cup U \cup \{\alpha, \infty\}$ in the columns of G_1, G_3 and G_5 and also once in the columns of G_2, G_4 and G_5. Since G_3, G_4 and G_5 are appended to these rows such that their columns belong to the sets C_3, C_4 and C_5 respectively, thus these rows satisfy the condition that every element of V' appears once in the columns of $C_1 \cup C_3 \cup C_5$ and once in the columns of $C_2 \cup C_4 \cup C_5$.

Now we consider the last $k+1$ rows of B except for the last row. The part of this row that intersects C_1 consist of a row from each of $R_{b(1)}, \ldots R_{b(w)}$ and then a row of G_1. The part of this row that intersects C_2 consist of a row from each of $R_{d(1)}, \ldots R_{d(w)}$ and then a row of G_2. Every row of R_{xy} contains all the elements of $M \times \{x, y\}$. Since $b(1), \ldots b(w)$ are all the pairs of T and $d(1), \ldots d(w)$ are all the pairs of S, thus $\bigcup_{i=1}^{w} b(i) = \bigcup_{i=1}^{w} d(i) = (V - V_n) \cup (\overline{V} - \overline{V}_n)$. Therefore each of these row contains all the elements of $M \times ((V - V_n) \cup (\overline{V} - \overline{V}_n))$ once in C_1 and once in C_2. Since G_3, G_4 and G_5 are appended to these rows such that their columns belong to the sets C_3, C_4 and C_5 respectively thus these rows satisfy the condition that every element of V' appears once in the columns of $C_1 \cup C_3 \cup C_5$ and once in the columns of $C_2 \cup C_4 \cup C_5$.

The last row of B intersects the set $C_1 \cup C_3 \cup C_5$ with the rows $D_1^4, \ldots D_{n-1}^4$, and last row of G_1 and G_3 and the pair $\{\alpha, \infty\}$. By definition of the D_is and G these rows contain every element of V'. The last row intersects the set $C_2 \cup C_4 \cup C_5$ with the rows $D_1^5, \ldots D_{n-1}^5$, and last row of G_2 and G_4 and the pair $\{\alpha, \infty\}$. By definition of the D_is and G these rows contain every element of V'.

Therefore every row of B satisfies the condition that every element of V' appears once in the columns of $C_1 \cup C_3 \cup C_5$ and once in the columns of $C_2 \cup C_4 \cup C_5$. Thus B is a PBTD$(mnt + k + 1)$. $\square$

For the next construction we need to define an intransitive starter and adder over $\mathbb{Z}_{2n}$ for a PBTD$(n + m)$ written on the symbol set $\mathbb{Z}_{2n} \cup \{\infty_i | i = 1, 2, \ldots, 2m\}$, where

$n > 2m$. We begin with some additional notation. Let $B_i = \{x_i, y_i\}$ for $i = 1, \dots, n-2m$, $B_i = \{x_i = \infty_{i-n+2m}, y_i\}$ for $i = n-2m+1, \dots, n$, $R_j = \{u_{j1}, u_{j2}\}$ for $j = 1, \dots, m$ and $C_j = \{v_{j1}, v_{j2}\}$ for $j = 1, \dots, m-1$. We also define $x + \infty_i = \infty_i$, for $x \in \mathbb{Z}_{2n}$ and $1 \leq i \leq 2m$. If A is the pair $\{\alpha, \beta\} \subset \mathbb{Z}_2 n \cup \{\infty_i | i = 1, 2, \dots, 2m\}$ and $s \in \mathbb{Z}_{2n}$ then we define $A + s$ to be the pair $\{\alpha + s, \beta + s\}$.

Definition 3.13 *Let* $n \equiv 1 \pmod 2$*. An* intransitive starter for a PBTD$(n+m)$ defined on $\mathbb{Z}_{2n} \cup \{\infty_1, \infty_2, \dots, \infty_{2m}\}$ *is a triple* (S, R, C) *where* $S = \{B_i | i = 1, \dots, n\}$*,* $R = \{R_j | j = 1, \dots, m\}$ *and* $C = \{C_j | j = 1, \dots, m-1\}$ *satisfying the following properties:*

1. $\displaystyle\bigcup_{B \in S \cup R} B = \mathbb{Z}_{2n} \cup \{\infty_1, \dots, \infty_{2m}\}$.
2. *Let* $D_0 = \{0, n\}$. $\{\pm(x_i - y_i) | i = 1, \dots n - 2m\} \cup \{\pm(u_{j1} - u_{j2}) | j = 1, \dots, m\} \cup \{\pm(v_{j1} - v_{j2}) | j = 1. \dots, m-1\} = \mathbb{Z}_{2n} - D_0$.
3. $\{\pm(v_{j1} - v_{j2}) | j = 1 \dots, m-1\} \cap \{0, 2, 4, \dots, 2(n-1)\} = \emptyset$.
4. $\{\pm(u_{j1} - u_{j2}) | j = 1 \dots, m\} \cap \{0, 2, 4, \dots, 2(n-1)\} = \emptyset$.

Let H *be the subgroup* $\{0, 2, 4, \dots, 2n-2\} \subset \mathbb{Z}_{2n}$*. An* adder *for the intransitive starter* (S, R, C) *is a bijection* $a : S \to H$ *such that*

$$\bigcup_{i=1}^{n} (B_i + a(x_i, y_i)) \bigcup_{i=1}^{m-1} C_i = (\mathbb{Z}_{2n} \cup \{\infty_i | i = 1, 2, \dots, 2m\}) \setminus D_0.$$

We now present the construction.

Theorem 3.21 *[6] Suppose there exists an intransitive starter* (S, C, R) *and a corresponding adder for a PBTD*$(n+m)$ *defined on* $\mathbb{Z}_{2n} \cup \{\infty_1 \dots, \infty_{2m}\}$ *and suppose there exists a PBTD*(m)*. Then there exists a PBTD*$(n+m)$*.*

Proof We begin by constructing an $n \times 2n$ array, G_1. Label its rows $0, 1, \dots, n-1$ and its columns $0, 1, \dots, 2n-1$. For $i = 1, 2, \dots, n$, $j = 0, 1, \dots, n-1$ and $\ell = 0, 1$, place $B_i + a(x_i, y_i) + \ell n + j$, in cell $(j, a(x_i, y_i) + \ell n + j)$, where the second argument is taken modulo $2n$. Let H be the subgroup $\{0, 2, 4, \dots, 2n-2\} \subset \mathbb{Z}_{2n}$. Since the range of a is H and n is odd, every cell in G_1 is filled.

Next we add m new rows to G_1. We label these rows $n, n+1, \dots, n+m-1$. For $i = 0, 1, \dots, m-1$ and $j = 0, 1, \dots, 2n-1$ place $R_{i+1} + j$ in cell $(n+i, j)$. We call the resulting $(n+m) \times 2n$ array G_2.

Next we construct $m-1$ arrays of size $n \times 2$. We call these arrays $G_{1,3}, G_{2,3}, \dots, G_{m-1,3}$. Label the rows of each array $0, 1, \dots, n-1$ and the columns $0, 1$. For $\ell = 0, 2, 4, \dots, n-1$, place $C_i + \ell$ in cell $(\ell, 0)$ of array $G_{i,3}$ and $C_i + \ell + n$ in cell $(\ell, 1)$ of $G_{i,3}$. For $\ell = 1, 3, 5, \dots, n-2$, place $C_i + \ell$ in cell $(\ell, 1)$ of array $G_{i,3}$ and $C_i + \ell + n$ in cell $(\ell, 0)$ of $G_{i,3}$. Let $G_3 = [G_{1,3} G_{2,3} \dots G_{m-1,3}]$. G_3 is an $n \times 2(m-1)$ array.

Next we construct a $n \times 1$ array which we will call G_4. We label the rows $0, 1, \dots, n-1$. In row ℓ we place $D_0 + \ell$.

Let P be a PBTD(m) defined on the set $\{\infty_1, \dots, \infty_{2m}\}$. Label the columns of P, $0, 1, \dots, 2m-2$. Arrange the columns of P so that the even numbered columns are an H$(m, 2m)$ and the odd numbered columns along with the last column are also an H$(m, 2m)$. Using all the arrays we previously constructed, we construct a $(n+m) \times (2n + 2m - 1)$ array as outlined below which we will call G.

$$G = \begin{array}{|c|c|c|} \hline & G_3 & G_4 \\ \cline{2-3} G_2 & \multicolumn{2}{c|}{P} \\ \hline \end{array}$$

We label the columns of G, $0, 1, \dots, 2n + 2m - 2$ and the rows, $0, 1, \dots, n + m - 1$. We claim that G is a PBTD$(n+m)$, where all the even numbered columns form an

H$(m+n, 2m+2n)$ and all the odd numbered columns plus the last column form an H$(m+n, 2m+2n)$. G is an $(n+m) \times (2n+2m-1)$ array defined on $V = \mathbb{Z}_{2n} \cup \{\infty_1 \ldots, \infty_{2m}\}$, which is a set of cardinality $2n+2m$. Therefore G has the correct dimensions and is defined on a set of the correct cardinality.

First we need to show that every unordered pair of distinct elements of V appears in G. Since P is a PBTD(m) defined on $\{\infty_1 \ldots, \infty_{2m}\}$, every unordered pair of distinct elements from $\{\infty_1 \ldots, \infty_{2m}\}$ appears in G. The rest of the pairs which appear in G are $B_i + a(x_i, y_i) + k$, where $1 \leq i \leq n$ and $0 \leq k \leq 2n-1$, $R_i + k$ where $1 \leq i \leq m$ and $0 \leq k \leq 2n-1$, $C_i + k$ where $1 \leq i \leq m-1$ and $0 \leq k \leq 2n-1$ and $D_0 + k$ where $0 \leq k \leq n-1$. Since $\{\pm(x_i - y_i) | i = 1, \ldots n-2m\} \cup \{\pm(u_{j1} - u_{j2}) | j = 1, \ldots, m\} \cup \{\pm(v_{j1} - v_{j2}) | j = 1. \ldots, m-1\} = \mathbb{Z}_{2n} - D_0$ and the difference between the elements in D_0 is n and since we take all possible translates of the B_is, R_is, C_is and D_0, thus all unordered pairs of distinct elements of $\mathbb{Z}_{2n}$ appear in G. Finally, since we take all possible translates of the B_is, we also have all unordered pairs where one element is from $\mathbb{Z}_{2n}$ and the other is from $\{\infty_1 \ldots, \infty_{2m}\}$ appearing in G. Thus all unordered pairs of distinct elements of V appear in G.

Next we will show that all of the columns of G are Latin. The kth column of G_2 consists of the pairs $B_i + k$, where $i = 1, \ldots, n$ and $R_i + k$ where $i = 1, \ldots, m$. Since $\bigcup_{B \in S \cup R} B = \mathbb{Z}_{2n} \cup \{\infty_1, \ldots, \infty_{2m}\}$, thus these columns are Latin.

The array G_3 consists of the arrays $G_{1,3}, G_{2,3}, \ldots, G_{m-1,3}$. Consider the first column of $G_{i,3}$. It consists of the pairs $C_i + 2k$ where $k = 0, 1, \ldots, n-1$. Since $\{\pm(v_{j1} - v_{j2}) | j = 1 \ldots, m-1\} \cap \{0, 2, 4, \ldots, 2(n-1)\} = \emptyset$, this column contains every element of $\mathbb{Z}_{2n}$. Now consider the second column of $G_{i,3}$. It consists of the pairs $C_i + 2k + 1$ where $k = 0, 1, \ldots, n-1$. So just like the previous column, this column contains every element of $\mathbb{Z}_{2n}$.

The column G_4 consists of the pairs $D_0 + k$ where $k = 0, 1, \ldots, n-1$. Since $D_0 =$

$\{0, n\}$, this column contains every element of $\mathbb{Z}_{2n}$.

Since P is a PBTD(m) defined on $\{\infty_1 \dots, \infty_{2m}\}$ therefore each column of P contains every element of $\{\infty_1 \dots, \infty_{2m}\}$. Therefore, the columns of G that intersect P consist of a column of either G_3 or G_4, which contain all the elements of $\mathbb{Z}_{2n}$, and a column of P which contains all the elements of $\{\infty_1 \dots, \infty_{2m}\}$. Thus these columns are Latin.

Lastly we need to show that every element of V appears once in the even numbered columns of G and once in the odd numbered columns along with the last column. First we consider the kth row of G that does not intersect the subarray P, where k is an even, non-negative integer. Since n is odd and since every element in A is even, the pairs that appear in the even numbered columns of this row are $B_i + a(x_i, y_i) + k$, for $i = 1, \dots, n$, $C_i + k$ for $i = 1, \dots, m-1$, and $D_0 + k$. Since $\bigcup_{i=1}^{n}(B_i + a(x_i, y_i)) \cup \bigcup_{i=1}^{m-1} C_i = \mathbb{Z}_{2n} \cup \{\infty_1, \dots, \infty_{2m}\} - D_0$, thus every element of V appears in the even numbered columns of this row. The pairs that appear in the odd numbered columns along with the last column of this row are $B_i + a(x_i, y_i) + n + k$, for $i = 1, \dots, n$, $C_i + n + k$ for $i = 1, \dots, m-1$, and $D_0 + k$. Since $D_0 + n + k = D_0 + k$ and since $\bigcup_{i=1}^{n}(B_i + a(x_i, y_i)) \cup \bigcup_{i=1}^{m-1} C_i = \mathbb{Z}_{2n} \cup \{\infty_1, \dots, \infty_{2m}\} - D_0$, thus every element of V appears in the odd numbered columns along with the last column of this row. Now if k is an odd non-negative integer then the pairs that appear in the even numbered columns of the kth row are $B_i + a(x_i, y_i) + n + k$, for $i = 1, \dots, n$, $C_i + n + k$ for $i = 1, \dots, m-1$, and $D_0 + k$. and the pairs that appear in the odd numbered columns along with the last column of the kth row are $B_i + a(x_i, y_i) + k$, for $i = 1, \dots, n$, $C_i + k$ for $i = 1, \dots, m-1$, and $D_0 + k$. Thus in this row every element of V appears once in the even numbered columns of G and once in the odd numbered columns along with the last column.

Now we consider the rows of G which do intersect the subarray P, that is rows $n, \dots, n + m - 1$. Consider row $n + k$ where $0 \le k \le m - 1$. Since P is a PBTD(m)

defined on the set $\{\infty_1, \ldots, \infty_{2m}\}$ such that the even numbered columns of P are an H$(m, 2m)$ and the odd numbered columns of P along with the last column are also an H$(m, 2m)$, thus in this row, the even numbered columns of P contain every element of $\{\infty_1, \ldots, \infty_{2m}\}$ and the odd numbered columns of P along with the last column also contain every element of $\{\infty_1, \ldots, \infty_{2m}\}$. In this row, the pairs that appear in the even numbered columns of G_2 are $R_{k+1} + j$ for $j = 0, 2, \ldots, 2n-2$ and the pairs that appear in the odd numbered columns of G_2 are $R_{k+1} + j$ for $j = 1, 3, \ldots, 2n-1$. Since $\{\pm(u_{j1} - u_{j2}) | j = 1 \ldots, m\} \cap \{0, 2, 4, \ldots, 2(n-1)\} = \emptyset$, thus every element of $\mathbb{Z}_{2n}$ appears once in the even numbered columns of G_2 and once in the odd numbered columns. Thus in this row, every element of V appears once in the even numbered columns of G and once in the odd numbered columns along with the last column.

Therefore G is a PBTD$(n + m)$. □

Example 3.5 *Intransitive starters and adders for a PBTD(n+m) where* $(n, m) = (23, 5)$ *and* $(27, 7)$*, which are listed in [7].*

$(n, m) = (23, 5)$

$S\ \{x_i, y_i\}$	0 2	1 5	7 13	10 18	30 40	3 15	20 34
$a(x_i, y_i)$	2	0	4	12	24	34	8
$S\ \{x_i, y_i\}$	21 37	24 42	6 26	9 31	12 43	22 41	∞_1 16
$a(x_i, y_i)$	18	14	6	44	22	30	20
$S\ \{x_i, y_i\}$	∞_2 23	∞_3 27	∞_4 32	∞_5 33	∞_6 28	∞_7 39	∞_8 17
$a(x_i, y_i)$	36	40	38	10	32	42	28
$S\ \{x_i, y_i\}$	∞_9 14	∞_{10} 45					
$a(x_i, y_i)$	26	16					

R	35 36	29 38	8 25	44 19	4 11
C	41 44	26 31	16 27	20 33	
D_0	0 23				

$(n, m) = (27, 7)$

$S\ \{x_i, y_i\}$	0 2	1 5	3 9	4 12	30 40	17 29	7 21
$a(x_i, y_i)$	2	4	8	10	0	6	12
$S\ \{x_i, y_i\}$	22 38	33 51	8 28	23 45	20 44	6 32	∞_1 10
$a(x_i, y_i)$	14	16	52	24	22	38	18
$S\ \{x_i, y_i\}$	∞_2 11	∞_3 25	∞_4 53	∞_5 24	∞_6 26	∞_7 35	∞_8 27
$a(x_i, y_i)$	46	28	32	26	20	48	34
$S\ \{x_i, y_i\}$	∞_9 39	∞_{10} 52	∞_{11} 47	∞_{12} 34	∞_{13} 49	∞_{14} 50	
$a(x_i, y_i)$	36	50	44	40	30	42	
R	41 42	16 31	46 37	48 13	43 14	36 19	18 15
C	34 39	1 8	43 32	51 10	45 24	41 18	
D_0	0 27						

We now have all the necessary constructions to examine the existence of PBTD(n) for $n \equiv 0 \pmod 2, n \geq 6$. To use these constructions we need some existence results for complementary frames and pairs of OPILS. We give the necessary results here.

Lemma 3.10 *[15] There exists a skew frame of type* 1^m *for* $m \equiv 1 \pmod 2$ *for* $m \geq 7$.

Lemma 3.11 *[2] There exists a pair of OPILS of type* 1^m *for* $m \neq 2, 3, 6$.

Lemma 3.12 *[5, 7] The following designs exist.*

1. *A complementary frame of type 2^5 and a pair of OPILS of type 2^5 which share an ordered transversal.*

2. *A complementary frame of type 2^7 and a pair of OPILS of type 2^7 which share an ordered transversal.*

3. *A complementary frame of type 1^n and a pair of OPILS of type 1^n which share an ordered transversal, where $n \in \{7, 11, 23\}$.*

Lemma 3.13 *[7] There exists a complementary frame of type 5^5.*

We now construct the necessary base cases for the recursive construction for PBTD(n) where $n \equiv 0 \pmod 2$.

Theorem 3.22 *[9] Let $m \equiv 1 \pmod 2$, $m \geq 7$. If there exists a PBTD($n+1$) and a pair of orthogonal Latin squares side n, then there is a PBTD($mn+1$).*

Proof By Lemma 3.10 there exists a skew frame of type 1^m, for $m \equiv 1 \pmod 2$ for $m \geq 7$. By Lemma 3.11 there exists a pair of OPILS of type 1^m, for $m \equiv 1 \pmod 2$ for $m \geq 7$. Therefore by Theorem 3.19 there is a PBTD($mn+1$). □

Theorem 3.23 *[5, 7] There exist PBTD(n) for $n \in \{32, 38, 44, 58, 94\}$.*

Proof The existence of these PBTDs is proved using Theorem 3.20. The required designs are outlined in the table below. By Lemma 3.12, there exists the required complementary frames and OPILS which share an ordered transversal. Lemma 3.7 gives the existence of the necessary incomplete orthogonal arrays. By Theorem 3.10 and Example 3.3, there exist a PBTD(6) and a PBTD(8). A PBTD(5) and a PBTD(7) are given in Example 3.1 and Example 3.2 respectively.

Complementary Frame and OPILS of type t^n	IA	PBTD$(tm+1)$	PBTD$(tm+k+1)$	PBTD
2^5	$(4,1,4)$	7	8	32
1^7	$(7,2,4)$	6	8	38
2^7	$(4,1,4)$	7	8	44
1^{11}	$(7,2,4)$	6	8	58
1^{23}	$(5,1,4)$	5	6	94

□

Theorem 3.24 *[7] There exists a PBTD(26).*

Proof By Lemma 3.13 there exists a complementary frame of type 5^5. By Lemma 3.3 there exist a pair of OPILS of type 5^5. There exists a PBTD(6) by Theorem 3.10 and Example 3.3. Therefore by Theorem 3.5 there exists a PBTD(26). □

Theorem 3.25 *[5, 7, 10] Let* $n \equiv 0 \pmod 2$. *Then there exists a PBTD(n) for* $6 \leq n \leq 268$.

Proof The following table lists the constructions used for PBTD(n) for $n \equiv 0 \pmod 2$, $6 \leq n \leq 268$. Note that is still unknown whether a PBTD(n) exists for $n \in \{9, 11, 15\}$.

n	Construction		n	Construction	
6	Starter-Adder	3.10	80	$8 \cdot 10$	3.4
8	Starter-Adder	3.10	82	$9 \cdot 9 + 1$	3.22
10	Starter-Adder	3.10	84	$6 \cdot 14$	3.18
12	Starter-Adder	3.10	86	$17 \cdot 5 + 1$	3.22
14	Starter-Adder	3.10	88	$(5+1) \cdot 14 + 4$	3.11
16	Starter-Adder	3.10	90	$6 \cdot 15$	3.18
18	Starter-Adder	3.10	92	$13 \cdot 7 + 1$	3.22
20	Starter-Adder	3.10	94	$1 \cdot 4 \cdot 23 + 1 + 1$	3.23
22	Starter-Adder	3.10	96	$6 \cdot 16$	3.18
24	$6 \cdot 4$	3.18	98	$7 \cdot 14$	3.4
26	$5 \cdot 5 + 1$	3.24	100	$10 \cdot 10$	3.4
28	Intrans. Starter-Adder	3.21	102	$6 \cdot 17$	3.18
30	$6 \cdot 5$	3.18	104	$8 \cdot 13$	3.4
32	$2 \cdot 3 \cdot 5 + 1 + 1$	3.23	106	$21 \cdot 5 + 1$	3.22
34	Intrans. Starter-Adder	3.21	108	$6 \cdot 18$	3.18
36	$7 \cdot 5 + 1$	3.22	110	$(5+1) \cdot 17 + 8$	3.11
38	$1 \cdot 5 \cdot 7 + 2 + 1$	3.23	112	$7 \cdot 16$	3.4
40	$5 \cdot 8$	3.4	114	$6 \cdot 19$	3.18
42	$6 \cdot 7$	3.18	116	$23 \cdot 5 + 1$	3.22
44	$2 \cdot 3 \cdot 7 + 1 + 1$	3.23	118	$13 \cdot 9 + 1$	3.22
46	$9 \cdot 5 + 1$	3.22	120	$6 \cdot 20$	3.18
48	$6 \cdot 8$	3.18	122	$11 \cdot 11 + 1$	3.22
50	$5 \cdot 10$	3.4	124	$(9+1) \cdot 12 + 4$	3.11
52	$(5+1) \cdot 8 + 4$	3.11	126	$6 \cdot 21$	3.18
54	$6 \cdot 9$	3.18	128	$8 \cdot 16$	3.4
56	$7 \cdot 8$	3.4	130	$10 \cdot 13$	3.4
58	$1 \cdot 5 \cdot 11 + 2 + 1$	3.23	132	$6 \cdot 22$	3.18
60	$6 \cdot 10$	3.18	134	$19 \cdot 7 + 1$	3.22
62	$(5+1) \cdot 10 + 2$	3.11	136	$8 \cdot 17$	3.4
64	$8 \cdot 8$	3.4	138	$6 \cdot 23$	3.18
66	$13 \cdot 5 + 1$	3.22	140	$10 \cdot 14$	3.4
68	$(7+1) \cdot 8 + 4$	3.11	142	$(9+1) \cdot 14 + 2$	3.11
70	$7 \cdot 10$	3.4	144	$6 \cdot 24$	3.18
72	$6 \cdot 12$	3.18	146	$29 \cdot 5 + 1$	3.22
74	$(5+1) \cdot 12 + 2$	3.11	148	$21 \cdot 7 + 1$	3.22
76	$15 \cdot 5 + 1$	3.22	150	$6 \cdot 25$	3.18
78	$6 \cdot 13$	3.18	152	$8 \cdot 19$	3.4

n	Construction		n	Construction	
154	$7 \cdot 22$	3.4	212	$(7+1) \cdot 25 + 12$	3.11
156	$6 \cdot 26$	3.18	214	$(9+1) \cdot 21 + 4$	3.11
158	$(5+1) \cdot 25 + 8$	3.11	216	$6 \cdot 36$	3.18
160	$8 \cdot 20$	3.4	218	$7 \cdot 31 + 1$	3.22
162	$6 \cdot 27$	3.18	220	$10 \cdot 22$	3.4
164	$(9+1) \cdot 16 + 4$	3.11	222	$6 \cdot 37$	3.18
166	$33 \cdot 5 + 1$	3.22	224	$7 \cdot 32$	3.4
168	$6 \cdot 28$	3.18	226	$45 \cdot 5 + 1$	3.22
170	$10 \cdot 17$	3.4	228	$6 \cdot 38$	3.18
172	$19 \cdot 9 + 1$	3.22	230	$10 \cdot 23$	3.4
174	$6 \cdot 29$	3.18	232	$8 \cdot 29$	3.4
176	$8 \cdot 22$	3.4	234	$6 \cdot 39$	3.18
178	$(7+1) \cdot 22 + 2$	3.11	236	$47 \cdot 5 + 1$	3.22
180	$6 \cdot 30$	3.18	238	$7 \cdot 34$	3.4
182	$7 \cdot 26$	3.4	240	$6 \cdot 40$	3.18
184	$8 \cdot 23$	3.4	242	$(11+1) \cdot 20 + 2$	3.11
186	$6 \cdot 31$	3.18	244	$27 \cdot 9 + 1$	3.22
188	$11 \cdot 17 + 1$	3.22	246	$6 \cdot 41$	3.18
190	$10 \cdot 19$	3.4	248	$8 \cdot 31$	3.4
192	$6 \cdot 32$	3.18	250	$10 \cdot 25$	3.4
194	$(7+1) \cdot 24 + 2$	3.11	252	$6 \cdot 42$	3.18
196	$7 \cdot 28$	3.22	254	$11 \cdot 23 + 1$	3.22
198	$6 \cdot 33$	3.18	256	$8 \cdot 32$	3.4
200	$8 \cdot 25$	3.4	258	$6 \cdot 43$	3.18
202	$(19+1) \cdot 10 + 2$	3.11	260	$10 \cdot 26$	3.4
204	$6 \cdot 34$	3.18	262	$29 \cdot 9 + 1$	3.22
206	$41 \cdot 5 + 1$	3.22	264	$6 \cdot 44$	3.18
208	$8 \cdot 26$	3.4	266	$7 \cdot 38$	3.4
210	$6 \cdot 35$	3.18	268	$(5+1) \cdot 43 + 10$	3.11

□

Using the base cases and a recursive construction we prove the existence of PBTD(n) for $n \equiv 0 \pmod 2$, $n \geq 6$.

Theorem 3.26 *[10] Let $n \equiv 0 \pmod 2$, $n \geq 6$. There exists a PBTD(n).*

Proof By Theorem 3.25 there exists a PBTD(n) for $n \equiv 0 \pmod 2$ for $6 \leq n \leq 268$.

Let $n \equiv 0 \pmod 2$, $n \geq 270$. We can write $n = 6(4m+1)+k$ where $m \geq 11$ and $k \in \{0, 2, 4, \ldots, 22\}$. By Theorem 3.10 and Example 3.3 there exists a PBTD(6) generated by a starter-adder pair. By Theorem 3.7 and Theorem 3.17 there exists PBTD$(4m+1)$ and PBTD$(4m+1+k)$. By Lemma 2.2 there exists a pair of orthogonal Latin squares of order $4m+1$. By Lemma 3.7 there exists an IA$(4m+1+k, k, 4)$. Therefore by Theorem 3.11 there exists a PBTD(n).

Thus there exists a PBTD(n) for $n \equiv 0 \pmod 2$, $n \geq 6$. □

3.6 Existence of PBTD(n) for $n \geq 5$

We combine the results of the previous sections. The existence of PBTDs has been determined with three possible exceptions.

Theorem 3.27 *There exists a PBTD(n) for* $n \geq 5$ *except possibly for* $n \in \{9, 11, 15\}$.

Chapter 4

Conclusions

We have proved two main results about the existence of balanced tournament designs and partitioned balanced tournament designs. For balanced tournaments designs we have shown that that there exists a BTD(n) for n a positive integer, $n \neq 2$ and for $n = 2$ a BTD(n) does not exist. For partitioned balanced tournament designs we have shown that there exists a PBTD(n) for n a positive integer, $n \geq 5$, except possibly for $n \in \{9, 11, 15\}$. For $n \leq 4$ a PBTD(n) does not exist. For $n \in \{9, 11, 15\}$ it is still unknown whether a PBTD(n) exists.

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