Smooth center manifolds for impulsive delay differential equations*

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Abstract

The existence and smoothness of center manifolds and a reduction principle are proven for impulsive delay differential equations. Several intermediate results of theoretical interest are developed, including a variation of constants formula for linear equations in the phase space of right-continuous regulated functions, linear variational equation and smoothness of the nonautonomous process, and a Floquet theorem for periodic systems. Three examples are provided to illustrate the results.

1 Introduction

Center manifold theory has a rich history as one of the fundamental tools in the study of nonlinear dynamical systems. Broadly speaking, the application of the theory to a given dynamical system near its nonhyperbolic states permits a reduction of dimension that is locally characteristic of the behaviour of the fully nonlinear system.

The dynamics of infinite-dimensional systems has been a great source of motivation in the development of techniques in functional analysis. For instance, (strongly continuous) semigroups of operators are often used as the building blocks of center manifolds (and, indeed, other invariant manifolds) for infinite-dimensional dynamical systems. The body of literature on this topic is vast; for a brief exposure one may consult the works of Chicone [5], Chow and Lu [6], DaPrato and Lunardi [8], Krizstin [22], Veltz and Fogeras [29], as well as the textbooks [9, 13].

In recent years, there has been a surge of interest in the dynamics of impulsive differential equations with time delays, especially in neural networks, mathematical biology and ecology, as such systems frequently involve memory effects (discrete or distributed delays), and bursting or discontinuous controls (impulses). While large-scale emergent behaviour such as synchronization in neural networks can be introduced through pinning algorithms, there is little available in terms of low-dimensional analysis techniques to study the emergence of classical bifucation patterns. Indeed, analysis of specific nonlinear impulsive systems with delays appears to be mostly confined to more static notions such as well-posedness, permanence, existence of global attractors and binary stability-instability analysis of equilibrium points, with a look toward bifurcation toward permanence of a compact region of the phase space — see [11, 25, 30, 31] for some recent applications to biological systems. Most dynamic bifurcation analysis at present seems restricted to numerical studies. For instance, in [32], the largest Lyapunov exponent is used to numerically investigate bifurcations to chaotic attractors in a three-species food chain model with distributed delay and impulsive control.

In this paper, we establish the theoretical existence, smoothness and reduction principle of center manifolds for a fairly broad class of impulsive delay differential equations, thereby introducing a classical method of analysis to this growing field of study. It should be mentioned that in the literature, one typically refers to nonautonomous invariant manifolds (of which the center manifold is included) as *invariant fiber bundles*. These are appropriate generalization of invariant manifolds to explicitly time-varying systems that can be visualized as time-varying manifolds [1]. However, to avoid unnecessary verbiage and to draw a distinction between them and linear invariant fibre bundles, we will continue to refer to them as center manifolds.

The structure of the paper is as follows. In Section 2, we provide an imprecise statement of our main result and elaborate on several of its corollaries – namely, the existence of local center manifolds for fully nonautonomous delay differential equations and finite-dimensional impulsive systems. We also outline our method of proof. Section 3 provides background material on impulsive delay differential equations and some

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of the function spaces that will be needed, as definitions specific to our results. Section 4 is devoted to the development of a variation-of-constants formula for linear nonhomogeneous impulsive delay differental equations that is interesting in its own right, but will be needed extensively after. The existence of Lipschitz continuous center fiber bundles (local and global) is proven in Section 5. A reduction principle (attractivity properties and restricted dynamics equtions) is established in Section 6. A detour is taken to study periodic linear systems in Section 7, before establishing the smoothness of the center manifold in Section 8, where we also prove that a periodic system necessarily generates a periodic center manifold. Some examples are provided in Section 9.

2 Statement of results and methodology

This section will be devoted to an informal statement of the results of this paper, together with a broad overview of the proofs. We will ultimately be interested in semilinear impulsive delay differential equations of the form

$$\begin{aligned} \dot{x} &= A(t)x_t + f(t, x_t), \quad t \neq \tau_k \\ \Delta x &= B_k x_{t^-} + g_k(x_{t^-}), \quad t = \tau_k, \end{aligned} \tag{1}$$

where $A(t) : \mathcal{RCR} \to \mathbb{R}^n$ and $B(k) : \mathcal{RCR} \to \mathbb{R}^n$ are for each $t \in \mathbb{R}$ and $k \in \mathbb{Z}$, bounded linear operators acting on the Banach space $\mathcal{RCR}$ of uniformly bounded functions $\phi : [-r, 0] \to \mathbb{R}^n$ that are continuous from the right and have limits on the left, with $r > 0$ finite. Also, $f : \mathbb{R} \times \mathcal{RCR} \to \mathbb{R}^n$ and $g : \mathbb{Z} \times \mathcal{RCR} \to \mathbb{R}^n$ are sufficiently smooth and vanishing with vanishing first derivatives at the origin $0 \in \mathcal{RCR}$, and $\{\tau_k : k \in \mathbb{Z}\}$ is a sequence of impulse times. We do not require global Lipschitzian conditions on the vector field f or jump functional g.

2.1 Statement of the result

Rather imprecisely, the main result of our paper is as follows.

Theorem. *Under "reasonable assumptions", there exists a Lipschitz function $\mathcal{C} : \mathcal{RCR}_c \to \mathcal{RCR}$, with domain consisting of a time-varying subset $\mathcal{RCR}_c \subset \mathcal{RCR}$, with the property that every sufficiently small solution of* (1) *with limited two-sided exponential growth is contained within the graph of $\mathcal{C}$: the* local center manifold. *Moreover, in the absence of unstable components in the linear part of* (1)*, the local center manifold attracts nearby solutions. Under certain conditions, the function $\mathcal{C} : \mathcal{RCR}_c \to \mathcal{RCR}$ is smooth.*

The reasonable assumptions of the theorem include, in particular, a splitting of the phase space $\mathcal{RCR}$ into a time-varying internal direct sum $\mathcal{RCR}_s(t) \oplus \mathcal{RCR}_c(t) \oplus \mathcal{RCR}_u(t)$ of three closed subspaces, which behave like time-varying stable, center and unstable subspaces associated to the linear system

$$\begin{aligned} \dot{y} &= A(t)y_t, \quad t \neq \tau_k \\ \Delta y &= B_k y_{t^-}, \quad t = \tau_k. \end{aligned} \tag{2}$$

More precisely, this splitting is a decomposition of $\mathcal{RCR}$ as an internal direct sum of three mutually orthogonal closed stable, center and unstable fibre bundles over $\mathcal{RCR}$ with base space $\mathbb{R}$ (equivalently, nonautonomous sets $\mathcal{RCR}_i \subset \mathbb{R} \times \mathcal{RCR}$). In addition, the *evolution family* $U(t, s) : \mathcal{RCR} \to \mathcal{RCR}$ associated to the linear system (2) must satisfy certain invertibility and exponential boundedness conditions when restricted to each factor of the decomposition, defined through the *projection operators* $P_i : \mathcal{RCR} \to \mathcal{RCR}_i$ onto the stable, center, and unstable fibre bundles. We will later say that the linear part is *spectrally separated* if these conditions are satisfied.

While presented somewhat abstractly, the spectral conditions are satisfied in several important special cases. For instance, they are satisfied when (2) is periodic, as proven in Section 7. The center manifold is also smooth in this case.

2.2 Corollary: center manifolds for finite-dimensional impulsive systems and systems with memoryless linear part

Our theorem stated imprecisely in Section 2.1 immediately grants existence and smoothness of local center manifolds or invariant fiber bundles under similar reasonable assumptions for ordinary impulsive differential equations in Euclidean space,

$$\begin{aligned} \dot{x} &= f(t,x), \quad t \neq \tau_k \\ \Delta x &= g_k(x), \quad t = \tau_k. \end{aligned}$$

It should be noted that there are numerous examples of center manifold theory for difference equations being applied to study periodic systems of impulsive ordinary differential equations – see [7] for a survey of this method. Despite this, it appears yet to be proven in the literature that such systems possess C^k-smooth invariant center fiber bundles in general. One result [4] is applicable for impulsive differential equations in Banach spaces, but only holds for small nonlinearities and grants C^1 smoothness. We thus prove prove C^k smoothness in Euclidean space.

Another useful corollary is the existence and smoothness of the center manifold for impulsive delay systems when the linear part is memoryless. That is, systems of the form

$$\begin{aligned} \dot{x} &= A(t)x + f(t,x_t), \quad t \neq \tau_k \\ \Delta x &= B_k x + g_k(x_{t^-}), \quad t = \tau_k, \end{aligned}$$

where the nonlinearities vanish and have vanishing derivatives at zero. In this case, the verification of spectral separation can be done on the finite-dimensional linear part, instead of in the whole infinite-dimensional phase space. This greatly simplifies calculations.

2.3 Methodology

At its core, our approach to prove the existence and smoothness of local center manifolds is an adaptation of the Lyapunov-Perron method used to prove the existence of center manifolds for various classes of functional differential equations without impulses. This programme is carried out successfully in [9, 22, 18, 19], for example.

The Lyapunov-Perron method makes use of a variation-of-constants formula to reinterpret solutions of the differential equation in question as *mild solutions* of a semilinear integral equation. In the fully nonautonomous context, this method was used by Chicone [5] to prove a nonautonomous center manifold theorem by first appealing to the evolution semigroup. The evolution semigroup allows one to effectively translate the problem into an autonomous setting by enlarging the phase space. Semigroup theory then provides the requisite variation of constants formula.

To contrast to the approach of Chicone, we work directly with the evolution family associated to (2) and prove a variation of constants formula that is reminiscent of a classical formula derived by Jack Hale for functional differential equations [12]. In the aforementioned reference, Hale proves that solutions of the inhomogeneous delay differential equation $\dot{x} = Ax_t + h(t)$ satisfy the formal variation of constants formula

$$x_t = T(t-s)x_s + \int_s^t T(t-\mu)\chi_0 h(\mu)d\mu,$$

where $T(t) : X \to X$ is the strongly continuous semigroup associated to the autonomous system $\dot{x}(t) = Ax_t$, the phase space is $X = C([-r,0],\mathbb{R}^n)$, and $\chi_0 : [-r,0] \to \mathbb{R}^{n\times n}$ is defined by $\chi_0(0) = I$ and $\chi_0(\theta) = 0$ for $\theta < 0$. Strictly speaking, the formula is ill-defined because $\chi_0 h(\mu)$ is not in the domain of $T(t-\mu)$.

The inconsistencies in Hale's variation of constants formula can be resolved in several ways, including adjoint semigroup theory and integrated semigroup theory [14]. We opt for an arguably more elementary approach, which is similar to the construction used in [3, 24]. Namely, we work with the phase space $\mathcal{RCR}$ of right-continuous regulated functions at the outset and prove that the nonhomogeneous impulsive functional differential equation

$$\begin{aligned} \dot{x} &= A(t)x_t + h(t), \quad t \neq \tau_k \\ \Delta x &= B_k x_{t^-} + r(k), \quad t = \tau_k \end{aligned} \tag{3}$$

satisfies a forward global existence and uniqueness of solutions property, and that its associated homogeneous equation generates an evolution family $U(t,s):\mathcal{RCR}\to\mathcal{RCR}$ that is sufficiently regular to define and prove the variation of constants formula

$$x_t = U(t,s)x_s + \int_s^t U(t,\mu)\chi_0 h(\mu)d\mu + \sum_{s<\tau_i\leq t} U(t,\tau_i)\chi_0 r(i),$$

where the integral is interpreted in the Pettis (weak) sense. The correctness of this formula is proven in Section 4.

It is interesting to note that the evolution family $U(t,s)$ generally fails to be strongly continuous, further necessitating the interpretation of the integral in the weak sense. Indeed, the integrand is not even Bochner measurable, which makes investigations into strong integrability quite difficult.

The weak integral behaves well with respect to composition of bounded linear operators, and as such commutes with the projection operators P_i onto the stable, center, and unstable fibre bundles associated to the linearization. This fact is later used to construct, for each $s\in\mathbb{R}$, a bounded linear operator

$$\mathcal{K}_s^\eta : (h,g)\mapsto \mathcal{K}_s^\eta(f,g)$$

mapping inhomogeneities (h,r) satisfying bounded exponential growth conditions onto the unique bounded-growth solutions of the inhomogeneous equation (3) whose components in the center fibre bundle vanish at time s.

The rest of the proof of existence follows loosely the approach taken in, for instance [9, 22, 18, 19]. albeit with some modifications. Namely, one cuts off the nonlinearities f and g away from the origin separately in the center and hyperbolic directions and constructs a nonlinear fixed-point equation for each $s\in\mathbb{R}$,

$$u = U(\cdot,s)\varphi + \mathcal{K}_s^\eta\circ R_\delta(u),$$

where $\varphi\in\mathcal{RCR}_c(s)$ is a component in the center bundle in the appropriate fiber and R_δ is the Nemitsky operator associated to the external direct sum of the nonlinearities f and g following the cutoff procedure. The fixed point $u_s = u_s(\varphi)$ of this equation is then used to define the global center manifold at time s. The fibre bundle defined by "gluing" these manifolds together along the real line defines the all-time global center manifold, and a suitable restriction generates the invariant center manifold. The proof of invariance and attraction is similar in spirit to the autonomous delay differential equations case.

To obtain smoothness, the nonlinearities must be smoothed in a way that makes the Nemitsky operator smooth in the part of the phase space in which the center manifold resides. As the center manifold is time-varying, the infinite-dimensionality of the space in which it resides makes the smooth renorming of the space a nontrivial problem to solve. To resolve this, we specialize to linearizations whose evolution families satisfy a particular decomposability condition that is formally analogous to a Floquet decomposition on the center fiber bundle. Then, we prove the smoothness of the center manifold by appealing to methods of contractions on scales of Banach spaces; see [28, 27, 9] for background on these techniques.

2.4 Notation

The following notation is common to the manuscript. For a subset $Z\subset\mathbb{R}$, the symbol χ_Z will always denote the identity-valued indicator function:

$$\chi_Z(\theta) = \begin{cases} 0, & \theta\notin Z \\ I, & \theta\in Z, \end{cases}$$

with I the identity on $\mathbb{R}^n$. The domain of χ_Z will be either stated or implied. The cardinality of a set X will be denoted $\#S$. If V is a normed vector space, the set of bounded linear maps on V will be denoted $\mathcal{L}(V)$, and if $V_1,\dots,V_p$ are all normed spaces, then the set of bounded p-linear maps from $V_1\times\cdots\times V_p$ into V will be denoted $\mathcal{L}^p(V_1\times\cdots\times V_p, V)$. For a function $f:\mathcal{T}\times X\to Y$ with $\mathcal{T}\subseteq\mathbb{R}$ and Banach spaces X,Y, $D^kf(t,x)\in\mathcal{L}^k(X\times\cdots\times X,X)$ denotes the kth Fréchet derivative of f in its second variable at the point $(t,x)\in\mathcal{T}\times X$.

3 Background material

In this section we will collect necessary results on the linear inhomogeneous impulsive retarded functional differential equation

$$\dot{x} = L(t)x_t + h(t), \qquad t \neq \tau_k \tag{4}$$
$$\Delta x = B_k x_{t^-} + r_k, \qquad t = \tau_k, \tag{5}$$

with the impulse condition $\Delta x(t) = x(t) - x(t^-)$, where $x_{t^-}(\theta) = x(t+\theta)$ for $-r \leq \theta < 0$ and $x_{t^-}(0) = \lim_{\theta \to 0^-} x(t+\theta)$, and $r > 0$ is finite. We will be working exclusively with spaces of right-continuous regulated functions; denote

$$\mathcal{RCR}(I, X) = \left\{ f : I \to X : \forall t \in I, \lim_{s \to t^+} f(s) = f(t) \text{ and } \lim_{s \to t^-} f(s) \text{ exists} \right\},$$

where $X \subseteq \mathbb{R}^n$ and $I \subseteq \mathbb{R}$. When X and I are closed,

$$\mathcal{RCR}_b(I, X) := \{ f \in \mathcal{RCR}(I, X) : ||f|| < \infty \}$$

is a Banach space with the norm $||f|| = \sup_{x \in I} |f(x)|$. One may consult [17] for background on regulated functions. We will write $\mathcal{RCR} := \mathcal{RCR}([-r, 0], \mathbb{R}^n)$ when there is no ambiguity, and note that since $\mathcal{RCR}_b([-r, 0], \mathbb{R}^n) = \mathcal{RCR}([-r, 0], \mathbb{R}^n)$, we may identify $\mathcal{RCR}$ with its associated Banach space. The following assumptions will be needed throughout.

H.1 The representation

$$L(t)\phi = \int_{-r}^{0} [d_\theta \eta(t, \theta)] \phi(\theta)$$

holds, where the integral is taken in the Lebesgue-Stieltjes sense, the function $\eta : \mathbb{R} \times [-r, 0] \to \mathbb{R}^{n \times n}$ is jointly measurable and is of bounded variation and right-continuous on $[-r, 0]$ for each $t \in \mathbb{R}$, and such that $|L(t)\phi| \leq \ell(t)||\phi||$ for some $\ell : \mathbb{R} \to \mathbb{R}$ locally integrable.

H.2 The sequence τ_k is monotonically increasing with $|\tau_k| \to \infty$ as $|k| \to \infty$, and the representation

$$B_k \phi = \int_{-r}^{0} [d_\theta \gamma_k(\theta)] \phi(\theta)$$

holds for $k \in \mathbb{Z}$ for functions $\gamma_k : [-r, 0] \to \mathbb{R}^{n \times n}$ of bounded variation and right-continuous, such that $|B_k| \leq b(k)$.

Remark 3.0.1. *Hypothesis H.1–H.2 could in principle be weakened. However, insofar as applied impulsive differential equations are concerned, hypothesis H.1 is sufficient. Indeed, H.1 includes the case of discrete time-varying delays: the linear delay differential equation*

$$\dot{x} = \sum_{k=1}^{m} A_k(t) x(t - r_k(t))$$

with r_k continuous, is associated to a linear operator satisfying condition H.1 with $\eta(t, \theta) = \sum A_k(t) H_{-r_k(t)}(\theta)$, where $H_z(\theta) = 1$ if $\theta \geq z$ and zero otherwise. It also obviously includes a large class of distributed delays.

In what follows, we will introduce several properties of regulated functions that will be needed throughout the paper, in addition to impulsive integral inequalities and some definitions.

3.1 Properties of regulated functions

We will require some elementary properties of regulated functions. The first result will be useful in proving boundedness of evolution families associated to linear, homogeneous impulsive equations. We have not seen the following elementary result in the literature, so it will be proven here for completeness.

Lemma 3.1.1. *Let $r > 0$ be finite and let $\phi \in \mathcal{RCR}([a,b],\mathbb{R}^n)$ for some $b \geq a + r$. With $\phi_t : [-r, 0] \to \mathbb{R}^n$ defined in the usual way, $t \mapsto ||\phi_t||$ is an element of $\mathcal{RCR}([a + r, b], \mathbb{R})$.*

Proof. Let $t \in [a + r, b]$ be fixed. We will only prove right-continuity, since the proof of the existence of left limits is similar. It suffices to prove that for any decreasing sequence $s_n \downarrow 0$, we have $||\phi_{t+s_n}|| \to ||\phi_t||$. Let $\epsilon > 0$ be given. By right-continuity of ϕ, for all $\epsilon > 0$, there exists $\delta > 0$ such that, if $0 < \mu < \delta$, then $|\phi(t + \mu) - \phi(t)| < \epsilon$. Therefore,

$$\begin{aligned}||\phi_{t+s_n}|| = \sup_{\mu\in[-r,0]} |\phi(t + \mu)| &\leq \sup_{\mu\in[-r,s_n]} |\phi(t + \mu)| \leq \max\{||\phi_t||, \sup_{\mu\in[0,s_n]} |\phi(t + \mu)|\} \\ &\leq \max\{||\phi_t||, |\phi(t)| + \epsilon\} \leq ||\phi_t|| + \epsilon,\end{aligned}$$

provided $s_n < \delta$. On the other hand, since ϕ is bounded, there exists some sequence $x_n \in [-r, 0]$ such that $|\phi_t(x_n)| \to ||\phi_t||$. By passing to a subsequence, we may assume $x_n \to \hat{x} \in [-r, 0]$. If $\hat{x} > -r$, then we have

$$||\phi_{t+s_n}|| \geq \sup_{\mu\in[-r+s_n,0]} |\phi(t + \mu)| = |\phi(\hat{x})| = ||\phi_t||$$

provided $s_n < -\hat{x}$, while if $\hat{x} = -r$, we notice that the sequence $x'_n = t + -r + s_n$ converges to $t + \hat{x}$, so that for all $\epsilon > 0$, there exists $N_3 > 0$ such that for $n \geq N$,

$$||\phi_{t+s_n}|| \geq |\phi(t + s_n) \geq ||\phi_t|| - \epsilon.$$

Therefore, if we let $s_{N_1} < \delta$ and $s_{N_2} < -\hat{x}$, then by setting $N = \max\{N_1, N_2, N_3\}$, it follows by the above three inequalities that for $n \geq N$,

$$-\epsilon \leq ||\phi_{t+s_n}|| - ||\phi_t|| \leq \epsilon.$$

We conclude $||\phi_{t+s_n}||$ converges to $||\phi_t||$. □

We will eventually need spaces of function $f : I \to X$ that are differentiable from the right and whose right-hand derivatives are elements of $\mathcal{RCR}(I, X)$. Specifically, denote the right-hand derivative by

$$d^+f(t) = \lim_{\epsilon\to0^+} \frac{f(t + \epsilon) - f(t)}{\epsilon}$$

and introduce the space

$$\mathcal{RCR}^1(I, X) = \{f \in \mathcal{RCR}(I, X) : d^+f \in \mathcal{RCR}(I, \mathbb{R}^n)\}.$$

This space is clearly complete with respect to the norm $||f||_1 = ||f|| + ||d^+f||$ when restricted to the subspace consisting of functions that are $|| \cdot |||_1$-bounded.

The next result concerns a dense subspace of $\mathcal{RCR}$. The proof is available in [17]

Lemma 3.1.2. *Let I be compact. For all $f \in \mathcal{RCR}(I, X)$, there exists a sequence of step functions $f_n : I \to X$ such that $||f_n - f|| \to 0$.*

We will need a characterization of the continuous dual of the space $\mathcal{RCR}$, denoted $\mathcal{RCR}^*$. A result from Tvrdy [26] provides such for the dual of the space of regulated left-continuous scalar-valued functions, and for our purposes the obvious modification that is needed is the following.

Lemma 3.1.3. *$F \in \mathcal{RCR}^*$ if and only if there exists $q \in \mathbb{R}^n$ and $p : [-r, 0] \to \mathbb{R}^n$ of bounded variation such that*

$$F(x) = q^*x(0) + \int_{-r}^{0} p^*(t)dx(t), \tag{6}$$

where the integral is a Perron-Stieltjes integral.

We will need a few convergence and boundedness results for Perron-Stieltjes integrals involving right-continuous regulated functions and functions of bounded variation. Symmetric arguments to those appearing in [26] obviously yield the following results; see Theorem 2.8 and Corollary 2.10 therein.

Lemma 3.1.4. *Let $f : [a,b] \to \mathbb{R}^n$ be of bounded variation and $g \in \mathcal{RCR}([a,b],\mathbb{R}^n)$. The integral $\int_a^b f^*(t)dg(t)$ exists in the Perron-Stieltjes sense, and*

$$\left|\int_a^b f^*(t)dg(t)\right| \leq (|f(a)| + |f(b)| + \mathrm{var}_a^b f)||g||, \tag{7}$$

where $\mathrm{var}_a^b f$ denotes the total variation of f on the interval $[a,b]$.

Lemma 3.1.5. *Let $h_n \in \mathcal{RCR}([a,b],\mathbb{R}^n)$ and $h \in \mathcal{RCR}([a,b],R^n)$ be such that $||h_n - h|| \to 0$ as $n \to \infty$. For any $f : [a,b] \to \mathbb{R}^n$ of bounded variation, the Perron-Stieltjes integrals $\int_a^b f^*(t)dh(t)$ and $\int_a^b f^*(t)dh_n(t)$ exist and*

$$\lim_{n\to\infty} \int_a^b f^*(t)dh_n(t) = \int_a^b f^*(t)dh(t). \tag{8}$$

3.2 Integral inequalities

We next provide two inequalities. The first is an impulsive Gronwall-Bellman inequality for regulated functions. The result is similar to Lemma 2.3 of [2], and the proof is omitted. The second one concerns an elementary estimation of sums of continuous functions at impulses, when the sequence of impulses satisfies a separation condition.

Lemma 3.2.1. *Suppose $x \in \mathcal{RCR}([s,\alpha],\mathbb{R})$ satisfies the inequality*

$$x(t) \leq C + \int_s^t (p(\mu)x(\mu) + h(\mu))d\mu + \sum_{s<\tau_i\leq t} (b_i x(\tau_i^-) + g_i) \tag{9}$$

for some nonnegative integrable function p, integrable and bounded h, nonnegative constants b_i, g_i and c, and all $t \in [s,\alpha]$. For $t \geq s$, define

$$z(t,s) = \exp\left(\int_s^t p(\mu)d\mu\right) \prod_{s<\tau_i\leq t} (1 + b_i).$$

Then, $\mu \mapsto z(t,\mu)$ is integrable and the following inequality is satisfied.

$$x(t) \leq Cz(t,s) + \int_s^t z(t,\mu)h(\mu)d\mu + \sum_{s<\tau_i\leq t} z(t,\tau_i)g_i. \tag{10}$$

Lemma 3.2.2. *Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and suppose $\{\tau_k\}$ satifies $\tau_{k+1} - \tau_k \geq \xi$.*

1. *If f is increasing, then $\sum_{s<\tau_i\leq t} f(\tau_i) \leq \frac{1}{\xi}\int_s^{t+\xi} f(\mu)d\mu$.*

2. *If f is decreasing, then $\sum_{s<\tau_i\leq t} f(\tau_i) \leq \frac{1}{\xi}\int_{s-\xi}^{t} f(\mu)d\mu$.*

Proof. Let $\{\tau_0,\dots,\tau_N\} = \{\tau_k : k \in \mathbb{Z}\} \cap (s,t]$. If f is increasing, then

$$\sum_{s<\tau_i\leq t} f(\tau_i) = \sum_{i=0}^N f(\tau_i) = \frac{1}{\delta}\sum_{i=0}^N f(\tau_i)\xi \leq \frac{1}{\xi}\sum_{i=0}^N f(\tau_0 + i\xi)\xi \leq \frac{1}{\xi}\int_s^{t+\xi} f(\mu)d\mu.$$

The decreasing case is similar. ☐

3.3 Evolution families, nonautonomous sets and processes

Definition 3.3.1. *Let X be a Banach space. An* evolution family on X *is a collection of bounded linear operators $\{U(t,s)\}_{t\geq s\in\mathbb{R}}$ on X that satisfy $U(t,s) = U(t,v)U(v,s)$ for all $t \geq v \geq s$ and $U(t,t) = I_X$.*

Nonlinear variants of evolution families are more appropriately defined in terms of nonautonomous sets, which are a specific type of fiber bundle. This is because insofar as evolution families may serve to define solutions to linear Cauchy problems, the interval of existence of a nonlinear Cauchy problems will typically depend on the initial data. The following definition is borrowed from Kloeden and Rasmussen [21], with slightly different notation.

Definition 3.3.2. *If X is a Banach space, a subset $\mathcal{M} \subseteq \mathbb{R} \times X$ is a* nonautonomous set over X*. For each $t \in \mathbb{R}$, the set*

$$\mathcal{M}(t) = \{x : (t,x) \in \mathcal{M}\}$$

*is called the t-*fiber of $\mathcal{M}$.

Definition 3.3.3. *A* process on X *is a pair $(S,\mathcal{M})$ where $\mathcal{M}$ is a nonautonomous set over $\mathbb{R} \times X$ and $S : \mathcal{M} \to X$, whose action we denote by $S(t,(s,x)) = S(t,s)x$, and satisfies the following.*

- $\{t\} \times X \subset \mathcal{M}(t)$ *and* $S(t,t) = I_X$ *for all* $t \in \mathbb{R}$.
- $S(t,s)x = S(t,v)S(v,s)x$ *whenever* $(s,x) \in \mathcal{M}(v)$ *and* $(v,S(v,s)x) \in \mathcal{M}(t)$.

Note that the above definition is different, for example, than the one for process appearing in [21], where processes are defined first as (partial) mappings, independent of nonautonomous sets. The reason for our distinction here is that we want to make precise the notion that a process $S(t,s)$ need not be defined on the entire Banach space X for every pair of time arguments, the way evolution families $U(t,s)$ are.

Definition 3.3.4. *If $\mathcal{M}$ is a nonautonomous set over X and Y is another Banach space, we will say a function $f : \mathcal{M} \to Y$ has a given regularity property (eg. continuous, Lipschitz continuous, smooth) if $f(t,\cdot) : \mathcal{M}_t \to Y$ has the same regularity property. That is, regularity is defined* fibrewise.

Definition 3.3.5. *Let $U(t,s) : X \to X$ be a family of bounded linear operators defining a forward process on a Banach space X — that is, $U(t,s) = U(t,v)U(v,s)$ for all $t \geq v \geq s$ and $U(t,t) = I_X$. We say that U is* spectrally separated *if there exists a triple (P_s, P_c, P_u) of bounded projection-valued functions $P_i : \mathbb{R} \to \mathcal{L}(X)$ with $P_s + P_c + P_u = I$ such that the following hold.*

1. *There exists a constant N such that* $\sup_{t\in\mathbb{R}} (||P_s(t)|| + ||P_c(t)|| + ||P_u(t)||) = N < \infty$.
2. *The projectors are mutually orthogonal; $P_i(t)P_j(t) = 0$ for $i \neq j$.*
3. $U(t,s)P_i(s) = P_i(t)U(t,s)$ *for all $t \geq s$ and $i \in \{s,c,u\}$.*
4. *Define $U_i(t,s)$ as the restriction of $U(t,s)$ to $X_i(s) = \mathcal{R}(P_i(s))$. The operators $U_c(t,s) : X_c(s) \to X_c(t)$ and $U_u(t,s) : X_u(s) \to X_u(t)$ are invertible and we denote $U_c(s,t) = U_c(t,s)^{-1}$ and $U_u(s,t) = U_u(t,s)^{-1}$ for $s \leq t$.*
5. *The operators U_c and U_u define all-time processes on the family of Banach spaces $X_c(\cdot)$ and $X_u(\cdot)$. Specifically, the following holds for all $t,s,v \in \mathbb{R}$.*

$$U_c(t,s) = U_c(t,v)U_c(v,s), \qquad U_u(t,s) = U_u(t,v)U_u(v,s).$$

6. *There exist real numbers $a < 0 < b$ such that for all $\epsilon > 0$, there exists $K \geq 1$ such that*

$$||U_u(t,s)|| \leq Ke^{b(t-s)}, \qquad t \leq s \tag{11}$$
$$||U_c(t,s)|| \leq Ke^{\epsilon|t-s|}, \qquad t,s \in \mathbb{R} \tag{12}$$
$$||U_s(t,s)|| \leq Ke^{a(t-s)}, \qquad t \geq s. \tag{13}$$

The above definition is a time-varying version of the spectral decomposition hypotheses associated to the center manifold theorem for autonomous delay differential equations appearing in [9].

Definition 3.3.6. *Let $U(t,s) : X \to X$ be spectrally separated. The nonautonomous sets*

$$X_i = \{(t,x) : t \in \mathbb{R}, x \in X_i(t)\}$$

for $i \in \{s,c,u\}$ are termed respectively the stable, center, and unstable bundles *associated to $U(t,s)$.*

3.4 Existence and uniqueness of solutions for the linear equation

In order to eventually prove the existence of the center manifold in Section 5, we will need to first verify existence, uniqueness and continuability of solutions.

Lemma 3.4.1. *Let $h \in \mathcal{RCR}(\mathbb{R},\mathbb{R}^n)$ and let hypotheses H.1–H.2 hold. For all $\phi \in \mathcal{RCR}$ and $s \in \mathbb{R}$, there exists a unique function $x \in \mathcal{RCR}([s-r,\infty),\mathbb{R}^n)$ satisfying $x_s = \phi$ and the integral equation*

$$x(t) = \begin{cases} \phi(0) + \int_s^t [L(\mu)x_\mu + h(\mu)]d\mu + \sum_{s<\tau_i\leq t}[B_i x_{\tau_i^-} + r_i], & t > s \\ \phi(t-s), & s-r \leq t \leq s. \end{cases} \tag{14}$$

The above lemma follows by hypotheses H.1–2, the Banach fixed-point theorem, Lemma 3.2.1 and typical continuation methods. It could also be proven by identifying the equation with a generalized ordinary differential equation, as in [10]. Under stronger assumptions, one may look at the proof of Theorem 5.1.1 for local existence and uniqueness. Note here that h may be unbounded on the real line; however, since it is regulated it is bounded on every compact set [17].

Consider now the homogeneous equation

$$\dot{x} = L(t)x_t, \qquad t \neq \tau_k \tag{15}$$

$$\Delta x = B_k x_{t^-}, \qquad t = \tau_k. \tag{16}$$

Definition 3.4.1. *Let hypotheses H.1–H.2 hold. For a given $(s,\phi) \in \mathbb{R} \times \mathcal{RCR}$, let $t \mapsto x(t;s,\phi)$ denote the unique solution of (15)–(16) satisfying $x_s(\cdot;s,\phi) = \phi$. The function $U(t,s) : \mathcal{RCR} \to \mathcal{RCR}$ defined by $U(t,s)\phi = x_t(\cdot,s,\phi)$ for $t \geq s$ is the* evolution family *associated to the homogeneous equation (15)–(16).*

Lemma 3.4.2. *The evolution family satisfies the following properties.*

1) For $s \leq t$, $U(t,s) : \mathcal{RCR} \to \mathcal{RCR}$ is a bounded linear operator. In particular,

$$||U(t,s)|| \leq \exp\left(\int_s^t \ell(\mu)d\mu\right) \prod_{s<\tau_i\leq t}(1+b(i)). \tag{17}$$

2) For $s \leq v \leq t$, $U(t,s) = U(t,v)U(v,s)$.

3) For all $\theta \in [-r,0]$, $s \leq t+\theta$ and $\phi \in \mathcal{RCR}$, $U(t,s)\phi(\theta) = U(t+\theta,s)\phi(0)$.

4) For all $\tau_k > s$, one has $U(\tau_k,s) = (I+\chi_0 B_k)U(\tau_k^-,s)$.[1]

5) Let $C(t,s)$ denote the evolution family on $\mathcal{RCR}$ associated to the "continuous" equation $\dot{x} = L(t)x_t$. The following factorization holds:

$$U(t,s) = \begin{cases} C(t,s), & [s,t] \cap \{\tau_k\}_{k\in\mathbb{Z}} \in \{\{s\},\emptyset\} \\ C(t,\tau_k) \circ (I+\chi_0 B_k) \circ U(\tau_k^-,s), & t \geq \tau_k > s \end{cases} \tag{18}$$

[1] Note here that the left limit is the uniform one-point limit. Namely, $U(\tau_k^-,s)\phi(\theta) = U(\tau_k,s)\phi(\theta)$ for $\theta < 0$, while $U(\tau_k^-,s)\phi(0) = U(\tau_k,s)\phi(0^-)$.

Proof. Property 2) and 3) are immediate consequences of the uniqueness assertion of Lemma 3.4.1 and the definition of the evolution family. For property 1), we obtain linearity by noticing that $\phi \mapsto x(t; s, \phi)$ is linear in ϕ for each $t \geq s$ and, consequently, $\phi \mapsto x_t(\cdot; s, \phi)$ is is also linear. To obtain boundedness, we notice that by virtue of the integral equation (14), $U(t, s)\phi(\theta)$ satisfies

$$\begin{aligned}|U(t,s)\phi(\theta)| &\leq ||\phi|| + \int_s^{t+\theta} |L(\mu)U(\mu,s)\phi| d\mu + \sum_{s<\tau_i\leq t+\theta} |B_i U(\tau_i^-, s)\phi| \\ &\leq ||\phi|| + \int_s^t \ell(\mu)||U(\mu,s)\phi||d\mu + \sum_{s<\tau_i\leq t} b(i)||U(\tau_i^-, s)\phi||.\end{aligned}$$

Since the upper bounds are independent of θ, denoting $X(t) = U(t, s)\phi$, we obtain

$$||X(t)|| \leq ||\phi|| + \int_s^t \ell(\mu)||X(\mu)||d\mu + \sum_{s<\tau_i\leq t} b(i)||X(\tau_i^-)||.$$

By Lemma 3.1.1, $t \mapsto ||X(t)|$ is an element of $\mathcal{RCR}([s-r,\infty),\mathbb{R})$. Invoking Lemma 3.2.1, we obtain the desired boundedness (17) of the evolution family and property 1) is proven. Finally, since

$$\begin{aligned}U(\tau_k, s)\phi(0) &= \phi(0) + \int_s^{\tau_k} L(\mu)U(\mu,s)\phi d\mu + \sum_{s<\tau_i\leq\tau_k} B_i U(\tau_i^-, s)\phi \\ &= U(\tau_k^-, s)\phi(0) + B_k U(\tau_k^-, s)\phi\end{aligned}$$

and $U(\tau_k^-, s)\phi(\theta) = U(\tau_k, s)\phi(\theta)$ for $\theta < 0$, we readily obtain property 4). The verification of property 5) follows by existence and uniqueness of solutions and property 4). □

4 The variation of constants formula

Existence, uniqueness and continuability of solutions of the linear inhomogeneous equation (4)–(5) has been granted by Lemma 3.4.1. From this result we directly obtain a decomposition of solutions.

Lemma 4.0.1. *Let $h \in \mathcal{RCR}(\mathbb{R}, \mathbb{R}^n)$ and let H.1–H.2 hold. Denote $t \mapsto x(t; s, \phi; h, r)$ the solution of the linear inhomogeneous equation (4)–(5) for inhomogeneities $h = h(t)$ and $r = r_k$, satisfying the initial condition $x_s(\cdot; s, \phi; h, r) = \phi$. The following decomposition is valid:*

$$x(t; s, \phi; h, r) = x(t; s, \phi; 0, 0) + x(t; s, 0; h, 0) + x(t; s, 0; 0, r) \tag{19}$$

With this decomposition, we will now proceed with the derivation of the variation of constants formula. We prove a pointwise formula in Section 4.1 before lifting the formula into the space $\mathcal{RCR}$ in Section 4.2

4.1 Pointwise variation of constants formula

The following lemmas prove representations of the inhomogeneous impulsive term $x_t(\cdot; s, 0; 0, r)$ and the inhomogeneous continuous term $x_t(\cdot; s, 0; h, 0)$.

Lemma 4.1.1. *Under hypotheses H.1–H.2, one has*

$$x_t(\cdot; s, 0; 0, r) = \sum_{s<\tau_i\leq t} U(t, \tau_i)\chi_0 r_i \tag{20}$$

Proof. Denote $x(t) = x(t; s, 0; 0, r)$. Clearly, for $t \in [s, \min\{\tau_i : \tau_i > s\}$, one has $x_t = 0$. Assume without loss of generality that $\tau_0 = \min\{\tau_i : \tau_i > s\}$. Then $x_{\tau_0} = \chi_0 r_0$ due to (14). From Lemma 3.4.1 and 3.4.2, we have $x_t = U(t,\tau_0)\chi_0 r_0$ for all $t \in [\tau_0, \tau_1)$, so we conclude that (20) holds for all $t \in [s, \tau_1)$. Supposing by induction that $x_t = \sum_{s<\tau_i\le t} U(t,\tau_i)\chi_0 r_i$ for all $t \in [s, \tau_k)$ for some $k \ge 1$, we have

$$\begin{aligned} x_{\tau_k} &= x_{\tau_k^-} + \chi_0 B_k x_{\tau_k^-} + \chi_0 r_k \\ &= U(\tau_k, \tau_{k-1}) x_{\tau_{k-1}} + \chi_0 r_k \\ &= U(\tau_k, \tau_{k-1}) \textstyle\sum_{s<\tau_i\le\tau_{k-1}} U(\tau_{k-1}, \tau_i)\chi_0 r_i + \chi_0 r_k \\ &= \textstyle\sum_{s<\tau_i\le\tau_k} U(t,\tau_i)\chi_0 r_i. \end{aligned}$$

Equality (20) then holds for $t \in [\tau_k, \tau_{k+1})$ by applying Lemma 3.4.2. The result follows by induction. $\square$

Lemma 4.1.2. *Let $h \in \mathcal{RCR}(\mathbb{R}, \mathbb{R}^n)$. Under hypotheses H.1–H.2, one has*

$$x_t(\theta; s, 0; h, 0) = \int_s^t U(t,\mu)[\chi_0 h(\mu)](\theta) d\mu, \tag{21}$$

where the integral is defined for each θ as the integral of the function $\mu \mapsto U(t,\mu)[\chi_0 h(\mu)](\theta)$ in $\mathbb{R}^n$.

Proof. The proof of this lemma is adapted from the proof of Theorem 16.3 of [12]. Let us denote $x(t;s)h = x(t;s,0;h,0)$. First, we note that operator $x(t,s) : \mathcal{RCR}([s,t],\mathbb{R}^n) \to \mathbb{R}^n$ is linear (a consequence of Lemma 3.4.1) for each fixed $s \le t$, and that it admits an extension to a linear operator $\tilde{x}(t,s) : \mathcal{L}_1^{loc}([s,t],\mathbb{R}^n) \to \mathbb{R}^n$. We do not prove this claim, since the proof is essentially identical to how one would prove Lemma 3.4.1. For $w \in [s,t]$ and denoting $\tilde{x}_t = [\tilde{x}(\cdot, s)h]_t$ for brevity, we see that

$$\begin{aligned} |\tilde{x}_w(\theta)| &\le \int_s^{w+\theta} |L(\mu)\tilde{x}_\mu| d\mu + \int_s^{w+\theta} |h(\mu)| d\mu \\ &\le |h|_1 + \int_s^t \ell(\mu)||x_\mu|| d\mu, \end{aligned}$$

which implies the uniform inequality $||x_t|| \le |h|_1 + \int_s^t \ell(\mu)||\tilde{x}||_\mu d\mu$. Applying Lemma 3.2.1 yields $||\tilde{x}_t|| \le e^{|\ell|_1}|h|_1$, where $|\cdot|_1$ denotes the $\mathcal{L}_1[s,t]$ norm. Thus, $|\tilde{x}(t,s)h| = |\tilde{x}_t(0)| \le e^{|\ell|_1}|h|_1$, so $\tilde{x}$ is bounded. By classical results of functional analysis, there exists an integrable, essentially bounded $n \times n$ matrix function $\mu \mapsto V(t,s,\mu)$ such that

$$\tilde{x}(t,s)h = \int_s^t V(t,s,\mu)h(\mu)d\mu. \tag{22}$$

First we show that $V(t,s,\mu)$ is independent of s. Let $\alpha \in [s,t]$ and let $k \in \mathcal{L}_1([s,t],\mathbb{R}^n)$ be such that $k = 0$ on $[s,\alpha]$. Then $\tilde{x}(t,s)k = x(t,\alpha)k$ and $x(t,\mu)k = 0$ for $\mu \in [s,\alpha]$. Thus,

$$\int_\alpha^t [V(t,s,\mu) - V(t,\alpha,\mu)]k(\mu)d\mu = 0$$

for all $k \in \mathcal{L}_1([\alpha,t],\mathbb{R}^n)$. Thus, $V(t,s,\mu) = V(t,\alpha,\mu)$ almost everywhere on $[\alpha,t]$. Since α is arbitrary, we have that $V(t,s,\mu)$ is independent of s.

Define $V(t,s) = V(t,s,\cdot)$ for any $t \ge s$ and $V(t,s) = 0$ for $s < t$. Let us denote $\tilde{x}(t) = \tilde{x}(t,s)h$ and $V_{\tau_i^-}(\theta,s) = V(\tau_i + \theta, s)$ when $\theta < 0$ and $V_{\tau_i^-}(0,s) = V(\tau_i^-, s)$. From the integral equation (14) and the representation (22), we have

$$
\begin{aligned}
&\int_s^t V(t,\mu)h(\mu)d\mu \\
&\quad= \int_s^t L(\mu)\tilde{x}_\mu d\mu + \sum_{s<\tau_i\le t} B_i\tilde{x}_{\tau_i^-} + \int_s^t h(\mu)d\mu \\
&\quad= \int_s^t \int_{-r}^0 [d_\theta\eta(\mu,\theta)]\tilde{x}(\mu+\theta)d\mu + \sum_{s<\tau_i\le t}\int_{-r}^0 [d_\theta\gamma_i(\theta)]\tilde{x}_{\tau_i^-}(\theta) + \int_s^t h(\mu)d\mu \\
&\quad= \int_s^t \int_{-r}^0 [d_\theta\eta(\mu,\theta)]\int_s^{\mu+\theta} V(\mu+\theta,\nu)h(\nu)d\nu d\mu + \sum_{s<\tau_i\le t}\int_{-r}^0 [d_\theta\gamma_k(\theta)]\int_s^{\tau_i+\theta} V_{\tau_i^-}(\theta,\nu)h(\nu)d\nu + \int_s^t h(\mu)d\mu \\
&\quad= \int_s^t \int_{-r}^0 [d_\theta\eta(\mu,\theta)]\int_s^{\mu} V(\mu+\theta,\nu)h(\nu)d\nu d\mu + \sum_{s<\tau_i\le t}\int_{-r}^0 [d_\theta\gamma_k(\theta)]\int_s^{\tau_i} V_{\tau_i^-}(\theta,\nu)h(\nu)d\nu + \int_s^t h(\mu)d\mu \\
&\quad= \int_s^t \int_\nu^t \int_{-r}^0 [d_\theta\eta(\nu,\theta)]V(\mu+\theta,\nu)h(\nu)d\mu d\nu + \sum_{s<\tau_i\le t}\int_s^{\tau_i}\int_{-r}^0 [d_\theta\gamma_i(\theta)]V_{\tau_i^-}(\theta,\nu)h(\nu)d\nu + \int_s^t h(\mu)d\mu \\
&\quad= \int_s^t \left[\int_\mu^t \int_{-r}^0 [d_\theta\eta(\nu,\theta)]V(\nu+\theta,\mu)h(\mu)d\nu + \sum_{s<\tau_i\le t}\chi_{(-\infty,\tau_i]}(\mu)\int_{-r}^0 [d_\theta\gamma_k(\theta)]V_{\tau_i^-}(\theta,\mu)h(\mu) + h(\mu)\right]d\mu \\
&\quad= \int_s^t \left[\int_\mu^t \int_{-r}^0 [d_\theta\eta(\nu,\theta)]V(\nu+\theta,\mu)d\nu + \sum_{s<\tau_i\le t}\chi_{(-\infty,\tau_i]}(\mu)\int_{-r}^0 [d_\theta\gamma_k(\theta)]V_{\tau_i^-}(\theta,\mu) + I\right]h(\mu)d\mu \\
&\quad= \int_s^t \left[I + \int_\mu^t L(\mu)V_\nu(\cdot,\mu)d\nu + \sum_{s<\tau_i\le t} B_iV_{\tau_i^-}(\cdot,\mu)\right]h(\mu)d\mu.
\end{aligned}
$$

Since the above holds for all $h \in \mathcal{L}_1([s,t],\mathbb{R}^n)$, we have that the *fundamental matrix* $V(t,s)$ satisfies

$$
V(t,s) = \begin{cases} I + \displaystyle\int_s^t L(\mu)V_\mu(\cdot,s)d\mu + \sum_{s<\tau_i\le t} B_iV_{\tau_i^-}(\cdot,s), & t \ge s \\ 0 & t < s. \end{cases} \tag{23}
$$

almost everywhere. By uniqueness of solutions (Lemma 3.4.1, it follows that $V(t,s)\xi = U(t,s)[\chi_0\xi](0)$ for all $\xi \in \mathbb{R}^n$. Since $\tilde{x}(t,s)$ is an extension of $x(t,s)$ to the larger space $\mathcal{L}_1([s,t],\mathbb{R}^n)$, representation (22) holds for $h \in \mathcal{RCR}([s,t],\mathbb{R}^n)$. Thus, for all $t \ge s$,

$$
\begin{aligned}
x_t(\theta;s,0;h,0) &= x(t+\theta,s)h \\
&= \int_s^{t+\theta} V(t+\theta,\mu)h(\mu)d\mu \\
&= \int_s^t V(t+\theta,\mu)h(\mu)d\mu \\
&= \int_s^t U(t+\theta,\mu)[\chi_0h(\mu)](0)d\mu \\
&= \int_s^t U(t,\mu)[\chi_0h(\mu)](\theta)d\mu,
\end{aligned}
$$

which is what was claimed by equation (21). □

With Lemma 4.0.1 through Lemma 4.1.2 at hand, we arrive at the variation of constants formula.

Lemma 4.1.3. *Let $h \in \mathcal{RCR}(\mathbb{R},\mathbb{R}^n)$. Under hypotheses H.1–H.2, one has the variation of constants formula*

$$x_t(\theta;s,\phi;h,r) = U(t,s)\phi(\theta) + \int_s^t U(t,\mu)[\chi_0 h(\mu)](\theta)d\mu + \sum_{s<\tau_i\leq t} U(t,\tau_i)[\chi_0 r_i](\theta). \tag{24}$$

4.2 Variation of constants formula in the space $\mathcal{RCR}$

The goal of this section will be to reinterpret the variation of constants formula (24) in such a way that the integral appearing therein may be thought of as a weak integral in the space $\mathcal{RCR}$. Specifically, we will show that the integral may be regarded as a *Gelfand-Pettis integral.* This form has several advantages, the most important being it will allow us to later commute bounded projection operators with the integral sign. We recall the following definition, which appears in [23].

Definition 4.2.1. *Let X be a Banach space and (S,Σ,μ) a measure space. We say that $f : S \to X$ is* Pettis integrable *if there exists a set function $I_f : \Sigma \to X$ such that*

$$\varphi^* I_f(E) = \int_E \varphi^* f d\mu$$

for all $\varphi^* \in X^*$ and $E \in \Sigma$. I_f is the indefinite Pettis integral of f, *and $I_f(E)$ the* Pettis integral of f on E.

By abuse of notation, we will often write $I_f(E) = \int_E f d\mu$ when there is no ambiguity. For our purposes, the following proposition will be of primary usefulness. Its proof is elementary and can be found in numerous textbooks on functional analysis and integration.

Proposition 4.2.1. *The pettis integral posesses the following properties.*

- *If f is Pettis integrable, then its indefinite Pettis integral is unique.*
- *If $T : X \to X$ is bounded, then $T\left(\int_E f d\mu\right) = \int_E (Tf)\, d\mu$ whenever one of the integrals exists.*
- *If $\mu(A\cap B) = 0$, then $\int_{A\cup B} f d\mu = \int_A f d\mu + \int_B f d\mu$.*
- $|| \int_E f d\mu|| \leq \int_E ||f|| d\mu$

Lemma 4.2.1. *Let $h \in \mathcal{RCR}(\mathbb{R},\mathbb{R}^n)$ and let H.1–H.2 hold. The function $U(t,\cdot)[\chi_0 h(\cdot)] : [s,t] \to \mathcal{RCR}$ is Pettis integrable for all $t \geq s$ and*

$$\left[\int_s^t U(t,\mu)[\chi_0 h(\mu)]d\mu\right](\theta) = \int_s^t U(t,\mu)[\chi_0 h(\mu)](\theta)d\mu. \tag{25}$$

Proof. By Lemma 3.1.3 and the uniqueness assertion of Proposition 4.2.1, if we can show for all $p : [-r,0] \to \mathbb{R}^n$ of bounded variation the equality

$$\int_{-r}^0 p^*(\theta)d\left[\int_s^t U(t,\mu)[\chi_0 h(\mu)](\theta)d\mu\right] = \int_s^t \left[\int_{-r}^0 p^*(\theta)d\Big[U(t,\mu)[\chi_0 h(\mu)](\theta)\Big]\right]d\mu$$

holds, then Lemma 4.2.1 will be proven. Note that the above is equivalent to

$$\int_{-r}^0 p^*(\theta)d\left[\int_s^t V(t+\theta,\mu)h(\mu)d\mu\right] = \int_s^t \left[\int_{-r}^0 p^*(\theta)d\Big[V(t+\theta,\mu)h(\mu)\Big]\right]d\mu. \tag{26}$$

We prove the lemma first when h is a step function. In this case, a consequence of equation (23) is that $\theta \mapsto V(t+\theta,\mu)h(\mu)$ and $\mu \mapsto V(t+\theta,\mu)h(\mu)$ are piecewise continuous, while Lemma 3.4.1 and Lemma 4.1.2 imply $\theta \mapsto \int_s^t V(t+\theta,\mu)h(\mu)d\mu$ is also piecewise continuous, all with at most finitely many discontinuities on any given bounded set. Conseqently, both integrals in (26) can be regarded as a Lebesgue-Stieltjes integrals, with Fubini's theorem granting the desired equality.

When $h \in \mathcal{RCR}(\mathbb{R}, \mathbb{R}^n)$ is an arbitrary right-continuous regulated function, we approximate its restriction to the inverval $[s, t]$ by a convergent sequence of step functions h_n by Lemma 3.1.2. Equation (26) is then satisfied with h replaced with h_n. Define the functions

$$J_n(\theta) = \int_s^t V(t+\theta, \mu) h_n(\mu) d\mu, \qquad K_n(\mu) = \int_{-r}^0 p^*(\theta) d\Big[V(t+\theta, \mu) h_n(\mu)\Big],$$
$$J(\theta) = \int_s^t V(t+\theta, \mu) h(\mu) d\mu, \qquad K(\mu) = \int_{-r}^0 p^*(\theta) d\Big[V(t+\theta, \mu) h(\mu)\Big],$$

so that $\int_{-r}^0 p^*(\theta) dJ_n(\theta) = \int_s^t K_n(\mu) d\mu$. By Lemma 3.4.2 and elementary integral estimates, $J_n \to J$ uniformly. The conditions of Lemma 3.1.5 are satisfied, and we have the limit

$$\int_{-r}^0 p^*(\theta) dJ_n(\theta) \to \int_{-r}^0 p^*(\theta) dJ(\theta).$$

Conversely, for each $\mu \in [s, t]$, Lemma 3.1.4 applied to the difference $K_n(\mu) - K(\mu)$ yields, together with Lemma 3.4.2,

$$|K_n(\mu) - K(\mu)| \leq (|p(0)| + |p(-r)| + \mathrm{var}_{-r}^0 p) \left(\int_s^t \exp\left(\int_y^t \ell(\nu) d\nu \right) dy \right) ||h_n - h||.$$

Thus, $K_n \to K$ uniformly, and the bounded convergence theorem implies $\int_s^t K_n(\mu) d\mu \to \int_s^t K(\mu) d\mu$. Therefore, equation (26) holds, and the lemma is proven. □

With Lemma 4.1.3 and Lemma 4.2.1 at hand, we obtain the variation of constants formula for the linear inhomogeneous equation (4)–(5) in the Banach space $\mathcal{RCR}$.

Theorem 4.2.1. *Let H.1–H.2 hold, and let $h \in \mathcal{RCR}(\mathbb{R}, \mathbb{R}^n)$. The unique solution $t \mapsto x_t(\cdot; s, \phi; h, r) \in \mathcal{RCR}$ of the linear inhomogeneous impulsive system* (4)–(5) *with initial condition $x_s(\cdot; s, \phi; h, r) = \phi$, satisfies the variation-of-constants formula*

$$x_t(\cdot; s, \phi; h, r) = U(t, s)\phi + \int_s^t U(t, \mu)[\chi_0 h(\mu)] d\mu + \sum_{s < \tau_i \leq t} U(t, \tau_i)[\chi_0 r_i], \tag{27}$$

where the integral is interpreted in the Pettis sense and may be evaluated pointwise using (25)*.*

As a simple corollary, we can express any solution $t \mapsto x_t$ defined on $[s, \infty)$ as the solution of an integral equation.

Corollary 4.2.1.1. *Let H.1–H.2 hold, and let $h \in \mathcal{RCR}(\mathbb{R}, \mathbb{R}^n)$. Any solution $t \mapsto x_t \in \mathcal{RCR}$ of the linear inhomogeneous impulsive system* (4)–(5) *defined on the interval $[s, \infty)$ satisfies for all $t \geq s$ the equation*

$$x_t = U(t, s) x_s + \int_s^t U(t, \mu)[\chi_0 h(\mu)] d\mu + \sum_{s < \tau_i \leq t} U(t, \tau_i)[\chi_0 r_i]. \tag{28}$$

5 Existence of Lipschitz continuous center manifolds

This section will be devoted to the existence of center manifolds, a reduction principle, and the derivation of abstract impulsive differential equations restricted to the center manifold.

5.1 Preliminary definitions and mild solutions of an abstract integral equation

At this stage it is appropriate to introduce several exponentially weighted Banach spaces that will be needed to construct the center manifolds. First, denote $PC(\mathbb{R}, \mathbb{R}^n)$ the set of functions $f : \mathbb{R} \to \mathbb{R}^n$ that are continuous everywhere except for at times $t \in \{\tau_k : k \in \mathbb{Z}\}$ where they are continuous from the right and have limits on the left.

$$\begin{aligned}
\mathcal{PC}^\eta &= \{\phi : \mathbb{R} \to \mathcal{RCR} : \phi(t) = f_t, f \in PC(\mathbb{R}, \mathbb{R}^n), ||\phi||_\eta = \sup_{t\in\mathbb{R}} e^{-\eta|t|}||\phi(t)|| < \infty\} \\
B^\eta(\mathbb{R}, \mathcal{RCR}) &= \{f : \mathbb{R} \to \mathcal{RCR} : ||f||_\eta = \sup_{t\in\mathbb{R}} e^{-\eta|t|}||f(t)|| < \infty\} \\
PC^\eta(\mathbb{R}, \mathbb{R}^n) &= \{f \in PC(\mathbb{R}, \mathbb{R}^n) : ||f||_\eta = \sup_{t\in\mathbb{R}} e^{-\eta|t|}||f(t)|| < \infty\} \\
B^\eta_{\tau_k}(\mathbb{Z}, \mathbb{R}^n) &= \{f : \mathbb{Z} \to \mathbb{R}^n : ||f||_\eta = \sup_{k\in\mathbb{Z}} e^{-\eta|\tau_k|}|f_k| < \infty\}.
\end{aligned}$$

Also, if $\mathcal{M} \subset \mathbb{R} \times \mathcal{RCR}$ is a nonautonomous set over $\mathcal{RCR}$, we define the space $\mathcal{PC}^\eta(\mathbb{R}, \mathcal{M})$ of piecewise-continuous functions taking values in $\mathcal{M}$ by

$$\mathcal{PC}^\eta(\mathbb{R}, \mathcal{M}) = \{f \in \mathcal{PC}^\eta : f(t) \in \mathcal{M}(t)\}.$$

If X^η is one of the above spaces, then the normed space $X^{\eta,s} = (X^\eta, ||\cdot||_{\eta,s})$ with norm

$$||F||_{\eta,s} = \begin{cases} \sup_{t\in\mathbb{R}} e^{-\eta|t-s|}||F(t)||, & \text{dom}(F) = \mathbb{R} \\ \sup_{k\in\mathbb{Z}} e^{-\eta|\tau_k-s|}||F(k)||, & \text{dom}(F) = \mathbb{Z}, \end{cases}$$

is complete.

Our attention shifts now to the semilinear system

$$\dot{x} = L(t)x_t + f(t, x_t), \qquad t \neq \tau_k \tag{29}$$

$$\Delta x = B_k x_{t^-} + g_k(x_{t^-}), \qquad t = \tau_k, \tag{30}$$

for nonlinearities $f : \mathbb{R} \times \mathcal{RCR} \to \mathbb{R}^n$ and $g_k : \mathcal{RCR} \to \mathbb{R}^n$. Additional assumptions on the nonlinearities, evolution family and sequence of impulses may include the following.

- H.3 For each $\phi \in \mathcal{RCR}([\alpha - r, \beta], \mathbb{R}^n)$, the function $t \mapsto f(t, \phi_t)$ is an element of $\mathcal{RCR}([\alpha, \beta], \mathbb{R}^n)$.
- H.4 The evolution family $U(t, s) : \mathcal{RCR} \to \mathcal{RCR}$ associated to the homogeneous equation (15)–(16) is spectrally separated.
- H.5 $\phi \mapsto (t, \phi)$ and $\phi \mapsto g_k(\phi)$ are C^m for some $m \geq 1$ for each $t \in \mathbb{R}$ and $k \in \mathbb{Z}$, and there exists $\delta > 0$ such that for each $j = 0, \ldots, m$, there exists $c_j : \mathbb{R} \to \mathbb{R}^+$ locally bounded and a positive sequence $\{d_j(k) : k \in \mathbb{Z}\}$ such that

$$\begin{aligned}
||D^j f(t, \phi) - D^j f(t, \psi)|| &\leq c_j(t)||\phi - \psi||, \\
||D^j g_k(\phi) - D^j g_k(\psi)|| &\leq d_j(k)||\phi - \psi||.
\end{aligned}$$

  for $\phi, \psi \in B_\delta(0) \subset \mathcal{RCR}$. Also, there exists $q > 0$ such that $||D^j f(t, \phi)|| \leq qc_j(t)$ and $||D^j g_k(\phi)|| \leq qd_j(k)$ for $\phi \in B_\delta(0)$.
- H.6 $f(t, 0) = g_k(0) = 0$ and $Df(t, 0) = Dg_k(0)$ for all $t \in \mathbb{R}$ and $k \in \mathbb{Z}$.
- H.7 There exists a constant $\xi > 0$ such that $\tau_{k+1} - \tau_k \geq \xi$ for all $k \in \mathbb{Z}$.

Definition 5.1.1. *A* mild solution *of the semilinear equation* (29)–(30) *is a function* $x : [s, T] \to \mathcal{RCR}$ *such that for all* $s \leq t < T$*, the function* $x(t) = x_t$ *satisfies the integral equation*

$$x(t) = U(t, s)x(s) + \int_s^t U(t, \mu)[\chi_0 f(\mu, x(\mu))]d\mu + \sum_{s<\tau_i\leq t} U(t, \tau_i)[\chi_0 g(\tau_i, x(\tau_i^-))], \tag{31}$$

and $x(t)(\theta) = x(t+\theta)(0)$ *whenver* $\theta \in [-r.0]$ *satisfies* $t+\theta \in [s, T]$*, where* U *is the evolution family associated to the homogeneous equation* (15)–(16) *and the integral is interpreted in the Pettis sense.*

Remark 5.1.1. *The right-hand side of equation* (31) *is well-posed under conditions H.1–H.3 in the sense that it defines for* $s \le t < T$*, a nonlinear operator from* $\mathcal{RCR}([s-r,t],\mathbb{R}^n)$ *into* $\mathcal{RCR}$*. Note also that for a function* $x:[s,T]\to\mathcal{RCR}$*, we denote* $x(\tau_i^-)(\theta) = x(\tau_i^-)(\theta)$ *for* $\theta < 0$ *and* $x(\tau_i^-)(0) = x(\tau_i)(0^-)$*.*

If $x:[s-r,T)\to\mathbb{R}^n$ is a *classical solution* — that is, x is differentiable from the right, continuous except at impulse times τ_k, continuous from the right on $[s-r,T]$ and its derivative satisfies the differential equation (29)–(30) — then $t\mapsto x_t$ is a mild solution. This can be seen by defining the inhomogeneities $h(t)\equiv f(t,x_t)$ and $r_k \equiv g_k(x_{\tau_k^-})$, solving the equivalent linear equation (4)–(5) with these inhomogeneities and intial condition $(s,x_s)\in\mathbb{R}\times\mathcal{RCR}$ in the mild sense, and applying Corollary 4.2.1.1. For this reason, we will work with equation (31) exclusively from now on.

Additionally, one should note that the assumption H.5 implies that the nonlinearities are uniformly locally Lipschitz continuous. Together with the other assumptions, this implies the local existence and uniqueness of mild solutions through a given $(s,\phi)\in\mathbb{R}\times\mathcal{RCR}$. Namely, we have the following lemma, which may be seen as a local, nonlinear version of (3.4.1).

Lemma 5.1.1. *Under assumptions H.1–H.5, for all* $(s,\phi)\in\mathbb{R}\times D$*, there exists a unique mild solution* $x^{(s,\phi)}:[s,s+\alpha)\to\mathcal{RCR}$ *of* (31) *for some* $\alpha=\alpha(s,\phi)>0$*, satisfying* $x(s)=\phi$*. Also, if one defines the nonautonomous set*

$$\mathcal{M} = \bigcup_{\phi\in\mathcal{RCR}}\bigcup_{s\in\mathbb{R}}\bigcup_{t\in[s,s+\alpha)}\{t\}\times\{s\}\times\{\phi\},$$

then $S:\mathcal{M}\to\mathcal{RCR}$ *with* $S(t,s)x = x^{(s,\phi)}(t)$ *is a process on* $\mathcal{RCR}$*.*

Combining the discussion following Definition 5.1.1 with Lemma 5.1.1, it follows that $S(t,s)$ satisfies the following abstract integral equation wherever it is defined.

$$S(t,s)\phi = U(t,s)\phi + \int_s^t U(t,\mu)\chi_0 f(\mu,S(\mu,s)\phi)d\mu + \sum_{s<\tau_i\le t} U(t,\tau_i)\chi_0 g(\tau_i,S(\tau_i^-,s)\phi). \tag{32}$$

Of use later will be a result concerning the smoothness of the process $S:\mathcal{M}\to\mathcal{RCR}$. This result is interesting in its own right and will be useful later in proving the periodicity of center manifolds; see Theorem 8.3.1.

Theorem 5.1.1. *Under hypotheses H.1–H.6, the process* $S:\mathcal{M}\to\mathcal{RCR}$ *is* C^{k+1}*. Also,* $S'(t,s) = DS(t,s)\phi$ *satisfies for* $t\ge s$ *the abstract integral equation*

$$S'(t,s) = U(t,s) + \int_s^t U(t,\mu)\chi_0 Df(\mu,S(\mu,s)\phi)S'(\mu,s)d\mu + \sum_{s<\tau_i\le t} U(t,\tau_i)\chi_0 Dg(\tau_i,S(\tau_i^-,s)\phi)S'(\tau_i^-,s). \tag{33}$$

Proof. We will prove only that S is C^1, the proof of higher-order smoothness being an essentially identical albeit notationally cumbersome extension thereof. Let $s\in\mathbb{R}$ be fixed. Let $\psi\in\mathcal{RCR}$ be given. For given $\nu>0$, denote by $B_\nu(\psi)$ the closed ball centered at ψ with radius ν in $\mathcal{RCR}$.

Introduce for given $\epsilon,\delta,\nu>0$ the normed vector space $(X_{\epsilon,\delta,\nu},||\cdot||)$, where $X_{\epsilon,\delta,\nu}$ consists of the functions $\phi:[s-r,s+\epsilon]\times B_\delta(\psi)\to B_\nu(\psi)$ such that $x\mapsto\phi(t,x)$ is continuous for each t, $\phi(t,x)(\theta)=\phi(t+\theta,x)(0)$ whenever $\theta\in[-r,0]$ and $[t+\theta,t]\subset[s-r,s+\epsilon]$, and $|\phi||<\infty$ for the norm given by

$$||\phi||_{\epsilon,\delta,\nu} = \sup_{\substack{t\in[s-r,s+\epsilon]\\ ||x-\psi||\le\delta}} ||\phi(t,x)||.$$

It can be easily verified that $(X_{\epsilon,\delta,\nu},||\cdot||)$ is a Banach space. With $\mathcal{L}(\mathcal{RCR})$ the bounded linear operators on $\mathcal{RCR}$, introduce also the space $(\mathbf{X}_{\epsilon,\delta},||\cdot||)$ consisting of functions $\Phi:[s-r,s+\epsilon]\times\mathcal{RCR}\to\mathcal{L}(\mathcal{RCR})$ such that $x\mapsto\Phi(t,x)$ is continuous for each t, $\Phi(t,x)h(\theta)=\Phi(t+\theta,x)h(0)$ for all $h\in\mathcal{RCR}$, and $||\Phi||<\infty$, where the norm is $||\Phi(t,x)|| = \sup_{||h||=1}||\Phi(t,x)h||_{\epsilon,\delta,\nu}$. Clearly, $(\mathbf{X}_{\epsilon,\delta},||\cdot||)$ is complete.

Define a pair of nonlinear operators

$$\Lambda_1 : X_{\epsilon,\delta,\nu} \to X_{\epsilon,\delta,\nu},$$

$$\Lambda_1(\phi)(t,x) = \chi_{[s-r,s)}(t)x(t-s) + \chi_{[s,s+\epsilon]}(t)\left[U(t,s)x(s) + \int_s^t U(t,s)\chi_0 f(\mu,\phi(\mu,x))d\mu + \sum_{s<\tau_i\leq t} U(t,\tau_i)\chi_0 g(\tau_i,\phi(\tau_i^-,x))\right]$$

$$\Lambda_2 : X_{\epsilon,\delta} \times \mathbf{X}_{\epsilon,\delta} \to \mathbf{X}_{\epsilon,\delta}$$

$$\Lambda_2(\phi,\Phi)(t,x)h = \chi_{[s-r,s)}(t)I_{\mathcal{RCR}}h + \chi_{[s,s+\epsilon]}(t)\left[U(t,s)h + \int_s^t U(t,\mu)\chi_0 Df(\mu,\phi(\mu,x))\Phi(\mu,x)hd\mu + \right.$$
$$\left. + \sum_{s<\tau_i\leq t} U(t,\mu)\chi_0 Dg(\tau_i,\phi(\tau_i^-,x)))\Phi(\tau_i^-,x)h\right],\ h\in\mathcal{RCR}.$$

By choosing ϵ and δ small enough, Λ_1 can be shown to be a uniform contraction. Indeed, if we denote $\kappa = \sup_{||x-\psi||\leq 2\delta}||x||$, the mean-value theorem grants the estimate

$$\begin{aligned}||\Lambda_1(\phi)-\Lambda_1(\gamma)|| &\leq \kappa \sup_{t\in[s,s+\epsilon]}\left(\int_s^t ||U(t,\mu)||c_1(\mu)d\mu + \sum_{s<\tau_i\leq t}||U(t,\tau_i)||d_1(i)\right)||\phi-\gamma|| \\ &\equiv \kappa L_\epsilon||\phi-\gamma||\end{aligned}$$

We can always obtain a uniform contraction by taking ϵ small enough. Also, note that $t\mapsto \Lambda_1(\phi)(t,x)\in\mathcal{RCR}$, $x\mapsto\Lambda_1(\phi,x)$ is continuous and $\Lambda_1(\phi)(t,x)(\theta) = \Lambda_1(\phi)(t+\theta,x)(0)$. To ensure the appropriate boundedness, if we denote $\overline{\kappa} = \sup_{||x-\psi||\leq\delta}k_0(x)$, the estimate

$$\begin{aligned}||\Lambda_1(\phi)-\psi|| &\leq ||\phi-\psi|| + \overline{\kappa}\sup_{t\in[s,s+\epsilon]}\left(\int_s^t||U(t,\mu)||c_0(\mu)d\mu + \sum_{s<\tau_i\leq t}||U(t,\tau_i)||d_0(i)\right) \\ &\equiv \delta + \overline{\kappa}M_\epsilon\end{aligned}$$

implies it is sufficient to choose $\epsilon,\delta,\nu>0$ small enough so that $\delta+\overline{\kappa}M_\epsilon<\nu$. This can always be done because $M_\epsilon\to 0$ as $\epsilon\to 0$ due to H.5 and Lemma 3.4.2.

The continuity of $\phi\mapsto\Lambda_2(\phi,\Phi)$ for fixed $\Phi\in\mathbf{X}_{\epsilon,\delta}$ follows by the estimate

$$\begin{aligned}||\Lambda_2(\phi,\Phi)-\Lambda_2(\gamma,\Phi)|| \leq \Bigg(&\int_s^{s+\epsilon}||U(s+\epsilon,\mu)||c_1(\mu)||(\phi(\mu,x)-\gamma(\mu,x)||d\mu \\ &+ \sum_{s<\tau_i\leq s+\epsilon}||U(s+\epsilon,\tau_i)||d_1(i)||\phi(\tau_i^-,x)-\gamma(\tau_i^-,x)||\Bigg)||\Phi||.\end{aligned}$$

Also, for each $\phi\in B_\delta(\psi)$ it is readily verified that $||\Lambda_2(\phi,\Phi)-\Lambda_2(\phi,\Gamma)||\leq\kappa L_\epsilon||\Phi-\Gamma||$, which by previous choices of $\epsilon,\delta,\nu>0$ indicates that $\Phi\mapsto\Lambda_2(\phi,\Phi)$ is a uniform contraction.

We are ready to prove the statement of the theorem. Denote by (x_n,x_n') the iterates of the map $\Lambda : X_{\epsilon,\delta,\nu}\times\mathbf{X}_{\epsilon,\delta,\nu}\to X_{\epsilon,\delta,\nu}\times\mathbf{X}_{\epsilon,\delta,\nu}$ defined by $\Lambda(x,x') = (\Lambda_1(x),\Lambda_2(x,x'))$ and initialized at (x_0,x_0') with $x_0(t,x)=x$ and $x_0'(t,x)=I_{\mathcal{RCR}}$. The fiber contraction theorem [16] implies convergence $(x_n,x_n')\to(x,x')$. Note also that $Dx_0=x_0'$. If we suppose $Dx_n=x_n'$ for some $n\geq 0$, then for $t\geq s$, Lemma 4.2.1 implies that for each $\theta \in [-r, 0]$,

$$\begin{aligned}
Dx_{n+1}(t,\phi)(\theta) &= D\left[U(t,s)x_n(s,\phi)(\theta) + \int_s^t U(t,\mu)\chi_0 f(\mu, x_n(\mu,\phi))(\theta)d\mu + \sum_{s<\tau_i\le t} U(t,\tau_i)\chi_0 g(\tau_i, x_n(\tau_i^-,\phi))(\theta)\right] \\
&= D\left[U(t,s)x_n(s,\phi)(\theta) + \int_s^t V(t+\theta,\mu) f(\mu, x_{n+1}(\mu,\phi))d\mu + \sum_{s<\tau_i\le t} V(t+\theta,\tau_i) g(\tau_i, x_{n+1}(\tau_i^-,\phi))\right] \\
&= U(t,s)x_n'(s,\phi)(\theta) + \int_s^t V(t+\theta,\mu) Df(\mu, x_n(\mu,\phi))x_n'(\mu,\phi)d\mu \\
&\quad + \sum_{s<\tau_i\le t} V(t+\theta,\tau_i) Dg(\tau_i, x_n(\tau_i^-,\phi))x_n'(\tau_i^-,\phi) \\
&= \Lambda_2(x_n, x_n')(t,\phi)(\theta) \\
&= x_{n+1}'(t,\phi)(\theta),
\end{aligned}$$

while for $t < s$, it is easily checked that $Dx_{n+1}(t,\phi) = x_{n+1}'(t,\phi)$. This proves that $Dx_{n+1}(\theta) = x_{n+1}'(\theta)$ pointwise in θ. To prove the result uniformly, we note that the difference quotient can be written for $t \ge s$ as

$$\begin{aligned}
&\frac{1}{||h||}\big(x_{n+1}(t,\phi+h) - x_{n+1}(t,\phi) - x_{n+1}'(t,\phi)h\big) \\
&\qquad = \int_s^t U(t,\mu)\chi_0 \frac{1}{||h||}\big(f(\mu, x_n(\mu,\phi+h)) - f(\mu, x_n(\mu,\phi)) - Df(\mu, x_n(\mu,\phi))Dx_n(\mu,\phi)h\big)d\mu \\
&\qquad\quad + \sum_{s<\tau_i\le t} U(t,\tau_i)\chi_0 \frac{1}{||h|||}\big(g(\tau_i, x_n(\tau_i^-,\phi+h)) - g(\tau_i, x_n(\tau_i^-,\phi)) - Dg(\tau_i, x_n(\tau_i^-,\phi))Dx_n(\tau_i^-,\phi)h\big).
\end{aligned}$$

Since x_n is differentiable by the induction hypothesis, the integrand and summand converge uniformly to zero as $h \to 0$. Thus, x_{n+1} is differentiable and $Dx_{n+1} = x_{n+1}'$, so by induction $Dx_n = x_n'$ for each n. Also, by construction, x_n' is continuous for each n and, being the uniform limit of continuous functions, $x' = \lim_{n\to\infty} x'$ is continuous. By the fundamental theorem of calculus,

$$\begin{aligned}
\frac{x(\phi+h) - x(\phi) - x'(\phi)h}{||h||} &= \lim_{n\to\infty} \frac{x_n(\phi+h) - x_n(\phi) - Dx_n(\phi)h}{||h||} \\
&= \lim_{n\to\infty} \int_0^1 \frac{1}{||h||}\left[x_n'(\phi + (\lambda-1)h) - x_n'(\phi)\right] hd\lambda \\
&= \int_0^1 \frac{1}{||h||}\left[x'(\phi + (\lambda-1)h) - x'(\phi)\right] hd\lambda \to 0
\end{aligned}$$

as $h \to 0$. By definition, x is differentiable and $Dx = x'$.

If we define $y(t)\phi = x(t,\phi)$ for the fixed point $x : [s-r, s+\epsilon] \times B_\delta(\psi) \to B_\nu(\psi)$, then y satisfies $y(t)\phi = S(t,s)\phi$ for $(t,\phi) \in [s, s+\epsilon] \times B_\delta(\psi)$. This can be seen by comparing the fixed point equation $y(t) = \Lambda_1(y)(t,\phi)$ with the abstract integral equation (32). We conclude that S is C^1 (fibrewise). The correctness of equation (33) follows by comparing to the fixed point equation associated to Λ_2. □

5.2 Bounded solutions of the inhomogeneous linear equation

In this section we will identify a pseudoinverse for η-bounded solutions of the inhomogeneous linear equation

$$x(t) = U(t,s)x(s) + \int_s^t U(t,\mu)[\chi_0 F(\mu)]d\mu + \sum_{s<\tau_i\le t} U(t,\tau_i)[\chi_0 G_i], \qquad -\infty < s \le t < \infty. \tag{34}$$

As defined in Definition 3.3.5, we recall now that $\mathcal{RCR}_c(t) = \mathcal{R}(P_c(t))$, where P_c is the projection onto the center bundle of the linear part of (29)–(30).

Lemma 5.2.1. *Let $\eta \in (0, \min\{-a, b\})$ and let H.1, H.2 and H.4 hold. Then,*

$$\mathcal{RCR}_c(\nu) = \{\varphi \in \mathcal{RCR} : \exists x \in \mathcal{PC}^\eta,\ x(t) = U(t,s)x(s),\ x(\nu) = \varphi\}. \tag{35}$$

Proof. If $\varphi \in \mathcal{RCR}_c(\nu)$, then $P_c(\nu)\varphi = \varphi$ and the function $x(t) = U(t,\nu)P_c(\nu)\varphi = U_c(t,\nu)\varphi$ is defined for all $t \in \mathbb{R}$, satisfies $x(t) = U(t,s)x(s)$, $x(\nu) = \varphi$, $x(t)(\theta) = x(t+\theta)(0)$, and by chosing $\epsilon < \eta$, there exists $K > 0$ such that

$$e^{-\eta|t|}||x(t)|| \le Ke^{\epsilon|\nu|}e^{-(\eta-\epsilon)|t|}||\varphi|| \le Ke^{\epsilon|\nu|}||\varphi||.$$

Finally, as $x(t) = [U(t,s)x(s)(0)]_t$ for all $t \in \mathbb{R}$, we conclude $x \in \mathcal{PC}^\eta$.

Conversely, suppose $\varphi \in \mathcal{RCR}$ admits some $x \in \mathcal{PC}^\eta$ such that $x(t) = U(t,s)x(s)$ and $x(\nu) = \varphi$. Let $||x||_\eta = \overline{K}$. We will show that $P_s(\nu)\varphi = P_u(\nu)\varphi = 0$, so that $\varphi = I\varphi = (P_c(\nu) + P_s(\nu) + P_u(\nu))\varphi = P_c(\nu)\varphi$, from which we will conclude $\varphi \in \mathcal{RCR}_c(\nu)$.

By spectral separation, we have for all $\rho < \nu$,

$$\begin{aligned} e^{-\eta|\rho|}||P_s(\nu)\varphi|| &= e^{-\eta|\rho|}||U_s(\nu,\rho)P_s(\rho)x(\rho)|| \\ &\le e^{-\eta|\rho|}Ke^{a(\nu-\rho)}||P_s(\rho)|| \cdot ||x(\rho)|| \\ &\le K\overline{K}e^{a(\nu-\rho)}||P_s(\rho)||, \end{aligned}$$

which implies $||P_s(\nu)\varphi|| \le K\overline{K}e^{a\nu}||P_s(\rho)||\exp(\eta|\rho| - a\rho)$. Since $\eta < -a$ and $\rho \mapsto ||P_s(\rho)||$ is bounded, taking the limit as $\rho \to -\infty$ we obtain $||P_s(\nu)\varphi||| \le 0$. Similarly, for $\rho > \nu$, we have

$$\begin{aligned} e^{-\eta|\rho|}||P_u(\nu)\varphi|| &= e^{-\eta|\rho|}||U_u(\nu,\rho)P_u(\rho)x(\rho)|| \\ &\le e^{-\eta|\rho|}Ke^{b(\nu-\rho)}||P_u(\rho)|| \cdot ||x(\rho)|| \\ &\le K\overline{K}e^{b(\nu-\rho)}||P_u(\rho)||, \end{aligned}$$

which implies $||P_u(\nu)\varphi|| \le K\overline{K}e^{b\nu}||P_u(\rho)||\exp(\eta|\rho| - b\rho)$. Since η <b and $\rho \mapsto ||P_u(\rho)||$ is bounded, taking the limit $\rho \to \infty$ we obtain $||P_u(\nu)\varphi|| \le 0$. Therefore, $P_s(\nu)\varphi = P_u(\nu)\varphi = 0$, and we conclude that $P_c(\nu)\varphi = \varphi$ and $\varphi \in \mathcal{RCR}_c(\nu)$. □

Lemma 5.2.2. *Let conditions H.1, H.2 and H.4 be satisfied. Let $h \in \mathcal{RCR}(\mathbb{R}, \mathbb{R}^n)$. The integrals*

$$\int_s^t U(t,\mu)P_c(\mu)[\chi_0 h(\mu)]d\mu, \qquad \int_v^t U(t,\mu)P_u(\mu)[\chi_0 h(\mu)]d\mu$$

Proof. The nontrivial cases are where $t \le s$ and $t \le v$. For the former, defining $H(\mu) = \chi_0 h(\mu)$ we have the string of equalities

$$\begin{aligned} U_c(t,s)P_c(s)\int_t^s U(s,\mu)H(\mu)d\mu &= U_c(t,s)\int_t^s U_c(s,\mu)P_c(\mu)H(\mu)d\mu \\ &= \int_t^s U_c(t,\mu)P_c(\mu)H(\mu)d\mu \\ &= \int_t^s U(t,\mu)P_c(\mu)H(\mu)d\mu \\ &= -\int_s^t U(t,\mu)P_c(\mu)H(\mu)d\mu. \end{aligned}$$

The first integral on the left exists due to Lemma 4.2.1 and Proposition 4.2.1. The subsequent equalities follow by Proposition 4.2.1 and the definition of spectral separation. The case $t \le v$ for the other integral is proven similarly.

□

Define the (formal) family of linear operators $\mathcal{K}_s^\eta : \mathcal{PC}^\eta(\mathbb{R},\mathbb{R}^n) \oplus B^\eta_{\tau_k}(\mathbb{Z},\mathbb{R}^n) \to B^\eta(\mathbb{R},\mathcal{RCR})$ by the equation

$$\begin{aligned}\mathcal{K}_s^\eta(F,G)(t) = &\int_s^t U(t,\mu)P_c(\mu)[\chi_0 F(\mu)]d\mu - \int_t^\infty U(t,\mu)P_u(\mu)[\chi_0 F(\mu)]d\mu + \int_{-\infty}^t U(t,\mu)P_s(\mu)[\chi_0 F(\mu)]d\mu \\ &+ \sum_s^t U(t,\tau_i)P_c(\tau_i)[\chi_0 G_i]d\tau_i - \sum_t^\infty U(t,\tau_i)P_u(\tau_i)[\chi_0 G_i]d\tau_i + \sum_{-\infty}^t U(t,\tau_i)P_s(\tau_i)[\chi_0 G_i]d\tau_i,\end{aligned} \tag{36}$$

indexed by $s \in \mathbb{R}$, where the external direct sum $\mathcal{PC}^{\eta,s}(\mathbb{R},\mathbb{R}^n) \oplus B^{\eta,s}_{\tau_k}(\mathbb{Z},\mathbb{R}^n)$ is identified as a Banach space with norm $||(f,g)||_{\eta,s} = ||f||_{\eta,s} + ||g||_{\eta,s}$, and the summations are defined as follows:

$$\sum_a^b F(\tau_i)d\tau_i = \begin{cases} \displaystyle\sum_{a<\tau_i\le b} F(\tau_i), & a \le b \\ \displaystyle -\sum_b^a F(\tau_i)d\tau_i, & b < a. \end{cases}$$

Lemma 5.2.3. *Let H.1, H.2, H.4 and H.7 hold, and let $\eta \in (0,\min\{-a,b\})$.*

1. *The function $\mathcal{K}_s^\eta : \mathcal{PC}^{\eta,s}(\mathbb{R},\mathbb{R}^n) \oplus B^{\eta,s}_{\tau_k}(\mathbb{Z},\mathbb{R}^n) \to B^{\eta,s}(\mathbb{R},\mathcal{RCR})$ with $\eta \in (0,\min\{-a,b\})$ and defined by formula (36) is linear and bounded. In particular, the norm satisfies*
$$||\mathcal{K}_s^\eta|| \le C\left[\frac{1}{\eta-\epsilon}\left(1+\frac{e^{(\eta-\epsilon)\xi}}{\xi}\right) + \frac{1}{-a-\eta}\left(1+\frac{2e^{(\eta-a)\xi}}{\xi}\right) + \frac{1}{b-\eta}\left(1+\frac{2e^{(b+\eta)\xi}}{\xi}\right)\right] \tag{37}$$
for some constants C and ϵ independent of s.
2. *$\mathcal{K}_s^\eta$ has range in $\mathcal{PC}^{\eta,s}$ and $v = \mathcal{K}_s^\eta(F,G)$ is the unique solution of (34) in $\mathcal{PC}^{\eta,s}$ satisfying $P_c(s)v(s) = 0$.*
3. *The expression $\mathcal{K}_*(F,G)(t) = (I - P_c(t))K_s^0(F,G)(t)$ uniquely defines, independent of s, a bounded linear map*
$$\mathcal{K}_* : \mathcal{PC}^0(\mathbb{R},\mathbb{R}^n) \oplus B^0_{\tau_k}(\mathbb{Z},\mathbb{R}^n) \to \mathcal{PC}^0.$$

Proof. Let $\epsilon < \min\{\min\{-a,b\}-\eta,\eta\}$. To show that $\mathcal{K}_s^\eta$ is well-defined, we start by mentioning that all improper integrals and inifnite sums appearing on the right-hand side of (36) can be interpreted as limits of well-defined finite integrals and sums, due to Lemma 4.2.1, Lemma 5.2.2 and Proposition 4.2.1. For brevity, write

$$\mathcal{K}_s^\eta(F,G) = \left(K_1^{u,f} - K_1^{c,F} + K_1^{u,F}\right) + \left(K_2^{u,G} - K_2^{c,G} + K_2^{s,G}\right),$$

where each term corresponds to the one in (36) in order of appearance.

We start by proving the convergence of the improper integrals. Denote

$$I(v) = \int_t^v U(v,\mu)P_u(\mu)[\chi_0 F(\mu)]d\mu,$$

and let $v_k \nearrow \infty$. We have, for $m > n$ and n sufficiently large so that $v_m > 0$,

$$\begin{aligned}||I(v_m) - I(v_n)|| &\le \int_{v_n}^{v_m} KNe^{b(t-\mu)}|F(\mu)|d\mu \\ &\le \int_{v_n}^{v_m} KNe^{b(t-\mu)}e^{\eta\mu}||F||_\eta d\mu \\ &= KN||F||_\eta e^{bt}\int_{v_n}^{v_m} e^{\mu(\eta-b)}d\mu \\ &= \frac{KN||F||_\eta}{b-\eta}e^{bt}\left(e^{-v_n(b-\eta)} - e^{-v_m(b-\eta)}\right) \\ &\le \frac{KN||F||_\eta}{b-\eta}e^{bt}e^{-v_n(b-\eta)}.\end{aligned}$$

Therefore, $I(v_k) \in \mathcal{RCR}$ is Cauchy, and thus converges; namely, it converges to the improper integral $K^{u,F}(t)$. One can similarly prove that $K^{s,F}(t)$ converges. For the infinite sums, we employ similar estimates; if we denote $S = \sum_{t<\tau_i<\infty} ||U_u(t,\tau_i)[\chi_0 G_i]||$ and assume without loss of generality that $\tau_0 = 0$, a fairly crude estimate (that we will later improve) yields

$$\begin{aligned} S &\leq \sum_{t<\tau_i<\infty} KNe^{b(t-\tau_i)}e^{\eta|\tau_i|}||G||_\eta \\ &= \sum_{-|t|<\tau_i\leq 0} KN||G||_\eta e^{bt}e^{|\tau_i|(b+\eta)} + \sum_{0<\tau_k<\infty} KN||G||_\eta e^{bt}e^{-(b-\eta)\tau_i} \\ &\leq KNe^{bt}\left(\frac{|t|}{\xi}e^{|t|(b+\eta)} + \frac{1}{1-e^{-(b-\eta)\xi}}\right)||G||_\eta. \end{aligned}$$

Thus, $K^{u,G}(t)$ converges uniformly. One can show by similar means that $K^{s,F}(t)$ and $K^{s,G}(t)$ both converge. Therefore, $\mathcal{K}_s^\eta(F,G)(t) \in \mathcal{RCR}$ exists. We can now unambigiously state that $\mathcal{K}_s^\eta$ is clearly linear.

Our next task is to prove that $||K_s^\eta(F,G)]||_{\eta,s} \leq Q||(F,G)||_{\eta,s}$ for constant Q satisfying the estimate of equation (37). We will prove the bounds only for $||K^{u,F}||_{\eta,s}$, $||K^{u,G}||_{\eta,s}$, $||K^{c,F}||_{\eta,s}$ and $||K^{c,G}||_{\eta,s}$; the others follow by similar calculations. For $t < s$, we we have

$$\begin{aligned} e^{-\eta|t-s|}||K^{u,F}(t)|| &\leq e^{-\eta|t-s|}\int_t^\infty KNe^{b(t-\mu)}|F(\mu)|d\mu \\ &\leq e^{\eta(t-s)}KN\left[\int_t^s e^{b(t-\mu)}e^{\eta|\mu-s|}||F||_{\eta,s}d\mu + \int_s^\infty e^{b(t-\mu)}e^{\eta|\mu-s|}||F||_{\eta,s}d\mu\right] \\ &= e^{\eta(t-s)}KN||F||_{\eta,s}\left[\int_t^s e^{b(t-\mu)}e^{\eta(s-\mu)}d\mu + \int_s^\infty e^{b(t-\mu)}e^{\eta(\mu-s)}d\mu\right] \\ &= e^{\eta(t-s)}KN||F||_{\eta,s}\left[e^{bt+\eta s}\frac{e^{-(b+\eta)t}-e^{-(b+\eta)s}}{b+\eta} + e^{bt-\eta s}\frac{e^{-(b-\eta)s}}{b-\eta}\right] \\ &\leq KN||F||_{\eta,s}\frac{1}{b-\eta} \end{aligned}$$

The above inequality is also satisfied for $t \geq 0$, and we conclude $||K^{u,F}||_{\eta,s} \leq KN(b-\eta)^{-1}||(F,G)||_{\eta,s}$. Next, for $t < s$,

$$\begin{aligned} e^{-\eta|t-s|}||K^{u,G}(t)|| &\leq e^{-\eta|t-s|}\sum_{t<\tau_i<\infty} KNe^{b(t-\tau_i)}|G_i| \\ &\leq e^{\eta(t-s)}KN\left[\sum_{t<\tau_i<s} e^{b(t-\tau_i)}e^{\eta|\tau_i-s|}||G||_{\eta,s} + \sum_{s\leq\tau_i<\infty} e^{b(t-\tau_i)}e^{\eta|\tau_i-s|}||G||_{\eta,s}\right] \\ &\leq e^{\eta(t-s)}KN||G||_{\eta,s}\frac{1}{\xi}\left[\int_{t-\xi}^s e^{b(t-\mu)}e^{\eta(s-\mu)}d\mu + \int_{s-\xi}^\infty e^{b(t-\mu)}e^{\eta(\mu-s)}d\mu\right] \\ &\leq e^{\eta(t-s)}\frac{KN||G||_{\eta,s}}{\xi}\left[e^{bt+\eta s}\frac{e^{-(b+\eta)(t-\xi)}-e^{-(b+\eta)x}}{b+\eta} + e^{bt-\eta s}\frac{e^{-(b-\eta)(s-\xi)}}{b-\eta}\right] \\ &\leq \frac{2KN||G||_{\eta,s}}{\xi(b-\eta)}\cdot e^{(b+\eta)\xi}, \end{aligned}$$

where we have made use of Lemma 3.2.2 to estimate the sums. The same conclusion is valid for $t \geq s$, and it follows that $||K^{u,G}||_{\eta,s} \leq 2KNe^{(b+\eta)\xi}(\xi(b-\eta))^{-1}||(F,G)||_{\eta,s}$. Next, for $t \leq s$,

$$\begin{aligned} e^{-\eta|t-s|}||K^{c,G}(t)|| &\leq e^{\eta(t-s)}KN||G||_{\eta,s}\sum_{t<\tau_i\leq s} e^{\epsilon(\tau_i-t)}e^{\eta(s-\tau_i)} \\ &\leq e^{\eta(t-s)}\frac{KN||G||_{\eta,s}}{\xi}\int_{s-\xi}^t e^{\epsilon(\mu-t)}e^{\eta(s-\mu)}d\mu \end{aligned}$$

$$
\begin{aligned}
&= e^{\eta(t-s)} \frac{KN||G||_{\eta,s}}{\xi(\eta-\epsilon)} \left(e^{\epsilon(s-\xi-t)} e^{\eta\xi} - e^{-\eta(t-s)} \right) \\
&\leq \frac{KN||G||_{\eta,s}}{\xi(\eta-\epsilon)} e^{(\eta-\epsilon)\xi}
\end{aligned}
$$

This estimate continues to hold for all $t, s \in \mathbb{R}$. To compare to the integral term, for $s \leq t$ we have

$$
\begin{aligned}
e^{-\eta|t-s|}||K^{c,F}(t)|| &\leq e^{-\eta(t-s)} KN||F||_{\eta,s} \int_s^t e^{\epsilon(t-\mu)} e^{\eta(\mu-s)} d\mu \\
&= e^{-\eta(t-s)} KN||F||_{\eta,s} \frac{1}{\eta-\epsilon} \left(e^{\eta(t-s)} - e^{\epsilon(t-s)} \right) \\
&\leq \frac{KN||F||_{\eta,s}}{\eta-\epsilon}
\end{aligned}
$$

and this estimate persists for all $t, s \in \mathbb{R}$. Similar estimates for the other integrals and sums appearing in (36) ultimately result in the bound appearing in (37). This proves part 1.

To prove part 2, denote $v = \mathcal{K}_s^\eta(F, G)$. It is clear from the definition of v, the orthoginality of the projection operators and Proposition 4.2.1 that $P_c(s)v(s) = 0$. Also, for all $-\infty < z \leq t < \infty$, denoting $\overline{F} = \chi_0 F$ and $\overline{G}_i = \chi_0 G$, we have

$$
\begin{aligned}
&U(t,z)v(z) + \int_z^t U(t,\mu)\overline{F}(\mu)d\mu + \sum_z^t U(t,\tau_i)\overline{G}_i d\tau_i \\
&\quad = U(t,z)v(z) + \int_z^t U(t,\mu)P_c(\mu)\overline{F}(\mu)d\mu - \int_t^z U(t,\mu)P_u(\mu)\overline{F}(\mu)d\mu + \int_z^t U(t,\mu)P_u(\mu)\overline{F}(\mu)d\mu \\
&\quad\quad + \sum_z^t U(t,\tau_i)P_c(\tau_i)\overline{G}_i d\tau_i - \sum_t^z U(t,\tau_i)P_u(\tau_i)\overline{G}_i d\tau_i + \sum_z^t U(t,\tau_i)P_u(\tau_i)\overline{G}_i d\tau_i \\
&\quad = \int_s^t U(t,\mu)P_c(\mu)\overline{F}(\mu)d\mu - \int_t^\infty U(t,\mu)P_u(\mu)\overline{F}(\mu)d\mu + \int_{-\infty}^t U(t,\mu)P_s(\mu)\overline{F}(\mu)d\mu \\
&\quad\quad + \sum_s^t U(t,\tau_i)P_c(\tau_i)\overline{G}_i d\tau_i - \sum_t^\infty U(t,\tau_i)P_u(\tau_i)\overline{G}_i d\tau_i + \sum_{-\infty}^t U(t,\tau_i)P_s(\tau_i)\overline{G}_i d\tau_i \\
&\quad = v(t),
\end{aligned}
$$

so that $t \mapsto v(t)$ solves the integral equation (34). This also demonstrates that $v \in \mathcal{PC}^\eta$. To show that it is the only solution in $\mathcal{PC}^\eta$ satisfying $P_c(s)v(s) = 0$, suppose there is another $r \in PC^\eta$ that satisfies $P_c(s)r(s) = 0$. Then the function $w := v - r$ is an element of $\mathcal{PC}^\eta$ that satisfies $w(t) = U(t,z)w(z)$ for $-\infty < z \leq t < \infty$. By Lemma 5.2.1, we have $w(s) \in \mathcal{RCR}_c(s)$. But since $P_c(s)w(s) = 0$ and $P_c(s)$ is the identity on $\mathcal{RCR}_c(s)$, we obtain $w(s) = 0$. Therefore, $w(t) = U(t,s)0 = U_c(t,s)0 = 0$ for all $t \in \mathbb{R}$, and we conclude $v = r$, proving the uniqueness assertion.

For assertion 3, we compute first

$$
\begin{aligned}
\mathcal{K}_*(F,G)(t) = &\int_{-\infty}^t U(t,\mu)P_s(\mu)[\chi_0 F(\mu)]d\mu - \int_t^\infty U(t,\mu)P_u(\mu)[\chi_0 F(\mu)]d\mu \\
&\sum_{-\infty}^t U(t,\tau_i)P_s(\tau_i)[\chi_0 G_i]d\tau_i - \sum_t^\infty U(t,\tau_i)P_u(\tau_i)[\chi_0 G_i]d\tau_i.
\end{aligned}
$$

Routine estimation using inequalities (11)–(13) together with Lemma 3.2.2 produces the bound

$$
||\mathcal{K}_*(F,G)(t)|| \leq KN \left(\frac{-1}{a} + \frac{1}{b} - \frac{e^{-a\xi}}{a\xi} + \frac{e^{b\xi}}{b\xi} \right) ||(F,G)||,
$$

and as the bound is independent of t, s, the result is proven. $\square$

5.3 Modifications of the nonlinearities

Let $\xi : \mathbb{R}_+ \to \mathbb{R}$ be a C^∞ bump function satisfying

i) $\xi(y) = 1$ for $0 \le y \le 1$,

ii) $0 \le \xi(y) \le 1$ for $1 \le y \le 2$,

iii) $\xi(y) = 0$ for $y \ge 2$.

We modify the nonlinearities of (29)–(30) in the center and hyperbolic directions separately. For $\delta > 0$, we let

$$F_\delta(t,x) = f(t,x)\xi\left(\frac{||P_c(t)x||}{N\delta}\right)\xi\left(\frac{||(P_s(t)+P_u(t))x||}{N\delta}\right) \tag{38}$$

$$G_\delta(k,x) = g_k(x)\xi\left(\frac{||P_c(\tau_k)x||}{N\delta}\right)\xi\left(\frac{||(P_s(\tau_k)+P_u(\tau_k))x||}{N\delta}\right), \tag{39}$$

The proof of the following lemma and corollary will be omitted. They can be proven by emulating the proof of [Lemma 6.1 ,[18]] and taking into account the uniform boundedness of the projectors P_i; see property 1 of Definition 3.3.5.

Lemma 5.3.1. *Let $f(t,\cdot)$ and $g_k(\cdot)$ be uniformly (in $t \in \mathbb{R}$ and $k \in \mathbb{Z}$) Lipschitz continuous on the ball $B_{\mathcal{RCR}}(\delta, 0)$ in $\mathcal{RCR}$ with mutual Lipschitz constant $L(\delta)$, and let $f(t,0) = g_k(0) = 0$. The functions*

$$F_\delta : \mathbb{R} \times \mathcal{RCR} \to \mathbb{R}^n, \qquad G_\delta : \mathbb{Z} \times \mathcal{RCR} \to \mathbb{R}^n$$

are globally, uniformly (in $t \in \mathbb{R}$ and $k \in \mathbb{Z}$) Lipschitz continuous with mutual Lipschitz constant L_δ that satisfies $L_\delta \to 0$ as $\delta \to 0$

Corollary 5.3.0.1. *The substitution operator*

$$R_\delta : \mathcal{PC}^{\eta,s} \to B^{\eta,s}(\mathbb{R},\mathbb{R}^n) \oplus B^{\eta,s}_{\tau_k}(\mathbb{Z},\mathbb{R}^n)$$

defined by $R_\delta(x)(t,k) = (F_\delta(t,x(t)), G_\delta(k,x(\tau_k)))$ is globally Lipschitz continuous with Lipschitz constant $\tilde{L}_\delta$ that satisfies $\tilde{L}_\delta \to 0$ as $\delta \to 0$. Moreover, the Lipschitz constant is independent of η, s.

Corollary 5.3.0.2. $||(F_\delta(t,x), G_\delta(k,x))|| \le 4\delta L_\delta$ *for all $x \in \mathcal{RCR}$ and $(t,k) \in \mathbb{R} \times \mathbb{Z}$.*

5.4 The center manifold

Let $\epsilon < \eta \in (0, \min\{-a, b\})$ and define a mapping $\mathcal{G}_s : \mathcal{PC}^{\eta,s} \times \mathcal{RCR}_c(s) \to \mathcal{PC}^{\eta,s}$ by

$$\mathcal{G}_s(u,\varphi) = U(\cdot,s)\varphi + \mathcal{K}^\eta_s(R_\delta(u)). \tag{40}$$

Note that by Lemma 5.2.3 and Corollary 5.3.0.1, the operator is well-defined, $\mathcal{K}^\eta_s$ is bounded and R_δ is globally Lipschitz continuous for each $\delta > 0$, provided H.1–H.7 hold. Choose δ small enough so that

$$\tilde{L}_\delta ||K^\eta_s||_\eta < \frac{1}{2}. \tag{41}$$

Notice that δ can be chosen so that (41) can be satisfied independent of s, due to Lemma 5.2.3. If $||\varphi|| < r/(2K)$ then $\mathcal{G}_s(\cdot,\varphi)$ leaves $\overline{B(r,0)} \subset \mathcal{PC}^{\eta,s}$ invariant. Moreover, $\mathcal{G}_s(\cdot,\varphi)$ is Lipschitz continuous with Lipschitz constant $\frac{1}{2}$. One may notice that r is arbitrary. We can now prove the following:

Theorem 5.4.1. *Let conditions H.1–H.7 hold. If δ is chosen as in (41), then there exists a globally Lipschitz continuous mapping $u^*_s : \mathcal{RCR}_c(s) \to \mathcal{PC}^{\eta,s}$ such that $u_s = u^*_s(\varphi)$ is the unique solution in $\mathcal{PC}^{\eta,s}$ of the equation $u_s = \mathcal{G}_s(u_s,\varphi)$.*

Proof. The discussion preceding the statement of Theorem 5.4.1 indicates that $\mathcal{G}_s(\cdot, \varphi)$ is a contraction mapping on $\overline{B(r,0)} \subset \mathcal{PC}^{\eta,s}$ for every $r > ||\varphi||2K$. Since the latter is a closed subspace of the Banach space $\mathcal{PC}^{\eta,s}$, the contraction mapping principle implies the existence of the function u_s^*. To show that it is a Lipschitz continuous, we note

$$\begin{aligned}||u_s^*(\varphi) - u_s^*(\psi)||_{\eta,s} &= ||\mathcal{G}_s(u_s^*(\varphi), \varphi) - \mathcal{G}_s(u_s^*(\psi), \psi))||_{\eta,s} \\ &\leq K||\varphi - \psi|| + \frac{1}{2}||u_s^*(\varphi) - u_s^*(\psi)||_{\eta,s}.\end{aligned}$$

Therefore, u_s^* is Lipschitz continuous with Lipschitz constant $2K$. □

Definition 5.4.1 (Lipschitz center manifold)**.** *The center manifold, $\mathcal{W}_c \subset \mathbb{R} \times \mathcal{RCR}$, is the nonautonomous set whose t-fibers for $t \in \mathbb{R}$ are given by*

$$\mathcal{W}_c(t) = \operatorname{Im}\{\mathcal{C}(t, \cdot)\}, \tag{42}$$

where $\mathcal{C} : \mathcal{RCR}_c \to \mathcal{RCR}$ is the (fiberwise) Lipschitz map defined by $\mathcal{C}(t, \phi) = u_t^*(\phi)(t)$.

The construction above implies the center manifold is fiberwise Lipschitz. We can prove a stronger result, namely that the Lipschitz constant can be chosen independent of the given fiber.

Corollary 5.4.1.1. *There exists a constant $L > 0$ such that $||\mathcal{C}(t, \phi) - \mathcal{C}(t, \psi)|| \leq L||\phi - \psi||$ for all $t \in \mathbb{R}$ and $\phi, \psi \in \mathcal{RCR}_c(t)$.*

Proof. Denote $u^\phi = u_t(\phi)$ and $u^\psi = u_t(\psi)$. A preliminary estimation appealing to the fixed-point equation (40) yields

$$||\mathcal{C}(t, \phi) - \mathcal{C}(t, \psi)|| \leq ||\phi - \psi|| + ||(\mathcal{K}_t^\eta(R_\delta u^\phi) - \mathcal{K}_t^\eta(R_\delta u^\psi))(t)||.$$

By Corollary 5.3.0.2, each of $R_\delta u^\phi$ and $R_\delta u^\psi$ are uniformly bounded, so Lemma 5.2.3 implies the existence of a constant $c > 0$ such that

$$\begin{aligned}||(\mathcal{K}_t^\eta(R_\delta u^\phi) - \mathcal{K}_t^\eta(R_\delta u^\psi))(t)|| &\leq c||(R_\delta u^\phi - R_\delta u^\psi)(t)|| \\ &\leq c \sup_{s \in \mathbb{R}} ||(R_\delta u^\phi - R_\delta u^\psi)(s)||e^{-\eta|t-s|} \\ &\leq c\tilde{L}_\delta ||u^\phi - u^\psi||_{\eta,t} \\ &\leq c\tilde{L}_\delta 2K||\phi - \psi||,\end{aligned}$$

and in the last line we used the Lipschitz constant from Theorem 5.4.1. Combining this result with the previous estimate for $||\mathcal{C}(t, \phi) - \mathcal{C}(t, \psi)||$ yields the uniform Lipschitz constant. □

Recall that by Lemma 5.1.1, there is a process $(S, \mathcal{M})$ on $\mathcal{RCR}$ such that $t \mapsto S(t, s)\phi$ is the unique mild solution of (31) through the initial conditon (s, ϕ) defined on an interval $[s, s + \alpha)$. With this in mind, the center manifold is locally positively invariant with respect to $\mathcal{S}$.

Theorem 5.4.2 (Center manifold: invariance and inclusion of bounded orbits)**.** *Let conditions H.1–H.7 hold. The center manifold $\mathcal{W}_c$ enjoys the following properties.*

1. *$\mathcal{W}_c$ is locally positively invariant: if $(s, \phi) \in \mathcal{W}_c$ and $||S(t, s)\phi|| < \delta$ for $t \in [s, T]$, then $(t, S(t, s)\phi) \in \mathcal{W}_c$ for $t \in [s, T]$.*

2. *If $(s, \phi) \in \mathcal{W}_c$, then $S(t, s)\phi = u_t^*(P_c(t)S(t, s)\phi)(t) = \mathcal{C}(t, P_c(t)S(t, s)\phi)$*

3. *If $x : \mathbb{R} \to \mathcal{RCR}$ is a mild solution of* (31) *satisfying $||x||_\eta < \delta$, then $(t, x(t)) \in W_c$ for all $t \in \mathbb{R}$.*

4. *$\mathbb{R} \times \{0\} \subset \mathcal{W}_c$ and $\mathcal{C}(t, 0) = 0$ for all $t \in \mathbb{R}$.*

Proof. Let $(s,\phi) \in \mathcal{W}_c$ and denote $x(t) = S(t,s)\phi$, with $||x|| < \delta$. Since $(s,\phi) \in \mathcal{W}_c$, there exists $\varphi \in \mathcal{RCR}_c(s)$ such that $\phi = u_s^*(\varphi)(s)$. Define $\hat{x} = u_s^*(\varphi)$. Then, it follows that $\varphi = P_c(s)\phi$, $\hat{x}(s) = \phi = P_c(s)\phi + K_s^\eta(R(\hat{x}))(s)$, and

$$\begin{aligned}\hat{x}(t) &= U(t,s)\varphi + \mathcal{K}_s^\eta(R_\delta(\hat{x}))(t)\\ &= U(t,s)\varphi + \left[U(t,s)K_s^\eta(R_\delta(\hat{x}))(s) + \int_s^t U(t,\mu)\chi_0 R_\delta(\mu,\hat{x}(\mu))d\mu + \sum_{s<\tau_i\leq t} U(t,\tau_i)\chi_0 G_\delta(i,\hat{x}(\tau_i^-))\right]\\ &= U(t,s)\hat{x}(s) + \int_s^t U(t,\mu)\chi_0 R_\delta(\mu,\hat{x}(\mu))d\mu + \sum_{s<\tau_i\leq t} U(t,\tau_i)\chi_0 G_\delta(i,\hat{x}(\tau_i^-))\end{aligned}$$

for all $t \in [s,T]$. But since $||x(t)|| < \delta$ on $[s,T]$, uniqueness of mild solutions (Lemma 3.4.1 with Theorem 4.2.1) implies that $x = \hat{x}|_{[s,T]}$.

Let $v \in [s,T]$ and define $z : \mathbb{R} \to \mathcal{RCR}$ by $z = \hat{x} - U(\cdot,v)P_c(v)\hat{x}(v)$. One can easily verify that

$$z(t) = U(t,v)z(v) + \int_v^t U(t,\mu)U(t,\mu)\chi_0 R_\delta(\mu,\hat{x}(\mu))d\mu + \sum_{v<\tau_i\leq t} U(t,\tau_i)\chi_0 G_\delta(i,\hat{x}(\tau_i^-))$$

for all $t \in [v,\infty)$ and that $P_c(v)z(v) = 0$. By Lemma 5.2.3, $z = \mathcal{K}_v^\eta(R_\delta(\hat{x}))|_{[v,\infty)}$, so that we may write

$$\hat{x} = U(\cdot,v)P_c(v)\hat{x}(v) + \mathcal{K}_v^\eta(R_\delta(\hat{x})) = u_v^*(P_c(v)\hat{x}(v)).$$

Therefore, $\hat{x}(v) = u_v^*(P_c(v)\hat{x}(v))(v)$, and since $x(v) = \hat{x}(v)$, this proves that $(v,x(v)) \in \mathcal{W}_c$ and, through essentially the same proof, that

$$x(v) = u_v^*(P_c(v)x(v))(v) = \mathcal{C}(v,x(v))(v).$$

The proofs of the other two assertions of the theorem follow by similar arguments, and are omitted. □

The modification of the nonlinearity R_δ results in the function u_s^* that defines the center manifold having a uniformly small hyperbolic part. Namely, we have the following lemma.

Lemma 5.4.1. *Define $\widehat{P}_c : \mathcal{PC}^\eta \to \mathcal{PC}^\eta(\mathbb{R},\mathcal{RCR}_c)$ by $\widehat{P}_c\phi(t) = P_c(t)\phi(t)$. If $\delta > 0$ is sufficiently small, then $||(I-\widehat{P}_c)u_s^*||_0 < \delta$.*

Proof. Recall that u_s^* satisfies the fixed-point equation $u_s^* = U(\cdot,s)\varphi + \mathcal{K}_s^\eta(R_\delta(u_s^*))$. Thus, with $\widehat{P}_h = I - \widehat{P}_c$,

$$\widehat{P}_h u_s^* = \widehat{P}_h \circ \mathcal{K}_s^\eta(R_\delta(u_s^*))$$

because $U(t,s)$ is an isomorphism of $\mathcal{RCR}_c(s)$ onto $\mathcal{RCR}_c(t)$ and $\varphi \in \mathcal{RCR}_c(s)$. By Corollary 5.3.0.2, we have for all $t \in \mathbb{R}$ that $||R_\delta(u_s^*(t))|| \leq 4\delta L_\delta$, which implies $R_\delta(u_s^*) \in B^0(\mathbb{R},\mathbb{R}^n) \oplus B_{\tau_k}^0(\mathbb{Z},\mathbb{R}^n)$. We obtain the claimed result by applying the second conclusion of Lemma 5.2.3 and taking δ sufficiently small, recalling from Corollary 5.3.0.1 that $L_\delta \to 0$ as $\delta \to 0$. □

6 Reduction principle

The results of this section concern the dynamics on and near the center manifold. Section 6.1 pertains to the dynamical behaviour of the solutions of the impulsive functional differential equation (29)–(30) on its center manifold. The dynamics near the center manifold are elucidated in Section 6.2.

6.1 Invariance equations for center bundle components of orbits in $\mathcal{W}_c$

On the center manifold, components of mild solutions on the center bundle are decoupled, as stated in the following lemma. To begin, let S_δ denote the forward process associated to the nonlinear integral equation be a mild solution of the integral equation

$$y(t) = U(t,s)y(s) + \int_s^t U(t,\mu)[\chi_0 F_\delta(\mu,y(\mu))]d\mu + \sum_{s<\tau_i\leq t} U(t,\tau_i)[\chi_0 G_\delta(i,y(\tau_i^-))].$$

Note that the above is the smoothed version of (31). The proof of the following follows by direct calculation and the second orbit property of the center manifold, as stated in Theorem 5.4.2.

Lemma 6.1.1 (Dynamics on the center manifold: integral equation)**.** *Let $y : \mathbb{R} \to \mathcal{RCR}$ satisfy $y(t) \in \mathcal{W}_c(t)$ with $y(t) = S_\delta(t,s)y(s)$. Consider the projection of y onto the center bundle: $w(t) = P_c(t)y(t)$. The projection satisfies the integral equation*

$$w(t) = U(t,s)w(s) + \int_s^t U(t,\mu)P_c(\mu)\chi_0 F_\delta(\mu, \mathcal{C}(\mu, w(\mu)))d\mu + \sum_{s<\tau_i\leq t} U(t,\tau_i)P_c(\tau_i)\chi_0 G_\delta(i, \mathcal{C}(\tau_i, w(\tau_i^-))) \tag{43}$$

When a solution on the center manifold is defined by a classical solution, we can show that its projection satisfies a particular impulsive differential equation. This identification carries over to solutions that merely have enough smoothness to ensure that their right-hand derivatives exist and are elements of the space $\mathcal{RCR}(\mathbb{R},\mathbb{R}^n)$.

Theorem 6.1.1 (Dynamics on the center manifold: abstract impulsive differential equation)**.** *Let $y \in \mathcal{RCR}^1(\mathbb{R},\mathbb{R}^n)$ satisfy $y_t \in \mathcal{W}_c(t)$ with $y_t = S_\delta(t,s)y_s$. Consider the projection $w(t) = P_c(t)y_t$ and define the linear operators $\mathcal{L}(t) : \mathcal{RCR}^1 \to \mathcal{RCR}$ and $\mathcal{B}_k : \mathcal{RCR}^1((-r,0],\mathbb{R}^n) \to \mathcal{L}^\infty((-r,0],\mathbb{R}^n)$ by*

$$\mathcal{L}(t)\phi = \begin{cases} L(t)\phi, & \theta = 0 \\ d^+\phi(\theta), & \theta < 0 \end{cases}, \qquad \mathcal{B}_k\phi(\theta) = \begin{cases} B_k\phi, & \theta = 0 \\ \phi(\theta) - \phi(\theta^-), & \theta < 0 \end{cases} \tag{44}$$

Then $w : \mathbb{R} \to \mathcal{RCR}^1$ satisfies, pointwise, the abstract impulsive differential equation

$$d^+w(t) = \mathcal{L}(t)w(t) + P_c(t)\chi_0 F_\delta(t, \mathcal{C}(t, w(t))), \qquad t \neq \tau_k \tag{45}$$

$$\Delta w(\tau_k) = \mathcal{B}_k w(\tau_k^-) + P_c(\tau_k)\chi_0 G_\delta(k, \mathcal{C}(\tau_k, w(\tau_k^-))), \qquad t = \tau_k, \tag{46}$$

where $\Delta w(\tau_k)(\theta) := \lim_{\epsilon\to 0^+}[w(\tau_k)(\theta) - w(\tau_k - \epsilon)(\theta)]$ is the non-uniform pointwise jump at time τ_k, defined for $\theta \in (-r, 0]$.

Proof. For brevity, denote $F(\mu) = F_\delta(\mu, \mathcal{C}(\mu, w(\mu)))$, $\overline{F}(\mu) = \chi_0 F(\mu)$, $\mathbf{F}(\mu) = P_c(\mu)\chi_0 F(\mu)$ and analogously for G_δ. We begin by noting that equation (43) allows us to write finite difference $w_\epsilon(t) = w(t+\epsilon) - w(t)$ as

$$\begin{aligned} w_\epsilon(t) = {} & [U(t+\epsilon,s) - U(t,s)]w(s) + (U(t+\epsilon,t) - I)\int_s^t U(t,\mu)\mathbf{F}(\mu)d\mu + U(t+\epsilon,t)P_c(t)\int_t^{t+\epsilon} U(t,\mu)\overline{F}(\mu)d\mu \\ & + (U(t+\epsilon,t) - I)\sum_{s<\tau_i\leq t} U(t,\tau_i)\mathbf{G}(i) + U(t+\epsilon,t)\sum_{t<\tau_i\leq t+\epsilon} U(t,\tau_i)\mathbf{G}(i) \end{aligned} \tag{47}$$

First, we show that $d^+U(t,s)\phi = \mathcal{L}(t)U(t,s)\phi$ pointwise for $\phi \in \mathcal{RCR}$. For $\theta = 0$, we have

$$\frac{1}{\epsilon}(U(t+\epsilon,s)\phi(0) - U(t,s)\phi(0)) = \frac{1}{\epsilon}\int_t^{t+\epsilon} L(\mu)U(\mu,s)\phi d\mu,$$

which converges to $L(t)U(t,s)\phi$ as $\epsilon \to 0^+$. For $\theta < 0$ and $\epsilon > 0$ sufficiently small,

$$\frac{1}{\epsilon}(U(t+\epsilon,s)\phi(\theta) - U(t,s)\phi(\theta)) = \frac{1}{\epsilon}(\phi(t+\epsilon+\theta-s) - \phi(t+\theta-s)) \longrightarrow d^+\phi(t+\theta-s) = d^+U(t,s)\phi(\theta).$$

Therefore, $d^+U(t,s)\phi = \mathcal{L}(t)U(t,s)\phi$ pointwise, as claimed. Since $U(t,t) = I$, this also proves the pointwise convergence

$$\frac{1}{\epsilon}(U(t+\epsilon,t) - I)\phi \to \mathcal{L}(t)\phi.$$

Next, we show that

$$\frac{1}{\epsilon}U(t+\epsilon,t)P_c(t)\int_t^{t+\epsilon} U(t,\mu)\overline{F}(\mu)d\mu \to P_c(t)\overline{F}(t) = \mathbf{F}(t) \tag{48}$$

pointwise as $\epsilon \to 0^+$. We do this by first proving that the sequence $x_n := \frac{1}{\epsilon_n} U(t+\epsilon_n, t) P_c(t) \int_t^{t+\epsilon_n} U(t,\mu)\overline{F}(\mu)d\mu$ is pointwise Cauchy for each sequence $\epsilon_n \to 0^+$. Assuming without loss of generality that ϵ_n is strictly decreasing, we have for all $n \geq m$,

$$x_n - x_m = \left[\frac{1}{\epsilon_n}U(t+\epsilon_n,t) - \frac{1}{\epsilon_m}U(t+\epsilon_m,t)\right] P_c(t) \int_t^{t+\epsilon_n} U(t,\mu)\chi_0 F(\mu)d\mu$$
$$+ \frac{1}{\epsilon_m}U(t+\epsilon,t)\int_{t+\epsilon_m}^{t+\epsilon_n} U_c(t,\mu)P_c(\mu)\chi_0 F(\mu)d\mu$$

Both integrals can be made arbitrarily small in norm by taking $n, m \geq N$ and N large enough. Since $\frac{1}{\epsilon}U(t+\epsilon,t)$ is pointwise convergent as $\epsilon \to$, we obtain that the sequence x_n is pointwise Cauchy, and is hence pointwise convergent. Direct calculation of the limit in the pointwise sense yields (48). Combining all of the above results with equation (47) gives the pointwise equality

$$d^+w(t) = \mathcal{L}(t)U(t,s)w(s) + \mathcal{L}(t)\int_s^t U(t,\mu)\mathbf{F}(\mu)d\mu + \mathbf{F}(t) + \mathcal{L}(t)\sum_{s<\tau_i\leq t} U(t,\tau_i)\mathbf{G}(i) = \mathcal{L}(t)w(t) + \mathbf{F}(t),$$

which is equivalent to (45).

To obtain the difference equation (46), we similarly identify $w_\epsilon(\tau_k)(\theta) := w(\tau_k)(\theta) - w(\tau_k - \epsilon)(\theta)$ with the decomposition

$$w_\epsilon(\tau_k) = [U(\tau_k,s) - U(\tau_k-\epsilon,s)]w(s) + \int_{\tau_k-\epsilon}^{\tau_k} U(t,\mu)\mathbf{F}(\mu)d\mu + \int_s^{\tau_k-\epsilon}[U(\tau_k,\mu) - U(\tau_k-\epsilon,\mu)]\mathbf{F}(\mu)d\mu$$
$$+ \sum_{\tau_k-\epsilon<\tau_i\leq\tau_k} U(\tau_k,\tau_i)\mathbf{G}(i) + \sum_{s<\tau_i\leq\tau_k-\epsilon}[U(\tau_k,\tau_i) - U(\tau_k-\epsilon,\tau_i)]\mathbf{G}(i)$$

Using Lemma 3.4.2 and Lemma 4.2.1, the above is seen to converge pointwise as $\epsilon \to 0^+$, with limit

$$\Delta w(\tau_k) = \mathcal{B}_k U(\tau_k^-,s)w(s) + \mathcal{B}_k\int_s^{\tau_k} U(\tau_k^-,\mu)\mathbf{F}(\mu)d\mu + \mathbf{G}(k) + \mathcal{B}_k\sum_{s<\tau_i<\tau_k} U(\tau_k^-,\tau_i)\mathbf{G}(i) = \mathcal{B}_k w(\tau_k^-) + \mathbf{G}(k),$$

which is equivalent to the difference equation (46). □

6.2 Attractivity of the center manifold

The final result of this section concerns the attractivity properties of the center manifold. In particular, we obtain the usual conclusion that in the absence of an unstable bundle, the center manifold is (nonuniformly) attracting.

Theorem 6.2.1 (Attraction of the center manifold). *For $\nu > 0$ and $S \in \mathbb{R}$, there exist positive constants C and δ such that*

1. *If u and v are mild solutions of (31) on the interval $I = [s-T, s]$ for $T > 0$ and $s \leq S$ satisfying*
   - *$(I - P_s(j))u(j) = (I - P_s(j))v(j)$ for either $j = s$ or $j = s - T$;*
   - *$||u(t)|| \leq \delta$ and $||v(t)|| \leq \delta$ for all $t \in I$,*

   then, $||P_s(s)[u(s) - v(s)]|| \leq C||P_s(s-T)[u(s-T) - v(s-T)]||e^{(a+\nu)T}$.
2. *If u and v are mild solutions on the interval $I = [s, s+T]$ for $T > 0$ and $s \geq S$ satisfying*
   - *$(I - P_u(j))u(j) = (I - P_u(j))v(j)$ for either $j = s$ or $j = s + T$;*
   - *$||u(t)|| \leq \delta$ and $||v(t)|| \leq \delta$ for all $t \in I$,*

   then, $||P_u(s)[u(s) - v(s)]|| \leq C||P_u(s+T)[u(s+T) - v(s+T)]||e^{-(b-\nu)T}$.

Proof. We begin by proving the first assertion, and will prove only the case where $(I-P_s(s))(u(s)-v(s))=0$, as the other case is similar. Let $L=L(\delta)$ be the Lipschitz constant of the nonlinearity $\tilde{R}_\delta$, and denote

$$z_-(t)=||P_s(t)(u-v)(t)||, \qquad z_+(t)=||(I-P_s(t))(u-v)(t)||.$$

Let $t\in I$. Routine integral estimation with the spectral separation assumptions result in the estimate

$$z_-(s)e^{at}\le Kz_-(t)+\int_t^s KNLe^{a(t-\mu)}(z_+(\mu)+z_-(\mu))d\mu+\sum_t^s KNLe^{a(t-\tau_i)}(z_+(\tau_i^-)+z_i(\tau_i^-))d\tau_i. \quad (49)$$

Also, the spectral separation guarantee that the expression

$$(I-P_s(t))u(t)=U(t,s)[P_c(s)+P_u(s)]u(s)+\int_s^t U(t,\mu)[P_c(\mu)+P_u(\mu)]\chi_0F_\delta(\mu,u(\mu))d\mu$$
$$+\sum_s^t U(t,\tau_i)[P_c(\tau_i)+P_u(\tau_i)]\chi_0G_\delta(i,u(\tau_i^-))d\tau_i$$

is well-defined even when $t\le s$, and similarly for v. Using the fact that $(I-P_s(s))(u-v)(s)=0$, we get the estimate

$$z_+(t)\le\int_t^s KNLe^{b(t-\mu)}(z_-(\mu)+z_+(\mu))d\mu+\int_t^s KNLe^{b(t-\tau_i)}(z_-(\tau_i^-)+z_+(\tau_i^-))d\tau_i.$$

Some routine changes of variables and Lemma 3.2.1 then imply

$$z_+(t)\le KNL\left[\int_t^s e^{(b-KL)(t-\mu)}z_-(\mu)d\mu+\sum_t^s e^{(b-KL)(t-\tau_i)}z_i(\tau_i^-)d\tau_i\right].$$

Substituting the above into (49) results in the somewhat bulky expression

$$z_-(s)e^{at}\le Kz_-(t)+\int_t^s KNLe^{a(t-\mu)}z_-(\mu)d\mu+\sum_t^s KNLe^{a(t-\tau_i)}z_-(\tau_i^-)d\tau_i$$
$$+\int_t^s(KNL)^2\left[\int_\mu^s e^{a(t-\mu)+(b-KNL)(\mu-\eta)}z_-(\eta)d\eta+\sum_\mu^s e^{a(t-\mu)+(b-KNL)(\mu-\tau_i)}z_-(\tau_i)d\tau_i\right]d\mu$$
$$+\sum_t^s(KNL)^2\left[\int_\mu^s e^{a(t-\tau_i)+(b-KNL)(\tau_i-\mu)}z_-(\mu)d\mu+\sum_{\tau_i}^s e^{a(t-\tau_i)+(b-KNL)(\tau_i-\tau_k)}z_-(\tau_k^-)d\tau_k\right]d\tau_i.$$

Applying Fubini's Theorem and estimating sums via Lemma 3.2.2 yields

$$z_-(s)e^{at}\le Kz_-(t)+\int_t^s(KNL+\kappa)e^{a(t-\mu)}z_-(\mu)d\mu+\sum_t^s(KNL+\kappa)e^{a(t-\tau_i)}z_-(\tau_i^-)d\tau_i,$$
$$\kappa=\frac{(KNL)^2}{b-a-KNL}\left(1+\frac{e^{(b-a-KNL)\xi}}{\xi}\right),$$

where ξ is the constant appearing in assumption H.7. Note that $\kappa=\kappa(\delta)$ is positive provided δ is chosen small enough. More changes of variables and use of Lemma 3.2.1 eventually lead us to the inequality

$$z_-(s)e^{-a(s-t)}Ke^{-as}\le z_-(t)\exp\left(-e^{-as}(KNL+\kappa)(t-s)\right),$$

which upon substituting $t=s-T$ and rearranging grants

$$z_-(s)\le Ke^{-as}\cdot\exp\left((a+e^{-as}(KNL+\kappa))T\right)z_-(s-T).$$

Choosing $C=Ke^{-aS}$ and δ small enough so that $e^{-as}(KNL(\delta)+\kappa(\delta))\le\nu$ results in the desired inequality.

An analogous argument proves the second assertion of the theorem. The proof is omitted. $\square$

7 Linear theory for periodic systems

Insofar as impulsive systems are concerned, the simplest class of linearization one can encounter is one that is periodic. In this section, we therefore study the linear system (15)–(16) satisfying conditions H.1, H.2 under the periodic constraints

$$L(t+T) = L(t), \qquad B_{k+q} = B_k, \qquad \tau_{k+q} = \tau_k + T, \tag{50}$$

for some $T > 0$ and $q \in \mathbb{N}$. Ultimately, we will prove that the evolution family associated to the such a periodic system is spectrally separated, thereby proving that under suitable regularity conditions on the nonlinearities, the semilinear system (29)–(30) posesses a center manifold provided the linear part is periodic.

7.1 The monodromy operator and compactness

We must discuss the interrelation between the period T and the range r of the delay. If $r < T$, it will be convenient to reinterpret the periodic system [(15)–(16)+(50)] as having the phase space $\mathcal{RCR}([-T,0],\mathbb{R}^n)$. This can always be done, since each of $L(t)$ and B_k extend in an obvious, trivial way to $\mathcal{RCR}([-T,0],\mathbb{R}^n)$. In the opposite case, where $r \geq T$ we let $j \in \mathbb{N}$ satisfy $r \leq jT$ and extend the phase space to $\mathcal{RCR}([-jT,0],\mathbb{R}^n)$. In both cases, the following proposition is true.

Proposition 7.1.1. *There exists $j \in \mathbb{N}$ minimal such that $r \leq jT$, and the evolution family $U(t,s)$ on $\mathcal{RCR}$ associated to the periodic system [(15)–(16)+(50)] extends uniquely to an evolution family $\tilde{U}(t,s)$ on $\mathcal{RCR}([-jT,0],\mathbb{R}^n)$ satisfying the identity*

$$\tilde{U}(t,s)\phi(\theta) = U(t,s)\psi(\theta)$$

for all $\phi \in \mathcal{RCR}([-jT,0],\mathbb{R}^n)$ and $\theta \in [-r,0]$, where $\psi = \phi|_{[-r,0]}$. In particular, $U(t,s) = \pi_\rightarrow \tilde{U}(t,s)\pi_\leftarrow$ where the linear maps $\pi_\leftarrow : \mathcal{RCR} \to \mathcal{RCR}([-jT,0,\mathbb{R}^n)$ and $\pi_\rightarrow : \mathcal{RCR}([-jT,0,\mathbb{R}^n) \to \mathcal{RCR}$ are

$$\pi_\leftarrow \phi(\theta) = \begin{cases} \phi(\theta), & \theta \in [-r,0], \\ 0, & \theta \in [-jT, r) \end{cases} \qquad \pi_\rightarrow \phi = \phi|_{[-r,0]}.$$

Following the above proposition, we denote $\mathcal{RCR}_j = \mathcal{RCR}([-jT,0],\mathbb{R}^n)$. For each $t \in \mathbb{R}$, define the operators $\tilde{V}_t : \mathcal{RCR}_j \to \mathcal{RCR}_j$ and $V_t : \mathcal{RCR} \to \mathcal{RCR}$ by

$$\tilde{V}_t = \tilde{U}(t+jT,t), \qquad V_t = U(t+jT,t).$$

Lemma 7.1.1. *$\tilde{V}_t$ is compact for each $t \in \mathbb{R}$.*

Proof. Let PC_S denote the set of functions $f : [-jT,0] \to \mathbb{R}^n$ that are continuous except at points $s \in S$, where they are right-continuous and posess limits on the left. For all $k \in \mathbb{N}$, the identity $\tilde{V}_t^k = U(t+kT,t)$ holds. This follows by existence and uniqueness (Lemma 3.4.1) of solutions. Moreover, if $\phi \in \mathcal{RCR}_j$, then $\tilde{V}_t\phi$ is continuous except at times $\theta_n \in [-jT,0]$ such that $t + jT + \theta_n \in \{\tau_k : k \in \mathbb{Z}\}$. At such times, $\tilde{V}_t\phi$ is continuous from the right and has limits on the left. Let $\Theta = \{\theta_1, \dots, \theta_N\}$ denote the set of all such discontinuity points; note that $N = jq$ is indeed finite. Therefore, if $\mathcal{B} \subset \mathcal{RCR}_j$ is bounded, then $Y := \tilde{V}_t(\mathcal{B}) \subset PC_\Theta$, the latter of which is complete with respect to the supremum norm.

By [2], a subset of $Y \subset PC_\Theta$ is precompact if and only if it is uniformly bounded and quasiequicontinuous – that is, for all $\epsilon > 0$, there exists $\delta > 0$ such that if $t_1, t_2 \in [\theta_{k-1}, \theta_k) \cap [-jT,0]$ satisfy $|t_1 - t_2| < \delta$, then $||x(t_1) - x(t_2)|| < \epsilon$ for all $x \in Y$. Uniform boundedness follows by equation (17). To obtain quasiequicontinuity, let $t_1 > t_2$ and $t = 0$ without loss of generality. We note that for all $\tilde{V}_t x \in Y$,

$$\begin{aligned} ||\tilde{V}_t x(t_1) - \tilde{V}_t x(t_2)|| &= ||U(jT+t_1, jT+t_2)U(jT+t_2,0)x(0) - U(jT+t_2,0)x(0)|| \\ &= ||\chi_0 \circ [U(jT+t_1,jT+t_2) - I]U(jT+t_2,0)x|| \\ &\leq \int_{jT+t_1}^{jT+t_2} \ell(\mu)d\mu \left(e^{\int_0^{jT}\ell(\mu)d\mu} \prod_{k=1}^{jq}(1+b(k)) \right) C \\ &\equiv K \int_{jT+t_1}^{jT+t_2} \ell(\mu)d\mu, \end{aligned}$$

where $||x|| \leq C$ for all $x \in \mathcal{B}$, and the inequality on the third line follows by Lemma 3.4.2 and the integral form of solutions provided by equation (14). Choosing δ so that $\int_{jT+t_1}^{jT+t_2} \ell(\mu)d\mu < \epsilon/K$ for $|t_1 - t_2| < \delta$ whenever $t_1, t_2 \in [-jT, 0]$ we obtain the required quasiequicontinuity of Y. Therefore, $Y = \tilde{V}_t(\mathcal{B})$ is precompact, so $\tilde{V}_t$ is compact. □

Lemma 7.1.2. *V_t is compact for each $t \in \mathbb{R}$.*

Proof. By Proposition 7.1.1, we have $V_t = \pi_{\rightarrow}\tilde{V}_t\pi_{\leftarrow}$. The boundedness of each of $\pi_{\leftarrow}$ and $\pi_{\leftarrow}$ together with the compactness of $\tilde{V}_t$ grants the compactness of V_t. □

The compactness of the *monodromy operator* V_t provides us with several useful results from the spectral theory of compact operators; see the monograph [20] for details.

Theorem 7.1.1. *Let $t \in \mathbb{R}$ and let σ_t denote the spectrum of $W_t := (V_t)_{\mathbb{C}}$, the complexification of V_t.*

1. *If $\lambda \in \sigma_t$ is nonzero, then λ and $\overline{\lambda}$ are eigenvalues of W_t.*

2. *The generalized eigenspace $M_{\lambda,t} \subset \mathcal{RCR}_{\mathbb{C}}$ associated to the eigenvalue $\lambda \in \sigma_t$ is finite-dimensional and invariant under W_t.*

3. *The Riesz projection*
$$P_{\lambda,t}(\lambda) = \frac{1}{2\pi i}\int_{\gamma}(\xi I - W_t)^{-1}d\xi$$
is a projection onto $M_{\lambda,t}$.

4. *If $\Lambda \subset \sigma_t$, then*
$$P_{\Lambda,t} = \sum_{\lambda\in\Lambda} Res_{z=\lambda}(zI - W_t)^{-1}$$
is a projection onto
$$M_{\Lambda,t} = \bigoplus_{\lambda\in\Lambda} M_{\lambda,t}.$$

5. *The projections $P_{\Lambda,t}$ commute with W_t and if Λ_1 and Λ_2 are disjoint, then $P_{\Lambda_1,t}P_{\Lambda_2,t} = 0$.*

6. *σ_t is bounded and $0 \in \sigma_t$ is the only accumulation point.*

Proof. The first five assertions are consequences of the spectral theory of compact operators, as is the fact that $0 \in \sigma_t$ (since $\mathcal{RCR}_{\mathbb{C}}$ is infinite-dimensional). Suppose then that σ_t were not bounded. Then there would exist $\lambda_n \in \sigma_t$ with $|\lambda_n| \to \infty$ and an associated sequence of unit-norm eigenvectors $x_n \in \mathcal{RCR}_{\mathbb{C}}$. Then, $||W_t(x_n + \overline{x_n})|| = 2|\lambda| \to \infty$, but
$$||W_t(x_n + \overline{x_n})|| = ||V_t(2\text{Re}(x_n))|| \leq 2||U(t + jT, t)||$$
and the latter is bounded by Lemma 3.4.2. This completes the proof. □

We also have the following theorem concerning eigenvalues of distinct monodromy operators and their generalized eigenspaces, whose proof follows entirely verbatim the proof of [Theorem 3.3 [9]].

Theorem 7.1.2. *Let $t, s \in \mathbb{R}$ be given with $t \geq s$ and let $\lambda \in \mathbb{C} \setminus \{0\}$.*

- *$\lambda \in \sigma_t$ if and only if $\lambda \in \sigma_s$*

- *The restriction of $U_{\mathbb{C}}(t, s)$ to $M_{\lambda,s}$ is a topological isomorphism onto $M_{\lambda,t}$.*

Due to the uniqueness of the eigenvalues across all of the monodromy operators, the following definition is appropriate.

Definition 7.1.1. *The* Floquet multipliers *of the evolution family $U(t, s)$ are the eigenvalues $0 \neq \lambda \in \sigma_0$ of the mondodromy operator W_0. The* Floquet spectrum *of the evolution family $U(t, s)$ is denoted $\sigma(U) := \sigma_0$.*

The projections of Theorem 7.1.1 take values in the complexified spaces $M_{\Lambda,t} \subset \mathcal{RCR}_{\mathbb{C}}$. To obtain real projections, it suffices to ensure that all conjugate multipliers are included in the set Λ.

Corollary 7.1.2.1. *Let $0 \notin \Lambda \subset \sigma(U)$. If $\Lambda = \overline{\Lambda}$, the projection $P_{\Lambda,t} : \mathcal{RCR}_{\mathbb{C}} \to \mathcal{RCR}_{\mathbb{C}}$ is the complexification of a projection operator on $\mathcal{RCR}$.*

By definition of complexification, if $x \in \mathcal{RCR}_{\mathbb{C}}$ is real (that is, $x = \xi + i0$ for some $\xi \in \mathcal{RCR}$), then $P_{\Lambda,t}x$ is also real. By abuse of notation, we will identify the complexification of said operator with itself whenever no confusion should arise. That is, we say that

$$P_{\Lambda,t} : \mathcal{RCR} \to \mathcal{RCR}$$

is also a projection, and is identified with its complexification.

7.2 Spectral separation of $U(t,s)$

Define the time-varying projectors

$$P_u(t) = P_{\Lambda_c,t}, \qquad P_c(t) = P_{\Lambda_c,t}, \qquad P_s(t) = I - P_u(t) - P_c(t) \tag{51}$$

where $\Lambda_u = \{\lambda \in \sigma(U) : |\lambda| > 1\}$ and $\Lambda_c = \{\lambda \in \sigma(U) : |\lambda| = 1\}$. Since these sets are self-conjugate (ie. $\Lambda = \overline{\Lambda}$), the first two operators above define, by Corollary 7.1.2.1, projections on $\mathcal{RCR}$. The third one is a complementary projector.

Lemma 7.2.1. *The projectors $P_i(t)$ for $i \in \{s, c, u\}$ are jT-periodic.*

Proof. Since $P_u(t)$ is the projector through the spectral subset Λ_u associated to the complexified operator $U_{\mathbb{C}}(t + jT, t)$, it follows that $P_u(t + kjT)$ is the projector through the same subset, associated to $U_{\mathbb{C}} = U_{\mathbb{C}}(t + jT + kjT, t + kjT)$, for all $k \in \mathbb{Z}$. By uniqueness of solutions and the periodicity condition, the latter is equal to $U_{\mathbb{C}}(t + jT, t)$, from which it follows that $P_u(t) = P_u(t + kjT)$, and mutis mutandis for the other projectors. ☐

Lemma 7.2.2. $\mathcal{R}(P_s(t)) = \bigcap_{\lambda \in \Lambda_c \cup \Lambda_u} \mathcal{N}(P_{\lambda,t})$.

Proof. Denote $P_{cu} = P_c + P_u$ and $\Lambda_{cu} = \Lambda_c \cup \Lambda_u$, so that $\mathcal{R}(P_s(t)) = \mathcal{N}(P_{cu}(t))$. If $x \in \cap_{\lambda \in \Lambda_{cu}} \mathcal{N}(P_{\lambda,t})$, then $P_{su}(t)x = \sum_{\lambda \in \Lambda_{cu}} P_{\lambda,t}x = 0$, which shows that $\cap_{\lambda \in \Lambda_{cu}} N(P_{\lambda,t}) \subseteq \mathcal{R}(P_s(t))$. To obtain the second inclusion, let $x \in \mathcal{RCR}_{\mathbb{C}}$ be such that $x \in \mathcal{N}(P_{cu}(t))$. For all $\lambda \in \Lambda_{cu}$, we have

$$P_{\lambda,t}x = P_{\lambda,t}^2 + \sum_{\mu \in \Lambda_{cu} \setminus \{\lambda\}} P_{\lambda,t}P_{\mu,t}x = P_{\lambda,t}P_{cu}(t)x = 0,$$

where the second equality is due to Theorem 7.1.1 and the conclusion is because $x \in \mathcal{N}(P_{cu}(t))$. ☐

Lemma 7.2.3. *The restriction of W_t to the subspace $\mathcal{R}(P_s(t))$ is compact, and $\sigma(W_t) \subset B_1(0)$.*

Proof. With the same notation as in the previous proof, since the generalized eigenspaces $M_{\Lambda_{cu},t}$ are invariant under W_t, the same is true for the (closed) complement, $\mathcal{R}(P_s(t))$. Denote $\tilde{W}_t$ the restriction of W_t to said complement. Suppose by way of contradiction $\xi \in$ is a (generalized) eigenvector of $\tilde{W}_t$ with eigenvalue λ with $|\lambda| \geq 1$. Then $(W_t - \lambda I)^k \xi = (\tilde{W}_t - \lambda I)\xi = 0$, so ξ is in fact a (generalized) eigenvector of W_t with eigenvalue λ and $|\lambda| \geq 1$. Consequently, $\xi \in \mathcal{R}(P_{cu}(t))$, which is a contradiction since $\mathcal{R}(P_s(t)) \cap \mathcal{R}(P_{cu}(t)) = \{0\}$. ☐

Theorem 7.2.1. *The evolution family $U(t,s) : \mathcal{RCR} \to \mathcal{RCR}$ associated to the periodic system [(15)–(16),(50)] is spectrally separated, with projectors (P_s, P_c, P_u) defined as in equation* (51)*. Also, $\mathcal{RCR}_c$ and $\mathcal{RCR}_u$ are finite-dimensional.*

Proof. We prove the theorem by verifying properties 1–5 of Definition 3.3.5 explicitly.

1. Since $P_s + P_u + P_c = I$, we have the estimate $||P_s|| \leq 1 + ||P_u|| + ||P_c||$. Thus, to prove property 1, it suffices to prove that $||P_u(t)||$ and $||P_c(t)||$ are uniformly bounded. We will prove only uniform boundedness

of $P_c(t)$, since the argument is identical for $P_u(t)$. Also, by periodicity (Lemma 7.2.1), it suffices to prove uniform boundedness on $[0, jT]$.

Assume for the moment that property 6 is satisfied (it will be proven later, independently of property 1). Suppose by way of contradiction that there exists $x_n \in \mathcal{RCR}$ and a (without loss of generality) increasing, convergent sequence $t_n \in [0, jT]$ with $||x_n|| = 1$ such that $||P_c(t_n)x_n|| = n$. Define $y_n = U_c(t_n, t_0)^{-1}x_n$. Then, we have

$$n = ||P_c(t_n)x_n|| = ||P_c(t_n)U_c(t_n, t_0)y_n|| = ||U_c(t_n, t_0)P_c(t_0)y_n|| \leq C||U_c(t_n, t_0)^{-1}|| \tag{52}$$

for some constant $C > 0$; see Lemma 3.4.2.

We will now prove that $U_c(t_n, t_0)^{-1}$ is uniformly bounded above; that is, there exists a constant $D > 0$ such that $||U_c(t_n, t_0)^{-1}|| \leq D$ for all $n \geq 0$. This will provide a contradiction to the inequality $n \leq C||U_c(t_n, t_0)^{-1}||$ in (52). Clearly, $U_c(t_n, t_0)^{-1}$ being uniformly bounded above is equivalent to $U_c(t_n, t_0)$ being uniformly bounded below; that is, there exists $E > 0$ such that $||U_c(t_n, t_0)x|| \geq E||x||$ for all $x \in \mathcal{R}(P_c(t_0))$.

Suppose $U_c(t_n, t_0)$ is not uniformly bounded below; that is, there exists $y_n \in \mathcal{R}(P_c(t_0))$ with $||y_n|| = 1$ such that $||U_c(t_n, t_0)y_n|| \to 0$. Since $\mathcal{R}(P_c(t_0))$ is finite-dimensional, we may pass to a subsequence (also denoted y_n) that is convergent by the Bolzano-Weierstrass theorem. Let $y_n \to y$ and $t_n \to t$. We have

$$||U_c(t, t_0)y|| \leq ||U_c(t, t_0)y - U_c(t, t_0)y_n|| + ||U_c(t, t_0)y_n - U_c(t_n, t_0)y_n|| + ||U_c(t_n, t_0)y_n||.$$

As $n \to \infty$, the first and third normed terms converge to zero. As for the term in the middle, we know that $U_c(t_n, t_0)y_n \to 0$ uniformly and, by passing to yet another subsequence (since $U_c(t, t_0)$ has finite-dimensional range) again denoted y_n, we may assume that $U_c(t, t_0)y_n$ also converges uniformly. Consequently, the term in the middle converges uniformly, and as such we may compute its limit pointwise. By Lemma 3.4.1, we have

$$||[U_c(t, t_0)y_n - U_c(t_n, t_0)y_n](\theta)|| \leq \left(\int_{t_n+\theta}^{t+\theta} \ell(\mu)d\mu + \sum_{t_n+\theta<\tau_i\leq t+\theta} b(i) \right) C$$

for n large enough, where $||U_c(s, t_0)|| \leq C$ for all $s \in [t_0, t]$ exists due to Lemma 3.4.2. Therefore, the pointwise (and uniform) limit is zero, and we conclude $U_c(t, t_0)y = 0$, which is a contradiction because $U_c(t, t_0)$ is an isomorphism by Theorem 7.1.2 and $||y|| = 1$.

2. This follows by property 5 of Theorem 7.1.1.

3. By following [[9], XIII – Theorem 3.3], we can show that $P(t)U(t+jT, s+kjT) = U(t+jT, s+kjT)P(s)$ for some $k \in \mathbb{N}$ chosen so that $s + (k-1)jT \leq t < s + kjT$, for each of the projectors $P \in \{P_u, P_c, P_s\}$. This implies $P(t)U(t, s + qjT) = U(t, s + qjT)P(s)$ for $q = k - 1$. Thus,

$$\begin{aligned}
P(t)U(t, s) &= P(t)U(t, s + qjT)U(s + qjT, s) \\
&= U(t, s + qjT)P(s)U(s + jT, s)^q \\
&= U(t, s + qjT)P(s)^q U(s + jT, s)^q \\
&= U(t, s + qjT)U(s + jT, s)^q P(s)^q \\
&= U(t, s)P(s),
\end{aligned}$$

where we have used the fact that $P(s)$ is a projector and commutes with $U(s + jT, s)$.

4. This follows from Theorem 7.1.2.

5. When $t \geq v \geq s$, the identity $U_c(t, s) = U_c(t, v)U_c(v, s)$ holds by properties of the evolution family U. When $t \geq s \geq v$, we find $I = U_c(t, v)^{-1}U_c(t, s)U_c(s, v)$, which implies

$$U_c(v, s) = U_c(v, t)U_c(t, s). \tag{53}$$

Also,

$$U_c(t, s) = U_c(t, v)U_c(t, v)^{-1}U_c(t, s) = U_c(t, v)[U_c(v, t)U_c(t, s)] = U_c(t, v)U_c(v, s). \tag{54}$$

Equation (53) implies $U_c(t,s) = U_c(t,v)U_c(v,s)$ for $v \geq s \geq t$, while (54) grants it for $t \geq s \geq v$. If $v \geq t \geq s$, then

$$U_c(t,s) = U_c(t,v)U_c(t,v)^{-1}U_c(t,s) = U_c(t,v)U_c(v,t)U_c(t,s) = U_c(t,v)U_c(v,s).$$

If $s \geq t \geq v$, then

$$U_c(t,s) = U_c(s,t)^{-1} = [U_c(s,v)U(v,t)_c]^{-1} = U_c(t,v)U_c(v,s).$$

Similarly, the desired equality holds if $s \geq v \geq t$. We have proven that $U_c(t,s) = U_c(t,v)U_c(v,s)$ for all $t, v, s \in \mathbb{R}$. The proof is identical for U_u.

6. This section is split into two parts, where we prove the estimates for U_c and U_s separately. The proof for U_u is similar to the center (U_c) case, and is omitted.

Center part: U_c. Let $\epsilon > 0$ be given. Recall that $U_c(t,s)$ is the restriction of $U(t,s)$ to $\mathcal{R}(P_c(s))$, so by Lemma 3.4.2 and periodicity, there exists $K > 0$ such that, for any $s \in \mathbb{R}$, we have $||U_c(t,s)|| \leq K$ provided $t \in [s, s+jT]$. As $U_c(s+jT,s)$ is compact and all of its eigenvalues satisfy $|\lambda| = 1$, Gelfand's (spectral radius) formula implies there exists an integer $k > 0$ such that $||U_c(s+jT,s)^k|| < 1 + \epsilon jT$. If we let m_t be the greatest integer such that $s + m_t kjT \leq t$ and $m_t^* \in \{0, \dots, k-1\}$ the greatest integer such that $s + m_t kjT + m_t^* jT \leq t$, then a trivial modification of the proof of [[9], XIII – Theorem 2.4] results in

$$||U_c(t,s)|| \leq K||U_c(s+jT,s)^k||^{m_t} \leq K(1+\epsilon jT)^{\frac{t-s}{jT}} \leq Ke^{\epsilon(t-s)}.$$

The proof is similar when $t \leq s$, and we obtain $||U_c(t,s)|| \leq Ke^{\epsilon|t-s|}$.

Stable part: U_s. Let $t \geq s$. As in the proof for the center part, we have $||U_s(t,s)|| \leq K$ provided $t \in [s, s+jT]$. Otherwise, since $U_s(s+jT,s)$ is compact by Lemma 7.2.3 and its spectrum is contained within the complex unit ball, there exists $k > 0$ such that $||U_s(s+jT,s)^k|| \leq (1 + ajT)$ for some $a < 0$. The rest of the proof follows by the same reasoning as the proof for the center part, and we obtain $||U(t,s)|| \leq Ke^{a(t-s)}$ as required.

Finally, $\mathcal{RCR}_c$ and $\mathcal{RCR}_u$ are finite-dimensional because Theorem 7.1.1 guarantees that the invariant subspaces $M_{\Lambda_c,t}$ and $M_{\Lambda_u,t}$ are finite-dimensional provided Λ_c and Λ_u are finite each finite sets — which they are because the eigenvalues of W_t can only accumulate at zero. The analogous result for $\mathcal{RCR}_c(t)$ and $\mathcal{RCR}_u(t)$ follows by Corollary 7.1.2.1. □

7.3 Floquet decomposition for $U_c(t,s)$

Of use in subsequent sections is the fact that, when restricted to the nonautonomous set $\mathcal{RCR}_c$, the evolution family $U(t,s)$ is essentially determined by the flow of a finite-dimensional ordinary differential equation. The following theorem makes this concrete; see [Theorem 4.5, [9]] for the analogous result for delay differential equations.

Theorem 7.3.1. *Denote $\mathcal{RCR}_c^{\mathbb{C}}(t)$ the complexification of $\mathcal{RCR}_c(t)$. There exists $Q \in \mathcal{L}(\mathcal{RCR}_c^{\mathbb{C}}(0))$ and $\alpha : \mathbb{R} \to \mathcal{L}(\mathcal{RCR}_c^{\mathbb{C}}(0), \mathcal{RCR}_c^{\mathbb{C}})$ with the following properties.*

- *α is jT-periodic, $\alpha(t) : \mathcal{RCR}_c^{\mathbb{C}}(0) \to \mathcal{RCR}_c^{\mathbb{C}}(t)$ is invertible, and there exists $\beta \geq 1$ such that for all $\phi \in \mathcal{RCR}_c^{\mathbb{C}}(0)$,*

$$\beta^{-1}||\phi|| \leq \sup_{t \in \mathbb{R}} ||\alpha(t)\phi|| \leq \beta||\phi||.$$

- *$U_c(t,0)\phi = \alpha(t)e^{tQ}\phi$ for all $\phi \in \mathcal{RCR}_c$.*

Proof. Define $Q = \log U_c(jT,0)$, where we choose the logarithm to be branch that avoids the (finite set of nonzero) eigenvalues of $U_c(jT,0)$. Defining $\alpha(t) = U(t,0)e^{-tQ}$, one may verify (compare to Proposition 4.4 and Theorem 4.5 of [9]) that α is periodic and $U_c(t,0)$ satisfies the claimed decomposition. Uniform boundedness of α above and below follows by its periodicity and boundedness of $U_c(t,0)$ on $[-jT, jT]$; see the proof of Theorem 7.2.1. $\alpha(t)$ is clearly invertible. □

8 Smoothness of center manifolds

In Section 5, we proved the existence of invariant center manifolds associated to the abstract integral equation (31). These invariant manifolds are images of a uniformly Lipschitz continuous function $C : \mathcal{RCR}_c \to \mathcal{RCR}$. In this secton, we will ultimately prove that this function is smooth. To accomplish this, we will need to introduce an additional regularity assumption on the nonlinear parts of vector field and jump map.

H.8 The functions c_j and sequences $\{d_j(k) : k \in \mathbb{Z}\}$ introduced in H.5 are bounded.

Note that H.8 is a purely nonautonomous property and is trivially satisfied if the vector field and jump functions are autonomous. We will also assume that center part $U_c(t, s)$ of the linear evolution family associated to the linear part of the semilinear equation (29)–(30) satisfies the following decomposability condition, which is can be seen as a nonautonomous Floquet decomposition.

H.9 There exists $\alpha : \mathbb{R} \to \mathcal{L}(\mathcal{RCR}_c)$ and $Q \in \mathcal{L}(\mathcal{RCR}_c(0))$ such that $\alpha(t) : \mathcal{RCR}_c(0) \to \mathcal{RCR}_c(t)$ is invertible for all $t \in \mathbb{R}$, there exists $\beta \geq 1$ such that $\beta^{-1}||\phi|| \leq \sup_{t\in\mathbb{R}} ||\alpha(t)\phi|| \leq \beta||\phi||$ for all $\phi \in \mathcal{RCR}_c(0)$, and the decomposition $U_c(t, 0) = \alpha(t)e^{tQ}$ is satisfied.

Remark 8.0.1. *By Theorem 7.3.1, a semilinear equation with linear part satisfying the periodic constraints* (50) *automatically fulfills condition H.9.*

From this point on, we will assume conditions H.1–H.9 are satisfied. The rest of this section will utilize several techniques from the theory of contraction mappings on scales of Banach spaces. In particular, many of the proofs that follow are adapted from [27, 9, 18], albeit adapted somewhat so as to manage the explicitly nonautonomous and impulsive characteristics of the problem. In order to avoid unnecessary technical details, some results will be stated without proof.

8.1 Modification of nonlinearities and fixed-point equation for the center manifold

By H.9, let $U_c(t, 0) = \alpha(t)e^{tQ}$. Define $\Gamma : \mathbb{R} \to \mathcal{L}(\mathcal{RCR}, \mathcal{RCR}_c^{\mathbb{C}}(0))$ by

$$\Gamma(t) = \alpha(t)^{-1}P_c(t).$$

Let $A = \sup_{t\in\mathbb{R}} ||\alpha^{-1}(t)||$, so that in particular $||\Gamma(t)|| \leq AN$.

We introduce a different modification of the nonlinearity than the one used in Section 5.3. For a smooth cutoff function ξ, define the smoothed nonlinearities

$$\tilde{F}_\delta(t, \phi) = f(t, \phi)\xi\left(\frac{||\Gamma(t)\phi||}{AN\delta}\right)\xi\left(\frac{||(I - P_c(t))\phi||}{N\delta}\right)$$
$$\tilde{G}_\delta(k, \phi) = g_k(\phi)\xi\left(\frac{||\Gamma(\tau_k)\phi||}{AN\delta}\right)\xi\left(\frac{||(I - P_c(\tau_k))\phi||}{N\delta}\right),$$

where by a suitable renorming, we may assume without loss of generality that $||\cdot||$ is smooth on $\mathcal{RCR}_c(0)\backslash\{0\}$. This follows because $\mathcal{RCR}_c(0)$ is finite dimensional.

The following lemma and corollary now follow from the properties of the Floquet-like decomposition $U_c(t, 0) = \alpha(t)e^{tQ}$. For a proof, see similar results in [18, 19].

Lemma 8.1.1. *Let $f(t, \cdot)$ and $g_k(\cdot)$ be uniformly (in $t \in \mathbb{R}$ and $k \in \mathbb{Z}$) Lipschitz continuous on the ball $B_{\mathcal{RCR}}(\delta, 0)$ in $\mathcal{RCR}$ with mutual Lipschitz constant $L(\delta)$, and let $f(t, 0) = g_k(0) = 0$. The functions*

$$\tilde{F}_\delta : \mathbb{R} \times \mathcal{RCR} \to \mathbb{R}^n, \qquad \tilde{G}_\delta : \mathbb{Z} \times \mathcal{RCR} \to \mathbb{R}^n$$

are globally, uniformly (in $t \in \mathbb{R}$ and $k \in \mathbb{Z}$) Lipschitz continuous with mutual Lipschitz constant L_δ that satisfies $L_\delta \to 0$ as $\delta \to 0$.

Corollary 8.1.0.1. *The substitution operator*

$$\tilde{R}_\delta : \mathcal{PC}^\eta \to B^\eta(\mathbb{R},\mathbb{R}^n) \oplus B^\eta_{\tau_k}(\mathbb{Z},\mathbb{R}^n)$$

defined by $R_\delta(x)(t,k) = (\tilde{F}_\delta(t,x(t)), \tilde{G}_\delta(k,x(\tau_k)))$ is globally Lipschitz continuous with Lipschitz constant $\tilde{L}_\delta$ that satisfies $\tilde{L}_\delta \to 0$ as $\delta \to 0$. Moreover, the Lipschitz constant is independent of η, s.

We can repeat the construction of the Lipschitz center manifold of Section 5.4, essentially verbatim. Namely, we define a family of nonlinear maps $\tilde{\mathcal{G}}^{\eta,s}_\delta : \mathcal{PC}^{\eta,s} \times \mathcal{RCR}_c(s) \to \mathcal{PC}^{\eta,s}$ by

$$\tilde{\mathcal{G}}^{\eta,s}_\delta(u,\varphi)) = U(\cdot,s)\varphi + \mathcal{K}^\eta_s(\tilde{R}_\delta(u)) \tag{55}$$

and obtain for $\delta > 0$ sufficiently small a unique fixed point $\tilde{u}_{\eta,s} : \mathcal{RCR}_c(s) \to \mathcal{PC}^{\eta,s}$ satisfying the equation $\tilde{\mathcal{G}}^{\eta,s}_\delta(\tilde{u}_{\eta,s},\cdot) = \tilde{u}_{\eta,s}$. The fixed point may be used to define the center manifold in the same way as Definition 5.4.1, and it enjoys the same invariance properties as outlined in Theorem 5.4.2 and the reduction principles of Section 6. Importantly, we have the following extension of Lemma 5.4.1, whose proof is omitted.

Lemma 8.1.2. *There exists $\delta > 0$ such that the fixed point $\tilde{u}_{\eta,s}$ of (55) satisfies $||(I - \widehat{P}_c)\tilde{u}_{\eta,s}||_0 < \delta$ for all $s \in \mathbb{R}$, where the operator $\widehat{P}_c : \mathcal{PC}^\eta \to \mathcal{PC}^\eta(\mathbb{R},\mathcal{RCR}_c)$ is defined as in Lemma 5.4.1.*

From this point on, our attention shifts to proving the smoothness of $\tilde{u}_{\eta,s} : \mathcal{RCR}_c(s) \to \mathcal{PC}^{\eta,s}$ as defined by the fixed point of (55). We begin with some notation. Introduce

$$\Sigma^\eta_{\tau_k} = \{f : \mathbb{Z} \to \mathcal{RCR} : ||f||_\eta = \sup_{k\in\mathbb{Z}} e^{-\eta|\tau_k|}||f_k||_\eta < \infty\},$$

the space of η-bounded $\mathcal{RCR}$-valued sequences. Then, define $\Sigma^\infty_{\tau_k} = \cup_{\eta>0}\Sigma^\eta_{\tau_k}$. Let

$$V^\eta = \{u \in \mathcal{PC}^\eta : ||(I - \widehat{P}_c)u||_0 < \infty\},$$

where $\widehat{P}_c$ is the projection operator from Lemma 8.1.2. Equipped with the norm

$$||u||_{V^{\eta,s}} = ||\mathcal{P}_c u||_{\eta,s} + ||(I - \mathcal{P}_c)u||_0,$$

the space $V^{\eta,s}$ is complete, where the s-shifted definitions are as outlined at the beginning of Section 5.1 .

Let $\delta > 0$ be chosen as in Lemma 8.1.2 and define

$$V^\eta_\delta = \{u \in V^\eta : ||(I - \widehat{P}_c)u||_0 < N\delta\}$$

and define $V^\eta_\delta(t) \subset \mathcal{RCR}$ by $V^\eta_\delta(t) = \{u(t) : u \in V^\eta_\delta\}$. Also, define the set $V^\infty_\delta = \cup_{\eta>0}V^\eta_\delta$. Set $B^\eta = PC^\eta(\mathbb{R},\mathbb{R}^n) \oplus B^\eta_{\tau_k}(\mathbb{Z},\mathbb{R}^n)$ and $B^\infty = \cup_{\eta>0}B^\eta$. Finally, the bounded p-linear maps from $X_1 \times \cdots \times X_p$ to Y for Banach spaces X_i and Y will be denoted $\mathcal{L}^p(X_1 \times \cdots X_p, Y)$. We may write simply $\mathcal{L}^p$ if there is no confusion.

By construction of the modified nonlinearity $\tilde{R}_\delta$ and the choice of δ from Lemma 8.1.2, the functions $u \mapsto \tilde{F}_\delta(t,u)$ and $u \mapsto \tilde{G}_\delta(k,u)$ are C^m on $V^\eta_\delta(t)$ and $V^\eta_\delta(\tau_k)$ respectively, for all $t \in \mathbb{R}$ and $k \in \mathbb{Z}$. We are therefore free to define

$$(\tilde{F}^{(p)}_\delta u)(t) = D^p\tilde{F}_\delta(t,u(t)), \qquad (\tilde{G}^{(p)}_\delta)(k) = D^p\tilde{G}_\delta u(j,u(\tau_k)),$$

for $1 \le p \le m$, where D^p denotes the pth Fréchet derivative with respect to the second variable. If we denote $\Pi^\eta = \mathcal{PC}^\eta \oplus \Sigma^\eta_{\tau_k}$, then for each $u \in V^\infty_\delta$ we can define a p-linear map $\tilde{R}^{(p)}_\delta(u) : \Pi^\infty \times \cdots \times \Pi^\infty \to B^\infty$ by the equation

$$\tilde{R}^{(p)}_\delta(u)((v_1,w_1),\ldots,(v_p,w_p))(t,k) = (F^{(p)}_\delta u(t)(v_1(t),\ldots,v_p(t)), G^{(p)}_\delta u(k)(w_1(t),\ldots,w_p(t))). \tag{56}$$

For $p = 0$, we define $\tilde{R}^{(0)}_\delta = \tilde{R}_\delta$ to be equal to the substitution operator as defined in Corollary 8.1.0.1. From here on, we assume conditions H.1–H.3, H.5–H.8 and H*4 are satisfied.

8.2 Smoothness of the modified nonlinearity

In this section we elaborate on various properties of the substitution operator $\tilde{R}_\delta$ and its formal derivative $\tilde{R}_\delta^{(p)}$ introduced in equation (56). First, we will need a result that states that condition H.5 holds also for the modified nonlinearities when restricted to V_δ^∞.

Lemma 8.2.1. *For $j = 1, \ldots, m$, there exist constants $\tilde{c}_j, \tilde{d}_j, \tilde{q} > 0$ such that*

$$||D^j \tilde{F}_\delta(t,\phi) - D^j \tilde{F}_\delta(t,\psi)|| \le \tilde{c}_j ||\phi - \psi||, \qquad ||D^j \tilde{F}_\delta(t,\phi)|| \le \tilde{q}\tilde{c}_j \qquad \phi,\psi \in V_\delta^\infty(t)$$
$$||D^j \tilde{G}_\delta(k,\phi) - D^j \tilde{G}_\delta(k,\psi)|| \le \tilde{d}_j ||\phi - \psi||, \qquad ||D^j \tilde{G}(k,\phi)|| \le \tilde{q}\tilde{d}_j \qquad \phi,\psi \in V_\delta^\infty(\tau_k).$$

Proof. We prove only the Lipschitzian property for $D^j F_\delta$, since the boundedness and corresponding results for $D^j G_\delta$ are proven similarly. Denote

$$X(t,\phi) = \xi\left(\frac{||\Gamma(t)\phi||}{AN\delta}\right)\xi\left(\frac{||(I - P_c(t))\phi||}{N\delta}\right).$$

When $\phi, \psi \in V_\delta^\infty(t)$, X is m-times continuously differentiable and its derivative is globally Lipschitz continuous. Moreover, the Lipschitz constant can be chosen independent of t because of the uniform boundedness (property 1) of the projection operator $P_c(t)$ and the operator $\Gamma(t)$. Let Lip_X^k denote the Lipschitz constant for $D^k X(t,\cdot)$. Then,

$$\begin{aligned} D^j \tilde{F}_\delta(t,\phi) - D^j \tilde{F}_\delta(t,\psi) &= D^j\left[f(t,\phi)X(t,\phi) - f(t,\psi)X(t,\phi)\right] \\ &= \sum_{N_1,N_2 \in P_2(j)} D^{\#N_1} f(t,\phi) D^{\#N_2} X(t,\phi) - D^{\#N_1} f(t,\psi) D^{\#N_2} X(t,\psi) \\ &= \sum_{N_1,N_2 \in P_2(j)} D^{\#N_1}[f(t,\phi) - f(t,\psi)] D^{\#N_2} X(t,\phi) + D^{\#N_1} f(t,\psi) D^{\#N_2}[X(t,\phi) - X(t,\psi)], \end{aligned}$$

where $P_2(j)$ denotes the set of partitions of length two from the set $\{1, \ldots, j\}$ and $\#Y$ is the cardinality of Y. Restricted to the ball $B_{2\delta}(0)$, the Lipschitz constants for $D^j f(t,\cdot)$ and the boundedness estimates from H.5 then imply the estimate

$$||D^j \tilde{F}_\delta(t,\phi) - D^j \tilde{F}_\delta(t,\psi)|| \le \left(\sum_{N_1,N_2 \in P_2(j)} (1+q) c_{\#N_1}(t) \text{Lip}_X^{\#N_2}\right) ||\phi - \psi||.$$

As each of c_j and d_j are bounded, the Lipschitz constant admits an upper bound. Outside of $B_{2\delta}(0)$, X and all of its derivatives are identically zero. □

Lemma 8.2.2. *Let $1 \le p \le m$, $\mu_i > 0$ for $i = 1, \ldots, p$, $\mu = \mu_1 + \cdots + \mu_p$ and $\eta \ge \mu$. Then we have $\tilde{R}_\delta^{(p)}(u) \in \mathcal{L}^p(\Pi^{\mu_1} \times \cdots \times \Pi^{\mu_p}, B^\eta)$ for all $u \in V_\delta^\infty$, with*

$$\begin{aligned} ||\tilde{R}_\delta^{(p)}(u)||_{\mathcal{L}^p} &\le \sup_{t\in\mathbb{R}} ||\tilde{F}_\delta^{(p)} u(t)|| e^{-(\eta-\mu)|t|} + \sup_{k\in\mathbb{Z}} ||\tilde{G}_\delta^{(p)} u(k)|| e^{-(\eta-\mu)|\tau_k|} \\ &= ||\tilde{R}_\delta^{(p)}(u)||_{\eta-\mu}. \end{aligned}$$

Also, $u \mapsto \tilde{R}_\delta^{(p)}(u)$ is continuous as a mapping $\tilde{R}_\delta^{(p)} : V_\delta^\sigma \to \mathcal{L}^p(\Pi^{\mu_1} \times \cdots \times \Pi^{\mu_p}, B^\eta)$ if $\eta > \mu$, for all $\sigma > 0$.

Proof. It is easy to verify that $\tilde{R}_\delta^{(p)}(u)$ is p-linear. For boundedness,

$$\begin{aligned} ||\tilde{R}_\delta^{(p)}(u)||_{\mathcal{L}^p} &= \sup_{\substack{t\in\mathbb{R} \\ ||v||_{\vec{\mu}}=1}} ||\tilde{F}_\delta^{(p)} u(t)(v_1(t), \ldots, v_p(t))|| e^{-\eta|t|} + \sup_{\substack{k\in\mathbb{Z} \\ ||w||_{\vec{\mu}}=1}} ||\tilde{G}_\delta(p) u(k)(w_1(\tau_k), \ldots, w_p(\tau_k))|| e^{-\eta|\tau_k|} \\ &\le \sup_{\substack{t\in\mathbb{R} \\ ||v||_{\vec{\mu}}=1}} ||\tilde{F}_\delta^{(p)} u(t)|| \cdot \left[||v_1(t)|| \cdots ||v_p(t)||\right] e^{-\eta|t|} + \sup_{\substack{k\in\mathbb{Z} \\ ||w||_{\vec{\mu}}=1}} ||\tilde{G}_\delta^{(p)} u(k)|| \cdot \left[||w_1(\tau_k)|| \cdots ||w_p(\tau_k)||\right] e^{-\eta|\tau_k|} \\ &= \sup_{t\in\mathbb{R}} ||\tilde{F}_\delta^{(p)} u(t)|| e^{-(\eta-\mu)|t|} + \sup_{k\in\mathbb{Z}} ||\tilde{G}_\delta^{(p)} u(k)|| e^{-(\eta-\mu)|\tau_k|}. \end{aligned}$$

The latter is finite by Lemma 8.2.1 whenever $\eta \geq \mu$. In particular, the latter lemma implies that for all $\phi \in V_\delta^\infty$, one has $\sup_{t\in\mathbb{R}} ||D^j \tilde{F}_\delta(t,\phi(t))|| \leq \tilde{q}\tilde{c}_j$, and similar for $\tilde{G}_k$. This uniform boundedness can then be used to prove the continuity of $u \mapsto \tilde{R}_\delta^{(p)}(u)$ when $\eta > \mu$; the proof follows that of [Lemma 7.3 [18]] and is omitted here. □

The proof of the following lemmas are essentially identical to the proofs of [Corollary 7.5, Corollary 7.6, Lemma 7.7 [18]] and are omitted here. The trivial modification stems from in our case, the nonlinearity consists of two independent components defined by F_δ and G_δ, as well as the time dependence. This latter complication is effectively resolved by Lemma 8.2.1.

Lemma 8.2.3. *Let $\eta_2 > k\eta_1 > 0$, $1 \leq p \leq k$. Then, $\tilde{R}_\delta : V_\delta^{\eta_1} \to \mathcal{L}^p(\Pi^{\eta_1} \times \cdots \times \Pi^{\eta_1}, B^{\eta_2})$ is C^k and $D^p\tilde{R}_\delta = \tilde{R}_\delta^{(p)}$.*

Lemma 8.2.4. *Let $1 \leq p \leq m$, $\mu_i > 0$ for $i = 1,\ldots,p$, $\mu = \mu_1 + \cdots + \mu_p$ and $\eta \geq \mu$. Then, $\tilde{R}_\delta^{(p)} : V_\delta^\sigma \to \mathcal{L}^p(\Pi^{\mu_1} \times \cdots \times \Pi^{\mu_p}, B^\eta)$ is C^{k-p} provided $\eta > \mu + (k-p)\sigma$.*

Lemma 8.2.5. *Let $1 \leq p \leq k$, $\mu_i > 0$ for $i = 1,\ldots,p$, $\mu = \mu_1 + \cdots \mu_p$ and $\eta > \mu + \sigma$ for some $\sigma > 0$. Let $X : \mathcal{RCR}_c(s) \to V_\delta^\sigma$ be C^1. Then, $\tilde{R}_\delta^{(p)} \circ X : \mathcal{RCR}_c(s) \to \mathcal{L}^p(\Pi^{\mu_1} \times \cdots \times \Pi^{\mu_p}, B^\eta)$ is C^1 and*

$$D\left(\tilde{R}_\delta^{(p)} \circ X\right)(\phi)(v_1,\ldots,v_p,\psi) = \tilde{R}_\delta^{(p+1)}(X(\phi))(v_1,\ldots,v_p,X'(\phi)\psi).$$

Theorem 8.2.1. *Let $\mathcal{J}_s^{\eta_2,\eta_1} : \mathcal{PC}^{\eta_1,s} \to \mathcal{PC}^{\eta_2,s}$ denote the (continuous) embedding operator for $\eta_1 \leq \eta_2$. Let $[\tilde{\eta},\overline{\eta}] \subset (0,\min\{-a,b\})$ be such that $k\tilde{\eta} < \overline{\eta}$. Then, for each $p \in \{1,\ldots,k\}$ and $\eta \in (p\tilde{\eta},\overline{\eta}]$, the mapping $J_s^{\eta\tilde{\eta}} \circ \tilde{u}_{\tilde{\eta},s} : \mathcal{RCR}_c(s) \to \mathcal{PC}^{\eta,s}$ is of class C^k provided $\delta > 0$ is sufficiently small.*

Proof. The proof here follows the same lines as [Theorem 7.7, [9]]. To begin, we choose $\delta > 0$ small enough so that Lemma 8.1.2 is satisfied in addition to having $\tilde{L}_\delta||\mathcal{K}_s^\eta|| < \frac{1}{4}$ for all $\eta \in [\tilde{\eta},\overline{\eta}]$. By Lemma 5.2.3 and Corollary 5.3.0.1, this can always be done in such a way that the inequality holds for all $s \in \mathbb{R}$.

We proceed by induction on k. For $p = 1 = k$, we let $\eta \in (\tilde{\eta},\overline{\eta}]$ and show that Lemma 10.0.1 applies with

$$Y_0 = V_\delta^{\tilde{\eta},s}, \qquad Y = \mathcal{PC}^{\tilde{\eta},s}, \qquad Y_1 = \mathcal{PC}^{\eta,s}, \qquad \Lambda = \mathcal{RCR}_c(s)$$
$$f(u,\varphi) = \tilde{\mathcal{G}}_\delta^{\tilde{\eta},s}(u,\varphi), \qquad f^{(1)}(u,\varphi) = \mathcal{K}_s^{\tilde{\eta}} \circ \tilde{R}_\delta^{(1)}(u), \qquad f_1^{(1)}(u,\varphi) = \mathcal{K}_s^\eta \circ \tilde{R}_\delta^{(1)}(u),$$

with embeddings $J = \mathcal{J}_s^{\eta\tilde{\eta}}$ and $J_0 : V_\delta^{\tilde{\eta},s} \hookrightarrow \mathcal{PC}^{\tilde{\eta},s}$. To check condition b1 we must first verify the C^1 smoothness of

$$V_\delta^{\tilde{\eta},s} \times \mathcal{RCR}_c(s) \ni (u,\varphi) \mapsto g(u,\varphi) = \mathcal{J}_s^{\eta\tilde{\eta}}\left(U(\cdot,s)\varphi + \mathcal{K}_s^{\tilde{\eta}} \circ \tilde{R}_\delta(J_0u)\right).$$

The embedding operator $\mathcal{J}_s^{\eta\tilde{\eta}}$ is itself C^1, as is $\varphi \mapsto U(\cdot,s)\varphi$ and $J_0u \mapsto \tilde{R}_\delta(J_0u)$, the latter due to Lemma 8.2.3. C^1 smoothness of g then follows by continuity of the linear embedding J_0. Verification of the equalities $D_1g(u,\varphi)\xi = Jf^{(1)}(J_0u,\varphi)J_0\xi$ and $Jf^{(1)}(J_0u,\varphi)\xi = f_1^{(1)}(J_0u,\varphi)J\xi$ is straightforward. Condition b2 follows by boundedness of the embedding operators and the the small Lipschitz constant for $\tilde{\mathcal{G}}_\delta^{\tilde{\eta},s}$. For condition b3, the fixed point is $\tilde{u}_{\tilde{\eta},s} : \mathcal{RCR}_c(s) \to \mathcal{PC}^{\tilde{\eta},s}$, and we may factor it as $\tilde{u}_{\tilde{\eta},s} = J_0 \circ \Phi$ with $\Phi : \mathcal{RCR}_c(s) \to V_\delta^{\tilde{\eta},s}$ defined by $\Phi(\varphi) = \tilde{u}_{\tilde{\eta},s}(\varphi)$; the latter is continuous by Theorem 5.4.1 and the factorization is justfied by Lemma 8.1.2. To check condition b4 we must verify that

$$V_\delta^{\tilde{\eta},s} \times \mathcal{RCR}_c(s) \ni (u,\varphi) \mapsto f_0(u,\varphi) = \tilde{\mathcal{G}}_\delta^{\tilde{\eta},s}(J_0u,\varphi)$$

has a continuous partial derivative in its second variable – this is clear since f_0 is linear in φ. Finally, condition b5 requires us to verify that the map $(u,\varphi) \mapsto \mathcal{J}_s^{\eta\tilde{\eta}} \circ \mathcal{K}_s^{\tilde{\eta}} \circ \tilde{R}_\delta^{(1)}(J_0u)$ is continuous from $V_\delta^{\tilde{\eta},s} \times \mathcal{RCR}_c(s)$ into $\mathcal{L}(\mathcal{RCR}_c(s),\mathcal{PC}^{\eta,s})$, but this once again follows by the continuity of the embedding operators and the smoothness of $\tilde{R}_\delta$ from Lemma 8.2.3.

The conditions of Lemma 10.0.1 are satisfied, and we conclude that $\mathcal{J}^{\eta\tilde{\eta}} \circ \tilde{u}_{\tilde{\eta},s}$ is of class C^1 and that the derivative $D(\mathcal{J}^{\eta\tilde{\eta}} \circ \tilde{u}_{\tilde{\eta},s}) \in \mathcal{L}(\mathcal{RCR}_c(s),\mathcal{PC}^{\eta,s})$ is the unique solution $w^{(1)}$ of the equation

$$w^{(1)} = \mathcal{K}_s^{\tilde{\eta}} \circ \tilde{R}_\delta^{(1)}(\tilde{u}_s^{\tilde{\eta}}(\varphi))w^{(1)} + U(\cdot,s) := F_1(w^{(1)},\varphi). \tag{57}$$

The mapping $F_1 : \mathcal{L}(\mathcal{RCR}_c(s), \mathcal{PC}^{\eta,s}) \times \mathcal{RCR}_c(s) \to \mathcal{L}(\mathcal{RCR}_c(s), \mathcal{PC}^{\eta,s})$ is a uniform contraction for $\eta \in [\tilde{\eta}, \overline{\eta}]$ – indeed, $F_1(\cdot, \varphi)$ is Lipschitz continuous with Lipschitz constant $\tilde{L}_\delta \cdot ||\mathcal{K}_s^\eta|| < \frac{1}{4}$; this follows from Lemma 8.2.2 and is independent of s. Thus, $\tilde{u}_s^{(1)}(\varphi) \in \mathcal{L}(\mathcal{RCR}_c(s), \mathcal{PC}^{\tilde{\eta},s}) \hookrightarrow \mathcal{L}(\mathcal{RCR}_c(s), \mathcal{PC}^{\eta,s})$ for $\eta \geq \tilde{\eta}$. Moreover, $\tilde{u}_s^{(1)} : \mathcal{RCR}_c(s) \to \mathcal{PC}^{\eta,s}$ is continuous if $\eta \in (\tilde{\eta}, \overline{\eta}]$.

Now, let $1 \leq p \leq k$ for $k \geq 1$ and suppose that for all $q \in \{1, \dots, p\}$ and all $\eta \in (q\tilde{\eta}, \overline{\eta}]$, the mapping

$$\mathcal{J}_s^{\eta\tilde{\eta}} \circ \tilde{u}_{\tilde{\eta},s} : \mathcal{RCR}_c(s) \to \mathcal{PC}^{\eta,s}$$

is of class C^q with $D^q(\mathcal{J}_s^{\eta\tilde{\eta}} \circ \tilde{u}_s^{\tilde{\eta}}) = \mathcal{J}_s^{\eta\tilde{\eta}} \circ \tilde{u}_{\tilde{\eta},s}^{(q)}$ and $\tilde{u}_{\tilde{\eta},s}^{(q)}(\varphi) \in \mathcal{L}^q(\mathcal{RCR}_c(s), \mathcal{PC}^{q\tilde{\eta},s})$ for each $\varphi \in \mathcal{RCR}_c(s)$, such that the mapping

$$\mathcal{J}_s^{\eta\tilde{\eta}} \circ \tilde{u}_{\tilde{\eta},s}^{(q)} : \mathcal{RCR}_c(s) \to \mathcal{L}^q(\mathcal{RCR}_c(s), \mathcal{PC}^{\eta,s})$$

is continuous for $\eta \in (q\tilde{\eta}, \overline{\eta}]$. Suppose additionally that $\tilde{u}_{\tilde{\eta},s}^{(q)}(\varphi)$ is the unique solution $w^{(p)}$ of an equation

$$w^{(p)} = \mathcal{K}_s^{\tilde{\eta}p} \circ \tilde{R}_\delta^{(1)}(\tilde{u}_{\tilde{\eta},s}(\varphi))w^{(p)} + H_{\tilde{\eta}}^{(p)}(\varphi) := F_{\tilde{\eta}}^{(p)}(w^{(p)}, \varphi), \tag{58}$$

with $H^1 = U(\cdot, s)$ and $H_x^{(p)}(\varphi)$ for $p \geq 2$ is a finite sum of terms of the form

$$\mathcal{K}_s^{px} \circ \tilde{R}_\delta^{(q)}(\tilde{u}_{\tilde{\eta},s}(\varphi))(\tilde{u}_{\tilde{\eta},s}^{(r_1)}(\varphi), \cdots, \tilde{u}_{\tilde{\eta},s}^{(r_q)}(\varphi))$$

with $2 \leq q \leq p$, $1 \leq r_i < p$ for $i = 1, \dots, q$, such that $r_1 + \cdots + r_q = p$. Under such assumptions, the mapping $F_{\tilde{\eta}}^{(p)} : \mathcal{L}^p(\mathcal{RCR}_c(s), \mathcal{PC}^{\eta,s}) \times \mathcal{RCR}_c(s) \to \mathcal{L}^p(\mathcal{RCR}_c(s), \mathcal{PC}^{\eta,s})$ is a uniform contraction for all $\eta \in [p\tilde{\eta}, \overline{\eta}]$; see Lemma 8.2.2.

Next, choose some $\eta \in ((p+1)\tilde{\eta}, \overline{\eta}]$, $\sigma \in (\tilde{\eta}, \eta/(p+1)]$ and $\mu \in ((p+1)\sigma, \eta)$. We will verify the conditions of Lemma 10.0.1 with the spaces and functions

$$\begin{aligned}
Y_0 &= \mathcal{L}^p(\mathcal{RCR}_c(s), \mathcal{PC}^{p\sigma,s}), \qquad && Y = \mathcal{L}^p(\mathcal{RCR}_c(s), \mathcal{PC}^{\mu,s}), \quad Y_1 = \mathcal{L}^p(\mathcal{RCR}_c(s), \mathcal{PC}^{\eta,s}) \\
f(u,\varphi) &= \mathcal{K}_s^\mu \circ \tilde{R}_\delta^{(1)}(\tilde{u}_{\tilde{\eta},s}(\varphi))u + H_{\mu/p}^{(p)}(\varphi), && \Lambda = \mathcal{RCR}_c(s), \\
f^{(1)}(u,\varphi) &= \mathcal{K}_s^\mu \circ \tilde{R}_\delta^{(1)}(\tilde{u}_{\tilde{\eta},s}(\varphi)) \in \mathcal{L}(Y), \\
f_1^{(1)}(u,\varphi) &= \mathcal{K}_s^\eta \circ \tilde{R}_\delta^{(1)}(\tilde{u}_{\tilde{\eta},s}(\varphi)) \in \mathcal{L}(Y_1),
\end{aligned}$$

We begin with the verification of condition b1. We must check that

$$\mathcal{L}^p(\mathcal{RCR}_c(s), \mathcal{PC}^{p\sigma,s}) \times \mathcal{RCR}_c(s) \ni (u,\varphi) \mapsto \mathcal{J}^{\eta\mu} \circ \mathcal{K}_s^\mu \circ \tilde{R}_\delta^{(1)}(\tilde{u}_{\tilde{\eta},s}(\varphi))u + \mathcal{J}^{\eta\mu} \circ H_{\mu/p}^{(p)}(\varphi)$$

is of class C^1, where now $\mathcal{J}^{\eta_2\eta_1} : \mathcal{L}^p(\mathcal{RCR}_c(s), \mathcal{PC}^{\eta_1,s}) \hookrightarrow \mathcal{L}^p(\mathcal{RCR}_c(s), \mathcal{PC}^{\eta_2,s})$. The above mapping is C^1 with respect to $u \in \mathcal{L}^p(\mathcal{RCR}_c(s), \mathcal{PC}^{p\sigma,s})$ since it is linear in this variable. With respect to $\varphi \in \mathcal{RCR}_c(s)$, we have that $\varphi \mapsto \mathcal{J}^{\eta\mu}\mathcal{K}_s^\mu \circ \tilde{R}_\delta^{(1)}(\tilde{u}_{\tilde{\eta},s}(\varphi))$ is C^1: this follows by Lemma 8.2.5 with $\mu > (p+1)\sigma$ and the C^1 smoothness of $\varphi \mapsto \mathcal{J}^{\sigma\tilde{\eta}} \circ \tilde{u}_{\tilde{\eta},s}(\varphi)$ with $\sigma > \tilde{\eta}$. For the C^1 smoothness of the portion $\varphi \mapsto \mathcal{J}^{\eta\mu}H_{\mu/p}^{(p)}(\varphi)$, we get differentiability from Lemma 8.2.5; we have that the derivative of $\varphi \mapsto H_{\mu/p}^{(p)}(\varphi)$ is a sum of terms of the form

$$\mathcal{K}_s^\mu \circ \tilde{R}_\delta^{(q+1)}(\tilde{u}_{\tilde{\eta},s}(\varphi))(\tilde{u}_{\tilde{\eta},s}^{(r_1)}(\varphi), \dots, \tilde{u}_{\tilde{\eta},s}^{(r_q)}(\varphi)) + \sum_{j=1}^{q} \mathcal{K}_s^\mu \circ \tilde{R}_\delta^{(q)}(\tilde{u}_{\tilde{\eta},s}(\varphi))(\tilde{u}_{\tilde{\eta},s}^{(r_1)}(\varphi), \dots, \tilde{u}_{\tilde{\eta},s}^{(r_j+1)}(\varphi), \dots, \tilde{u}_{\tilde{\eta},s}^{(r_q)}(\varphi)),$$

and each $\tilde{u}_{\tilde{\eta},s}^{(j)}$ is understood as a map into $\mathcal{PC}^{j\sigma,s}$. Applying Lemma 8.2.2 with $\mu > (p+1)\sigma$ grants continuity of $DH_{\mu/p}^{(p)}(\varphi)$ and, subsequently, to $\mathcal{J}^{\eta\mu}DH_{\mu/p}^{(p)}(\varphi)$. The other embedding properties of condition b1 are easily checked. Condition b4 can be proven similarly.

The Lipschitz condition and boundedness of b2 follows by the choice of $\delta > 0$ at the beginning and the uniform contractivity of H_p described above. Condition b3 is proven by writing

$$\mathcal{J}^{\eta\mu} \circ \mathcal{K}_s^\mu \circ \tilde{R}_\delta^{(1)}(\tilde{u}_{\tilde{\eta},s})(\varphi)) = \mathcal{K}_s^\eta \circ \tilde{R}_\delta^{(1)}(\tilde{u}_{\tilde{\eta},s}(\varphi))$$

and applying Lemma 8.2.2 together with the C^1 smoothness of $\tilde{u}_{\tilde{\eta},s}$ to obtain the continuity of $\varphi \mapsto \tilde{R}^{(1)}_\delta(\tilde{u}_{\tilde{\eta},s}) \in \mathcal{L}(Y, Y_1)$. This also proves the final condition b5 of Lemma 10.0.1, and we conclude that $\tilde{u}^{(p)}_{\tilde{\eta},s} : \mathcal{RCR}_c(s) \to \mathcal{L}^p(\mathcal{RCR}_c(s), \mathcal{PC}^{\eta,s})$ is of class C^1 with $\tilde{u}^{(p+1)}_{\tilde{\eta},\mu} = D\tilde{u}^{(p)}_{\tilde{\eta},s} \in \mathcal{L}^{(p+1)}(\mathcal{RCR}_c(s), \mathcal{PC}^{\eta,\mu})$ given by the unique solution $w^{(p+1)}$ of the equation

$$w^{(p+1)} = \mathcal{K}^\mu_s \circ \tilde{R}^{(1)}_\delta(\tilde{u}_{\tilde{\eta},s}(\varphi))w^{(p+1)} + H^{(p+1)}_{\mu/(p+1)}(\varphi) \tag{59}$$

where $H^{(p+1)}_{\mu/(p+1)}(\varphi) = \mathcal{K}^\mu_s \circ \tilde{R}^{(2)}_\delta(\tilde{u}_{\tilde{\eta},s}(\varphi))(\tilde{u}^{(p)}_{\tilde{\eta},s}(\varphi), \tilde{u}^{(1)}_{\tilde{\eta},s}(\varphi)) + DH^{(p)}_{\mu/p}(\varphi)$. Similar arguments to the proof of the case $k = 1$ show that the fixed point $w^{(p+1)}$ is also contained in $\mathcal{L}^{(p+1)}(\mathcal{RCR}_c(s), \mathcal{PC}^{\tilde{\eta}(p+1),s})$, and the proof is complete. □

Corollary 8.2.1.1. *$\mathcal{C} : \mathcal{RCR}_c \to \mathcal{RCR}$ is C^k and tangent at the origin to the center bundle $\mathcal{RCR}_c$. More precisely, $\mathcal{C}(t, \cdot) : \mathcal{RCR}_c(t) \to \mathcal{RCR}$ is C^k and $D\mathcal{C}(t, 0)\phi = \phi$ for all $\phi \in \mathcal{RCR}_c(t)$.*

Proof. Let $\tilde{\eta}, \eta$ be as in the proof of Theorem 8.2.1. Define the evaluation map $ev_t : \mathcal{PC}^\eta \to \mathcal{RCR}$ by $ev_t(f) = f(t)$. Since we can decompose the center manifold as

$$\mathcal{C}(t, \phi) = ev_t(\tilde{u}_t(\phi)) = ev_t(\mathcal{J}^{\eta\tilde{\eta}}_t \tilde{u}_t(\phi)),$$

boundedness of the linear evaluation map on the space $\mathcal{PC}^{\eta,t}$ then implies the C^k smoothness of $\mathcal{C}(t, \cdot)$. To obtain the tangent property, we remark that

$$D\mathcal{C}(t, 0)\phi = ev_t \left(D \left(\mathcal{J}^{\eta\tilde{\eta}}_t \circ \tilde{u}_t(0) \right) \phi \right) = ev_t \left(\tilde{u}^{(1)}_{\eta,t}(0)\phi \right).$$

From equation (57) and Theorem 5.4.2, we obtain $\tilde{u}^{(1)}_{\eta,t}(0) = U(\cdot, t)$, from which it follows that $D\mathcal{C}(t, 0)\phi = \phi$, as claimed. □

As a secondary corollary, we can prove that each derivative of the center manifold is uniformly Lipschitz continuous. The proof is similar to that of Corollary 5.4.1.1 if one takes into account the representation of the derivatives $\tilde{u}^{(p)}_{\tilde{\eta},s}$ as solutions of the fixed-point equations (59), whose right-hand side is a contraction with Lipschitz constant independent of s.

Corollary 8.2.1.2. *For each $p \in \{1, \ldots, k\}$, there exists a constant $L(p) > 0$ such that the center manifold satisfies $||D^pC(t, \phi) - D^pC(t, \psi)|| \le L(p)||\phi - \psi||$ for all $t \in \mathbb{R}$ and $\phi, \psi \in \mathcal{RCR}_c(t)$.*

From Remark 8.0.1 we readily obtain the smoothness of the center manifold in the case where the semilinear equation is periodic. In particular, in such a situation many some of the assumptions H.1–H.9 are satisfied automatically and can be ignored.

Corollary 8.2.1.3. *Suppose the semilinear equation* (29)–(30) *satisfies the following conditions.*

- P.1 *The equation is periodic with period T and c impulses per period. That is, $A(t + T) = A(t)$ and $f(t + T, \cdot) = f(t)$ for all $t \in \mathbb{R}$, and $B_{k+c} = B_k$, $g_{k+c} = g_k$ and $\tau_{k+c} = \tau_k + T$ for all $k \in \mathbb{Z}$.*
- P.2 *Conditions H.1–H.3 and H.5–H.6 are satisfied.*

Then, the conclusions of Corollary 8.2.1.1 and Corollary 8.2.1.2 hold.

Finally, when the impulsive differential equation has a linear part with no delay,the center manifold is smooth.

Corollary 8.2.1.4. *Consider the semilinear impulsive system*

$$\begin{aligned} \dot{x} &= A(t)x + f(t, x_t), \quad t \neq \tau_k \\ \Delta x &= B_k x + g_k(x_{t^-}), \quad t = \tau_k. \end{aligned}$$

If conditions H.1–H.3, H.5–H.8 are satisfied and, additionally, the linearization is spectrally separated when considered as a finite-dimensional impulsive system with phase space $\mathbb{R}^n$, then this system posessses a C^k center manifold.

Proof. Under the spectral separation assumption, there exist projectors P_s, P_c, P_u onto the center, stable and unstable subspaces respectively of the linearization, considered as a finite-dimensional system. Let $X(t,s)$ be its corresponding Cauchy matrix, and define the projectors on $\mathcal{RCR}_c$,

$$\mathcal{P}_i(t)\phi(\theta) = X(t+\theta, t)P_i(t)\phi(0),$$

for $i \in \{c, u\}$, and $\mathcal{P}_s = I - \mathcal{P}_c - \mathcal{P}_u$. It can be verified that, considered as an impulsive delay differential equation with phase space $\mathcal{RCR}$, the the linearization is spectrally separated, with $\mathcal{P}_i$ the projectors onto the invariant fiber bundles.

Let $||X(t,s)P_c(s)|| \leq Ke^{\epsilon|t-s|}$ by spectral separation. The evolution family restricted to the center fiber bundle admits the representation

$$U_c(t,s)\phi(\theta) = X(t+\theta, s)\phi(0),$$

so that we can write $U_c(t,0) = \alpha(t)e^{Qt}$ with $\alpha(t) = X(t+\cdot, 0)e^{-\epsilon It}$ and $Q = \epsilon I$. It follows that $||\alpha(t)|| \leq K$, so H.9 is satisfied. The result follows by Corollary 8.2.1.1. □

8.3 A periodic center manifold

In this section we will prove that the center manifold is itself a periodic function, provided the conditions P.1–P.2 of Corollary 8.2.1.3 are satisfied. We begin with a preparatory lemma.

Lemma 8.3.1. *Define the operator $N_s : \mathcal{RCR}_c(s) \to \mathcal{RCR}_c(s)$ by*

$$N_s(\phi) = P_c(s)S(s+jT, s)\mathcal{C}(s, \phi).$$

This operator is well-defined and invertible in a neighbourhood of $0 \in \mathcal{RCR}_c(s)$. Moreover, the neighbourhood can be written $U \cap \mathcal{RCR}_c(s)$ for some open neighbourhood $U \subset \mathcal{RCR}$ of $0 \in \mathcal{RCR}$, independent of s.

Proof. To show that N_s is invertible in a neighbourhood of the origin we will use the inverse function theorem. The Fréchet derivative of N_s at 0 is given by

$$\begin{aligned} DN_s(0)\phi &= P_c(s) \circ DS(s+jT, s)(0) \circ D\mathcal{C}(s,0)\phi \\ &= P_c(s+jT) \circ U(s+jT, s)\phi \\ &= U_c(s+jT, s)\phi, \end{aligned}$$

where we used Corollary 8.2.1.1 to calculate $D\mathcal{C}(s,0)$ and Theorem 5.1.1 to calculate $DS(s+jT,s)(0)$. Since $U(s+jT,s)$ is an isomorphism (Theorem 7.2.1) of $\mathcal{RCR}_c(s)$ with $\mathcal{RCR}_c(s+jT) = \mathcal{RCR}_c(s)$, we obtain the claimed local invertibility.

To show that the neighbourhood may be written as claimed, we notice that $DN_s(x)$ is uniformly convergent (in the variable s) as $x \to 0$ to $DN_s(0)$. Indeed, we have the estimate

$$||DN_s(x) - DN_s(0)|| \leq ||U_c(s+jT, s)P_c(s)|| \cdot ||D\mathcal{C}(s,x) - D\mathcal{C}(s,0)||,$$

and the Lipschitz property of Corollary 8.2.1.2 together with uniform boundedness of the projector $P_c(s)$ and center monodromy operator $U_c(s+jT, s)$ grants the uniform convergence as $x \to 0$. As a consequence, the implicit function may be defined on a neighbourhood that does not depend on s. □

Theorem 8.3.1. *There exists $\delta > 0$ such that $\mathcal{C}(s+jT, \phi) = \mathcal{C}(s, \phi)$ for all $s \in \mathbb{R}$ whenever $||\phi|| \leq \delta$.*

Proof. By Lemma 8.3.1, there exists $\delta > 0$ such that if $||\phi|| \leq \delta$, we can write $\phi = N_s(\psi)$ for some $\psi \in \mathcal{RCR}_c(s)$. By Theorem 5.4.2 and the periodicity condition P.1,

$$\begin{aligned} \mathcal{C}(s+jT, \phi) &= \mathcal{C}(s+jT, N_s(\psi)) \\ &= \mathcal{C}(s+jT, P_c(s+jT)S(s+jT, s)\mathcal{C}(s,\psi)) \\ &= S(s+jT, s)\mathcal{C}(s,\psi) \\ &= S(s, s-jT)\mathcal{C}(s,\psi) \\ &= \mathcal{C}(s, P_c(s)S(s, s-jT)\mathcal{C}(s,\psi)) \\ &= \mathcal{C}(s, P_c(s)S(s+jT, s)\mathcal{C}(s,\psi)) \\ &= \mathcal{C}(s, N_s(\psi)) = \mathcal{C}(s,\phi), \end{aligned}$$

where the identity $S(s+jT,s)=S(s,s-jT)$ follows due to periodicity and uniqueness of solutions.

□

9 Examples

This section contains three examples illustrating our main results. The first, in Section 9.1, is an explicit derivation of the dynamics on the center manifold for a class of scalar nonautonomous impulsive delay differential equations. Section 9.2 considers a particular case of such a system: the delayed logistic equation with linear impulsive harvesting. Finally, in Section 9.3 we consider an abstract mathematical model of infectious disease with pulse vaccination, incubation period and temporary immunity.

9.1 Reduction principle for a scalar nonautonomous system

In this section we compute to leading order the reduced vector field on the center manifold associated to a class of scalar impulsive delay differential equations. Our prototype system is of the form

$$\dot{x} = a(t)x + f(t,x_t), \qquad t \neq \tau_k \tag{60}$$

$$\Delta x = b_k x + g_k(x_{t^-}), \qquad t = \tau_k. \tag{61}$$

Here we do not assume any periodicity conditions, but we will assume that $f(t,\cdot)$ and $g_k(\cdot)$ are C^2 with $f(t,0)=g_k(0)=0$ and $D_2 f(t,0)=D_1 g_k(0)=0$, so that the linearization of the above system takes the form of the finite-dimensional system

$$\dot{y} = a(t)y, \qquad t \neq \tau_k \tag{62}$$

$$\Delta y = b_k y, \qquad t = \tau_k. \tag{63}$$

Theorem 9.1.1. *Let the nonlinearities f and g be C^2 in $\mathcal{RCR}$ and satisfy the conditions H.3, H.5, H.6 and H.8, and let the sequence of impulses satisfy H.7. Suppose $b_k > -1$ for all $k\in\mathbb{Z}$ and the linear term $a:\mathbb{R}\to\mathbb{R}$ is locally integrable. Define the function $\kappa:\mathbb{R}^2\to\mathbb{R}$ by*

$$\kappa(t,s) = \int_s^t a(\mu)d\mu + \sum_s^t \log(1+b_i)d\tau_i. \tag{64}$$

and suppose $\gamma = \limsup_{|t-s|\to\infty}|\kappa(t,s)| < \infty$. Then, system (60)–(61) has a one-dimensional C^2 center manifold, the unstable subspace is trivial, and the dynamics restricted to the center manifold are given to leading order by the scalar impulsive differential equation

$$\dot{z} = a(t)z + \mathbf{f}(t)z^2 + O(z^3), \qquad t \neq \tau_k \tag{65}$$

$$\Delta z = b_k z + \mathbf{g}_k z^2 + O(z^3), \qquad t = \tau_k, \tag{66}$$

with quadratic coefficients $\mathbf{f}(t) = \frac{1}{2}D_2^2 f(t,0)\mathbf{X}(t)^2$ and $\mathbf{g}_k = \frac{1}{2}D_1^2 g_k(0)\mathbf{X}(\tau_k^-)^2$, where $\mathbf{X}\in\mathcal{PC}$ is defined by $\mathbf{X}(t)(\theta) = \exp(\kappa(t+\theta,t))$ and the limit $\mathbf{X}(\tau_k^-)$ is the usual pointwise (at $\theta=0$) limit.

Proof. The condtion $\gamma<\infty$ implies that (62)–(63), considered as a finite-dimensional impulsive system, is spectrally separated with projector $P_c = I$, with evolution family $X(t,s)$. By Corollary 8.2.1.4 we obtain the C^2 smoothness of the center manifold. By the proof of the aforementioned corollary, we have the representation

$$U_c(t,s)\phi(\theta) = X(t+\theta,s)\phi(0), \qquad \mathcal{P}_c(t)\phi = X(t+\cdot,t)\phi(0)$$

for the evolution family in $\mathcal{RCR}_c$, and the associated projector.

To compute the dynamics on the center manifold, we use Theorem 6.1.1 keeping in mind $\mathcal{RCR}_c(t) = \text{span}\{X(t+\cdot,t)\}$. Equation (45)–(46) implies that the dynamics on the center manifold are given to leading order by the impulsive partial differential equation

$$d^+w(t) = \mathcal{A}(t)w(t) + X(t+\cdot,t)\frac{1}{2}\chi_0 D_2^2 f(t,0)w(t)^2 + O(w^3)$$

$$\Delta w(\tau_k) = \mathcal{B}_k w(\tau_k^-) + X(\tau_k+\cdot,\tau_k)\frac{1}{2}\chi_0 D_1^2 g_k(0)w(\tau_k^-)^2 + O(w^3)$$

with the linear differential- and difference operators

$$\mathcal{A}(t)\phi(\theta) = \chi_0(\theta)a(t)\phi(0) + (1-\chi_0(\theta))\partial_\theta\phi(t)(\theta), \qquad \mathcal{B}_k\phi(\theta) = \chi_0(\theta)b_k\phi(0) + (1-\chi_0(\theta))[\phi(\theta)-\phi(\theta^-)],$$

and the expansion being justified by Corollary 8.2.1.2. Introducing coordinates $w(t)(\theta) = X(t+\theta,t)z(t)$ for some $z(t)\in\mathbb{R}$, we remark that by setting $\theta=0$ we obtain the identity $z(t) = w(t)(0)$, from which it follows that $z\in\mathcal{RCR}^1(\mathbb{R},\mathbb{R})$ satisfies the impulsive differential equation (65)–(66) – this last may be obtained by evaluating the above impulsive partial differential equation at $\theta=0$ and exploiting the bilinearity of the second differentials and the representation $w(t) = \mathbf{X}(t)z(t)$. □

9.2 Dynamics of a delayed logistic impulsive harvesting model

As an application of Theorem 9.1.1, let us consider the delay logistic growth model for a single species, subject to linear harvesting:

$$\begin{aligned} \dot{x} &= r(t)x\left(1-\frac{x(t-\tau)}{K(t)}\right), && t\neq t_k \\ \Delta x &= -\rho x(t_k^-), && t=t_k. \end{aligned}$$

Here, the functions $r(t)$ and $K(t)$ are taken to be positive. Assuming the sequence t_k is T periodic with c impulses per period and the intrinsic growth rate $r(t)$ is also T periodic, the linearization at the origin for the above system has a one-dimensional center fiber bundle whenever the harvesting effort $\rho\in(0,1)$ satisfies the equation

$$(1-\rho)^c\exp\left(\int_0^T r(s)ds\right) = 1. \tag{67}$$

Much of the results of the aforementioned section apply if the parameter is taken as an additional state variable, so that one may study bifurcaions near the critical parameter ρ where equation (67) is satisfied. We will not do this here, however, as such computational aspects and bifurcation theory are not in the scope of this article.

When (67) is satisfied, applying Theorem 9.1.1 yields the dynamics on the center manifold described by the impulsive differential equation

$$\begin{aligned} \dot{z} &= r(t)z - \frac{r(t)}{K(t)}\exp\left(-\int_{t-\tau}^t a(s)ds - \sum_{t-\tau<t_k\leq t}\log(1-p)\right)z^2 + O(z^3), && t\neq t_k \\ \Delta z &= -\rho z(t_k^-) + O(z^3), && t=t_k. \end{aligned}$$

9.3 Dynamics near disease-free solutions in a susceptible-infected model with pulse vaccination, incubation period and temporary immunity

Consider the following model of disease transmission

$$\dot{S} = -g(t,I(t))S(t) + W(R_t) \qquad t\neq t_k \tag{68}$$

$$\dot{I} = g(t,I(t-\tau))S(t-\tau) - hI(t), \qquad t\neq t_k \tag{69}$$

$$\dot{R} = hI(t) - W(R_t), \qquad t\neq t_k \tag{70}$$

$$\Delta S = -\rho S(t_k^-), \qquad t=t_k \tag{71}$$

$$\Delta R = \rho S(t_k^-), \qquad t=t_k. \tag{72}$$

S, I and R denote the susceptible, infected and immune/vaccinated cohorts, respectively. The function $g(t,I)$ is the per capita infection rate and $\tau>0$ is the incubation period. Infected individuals clear their infection at rate h and are then immune, before subsequently losing their immunity rate $W(R_t)$ for some functional $W:\mathcal{RCR}([-r,0],\mathbb{R})\to\mathbb{R}$. A proportion $\rho\in(0,1)$ of susceptible individuals are successfully vaccinated at times t_k, and these individuals subsequently obtain temporarity immunity subject to waning as if they had just cleared their infection. We make the following assumptions.

SIR.1 The per capita infection rate $g(t,\cdot)$ is C^k in its second variable, uniformly in t, periodic in its first variable and satisfies $g(t,0)=0$ and $g(t,I)>0$ for $I>0$.

SIR.2 The waning rate $W:\mathcal{RCR}([-r,0],\mathbb{R})\to\mathbb{R}$ is C^k smooth, globally Lipschitz continuous, and satisfies $W(R)\geq 0$ for $R\geq 0$ and $W(0)=0$.

SIR.3 The sequence of vaccination times t_k is periodic with period T and c impulses per period.

The first step in our analysis will be to study the invariant disease-free subsapce – that is, the region of phase space where $I\equiv 0$ – and identify a one-parameter family of bounded (periodic) solutions. We will then translate this family of solutions to the origin and obtain an impulsive delay differential equation that posesses a C^k center manifold on which all bounded solutions must reside.

On the infection-free subspace the dynamics are given by

$$\dot{S}=W(R_t) \qquad t\neq t_k \tag{73}$$

$$\dot{R}=-W(R_t), \qquad t\neq t_k \tag{74}$$

$$\Delta S=-\rho S(t_k^-), \qquad t=t_k \tag{75}$$

$$\Delta R=\rho S(t_k^-), \qquad t=t_k. \tag{76}$$

Lemma 9.3.1. *Equation* (73)–(76) *admits a one-parameter family of periodic solutions* (S^N,R^N) *satisfying the equation* $S^N+R^N=N$ *for* $N>0$*. If the waning rate has a sufficiently small Lipschitz constant, there is only one such family of periodic solutions and it is globally attractive.*

Proof. It is easily verified that the positive orthant of (73)–(76)is positively invariant and that the lines $S+R=N$ are invariant for fixed $N>0$. Taking N as a parameter, we may write

$$\begin{aligned}\dot{S}&=W(N-S_t), \quad t\neq t_k\\ \Delta S&=-\rho S(t_k^-), \qquad t=t_k.\end{aligned} \tag{77}$$

From the above discussion, the closed convex set $K=\{\phi\in\mathcal{RCR}:0\leq\phi\leq N\}$ is positively invariant under the process $S(t,s)$ associated to (77). Without loss of generality, assume $T\geq r$. The monodromy operator $M:K\to K$ with $M\phi=S(T,0)\phi$ is therefore well-defined. M is continuous by Theorem 5.1.1 and from the Lipschitz assumption on the waning rate W, it is easy to verify that $M(K)$ is uniformly bounded and quasiequicontinuous; see the similar proof of Lemma 7.1.1. Thus, $M(K)$ is precompact [2], and the Schauder fixed point theorem implies the existence of a fixed point X lying in K. Define $Y=N-X\in K$. The invariance of $S+R=N$ for (73)–(76) and X being a fixed point of the monodromy operator allows us to conclude that the solution (S^N,R^N) through (X,Y) is a periodic solution satisfying $S^N+R^N=N$.

To obtain uniqueness for waning rates with small Lipschitz constants, we note that $S(t,0)$ satisfies the integral equation

$$S(t,0)\phi=\int_0^t U(t,s)W(N-S(s,0)\phi))ds,$$

where for $t\geq 0$, $U(t,0)\phi(\theta)=\phi(0)\prod_{0<t_k\leq t}(1-p)$ satisfies an exponential estimate of the form $||U(t,0)||\leq e^{-ct}$ for some $c>0$. If W has Lipschitz constant γ, then denoting $Z(t)=S(t,0)\phi-S(t,0)\psi$ for given $\phi,\psi\in\mathcal{RCR}$, we obtain the integral estimate

$$e^{ct}||Z(t)||\leq e^{ct}||\phi-\psi||+\int_0^t\gamma e^{cs}||Z(s)||ds,$$

which together with the Gronwall inequality implies the Lipschitzian estimate

$$||S(T,0)\phi-S(T,0)\psi|||\leq\frac{\gamma}{c+\gamma}\left(e^{\gamma T}-e^{-cT}\right)||\phi-\psi||.$$

The monodromy operator $S(T,0):\mathcal{RCR}\to\mathcal{RCR}$ is a contraction when γ is small enough. Iterating from $0<S_0\in\mathcal{RCR}$, the fixed point is positive due to the positive invariance of the convex set K. A similar construction to the above yields the unique positive periodic solution satisfying $S^N+R^N=N$. Global attraction follows because $S(T,0)$ is a contraction. □

Next, we introduce the change of variables $S = z_1 + S^N$ and $R = z_2 + R^N$ to translate the positive disease-free periodic solution of Lemma 9.3.1 to the origin. Doing so, writing the equation in an explicitly semilinear form and denoting $I = z_3$, we obtain

$$\begin{aligned}
\dot{z}(t) &= \begin{bmatrix} -\partial_2 g(t,0)\chi_0 & 0 & DW(R_t^N) \\ \partial_2 g(t-r,0)\chi_{-\tau} & -h\chi_0 & 0 \\ 0 & h\chi_0 & -DW(R_t^N) \end{bmatrix} z_t + \begin{bmatrix} -\mathbf{g}(t,0,z_t) + \mathbf{W}^N(t,z_t) \\ \mathbf{g}(t,-r,z_t) \\ -\mathbf{W}^N(t,z_t) \end{bmatrix}, & t \neq t_k \\
\Delta z_1(t) &= -\rho z_1(t_k^-), & t = t_k \\
\Delta z_3(t) &= \rho z_1(t_k^-), & t = t_k,
\end{aligned} \tag{78}$$

where $z_t = ((z_1)_t, (z_2)_t, (z_3)_t) \in \mathcal{RCR}([-r,0],\mathbb{R}^3)$ and we introduce auxiliary functions

$$\begin{aligned}
\mathbf{W}^N(t,\phi) &= W(\phi_3 + R_t^N) - W(R_t^N) - DW(R_t^N)\phi_3 \\
\mathbf{g}(t,\theta,\phi) &= g(t+\theta,\phi_2(\theta))\phi_1(\theta) - \partial_2 g(t+\theta,0)\phi_1(\theta).
\end{aligned}$$

Note that the origin $(0,0,0) \in \mathcal{RCR}$ is now an equilibrium point. The dimension of the center manifold of the semilinear system (78) is equal to that of the center fiber bundle of its linear part. Since the latter is finite-dimensional (by Theorem 7.2.1), we obtain the following description of the the bounded solutions of the original system due to Theorem 5.4.2 and Corollary 8.2.1.3.

Corollary 9.3.0.1. *Every solution of the SIR model* (68)–(72) *with a uniformly small infected component* $I \ll 1$ *and near-equilibrium disease-free components* $|S + R - N| \ll 1$ *for some constant* N*, is contained within a finite-dimensional global center manifold, up to translation by a particular disease-free periodic orbit of constant population size. The center manifold is* C^k *and periodic.*

Further analysis is necessary to determine the dimension of the center manifold or compute the dynamics restricted to it. Such investigations are beyond the scope of this article, as they necessitate a method to analytically compute the portion of the Floquet spectrum on the unit circle for linear periodic impulsive systems with delays.

10 Conclusion

We proved (Theorem 5.4.1) the existence of a Lipschitz continuous global center manifold for the impulsive delay differential equation (1) under the assumption of C^1 smoothness of the nonlinear portion of vector field and jump map, in addition to a spectral separation condition on the linearization and some technical requirements concerning the uniform boundedness in time for the nonlinearities in a neighbourhood of zero. The center manifold is defined through a function $\mathcal{C} : \mathcal{RCR}_c \to \mathcal{RCR}$ where $\mathcal{RCR}_c$ is the center fibre bundle associated to the linearization. Under the conditions of the theorem, the center manifold is in fact uniformly Lipschitz continuous (Corollary 5.4.1.1), in that the Lipschitz constant can be chosen independent of the fiber $\mathcal{RCR}_c(t)$. Under the same assumptions, we proved the a reduction principle in Section 6. The invariance of the center manifold was proven (Theorem 5.4.2), as was its attraction (Theorem 6.2.1). The reduced flow and integral equations on the center manifold were also derived (Section 6.1).

In order to establish the existence of the center manifold, it was necessary to derive a variation of constants formula (Theorem 4.2.1) associated to the inhomogeneous linear equation (4)–(5). The integral equation defining mild solutions of the nonlinear equation (1)–(2) was then defined in Section 5.1, where it was proven that the nomautonomous process associated to the aforementioned integral equation is smooth (Theorem 5.1.1).

To prove the smoothness of the center manifold (Corollary 8.2.1.1) and its tangency at the origin to the center fiber bundle $\mathcal{RCR}_c$, it was necessary to smooth the nonlinearities in a slightly different way. We also imposed the additional assumption H.9 that the flow on the center fiber bundle of the linearization admitted a decomposition as an exponential composed with a uniformly (in time) bounded linear operator. In this case, each derivative of $\mathcal{C}$ is Lipschitz continuous, with Lipschitz constant independent (Corollary 8.2.1.2) of the fiber $\mathcal{RCR}_c(t)$.

Many of the technical requirements needed for our proof of the existence and smoothness of center manifolds are automatically satisfied when the linear part of the system in question is does not feature delays

(Corollary 8.2.1.4), although this class of models is fairly small. Since so many nonlinear models feature periodicity, we made a concerted effort to develop enough of the Floquet theory for periodic linear impulsive delay differential equations (Section 7) to guarantee the existence and smoothness of center maoifolds under the assumption that the linear part is periodic (Corollary 8.2.1.3). In this case, we obtain the expected result that the center manifold is itself a periodic function (Theorem 8.3.1) provided the entire quasilinear system is periodic.

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References

References

[1] B. Aulbach. The fundamental existence theorem on invariant fiber bundles. *Journal of Difference Equations and Applications*, 3(5-6):267–312, 1998.

[2] D.D. Bainov and P.S. Simeonov. *Impulsive Differential Equations: Periodic Solutions and Applications*. Longman Scientific & Technical, 1993.

[3] L. Barreira, M. Fan, C. Valls, and J. Zhang. Parameter dependence of stable manifolds for delay equations with polynomial dichotomies. *Journal of Dynamics and Differential Equations*, 24(1):101–118, Mar 2012.

[4] L. Barreira and C. Valls. Center manifolds for impulsive equations under nonuniform hyperbolicity. *Nonlinear Analysis: Theory, Methods & Applications*, 74(5):1616 – 1627, 2011.

[5] C. Chicone and Y. Latushkin. Center manifolds for infinite dimensional nonautonomous differential equations. *Journal of Differential Equations*, 141(2):356 – 399, 1997.

[6] S.-N.' Chow and K. Lu. Invariant manifolds for flows in banach spaces. *Journal of Differential Equations*, 74(2):285 – 317, 1988.

[7] K.E.M. Church and X. Liu. Bifurcation analysis and application for impulsive systems with delayed impulses. *to appear in International Journal of Bifurcation and Chaos*, December 2017.

[8] G. Da Prato and A. Lunardi. Stability, instability and center manifold theorem for fully nonlinear autonomous parabolic equations in banach space. *Archive for Rational Mechanics and Analysis*, 101(2):115–141, Jun 1988.

[9] S.A. van Gils, S.M. Verduyn Lunel, O. Dieckmann, H.-O. Walther. *Delay Equations: Functional-, Complex-, and Nonlinear Analysis*. Springer-Verlag New York, 1995.

[10] M. Federson and S. Schwabik. Generalized ODEs approach to impulsive retarded differential equations. *Differential and Integral Equations*, 19(11):1201–1234, 2006.

[11] H. Guo and L. Chen. The effects of impulsive harvest on a predator–prey system with distributed time delay. *Communications in Nonlinear Science and Numerical Simulation*, 14(5):2301 – 2309, 2009.

[12] J. Hale. *Functional Differential Equations*. Applied Mathematical Sciences. Springer-Verlag New York, 1971.

[13] M. Haragus and G. Iooss. *Local Bifurcations, Center Manifolds, and Normal Forms in Infinite-Dimensional Dynamical Systems*. Springer-Verlag London, 2011.

[14] M.L. Hbid and K. Ezzinbi. Variation of constant formula for delay differential equations. In O. Arino, M.L. Hbid, and E. Ait Dads, editors, *Delay Differential Equations and Applications*, pages 143–159. Springer Netherlands, Dordrecht, 2006.

[15] W. He, F. Qian, and J. Cao. Pinning-controlled synchronization of delayed neural networks with distributed-delay coupling via impulsive control. *Neural Networks*, 85(Supplement C):1 – 9, 2017.

[16] M. Hirsch and C.C. Pugh. Stable manifolds for hyperbolic sets. *Bull. Amer. Math. Soc.*, 75:149–152, 1969.

[17] C.S. Hönig. *Volterra Stieltjes-Integral Equations.* North-Holland Publishing Company, 1975.

[18] H. J. Hupkes and S. M. Verduyn Lunel. Center manifold theory for functional differential equations of mixed type. *Journal of Dynamics and Differential Equations*, 19(2):497–560, Jun 2007.

[19] H.J. Hupkes and S.M. Verduyn Lunel. Center manifolds for periodic functional differential equations of mixed type. *Journal of Differential Equations*, 245(6):1526–1565, sep 2008.

[20] T. Kato. *Perturbation Theory for Linear Operators.* Springer-Verlag Berlin Heidelberg, 2 edition, 1995.

[21] P.E. Kloeden and M. Rasmussen. *Nonautonomous Dynamical Systems.* American Mathematical Society, 2011.

[22] T. Krisztin. C1-smoothness of center manifolds for differential equations with statedependent delay. In H. Brunner et al., editor, *Nonlinear Dynamics and Evolution Equations*, number 48, pages 213 – 226. Fields Inst. Commun., Amer. Math. Soc., Providence, 2006.

[23] K. Musial. Pettis integration. *Suppl. Rend. Circ. Mat. Palermo, Ser. II*, 10:133–142, 1985.

[24] L. Pan. Parameter dependence of stable invariant manifolds for delay differential equations under (μ, ν)-dichotomies. *Journal of Mathematics*, 2014(989526, 9 pages.).

[25] Y. Pei, S. Liu, C. Li, and L. Chen. The dynamics of an impulsive delay si model with variable coefficients. *Applied Mathematical Modelling*, 33(6):2766 – 2776, 2009.

[26] M. Tvrdý. Regulated functions and the perron-stieltjes integral. *Časopis Pro Pěstování Matematiky*, 114(2):187–209, 1989.

[27] A Vanderbauwhede and S.A Van Gils. Center manifolds and contractions on a scale of banach spaces. *Journal of Functional Analysis*, 72(2):209 – 224, 1987.

[28] A. Vanderbauwhede and G. Iooss. *Center Manifold Theory in Infinite Dimensions*, pages 125–163. Springer Berlin Heidelberg, Berlin, Heidelberg, 1992.

[29] R. Veltz and O. Faugeras. A center manifold result for delayed neural fields equations. *SIAM Journal on Mathematical Analysis*, 45(3):1527–1562, 2013.

[30] L. Yang and S. Zhong. Dynamics of an impulsive diffusive ecological model with distributed delay and additive allee effect. *Journal of Applied Mathematics and Computing*, 48(1):1–23, Jun 2015.

[31] T. Zhang, W. Ma, and X. Meng. Global dynamics of a delayed chemostat model with harvest by impulsive flocculant input. *Advances in Difference Equations*, 2017(1):115, Apr 2017.

[32] M. Zhao, X. Wang, H. Yu, and J. Zhu. Dynamics of an ecological model with impulsive control strategy and distributed time delay. *Mathematics and Computers in Simulation*, 82(8):1432 – 1444, 2012.

Appendix

This section contains a lemma on smoothness of compositions of mappings on scales of Banach spaces. The lemma and several modifications thereof appear in [9, 27, 19] among others. To keep the article mostly self-contained, we include it here.

Lemma 10.0.1 (Lemma 6.7 [9])**.** *Let Y_0, Y, Y_1 be Banach spaces with continuous embeddings $J_0 : Y_0 \hookrightarrow Y$ and $J : Y \hookrightarrow Y_1$ and let Λ be another Banach space. Consider the fixed-point equation $y = f(y, \lambda)$ for $f : Y \times \Lambda \to Y$. Suppose the following conditions hold.*

b1) The function $g : Y_0 \times \Lambda \to Y_1$ defined by $(y_0, \lambda) \mapsto g(y_0, \lambda) = Jf(J_0 y_0, \lambda)$ is of class C^1 and there exist mappings

$$f^{(1)} : J_0 Y_0 \times \Lambda \to \mathcal{L}(Y),$$
$$f_1^{(1)} : J_0 Y_0 \times \Lambda \to \mathcal{L}(Y_1)$$

such that $D_1 g(y_0, \lambda)\xi = Jf^{(1)}(J_0 y_0, \lambda) J_0 \xi$ for all $(y_0, \lambda, \xi) \in Y_0 \times \Lambda \times Y_0$ and $Jf^{(1)}(J_0 y_0, \lambda) y = f_1^{(1)}(J_0 y_0, \lambda) Jy$ for all $(y_0, \lambda, y) \in Y_0 \times \Lambda \times Y$.

b2) There exists $\kappa \in [0, 1)$ such that $f(\cdot, \lambda) : Y \to Y$ is Lipschitz continuous with Lipschitz constant κ, and each of $f^{(1)}(\cdot, \lambda)$ and $f_1^{(1)}(\cdot, \lambda)$ is uniformly bounded by κ.

b3) Under the previous condition, the unique fixed point $\Psi : \Lambda \to Y$ satisfying the equation $\Psi(\lambda) = f(\Psi(\lambda), \lambda)$ itself satisfies $\Psi = J_0 \circ \Phi$ for some continuous $\Phi : \Lambda \to Y_0$.

b4) $f_0 : Y_0 \times \Lambda \to Y$ defined by $(y_0, \lambda) \mapsto f_0(y_0, \lambda) = f(J_0 y_0, \lambda)$ has a continuous partial derivative

$$D_2 f : Y_0 \times \Lambda \to \mathcal{L}(\Lambda, Y)$$

b5) The mapping $(y, \lambda) \mapsto J \circ f^{(1)}(J_0 y, \lambda)$ from $Y_0 \times \Lambda$ into $\mathcal{L}(Y, Y_1)$ is continuous.

Then, the mapping $J \circ \Psi$ is of class C^1 and $D(J \circ \Psi)(\lambda) = J \circ \mathcal{A}(\lambda)$ for all $\lambda \in \Lambda$, where $\mathcal{A} = \mathcal{A}(\lambda)$ is the unique solution of the fixed point equation $A = f^{(1)}(\Psi(\lambda), \lambda)A + D_2 f_0(\Phi(\lambda), \lambda)$.