The Vertex Algebra Vertex

by

Miroslav Rapčák

A thesis
presented to the University of Waterloo
in fulfillment of the
thesis requirement for the degree of
Doctor of Philosophy
in
Physics

Waterloo, Ontario, Canada, 2019

Examining Committee Membership

The following served on the Examining Committee for this thesis. The decision of the Examining Committee is by majority vote.

Supervisor:	Davide Gaiotto Perimeter Institute for Theoretical Physics
Co-supervisor:	Jaume Gomis Perimeter Institute for Theoretical Physics
Committee Member:	Niayesh Afshordi University of Waterloo, Department of Physics and Astronomy
Committee Member:	Kevin Costello Perimeter Institute for Theoretical Physics
Internal/External Member:	Benoit Charbonneau University of Waterloo, Department of Pure Mathematics
External Examiner:	Rajesh Gopakumar International Centre for Theoretical Sciences in Bangalore

Author's Declaration:

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Abstract

This thesis reviews some aspects of a large class of vertex operator algebras labelled by (p, q) webs colored by non-negative integers associated to faces of the web diagrams [1, 2, 3, 4]. Such vertex operator algebras conjecturally correspond to two mutually dual setups in gauge theory. First, they appear as a subsector of local operators living on two-dimensional junctions of half-BPS interfaces in four-dimensional $\mathcal{N} = 4$ super Yang-Mills theory. Secondly, they are AGT dual to $\mathcal{N} = 2$ gauge theories supported on four-cycles inside toric Calabi-Yau three-folds.

We review four (conjecturally equivalent) definitions of the vertex algebra vertex Y_{N_1,N_2,N_3} associated to the simplest trivalent (p, q) web. The algebra can be defined in terms of 1. the quantum Hamiltonian (BRST) reduction, 2. truncations of the $\mathcal{W}_{1+\infty}$ algebra, 3. the kernel of screening charges and 4. the generalized Miura transformation. Equivalence of such definitions is strongly supported by matching various properties of the algebra such as characters and highest weights of their modules, central charges, duality properties and many others.

The algebra Y_{N_1,N_2,N_3} plays the role of a building block of more complicated algebras that can be characterized as extensions of Y_{N_1,N_2,N_3}'s associated to each vertex of the (p, q) web by bimodules associated to each internal line. This procedure mimics the topological-vertex-like counting of D0-D2-D4 bound states in toric three-folds that motivates the name "The Vertex Algebra Vertex". Conjecturally, there exists an unique (or at least canonical) vertex operator algebra characterized by such an extension. We review different definitions of a larger class of algebras corresponding to truncations of a $\mathfrak{gl}(N_1|N_2)$ analogue of the $\mathcal{W}_{1+\infty}$ algebra and their shifted versions. Finally, we discuss various generalizations of the Feigin-Frenkel duality coming from the S-duality of the $\mathcal{N} = 4$ super Yang-Mills setup, stable equivalence (equivalence of vertex algebras up to decoupled free fields) and its interpretation in terms of brane transitions, ortho-symplectic version of the algebras and the structure of their modules.

Even though this thesis is largely a compilation of the above-mentioned papers, it contains many novel remarks that are not contained in the original work.

Acknowledgements

I would like to express my deepest gratitude to my supervisor Davide Gaiotto for his guidance, support and for being a constant source of new ideas. I am thankful to Matěj Kudrna, Tomáš Procházka, Yan Soibelman, Martin Schnabl, Yaping Yang, Gufang Zhao for numerous explanations during our exciting collaborations.

I am grateful to Niayesh Afshordi, Mina Aganagic, Mikhail Bershtein, Lakshya Bhardwaj, Dylan Butson, Benoit Charbonneau, Kevin Costello, Thomas Creutzig, Tudor Dimofte, Nadav Drukker, Igor Frenkel, Matthias Gaberdiel, Jaume Gomis, Sarah Harrison, Rajesh Gopakumar, Sergei Gukov, Koichi Harada, Justin Hilburn, Ondřej Hulík, Nafiz Ishtiaque, Taro Kimura, Shota Komatsu, Wei Li, Ying-Hsuan Lin, Pietro Longhi, Dalimil Mazáč, Antun Milas, Seyed Faroogh Moosavian, Nikita Nekrasov, Tadashi Okazaki, Vasily Pestun, Jörg Teschner, Oleksandr Tsymbaliuk, Pedro Vieira, Ben Webster, Edward Witten, Junya Yagi, Masahito Yamazaki, Philsang Yoo, Alexander Zamolodchikov, Yegor Zenkevich and Barton Zwiebach for interesting discussions and support at various stages of my PhD studies.

I am thankful to my wife Lenka who was willing to move with me to Canada and who has been a constant support in my life. I would hardly be able to finish my PhD without her. I would like to thank my parents for their encouragement and my friends for playing such an extremely important role in my life.

Finally, I would like to thank Perimeter Institute for providing me with an inspiring environment during the past five years.

I only wish department of Physics and Astronomy at University of Waterloo treated its graduate students with some respect.

Table of Contents

List of Figures

1. Introduction

Vertex Operator Algebras (VOA) formalize the notion of local operators on a Riemann surface with the following two properties. First, correlation functions are meromorphic functions of insertion points with singularities associated to colliding operators. Secondly, the algebra contains the Virasoro algebra associated to the holomorphic component of the stress-energy tensor [6, 7, 8, 9]. Since their discovery, VOAs have played an important role in string theory and the theory of critical phenomena in two-dimensional condensed matter systems. Due to the rich structure but computational manageability, VOAs turn out to be often related to deep but solvable problems and they keep appearing in other contexts in both physics and mathematics (see for example [10, 11, 12, 13, 14, 15, 16, 17]).

Starting with the work of Alday-Gaiotto-Tachikawa, VOAs have been shown to be closely related to four-dimensional supersymmetric gauge theories [14, 18]. AGT conjectured a correspondence between four-dimensional $\mathcal{N} = 2$ gauge theories coming from a compactification of N M5-branes [19, 20] on a Riemann surface and VOAs known under the name $\mathcal{W}_N \times \widehat{\mathfrak{gl}(1)}$ algebras [21, 22].

This thesis is a review of recent developments from the series of papers [1, 2, 3, 4] (see also [23, 24, 25, 26, 27, 28, 29, 30, 31, 30, 32, 33, 34, 35, 36, 37]) that considerably extend the family of vertex operator algebras corresponding to some setups in gauge theory. The gauge-theoretical viewpoint provides an intuitive way to study such VOAs and the rigidity of VOAs allows us to probe many non-trivial gauge-theoretical questions exactly.

1.1 Colored (p, q) webs

Vertex operator algebras from [1, 2, 3, 4] are labelled by a colored (p, q) web such as the one in the figure 1.1. Each vertex of the diagram is a trivalent fan of $(p_1, q_1), (p_2, q_2)$ and $(-p_1 - p_2, -q_1, -q_2)$ vectors satisfying $p_1q_2 - p_2q_1 = 1$. For each such (p, q) web, colored

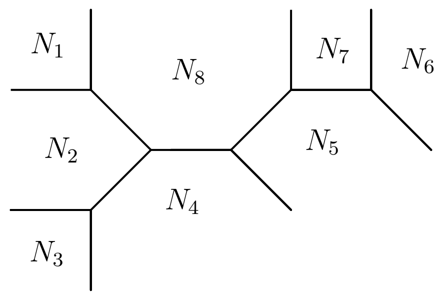


Figure 1.1: A generic (p, q) web colored by non-negative integers N_i to which we associate a vertex operator algebra.

by non-negative integers at each face of the diagram, we conjecturally associate a family of vertex operator algebras parametrized by a complex parameter $\Psi \in \mathbb{CP}^1$.

To the simplest trivalent junction of $(1, 0)$, $(0, 1)$ and $-(1, 1)$ lines with integral parameters N_1, N_2, N_3 shown in the figure 1.2, we associate an algebra Y_{N_1,N_2,N_3} generalizing the well-known $Y_{0,0,N} = \mathcal{W}_N \times \widehat{\mathfrak{gl}(1)}$. This four-parameter family of algebras will be called the vertex algebra vertex and it will serve as a building block of more complicated VOAs associated to more general colored (p, q) webs. Such general algebras can be characterized as extensions of tensor products of Y_{N_i,N_j,N_k} associated to vertices of the web diagram by bimodules associated to internal lines of the web diagram. Even though we do not have a precise definition of the algebra associated to a generic colored (p, q) web, we conjecture that such an extension is unique or at least canonical.

1.2 Dual setups

The (p, q) web VOAs are expected to naturally appear in two mutually dual gauge-theoretical setups. Correspondingly, the web diagrams above can be given two mutually dual interpretations.

First, we can look at the (p, q) web as a projection of a ten-dimensional type IIB string theory setup to the 23-plane. Different (p, q) lines then correspond to (p, q)-fivebranes[1] spanning directions 01456 and one extra direction in the 23-plane specified by the web diagram. The web diagram can be thus thought of as a diagram labeling a junction of fivebranes [38, 39]. The integers N_i then correspond to numbers of D3-branes attached to the web from different corners and spanning directions 0123.

[1] $(1, 0)$ can be identified with the D5-brane and $(0, 1)$ with the NS5-brane.

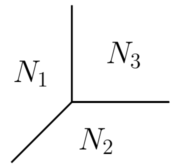


Figure 1.2: The simplest colored (p,q) web (the vertex algebra vertex) associated to the algebra Y_{N_1,N_2,N_3}.

The string theory configuration has a low energy effective description in terms of a four-dimensional supersymmetric gauge theory in the presence of junctions of domain walls. The field theory describing the low energy behavior of N D3-branes is known to be $\mathcal{N}=4$ super Yang-Mills theory with the gauge group $U(N)$. The configuration at hand thus leads to a system of $U(N_i)$ supersymmetric gauge theories associated to directions 0123 at different corners, mutually coupled by domain walls as indicated by the colored (p,q) web.

Throughout the text, we are going to restrict to a subclass of operators in the physical theory that live in the cohomology of the Geometric Langlands (GL) supercharge[2] from [40]. GL-twisted theories depend on a complex parameter $\Psi \in \mathbb{CP}^1$ that is a non-trivial combination of the complexified gauge coupling of the $\mathcal{N}=4$ super Yang-Mills theory and a parameter parametrizing different choices of the GL supercharge. The parameter Ψ plays the role of a structure constant of VOAs discussed in this thesis.

In the infra-red (when zooming out the (p,q) web), the configuration at hand generally looks like a star-shaped junction of non-trivial interfaces. It turns out that local operators in the cohomology of the GL supercharge for generic values of Ψ live at the two-dimensional junction and form generators of a VOA [41, 42, 43, 44, 45, 46, 47, 1].

The colored (p,q) web can be given a very different interpretation. One can start with M-theory on $\mathbb{R}^4 \times S^1 \times CY^3$, where CY^3 is a toric Calabi-Yau three-fold with (p,q) web being its toric diagram[48, 49, 50]. Such a Calabi-Yau three-fold can be thought of as a $\mathbb{R} \times T^2$ fibration over the $\mathbb{R}^3$ base with the toric diagram specifying loci where different (p,q) cycles of the fibration degenerate. Faces of the toric diagram can be associated with four-cycles inside CY^3 that are fixed under the toric action. The interpretation of the integral numbers in the colored (p,q) web are simply multiplicities of M5-branes wrapping such four-cycles and sharing an extra Riemann surface $\Sigma \subset \mathbb{R}^4 \times S^1$ in orthogonal directions [51, 52, 53, 54, 55].

[2]To preserve the GL-supercharge in the presence of (p,q) interfaces, a deformation of the interfaces is needed as we will briefly discuss in section 3.1.2. A precise form of such deformations (and even the possibility to move away of the family of GL-twists) still remains to be explored in detail.

Toric Calabi-Yau three-folds admit a two-parameter Ω-deformation with parameters $\epsilon_1, \epsilon_2, \epsilon_3$ satisfying $\epsilon_1 + \epsilon_2 + \epsilon_3 = 0$ [56, 57, 42, 54, 55]. One can argue that the system of M5-branes in the presence of such an Ω-background have an effective description in terms of a theory on Σ giving rise to a vertex operator algebra. It turns out that in all examples, the resulting algebra actually depends only on the ratio $\Psi = -\epsilon_1/\epsilon_2$.

The two setups discussed above can be related along the lines of [48] and the relation will be reviewed further in section 3.2.1. The two perspectives give orthogonal viewpoints on the VOAs at hand. The type IIB perspective turns out to be convenient in addressing relations to various GL-like dualities as discussed in [24, 26, 27]. The type IIA perspective provides us with a connection with the topological-vertex-like counting of D0-D2-D4 brane bound states from [51, 52, 53]. In particular, BPS characters of [51, 52, 53] can be identified with vacuum characters of the involved algebras. Furthermore, the type IIA perspective points towards a generalization of the AGT correspondence [14, 18]. AGT associates the algebra $\mathcal{W}_N \times \widehat{\mathfrak{gl}(1)}$ to an $\mathcal{N} = 2$ gauge theory on $\mathbb{C}^2$. More concretely, $\mathcal{W}_N \times \widehat{\mathfrak{gl}(1)}$ can be shown to act on the cohomology of the moduli space of instantons on $\mathbb{C}^2$ as discussed in [58, 57, 16, 59, 60]. The Nekrasov partition function [57] can be then identified with conformal blocks of the algebra. Note that $\mathbb{C}^2$ can be embedded inside $\mathbb{C}^3$ leading to the geometric configuration associated to $Y_{0,0,N}$. Turning on N_i M5-branes at each of the three coordinate planes leads to a natural generalization of the AGT conjecture for these spiked-instanton configurations [54, 55]. One can indeed show [4] that the algebra Y_{N_1,N_2,N_3} plays the role of the AGT dual in the generalization of the AGT correspondence to toric divisors inside $\mathbb{C}^3$ [54, 55]. Presumably the story have further generalizations beyond this example with (p, q) web VOAs being the VOA duals of gauge theories supported on intersecting four-cycles inside toric Calabi-Yau three-folds and possibly even more generally [54, 30, 4].

1.3 Four constructions of Y_{N_1,N_2,N_3}

A concrete definition of the VOA associated to a general colored (p, q) web is currently not known but there exist large classes of colored webs whose VOA do have such a definition. Furthermore, in some cases, more (conjecturally equivalent) definitions exist.

Let us first restrict to the simplest trivalent junction and the algebra Y_{N_1,N_2,N_3}. Using the results of [44, 43, 46, 47], one can argue that the path integral of the $\mathcal{N} = 4$ SYM in the type IIB setup localizes to the path integral of a pair of complexified Chern-Simons theories with gauge groups $U(N_1|N_3)$ and $U(N_1|N_2)$ coupled together along a two-dimensional interface, possibly involving some extra 2d fields. Along the lines of [41, 61], one can argue that local operators at the interface give rise to a VOA that can be identified with a BRST reduction (motivated by the defect conditions coming from D3-branes ending on a D5-brane from [62, 63]) of two coppies of Kac-Moody algebra $\widehat{\mathfrak{gl}(N_1|N_3)}$ and $\widehat{\mathfrak{gl}(N_1|N_2)}$ extended by extra free fields. The type IIB perspective thus gives a natural proposal for a BRST definition of the algebras Y_{N_1,N_2,N_3} that combine the quantum Drinfeld-Sokolov reduction [64, 65, 66, 67, 68, 69] and the BRST coset [70, 71].

Another perspective comes from the analysis of the large N_i limit of the vacuum character of Y_{N_1,N_2,N_3} and the type IIA picture. It turns out that the large N_i limit of the vacuum character can be identified with the MacMahon function[3]. MacMahon function can be furthermore identified with the character of a VOA generated by fields $W_1, W_2, W_3, \ldots$ with W_i being a generator of conformal weight i. As discussed in [74, 75, 5, 76], there exists a two-parameter family of algebras satisfying associativity of the operator product expansion with such a conformal-weight content. It turns out that there exist specializations of the parameters to one-dimensional lines inside the two-dimensional space of algebras for which $\mathcal{W}_{1+\infty}$ contains an ideal $\mathcal{I}_{N_1,N_2,N_3}$. This ideal can be factored out that leads to the algebra Y_{N_1,N_2,N_3} with Ψ parametrizing the corresponding truncation line. This gives an alternative definition of the algebra Y_{N_1,N_2,N_3}.

There exists a well-known construction of the algebra $\mathcal{W}_N \times \widehat{\mathfrak{gl}(1)}$ as a subalgebra of the tensor product of N coppies of $\widehat{\mathfrak{gl}(1)}$ current algebras (Heisenberg VOAs) [22]. Such a free field realization can be obtained by multiplying N Miura differential operators $\mathcal{L}_i^{(3)} = \epsilon_3 \partial + J_i(z)$, where $J_i(z)$ is the i'th $\widehat{\mathfrak{gl}(1)}$ current. It turns out that there exists a generalization of the construction to an arbitrary Y_{N_1,N_2,N_3} by introducing pseudo-differential operators $\mathcal{L}_i^{(1)}$ and $\mathcal{L}_i^{(2)}$ and taking a product of N_i factors of each type [3]. This free field realization turns out to be useful when discussing generic modules of the algebras and more importantly when comparing with the algebras coming from the geometric action of the cohomological Hall algebra [73] of $\mathbb{C}^3$ used in the proof of the AGT correspondence [4] for spiked instantons [54, 55] associated to divisors in $\mathbb{C}^3$.

There exists yet another characterization of the free-field realization as an intersection of kernels of screening charges discussed in [28, 29, 3]. For each ordering of Miura factors, one can associate a screening charge to each neighbouring pair of $\widehat{\mathfrak{gl}(1)}$ currents and identify the free field realization as an intersection of kernels of such screening charges.

[3]Note that the MacMahon function is also the generating function of Donaldson-Thomas invariants of $\mathbb{C}^3$ [72, 73] and we can see first hints of the geometric side.

1.4 Glued algebras

Let us now discuss algebras associated to more complicated (p,q) webs. As mentioned above, zooming out a (p,q) web leads to a single junction with local operators giving rise to a more general VOA. From the resolved point of view, the local operators of the infrared can have different origin. They can be local operators associated to vertices of the (p,q) web or line operators stretched between them (if we have also non-trivial compact faces, one might have to consider also surface operators spanning these internal faces). The resulting VOA is then expected to be an extension of the tensor product of Y-algebras associated to such vertices by bi-modules[4] associated to the internal lines. It is tempting to conjecture that such extensions are uniquely fixed by Jacobi identities (associativity of OPEs).

In some cases, one can be more specific and give a concrete definition of the glued algebra. A large class of examples come from considering a general number of $(1,0)$ lines ending from the left and from the right on sequence of $(n,1)$ lines. If the colors decrease from the up to the bottom on both sides of the $(n,1)$ lines, such as in the figure 4.5, the analysis of boundary conditions from [62, 63] allow us to find a proposal for the BRST definition of the algebra [2]. One can furthermore check (at least in examples and in the large N_i limit) that characters of such reductions agree with characters predicted form gluing.

Similarly, one can consider the large N_i limit of the vacuum character and look for a definition of the glued algebras in terms of truncations of universal infinity algebras. It turns out that in the example above, the limit depends only on a half-integral parameter ρ_i associated to each finite (p,q) segment. The character agrees with a $\mathfrak{gl}(n|m)$ version of the algebra $\mathcal{W}_{1+\infty}$ containing a $n|m$ super-matrix of generators at each spin with n/m being the numbers of $(1,0)$-branes ending from the left/right. For example the (p,q) web from the figure 4.5 leads to an algebra of type $\mathfrak{gl}(1|3)$. Fixing values of ρ_i fixes all the parameters N_i up to three. It is tempting to conjecture that there again exists a unique two-parameter family of algebras with a given spin content (containing $\mathfrak{gl}(n|m)$ as a subalgebra and having all the fields at higher weights in its adjoint representation) with the web VOAs corresponding to its truncations. On the other hand, fixing general values of ρ_i leads to an algebra with generators associated to the roots of $\mathfrak{gl}(n|m)$ starting at level $1+\rho_i$ with

[4] As we discuss bellow, line operators ending at a junction lead naturally to a degenerate module of the algebra. Lines stretched between two vertices then correspond to bimodules. The necessary bimodules are expected to be labelled by tensor representations of $\mathfrak{gl}(N_i|N_j)$ Lie super-algebras, i.e. representations that can be obtained from tensor products of the fundamental and the anti-fundamental representation.

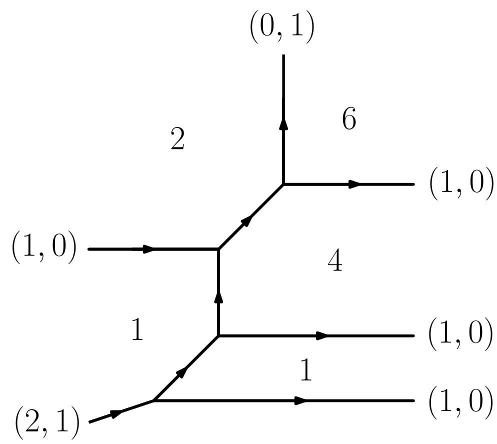


Figure 1.3: An example of a diagram with a known BRST definition.

i labeling the root. For example, the $\mathcal{N}=2$ super $\mathcal{W}_{1+\infty}$ algebra [77] should correspond to the $\mathfrak{gl}(1|1)$ configuration with shift[5] $\rho = 1/2$. It is natural to speculate that the picture of truncations naturally extends also to other brane diagrams.

Finally, we expect that the free-field realization can be extended to glued algebras. Having a free-field realization of the algebras associated to each vertex, one can construct a field associated to each highest weight of the added bi-modules as an exponential vertex operator of free bosons (or their descendant). One can then consider an algebra extended by such vertex operators.[6] There are two technical difficulties in the construction. First, it turns out that there exist more possible choices of vertex operators with correct highest weights. Secondly, OPEs are generally not reproduced correctly unless we include integrals of screening currents. We will review these issues briefly in later sections but their proper investigation (along the lines of [78]) is still to be performed.

1.5 Some properties

The gauge theory perspective motivates various constructions of a large class of vertex operator algebras relevant to different gauge-theoretical setups. It also provides us with many non-trivial insights into the properties of VOAs.

We have already mentioned that local operators at the two-dimensional junction in

[5]Note in particular that the two fermionic currents $G^{\pm}$ have the conformal weight $\frac{3}{2} = 1 + \frac{1}{2}$.
[6]In particular, it turns out that the vertex operators have only $\mathbb{Z}_2$ braiding.

the type IIB perspective give rise to a VOA. Local operators are not the only operators surviving the GL twist. One can also consider line operators (Wilson, 't Hooft or more general lines) supported at the interfaces [40, 46, 47]. Line operators ending at a junction then naturally give rise to modules for the VOA generated by a fusion of the endpoint with local operators at the intersection. These line operators are expected to be labeled by tensor representations (representations that can be constructed from tensor products of the fundamental and the anti-fundamental representation) of the Lie super-algebra $\mathfrak{gl}(N_i|N_j)$ for line operators living on a (p,q) interface between theories of gauge groups $U(N_i)$ and $U(N_j)$.

On the other hand, one can consider Gukov-Witten (GW) surface operators sitting at a point of the junction but spanning faces of the (p,q) web [79, 46, 47, 1, 3]. It turns out that N_i complex parameters valued on a torus survive the GL twist of the $U(N_i)$ gauge theory. Furthermore, the torus is lifted to a whole complex plane (modulo Weyl transformations permuting the parameters) by the choice of the boundary condition specifying how the defect ends at the interface. Fusion of line operators living at the interfaces with boundary lines of surface defects then change such a boundary condition.

Let us discuss how this physical picture fits into the VOA story. Gauge theory suggests an existence of a continuous family of modules parametrized by as many complex numbers as the sum of integral parameters in the given colored web diagram, e.g. $N_1+N_2+N_3$ in the case of Y_{N_1,N_2,N_3}. Furthermore, whenever we specialize parameters in such a way that some of the torus-valued GW parameters vanish, one gets a degenerate module (corresponding to line operators in a representation of the gauge group preserved by the specialized GW defect). In particular, if all the parameters are specialized, one gets modules parametrized by line operators associated to each semi-infinite line in a given (p,q) web. Moreover, modules associated to different lines are expected to braid trivially.

Apart from an intuitive understanding of modules, the physical picture predicts various dualities generalizing the Feigin-Frenkel duality [67]. Feigin and Frenkel realised that the Drinfeld-Solokov reduction of the $\widehat{\mathfrak{sl}(N)}$ Kac-Moody algebra at shifted levels $\Psi = k + N$ and $\frac{1}{\Psi}$ both lead to the same $\mathcal{W}_N$ algebra. This duality can be understood as a consequence of the S-duality. Note that the type IIA setup is manifestly invariant under the relabeling of the cycles of the torus. On the other hand, such a relabeling (corresponding to $SL(2,\mathbb{Z})$ transformations of the charges (p,q) and the modular parameter Ψ) gives rise to a non-trivial transformation of boundary conditions in the type IIB setup [63]. If a brane diagram preserves some subgroup G of the $SL(2,\mathbb{Z})$ transformations, there will be a non-trivial duality action on the family of corresponding vertex operator algebras giving rise to dual BRST constructions of the corresponding algebra. For example in the case of Y_{N_1,N_2,N_3}, the web diagram preserves an S_3 subgroup of $SL(2,\mathbb{Z})$ permuting the (p,q) lines. This gives rise to an S_3 duality group action on Y_{N_1,N_2,N_3}. Form the $\mathcal{W}_{1+\infty}$ algebra perspective, the S_3 action can be identified with the triality[7] discovered in [75]. Another example of a diagram preserving some subalgebra of the $SL(2,\mathbb{Z})$ duality group would be the diagram associated to the $\mathfrak{gl}(1|1)$ version of $\mathcal{W}_{1+\infty}$ (and its shifted versions) that leads to a $\mathbb{Z}_2 \times \mathbb{Z}_2$ duality action. This action can be identified for $\rho = \frac{1}{2}$ with the one of [77] and it is enhanced to S_4 if $\rho = 0$.

Let us note that the algebra in the infra-red picture depends on a specific resolution. It was conjectured and checked in examples in [2] that different resolutions give rise to the same algebra up to the contribution of decoupled b, c and β, γ ghost systems. Transition of branes then leads to various (sometimes highly nontrivial) identities between vertex operator algebras. Equivalence of algebras up to free-field contributions goes under the name stable equivalence [81] and is an active area of research.

Finally, let us mention that introducing orientifolds parallel to D3-branes in the type IIB setup leads to an otrhosymplectic version of the algebras [1]. Calculation of characters in the large N_i hits that the algebras might be identified with quotients of the even version of the $\mathcal{W}_{1+\infty}$ from [82, 83]. Algebras associated to more complicated (p,q) webs in the presence of the orientifold planes should come from gluing of these algebras.

[7]The algebra $\mathcal{W}_{1+\infty}$ is known to be isomorphic to the affine Yangian of $\mathfrak{gl}(1)$. The triality in the representation of [80] simply permutes the deformation parameters h_i.

2. VOA preliminaries

This section reviews basics of vertex operator algebras and introduces the notation in terms of fields and operator product expansion. Physically oriented readers are advised to consult [6, 7, 8] and mathematically oriented readers are recommended to see [11, 9] for further details.

2.1 Vertex operator algebras

Roughly speaking, VOAs are algebras of fields $\mathcal{O}_\alpha(z)$ depending on a position $z \in \mathbb{C}$ together with a set of operator product expansions specifying singular behavior of correlation functions when two fields collide. From two fields $\mathcal{O}_\alpha(z)$ and $\mathcal{O}_\beta(w)$, one can form new operators by considering derivatives $\partial^n \mathcal{O}_\alpha(z)$ and the normal ordered product[1]

$$(\mathcal{O}_\alpha \mathcal{O}_\beta)(w) = \oint_w \frac{dz}{2\pi i} \frac{\mathcal{O}_\alpha(z)\mathcal{O}_\beta(w)}{z-w}. \tag{2.1}$$

The derivative satisfies

$$\partial(\mathcal{O}_\alpha \mathcal{O}_\beta)(w) = (\partial\mathcal{O}_\alpha \mathcal{O}_\beta)(w) + (\mathcal{O}_\alpha \partial\mathcal{O}_\beta)(w). \tag{2.2}$$

Vertex operator algebras admit a $\mathbb{Z} \times \mathbb{Z}_2$ grading, where the $\mathbb{Z}$ grading is called the conformal weight $h(\mathcal{O}_\alpha)$ and satisfy

$$\begin{aligned} h((\mathcal{O}_\alpha \mathcal{O}_\beta)) &= h(\mathcal{O}_\alpha) + h(\mathcal{O}_\beta), \\ h(\partial\mathcal{O}_\alpha) &= h(\mathcal{O}_\alpha) + 1. \end{aligned} \tag{2.3}$$

[1] The normally-ordered product can be thought of as an operator coming from a collision of two operators while subtracting the singular terms in the operator product expansion.

The $\mathbb{Z}_2$ grading[2] $|\mathcal{O}_\alpha|$ distinguishes fermionic and bosonic fields. The grading $\mathbb{Z}_2$ is additive under the normal ordering $|(\mathcal{O}_\alpha \mathcal{O}_\beta)| = |\mathcal{O}_\alpha| + |\mathcal{O}_\beta|$ and does not change under the derivative $|\partial \mathcal{O}_\alpha| = |\mathcal{O}_\alpha|$. Fields $\mathcal{O}_\alpha(z)$ and $\mathcal{O}_\beta(w)$ satisfy the operator product expansion (OPE)

$$\mathcal{O}_\alpha(z)\mathcal{O}_\beta(w) = \sum_{k=0}^{h(\mathcal{O}_\alpha)+h(\mathcal{O}_\beta)} \frac{\{\mathcal{O}_\alpha \mathcal{O}_\beta\}_k(w)}{(z-w)^k}, \tag{2.4}$$

where $\{\mathcal{O}_\alpha \mathcal{O}_\beta\}(w)$ is a field of conformal weight k. In particular, we have

$$\{\mathcal{O}_\alpha \mathcal{O}_\beta\}_0(z) = (\mathcal{O}_\alpha \mathcal{O}_\beta)(z). \tag{2.5}$$

OPEs furthermore satisfy

$$\mathcal{O}_\alpha(z)\mathcal{O}_\beta(w) = (-1)^{|\mathcal{O}_\alpha|+|\mathcal{O}_\beta|}\mathcal{O}_\beta(w)\mathcal{O}_\alpha(z). \tag{2.6}$$

Sometimes, we omit the regular term in the OPE and exchange the equal sign by $\sim$.

A vertex operator algebra is called strongly generated by fields $W_\alpha(z)$ with α labeling different strong generators, if all the other fields are generated from $W_\alpha(z)$ by derivatives and normally-ordered products. The algebra structure is given by operator product expansions of strong generators

$$W_\alpha(z)W_\beta(w) = \sum_{k=0}^{h(W_\alpha)+h(W_\beta)} \frac{\{W_\alpha W_\beta\}_k(w)}{(z-w)^k}. \tag{2.7}$$

Knowing the system of OPEs of strong generators, one can find OPEs of a general field by using contractions defined as

$$\overbrace{\mathcal{O}_\alpha(z)\mathcal{O}}_\beta(w) = \sum_{k=1}^{h(\mathcal{O}_\alpha)+h(\mathcal{O}_\beta)} \frac{\{\mathcal{O}_\alpha \mathcal{O}_\beta\}_k(w)}{(z-w)^k} \tag{2.8}$$

and the generalized Wick's theorem

$$\mathcal{O}_\alpha(z)(\mathcal{O}_\beta\mathcal{O}_\gamma)(w) = \oint_w \frac{dx}{2\pi i}\frac{1}{x-w}\left(\overbrace{\mathcal{O}_\alpha(z)\mathcal{O}}_\beta(x)\mathcal{O}_\gamma(w) + \overbrace{\mathcal{O}_\alpha(z)\mathcal{O}_\beta(x)\mathcal{O}}_\gamma(w)\right). \tag{2.9}$$

The operator ∂ acts simply as a derivative wrt. z or w respectively on the right hand side of OPE.

[2] VOAs with such $\mathbb{Z}_2$ grading are sometimes called vertex operator super-algebras. We will not distinguish these notions in this thesis.

The above data satisfy various properties. First, there exist the identity field $\mathbb{1}$ with trivial OPEs and a field T called the stress-energy tensor that is a weight-two bosonic field with OPE

$$T(z)T(w) \sim \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(z)}{z-w} \tag{2.10}$$

where $c \in \mathbb{C}$ is called the central charge. All generators have the following OPE with the stress-energy tensor[3]

$$T(z)W_\alpha(w) \sim \cdots + \frac{h(W_\alpha)W_\alpha(w)}{(z-w)^2} + \frac{\partial W_\alpha(w)}{z-w}. \tag{2.11}$$

Secondly, the system of OPEs satisfy the associativity constraint

$$\{\mathcal{O}_\alpha\{\mathcal{O}_\beta\mathcal{O}_\gamma\}_j\}_k - (-1)^{|\mathcal{O}_\alpha|+|\mathcal{O}_\beta|}\{\mathcal{O}_\beta\{\mathcal{O}_\alpha\mathcal{O}_\gamma\}_k\}_j = \sum_{l>0}\binom{k-1}{l-1}\{\{\mathcal{O}_\alpha\mathcal{O}_\beta\}_l\mathcal{O}_\gamma\}_{j+k-l}. \tag{2.12}$$

This constraint might be satisfied only modulo some composite fields forming an ideal $\mathcal{I}$ of the OPE algebra as we will see for example in the algebra $Y_{0,1,2}[\Psi]$. The algebra $Y_{0,1,2}[\Psi]$ is going to be strongly generated by five fields satisfying the associativity constraint up to two composite fields together with the normally-ordered product of their derivatives. The algebra is consisted if we mode out these fields. The fields in $\mathcal{I}$ are sometimes going to be called null.

The VOA data naturally lead to an associative algebra of modes. We can expand the strong generators into the modes as

$$W_\alpha(z) = \sum_{n=-\infty}^{\infty}\frac{W_{\alpha,i}}{z^{n+h(W_\alpha)}}, \tag{2.13}$$

i.e.

$$W_{\alpha,i} = \oint\frac{dz}{2\pi i}W_\alpha(z)z^{n+h(W_\alpha)-1}. \tag{2.14}$$

The commutation relations of modes can be determined from the OPE by the standard contour deformation argument

$$[W_{\alpha,i}, W_{\beta,j}] = \oint_0\frac{dw}{2\pi i}\oint_w\frac{dz}{2\pi i}z^{i+h(W_\alpha)}w^{j+h(W_\beta)}W_\alpha(z)W_\beta(w) \tag{2.15}$$

[3] A field W_α with all the other terms corresponding to the dots vanishing are called primary.

with the corresponding OPE inserted on the right hand side. The modes of the normally-ordered products and derivatives can be explicitly written as

$$\begin{aligned}(\partial\mathcal{O}_\alpha)_i &= -(h(\mathcal{O}_\alpha)+i)\mathcal{O}_{\alpha,i}\\ (\mathcal{O}_\alpha\mathcal{O}_\beta)_i &= \sum_{j\leq -h(\mathcal{O}_\alpha)}\mathcal{O}_{\alpha,j}\mathcal{O}_{\beta,i-j}+\sum_{j>-h(\mathcal{O}_\alpha)}\mathcal{O}_{\beta,i-j}\mathcal{O}_{\alpha,j}.\end{aligned} \tag{2.16}$$

The corresponding associative algebra is the universal enveloping algebra of the modes (or its completion containing the infinite sums from the normal ordered products) of the generating fields satisfying the above commutation relations modded by the ideal corresponding to modes of null fields. Note that the associativity of OPE becomes the Jacobi identity of the associative algebra.

2.2 Examples

Virasoro algebra First, one can consider the VOA with the stress-energy tensor $T(z)$ being the only generator of the algebra. Expanding into modes

$$T(z)=\sum_{i=-\infty}^{\infty}\frac{L_i}{z^{i+2}} \tag{2.17}$$

and using the contour integral formula (2.15), one gets the Virasoro algebra with commutation relations

$$[L_i,L_j]=(i-j)L_{i+j}+\frac{c}{12}(i^3-i)\delta_{i+j,0}. \tag{2.18}$$

Heisenberg VOA Second, probably the first non-trivial example is the Heisenberg VOA (or the $\widehat{\mathfrak{gl}(1)}$ Kac-Moody algebra) that is an algebra generated by a single field $J(z)$ with OPE of the form

$$J(z)J(w)\sim\frac{1}{(z-w)^2}. \tag{2.19}$$

One can show that there exists a stress-energy tensor

$$T(z)=\frac{1}{2}(JJ)(z) \tag{2.20}$$

with $J(z)$ being a primary field of weight one, i.e. $h(J)=1$. If we decompose the field into the modes

$$J(z)=\sum_{n=-\infty}^{\infty}\frac{J_n}{z^{n+1}}, \tag{2.21}$$

we get an infinite system of Heisenberg algebras

$$[J_n,J_m]=n\delta_{n,-m}. \tag{2.22}$$

Kac-Moody algebra More generally, one can associate Kac-Moody algebra the $\widehat{\mathfrak{g}}_k$ to any simple Lie algebra (or super-algebra) $\mathfrak{g}$ generated by currents J_α with α labeling generators of $\mathfrak{g}$ and satisfying

$$J_\alpha(z)J_\beta(w)\sim\frac{kg_{\alpha\beta}}{(z-w)^2}+\frac{\sum_\gamma f_{\alpha\beta}^\gamma J_\gamma(w)}{z-w}. \tag{2.23}$$

In this expression, $k\in\mathbb{C}$ is called the level, $g_{\alpha\beta}$ is the Killing form of the Lie algebra and $f_{\alpha\beta}^\gamma$ are the structure constants $[g_\alpha,g_\beta]=\sum_\gamma f_{\alpha\beta}^\gamma g_\gamma$. The algebra contains the standard Sugawara stress-energy tensor

$$T=\frac{1}{2(k+h)}\sum_{\alpha,\beta}g^{\alpha\beta}(J_\alpha J_\beta) \tag{2.24}$$

with h the dual Coxeter number of $\mathfrak{g}$ and $g^{\alpha\beta}$ the inverse of $g_{\alpha\beta}$.

It is convenient to introduce the notation $\widehat{\mathfrak{gl}(N|M)}_\Psi$ for a product of the $\widehat{\mathfrak{gl}(N|M)}_{\Psi-N+M}$ Kac-Moody algebra and the Heisenberg VOA $\widehat{\mathfrak{gl}(1)}_{(N-M)\Psi}$. OPEs of the currents J_b^a (with a,b labeling row and column indices of the $\mathfrak{gl}(N|M)$ super-matrix) are given by

$$\begin{aligned}J_b^a(z)J_d^c(0)\sim\ &\frac{(-1)^{p(b)p(c)}(\Psi-M+N)\delta_d^a\delta_b^c+\delta_b^a\delta_d^c}{z^2}+\\&+\frac{(-1)^{p(a)p(b)+p(c)p(d)+p(c)p(b)}\delta_d^aJ_b^c(0)-(-1)^{p(b)p(c)}\delta_b^cJ_d^a}{z}\end{aligned} \tag{2.25}$$

where $p(a)=0$ for $a=1,\dots,M$ and $p(a)=1$ otherwise.

Note that for $\widehat{\mathfrak{gl}(N|0)}_\Psi$ the algebra reduces to the Kac-Moody algebra $\widehat{\mathfrak{gl}(N)}_{\Psi-N}\times\widehat{\mathfrak{gl}(1)}_{N\Psi}$ with OPEs given by

$$J_b^a(z)J_d^c(w)\ \sim\ \frac{(\Psi-N)\delta_d^a\delta_b^c+\delta_b^a\delta_d^c}{(z-w)^2}+\frac{\delta_d^aJ_b^c(w)-\delta_b^CJ_D^a(w)}{z-w} \tag{2.26}$$

and similarly $\widehat{\mathfrak{gl}(0|M)}_\Psi$ reduces to the algebra $\widehat{\mathfrak{gl}(M)}_{-\Psi+N} \times \widehat{\mathfrak{gl}(1)}_{-N\Psi}$. In the expressions above, we choose a certain normalization of the diagonal $\widehat{\mathfrak{gl}(1)}$ current that will be important in later discussions of VOA modules.

β,γ and b,c ghosts Finally, there is the β,γ system and its fermionic version b,c system generated by fields $\beta(z),\gamma(z)$ and $b(z),c(z)$ respectively. They satisfy OPE

$$\beta(z)\gamma(w) \sim \frac{1}{z-w}, \qquad b(z)c(w) \sim \frac{1}{z-w}. \tag{2.27}$$

There exists a one parameter family of stress-energy tensors

$$T = (\partial\beta\gamma) - \lambda\partial(\beta\gamma), \qquad T = (\partial bc) - \lambda\partial(bc) \tag{2.28}$$

with the corresponding central charge

$$c = \pm(3(2\lambda-1)^2 - 1) \tag{2.29}$$

with the plus sign for β,γ ghosts and minus sign for b,c ghosts. The b,c system with the choice of the stress-energy tensor such that $c=\frac{1}{2}$ is also called the free fermion (ψ,χ) and the β,γ system is sometimes called the symplectic-boson pair (X,Y).

We introduce the notation $\mathcal{S}^{N|M}$ for a system of N pairs of symplectic bosons (X_a, Y^a) where $a=1,\dots N$ and M free fermions (χ_i,ψ^i) for $i=1,\dots,M$ with OPEs given by

$$X_a(z)Y^b(w) \sim \frac{\delta_a^b}{z-w}, \qquad \chi_i(z)\psi^j(w) \sim \frac{\delta_i^j}{z-w}. \tag{2.30}$$

Note that the algebra $\mathcal{S}^{N|M}$ contains $\widehat{\mathfrak{gl}(N|M)}_{N-M-1}$ subalgebra generated by bilinears

$$J = \begin{pmatrix} X_a Y^b & X_a\psi^i \\ \chi_j Y^b & \chi_j\psi^i \end{pmatrix}. \tag{2.31}$$

Similarly, exchanging the role of bosons and fermions, we introduce a notation $\overline{\mathcal{S}}^{N|M}$ for a system M symplectic bosons (X_i, Y^i) and N free fermions (χ_a,ψ^a) with OPEs given by

$$X_i(z)Y^j(w) \sim \frac{\delta_i^j}{z-w}, \qquad \chi_a(z)\psi^b(w) \sim \frac{\delta_a^b}{z-w}. \tag{2.32}$$

From their bilinears, we can construct $\widehat{\mathfrak{gl}(N|M)}_{N-M+1}$ algebra generated by bilinears

$$J = \begin{pmatrix} \chi_a \psi^b & \chi_a Y^i \\ X_j \psi^b & X_j Y^i \end{pmatrix}. \tag{2.33}$$

We also introduce a notation $\mathcal{F}$ for the algebra of a free fermion $\mathcal{S}^{0|1}$ and $\mathcal{B}$ for the algebra of a free symplectic boson $\mathcal{S}^{1|0}$.

Occasionally, we also use notation $\mathcal{S}_i^{N|M}$ for the system $\mathcal{S}^{N|M}$ with a choice of the stress-energy tensor such that the fields have the shifted conformal dimension $(\frac{1}{2}+i, \frac{1}{2}-i)$. Corresponding stress-energy tensor have central charge

$$(N-M)(12(1+i)^2-1). \tag{2.34}$$

We analogously define $\overline{\mathcal{S}}_i^{N|M}$, $\mathcal{B}_i$ and $\mathcal{S}_i$.

2.3 Modules

Let us briefly discuss some aspects of VOA modules. By a module of a VOA strongly generated by W_α, we mean a set of fields $M_\beta(w)$ with a set of OPEs with the generating fields $W_\alpha(z)$ that are consistent with the algebra structure, i.e. they satisfy the associativity constraint with two insertions of $W_\alpha(z)$ and a single insertion of $M_\alpha(w)$ and they have trivial OPE with the null fields of the algebra. Note that OPEs of the fields $M_\alpha(w)$ themselves are not part of the data.

For each VOA, there exists a natural module called the vacuum module that comes from the action of the algebra on itself. The vacuum module has also a simple description in terms of the associative algebra of modes. Such algebra decomposes into the subalgebra of positive, negative and zero modes. One can consider a one-dimensional module of the subalgebra of positive and zero modes on which all such modes act by zero. One can then generate the module of the full algebra by an action of all the negative modes modulo the commutation relations and the relations coming from the action of positive modes of null fields. Such an induced module is not irreducible and one needs to further set $W_{\alpha,i} = 0$ for $i < h(W_\alpha)$ to get the desired vacuum module.

One can generalize the construction of the vacuum module slightly by considering a non-trivial representation of the zero-mode subalgebra (with the zero mode of the null fields acting by zero), extending the action to the subalgebra of non-negative modes by letting all the positive modes act by zero and inducing the module of the whole algebra in a similar way as in the case of the vacuum module. Such a module is generically not going to be irreducible and passing to an irreducible quotient, one needs to again remove some states. We will see many non-trivial modules of this form (and even more complicated examples) in later sections.

Heisenberg VOA For example, one can define a module of the Heisenberg VOA by introducing a field $M[\alpha](w)$ with OPE

$$J(z)M[\alpha](w) \sim \frac{\alpha M[\alpha](w)}{z-w}. \tag{2.35}$$

The module is then generated by a normally-ordered product of $M[\alpha](w)$ with normally-ordered products of derivatives $\partial^n J(z)$. One can show that in terms of modes, the module can be identified with a highest weight module generated from the state $|\alpha\rangle$ anihilated by all the positive modes $J_n|\alpha\rangle = 0$ for $n > 0$ and satisfying $J_0|\alpha\rangle = \alpha|\alpha\rangle$. Note that the vacuum module can be identified simply with the $\alpha = 0$ case.

Virasoro algebra Similarly, modules of the stress-energy tensor VOA can be generated from $M[h](z)$ with OPE

$$T(z)M[h](z) \sim \frac{hM[h](z)}{(z-w)^2} + \frac{\partial M[h](z)}{z-w}. \tag{2.36}$$

The vacuum module then corresponds to $h = 0$ and $\partial M[h] = 0$ that corresponds to setting $L_{-1}|0\rangle = 0$.

Ising model As an illustration of possible restrictions that arise in the presence of null fields that are modded out, let us consider the $c = 1/2$ Virasoro algebra. It is well-known [84] that the vacuum representation contains a singular vector at level 6 with the corresponding primary field

$$\phi_6 = 128(T(TT)) + 186(\partial T\partial T) - 264(\partial^2 TT) - 9\partial^4 T. \tag{2.37}$$

The requirement of vanishing of the null field constrains possible modules for the VOA. In our case, let us consider a generic primary field $M[h](z)$ of dimension h as above and require the operator product expansion of ϕ_6 and $M[h](z)$ to vanish. The most singular (sixth order) term is precisely the zero mode of ϕ_6 acting on the highest weight

$$\frac{1}{2\pi i}\oint dz z^5 \phi_6(z)|h\rangle = 4h(2h-1)(16h-1)|h\rangle = 0 \tag{2.38}$$

and the variety of highest weights consists of three points $h = 1/2$, $h = 1/16$, and $h = 0$. These are the allowed primary fields of the Ising model.

It is interesting to look also at the conditions following from the vanishing of the lower order poles in the OPE

$$\begin{aligned}\phi_6(z)M[h](w) \sim\ & \frac{4h(2h-1)(16h-1)M[h](w)}{(z-w)^6} + \frac{12(2h-1)(16h-1)\partial M[h](w)}{(z-w)^5} \\ &+ \frac{48h(8h-17)(TM[h])(w)}{(z-w)^4} + \frac{6(64h+7)\partial^2 M[h](w)}{(z-w)^4} + \mathcal{O}((z-w)^{-3}).\end{aligned} \tag{2.39}$$

The quintic pole vanishes for $h = 1/2$ and $h = 1/16$, while for $h = 0$ it requires $\partial M[0] = 0$ which is the usual singular vector of the vacuum representation at level 1 (translation invariance of the vacuum).

Let us look at the quartic pole more closely. For $h = 0$ it does not give us anything new while for $h = 1/2$ it requires

$$4(TM[1/2])(z) - 3\partial^2 M[1/2](z) \tag{2.40}$$

to be zero and for $h = 1/16$

$$3(TM[1/16])(z) - 4\partial^2 M[1/16](z) \tag{2.41}$$

to be zero. These are just the singular vectors of $h_{2,1}$ and $h_{1,2}$ Virasoro primaries. We could proceed further and find other relations coming from the lower order poles.

2.4 Constructions

Definition of VOAs is very non-trivial and it is not a simple task to construct new VOAs. Let us now mention few well-known constructions that will be used intensively throughout the text.

BRST reduction The fist construction goes under the name BRST (or the quantum Hamiltonian) reduction. If we have an algebra that contains a fermionic field J_{BRST} of conformal weight one and trivial OPE with itself, one can restrict to the cohomology of the zero mode of such a generator

$$Q_{BRST} = \oint \frac{dz}{2\pi i} J_{BRST}(z). \tag{2.42}$$

Associativity of OPEs ensure that OPE of Q_{BRST}-closed operators contains only Q_{BRST}-closed operators and OPE of a Q_{BRST}-exact operator with any other operator is Q_{BRST} exact. The restriction to the Q_{BRST} cohomology thus makes sense. We will see various examples of such BRST reductions called the Drinfeld-Sokolov reduction [64, 65, 66, 66, 67, 68] or the BRST cosets [85, 86, 70, 71].

Free field realization Secondly, one can consider algebras constructed as subalgebras of tensor products of free-field algebras such as the Heisenberg VOA, β, γ ghosts and b, c ghosts. The corresponding construction goes under the name free field realization.

As a simple example, note that the stress-energy tensor $T(z)$ of central charge $c = 1 - 12\alpha_0^2$ can be constructed in terms of a single Heisenberg VOA as

$$T(z) = \frac{1}{2}(JJ)(z) + \alpha_0 \partial J(z). \tag{2.43}$$

We will see many examples of free field realizations that will be characterized in terms of an intersection of kernels of operators called screening charges [87] or coming from a product of differential operators containing fields as coefficients called the Miura transformation [22].

Extensions by modules Thirdly, one can consider extensions of known VOAs by their modules. In particular, it turns out that we can sometimes give a VOA structure to a VOA together with a collection of modules by specifying OPEs of the fields associated to modules. The simplest example dates back to the early days of VOAs [11] and corresponds to various lattice extensions of Heisenberg VOAs. In recent years, it turned out that the construction is much more general [88, 24, 2] and is an active area of research.

Bootstrap Finally, one can take a very pedestrian way to construct new VOAs using bootstrap. Starting with some assumptions on the VOA (with a prototypical example of looking for extensions of the stress-energy tensor by primary fields of prescribed conformal weights), one can write the most general ansatz for OPEs that is consistent with such assumptions and solve the conditions coming from the requirement of the associativity of OPEs. For example, if we consider the algebra to be an extension of $T(z)$ by a single primary field of conformal weight three $W_3(z)$, it turns out that the associativity constraint fully fixes the OPE of $W_3(z)$ with itself and leads to the algebra called $\mathcal{W}_3$ [21] that depends on a single continuous parameter c. Performing a similar analysis with adding a primary field for each weight $3, 4, 5, \ldots$, one finds a two-parameter family of algebras [74, 75, 5, 76] called $\mathcal{W}_\infty$. One can see that the bootsrap approach is very restrictive and might serve as a great tool to construct new VOAs.

3. The vertex

In this section, we discuss the algebra $Y_{N_1,N_2,N_3}[\Psi]$ associated to the simplest, trivalent (p,q) web. We review the type IIA and the type IIB perspective and give four conjecturally equivalent definitions of the algebra in terms of a BRST reduction, truncations of $\mathcal{W}_{1+\infty}$ algebra, a kernel of screening charges and a Miura transformation.

3.1 Type IIB perspective

The first perspective originates from the following string theory setup. Type IIB string theory contains six-dimensional objects labelled by co-prime integers (p,q) with $(1,0)$ corresponding to the D5-brane, $(0,1)$ to the NS5-brane and (p,q) to a brane of a general D5 and NS5 charge. One can consider a web of such branes that preserves quarter of the supersymmetry as first discussed in [38, 39]. The (p,q) diagram then describes a projection of the ten-dimensional string-theoretical setup to the 23-plane with the lines specifying the position of branes and its slope equal to the ratio p/q.

We can then attach N_i D3-branes to such a web of (p,q)-branes from various corners[1] as indicated by the colored (p,q) web. The low energy dynamics of N_i D3-branes is known to have an effective description in terms of 4d $\mathcal{N}=4$ super Yang-Mills (SYM) theory with the gauge group $U(N_i)$. D3-branes ending on (p,q) fivebranes then imply the existence of a family of half-BPS interfaces[2] $\mathcal{B}_{(p,q)}$ for 4d $\mathcal{N}=4$ SYM with unitary gauge groups, parameterized by two integers (p,q) defined up to an overall sign. Concretely, these

[1] This motivated the name "corner algebras" in [1].

[2] The main property of these interfaces is that they are covariant under the action of $PSL(2,Z)$ S-duality transformations of $\mathcal{N}=4$ SYM, which act in the obvious way on the integers (p,q). This action leads to conjectures of dual descriptions of VOAs and served in [1] as a strong test of the correct identification of the algebra $Y_{N_1,N_2,N_3}[\Psi]$.

interfaces arise as the field theory limit of a setup involving two sets of D3 branes ending on a single (p,q)-fivebrane from opposite sides [62, 89, 63].

3.1.1 Gauge theory setup

Most of the $\mathcal{B}_{(p,q)}$ interfaces do not admit a straightforward, weakly coupled definition. Rather, they involve some intricate 3d SCFT coupled to the $U(N)$ and $U(M)$ gauge theories on the two sides of the interface. The exceptions are $\mathcal{B}_{(1,0)}$ and $\mathcal{B}_{(p,1)}$ interfaces.

The $\mathcal{B}_{(1,0)}$ interface, also denoted as a D5 interface, has a definition which depends on the relative value of N and M:

- If $N = M$, a D5 interface breaks the $U(N)_L \times U(N)_R$ gauge symmetry of the bulk theories to a diagonal $U(N)$. A set of 3d hypermultiplets transforming in a fundamental representation of $U(N)$ is coupled to the $U(N)$ gauge fields. Concretely, the 4d fields on the two sides of the interface are identified at the interface, up to some discontinuities involving bilinears of the 3d fields.
- If $N > M$, a D5 interface breaks the $U(M)_L \times U(N)_R$ gauge symmetry of the bulk theories to a block-diagonal $U(M)$. Concretely, $U(N)_R$ is broken to a block-diagonal $U(N-M)_R \times U(M)_R$ and $U(N-M)_R \times U(M)_L \times U(M)_R$ is further broken to the diagonal $U(M)$. The breaking of $U(N-M)_R$ involves a Nahm pole boundary condition of rank $N - M$. No further matter fields are needed at the interface.
- If $M > N$, a D5 interface breaks the $U(M)_L \times U(N)_R$ gauge symmetry of the bulk theories to a $U(N)$, including a Nahm pole of rank $M - N$.

The $\mathcal{B}_{(0,1)}$ interface, also denoted as an NS5 interface, has a uniform definition for all N and M [63]: the gauge groups are unbroken at the interface and coupled to 3d hypermultiplets transforming in a bi-fundamental representation of $U(M) \times U(N)$. The $\mathcal{B}_{(p,1)}$ interface is obtained from a $\mathcal{B}_{(0,1)}$ interface by adding q units of Chern-Simons coupling on one side of the interface, $-q$ on the other side.

A well known property of (p,q)-fivebranes is that they can form quarter-BPS webs [38, 39], configurations with five-dimensional super-Poincare invariance involving fivebrane segments and half-lines drawn on a common plane, with slope determined by the phase of their central charge. For graphical purposes, the slope can be taken to be p/q, though the actual slope depends on the IIB string coupling τ and is the phase of $p\tau + q$. The simplest example of brane web is the junction of three semi-infinite branes of type $(1,0)$, $(0,1)$ and $(1,1)$.

These configurations preserve four super-charges, organized in a $(0,4)$ 2d supersymmetry algebra. One may thus consider a setup with N_1,N_2,N_3 D3 branes respectively filling the faces of the junction in between the $(1,1)$ and $(1,0)$ fivebranes, the $(1,0)$ and $(0,1)$ fivebranes and the $(0,1)$ and $(1,1)$ fivebranes.

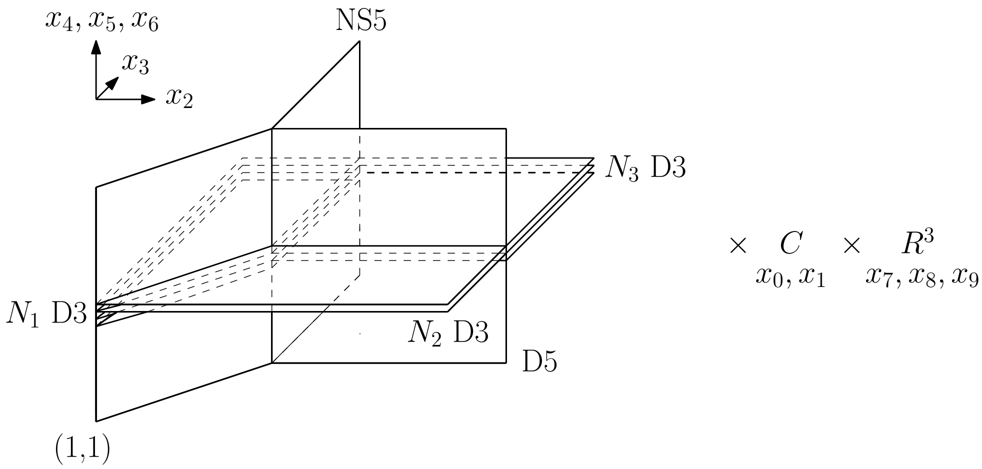


Figure 3.1: The brane system engineering our Y-junction for four-dimensional $\mathcal{N} = 4$ SYM. The three fivebranes extend along the 01456 directions together with a ray in the 23 plane. The stacks of D3-branes extend along the 01 directions and fill wedges in the 23 plane. Notice the $SO(3)_{456} \times SO(3)_{789}$ isometry of the system, which becomes the R-symmetry of a $(0,4)$ 2d super-symmetry algebra.

We will next conjecture the field theory description of the junction. Our conjecture is motivated by some matching of 2d anomalies and consistency with the GL-twisted description discussed later in this section.

The $N_1 = 0, N_2 = N_3$ junctions At first, we can set $N_2 = N_3$ and $N_1 = 0$. That means we have a $U(N_3)$ gauge theory defined on the $x_2 > 0$ half-space with Neumann boundary conditions at $x_2 = 0$. The boundary conditions are deformed by an unit of Chern-Simons boundary coupling on the $x_3 < 0$ half of the boundary. We also have an interface at $x_3 = 0$, where the $U(N_3)$ gauge theory is coupled to a set of N_3 3d hypermultiplets transforming in a fundamental representation of the gauge group.

The interface meets the boundary at $x_2 = x_3 = 0$. The hypermultiplets must have some boundary conditions at the origin of the plane, preserving $(0,4)$ supersymmetry. There is a

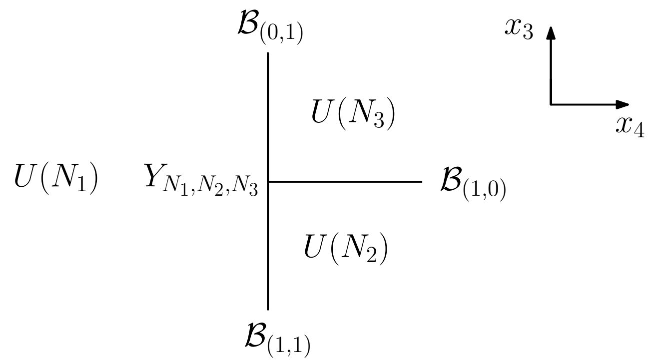


Figure 3.2: The gauge theory image of a Y-junction on the 23 plane. We denote the specific junction as Y_{N_1,N_2,N_3}. The $Y_{N_1,N_2,N_3}[\Psi]$ VOA will arise as a deformation of the algebra of BPS local operators at the junction.

known example of such a boundary condition, involving Neumann boundary conditions for all the scalar fields. We expect it to appear in the field theory limit of the junction setup. The choice of Neumann b.c. is natural for the following reasons: the relative motion of the D3 branes on the two sides of the D3 interface involves the 3d hypermultiplets acquiring a vev. The junction allows for such a relative motion to be fully unrestricted and thus the 3d hypermultiplets boundary conditions should be of Neumann type.

The $(0,4)$ boundary conditions for the hypermultiplets have an important feature: they set to zero the left-moving half of the hypermultiplet fermions at the boundary. Such a chiral boundary condition has a 2d gauge anomaly which is cancelled by anomaly inflow from the boundary $U(N_3)$ Chern-Simons coupling along the negative imaginary axis. This anomaly will reappear in a similar role in the next section.

The $N_1 = 0, N_2 \neq N_3$ junctions Next, we can consider $N_3 = N_2 + 1$ and $N_1 = 0$. Now we do not have 3d matter along the positive real axis, but the gauge group drops from $U(N_2+1)$ to $U(N_2)$ across the boundary. The four-dimensional gauginoes which belong to the $U(N_2+1)$ Lie algebra but not to the $U(N_2)$ subalgebra live on the upper right quadrant of the junction plane with non-trivial boundary conditions on the two sides. They may in principle contribute a 2d $U(N_2)$ gauge anomaly at the corner. It is a bit tricky to compute it, but we will recover it from a vertex algebra computation in Section 3.1.2. Again, we expect it to cancel the anomaly inflow from the boundary $U(N_2)$ Chern-Simons coupling along the negative imaginary axis.

Similar considerations for general $N_3 \neq N_2$ and $N_1 = 0$, though the positive real axis now supports a partial Nahm pole boundary condition along with the reduction from $U(N_1)$ to $U(N_2)$ or vice-versa. Again, we will describe the corresponding anomalies and their cancellations in Section 3.1.2.

The $N_1 > 0, N_2 = N_3$ junctions Next, we can set $N_2 = N_3$ but take general N_1. That means we have an $U(N_3)$ gauge theory defined on the $x_2 > 0$ half-space and an $U(N_1)$ gauge theory defined on the $x_2 < 0$ half-space. Both boundary conditions are deformed by an unit of Chern-Simons boundary coupling on the $x_3 < 0$ half of the boundary, with opposite signs for the two gauge groups. At the common boundary at $x_2 = 0$, the gauge fields are coupled to 3d $N_1 \times N_3$ bi-fundamental hypermultiplets. We also have an interface at $x_3 = 0$, $x_2 > 0$, where the $U(N_3)$ gauge theory is coupled to a set of N_3 3d hypermultiplets transforming in a fundamental representation of the gauge group.

The interfaces meet at $x_2 = x_3 = 0$. The fundamental hypermultiplets should be given a boundary condition at the origin which preserve $(0, 4)$ symmetry. The boundary condition may involve the bi-fundamental hypermultiplets restricted to the origin and, potentially, extra 2d degrees of freedom defined at the junction only.

We can attempt to define the boundary condition starting from the basic $(0, 4)$ Neumann b.c. for the fundamental hypermultiplets and adding extra couplings at the origin. These couplings will not play a direct role for us but help us conjecture the correct choice of auxiliary 2d degrees of freedom needed at the corner in order to reproduce the field theory limit of the brane setup.

Indeed, the values at $x_2 = x_3 = 0$ of the bi-fundamental and fundamental hypers behave as $(0, 4)$ hypermultiplets and $(0, 4)$ twisted hypermultiplets respectively. There is a known way to couple these types of fields in a $(0, 4)$-invariant way, but it requires the addition of an extra set of $(0, 4)$ fields: Fermi multiplets transforming in the fundamental representation of $U(N_1)$ which can enter in a cubic fermionic superpotential with the hypermultiplets and twisted hypermultiplets [90].

This coupling is known to occur in similar situations involving multiple D-branes ending on an NS5-brane [91]. The Fermi multiplets should arise from D3-D5 strings and the coupling from a disk amplitude involving D3-D5, D3-D3' and D3'-D5 strings in the presence of an NS5 brane.

The $U(N_1)$ fundamental Fermi multiplets also play another role: they consist of 2d left-moving fermions, whose anomaly compensates the inflow from the boundary $U(N_1)$ Chern-Simons coupling along the negative imaginary axis.

The $N_1 > 0, N_2 \neq N_3$ junctions Next, we can consider $N_3 = N_2 + 1$ and general N_1. Now the number of hypermultiplets along the imaginary axis drops from $N_1 \times N_3$ to $N_1 \times N_2$ across the origin of the junction's plane. We can glue together $N_2 \times N_1$ of them according to the embedding of $U(N_2)$ in $U(N_3)$ along the real axis, but we need a boundary condition for the remaining N_1 hypermultiplets.

Neumann boundary conditions for these N_1 hypermultiplets would contribute an anomaly of the wrong side to cancel the inflow from the boundary $U(N_1)$ Chern-Simons coupling along the negative imaginary axis. The opposite choice of boundary conditions, i.e. Dirichlet b.c. for the scalar fields, imposes the opposite boundary condition on the hypermultiplet's fermions and seems a suitable choice. We will thus not need to add extra 2d Fermi multiplets at the corner.[3]

Similar considerations for general $N_2 \neq N_3$ and general N_1, though the positive real axis now supports a partial Nahm pole boundary condition along with the reduction from $U(N_3)$ to $U(N_2)$ or vice-versa. The boundary conditions at the corner for the $|N_3 - N_2| \times N_1$ hypermultiplets which do not continue across the corner will be affected by the Nahm pole. We will refrain from discussing them in detail here and focus on the GL-twisted version in the next section.

3.1.2 From junctions to interfaces in Chern-Simons theory

The analysis of [43] gives a prescription for how to embed calculations in (analytically continued) Chern-Simons theory into GL-twisted four-dimensional $\mathcal{N} = 4$ Super-Yang-Mills theory.

Concretely, a Chern-Simons calculation on a three-manifold M_3 maps to a four-dimensional gauge theory calculation on $M_3 \times \mathbb{R}^+$ with a specific boundary condition which deforms the standard supersymmetric Neumann boundary conditions. The (analytically continued) Chern-Simons level is related to the coupling Ψ of the GL-twisted $\mathcal{N} = 4$ SYM as [46]

$$k + h = \Psi \tag{3.1}$$

Assuming that the deformed Neumann boundary conditions transform in a manner analogous to the undeformed ones, that means the S transformation will map the Chern-

[3]Notice that one can obtain such boundary conditions starting from Neumann boundary conditions and coupling them to $(0,4)$ 2d Fermi multiplets, which get eaten up in the process. It would be nice to follow in detail in the field theory the process of separating a $D3$ brane segment from the $N = M$ system and flowing to the $N_3 = N_2 + 1$ system, by giving a vev to the fundamental hypermultiplets which induces a bilinear coupling of the 2d Fermi multiplets

Simons setup to a different setup involving a deformed Nahm pole boundary condition. This was a basic step in the gauge-theory description of categorified knot invariants in [46].

In general, we expect the $\mathcal{B}_{(p,q)}$ boundary conditions to admit deformations $\tilde{\mathcal{B}}_{(p,q)}$ compatible with the GL twist, such that $\tilde{\mathcal{B}}_{(0,1)}$ coincides with deformed Neumann boundary conditions and $PSL(2,Z)$ duality transformations act in the obvious way on the integers (p,q).

The general formalism of [43] relates a variety of analytically continued path integrals in d dimensions and topological field theory calculations in $d+1$ dimensions, possibly including local observables or defects. Intuitively, observables which are functions of the d-dimensional fields will map to the same functions applied to the boundary values of $(d+1)$-dimensional fields, but modifications of the d-dimensional path integral may propagate to modifications of the $(d+1)$-dimensional bulk. Extra degrees of freedom added in the d-dimensional setup may remain at the boundary of the $(d+1)$-dimensional bulk or analytically continued to extra degrees of freedom in the bulk.

A simple, rather trivial example of this flexibility is the observation that one can split off a well-defined multiple of the Chern-Simons action before analytic continuation, giving rise to a bulk theory with coupling $\Psi+q$ with a $\tilde{\mathcal{B}}_{(q,1)}$ boundary condition.

A more important example is analytically continued Chern-Simons theory defined on a manifold with boundary, $M_3 = C\times\mathbb{R}^+$, with some boundary condition B_{3d}. This setup will map to a calculation involving four-dimensional gauge theory on a corner geometry $C\times\mathbb{R}^+\times\mathbb{R}^+$. One of the two sides of the corner will have deformed Neumann boundary condition $\tilde{\mathcal{B}}_{(0,1)}$. The other side will have some boundary condition B_{4d} which can be derived from the boundary condition B_{3d} in a systematic fashion. At the corner, the two boundary conditions will be intertwined by some interface which is also derived from the boundary condition B_{3d}.

The simplest possibility is to consider holomorphic Dirichlet boundary conditions D^{3d} in Chern-Simons theory, given by $A_{\bar{z}}=0$ at the boundary. It is well known that these boundary conditions support Kac-Moody currents $J = A_z|_\partial$ of level $\Psi - h$, given by the holomorphic part of the connection restricted to the boundary. These boundary conditions will lift to a deformation of Dirichlet boundary conditions in SYM.

A slightly more refined possibility is to consider a generalization of holomorphic Dirichlet boundary conditions D^{3d}_ρ which is labelled by an $\mathfrak{su}(2)$ embedding in the gauge Liea algebra [92, 93]. These boundary conditions require the boundary gauge field to be a generalized oper of type ρ. They are expected to support the vertex operator algebras $\mathcal{DS}_\rho[\widehat{\mathfrak{g}}_{\Psi-h}]$ obtained from the $\widehat{\mathfrak{g}}_{\Psi-h}$ Kac-Moody algebra by a Quantum Drinfeld-Sokolov reduction. In particular, for the regular $\mathfrak{su}(2)$ embedding inside $\mathfrak{gl}(N)$, one obtains the standard $\mathcal{W}_N[\Psi] \times \widehat{\mathfrak{gl}(1)}$ algebras. These boundary conditions will lift to a deformation of the regular Nahm pole boundary conditions in SYM.

The regular Nahm pole boundary condition in SYM is precisely $\mathcal{B}_{(1,0)}$. That means the Chern-Simons setup leading to the standard $\mathcal{W}_N[\Psi] \times \widehat{\mathfrak{gl}(1)}$ algebras lifts to a corner geometry in SYM with $\tilde{\mathcal{B}}_{(1,0)}$ on one edge and a boundary condition we expect to coincide with $\tilde{\mathcal{B}}_{(0,1)}$ on the other edge. This is supported by the analysis of [42], which reduced the problem on a compact Riemann surface C and found conformal blocks for the corresponding $\mathcal{W}_N$-algebras.

At this point it is natural to seek configurations in Chern-Simons theory which could be uplifted to a deformation of the junctions in the previous section for general N_1, N_2, N_3, involving $\tilde{\mathcal{B}}_{(1,0)}$, $\tilde{\mathcal{B}}_{(0,1)}$ and $\tilde{\mathcal{B}}_{(1,1)}$ interfaces.

We take the same coupling Ψ uniformly in the whole plane of the Y_{N_1,N_2,N_3} junction and the T-shaped configuration of Figure 3.2: the construction of [43] applied along the x_2 direction maps the four-dimensional gauge theory with $\tilde{\mathcal{B}}_{(0,1)}$ boundary conditions at $x_3 > 0$ to a Chern-Simons theory with $k + h = \Psi$ and the four-dimensional gauge theory with $\tilde{\mathcal{B}}_{(1,1)}$ boundary conditions at $x_3 < 0$ to a Chern-Simons theory with $k + h = \Psi - 1$. The interface at $x_3 = 0$ together with the junction will encode some two-dimensional interface between the two Chern-Simons theories, as described in the following.

The $N_1 = 0$ and $N_3 > N_2$ junctions At first, we can take $N_1 = 0$ and $N_3 > N_2$. In order to re-produce the (deformation of the) bulk Nahm pole, we can consider the following interface between $U(N_3)$ and $U(N_2)$ Chern-Simons theories at levels $\Psi - N_3$ and $\Psi - N_2 - 1$. First, we take the boundary condition $D^{3d}_{N_3-N_2,1,\cdots,1}$ for the former CS theory, defined by the same $SU(2)$ embedding in $U(N_3)$ as the Nahm pole we need to realize, which decomposes the fundamental of $U(N_3)$ into a dimension $N_3 - N_2$ irrep together with N_2 copies of the trivial representation. This boundary condition preserves an $U(N_2)$ subgroup of the $U(N_3)$ gauge group, which we couple to the $U(N_2)$ gauge fields on the other side of the interface.

Classically, the $U(N_3)$ connection at the interface decomposes into blocks

$$A^{U(N_3)}|_\partial = \begin{pmatrix} *_{(N_3-N_2)\times(N_3-N_2)} & *_{(N_3-N_2)\times N_2} \\ *_{N_2\times(N_3-N_2)} & A^{U(N_2)}|_\partial \end{pmatrix} \tag{3.2}$$

with one block identified with the $U(N_2)$ connection and the other blocks subject to the oper boundary condition.

In order for this interface to make sense quantum mechanically, the anomaly of the $\mathfrak{gl}(N_2)$ currents in the VOA $\mathcal{DS}_{N_3-N_2,1,\cdots,1}[\widehat{\mathfrak{gl}(N_3)}_\Psi] \equiv \mathcal{DS}_{N_3-N_2}[\widehat{\mathfrak{gl}(N_3)}_\Psi]$ must be cancelled by anomaly inflow from the expected level $\Psi - N_2 - 1$ of the $U(N_2)$ Chern-Simons theory.[4] We will demonstrate this fact for general $N_3 - N_2$ later on with a detailed Quantum Drinfeld-Sokolov reduction. Essentially, the naive level $\Psi - N_3$ is shifted to $\Psi - N_2 - 1$ by boundary ghost contributions. For $N_3 = N_2 + 1$ it is almost obvious: the $\mathfrak{gl}(N_2)_\Psi$ currents in $\mathfrak{gl}(N_3)_\Psi$ currents have anomaly $\Psi - N_3 = \Psi - N_2 - 1$, just as expected.

The $N_1 = 0$ and $N_2 = N_3$ junctions Next, we can take $N_1 = 0$ and $N_2 = N_3$. Recall that the bulk setup involves fundamental hypermultiplets extended along the $\tilde{\mathcal{B}}_{(1,0)}$ interface. It turns out (see [1, 94] for details) that the topological twist of these 3d degrees of freedom implements an analytically continued two-dimensional path integral for a theory of free chiral symplectic bosons or sometimes called β, γ system, where here the dimension of both β and γ are $1/2$, so that they can be treated on the same footing. Each hypermultiplet provides a single copy of the symplectic bosons VOA.

Thus we will consider a simple interface between $U(N_3)_{\Psi-N_3}$ and $U(N_3)_{\Psi-N_3-1}$ Chern-Simons theories: we identify the gauge fields across the interface, but couple them to the theory $\mathcal{S}^{N_3}$ of N_3 β, γ systems transforming in a fundamental and anti-fundamental representation of $\mathfrak{gl}(N_3)$. This VOA includes $\mathfrak{gl}(N_3)$ currents $J^a_b = \beta^a \gamma_b$ whose anomalies precisely compensate the shift of Chern-Simons levels. This is just another manifestation of the corner anomaly cancellation discussed in the previous section.

The $N_1 > 0$ junctions Next, we can consider general N_1. Now we will have $\tilde{\mathcal{B}}_{(0,1)}$ and $\tilde{\mathcal{B}}_{(1,1)}$ interfaces between $U(N_1)$ and $U(N_3)$ gauge theories. According to [47], a $\tilde{\mathcal{B}}_{(0,1)}$ interface between $U(N_1)$ and $U(N_2)$ GL-twisted gauge theories will map to a $U(N_3|N_1)$ Chern-Simons theory at level $\Psi - N_3 + N_1$. We can thus proceed as before and consider interfaces between $U(N_3|N_1)$ and $U(N_2|N_1)$ Chern-Simons theories at levels $\Psi - N_3 + N_1$ and $\Psi - N_2 + N_1 - 1$.

If $N_2 \neq N_3$, the interface should be a super-group generalization $D^{3d}_{N-M,1,\cdots,1|1,\cdots,1}$ of the Nahm-pole-like boundary condition, preserving an $U(N_2|N_1)$ subgroup of the gauge group which can be coupled to the Chern-Simons gauge fields on the other side of the interface. The oper-like boundary conditions have an obvious generalization to supergroups, with $\mathfrak{sl}(2)$ embedding into the bosonic subalgebra. It would be interesting to determine the

[4] We remind the reader again that the VOA we denote as $\widehat{\mathfrak{gl}(N_3)}_\Psi$ has an $\widehat{\mathfrak{sl}(N_3)}_\Psi$ current subalgebra.

corresponding boundary condition on the bi-fundamental hypermultiplets present on the $\tilde{\mathcal{B}}_{(0,1)}$ and $\tilde{\mathcal{B}}_{(1,1)}$ interfaces.

If $N_2 = N_3$, we need to generalize the symplectic boson VOA to something which admits an action of $U(N_3|N_1)$ with appropriate anomalies. The obvious choice is to add at the interface both N copies of the chiral symplectic bosons VOA and N_1 chiral complex fermions. The fermions do not need to be uplifted to 3d fields and can instead be identified in four-dimensions with the $(0,4)$ Fermi multiplets at the origin of the junction.

The symplectic bosons and fermions combine into a fundamental representation of $\mathfrak{gl}(N_3|N_1)$ and define together a VOA $\mathcal{S}^{N_3|N_1}$ which includes the required $\widehat{\mathfrak{gl}(N_3|N_1)}$ currents. [5]

For concreteness, let us conclude this section by explicitly writing the oper boundary condition in the Chern-Simons theory for $N_1 = 1, N_2 = 1, N_3 = 4$:

$$\left(\begin{array}{ccc|c|c} * & 1 & 0 & 0 & 0 \\ * & * & 1 & * & * \\ * & * & * & * & * \\ \hline * & * & 0 & & \\ \hline * & * & 0 & & \end{array}\right)\begin{array}{l} \\ N-M \\ \\ M \\ L \end{array} \quad . \tag{3.3}$$

3.1.3 From Chern-Simons theory to VOA's

In the gauge theory constructions of Section 3.1.2, we have encountered a variety of boundary conditions and interfaces for (analytically continued) Chern-Simons theory. In this section we discuss the chiral VOA of local operators located at these boundaries or interfaces.

Kac-Moody algerbas The best known example, of course, is the relation between Chern-Simons theory and the Kac-Moody models [41]: a Chern-Simons theory with gauge group G and level k defined on a half-space with appropriate orientation and an anti-chiral Dirichlet boundary condition $A_{\bar{z}} = 0$ supports at the boundary a VOA $\widehat{\mathfrak{g}}_k$ based on the

[5] Notice that the coupling of the $\mathcal{S}^{N_3|N_1}$ VOA to the 3d Chern Simons theory induces a discontinuity of A_z across the interface proportional to the Kac-Moody currents in the VOA. In particular, the discontinuity of the odd currents in $\widehat{\mathfrak{gl}(N_3|N_1)}$ is proportional to products of a 2d symplectic boson and a 2d fermion. This must correspond to the effect of the junction coupling between the $(0,4)$ Fermi multiplets and the restrictions of the fundamental and bi-fundamental hypermultiplets to the junction.

Lie algebra $\mathfrak{g}$ of the gauge group G, with currents J of level k which are proportional the restriction of A_z to the boundary.[6]

Dirichlet boundary conditions are associated to a full reduction of the gauge group at the boundary: gauge transformations must go to the identity at the boundary and constant gauge transformations at the boundary become a global symmetry of the boundary local operators. For our purposes, we need to consider a more general situation, where the gauge group is only partially reduced and may be coupled at the boundary to extra two-dimensional degrees of freedom.

Coset First, we should ask if Neumann boundary conditions could be possible, so that the gauge group is fully preserved at the boundary. In the absence of extra 2d matter fields, this is not possible, because of the boundary gauge anomaly inflowing from the bulk Chern-Simons term. We would like to claim that Neumann boundary conditions are possible if extra 2d matter fields are added, say a 2d chiral CFT T^{2d} equipped with chiral, $\mathfrak{g}$-valued currents J^{2d} of level $-k-2h$.

Indeed, we can produce Neumann boundary conditions by coupling auxiliary two-dimensional chiral gauge fields to the combination of T^{2d} and standard Dirichlet boundary conditions. The level of T^{2d} is chosen in such a way to cancel the naive bulk anomaly inflow when combined with the ghost contribution to the boundary anomaly. The effect if coupling two-dimensional gauge fields to VOA is well understood from the study of coset conformal field theory [70, 71].

The VOA of boundary local operators should be built from the combination of $\widehat{\mathfrak{g}}_k$, T^{2d} and a bc ghost system $bc^{(coset)}$ valued in the Lie algebra $\mathfrak{g}$, taking the cohomology of the BRST charge

$$Q^{(coset)}_{\mathrm{BRST}} = \oint dz \mathrm{Tr}\left[\frac{1}{2} : b(z)[c(z), c(z)] : + c(z)(J(z) + J(z)^{2d})\right] + Q^{2d}_{\mathrm{BRST}} \tag{3.4}$$

which implements quantum-mechanically the expected boundary conditions $J(z)+J(z)^{2d} = 0$. We included Q^{2d}_{BRST} to account for the possibility that T^{2d} itself was defined in a BV formalism. We will denote such procedure as a $\mathfrak{g}$-BRST reduction.

The relation to coset constructions is related to the observation that the BRST cohomology includes the sub-algebra of local operators in T^{2d} which are local with the currents J^{2d}. In other words, the boundary VOA includes the current algebra *coset*

$$\frac{T^{2d}}{\widehat{\mathfrak{g}}_{-k-2h}} \tag{3.5}$$

[6]The proportionality factor is k.

which generalizes the idea that Neumann boundary conditions support local gauge-invariant operators in T^{2d}.

Interfaces can be included in this discussion by a simple folding trick. The change in orientation maps $k \to -2h - k$. Thus we can consider a Neumann-type interface between G_k and $G'_{k'}$ Chern-Simons theories coupled to a 2d chiral CFT T^{2d} equipped with chiral, $\mathfrak{g} \times \mathfrak{g}'$-valued currents of levels $-k-2h$ and k'.

The interface VOA will be the $\mathfrak{g} \oplus \mathfrak{g}'$-BRST reduction of the combination of $\mathfrak{g}_k$, $\mathfrak{g}'_{-k'-2h'}$, T^{2d} and $\mathrm{bc}^{\mathfrak{g}\oplus\mathfrak{g}'}$. This implements a coset

$$\frac{T^{2d}}{\widehat{\mathfrak{g}}_{-k-2h} \otimes \widehat{\mathfrak{g}}'_{k'}} \tag{3.6}$$

The construction above has an obvious generalization to mixed boundary conditions, where the gauge group is reduced to a subgroup H at the boundary and coupled with extra degrees of freedom T^{2d} equipped with chiral, $\mathfrak{h}$-valued currents J^{2d} of level $-k_H - 2h_H$. The boundary VOA will consist of the $\mathfrak{h}$-BRST reduction of the combination of $\widehat{\mathfrak{g}}_k$, T^{2d} and $\mathrm{bc}^{(coset)}$.

The simplest example of this construction is a trivial interface between G_k and G_k Chern-Simons theories. The interface breaks the $G \times G$ gauge groups to the diagonal combination, gluing together the gauge fields on the two sides. The VOA of local operators should be the BRST cohomology of $\widehat{\mathfrak{g}}_k \times \widehat{\mathfrak{g}}_{-k-2h}$ combined with one set $\mathrm{bc}^{\mathfrak{g}}$ of bc ghosts valued in the Lie algebra of G. This BRST cohomology is trivial: the trivial interface in Chern-Simons theory supports no local operators except for the identity.

A more interesting example is an interface where the G_k Chern-Simons theory is coupled to some 2d degrees of freedom T^{2d} equipped with chiral, $\mathfrak{g}$-valued currents J^{2d} of level k'. Notice that the levels on the two sides of the interface should be k and $k+k'$.

Then the interface VOA will be given by the BRST cohomology of $\widehat{\mathfrak{g}}_k \times T^{2d} \times \widehat{\mathfrak{g}}_{-k-k'-2h}$ combined with one set of $b\,c$ ghosts valued in the Lie algebra of G. This can be interpreted as either of two conjecturally equivalent cosets

$$\frac{\widehat{\mathfrak{g}}_k \times T^{2d}}{\widehat{\mathfrak{g}}_{k+k'}} \stackrel{?}{=} \frac{\widehat{\mathfrak{g}}_{-k-k'-2h} \times T^{2d}}{\widehat{\mathfrak{g}}_{-k-2h}} \tag{3.7}$$

An example of this was discussed in [95] with T^{2d} taken to be a set of chiral fermions transforming in the fundamental representation of $\mathfrak{sl}(N)$, resulting in the coset

$$\frac{\widehat{\mathfrak{sl}(N)}_k \times \widehat{\mathfrak{sl}(N)}_1}{\widehat{\mathfrak{sl}(N)}_{k+1}} \tag{3.8}$$

which is a well-known realization of the $\mathcal{W}_N[k+N]$ VOA.

Drinfeld-Sokolov reduction A second important topic we need to discuss is the Quantum Drinfeld-Sokolov reduction $\mathcal{DS}_\rho[\widehat{\mathfrak{g}}_k]$ of the Kac-Moody algebra $\widehat{\mathfrak{g}}_k$, the VOA which appear at "oper-like" boundary conditions for a G_k Chern-Simons theory, labelled by an $\mathfrak{sl}(2)$ embedding ρ.

As a starting point, we may recall the construction for $\mathfrak{sl}(2)$ gauge group and the regular $\mathfrak{sl}(2)$ embedding [92]. The classical boundary condition takes the schematic form

$$A_{\bar{z}} = \begin{pmatrix} \frac{1}{2}a^K_{\bar{z}} & 0 \\ * & -\frac{1}{2}a^K_{\bar{z}} \end{pmatrix} \qquad A_z = \begin{pmatrix} * & 1 \\ * & * \end{pmatrix} \tag{3.9}$$

where the $*$ denotes elements which are not fixed by the boundary condition and $\frac{1}{2}a^K_{\bar{z}}$ is the connection on the canonical bundle.

Gauge-transformations can be used to locally gauge-fix the holomorphic connection to

$$A_z = \begin{pmatrix} 0 & 1 \\ t(z) & 0 \end{pmatrix} \tag{3.10}$$

with $t(z)$ behaving as a classical stress tensor.

Quantum mechanically, one proceeds as follows [64, 65, 66]. The stress tensor of the usual $\widehat{\mathfrak{sl}(2)}_k$ currents is shifted by the current ∂J^3 associated to the Cartan element, in such a way that $J^+(z)$ acquires conformal dimension 0 and $J^-(z)$ acquires conformal dimension 2. Furthermore, a single pair of $b\,c$ ghosts is added, allowing us to define a BRST charge

$$Q^{(DS)}_{\mathrm{BRST}} = \oint dz c(z)(J^+(z)-1) \tag{3.11}$$

enforcing the $J^+(z)=1$ constraint. The total stress tensor

$$T = T_{\widehat{\mathfrak{sl}(2)}_k} - \partial J^3 - b\partial c \tag{3.12}$$

is in the BRST cohomology and generates it. It has central charge

$$\frac{3k}{k+2} - 6k - 2 = 13 - \frac{6}{k+2} - 6(k+2) = 1 + 6(b+b^{-1})^2 \tag{3.13}$$

with $b^2 = -(k+2) = -\Psi$.

The construction generalizes as follows [66, 67, 68]. Take the t^3 element in the $\mathfrak{su}(2)$ embedding ρ. The Lie algebra $\mathfrak{g}$ decomposes into eigenspaces of t^3 as

$$\mathfrak{g} = \oplus_i \mathfrak{g}_{i/2} \tag{3.14}$$

The raising generator t^+ of ρ is an element in $\mathfrak{g}_1$. Naively, we want to set to zero all currents of positive degree under t^3 except for the one along t^+, which should be set to 1. We cannot quite do so because if we set to zero all currents in $\mathfrak{g}_{1/2}$ we will also set to zero their commutator, including the current along the t^+ direction. The commutator together with the projection to t^+ gives a symplectic form on $\mathfrak{g}_{1/2}$ and we are instructed to only set to zero some Lagrangian subspace $\mathfrak{g}^+_{1/2}$ in $\mathfrak{g}_{1/2}$.

Then $\mathcal{DS}_\rho[\widehat{\mathfrak{g}}_k]$ is defined as the BRST cohomology of a complex which is almost the same as the one we would use to gauge the triangular sub-group

$$\mathfrak{n} = \mathfrak{g}^+_{1/2} \oplus \bigoplus_{i>1} \mathfrak{g}_{i/2} \tag{3.15}$$

In particular, we add to $\widehat{\mathfrak{g}}_k$ a set of $b\,c$ ghosts valued respectively in $\mathfrak{n}$ and $\mathfrak{n}^*$.

The main difference is that we will shift the stress tensor by the t^3 component of ∂J and by a similar ghost contribution $[b, t^3] \cdot c$ in such a way that currents and b-ghosts in $\mathfrak{g}_{i/2}$ have conformal dimension $1 - i/2$. This allows us to add the crucial extra term setting the t^+ component of J to 1:

$$Q^{(DS)}_{\text{BRST}} = \oint dz \text{Tr}\left[\frac{1}{2} : b(z)[c(z), c(z)] : + c(z)J(z)\right] - t^+ \cdot c(z) \tag{3.16}$$

In general, if the $\mathfrak{gl}(2)$ embedding ρ commutes with some subalgerba $\mathfrak{h}$ of $\mathfrak{g}$, the currents in $\mathfrak{h}$ can be corrected by ghost contributions to give $\mathfrak{h}$ currents in $\mathcal{DS}_\rho[\widehat{\mathfrak{g}}_k]$. The ghost contributions will shift the level away from the value inherited from $\widehat{\mathfrak{g}}_k$.

The oper-like boundary conditions can be further modified by gauging subgroups of h coupled to appropriate 2d degrees of freedom and/or promoted to interfaces by identifying the H subgroup of the G connection with an H connection on the other side of the interface. This will lead to further $\widehat{\mathfrak{h}}$-BRST cosets involving $\mathcal{DS}_\rho[\widehat{\mathfrak{g}}_k]$ as an ingredient.

3.1.4 BRST definition of $Y_{N_1,N_2,N_3}[\Psi]$

After describing junction conditions and VOAs associated to different junctions of Chern-Simons gauge theories, it is simple to write down a proposal for a BRST definition of

Y-algerbas as a combination of the Drinfeld-Sokolov reduction and the coset construction of a super Kac-Moody algebra. Schematically, they are defined as[7]

$$
\begin{aligned}
Y_{N_1,N_2,N_3}[\Psi] &= \frac{\mathcal{DS}_{N_3-N_2}[\widehat{\mathfrak{gl}(N_3|N_1)}_\Psi]}{\widehat{\mathfrak{gl}(N_2|N_1)}_{\Psi-1}} && \text{for } N_3 > N_2 \\
Y_{N_1,N_2,N_3}[\Psi] &= \frac{\widehat{\mathfrak{gl}(N_3|N_1)}_\Psi \times \mathcal{S}^{N_3|N_1}}{\widehat{\mathfrak{gl}(N_3|N_1)}_{\Psi-1}} \\
Y_{N_1,N_2,N_3}[\Psi] &= \frac{\mathcal{DS}_{N_2-N_3}[\widehat{\mathfrak{gl}(N_2|N_1)}_{-\Psi+1}]}{\widehat{\mathfrak{gl}(N_3|N_1;)}_{-\Psi}} && \text{for } N_3 < N_2
\end{aligned}
\tag{3.17}
$$

where $\mathcal{DS}_{N_3-N_2}$ denotes the Drinfeld-Sokolov reduction with respect to the principal embedding inside the $(N_3-N_2)\times(N_3-N_2)$ diagonal block of $\mathfrak{gl}(N_3|N_1)$ and by the division by $\widehat{\mathfrak{gl}(N_2|N_1)}_{\Psi-1}$ we mean the BRST coset to be defined later. $\mathcal{S}^{N_3|N_1}$ labels the set of N symplectic bosons and N_1 free fermions that contains a $\widehat{\mathfrak{gl}(N_3|N_1)}_{N_3-N_1-1}$ subalgebra formed from the field bilinears. More concretely, for parameters in the range $N_3 > N_2$, $Y_{N_1,N_2,N_3}[\Psi]$ is defined as the BRST reduction of

$$
\widehat{\mathfrak{gl}(N_3|N_1)}_\Psi \times \widehat{\mathfrak{gl}(N_2|N_1)}_{-\Psi+1} \times gh^{(DS)} \times gh^{(coset)} \tag{3.18}
$$

by two successive BRST reductions. In the expression above, we have introduced $gh^{(DS)}$ for (super)ghosts needed for the Drinfeld-Sokolov reduction implemented by $Q^{(DS)}_{BRST}$ and $gh^{(coset)}$ for (super)ghosts associated to the BRST coset implemented by $Q^{(coset)}_{BRST}$.

$Q^{(DS)}_{BRST}$ can be defined in the following three steps (assuming $N > M$):

1. Pick the principal $\mathfrak{sl}(2)$ embedding inside the $\mathfrak{gl}(N_3-N_2)$ subalgebra associated to the $(N_3-N_2)\times(N_3-N_2)$ block inside $\mathfrak{gl}(N_3|N_1)$. The corresponding Cartan generator of such embedding can be taken to be of the form

$$
\mathfrak{h} = \frac{N_3-N_2-1}{2}E_{11} + \frac{N_3-N_2-3}{2}E_{22} + \cdots + \frac{N_2-N_3+1}{2}E_{N_3-N_2,N_3-N_2} \tag{3.19}
$$

[7]Throughout the paper, we use the notation $\widehat{\mathfrak{gl}(N_3|N_1)}_\Psi = \widehat{\mathfrak{gl}(1)}_{(N_3-N_1)\Psi} \times \widehat{\mathfrak{sl}(N_3|N_1)}_{\Psi-N_3+N_1}$, where $\Psi - N_3 + N_1$ is the level of the $\widehat{\mathfrak{sl}(N_3|N_1)}$ Kac-Moody subalgebra, i.e. Ψ is the level relative to the critical level. Although $\widehat{\mathfrak{gl}(1)}$ current algebra does not have any intrinsic level, we use the subscript to indicate the normalization of the $\widehat{\mathfrak{gl}(1)}$ current with respect to which the electric modules have integral dimensions.

where E_{ij} is a generator of the $\mathfrak{gl}(N_3|N_1)$ Lie algebra associated to the matrix with one at the position i, j. The generator $\mathfrak{h}$ provides us with a grading that we use in the next step.

2. Decompose the adjoint representation of $\mathfrak{gl}(N_3|N_1)$ into subspaces of $\mathfrak{h}$-charge greater then, equal to, and smaller than one half: $g_{<\frac{1}{2}} \oplus g_{\frac{1}{2}} \oplus g_{>\frac{1}{2}}$. Introduce fermionic bc ghosts for each bosonic element and bosonic $\beta\gamma$ ghosts for each fermionic element in $g_{>\frac{1}{2}}$ and for half of the elements in $g_{\frac{1}{2}}$.[8] This system of (super)ghosts is labeled by $gh^{(DS)}$.[9]

3. Define a nilpotent BRST charge $Q^{(DS)}_{BRST}$ constraining $g_{>\frac{1}{2}}$ and half of $g_{\frac{1}{2}}$ generators to a fixed value

$$Q^{(DS)}_{BRST} = \oint dz \left[(J_i - t^+_i) c^i + \frac{1}{2} f^k_{ij} b_k c^i c^j \right] \tag{3.20}$$

where t^+ is the raising operator of the $SU(2)$ embedding. In our conventions, this generator has all the entries vanishing except of those above the diagonal that are set to one. f^k_{ij} are the structure constants of the algebra of constraints (restrictions of the structure constants of the $\mathfrak{gl}(N|L)$ Lie algebra).

The coset BRST reduction is then performed by adding (super)ghosts of conformal dimension $h(c^i) = h(\gamma^j) = h(b_i) - 1 = h(\beta_j) - 1 = 0$, one for each generator of $\mathfrak{gl}(N_2|N_1)$. We denote this (super)ghost system by $gh^{(coset)}$ and study the cohomology with respect to

$$Q^{(coset)}_{BRST} = \oint dz \left[(J^1_j - J^2_j) c^j + \frac{1}{2} f^l_{jk} b_l c^j c^k \right]. \tag{3.21}$$

Here J^α_j are the currents of the two copies of the $\widehat{\mathfrak{gl}(N_2|N_1)}$ algebra being sewed and f^k_{ij} are the structure constants of $\mathfrak{gl}(N_2|N_1)$.[10] For the notational simplicity we wrote the formula as if there were only bosonic generators and fermionic ghosts, but the generalization should be obvious.

[8]This half of the elements needs to to be picked such that they form a Lagrangian subspace inside $g_{\frac{1}{2}}$ with respect to the symplectic pairing given by the standard invariant two-form of $SU(N)$.

[9]The conformal dimensions of such ghosts are $h(c^i) = h(\gamma^i) = 1 - h(b_i) = 1 - h(\beta_i) = 1 - h(J_i) = -H(J_i)$ where $H(J_i)$ is the H-charge of the element J_i. This assignment of conformal dimensions ensures that the BRST charge has degree one with respect to the modified stress-energy tensor of the Drinfeld-Sokolov reduction and it is useful to count the contribution from the ghosts in the total stress-energy tensor.

[10]The upper index $\alpha = 1, 2$ runs over the two copies of algebra while indices j run over the generators of the adjoint representation of $\mathfrak{gl}(N_2|N_1)$.

In the case when $N_3 - N_2 = 1$, the $\mathcal{DS}_1$ is a trivial operation and can be omitted. On the other hand, if $N_3 = N_2$, one needs to add symplectic bosons $\mathcal{S}^{N_3|N_1}$ in the fundamental representation of $\mathfrak{gl}(N_3|N_1)$. These are known to contain a conformally embedded $\widehat{\mathfrak{gl}(N_3|N_1)}_{N_3-N_2-1}$ Kac-Moody algebra formed by their bilinears. The resulting Y-algebra can be identified with the BRST reduction of

$$\widehat{\mathfrak{gl}(N_3|N_1)}_\Psi \times \mathcal{S}^{N_3|N_1} \times \widehat{\mathfrak{gl}(N_3|N_1)}_{-\Psi+1} \times gh^{(coset)} \tag{3.22}$$

by the BRST charge

$$Q^{(coset)}_{BRST} = \oint dz \left[c^i (J^1_i - J^2_i - J^{\mathcal{S}}_i) + \frac{1}{2} f^k_{ij} b_k c^i c^j \right] \tag{3.23}$$

where $J^{\mathcal{S}}$ are the $\widehat{\mathfrak{gl}(N_3|N_1)}$ currents obtained from the bilinears in $\mathcal{S}^{N_3|N_1}$ fields. Intuitively, this BRST operator couples the symplectic bosons to the two Chern-Simons theories connected by the interface.

In the following, we will use the unified notation

$$\mathcal{DS}_{N_3-N_2}[\widehat{\mathfrak{gl}(N_3|N_1)}_\Psi] \tag{3.24}$$

for any non-negative $N_3 - N_2$ that is defined by the DS-reduction described above for $N_3 - N_2 > 1$, that is trivial in the case of $N_3 - N_2 = 1$, and that produces

$$\mathcal{DS}_0[\widehat{\mathfrak{gl}(N_3|N_1)}_\Psi] = \widehat{\mathfrak{gl}(N_3|N_1)}_\Psi \times \mathcal{S}^{N_3|N_1} \tag{3.25}$$

in the case that $N_3 = N_2$.

3.2 Type IIA perspective

3.2.1 Divisors in toric three-folds

Let us describe a dual M-theory configuration associated to the trivalent junction leading to the $Y_{N_1,N_2,N_3}[\Psi]$ algebra. Consider three stacks of N_1, N_2 and N_3 $M5$-branes supported on $\mathbb{C}^2_{x_1,x_2}$, $\mathbb{C}^2_{x_1,x_3}$ and $\mathbb{C}^2_{x_2,x_3}$ inside $\mathbb{C}^3_{x_1,x_2,x_3}$, i.e. a configuration associated to the divisor $N_1\mathbb{C}^2_{x_2,x_3} + N_2\mathbb{C}^2_{x_1,x_3} + N_3\mathbb{C}^2_{x_1,x_2}$ with $N_i \in \mathbb{Z}_{\geq 0}$, and wrapping an extra Riemann surface Σ inside $S^1 \times \mathbb{R}^4$. Compactifying on the extra Riemann surface Σ shared by all the branes, the configuration has a low-energy description in terms of $U(N_i)$ gauge theories supported

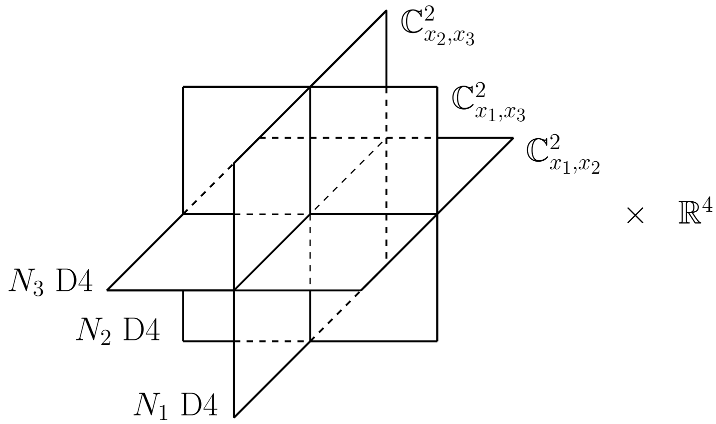


Figure 3.3: Configuration of branes in type IIA string theory associated to spiked instantons. Three stacks of D4-branes span the three four-cycles inside $\mathbb{C}^3$ fixed under the T^2 action discussed in the main text together with one orthogonal direction in $\mathbb{R}^4$.

at the three irreducible components of the divisor, namely on $\mathbb{C}^2_{x_1,x_2}$, $\mathbb{C}^2_{x_1,x_3}$ and $\mathbb{C}^2_{x_2,x_3}$, mutually interacting along their intersections $\mathbb{C}_{x_1}$, $\mathbb{C}_{x_2}$, $\mathbb{C}_{x_3}$ via bi-fundamental 2d fields. This setup can be identified with a restriction of the more general spiked-instanton setup of $M5$-branes intersecting inside $\mathbb{C}^4$ from [54, 55]. The type IIA setup can be obtained by a compactification of the circle S^1 leading to type IIA string theory on $\mathbb{C}^3 \times \mathbb{R}^4$ with M5-branes (originally wrapping the M-theory circle) becoming D4-branes.

The above M-theory setup can be related [48, 42, 96, 2, 4] to the type IIB configuration from above using the duality between the M-theory on a torus and type IIB string theory in the presence of a web of (p,q)-branes. In the example at hand, $\mathbb{C}^3 = \mathbb{R}^6$ endowed with the standard symplectic structure has the natural Hamiltonian action of $T^3 = U(1)^3$, whose moment map realizes $\mathbb{C}^3$ as a singular Lagrangian T^3- fibration over the first octant in $\mathbb{R}^3$. The action of the 2-dimensional subtorus $T^2 \subset T^3$ preserving the canonical bundle is generated by the following rotations $(e^{it_1}z_1, z_2, e^{-it_1}z_3)$ and $(z_1, e^{it_2}z_2, e^{-it_2}z_3)$. The moment map of this T^2 action from $\mathbb{C}^3$ to $\mathbb{R}^2$ is given by $\mu_1 = |z_1|^2 - |z_3|^2$ and $\mu_2 = |z_2|^2 - |z_3|^2$. The directions in which the T^2 torus fibration, when projected to $\mathbb{R}^2$, are as follows. The t_1 action degenerates for $z_1 = z_3 = 0$, corresponding to the $\mu_1 = 0, \mu_2 > 0$, the t_2 action degenerates at $\mu_2 = 0$ and $\mu_1 > 0$ and finally $t_1 + t_2$ degenerates at $\mu_1 = \mu_2 < 0$. The degeneration of the fibers in the μ_1, μ_2 plane is shown in the figure 4.12 on the left.

From the dual point of view, one gets a type IIB theory on $\mathbb{R}^8 \times T^2$ with one of the cycle $S_1 \subset T^2$ coming from the toric fibration of the Calabi-Yau 3-fold and the other cycle corresponding to the M-theory circle $S_2 \subset T^2$. Singularities of the torus fibration

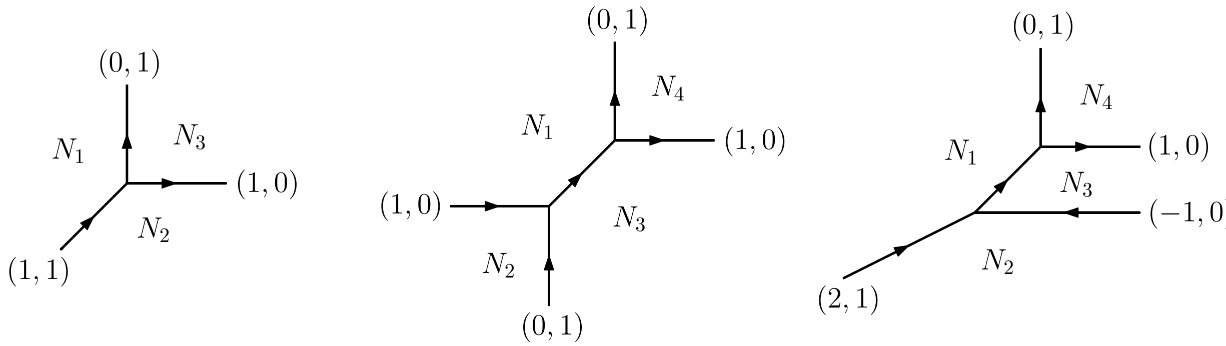


Figure 3.4: Toric diagram associated to $\mathbb{C}^3$ (left), $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{CP}^1$ (middle) and $\mathcal{O}(-2) \oplus \mathcal{O} \to \mathbb{CP}^1$ (right). The lines show loci where (p, q)-cycles of the torus T^2 degenerate.

correspond to (p, q)-branes spanning orthogonal directions with p and q labeling the degenerating circle. The geometry of the Calabi-Yau 3-fold thus maps to a web of (p, q)-branes. $M5$-branes associated to the faces in the toric diagram map to $D3$-branes attached to (p, q)-branes from the three corners. This is exactly the type IIB setup described above leading to $Y_{N_1,N_2,N_3}[\Psi]$. Furthermore, as discussed in [4], the parameter Ψ is related to the Ω-deformation parameters as $\Psi = -\epsilon_1/\epsilon_2$.

The example of $\mathbb{C}^3$ has a natural generalization for an arbitrary toric Calabi-Yau 3-fold given by a toric diagram specifying loci where the torus cycles degenerate. The two simplest examples are shown in the figure 4.12 and correspond to the bundles $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{CP}^1$ and $\mathcal{O}(-2) \oplus \mathcal{O} \to \mathbb{CP}^1$ respectively. Smooth components of a general toric divisor can be identified with faces of the corresponding toric diagram and the colored (p, q) web thus labels such a divisor. Compactifying on the extra Rieman surface, one gets a system of four-dimensional theories with gauge groups $U(N_i)$ supported on various four-cycles inside the toric Calabi-Yau three-fold and mutually coupled along their intersections.

3.2.2 AGT for spiked instantons

Note that $Y_{N_1,N_2,N_3}[\Psi]$ becomes the standard $\mathcal{W}_{N_i} \times \widehat{\mathfrak{gl}(1)}$ algebra if two of the remaining parameters $N_j = 0$ for $j \neq i$ vanish. Moreover, the gauge theory configuration reduces to a simple configuration of the $U(N_i)$ gauge theory on $\mathbb{C}^2$ in the presence of Ω-background. This is the standard setup of the AGT correspondence. Indeed, Alday-Gaiotto-Tachikawa [14, 18] relate the Nekrasov partition function [56] of a $U(N_i)$ gauge theory on $M_4 = \mathbb{C}^2$

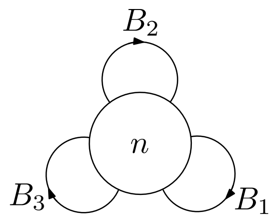


Figure 3.5: Quiver for the cohomological Hall algebra of $\mathbb{C}^3$.

in the presence of the Ω-background with conformal blocks of the $\mathcal{W}_N$ algebra living on the extra Riemann surface Σ. It thus seems natural to generalize the AGT correspondence and conjecture that $Y_{N_1,N_2,N_3}[\Psi]$ is the VOA dual to the spiked-instanton configuration of Nekrasov [54, 55]. In a greater generality, one might expect that the general (p,q) web VOA (to be discussed in a grater detail in the next chapter) is the AGT dual to a more complicated gauge-theoretical configuration associated to a toric divisor inside a toric Calabi-Yau three-fold.

A key step in the proof of the standard AGT correspondence is the construction of the action of $\mathcal{W}_N[\Psi] \times \widehat{\mathfrak{gl}(1)}$ on the equivariant cohomology of the moduli space of $U(N)$ instantons on $\mathbb{C}^2$ with equivariant parameters ϵ_1, ϵ_2 [59, 16, 60]. Such an action descent from the action of the cohomological Hall algebra associated to $\mathbb{C}^2$ by taking its Drinfeld double that can be identified with the affine Yangian of $\mathfrak{gl}(1)$. Algebras $\mathfrak{W}_N \times \widehat{\mathfrak{gl}(1)}$ then appear as specializations associated to a choice of the rank of the gauge group.

Generalization to $\mathbb{C}^3$ is conceptually straightforward but rather technical. A proper exposition would require an introduction of new techniques that would move us too far from our discussion of VOAs. Let us at least briefly review the idea of the proof from [4]. Interested reader is referred to [4] and references therein.

In analogy with the $\mathbb{C}^2$ case, one can consider the cohomological Hall algebra [73] associated to $\mathbb{C}^3$. The cohomological Hall algebra is an algebra defined on the equivariant cohomology of representations of the triple quiver from the figure 3.5 lying in the critical locus of the potential

$$W = \mathrm{Tr}\,[B_1[B_2, B_3]] \tag{3.26}$$

and satisfying stability conditions that we are going to omit in our discussion. The quiver 3.5 can be thought of as a quiver of an effective quantum mechanics describing the low-energy dynamics of a stack of n D1-branes. The three loops with the corresponding $n \times n$ complex-valued matrices B_1, B_2, B_3 are associated to the three complex directions along

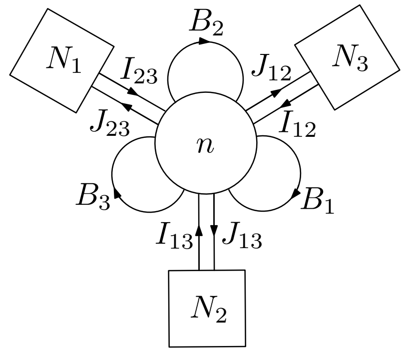


Figure 3.6: Quiver for spiked instantons associated to the toric divisor $N_1\mathbb{C}^2_{x_2,x_3}+N_2\mathbb{C}^2_{x_1,x_3}+N_3\mathbb{C}^2_{x_1,x_2}$ inside $\mathbb{C}^3$.

$\mathbb{C}^3$. The potential W is the potential of the quiver quantum mechanics. The equivariant cohomology then corresponds to the space of vacua of such a quantum mechanics.

There exists a natural algebra structure on the cohomology if we consider a tensor sum of equivariant cohomologies over all $n > 0$. Having a tripple of matrices (B_1, B_2, B_3) of rank n and a triple of matrices $(\tilde{B}_1, \tilde{B}_2, \tilde{B}_3)$ of rank m, we can embed them diagonally inside a matrix of rank $n+m$. This naturally lead to the restriction map p and the embedding η of the form

$$\begin{pmatrix} B_i & 0 \\ 0 & \tilde{B}_i \end{pmatrix} \overset{p}{\leftarrow} \begin{pmatrix} B_i & * \\ 0 & \tilde{B}_i \end{pmatrix} \overset{\eta}{\rightarrow} \begin{pmatrix} B_i & * \\ * & \tilde{B}_i \end{pmatrix}. \tag{3.27}$$

These maps induce a multiplication on the tensor sum of equivariant cohomologies of the above space by composing the pull-back of p with the pushforward of $* = \eta$, i.e. $\eta_* p^*$. It can be checked that the conditions of being in a critical locus and the stability conditions are consistent with such maps. Note that the multiplication produces a rank $n+m$ configuration. Physically, we can thus expect that the multiplication describe a process that is associated to a fusion of a bound state of n and m D0-branes into a stack of $n+m$ D0-branes. Indeed, one of the motivations of [73] for the cohomological Hall algebra was to give a precise definition of the BPS algebra associated to such processes [97].

Repeating the above argument for a system of n D0-branes bound to the intersecting D4-branes, one arrives to the framed quiver from the figure . Apart from the three loops B_1, B_2, B_3, we have also degrees of freedom coming from the D0-D4 strings with $N_i \times n$ matrices I_i, J_i for each $i = 1, 2, 3$. The potential of the corresponding quantum mechanics is

$$\mathrm{Tr}\,[B_1[B_2, B_3]] + I_1 B_1 J_1 + I_2 B_2 J_2 + I_3 B_3 J_3. \tag{3.28}$$

The vacua then correspond to the equivariant cohomology of such matrices lying in the critical locus of the potential coming from the variation with respect to all the involved fields. The variation with respect to I_i, J_i modify the stability conditions and the variation with respect to B_i lead to modified moment-map equations. The more general quiver is expected to parametrized spiked instantons[11] [54, 55] associated to intersecting theories with n corresponding to the instanton number.

An analogous correspondence as the one in (3.27) can be also written down in the case of the spiked instanton moduli, where the restriction p restrict to a product of a representation of the framed quiver of dimension m and unframed quiver of dimension n and the embedding η is inside a framed-quiver representation of dimension $n+m$. This correspondence induces an action of the cohomological Hall algebra on the equivariant cohomology of the moduli space of spiked instantons that increases the instanton number m. Physically, the multiplication should capture a process of forming a bound-state of $n+m$ D0-branes with D4-branes from a bound state of m D0-branes bound to D4-branes and a stack of n D0-branes.

In order to describe reverse processes that decrease the instanton number (corresponding to a decay of a bound state), we need to extend the cohomological Hall algebra. At the level of VOAs, such an extension corresponds to an extension of the subalgebra of positive modes by an algebra of zero and negative modes. It turns out that there exists an extension of the cohomological Hall algebra that naturally acts on the moduli space known as the affine Yangian of $\mathfrak{gl}(1)$. The affine Yangian of $\mathfrak{gl}(1)$ is known to be isomorphic (as an associative algebra) to a VOA called $\mathcal{W}_{1+\infty}$ that is closely related to $Y_{N_1,N_2,N_3}[\Psi]$.

One can actually proof an equivalence of the module of the affine Yangian with the generic module (to be discussed in the last section) by matching free field realizations along the lines of [16]. It is a well-known fact that the action of the affine Yangian of $\mathfrak{gl}(1)$ on the moduli space associated to the rank-one moduli space $N_3 = 1, N_1 = N_2 = 0$ can be identified with the action of the Heisenberg algebra on a generic module. Action on the modules associated to other directions $N_2 = 1, N_1 = N_3 = 0$ and $N_1 = 1, N_2 = N_3 = 0$ can be simply obtained by a permutation of the equivariant parameters ϵ_i. Furthermore, using a coproduct of the affine Yangian $N_1 + N_2 + N_3$-times and letting N_i affine Yangian factors act on a rank-one moduli space associated to the direction i, one gets a new module depending on parameters $\epsilon_1, \epsilon_2, \epsilon_3$ and $N_1 + N_2 + N_3$ parameters associated to the highest weights of each Heisenberg factor. This gives a free field realization of more complicated modules. One can also show that there exists a so-called hyperbolic localization map that

[11] It turns out that in the $N_2 = N_3 = 0$ case (or its permutations), the moduli space reduces to the standard ADHM moduli parametrizing instantons on $\mathbb{C}^2$.

decomposes the general moduli spaces with dimensions (N_1, N_2, N_3) of framing vectors into a product of $N_1 + N_2 + N_3$ rank-one moduli spaces. Furthermore, using the hyperbolic localization, it can be shown that the action of the affine Yangian can be identified with the free field realization above. It is just a straightforward calculation to check that the free field realization agrees with the one defined later in this chapter.

The parameters ϵ_i parametrizing OPEs of the resulting algebra come from equivariant parameters associated to the rotation of the coordinates $\mathbb{C} \subset \mathbb{C}^3$ and the $N_1 + N_2 + N_3$ highest weights of the modules come from the equivariant parameters associated the $U(n)$ action on matrices I_i, J_i associated to the framing nodes.

3.2.3 D0-D2-D4 brane counting

Apart from the generalization of the AGT for spiked instantons, the D4-D2-D0 configuration suggests a relation to the BPS counting of [51, 52, 53]. From the point of view of the theory on intersecting D4-branes, D2- and D0- branes modify the gauge bundle of the corresponding effective gauge theories. Fixing numbers D0 and D2 branes then corresponds to restricting the path integral to a particular instanton sector. Counting of BPS states for such configurations in a fixed instanton sector has been performed in [51, 52, 53]. BPS indices can be arranged into a generating function with parameter q corresponding to the fugacity for the D0 charge and Q corresponding to the fugacity for the D2 charge. It is natural to expect that such characters are going to appear naturally also from VOAs.

We will later argue that vacuum characters of $Y_{N_1,N_2,N_3}[\Psi]$ indeed agree with those of Jafferis in [51] who proposed the same box-counting interpretation for the corresponding D0-D2-D4 brane counting. Moreover, we will see that the gluing proposal at the level of vacuum characters (to be discussed in the next chapter) matches the one proposed in [52, 53]. Gluing at the level of full algebras seems to categorify these BPS counting problems.

3.2.4 $\mathcal{W}_{1+\infty}$ and topological vertex

We will now discuss an alternative definition of $Y_{N_1,N_2,N_3}[\Psi]$ in terms of truncations of the $\mathcal{W}_{1+\infty}$ algebra. The identification of $Y_{N_1,N_2,N_3}[\Psi]$ with truncations of $\mathcal{W}_{1+\infty}$ is supported by the following consistency checks:

1. Agreement of the vacuum character in the large N_i limit and the appearance of the first null state of $Y_{N_1,N_2,N_3}[\Psi]$ with respect to the large N_i limit character.

2. Factorization of the central charge that allows us to naturally identify parameters $\lambda_1, \lambda_2, \lambda_3$ of the $\mathcal{W}_{1+\infty}$ algebra.

3. Triality symmetry inherited from the S-duality.

Apart from these three, there are many other consistency checks in explicit examples discussed in [1, 2, 76].

The vacuum character The vacuum character of $Y_{N_1,N_2,N_3}[\Psi]$ for $N_3 \geq N_2$ can be computed using the BRST prescription along the lines of [1] and give the following contour-integral representation (see also [36])

$$\chi\left[Y_{N_1,N_2,N_3}[\Psi]\right] = \chi_{\mathcal{W}_{N_3-N_2}}(q) \oint dV_{N_2|N_1} \chi^{N_2|N_1}_{\frac{N_3-N_2}{2}}(q, x_i, y_i). \tag{3.29}$$

In this expression, $\chi_{\mathcal{W}_{N_3-N_2}}$ is the character of the $\mathcal{W}_{N_3-N_2} \times \widehat{\mathfrak{gl}(1)}$ algebra, $\oint dV_{N_2,N_1}$ is the Vandermonde projector (invariant integration) that projects to $\mathfrak{gl}(N_1|N_2)$ invariant combinations of fields, and $\chi^{N_2|N_1}_{\frac{N_3-N_2}{2}}(q, x_i, y_i)$ is the character of a system of symplectic bosons in the fundamental representation of $\mathfrak{gl}(N_2|n_1)$ and with the level shifted by $\frac{N_3-N_2}{2}$ that comes from the DS-reduction of the off-diagonal blocks of $\widehat{\mathfrak{gl}(N_3|N_1)}_\Psi$. All of these ingredients are reviewed in appendix A.2.

Let us give a combinatorial description of the fields in Y_{0,N_2,N_3}. The Drinfeld-Sokolov reduction $\mathcal{DS}_{N_3-N_2}$ produces $N_3 - N_2$ fields W_i of conformal weights $i = 1, 2, \ldots, N_3 - N_2$ together with N_2 U, V fields with conformal weight $\frac{N_3-N_2}{2}$ and transforming in fundamental (anti-fundamental) representation of the preserved $\mathfrak{gl}(N_2)$ algebra. Fields preserved by the coset are then labelled by $\mathfrak{gl}(N_2)$-invariant words built from the following letters: $\partial^n W_i$ singlets of $U(N_2)$ of weight $i + n$, with $1 \leq i \leq N_3 - N_2$, $\partial^n U$ fundamentals of $\mathfrak{gl}(N_2)$ of weight $n + \frac{N_3-N_2+1}{2}$ and $\partial^n V$ anti-fundamentals of $\mathfrak{gl}(N_2)$ of weight $n + \frac{N_3-N_2+1}{2}$.

Equivalently, we can quotient the collection of words built from singlets $\partial^n W_i$ and bilinears $\partial^n U \cdot \partial^m V$ by the relations satisfied by products of bilinears. If we ignore these relations, the $\partial^n U \cdot \partial^m V$ give 1 generator of weight $N_3 - N_2 + 1$, 2 of weight $N_3 - N_2 + 2$, etc. and combine with the $\partial^n W_i$ to give a $W_{1+\infty}$-like set of generators.

The first non-trivial relation should be $\det_{(N_2+1)\times(N_2+1)}(\partial^n U \cdot \partial^m V) = 0$, occurring at level $N_3 - N_2 + 1 + N_3 - N_2 + 3 + \cdots N_3 + N_2 + 1 = (N_3 + 1)(N_2 + 1)$.

For the algebra $Y_{N_1,0,N_3}[\Psi]$, we can again give a combinatorial description of its generators. The Drinfeld-Sokolov reduction $\mathcal{DS}_{N_3}$ produces W_i generators of weight $i =$

$1, 2, \ldots, N_3$ together with fermionic fields A, B of weight $\frac{N_3+1}{2}$ transforming in the fundamental (anti-fundamental) representation of the preserved $\mathfrak{gl}(N_1)$ algebra. The fields surviving the coset are then labelled by $\mathfrak{gl}(N_1)$-invariant words built from the following letters: $\partial^n W_i$ singlets of $\mathfrak{gl}(N_1)$ of weight $i+n$, with $1 \leq i \leq N_3$, $\partial^n A$ fermionic fundamentals of $\mathfrak{gl}(N_1)$ of weight $n + \frac{N_3+1}{2}$ and $\partial^n B$ fermionic anti-fundamentals of $\mathfrak{gl}(N_1)$ of weight $n + \frac{N+1}{2}$.

Equivalently, we can quotient the collection of words built from singlets $\partial^n W_i$ and bilinears $\partial^n A \cdot \partial^m B$ by the relations satisfied by products of bilinears. If we ignore the relations which occur at finite N_1, the combinations of the form $\partial^n A \cdot \partial^m B$ give 1 generator of weight $N_3 + 1$, 2 of weight $N_3 + 2$, etc. and combine with the $\partial^n W_i$ to give a $W_{1+\infty}$-like set of generators. The first non-trivial relation should be $(A \cdot B)^{N_1+1} = 0$, occurring at level $(N_1 + 1)(N_3 + 1)$.

The combinatorial description of the generators for $Y_{0,N_2,N_3}[\Psi]$ and $Y_{N_1,0,N_3}[\Psi]$ has an obvious generalization: they should be labelled by $\mathfrak{gl}(N_2|N_1)$-invariant words built from the following letters: $\partial^n W_i$ singlets of $\mathfrak{gl}(N_2|N_1)$ of weight $i+n$, with $1 \leq i \leq N_3 - N_2$, $\partial^n \mathcal{U}$ fundamentals of $\mathfrak{gl}(N_2|N_1)$ (i.e. sets of N_2 bosons and N_1 fermions) of weight $n + \frac{N_3-N_2+1}{2}$ and $\partial^n \mathcal{V}$ anti-fundamentals of $\mathfrak{gl}(N_2|N_1)$ (i.e. sets of N_2 bosons and N_1 fermions) of weight $n + \frac{N_3-N_2+1}{2}$

Equivalently, we can quotient the collection of words built from singlets $\partial^n W_i$ and bilinears $\partial^n \mathcal{U} \cdot \partial^m \mathcal{V}$ by the relations satisfied by products of bilinears. If we ignore the relations which occur at finite N_1 and N_2, the combinations of the form $\partial^n \mathcal{U} \cdot \partial^m \mathcal{V}$ give 1 generator of weight $N_3 - N_2 + 1$, 2 of weight $N_3 - N_2 + 2$, etc. and combine with the $\partial^n W_i$ to give a $W_{1+\infty}$-like set of generators.

The first non-trivial relation should involve a mixed symmetrization of the $\partial^n \mathcal{U}$ labels in a product of bilinears which vanishes for fundamentals of $\mathfrak{gl}(N_2|N_1)$. The representations $R_{a,s}$ of $\mathfrak{gl}(N_2|N_1)$ labelled by rectangular Young Tableaux obtained from mixed symmetrization of fundamentals of $\mathfrak{gl}(N_2|N_1)$ are non-vanishing for (a, s) inside the "$\mathfrak{gl}(N_2|N_1)$-hook", the difference between the positive quadrant and the shifted quadrant with $s = N_1 + 1$, $a = N_2 + 1$.

The first non-trivial vanishing condition occurs for R_{N_2+1,N_1+1}. This is a modification of the $(N_1 + 1)$-th power of the determinant $\det_{(N_2+1)\times(N_2+1)} (\partial^n \mathcal{U} \cdot \partial^m \mathcal{V})$ and should have weight $(N_1 + 1)(N_2 + 1)(N_3 + 1)$.

We now formulate the following conjecture: the generators of Y_{N_1,N_2,N_3} are in one-to-one correspondence with 3d partitions (as in the crystal melting story [72]) restricted to lie in the difference between the positive octant and the shifted positive octant with origin at N_1,

n_2, N_3. (See figure 3.7 for the $Y_{2,1,1}[\Psi]$ example.) Notice that unrestricted 3d partitions are counted by the McMahon function

$$\chi_\infty(q) = \frac{1}{\prod_{n>0}(1-q^n)^n} \tag{3.30}$$

which also counts generators of $W_{1+\infty}$ and is the generating function of Donaldson-Thomas invariants of $\mathbb{C}^3$ appearing in the topological vertex literature [49, 50]. This conjecture has been chect for a large class of $Y_{N_1,N_2,N_3}[\Psi]$ in [1], agrees with the D4-D2-D0 brane counting of [51] and (as we will see bellow), there exist quotients of the $\mathfrak{gl}(1)$ affine Yangian with correct vacuum character.

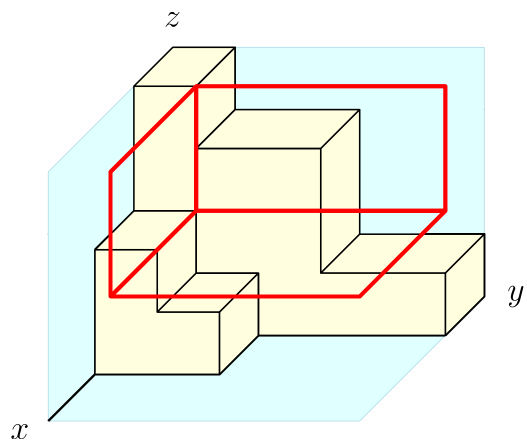


Figure 3.7: Example of a 3d partition for algebra $Y_{2,1,1}[\Psi]$. All the boxes of all allowed partitions are constrained to lie between the corner with a vertex at the origin and shifted (red) corner with vertex at $(2,1,1)$.

Truncations of $\mathcal{W}_{1+\infty}$

The vertex operator algebra $\mathcal{W}_\infty$ is an algebra obtained by extending the Virasoro algebra by independent primary fields of each integral spin ≥ 3, so that the generators are

$$T, W_3, W_4, W_5 \ldots \tag{3.31}$$

Imposing the conditions of associativity, [74, 75] concluded that there exists a two parameter family of such algebras, one parameter being the central charge c and the other one can be chosen to be

$$x^2 = \frac{(C_{33}^4)^2 C_{44}^0}{(C_{33}^0)^2} \tag{3.32}$$

where C^l_{jk} are the OPE coefficients (C^l_{jk} is the coefficient of primary operator W_l in the OPE of W_j and W_k)[12].

It is convenient to add a decoupled $\widehat{\mathfrak{gl}(1)}$ current into the algebra and define $\mathcal{W}_{1+\infty} \equiv \mathcal{W}_\infty \times \widehat{\mathfrak{gl}(1)}$. At special curves in the two-parameter space of such algebras, $\mathcal{W}_{1+\infty}$ develops an ideal $\mathcal{I}$. Quotienting this ideal out, one obtains a truncation of $\mathcal{W}_{1+\infty}$. According to [75], some of such truncations can be identified with $\mathcal{W}_N \times \widehat{\mathfrak{gl}(1)}$ algebras generated by fields up to spin N. The structure of truncations of $\mathcal{W}_{1+\infty}$ was further analyzed in [5] and later in [2] where new truncations were discovered. It turns out that Y-algebra can be identified with these more general truncations of $\mathcal{W}_{1+\infty}$.

As pointed out in [5], there exists an useful parametrization of the structure constants in terms of a triple of parameters λ_i satisfying

$$\frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3} = 0 \tag{3.33}$$

in terms of which the central charge and parameter (3.32) are given by

$$\begin{aligned} c_\infty &= (\lambda_1 - 1)(\lambda_2 - 1)(\lambda_3 - 1) \\ x^2 &= \frac{144(c+2)(\lambda_1 - 3)(\lambda_2 - 3)(\lambda_3 - 3)}{(\lambda_1 - 2)(\lambda_2 - 2)(\lambda_3 - 2)}. \end{aligned} \tag{3.34}$$

Modifying the stress energy tensor in such a way that the current J has conformal weight one, the central charge get shifted by one $c_{1+\infty} = c_\infty + 1$. The reason for introducing this parametrization is that for $\lambda_j = N$ where N is any positive integer, the algebra truncates to $\mathcal{W}_N \times \widehat{\mathfrak{gl}(1)}$. Although the structure constants of the algebra in the primary basis are manifestly invariant under S_3 transformation permuting the parameters λ_j, this triality symmetry acts non-trivially on representations. We might as well analytically continue the structure constants of $\mathcal{W}_N \times \widehat{\mathfrak{gl}(1)}$ as a function of the rank parameter N (since with a suitable choice of normalization they are just rational functions of N and c) and find following Gaberdiel and Gopakumar [75] that for a fixed value of the central charge c, there are generically three different values λ_j of N for which we get the same structure constants.

The local fields of the $\mathcal{W}_{1+\infty}$ algebra can be labeled by 3d partitions where the conformal dimension of fields is given by the number of boxes of the corresponding partition[13]. At

[12]Although starting from spin 6 the primary operators are not uniquely determined even up to an overall rescaling, there is no such problem with primaries of spin 3 or 4.

[13]This simple combinatorial interpretation is one of the main reasons for considering the additional $U(1)$ factor instead of restricting purely to $\mathcal{W}_{1+\infty}$.

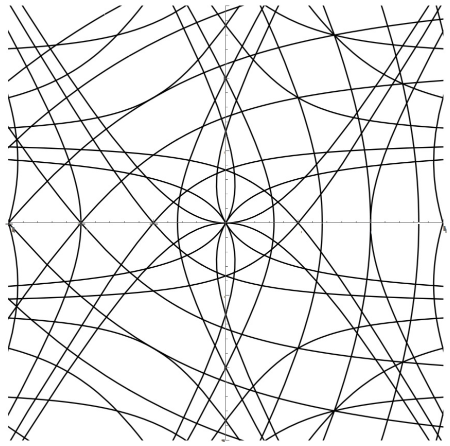


Figure 3.8: Truncation curves parametrized by (N_1, N_2, N_3) such that $(N_1+1)(N_2+1)(N_3+1) \leq 6$. This restriction means that the first generator that we are removing appears at level ≤ 6 in the vacuum module. We use the parametrization from [5] where the two axes are related to λ_i parameters by $x = \frac{1}{3}(2\lambda_1 - \lambda_2 - \lambda_3)$, $y = \frac{1}{\sqrt{3}}(\lambda_2 - \lambda_3)$ which manifestly shows the S_3 triality symmetry. At the points where two curves cross, we find the minimal models of $\mathcal{W}_{1+\infty}$ algebra if we quotient out by the maximal ideal which in particular contains the two ideals coming from the two curves that meet.

special curves in the two-dimensional parameter space of $\mathcal{W}_{1+\infty}$ algebras, the generators associated to 3d partitions having a box at coordinates (N_1+1, N_2+1, N_3+1)[14] form an ideal $\mathcal{I}_{N_1,N_2,N_3}$. In other words, $\mathcal{I}_{N_1,N_2,N_3}$ contains all the configurations, where the boxes do not fit between the corner and its copy shifted by (N_1, N_2, N_3). The curve in the parameter space for which $\mathcal{I}_{N_1,N_2,N_3}$ appears is given by

$$\frac{N_1}{\lambda_1} + \frac{N_2}{\lambda_2} + \frac{N_3}{\lambda_3} = 1. \tag{3.35}$$

Note that due to (3.33), the ideals $\mathcal{I}_{N_1,N_2,N_3} \supset \mathcal{I}_{N_1+k,N_2+k,N_3+k}$ are associated to the same curve. Derivation of the formula (3.35) using the isomorphism of the algebra with the affine Yangian of $\mathfrak{gl}(1)$ along the lines of [98, 2] can be found in the appendix A.1. If we quotient by the ideal $\mathcal{I}_{N_1,N_2,N_3}$, we recover an algebra with generators associated to 3d partitions living between the corner at the origin and the corner shifted by (N_1, N_2, N_3). Each truncation curve has a corresponding maximal truncation which we get by factoring out

$$\mathcal{I}_{(N_1,N_2,N_3)-\max(N_1,N_2,N_3)(1,1,1)}, \tag{3.36}$$

[14]Here we use the convention that the box corresponding to $J_{-1}|0\rangle$ is at position $(1, 1, 1)$.

or in other words choosing one of (N_1, N_2, N_3) to be zero. These are the truncations discussed in [5] and they correspond to quotients that are irreducible for generic values of the central charge. For illustration, few truncation curves are depicted in figure 3.8.

Identification of Y_{N_1,N_2,N_3} with $\mathcal{W}_{1+\infty}$

We can now see that the vacuum character of Y_{N_1,N_2,N_3} agrees with the vacuum character of the truncation $\mathcal{W}_{1+\infty}/\mathcal{I}_{N_1,N_2,N_3}$. In this section we discuss few pieces of evidence supporting this identification.

The central charge of Y_{N_1,N_2,N_3} can be determined from the BRST definition [1] and one finds the following expression

$$
\begin{aligned}
c_{N_1,N_2,N_3}[\Psi] = {}& \frac{1}{\Psi}(N_1 - N_3)\left((N_1 - N_3)^2 - 1\right) + \Psi(N_2 - N_3)\left((N_2 - N_3)^2 - 1\right) + \\
& + \frac{1}{\Psi - 1}(N_2 - N_1)((N_2 - N_1)^2 - 1) + (2N_3 + N_2 - 3N_1)(N_3 - N_2)^2 + N_1 - N_3. \qquad (3.37)
\end{aligned}
$$

Note that this is invariant under replacements

$$
\begin{array}{ll}
\Psi \leftrightarrow \frac{1}{\Psi} & N_1 \leftrightarrow N_2 \\
\Psi \leftrightarrow 1 - \Psi & N_2 \leftrightarrow N_3
\end{array} \qquad (3.38)
$$

which generate the S_3 group of transformations that will be further discussed in tha last section. The group acts by permutations on (N_1, N_2, N_3) and on Ψ by fractional linear transformations permuting $(0, 1, \infty)$. This motivates us to introduce another parametrization

$$
\begin{aligned}
\lambda_1 &= N_1 - \Psi N_2 - (1 - \Psi)N_3 \\
\lambda_2 &= -\frac{N_1 - \Psi N_2 - (1 - \Psi)N_3}{\Psi} \\
\lambda_3 &= \frac{N_1 - \Psi N_2 - (1 - \Psi)N_3}{\Psi - 1}
\end{aligned} \qquad (3.39)
$$

satisfying

$$
\frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3} = 0 \qquad (3.40)
$$

just like in the case of $\mathcal{W}_{1+\infty}$. Furthermore, the expression for the central charge (3.37) can be rewritten in the form

$$
c_{1+\infty} = (\lambda_1 - 1)(\lambda_2 - 1)(\lambda_3 - 1) + 1. \qquad (3.41)
$$

which is equal to the central charge in $\mathcal{W}_{1+\infty}$ (3.37).[15] The S_3 triality action of [1] acts simply by permutations of the parameters λ_i and the central charge is manifestly triality invariant in this parametrization. It is convenient for what follows to introduce h parameters (h_1, h_2, h_3) by

$$\Psi = -\frac{h_2}{h_1} \qquad 0 = h_1 + h_2 + h_3 \tag{3.42}$$

(note that they are determined by Ψ only up to an overall scale factor). In terms of these, the parameters λ_i can be written in more symmetric form

$$\lambda_i = \frac{N_1 h_1 + N_2 h_2 + N_3 h_3}{h_i}. \tag{3.43}$$

Parameters h_i naturally appear as parameters of the affine Yangian of $\mathfrak{gl}(1)$ and can also be identified with the Nekrasov Ω-deformation parameters in the cohomological Hall algebra construction [4].

Note the special points where the two truncation curves intersect. Algebras with such values of parameters contain further null-states that can be factorized. From the point of view of Y-algebras, these points correspond to DS-reduction and coset of Kac-Moody algebras at rational levels. At rational levels the Kac-Moody algebras contain null states and to take them into account one should use Kac-Weyl characters to calculate the characters of the final algebra. At least in the case of $\mathcal{W}_N$ algebras these are known to lead to minimal models [99]. It would be nice to generalize this construction to all Y-algebras. In our considerations we will always consider Ψ to be generic corresponding to a generic points of the truncation curve (N_1, N_2, N_3).

3.2.5 Generalized Miura transformation

As motivated above, we also expect an existence of a free-field realization of a general Y_{N_1,N_2,N_3} algebra. Such a free field realization can be characterized either in terms of the Miura transformation or the intersection of kernels of screening charges. Let us now give a generalization of the well-known Miura transformation for $Y_{0,0,N}$ of [22, 7] to general Y_{N_1,N_2,N_3}.

[15] Note the shift by one due to the presence of the $\widehat{\mathfrak{gl}(1)}$ factor.

Review of $Y_{0,0,N}$ Firstly, we review the standard Miura transformation for $Y_{0,0,N}$. Consider a set of N $\widehat{\mathfrak{gl}(1)}$ currents $J_j(z)$ with OPEs

$$J_j(z)J_k(w) \sim \frac{\delta_{jk}}{(z-w)^2} \tag{3.44}$$

and define operators $U_k(z)$ via

$$(\alpha_0\partial + J_1(z)) \cdots (\alpha_0\partial + J_N(z)) \equiv \prod_{j=1}^{N} \mathcal{L}_j^{(3)}(z) = \sum_{k=0}^{N} U_k(z)(\alpha_0\partial)^{N-k}, \tag{3.45}$$

where we have an identification $\alpha_0 = h_3$. Operators U_k and their normal ordered products and derivatives form a closed algebra under operator product expansion [100].

The general case One can extend the Miura transformation to the case where there are nodes of different types. For that it is important to remember that we have three types of nodes corresponding to three different free field representations of $\mathcal{W}_{1+\infty}$ corresponding to $\lambda_1 = 1$, $\lambda_2 = 1$ or $\lambda_3 = 1$ (as well as their conjugates). The usual Miura transformation in our conventions has all nodes of type 3 with $\lambda_3 = 1$. We will see that the usual procedure works even in the case of $\lambda_1 = 1$ or $\lambda_2 = 1$ but we have to replace the elementary factor

$$\mathcal{L}^{(3)}(z) \equiv \alpha_0\partial + J^{(3)}(z) \tag{3.46}$$

by a pseudo-differential operator with an infinite number of coefficients which are local fields. This generalization is common in the context of integrable hierarchies of differential equations (e.g. KdV or KP hierarchies), [101, 102].

Let us first consider what happens in the case that $\lambda_1 = 1$. In this situation, there exists a free field representation of $\mathcal{W}_{1+\infty}$ associated to a single free boson $\phi^{(1)}$, but in the quadratic U-basis (which is itself associated to the third direction), there is an infinite number of non-trivial U_j generators, all expressed in terms of $\phi^{(1)}$. Choosing for convenience the parametrization as in [98]

$$\begin{aligned} h_1 &= h \\ h_2 &= -\frac{1}{h} \\ h_3 &= \frac{1}{h} - h = \alpha_0 \\ \psi_0 &= \lambda_3 = N \end{aligned} \tag{3.47}$$

we need to require

$$1 = \lambda_1^{(1)}, \tag{3.48}$$

i.e.

$$N^{(1)} = \lambda_3^{(1)} = -\frac{h^2}{h^2-1} = -\frac{h_1}{h_1 h_2 h_3}. \tag{3.49}$$

From the Miura transformation point of view, this $N^{(1)}$ is the order of the pseudo-differential operator corresponding to the $\phi^{(1)}$ representation. In the following, it will be useful to choose the normalization coefficient of the two-point function of the current $J^{(1)} \equiv \partial\phi^{(1)}$ to be $N^{(1)}$,

$$J^{(1)}(z)J^{(1)}(w) \sim \frac{N^{(1)}}{(z-w)^2}. \tag{3.50}$$

Having fixed all the parameters of algebra, we can now find the expressions for $U_j^{(1)}(z)$ fields in terms of $J^{(1)}$, requiring just the commutation relations spelled out in [5]. They are uniquely determined up to the conjugation $J^{(1)} \leftrightarrow -J^{(1)}$ symmetry. Fixing a positive sign, the expressions for the first few fields are

$$\begin{aligned}
U_1^{(1)} &= J^{(1)} \\
U_2^{(1)} &= \left(2-\frac{1}{h^2}\right)\left(\frac{(J^{(1)}J^{(1)})}{2}+\frac{h\partial J^{(1)}}{2}\right) \\
U_3^{(1)} &= \left(2-\frac{1}{h^2}\right)\left(3-\frac{2}{h^2}\right)\left(\frac{(J^{(1)}(J^{(1)}J^{(1)}))}{6}+\frac{h(\partial J^{(1)}J^{(1)})}{2}+\frac{h^2\partial^2 J^{(1)}}{6}\right) \\
U_4^{(1)} &= \left(2-\frac{1}{h^2}\right)\left(3-\frac{2}{h^2}\right)\left(4-\frac{3}{h^2}\right)\left(\frac{(J^{(1)}(J^{(1)}(J^{(1)}J^{(1)})))}{24}+\right. \\
&\quad \left.+\frac{h(\partial J^{(1)}(J^{(1)}J^{(1)}))}{4}+\frac{h^2(\partial J^{(1)}\partial J^{(1)})}{8}+\frac{h^2(\partial^2 J^{(1)}J^{(1)})}{6}+\frac{h^3\partial^3 J^{(1)}}{24}\right)
\end{aligned} \tag{3.51}$$

The expressions for higher $U_j^{(1)}$ fields are uniquely determined from the OPE of $U_3^{(1)}U_{j-1}^{(1)}$. But even the general pattern is not very difficult to understand: first of all, each $U_j^{(1)}$ has an overall multiplicative factor

$$\prod_{k=1}^{j-1}\left[1+k\left(1-\frac{1}{h^2}\right)\right] = \prod_{k=1}^{j-1}\left(1-\frac{k}{N^{(1)}}\right). \tag{3.52}$$

Next, there is a sum of all dimension j operators that we can construct out of a free boson. The power of h in each term counts the number of derivatives appearing in the operator and the combinatorial factors can be most easily seen using the operator-state correspondence:

$$\begin{aligned}
U_1^{(1)} &\to a_{-1} \\
U_2^{(1)} &\to \frac{a_{-1}^2}{2} + \frac{ha_{-2}}{2} \\
U_3^{(1)} &\to \frac{a_{-1}^3}{6} + \frac{ha_{-1}a_{-2}}{2} + \frac{h^2 a_{-3}}{3} \\
U_4^{(1)} &\to \frac{a_{-1}^4}{24} + \frac{ha_{-1}^2 a_{-2}}{4} + \frac{h^2 a_{-2}^2}{8} + \frac{h^2 a_{-1}a_{-3}}{3} + \frac{h^3 a_{-4}}{4}
\end{aligned} \tag{3.53}$$

These are exactly the coefficients appearing in Newton's identities if we think of U_j to be the homogeneous symmetric polynomials and a_j to be the power sum symmetric polynomials. One can thus also write a closed-form formula

$$U_j^{(1)} = \prod_{k=1}^{j-1} \left(1 - \frac{k}{N^{(1)}}\right) \sum_{m_1+2m_2+\ldots+jm_j=j} \prod_{k=1}^{j} \frac{1}{m_k! k^{m_k}} \left(\frac{h^{k-1}}{(k-1)!} \partial^{k-1} J^{(1)}\right)^{m_k} \tag{3.54}$$

where everything is normal ordered. The total Miura operator representing the $\phi^{(1)}$ node of the diagram (see figure 3.9) is thus given by the pseudo-differential operator

$$\mathcal{L}^{(1)}(z) \equiv (\alpha_0 \partial)^{\frac{h_1}{h_3}} + \sum_{j=1}^{\infty} U_j^{(1)}(z)(\alpha_0 \partial)^{\frac{h_1}{h_3}-j}. \tag{3.55}$$

In the case of representation of type $\phi^{(2)}$ the calculation is entirely analogous and in fact we can just make a replacement $h \leftrightarrow -\frac{1}{h}$. We require $\lambda_2^{(2)} = 1$ and so in this case

$$N^{(2)} = \lambda_3^{(2)} = \frac{1}{h^2-1} = -\frac{h_2}{h_1 h_2 h_3}. \tag{3.56}$$

The current is again normalized such that the quadratic pole of the $J^{(2)}J^{(2)}$ OPE is equal to this value of $N^{(2)}$. Choosing the sign of $U_1^{(2)}$, all other $U_j^{(2)}$ operators are uniquely determined and we find

$$\begin{aligned}
U_1^{(2)} &= J^{(2)} \\
U_2^{(2)} &= (2-h^2)\left(\frac{(J^{(2)}J^{(2)})}{2} - \frac{\partial J^{(2)}}{2h}\right) \\
U_3^{(2)} &= (2-h^2)(3-2h^2)\left(\frac{(J^{(2)}(J^{(2)}J^{(2)}))}{6} - \frac{(\partial J^{(2)}J^{(2)})}{2h} + \frac{\partial^2 J^{(2)}}{6h^2}\right) \\
U_4^{(2)} &= (2-h^2)(3-2h^2)(4-3h^2)\left(\frac{(J^{(2)}(J^{(2)}(J^{(2)}J^{(2)})))}{24} - \right. \\
&\quad \left. - \frac{(\partial J^{(2)}(J^{(2)}J^{(2)}))}{4h} + \frac{(\partial J^{(2)}\partial J^{(2)})}{8h^2} + \frac{(\partial^2 J^{(2)}J^{(2)})}{6h^2} - \frac{\partial^3 J^{(2)}}{24h^3}\right)
\end{aligned} \tag{3.57}$$

The formula for $U_j^{(2)}$ is now

$$U_j^{(2)} = \prod_{k=1}^{j-1}\left(1 - \frac{k}{N^{(2)}}\right) \sum_{m_1+2m_2+\ldots+jm_j=j} \prod_{k=1}^{j} \frac{1}{m_k! k^{m_k}} \left(\frac{(-1)^{k-1}}{(k-1)! h^{k-1}} \partial^{k-1} J^{(2)}\right)^{m_k} \tag{3.58}$$

and the Miura pseudo-differential operator representing a node of type $\phi^{(2)}$ is

$$\mathcal{L}^{(2)}(z) \equiv (\alpha_0 \partial)^{\frac{h_2}{h_3}} + \sum_{j=1}^{\infty} U_j^{(2)}(z)(\alpha_0 \partial)^{\frac{h_2}{h_3}-j}. \tag{3.59}$$

It has been later noticed in [37] that the above pseudo-differential operators have actually a simpler and uniform description in terms of the following operator

$$\mathcal{L}^{(\kappa)} = \exp\left[-\frac{i}{h_\kappa}\phi^{(\kappa)}\right](h_3\partial)^{\frac{h_\kappa}{h_3}} \exp\left[\frac{i}{h_\kappa}\phi^{(\kappa)}\right], \tag{3.60}$$

where the products are normally ordered and we have introduced auxiliary fields ϕ^κ with a logarithmic OPE

$$\phi^\kappa(z)\phi^\kappa(w) \sim -\frac{h_\kappa}{h_1 h_2 h_3} \log(z-w) \tag{3.61}$$

such that we can identify $J^{(\kappa)} = \partial \phi^{(\kappa)}$.

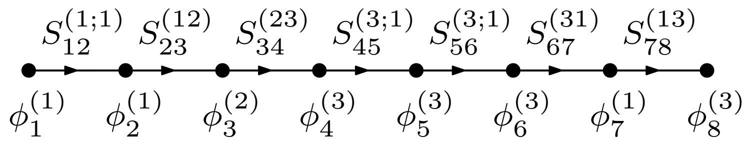


Figure 3.9: An example of the ordering of free bosons for $Y_{3,1,4}$. The algebra can be found by multiplying the Miura pseudo-differential operators in the order $\mathcal{L}_1^{(1)}(z)\mathcal{L}_2^{(1)}(z)\ldots\mathcal{L}_7^{(1)}(z)\mathcal{L}_8^{(3)}(z)$ as shown in the figure. Alternatively, one can construct the free field realization as an intersection of kernels of screening charges $S_{12}^{(1;1)}, S_{23}^{(12)}, \ldots S_{78}^{13}$ associated to the lines of the chain of free bosons.

We can use these newly constructed building blocks to find a free field representation of any Y_{N_1,N_2,N_3} algebra: pick an arbitrary ordering of N_j bosons of type $\phi^{(j)}$ as shown in the figure 3.9 for a particular ordering of the $Y_{3,1,4}$ algebra and multiply the corresponding Miura operators $\mathcal{L}_j^{(\kappa_j)}$. Commuting all the derivatives to the right (recall that even for non-integer powers of derivative the generalization of Leibniz rule still applies), we find in the end a pseudo-differential operator of the form

$$\mathcal{L}(z) = (h_3\partial)^{\frac{N_1h_1+N_2h_2+N_3h_3}{h_3}} + \sum_{j=1}^{\infty} U_j(z)(h_3\partial)^{\frac{N_1h_1+N_2h_2+N_3h_3}{h_3}-j} \tag{3.62}$$

where U_j are certain normal ordered differential polynomials in the free boson fields. The statement is that the fields $U_j(z)$, their normal ordered products and derivatives form a closed subalgebra of the algebra of $N_1+N_2+N_3$ free bosons which represents Y_{N_1,N_2,N_3} in terms of free bosons. Furthermore, OPEs of these U_j fields are still those of the quadratic U-basis with structure constants given in [5].

3.2.6 Screening charges

To each ordering of N_κ free bosons $\phi_i^{(\kappa)}$ of type κ with the corresponding currents $J_i^{(\kappa)} = \partial\phi_i^{(\kappa)}$ normalized as

$$J_{i_1}^{(\kappa_1)}(z)J_{i_2}^{(\kappa_2)}(w) \sim -\frac{h_\kappa}{h_1h_2h_3}\frac{\delta^{\kappa_1,\kappa_2}}{(z-w)^2}, \tag{3.63}$$

we have the associated free field realization of the algebra Y_{N_1,N_2,N_3}. On the other hand, the authors of [28, 29] construct a free field realization of the same algebra as a kernel of

$N_1 + N_2 + N_3 - 1$ screening charges acting on the tensor product of the current algebras above. Let us define screening charges for each such ordering and check that they are of the form of [29].

Consider a fixed ordering of free bosons such as the one in the figure 3.9. One associates a screening charge to each neighboring free bosons (lines connecting two nodes of the chain). If the two free bosons are of the same type, say $\kappa_i = \kappa_{i+1} = 3$, the corresponding screening current can be chosen to be either

$$S^{(3;1)}_{i,i+1} = \oint dz \exp\left[-h_1 \phi^{(3)}_i + h_1 \phi^{(3)}_{i+1}\right] \tag{3.64}$$

or

$$S^{(3;2)}_{i,i+1} = \oint dz \exp\left[-h_2 \phi^{(3)}_i + h_2 \phi^{(3)}_{i+1}\right]. \tag{3.65}$$

These two can be determined from the requirement that the zero mode of the exponential vertex operator commutes with the free field realization of the spin one and the spin two fields in the Virasoro algebra $Y_{0,0,2}$. One gets similar expressions for the other three types with the h_i parameters permuted. To a pair of free bosons of different type (say ordering $\phi^{(3)}_i \times \phi^{(2)}_{i+1}$), one associates instead the screening charge[16]

$$S^{(32)}_{i,i+1} = \oint dz \exp\left[-h_2 \phi^{(3)}_i + h_3 \phi^{(2)}_{i+1}\right] \tag{3.66}$$

and similarly for the other five combinations.

The screening charge $S_{i,i+1}$ maps the vacuum representation of the product of the current algebras generated by $J^{(\kappa)}_i = \partial \phi^{(\kappa)}_i$ to a module with the highest weight vector $j_{i,i+1}(0)|0\rangle$, where $j_{i,i+1}$ is the screening current associated to the screening charge $S_{i,i+1}$. The algebra Y_{N_1,N_2,N_3} can be defined as an intersection of kernels of screening charges

$$Y_{N_1,N_2,N_3} = \bigcap_{i=1}^{N_1+N_2+N_3-1} \ker S_{i,i+1}. \tag{3.67}$$

Consider now a triple of free bosons neighbouring in the chain and let us compute the matrix of inner products of the corresponding two exponents of the screening currents with

[16] The commutation with the spin one and the spin two field gives two possible solutions as in the case of the Virasoro algebra but only one is preserved by the requirement of commutativity with the spin three generator.

respect to the metric given by the normalization of two-point function

$$g_{jk} = -\frac{h_{\kappa_j}}{h_1 h_2 h_3}\delta_{jk} \tag{3.68}$$

We will see that the different choices of ordering and different choices of the screening currents (3.64) and (3.65) lead to different matrices from [29].

If all the three free bosons are of the same type $\phi_1^{(3)} \times \phi_2^{(3)} \times \phi_3^{(3)}$, one can pick either both screening charges to be of the same type (3.64) or (3.65) or one of the first type and the second one of the second type. In these four cases, one gets respectively the following two matrices

$$-\frac{h_1}{h_2}\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}, \qquad -\frac{h_1}{h_2}\begin{pmatrix} 2 & -\frac{h_2}{h_1} \\ -\frac{h_2}{h_1} & 2 \end{pmatrix}, \tag{3.69}$$

together with matrices with the parameters $h_1 \leftrightarrow h_2$ interchanged. These two matrices are of the form 1 and 2 from (2.24) of [29].

If one of the three free bosons is of a different type than the other two, say 332, one has two possible orderings. In the first case, $\phi_1^{(3)} \times \phi_2^{(3)} \times \phi_3^{(2)}$, one has again a choice between the screening currents (3.64) and (3.65) leading to the following two overlap matrices

$$\begin{pmatrix} -2\frac{h_1}{h_2} & 1 \\ 1 & 1 \end{pmatrix}, \qquad \begin{pmatrix} -2\frac{h_2}{h_1} & \frac{h_2}{h_1} \\ \frac{h_2}{h_1} & 1 \end{pmatrix} \tag{3.70}$$

that are of the form 4 and 3 of [29]. The last, symmetric ordering $\phi_1^{(3)} \times \phi_2^{(2)} \times \phi_3^{(3)}$ gives an overlap matrix of the form

$$\begin{pmatrix} 1 & \frac{h_3}{h_1} \\ \frac{h_3}{h_1} & 1 \end{pmatrix} \tag{3.71}$$

which is of the form 5. Finally, if all the bosons are of a different type, one gets the matrix of overlaps

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}. \tag{3.72}$$

Comparing the free field realizations of $Y_{0,0,2}$ and $Y_{0,1,1}$ from the Miura transformation and from the kernel of screening charges together with the triality symmetry permuting the Y-algebra labels, one can see that the two free field realizations are the same.

4. Webs

4.1 Modules

Before continuing with the discussion of VOAs associated to a general colored (p,q) web let us pause and discuss the physics and the structure of degenerate modules of Y_{N_1,N_2,N_3}.

4.1.1 Type IIB perspective and BRST construction

The standard $\mathcal{W}_N$ algebras have maximally degenerate modules $M_{\lambda,\mu}$ labelled by a pair of dominant weights of $\mathfrak{sl}(N)$. These modules are expected to arise in the gauge theory construction from local operators at the corner which are attached to a boundary Wilson line of weight λ along the NS5 boundary and a boundary 't Hooft line of weight μ along the D5 boundary. These two line defects are correspondingly exchanged by S-duality.

If we denote $W_\lambda = M_{\lambda,0}$ and $H_\mu = M_{0,\mu}$, then the following facts hold true:

- The W_λ have the same fusion rules

$$W_\lambda \times W_{\lambda'} \sim \sum_{\lambda''} c^{\lambda''}_{\lambda,\lambda'} W_{\lambda''} \tag{4.1}$$

  as finite-dimensional $\mathfrak{sl}(N)$ irreps. They have non-trivial braiding and fusion matrices which are closely related to these of $\mathfrak{sl}_{\Psi-N}$. Conformal blocks with W_λ insertions satisfy BPZ differential equations.

- The H_μ also have the same fusion rules

$$H_\mu \times H_{\mu'} \sim \sum_{\mu''} c^{\mu''}_{\mu,\mu'} H_{\mu''} \tag{4.2}$$

as finite-dimensional $\mathfrak{sl}(N)$ irreps. They have non-trivial braiding and fusion matrices which are closely related to these of $\mathfrak{sl}(N)_{\Psi^{-1}-N}$. Conformal blocks with H_μ insertions satisfy BPZ differential equations

- The W_λ and H_μ vertex operators are almost mutually local. They are local if we restricts the weights to those of GL-dual groups. The fuse in a single channel $M_{\lambda,\mu}$.

We expect analogous statements for maximally degenerate modules of $Y_{N_1,N_2,N_3}[\Psi]$, involving local operators sitting at the end to three boundary lines, one for each component of the gauge theory junction. These modules should thus carry three labels, permuted by the S^3 triality symmetry, corresponding to the possible labels of BPS line defects living on the $\tilde{\mathcal{B}}_{(p,q)}$ boundary conditions. It is known [47] that such line defects include analogues of Wilson lines, labelled by data akin to dominant weights of $\mathfrak{gl}(N_1|N_2)$, $\mathfrak{gl}(N_2|N_3)$, $\mathfrak{gl}(N_3|N_1)$ respectively.

In particular, we expect the following to be true: if we denote W_λ, H_μ and D_σ the modules associated to either type of boundary lines

- The W_λ should have the same fusion rules as finite-dimensional $\mathfrak{gl}(N_3|N_1)$ irreps, with appropriate non-trivial braiding and fusion matrices and BPZ-like differential equations.

- The H_μ should have the same fusion rules as finite-dimensional $\mathfrak{gl}(N_2|N_3)$ irreps, with appropriate non-trivial braiding and fusion matrices and BPZ-like differential equations.

- The D_σ should have the same fusion rules as finite-dimensional $\mathfrak{gl}(N_1|N_2)$ irreps, with appropriate non-trivial braiding and fusion matrices and BPZ-like differential equations.

- The W_λ, H_μ and D_σ vertex operators should be mutually local and fuse together into a single channel $M_{\lambda,\mu,\nu}$

In the BRST constructions for $Y_{N_1,N_2,N_3}[\Psi]$, the data for $\mathfrak{gl}(N_3|N_1)$ and $\mathfrak{gl}(N_3|N_2)$ representations appears rather naturally, as one may implement the BRST reduction starting from Weyl modules of the current algebras built from irreducible representations of the zero-mode algebra, up to subtleties in relating weights and representations for supergroups.

The data of $\mathfrak{gl}(N_2|N_3)$ is much harder to uncover, though in principle it can be done with the help of the gauge theory description in [47]. In general, the line defect along the D5

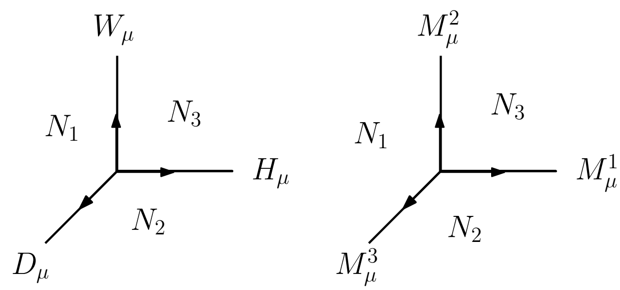


Figure 4.1: Modules $H_\mu, W_\lambda, D_\sigma$ associated to the three classes of boundary lines. We also include the notation from [2] that relabelled $M^1_\mu = H_\mu, M^2_\mu = W_\mu, M^3_\mu = D_\mu$ in a triality covariant way.

interface will map to some disorder local operator at the interface between Chern-Simons theories.

It is straightforward to modify the vacuum character calculations in order to compute the characters of degenerate modules of type W or D: essentially, one just inserts characters of finite-dimensional irreducible representations in the contour integrals, with fugacities associated to DS-reduced directions specialized to the appropriate powers of q.

In the following, we will illustrate the calculation on the example of $M^2_\nu \otimes M^3_\mu$ modules of $Y_{0,N_2,N_3}[\Psi]$ algebras such that we do not have to take into account the complications associated to representations of Lie superalgebras. These modules are labeled by tensor representations of $\mathfrak{gl}(N_2)$ and $\mathfrak{gl}(N_3)$.

Representations of $\mathfrak{gl}(N_2)$ are labeled by a set of integers $(\mu_1, \mu_2, \ldots, \mu_{N_2})$, where $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{N_2}$ (note that we do not restrict these to be non-negative and look at all the irreducible representation one can get from the tensor product of the fundamental and the anti-fundamental representation). Choosing a normalization of the $\widehat{\mathfrak{gl}(1)}$ current such that

$$J(z)J(w) \sim \frac{\lambda_1+\lambda_2}{(z-w)^2} = -\frac{\lambda_1\lambda_2}{\lambda_3}\frac{1}{(z-w)^2}, \tag{4.3}$$

the charge of M^3_μ and its conformal dimension are given by [1]

$$
\begin{aligned}
j(M^3_\mu) &= \sum_{j=1}^{M} \mu_j, &(4.4)\\
h(M^3_\mu) &= -\frac{\lambda_3}{2\lambda_1}\sum_{j=1}^{M}\mu_j^2 - \frac{\lambda_3}{2\lambda_2}\sum_{j=1}^{M}(2j-M-1)\mu_j + \frac{N}{2}\sum_{j=1}^{M}|\mu_j|. &(4.5)
\end{aligned}
$$

The characters of M^2_ν and M^3_μ modules of $Y_{0,N_2,N_3}[\Psi]$ can be calculated according to [1, 2] in a similar way as the vacuum character. The only modification is to insert a corresponding Schur polynomial $s_\mu(x_i)$ and $s_\nu(x_i)$ into the formula, i.e. in the case of $N_3 > N_2$ and M^3_μ representation, the character is given by

$$
\chi_{0,N_2,N_3}(M^3_\mu) = \chi_{\mathcal{W}_{N_3-N_2}}(q)\oint dV_{N_2}\chi^{N_2|0}_{\frac{N_3-N_2}{2}}(q,x_i)s_\mu(x_i). \tag{4.6}
$$

In the case of M^2_ν modules, one needs to first perform the DS reduction by substituting $x_j = q^{\frac{1}{2}(2j-N_2-1)}$ for $j \leq N_3 - N_2$ and then insert into the integral

$$
\chi_{0,N_2,N_3}(M^2_\nu) = \chi_{\mathcal{W}_{N_3-N_2}}(q)\oint dV_{N_2}\chi^{N_2|0}_{\frac{N_3-N_2}{2}}(q,x_i)s_\nu\left(x_j \to q^{\frac{1}{2}(2j-N_2-1)}, x_i\right). \tag{4.7}
$$

One can similarly calculate characters of modules with two asymptotics $M^2_\mu \otimes M^3_\nu$ by first doing the DS reduction substitution and then inserting into the integral formula both characters.

For positive values of μ_j, these characters have a nice box-counting interpretation that was discussed above. The conjugate representations have the sign and the order of μ's reversed and they have the same character and the same conformal dimension. Furthermore, if we split μ into positive and negative parts, $\mu = \mu_+ + \mu_-$, we see that both the $\mathfrak{gl}(1)$ charge and the dimension are additive under this splitting. As discussed in [33], the general character have also a box-counting interpretation coming from shifts of μ_i charges.

There is also an obvious extension of the 3d partition counting problem we associated to the vacuum characters of $Y_{N_1,N_2,N_3}[\Psi]$: one may consider 3d partitions with semi-infinite cylindrical ends modeled on 2d partitions λ, μ, ν, as in the definition of the topological string vertex [72].

[1] Checks of these formulas for $Y_{N,0,0}$, $Y_{0,1,0}$, $Y_{0,1,1}$, $Y_{0,2,1}$, $Y_{0,1,2}$ can be found in the appendix of [2].

The crucial observation is that the restriction for the 3d partition to lie in the region R_{N_1,N_2,N_3} forces λ, μ, ν to lie respectively in R_{N_2,N_3}, R_{N_3,N_1} and R_{N_1,N_2}. Thus λ, μ, ν have precisely the same form as the data labeling our degenerate modules $M_{\lambda,\mu,\nu} = W_\mu \times H_\lambda \times D_\nu$.

We are thus lead to the conjecture that the counting of 3d partitions with semi-infinite ends restricted to R_{N_1,N_2,N_3} computes the character of $M_{\lambda,\mu,\nu}$ for Y_{N_1,N_2,N_3} for λ, μ, ν covariant representations (such that appears in a tensor product of the fundamental representation) of the corresponding Lie super algebras. For a general tensor representation (obtained from tensor products of both the fundamental and the anti-fundamental representation), a shifting is required as discussed in [33]. Further investigation of the relation with the representation theory of Lie superalgebras is desirable.

4.1.2 Type IIA perspective and truncations of $\mathcal{W}_{1+\infty}$

The above conjecture identifies the characters of degenerate modules of Y_{N_1,N_2,N_3} with functions counting box configuration with prescribed asymptotics. This partition functions naturally appear in the geometric context from counting D4-D0 bound states in $\mathbb{C}^3$ in the presence of extra D2 branes supported on the lines $\mathbb{C}_{x_1}, \mathbb{C}_{x_2}, \mathbb{C}_{x_3}$ [52, 53, 51] or its unrestricted analogue leading to the topological vertex [49, 50, 72]. Indeed the line operators supported at the interfaces map to D2-branes with correct support under the duality of [48]. Moreover, it is well-known that the affine Yangian of $\mathfrak{gl}(1)$ admits modules parametrized by three assymptotic Young tableaux (see for example [98]) and it is natural to expect that the truncation condition simply restricts the possible Young tableaux.

In the special case that the three asymptotic representations are covariant representations (i.e. contained in the tensor power of the fundamental representation), we can use the box counting interpretation of the topological vertex to find the conformal dimensions, other highest weight charges and characters [72, 103, 80, 98]. In this case, the representations (λ, μ, ν) can be labeled by three Young diagrams. The states in the module of Y_{N_1,N_2,N_3} are then in one-to-one correspondence with the plane partitions which have nontrivial asymptotics given by the Young diagrams (λ, μ, ν) and further restricted such that the box at coordinates $(N_1 + 1, N_2 + 1, N_3 + 1)$ is not present.

The highest weight state corresponds to the configuration with minimal (but infinite) number of boxes compatible with the asymptotics. The states at level l are in one-to-one correspondence with plane partitions obtained by adding l boxes to this minimal configuration (always in a way such that the resulting configuration of boxes is a plane partition). This identification allows us to write down the character purely in terms of a combinatorial counting. The conformal dimension of the module can be similarly computed

[98] by first computing the generating function of the conserved charges of Y_{N_1,N_2,N_3} and extracting the eigenvalue of the L_0 generator from it. The result for the representation with an asymptotic Young diagram in the third direction is[2]

$$j(M^3_\mu) = \sum_j \mu_j \tag{4.8}$$

$$\begin{aligned} h(M^3_\mu) &= -\frac{\lambda_3}{2\lambda_1}\sum_j \mu_j^2 - \frac{\lambda_3}{2\lambda_2}\sum_j (2j-1)\mu_j + \frac{\lambda_3}{2}\sum_j \mu_j \\ &= -\frac{\lambda_3}{2\lambda_1}\sum_j \mu_j^2 - \frac{\lambda_3}{2\lambda_2}\sum_j (\mu^T)_j^2 + \frac{\lambda_3}{2}\sum_j \mu_j. \end{aligned} \tag{4.9}$$

In particular, the conformal dimension of the minimal representation is given by

$$h(\square_3) = \frac{1+\lambda_3}{2} \tag{4.10}$$

irrespectively of the truncation that we are considering. Its complex conjugate representation has an opposite $\widehat{\mathfrak{gl}(1)}$ charge but the same conformal dimension. By fusing these, we can in principle obtain an arbitrary maximally degenerate representation of the type we need for the gluing procedure.

The main disadvantage of the approach using box counting is that we have only access to representations whose asymptotics are those obtained from the fundamental representation (i.e. covariant representations) and it is not clear how to generalize these results directly to the case of fusions of both fundamental and anti-fundamental representations. The second disadvantage is the lack of useful closed-form formulas for the characters of the modules, but see [28] for the case where one of the parameters (N_1, N_2, N_3) vanishes.

4.1.3 BRST construction

4.1.4 Free field realization

Vertex operators Above, we have identified the algebra Y_{N_1,N_2,N_3} as a subalgebra of the tensor product of $N_1+N_2+N_3$ Heisenberg algebras. Here, we will discuss how to realize degenerate modules as zero modes of exponential vertex operators of the form

$$|q^1, \ldots, q^{N_1+N_2+N_3}\rangle = \exp\left(\sum_{j=1}^{N_1+N_2+N_3} q^j \phi_j\right)|0\rangle. \tag{4.11}$$

[2]The normalization of $\widehat{\mathfrak{gl}(1)}$ current is discussed later.

acting on the vacuum state $|0\rangle$. If we act on this state with the zero mode of current $J_j = \partial\phi_j$, we find

$$J_{j,0}|q^1,\dots,q^N\rangle = g_{jk}q^k|q^1,\dots,q^N\rangle \equiv q_j|q^1,\dots,q^N\rangle \tag{4.12}$$

where g_{jk} is the metric extracted from the two-point functions of the currents,

$$J_j(z)J_k(w) \sim \frac{g_{jk}}{(z-w)^2} \sim -\frac{h_{\kappa(j)}}{h_1h_2h_3}\frac{\delta_{jk}}{(z-w)^2}. \tag{4.13}$$

Our conventions for charges are such that q^j are the charges that appear in the exponents of vertex operators while q_j are the coefficients of the first order poles of OPE with currents J_j.

For example, the zero more of the $\widehat{\mathfrak{gl}(1)}$ current

$$U_1(z) = \sum_{j=1}^{N_1+N_2+N_3} J_j(z) \tag{4.14}$$

acts on the highest weight state by

$$U_{1,0}|q^1,\dots,q^{N_1+N_2+N_3}\rangle = \left(\sum_{j=1} q_j\right)|q^1,\dots,q^{N_1+N_2+N_3}\rangle \tag{4.15}$$

and analogously for the stress-energy tensor

$$T \;=\; W_2 = -\frac{1}{2}\sum_j \frac{h_1h_2h_3}{h_{\kappa_j}}(J_jJ_j)(z) + \frac{1}{2}\sum_{j<k} h_{\kappa_k}\partial J_j - \frac{1}{2}\sum_{j>k} h_{\kappa_k}\partial J_j \tag{4.16}$$

and other generators of the algebra. Using such expressions and the knowledge of the $W_{i,0}$ eigenvalues of the highest weight states that can be determined from box counting in the affine Yangian language [98] or the BRST calculation of [1], one can in principle identify values of q^i for each degenerate module.

Such an identification is not unique and more representatives of the same module might exist. Even more representative exist if we look at free-field realizations in terms of the free-boson descendants. Let us briefly discuss the situation for the fundamental and the anti-fundamental representation of the algebra $Y_{N_1,N_2,0}$. For more details see section 5.4.

As discussed above, there exist $(N_1+N_2)!$ free field realizations of any $Y_{N_1,N_2,0}[\Psi]$ algebra associated to different orderings of the free bosons. It turns that for a given ordering, there exist at most $(N_1 + N_2)!$ free field realization of both the fundamental and the anti-fundamental representation, but in generally not all of them.

It might be puzzling that we find more than one free field realization of the same $Y_{N_1,N_2,0}$ module since it is not clear that all of these have the correct fusion and braiding properties and lead equivalent OPEs of degenerate modules. We expect the analysis of [87, 78] to lead to a resolution of the puzzle. To determine the braiding and the fusion, one needs to determine three-point functions of all the degenerate modules. Choosing a particular free-field realization of degenerate modules within a given three-point function leads generically to a zero value if we do not insert a correct number of screening charges. After such an insertion, we expect (and check in a very limited number of examples) that all the free field realizations lead to zero value or equal fusion and braiding properties. Note also that the free field realization gives an explicit construction of all the conformal blocks in terms contour integrals of meromorphic functions with possible branch-cuts.

Simple realization Before discussing the fusion and braiding and checking the independence on the choice of the free field representative, let us mention one simple realization of the fundamental and the anti-fundamental representation that exists for every free field realization.

Based on examples, we conjecture that one can realize the fundamental representation in the first direction as a descendant of the exponential $\exp[h_3\phi_i^{(2)}]$, where $\phi_i^{(2)}$ is the left-most free boson of the second type in a given ordering. The level of the descendant equals the number of free bosons of the first type on the left of such $\phi_i^{(2)}$. The anti-fundamental field is given by a descendant of $\exp[-h_3\phi_j^{(2)}]$, where $\phi_j^{(2)}$ is the right-most free boson of the second type and the level is given by the number of free bosons of the first type on the right of $\phi_j^{(2)}$. Similar simple realizations can be found also for representations in the second and third direction: a simple box in the second direction is associated to the left-most free boson of the first or third type and the level is given by the number of bosons of the second type on the left of it. For $N_3 = 0$ the box and antibox in the third direction correspond to the first and last boson and are always on level 0 (there are no obstructions since we have no bosons of the third type). The charge q appearing in the exponential is given by h_σ for box and $-h_\sigma$ for the anti-box and σ is such that the triple (σ, τ, π) in $h_\sigma, \phi^{(\tau)}$ and the direction π is a permutation of (123).

$Y_{0,0,2}$ example Let us start by an illustration how things work in the case of the Virasoro algebra in ordering $\mathcal{L}_1^{(3)}\mathcal{L}_2^{(3)}$. The two available screening currents are

$$\begin{aligned} j_1 &= \exp\left[-h_1\left(\phi_1^{(3)}-\phi_2^{(3)}\right)\right] \\ j_2 &= \exp\left[-h_2\left(\phi_1^{(3)}-\phi_2^{(3)}\right)\right] \end{aligned} \tag{4.17}$$

The two realizations of the identity, the fundamental representation and the anti-fundamental representation in the first and the second direction are

$$\begin{aligned} M_{\mathbb{1}}^1 &= \mathbb{1}, & M_{\mathbb{1}}^2 &= \exp\left[h_3\left(\phi_1^{(3)}-\phi_2^{(3)}\right)\right], \\ M_{\Box_1}^1 &= \exp\left[h_2\phi_1^{(3)}\right], & M_{\Box_1}^2 &= \exp\left[h_2\phi_2^{(3)}+h_3\left(\phi_1^{(3)}-\phi_2^{(3)}\right)\right], \\ M_{\bar{\Box}_1}^1 &= \exp\left[-h_2\phi_2^{(3)}\right], & M_{\bar{\Box}_1}^2 &= \exp\left[-h_2\phi_1^{(3)}+h_3\left(\phi_1^{(3)}-\phi_2^{(3)}\right)\right], \\ M_{\Box_2}^1 &= \exp\left[h_1\phi_1^{(3)}\right], & M_{\Box_2}^2 &= \exp\left[h_1\phi_2^{(3)}+h_3\left(\phi_1^{(3)}-\phi_2^{(3)}\right)\right], \\ M_{\bar{\Box}_2}^1 &= \exp\left[-h_1\phi_2^{(3)}\right], & M_{\bar{\Box}_2}^2 &= \exp\left[-h_1\phi_1^{(3)}+h_3\left(\phi_1^{(3)}-\phi_2^{(3)}\right)\right]. \end{aligned} \tag{4.18}$$

We see that there indeed exists the simple free field realization of the identity, the fundamental and the anti-fundamental representation.

Let us now check that two-point functions of different realizations of the identity and the two-point function of the fundamental with the anti-fundamental field are independent of the choice of the free field realization. To check all the three-point functions, one would have to relate normalizations of different realizations of all the degenerate modules and then compare all the three point funcions. Because we do not aim to do the comparison here, we disregard such normalizations and only check the braiding properties.

The charge of the identity realized by $M_{\mathbb{1}}^2$ cannot be subtracted by insertions of the screening charges and thus vanishes. The true identity $\mathbb{1}$ is the only realization of the vacuum module giving a non-zero one-point function.

The only combination that gives a non-vanishing two-point function of the fundamental and the anti-fundamental representation comes from the first realizations and give

$$
\begin{aligned}
\langle M^1_{\square_2}(z) M^1_{\bar{\square}_2}(w)\rangle_{Y_{0,0,2}} &\propto \oint_z d\tilde{z} \langle j_1(\tilde{z}) M^1_{\square_2}(z) M^1_{\bar{\square}_2}(w)\rangle \\
&\propto \oint_z d\tilde{z}\, (\tilde{z}-z)^{\frac{h_1}{h_2}} (\tilde{z}-w)^{\frac{h_1}{h_2}} \propto \oint_0 d\tilde{z}\, \tilde{z}^{\frac{h_1}{h_2}} (\tilde{z}+z-w)^{\frac{h_1}{h_2}} \\
&\propto (z-w)^{2\frac{h_1}{h_2}} \oint_0 d\tilde{z} \left(\frac{\tilde{z}}{w-z}\right)^{\frac{h_1}{h_2}} \left(1-\frac{\tilde{z}}{w-z}\right)^{\frac{h_1}{h_2}} \\
&\propto (z-w)^{2\frac{h_1}{h_2}+1},
\end{aligned}
\tag{4.19}
$$

where $\langle \dots \rangle_{Y_{N_1,N_2,N_3}}$ denotes the correlation function with possible insertions of the screening charges of Y_{N_1,N_2,N_3} that cancel the charge of the exponential factors. The exponent is exactly (up to the minus sign) the sum of conformal dimensions of the fundamental and the anti-fundamental representation which is the expected z-dependence of the two-point function.

$Y_{1,1,0}$ example The second example is the first non-trivial case that contains free field realizations of degenerate modules at higher levels and at the same time there is a mismatch between the number of free field realizations of the fundamental and the anti-fundamental representation. One gets the following realizations of the identity, the fundamental and the anti-fundamental field in the first and second direction for the ordering $\phi_1^{(1)} \times \phi_2^{(2)}$ of the free bosons

$$
\begin{aligned}
M^1_{\mathbb{1}} &= \mathbb{1}, & M^2_{\mathbb{1}} &= \exp\left[h_2\phi_1^{(1)} - h_1\phi_2^{(2)}\right] \\
M^1_{\square_1} &= \exp\left[h_2\phi_1^{(1)} + (h_3-h_1)\phi_2^{(2)}\right], & M^2_{\square_1} &= (h_2 J_1^{(1)} - h_1 J_2^{(2)}) \exp\left[h_3\phi_2^{(2)}\right], \\
M_{\bar{\square}_1} &= \exp\left[-h_3\phi_2^{(2)}\right], \\
M_{\square_2} &= \exp\left[h_3\phi_1^{(1)}\right], \\
M^1_{\bar{\square}_2} &= \exp\left[(h_2-h_3)\phi_1^{(1)} - h_1\phi_2^{(2)}\right], & M^2_{\bar{\square}_2} &= (h_2 J_1^{(1)} - h_1 J_2^{(2)}) \exp\left[-h_3\phi_1^{(1)}\right].
\end{aligned}
\tag{4.20}
$$

and the following screening current

$$
j = \exp\left[-h_2\phi_1^{(1)} + h_1\phi_2^{(2)}\right]. \tag{4.21}
$$

Note that there is only a single realization of the fundamental field and one of the realizations (the simple one) of the anti-box is at level one.

Let us first check that one-point function of the identity realized as $M^2_{\mathbb{1}}$ equals the vacuum amplitude

$$\left\langle M^2_{\mathbb{1}}(z)\right\rangle_{Y_{0,1,1}} \propto \oint_z d\tilde{z}\ \langle j(\tilde{z}) M^2_{\mathbb{1}}(z)\rangle \propto \oint_z d\tilde{z}\ (\tilde{z}-z)^{\frac{h_2}{h_3}+\frac{h_1}{h_3}} \propto 1. \tag{4.22}$$

Similarly for the two-point function with two contour integrations, one gets

$$\begin{aligned}\left\langle M^2_{\mathbb{1}}(z) M^2_{\mathbb{1}}(w)\ \right\rangle_{Y_{0,1,1}} &\propto \oint_z d\tilde{z}_2 \oint_w d\tilde{z}_1 \langle j(\tilde{z}_1) j(\tilde{z}_2) M^2_{\mathbb{1}}(z) M^2_{\mathbb{1}}(w)\rangle \\ &\propto \oint_z d\tilde{z}_2 \oint_w d\tilde{z}_1 \frac{(\tilde{z}_1-\tilde{z}_2)(z-w)}{(\tilde{z}_1-z)(\tilde{z}_1-w)(\tilde{z}_2-z)(\tilde{z}_2-w)} \\ &\propto \oint_z d\tilde{z}_2 \frac{(w-\tilde{z}_2)}{(\tilde{z}_2-z)(\tilde{z}_2-w)} \propto 1. \end{aligned} \tag{4.23}$$

Let us now show that the two-point function of both realizations of the anti-fundamental representation with the fundamental representations are also equal

$$\begin{aligned}\langle M^1_{\square_1}(z) M^1_{\bar{\square}_1}(w)\rangle_{Y_{0,1,1}} &\propto \oint_z d\tilde{z}\ \langle j(\tilde{z}) M^1_{\square_1}(z) M_{\bar{\square}_1}(w)\rangle \\ &\propto \oint_z d\tilde{z}\ (\tilde{z}-z)^{-2}(\tilde{z}-w)(z-w)^{\frac{h_1}{h_3}-1} = (z-w)^{\frac{h_3}{h_1}-1}. \end{aligned} \tag{4.24}$$

One gets the same expression from the other realization

$$\begin{aligned}\left\langle M^2_{\square_1}(z) M_{\bar{\square}_1}(w)\right\rangle_{Y_{0,1,1}} &\propto \left\langle J^{(2)}_2 \exp\left[h_3\phi^{(2)}_2\right](z) \exp\left[-h_3\phi^{(2)}_1\right](w)\right\rangle \\ &\propto (z-w)^{\frac{h_1}{h_3}-1}, \end{aligned} \tag{4.25}$$

where the -1 factor comes from the contraction with $J^{(2)}_2$.

4.2 Motivation for gluing

4.2.1 Type IIB perspective

It is natural to consider gauge theory configurations involving a more intricate junctions involving several semi-infinite interfaces converging to a single two-plane. It is also natural to consider intricate webs, involving finite interface segments as well as semi-infinite ones.

Web configurations would break scale invariance. In the IR, they would approach a single junction.

Conversely, one may consider webs with several simpler junction as a regularization of an intricate junction. If all junctions are dual to our basic Y-junctions, this may become a computational tool to determine the VOAs at generic junctions.

There is a precedent to this: complicated half-BPS interfaces in $\mathcal{N} = 4$ SYM can often be decomposed as a sequence of simpler interfaces, with a smooth limit sending to zero the relative distances between the interfaces. This is an important computational tool, as it allows one to apply S-duality transformations to well-understood individual pieces and then assemble them to the S-dual of the original, intricate interface.

A concrete example could be a Nahm pole associated to a generic $\mathfrak{su}(2)$ embedding ρ, realized as a sequence of individual simple Nahm pole interfaces. This is a smooth resolution, as long as the individual interfaces are ordered in a specific way [62]. The S-dual configuration is a sequence of bi-fundamental interfaces building up a complicated three-dimensional interface gauge theory with a good IR limit [63].

One may want to follow that example for junctions, say to decompose a Y-junction of complicated interfaces into a web of simpler Y-junctions. This idea raises a variety of hard questions, starting from figuring out criteria for a smooth IR limit of an interface web. Furthermore, the same configuration may be the limit of many different inequivalent webs an issue that will be discussed in the section 5.2.

In general, local operators at the final junction may arise either from local operators at each elementary junction in the web or from extended operators, such as a finite line defect segment joining two consecutive junctions. Thus we may hope that the final VOA will be an extension of the product of the VOA's at the vertices of the web, including products of degenerate modules associate to the finite line defect segment.

This picture is supported by the observation that although the dimensions of degenerate modules are not integral, the sum of the dimensions of the local operators at the two ends of a finite line defect segment will be integral. For example, a finite Wilson line W_μ on a finite segment of NS5 interface supports two local operators at the endpoints which have dimensions which differ by integral amounts from $\Delta_\mu[\Psi]$ and $\Delta_\mu[-\Psi] = -\Delta_\mu[\Psi]$ respectively, where $\Delta_\mu[\Psi]$ is the dimension of the μ vertex operator in the $\widehat{\mathfrak{gl}(N_3|N_1)}_\Psi$ Kac-Moody algebra. This is not quite a full definition of the final interface VOA, but it strongly restricts its form.[3]

[3]It may be possible to formalize this procedure as a sort of tensor product of VOAs over a common braided monoidal category.

A striking observation is the formal resemblance between this idea and the way the topological vertex is used to assemble the topological string partition function of general toric Calabi-Yau, by summing up over a choice of partition μ for each internal leg of the toric diagram [72] or its D4-D2-D0 brane-counting analogue [49, 52, 51].

The simplest possible situation for us is a web which can be interpreted as a collection of D5-branes ending on a NS5-brane: a sequence of $(q_i, 1)$ fivebrane segments with Y-junctions to semi-infinite D5-branes coming from the left or the right. Such a configuration can be lifted directly to a sequence of interfaces in 3d Chern-Simons theory. If the 3d interfaces have a good collision limit, one can derive directly the junction VOA. This situation also allows one to start probing questions about the extension structure of the final VOA and the equivalence between different web resolutions of the same interface that leads to the notion of the stable equivalence of VOAs discussed further in the section 5.2.

The simplest possibility we can discuss is that of an infinite D5 interface crossing an infinite NS5 interface. The four-way junction has two obvious resolutions, akin to the toric diagram of the conifold, involving either a $(1,1)$ or a $(1,-1)$ finite interface segment.

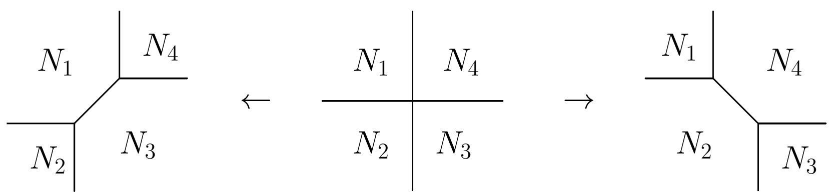


Figure 4.2: Two possible resolutions of the configuration of D5-brane (horizontal) crossing NS5-brane (vertical). First resolution includes a finite segment of $(1,1)$-brane whereas the second resolution includes a $(1,-1)$-brane segment. N_1, N_2, n_3, N_4 D3-branes are attached to fivebranes leading to webs of interfaces between $U(N_1), U(N_2), U(N_3), U(N_4)$ theories.

We can denote the choices of gauge group in the four quadrants as N_1, N_2, N_3, N_4, counterclockwise from the top left quadrant. For some values of N_1, N_2, N_3, N_4, the two resolutions produce obviously the same 3d interface in the scaling limit and then the same VOA. For example, if $N_1 = N_2$ and $N_3 = N_4$ then the CS theory interface results from the collision of interfaces which support some 2d matter coupled to the $U(N_3|N_1)$ CS gauge fields. The two resolutions give the same two interfaces in different order, and the collision/scaling limit is obviously the same: an interface which supports both 2d matter fields at the same location. For general values of $N_1 = N_2$ and $N_3 = N_4$ the VOAs seem to differ by contributions of free fields.

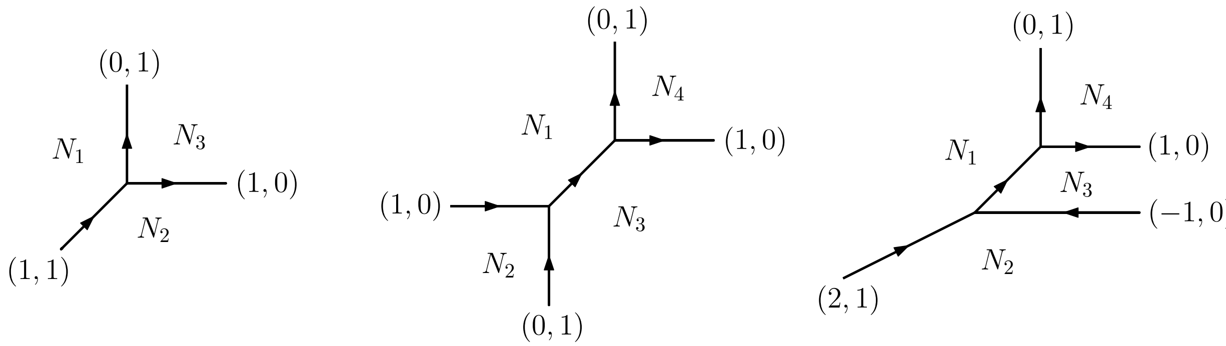


Figure 4.3: Toric diagram associated to $\mathbb{C}^3$ (left), $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{CP}^1$ (middle) and $\mathcal{O}(-2) \oplus \mathcal{O}\mathbb{CP}^1$ (right). The lines show loci where (p, q)-cycles of the torus T^2 degenerate.

4.2.2 Type IIA perspective

The example of the simple trivalent junction associated to $\mathbb{C}^3$ has a natural generalization for an arbitrary toric Calabi-Yau 3-fold given by a toric diagram specifying loci where the torus cycles degenerate. The two simplest examples are shown in the figure 4.3 and correspond to the bundles $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{CP}^1$ and $\mathcal{O}(-2) \oplus \mathcal{O}\mathbb{CP}^1$ respectively. The extensions described above have now different interpretation. Note that both examples above contain a compact two-cycle $\mathbb{CP}^1$ that is fixed under the toric action. One can then treat D2-branes wrapping compact two-cycles as dynamical objects. Moreover, there are two points (the south and the north pole of $\mathbb{CP}^1$) that are fixed under the toric action. Assuming that the vertex operator algebra have an origin from the algebra of BPS states associated to such brane configurations, one can expect that a general bound state of D2-branes dressed by D0-branes located at the two fixed points are going to play a role. This resembles the type IIB configuration with D2-branes (associated to finite internal lines of the web diagram) corresponding to the highest weight vectors of the added bi-modules and dressing by D0-branes corresponding to a generation of the descendants. This picture is consistent at the level of characters, where one can argue that the gluing prescription for VOAs agrees with the one of [49, 52, 51].

Moreover, one can also expect that both the free-field realization construction and the truncation picture generalizes to the glued algebras. In particular the BPS algebra (cohomological Hall algebra) associated to the given toric three-fold is independent of the choice of the divisor. Based on the analysis of the vacuum character of the glued algebras, we can conjecture that there exists a Drinfeld double of the BPS algebra at least in the examples 4.3 and they can be identified with shifted affine Yangians of $\mathfrak{gl}(1|1)$ and $\mathfrak{gl}(2)$

respectively. Shifts are determined by the intersection number of the corresponding divisor D with the $\mathbb{CP}^1$ associated to the internal line. In particular for the last two examples from figure 4.3, we expect shift

$$\#(D \cap \mathbb{CP}^1) = N_2 + N_4 - N_1 - N_3, \qquad \#(D \cap \mathbb{CP}^1) = N_2 + N_4 - 2N_3, \tag{4.26}$$

where the divisors are

$$\begin{aligned} &N_2(\text{Fiber over the north pole of } \mathbb{CP}^1) + N_4(\text{Fiber over the south pole of } \mathbb{CP}^1) \\ &\qquad + N_1(\mathcal{O}(-1)_1 \to \mathbb{CP}^1) + N_3(\mathcal{O}(-1)_2 \to \mathbb{CP}^1) \end{aligned} \tag{4.27}$$

in the case $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ (middle) and analogously for the case $\mathcal{O}(-2) \oplus \mathcal{O}$ (right)

$$\begin{aligned} &N_2(\text{Fiber over the north pole of } \mathbb{CP}^1) + N_4(\text{Fiber over the south pole of } \mathbb{CP}^1) \\ &\qquad + N_1(\mathcal{O}(-2) \to \mathbb{CP}^1) + N_3(\mathcal{O} \to \mathbb{CP}^1). \end{aligned} \tag{4.28}$$

Similarly, one can consider (p, q)-webs of n D5-branes ending from the left and m D5-branes ending from the right on a sequence of $(n, 1)$-branes for varying integer n such that the (p, q) charges are conserved at each vertex. These configurations are expected to lead to shifted Yangians of $\widehat{\mathfrak{gl}(n|m)}$ with shifts determined by the intersection numbers of D with various $\mathbb{CP}^1$'s associated to internal lines. The Yangians associated to more complicated web-diagrams are highly unexplored.

Both Yangians enjoy a co-product structure and the brane-separation argument holds also in this case. This suggests that the free-field realization should also admit a generalization in this case. On the other hand, many technical difficulties appear. For example, we do not expect that Heisenberg algebras themselves are sufficient in this case. As we will see, particular lattice extensions by exponential vertex operators seem to be required. Also, it is not clear what is the role of screening generators and different realization of the bi-fundamental fields. Further investigation of these issues is needed.

4.3 BRST construction

In some cases, one can give a BRST definition of the glued algebra. Gluing construction can then shed new light on the structure of the algebras obtained by such reductions. Configurations we discuss in this section are associated to diagrams with D5-branes ending from both left and right on a linear chain of $(n, 1)$ branes. The most general configuration that we will be able to give a BRST definition is such that the diagram can be cut into two halves that satisfy the following condition: The number of D3-branes is non-increasing if we follow the upper part of the diagram from the top to the bottom and the number of D3-branes is non-increasing if we follow the lower part of the diagram from the bottom up.

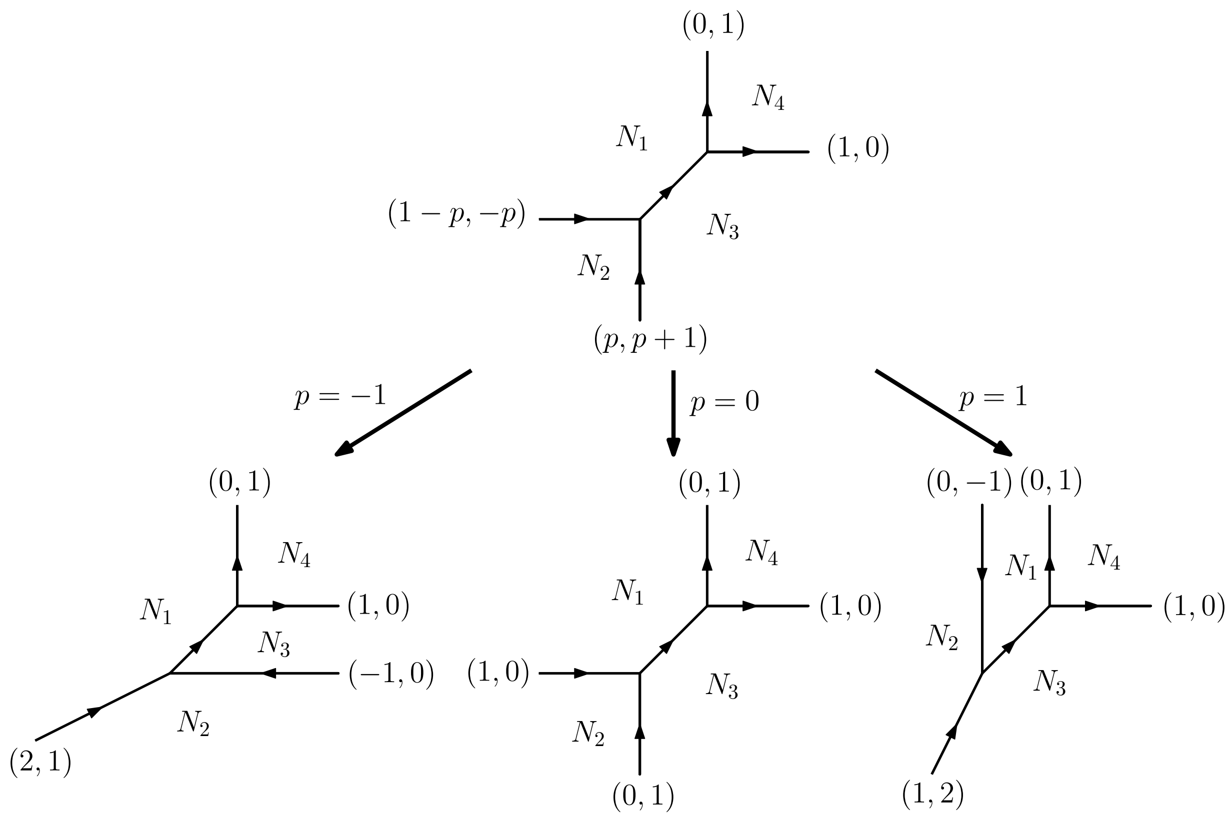
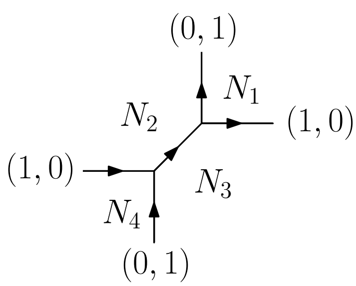


Figure 4.4: Configurations containing a simple finite leg segment. The $p = \pm 1$ cases are related by S-duality and it acts within the family of $p = 0$ algebras.

Let us motivate the BRST construction. We expect that a proper justification along these lines can be done analogously to [104, 43, 46, 47]. In the Kapustin-Witten twisted theory, the path integral in this configuration localizes to the path integral of the complexified CS theory supported at $(n, 1)$ branes connected by a nontrivial interface descending from D3-branes ending on D5-branes. In the IR, the finite internal fivebrane segments shrink and we can view the configuration as a single interface between the upper and the lower CS theory. The half-BPS boundary conditions in the $\mathcal{N} = 4$ super Yang-Mills theory descending from D3-branes ending on fivebranes were analyzed in [62]. These boundary conditions can be translated to the boundary conditions of the bosonic blocks of the complexified super Chern-Simons theory. The boundary condition on the off-diagonal blocks (descending from boundary conditions on the 3d bifundamental hypermultiplets supported at the $(n, 1)$ interface) requires some guesswork and will be discussed later. We conjecture the corresponding VOA to be a BRST reduction of the super Kac-Moody algebra induced at the interface from the upper and lower CS theories by a BRST charge implementing the boundary conditions.

One can see that requiring a single internal edge (or equivalently two vertices), only the configurations from the figure 4.4 appear. Note that the S-duality action maps the families $p = -1 \leftrightarrow p = 1$. We give a BRST definition for the $p = -1$ case. The $p = 1$ algebras can be identified with those by S-duality. On the other hand, $p = 0$ example is self-dual under the S-duality action and we expect the corresponding algebras to have dual BRST descriptions in general. This section gives a BRST definition of almost all algebras associated to diagrams with these configurations. The only exceptions are the configurations for $p = 0$ with D3-branes satisfying the following four conditions $N_1 > N_2, N_3 > N_4, N_3 > N_2, N_1 > N_4$.

4.3.1 Algebras of type $1|1$ (resolved conifold diagram)



In this section we want to discuss the junction of two Y-algebras that corresponds to the resolved conifold diagram as in the figure on the left. We first introduce a convenient notation $\mathcal{W}^{1|1}_{N_1,\bar{N}_2,N_3,\bar{N}_4}[\Psi]$ for these algebras. The label $\mathcal{W}^{1|1}$ refers to algebras associated to a linear chain of $(n, 1)$ five-branes to which one D5-brane is attached from the left and the other one is attached from the right. Furthermore, we overline the numbers $\bar{N}_2$ and $\bar{N}_4$ of D3-branes ending on the $(n, 1)$-branes from the left and we leave N_1 and N_2 for the D3-branes ending from the right. This labeling will be used also for more complicated diagrams with a linear chain.

From the gluing point of view the algebra is a conformal extension of

$$\mathcal{W}^{1|1}_{N_1,\bar{N}_2,N_3,\bar{N}_4}[\Psi] \supset Y_{N_2,N_3,N_1}[\Psi] \times Y_{N_3,N_2,N_4}[\Psi] \tag{4.29}$$

by bimodules labeled by representations of $\mathfrak{gl}(N_3|N_2)$. It turns out that the conformal weight of the fundamental and the anti-fundamental bi-module equals a half-integral number given by

$$h(\square) = 1 + \rho = 1 + \frac{N_1 + N_4 - N_2 - N_3}{2} \tag{4.30}$$

that is the level at which the first field needs to be added. Moreover, at the level of characters, the BRST construction result must agree with the gluing proposal

$$\chi[\mathcal{W}^{1|1}_{N_1,\bar{N}_2,N_3,\bar{N}_4}] = \sum_\mu \chi[Y_{N_2,N_3,N_1}[\Psi]](M^3_\mu)\ \chi[Y_{N_3,N_2,N_4}[\Psi]](M^3_\mu). \tag{4.31}$$

BRST construction In the Kapustin-Witten twisted theory, the path integral of the configuration again localizes to the path integral of the complexified $U(N_1|N_2)_\Psi$ and $U(N_3|N_4)_\Psi$ Chern-Simons theories connected by a nontrivial boundary condition that is a combination of oper boundary condition and continuity condition. The BRST definition of the VOAs is then a reduction that implements the boundary condition on the two $\widehat{\mathfrak{gl}(N_1|N_2)}_\Psi$ and $\widehat{\mathfrak{gl}(N_3|N_4)}_{-\Psi}$ Kac-Moody factors coming from the restriction of the gauge fields of the upper and lower CS theory to the interface.

Implementing the constraints coming from the boundary conditions for $N_1 > N_3$ and $N_2 > N_3$ by a BRST reduction, one expects the final VOA to be a combination of the Drinfeld-Sokolov reduction of $\widehat{\mathfrak{gl}(N_1|N_2)}_\Psi$ with respect to the principal $\mathfrak{sl}(2)$ embedding inside the $(N_1-N_3)\times(N_1-N_3)$ block in the $\mathfrak{gl}(N_1)$ bosonic part of $\mathfrak{gl}(N_1|N_2)$, the Drinfeld-Sokolov reduction with respect to the principal embedding in the $(N_2-N_4)\times(N_2-N_4)$ block in the other $\mathfrak{gl}(N_2)$ bosonic part and the coset with respect to the remaining $\widehat{\mathfrak{gl}(M|N)}_\psi$ Kac-Moody algebra. In analogy with the construction of [1], one writes for such a combined BRST reduction

$$\mathcal{W}^{1|1}_{N_1,\bar{N}_2,N_3,\bar{N}_4}[\Psi] = \frac{\overline{\mathcal{DS}}_{N_2-N_4}\left[\mathcal{DS}_{N_1-N_3}[\widehat{\mathfrak{gl}(N_1|N_2)}_\Psi]\right]}{\widehat{\mathfrak{gl}(N_3|N_4)}_\Psi}. \tag{4.32}$$

In expressions of this form, we need to be careful what we mean by a sequence of Drinfeld-Sokolov reductions. There are two natural definitions. The first natural choice would be to pick a grading associated to the sum of the Cartan elements of the two sl_2 embeddings and constrain the fields with positive weight with respect to this combined element as in the case of the standard DS-reduction. We can see that this choice would be symmetric with respect to both trivalent junctions of the diagram. This would not match the predictions from the gluing suggesting that this is not the right thing to do. We expect the symmetric variant to be related to the unresolved configuration. In particular, note that the $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry of the resolved conifold diagram get enhanced to S_4 that is consistent with the S_4 symmetry of the central charge derived from the BRST construction. We leave details for the future.

The other possibility is to slightly modify the standard construction by doing the reduction in two steps. First, we need to constrain the components with positive weight with respect to the first embedding associated to the $\mathfrak{gl}(N_1 - N_3)$ block (DS-reduction associated to the upper vertex as in the case of Y-algberas). Classically (and at the level of characters), this first constraint decomposes $\mathfrak{gl}\widehat{(N_1|N_2)}_\Psi$ fields as

$$\mathcal{DS}_{N_1-N_3} : \mathfrak{gl}\widehat{(N_1|N_2)}_\Psi \;\to\; \mathcal{W}_{N_1-N_3} \times \mathfrak{gl}\widehat{(N_3|N_2)}_{\Psi-1} \times \mathcal{S}^{N_3|N_1}_{\frac{N_1-N_3}{2}} \tag{4.33}$$

where $\mathcal{W}_{N_1-N_3}$ denotes the fields with the spin content of the $\mathcal{W}_{N_1-N_3} \times \widehat{\mathfrak{gl}(1)}$ algebra and $\mathcal{S}^{N_3|N_2}_{\frac{N_1-N_3}{2}}$ a system of N_3 symplectic bosons and N_2 fermions with the conformal dimension $\frac{N_1-N_3+1}{2}$. The first reduction produces an algebra containing the $\mathfrak{gl}\widehat{(N_3|N_2)}_{\Psi-1}$ Kac-Moody algebra as a subalgebra coming from the the $\mathfrak{gl}\widehat{(K|L)}_\Psi$ currents modified by off-diagonal ghosts. In the second step, one needs to constrain the fields of the Kac-Moody algebra $\mathfrak{gl}\widehat{(N_3|N_2)}_{\Psi-1}$ with shifted level by setting to zero fields with positive weight[4] with respect to the Cartan element of the sl_2 embedding associated to the second vertex. The algebra decomposes classically (at the level of characters) as

$$\overline{\mathcal{DS}}_{N_2-N_4} : \mathfrak{gl}\widehat{(N_3|N_2)}_{\Psi-1} \;\to\; \mathcal{W}_{N_2-N_4} \times \mathfrak{gl}\widehat{(N_3|N_4)}_\Psi \times \overline{\mathcal{S}}^{N_3|N_4}_{\frac{n_2-N_4}{2}} \tag{4.34}$$

where $\overline{\mathcal{S}}^{N_3|N_4}_{\frac{N_2-N_3}{2}}$ now contains N_3 fermionic and N_4 bosonic generators that refers to the fact that corresponding D5-brane ends from the opposite direction. The $\mathcal{S}^{N_3|N_1}_{\frac{N_1-N_3}{2}}$ fields from the first step are left unconstrained but the modification term that needs to be added to the stress-energy tensor in the second step splits them into fields

$$\overline{\mathcal{DS}}_{N_2-N_4} : \mathcal{S}^{N_3|N_1}_{\frac{N_1-N_3}{2}} \;\to\; \mathcal{S}^{N_3|N_4}_{\frac{N_1-N_3}{2}} \times \prod_{i=\rho+\frac{1}{2}}^{\rho+N_2-N_4-\frac{1}{2}} \mathcal{F}_i \tag{4.35}$$

where $N_2 - N_4$ components were split into fermionic fields $\mathcal{F}_i$ with dimensions

$$\rho+1, \rho+2, \ldots, \rho+L-N. \tag{4.36}$$

[4]Remember that only half of the fields with weight $\frac{1}{2}$ need to be constrained as in the case of Y-algebras.

In the case when DS-reduction is with respect to a one dimensional block (i.e. $N_1 - N_3 = 1$ or $N_2 - N_4 = 1$), no constraints need to be imposed remembering that fields from the off-diagonal block of the first reduction are not constrained in the second step. Similarly if $N_1 - N_3 = 0$ or $N_2 - N_4 = 0$ vanishes, instead of constraining the fields, one needs to introduce extra $\mathcal{S}^{N_3|N_2}$ or $\overline{\mathcal{S}}^{N_3|N_4}$ fields into the system, similarly as in the case of the trivalent vertex and use currents modified by bilinears in these extra fields in the BRST reductions of the following steps.

After the two DS-reductions, the algebra still contains a $\mathfrak{gl}\widehat{(N_3|N_4)}_\Psi$ factor as a subalgebra and one should take a coset with respect to this factor. By coset, we mean an equivariant BRST reduction with respect to the BRST charge that glues this $\mathfrak{gl}\widehat{(N_3|N_4)}_\Psi$ subalgebra with the $\mathfrak{gl}\widehat{(N_3|N_4)}_{-\Psi}$ algebra induced from the bottom Chern-Simons theory. Note that the shifted levels of the two factors are opposite which is a necessary condition for the BRST charge to be nilpotent.

An analogous definition of the algebra can be given in the case $N_3 > N_1$ and $N_4 > N_2$ with $N_1 \leftrightarrow N_3$, $N_2 \leftrightarrow N_4$ and the two DS-reductions interchanged (this correspond to rotation of the diagram by 180°).

We can also define a similar reduction for the case when the number of D3-branes decreases from the top and from the bottom until the two series of D3-brane numbers meet. In the case of the resolved conifold diagram, this corresponds to $N_1 > N_3$ and $N_4 > N_2$. One can then read the boundary conditions from both sides, show that the resulting algebras contain two Kac-Moody algebras of opposite level and then equivariantly glue these factors. The resulting algebra is the BRST reduction of the system

$$\mathcal{DS}_{N_1-N_3}[\mathfrak{gl}\widehat{(N_1|N_2)}] \times \overline{\mathcal{DS}}_{N_4-N_2}[\mathfrak{gl}\widehat{(N_3|N_4)}_{-\Psi}] \times gh \tag{4.37}$$

that glues the two $\mathfrak{gl}\widehat{(N_3|N_2)}_{\Psi-1} \times \mathfrak{gl}\widehat{(N_3|N_2)}_{-\Psi+1}$ subalgebras. As usual, gh in the expression above denotes the ghosts needed to implement the gluing. Note that combining the fields in the fundamental representation of the remaining $\mathfrak{gl}(N_3|N_2)$ factors coming from the two DS-reduction into the $\mathfrak{gl}(N_3|N_2)$ invariant combinations gives rise to fields of dimensions starting with $\rho + 1$. This is consistent with the gluing picture and BRST reduction above although the origin of the fields is slightly different.

Central charge and characters Having defined the algebras by a BRST reduction, one can calculate the central charge straightforwardly. The result is[5]

$$\begin{aligned} c\left[\mathcal{W}^{1|1}_{N_1,\bar{N}_2,N_3,\bar{N}_4}[\Psi]\right] &= \Psi\big((L-N_4)((L-N_4)^2-1)-(N_1-M)((N_1-M)^2-1)\big) \\ &+\frac{1}{\Psi}\big((N_2-N_1)((N_2-N_1)^2-1)-(N_4-N_3)((N_4-N_3)^2-1)\big) \\ &+(N_2-N_4+N_3-N_1)(N_2^2+N_2N_4-4N_2N_3+N_2N_1-2N_4^2 \\ &+N_4N_3+2N_4N_1+N_3^2+N_3N_1-2N_1^2+1). \end{aligned} \tag{4.38}$$

The details of the computation are given in [2].

Having central charge of a general $1|1$ algebra, we can now test the predictions of the gluing construction. We conjectured that the central charge of the glued algebra is simply a sum of the central charges associated to its vertices and indeed

$$c\left[\mathcal{W}^{1|1}_{N_1,\bar{N}_2,N_3,\bar{N}_4}[\Psi]\right] = c\left[Y_{N_2,N_3,N_1}[\Psi]\right] + c\left[Y_{N_3,N_2,N_4}[\Psi]\right] \tag{4.39}$$

so the extension is conformal. Moreover, one can check that the central charge is invariant under the $\mathbb{Z}_2\times\mathbb{Z}_2$ duality action.

Characters The vacuum character of the resulting algebra can be also computed straightforwardly following the description outlined above. One finds a general expression

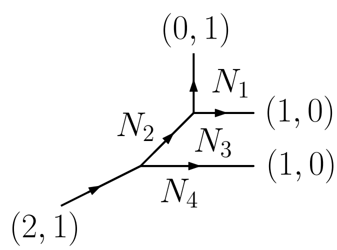
$$\chi\left[\mathcal{W}^{1|1}_{N_1,\bar{N}_2,N_3,\bar{N}_4}\right] = \chi_{\mathcal{W}_{N_1-N_3}}\chi_{\mathcal{W}_{N_2-N_4}} \prod_{r=\rho+\frac{1}{2}}^{\rho+N_2-N_4-\frac{1}{2}} \chi^{\mathcal{F}}_r \oint dV_{N_3,N_4}\chi^{N_4|N_3}_{\frac{N_1-N_3}{2}}(x_j,y_i)\chi^{N_1|N_3}_{\frac{N_2-N_4}{2}}(y_i,x_j) \tag{4.40}$$

Note that the variables x_i and y_j in the two symplectic boson factors interchange. We can identify the first two factors with the vacuum characters of $\mathcal{W}_{N_1-N_3}$ and $\mathcal{W}_{N_2-N_4}$ algebras coming from the diagonal blocks of DS-reduction, the factors $\chi^{\mathcal{F}}_i$ coming from the N_2-N_4 fermionic fields with a shifted level and the integral projecting to the $U(M|N)$ invariant combinations of the fundamental fields. Explicit expressions for these building blocks can be found in appendix A.2.

To write the characters of more general modules associated to Wilson lines supported at the two NS5-like interfaces one only needs to insert the corresponding Schur polynomials into the formula above in the same way as in the case of Y-algebras.

[5]Note that the pole at $\Psi=1$ in the formula disappeared and the poles at 0 and ∞ are multiplied with two factors associated to the two external legs with given asymptotics as expected from the orientation of the infinite fivebrane segments.

4.3.2 Algebras of type 0|2



In this section, we consider an analogous diagram as the one of the resolved conifold but now with both D5-branes ending from the right as shown in the figure. The discussion will be similar to the one of the previous section but let us highlight few differences.

The glued algebra is a conformal extension of two mutually commuting Y-algebras

$$\mathcal{W}^{0|2}_{N_1,\bar{N}_2,N_3,N_4}[\Psi] \supset Y_{N_2,N_3,N_1}[\Psi] \times Y_{N_2,N_4,N_3}[\Psi-1] \tag{4.41}$$

with gluing matter given by bimodules $M^3_\mu \times M^2_\mu$. Specializing the parameters from the section 4.4.2 to the case at hand, one finds $p=1$ and

$$h(\Box) = 1+\rho = 1+\frac{N_1+N_4-2N_3}{2} \tag{4.42}$$

(note that this is independent of N_2). In terms of the characters, we expect the BRST construction to produce

$$\chi\left[\mathcal{W}^{0|2}_{N_1,\bar{N}_2,N_3,N_4}\right] = \sum_\mu \chi[Y_{N_2,N_3,N_1}[\Psi]]\left(M^3_\mu\right)\chi[Y_{N_2,N_4,N_3}[\Psi-1]]\left(M^2_\mu\right) \tag{4.43}$$

and the central charge to be the sum of the central charges of the two Y-algebras.

BRST construction Looking at the system from the IR, the configuration looks like a junction of interfaces between $U(N_1)$, $U(N_2)$ and $U(N_4)$ gauge theories. After a topological twist, the path integral localizes to the path integral of the complexified Chern-Simons theories induced at the NS5 and $(2,1)$ interface glued together by a boundary condition descending from the boundary condition coming from D3-branes ending on fivebranes. This boundary condition can be extracted from the boundary conditions discussed in [62] in the case when $N_1 \geq N_3 \geq N_4$ or $N_4 \geq N_3 \geq N_1$ and is a combination of two oper boundary conditions and a continuity condition.

Let us first discuss the $N_1 \geq N_3 \geq N_4$ case. Imposing the boundary conditions as constraints on the Kac-Moody algebras descending from the upper and the lower CS theories using the BRST procedure leads to the following identification of the VOA

$$\mathcal{W}^{0|2}_{N_1,\bar{N}_2,N_3,N_4}[\Psi] = \frac{\mathcal{DS}_{N_3-N_4}\left[\mathcal{DS}_{N_1-N_3}[\widehat{\mathfrak{gl}(N_1|N_2)}_\Psi]\right]}{\widehat{\mathfrak{gl}(N_4|N_2)}_{\Psi-2}}. \tag{4.44}$$

Note that both DS-reduction are performed in the same block of the bosonic generators of $\widehat{\mathfrak{gl}(N_1|N_2)}_\Psi$. Analogously to the resolved conifold algebra, we perform the reduction in three steps. After the first reduction associated to the upper vertex, one obtains an algebra containing the $\widehat{\mathfrak{gl}(M|L)}_{\Psi-1}$ subalgebra. In the second step one uses the BRST charge implementing the DS reduction associated to the second vertex with the currents of the $\widehat{\mathfrak{gl}(M|L)}_{\Psi-1}$ algebra with the level shifted by one. Since the second reduction is performed in the same bosonic block of the algebra, the resulting algebra contains subalgebra $U(N|L;\Psi-2)$ with the level shifted by two. In the final step one glues equivariantly the components of the $\widehat{\mathfrak{gl}(N_4|N_2)}_{\Psi-2}$ subalgebra with the extra $\widehat{\mathfrak{gl}(N_4|N_2)}_{-\Psi+2}$ Kac-Moody algebra coming from the lower CS theory.

Under the two DS-reductions, the fields decompose in a similar way as in the case of the 1|1 algebra. The only difference is that the the $\mathcal{S}^{N_3|N_2}_{\frac{N_1-n_3}{2}}$ factor from the first reduction decomposes under the second reduction as

$$\mathcal{DS}_{N_3-N_4} : \mathcal{S}^{N_3|N_2}_{\frac{N_1-n_3}{2}} \;\rightarrow\; \mathcal{S}^{N_4|N_2}_{\frac{N_3-N_4}{2}} \times \prod_{i=\rho+\frac{1}{2}}^{\rho+N_3-N_4-\frac{1}{2}} \mathcal{B}_i \tag{4.45}$$

producing N_3-N_4 bosonic fields of the shifted dimension

$$\rho+1, \rho+2, \ldots, \rho+N_3-N_4. \tag{4.46}$$

Note again the appearance of the parameter ρ from (4.42). The fields with shifted dimensions (coming from the off-diagonal blocks containing fields charged under the Cartans of both sl_2 embeddings) are now bosonic. The same is true also for the $\mathfrak{gl}(N|L)$ invariant combinations of the symplectic bosons and fermions coming from the two BRST reductions. All the fields of the resulting algebra are bosonic in this case as expected.

An analogous definition can be given in the case of $N_4 \geq N_3 \geq N_1$ with the factors $N_1 \leftrightarrow N_4$ and $\Psi \rightarrow -\Psi+2$ exchanged (since this configuration can be obtained from the previous one by an $SL(2,\mathbb{Z})$ transformation).

One can also define BRST reduction in the case when $N_1 > N_3$ and $N_4 > N_3$ by performing the DS-reduction for the upper and to lower vertices independently,

$$\mathcal{DS}_{N_1-N_3}[\widehat{\mathfrak{gl}(N_1|N_2)}_\Psi] \times \mathcal{DS}_{N_4-N_3}[\widehat{\mathfrak{gl}(N_4|N_2)}_{-\Psi+2}] \tag{4.47}$$

and then gluing the $\widehat{\mathfrak{gl}(M|L)}_{\Psi-1}$ subalgebra of the first vertex with the $\widehat{\mathfrak{gl}(M|L)}_{-\Psi+1}$ of the second vertex using BRST (as in the resolved conifold case).

Central charge and characters The central charge is given by (more details of the computation are given in [2])[6]

$$
\begin{aligned}
c \;=\; & \frac{(N_4-N_2)((N_4-N_2)^2-1)}{\Psi-2}-\frac{(N_2-N_1)((N_2-N_1)^2-1)}{\Psi} \\
& +(((N_1-N_3)^2-1)(N_1-N_3)+((N_3-N_4)^2-1)(N_3-N_4))\Psi-N_3-N_1 \\
& +(N_4-N_3)^2(-3N_2+N_4+2N_3)+(N_3-N_1)^2(-3N_2+N_3+2N_1)+2N_2 \\
& -\left((N_4-N_3)^2-1\right)+(N_4-N_3)-N_3-N_1(N_4-N_3)^2(-3N_2+N_4+2N_3) \\
& +(N_3-N_1)^2(-3N_2+N_3+2N_1)+2N_2-\left((N_4-N_3)^2-1\right)(N_4-N_3).
\end{aligned}
\tag{4.48}
$$

It can again be shown to be equal to the sum of the two central charges of the two elementary vertices.

Characters The vacuum character is given by an integral formula

$$
\chi=\chi_{\mathcal{W}_{N_1-N_3}}\chi_{\mathcal{W}_{N_3-N_4}}\prod_{r=\rho+\frac{1}{2}}^{\rho+N_3-N_2-\frac{1}{2}}\chi_r^{\mathcal{B}}\oint dV_{N,M}\chi^{N_4|N_2}_{\frac{N_1-N_3}{2}}(x_i,y_j)\chi^{N_4|N_2}_{\frac{N_3-N_4}{2}}(x_i,y_j). \tag{4.49}
$$

The characters of the two modules associated to the line defects supported at the NS5 and the $(2,1)$ interface can be computed in a similar way with an extra insertion of the corresponding Schur polynomials.

4.3.3 Algebras of type $M|N$

In this section, we briefly discuss a generalization of the BRST reductions in the case of diagrams with D5-branes ending on $(n,1)$-branes from both left and right. We describe the BRST reduction of a general configuration with monotonic number of D3-branes. Example of such a configuration is given in the figure 4.5.

BRST construction Let $N_1 \geq N_2 \geq \cdots \geq N_n$ be the sequence of D3-branes on the left of the $(n,1)$-branes and $M_1 \geq M_2 \geq \cdots \geq M_n$ be the sequence of D3-branes attached from the right. There is a natural generalization of the construction from the previous two sections where $1|1$ and $0|2$ algebras were constructed using a sequence of DS-reductions

[6] The structure of poles in Ψ can be again read off from the diagram. Note that the pole at $\Psi = 2$ associated to the $(1,2)$ infinite five-brane appeared.

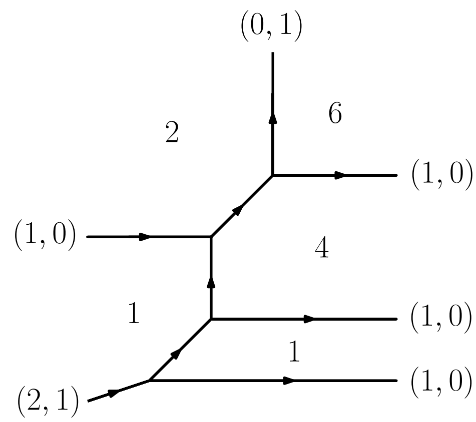


Figure 4.5: Example of a configuration of branes with a BRST definition.

and one coset construction. To find the expression for the BRST reduction, we follow the diagram from the top to the bottom. We start with the Kac-Moody algebra $\mathfrak{gl}(\widehat{M_1, N_1})_\Psi$. Each time a D5-brane ends from the right, the Drinfeld-Sokolov reduction $\mathcal{DS}_{M_i - M_{i+1}}$ needs to be performed (where i labels the D5-branes ending from the right). Similarly, each time a D5-brane ends on the chain of $(n, 1)$-branes from the left, the Drinfeld-Sokolov reduction $\overline{\mathcal{DS}}_{N_j - N_{j+1}}$ needs to be performed (here i labels the D5-branes ending on the left). Finally, one needs to take a coset with respect to the remaining $\mathfrak{gl}(\widehat{M_m|N_n})_{\Psi+M-N}$ super Kac-Moody algebra. For example, the diagram from 4.5 leads to the following algebra

$$\mathcal{W}_{6\bar{2}4\bar{1}10}[\Psi] = \frac{\mathcal{DS}_1[\mathcal{DS}_3[\overline{\mathcal{DS}}_1[\mathcal{DS}_2[\widehat{\mathfrak{gl}(6|2)}_\Psi]]]]}{\widehat{\mathfrak{gl}(1)}_{\Psi-2}} \tag{4.50}$$

where the $\mathcal{DS}_N$ and $\overline{\mathcal{DS}}_N$ are defined as in the case of $1|1$ and $0|2$ diagrams. Note that after each DS-reduction associated to the D5-brane ending from the right, the final algebra contains a Kac-Moody algebra with level shifted by minus one and after each DS-reduction associated to the D5-brane ending from the left, the final algebra contains a Kac-Moody algebra with level shifted by one. The final level one gets after performing all the DS-reductions is opposite to the level of the Kac-Moody algebra induced from the bottom CS theory.

Let us now summarize how the fields decompose under $\mathcal{DS}_{N-M}$ and $\overline{\mathcal{DS}}_{K-L}$ reductions at each step. The $\widehat{\mathfrak{gl}(N|K)}_\Psi$ Kac-Moody algebra factor decomposes as

$$\begin{aligned} \mathcal{DS}_{N-M} : \widehat{\mathfrak{gl}(N|K)}_\Psi &\to \mathcal{W}_{N-M} \times \widehat{\mathfrak{gl}(M|K)}_{\Psi-1} \times \mathcal{S}^{M|K}_{\frac{N-M}{2}} \\ \overline{\mathcal{DS}}_{K-L} : \widehat{\mathfrak{gl}(N|K)}_\Psi &\to \mathcal{W}_{K-L} \times \widehat{\mathfrak{gl}(N|L)}_{\Psi+1} \times \mathcal{S}^{N|L}_{\frac{K-L}{2}}. \end{aligned} \tag{4.51}$$

On the other hand, the fields $\mathcal{S}^{N|K}_k$ and $\overline{\mathcal{S}}^{N|K}_k$ from the previous steps decompose as

$$\begin{aligned} \mathcal{DS}_{N-M} : \mathcal{S}^{N|K}_k &\to \mathcal{S}^{M|K}_k \times \mathcal{B}_{k-\frac{N-M-1}{2}} \times \cdots \times \mathcal{B}_{k+\frac{N-M-1}{2}} \\ \mathcal{DS}_{N-M} : \overline{\mathcal{S}}^{N|K}_k &\to \overline{\mathcal{S}}^{M|K}_k \times \mathcal{F}_{k-\frac{N-M-1}{2}} \times \cdots \times \mathcal{F}_{k+\frac{N-M-1}{2}} \\ \overline{\mathcal{DS}}_{K-L} : \mathcal{S}^{N|K}_k &\to \mathcal{S}^{N|L}_k \times \mathcal{F}_{k-\frac{K-L-1}{2}} \times \cdots \times \mathcal{F}_{k+\frac{K-L-1}{2}} \\ \overline{\mathcal{DS}}_{K-L} : \overline{\mathcal{S}}^{N|K}_k &\to \overline{\mathcal{S}}^{N|L}_k \times \mathcal{B}_{k-\frac{K-L-1}{2}} \times \cdots \times \mathcal{B}_{k+\frac{K-L-1}{2}}. \end{aligned} \tag{4.52}$$

The decomposition is shown explicitly for the example above in the figure 4.6.

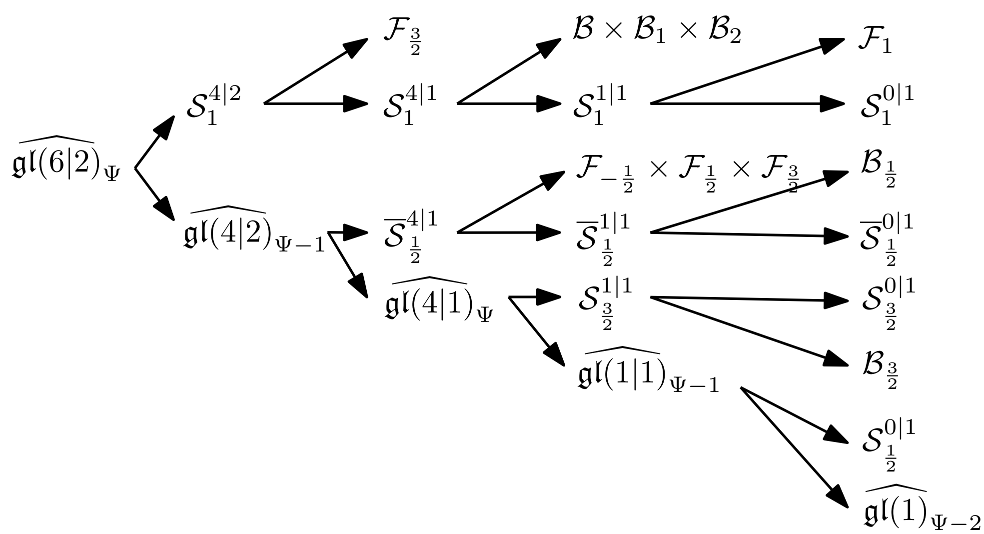


Figure 4.6: The structure of DS-reductions from the example 4.50.

A similar BRST reduction can be defined in the case when the diagram can be cut into two halves where in the upper half, the number of D3-branes decreases from the top to the bottom and in the lower half it decreases from the bottom to the top. The BRST definition is then given by performing a sequence of DS-reductions on both the upper and the bottom Kac-Moody algebra and then gluing two remaining Kac-Moody subalgebras with opposite level by BRST construction.

It was already suggested in [1] that one can use a construction analogous to the topological vertex [49, 52, 51] to produce more complicated vertex operator algebras by gluing Y-algebras. Consider a web of (p, q)-branes with stacks of D3-branes attached to them from different sides as in Figure 4.7.[7]

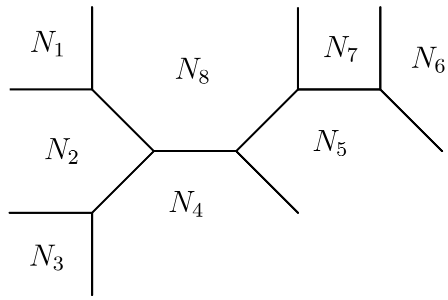


Figure 4.7: A generic (p, q)-web with stacks of N_i D3-branes attached. The gluing construction associates a vertex operator algebra to such a diagram. To each vertex in the diagram, one associates a Y-algebra and to each finite line segment one associates a class of bimodules for the two Y-algebras that are connected by the corresponding line segment. The final vertex operator algebra is a conformal extension of the product of Y-algebras by such bimodules and their fusions.

This configuration gives rise to a web of domain walls in the $\mathcal{N} = 4$ super Yang-Mills theory. In the topological twist of the theory, local operators inserted at trivalent junctions of the diagram give rise to Y-algebras. Looking at the configuration from the IR, the finite segments of fivebranes become infinitely small and the whole configuration can be thought of as a resolution of a single star shaped junction of more complicated domain walls. According to this picture, the line operators supported at finite segments and ending at the two trivalent junctions play the role of local operators of the IR junction and should be added to the final vertex operator algebra. The line operators living at interfaces and ending at their junctions will be associated to modules for Y-algebras. Operators one needs to add to the collection of Y-algebras correspond to bimodules associated to such line operators and their fusions. It turns out that these bimodules have (half-) integral conformal dimension with respect to the total stress-energy tensor (sum of the stress-energy tensors associated to each trivalent junction) and can indeed be added to the vertex operator algebra. In this section, we explain this construction in detail.

[7]Throughout the paper, we consider only webs corresponding to toric diagrams of Calabi-Yau threefolds without compact four-cycles, i.e. tree-like diagrams. The construction should be possible in general but in the presence of the closed faces, generic modules associated the Gukov-Witten defects [79] stretched within the internal faces can also be added to the VOA.

4.4 Gluing proposal

Now, we will review general structure of the gluing construction.

4.4.1 The vertex

We start with a description of the basic building blocks of our construction. The algebra of local operators associated to the trivalent junction of D5, NS5 and (1,1)-brane can be identified with $Y_{N_1,N_2,N_3}[\Psi]$ algebra reviewed above. In order to allow more general gluing, it proves useful to consider a larger family of trivalent junctions that will then serve as building blocks in the gluing construction. Luckily, one can obtain a larger class of such vertices by applying S-duality transformations to the basic D5-NS5-(1,1) junction. In the topological vertex literature, this operation is related to the change of framing.

S-duality acts on an $A^T \equiv (p,q)^T$ five-brane by a left multiplication by an $SL(2,\mathbb{Z})$ matrix

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad \text{for} \qquad ad - bc = 1. \tag{4.53}$$

The corresponding transformation of the coupling parameter Ψ is

$$\Psi \to \frac{a\Psi + b}{c\Psi + d}. \tag{4.54}$$

In terms of ϵ parameters, the transformation is implemented by the left multiplication of $(\epsilon_1, \epsilon_2)^T$ by matrix

$$\left(M^{-1}\right)^T = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix} \tag{4.55}$$

such that the combination

$$\epsilon^T A \equiv \begin{pmatrix} \epsilon_1 & \epsilon_2 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} \tag{4.56}$$

stays invariant.

Using these $SL(2,\mathbb{Z})$ transformations, one can map a trivalent junction of $A_j = (p_j, q_j)$, $j = 1,2,3$, defects (satisfying conservation of charges and the condition that ensures existence of such a transformation)

$$\begin{aligned} 0 &= A_1 + A_2 + A_3 \\ 1 &= A_1 \wedge A_2 \equiv p_1 q_2 - p_2 q_1 \end{aligned} \tag{4.57}$$

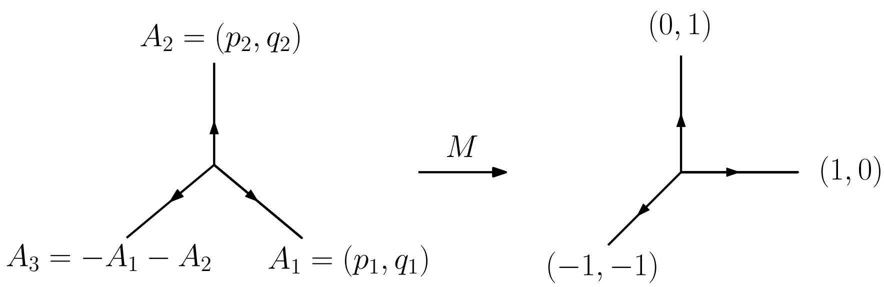


Figure 4.8: Transformation relating generic vertex of interest with the one used in identification of Y-algebras.

to the configuration used in the definition of Y-algebras by

$$M = \begin{pmatrix} q_2 & -p_2 \\ -q_1 & p_1 \end{pmatrix}. \tag{4.58}$$

In other words, to each such trivalent junction of A_1, A_2, and A_3 defects as shown in the figure 4.8 and the coupling parameter Ψ, one associates the algebra

$$Y^{A_1,A_2,A_3}_{N_1,N_2,N_3}[\Psi] = Y^{\binom{1}{0},\binom{0}{1},\binom{-1}{-1}}_{N_1,N_2,N_3}\left[-\frac{q_2\Psi - p_2}{q_1\Psi - p_1}\right] \equiv Y_{N_1,N_2,N_3}\left[-\frac{q_2\Psi - p_2}{q_1\Psi - p_1}\right]. \tag{4.59}$$

In terms of ϵ parameters $\epsilon = (\epsilon_1, \epsilon_2)^T$ and the fivebrane charges

$$Y^{A_1,A_2,A_3}_{N_1,N_2,N_3}[\epsilon_j] = Y_{N_1,N_2,N_3}\left[\epsilon^T A_j\right] \equiv Y_{N_1,N_2,N_3}[h_j] \tag{4.60}$$

where $h_j = \epsilon^T A_j = p_j\epsilon_1 + q_j\epsilon_2$. Note that the necessary consistency requirement $\tilde{\epsilon}_1 + \tilde{\epsilon}_2 + \tilde{\epsilon}_3 = 0$ follows from the charge conservation $A_1 + A_2 + A_3 = 0$ at the trivalent junction.[8] In terms of the invariant λ-parameters parametrizing the structure constants of Y (3.39) we have

$$\lambda_j = \frac{N_1h_1 + N_2h_2 + N_3h_3}{h_j} = \frac{\epsilon^T(N_1A_1 + N_2A_2 + N_3A_3)}{\epsilon^T A_j}. \tag{4.61}$$

This is insensitive to rescalings of ϵ and A_j parameters. λ_j determined in this way satisfy both (3.33) and (3.35).

[8]Note also that identification is possible for any values of A_1 and A_2 not only those related to the junction of NS5- and D5-branes by S-duality. One is tempted to identify generic vertex with such algebra. This naive guess would not be consistent with gluing proposal since bimodules added in gluing construction would not be (half-) integral.

There exists a natural $\mathbb{Z}_2$ sign of the $SL(2,\mathbb{Z})$ transformations. By taking a $\mathbb{Z}_2$ reduction of an $SL(2,\mathbb{Z})$ transformation matrix, we obtain an element of $SL(2,\mathbb{Z}_2) \simeq S_3$ and taking the sign of the corresponding permutation gives us a homomorphism $SL(2,\mathbb{Z}) \to \mathbb{Z}_2$. Concretely, we can map

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto (-1)^{ac+ad+bd+1}. \tag{4.62}$$

obtaining the required sign. Choosing our canonically oriented vertex to have the + orientation, any other vertex can be assigned an orientation given by the sign of the $SL(2,\mathbb{Z})$ transformation mapping the canonically oriented vertex to the vertex we are considering. Concretely, the orientation is given by

$$\operatorname{sgn}\left[Y^{A_1,A_2,A_3}_{N_1,N_2,N_3}\right] = (-1)^{p_1p_2+q_1q_2+p_1q_2+1} = (-1)^{p_1p_2+q_1q_2+p_2q_1}. \tag{4.63}$$

4.4.2 The edge

Let us first consider gluing two vertices as in the figure 4.9 where the numbers are subject to constraints

$$\begin{aligned} A_1 + A_2 &= A_1' + A_2' \\ A_1 \wedge A_2 &= 1 \\ A_1' \wedge A_2' &= 1 \end{aligned} \tag{4.64}$$

The first equation is simply the condition of the conservation of charges and the remaining conditions come from the requirement that both vertices are S-dual to the elementary junction of NS5, D5 and (1,1)-brane. One can always change the orientation of the ingoing and the outgoing legs and change the signs of corresponding (p,q) charges in order to obtain the configuration in 4.9.

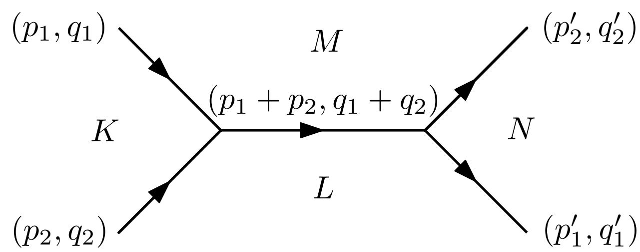


Figure 4.9: Any junction of two Y-diagrams can be put into this form by reversing the orientation of the legs and changing the signs of the corresponding labels. The parameters are subject to the constraints from (4.64).

Using the S-duality transformation and the fact that all the building blocks are S-dual to the triple junction of D5, NS5, and (1,1)-brane, one can transform our system uniquely to a new configuration depicted in figure 4.10 by the transformation

$$M = \begin{pmatrix} q_2 & -p_2 \\ -q_1 & p_1 \end{pmatrix}. \tag{4.65}$$

We used the fact that conditions (4.64) let us express all pairings in terms of one remaining invariant parameter (measuring the relative framing of the two vertices)

$$p \equiv -A_2 \wedge A_2' = 1 + A_2 \wedge A_1' = -1 + A_1 \wedge A_2' = -A_1 \wedge A_1'. \tag{4.66}$$

The first vertex is by definition positively oriented, while the orientation of the second vertex can be easily read off from (4.63) and we find it to be equal to $(-1)^p$.

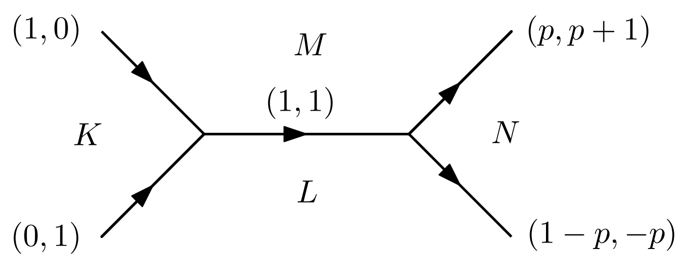


Figure 4.10: By $SL(2,\mathbb{Z})$ transformation, one can put diagram 4.9 to this form where parameter p is given by combination 4.66.

By looking at the two Y-vertices in diagram 4.9 or 4.10, one can deduce that the final algebra will be a conformal extension of

$$Y_{L,M,K}^{-A_1,-A_2,A_1+A_2}[\Psi] \times Y_{M,L,N}^{A_1',A_2',-A_1'-A_2'}[\Psi] \tag{4.67}$$

by a collection of M_μ^3 bimodules of the two Y-algebras on the right hand side.

Conformal dimension of gluing fields We can now check that dimension of the bimodules are (half-)integral. This can be easily seen from the transformed diagram 4.10. Remember that the total stress-energy tensor of the glued algebra is given by a sum of stress-energy tensors associated to the vertices. In particular, the conformal dimension of a bimodule is the sum of the two dimensions coming from the two endpoints,

$$h_\mu = h(M_\mu^3) + h'(M_\mu^3). \tag{4.68}$$

In the special case that the exchanged representation is the fundamental one, we find

$$h_{\square} = 1 + \frac{\lambda_3 + \lambda_3'}{2} = 1 + \frac{K + N - L - M}{2} + \frac{p(M - L)}{2} \equiv 1 + \rho. \tag{4.69}$$

Note in particular that all the dependence on continuous parameters like Ψ has canceled. The resulting dimension is always (half-)integral. The parameter ρ introduced in this formula will be important in later sections.

Specializing now to the case $L = 0$, we can be more general and write the expression for arbitrary line operator in representation μ:

$$h_\mu = \frac{1+p}{2}\sum_{j=1}^{M} \mu_j^2 + \frac{1-p}{2}\sum_{j=1}^{M}(2j - M - 1)\mu_j + \frac{K+N}{2}\sum_{j=1}^{M}|\mu_j|. \tag{4.70}$$

Analogously, in the case that $M = 0$, we have

$$h_\mu = \frac{1-p}{2}\sum_{j=1}^{L} \mu_j^2 + \frac{1+p}{2}\sum_{j=1}^{L}(2j - L - 1)\mu_j + \frac{K+N}{2}\sum_{j=1}^{L}|\mu_j| \tag{4.71}$$

This is again independent of the continuous parameter Ψ and is (half-)integral.

Gluing in terms of λ parameters If we fix the discrete parameter ρ which determines the dimension of the gluing matter and the fivebrane charges A_j and A_j', we can write explicitly the gluing conditions for Y-algebras directly in terms of λ_j and λ_j' parameters. Let us first introduce a vector in fivebrane charge space characterizing the first vertex

$$\sigma = \frac{A_2}{\lambda_3} - \frac{A_3}{\lambda_2} \tag{4.72}$$

and similarly for the second vertex. Using the fivebrane charge conservation

$$A_1 + A_2 + A_3 = 0 \tag{4.73}$$

and (3.40) we find that the definition of σ is cyclic invariant,

$$\lambda_1\lambda_2\lambda_3\sigma = \lambda_2(\lambda_3 A_3 - \lambda_1 A_1) = \lambda_3(\lambda_1 A_1 - \lambda_2 A_2). \tag{4.74}$$

The usefulness of σ lies in the fact that it encodes the λ_j parameters of the vertex, i.e.

$$\sigma \wedge A_j = \frac{1}{\lambda_j}. \tag{4.75}$$

Furthermore, eliminating the number of D3-branes and parameter Ψ from the gluing conditions, the gluing condition translates to a simple condition

$$\sigma \wedge \sigma' = 0 \tag{4.76}$$

satisfied by σ and σ' associated to glued vertices. This condition means that the vectors σ associated to the neighbouring vertices are parallel. We can use this and the definition of ρ to determine λ'_j once we know λ_j, ρ, A_j and A'_j,

$$\begin{aligned} 2\rho &= \lambda_3 + \lambda'_3 \\ 0 &= (\lambda_1 A_1 - \lambda_2 A_2) \wedge (\lambda'_1 A'_1 - \lambda'_3 A'_3) \\ 0 &= (\lambda_1 A_1 - \lambda_2 A_2) \wedge (\lambda'_2 A'_2 - \lambda'_3 A'_3) \end{aligned} \tag{4.77}$$

which is a linear system of equations and can be easily solved for λ'.

Statistics of the gluing matter Gluing fields turn out to have either bosonic or fermionic character depending on the relative $\mathbb{Z}_2$ sign (4.63) of the two vertices that we are gluing (and not whether the dimension of the gluing matter is integral or half-integral). We expect to have fermionic fields if the two vertices have the same sign and bosonic fields if the sign is opposite. In terms of the framing factor p we will have bosons for p odd and fermions for p even. This is indeed consistent examples with values $p = -1, 0, 1$ that we discuss in greater detail in later chapters.

4.4.3 Gluing in general

Let us consider an arbitrary of (p, q)-webs composed of the trivalent junctions glued by five-brane edges as discussed in the previous sections and let us attach stacks of D3-branes to the faces of the diagram. This configuration gives rise to a web of domain walls in $\mathcal{N} = 4$ SYM that we want to associate a vertex operator algebra to. The vertex operator algebra will be a conformal extension of a tensor product of mutually commuting Y-algebras associated to the vertices in the diagram by the bimodules associated to line operators inserted at the finite five-brane segments and their fusions. To each such segment, one can associate a parameter ρ_i as in the case of a single edge.

One can make following conjectures about the resulting algebra:

1. The total stress-energy tensor of the resulting algebra is the sum of stress-energy tensors of the individual vertices.

2. As consequence of this, the central charge of the resulting algebra is the sum of the central charges associated to all vertices.

3. The characters of modules associated to a collection of edges can be computed as a sum of products of Y-algebra characters, where the sum runs over representations of a tensor product of Lie (super-) algebras associated to the internal edges. For example in the case of two vertices we have

$$\chi = \sum_{\mu} \chi\left[Y^{(1)}\right](M^3_\mu)\chi\left[Y^{(2)}\right](M^3_\mu) \tag{4.78}$$

4. These modules can be obtained by fusion of elementary bimodules associated to the line operators in the fundamental and anti-fundamental representation supported at the internal edge. The dimension of these representations is given by (4.69).

5. To each external leg, one can associate a family of modules labeled by representations of the supergroup associated to the corresponding leg. Different families associated to non-parallel legs braid trivially, i.e. have conformal dimension that differs by an integer.

6. If the (p, q)-web is invariant under a subgroup of $SL(2, \mathbb{Z})$ transformation, the glued algebra will turn out to have dual BRST realization. If the algebra is realized as a truncation of an infinite algebra, there will be corresponding duality action on the parameter space of the corresponding infinite algebra.

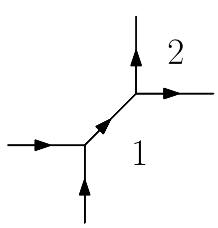
In the following we will illustrate the general discussion of the gluing construction on few concrete examples.

4.4.4 Example - $\mathcal{N} = 2$ superconformal algebra

Let us start with $\mathcal{N} = 2$ superconformal algebra. This algebra is obtained by extending the Virasoro algebra by a $U(1)$ current J and two oppositely charged spin $\frac{3}{2}$ fermionic primaries $G^{\pm}$. The $\widehat{\mathfrak{gl}(1)}$ current J generates the $SO(2)$ R-symmetry rotating the supercharges. The operator product expansions are given by

$$\begin{aligned}
T(z)T(w) &\sim \frac{c/2}{(z-w)^4}+\frac{2T(w)}{(z-w)^2}+\frac{\partial T(w)}{z-w}\\
T(z)G^\pm(w) &\sim \frac{3/2G^\pm(w)}{(z-w)^2}+\frac{\partial G^\pm(w)}{z-w}\\
T(z)J(w) &\sim \frac{J(w)}{(z-w)^2}+\frac{\partial J(w)}{z-w}\\
J(z)J(w) &\sim \frac{c/3}{(z-w)^2}\\
J(z)G^\pm(w) &\sim \frac{\pm G^\pm(w)}{z-w}\\
G^+(z)G^-(w) &\sim \frac{2c}{3(z-w)^3}+\frac{2J(w)}{(z-w)^2}+\frac{2T(w)}{z-w}+\frac{\partial J(w)}{z-w}\\
G^\pm(z)G^\pm(w) &\sim reg.
\end{aligned} \tag{4.79}$$

The central charge c is the only free continuous parameter.



From Poincaré-Birkhoff-Witt theorem we see that the vacuum character of this algebra at a generic central charge is given by

$$\prod_{n=0}^{\infty}\frac{\left(1+q^{\frac{3}{2}+n}\right)^2}{(1-q^{1+n})\,(1-q^{2+n})} \tag{4.80}$$

Up to an $\widehat{\mathfrak{gl}(1)}$ factor, this is exactly what one would obtain from the gluing construction starting from the diagram on the left. We can thus attempt to decompose the $\mathcal{N}=2$ SCA $\times\widehat{\mathfrak{gl}(1)}$ into elementary building blocks that enter the gluing construction. First, we decouple the $\widehat{\mathfrak{gl}(1)}$ currents to isolate the $\mathcal{W}_\infty$ stress-energy tensor that lives at the $(2,0,1)$ vertex. The unique combination commuting with $J(z)$ is

$$T_0(z)=T(z)-\frac{3}{2c}(JJ)(z). \tag{4.81}$$

Similarly, we can find spin 3 and spin 4 primaries that commute with $J(z)$ (they are determined uniquely up to a rescaling). We can next compute the combination of OPE coefficients

$$\frac{(C_{33}^4)^2C_{44}^0}{(C_{33}^0)^2}=\frac{12(c+1)(c+9)^2(5c-9)}{(c-1)(c+6)(2c-3)(5c+17)} \tag{4.82}$$

and assuming OPEs to be those of $\mathcal{W}_\infty$, this together with $c_\infty = c-1$ lets us determine the λ parameters associated to $(2,0,1)$ vertex to be

$$\begin{aligned}
\lambda_1^{(1)} &= -\frac{2c}{c-3} = \frac{-2\epsilon_1-\epsilon_2}{\epsilon_1} = \Psi - 2 \\
\lambda_2^{(1)} &= -\frac{c}{3} = \frac{-2\epsilon_1-\epsilon_2}{\epsilon_2} = \frac{2-\Psi}{\Psi} \\
\lambda_3^{(1)} &= \frac{2c}{c+3} = \frac{-2\epsilon_1-\epsilon_2}{-\epsilon_1-\epsilon_2} = \frac{2-\Psi}{1-\Psi}
\end{aligned} \tag{4.83}$$

which is what we could directly read off from the diagram. The identification between parameters is

$$c = 3 - \frac{6}{\Psi}, \qquad \Psi = -\frac{6}{c-3}. \tag{4.84}$$

We can also determine the λ parameters of the second vertex

$$\begin{aligned}
\lambda_1^{(2)} &= 1 \\
\lambda_2^{(2)} &= \frac{c-3}{6} = \frac{\epsilon_1}{\epsilon_2} = -\frac{1}{\Psi} \\
\lambda_3^{(2)} &= -\frac{c-3}{c+3} = \frac{\epsilon_1}{-\epsilon_1-\epsilon_2} = \frac{1}{\Psi-1}
\end{aligned} \tag{4.85}$$

which is consistent with the $\widehat{\mathfrak{gl}(1)}$ degree of freedom coming from the second vertex.

Finally, let us identify the gluing matter. The fields of the lowest dimension that do not come from the vertices are are the fields $G^\pm(z)$. Following the choice of the normalization of $\widehat{\mathfrak{gl}(1)}$ factors

$$J^{(1)}(z)J^{(1)}(w) \sim -\frac{c(c+3)}{3(c-3)}\frac{1}{(z-w)^2}, \qquad J^{(2)}(z)J^{(2)}(w) \sim \frac{c+3}{6}\frac{1}{(z-w)^2}, \tag{4.86}$$

we know that the basic gluing fields will have charges ± 1 with respect to these. We define a rotated basis of $\widehat{\mathfrak{gl}(1)}$ currents

$$J(z) \equiv \frac{c-3}{3(c-1)}J^{(1)}(z) - \frac{2c}{3(c-1)}J^{(2)}(z), \qquad \tilde{J}(z) \equiv J^{(1)}(z) + J^{(2)}(w) \tag{4.87}$$

such that $J(z)$ is the conventionally normalized R-current in $\mathcal{N}=2$ SCA and that $\tilde{J}(z)$ decouples. The other primary gluing fields are given by the normal ordered products

$$G^\pm_{(k)}(z) \equiv (\partial^{k-1}G^\pm(\partial^{k-2}G^\pm(\cdots(\partial G^\pm G^\pm)\cdots)))(z). \tag{4.88}$$

Their $U(1)$ charges are given by

$$j^{(1)}(G^{\pm}_{(k)}) = \pm k, \quad j^{(2)}(G^{\pm}_{(k)}) = \mp k, \quad j(G^{\pm}_{(k)}) = \pm k, \quad \tilde{j}(G^{\pm}_{(k)}) = 0. \tag{4.89}$$

The conformal dimensions are

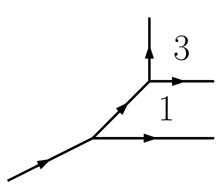
$$h^{(1)}_{1+\infty}(G^{\pm}_{(k)}) = \frac{(c-3)k^2 + 2(c+3)k}{2(c+3)}, \quad h^{(2)}_{1+\infty}(G^{\pm}_{(k)}) = \frac{3k^2}{c+3}, \quad h_{1+\infty}(G^{\pm}_{(k)}) = \frac{k(k+2)}{2}. \tag{4.90}$$

as predicted by (4.70)

4.4.5 Example - $\mathcal{W}_3^{(2)}$

As another example, consider the Bershadsky-Polyakov algebra $\mathcal{W}_3^{(2)}$ [105, 106]. It has the same matter content as $\mathcal{N}=2$ SCA except for the fact that the spin $\frac{3}{2}$ fields are bosons instead of being fermions. The operator product expansions are now

$$\begin{aligned}
T(z)T(w) &\sim -\frac{(2k+3)(3k+1)}{2(k+3)(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} \\
T(z)J(w) &\sim \frac{J(w)}{(z-w)^2} + \frac{\partial J(w)}{z-w} \\
J(z)J(w) &\sim \frac{2k+3}{3(z-w)^2} \\
T(z)G^{\pm}(w) &\sim \frac{\frac{3}{2}G^{\pm}(w)}{(z-w)^2} + \frac{\partial G^{\pm}(w)}{z-w} \\
J(z)G^{\pm}(w) &\sim \frac{\pm G^{\pm}(w)}{z-w} \\
G^{\pm}(z)G^{\pm}(w) &\sim reg. \\
G^{+}(z)G^{-}(w) &\sim \frac{(k+1)(2k+3)}{(z-w)^3} + \frac{3(k+1)J(w)}{(z-w)^2} - \frac{(k+3)T(w)}{z-w} \\
&\quad + \frac{3(JJ)(w)}{z-w} + \frac{3(k+1)\partial J(w)}{2(z-w)}.
\end{aligned} \tag{4.91}$$



From the gluing construct we see that the vacuum character

$$\prod_{n=0}^{\infty} \frac{1}{(1-q^{1+n})^2(1-q^{\frac{3}{2}+n})^2(1-q^{2+n})} \tag{4.92}$$

of $\widehat{\mathfrak{gl}(1)} \times \mathcal{W}_3^{(2)}$ equals that of the diagram on the left. We can try to see if this identification works even at the level of operator product expansions. Similarly to the discussion in the previous section, we can first decouple the $\widehat{\mathfrak{gl}(1)}$ factors and find the stress-energy tensor

$$T_\infty^{(1)}(z) = T(z) - \frac{3}{2(2k+3)}(JJ)(z) \tag{4.93}$$

with the central charge

$$c_\infty^{(1)} = -\frac{6(k+1)^2}{k+3}. \tag{4.94}$$

Analogously, we can construct primary spin 3 and spin 4 currents commuting with $\widehat{\mathfrak{gl}(1)}$ factors,

$$\begin{aligned} W_3^{(1)} &= (G^+G^-) + \frac{3(k+3)}{2k+3}(TJ) + \frac{k+3}{2}\partial T - \frac{9(k+2)}{(2k+3)^2}(J(JJ)) \\ &\quad -3(J\partial J) - \frac{k^2+4k+6}{2k+3}\partial^2 J \\ W_4^{(1)} &= (J(G^+G^-)) + \ldots, \end{aligned} \tag{4.95}$$

compute their three-point functions

$$\begin{aligned} C_{33}^0 &= -\frac{(4k+9)(2k+1)(k+3)(k+1)^2}{2k+3} \\ C_{44}^0 &= -\frac{(5k+12)(4k+9)(3k+5)(2k+1)(k+1)^2k^2}{3(15k^2+19k-18)} \\ C_{33}^4 &= \frac{12(k+3)^2}{2k+3} \end{aligned} \tag{4.96}$$

and finally find the invariant combination of structure constants

$$\frac{C_{33}^4 C_{44}^0}{(C_{33}^0)^2} = -\frac{48k^2(k+3)^2(3k+5)(5k+12)}{(k+1)^2(2k+1)(4k+9)(15k^2+19k-18)}. \tag{4.97}$$

Equating this to (3.37), we can determine the $\lambda_j^{(1)}$ parameters (assuming that the commutant of $\widehat{\mathfrak{gl}(1)}$ currents is $\mathcal{W}_\infty$) to be

$$\begin{aligned}
\lambda_1^{(1)} &= 2k+3 = \frac{\epsilon_2+3\epsilon_3}{\epsilon_1} = 2\Psi - 3 \\
\lambda_2^{(1)} &= -\frac{2k+3}{k+3} = \frac{\epsilon_2+3\epsilon_3}{\epsilon_2} = \frac{-2\Psi+3}{\Psi} \\
\lambda_3^{(1)} &= \frac{2k+3}{k+2} = \frac{\epsilon_2+3\epsilon_3}{\epsilon_3} = \frac{-2\Psi+3}{1-\Psi}.
\end{aligned} \tag{4.98}$$

We identified

$$\Psi = k+3. \tag{4.99}$$

We can read-off the λ-parameters of the second vertex from the diagram (again we cannot determine them from the algebra because of no continuous parameters associated to the affine $\widehat{\mathfrak{gl}(1)}$ algebra)

$$\lambda_1^{(2)} = \frac{-2\epsilon_1-\epsilon_2}{-2\epsilon_1-\epsilon_2} = 1 \tag{4.100}$$

$$\lambda_2^{(2)} = \frac{-2\epsilon_1-\epsilon_2}{\epsilon_1} = -2+\Psi = k+1 \tag{4.101}$$

$$\lambda_3^{(2)} = \frac{-2\epsilon_1-\epsilon_2}{\epsilon_1+\epsilon_2} = \frac{-2+\Psi}{1-\Psi} = -\frac{k+1}{k+2}. \tag{4.102}$$

$\widehat{\mathfrak{gl}(1)}$ currents Now we can turn our attention to the identification of the $\widehat{\mathfrak{gl}(1)}$ currents. We take a linear combination of $\widehat{\mathfrak{gl}(1)}$ currents

$$J(z) = \frac{k+3}{3(k+2)} J^{(1)}(z) - \frac{2k+3}{3(k+2)} J^{(2)}(z) \tag{4.103}$$

$$\tilde{J}(z) = J^{(1)}(z) + J^{(2)}(z) \tag{4.104}$$

such that $\tilde{J}(z)$ decouples from $\mathcal{W}_3^{(2)}$.

Gluing matter The gluing fields in the case of $\mathcal{W}_3^{(2)}$ are given by powers of $G^\pm(z)$,

$$G^\pm_{(n)}(z) \equiv (G^\pm(G^\pm(\cdots(G^\pm G^\pm)\cdots)))(z). \tag{4.105}$$

Their $\widehat{\mathfrak{gl}(1)}$ charges are

$$j^{(1)}(G^\pm_{(n)}) = \pm n, \quad j^{(2)}(G^\pm_{(n)}) = \mp n, \quad j(G^\pm_{(n)}) = \pm n, \quad \tilde{j}(G^\pm_{(n)}) = 0. \tag{4.106}$$

and the conformal dimensions are

$$h^{(1)}_{1+\infty}(G^{\pm}_{(n)}) = \frac{n(3k+6-n)}{2k+4}, \quad h^{(2)}_{1+\infty}(G^{\pm}_{(n)}) = \frac{n^2}{2k+4}, \quad h_{1+\infty}(G^{\pm}_{(n)}) = \frac{3n}{2}. \tag{4.107}$$

which is consistently with (4.70). Note that because of the bosonic nature of the gluing fields, the quadratic terms proportional to n^2 in $h^{(1)}_{1+\infty}$ and $h^{(2)}_{1+\infty}$ cancel so that the total dimension of the gluing fields grows linearly with n.

4.5 Truncations

In this section we study what happens as the number of attached D3-branes grows to infinity. We have already discussed the simplest case of the algebra $Y_{N_1,N_2,N_3}[\Psi]$. As N_1, N_2 or N_3 goes to infinity, the vacuum character approaches the MacMahon function which counts the plane partitions without any further constraints. Combinatorially, the requirement of the absence of a box at position (N_1+1, N_2+1, N_3+1) which leads to truncation of the algebra disappears. The operator product expansions for the spin content given by MacMahon functions were studied in [75] and the result is a two-parametric family of algebras $\mathcal{W}_{1+\infty}$ parametrized by λ_j with constraint (3.33). The central charge and the OPE structure constants in the primary basis are determined in terms of λ_j as in (3.37). Y_{N_1,N_2,N_3} algebras can be recovered by recalling that if (3.35) is satisfied the algebra $\mathcal{W}_{1+\infty}$ develops an ideal such that Y_{N_1,N_2,N_3} is the quotient of $\mathcal{W}_{1+\infty}$ by this ideal. In this section, we generalize this point of view to other algebras that we constructed by the gluing procedure.

4.5.1 Resolved conifold - $\mathcal{W}^{\rho}_{1|1\times\infty}$ algebras

As a first example, let us see what are the possible limits of the conifold $\mathcal{W}^{1|1}_{K,\bar{L},M,\bar{N}}$ algebras as the number of D3 branes approaches infinity. Compared to the Y_{N_1,N_2,N_3} junction, the conifold configuration has another stack of D3-branes so one might naively expect a three-parametric family of algebras. We will see that in the infinite numbers of branes limit, one recovers different characters for each choice of the discrete parameter

$$\rho = \frac{N+L-K-M}{2} \tag{4.108}$$

that we keep fixed as we take the limit. For each choice of ρ, there are two continuous independent λ-parameters from one of the vertices as in the case of Y_{N_1,N_2,N_3}. The λ-parameters of the second vertex can be then determined in terms of the discrete parameter ρ and the gluing conditions (4.77). We thus obtain a family of algebras $\mathcal{W}^{\rho}_{1|1\times\infty}$ associated to the conifold diagram, labeled by one discrete parameter ρ (associated to the edge) and two continuous parameters parametrizing the structure constants of the algebra. We expect to be able to recover $\mathcal{W}^{1|1}_{K,\bar{L},M,\bar{N}}$ algebras as truncations of the $\mathcal{W}^{\rho}_{1|1\times\infty}$ family.

BRST computation of the character Let us now explicitly verify the claims of the previous section by computing the vacuum character of $\mathcal{W}^{\rho}_{1|1\times\infty}$,

$$\chi\left[\mathcal{W}^{\rho}_{1|1\times\infty}\right] = \prod_{n=1}^{\infty} \frac{(1+q^{n+\rho})^{2n}}{(1+q^n)^{2n}}. \tag{4.109}$$

Let us first see how the character (4.109) appears from the BRST definition of the algebra for $K \geq M$ and $L \geq N$. When computing the vacuum character, there are various contributions coming from the different blocks of $\mathfrak{gl}(K|L)$. Firstly, there are characters of W_{K-M} and W_{L-N} coming from the two diagonal blocks. Secondly there is a sequence of pairs of $L-N$ fermionic fields with conformal weights

$$\rho+1, \rho+2, \ldots \rho+L-N \tag{4.110}$$

coming from the fermionic off-diagonal blocks that are influenced by both DS-reductions. Apart from these, there are $\mathfrak{gl}(M|N)$ invariant combinations of $\mathcal{S}^{M|N}_{\frac{K-M}{2}}$ and $\overline{\mathcal{S}}^{M|N}_{\frac{L-N}{2}}$. These can be identified with products of bilinears of their generators. If we forget about the relations satisfied by the products of bilinears (which is a condition that disappears in the infinite number of branes limit), $\mathcal{S}^{M|N}_{\frac{K-M}{2}}$ fields form an infinite tower of generators of each integral spin starting with $K-M+1$. This sequence continues the one of $\mathcal{W}_{K-M}$ and together they form one factor of $\mathcal{W}_{1+\infty}$. Similarly, $\mathfrak{gl}(M|N)$ invariant combinations of $\overline{\mathcal{S}}^{M|N}_{\frac{L-N}{2}}$ continues the sequence of fields of $\mathcal{W}_{L-N}$ to produce the second factor of the $\mathcal{W}_{1+\infty}$ vacuum character. Finally, the bilinears mixing $\mathcal{S}^{M|N}_{\frac{K-M}{2}}$ and $\overline{\mathcal{S}}^{M|N}_{\frac{L-N}{2}}$ form an infinite tower starting at conformal dimension $\rho+L-N+1$. Note that these fields are fermionic since a bosonic field gets combined with a fermionic field and these combinations continue the $L-N$ fermionic fields discussed above. One can see that total character is indeed given by (4.109).

The BRST proposal for $K \geq M$ and $N \geq L$ produces the same character by a slightly different argument. The two $\mathcal{W}_{K-M}$ and $\mathcal{W}_{N-L}$ blocks get extended by bilinears of $\mathcal{S}^{M|L}_{\frac{K-M}{2}}$ and $\overline{\mathcal{S}}^{M|L}_{\frac{K-M}{2}}$ respectively. In this case there are no off-diagonal blocks that would be influenced by both DS-reductions but $\mathfrak{gl}(M|K)$ invariant combinations combining the fields coming from both $\mathcal{S}^{M|L}_{\frac{K-M}{2}}$ and $\overline{\mathcal{S}}^{M|L}_{\frac{K-M}{2}}$ give rise to fermions with each integral spin starting at spin $1+\rho$ and we can draw the same conclusion as in the previous case.

Character from gluing From the point of view of gluing, the character formula (4.109) can be obtained by a small modification of the standard sums used in topological vertex calculations. In the limit of infinite numbers of branes $K, L, M, N \to \infty$, the relevant tensor representations of $U(\infty)$ decouple into contravariant representations (contained in tensor powers of the fundamental representation) times covariant representations (contained in tensor powers of the anti-fundamental representation). Moreover, the pit conditions truncating the two trivalent vertex algebras disappear to infinity and the characters involved considerably simplify.

We can use this example to illustrate the gluing at the level of $\mathcal{W}_{1+\infty}$ algebras. First of all, the λ parameters associated to two vertices are connected via

$$\lambda_j' = \lambda_j \frac{\lambda_3}{2\rho - \lambda_3} \tag{4.111}$$

as follows from (4.77). We want to sum over all characters of $\mathcal{W}_{1+\infty}$ labeled by the representations of the line operators stretched along the edge. In the limit of large number of D3-branes, these are parametrized by a pair of Young diagram labels (μ, ν), the first labeling the contravariant part and the second labeling the covariant part of the $U(\infty)$ representation. The corresponding $\mathcal{W}_{1+\infty}$ character factorizes and is equal to

$$\chi_{(\mu,\nu)} = q^{h_\mu + h_\nu} \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^n} P_\mu(q) P_\lambda(q) \tag{4.112}$$

where the power of q in the prefactor is the conformal dimension of the representation and where $P_\mu(q)$ is the quantum dimension of the representation (normalized to be a polynomial in q starting with $1+\ldots$),

$$P_\lambda(q) = \prod_{\square \in \lambda} \frac{1}{1 - q^{\mathrm{hook}(\square)}} \tag{4.113}$$

(see [49, 72, 98]). The full vacuum character for the conifold algebra is now obtained by summing over all the representations of the line operators and taking the product of characters of algebras associated to both vertices

$$\chi_{\mathcal{W}^{\rho}_{1|1\times\infty}} = \sum_{\mu,\nu\geq 0} z^{|\mu|-|\nu|}\chi_{(\mu,\nu)}\chi'_{(\mu,\nu)} \tag{4.114}$$

$$= \prod_{n=1}^{\infty}\frac{1}{(1-q^n)^{2n}} \times \sum_{\mu,\nu\geq 0}\left(q^{h_\mu+h'_\mu}z^{|\mu|}P^2_\mu(q)\right)\times\left(q^{h_\nu+h'_\nu}z^{-|\nu|}P^2_\nu(q)\right). \tag{4.115}$$

We turned on the fugacity parameter z for the $U(1)$ current associated to one of the two Y-algebra vertices which refines the character. Now we need to evaluate

$$\sum_{\mu\geq 0}\left(q^{h_\mu+h'_\mu}z^{|\mu|}P^2_\mu(q)\right) = \sum_{\mu\geq 0}\left(q^{\frac{1}{2}\sum_j \mu_j^2+\frac{1}{2}\sum_j(2j-1)\mu_j+\rho\sum_j\mu_j}z^{\sum_j\mu_j}P^2_\mu(q)\right). \tag{4.116}$$

This sum is a typical example of sums studied in the topological vertex computations and we find

$$\sum_{\mu\geq 0}\left(q^{h_\mu+h'_\mu}z^{|\mu|}P^2_\mu(q)\right) = \prod_{n=1}^{\infty}(1+zq^{\rho+n})^n. \tag{4.117}$$

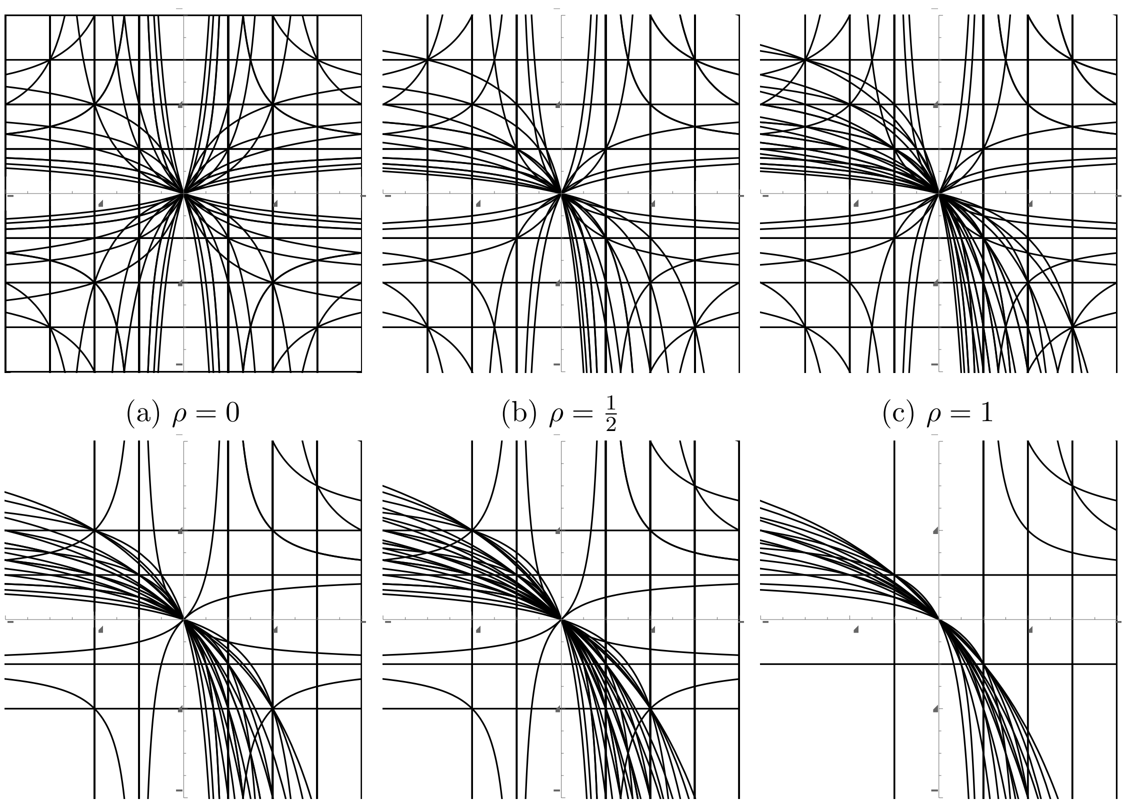
This again reproduces the formula (4.109), this time with the additional fugacity parameter z.

Let us now consider two special values of the parameter ρ. In the case when $\rho = 0$, one gets in the large N limit the character of $\mathcal{W}^{1|1}_{1+\infty}$ (an algebra generated by 2×2 matrix of generators for each integral spin). This algebra appeared in [107] as an example in the context of categorified Donaldson-Thomas invariants and corresponding counting of D0-D2-D6 bound states. We devote the next section to the example of $\rho = \frac{1}{2}$ that can be identified with $\mathcal{N}=2$ super-$\mathcal{W}_\infty$.

$\mathcal{N}=2$ super-$\mathcal{W}_\infty$ In the case of $\rho=\frac{1}{2}$, the character can be identified with the vacuum character of $\mathcal{N}=2$ super-$\mathcal{W}_\infty\times\widehat{\mathfrak{gl}(1)}$ of [77].[9] The authors extended the $\mathcal{N}=2$ superconformal algebra by a simple tower of higher spin $\mathcal{N}=2$ supermultiplets with spins of lowest components being $2,3,\ldots$. Imposing the Jacobi identities, a two-parameter family of such algebras was found. For special values of parameters, a truncation of this algebra admits a coset construction using the Kazama-Suzuki coset

$$\frac{\widehat{\mathfrak{gl}(N+1)}_{\Psi^{-1}}\times\mathcal{S}^{N|0}}{\widehat{\mathfrak{gl}(N)}_{\Psi^{-1}}} \tag{4.118}$$

[9] See also [108] for the special case of parameters where the algebra becomes linear.



(a) $\rho = 0$ (b) $\rho = \frac{1}{2}$ (c) $\rho = 1$

(d) $\rho = \frac{3}{2}$ (e) $\rho = 2$ (f) $\rho = \frac{5}{2}$

Figure 4.11: First few truncation curves in the (μ_1, μ_3) parametrization for $0 \leq \rho \leq \frac{5}{2}$ and the number of branes $K + L + M + N \leq 8$. Note that the figures are invariant under the reflection $\mu_1 \leftrightarrow \mu_2$ corresponding to the S-duality action.

and a construction using the Drinfeld-Sokolov reduction of $\mathfrak{gl}(\widehat{N+1|}N)$. Both of these realizations can be identified with the BRST constructions that we propose for a special choice of discrete parameters K, L, M, N and turn out to be related by S-duality of our diagram

$$\mathcal{W}^{1|1}_{N+1,\bar{N},0,\bar{0}}[\Psi] \quad \leftrightarrow \quad \mathcal{W}_{N+1,\bar{0},N,\bar{0}}\left[\frac{1}{\Psi}\right]. \tag{4.119}$$

In particular, for $N = 1$, we get the $\mathcal{N} = 2$ superconformal algebra that is an extension of the Virasoro algebra by a spin 1 current and two oppositely charged spin $\frac{3}{2}$ fermions. Together with the stress-energy tensor, these four fields form a $\mathcal{N} = 2$ supermultiplet with lowest component having spin 1. In [109], the $\mathcal{N} = 2$ SCA was extended by adding a $\mathcal{N} = 2$ supermultiplet with lowest spin 2. It is natural to conjecture that in general the $\mathcal{N} = 2$ extension of $\mathcal{W}_N$ is given by the $\mathcal{L}^{1|1}_{N+1,\bar{N},0,\bar{0}}$ algebra and that all the other configurations with $\rho = \frac{1}{2}$ are other truncations of $\mathcal{N} = 2$ $\mathcal{W}_\infty$.

Candu and Gaberdiel introduce a parameter μ with the property that setting $\mu = -N$, we recover the truncations discussed above and parametrize the full algebra in terms of μ and the central charge c. Analogously to the triality symmetry of $\mathcal{W}_{1+\infty}$, at each generic fixed value of the central charge c there are four different values of μ which give identical OPEs in the primary basis. These values of μ are [77]

$$\mu_1 = \mu, \qquad \mu_2 = \frac{(c-1)\mu}{c+3\mu}, \qquad \mu_3 = \frac{c+3\mu}{3(\mu-1)}, \qquad \mu_4 = -\frac{c}{3\mu}.$$

Since $\mathcal{N} = 2$ $\mathcal{W}_\infty$ has apart from the stress-energy tensor an extra spin 2 primary field commuting with the $\widehat{\mathfrak{gl}(1)}$ factor, we can find a linear combination of the spin 2 fields which give us two independent commuting Virasoro subalgebras. Their central charges are

$$\begin{aligned} c_1 &= \frac{c(\mu+1)(c+6\mu-3)}{3(c+3\mu)^2} \\ c_2 &= -\frac{(c-3\mu)(c(\mu-2)-3\mu)}{3(c+3\mu^2)} \\ c &= c_1+c_2+1. \end{aligned} \tag{4.120}$$

Note that we have

$$\begin{aligned} c_1 &= \frac{(1+\mu_1)(1+\mu_3)(\mu_1\mu_3-\mu_1-\mu_3)}{\mu_1+\mu_3}, \\ c_2 &= \frac{(1+\mu_2)(1+\mu_4)(\mu_2\mu_4-\mu_2-\mu_4)}{\mu_2+\mu_4} \end{aligned} \tag{4.121}$$

so defining

$$\mu_1 = -\lambda_1, \quad \mu_2 = -\lambda_1', \quad \mu_3 = -\lambda_2, \quad \mu_4 = -\lambda_2' \tag{4.122}$$

we can rewrite the partial central charges c_1 and c_2 in the standard form

$$\begin{aligned} c_1 &= (\lambda_1 - 1)(\lambda_2 - 1)(\lambda_3 - 1) \\ c_2 &= (\lambda_1' - 1)(\lambda_2' - 1)(\lambda_3' - 1). \end{aligned} \tag{4.123}$$

We can thus identify the parameters λ_j and λ_j' with the λ-parameters associated to the two vertices. This hints that $\mathcal{N} = 2$ super $\mathcal{W}_\infty$ algebra indeed contains two mutually commuting $\mathcal{W}_\infty$ algebras as subalgebras and gives a picture consistent with the gluing[10]. The duality transformations of the algebra that can be identified with the $\mathbb{Z}_2 \times \mathbb{Z}_2$ duality action given by transformations

$$\Psi \leftrightarrow \Psi, \qquad K \leftrightarrow M, \qquad L \leftrightarrow N \tag{4.124}$$

and

$$\Psi \leftrightarrow \frac{1}{\Psi}, \qquad K \leftrightarrow M \tag{4.125}$$

that can be identified with permutation of parameters

$$\begin{aligned} \mu_1 \leftrightarrow \mu_2, \qquad \mu_3 \leftrightarrow \mu_4 \\ \mu_1 \leftrightarrow \mu_3, \qquad \mu_2 \leftrightarrow \mu_4. \end{aligned} \tag{4.126}$$

Note that the parametrization and the whole construction works for arbitrary value of ρ and we expect an existence of a two continuous parameter families of $\mathbb{Z}_2 \times \mathbb{Z}_2$ algebras for each choice of ρ such that $\mathcal{W}^{1|1}_{K,\bar{L},M,\bar{N}}$ are their truncations. The structure of these truncations in the (μ_1, μ_2) parameter space is shown in the figure 4.11 for various values of ρ. You can see that figures are indeed invariant under $\mu_1 \leftrightarrow \mu_2$. The points where two truncation curves intersect correspond to the BRST reductions at rational levels and we expect them to correspond to minimal models of $\mathcal{W}^{\rho}_{1|1\times\infty}$ algebras.

4.5.2 Truncations of $\mathcal{W}^{\rho}_{2\times\infty}$

Let us now consider the case of algebras of the type $0|2$ with the corresponding parameter ρ fixed. Sending parameters $K, L, M, N \to \infty$ to infinity, relations satisfied by product

[10] A similar observation was made by [25, 32] where the authors study the gluing of $\mathcal{N} = 2$ $\mathcal{W}_\infty$ from the Yangian point of view.

of bilinears in the BRST calculation of the character disappear and one gets a character analogous to $\mathcal{W}^{\rho}_{1|1\times\infty}$. The only difference is that all the invariant combinations are bosonic and the final character is given by

$$\chi\left[\mathcal{W}^{\rho}_{0|2\times\infty}\right]=\prod_{n=1}^{\infty}\frac{1}{(1-q^{n})^{2n}(1-zq^{n+\rho})^{n}(1-z^{-1}q^{n+\rho})^{n}}. \tag{4.127}$$

The same formula can be obtained from the gluing construction in the same way as in the resolved conifold case. We just need to use

$$\sum_{\mu\geq 0}\left(q^{h_{\mu}+h'_{\mu}}z^{|\mu|}P^{2}_{\mu}(q)\right)=\sum_{\mu\geq 0}\left(q^{\sum_{j}(2j-1)\mu_{j}+\rho\sum_{j}\mu_{j}}z^{\sum_{j}\mu_{j}}P^{2}_{\mu}(q)\right)=\prod_{n=1}^{\infty}\frac{1}{(1-zq^{\rho+n})^{n}}. \tag{4.128}$$

Algebras discussed in this section can be identified with truncations of $\mathcal{W}^{\rho}_{0|2\times\infty}$. Note that $\rho=0$ case again coincides with algebras studied in [107] in the context of counting D6-D2-D0 bound states on the resolution of $\mathbb{C}^{2}/\mathbb{Z}_{2}\times\mathbb{C}$.

4.5.3 Truncations of $\mathcal{W}^{\rho_i}_{M|N\times\infty}$

All examples discussed so far in this section can be identified with truncations of some infinite algebra. In the BRST reduction described in above, one generates W_N algebra and symplectic bosons in fundamental representation of the reduced group associated to each vertex. Moreover, at each vertex, symplectic bosons generated in the previous step decomposes into the fields of shifted dimensions and symplectic bosons in fundamental representation of the reduced group. Example of such process for first three reductions from the example (4.50) is diagrammatically captured in 4.6. After projecting to the coset invariant combinations, one can argue that in the infinite number of branes limit, one obtains the character of the form

$$\chi[\mathcal{W}^{\rho_i}_{M|N\times\infty}]=\left(\prod_{n=1}^{\infty}\frac{1}{(1-q^{n})^{n}}\right)^{N+M}\prod_{i=1}(1\pm q^{n+\rho_i})^{\pm 2n}\prod_{i>j}(1\pm q^{n+\rho_i+\rho_j})^{\pm 2n} \tag{4.129}$$

where the products run over all internal edges and one chooses the + sign if both branes of corresponding finite segment ends from the same side and the − sign otherwise. The same character follows from the gluing construction: for example, in the $U(3)$ case we use the fact that

$$\prod_{n=1}^{\infty}\frac{1}{(1-\frac{z_1}{z_2}q^n)^n(1-\frac{z_1}{z_3}q^n)^n(1-\frac{z_2}{z_3}q^n)^n}=$$
$$=\sum_{\mu,\nu}q^{||\mu||^2+||\nu||^2-(\mu,\nu)}\left(\frac{z_1}{z_2}\right)^{|\mu|}\left(\frac{z_2}{z_3}\right)^{|\nu|}P(\cdot,\mu)P(\mu,\nu)P(\nu,\cdot) \quad (4.130)$$

where $P(\mu,\nu)$ are the the box-counting functions [98] related up to an overall factor to the topological vertex $C(\mu,\nu,\cdot)$. For the total character we thus find

$$\chi^{\rho=0}_{3\times\infty}=\prod_{n=1}^{\infty}\prod_{j,k=1}^{3}\frac{1}{(1-z_jz_k^{-1}q^n)^n} \quad (4.131)$$

as expected for $\mathcal{W}_{3\times\infty}$.

Algebras coming from gluing or BRST construction for finite number of branes can be identified with truncations of $\chi^{\rho_i}_{N|M\times\infty}$. For fixed values ρ_i, there are three integral parameters left unfixed. These parameters parametrize truncation lines of $\chi^{\rho_i}_{N|M\times\infty}$ inside the conjecturally two parameter family of algebras. Shifting all the numbers of branes by a constant value again corresponds to a different truncation above the same truncation curve.

4.6 Free field realization

Starting with $Y_{N_1,N_2,N_3}[\Psi]$ as a building block, one can construct more complicated VOAs associated to an arbitrary (p,q) web of five-branes and D3-branes attached to them at various faces. The resulting vertex operator algebra is an extension of tensor product of Y-algebras associated to each vertex by bi-modules (and their fusion) associated to each internal line of the web diagram. Existence of such an extension[11] was conjectured in [2] but no explicit construction of OPEs between gluing bi-modules was proposed. The free field realization discussed above seems to provide us with a way to determine OPEs of such

[11] There exists a large list of special examples appearing in various contexts in the literature. The story of extensions of VOAs dates back to the early days of VOAs, where authors of [10, 11] constructed lattice extensions of the free boson VOA. Extensions of $\mathcal{W}_N$ algebras are discussed for example in [110, 111] and many other places. More recently, gluing at the level of affine Yangians was initiated in [25, 32] and at the level of quantum toroidal algebras appears in [112].

bi-modules. In the two explicit examples bellow, we will indeed see that this is indeed the case. Note also that such construction leads to an algorithmic way to determine a free field realization of the glued algebra. We expect some of the free field realizations to be related via bosonisation to well-known free-field realizations, such as the Wakimoto realization of Kac-Moody algebras [113, 114].

Let us briefly review the gluing construction in the case of a single edge. The generalization to more complicated configurations is straightforward and will be briefly discussed later. Consider a (p,q)-brane configuration from the figure 4.12. The resulting VOA is an extension of the product

$$Y^{-\tilde{A}_1,-\tilde{A}_2,\tilde{A}_1+\tilde{A}_2}_{N_2,N_4,N_3}[\Psi]\otimes Y^{A_1,A_2,-A_1-A_2}_{N_4,N_2,N_1}[\Psi] \tag{4.132}$$

where $Y^{A_1,A_2,A_3}_{N_1,N_2,N_3}[\Psi]$ is related to the standard algebra $Y_{N_1,N_2,N_3}[\Psi]$ by an $SL(2,\mathbb{Z})$ transformation of parameters

$$Y^{A_1,A_2,A_3}_{N_1,N_2,N_3}[\Psi]=Y_{N_1,N_2,N_3}\left[-\frac{q_2\Psi-p_2}{q_1\Psi-p_1}\right]. \tag{4.133}$$

The parameters h_i of the algebra can be easily determined from

$$h_i=A_i\cdot\epsilon, \tag{4.134}$$

where we have introduced the vector $\epsilon=(\epsilon_1,\epsilon_2)$ and A_i are the (p,q) charges of the ith interface with the arrow pointing out of the vertex. Note that ϵ_i are universal parameters and in the case of the standard trivalent junction of NS5, D5 and $(1,1)$ branes, one has the identification $h_i=\epsilon_i$ with $\epsilon_3=-\epsilon_1-\epsilon_2$[12]. The extension is then generated by fusions of the tensor product of the fundamental representation associated to the first vertex and anti-fundamental representation associated to the second vertex and vice versa.

In the free field realization, the fundamental and the anti-fundamental representation have a simple realization in terms of an exponential vertex operator and its descendant. For simplicity of the discussion, we will restrict to the case $N_4=0$ and identify only the simple realization of the fundamental and the anti-fundamental representation for the following ordering

$$\mathcal{L}^{(2)}_1\cdots\mathcal{L}^{(2)}_{N_2}\mathcal{L}^{(3)}_{N_2+1}\cdots\mathcal{L}^{(3)}_{N_1+N_2} \tag{4.135}$$

[12]If we consider gluing of vertices, we need to distinguish ϵ-parameters and h-parameters. The ϵ-parameters are determined by Ψ while the h-parameters are associated to each vertex and they are related to ϵ_j by $SL(2,\mathbb{Z})$-transformation which brings the vertex to the standard one. [2]

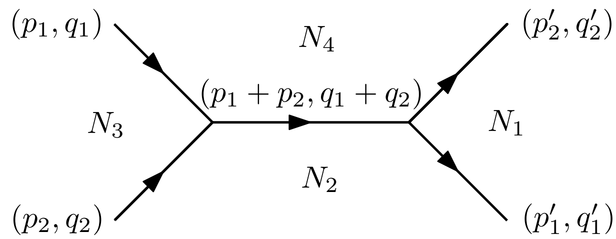


Figure 4.12: Gluing of two vertices.

of free bosons in the right vertex and

$$\tilde{\mathcal{L}}_1^{(3)} \cdots \tilde{\mathcal{L}}_{N_2}^{(3)} \tilde{\mathcal{L}}_{N_2+1}^{(1)} \cdots \tilde{\mathcal{L}}_{N_2+N_3}^{(1)} \tag{4.136}$$

in the left vertex. The generalization for $N_4 \neq 0$, a general ordering and 'non-simple' realizations is straightforward but the formulas become more involved.

The gluing fields are the generated from the fundamental and the anti-fundamental representation associated to lines supported at the internal interface generated by

$$M_{\square} = M^3_{\square} \otimes \tilde{M}^3_{\bar{\square}}, \qquad M_{\bar{\square}} = M^3_{\bar{\square}} \otimes \tilde{M}^3_{\square} \tag{4.137}$$

where $M^3_{\square}$ and $M^3_{\bar{\square}}$ are the primaries associated to the fundamental and the anti-fundamental module associated to the third direction of the right vertex and $\tilde{M}^3_{\square}$ and $\tilde{M}^3_{\bar{\square}}$ associated to the left vertex. The simple free field realizations in the given ordering are of the form

$$\begin{aligned} M^3_{\square} &= \exp\left[h_1\ \phi_1^{(2)}\right] \qquad & M^3_{\bar{\square}} &= f(J) \exp\left[-h_1\ \phi_{N_2}^{(2)}\right] \\ \tilde{M}^3_{\square} &= \exp\left[-\tilde{h}_2\ \tilde{\phi}_1^{(3)}\right] \qquad & \tilde{M}^3_{\bar{\square}} &= f(\tilde{J}) \exp\left[\tilde{h}_2\ \tilde{\phi}_{N_2}^{(3)}\right] \end{aligned} \tag{4.138}$$

where $f(J)$ is a level N_1 and $f(\tilde{J})$ is a level N_3 field of the free boson. Even though we lack a closed form expression for $f(J)$ and $f(\tilde{J})$, they can be easily determined from the requirement that $M^3_{\bar{\square}}$ and $\tilde{M}^3_{\bar{\square}}$ are primary fields of correct W-charges. All the other bi-fundamental fields can be constructed from the fusion of $M_{\square}$ and $M_{\bar{\square}}$.

In configurations with more internal finite interfaces, one can introduce corresponding fundamental and anti-fundamental representations associated to each finite segment and extend the tensor product of Y-algebras by fusion of all such generators. We will illustrate the gluing procedure on two examples bellow.

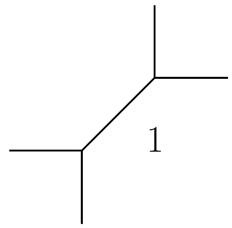


Figure 4.13: The simplest example of gluing of two $\widehat{\mathfrak{gl}}(1)$ Kac-Moody algebras.

4.6.1 Gluing two $\widehat{\mathfrak{gl}(1)}$'s

Let us consider the first example of gluing of two $\widehat{\mathfrak{gl}(1)}_\Psi$ Kac-Moody algebras as shown in the figure 4.13. Let $\phi \equiv \phi_1^{(2)}$ be the free boson associated to the right vertex and $\tilde{\phi} \equiv \tilde{\phi}_1^{(1)}$ be the one associated to the second one. We normalize them such that $J = \partial\phi_1^{(2)}$ and $\tilde{J} = \partial\tilde{\phi}_1^{(2)}$ have the following OPE

$$J(z)J(w) \sim -\frac{1}{\epsilon_1\epsilon_3}\frac{1}{(z-w)^2}, \qquad \tilde{J}(z)\tilde{J}(w) \sim -\frac{1}{\epsilon_2\epsilon_3}\frac{1}{(z-w)^2}. \tag{4.139}$$

Generators that need to be added to the algebra can be identified with the fusion of the following vertex operators realizing the fundamental and anti-fundamental representation

$$M_\square = \exp\left[\epsilon_1\phi - \epsilon_2\tilde{\phi}\right], \qquad M_{\bar{\square}} = \exp\left[-\epsilon_1\phi + \epsilon_2\tilde{\phi}\right]. \tag{4.140}$$

One can easily check that the two generators have correct charges with respect to the two $\mathfrak{gl}(1)$ subalgebras and that the conformal weight with respect to the sum of the two stress-energy tensors is $1/2$. Moreover, the free field realization gives also an explicit realization of the OPE between the added fields $M_\square$ and $M_{\bar{\square}}$ that has the following simple form

$$M_\square(z)M_{\bar{\square}}(w) \sim \frac{1}{z-w} \tag{4.141}$$

with all the other OPEs trivial. The exponent was determined from the product of the two exponents (with the metric determined by the normalization of the free bosons)

$$-\frac{(-\epsilon_1)\epsilon_1}{\epsilon_1\epsilon_3} - \frac{\epsilon_2(-\epsilon_2)}{\epsilon_2\epsilon_3} = -1. \tag{4.142}$$

One can immediately see that the BRST definition of the algebra is reproduced. In particular, the added fields $M_\square$ and $M_{\bar{\square}}$ form the free fermion pair and the combination $J+\tilde{J}$ can be identified with the decoupled $\widehat{\mathfrak{gl}(1)}$ Kac-Moody algebra. The relation between free fermions and the vertex operators $M_\square$, $M_{\bar{\square}}$ is the well known bosonization.

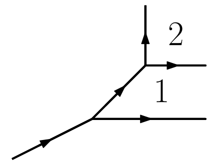


Figure 4.14: The web diagram associated to the $\widehat{\mathfrak{gl}}(2)$ Kac-Moody algebra.

4.6.2 $\widehat{\mathfrak{gl}(2)}$ from gluing

Let us now discuss the structure of glued generic modules for the algebra $\widehat{\mathfrak{gl}}(2)$ associated to the figure 4.14. This example will serve as a prototype for a more general configuration whose GW-defects give rise to VOA modules induced from the Gelfand-Tsetlin modules of the zero-modes algebra.

$Y_{0,1,2}$ **vertex** First, let us construct the free field realization of the algebra $Y_{0,1,2}$. The algebra has a free field realization in terms of three free bosons normalized as

$$\begin{aligned} J_1^{(2)}(z)J_1^{(2)}(w) &\sim -\frac{1}{\epsilon_1\epsilon_3}\frac{1}{(z-w)^2}, \\ J_2^{(3)}(z)J_2^{(3)}(w) &\sim -\frac{1}{\epsilon_1\epsilon_2}\frac{1}{(z-w)^2}, \\ J_3^{(3)}(z)J_3^{(3)}(w) &\sim -\frac{1}{\epsilon_1\epsilon_2}\frac{1}{(z-w)^2}. \end{aligned} \tag{4.143}$$

The generators of the algebra $Y_{0,1,2}$ were already found previously. For the purpose of our discussion, let us recall the $\widehat{\mathfrak{gl}(1)}$ field

$$J = J_1 + J_2 + J_3. \tag{4.144}$$

Let us now discuss the free field realization of the fundamental and the anti-fundamental module in the third direction that will play the role of J^+ and J^- generators after tensoring with the corresponding modules of the other vertex. The fundamental field can be realized as

$$M_\Box = \exp\left[\epsilon_1\phi_1^{(2)}\right]. \tag{4.145}$$

Note that ϵ_1 is precisely the charge predicted by the generating function of the ψ-charges and all the W-charges of the representation match. The anti-fundamental field is more complicated since it appears at level two (there are two of free bosons of the third type to the left of $\phi_3^{(3)}$). One finds the following expression for the fundamental field

$$M_{\bar{\square}} = \left(-\frac{\epsilon_1\epsilon_2}{\epsilon_3}J_3J_2 + \epsilon_1 J_1(J_2+J_3) - \frac{\epsilon_1\epsilon_3}{\epsilon_2}J_1J_1 - \partial J_2 + \frac{\epsilon_3}{\epsilon_2}\partial J_1\right)\exp\left[-\epsilon_1\phi_1^{(2)}\right]. \quad (4.146)$$

Note that this asymmetric form of the fundamental and the anti-fundamental field is related to our asymmetric choice of the free-boson ordering. The symmetric choice would to lead to both J^+ and J^- at level one. We expect the two choices to correspond to the symmetric and the asymmetric Wakimoto realizations. The symmetric Wakimoto realization is a free field realization of $\widehat{\mathfrak{gl}(2)}$ in terms of two free bosons and parafermionic fields. Parafermionic fields can be bosonized and we expect to find our symmetric free field realization. Similarly, one can bosonize the β,γ system of the Wakimoto realization in terms of two free bosons and a β,γ system and we expect to recover our non-symmetric free field realization. Detailed discussion of the relation with Wakimoto realization is beyond the scope of this paper.

$Y_{0,0,1}$ **vertex** Let us normalize the free boson $\tilde{J} = \partial\tilde{\phi}_1^{(2)}$ of the second vertex as

$$\tilde{J}(z)\tilde{J}(w) \sim \frac{1}{\epsilon_1\epsilon_3}\frac{1}{(z-w)^2}. \quad (4.147)$$

The fundamental and the anti-fundamental representations associated to the second direction are then

$$\tilde{M}_{\square} = \exp\left[\epsilon_1\phi(z)\right], \qquad \tilde{M}_{\bar{\square}} = \exp\left[-\epsilon_1\phi(z)\right]. \quad (4.148)$$

Glued algebra Having identified the fields and the relevant fundamental and the anti-fundamental representation of each vertex, one can now easily construct the glued VOA. The Cartan elements of the $\widehat{\mathfrak{gl}}(2)$ Kac-Moody algebra can be fixed by requiring the correct OPE between them and with the fields $J_{12} \propto M_{\bar{\square}}$ and $J_{21} \propto M_{\square}$. One finds

$$J_{11} = \epsilon_3\tilde{J}, \qquad J_{22} = -\frac{\epsilon_2}{\epsilon_3}J + \epsilon_1\tilde{J}. \quad (4.149)$$

The normalization of generators J_{12} and J_{21} can be found from their OPE. One finds

$$J_{12} = \frac{\epsilon_2\epsilon_3}{\epsilon_1}M_{\bar{\square}}, \qquad J_{21} = M_{\square}.$$

Note that the OPE of the exponential factors is trivial and both the second order and the first order pole come from the OPE of the J_i fields with the exponential factor of the anti-fundamental field. All the OPEs of $\widehat{\mathfrak{gl}(2)}$ Kac-Moody algebra are reproduced.

5. Some advanced topics

5.1 Dualities from S-duality

Let us add few comments on dualities mentioned many times throughout the text. The (p,q) webs enjoy the action of $SL(2,\mathbb{Z})$ duality transformations. Many calculations in the geometric picture based on the affine Yangian and free-field realizations are manifestly triality covariant. The $SL(2,\mathbb{Z})$ action in this picture corresponds simply to relabeling of complex coordinates parametrizing the toric three-fold and relabeling cycles of the torus fibration.

On the other hand, in the picture based on interfaces in $\mathcal{N}=4$ SYM, we can see that $SL(2,\mathbb{Z})$ transformation acts non-trivially on the boundary conditions at hand. In some cases, starting with a configuration with known BRST definition, there might exist a transformation that produces another configuration with a known BRST definition. One then expects that both BRST reductions lead to an equivalent VOA. Using S-dualities, we can thus predict many new dual constructions[1] of (p,q)-web VOAs.

For example, in the $Y_{N_1,N_2,N_3}[\Psi]$ case, there exists an S_3 subalgebra of $SL(2,\mathbb{Z})$ that preserves the (p,q) web and simply permutes its legs. This action descends to dualities of the corresponding VOA. Our definition is manifestly symmetric under the reflection $\Psi \leftrightarrow 1-\Psi$ accompanied by the exchange $N_3 \leftrightarrow N_2$. A non-trivial conjecture comes from the “S-duality” transformation $\Psi \leftrightarrow \Psi^{-1}$ accompanied by the exchange $N_1 \leftrightarrow N_2$. The two transformations combine into an S_3 triality symmetry which acts by permuting the three integral labels N_1,N_2,N_3 while acting on the coupling Ψ by appropriate $PSL(2,Z)$ duality transformations. In particular, we have cyclic rotations:

$$Y_{N_1,N_2,N_3}[\Psi] = Y_{N_3,N_1,N_2}[\frac{1}{1-\Psi}] = Y_{N_2,N_3,N_1}[1-\frac{1}{\Psi}] \qquad (5.1)$$

[1] This duality action played an important role in testing the proposal for the BRST definition of Y_{N_1,N_2,N_3} in [1].

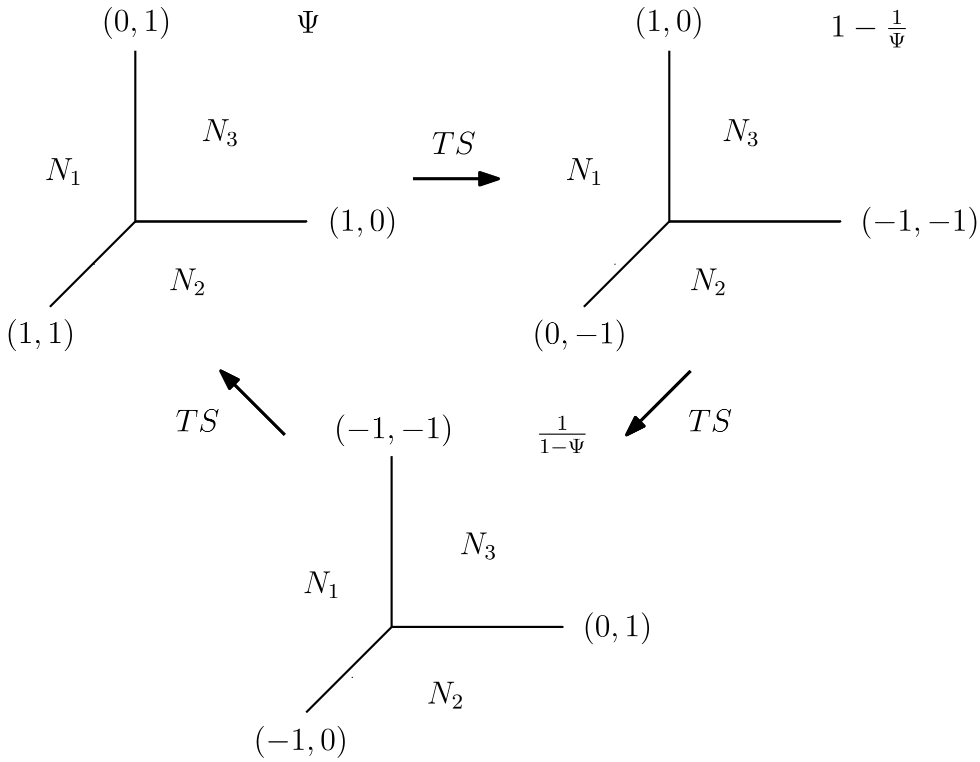


Figure 5.1: The dualities which motivate the identification (5.1) of the VOA $Y_{N_1,N_2,N_3}[\Psi]$, $Y_{N_3,N_1,N_2}[\frac{1}{1-\Psi}]$ and $Y_{N_2,N_3,N_1}[1-\frac{1}{\Psi}]$.

An alternative, instructive way to describe the S_3 symmetry is to use the parameters ϵ_i which satisfy

$$\epsilon_1+\epsilon_2+\epsilon_3=0 \qquad\qquad \Psi=-\frac{\epsilon_2}{\epsilon_1} \tag{5.2}$$

Then the S_3 symmetry acts on

$$Y^{\epsilon_1,\epsilon_2,\epsilon_3}_{N_1,N_2,N_3}\equiv Y_{N_1,N_2,N_3}[-\frac{\epsilon_2}{\epsilon_1}] \tag{5.3}$$

by a simultaneous permutation of the ϵ_i and N_i labels.

Finally, note that in terms of the parameters λ_i of the $\mathcal{W}_{1+\infty}$ algebra, the triality transformation acts by permutation that is consistent with the triality action of [75, 5]

We can illustrate this type of relations for $Y_{0,0,N}$. The $Y_{0,0,N}[\Psi]$ VOA is defined as the regular quantum Drinfeld-Sokolov reduction of $\widehat{\mathfrak{gl}(N)}_\Psi$ and thus coincides with the standard

W-algebra $\mathcal{W}_N[\Psi] \times \widehat{\mathfrak{gl}(1)}$ with parameter $b^2 = -\Psi$[2] The W_N algebra has a symmetry $b \to b^{-1}$ known as Feigin-Frenkel duality, demonstrating immediately the expected S-duality relation between $Y_{0,0,N}[\Psi]$ and $Y_{0,0,N}[\Psi^{-1}]$.

On the other hand, our definition of $Y_{N,0,0}[1-\Psi^{-1}]$ involves a BRST reduction of a product of elementary VOAs

$$\widehat{\mathfrak{gl}(N)}_{-\frac{1}{1-\Psi}} \times \widehat{\mathfrak{gl}(N)}_{\frac{\Psi}{1-\Psi}} \times \mathcal{S}^{0|N} \times gh^{(coset)}, \tag{5.4}$$

where $\mathcal{S}^{0|N}$ denotes the VOA of N complex free fermions transforming in a fundamental representation of $\mathfrak{gl}(N)$ and $gl^{(coset)}$ a bc ghost system valued in the $\mathfrak{gl}(N)$ Lie algebra.

The BRST complex is essentially a symmetric description of a coset construction, which is essentially the analytic continuation of the well-known coset definition of the $\mathcal{W}_N$ algebra. See e.g. [115] for a review and further references on this "triality" enjoyed by $\mathcal{W}_N$ algebras.

The general case leads to many new dualities for VOAs Y_{N_1,N_2,N_3}.

Similarly, the resolved conifold diagram preserves the $\mathbb{Z}_2 \times \mathbb{Z}_2$ subgroup leading to the following duality action

$$\begin{aligned} \mathcal{W}^{1|1}_{K,\bar{L},M,\bar{N}}[\Psi] \quad &\leftrightarrow \quad \mathcal{W}^{1|1}_{K,\bar{M},L,\bar{N}}\left[\frac{1}{\Psi}\right] \\ \mathcal{W}^{1|1}_{K,\bar{L},M,\bar{N}}[\Psi] \quad &\leftrightarrow \quad \mathcal{W}^{1|1}_{N,\bar{L},M,\bar{K}}\left[\frac{1}{\Psi}\right]. \end{aligned} \tag{5.5}$$

In the case where both left and right algebras have a known BRST definition, we are led to many new dualities. Similarly, we expect that the triality of $\mathcal{W}_{1+\infty}$ becomes $\mathbb{Z}_2 \times \mathbb{Z}_2$ duality for $\mathcal{W}^{\rho}_{1|1\times\infty}$ generalizing the story of [77] for $\rho = \frac{1}{2}$.

Note also that for $\rho = 0$ the algebra is conjectured to be independent of the resolution of the X-diagram (an issue discussed in the next section). One can then consider the unresolved diagram and see that its symmetry is enhanced to S_4 that permutes the numbers K, L, M, N. We expect these algebras to be symmetric under such an enhanced symmetry.

Moreover, we expect that for general ρ, the S_4 symmetry is broken to $\mathbb{Z}_2 \times \mathbb{Z}_2$ only in a very mild way. It turns out that different resolutions lead to the same algebra up to the contribution of extra b, c ghosts (see the next section). We can then expect that the S_4 duality will indeed be present for general ρ but only up to such contributions (we say that the S_4 duality holds up to the spable equivalence [81]). Moreover, note the assymetry

[2]Recall our choice of notation $\widehat{\mathfrak{gl}(N)}_\Psi$ in terms of $\widehat{\mathfrak{sl}(N)}_{\Psi-N}$ and the $\widehat{\mathfrak{gl}(1)}$ current

in the definition of the type $1|1$ algebra originating from the two successive DS-reductions with the two orders associated to the two possible resolutions. It is natural to expect that the symmetric situation (a single BRST reduction with respect to the $\mathfrak{sl}(2)$ embedding that is a sum of the two $\mathfrak{sl}(2)$ embeddings) leads to an algebra naturally associated to the unresolved diagram. Such a tetravalent junction preserves the full S_4 symmetry and one expects this symmetry to descent to the corresponding VOA. It is easy to check that the central charge of the algebras S_4-invariant.

5.2 Stable equivalence and flips

In this section, we comment on the flip transition that plays an important role in the literature related to the BPS counting. At the level of diagrams the flip transition corresponds to the sliding of fivebranes. We conjecture that the algebras associated to diagrams related by a flip transformation differ by a trivial algebra of decoupled bc and $\beta\gamma$ ghosts. These flip transitions and independence of the algebra on the resolution of the diagram leads to the notion of the stable equivalence of VOAs [81].

As a test of this conjecture we argue that central charges of the algebras related by a flip differ only by a contribution of bc and $\beta\gamma$ ghosts and the large N_i limit calculation of the character is consistent as well. Many other checks have been performed in [2] and some situations have been rigorously proved [116, 81].

5.2.1 Flip of algebras of type $1|1$

Let us start with a flip in the resolved conifold diagram. This transition exchanges the order of the two D5-branes as shown in the figure 5.2. The related algebras are

$$\mathcal{W}^{1|1}_{K\bar{L},M,\bar{N}}[\Psi] \quad \leftrightarrow \quad \mathcal{W}^{1|1}_{L,\bar{K},N,\bar{M}}[-\Psi] \tag{5.6}$$

where the minus sign is a consequence of the parity transformation relating the right hand side of the figure 5.2 to the standard resolved configuration.

The central charges of the flipped algebras differ by a Ψ-independent factor

$$c\left[\mathcal{W}^{1|1}_{K\bar{L},M,\bar{N}}[\Psi]\right] - c\left[\mathcal{W}^{1|1}_{L,\bar{K},N,\bar{M}}[-\Psi]\right] = 4\rho(2\rho^2-1). \tag{5.7}$$

This is exactly the contribution coming from free (b,c) systems with conformal dimensions

$$(\rho+1,-\rho),(\rho+2,-\rho),\ldots,(-\rho,\rho+1) \tag{5.8}$$

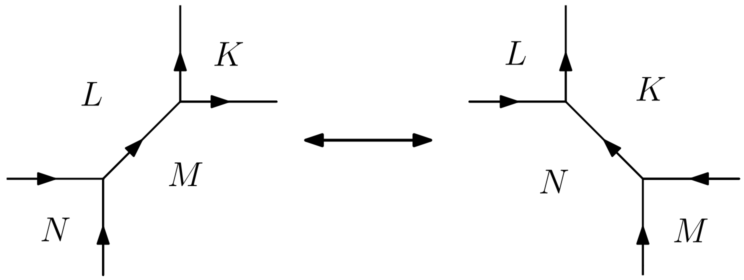


Figure 5.2: Flip transition for type 1|1 (resolved conifold) algebras.

since (for $\rho < 0$)

$$\sum_{h=\rho+1}^{-\rho} c[h] = 4\rho(2\rho^2 - 1) \tag{5.9}$$

where $c[h]$ is the central charge of the stress-energy tensor of the b, c ghost system with respect to which $h_c = h$ and $h_b = 1 - h$ are the conformal weights of the c, b fields.

At the level of characters, the difference is by a factor of (for ρ negative)

$$\prod_{m=\rho+\frac{1}{2}}^{-\rho+\frac{1}{2}} \chi_m^{\mathcal{F}} = \prod_{n=0}^{\infty} \prod_{m=\rho+1}^{-\rho+1} \left(1 + q^{n+m}\right) \tag{5.10}$$

as can be most easily seen from the BRST construction. The only difference at the level of the BRST reduction is in the two off-diagonal blocks whose elements are charged under the Cartan elements of both $\mathfrak{sl}(2)$ embeddings. The contributions from the $\mathcal{W}_{K-M}$ and $\mathcal{W}_{L-N}$ factors are present in the characters of both algebras. The integral projecting on the $\mathfrak{gl}(N|M)$ invariant combinations is the same as well since all the fields under the integral originate from the off-diagonal blocks charged with respect to only one of the two sl_2 embeddings. The only difference is thus in the product of $\chi_i^{\mathcal{F}}$ factors given above.

Example - Flip of $\widehat{\mathfrak{gl}(1)}_\Psi$ The flip of the $\widehat{\mathfrak{gl}(1)}_\Psi$ algebra is the simplest but also trivial example of a flip transition since the algebra is simply

$$\mathcal{DS}_0[\widehat{\mathfrak{gl}(1)}_\Psi] = \widehat{\mathfrak{gl}(1)}_\Psi \times \mathcal{F} \tag{5.11}$$

from the definition. It automatically contains a decoupled fermion $\mathcal{F}$.

Example - Flip of Virasoro $\times\widehat{\mathfrak{gl}(1)}_\Psi$ A non-trivial example is the flip of the $\mathcal{W}^{1|1}_{2,\bar{0},0,\bar{0}}[\Psi]$ algebra, i.e. $\mathcal{W}^{1|1}_{0,\bar{2},0,\bar{0}}[-\Psi]$. The BRST definition of the algebra is in terms of a reduction of

$$\widehat{\mathfrak{gl}(2)}_\Psi \times \{\chi_1, \psi_1\} \times \{\chi_2, \psi_2\} \times \{b, c\} \tag{5.12}$$

implemented by the BRST charge

$$Q = \oint dz (J_{12} + \chi_1 \psi_2 - 1) c \tag{5.13}$$

that can be identified with the BRST reduction associated to the principal $\mathfrak{sl}(2)$ embedding inside $\mathfrak{gl}(2)$ but with the current modified by a fermionic bilinear.

The cohomology is generated by fields

$$\begin{aligned}
J &= J_{11} + J_{22} \\
\tilde{\psi}_1 &= \psi_1 + cb\psi_2 + J_{11}\psi_2 \\
\tilde{\chi}_1 &= \chi_1 \\
\tilde{\psi}_2 &= \psi_2 \\
\tilde{\chi}_2 &= \chi_2 - cb\chi_1 + J_{11}\chi_1 \\
T &= \frac{1}{2\Psi}\left(J_{11}J_{11} + 2J_{12}J_{21} + J_{22}J_{22}\right) \\
&\quad - c\partial b - \psi_1 \partial\chi_1 + \partial\psi_2 \chi_2 + \frac{\Psi - 1}{2\Psi}(J'_{11} - J'_{22})
\end{aligned} \tag{5.14}$$

where T is the stress-energy tensor with the central charge

$$c = 10 - 6\Psi - \frac{6}{\Psi} \tag{5.15}$$

as expected. The fields $(\tilde{\psi}_1, \tilde{\chi}_1)$ and $(\tilde{\psi}_2, \tilde{\chi}_2)$ have OPEs of a pair of free fermions. Their OPEs with J are

$$J(z)\tilde{\psi}_1 \sim -\frac{\Psi\tilde{\psi}_2}{(z-w)^2}, \quad J(z)\tilde{\chi}_1 \sim \frac{\Psi\tilde{\chi}_2}{(z-w)^2}. \tag{5.16}$$

$\tilde{\chi}_1$ and $\tilde{\psi}_2$ are primaries of conformal dimension 0 while $\tilde{\chi}_2$ and $\tilde{\psi}_1$ having OPEs with the stress-energy tensor of the form

$$\begin{aligned}
T(z)\tilde{\chi}_2(w) &\sim \frac{(1-\Psi)\tilde{\chi}_1}{(z-w)^3} + \frac{\tilde{\chi}_2(w)}{(z-w)^2} + \frac{\partial\tilde{\chi}_2(w)}{z-w} \\
T(z)\tilde{\psi}_1(w) &\sim -\frac{(1-\Psi)\tilde{\psi}_2}{(z-w)^3} + \frac{\tilde{\psi}_1(w)}{(z-w)^2} + \frac{\partial\tilde{\psi}_1(w)}{z-w}
\end{aligned} \tag{5.17}$$

Note that one can modify both the J current and the stress-energy tensor T as

$$\begin{aligned} J &\rightarrow J + \Psi\partial\tilde{\chi}_1\tilde{\psi}_2 \\ T &\rightarrow T - \partial\tilde{\chi}_1\tilde{\psi}_1 - \partial\tilde{\psi}_2\tilde{\chi}_2 - \Psi\partial\tilde{\chi}_1\tilde{\chi}_1\tilde{\psi}_2\partial\tilde{\psi}_2 + \frac{\Psi-1}{2}\partial^2(\tilde{\chi}_1\tilde{\psi}_2). \end{aligned} \tag{5.18}$$

After such a modification, the free fields decouple and one is left with the Virasoro algebra times a $U(1)$ algebra. The central charge of the modified stress-energy tensor

$$14 - 6\Psi - \frac{6}{\Psi} \tag{5.19}$$

is the same as the central charge of the algebra before the flip.

5.2.2 Flip of algebras of type $0|2$

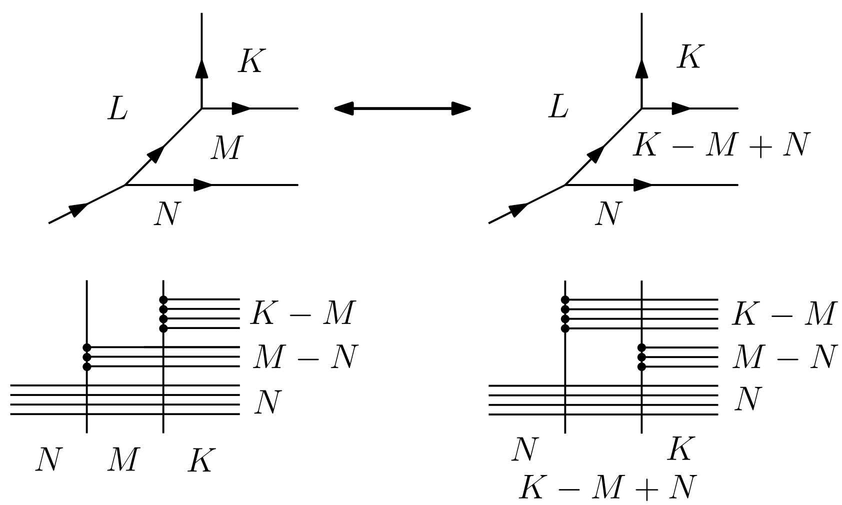


Figure 5.3: Flip transition for type $0|2$ algebras in the case of $K \geq M \geq N$. A fixed number of D3-branes is attached to each D5-brane. The crossing of D5-branes acts non-trivially on the number of D3-branes at the internal face.

Flipping D5-branes in the $0|2$ diagram results in the change of numbers of D3-branes between the two fivebranes as shown in the figure 5.3. The algebras related by such a flip are

$$\mathcal{W}^{0|2}_{K,\bar{L},M,N}[\Psi] \;\leftrightarrow\; \mathcal{W}^{0|2}_{K,\bar{L},K-M+N,N}[\Psi]. \tag{5.20}$$

One can show analogously to the resolved conifold diagram that the central charges and characters again differ by a contribution of $2|\rho|$ copies of the (β,γ) systems with correct conformal dimensions. The only difference in this case is the different expression for the parameter ρ and the bosonic nature of the decoupled fields.

Example $\mathcal{W}^{0|2}_{0,\bar{0},2,3}$ Let us show that the algebra $\mathcal{W}^{0|2}_{0,\bar{0},2,3}$ associated to the flip of the $\mathcal{W}^{(2)}_3 \times \widehat{\mathfrak{gl}(1)}$ algebra contains $\mathcal{W}^{(2)}_3 \times U(1)$ as a subalgebra together with a decoupled free fermion.

The algebra $\mathcal{W}^{0|2}_{0,\bar{0},2,3}$ is defined as a BRST reduction of $\widehat{\mathfrak{gl}(3)}_\Psi \times \{b_{12}, c_{12}\}$ by the BRST charge

$$Q_2 = \oint dz(J_{12}-1)c_{12}. \tag{5.21}$$

The cohomology contains the currents

$$\begin{aligned} J_1 &= \frac{J_{11}+J_{22}}{\Psi} - \frac{1-\Psi}{\Psi}J_{33}, \\ J_2 &= J_{11}+J_{22} \end{aligned} \tag{5.22}$$

that are mutually local and they are normalized according to (4.5) as

$$J_1(z)J_1(w) \sim \frac{\Psi+\frac{3}{\Psi}-4}{(z-w)^2}, \qquad J_1(z)J_1(w) \sim \frac{2(\Psi-1)}{(z-w)^2}. \tag{5.23}$$

Apart from these currents, the reduced algebra contains generators of dimension $\frac{1}{2}$ given by

$$\begin{aligned} G_1^+ &= J_{13}, \\ G_1^- &= J_{32} \end{aligned} \tag{5.24}$$

and of dimension $\frac{3}{2}$ of the form

$$\begin{aligned} G_2^+ &= J_{23} + (\Psi-2)J_{11}J_{13} + (\Psi-3)J_{13}J_{22} + J_{13}bc, \\ G_2^- &= J_{31} + (\Psi-3)J_{11}J_{32} + (\Psi-2)J_{22}J_{32} - J_{32}bc \end{aligned} \tag{5.25}$$

together with the stress-energy tensor

$$T = \frac{1}{2\Psi}\sum_{i,j=1,2,3} J_{ij}J_{ji} + \frac{1}{2}\partial J_{11} - \frac{1}{2}\partial J_{22} + \partial b_{12}c_{12} \tag{5.26}$$

with the central charge

$$c = 25 - \frac{24}{\Psi} - 6\Psi. \tag{5.27}$$

The superscripts $\pm$ in the expressions above denote the charge of the gluing fields with respect to J_1 and J_2 currents.

Let us discuss OPEs of the algebra. $G_1^\pm$ form subalgebra of symplectic bosons with OPE

$$G_1^+(z)G_1^-(w) \sim \frac{1}{z-w}. \tag{5.28}$$

The operator product expansions between J_i currents and $G_2^\pm$ fields are

$$J_1(z)G_2^\pm(w) \sim \pm\frac{G_2^\pm}{z-w}, \qquad J_2(z)G_2^\pm(w) \sim \frac{(\Psi-1)(2\Psi-5)G_1^\pm}{(z-w)^2} \pm \frac{G_2^\pm}{z-w}, \tag{5.29}$$

OPEs between $G_1^\pm$ and $G_2^\pm$ fields are

$$\begin{aligned} G_1^\pm(z)G_2^\pm(w) &\sim \pm\frac{(2-\Psi)G_1^\pm G_1^\pm}{z-w} \\ G_1^\pm(z)G_2^\mp(w) &\sim \pm\frac{1}{z-w}\left((3-\Psi)G_1^-G_1^+ + \frac{\Psi}{\Psi-1}J_1 + \frac{1-3\Psi+\Psi^2}{\Psi-1}J_2\right). \end{aligned} \tag{5.30}$$

Finally for OPEs between $G_2^\pm$ fields we find

$$\begin{aligned} G_2^\pm(z)G_2^\pm(w) &\sim \pm\left(2\Psi^3-13\Psi^2+27\Psi-17\right)\left(\frac{G_1^\pm G_1^\pm}{(z-w)^2} + \frac{G_1^\pm\partial G_1^\pm}{z-w}\right) \\ G_2^+(z)G_2^-(w) &\sim \frac{(\Psi-1)(\Psi-3)(2\Psi-5)}{(z-w)^3} \\ &+ \frac{1}{(z-w)^2}\left(\frac{2(2\Psi-\Psi^2)}{\Psi-1}J_1 + \frac{(\Psi-3)(\Psi-2)}{\Psi-1}J_2 + (2\Psi^3-11\Psi^2+16\Psi-3)G_1^+G_1^-\right) \\ &+ \frac{1}{z-w}\Bigg(\Psi T + 2(2\Psi-5)J_2G_1^+G_1^- - \frac{\Psi^2}{2(\Psi-1)^2}J_1J_1 + \frac{\Psi(6-7\Psi+2\Psi^2)}{(\Psi-1)^2}J_1J_2 \\ &+ \frac{(2\Psi^4-12\Psi^3+21\Psi^2-8\Psi-4)}{2(\Psi-1)^2}J_2J_2 + (1-\Psi)G_1^+G_2^- \\ &+ 2(\Psi^3-6\Psi^2+10\Psi-3)\partial G_1^+G_1^- + (\Psi^2-4\Psi+3)G_1^+\partial G_1^- \\ &+ \frac{2\Psi-\Psi^2}{\Psi-1}\partial J_1 - \frac{\Psi^2-5\Psi+6}{2(\Psi-1)}\partial J_2\Bigg). \end{aligned} \tag{5.31}$$

If we redefine the generators of the algebra as

$$\begin{aligned} J_1 &\to J_1 + G_1^+ G_1^- \\ J_2 &\to J_2 + G_1^+ G_1^- \\ G_2^+ &\to G_1^+ - (G_1^+)^2 G_1^- + \frac{\Psi J_1 G_1^+}{1-\Psi} + \frac{(1-3\Psi+\Psi^2) J_2 G_1^+}{1-\Psi} + (\Psi-1)\partial G_1^+ \\ G_2^- &\to G_1^- - G_1^+ (G_1^-)^2 + \frac{\Psi J_1 G_1^-}{1-\Psi} + \frac{(1-3\Psi+\Psi^2) J_2 G_1^-}{1-\Psi} - (\Psi-1)\partial G_1^- \\ T &\to T - \frac{1}{2}\partial(G_1^+ G_1^-) \end{aligned} \tag{5.32}$$

we discover that the currents $G_1^\pm$ form a symplectic boson pair and they decouple. The remaining algebra can be identified with the algebra $\mathcal{W}_3^{(2)} \times \widehat{\mathfrak{gl}(1)}$ with the stress-energy tensor of the correct central charge

$$c = 26 - \frac{24}{\Psi} - 6\Psi. \tag{5.33}$$

5.2.3 Flip in a general diagram

In a general tree diagram, the flip is a local transition that influences only the vertices associated to the flipped leg and the edge along which the leg flipped. In particular, this means that both the vacuum character and the central charge of the full algebra differ by a contribution of free fields that can presumably be decoupled and the change in these quantities can be read-off locally.

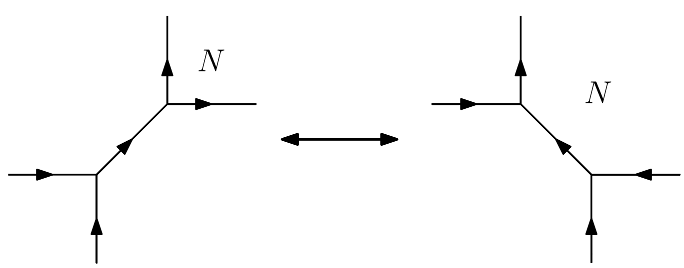


Figure 5.4: Flip leading to the stable equivalence $\mathcal{DS}_N[\widehat{\mathfrak{gl}(N)}_\Psi \times \mathcal{F}^{\mathfrak{gl}(N)}] \simeq \mathcal{DS}_N[\widehat{\mathfrak{gl}(N)}_\Psi]$.

Note also that both the central charge and the vacuum character remain the same in the case of vanishing $\rho = 0$. At the level of BRST reduction, one can indeed see that the two reductions can be related by a unitary transformation of the current algebra generators. It is natural to expect that the two algebras related by a flip are equal in arbitrary diagram as long as $\rho = 0$.

The equivalence of VOAs up to the contribution of free fields has been called by Creutzig the stable equivalence [81] and will be denoted as $\simeq$. For example, considering the configurations of type $M|N$, the brane transitions lead to the conjecture of stable equivalence of algebras related by a permutation of the $\mathcal{DS}$ reductions in their BRST definition, schematically

$$\frac{\mathcal{DS}[\ldots\mathcal{DS}_n[\mathcal{DS}_m[\ldots\mathcal{DS}[\widehat{\mathfrak{gl}(K|L)}]\ldots]}{\widehat{\mathfrak{gl}(M|N)}} \simeq \frac{\mathcal{DS}[\ldots\mathcal{DS}_m[\mathcal{DS}_n[\ldots\mathcal{DS}[\widehat{\mathfrak{gl}(K|L)}]\ldots]}{\widehat{\mathfrak{gl}(M|N)}}. \quad (5.34)$$

In particular considering the flip from the figure 5.4, one recovers the relation from [116, 81]:

$$\mathcal{DS}_N[\mathfrak{gl}(N)\times\mathcal{F}^{\mathfrak{gl}(N)}] \simeq \mathcal{DS}_N[\mathfrak{gl}(N)] \quad (5.35)$$

that generalizes to all the other Drinfeld-Sokolov reductions $\mathcal{DS}_\rho$ by modifying the boundary condition on the right corresponding to a more general diagram of type $1|M$.

5.3 Ortho-symplectic algebras

5.3.1 Branes and O3-planes

In this section, we describe an analogue of the above construction to a Y-junction of defects in $\mathcal{N} = 4$ SYM with orthogonal and symplectic gauge groups. Theories with these gauge groups can be realized by $D3$-branes sitting on an O3-plane. The gauge theory perspective on boundary conditions and interfaces associated to fivebranes in the presence of O3-planes was developed in [63], building on a broad literature in string theory [117, 118, 119] and gauge theory [120, 121, 122].

There are four $O3$-planes in type IIB string theory. When superimposed to a stack of D3 brane, they give rise to four possible choices of gauge groups: $O3^-$ planes give an $SO(2n)$ gauge theory, $\tilde{O}3^-$ planes give an $SO(2n+1)$ gauge theory, $O3^+$ planes give an $Sp(2n)$ gauge theory and $\tilde{O}3^+$ planes give a gauge theory denoted as $Sp(2n)'$, which is the same as $Sp(2n)$ but has a different convention for the θ angle, so that $\theta = 0$ in $Sp(2n)'$ is the same as $\theta = \pi$ in $Sp(2n)$.

The $O3^-$ plane is unaffected by duality transformations. Correspondingly, $SO(2n)$ $\mathcal{N}=4$ SYM has a $PSL(2,Z)$ S-duality group. The remaining three types of $O3$ planes are exchanged by duality transformations. A T transformation clearly maps $Sp(2n) \leftrightarrow Sp(2n)'$ and relates $O3^+$ and $\tilde{O}3^+$. It leaves $\tilde{O}3^-$ invariant. On the other hand, an S transformation exchanges the $Sp(2n)$ and $SO(2n+1)$ gauge groups and the $\tilde{O}3^-$ and $O3^+$ planes, while it maps $Sp(2n)'$ to itself and leaves $\tilde{O}3^+$ invariant.

The story is further complicated by the fact that the elementary interfaces in the presence of O3 planes are associated to "half-fivebranes" that are Z_2 projections of ordinary fivebranes. The type of O3 planes jumps across these interfaces. As a consequence, half-NS5 interfaces must interpolate between $SO(2n)$ and $Sp(2m)$ or between $SO(2n+1)$ and $Sp(2m)'$:

$$\begin{array}{c|c} O3^- & O3^+ \\ SO(2n) & Sp(2m) \\ \multicolumn{2}{c}{NS5} \end{array} \qquad\qquad \begin{array}{c|c} \tilde{O}3^- & \tilde{O}3^+ \\ SO(2n+1) & Sp'(2m) \\ \multicolumn{2}{c}{NS5} \end{array}$$

while half-D5 interfaces must interpolate between $SO(2n)$ and $SO(2m+1)$ or $Sp(2n)$ and $Sp(2m)'$:

$$\begin{array}{c|c} O3^- & \tilde{O}3^- \\ SO(2n) & SO(2m+1) \\ \multicolumn{2}{c}{D5} \end{array} \qquad\qquad \begin{array}{c|c} O3^+ & \tilde{O}3^+ \\ Sp(2n) & Sp'(2m) \\ \multicolumn{2}{c}{D5} \end{array}$$

The gauge theory description of the interfaces is very similar to the unitary cases, except that the orbifold projection cuts in half the interface degrees of freedom. Half-NS5 interfaces support "half-hypermultiplets" transforming as bi-fundamentals of $SO \times Sp$. [3]

Half-D5 interfaces between orthogonal groups involve a Nahm pole of odd rank [4]. Half-

[3]Notice that half-hypermultiplets must transform in a symplectic representation, precluding such elementary interfaces for $SO \times SO$ or $Sp \times Sp$. Furthermore, half-hypermultiplets have a potential anomaly which has to be cancelled by inflow from the bulk, constraining the choice of $Sp(2n)$ vs $Sp(2n)'$ as predicted by string theory.

[4]Notice that the rank of the Nahm pole must be odd for the $\mathfrak{sl}(2)$ embedding to exist in an orthogonal group

D5 interfaces between symplectic groups involve a Nahm pole of even rank or a half-hypermultiplet in the fundamental representation of Sp. [5]

The half-$(1,1)$-type interfaces work in a similar manner as half-NS5 interfaces, except that the role of Sp' and Sp is reversed because of the extra interface Chern-Simons terms.

The relation between the four-dimensional gauge theory setup and analytically continued Chern-Simons theory works in the same manner as in the unitary case, up to matter of conventions for the levels of the corresponding Chern-Simons theories.

We use conventions where κ is the level of the $\widehat{\mathfrak{so}}$ currents and $-\kappa/2$ the level of the $\widehat{\mathfrak{sp}}$ currents. The dual Coxeter number for $\mathfrak{so}(n)$ is $n-2$ and for $\mathfrak{sp}(2m)$ is $m+1$. The critical level for $\widehat{\mathfrak{osp}(n|2m)}$ is $2-n+2m$. A half-NS5 interface in the presence of gauge theory parameter Ψ will result in an $\widehat{\mathfrak{osp}(n|2m)}_{\pm\Psi-n+2m+2}$ theory, depending on which side of the interface the SO and Sp or Sp' groups lie.

The relation between Nahm poles and DS reductions will be the same as before. Furthermore, half-hypermultiplets in the fundamental representation of $Sp(2m)$ will map to symplectic bosons which support $\widehat{\mathfrak{sp}(2m)}_{-\frac{1}{2}}$ currents. Adding n Majorana chiral fermions will promote that to $\widehat{\mathfrak{osp}(n|2m)}_1$ currents.

5.3.2 Definition of ortho-symplectic Y-algebras

Depending on the choice of O3 plane in the top right corner, the Y-junction setup for orthogonal and symplectic gauge groups gives rise to four classes of ortho-symplectic Y-algebras: $Y^{\pm}_{L,M,N}[\Psi]$ and $\tilde{Y}^{\pm}_{L,M,N}[\Psi]$.

Because of the duality properties of O3 planes, $\tilde{Y}^{+}_{L,M,N}[\Psi]$ will have the same triality properties as $Y_{L,M,N}[\Psi]$. Instead, triality will map into each other $Y^{\pm}_{L,M,N}[\Psi]$ and $\tilde{Y}^{-}_{L,M,N}[\Psi]$, up to the usual S_3 action on labels and coupling.

In particular, the definition of the algebras will imply

$$Y^{+}_{L,M,N}[\Psi] = Y^{+}_{L,N,M}[1-\Psi] \qquad Y^{-}_{L,M,N}[\Psi] = \tilde{Y}^{-}_{L,N,M}[1-\Psi] \tag{5.36}$$

and the non-trivial S-duality conjecture is

$$Y^{+}_{L,M,N}[\Psi] = \tilde{Y}^{-}_{M,L,N}[\frac{1}{\Psi}] \qquad Y^{-}_{L,M,N}[\Psi] = Y^{-}_{M,L,N}[\frac{1}{\Psi}] \tag{5.37}$$

[5] Notice that the rank of the Nahm pole must be even for the $\mathfrak{su}(2)$ embedding to exist in an orthogonal group. Also, the type of Sp theory must jump across the interface for the same anomaly inflow constraint mentioned in the previous footnote.

etcetera.

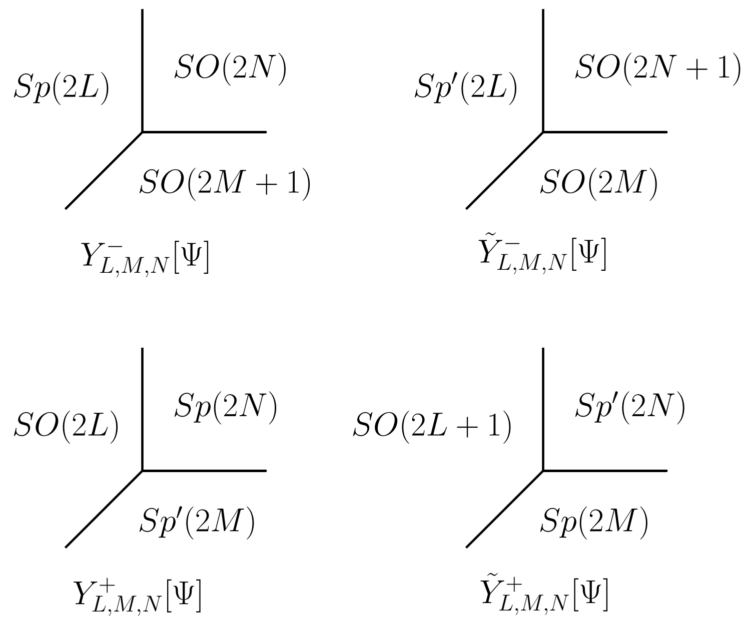


Figure 5.5: Configurations defining ortho-symplectic Y-algebras.

We will give now a brief definition of these vertex algebras. The VOAs $Y^-_{L,M,N}[\Psi]$ corresponding to the first figure in 5.5 are defined as follows. There are a super Chern-Simons theory with gauge groups $OSp(2N,2L)$ and $OSp(2M+1,2L)$ induced at the NS5 interfaces. For $L=0$, $N=M$ or $N=M+1$, there is no Nahm-pole present and corresponding Y-algebra is a BRST reduction of

$$\begin{aligned}
&\widehat{\mathfrak{so}(2M)}_{\Psi-2M+2}\times\widehat{\mathfrak{so}(2M+1)}_{-\Psi-2M+2}\\
&\widehat{\mathfrak{so}(2M+2)}_{\Psi-2M}\times\widehat{\mathfrak{so}(2M+1)}_{-\Psi-2M+2}
\end{aligned}\tag{5.38}$$

that lead to cosets

$$\begin{aligned}
Y^-_{0,M,M}[\Psi] &= \frac{\widehat{\mathfrak{so}(2M+1)}_{-\Psi-2M+2}}{\widehat{\mathfrak{so}(2M)}_{-\Psi-2M+2}}\\
Y^-_{0,M,M+1}[\Psi] &= \frac{\widehat{\mathfrak{so}(2M+2)}_{\Psi-2M}}{\widehat{\mathfrak{so}(2M+1)}_{\Psi-2M}}.
\end{aligned}\tag{5.39}$$

For $L=0$ and $N>M+1$, the VOA is defined as a BRST reduction of the DS-reduction by the $(2N-2M-1)\times(2N-2M-1)$ block

$$\mathcal{DS}_{2N-2M-1}[\widehat{\mathfrak{so}(2N)}_{\Psi-2N+2}]\times\widehat{\mathfrak{so}(2M+1)}_{-\Psi-2M+2} \tag{5.40}$$

i.e. coset

$$Y^{-}_{0,M,N}[\Psi]=\frac{\mathcal{DS}_{2N-2M-1}[\widehat{\mathfrak{so}(2N)}_{\Psi-2N+2}]}{\widehat{\mathfrak{so}(2M+1)}_{\Psi-2M}}. \tag{5.41}$$

and similary for $N<M$

$$Y^{-}_{0,M,N}[\Psi]=\frac{\mathcal{DS}_{2M+1-2N}[\widehat{\mathfrak{so}(2M+1)}_{\Psi-2M}]}{\widehat{\mathfrak{so}(2N)}_{\Psi-2N+2}}. \tag{5.42}$$

For $L\neq 0$, levels of the super Chern-Simons theories are $\Psi-2N+2L+2$ and $-\Psi-2M+2L$ respectively. In the four cases described above, one gets BRST reductions of similar combinations of DS-reduced and not reduced theory leading to

$$\begin{aligned}
Y^{-}_{L,M,M}[\Psi] &= \frac{\widehat{\mathfrak{osp}(2M+1|2L)}_{-\Psi-2M+2+2L}}{\widehat{\mathfrak{osp}(2M|2L)}_{-\Psi-2M+2+2L}},\\
Y^{-}_{L,M,M+1}[\Psi] &= \frac{\widehat{\mathfrak{osp}(2M+2|2L)}_{\Psi-2M+2L}}{\widehat{\mathfrak{osp}(2M+1|2L)}_{\Psi-2M+2L}},\\
Y^{-}_{L,M,N}[\Psi] &= \frac{\mathcal{DS}_{2N-2M-1}[\widehat{\mathfrak{osp}(2N|2L)}_{\Psi-2N+2L+2}]}{\widehat{\mathfrak{osp}(2M+1|2L)}_{\Psi-2M+2L}} \qquad N>M+1,\\
Y^{-}_{L,M,N}[\Psi] &= \frac{\mathcal{DS}_{2M+1-2N}[\widehat{\mathfrak{osp}(2M+1|2L)}_{\Psi-2M+2L}]}{\widehat{\mathfrak{so}(2N|2L)}_{\Psi-2N+2+2L}} \qquad N<M.
\end{aligned} \tag{5.43}$$

The VOA $\tilde{Y}^{-}_{L,M,N}[\Psi]$ corresponding to the second configuration in 5.5 are defined simply as

$$\tilde{Y}^{-}_{L,M,N}[\Psi]=Y^{-}_{L,N,M}[1-\Psi]. \tag{5.44}$$

Let us now define the VOAs $Y^{+}_{L,M,N}[\Psi]$ corresponding to the bottom left diagram in 5.5. Let $L=0$ and $N=M$. An $Sp(2N)$ Chern-Simons theory is induced at the vertical boundary with shift in the level by $\frac{1}{2}$. The anomaly mismatch compensated by a (half)-symplectic boson in a fundamental representation of $Sp(2N)$. The VOA is then identified with the BRST reduction of

$$\widehat{\mathfrak{sp}(2N)}_{\frac{\Psi}{2}-N-1} \times \mathcal{S}^{\mathfrak{sp}(2N)} \times \widehat{\mathfrak{sp}(2N)}_{-\frac{\Psi}{2}-N-\frac{1}{2}} \tag{5.45}$$

i.e. the coset

$$Y^{+}_{0,N,N}[\Psi] = \frac{\widehat{\mathfrak{sp}(2N)}_{\frac{\Psi}{2}-N-1} \times \mathcal{S}^{\mathfrak{sp}(2N)}}{\widehat{\mathfrak{sp}(2N)}_{\frac{\Psi}{2}-N-\frac{3}{2}}}. \tag{5.46}$$

If $M \neq N$, there are no symplectic bosons present but Nahm-pole boundary conditions appears leading for $N > M$ to

$$Y^{+}_{0,M,N}[\Psi] = \frac{\mathcal{DS}_{2N-2M}[\widehat{\mathfrak{sp}(2N)}_{\frac{\Psi}{2}-N-1}]}{\widehat{\mathfrak{sp}(2M)}_{\frac{\Psi}{2}-M-\frac{3}{2}}} \tag{5.47}$$

and for $N < M$ to

$$Y^{+}_{0,M,N}[\Psi] = \frac{\mathcal{DS}_{2M-2N}[\widehat{\mathfrak{sp}(2M)}_{-\frac{\Psi}{2}-M-\frac{1}{2}}]}{\widehat{\mathfrak{sp}(2N)}_{-\frac{\Psi}{2}-N-1}}. \tag{5.48}$$

If $L \neq 0$, one gets analogous expression with super-groups and dual super-Coxeter numbers:

$$\begin{aligned} & Y^{+}_{L,N,N}[\Psi] = \frac{\widehat{\mathfrak{osp}(2L|2N)}_{-\Psi+2N-2L+2} \times \mathcal{S}^{\mathfrak{osp}(2L|2N)}}{\widehat{\mathfrak{osp}(2L|2N)}_{-\Psi+2N-2L+3}}, \\ N > M \qquad & Y_{L,M,N}[\Psi] = \frac{\mathcal{DS}_{2N-2M}[\widehat{\mathfrak{osp}(2L|2N)}_{-\Psi+2N-2L+2}]}{\widehat{\mathfrak{osp}(2L|2M)}_{-\Psi+2M-2L+3}} \\ N < M \qquad & Y_{L,M,N}[\Psi] = \frac{\mathcal{DS}_{2M-2N}[\widehat{\mathfrak{osp}(2L|2M)}_{-\Psi+2M-2L+1}]}{\widehat{\mathfrak{osp}(2L|2N)}_{-\Psi+2M-2L+2}}. \end{aligned} \tag{5.49, 5.50}$$

The last diagram of 5.5 gives rise to $\tilde{Y}^{+}_{L,M,N}[\Psi]$:

$$\tilde{Y}^{+}_{L,N,N}[\Psi] = \frac{\widehat{\mathfrak{osp}(2L+1|2N)}_{-\Psi+2N-2L+1} \times \mathcal{S}^{\mathfrak{osp}(2L+1|2N)}}{\widehat{\mathfrak{osp}(2L+1|2N)}_{-\Psi+2N-2L+2}},$$

$$N > M \qquad \tilde{Y}^{+}_{L,M,N}[\Psi] = \frac{\mathcal{DS}_{2N-2M}\left[\widehat{\mathfrak{osp}(2L+1|2N)}_{-\Psi+2N-2L+1}\right]}{\widehat{\mathfrak{osp}(2L+1|2N)}_{-\Psi+2M-2L+2}} \tag{5.51}$$

$$N < M \qquad \tilde{Y}^{+}_{L,M,N}[\Psi] = \frac{\mathcal{DS}_{2M-2N}\left[\widehat{\mathfrak{osp}(2L+1|2M)}_{-\Psi+2M-2L}\right]}{\widehat{\mathfrak{osp}(2L+1|2N)}_{-\Psi+2M-2L+1}}. \tag{5.52}$$

where $\mathcal{S}^{\mathfrak{osp}(n|2N)}$ denotes a combination of N symplectic bosons and n real fermions which supports bilinear $\widehat{\mathfrak{osp}(n|2N)}$ currents.

5.3.3 Central charge

Central charge of orthosymplectic Y-algebras are given by (see appendix of [3] for a detailed calculation)

$$\begin{aligned} c^{-}_{L,M,N}[\Psi] &= \tilde{c}^{-}_{L,N,M}[1-\Psi] \\ &= -\frac{(2(L-M)-1)(2(L-M)+1)(L-M)}{\Psi-1} \\ &\quad +\frac{2(2(L-N)+1)(L-N+1)(L-N)}{\Psi} \\ &\quad +2\Psi(2(M-N)+1)(M-N+1)(M-N) \\ &\quad -2L(1+6M^2+M(6-12N)-6N+6N^2) \\ &\quad +4M^3-3M(1-2N)^2+N(5-12N+8N^2) \end{aligned} \tag{5.53}$$

and

$$\begin{aligned} c^{+}_{L,M,N}[\Psi] &= \tilde{c}^{+}_{L-\frac{1}{2},M,N}[\Psi] \\ &= -\frac{2(M-L)(2(M-L)+1)(M-L+1)}{1-\Psi} \\ &\quad -\frac{2(N-L)(2(N-L)+1)(N-L+1)}{\Psi} \\ &\quad +\Psi(2(M-N)-1)(2(M-N)+1)(M-N) \\ &\quad +L(1-12(M-N)^2)-N+2(M-N)^2(3+2M+4N). \end{aligned} \tag{5.54}$$

One can check that the expressions above are indeed invariant under transformations 5.37. Note also that S_3 action preserves $\tilde{Y}^+$-algebras and we can indeed write their central charge $\tilde{c}^+_{L,M,N}[\Psi]$ in S_3 invariant way

$$\begin{aligned}
\tilde{c}^+_{L,M,N}[\Psi] &= \frac{1}{2}\frac{1}{\Psi}(L-N)(4(L-N)^2-1)+\frac{1}{2}(1-\frac{1}{\Psi})(N-L)(4(N-L)^2-1)\\
&\frac{1}{2}\Psi(M-N)(4(M-N)^2-1)+\frac{1}{2}(1-\Psi)(N-M)(4(N-M)^2-1)\\
&\frac{1}{2}\frac{1}{1-\Psi}(L-M)(4(L-M)^2-1)+\frac{1}{2}\frac{\Psi}{\Psi-1}(M-L)(4(M-L)^2-1)\\
&-2(L+M-2N)(L-2M+N)(-2L+M+N)+\frac{1}{2}.
\end{aligned} \tag{5.55}$$

Note that the centra charges factorize nicely

$$\frac{((2(M-N)+1)\Psi+2(L-N)-1)(2(M-N+1)\Psi+2(N-L-1))((M-N)\Psi+N-L)}{\Psi(\Psi-1)},$$
$$\frac{((2(M-N)+1)\Psi-2(L-N-1))((2(N-M)-1)\Psi+2(L-N)-1)((N-M)\Psi+L-N)}{\Psi(\Psi-1)} \tag{5.56}$$

5.3.4 Relation to the even $\mathcal{W}_\infty$

Let us conclude this section by few comments related to the orthosymplectic algebras. First, one can check that characters of the algebra in the large L, M, N limit equal the one of the even $\mathcal{W}_\infty$ algebra with even-spin content $W_2, W_4, W_6, \ldots$. It is known that there again exists a two-parameter family of algebras [82, 83] with such a spin content. It is natural to expect that the finite L, M, N algebras will be truncations of such an universal algebra. The problem is that some cosets produce an extra generator that does not fit the spin content. The conformal weight of such an extra generator goes to infinity in the large L, M, N limit. It might still be possible that this extra generator can be either decoupled or projected out of the algebra by some orbifold. Note also that the more complicated duality structure from [82] corresponds to the more complicated dualities of the webs discussed above. Finally, from the string theory perspective, the transition to the ortho-symplectic algebras correspond to taking orientifolds of the original setup. It would be interesting to explore a possibility that the even $\mathcal{W}_\infty$ algebra can be also obtained from some projection of the standard $\mathcal{W}_\infty$ algebra.

5.4 Modules

5.4.1 Gauge theory origin

Line operators Apart from the local operators living at the two-dimensional corner, line operators supported at each of the three interfaces are part of the twisted theory as well. Consider line operators supported at one of the three interfaces, going from the infinity and ending at the corner at point $z \in \mathbb{C}$. The endpoint z determines the insertion of the corresponding vertex operator from the CFT point of view. The process of fusing local operators living at the corner with the line endpoint generates a module for $Y_{N_1,N_2,N_3}[\Psi]$.

Line operators supported at the NS5 interface can be identified with the Wilson lines associated to a finite-dimensional representation μ of the Lie super-algebra $\mathfrak{gl}(N_1|N_3)$ as discussed in [47]. Similarly, line operators at the D5-interface are 't Hooft operators associated to $\mathfrak{gl}(N_3|N_2)$ representations and line operators at the (1,1)-interface are Wilson line operators associated to representations of $\mathfrak{gl}(N_2|N_1)$. These modules play the role of degenerate modules of $Y_{N_1,N_2,N_3}[\Psi]$. The algebra $Y_{N_1,N_2,N_3}[\Psi]$ has a natural grading by spin and degenerate modules are characterized by the fact that they contain less states in some graded component compared to a generic module.

Gukov-Witten defects Apart from the line operators discussed above, Gukov-Witten (GW) surface defects [79] also survive the GL twist. Inserting such a GW defect at a point $z \in \mathbb{C}$ and attaching it to one of the corners of the Y-shaped junction, one gets a new (continuous) family of modules for the corner VOA.

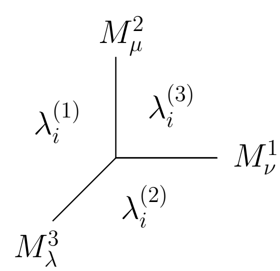


Figure 5.6: Line operators M^i_μ labeled by finite representations of the gauge groups are supported at interfaces and give rise to degenerate modules of VOA. GW defects attached to the corners of the diagram are labeled by $\lambda_i^{(1)}$, $\lambda_i^{(2)}$, $\lambda_i^{(3)}$ in N_1, N_2 and N_3 complex tori with modular parameters Ψ and give rise to generic modules.

GW defects in the $U(N)$ gauge theory are labeled according to [79, 46, 47] by four real parameters[6] $(\alpha, \beta, \gamma, \eta) \in (T, \mathfrak{t}, \mathfrak{t}, T)$, where T is the Cartan of the gauge group $U(N)$ and $\mathfrak{t}$ the Cartan subalgebra of the Lie algebra $\mathfrak{u}(N)$. In the GL-twisted theory, parameters β and γ were argued in [47] to deform the integration contour of the complexified Chern-Simons theory. On the other hand, the combination

$$\lambda = \Psi\alpha - \eta \tag{5.57}$$

parametrizes the monodromy of the complexified gauge connection $\mathcal{A} = A + \omega\phi$ around the defect, i.e.

$$\mathcal{A}(z) \sim \frac{\mathrm{diag}(\lambda_1, \dots, \lambda_N)}{z} \tag{5.58}$$

near the defect at the origin $z = 0$. The parameter ω is related to Ψ in such a way that $\mathcal{A}$ is a closed combination at the interface (modulo a gauge transformation). Since both α and η live in the Cartan subgroup $\alpha, \eta \in (S^1)^N$ of the gauge group $U(N)$, we see that the corresponding monodromies (and Gukov-Witen defects in the GL-twisted theory) are labeled by points in N complex tori of modular parameter Ψ.

Let us discuss S-duality transformation of the GW parameters identified in [79]. The pair (β, γ) transforms as

$$S : (\beta, \gamma) \to |\tau|(\beta, \gamma) \tag{5.59}$$

under the S-transformation and it is unaffected by the T-transformation. On the other hand, the pair (α, η) relevant to us transforms as

$$S : (\alpha, \eta) \to (\eta, -\alpha), \qquad T : (\alpha, \eta) \to (\alpha, \eta - \alpha). \tag{5.60}$$

The complex parameter λ of the twisted theory transforms as

$$S : \lambda = \Psi\alpha - \eta \to \lambda' = \alpha - \frac{1}{\Psi}\eta, \qquad T : \lambda \to \lambda. \tag{5.61}$$

We see that λ is invariant under the T-transformation and the S-transformation simply multiplies the Gukov-Witten parameter by $1/\Psi$ and exchanges the role of α and η. In later sections, we will see that this transformation is consistent with the triality covariance of Y_{N_1,N_2,N_3}.

[6] In general, the parameter η lives in the Cartan subalgebra of the Langlands dual gauge group $T^\vee$. Since $U(N)$ is left invariant under the Langlands duality, we do not distinguish them in this work.

When a GW defect ends at an interface, one needs to further specify a boundary condition for the defect. We will see later that the choice of the boundary condition lifts $\lambda^{(\kappa)}$ for $\kappa = 1, 2, 3$ living in the N_κ complex-dimensional torus of in each corner to $\tilde{\lambda}^{(\kappa)} \in \mathbb{C}^{N_\kappa}$. The boundary line of the surface operator can be fused with line operators discussed above. Such a fusion changes the boundary condition for the GW defect. For example in the $Y_{0,0,1}$ configuration, line operators supported at the NS5 interface produce a defect with charge $n \in \mathbb{Z}$ that lifts the parameter η and the line defect supported at the other interface creates a vortex of monodromy $\Psi m \in \Psi\mathbb{Z}$ lifting the parameter α. Similarly in the other two corners, the fundamental domain of the torus is lifted to the full $\mathbb{C}$ by modules coming from line operators at the corresponding two boundaries.

For generic values of GW-parameters, the defect breaks the gauge group to the maximal torus at the defect. Corresponding modules are going to be associated to generic modules for the corner VOA. For special values of parameters, a Levi subgroup of the gauge group is preserved and we expect the corresponding representations to be (partially) degenerate, i.e. the associated Verma module contains some null states. For example, if two of the monodromy parameters are specialized, the next-to-minimal Levi subgroup $U(2)\times U(1)^{N_i-2}$ is preserved. One can decorate such a configuration by line operators in some representation of the preserved $SU(2)$ gauge group. In the parameter space of the lifted GW parameters, one gets a discrete set of codimension one walls corresponding to degenerate modules for each pair of Cartan elements. The full parameter space of generic modules thus has a chamber-like structure with the modules degenerating at the walls. At the intersection of more walls, we expect further degeneration to appear. These intersections correspond to larger Levi subgroups. In the case that GW parameters are maximally specialized, we have a trivial interface (there are no singularities in the bulk) and we expect the corresponding modules to be maximally degenerate. The corresponding modules are labeled by finite representations of gauge groups (labeling line operators at the interfaces).

Finally, let us note that throughout the discussion above, one needs to mod out Weyl groups of $U(N_i)$ since modules related by the Weyl transformations are gauge equivalent.

$Y_{0,0,1} = \widehat{\mathfrak{gl}(1)}$ example Let us illustrate how above gauge theory elements fit nicely with the simplest example $Y_{0,0,1} = \widehat{\mathfrak{gl}(1)}$. This example is extremely important since all the other algebras can be obtained from a fusion (coproduct) combined with the triality transformation of this simple algebra.

The insertion of the complexified gauge connection $\mathcal{A}$ at the corner can be identified with the $\widehat{\mathfrak{gl}(1)}$ current J normalized as

$$J(z)J(w) \sim \frac{\Psi}{(z-w)^2}. \tag{5.62}$$

In [1], line operators supported at the NS5-boundary were identified with electric modules of charge $n \in \mathbb{Z}$ and conformal dimension $\frac{1}{2\Psi}n^2$. Line operators at the D5-boundary were identified with magnetic operators with monodromy $\Psi m \in \Psi\mathbb{Z}$ and conformal dimension $\frac{\Psi}{2}m^2$. On the other hand, GW defects are parametrized by a complex torus with the modular parameter Ψ parametrizing the monodromy for the complexified gauge connection in the bulk. If the GW defect ends at the NS5 boundary, one can fuse the end line of the defect with line operators supported at the boundary. Such a line operator shifts the charge by 1 and lifts the torus of the Gukov-Witten defect in the real direction. Similarly, fusing with modules supported at the D5-boundary lifts it in the Ψ direction tessellating $\mathbb{C}$ as shown in the figure 5.7. The GW parameter λ thus lifts to $\tilde{\lambda} \in \mathbb{C}$ that can be identified with the J_0 eigenvalue. The fusion with an electric module shifts it by one $\tilde{\lambda} \to \tilde{\lambda}+1$, whereas the fusion with a magnetic module shifts it by Ψ, i.e. $\tilde{\lambda} \to \tilde{\lambda}+\Psi$. The module coming from the GW defect has charge $\tilde{\lambda}$ and conformal dimesion $\frac{1}{2\Psi}\tilde{\lambda}^2$.

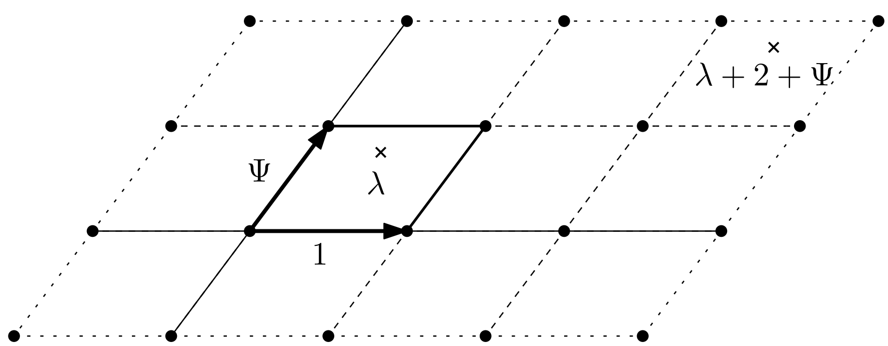


Figure 5.7: The lattice structure of modules of the $\widehat{\mathfrak{gl}(1)}$ algebra. GW-defects are labeled by a point in the torus of modular parameter Ψ. Fusion with electric and magnetic modules of charges n and Ψn lift the torus along the full complex plane parametrizing generic module of the algebra. Lattice points correspond to dyon modules of the algebra and the position in the fundamental domain corresponds to the GW-parameter. For example, the modules of charge $\tilde{\lambda} = \lambda + 2 + \Psi$ and λ in the fundamental domain are related by fusion with the electric module of charge 2 and the magnetic module of charge 1.

Note that the S-duality transformation exchanges NS5-brane and D5-brane and the orientation of the diagram gets reversed. The transformed level of the algebra is $1/\Psi$ and the transformed lifted GW parameter becomes $\tilde{\lambda}/\Psi$. This is consistent both with the transformation of degenerate modules and the unlifted GW parameter. Note that conformal dimension of the generic module is invariant under the S-duality transformation and so is the charge if we renormalize $\tilde{J} = \frac{1}{\sqrt{\Psi}} J$. The roles of α and η interchange.

Let us show that transformations of parameters are also consistent with the triality relation

$$Y_{0,0,1}[\Psi] = Y_{0,1,0}\left[\tilde{\Psi} = 1 - \frac{1}{\Psi}\right]. \tag{5.63}$$

The insertion of $\mathcal{A}$ at the corner of $Y_{0,1,0}[\tilde{\Psi}]$ leads to the $\widehat{\mathfrak{gl}(1)}$ Kac-Moody algebra normalized as

$$J(z)J(w) \sim \frac{1-\tilde{\Psi}}{(z-w)^2}. \tag{5.64}$$

Consider a GW defect with the parameter $\tilde{\lambda}^{(2)}$. The charge of the corresponding module with respect to the normalized current $J/\sqrt{\tilde{\Psi}-1}$ equals

$$\frac{\tilde{\lambda}^{(2)}}{\sqrt{1-\tilde{\Psi}}} = \frac{\tilde{\lambda}^{(2)}}{\sqrt{1-1+\frac{1}{\Psi}}} = \sqrt{\Psi}\tilde{\lambda}^{(2)}. \tag{5.65}$$

Comparing it with the charge with respect to the normalized current of $Y_{0,0,1}[\Psi]$ that equals $\tilde{\lambda}^{(3)}/\sqrt{\Psi}$, we see that the two GW parameters must be indeed related by $\tilde{\lambda}^{(3)} = \tilde{\lambda}^{(2)}/\Psi$ consistently with the above discussion.

Reparametrization of GW defects The trivalent junction of interest is invariant under the S_3 subgroup of the $SL(2,\mathbb{Z})$ group of S-duality transformations. To get manifestly triality invariant parametrization of the algebra and its modules, let us introduce parameters h_1, h_2, h_3 by

$$\Psi = -\frac{h_2}{h_1}, \qquad h_1 + h_2 + h_3 = 0. \tag{5.66}$$

Note that the parameters h_i are determined up to the overall rescaling. The VOA is independent on such a rescaling. Up to the rescaling, one can relate parameters h_i and Ψ for example as

$$h_1 = \frac{1}{\sqrt{\Psi}}, \qquad h_2 = -\sqrt{\Psi}, \qquad h_3 = \sqrt{\Psi} - \frac{1}{\sqrt{\Psi}}. \tag{5.67}$$

Instead of the lifted GW parameter $\tilde{\lambda}^{(3)}$ parametrizing defects in the third corner, one can consider the combination

$$x^{(3)} = \frac{1}{\sqrt{\Psi}}\tilde{\lambda}^{(3)} = h_1\alpha^{(3)} + h_2\eta^{(3)} \tag{5.68}$$

and similar combinations in the other two corners

$$\begin{aligned} x^{(2)} &= h_3\alpha^{(2)} + h_1\eta^{(2)} \\ x^{(1)} &= h_2\alpha^{(1)} + h_3\eta^{(1)}. \end{aligned} \tag{5.69}$$

In the $Y_{0,0,1}$ example, we can identify the parameter $x^{(3)}$ with the coefficient in the exponent of the vertex operator[7]

$$\exp\left[x^{(3)}\phi(w)\right] \tag{5.70}$$

in the free field realization of the module with the current $J^{(3)} = \partial\phi^{(3)} = J/\sqrt{\Psi}$ normalized as

$$J^{(3)}(z)J^{(3)}(w) \sim -\frac{1}{h_1h_2}\frac{1}{(z-w)^2}. \tag{5.71}$$

In this parametrization, the electric module M^2 of unit charge corresponds to $\alpha^{(3)} = 1$ whereas the magnetic module to $\eta^{(3)} = 1$.

In the other two frames $Y_{1,0,0}$ and $Y_{0,1,0}$ with currents $J^{(\kappa)} = \partial\phi^{(\kappa)}$ normalized as

$$J^{(\kappa)}(z)J^{(\kappa)}(w) \sim -\frac{h_\kappa}{h_1h_2h_3}\frac{1}{(z-w)^2}, \tag{5.72}$$

parameters $x^{(\kappa)}$ are again exponents of the corresponding vertex operator. We will later see that parameters $x_i^{(\kappa)}$ can be identified with shifts of exponents of $N_1+N_2+N_3$ vertex operators also for general Y_{N_1,N_2,N_3}.

In the parametrization using h_i, the triality tranformation simply permutes h_κ together with parameters $\alpha^{(\kappa)}, \eta^{(\kappa)}$. The invariance of the charge of the current normalized to identity is manifest.

[7]In the following we will drop the normal ordering symbols and we assume all the exponential vertex operators are normal ordered.

5.4.2 Generic modules

Let us turn to the discussion of generic modules of $Y_{N_1,N_2,N_3}[\Psi]$ associated to Gukov-Witten defects. We start with a review of the algebra of zero modes and how to parametrize modules of a VOA induced from modules of the zero mode algebra. We review a compact way to parametrize highest weights in terms of Yangian generating functions $\psi(u)$. The, we describe a general structure of the variety of highest weights parametrizing generic representations of Y_{N_1,N_2,N_3} in the primary basis. The next two sections state the generating function for such representations and related its parameters with Gukov-Witten parameters and parameters of Fock modules in the corresponding free field realization of modules. Finally, a simple example of the variety of highest weights.

Zero modes and generic modules A rich class of $Y_{N_1,N_2,N_3}[\Psi]$ representations can be induced from representations of the subalgebra of zero modes

$$X_0 = \frac{1}{2\pi i}\oint dz z^{h(X)-1} X(z) \tag{5.73}$$

for X a field of spin $h(X)$. Starting with a highest-weight vector anihilated by all positive modes, one can show that the algebra of zero modes of truncations of $\mathcal{W}_{1+\infty}$ acting on the highest weight vector is commutative [76]. We can thus define a one-dimensional module for the zero-mode algebra by prescribing how zero modes of the strong generators W_j act. If there are relations in the space of fields (which show as singular vectors of the vacuum Verma module), the zero mode of the corresponding null fields must vanish when acting on the highest weight state. The existence of null fields thus constrains possible highest weights leading to a variety of highest weights.

Let us add few comments:

1. In the math literature, the algebra of zero modes acting on the highest weight state appears under the name of the Zhu algebra[8] [124]. If the Zhu algebra is commutative (as in the $Y_{N_1,N_2,N_3}[\Psi]$ case [76]) the variety of highest weights is the spectrum of the Zhu algebra.

2. Not all the modules produced by gluing are induced from the algebra of zero modes with trivial action of the positive modes on the highest weight vectors. Gluing of

[8]The Zhu commutative product is defined as a modified normal ordered product $[X]\star[Y] = (X,Y) +$ corrections. The corrections are the commutators $[Y_1, Y_{-1}] + [Y_2, Y_{-2}] + \dots$ from the mode expansion of the normal ordered product acting on the highest weight state. For a more precise comparison see [123].

highest weight modules of $Y_{N_1,N_2,N_3}[\Psi]$ leads in general to irregular modules of the glued algebra. We will later illustrate this phenomenon on the simplest example of the $\widehat{\mathfrak{gl}}(2)$ Kac-Moody algebra.

3. Even in the case when the module of the glued algebra has a trivial action of positive modes on the space of highest weights, the space of highest weights itself generically forms an infinite-dimensional representation of the zero mode algebra.

Generating function of highest weights for Y_{N_1,N_2,N_3} Generic highest weight modules of a VOA with a commutative algebra of zero modes are parametrized by the action of such zero modes on the highest weight state. For example, modules of the $\widehat{\mathfrak{gl}(1)} \times \mathcal{W}_N \equiv Y_{0,0,N}$ algebra are labeled by N highest weights, i.e. eigenvalues of W_i zero modes for $1, 2, 3, \ldots N$. Analogously, a generic representation of $\mathcal{W}_{1+\infty}$ is specified by an infinite set of higher spin charges of the highest weight state, one for each independent generator of spin $1, 2, 3, \ldots$. To label a generic highest weight representation of $\mathcal{W}_{1+\infty}$ and its truncations, it is convenient to introduce a generating function of the highest weight charges.

We will not be able to write down explicitly the generating function of highest weights in the primary basis of the algebras. Instead, we will see that the modules can be easily parametrized using the Yangian description in terms of generators ψ_i, f_i, e_i from [80, 98]. We will specify the module by the eigenvalues of the commuting ψ_i generators on the highest weight state encoded in the generating function

$$\psi(u) = 1 + h_1 h_2 h_3 \sum_{j=0}^{\infty} \frac{\psi_j}{u^{j+1}}. \tag{5.74}$$

Another possibility to encode the highest weight charges is in terms of the generating function of U-charges of the quadratic basis[9] of $\mathcal{W}_{1+\infty}$. U-basis is particularly useful for description of $Y_{0,0,N}$ with the generating function given by

$$\mathcal{U}(u) = \sum_{k=0}^{N} \frac{u_k}{(-u)(-u+\alpha_0)\cdots(-u+(k-1)\alpha_0)} \tag{5.75}$$

where u_j are the eigenvalues of zero modes of the U_j-generators of $Y_{0,0,N}$ and $u_0 \equiv 1$. The generating function is a ratio of two N-th order polynomials in u-plane, so we may factorize

[9] OPEs of the $\mathcal{W}_{1+\infty}$ algebra in the U-basis contain only quadratic non-linearities with all the structure constants fixed in [5].

it and write

$$\mathcal{U}(u) = \prod_{j=1}^{N} \frac{u - \Lambda_j - (j-1)\alpha_0}{u - (j-1)\alpha_0}. \tag{5.76}$$

As shown in [98], the transformation between generating function $\mathcal{U}(u)$ and $\psi(u)$ is given by

$$\psi(u) = \frac{u - N\alpha_0}{u} \frac{\mathcal{U}(u-\alpha_0)}{\mathcal{U}(u)} \tag{5.77}$$

if we identify the parameters as

$$h_1 h_2 = -1, \qquad h_3 = \alpha_0, \qquad \psi_0 = N. \tag{5.78}$$

These relations allow us to translate between ψ_j charges of the highest weight state and the corresponding u_j charges.

Plugging in the product formula for $\mathcal{U}$, we find

$$\psi(u) = \frac{(u - \Lambda_1 - \alpha_0)(u - \Lambda_2 - 2\alpha_0) \cdots (u - \Lambda_N - N\alpha_0)}{(u - \Lambda_1)(u - \Lambda_2 - \alpha_0) \cdots (u - \Lambda_N - (N-1)\alpha_0)}. \tag{5.79}$$

Defining

$$x_j = \Lambda_j + (j-1)h_3 \tag{5.80}$$

we can rewrite this as

$$\psi(u) = \prod_{j=1}^{N} \frac{u - x_j - h_3}{u - x_j}, \tag{5.81}$$

i.e. the parameters x_j specify the positions of poles of $\psi(u)$ in the spectral parameter plane while the zeros are at positions $x_j + h_3$. Using the variables x_j, we have a manifest permutation symmetry of the generating function, while the shifted variables Λ_j are chosen such that the vacuum representation has $\Lambda_j = 0$.

Zero mode algebra of Y_{N_1,N_2,N_3} The algebras Y_{N_1,N_2,N_3} are finitely (generically non-freely) generated vertex operator algebras by fields $W_1, W_2, \ldots, W_n$, where

$$n = (N_1+1)(N_2+1)(N_3+1) - 1. \tag{5.82}$$

The finite generation can be seen from the structure of null states of the algebra. The first state of $\mathcal{W}_{1+\infty}$ that needs to be removed in order to get the algebra Y_{N_1,N_2,N_3} appears at level $n+1$. Assuming that the coefficient in front of W_{n+1} does not vanish, one can use this null field to eliminate the W_{n+1} field from the OPEs. At the next level, three more null fields appear. Two of them are the derivative of the null field at level $n+1$ and its normal ordered product with W_1 but one also gets one extra condition. This condition can be used to remove the field W_{n+2}. One can continue this procedure and (assuming that there are enough conditions at each level) one can remove all W_i for $i > n$ from OPEs.

In this way, one can solve many null state conditions by restricting to a finite number of W-generators but generically (apart from the case of $Y_{N,0,0}, Y_{0,N,0}, Y_{0,0,N}$) some null states remain. These are going to be composite primary fields formed by the restricted set of W-generators and need to be removed as well. The first constraint appears generically already at level $n+2$. For large enough values of N_i, one can see from the box-counting that there are be 12 null states at this level but only $\partial^2 W_n$, $(W_n \partial J)$, $(J\partial W_n)$, $(J(JW_n))$, (TW_n), ∂W_{n+1}, (JW_{n+1}), W_{n+2} are removed by the above argument. One has still 4 constrains that lead to a non-trivial conditions on the algebra of zero modes. Note that for small values of of N_i, there will be less states at this level as can be easily seen from the box-counting and as we will see in examples below. We will also see that some constraints will be trivially satisfied and only some of them are actually non-trivial.

One can see that for generic values of N_1, N_2, N_3 the problem outlined above becomes rather complex. The null states have been fully identified only in the case $Y_{0,1,1}$ and $Y_{0,1,2}$ in the literature [125, 126] and lead to nontrivial constraints on the allowed highest weights[10]. From the discussion above, one can still draw the conclusion what will be the general structure of the variety of highest weights. As argued above, the possible highest weights are given by a subvariety inside the space of the highest weights of zero modes

$$(W_i)_0|w_i\rangle = w_i|w_i\rangle. \tag{5.83}$$

The highest weights are constrained by the existence of null states X^i_{null} and we conjecture that the resulting variety of highest weights of the algebra of zero modes

$$(X^i_{null})_0|w_i\rangle = f^i(w_i)|w_i\rangle = 0 \tag{5.84}$$

is $N_1+N_2+N_3$ dimensional subvariety inside $\mathbb{C}^n$. Although we will not be able to explicitly construct the null states in general in terms of primary fields, we will give an explicit parametrization of the variety by generalizing the generating function of ψ_i charges of the $\mathcal{W}_N$ algebra. The conjecture for the dimensionality comes from the existence of $N_1+N_2+N_3$ continuous parameters of surface defects available in the configuration. The number $N_1+N_2+N_3$ can be also guessed from the free field realization of the algebra Y_{N_1,N_2,N_3}

[10] The cases $Y_{N,N,0}$, $Y_{N,N-1,0}$ and $Y_{N,N-2,0}$ have been considered rigorously in math literature [127, 128, 129, 130, 76]

inside $Y_{1,0,0}^{\otimes N_1} \otimes Y_{0,1,0}^{\otimes N_2} \otimes Y_{0,0,1}^{\otimes N_3}$, where modules of each of the factors are parametrized by N_1, N_2 and N_3 parameters respectively. The dimensionality indeed matches in examples of $Y_{0,1,1}$ and $Y_{0,1,2}$ from the literature.

Note that the above discussion also implies that the character of the module with generic highest weights counts $N_1 + N_2 + N_2$-tuples of partitions, i.e.

$$\chi_{N_1+N_2+N_3}(q) = \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^{N_1+N_2+N_3}}. \tag{5.85}$$

A general state of a generic module of the algebra can be constructed by an action of negative modes W_i on the highest weight state subject to the null state conditions. As in the case of zero modes, where the null states were used to carve out an $N_1+N_2+N_3$ dimensional subvariety, one can use negative modes of the null conditions to remove appropriate states at higher levels. Only $N_1 + N_2 + N_2$ of the modes at each level are independent, giving rise to the above character.

Generating function for Y_{N_1,N_2,N_3} As we have just seen, truncations Y_{N_1,N_2,N_3} are finitely generated by $W_1, \ldots, W_n$ where n is given by (5.82). In particular, generic representations have a finite number of states at level one. Following the usual notion of quasi-finite representations of linear $\mathcal{W}_{1+\infty}$ [131, 132], it was argued in [98] that a highest weight representation of $\mathcal{W}_{1+\infty}$ has a finite number of states at level 1 if and only if generating function $\psi(u)$ equals a ratio of two Drinfeld polynomials of the same degree. This is indeed true for $Y_{0,0,N}$. We will now generalize the formula (5.81) to a generating function $\psi(u)$ that parametrize generic representations for all Y_{N_1,N_2,N_3}. In particular, we conjecture that the complicated variety parametrizing modules of the algebra Y_{N_1,N_2,N_3} can be simply parametrized.

Such a parametrization of the variety of highest weights is natural the from point of view of the coproduct structure of the affine Yangian, but also from free field realization viewpoint and the gauge theory perspective. After stating these motivations, we write down an explicit formula for the generating function of ψ_i charges for arbitrary Y_{N_1,N_2,N_3} in 5.88. A parametrization of the variety of highest weights can be recovered after changing the variables from the affine Yangian generators ψ_i to the zero modes of W_i generators as discussed in [3].

Free field realization Both the Miura transformation for Y_{N_1,N_2,N_3} and the definition in terms of a kernel of screening charges give an embedding of the algebras of the form

$$Y_{N_1,N_2,N_2} \subset Y_{N_1,0,0} \times Y_{0,N_2,0} \times Y_{0,0,N_3} \subset Y_{1,0,0}^{\otimes N_1} \times Y_{0,1,0}^{\otimes N_2} \times Y_{0,0,1}^{\otimes N_3}. \tag{5.86}$$

Each factor $Y_{0,0,1}$ in the free field realization above can be identified with one multiplicative factor in (5.81). The full free field realization therefore suggests that the generating function of a generic module of Y_{N_1,N_2,N_3} should be simply a product of three W_N factors corresponding to $Y_{N_1,0,0}$, $Y_{0,N_2,0}$ and $Y_{0,0,N_3}$. Note that the parameter α_0 remains the same in the fusion procedure.

Yangian point of view Using the map between $\mathcal{W}_{1+\infty}$ modes and Yangian generators [98], we can translate the fusion to Yangian variables. The coproduct of ψ_j generators with $j \geq 3$ is no longer a finite linear combination of other generators and their products, but involves an infinite sum. This is related to the non-local terms that enter the map between VOA description and the Yangian description. Fortunately, when acting on a highest weight state (corresponding to a primary field via the operator-state correspondence) these additional terms drop out and we obtain a simple formula

$$\psi(u) = \psi^{(1)}(u)\psi^{(2)}(u) \tag{5.87}$$

analogous to the usual ones in finite Yangians.[11] This coproduct of the affine Yangian also suggests a simple form of the generating function in terms of a product of three W_{N_i} factors associated to each corner. The compatibility of parameters in this case requires that h_1, h_2 and h_3 parametrizing the algebra are the same while the ψ_0 is additive under the fusion. In terms of λ-parameters this is the same condition as found above.

Gauge theory and brane picture The gauge theory setup suggests that the modules should be parametrized linearly. The GW parameters that label modules live in the $N_1 + N_2 + N_3$ dimensional tori (modulo Weyl group) that we expect to be lifted to $\mathbb{C}^{N_1+N_2+N_3}$ by boundary conditions imposed on the GW defect ending at the interfaces. Moreover, this picture suggests that generically the contribution from GW-parameters in each corner should be independent.

The coproduct from the point of view of the gauge theory corresponds to increasing the rank of gauge groups in the three corners of the diagram. One can look at it as an inverse process to Higgsing the theory that corresponds to separation of D3-branes and reduces the gauge group. This procedure can be performed in each corner suggesting that the coproduct of $\mathcal{W}_N$ should have a natural generalization for $Y_{N_1,N_2,N_3}[\Psi]$. The process

[11] Since the Yangian has a non-trivial automorphisms, like the spectral shift automorphism translating the parameter u, we can precompose this with the coproduct if needed to obtain slightly more general coproducts. This is actually what is needed if we want the fusion of two vacuum representations to produce a vacuum representation.

is independent on the gauge coupling suggesting that Ψ is constant in agreement with the other pictures discussed above.

Generating function The discussion above motivates us to write down an explicit formula for the generating function of ψ_i charges for Y_{N_1,N_2,N_3} acting on the highest weight state by simply multiplying contributions from $\mathcal{W}_N$ algebras from each corner

$$\psi(u) = \prod_{j=1}^{N_1} \frac{u - x_j^{(1)} - h_1}{u - x_j^{(1)}} \prod_{j_2=1}^{N_2} \frac{u - x_j^{(2)} - h_2}{u - x_j^{(2)}} \prod_{j_3=1}^{N_3} \frac{u - x_j^{(3)} - h_3}{u - x_j^{(3)}}. \tag{5.88}$$

Note that the expression is manifestly triality invariant, depends on the correct number of parameters and the truncation curves are reproduced correctly. In particular, extracting ψ_0 from the expression above, one gets

$$h_1 h_2 h_3 \psi_0 = -N_1 h_1 - N_2 h_2 - N_3 h_3. \tag{5.89}$$

Identifying the scaling-independent combinations[12]

$$\lambda_1 = -\psi_0 h_2 h_3, \qquad \lambda_2 = -\psi_0 h_1 h_3, \qquad \lambda_3 = -\psi_0 h_1 h_2, \tag{5.90}$$

one gets the correct expression

$$\frac{N_1}{\lambda_1} + \frac{N_2}{\lambda_2} + \frac{N_3}{\lambda_3} = 1 \tag{5.91}$$

satisfied by parameters of Y_{N_1,N_2,N_3}.

Parameters $x_i^{(\kappa)}$ can be identified with the lifted Gukov-Witten parameters in the third corner. This can be seen from the comparison of the $U(1)$ charge for $Y_{0,0,1}$ and the fact that each multiplicative factor corresponds to one such factor. The unlifted Gukov-Witten parameters themselves can be identified by modding out by the lattice $h_1 n + h_2 m$ for $n, m \in \mathbb{Z}$. We will later see that that $x_i^{(3)} = h_1 n + h_2 m$ corresponding to the trivial GW defect (and a possibly non-trivial line operator) corresponds to a degenerate module. We will also see that the fusion of a degenerate module with a generic module labeled by a parameter $x^{(3)}$ amounts to a shift of $x^{(3)}$ by a lattice vector.

Note also that the generating function is manifestly invariant under the Weyl group associated to the three gauge groups $U(N_i)$.

[12] The algebra is invariant under the simultaneous rescaling of ψ_0 and h_i, see [80, 98].

Free field realization The parameters $x_i^{(\kappa)}$ from the generating function of ψ_i charges that have been already related to the Gukov-Witten parameters can be also related to exponents in the expression for the vertex operators in free field realization. A highest weight vector in free field representation with generic charges can be obtained by acting on the vacuum state with the vertex operator

$$|q^1, \ldots, q^N\rangle = \exp\left(\sum_{j=1}^{N} q^j \phi_j\right) |0\rangle. \tag{5.92}$$

Acting on this state with the zero mode of current $J_j = \partial\phi_j$, we find

$$J_{j,0}|q^1, \ldots, q^N\rangle = g_{jk} q^k |q^1, \ldots, q^N\rangle \equiv q_j |q^1, \ldots, q^N\rangle \tag{5.93}$$

where g_{jk} is the metric extracted from the two-point functions of the currents,

$$J_j(z)J_k(w) \sim \frac{g_{jk}}{(z-w)^2} \sim -\frac{h_{\kappa(j)}}{h_1 h_2 h_3} \frac{\delta_{jk}}{(z-w)^2}. \tag{5.94}$$

Our conventions for charges are such that q^j are the charges that appear in the exponents of vertex operators (and in positions of zeros and poles of $\psi(u)$) while q_j are the coefficients of the first order poles of OPE with currents J_j. We reintroduce the $-h_1 h_2$ factors in order to make the expressions manifestly triality invariant and also of definite scaling dimension under the scaling symmetry of the algebra [98].

The $\widehat{\mathfrak{gl}(1)}$ current of $\mathcal{W}_{1+\infty}$ whose zero mode is ψ_1 is given by

$$U_1(z) = \sum_{j=1}^{N} J_j(z) \tag{5.95}$$

so ψ_1 acts on the highest weight state by

$$\psi_1 |q^1, \ldots, q^N\rangle = \left(\sum_{j=1}^{N} q_j\right) |q^1, \ldots, q^N\rangle. \tag{5.96}$$

To find the total stress-energy tensor of $\mathcal{W}_{1+\infty}$, we first use the Miura transform to find the free field representation of $U_2(z)$:

$$U_2(z) = \frac{1}{2} \sum_{j\geq 1} \left(1 - \frac{h_3}{h_{\kappa_j}}\right) \big((J_j J_j)(z) + h_{\kappa_j} \partial J_j(z)\big) + \sum_{j<k} (J_j J_k)(z) + \sum_{j<k} h_{\kappa_j} \partial J_k \tag{5.97}$$

from which we can find the total $\mathcal{W}_{1+\infty}$ stress-energy tensor

$$T_{1+\infty}(z) = -\frac{1}{2}\sum_j \frac{h_1h_2h_3}{h_{\kappa_j}}(J_jJ_j)(z) + \frac{1}{2}\sum_{j<k} h_{\kappa_k}\partial J_j - \frac{1}{2}\sum_{j>k} h_{\kappa_k}\partial J_j \tag{5.98}$$

Let us now Consider one free boson $\phi^{(\kappa)}(z)$ in κ-th direction associated to elementary Miura factor $R^{(\kappa)}$. It is easy to verify that the state created by the vertex operator

$$:\exp\left(q\phi^{(\kappa)}\right): \tag{5.99}$$

from the vacuum is a highest weight state with the generating function of highest weight charges $\psi(u)$ equal to

$$\psi^{(\kappa)}(u) = \frac{u-q-h_\kappa}{u-q}. \tag{5.100}$$

For a longer chain with more free bosons, we have an analogous product of the corresponding simple factors, but the spectral parameter is shifted between the nodes: $\psi(u)$ corresponding to $Y_{0,0,2}$ with ordering of fields $R(z) = R_1^{(3)}(z)R_2^{(3)}(z)$

$$\psi(u) = \frac{u-q^1-h_3}{u-q^1}\frac{u-q^2-2h_3}{u-q^2-h_3}. \tag{5.101}$$

Analogously, $\psi(u)$ corresponding to $Y_{1,1,0}$ with ordering of fields $R(z) = R_1^{(1)}(z)R_2^{(2)}(z)$ has

$$\psi(u) = \frac{(u-q^1-h_1)}{(u-q^1)}\frac{(u-q^2-h_1-h_2)}{(u-q^2-h_1)}. \tag{5.102}$$

In other words, the Miura factor on the left affects the factors that come on the right of it by shifting the u-parameter. The general formula for an arbitrary ordering

$$\mathcal{L}(z) = \mathcal{L}_1^{(\kappa_1)}(z)\cdots\mathcal{L}_{N_1+N_2+N_3}^{(\kappa_{N_1+N_2+N_3})}(z) \tag{5.103}$$

has the generating function of charges equal to

$$\psi(u) = \prod_{j=1}^{N_1+N_2+N_3} \frac{u-q^j-\sum_{k\leq j} h_{\kappa_k}}{u-q^j-\sum_{k<j} h_{\kappa_k}}. \tag{5.104}$$

We see that up to constant shifts and rescalings (depending on ordering of free fields) the zeros and poles of the generating function $\psi(u)$ of highest weight state correspond to zero modes q^j of the free bosons, in particular

$$x_j^{(\kappa(j))} = q^j + \sum_{k<j} h_{\kappa_k}. \tag{5.105}$$

Let us start with the analysis of domain walls of minimal degenerations associated to the next-to-minimal Levi subgroup.

As we discussed in connection with (5.88), the $N_1+N_2+N_3$ lifted GW parameters $x_i^{(\kappa)}$ correspond to positions of poles of the generating function $\psi(u)$ in the u-plane. The poles are determined up to a permutation of order of poles in each group. A natural question to ask is for which values of parameters $x_j^{(\kappa_j)}$ do we obtain a degenerate module.

$Y_{1,1,0}$ **- singlet algebra of symplectic fermion** The algebra $Y_{1,1,0}$ is the simplest truncation of $\mathcal{W}_{1+\infty}$ which is not a $\mathcal{W}_N$ algebra, although as we will see, it can be understood as (a simple quotient of) $\mathcal{W}_3$ algebra at a special value of the central charge. First of all, the $Y_{1,1,0}$ truncation requires

$$\frac{1}{\lambda_1}+\frac{1}{\lambda_2}=1 \tag{5.106}$$

as well as the usual constraint

$$\frac{1}{\lambda_1}+\frac{1}{\lambda_2}+\frac{1}{\lambda_3}=0. \tag{5.107}$$

From these constraints, we learn that $\lambda_3=-1$. Plugging this into the central charge formula, we find

$$c_\infty=-2 \tag{5.108}$$

independently of the value of λ_1.

Considering $Y_{1,1,0}$ algebra as truncation of $\mathcal{W}_{1+\infty}$, the first singular vector in the vacuum representation appear at level $4=2\cdot 2\cdot 1$. Generically, starting from spin 4 we can use these singular vectors to eliminate the higher spin generators of spin $4,5,\ldots$, obtaining an algebra that is generated by fields of spins 1, 2 and 3. Therefore we identify $Y_{1,1,0}$ with a quotient of the $\mathcal{W}_3$ algebra at $c=-2$ times a free boson. The OPEs of $\mathcal{W}_3$ are given by the Virasoro algebra coupled to a spin 3 current which has OPE

$$\begin{aligned}W_3(z)W_3(w) \quad\sim\quad & C_{33}^0\Bigg(\frac{\mathbb{1}}{(z-w)^6}-\frac{3T(w)}{(z-w)^4}-\frac{3\partial T(w)}{2(z-w)^3}-\frac{4(TT)(w)}{(z-w)^2}\\ &+\frac{3\partial^2T(w)}{4(z-w)^2}-\frac{4(\partial TT)(w)}{z-w}+\frac{\partial^3T(w)}{6(z-w)}\Bigg).\end{aligned} \tag{5.109}$$

We kept the normalization of W_3 generator free for later convenience. We could absorb the structure constant C_{33}^0 by rescaling the W_3 generator.

We are now interested in constraints on generic representations of $Y_{1,1,0}$. From the physical reasoning as well as from the free field representations, we would expect the generic representation of $Y_{1,1,0}$ to be parametrized by two continuous parameters, while the $\widehat{\mathfrak{gl}(1)} \times \mathcal{W}_3$ algebra have in general three highest weights. We thus need to find a singular vector in $\mathcal{W}_3$ that would reduce the number of parameters by one. From the general reasoning, we expect the first relation to appear at level 6. In fact, there are two singular primaries at level 6. We can see this by looking at characters: the character of the vacuum representation of $\widehat{\mathfrak{gl}(1)} \times \mathcal{W}_3$ is

$$\prod_{s=1}^{3}\prod_{j=0}^{\infty}\frac{1}{1-q^{s+j}} \simeq 1+q+3q^2+6q^3+12q^4+21q^5+40q^6+67q^7+117q^8+\ldots \tag{5.110}$$

while the vacuum representation of $Y_{1,1,0}$ has

$$\begin{aligned} \chi_{vac}(q) &= \sum_{j=0}^{\infty}\frac{q^j}{\prod_{k=1}^{j}(1-q^k)^2} = \frac{\sum_{j=0}^{\infty}(-1)^j q^{j(j+1)/2}}{\prod_{k=1}^{\infty}(1-q^k)^2} \\ &\simeq 1+q+3q^2+6q^3+12q^4+21q^5+38q^6+63q^7+106q^8+170q^9+\ldots \end{aligned} \tag{5.111}$$

We see that at level 6 there are two null states in $Y_{1,1,0}$ compared to the situation in $\widehat{\mathfrak{gl}(1)} \times \mathcal{W}_3$ at the generic value of the central charge. The first null state is the even quadratic primary composite field

$$N_{6e} = (W_3W_3) + C_{33}^0\left(\frac{8}{9}(T(TT)) + \frac{19}{36}(\partial T\partial T) + \frac{7}{9}(\partial^2 TT) - \frac{2}{27}\partial^4 T\right) \tag{5.112}$$

and the second one is the odd field

$$N_{6o} = 8(T\partial W_3) - 12(\partial T W_3) - \partial^3 W_3. \tag{5.113}$$

Requiring that the action of the zero mode of N_{6o} on the generic highest weight state vanishes gives us identical zero while the similar requirement for N_{6e} gives us a non-trivial constraint

$$0 = w_3^2 + \frac{C_{33}^0}{9}h^2(8h+1). \tag{5.114}$$

This is the constraint we were looking for. It reduces the dimension of the space of generic primaries from three to two which is in accordance with what we expect. In principle, we could proceed further by studying the singular vectors at higher levels and possibly discover new (independent) constraints. In order to show that (5.114) is necessary and sufficient, we will construct a free field realization of $Y_{1,1,0}$ and check that the generic modules can indeed by realized.

Free field realization from Miura Let us see what free field representation we find by applying the Miura transformation explained above. The total Miura operator is a product of two basic Miura factors associated to first and second asymptotic direction

$$\begin{aligned} R(z) &= \partial^{-1} + U_1(z)\partial^{-2} + U_2(z)\partial^{-3} + \ldots \\ &= \left[\mathbb{1} + U_1^{(1)}(\alpha_0\partial)^{-1} + U_2^{(1)}(\alpha_0\partial)^{-2} + \ldots\right](\alpha_0\partial)^{h_1/h_3} \times \\ &\quad \times \left[\mathbb{1} + U_1^{(2)}(\alpha_0\partial)^{-1} + U_2^{(2)}(\alpha_0\partial)^{-2} + \ldots\right](\alpha_0\partial)^{h_2/h_3} \end{aligned} \tag{5.115}$$

By commuting the derivatives to the right we find

$$\begin{aligned} U_1 &= U_1^{(1)} + U_1^{(2)} \\ U_2 &= U_2^{(1)} + U_2^{(2)} + U_1^{(1)}U_1^{(2)} + h_1\partial U_1^{(2)} \\ U_3 &= U_3^{(1)} + U_3^{(2)} + U_1^{(1)}U_2^{(2)} + U_2^{(1)}U_1^{(2)} + (h_1 - h_3)U_1^{(1)}\partial U_1^{(2)} \\ &\quad + h_1\partial U_2^{(2)} + \frac{h_1(h_1-h_3)}{2}\partial^2 U_1^{(2)} \end{aligned} \tag{5.116}$$

Plugging in expressions for U_j in terms of free bosons, we find

$$\begin{aligned} U_1 &= J_1 + J_2 \\ U_2 &= \frac{2h^2-1}{2h^2}(J_1J_1) + (J_1J_2) + \frac{2-h^2}{2}(J_2J_2) + \frac{2h^2-1}{2h}\partial J_1 + \frac{3h^2-2}{2h}\partial J_2 \\ U_3 &= \frac{(2h^2-1)(3h^2-2)}{6h^4}(J_1(J_1J_1)) + \frac{2h^2-1}{2h^2}(J_1(J_1J_2)) - \frac{h^2-2}{2}(J_1(J_2J_2)) \\ &\quad + \frac{(h^2-2)(2h^2-3)}{6}(J_2(J_2J_2)) + \frac{(2h^2-1)(3h^2-2)}{2h^3}(\partial J_1J_1) \\ &\quad + \frac{2h^2-1}{2h}(\partial J_1J_2) + \frac{5h^2-4}{2h}(J_1\partial J_2) - \frac{(h^2-2)(4h^2-3)}{2h}(\partial J_2J_2) \\ &\quad + \frac{(2h^2-1)(3h^2-2)}{6h^2}\partial^2 J_1 + \frac{11h^4-16h^2+6}{6h^2}\partial^2 J_2 \end{aligned} \tag{5.117}$$

in the normalization

$$J_1(z)J_1(w) \sim -\frac{h_1/h_1h_2h_3}{(z-w)^2}, \qquad J_2(z)J_2(w) \sim -\frac{h_2/h_1h_2h_3}{(z-w)^2} \tag{5.118}$$

and with conventions in (3.47). There is an infinite number of non-zero U_j operators with $j \geq 4$ but they can all be read off from OPE of U_j fields with $j = 1, 2, 3$. Finally using the transformations of appendix A.3 we find in the primary basis

$$\begin{aligned}
W_1 &= -J_1 - J_2 \\
W_2 &= \frac{1}{2h^2}(J_1J_1) + (J_1J_2) + \frac{h^2}{2}(J_2J_2) - \frac{1}{2h}\partial J_1 - \frac{h}{2}\partial J_2 \\
W_3 &= -\frac{h^2+1}{3h^4}(J_1(J_1J_1)) - \frac{h^2+1}{h^2}(J_1(J_1J_2)) - (h^2+1)(J_1(J_2J_2)) \\
&\quad -\frac{h^2(h^2+1)}{3}(J_2(J_2J_2)) + \frac{h^2+1}{2h^3}(\partial J_1J_1) + \frac{h^2+1}{2h}(\partial J_1J_2) \\
&\quad +\frac{h^2+1}{2h}(J_1\partial J_2) + \frac{h(h^2+1)}{2}(\partial J_2J_2) - \frac{h^2+1}{12h^2}\partial^2 J_1 - \frac{h^2+1}{12}\partial^2 J_2
\end{aligned} \tag{5.119}$$

with all other W_j currents, $j \geq 4$ vanishing (as they should). To compare to the previous discussion, where the current J was chosen to be $J_1 + J_2$ with unit normalization and T and W_3 were expressed in terms of the orthogonal combination, if we choose the orthogonal combination to be the current

$$J_- = -h^{-1}J_1 - hJ_2 \tag{5.120}$$

we exactly reproduce the formulas of the previous section up to an overall normalization.

5.4.3 Degenerate modules

Surface defects preserving Levi subgroups A generic Gukov-Witten defect breaks the gauge group at the defect to the maximal torus $U(1)^N$, but a larger symmetry group can be preserved if the GW-parameters are specialized. In particular, if the parameters $x_i^{(\kappa)}$ and $x_j^{(\kappa)}$ specifying the singularity of the ith and jth factors are equal $x_i^{(\kappa)} = x_j^{(\kappa)}$ (modulo the lattice), the next-to minimal Levi subgroup $U(2) \times U(1)^{N-2}$ is preserved by the configuration. On the VOA side, these specializations are going to correspond to degenerate modules. For a fixed value of the specialized GW parameters, one can still turn on a Wilson and 't Hooft operator in some representation of the preserved $U(2)$ at each boundary. Different choice of the line operators will label different degenerate modules. Similarly, if parameters in different corners are specialized, $U(1|1)$ supergroup is preserved at the boundary Chern-Simons theory by the defect and one gets different classes of degenerate modules as we will see below.

We can see that the parameter space parametrizing generic modules is divided into domains with a degeneration appearing at the boundaries of the domains. At intersections of such domain walls (where more parameters are specialized), we expect further degeneration of the module. These more complicated representations correspond to larger Levi subgroups decorated by line operators in a representation of the preserved Levi subgroup on the gauge theory side.

A maximal degeneration appears when $N-1$ parameters are specialized and the full gauge group $U(N)$ is preserved at the defect. Note that the value of the overall $U(1)$ charge does not affect the structure of modules and breaking of the gauge symmetry. On the other hand, maximally degenerate modules with generic values of the $U(1)$ charge still correspond to a nontrivial GW defect with a prescribed singularity for the $U(1)$ factor. Modules associated to line operators with a trivial GW defect correspond to maximal specializations of all the N parameters with quantized values of the $U(1)$ charge.

Minimal degenerations Let us start with the analysis of domain walls of minimal degenerations associated to the next-to-minimal Levi subgroup.

As we discussed in connection with (5.88), the $N_1+N_2+N_3$ lifted GW parameters $x_i^{(\kappa)}$ correspond to positions of poles of the generating function $\psi(u)$ in the u-plane. The poles are determined up to a permutation of order of poles in each group. A natural question to ask is for which values of parameters $x_j^{(\kappa_j)}$ do we obtain a degenerate module.

The discussion is easy at the first level. A generic module has $N_1+N_2+N_3$ states at this level. We can detect the appearance of a singular vector by studying the rank of the Shapovalov form

$$\langle hw|f_k e_j\rangle hw| = -\langle hw|\psi_{j+k}|hw\rangle \tag{5.121}$$

(where we used the basic commutation relation between e_j and f_j generators of Y). The matrix on the right is a Hankel matrix and we can use a variant of the basic theorem by Kronecker which tells us that (in general infinite dimensional) Hankel matrix has a finite rank if and only if the associated generating function

$$\sum_j \psi_j z^j \tag{5.122}$$

is a Taylor expansion of a rational function. Furthermore, the rank of the Hankel matrix is equal to one plus the degree of this rational function. In our case we have a slightly different version of this theorem because the coefficients ψ_j are Taylor coefficients of

$$\frac{\psi(u)-1}{\sigma_3} \tag{5.123}$$

but the result is the same: the number of vectors at level 1 in the irreducible module with highest weight charges $\psi(u)$ is equal to the degree (i.e. number of zeros counted with multiplicities) of $\psi(u)$. This is automatically consistent with the form of the generating function (5.88) which has generically $N_1 + N_2 + N_3$ zeros and poles. In this way we also rederive the result of [98] that the vacuum representation has exactly one zero and one pole. The distance between them is fixed by the parameters of the algebra. The absolute position of the zero in u-plane is determined by $U(1)$ charge of the highest weight vector and is translated under the spectral shift transformation.

This also refines the statement it [98] that the representation is quasi-finite (i.e. has only a finite number of states at each level) if and only if $\psi(u)$ is a rational function. In the case of Y_{N_1,N_2,N_3} the quasi-finiteness is automatically satisfied.

Applying the results of the previous discussion to the highest weight vector of the generic Y_{N_1,N_2,N_3} module with weights parametrized by (5.88), we conclude that we have a singular vector at level 1 if one of the following conditions is satisfied

$$x_j^{(\tau)} - x_k^{(\sigma)} = -h_\tau, \tag{5.124}$$

i.e. a zero of type j collides with a pole of type k.

At higher levels the discussion is not so simple because the commutation relations used to evaluate the ranks of Shapovalov matrices become more involved. But from the structure of the Shapovalov matrices, we expect the highest singular vectors to appear only if the distance between a zero and a pole of (5.88) is an integer linear combination of h_j parameters. If this assumption of locality (i.e. pairwise interaction between zeros and poles) is satisfied, we can learn more about the relation between the level where such a singular vector appears and the corresponding distance between the zero-pole pair. It is then enough to look at the case of the zero-pole pair of the same type in the algebra $Y_{0,0,2}$ and of different type in the case of $Y_{1,1,0}$.

The first case is simple - we are interested in singular vectors of the Virasoro algebra for which we have a known classification: for generic values of the central charge the Verma module has a singular vector at level rs if and only if the highest weight equals $\Delta_{r,s}$ [84]. The generating function of charges $\psi(u)$ is

$$\psi(u) = \frac{(u - x_1^{(3)} - h_3)(u - x_2^{(3)} - h_3)}{(u - x_1^{(3)})(u - x_2^{(3)})} \tag{5.125}$$

We can extract the conformal dimension Δ with respect to the T_∞ Virasoro subalgebra (decoupled from the $U(1)$ field)

$$\Delta = \frac{h_3^2 - \left(x_1^{(3)} - x_2^{(3)}\right)^2}{4h_1h_2}. \tag{5.126}$$

This is equal to $\Delta_{r,s}$ if and only if

$$x_1^{(3)} - x_2^{(3)} = sh_1 + rh_2, \qquad \text{or} \qquad x_2^{(3)} - x_1^{(3)} = sh_1 + rh_2. \tag{5.127}$$

Therefore, a singular vector of the algebra $Y_{0,0,2}$ appears at level rs if and only if the distance between two poles of the 3rd type is a positive or negative integer linear combination of h_1 and h_2. Similarly for the other two types of poles.

The Kac determinant and singular vectors of $\mathcal{W}_N$ are known as well [133, 7]. The singular vectors (zeros of the Kac determinant) at level rs (where $r, s \geq 1$ are integers) are labeled by roots of $SU(N)$. Choosing the standard ordering (J_1 the leftmost field in the Miura transformation), the equations for vanishing hyperplanes are

$$q^j - q^k + (j-k)h_3 = sh_1 + rh_2 \tag{5.128}$$

where $1 \leq j \neq k \geq N$ label the (positive and negative) roots of $SU(N)$. The poles of $\psi(u)$ are related to $U(1)$ charges q^j (still assuming the standard ordering and using the conventions of (3.47)) by

$$x_j^{(3)} = q^j + (j-1)h_3 \tag{5.129}$$

so we can rewrite the equations for vanishing hyperplanes as

$$x_j^{(3)} - x_k^{(3)} = sh_1 + rh_2. \tag{5.130}$$

This is exactly of the same form as the condition that we found in the case of the Virsoro algebra. We see is that the positive or negative roots in the $\mathcal{W}_N$ language determine which poles of $\psi(u)$ approach each other and the integers s and r determine the distance between these poles, quantized in the units of h_1 and h_2. Therefore, in the case of $\mathcal{W}_N$, we have an independent confirmation of the fact that the leading singular vectors in degenerate modules correspond to pairwise interactions between poles of $\psi(u)$.

In the gauge theory language, we see that (at least in the case of $\mathcal{W}_N$-algebras) degenerations appear when the GW parameters are specialized in such a way that a next-to-minimal Levi subgroup is preserved. The parameters r, s then label representations of the preserved $SU(2)$ subalgebra associated to the corresponding line operators supported at the two interfaces.

The remaining elementary case that we need to analyze is $Y_{1,1.0}$. In this case, the parameter space of generic modules is two-dimensional, so after decoupling the overall $U(1)$, we are left with a one-dimensional parameter space. Analogously to the case of the Virasoro algebra, there is no difference between minimally and maximally degenerate modules. We can look for degenerate modules in at least three possible ways: directly studying the Shapovalov form (Kac determinant), using box counting [98, 2] or using the BRST construction of the algebra [1].

A direct calculation (which we explicitly checked up to level 4) leads to the following condition: given $n \geq 1$, we have a leading singular vector at level n if

$$x_1^{(1)} - x_2^{(2)} = -h_1 - nh_3, \qquad \text{or} \qquad x_1^{(1)} - x_2^{(2)} = h_2 + nh_3. \tag{5.131}$$

Note that these two conditions are exchanged if we formally replace $n \leftrightarrow 1 - n$. We can thus use only one of the conditions with n running over all integers, but for non-positive values of n the level at which corresponding singular vector appears is $1 - n$.

In $Y_{1,1,0}$, there is no difference between the maximally degenerate and minimally degenerate modules. For the maximally degenerate modules we can use the box counting (plane partition) interpretation of modules.[13] The maximally degenerate modules of Y_{110} in this picture correspond to plane partitions (with possible asymptotics) which have no box at position $(2, 2, 1)$. In other words, they can be thought of as pairs of partitions glued together by the first column (assuming for the moment that there is no asymptotics in 3rd direction). The degenerate modules are labeled by two integers, the heights of asymptotic Young diagrams in 1st and 2nd directions. But only the difference of these two integers matters, the modules with the same difference of heights differ only by the overall $U(1)$ charge. Finally, the parameter n appearing in (5.131) can be identified with one plus the difference of the heights of the asymptotic Young diagrams. It is easy to check that this interpretation predicts the correct level of the null vector, the correct irreducible character and the conformal dimension.

Turning on a non-trivial asymptotics in 3rd direction decouples the pair of Young diagrams so the box counting predicts a generic module (i.e. character equal to the square of the free boson character). The conformal dimensions of these modules also don't produce any additional zero of the Shapovalov form, confirming the whole box-counting picture.

The same structure of maximally degenerate modules can also be seen from the BRST analysis of [1]. In particular, the BRST analysis of the algebra have not found any other degenerate modules and the degenerate ones appear exactly for the above values of generic parameters. From the gauge theory point of view, the value n can be identified with the

[13] In general the box counting works only for so called covariant modules which have asymptotics made of boxes (tensor products of the fundamental representation). In general it is important to consider a more general class of representations where there are both asymptotic boxes and anti-boxes. Fortunately in the case of $Y_{1,1,0}$ the anti-box in first direction is equivalent to a box in the second direction and vice versa, so the simple box counting picture is applicable.

difference of charges of the $U(1)$ line operators supported at the boundary 1 and 2. Turning on the Wilson line operator at the boundary 3 lifts the degeneration.

Free field representation of degenerate primaries Let us briefly comment on the realization of the degenerate modules of $Y_{1,1,0}$ in a given free field realization. The highest weight primaries of all the representations (including the generic ones) can be realized as simple exponential vertex operators with exponents given by the parameters q^j (related to $x_j^{(\kappa_j)}$ by constant shifts). It turns out that a half of the degenerate modules associated to the degenerations (5.131) can be also realized in terms of a free boson descendant of an exponential vertex operator. For example, in the $\phi_1^{(1)} \times \phi_2^{(2)}$ ordering, the modules in the 2nd direction specialized to $n = 1$ and $n = 2$ can be realized as

$$\left(h_2 J_1^{(1)} - h_1 J_2^{(2)}\right) \exp\left[q\phi_1^{(1)} + (q + h_3)\phi_2^{(2)}\right],$$
$$-\frac{1}{2}\left(\left(h_2 J_1^{(1)} - h_1 J_2^{(2)}\right)^2 - \partial\left(h_2 J_1^{(1)} - h_1 J_2^{(2)}\right)\right) \exp\left[q\phi_1^{(1)} + (q + 2h_3)\phi_2^{(2)}\right]. \quad (5.132)$$

Similarly, for any $n > 0$, one can realize the corresponding degenerate modules in terms of a level n descendant. The descendants are generally given in terms of Bell polynomials

$$\sum_{m_1 + 2m_2 + \cdots + nm_n = n} \prod_{k=1}^{n} \frac{1}{m_k! k^{m_k}} \left(\frac{-1}{(k-1)!}\partial^{k-1} J\right)^{m_k} \exp\left[q\phi_1^{(1)} + (q + nh_3)\phi_2^{(2)}\right] \quad (5.133)$$

for $J = h_2 J_1^{(1)} - h_1 J_2^{(2)}$. This is analogous to expressions for singular vectors in free field representations of the Virasoro algebra which are given in terms of Jack polynomials [134, 135]. In the case of Y_{110} these reduce to Schur polynomials whose special case are the Bell polynomials (5.133). Higher level specializations will be further discussed in the next section in the context of maximally degenerate representations but note that the issue is present already for the partially degenerate modules associated to specializations of GW parameters at different corners.

Maximally degenerate modules In the previous section, we have discussed the general structure of degenerations of Y-algebra modules and concentrated mostly on the minimally degenerate ones. On the other hand, we will now discuss briefly free field realization of the maximally degenerate modules associated to line operators supported at the interfaces, i.e. trivial GW defects. These modules play an important role in the gluing construction that allows to engineer more complicated VOAs by extensions of tensor products of Y_{N_1,N_2,N_3} algebras [1, 2, 30].

Here, we mostly concentrate on the free field realization of the identity operator together with the modules associated to the line operators in the fundamental and the anti-fundamental representation. All the other maximally degenerate representations can be obtained from the fusion of these two (and a shift of $\mathfrak{gl}(1)$ charge). We will further restrict to the case when $N_3 = 0$. The general case is a bit more complicated because of the appearance of continuous families of free field realizations. We will briefly comment on this issue later. Let us start with writing down the generating function $\psi(u)$ for such representations.

The generating function for the vacuum representation has a single factor

$$\psi_{\bullet}(u) = \frac{u + h_1 h_2 h_3 \psi_0}{u} = \frac{u - N_1 h_1 - N_2 h_2 - N_3 h_3}{u} \tag{5.134}$$

where we used the identity

$$h_1 h_2 h_3 \psi_0 = -N_1 h_1 - N_2 h_2 - N_3 h_3. \tag{5.135}$$

On the other hand the generating function for the fundamental representation in the first direction can be written as

$$\psi_{\square_1}(u) = \frac{(u + h_1 h_2 h_3 \psi_0)(u + h_1)}{(u - h_2)(u - h_3)} \tag{5.136}$$

and similarly for the fundamental representation in the other two directions [98, 2].

The generating function of the anti-fundamental representation can be obtained from the formula for the generating function $\psi(u)$ of a conjugate representation [25]

$$\bar{\psi}(u) = \psi^{-1}\left(-u - h_1 h_2 h_3 \psi_0\right). \tag{5.137}$$

This is a composition of the inverse anti-automorphism and the reflection in spectral parameter and produces an automorphism just as in the case of finite Yangians. The additional spectral shift is necessary in order to have self-conjugate vacuum representation. It is easy to verify that the effect of conjugation is to flip the sign of all odd primary highest weight charges. Note that there exists a conjugation automorphism of the whole affine Yangian (not just acting on the highest weight state), but the ψ_j generators transform in a more complicated way, mixing with e_j and f_j generators.

Applying the conjugation to the generating function (5.136), we get the generating function for the anti-fundamental representation

$$\psi_{\bar{\square}_1} = \frac{(u + h_2 + h_1 h_2 h_3 \psi_0)(u + h_3 + h_1 h_2 h_3 \psi_0)}{u(u - h_1 + h_1 h_2 h_3 \psi_0)} \tag{5.138}$$

and similarly for the other two directions.

5.4.4 Gluing and generic modules

Let us discuss how to glue generic modules and its interpretation in terms of the physics of GW defects. The highest weight vector of a generic module of a Y-algebra can be realized as an exponential vertex operator $\exp\left[Q^\mu \Phi_\mu\right]$, where we introduced a vector of free fields and a dual vector of charges

$$\begin{aligned} \Phi_\mu &= \left(\phi_1^{(2)}, \ldots, \phi_{N_2}^{(2)}, \phi_{N_2+1}^{(3)}, \ldots, \phi_{N_2+N_1}^{(3)}\right) \\ Q^\mu &= \left(q^1, \ldots, q^{N_2}, q^{N_2+1}, \ldots, q^{N_2+N_1}\right) \end{aligned} \tag{5.139}$$

and similarly for the other vertex

$$\begin{aligned} \tilde{\Phi}_\mu &= \left(\tilde{\phi}_1^{(1)}, \ldots, \tilde{\phi}_{N_2}^{(1)}, \tilde{\phi}_{N_2+1}^{(3)}, \ldots, \tilde{\phi}_{N_2+N_1}^{(3)}\right) \\ \tilde{Q}^\mu &= \left(\tilde{q}^1, \ldots, \tilde{q}^{N_2}, \tilde{q}^{N_2+1}, \ldots, \tilde{q}^{N_2+N_1}\right) \end{aligned} \tag{5.140}$$

A generic module of a glued algebra can be then realized as a tensor product of such exponentials associated to each vertex in the diagram.

Note that the parameters q^i and $\tilde{q}^i$ correspond to the same GW defect and the gauge theory setup suggests that they must be identified (up to shifts induced by line operators supported at the boundary A_1 and $\tilde{A}_2$), in particular

$$q^i \pm \tilde{q}^i = n^i h_3 \tag{5.141}$$

for some integers n^i and $h_3 = A_3 \cdot \epsilon = -\tilde{A}_3 \cdot \epsilon = (-A_1 - A_2) \cdot \epsilon$. The relative sign depends on the relative orientation of the two glued vertices. In [2], we defined the orientation of a vertex $Y_{N_1,N_2,N_3}^{A_1,A_2,A_3}$ as a sign given by $(-1)^{p_1 p_2 + q_1 q_2 + p_2 q_1}$. The relative orientation and the sign in the above equation[14] is given by a product of such factors in the two vertices. In particular, one gets -1 for the resolved conifold diagram and $+1$ for the toric diagram of $\mathbb{C}/\mathbb{Z}_2 \times \mathbb{C}$. We will see later in examples that this condition is necessary for the gluing bi-modules to be local with the GW modules.

Note that inclusion of bi-fundamental fields might change the algebra of zero modes that might become non-commutative. Moreover, we will see later that the modules are in general not even modules induced from the modules of the zero-mode algebra. GW modules associated to the commutative zero-mode algebra of Y_{N_1,N_2,N_3} are thus building blocks of modules for more complicated algebras with non-commutative algebra of zero modes.

[14] The sign would be opposite if we have glued the fundamental representation of the first vertex with the fundamental representation of the second vertex and similarly for the anti-fundamental representation.

Gluing two $\widehat{\mathfrak{gl}(1)}$'s Let us come back to the example of gluing of two $\widehat{\mathfrak{gl}(1)}$ Kac-Moody algebras as shown in the figure 4.13. Having an explicit description of the glued algebra in terms of free fields, we would like to discuss generic modules of the glued algebra. According to the discussion above, we expect the correct GW-defect module to be generated by descendants of

$$M\left[q,\tilde{q}\right]=\exp\left[q\phi+\tilde{q}\tilde{\phi}\right], \tag{5.142}$$

where the parameters $\beta,\tilde{\beta}$ are related by (5.141), i.e.

$$q-\tilde{q}=n\epsilon_3 \tag{5.143}$$

for some integer n. Note that this is exactly the condition following form the locality of $M[q,\tilde{q}]$ with the gluing bi-modules $M_{\square}$ and $M_{\tilde{\square}}$. In particular, requiring the OPE to be of the following form

$$M_{\square}(z)M[q,\tilde{q}](w)\sim\frac{\exp\left[q-\epsilon_1,\tilde{q}+\epsilon_2\right](w)}{(z-w)^n}+\dots \tag{5.144}$$

where n is an integer, one gets a constraint

$$\frac{\epsilon_1 q}{\epsilon_1\epsilon_3}-\frac{\epsilon_2\tilde{q}}{\epsilon_2\epsilon_3}=n \tag{5.145}$$

which is the same constraint as (5.143).

Note that the fusion with gluing fields preserve the constraint (5.143) and only shifts the coefficient n. Fields $M[q-\epsilon_1,\tilde{q}+\epsilon_2]$ and $M[q,\tilde{q}]$ are actually vectors of a single module. The only parameter of the module is thus the $\mathfrak{gl}(1)$ charge of the decoupled current $J+\tilde{J}$.

$\widehat{\mathfrak{gl}(2)}$ from gluing Let us now discuss the structure of glued generic modules for the algebra $\widehat{\mathfrak{gl}(2)}$ associated to the figure 4.14. Generic modules can be now constructed from

$$M[q^1,q^2,q^3,q^4]=\exp\left[q^1\phi_1^{(2)}+q^2\phi_2^{(3)}+q^3\phi_3^{(3)}+q^4\tilde{\phi}_1^{(2)}\right] \tag{5.146}$$

where q^1 and q^4 are constrained by the condition

$$q^1+q^4=\epsilon_3 n \tag{5.147}$$

for some integer n.

For each such module, it is simple to compute the action of the $\widehat{\mathfrak{gl}}(2)$ generators on each such vector. Depending on the number n in the constraint above, one gets different structure of the modules. For example, for $n > 1$, one gets

$$\begin{aligned} J_{21}(z)M[q^1,q^2,q^3,q^4](w) &\propto \frac{M[q^1+\epsilon_1,q^2,q^3,q^4-\epsilon_1]}{(z-w)^n}+\dots \\ J_{12}(z)M[q^1,q^2,q^3,q^4](w) &\propto \mathcal{O}((z-w)^{n-2}). \end{aligned} \tag{5.148}$$

For $n < 1$, the singularity is present in the OPE with J_{12} instead. We expect corresponding modules to be a special type of the irregular modules discussed in [136].

The most interesting situation appears when $n = 1$. In such a case both $M_{\square}$ and $M_{\bar{\square}}$ have a simple pole in the OPE with generic modules and one obtains

$$\begin{aligned} J_{11}(z)M[-q^4+\epsilon_3,q^2,q^3,q^4](w) &\sim \frac{q^4}{\epsilon_1}\frac{M[-q^4+\epsilon_3,q^2,q^3,q^4]}{z-w} \\ J_{22}(z)M[-q^4+\epsilon_3,q^2,q^3,q^4](w) &\sim -\frac{q^1+q^2+q^4+\epsilon_2}{\epsilon_1}\frac{M[-q^4+\epsilon_3,q^2,q^3,q^4]}{z-w} \\ J_{12}(z)M[-q^4+\epsilon_3,q^2,q^3,q^4](w) &\sim -\frac{(q^1+q^4)(q^2+q^4-\epsilon_3)}{\epsilon_1^2}\frac{M[-q^4+\epsilon_3-\epsilon_1,q^2,q^3,q^4+\epsilon_1]}{z-w} \\ J_{21}(z)M[-q^4+\epsilon_3,q^2,q^3,q^4](w) &\sim \frac{M[-q^4+\epsilon_3+\epsilon_1,q^2,q^3,q^4-\epsilon_1]}{z-w}. \end{aligned} \tag{5.149}$$

We can see that the zero modes of J_{12} and J_{21} shift the exponent of $M[-q^4+\epsilon_3,q^2,q^3,q^4]$. The representation of the zero-mode subalgebra is thus spanned by $M[-q^4+\epsilon_3+n\epsilon_1,q^2,q^3,q^4-n\epsilon_1]$ for $n \in \mathbb{Z}$.

We can actually show that the above action of zero modes generate a generic Gelfand-Tsetlin module of $\widehat{\mathfrak{gl}(2)}$.

Gelfand-Tsetlin modules for $\mathfrak{gl}(2)$ are parametrized by a triple of complex parameters

$$\begin{pmatrix} \lambda_{21} & & \lambda_{22} \\ & \lambda_{11} & \end{pmatrix} \tag{5.150}$$

where λ_{11} and $\lambda_{11}+n$ are vectors of the same module. For generic values of parameters, the Gelfand-Tsetlin module is spanned by vectors with Gelfand-Tsetlin table of the form

$$\begin{pmatrix} \lambda_{21} & & \lambda_{22} \\ & \lambda_{11}+n & \end{pmatrix} \tag{5.151}$$

for each $n \in \mathbb{Z}$. Generators $J_{11}, J_{22}, J_{12}, J_{21}$ act on such vectors as

$$\begin{aligned}
J_{11} \begin{pmatrix} \lambda_{21} & & \lambda_{22} \\ & \lambda_{11} & \end{pmatrix} &= \lambda_{11} \begin{pmatrix} \lambda_{21} & & \lambda_{22} \\ & \lambda_{11} & \end{pmatrix}, \\
J_{22} \begin{pmatrix} \lambda_{21} & & \lambda_{22} \\ & \lambda_{11} & \end{pmatrix} &= (1 + \lambda_{22} + \lambda_{21} - \lambda_{11}) \begin{pmatrix} \lambda_{21} & & \lambda_{22} \\ & \lambda_{11} & \end{pmatrix}, \\
J_{12} \begin{pmatrix} \lambda_{21} & & \lambda_{22} \\ & \lambda_{11} & \end{pmatrix} &= -(\lambda_{11} - \lambda_{21})(\lambda_{11} - \lambda_{22}) \begin{pmatrix} \lambda_{21} & & \lambda_{22} \\ & \lambda_{11} + 1 & \end{pmatrix}, \\
J_{21} \begin{pmatrix} \lambda_{21} & & \lambda_{22} \\ & \lambda_{11} & \end{pmatrix} &= \begin{pmatrix} \lambda_{21} & & \lambda_{22} \\ & \lambda_{11} - 1 & \end{pmatrix},
\end{aligned} \tag{5.152}$$

Comparing parameters $\lambda_{21}, \lambda_{22}, \lambda_{11}$ with the lifted Gukov-Witten parameters q_i from (5.149), one gets[15]

$$\begin{aligned}
\lambda_{11} &= \frac{q^4}{\epsilon_1}, \\
\lambda_{21} &= -\frac{q^3}{\epsilon_1}, \\
\lambda_{22} &= -\frac{q^2 - \epsilon_3}{\epsilon_1}.
\end{aligned} \tag{5.153}$$

Note that fusion of a vector of the generic module with J_{12} and J_{21} shifts q^4 by an integral multiple of ϵ_1, this corresponds exactly to the shift of parameter λ_{11} by an integer as expected. Note also that the parameters associated to a given face of the toric diagram correspond to Gelfand-Tsetlin parameters of a given row of the Gelfand-Tsetlin table.

5.4.5 Gelfand-Tsetlin modules for $\widehat{\mathfrak{gl}(N)}$ and their $\mathcal{W}$-algebras

$\widehat{\mathfrak{gl}(N)}$ Kac-Moody Algebras In the previous section, we have described the structure of generic modules for the $\widehat{\mathfrak{gl}(2)}$ Kac-Moody algebra. Let us now comment on the structure of generic modules for any $\widehat{\mathfrak{gl}(N)}$ Kac-Moody algebra and $\mathcal{W}$-algebras associated to their Drinfeld-Sokolov reduction.

The Kac-Moody algebra $\widehat{\mathfrak{gl}(N)}$ can be realized in terms of a web diagram in the figure 5.8. The lifted GW parameters associated to internal faces must be again equal up to

[15] There are actually two solutions related by an exchange of $\lambda_{21} \leftrightarrow \lambda_{22}$.

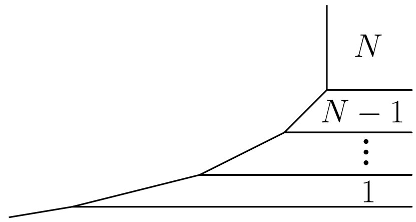


Figure 5.8: The web diagram associated to the $\widehat{\mathfrak{gl}}(N)$ Kac-Moody algebra.

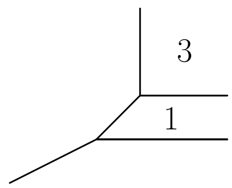


Figure 5.9: The web diagram associated to the $\mathcal{W}_3^{(2)} \times \widehat{\mathfrak{gl}}(1)$ algebra.

shifts induced by line operators supported at the $(1,0)$ interfaces, i.e. they differ by a multiple of $\epsilon_1 + m\epsilon_2$, where $-(m,1)$ are charges of the finite interface of the given face. In the same way as in the case of the $\widehat{\mathfrak{gl}(2)}$ Kac-Moody algebra, one should be able to choose of the shifts of the lifted GW parameters such that the OPEs of J_{ij} for $i > j$ with generic modules have OPE with a simple pole. Generic modules are then going to be parametrized by a Gelfand-Tsetlin table of $\frac{N(N+1)}{2}$ entries. For example, in the case of $\widehat{\mathfrak{gl}}(3)$, the Gelfand-Tsetlin table will be of the form

$$\begin{pmatrix} \lambda_{31} & & \lambda_{32} & & \lambda_{33} \\ & \lambda_{21} & & \lambda_{22} & \\ & & \lambda_{11} & & \end{pmatrix}. \tag{5.154}$$

The parameters in each line will be shifted and renormalized GW parameters associated to a given face. The full modules is then spanned by the vectors

$$\begin{pmatrix} \lambda_{31} & & \lambda_{32} & & \lambda_{33} \\ & \lambda_{21}+n_1 & & \lambda_{22}+n_2 & \\ & & \lambda_{11}+n_3 & & \end{pmatrix} \tag{5.155}$$

for any integers n_1, n_2, n_3. These shifts are generated by the fusion with bi-modules coming from line operators at each internal face.

$\mathcal{W}$-algebras The same structure of modules is expected also for similar configurations with different ranks of gauge groups. The corresponding algebra can be identified with a $\mathcal{W}$-algebra associated to the Drinfeld-Sokolov reduction of the $\widehat{\mathfrak{gl}(N)}$ Kac-Moody algebra possibly with extra symplectic bosons as discussed in [2]. The corresponding Gelfand-Tsetlin modules are parametrized by a generalized Gelfand-Tsetlin table with N_i complex numbers associated to each face with gauge group $U(N_i)$. Except of the N_1 corner parameters in the upper-right face, all the other parameters can be shifted by fusion with bi-modules added to the algebra.

For example the algebra $\mathcal{W}_3^{(2)} \times \widehat{\mathfrak{gl}(1)}$ associated to the diagram 5.9. Have the following Gelfand-Tsetlin table parameterizing generic modules

$$\begin{pmatrix} \lambda_{31} & \lambda_{32} & \lambda_{33} \\ & \lambda_{11} & \end{pmatrix}. \tag{5.156}$$

The full modules is spanned by such vectors with the parameter λ_{11} shifted by any integer.

6. Conclusion and outlook

This thesis reviews some aspects of a large class of vertex operator algebras labelled by colored (p, q) webs. The web diagrams can be given two different interpretations and the corresponding algebras appear in two mutually dual gauge-theoretical setups. The gauge theory perspective then provides many interesting conjectures and insights. Let me conclude by mentioning few interesting directions and open questions:

1. We have given four definitions of the algebra $Y_{N_1,N_2,N_3}[\Psi]$. Can we prove their equivalence?

2. Do all four definitions generalize to the ortho-symplectic version of $Y_{N_1,N_2,N_3}[\Psi]$?

3. Can we glue $\mathcal{W}_{1+\infty}$ algebras associated to any (p, q) web with parameters ρ associated to the internal lines? Is the corresponding algebra a unique two-parameter family with the web algebras being its truncations?

4. What are all the possible free-field realizations of degenerate modules of $Y_{N_1,N_2,N_3}[\Psi]$ for any choice of ordering?

5. How to determine relations of the gluing fields using the free field realization? Which free field representatives and what contour integrals of screening currents do we need?

6. Does the Miura transformation generalizes to other web algebras?

7. Does the gluing story generalize to double truncations associated to points where two truncation curves of the infinity algebra intersect?

8. What representations of Lie superalgebras $\mathfrak{gl}(N|M)$ are needed to produce all the modules needed in the gluing construction?

9. What specializations of Gelfand-Tsetlin modules are needed to find a BRST definition of generic modules of $Y_{N_1,N_2,N_3}[\Psi]$?

10. Can we proof the stable equivalence of algebras associated to different resolutions?

11. Can we associate VOAs to tetravalent and more complicated star-shaped junctions?

12. $\mathcal{W}_{1+\infty}$ is known to be isomorphic to the affine Yangian of $\mathfrak{gl}(1)$. Can we formulate the gluing proposal at the level of affine Yangians?

13. What is the q-deformation of the above story?

14. The algebras should naturally emerge from the cohomological Hall algebra of the toric three-fold. Can we extend the construction of $Y_{N_1,N_2,N_3}[\Psi]$ to any other (p,q) web algebra or beyond? This could be considered as a proof of AGT for gauge theories supported on divisors in toric Calabi-Yau three-folds. Can we go beyond toric manifolds or to higher dimension?

15. The cohomological Hall algebra naturally leads to generic modules of $Y_{N_1,N_2,N_3}[\Psi]$. Is there a geometric construction that produces degenerate modules or the MacMahon module of the corresponding $\mathcal{W}_{1+\infty}$ algebra?

16. The motivation for the BRST definition uses a non-trivial configuration of interfaces in $\mathcal{N}=4$ SYM. What is the exact form of junction conditions and what exactly is the deformation of the interfaces discussed in the main text? Why are the corresponding type IIA and the type IIB twists dual to each other?

17. Is there a holographic description of the algebras?

18. Can we use some of the results to find some consequences in the geometric Langlands program?

References

[1] Davide Gaiotto and Miroslav Rapčák. Vertex Algebras at the Corner. 2017.

[2] Tomáš Procházka and Miroslav Rapčák. Webs of W-algebras. 2017.

[3] Tomáš Procházka and Miroslav Rapčák. $\mathcal{W}$-algebra Modules, Free Fields, and Gukov-Witten Defects. 2018.

[4] Miroslav Rapčák, Yan Soibelman, Yaping Yang, and Gufang Zhao. Cohomological Hall algebras, vertex algebras and instantons. 2018.

[5] Tomáš Procházka. Exploring $\mathcal{W}_\infty$ in the quadratic basis. 11 2014.

[6] Paul H. Ginsparg. APPLIED CONFORMAL FIELD THEORY. In *Les Houches Summer School in Theoretical Physics: Fields, Strings, Critical Phenomena Les Houches, France, June 28-August 5, 1988*, pages 1–168, 1988.

[7] Peter Bouwknegt and Kareljan Schoutens. W symmetry in conformal field theory. *Phys. Rept.*, 223:183–276, 1993.

[8] P. Di Francesco, P. Mathieu, and D. Senechal. *Conformal Field Theory*. Graduate Texts in Contemporary Physics. Springer-Verlag, New York, 1997.

[9] E. Frenkel and D. Ben-Zvi. *Vertex algebras and algebraic curves*. 2004.

[10] Richard E. Borcherds. Vertex algebras, Kac-Moody algebras, and the monster. *Proc. Nat. Acad. Sci.*, 83:3068–3071, 1986.

[11] I. Frenkel, J. Lepowsky, and A. Meurman. *VERTEX OPERATOR ALGEBRAS AND THE MONSTER*. 1988.

[12] A. A. Belavin, Alexander M. Polyakov, and A. B. Zamolodchikov. Infinite Conformal Symmetry in Two-Dimensional Quantum Field Theory. *Nucl. Phys.*, B241:333–380, 1984. [,605(1984)].

[13] Matthias R. Gaberdiel and Rajesh Gopakumar. An AdS_3 Dual for Minimal Model CFTs. *Phys. Rev.*, D83:066007, 2011.

[14] Luis F. Alday, Davide Gaiotto, and Yuji Tachikawa. Liouville correlation functions from four-dimensional gauge theories. 06 2009.

[15] Christopher Beem, Madalena Lemos, Pedro Liendo, Wolfger Peelaers, Leonardo Rastelli, and Balt C. van Rees. Infinite Chiral Symmetry in Four Dimensions. *Commun. Math. Phys.*, 336(3):1359–1433, 2015.

[16] Olivier Schiffmann and Eric Vasserot. Cherednik algebras, W algebras and the equivariant cohomology of the moduli space of instantons on A2. 02 2012.

[17] Edward Frenkel. Lectures on the Langlands program and conformal field theory. In *Proceedings, Les Houches School of Physics: Frontiers in Number Theory, Physics and Geometry II: On Conformal Field Theories, Discrete Groups and Renormalization: Les Houches, France, March 9-21, 2003*, pages 387–533, 2007.

[18] Niclas Wyllard. A(N-1) conformal Toda field theory correlation functions from conformal N = 2 SU(N) quiver gauge theories. *JHEP*, 11:002, 2009.

[19] Edward Witten. Solutions of four-dimensional field theories via M theory. *Nucl. Phys.*, B500:3–42, 1997. [,452(1997)].

[20] Davide Gaiotto. N=2 dualities. *JHEP*, 08:034, 2012.

[21] A. B. Zamolodchikov. Infinite Additional Symmetries in Two-Dimensional Conformal Quantum Field Theory. *Theor. Math. Phys.*, 65:1205–1213, 1985. [Teor. Mat. Fiz.65,347(1985)].

[22] V. A. Fateev and Sergei L. Lukyanov. The Models of Two-Dimensional Conformal Quantum Field Theory with Z(n) Symmetry. *Int. J. Mod. Phys.*, A3:507, 1988. [507(1987)].

[23] Davide Gaiotto. Twisted compactifications of 3d n = 4 theories and conformal blocks. 11 2016.

[24] Thomas Creutzig and Davide Gaiotto. Vertex Algebras for S-duality. 2017.

[25] Matthias R. Gaberdiel, Wei Li, Cheng Peng, and Hong Zhang. The supersymmetric affine Yangian. *JHEP*, 05:200, 2018.

[26] Thomas Creutzig, Davide Gaiotto, and Andrew R. Linshaw. S-duality for the large $N = 4$ superconformal algebra. 2018.

[27] Edward Frenkel and Davide Gaiotto. Quantum Langlands dualities of boundary conditions, D-modules, and conformal blocks. 2018.

[28] Mikhai Bershtein, Boris L. Feigin, and Grigory Merzon. Plane partitions with a "pit": generating functions and representation theory. 06 2015.

[29] Alexey Litvinov and Lev Spodyneiko. On W algebras commuting with a set of screenings. *JHEP*, 11:138, 2016.

[30] Boris Feigin and Sergei Gukov. VOA[M_4]. 2018.

[31] Abhijit Gadde, Sergei Gukov, and Pavel Putrov. Fivebranes and 4-manifolds. 2013.

[32] Matthias R. Gaberdiel, Wei Li, and Cheng Peng. Twin-plane-partitions and $\mathcal{N} = 2$ affine Yangian. 2018.

[33] Koichi Harada and Yutaka Matsuo. Plane partition realization of (web of) $\mathcal{W}$-algebra minimal models. *JHEP*, 02:050, 2019.

[34] Thomas Creutzig and Yasuaki Hikida. Rectangular W-algebras, extended higher spin gravity and dual coset CFTs. *JHEP*, 02:147, 2019.

[35] Yegor Zenkevich. Higgsed network calculus. 2018.

[36] Davide Gaiotto and Tadashi Okazaki. Dualities of Corner Configurations and Supersymmetric Indices. 2019.

[37] Tomáš Procházka. Instanton R-matrix and W-symmetry. 2019.

[38] Ofer Aharony and Amihay Hanany. Branes, superpotentials and superconformal fixed points.

[39] Ofer Aharony, Amihay Hanany, and Barak Kol. Webs of (p, q) 5-branes, Five Dimensional Field Theories and Grid Diagrams.

[40] Anton Kapustin and Edward Witten. Electric-magnetic duality and the geometric langlands program.

[41] Edward Witten. Quantum Field Theory and the Jones Polynomial. *Commun. Math. Phys.*, 121:351–399, 1989.

[42] Nikita Nekrasov and Edward Witten. The Omega Deformation, Branes, Integrability, and Liouville Theory. 02 2010.

[43] Edward Witten. A new look at the path integral of quantum mechanics. 09 2010.

[44] Edward Witten. Analytic Continuation Of Chern-Simons Theory. *AMS/IP Stud. Adv. Math.*, 50:347–446, 2011.

[45] Davide Gaiotto and Edward Witten. Knot Invariants from Four-Dimensional Gauge Theory. *Adv. Theor. Math. Phys.*, 16(3):935–1086, 2012.

[46] Edward Witten. Fivebranes and knots. 01 2011.

[47] Victor Mikhaylov and Edward Witten. Branes and supergroups. 10 2014.

[48] Naichung Conan Leung and Cumrun Vafa. Branes and toric geometry. *Adv. Theor. Math. Phys.*, 2:91–118, 1998.

[49] Mina Aganagic, Albrecht Klemm, Marcos Marino, and Cumrun Vafa. The Topological vertex. *Commun. Math. Phys.*, 254:425–478, 2005.

[50] Amer Iqbal, Can Kozcaz, and Cumrun Vafa. The Refined topological vertex. *JHEP*, 10:069, 2009.

[51] Daniel Jafferis. Crystals and intersecting branes. 2006.

[52] Mina Aganagic, Daniel Jafferis, and Natalia Saulina. Branes, black holes and topological strings on toric Calabi-Yau manifolds. *JHEP*, 12:018, 2006.

[53] Mina Aganagic and Kevin Schaeffer. Refined Black Hole Ensembles and Topological Strings. *JHEP*, 01:060, 2013.

[54] Nikita Nekrasov. BPS/CFT correspondence II: Instantons at crossroads, moduli and compactness theorem. *Adv. Theor. Math. Phys.*, 21:503–583, 2017.

[55] Nikita Nekrasov and Naveen S. Prabhakar. Spiked Instantons from Intersecting D-branes. *Nucl. Phys.*, B914:257–300, 2017.

[56] Nikita Nekrasov and Albert S. Schwarz. Instantons on noncommutative R**4 and (2,0) superconformal six-dimensional theory. *Commun. Math. Phys.*, 198:689–703, 1998.

[57] Nikita A. Nekrasov. Seiberg-Witten prepotential from instanton counting. *Adv. Theor. Math. Phys.*, 7(5):831–864, 2003.

[58] Hiraku Nakajima and Kota Yoshioka. Lectures on instanton counting. In *CRM Workshop on Algebraic Structures and Moduli Spaces Montreal, Canada, July 14-20, 2003*, 2003.

[59] Davesh Maulik and Andrei Okounkov. Quantum groups and quantum cohomology. 11 2012.

[60] Alexander Braverman, Michael Finkelberg, and Hiraku Nakajima. Instanton moduli spaces and w-algebras. 06 2014.

[61] Gregory W. Moore and Nathan Seiberg. Taming the Conformal Zoo. *Phys. Lett.*, B220:422–430, 1989.

[62] Davide Gaiotto and Edward Witten. Supersymmetric Boundary Conditions in N=4 Super Yang-Mills Theory. 04 2008.

[63] Davide Gaiotto and Edward Witten. S-Duality of Boundary Conditions In N=4 Super Yang-Mills Theory. 07 2008.

[64] A. Alekseev and S. Shatashvili. Path integral quantization of the coadjoint orbits of the virasoro group and 2-d gravity. *Nuclear Physics B*, 323(3):719 – 733, 1989.

[65] Alexander M. Polyakov. Quantum Gravity in Two-Dimensions. *Mod. Phys. Lett.*, A2:893, 1987.

[66] Michael Bershadsky and Hirosi Ooguri. Hidden SL(n) Symmetry in Conformal Field Theories. *Commun. Math. Phys.*, 126:49, 1989.

[67] B. Feigin and E. Frenkel. Quantization of the Drinfeld-Sokolov reduction. *Phys. Lett.*, B246:75–81, 1990.

[68] F. A. Bais, T. Tjin, and P. van Driel. Covariantly coupled chiral algebras. *Nucl. Phys.*, B357:632–654, 1991.

[69] Victor G. Kac, Shi shyr Roan, and Minoru Wakimoto. Quantum reduction for affine superalgebras.

[70] Dimitra Karabali and Howard J. Schnitzer. BRST Quantization of the Gauged WZW Action and Coset Conformal Field Theories. *Nucl. Phys.*, B329:649–666, 1990.

[71] Stephen Hwang and Henric Rhedin. The BRST Formulation of G/H WZNW models. *Nucl. Phys.*, B406:165–186, 1993.

[72] Andrei Okounkov, Nikolai Reshetikhin, and Cumrun Vafa. Quantum Calabi-Yau and Classical Crystals.

[73] Maxim Kontsevich and Yan Soibelman. Cohomological Hall algebra, exponential Hodge structures and motivic Donaldson-Thomas invariants. *Commun. Num. Theor. Phys.*, 5:231–352, 2011.

[74] K. Hornfeck. W algebras of negative rank. *Phys. Lett.*, B343:94–102, 1995.

[75] Matthias R. Gaberdiel and Rajesh Gopakumar. Triality in minimal model holography. 05 2012.

[76] Andrew R. Linshaw. Universal two-parameter $\mathcal{W}_\infty$-algebra and vertex algebras of type $\mathcal{W}(2,3,\ldots,N)$. 2017.

[77] Constantin Candu and Matthias R. Gaberdiel. Duality in N=2 Minimal Model Holography. *JHEP*, 02:070, 2013.

[78] Giovanni Felder. BRST Approach to Minimal Models. *Nucl. Phys.*, B317:215, 1989. [Erratum: Nucl. Phys.B324,548(1989)].

[79] Sergei Gukov and Edward Witten. Gauge Theory, Ramification, And The Geometric Langlands Program. 2006.

[80] Alexander Tsymbaliuk. The affine Yangian of $gl(1)$ revisited. *Advances in Mathematics*, 304:583–645, 2017.

[81] Discussions with Thomas Creutzig.

[82] Constantin Candu, Matthias R. Gaberdiel, Maximilian Kelm, and Carl Vollenweider. Even spin minimal model holography. *JHEP*, 01:185, 2013.

[83] Shashank Kanade and Andrew R. Linshaw. Universal two-parameter even spin $\mathcal{W}_\infty$-algebra. 2018.

[84] Philippe Francesco, Pierre Mathieu, and David Sénéchal. *Conformal field theory*. Springer, 2012.

[85] P. Goddard, A. Kent, and David I. Olive. Virasoro Algebras and Coset Space Models. *Phys. Lett.*, 152B:88–92, 1985.

[86] P. Goddard, A. Kent, and David I. Olive. Unitary Representations of the Virasoro and Supervirasoro Algebras. *Commun. Math. Phys.*, 103:105–119, 1986.

[87] V. S. Dotsenko and V. A. Fateev. Conformal Algebra and Multipoint Correlation Functions in Two-Dimensional Statistical Models. *Nucl. Phys.*, B240:312, 1984. [,653(1984)].

[88] B. L. Feigin. Extensions of vertex algebras. Constructions and applications. *Usp. Mat. Nauk*, 72(4):131–190, 2017. [Russ. Math. Surveys72,no.4,707(2017)].

[89] Davide Gaiotto and Edward Witten. Janus Configurations, Chern-Simons Couplings, And The Theta-Angle in N=4 Super Yang-Mills Theory. 04 2008.

[90] David Tong. The holographic dual of ads3 x s3 x s3 x s1. 02 2014.

[91] Jaume Gomis, Bruno Le Floch, Yiwen Pan, and Wolfger Peelaers. Intersecting surface defects and two-dimensional cft. 10 2016.

[92] Herman L. Verlinde. Conformal Field Theory, 2-D Quantum Gravity and Quantization of Teichmuller Space. *Nucl. Phys.*, B337:652–680, 1990.

[93] Adel Bilal. W algebras from Chern-Simons theory. *Phys. Lett.*, B267:487–496, 1991.

[94] Kevin Costello and Davide Gaiotto. Vertex Operator Algebras and 3d $\mathcal{N} = 4$ gauge theories. 2018.

[95] Kevin Costello. M-theory in the omega-background and 5-dimensional non-commutative gauge theory. 10 2016.

[96] Junya Yagi. Compactification on the Ω-background and the AGT correspondence. *JHEP*, 09:101, 2012.

[97] Jeffrey A. Harvey and Gregory W. Moore. On the algebras of BPS states. *Commun. Math. Phys.*, 197:489–519, 1998.

[98] Tomáš Procházka. W-symmetry, topological vertex and affine Yangian. 12 2015.

[99] Edward Frenkel, Victor Kac, and Minoru Wakimoto. Characters and fusion rules for W algebras via quantized Drinfeld-Sokolov reductions. *Commun. Math. Phys.*, 147:295–328, 1992.

[100] SL Lukyanov. Quantization of the Gel'fand-Dikii brackets. *Functional Analysis and Its Applications*, 22(4):255–262, 1988.

[101] Tetsuji Miwa, Michio Jimbo, and Etsuro Date. *Solitons: Differential equations, symmetries and infinite dimensional algebras*, volume 135. Cambridge University Press, 2000.

[102] Boris Khesin and Ilya Zakharevich. Poisson - Lie group of pseudodifferential symbols. *Commun. Math. Phys.*, 171:475–530, 1995.

[103] Boris Feigin, Michio Jimbo, Tetsuji Miwa, Evgeny Mukhin, et al. Quantum toroidal $\mathfrak{gl}_1$-algebra: Plane partitions. *Kyoto Journal of Mathematics*, 52(3):621–659, 2012.

[104] Victor Mikhaylov. Teichmuller TQFT vs Chern-Simons Theory. 2017.

[105] Michael Bershadsky. Conformal field theories via Hamiltonian reduction. *Commun. Math. Phys.*, 139:71–82, 1991.

[106] Alexander M. Polyakov. Gauge Transformations and Diffeomorphisms. *Int. J. Mod. Phys.*, A5:833, 1990.

[107] Kevin Costello. M-theory in the Omega-background and 5-dimensional non-commutative gauge theory. 2016.

[108] E. Bergshoeff, C. N. Pope, L. J. Romans, E. Sezgin, and X. Shen. The Super W(infinity) Algebra. *Phys. Lett.*, B245:447–452, 1990.

[109] L. J. Romans. The $N = 2$ super-W_3 algebra. *Nucl. Phys.*, B369:403–432, 1992.

[110] A. A. Belavin, M. A. Bershtein, B. L. Feigin, A. V. Litvinov, and G. M. Tarnopolsky. Instanton moduli spaces and bases in coset conformal field theory. *Commun. Math. Phys.*, 319:269–301, 2013.

[111] Mikhail Bershtein, Boris Feigin, and Alexei Litvinov. Coupling of two conformal field theories and Nakajima-Yoshioka blow-up equations. *Lett. Math. Phys.*, 106(1):29–56, 2016.

[112] B. Feigin, M. Jimbo, T. Miwa, and E. Mukhin. Branching rules for quantum toroidal gl_n. *Adv. Math.*, 300:229–274, 2016.

[113] Minoru Wakimoto. Fock representations of the affine lie algebra A1(1). *Commun. Math. Phys.*, 104:605–609, 1986.

[114] B. L. Feigin and E. V. Frenkel. Affine Kac-Moody algebras and semiinfinite flag manifolds. *Commun. Math. Phys.*, 128:161–189, 1990.

[115] Matthias R. Gaberdiel and Rajesh Gopakumar. Minimal model holography. 07 2012.

[116] Tomoyuki Arakawa, Thomas Creutzig, and Andrew R. Linshaw. W-algebras as coset vertex algebras. 2018.

[117] Eric G. Gimon and Joseph Polchinski. Consistency conditions for orientifolds and d-manifolds.

[118] David Kutasov. Orbifolds and solitons.

[119] Ashoke Sen. Duality and orbifolds.

[120] Amihay Hanany and Alberto Zaffaroni. Issues on orientifolds: On the brane construction of gauge theories with so(2n) global symmetry.

[121] Bo Feng and Amihay Hanany. Mirror symmetry by o3-planes.

[122] Philip C. Argyres, Anton Kapustin, and Nathan Seiberg. On s-duality for non-simply-laced gauge groups.

[123] D. Brungs and W. Nahm. The Associative algebras of conformal field theory. *Lett. Math. Phys.*, 47:379–383, 1999.

[124] Yongchang Zhu. Modular invariance of characters of vertex operator algebras. 1995.

[125] Wei-qiang Wang. Classification of irreducible modules of $\mathcal{W}_3$ algebra with c=-2. *Commun. Math. Phys.*, 195:113–128, 1998.

[126] Tomoyuki Arakawa, Ching Hung Lam, and Hiromichi Yamada. Zhu's algebra, c_2-algebra and c_2-cofiniteness of parafermion vertex operator algebras, 2012.

[127] Linshaw, Andrew R. Invariant theory and the $W_{1+\infty}$ algebra with negative integral central charge. *arXiv preprint arXiv:0811.4067*, 2008.

[128] Thomas Creutzig and Andrew R. Linshaw. Cosets of affine vertex algebras inside larger structures. 2014.

[129] Andrew R. Linshaw. The Structure of the Kac-Wang-Yan Algebra. *Commun. Math. Phys.*, 345(2):545–585, 2016.

[130] Tomoyuki Arakawa, Thomas Creutzig, Kazuya Kawasetsu, and Andrew R. Linshaw. Orbifolds and Cosets of Minimal W-Algebras. *Commun. Math. Phys.*, 355(1):339–372, 2017.

[131] Victor Kac and Andrey Radul. Quasifinite highest weight modules over the Lie algebra of differential operators on the circle. *Commun. Math. Phys.*, 157:429–457, 1993.

[132] Victor Kac and Andrey Radul. Representation theory of the vertex algebra W(1+infinity). 1995.

[133] S. Mizoguchi. Determinant Formula and Unitarity for the W_3 Algebra. *Phys. Lett.*, B222:226–230, 1989.

[134] Katsuhisa Mimachi and Yasuhiko Yamada. Singular vectors of the virasoro algebra in terms of jack symmetric polynomials. *Communications in mathematical physics*, 174(2):447–455, 1995.

[135] Hidetoshi Awata, Yutaka Matsuo, Satoru Odake, and Jun'ichi Shiraishi. Excited states of Calogero-Sutherland model and singular vectors of the W(N) algebra. *Nucl. Phys.*, B449:347–374, 1995.

[136] Davide Gaiotto and Joel Lamy-Poirier. Irregular Singularities in the H_3^+ WZW Model. 2013.

[137] Masayuki Fukuda, Satoshi Nakamura, Yutaka Matsuo, and Rui-Dong Zhu. SH^c realization of minimal model CFT: triality, poset and Burge condition. *JHEP*, 11:168, 2015.

A. APPENDICES

A.1 Truncation curves

To derive the truncation curves, we proceed by determining the generating function $\psi(u)$ in the affine Yangian language [80, 98] for the first null state in the given module. The state in question is now represented by a cube of dimensions (N_1+1, N_2+1, N_3+1) and it is the first state that does not lie below the corner shifted by (N_1, N_2, N_3). We want to see for which values of parameters of the algebra is this state singular (a null vector).

First of all, the generating function of charges for a configuration of $N_1 \times N_2 \times N_3$ boxes is

$$\psi(u) = \frac{u+\psi_0 h_1 h_2 h_3}{u} \prod_{l=1}^{N_1} \prod_{m=1}^{N_2} \prod_{n=1}^{N_3} \varphi(u - lh_1 - mh_2 - nh_3) \tag{A.1}$$

$$= \frac{(u+\psi_0 h_1 h_2 h_3)(u - N_1 h_1 - N_2 h_2)(u - N_1 h_1 - N_3 h_3)(u - N_2 h_2 - N_3 h_3)}{(u - N_1 h_1)(u - N_2 h_2)(u - N_3 h_3)(u - N_1 h_1 - N_2 h_2 - N_3 h_3)} \tag{A.2}$$

The simple poles in this function are positions where boxes can be added or removed [137, 98]. In particular there is a simple pole at

$$u = N_1 h_1 + N_2 h_2 + N_3 h_3 \tag{A.3}$$

which means that the box at coordinates (N_1, N_2, N_3) can be generically removed. But for special values of parameters h_1, h_2 and h_3 this simple pole can be canceled by a simple zero at $u = -\psi_0 h_1 h_2 h_3$ and this is the equation for the truncation curve:

$$N_1 h_1 + N_2 h_2 + N_3 h_3 = -\psi_0 h_1 h_2 h_3. \tag{A.4}$$

Note that for (N_1, N_2, N_3) truncation we should consider the configuration of boxes with $(N_1+1) \times (N_2+1) \times (N_3+1)$ boxes, but because of the condition $h_1 + h_2 + h_3 = 0$ these give us the same truncation curve.

An alternative way to arrive at this result to is to study vanishing of F coefficient. Its vanishing means that the amplitude for removal of the box at coordinates (N_1+1, N_2+1, N_3+1) vanishes which is exactly the condition for this vector to be the highest weight vector of the submodule it generates. We find

$$F(\Lambda + \square \to \Lambda) \propto \sqrt{N_1 h_1 + N_2 h_2 + N_3 h_3 + \psi_0 h_1 h_2 h_3} \tag{A.5}$$

and the equation (3.35) follows from this immediately.

A.2 Characters

A.2.1 Building blocks

This section contains explicit formulæ for various terms appearing in the calculation of the characters using BRST construction of the algebras discussed in the text. The vacuum character of $\mathcal{W}_N$ algebra is given by

$$\chi_{\mathcal{W}_N}(q) = \prod_{m=0}^{\infty} \prod_{n=1}^{N} \frac{1}{1-q^{n+m}}. \tag{A.6}$$

The characters of the complex $\mathcal{S}_m^{M|L}$ of M symplectic bosons and L free fermions with the level shifted by m is

$$\chi_m^{M|L}(q, x_i, y_j) = \prod_{n=0}^{\infty} \prod_{i=1}^{M} \prod_{j=1}^{L} \frac{\left(1+y_j q^{n+\frac{1}{2}+m}\right)\left(1+y_j^{-1} q^{n+\frac{1}{2}+m}\right)}{\left(1-x_i q^{n+\frac{1}{2}+m}\right)\left(1-x_i^{-1} q^{n+\frac{1}{2}+m}\right)}. \tag{A.7}$$

The character of $\overline{\mathcal{S}}_m^{M|L}$ with M fermionic and L bosonic components has analogous character with $x_i \leftrightarrow y_i$ together with $M \leftrightarrow L$ interchanged.

The projection onto $U(M|L)$ invariant combinations is performed by integration with the Vandermonde measure

$$dV_{M,L} = \frac{1}{M!L!} \prod_{i=1}^{M} \frac{dx_i}{x_i} \prod_{j=1}^{L} \frac{dy_j}{y_j} \frac{\prod_{i_1>i_2}\left(1-\frac{x_{i_1}}{x_{i_2}}\right) \prod_{j_1>j_2}\left(1-\frac{y_{j_1}}{y_{j_2}}\right)}{\prod_i \prod_j \left(1+\frac{x_i}{y_j}\right)\left(1+\frac{y_j}{x_i}\right)}. \tag{A.8}$$

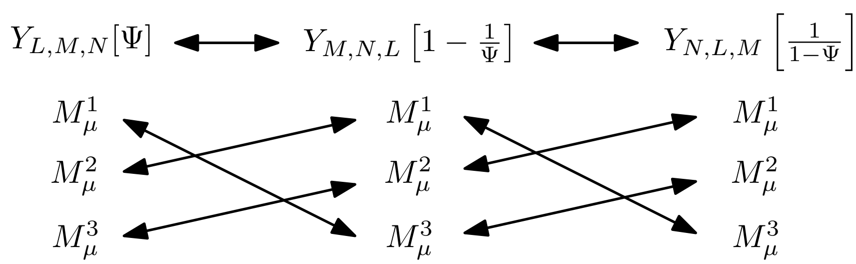
In the generic $U(M|L)$ case, the denominator needs to be expanded and regularized. In all the examples of this paper, we restrict to the case of $M = 0$ or $L = 0$ when the denominator vanishes and we do not have to deal with these technicalities.

In later sections, we also need the character of free fermions and symplectic bosons with shifted dimension

$$\chi_m^{\mathcal{F}} = \prod_{n=0}^{\infty} \left(1 + q^{n+\frac{1}{2}+m}\right),$$
$$\chi_m^{\mathcal{B}} = \prod_{n=0}^{\infty} \frac{1}{1 - q^{n+\frac{1}{2}+m}}. \tag{A.9}$$

A.2.2 S-duality transformations of modules

Triality transformation of modules of Y-algebras is given by following diagram:



S-duality then acts as

$$Y_{L,M,N}[\Psi] \;\leftrightarrow\; Y_{M,L,N}\left[\frac{1}{\Psi}\right], \qquad M_\mu^1 \;\leftrightarrow\; M_\mu^2. \tag{A.10}$$

A.3 Transformation between primary and quadratic bases

We list first few formulas relating the primary basis generators W_j to the generators U_j in quadratic basis.

$$
\begin{aligned}
W_1 &= -U_1 \\
W_2 &= -U_2 + \frac{N-1}{2N}(U_1U_1) + \frac{(N-1)\alpha_0}{2}U_1' \\
W_3 &= -U_3 + \frac{N-2}{N}(U_1U_2) - \frac{(N-1)(N-2)}{3N^2}(U_1(U_1U_1)) - \frac{(N-1)(N-2)\alpha_0}{2N}(U_1'U_1) \\
&\quad + \frac{(N-2)\alpha_0}{2}U_2' - \frac{(N-1)(N-2)\alpha_0^2}{12}U_1'' \\
W_4 &= -U_4 + \frac{(N-3)(N-2)(N-1)(5N+6)\alpha_0(\alpha_0^2N^2-\alpha_0^2N-1)}{2N^2(5\alpha_0^2N^3-5\alpha_0^2N-5N-17)}(U_1'(U_1U_1)) \\
&\quad + \frac{(N-3)(N-2)(N-1)(\alpha_0^2N^2-\alpha_0^2N-1)(2\alpha_0^2N^2+3\alpha_0^2N-3)}{4N^2(5\alpha_0^2N^3-5\alpha_0^2N-5N-17)}(U_1'U_1') \\
&\quad - \frac{(N-3)(N-2)(N-1)\alpha_0(5\alpha_0^2N^2+7\alpha_0^2N-5)}{2N(5\alpha_0^2N^3-5\alpha_0^2N-5N-17)}(U_1'U_2) \\
&\quad + \frac{(N-3)(N-2)(N-1)(5N+6)(\alpha_0^2N^2-\alpha_0^2N-1)}{4N^3(5\alpha_0^2N^3-5\alpha_0^2N-5N-17)}(U_1,(U_1,(U_1,U_1))) \\
&\quad - \frac{(N-3)(N-2)(5N+6)(\alpha_0^2N^2-\alpha_0^2N-1)}{N^2(5\alpha_0^2N^3-5\alpha_0^2N-5N-17)}(U_1(U_1U_2)) \\
&\quad + \frac{(N-3)(N-2)(5\alpha_0^2N^2+7\alpha_0^2N-5)}{2N(5\alpha_0^2N^3-5\alpha_0^2N-5N-17)}(U_2U_2) \\
&\quad + \frac{(N-3)(N-2)(N-1)(2\alpha_0^4N^4-5\alpha_0^2N^3-2\alpha_0^4N^2-7\alpha_0^2N^2-4\alpha_0^2N+5N-2)}{4N^2(5\alpha_0^2N^3-5\alpha_0^2N-5N-17)}(U_1''U_1) \\
&\quad + \frac{(N-3)(N-2)(N-1)\alpha_0(\alpha_0^4N^4-10\alpha_0^2N^3-\alpha_0^4N^2-14\alpha_0^2N^2-2\alpha_0^2N+10N-1)}{24N(5\alpha_0^2N^3-5\alpha_0^2N-5N-17)}U_1^{(3)} \\
&\quad - \frac{(N-3)(N-2)(2\alpha_0^4N^4-2\alpha_0^4N^2-5\alpha_0^2N^2-11\alpha_0^2N+3)}{4N(5\alpha_0^2N^3-5\alpha_0^2N-5N-17)}U_2'' \\
&\quad - \frac{(N-3)(N-2)\alpha_0}{2N}(U_1U_2') + \frac{(N-3)}{N}(U_1,U_3) + \frac{(N-3)\alpha_0}{2}U_3'
\end{aligned}
$$

We choose the normalization such that $W_j = -U_j + \ldots$. Since this choice of normalization is rather arbitrary, we should also specify the values of structure constants that fix the relative normalization of the charges:

$$
\begin{aligned}
C_{11}^0 &= N \\
C_{22}^0 &= \frac{1}{2}(N-1)(1-N(N+1)\alpha_0^2) \\
C_{33}^0 &= \frac{(N-1)(N-2)(1-N(N+1)\alpha_0^2)(4-N(N+2)\alpha_0^2)}{6N} \\
C_{44}^0 &= \frac{(N-1)(N-2)(N-3)(N+1)}{4N^2(5N^3\alpha_0^2-5N\alpha_0^2-5N-17)} \times (1-N(N+1)\alpha_0^2)(4-N(N+2)\alpha_0^2) \times \\
&\quad \times (9-N(N+3)\alpha_0^2)(1-N(N-1)\alpha_0^2) \\
C_{55}^0 &= \frac{(N-1)(N-2)(N-3)(N-4)(N+1)}{10N^3(7N^3\alpha_0^2-7N\alpha_0^2-7N-107)} \times (1-N(N+1)\alpha_0^2)(4-N(N+2)\alpha_0^2) \times \\
&\quad \times (9-N(N+3)\alpha_0^2)(16-N(N+4)\alpha_0^2)(1-N(N-1)\alpha_0^2)
\end{aligned}
$$

Acting on the highest weight state, the relation between charges becomes somewhat simpler

$$
\begin{aligned}
w_1 &= -u_1 \\
w_2 &= -u_2 + \frac{N-1}{2N}u_1^2 - \frac{(N-1)\alpha_0}{2}u_1 \\
w_3 &= -u_3 + \frac{N-2}{N}u_1u_2 - \frac{(N-1)(N-2)}{3N^2}u_1^3 - (N-2)\alpha_0 u_2 \\
&\quad + \frac{(N-1)(N-2)\alpha_0}{2N}u_1^2 - \frac{(N-1)(N-2)\alpha_0^2}{6}u_1
\end{aligned}
$$

$$
\begin{aligned}
w_4 \;=\; & -u_4 + \frac{N-3}{N}u_1u_3 + \frac{(N-3)(N-2)(5\alpha_0^2N^2+7\alpha_0^2N-5)}{2N(5\alpha_0^2N^3-5\alpha_0^2N-5N-17)}u_2^2 \\
& -\frac{(N-3)(N-2)(5N+6)(\alpha_0^2N^2-\alpha_0^2N-1)}{N^2(5\alpha_0^2N^3-5\alpha_0^2N-5N-17)}u_1^2u_2 \\
& +\frac{(N-3)(N-2)(N-1)(5N+6)(\alpha_0^2N^2-\alpha_0^2N-1)}{4N^3(5\alpha_0^2N^3-5\alpha_0^2N-5N-17)}u_1^4 - \frac{3(N-3)\alpha_0}{2}u_3 \\
& +\frac{(N-3)(N-2)\alpha_0(15\alpha_0^2N^3+2\alpha_0^2N^2-17\alpha_0^2N-15N-29)}{2N(5\alpha_0^2N^3-5\alpha_0^2N-5N-17)}u_1u_2 \\
& -\frac{(N-3)(N-2)(N-1)(5N+6)\alpha_0(\alpha_0^2N^2-\alpha_0^2N-1)}{2N^2(5\alpha_0^2N^3-5\alpha_0^2N-5N-17)}u_1^3 \\
& -\frac{(N-3)(N-2)(6\alpha_0^4N^4-6\alpha_0^4N^2-5\alpha_0^2N^2-19\alpha_0^2N-1)}{2N(5\alpha_0^2N^3-5\alpha_0^2N-5N-17)}u_2 \\
& +\frac{(N-3)(N-2)(N-1)(\alpha_0^2N^2-\alpha_0^2N-1)(6\alpha_0^2N^2+7\alpha_0^2N+1)}{4N^2(5\alpha_0^2N^3-5\alpha_0^2N-5N-17)}u_1^2 \\
& -\frac{(N-3)(N-2)(N-1)\alpha_0(\alpha_0^2N^2-\alpha_0^2N-1)(\alpha_0^2N^2+\alpha_0^2N+1)}{4N(5\alpha_0^2N^3-5\alpha_0^2N-5N-17)}u_1
\end{aligned}
$$

As one can see from these expressions, they are becoming increasingly complicated and it is unfortunate that no closed-form expression for the primary charges is known.