Entangling qubit registers via many-body states of ultracold atoms

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Inspired by the experimental measurement of the Rényi entanglement entropy in a lattice of ultracold atoms by Islam *et al.* [Nature (London) **528**, 77 (2015)], we propose a method to entangle two spatially separated qubits using the quantum many-body state as a resource. Through local operations accessible in an experiment, entanglement is transferred to a qubit register from atoms at the ends of a one-dimensional chain. We compute the operational entanglement, which bounds the entanglement physically transferable from the many-body resource to the register, and discuss a protocol for its experimental measurement. Finally, we explore measures for the amount of entanglement available in the register after transfer, suitable for use in quantum information applications.

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I. INTRODUCTION

Islam *et al.* [1] have performed a measurement of the Rényi entanglement entropy in a one-dimensional optical lattice of ^{87}Rb atoms by exploiting a many-body analog of the Hong-Ou-Mandel [2] photon interference effect. After interfering two proximate copies of an L-site lattice using the atomic control of a quantum gas microscope [3], a measurement of the parity of the site-resolved particle occupation number provides access to the state overlap of the two copies. If the initial copies are identical, this gives the purity of the state [4]. Hence, if a globally pure state is partitioned into spatial subregions, the many-body interference or parity measurement protocol localized to a subregion yields the Rényi entropy, a measure of entanglement between subregions [5]. This provides an experimental probe of a remarkable feature of quantum mechanics with no classical analog: Information may be encoded in a composite system in such a way that it is inaccessible from independent measurements of its component parts.

The advantage of measuring the Rényi entropy as in Ref. [1] is that it encodes the entanglement between subsystems in a scalar quantity that can be accessed through the expectation values of local operators [4]. This is in contrast to other entanglement measures calculated directly from the full density matrix, which is generally inaccessible in experiments without using full state tomography [6]. In particular, there is currently no scalable scheme for its reconstruction for N interacting itinerant particles. This fact makes the two-copy Rényi entropy $S_2(A) = -\ln_2(\mathrm{Tr}\,\rho_A^2)$ particularly well suited for exploration in a quantum many-body system bipartitioned into a spatial region A and its complement $\bar{A}$.

S_2 has proved fruitful for the general characterization of many-body phases and quantum phase transitions, e.g., through the exploration of its scaling with subsystem size [7]. Additionally, given that entanglement is a physical resource that can be used for quantum information processing [8,9], it is natural to ask whether this many-body entanglement can be harnessed for these tasks [10–13]. One route to exploit entanglement between spatial regions of a many-body state for quantum information processing is to transfer entanglement of the many-body system to an external register of localized qubits using local operations; in this way the many-body state acts as an entanglement reservoir for the quantum register. To quantify the entanglement that is usable, one must take into account physical restrictions that limit the amount of entanglement that may be transferred to the register. For itinerant particles, a superselection rule (SSR) due to particle number conservation provides one key limitation [14]. Further restrictions are imposed if one wants to entangle spatially separated qubits with only local operations on the many-body system [15].

In this paper we propose the general scheme shown in Fig. 1 and present an experimental protocol, using the basic capabilities of Islam *et al.* [1], to transfer some of the entanglement in a many-body state of ultracold atoms to two spatially separated qubits composing an external quantum register. We present this protocol within the context of the Islam experiment, but the general concept of entangling qubits using many-body states is relevant to many other systems [10–13]. We emphasize the importance of the *operational entanglement* as a bound on the transferable entanglement and discuss its measurement in the many-body state. The demonstration of this transfer would be proof of principle confirmation that a quantum register can be entangled in current experimental apparatuses for ultracold atoms.

II. ENTANGLEMENT IN THE BOSE-HUBBARD MODEL

The ^{87}Rb atoms of the Islam experiment are confined to move in a deep one-dimensional optical lattice. In their weakly interacting regime, the low-energy dynamics of the atoms are accurately governed by the lattice Bose-Hubbard Hamiltonian

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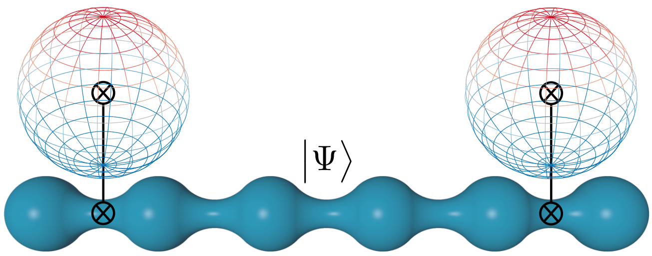

FIG. 1. Schematic setup whereby entanglement can be transferred from a quantum many-body state $|\Psi\rangle$ to a quantum register composed of two spatially separated qubits (Bloch spheres).

with N particles on L sites:

$$H = -J\sum_{i=1}^{L-1}(b_i^\dagger b_{i+1} + \text{H.c.}) + \frac{U}{2}\sum_{i=1}^{L} n_i(n_i - 1), \quad (1)$$

where $b_i^\dagger$ (b_i) creates (annihilates) a boson and $n_i = b_i^\dagger b_i$ counts the number of atoms on site i. In addition, J sets the rate of tunneling between sites, while U parametrizes the strength of the on-site repulsion between atoms. In an experiment, the interaction strength between ^{87}Rb atoms is fixed by their s-wave scattering length, while J can be tuned by manipulating the height of the optical lattice. In the thermodynamic limit at unit filling ($N = L$), Eq. (1) exhibits two distinct phases: a Mott insulator for $U/J \gg 1$ and a superfluid for $U/J \ll 1$, both of which are observed experimentally. A quantum phase transition separates these two phases at $(U/J)_c \approx 3.3$ [16–19].

The spatially delocalized nature of particles in the superfluid phase suggests that it should be significantly more entangled under a spatial bipartition than a Mott insulator with localized particles. This is manifest as an increase in S_2 accompanying the onset of delocalization at $U/J \sim O(1)$ observed in the experiment for $N = 4$ atoms [1]. The same experimental capabilities that allow for the measurement of the entanglement in an optical lattice can also be used to transfer entanglement to spatially separated qubits, which can be employed as a quantum register for information processing tasks via logic gates. This entanglement transfer procedure is limited by the SSR that forbids the creation of a coherent superposition of states with different local particle number [14]. Thus entanglement that arises *solely* due to particle fluctuations between subregions is not physically transferable to a register without a global phase reference [20].

A. Operational entanglement

To address this issue, Wiseman and Vaccaro [14] introduced the concept of operational entanglement, the amount of entanglement that can be extracted from a resource (many-body state) and transferred to a quantum register in the presence of a SSR. Conceptually, it is the weighted sum of the spatial entanglement when projecting onto states of fixed local particle number. For the two-copy Rényi entropy it is defined as

$$S_2^{\text{op}}(A) = \sum_n P_n S_2(A_n), \quad (2)$$

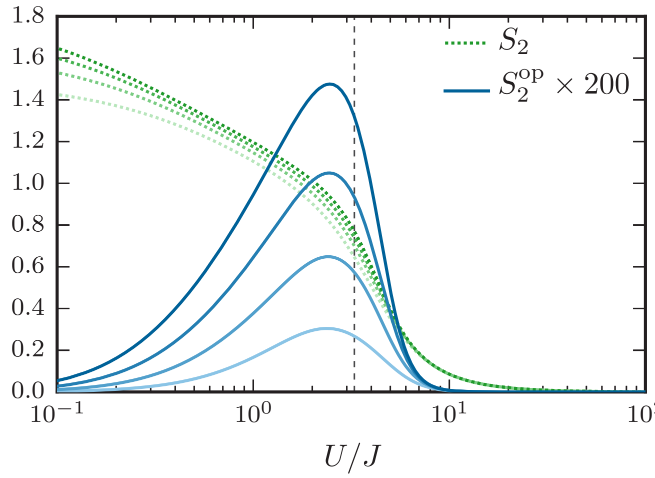

FIG. 2. Spatial second Rényi entropy S_2 and operational entanglement S_2^{op} for symmetric bipartitions $\ell = L/2$ of the Bose-Hubbard model. Curves increase in saturation for $L = N = 6,8,10,12$. The dashed vertical line indicates the location of the thermodynamic phase transition.

where $S_2(A_n)$ is the Rényi entropy evaluated for the reduced density matrix

$$\rho_{A_n} = \frac{1}{P_n}\hat{P}_n \rho_A \hat{P}_n \quad (3)$$

projected by $\hat{P}_n$ onto states of fixed local particle number n in subsystem A. The summation is over all possible local particle number states in the subregion with $n = 0,\ldots,N$, each having probability $P_n = \langle\Psi|\hat{P}_n|\Psi\rangle$. This projection is a local operation that can only decrease entanglement [15] so $S_2^{\text{op}} \leqslant S_2$.

Thus it is S_2^{op}, not S_2, that bounds the amount of entanglement that can be generated in the register using local operations and classical communication (LOCC). A measurement of S_2^{op} is possible with a simple modification of the experimental interference or parity measurement procedure [1,4]. This requires that a projection onto states of definite subsystem particle number n be made after interference, which can be achieved in the experimental system by measuring the total particle number in each of the $\bar{A}$ subsystems, $n_{\bar{A}}$ after the beam splitter operation is performed. Given that each copy is initialized to a state with N particles, P_n is the probability of having $n_{\bar{A}} = N - n$ in a single copy; $S_2(A_n)$ is computed from the parity measurements of instances $n_{\bar{A}} = N - n$ in both copies, which must be binned accordingly to compute S_2^{op}. Therefore, by collecting these n-resolved statistics of the Rényi entropy, Eq. (2) can be used to experimentally measure the operational entanglement.

In order to explore which parameter regime maximizes operational entanglement, we calculate S_2^{op} in the Bose-Hubbard model. Experiments on ^{87}Rb in the near future should be possible with $4 < N \lesssim 10$ and we study the ground states of systems with sizes of this order via exact numerical diagonalization of Eq. (1). In Fig. 2 we compare the two-copy Rényi entropy for a symmetric spatial bipartition to the operational entanglement for a range of U/J and N relevant for experiments. Unlike the entropy under a spatial bipartition, which is maximum deep in the superfluid phase (or the particle entanglement, which is maximum deep in the Mott phase [21]), S_2^{op} displays a peak at an intermediate value of the interaction. While for these system sizes the peak is not positioned directly at the thermodynamic-limit critical point $(U/J)_c \approx 3.3$, it appears to approach this value as L is increased.[1] This suggests that the appropriate experimental parameters for maximizing the transfer of many-body entanglement to a system of quantum registers will be those that tune the system to near the superfluid-Mott transition.

As shown in Fig. 2, S_2^{op} is necessarily smaller than S_2, as it does not include entanglement generated by particle fluctuations between subsystems that is not physically accessible due to the SSR. Additionally, S_2^{op} is reduced as interactions in Eq. (1) are strictly on site and occur at fixed subsystem occupation through second-order processes. Thus, the behavior of the physically accessible entanglement differs from S_2 both qualitatively and quantitatively.

III. EXTRACTING MANY-BODY ENTANGLEMENT

Given that the operational entanglement indicates that some of the entanglement between spatial subregions of the many-body ground state may be transferred to an external quantum register using LOCC, we now describe an experimental procedure to do so. This allows the many-body state to act as an entanglement resource for quantum information protocols. We concentrate on the minimal $L = N = 6$ Bose-Hubbard system where entanglement may be transferred to two spatially separated qubits. Each qubit is comprised of one atom occupying one of two neighboring lattice sites adjacent to the Bose-Hubbard chain; the two locations of the atom provide the logical states. Thus, the physical system we describe consists of ten total lattice sites, which must be doubled as shown in Fig. 3 if a two-copy Rényi measurement is to be made on the final entanglement between the qubits.

The starting point is the isolation of a 6×4 array of atoms that can be prepared deep in the Mott phase. This array includes the many-body entanglement resource, which will be partitioned into three spatial subregions with two sites each (labeled A,B,C), two qubit registers Q_A and Q_B, and a copy that will be employed to read out the amount of entanglement generated between Q_A and Q_B. To manipulate and measure entanglement in the system, we define a pairwise hopping unitary operator

$$U_{ij}(\phi) \equiv \exp[i\phi(b_i^\dagger b_j + \mathrm{H.c.})]. \quad (4)$$

This is a trivial generalization of the beam-splitter operation reported in Ref. [1] (where $\phi = \pi/4$) and $\phi = \pi/2$ corresponds to a SWAP gate between i and j within the $n_{i,j} = 0,1$ subspace. Additionally, this physical operation can be used to perform single-qubit rotations when applied within a single qubit. As U_{ij} will not generally preserve particle number within the resource and qubits (and thus not remain in the logical subspace of the qubits), subsystem resolved particle

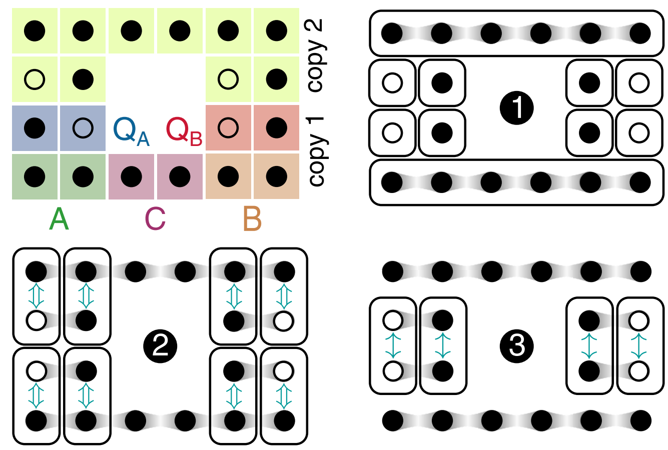

FIG. 3. Shown on the top left is an array of 20 optical lattice sites forming the two copies necessary to measure the second Rényi entropy. The remaining panels show the protocol (described in the text) to transfer entanglement from a many-body state in $A \cup B \cup C$ to spatially separated qubits Q_A and Q_B. Solid lines correspond to a large tunnel barrier, double arrows represent the application of a SWAP operation, and single arrows indicate performing many-body interference.

occupation number measurements must be used to postselect states that have exactly one particle in each of A and B.

Transfer of many-body entanglement to the register and its subsequent measurement can be accomplished via the three-step procedure depicted in Fig. 3.

① The optical lattice within the array is manipulated such that large barriers (as indicated by solid lines) isolate the many-body resource. Each qubit must be constructed with exactly one particle between its two sites, with the barrier between them remaining high throughout the experiment. The many-body resource can be prepared identically to Ref. [1] with the lattice strength tuned near the critical value $(U/J)_c$ to maximize the operational entanglement as discussed above.

② A SWAP operation (double arrow) $A \Leftrightarrow Q_A$ is performed by applying the unitary hopping operator

$$U_{1,1'}(\pi/2)U_{2,2'}(\pi/2), \quad (5)$$

where sites 1,2 are in region A, while $1',2'$ label adjacent sites in Q_A. This is repeated for $B \Leftrightarrow Q_B$ and the identical procedure is performed in the copy. Thus entanglement is transferred from the many-body resource to the spatially separated qubits.

③ To read out this entanglement, a beam-splitter operation (single arrow) is performed between the two copies of Q_A and Q_B, followed by a subsystem resolved particle number measurement where instances with one atom in each qubit are postselected, which is discussed below.

The above procedure will transfer many-body entanglement to a quantum register. As only A and B are swapped with the register, its density matrix ρ_{Q_A,Q_B} will generically be in a mixed state, even if the initial many-body state (ρ_{ABC}) was pure. Consequently, the mutual information

$$I_2(AB) = S_2(A) + S_2(B) - S_2(AB) \quad (6)$$

[1]Note, however, that the choice of bipartition is arbitrary and an exact scaling theory is unknown.

will have contributions from both classical correlations and quantum entanglement. Here $I_2(AB)$ is measurable in current experiments combined with postselection to conserve particle number in $Q_{A/B}$.

A. Postselection in the qubit subspace

Because the protocol discussed above requires a tripartite partitioning of each copy of the many-body state, postselection of experimental instances that are in the qubit subspace of $Q_{A/B}$ after the SWAP operation is nontrivial. In measuring $S_2(AB)$, postselecting to $n_C = N - 2$ is insufficient to solely select instances in the qubit subspace. We define the number of particles in each qubit to be $n_{Q_{A/B}}$ and $n_{\bar{Q}_{A/B}}$. Since the beam-splitter operation conserves the sum of the number of particles in each qubit and its copy, instances within the qubit subspace always result in $n_{Q_A} + n_{\bar{Q}_A} = 2$ and $n_{Q_B} + n_{\bar{Q}_B} = 2$. If the data are additionally postselected such that $n_C = n_{\bar{C}} = N - 2$, the only nonqubit instances arise from those of the form

$$|02\rangle_{Q_A}|00\rangle_{Q_B}|00\rangle_{\bar{Q}_A}|02\rangle_{\bar{Q}_B} \tag{7}$$

and eight related permutations, before the beam-splitter operation. As such states have a zero expectation value of parity after the beam splitter (or equivalently the expectation value of the SWAP operator is zero for such states), it is only necessary to determine the normalization of parity measurement, i.e., the probability of instances in the qubit subspace. For a given initial state, the qubit probability P_{QB} can be independently measured from separate experiments where no beam-splitter operation is performed and only the site-resolved number of particles is measured. Then the beam-splitter parity measurements are performed and postselected for states with $n_{Q_A} + n_{\bar{Q}_A} = 2$, $n_{Q_B} + n_{\bar{Q}_B} = 2$, and $n_C = n_{\bar{C}} = N - 2$; we define the result of this measurement to be Σ_{AB} and the probability of such instances to be $\tilde{P}_{AB}$. The desired value can be then computed as

$$S_2(Q_A Q_B) = -\ln\left(\frac{\tilde{P}_{AB}}{P_{\mathrm{QB}}}\Sigma_{AB}\right). \tag{8}$$

B. Quantifying entanglement between qubits

To quantify only the desired generation of quantum entanglement between the qubits, we compute various measures of mixed state entanglement for the reduced density matrix ρ_{AB} of the many-body ground state. Unlike for pure states, where the von Neumann entropy is the unique and appropriate entanglement measure, for mixed states, there is a variety of entanglement measures with different physical meanings. For example, the entanglement of formation E_F, roughly defined as the amount of entanglement required to form the mixed state, can be directly computed for any two-qubit density matrix [22]. The logarithmic negativity $E_{\mathcal{N}}$ depends on the sum of the negative eigenvalues of the density matrix after a partial transpose and thus is readily computable for any density matrix [23]. It provides an upper bound to the amount of entanglement that can be extracted from the mixed state using LOCC.

In Fig. 4 we have plotted $I_2(AB)$, E_F, and $E_{\mathcal{N}}$ of ρ_{AB} for the ground state of Eq. (1) in the six-site geometry of Fig. 3, projected onto states with a single particle occupying A and B. We find that all these measures peak near the quantum phase transition [24–26]. The peaks coincide with the parameter region of maximal operational entanglement desired for optimal transfer between resource and register. Further, $E_{\mathcal{N}} > 0$ is a necessary and sufficient condition for a two-qubit state to be inseparable [27] such that it can be distilled to form a maximally entangled state [28]. This implies that near the critical point the many-body resource has entanglement that can be extracted and distilled. Although there is no general relationship between I_2 and the entanglement measures E_F and $E_{\mathcal{N}}$, in this case we can compute the relationship exactly for the Bose-Hubbard model. Thus, measurement of I_2 in an experimental regime where the Bose-Hubbard parameters are known will provide an estimate of the entanglement that can be generated between the qubits through the relationship calculated in Fig. 4.

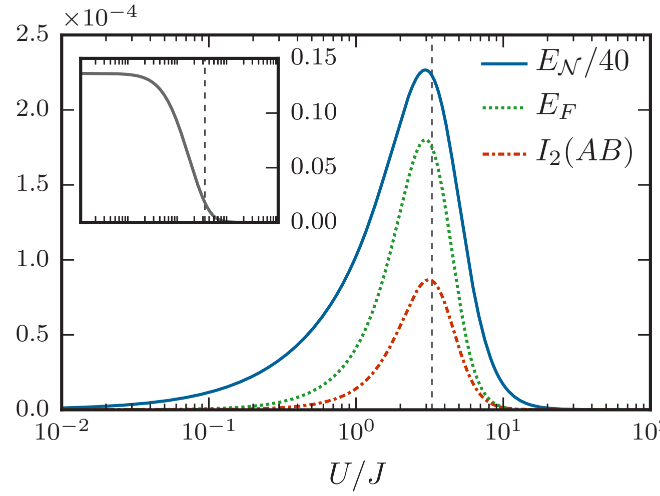

FIG. 4. Logarithmic negativity $E_{\mathcal{N}}$, entanglement of formation E_F, and two-copy Rényi mutual information $I_2(AB)$ of the spatially separated qubits obtained from the $L = N = 6$ Bose-Hubbard ground state. The inset shows the probability of projecting onto a state with a single particle in each of A and B. The dashed vertical line indicates the location of the thermodynamic phase transition.

IV. DISCUSSION

We have introduced an experimental procedure for the transfer of entanglement from a many-body resource to spatially separated qubits forming a register suitable for quantum information processing. Conservation of particle number limits the amount of entanglement transferable from the resource, as quantified by the operational entanglement. The precise control of the current generation of quantum emulator experiments enables the faithful creation of lattice Bose-Hubbard models using ultracold atoms. This allows us to quantify the operational entanglement using exact calculations and we find that the transferable entanglement is maximized near the quantum phase transition between the Mott insulator and superfluid phases. This is in contrast to the naive expectation that transfer should occur in the superfluid phase, where experiments have confirmed that the two-copy Rényi entanglement is largest [1].

We have introduced a measurement protocol to experimentally probe the entanglement transferred by this procedure that employs a variation of a many-body interference technique [1,4]. It is explicitly described for the transfer of entanglement from a six-site resource to a register composed of two two-site qubits, 20 lattice sites in total. It can be easily scaled to arbitrary size as experimental technology progresses. Our Bose-Hubbard calculations quantify the relationship between mutual information accessible by this protocol and well-known measures for entanglement in mixed states.

The ability to engineer a wealth of variations of the Bose-Hubbard model will open up exciting prospects for extensions and optimizations of our results, through inhomogeneous parameters, topologies, and dimensionality. The experimental implementation of our protocol will demonstrate the potential of using many-body states of ultracold atoms as an entanglement resource for quantum information processing.

Note added in proof. A recent manuscript [29] has reported experimental entanglement measurements on a system with $L = 6$ sites, the same number of atoms considered in here.

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