Rational approximations on smooth rational surfaces

by

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Author's Declaration

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

In this thesis, we study a conjecture made by D. McKinnon about rational approximations to rational points in algebraic varieties. The conjecture states that if a rational point P on a variety X lies on a rational curve, then the best approximations to P can be chosen to lie along a rational curve on X. According to the conditions of the conjecture, it is natural to study this problem on algebraic varieties that contain a dense subset of rational points. Motivated by this remark, we study the conjecture on smooth rational surfaces, which not only contain a dense set of rational points, but their classification is also well understood.

Given a point P on an algebraic variety and an ample divisor D, the approximation constant $\alpha_P(D)$ measures how well P can be approximated by rational points on the variety, with respect to a height function associated to D. In the study of the conjecture, it became clear that if a curve C contains the best approximations to P with respect to ample divisors D and D', then C turns out to be also a curve containing the best approximations for any divisor that is a linear combination of D and D'. This property motivated the study of the nef cone of the algebraic variety. Every ample divisor belongs to the interior of the nef cone and can be written as a linear combination of the generators of the nef cone.

By an exhaustive study of the effective and nef cones on a smooth rational surface, it is possible to find a curve that contains the best approximations to the point P with respect to an ample divisor, which can be written in terms of the generators of the nef cone. In this work we use the fact that a smooth rational surface is obtained by a finite number of blow-ups of a Hirzebruch surface or of the projective plane. The Hirzebruch surfaces are equipped with morphisms to the projective line $\mathbb{P}^1$ and to cones in some projective space. The study of the fibres of these morphisms provides good candidates of curves with best approximations and we rely on them to prove the conjecture for these cases.

We review the conjecture proved by McKinnon in the case of smooth rational surfaces of Picard rank 4. We explore some of the examples in this case to present the techniques using the nef cone of the variety, and then we extend the result for surfaces of bigger Picard ranks. Finally, we extend the result to surfaces obtained by blowing up an arbitrary number of times on smooth points of the reducible fibre of the map to $\mathbb{P}^1$.

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Dedication

To my grandma, the most courageous woman that I've ever met. Thanks for all your love and care, I know you will be always with me.

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Chapter 1

Introduction

The main question in the area of Diophantine approximations asks how well an irrational number can be approximated by rational numbers. Explicitly, given $x \in \mathbb{R}$, we want to study how closely we can approximate x by rational numbers $\frac{p}{q}$. It is known that since rational numbers are dense in $\mathbb{R}$, we can approximate x as closely as we want. However, if we put some constraints on these approximations, the question becomes more subtle.

In this context there are three main theorems that describe the historical progress on answering the approximation problem on the real numbers.

Theorem 1.1. *(**Dirichlet (1842)**). Let $x \in \mathbb{R}$, $x \notin \mathbb{Q}$. Then there are infinitely many rational numbers $\frac{p}{q}$ such that $|x - \frac{p}{q}| \leq \frac{1}{q^2}$.*

Theorem 1.1 tells us that we can always approximate an irrational number x, by a rational number $\frac{p}{q}$ within a distance of $\frac{1}{q^2}$. One of the applications of this theorem in number theory is its connections with arithmetic progressions and distribution of prime numbers. For more details see [12, Part D].

Theorem 1.2. *(**Liouville (1844)**). Let x be a real algebraic number of degree $d \geq 2$, then for any $\epsilon > 0$ there are only finitely many rational numbers $\frac{p}{q}$ such that $|x - \frac{p}{q}| \leq \frac{1}{q^{d+\epsilon}}$.*

Liouville's theorem implies that algebraic numbers cannot be closely approximated by rational numbers. Liouville used this result to prove the existence of trascendental numbers.

Theorem 1.3. ***(Roth (1955))****. Let x be a real algebraic number, then for any $\epsilon > 0$ there are only finitely many rational numbers $\frac{p}{q}$ such that $|x - \frac{p}{q}| \leq \frac{1}{q^{2+\epsilon}}$.*

It took about a century to complete the gap between the approximation exponents provided by Dirichlet and Liouville. There were improvements to the approximation exponent, but it was Roth who provided the argument that essentially the exponent 2 is the best possible. The proof of this fact by Klaus Roth made him the winner of the Fields medal in 1958.

These problems from number theory are reformulated in algebraic geometry. Given a rational point in an algebraic variety defined over a number field, how well can we approximate this point by rational points in the variety? We study this problem for the case of smooth surfaces birational to $\mathbb{P}^2$ and study where these approximations can be found.

In general, we can take an algebraic variety X over $\mathbb{Q}$ and a point $x \in X(\overline{\mathbb{Q}})$ and study how well we can approximate x by rational points in $X(\mathbb{Q})$. Roth's theorem can be rewritten in the context of algebraic varieties as it was described by D. McKinnon and M. Roth in [21]. The authors defined an invariant $\alpha_x(L)$ that measures how quickly rational points accumulate around x with respect to a divisor L. We briefly describe these constructions in the next chapter.

In the process of calculating the approximation constants in algebraic varieties, in particular when approximating a rational point x in $X(\mathbb{Q})$, it was possible to find sequences of best approximations to x lying along a curve. In [19], the author made the conjecture that sequences of best approximation to x can be chosen to lie in a curve as long as the point x lies in a rational curve defined over $\mathbb{Q}$.

In general, we will take an algebraic variety X defined over a number field k and any rational point point $x \in X(k)$. We want to study how "well" we can approximate x by a sequence of rational points in $X(k)$ different from x. To study this problem, first we need an interpretation of Roth's theorem in terms of varieties. So, we rewrite the statement of Roth's theorem as: For any $\epsilon > 0$, there are only finitely many numbers $\frac{p}{q}$ such that

$$q\left|x - \frac{p}{q}\right|^{\frac{1}{2}-\epsilon} \leq 1. \tag{1.1}$$

The factor q is known as the "complexity" of the rational point $\frac{p}{q}$ and the power $\frac{1}{2}$ is what we call the "approximation exponent" to describe the smallest exponent with the property that the inequality (1.1) has only finitely many solutions $\frac{p}{q} \in \mathbb{Q}$. Notice that now we are looking for the smallest exponent, since we moved the exponent to the distance and we got the reciprocal $\frac{1}{2}$ of the original exponent 2.

In order to interpret Roth's theorem on algebraic varieties, we need to describe the distance and the complexity in an algebraic variety. We briefly describe these notions here.

1.1 Complexity: Height functions

To study the complexity of a rational point in an algebraic variety, we use height functions. It is well known that given a divisor class D in an algebraic variety X, there is a height function $H_D : X(\overline{\mathbb{Q}}) \longrightarrow [0, \infty)$. For details in this construction and the properties of H_D see [12, Section B.3] or [15, Chapter 4]. In particular, if we choose an ample divisor class D, its associated height function H_D measures the complexity of the point on the variety X.

One important property for the height associated to an ample divisor class is the *finiteness property.* Explicitly, if D is an ample divisor on a projective variety X, the set $\{x \in X(\mathbb{Q}) : \ H_D(x) \leq B\}$ is finite for all $B \in \mathbb{R}$. This property is also known as the *Northcott property.*

The choice of height function H_D with respect to a divisor class D is not unique. However, all the choices of heights with respect to a divisor are equivalent. Concretely, given two choices of heights associated to a divisor D, H_D and H'_D, there exist real constants c and C with $0 < c \leq C$ such that $cH'_D \leq H_D \leq CH'_D$.

Example 1.4. *Let $X = \mathbb{P}^n_{\mathbb{Q}}$, let D be the divisor class of a hyperplane in X, and let $x = [x_0 : x_1 : \cdots : x_n] \in X(\mathbb{Q})$. We can choose the coordinates $x_i \in \mathbb{Z}$ such that* $\gcd(x_0, x_1, \cdots, x_n) = 1$. *Then*

$$H_D(x) = \max\{|x_0|, |x_1|, \cdots, |x_n|\}.$$

This definition of the height can be extended to number fields. More precisely, let k be a number field. Consider D a divisor class of a hyperplane in $\mathbb{P}^n(k)$ and $x = [x_0 : x_1 : \cdots : x_n] \in \mathbb{P}^n(k)$, then

$$H_D(x) = \prod_{v \in M_k} \max\{\|x_0\|_v, \|x_1\|_v, \cdots, \|x_n\|_v\}.$$

where M_k is the set of all places of k and all of them are normalized with respect to k. In this work all the heights will be multiplicative, relative to the number field k.

We will simply denote this height as H with no sub-index or $H_{\mathcal{O}(1)}$, where $\mathcal{O}(1)$ is the line bundle on $\mathbb{P}^n$ associated to a hyperplane.

Recall that an ample divisor D on a variety X defined over k, induces an embedding $\phi_D : X \to \mathbb{P}^n$ of X into some projective space $\mathbb{P}^n$. Thus, it is natural to define a height function H_D on $X(\overline{k})$ such that for each point $x \in X(\overline{k})$, $H_D(x) = H(\phi_D(x))$. Using this fact, one can construct a height function for each divisor in $\mathrm{Div}(X)$. See [12].

1.2 Distance functions

Let k be a number field, X be a variety over k, and $\upsilon \in M_k$ be a place in k. We also denote by υ the completion of υ in k_υ, where k_υ is the complete field under υ, i.e. k_υ is obtained from k by adding all the limits of Cauchy sequences in k with respect to the place υ.

There is a standard procedure to define a distance function d_υ on $X(k_\upsilon)$ by choosing an embedding $X \hookrightarrow \mathbb{P}^n_k$ and pulling back the distance function d_υ on $\mathbb{P}^n(k_\upsilon) \times \mathbb{P}^n(k_\upsilon)$. The following are common distances that are used in the literature. For properties of these υ-adic distances, see Silverman [24].

If υ archimedean: Let $x = [x_0 : x_1 : \cdots : x_n]$ and $y = [y_0 : y_1 : \cdots : y_n]$ be points in $\mathbb{P}^n(\mathbb{C})$.

$$d_\upsilon(x, y) = \left(1 - \frac{|\sum_{i=1}^n x_i \overline{y_i}|^2}{(\sum_{i=1}^n |x_i|^2)(\sum_{i=1}^n |y_i|^2)}\right)^{[k_\upsilon:\mathbb{R}]/2}$$

If υ non-archimedean: Let $x = [x_0 : x_1 : \cdots : x_n]$ and $y = [y_0 : y_1 : \cdots : y_n]$ be points in $\mathbb{P}^n(k_\upsilon)$.

$$d_\upsilon(x, y) = \frac{\max_{0 \le i < j \le n} \|x_i y_j - x_j y_i\|_\upsilon}{(\max_{0 \le i \le n} \|x_i\|_\upsilon)(\max_{0 \le j \le n} \|y_j\|_\upsilon)}$$

Notice that the distance defined depends on the embedding of X into projective space. However, it was proven in [20, Proposition 2.4] that the distances associated to a place υ are all equivalent. Moreover, the following proposition was proven in the same paper (Lemma 2.6):

Proposition 1.5. *Let $x \in X(\overline{k})$, and K be any finite extension of k over which x is defined. Let U be any open affine subset of X_K containing x. Let $u_1,\ u_2,\ \cdots, u_r$ be any elements of $\Gamma(U, O_{X_K})$ that generate the maximal ideal of x. Then for any sequence $\{x_i\} \subset U(K_v)$ such that $d_v(x_i, x) \to 0$ as $i \to \infty$, $d_v(x, x_i)$ is equivalent to* $\max(\|u_1(x_i)\|_v, \cdots, \|u_r(x_i)\|_v)$.

For our purposes of finding approximation constants, the choice of the distance will not change the values of the approximation constants. Since we will study sequences $\{x_i\} \in X(k)$ such that $d_v(x, x_i) \to 0$, we will use Proposition 1.5 to calculate the distances when needed.

1.3 The approximation constant

Recall that the *approximation exponent* is the smallest exponent for which the inequality (1.1) has only finitely many rational solutions. In order to generalize this exponent in the context of algebraic varieties, we define the last ingredient, called the *approximation constant* of a point on a variety X with respect to a divisor on X.

Definition 1.6. *Let D be a divisor on X and $x \in X(\overline{k})$. Consider a sequence $\{x_i\} \subset X(k)$ of distinct points such that $d_v(x, x_i) \to 0$. The approximation constant of $\{x_i\}$ with respect to D is*

$$\alpha_x(\{x_i\}, D) := \inf\{\gamma \in \mathbb{R} :\ d_v(x_i, x)^\gamma H_D(x_i) \text{ is bounded}\}.$$

We also define the approximation constant of x with respect to D as

$$\alpha_x(D) := \inf_{\{x_i\} \to x} \{\alpha_x(\{x_i\}, D)\}.$$

Where the last definition takes the infimum over all the possible sequences that converge to x with respect to a previously fixed υ-adic distance. If there is no such sequence $\{x_i\} \to x$ then we set $\alpha_x(D) = \infty$.

Example 1.7. *In this new language, consider $X = \mathbb{P}^1$ be the projective line over the algebraic closure of $\mathbb{Q}$, and let $x \in \mathbb{P}^1(\overline{\mathbb{Q}})$, and H be a hyperplane (i.e. a divisor of a point)* *in X.*

$$\begin{aligned} \textit{Liouville} &: \alpha_x(H) \geq \frac{1}{d}, \ \textit{where } x \in \mathbb{R}, \quad d = [\mathbb{Q}(x) : \mathbb{Q}] \\ \textit{Dirichlet} &: \alpha_x(H) \leq \frac{1}{2}, \ x \in \mathbb{R} \cap \overline{\mathbb{Q}}, \ x \notin \mathbb{Q} \\ \textit{Roth} &: \alpha_x(H) \geq \frac{1}{2}, \ x \in \mathbb{R} \cap \overline{\mathbb{Q}}. \end{aligned}$$

In the course of computing approximation constants on algebraic varieties, it was possible to find sequences of best approximation (i.e. sequences $\{x_i\} \subset X(k)$ for which $\alpha_x(D) = \alpha_x(\{x_i\}, D)$) lying along curves passing through the point being approximated. This idea gave origin to the notion of *curves of best approximation*, as those curves containing a sequence of best approximation. In chapter 2, we will give more details about this type of curves.

The following conjecture was studied in [19] for smooth rational surfaces of Picard rank at most 4.

Conjecture 1.8. *Let X be an algebraic variety defined over k and D be any ample divisor on X. Let x be any k-rational point on X such that there is some rational curve on X, defined over k and passing through x. Then, there exists a curve of best approximation to x on X with respect to D.*

The results of this thesis are analogs of the approximation of rational numbers by sequences of rational numbers. We restrict to this case because the Conjecture 1.8 is known to be false if instead, we approximate an algebraic point $P \in X(\overline{k})$ not in $X(k)$.

Additionally, notice that if we do not impose the condition of x lying in a rational curve is important, then the Conjecture 1.8 is false. For instance, abelian varieties contain a dense set of rational points, yet they do not contain rational curves.

The main purpose of this thesis is to study this conjecture on smooth rational surfaces. This type of varieties have two main characteristics that motivate the study of the conjecture in these cases. First, the fundamental property of this type of varieties is that they contain a dense set of rational points, so we are able to find infinite sequences of best approximation. Second, these types of varieties have a nice classification provided, for instance, in [3], [1], [18], [6]. All of these surfaces are obtained by doing a finite number of blow-ups of the ruled surfaces. In this work, we study this conjecture for smooth surfaces of bigger Picard ranks. By making use of the classification, in this thesis we prove:

Theorem A. *Let X be the blow-up of a surface of Picard rank 4 with only one reducible fibre F of the map to $\mathbb{P}^1$, at k smooth points of F. Then the conjecture is true for X.*

Theorem B. *Let X be the blow-up of a surface of Picard rank 4 with only one reducible fibre F of the map to $\mathbb{P}^1$, at a singular point of F. Then the conjecture is true for X.*

Theorem C. *Let $n \geq 2$, and let X be the blow-up of the Hirzebruch surface $\mathbb{F}_n$ at $k < n$ points, no two of which lie in the same fibre of the map to $\mathbb{P}^1$. Let Y be the blow-up of X at $s \leq k$ points all lying on different reducible fibres of the map to $\mathbb{P}^1$, then the conjecture is true for Y.*

One of the most important contributions of this work is the study of the effective and nef cones of the surfaces mentioned in the theorems above. Readers interested on using the nef and the effective cones of smooth rational surfaces can find in this work the explicit descriptions of the generators of these cones.

In chapter 2, we briefly describe some of the properties of the approximation constant that will be used later in this work. We also review the properties of curves of best approximation. In chapter 3, we review some of the properties of smooth rational surfaces, the effective cone, and the nef cone.

In chapter 4, we revisit some of the examples developed by D. McKinnon in [19] to understand the methods used in calculating the curves of best approximation. In chapter 5, we continue the work on proving the conjecture for smooth rational surfaces of Picard rank five and try to extend the results to smooth surfaces of bigger Picard rank by using the explicit description of the nef cone in each case.

Chapter 2

Properties of α and curves of best approximation

In this chapter we state and prove some of the properties of the constant α and the curves of best approximation, that are used later in this work. We state the main conjecture that will be studied for certain algebraic varieties. Most of the properties in this chapter were presented in [19], [21], and [20].

2.1 Properties of the approximation constant α

The following proposition was proven in [19, Proposition 2.6].

Proposition 2.1. *Let $x \in \mathbb{P}^n(k)$ and let H be a hyperplane in $\mathbb{P}^n$. Then $\alpha_x(H) = 1$ and any line L through x contains a sequence $\{x_i\} \subset L(k)$ for which $\alpha_x(H) = \alpha_x(\{x_i\}, H)$.*

Remark 2.2. *Proposition 2.1 is independent of the choice of the υ-adic distance on k. This is the main reason why in our definition of the approximation exponent we do not define $\alpha_x(D)$ relative to the distance. Conjecture 1.8 relies on the analog of approximating rational points by rational points on a projective variety over a number field k. Since the varieties studied in this work are considered embedded into some projective space, the results of this work are also independent of the choice of the distance.*

Proposition 2.3. *Let X be a projective variety over Spec(k), $x \in X(\overline{k})$, and D and D' be divisors on X, then*

1. *Let $\{x_i\} \subset X(k)$ be a sequence of points converging to x. Let $\{x_i'\}$ be a subsequence of $\{x_i\}$. Then, $\alpha_x(\{x_i'\}, D) \leq \alpha_x(\{x_i\}, D)$.*

2. *For any positive integer n, $\alpha_{x,X}(nD) = n \cdot \alpha_{x,X}(D)$.*

3. *For any a, b positive rational numbers,*

$$\alpha_{x,X}(aD + bD') \geq a\alpha_{x,X}(D) + b\alpha_{x,X}(D').$$

4. *Suppose that X is reducible over k and let X_1, X_2, $\cdots$, X_r be the irreducible components over k containing x. Then*

$$\alpha_{x,X}(D) = \min\{\alpha_{x,X_1}(D|_{X_1}), \cdots, \alpha_{x,X_r}(D|_{X_r})\}.$$

5. *If D is an ample divisor and $x \in X(k)$, then $\alpha_x(D) > 0$.*

Proof. The following proofs are given in [20]. We provide brief proofs of them here.

1. Notice that the sequence $d_v(x, x_i')^\gamma H_D(x_i')$ is a subsequence of $d_v(x, x_i)^\gamma H_D(x_i)$. Hence, any value for γ that makes the latter bounded will make the former bounded. Thus, $\alpha_x(\{x_i'\}, D) \leq \alpha_x(\{x_i\}, D)$.

2. This follows because $H_{nD} = H_D^n$ and thus $d_v(x, x_i)^\gamma H_{nD}(x_i) = (d_v(x, x_i)^{\frac{\gamma}{n}} H_D(x_i))^n$.

3. Let $\alpha_1 = \alpha_x(D)$ and $\alpha_2 = \alpha_x(D')$. We want to study the sequence:

$$\{d_v(x, x_i)^\gamma H_{aD+bD'}(x_i)\}.$$

   Recall that our height functions are multiplicative, so $H_{D+D'} = H_D H_{D'}$. Hence, we can rewrite this sequence as

$$d_v(x, x_i)^\gamma H_{aD+bD'}(x_i) = \left(d_v(x, x_i)^{\frac{\gamma}{2a}} H_D(x_i)\right)^a \left(d_v(x, x_i)^{\frac{\gamma}{2b}} H_{D'}(x_i)\right)^b.$$

   But, by definition of α_1, for the sequence $\left(d_v(x, x_i)^{\frac{\gamma}{2a}} H_D(x_i)\right)^a$ to be bounded from above, $\frac{\gamma}{2a} \geq \alpha_1$, and similarly for $\left(d_v(x, x_i)^{\frac{\gamma}{2b}} H_{D'}(x_i)\right)^b$ to be bounded, $\frac{\gamma}{2b} \geq \alpha_2$. Thus, $\gamma \geq a\alpha_1 + b\alpha_2$, so $\alpha_{x,X}(aD + bD') \geq a\alpha_{x,X}(D) + b\alpha_{x,X}(D')$, as desired.

4. Let $x \in X_n$ for some $n \in \{1, \cdots, r\}$. Since X_n is a subvariety of X, we can restrict the height H_D to a height $H_{D|_{X_n}}$ on X_n, and the distance on $X(\overline{k})$ to a distance on $X_n(\overline{k})$. Thus, for a sequence $\{x_i\} \subset X_n(k)$, $\alpha_{x,X_n}(\{x_i\}, D|_{X_n}) = \alpha_{x,X}(\{x_i\}, D)$. But, by definition of $\alpha_{x,X}(D)$, we see that

$$\alpha_{x,X}(D) \leq \min\{\alpha_{x,X_1}(D|_{X_1}), \cdots, \alpha_{x,X_r}(D|_{X_r})\}.$$

Conversely, for any sequence $\{x_i\} \subset X(k)$ in X that converges to x, pick a component X_n that contains x. Then, we can restrict the sequence $\{x_i\}$ to a subsequence $\{x_i'\} \subset X_n(k)$ that still converges to x. Then, by item 1 in this proposition, $\alpha_{x,X}(\{x_i'\}, D) \leq \alpha_{x,X}(\{x_i'\}, D)$. But since the subsequence $\{x_i'\}$ belongs to X_n, we see that

$$\alpha_{x,X_n}(D|_{x_n}) \leq \alpha_{x,X_n}(\{x_i'\}, D|_{x_n}) = \alpha_{x,X}(\{x_i'\}, D) \leq \alpha_{x,X}(\{x_i\}, D)$$

which implies that $\alpha_{x,X_n}(D|_{X_n}) \leq \alpha_{x,X}(D)$, as desired.

5. Let D be an ample divisor. Then there is some positive integer m such that mD is very ample. Hence, there is an embedding $\phi : X \hookrightarrow \mathbb{P}^r$ associated to mD for some $r \in \mathbb{N}$. Let $v \in M_k$, let $\{x_i\} \subset X(k)$ be a sequence that converges to x with respect to d_v, and let $\gamma \in \mathbb{R}$. Then,

$$d_v(x, x_i)^\gamma H_{mD}(x_i) \gg d_v(\phi(x), \phi(x_i))^\gamma H_L(\phi(x_i)) \tag{2.1}$$

where L is a hyperplane in $\mathbb{P}^r$. This inequality follows because ϕ is an isomorphism between X and its image $\phi(X) \subset \mathbb{P}^r$ and the fact that distances are equivalent under embeddings. Notice that the right hand side of inequality (2.1) is studying the approximation on $\mathbb{P}^r$ and by proposition 2.1 we know that the sequence is bounded if and only if $\gamma \geq 1$. Thus $\alpha_x(mD) \geq 1$. But by statement 2 in this proposition, we know that $\alpha_x(mD) = m\alpha_x(D)$, which implies that $\alpha_x(D) \geq \frac{1}{m} > 0$.

□

Remark 2.4. *If D is a divisor satisfying the Northcott property (i.e. a divisor for which there is only a finite number of points of bounded height with respect to D), then $\alpha_x(D) \geq 0$. This is because if for some negative power γ, then sequence $d_v(x, x_i)^\gamma H_D(x_i)$ would be bounded from above. This will imply that $H_D(x_i) \to 0$, which is impossible.*

2.2 Curves of best approximation

Throughout this work a *curve* will mean an irreducible algebraic variety of dimension one, and a *surface* will mean an algebraic variety of dimension two. If X is a surface, the curves on X are all the non-zero irreducible effective divisors on X.

Definition 2.5. *A sequence $\{x_i\} \to x$ whose approximation constant $\alpha_x(\{x_i\}, D)$ is equal to $\alpha_x(D)$ is called a* ***sequence of best approximation****. A curve C passing through x is called a* ***curve of best approximation*** *(we use the abbreviation "COBA") with respect to D if it contains a sequence of best approximation to x.*

Through the study of approximation constants on algebraic varieties, many examples showed that under certain conditions, the sequences with best approximation constants could be chosen to lie on curves.

Conjecture 2.6. *[19, McKinnon] Let $P \in X(k)$, where X is an algebraic variety defined over k and an ample divisor D on X. If $P \in C$ where C is rational curve in X defined over k, then, there exists a sequence on $X(k) \setminus \{P\}$ of best approximation to P with respect to D, and moreover that sequence can be chosen to lie on some rational curve through P.*

The purpose of this work is to verify this conjecture on rational surfaces. We want to study the conjecture on this type of varieties, since the set of rational points in these surfaces is dense, and the classification of these surfaces is also widely known.

Proposition 2.7. *Let X be an algebraic variety defined over k, $x \in X(k)$, C be a curve in X passing through x, and D be an ample divisor on X. Then, C is a curve of best approximation for x with respect to D if an only if $\alpha_x(D) = \alpha_x(D|_C)$.*

Proof. First notice that is is always true that $\alpha_x(D) \leq \alpha_x(D|_C)$, since every sequence in C that converges to x is also a sequence in X.

Assume that C is a COBA with respect to D. Then C contains a sequence of best approximation to x, call it $\{x_i\}$. Then

$$\alpha_x(D) = \alpha_x(\{x_i\}, D) = \alpha_x(\{x_i\}, D|_C) \geq \alpha_x(D|_C)$$

The second equality holds because $H_{D|_C}(x_i) = H_D(\iota(x_i))$, where $\iota : C \hookrightarrow X$ is the inclusion of C in X. Hence for the fixed sequence $\{x_i\}$ and a fixed $\gamma \in \mathbb{R}$, the sequence

$\{d_v(x,x_i)^\gamma H_D(x_i)\}$ is bounded if and only if the sequence $\{d_v(x,x_i)^\gamma H_D(\iota(x_i))\}$ is bounded.

Thus, if C is a COBA, then $\alpha_x(D) = \alpha_x(D|_C)$.

Conversely, assume that $\alpha_x(D) = \alpha_x(D|_C)$, then there exists a sequence $\{y_i\}$ in C such that $\alpha_x(D) = \alpha_x(D|_C) = \alpha_x(\{y_i\}, D|_C)$. But that sequence is also in X. Thus C is a COBA for x with respect to D. □

The statement of the following proposition was made in [19]. We provide the proof here.

Proposition 2.8. *Let X be a variety defined over a number field k. If C is a COBA for $x \in X(\overline{k})$ with respect to a divisor D, then C is rational.*

Proof. Notice that since C is a COBA with respect to D, $\#C(k)$ is infinite as it contains an infinite sequence of rational points. So C has genus 0 or 1. Assume C has genus 1. By Siegel's theorem (See [25, page 276]), if $x \in C(\overline{k})$,

$$\lim_{\substack{P \in C(k) \\ H_D(P) \to \infty}} \frac{\log(d(P,x))}{\log(H_D(P))} = 0. \tag{2.2}$$

Let $\alpha := \alpha_x(D) < \infty$. Then there is a sequence $\{x_i\} \in C(k)$ such that $d(x_i,x)^\alpha H_D(x_i) = O(1)$. So, there is a positive constant B such that

$$\alpha \log(d(x_i,x)) + \log(H_D(x_i)) < B$$
$$\alpha\left(\frac{\log(d(x_i,x))}{\log(H_D(x_i))}\right) + 1 < \frac{B}{\log(H_D(x_i))}.$$

Hence, if we take the limit over the rational points in C with $H_D(x_i) \to \infty$, and use equation (2.2), we see that the last inequality is impossible. Thus, C must have genus 0. □

Now we define what nef divisors are. These type of divisors will be relevant for this work since they are closely related with ample divisors. Any nef divisor is the limit of ample divisors. This relation will be explained in the next chapter. Here we define the notion of nef divisor to provide some of the properties of curves of best approximation with respect to nef divisors.

Definition 2.9. *Let X be an algebraic variety and let D be a divisor on X. Then D is called nef if its degree on every curve $C \subset X$ is non-negative. i.e. if $D \cdot C \geq 0$.*

The following results will be used when considering curves of best approximation with respect to nef divisors, in Chapter 5.

Theorem 2.10. *[21, Liouville-type theorem] Let X be an algebraic variety over Spec(k), $x \in X(k)$ any point, and $d = [K : k]$ where K is the field of definition of x.*

Let $X_K = X \times_k K$, $\widetilde{X}$ be the blowup of X_K at x with exceptional divisor E, and set $\pi : \widetilde{X} \longrightarrow X_K \longrightarrow X$. Let L be a nef divisor on X, and $\gamma \in \mathbb{Q}$ such that $L_\gamma := \pi^*(L) - \gamma E \in Eff(\widetilde{X})$. Finally let B' be the stable base locus of L_γ and set $B = \pi(B')$. Then, there is a positive real constant M such that for all $y \in X(k) - B(k)$, $d_v(x, y)^{\frac{\gamma}{d}} H_L(y) \geq M$, and

- *(a) For any sequence $\{x_i\} \subset X(k)$ approximating x, if infinitely many points of $\{x_i\}$ are outside B then $\alpha_x(\{x_i\}, L) \geq \frac{\gamma}{d}$.*
- *(b) If $\alpha_x(L) < \frac{\gamma}{d}$ then $x \in B$ and $\alpha_x(L) = \alpha_x(L|_B)$.*
- *(c) If $x \in B$ and $\alpha_x(L|_B) \geq \frac{\gamma}{d}$, then $\alpha_x(L) \geq \frac{\gamma}{d}$.*

Furthermore there is a subvariety Y of X such that $x \in Y$ and for all $y \in X(k)$, we have $d_v(x, y)^{\frac{\gamma}{d}} H_L(y) \geq M$ provided that $(L - \gamma E)|_Y$ is in the effective cone of Y.

Theorem 2.11. *[20, Thm. 2.16] Let C be a k-rational curve (possibly singular) and $\phi : \mathbb{P}^1 \longrightarrow C$ be its normalization map. Then for any ample divisor D on C and any $x \in C(\overline{k})$, we have*

$$\alpha_x(D) = \min_{q \in \phi^{-1}(x)} \left\{ \frac{d}{m_q r_q} \right\}$$

where $d = deg(D)$, m_q is the multiplicity of the branch of C through x corresponding to q, and

$$r_q = \begin{cases} 0 & \text{if } \kappa(q) \nsubseteq k_v \\ 1 & \text{if } \kappa(q) = k \\ 2 & \text{otherwise} \end{cases}$$

Where $\kappa(q)$ is the field of definition of the point q.

Proof. The proof is given in theorem in [20, Page 19]. $\square$

Remark 2.12. *If C in Theorem 2.11 is smooth, then ϕ is an isomorphism. So, there is only one point $q = \phi^{-1}(x)$, and $m_q = 1$. Moreover, if C is smooth and we want to approximate $x \in C(k)$, then $\kappa(q) = k$, thus $r_q = 1$. In this case $\alpha_{x,C}(D) = \deg(D)$.*

Remark 2.13. *Let $x \in X(k)$ and let C be a COBA for x with respect to a nef divisor D on X. Then, $\alpha_x(D) = \alpha_{x,C}(D|_C)$. We also know that since D is nef in X, $D|_C$ is effective in C. If $D|_C$ is not the zero divisor, $\deg(D|_C) > 0$, thus by Riemann Roch $D|_C$ is ample, so theorem 2.11 applies for the divisor $D|_C$ on C.*

Theorem 2.14. *Let X be a variety defined over k and $x \in X(k)$. Let D_1 and D_2 be two nef divisors on X, and let $D = a_1 D_1 + a_2 D_2$ where $a_1, a_2 \geq 0$. If C is a COBA for both D_1 and D_2 then C is also a COBA for D.*

Proof. By properties of α (Proposition 2.3, (3)) we know that $\alpha_x(D) \geq a_1\alpha_x(D_1) + a_2\alpha_x(D_2) = a_1\alpha_{x,C}(D_1|_C) + a_2\alpha_{x,C}(D_2|_C)$. Let us denote $\alpha_i := \alpha_x(D_i) = \alpha_{x,C}(D_i|_C)$ for $i = 1, 2$.

Since the intersection of divisors is a linear pairing, and D_1 and D_2 are nef divisors, D is also a nef divisor. Thus, by Theorem 2.11,

$$\alpha_x(D) \leq \alpha_{x,C}(D|_C) = \frac{\deg(D|_C)}{m_q}$$

where m_q is the maximal multiplicity corresponding to the point $q \in \phi^{-1}(x)$. Recall that $\deg(D|_C) = a_1\deg(D_1|_C) + a_2\deg(D_2|_C)$, which implies that

$$\alpha_{x,C}(D|_C) = a_1\frac{\deg(D_1|_C)}{m_q} + a_2\frac{\deg(D_2|_C)}{m_q} = a_1\alpha_1 + a_2\alpha_2.$$

Hence $\alpha_x(D) \leq \alpha_{x,C}(D|_C) = a_1\alpha_1 + a_2\alpha_2$ giving us the equality $\alpha_x(D) = a_1\alpha_1 + a_2\alpha_2$. So C is also a COBA for D as desired.

□

The following proposition which is a new contribution, will be used later in this work (chapter 5).

Proposition 2.15. *Let D be a base point free divisor on X with associated map $\phi_D : X \longrightarrow \mathbb{P}^n$, and let C be a smooth curve passing through a rational point $x \in X(k)$ such that $D \cdot C = 1$. If $x \notin C'$ for any C' with $D \cdot C' = 0$, then C is a COBA for x with respect to D.*

Proof. Let $\{x_i\}$ be a sequence in $X(k)$ and $\{x_i\} \to x$. We study the sequence $d(x, x_i)^\gamma H_D(x_i)$. Recall that $H_D(x_i) = H(\phi_D(x_i))$ is the height with respect to a hyperplane section in $\mathbb{P}^n$. Then there is some positive constant λ (depending on υ) such that

$$d_\upsilon(x, x_i)^\gamma H_D(x_i) \geq \lambda d_\upsilon(\phi_D(x), \phi_D(x_i))^\gamma H(\phi_D(x_i)).$$

Hence, approximating $x \in X(k)$ is equivalent to approximating $\phi_D(x) \in \mathbb{P}^n$. Also notice that since x does not belong to any curve contracted by x, we can pick the sequence $\{x_i\}$ such that the points x_i do not belong to any contracted curve as well, hence $d_\upsilon(\phi_D(x), \phi_D(x_i)) \neq 0$. Thus, the smallest value of γ so that the sequence is bounded is $\gamma = 1$.

Finally, notice that since $C \cdot D = 1$, C is isomorphic to $\phi_D(C)$ which is a line, and by Proposition 2.1 lines are curves of best approximation. Hence, C is a COBA for x with respect to D. □

Theorem 2.16. *Let X be a variety defined over k, and let D be a nef divisor on X and E be an effective divisor on X. Let $P \in X(k)$ be a rational point, and C be a curve of best D-approximation to P. If $C \cap E = \emptyset$, then C contains a sequence of best $(D + E)$-approximation to P.*

Proof. See [19, Theorem 3.3]. □

Note: Throughout this work we will fix a place $\upsilon \in M_k$ and find α with respect to the distance d_υ. In all the cases that we will study, the curves in which we look for sequences of approximation are smooth and thus α will be independent of the υ-adic distance.

Chapter 3

Birational Geometry of Rational surfaces

From now on we will consider X to be a smooth surface over the complex numbers.

In this section we will briefly describe convex cones in the Néron Severi Space $\mathrm{NS}(X) \otimes \mathbb{R}$ that have some information about the birational geometry of X. Particularly, we will study the convex nef cone of X. This cone contains all the ample divisors of X and that will help us to describe any ample divisor as a combination of nef divisors. We will study curves of best approximation for generators of the nef cone and then provide curves for an arbitrary ample divisor.

Definition 3.1. *Let $D_1, D_2 \in Div(X)$. We say that D_1 and D_2 are* ***numerically equivalent****, $D_1 \equiv D_2$, if they have the same intersection number $D_1 \cdot C = D_2 \cdot C$ for any curve $C \subset X$. The* ***Néron-Severi space*** *NS(X) of X is the $\mathbb{Q}$-vector space of $\mathbb{Q}$-divisors modulo numerical equivalence i.e. NS(X) = Div(X) $\otimes \mathbb{Q}/\equiv$.*

Note: In general numerical equivalence ($\equiv$) and linear equivalence ($\sim$) are different notions. However, we are interested in X being a rational smooth surface and in these structures, $\equiv$ and $\sim$ are equivalent. In this case $\mathrm{NS}(X) = \mathrm{Pic}(X) \otimes \mathbb{Q}$.

Theorem 3.2. *(Nakai-Moishezon Criterion). A divisor D on a surface X is ample if and only if $D^2 > 0$ and $D \cdot C > 0$ for all irreducible curves in X.*

The proof of this theorem can be found in [11] (V, 1.10).

Remark 3.3. *In the following chapters we will use the following facts that are consequences of the previous theorems.*

- *Notice that given two numerically equivalent divisors $D_1 \equiv D_2$, D_1 is ample if and only if D_2 is ample. Thus we can consider ampleness on classes of numerical equivalence. i.e. this notion makes sense in NS(X).*
- *Although the Nakai-Moishezon Criterion is related to divisors, it can be extended to $\mathbb{Q}$-divisors. Indeed, a $\mathbb{Q}$-divisor D is ample if there is a positive integer m that clears denominators and make $mD \in Div(X)$ an ample divisor.*
- *The criterion can also be extended to $\mathbb{R}$-divisors. We can set an $\mathbb{R}$-divisor to be ample if it is a positive $\mathbb{R}$-linear combination of ample divisors. Thus ampleness can be considered for divisor classes in NS$(X) \otimes \mathbb{R}$.*
- *Since the intersection of divisors is a linear operation, the set of ample divisor classes form a cone in NS(X), which is named the Ample cone of X and it is denoted by Amp(X).*

Similarly, the condition of being nef behaves well under numerical equivalence classes, and although the notion of nef refers to divisors on X, this notion can be extended to $\mathbb{Q}$-divisors and to $\mathbb{R}$-divisors, hence the nef condition can be studied in $\mathrm{NS}(X) \otimes \mathbb{R}$. In particular, the classes of nef divisors under numerical equivalence form a closed and convex cone, which we call the *nef cone of* X and we denote it by $\mathrm{Nef}(X)$. In particular we have the containment $\mathrm{Amp}(X) \subset \mathrm{Nef}(X)$. The following theorem gives us a more strict relation between these cones. The proof of the following theorem can be found in [16, 1.4.9].

Theorem 3.4. *(Kleiman's Criterion). The nef cone is the closure of the ample cone. The ample cone is the interior of the nef cone.*

In the cases that we will consider in the following chapter, we will study the nef cone of a rational surface. In many of the cases, the generators of the cone are base point free. The following proposition will be applied when we study base point free divisors that are generators of the nef cone.

Proposition 3.5. *Every base point free divisor is nef.*

Proof. Let D be a base point free divisor on X and let $C \subset X$ be an irreducible curve on X. We want to show that $C \cdot D \geq 0$. Recall that D is a codimension one subvariety and that its intersection with a curve could be negative only if it contains the curve. So it is enough to show that D does not contain C. Let $x \in C$ be a point in the curve. Since D is base point free, it corresponds to a morphism $\phi : X \longrightarrow \mathbb{P}^n$ to some projective space. Let H be a hyperplane in $\mathbb{P}^n$ that does not contain $\phi(x)$. Then $\phi^*(H)$ is a linearly equivalent divisor to D and it does not contain x as a base point. Then $C \nsubseteq \phi^*(H)$, thus $C \cdot D = C \cdot \phi^*(H) \geq 0$. $\square$

Definition 3.6. *Let $D = \sum_{i=1}^{k} n_i Y_i$ be a divisor, where Y_i is a divisor class of a curve, we say that D is effective, and we denote it by $D \geq 0$, if $n_i \geq 0$ for all $i \in \{1, \cdots, k\}$. The effective cone of X, denoted by Eff(X) is the convex cone in NS$(X) \otimes \mathbb{R}$, generated by the classes of all effective divisors in X.*

Remark 3.7. *If X is a surface, the effective divisors of X are the curves on X. Moreover the nef cone is the dual of the effective cone. We use the notation Eff$^{\vee}(X) =$ Nef(X).*

For the proof of the following facts, see for example [3, Chapter II], [23, Chapter 4, Section 3], or [11, Section V.3].

Theorem 3.8. *Let X be a surface and p be a point on X and let $\epsilon : \widetilde{X} \longrightarrow X$ be the blow-up of X at the point p, with exceptional curve E.*

1. *Let C be an irreducible curve on X passing through p with multiplicity m and let $\widetilde{C}$ be the strict transform of C under ϵ. Then,*

$$\epsilon^*(C) = \widetilde{C} + mE.$$

2. *Pic$(\widetilde{X}) \cong$ Pic$(X) \oplus \mathbb{Z}E \cong$ Pic$(X) \oplus \mathbb{Z}$.*

3. *Let D and D' be divisors on X. Then $(\epsilon^* D) \cdot (\epsilon^* D') = D \cdot D'$, $E \cdot (\epsilon^* D) = 0$, $E^2 = -1$.*

4. *$NS(\widetilde{X}) \cong NS(X) \oplus \mathbb{Z}E$.*

Corollary 3.9. *Let X be a surface and C be a curve on X. Let $\epsilon : \widetilde{X} \longrightarrow X$ be the blow-up map of X at a point p lying on C with multiplicity m. Let $\widetilde{C}$ be the strict transform of C under the map ϵ. Then, $\widetilde{C} \cdot \widetilde{C} = (C \cdot C) - m^2$.*

Proof. By the previous theorem, $\epsilon^*(C) = \widetilde{C} + mE$. Thus

$$\begin{aligned} C^2 &= \epsilon^*(C) \cdot \epsilon^*(C) \\ &= (\widetilde{C} + mE)^2 \\ &= \widetilde{C}^2 + 2m(E \cdot \widetilde{C}) + m^2 E^2 \\ &= \widetilde{C}^2 + m^2 \end{aligned}$$

So $\widetilde{C}^2 = C^2 - m^2$. □

The proof of the following facts can be found in [3, Chapter 2] or [1, Chapter VI]

Theorem 3.10. *(Elimination of indeterminacy) Let $\phi : S \dashrightarrow X$ be a rational map, where S is a surface and X is a projective variety. Then there exists a surface S' and a morphism $\eta : S' \longrightarrow S$ which is a composition of a finite number of blow-ups, and a morphism $f : S' \longrightarrow X$ such that $\phi \circ \eta = f$.*

Theorem 3.11. *(Universal property of blowing-up) Let $f : X \longrightarrow S$ be a birational morphism of surfaces, and suppose that the rational map f^{-1} is not defined at a point p in S. Then f factors as $f = \epsilon \circ g$, where $g : X \longrightarrow \widetilde{S}$ is a birational morphism and $\epsilon : \widetilde{S} \longrightarrow S$ is the blow-up at p.*

Remark 3.12. *The previous theorems justify the following facts.*

- *Any birational morphism $f : S \longrightarrow S'$ that is a composition of n blow-ups and an isomorphism gives the relation $NS(S) \cong NS(S') \oplus \mathbb{Z}^n$.*
- *Let $\epsilon : \widetilde{S} \longrightarrow S$ be the blow-up of S at a point on S and let $f : \widetilde{S} \longrightarrow X$ be a morphism, where X is a variety and f contracts E. Then f factors through S, i.e. there exists a morphism $g : S \longrightarrow X$ such that $f = g \circ \epsilon$.*

Definition 3.13. *Let X be a surface. We define $B(X)$ to be the set of isomorphism classes of surfaces birationally equivalent to X. Let S_1 and S_2 be surfaces in $B(X)$. Then we say that S_1 dominates S_2 if there is a birational morphism $S_1 \longrightarrow S_2$. We also say that the surface X is minimal if its class in $B(X)$ is minimal, so that every birational morphism $X \longrightarrow S'$ is an isomorphism.*

The proof of the following facts can be found in [3, Chapter 2] or [6, Chapter 3].

Proposition 3.14. *Every surface dominates a minimal surface.*

Theorem 3.15. ***(Castelnuovo's contractibility criterion).*** *Let S be a surface and $E \subset S$ be a curve isomorphic to $\mathbb{P}^1$ with $E^2 = -1$. Then E is an exceptional curve on S.*

3.1 Hirzebruch Surfaces

In this section we present a short introduction to these type of surfaces and some of the properties that will be relevant to our study. Some good references for the subject are [11, Charter 5, Section 2], [3, Chapter 4], and [5, Chapter 5, Section 5].

Hirzebruch surfaces are ruled surfaces over the base curve $\mathbb{P}^1$. They can be obtained by a sequence of blow-ups and blow-downs of the complex projective plane. However, we will consider them as the projectivization of a rank 2 vector bundle over $\mathbb{P}^1$. More precisely, we define the n-th Hirzebruch surface $\mathbb{F}_n$ as:

$$\mathbb{F}_n := \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n))$$

for $n \in \mathbb{N} \cup \{0\}$. We call $\pi : \mathbb{F}_n \longrightarrow \mathbb{P}^1$ the morphism associated to $\mathbb{F}_n$ as a vector bundle.

The following facts justify that every rank 2 vector bundle over $\mathbb{P}^1$ is decomposable and they justify the choices of the decomposition in the definition of $\mathbb{F}_n$. The proofs of these facts can be found in [3, Chapter III], [11, Chapter 5, Section 2], or [6, Chapter 12].

Proposition 3.16. *Every geometrically ruled surface over C is C-isomorphic to $\mathbb{P}_C(E)$ for some rank 2 vector bundle E over C. (The bundles $\mathbb{P}_C(E)$ and $\mathbb{P}_C(E')$ are C-isomorphic if there exists a line bundle L over C such that $E' = E \otimes L$).*

Proposition 3.17. *Every rank 2 vector bundle on $\mathbb{P}^1$ is decomposable, as the sum of two invertible sheaves. In particular every geometrically ruled surface over $\mathbb{P}^1$ is isomorphic to one of the surfaces $\mathbb{F}_n = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n))$ for $n \geq 0$.*

We denote by S_n for the section of the map $\pi : \mathbb{F}_n \longrightarrow \mathbb{P}^1$ that has self-intersection $-n$, and we denote by F_n a general fibre of π. When it is clear from the context or when the value of n is not relevant for our proofs, we will simply use the notation S and F, for S_n and F_n respectively.

The divisors F and S on $\mathbb{F}_n$ generate freely $\mathrm{Pic}(\mathbb{F}_n)$, which in this case coincides with $\mathrm{NS}(\mathbb{F}_n)$, and the intersection theory on $\mathrm{Pic}(\mathbb{F}_n)$ is given by $F \cdot F = 0$, $F \cdot S = 1$, $S \cdot S = -n$, and it extends by linearity.

Theorem 3.18. *Let S be a minimal rational surface. Then S is isomorphic to $\mathbb{P}^2$ or to one of the surfaces $\mathbb{F}_n$ for $n \neq 1$.*

Proof. See [3, Theorem V.10, Page 59] . □

This theory gives us an algorithm to study rational surfaces. Every rational surface is obtained by blowing up a finite number of times a minimal rational surface. We will follow this algorithm to study rational approximations in smooth rational surfaces in the next chapters.

Proposition 3.19. *Let $\phi : X \longrightarrow Y$ be a morphism and let $D \in Div(Y)$ be a base point free divisor. Then $\phi^*(D)$ is a base point free divisor in X.*

Proof. Since D is base point free, D corresponds to a morphism $f : Y \longrightarrow \mathbb{P}^n$ from Y to some projective space $\mathbb{P}^n$. This means that there is a hyperplane section H in $\mathbb{P}^n$ such that $f^*(H) = D$. Hence $\phi^*(D) = \phi^*(f^*(H))$ and $f \circ \phi : X \longrightarrow \mathbb{P}^n$ is a morphism. Thus $\phi^*(D)$ is base point free. □

3.2 The nef cone of smooth rational surfaces

In this work, the main ingredients for proving the conjecture rely on finding the generators of the nef cone of rational surfaces. The methods to find the effective cone and the nef cone for these type of varieties are given in [17], [18], or [6]. We provide an explicit method to find the effective and nef cones that we use further in this work as follows:

1. We prove that the Picard group of the smooth rational surfaces studied in this work is generated by the exceptional curves of the blow-ups and the section of the map $\pi : X \longrightarrow \mathbb{P}^1$. Since these generators of $\mathrm{Pic}(X)$ are effective, we choose them as potential generators of the effective cone. Call this cone σ. We know that $\sigma \subset \mathrm{Eff}(X)$.

2. Since X is a surface, we know that $\mathrm{Nef}(X) = \mathrm{Eff}(X)^\vee \subset \sigma^\vee$. So, to show that $\sigma^\vee = \mathrm{Nef}(X)$ we use two approaches:

   (a) Verify that each generator of $\sigma^\vee$ is nef. Then, $\mathrm{Nef}(X) = \sigma^\vee$. Thus, the initial assumption that σ was the effective cone was correct. In the examples that we will study, most of the generators of the cone $\sigma^\vee$ are in fact base point free and thus by Proposition 3.5 those generators will be nef.

(b) By Kleiman's Criterion, we know that the nef cone is the closure of the ample cone. So if we verify that any divisor in the interior of $\sigma^\vee$ is ample, we prove that $\sigma^\vee$ is the nef cone. We use the Nakai-Moishezon Criterion to study the intersection theory of the divisors in the interior of the cone $\sigma^\vee$.

The following fact will be used in Chapter 5, to determine the dual cone of a simplicial cone on $\mathrm{NS}(X) \otimes \mathbb{R}$.

Proposition 3.20. *Consider $\mathbb{R}^n$ with an inner product $\langle \cdot, \cdot \rangle$. Let $C = Cone(v_1, v_2, \cdots, v_n)$ be a simplicial cone in $\mathbb{R}^n$. Let $\{w_1, w_2, \cdots, w_n\}$ be a set of vectors in $\mathbb{R}^n$ such that $\langle v_i, w_j \rangle = \delta_{ij}$ where δ_{ij} is the Kronecker delta. Then $C^\vee = Cone(w_1, w_2, \cdots, w_n)$.*

Proof. Let $\tau := \mathrm{Cone}(w_1, w_2, \cdots, w_n)$. Notice that since $\langle v_i, w_j \rangle = \delta_{ij} \geq 0$ for all i, j, then $\tau \subseteq C^\vee$.

Assume for the sake of contradiction that τ is a proper subset of $C^\vee$. Then, there exists a vector $w \in C^\vee \setminus \tau$. By the separation theorem [2, page 105], there exists a vector $e \in \mathbb{R}^n$ such that

$$\begin{aligned} \langle e, w_i \rangle &\leq 0 \quad \text{for all } i \in \{1, 2, \cdots, n\} && (3.1) \\ \langle e, w \rangle &> 0 && (3.2) \end{aligned}$$

But, since C is simplicial, we can write $e = \sum_{i=1}^n \lambda_i v_i$ for some scalars $\lambda_i \in \mathbb{R}$. Thus $\langle w_i, e \rangle = \lambda_i \leq 0$ which implies that $-e \in C$, so $\langle -e, w \rangle \geq 0$. Hence $\langle e, w \rangle \leq 0$ which contradicts (4.2). Thus such w does not exists and the statement holds. $\square$

Chapter 4

Revisited Examples

In this section we verify the conjecture for some examples that were provided in [19]. Although all the cases were completely developed in that paper, we provide more detailed proofs here and we use these examples to present the strategy that will be used when proving the main conjecture in this thesis for rational surfaces of bigger Picard ranks. The propositions in this chapter are new contributions to that previous work.

The method to show the conjecture for these examples is given in the following steps:

1. Find the $\mathrm{Pic}(X)$ and the $\mathrm{NS}(X)$.

2. Find the Nef cone, $\mathrm{Nef}(X)$. To do this, we study the subcone σ of the effective cone $\mathrm{Eff}(X)$, generated by the exceptional curves in $\mathrm{Pic}(X)$. We verify if the dual cone $\sigma^\vee$ is our desired nef cone.

3. For each generator of $\mathrm{Nef}(X)$, we find a COBA C.

4. For each curve C in the previous step, we find a subcone Γ_C in which possibly C is a COBA for every divisor in this subcone. This gives us a division of $\mathrm{Nef}(X)$ into subcones in which possibly each curve in the previous step is a COBA.

5. We verify if for each generator of Γ_C, C is indeed a COBA. If so, then by Theorem 2.14, C will be a COBA for every divisor in Γ_C.

4.1 The case of $\mathbb{P}^2$ blown up at a point

Let $X = \widetilde{\mathbb{P}^2}$ the blow up of $\mathbb{P}^2$ at one point. Without loss of generality, we can choose the blown up point to be $p = [0:0:1]$. Hence X can be written as

$$X = \{([x:y:z],[u:v]) \in \mathbb{P}^2 \times \mathbb{P}^1 : \ xv = yu\} \subseteq \mathbb{P}^2 \times \mathbb{P}^1.$$

Consider the two projections of X, $\pi : X \longrightarrow \mathbb{P}^2$ and $\pi_1 : X \longrightarrow \mathbb{P}^1$ and let E be the exceptional divisor on X. Notice that $\pi : X \setminus E \to \mathbb{P}^2 \setminus \{p\}$ is an isomorphism. Hence $\mathrm{Pic}(X \setminus E) \cong \mathrm{Pic}(\mathbb{P}^2 \setminus \{p\})$. Also, since $\{p\}$ has codimension 2 on $\mathbb{P}^2$, we know that $\mathrm{Pic}(\mathbb{P}^2 \setminus \{p\}) \cong \mathrm{Pic}(\mathbb{P}^2) \cong \mathbb{Z}$. Let $L = \pi^*(L')$ to be the pullback by π of the class of a line L' in $\mathbb{P}^2$, then $\mathrm{Pic}(X \setminus E) \cong \mathbb{Z}L$. Using the exact sequence [11, page 133]:

$$0 \longrightarrow \mathbb{Z}E \longrightarrow \mathrm{Pic}(X) \longrightarrow \mathrm{Pic}(X \setminus E) \longrightarrow 0.$$

We find that $\mathrm{Pic}(X) \cong \mathbb{Z}L \oplus \mathbb{Z}E$.

Now we study the intersection theory on X. First notice that if L_1 and L_2 are lines in $\mathbb{P}^2$ avoiding p, then $\pi^*(L_1)$ and $\pi^*(L_2)$ are lines in X that do not intersect E and since $L_1 \cdot L_2 = 1$ and π is an isomorphism away from p, $L^2 = 1$ and $L \cdot E = 0$.

Claim 4.1. *If H is a line in $\mathbb{P}^2$ passing through $p = [0:0:1]$, then there exists a point $q \in \mathbb{P}^1$ such that $\pi^*(H) = \pi_1^*(q) + E$.*

Proof. Let H be a line through p. Without loss of generality, H is cut out by the equation $y = ax$ for some $a \in k \setminus \{0\}$. Then

$$\pi^{-1}(H) = \{([x:y:z],[u:v]) : \ y = ax \text{ and } xv = yu\} = E \cup \pi_1^{-1}([1:a])$$

From the equations $y = ax$ and $xv = uy$ we get $x(v - au) = 0$. Let us consider the cases in which this last equation holds.

If $x = 0$, we get the elements in the exceptional divisor E. So we can cover E by the affine pieces $E = (E \cap U_{z,u}) \cup (E \cap U_{z,v})$ where $U_{z,u} = \{z \neq 0, \ u \neq 0\}$ and $U_{z,v} = \{z \neq 0, \ v \neq 0\}$. Each affine piece is isomorphic to $\mathbb{A}^1$. So for a point $q \in E$, q lies in some affine neighborhood and using the isomosphism with $\mathbb{A}^1$, we get that the local ring $\mathcal{O}_q = k[x]_x$ and it has maximal ideal (x), so q vanishes in the equation $x(v - au) = 0$ with multiplicity 1.

If $(v - au) = 0$, we get the elements in $\pi_1^{-1}([1 : a])$ which is a line in $\mathbb{P}^2$ passing through p with slope $[1 : a]$ in X, i.e. $\pi_1^{-1}([1 : a]) = \{[x, y, z], [1 : a] : \ ax = y\}$ So by a similar argument this can be covered by two affine pieces isomorphic to $\mathbb{A}^1$ and again, a point in this line vanishes in $x(v-au)$ with multiplicity 1. So now we can conclude that as divisors $\pi^*(H) = \pi_1^*([1 : a]) + E$. $\square$

From the previous claim we have that $\pi_1^*(q) = L - E$ where q is a divisor (i.e. a point) on $\mathbb{P}^1$. Also notice that given two distinct points q_1 and q_2 in $\mathbb{P}^1$, their inverse image under π_1 are two distinct curves in X that do not intersect. So $(L - E)^2 = (\pi_1^*(q_1)) \cdot (\pi_1^*(q_2)) = 0$. Hence

$$0 = (L - E)^2 = L \cdot L - 2L \cdot E + E \cdot E$$

which implies that $E^2 = -1$. This gives us the complete structure of the intersection theory on X, which is summarized in the following table:

$\cdot$	L	E
L	1	0
E	0	-1

So, we can see that L and E are linearly independent since the matrix from the table is invertible. This tells us that $\mathrm{NS}(X) \otimes \mathbb{R} = \mathbb{R}L \oplus \mathbb{R}E$. Now, to find $\mathrm{Nef}(X)$, recall that $\mathrm{Nef}(X) = \overline{\mathrm{Amp}(X)}$ where $\mathrm{Amp}(X)$ is the cone of ample divisors of X. Since L and $L - E$ define the morphisms π and π_1 respectively, we see that L and $L - E$ are base point free and so, they are nef divisors. Also, notice that none of them are ample (for example E and $L - E$ are curves and $L \cdot E = 0 = (L - E)^2$). Then, these divisors are extreme rays of $\mathrm{Nef}(X)$ and since $\mathrm{NS}(X)$ is two dimensional, we have that $\mathrm{Nef}(X) = \mathrm{Cone}(L, L-E)$.

Now, given a point $x \in X(k)$, we want to find a COBA for each divisor $D \in \mathrm{Nef}(X)$. We start by finding a COBA for each of the generators of the nef cone. These approximations depend on the position of the point that we are approximating. So we need to consider some cases:

CASE 1: $x \notin E$. Recall that π is an isomorphism on $\mathbb{P}^2 \setminus \{p\}$. So if we choose a sequence $\{x_i\}$ on $X(k)$ converging to x, we can assume none of the elements in the sequence belong to E. Then, using the morphism π, this is equivalent to choose a sequence $\{\pi(x_i)\}$ on $\mathbb{P}^2(k)$ converging to $\pi(x)$. So we have

$$d(x, x_i)^\gamma H_L(x_i) = \lambda d(\pi(x), \pi(x_i))^\gamma H(\pi(x_i))$$

Where λ is some real constant, and H is the height on $\mathbb{P}^2$ described in Example 1.4. Thus, the problem is equivalent to the problem of approximating the rational point $\pi(x)$ in $\mathbb{P}^2$.

By Proposition 2.1, any line through $\pi(x)$ is a COBA with approximation constant 1. Let C_1 be a line in X passing through x. Then C_1 is a COBA with respect to L, and we know that $\alpha_x(L) = \alpha_{x,C_1}(L|_{C_1}) = C_1 \cdot L = 1$.

Now, we study the approximation of x with respect to $L-E$. Consider a sequence in $X(k)$, $\{x_i\} \to x$. Recall that $L-E$ is base point free, thus $H_{L-E} \geq O(1)$, so $d(x, x_i)^\gamma H_{L-E}(x_i) \geq d(x, x_i)^\gamma O(1)$. Since $d(x, x_i) \to 0$, $\alpha_x(L-E) \geq 0$. Consider $C = \pi_1^{-1}(\pi_1(x))$, then the sequence

$$d(x, x_i)^\gamma H_{L-E}(x_i) = d(x, x_i)^\gamma H(\pi_1(x_i)) = d(x, x_i)^\gamma H(\pi_1(x))$$

is bounded if $\gamma = 0$, hence $\alpha_x(L-E) = 0$ and C is COBA for x with respect to $L-E$. Notice that C is a line on X passing through x, which tells us that C is also a COBA for x with respect to L.

Finally, let $D = aL + b(L-E)$ for some $a, b \geq 0$ be a nef divisor, $D \in \mathrm{Nef}(X)$. By Theorem 2.14, C is also a COBA with respect to D. Additionally, by remarks on Theorem 2.11, $\alpha_x(D) = D \cdot C = (aL + b(L-E)) \cdot (L-E) = a$. We conclude the results on the following table:

Divisor	COBA	α_x
L	$C = \pi_1^{-1}(\pi_1(x)) \sim L - E$	1
$L - E$	$C = \pi_1^{-1}(\pi_1(x)) \sim L - E$	0
$aL + b(L-E)$	$C = \pi_1^{-1}(\pi_1(x)) \sim L - E$	a

CASE 2: $x \in E$. In this case, consider $C_1 = \pi^{-1}(p) = E$. By a similar argument as case 1 (with the curve C), we see that C_1 is a COBA for x with respect to L and that $\alpha_x(L) = L \cdot E = 0$. On the other hand, to approximate x with respect to $L-E$, we can use exactly the same argument as in case 1, this is because all the fibers of π_1 are lines (this is independent of the choice of the point x). So in this case $C = \pi_1^{-1}(\pi_1(x))$ is still a COBA with respect to $L-E$. Also, notice that E is not a COBA for x with respect to

$L - E$. This is because using the map π_1 we can see that $\alpha_{x,E}(L - E) = 1 > 0$.

In this case, we need to divide the nef cone into subcones that have a common COBA. Let us assume that E is a COBA for a divisor $D \in$ Nef(X), then $\alpha_x(D) = \alpha_{x,E}(D|_E) = D \cdot E$. In particular $\alpha_{x,E}(D) \leq \alpha_{x,C}(D|_C) = D \cdot (H - E)$, and if we write $D = aL + b(L - E)$, then $D \cdot E \leq D \cdot (L - E)$ implies that $b \leq a$. On the other hand, if we assume that C_1 is a COBA for D we have the reverse inequality $D \cdot E \geq D \cdot (L - E)$ which tells us that $b \geq a$. This analysis suggests that we can divide the nef cone into the subcones

$$\text{Nef}(X) = \langle L, 2L - E \rangle \cup \langle 2L - E, L - E \rangle \tag{4.1}$$

where a divisor in the first cone has E as a COBA and a divisor on the second cone has C as a COBA. Let us verify this.

Claim 4.2. *E is a COBA for x with respect to $2L - E$.*

Proof. Let $\{x_i\}_{i \in \mathbb{N}} \in E(k)$ be a sequence converging to x and consider the sequence

$$d(x, x_i)^\gamma H_{2L-E}(x_i) = \left(d(x, x_i)^\delta H_L(x_i)\right)\left(d(x, x_i)^\beta H_{L-E}(x_i)\right)$$

Where $\gamma = \delta + \beta$. Since E is COBA with respect to L, the first factor of the RHS of the equation is bounded with minimal value $\delta = 0$. Since E is not a fiber of π_1 and the image of E under π_1 is all $\mathbb{P}^1$, we see that the second factor of the RHS is bounded with minimal value $\beta = 1$. Then, the minimal value for γ is $\gamma = 1$. Notice that any sequence that has infinitely many points not belonging to the fiber of $\pi_1(x)$ will have approximation constant at least 1. Hence E is a COBA for $2L - E$. $\square$

Claim 4.3. *C is a COBA for x with respect to $2L - E$.*

Proof. Let $\{x_i\}_{i \in \mathbb{N}} \in C(k)$ be a sequence converging to x and consider the sequence

$$d(x, x_i)^\gamma H_{2L-E}(x_i) = \left(d(x, x_i)^\delta H_L(x_i)\right)\left(d(x, x_i)^\beta H_{L-E}(x_i)\right)$$

where $\gamma = \delta + \beta$. Since C is a COBA for $L - E$, we see that the second factor is bounded with minimal value of $\beta = 0$. By a similar argument as in case 1, we see that C, which is a line through x gives the approximation constant $\delta = 1$. Hence, the minimal value for γ is 1, which implies that C is also a COBA for x with respect to $2L - E$. $\square$

From Claims 4.2 and 4.3, we can conclude that the division of the nef cone in equation (4.1) is correct and finally we calculate the approximation constant in each case. If $D \in \langle L, 2L - E\rangle$, say $D = aL + b(2L - E)$ for some $a, b \geq 0$. Then $\alpha_x(D) = D \cdot E = b$. If $D \in \langle L - E, 2L - E\rangle$, say $D = c(L - E) + d(2L - E)$ for some $c, d \geq 0$. Then $\alpha_x(D) = D \cdot (L - E) = d$. We conclude the results in the following table:

Divisor	COBA	α_x
$D = aL + b(2L - E)$	E	b
$D = c(L - E) + d(2L - E)$	$C = \pi_1^{-1}(\pi_1(x)) \sim L - E$	d

In view of the claims 4.2 and 4.3, we prove the following proposition that applies to further cases.

Proposition 4.4. *Let D_1 and D_2 be base point free divisors on X with associated morphisms to projective spaces f_1 and f_2 respectively. Let $x \in X(k)$, and let C be an irreducible rational smooth component in the fibre of f_1 through x. If x does not lie on any curve contracted by both f_1 and f_2, and if $C \cdot D_2 = 1$, then C is COBA for x with respect to $D_1 + D_2$.*

Proof. Let $\{x_i\} \subset X(k)$ be a sequence that converges to x. We study the sequence

$$d_v(x, x_i)^\gamma H_{D_1+D_2}(x_i) \quad = \quad \left(d_v(x, x_i)^\delta H_{D_1}(x_i)\right) \left(d_v(x, x_i)^\beta H_{D_2}(x_i)\right)$$

where $\gamma = \delta + \beta$. Since C is contracted by f_1 and it contains x, C is COBA for x with respect to D_1 and minimal value of $\delta = 0$. Additionally, since $C \cdot D_2 = 1$, $f_2(C)$ is a line through $f_2(x)$ and in this case, approximating x by points in C is equivalent to approximating $f_2(x)$ by points in a line through $f_2(x)$ in a projective space. By Proposition 2.1, $d_v(x, x_i)^\beta H_{D_2}(x_i)$ is bounded with minimal value of $\beta = 1$, and C obtains that value. So, the minimal value of γ is $\gamma = 1$. Hence, as there is no curve through x that is contracted by both f_1 and f_2, there is no curve that attains a value of γ less than 1, which implies that C is COBA for $D_1 + D_2$. □

The case of blowing up the plane at two points can be found in the Appendix.

4.2 The case of $\mathbb{P}^2$ blown up at three non-collinear points

In this section we develop the example using the techniques from [19]. The proof of the conjecture in this case is roughly the same provided in [19]. Some of the generators of the cones provided there were missing. We consider this case including the new elements in the cones and complete the gap in that proof.

Let $X = BL_3\mathbb{P}^2$ the blow up of $\mathbb{P}^2$ at three non-collinear points. Without loss of generality, we can choose the blown-up points to be $p_1 = [0:0:1]$, $p_2 = [0:1:0]$ and $p_3 = [1:0:0]$. As before, we can describe X explicitly by

$$X = \{([x:y:z],[u:v],[s,t]),[r,w]) : \ xv = yu, xt = zs, yw = rz\} \subseteq \mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1.$$

We call π the projection of X to $\mathbb{P}^2$, and π_i the projection of X to the i-th copy of $\mathbb{P}^1$ for $i = 1, 2, 3$. Let E_i be the exceptional divisor corresponding to the projection π_i. Explicitly,

$$\begin{array}{rcl} E_1 & = & [0:0:1] \times \mathbb{P}^1 \times [0:1] \times [0:1] \\ E_2 & = & [0:1:0] \times [0:1] \times \mathbb{P}^1 \times [1:0] \\ E_3 & = & [1:0:0] \times [1:0] \times [1:0] \times \mathbb{P}^1 \end{array}$$

By a similar argument as in Section 4.1, we see that $\mathrm{Pic}(X) \cong \mathbb{Z}L \oplus \mathbb{Z}E_1 \oplus \mathbb{Z}E_2 \oplus \mathbb{Z}E_3$, and that $L \sim \pi^*(L')$ and $L - E_i \sim \pi_i^*(P)$, where L' is a line in $\mathbb{P}^2$ and P is the divisor of a point in $\mathbb{P}^1$. (By the correspondence between Cartier divisors and line bundles, it is common to use the notation for L and $L - E_i$ as line bundles and write $L = \pi^*(\mathcal{O}_{\mathbb{P}^2}(1))$ and $L - E_i = \pi_i^*(\mathcal{O}_{\mathbb{P}^1}(1))$, where $\mathcal{O}_{\mathbb{P}^n}(1)$ is the line bundle associated to a hyperplane in $\mathbb{P}^n$). The intersection theory on X is given by

$\cdot$	L	E_1	E_2	E_3
L	1	0	0	0
E_1	0	-1	0	0
E_2	0	0	-1	0
E_3	0	0	0	-1

Clearly all the generators of $\mathrm{Pic}(X)$ are linearly independent, hence $\mathrm{NS}(X) \oplus \mathbb{R} = \mathbb{R}L \oplus \mathbb{R}E_1 \oplus \mathbb{R}E_2 \oplus \mathbb{R}E_3$. Let L_{12} be the line joining the points p_1 and p_2, L_{23} be the line joining

p_2 and p_3, and let L_{13} be the line joining p_1 and p_3. Consider the inverse image of the points $[0:1]$ and $[1:0]$ in $\mathbb{P}^1$ under the projections π_1 and π_2 and π_3.

$$\begin{aligned}
\pi_1^{-1}([0:1]) &= \{([0:y:z],[0:1],[s:t],[r:w]) : zs=0, yw=rz\} \\
&= E_2 \cup \{([0:y:z],[0:1],[0:1],[y:z]) : z \neq 0\} \\
\pi_2^{-1}([0:1]) &= \{([0:y:z],[u:v],[0:1],[r:w]) :\ yu=0, yw=rz\} \\
&= E_1 \cup \{([0:y:z],[0:1],[0:1],[y:z]) :\ y \neq 0\} \\
\pi_3^{-1}([0:1]) &= \{([x:0:z],[u:v],[s:t],[0:1]) :\ xv=0, xt=sz\} \\
&= E_1 \cup \{([x:0:z],[1:0],[x:z],[0:1]) :\ x \neq 0\} \\
\pi_1^{-1}([1:0]) &= \{([x:0:z],[1:0],[s:t],[r:w]) :\ rz=0, xt=sz\} \\
&= E_3 \cup \{([x:0:z],[1:0],[x:z],[0:1]) : z \neq 0\} \\
\pi_2^{-1}([1:0]) &= \{([x:y:0],[u:v],[1:0],[r:w]) :\ yw=0, xv=yu\} \\
&= E_3 \cup \{([x:y:0],[x:y],[1:0],[1:0]) :\ y \neq 0\} \\
\pi_3^{-1}([1:0]) &= \{([x:y:0],[u:v],[s:t],[1:0]) :\ xt=0, xv=yu\} \\
&= E_2 \cup \{([x:y:0],[x:y],[1:0],[1:0]) :\ x \neq 0\}
\end{aligned}$$

Let S_{ij} be the strict transform of the line L_{ij}, $1 \leq i < j \leq 3$. Explicitly,

$$\begin{aligned}
S_{12} &= \{([0:y:z],[0:1],[0:1],[y:z]) :\ [y:z] \in \mathbb{P}^1\} \\
S_{13} &= \{([x:0:z],[1:0],[x:z],[0:1]) :\ [x:z] \in \mathbb{P}^1\} \\
S_{23} &= \{([x:y:0],[x:y],[1:0],[1:0]) :\ [x:y] \in \mathbb{P}^1\}
\end{aligned}$$

Notice that $\pi(S_{ij}) = L_{ij}$, $1 \leq i < j \leq 3$.

Claim 4.5. *$L_{ij} = S_{ij} + E_i + E_j$, $1 \leq i < j \leq 3$.*

Proof. Notice that S_{ij} is a line, and $L_{ij} \cdot S_{ij} = E_i \cdot S_{ij} = E_j \cdot S_{ij} = 1$, and $S_{ij} \cdot E_k = 0$, where $k \in \{1,2,3\}$, $k \neq i$, and $k \neq j$. Then, as a divisor, $S_{ij} = a_0 L + a_i E_i + a_j E_j + a_k E_k$. Using the intersection pairing in this case, we see that $a_0 = 1$, $a_i = a_j = -1$, and $a_k = 0$. Hence, the claim follows. $\square$

Claim 4.6. *Let $\sigma = Cone(E_1, E_2, E_3, S_{12}, S_{13}, S_{23})$. Then, Eff$(X) = \sigma$ and Nef$(X) = \sigma^\vee$.*

Proof. Since all the generators of σ are effective, we see that $\sigma \subseteq \text{Eff}(X)$, so $\text{Nef}(X) = \text{Eff}(X)^\vee \subseteq \sigma^\vee$. If we show that all generators of $\sigma^\vee$ are nef, the claim will follow. Using Polymake, we calculate that $\sigma^\vee = \text{Cone}(L, L - E_1, L - E_2, L - E_3, 2L - E_1 - E_2 - E_3)$. Recall that $L - E_i$ is the divisor associated to the morphism π_i, $i \in \{1, 2, 3\}$, so $L - E_i$ is base point free and thus, it is nef for all i. Moreover, L is the divisor associated to π, so by the same reason, L is nef. Finally, $2L - E_1 - E_2 - E_3$ is associated to the morphism $\phi : X \longrightarrow \mathbb{P}^2$ that contracts the three strict transforms S_{12}, S_{13}, and S_{23}. So it is also base point free and thus it is nef. We conclude that

$$\text{Eff}(X) = \langle E_1, E_2, E_3, S_{12}, S_{13}, S_{23} \rangle$$

$$\text{Nef}(X) = \langle L, L - E_1, L - E_2, L - E_3, T \rangle$$

where $T = 2L - E_1 - E_2 - E_3$. $\square$

Now, we consider the following cases:

Case 1: $x \in X(k) \setminus (E_1 \cup E_2 \cup E_3)$. This case is similar to case 1 in the previous example. Since x is the only element on the fibre of $\pi(x)$ through π, any line containing x is a COBA for x with respect to L, and the approximation constant is 1. Let $C_i = \pi_i^{-1}(\pi_i(x))$ for $i \in \{1, 2, 3\}$. Notice that if $x \notin S_{ij}$ for any $1 \leq i < j \leq 3$, then C_i is an irreducible component. Moreover, C_i is a COBA for $L - E_i$, and $C_i \sim L - E_i$. On the other hand, if $x \in S_{ij}$ for some i, j, then $C_i = E_j \cup S_{ij}$ and $C_j = E_i \cup S_{ij}$, but since $x \in S_{ij}$, we pick S_{ij} as a COBA. We use the same notation for the curves $C_i = C_j = S_{ij}$. Finally, consider approximations to x with respect to T. Notice that ϕ is an isomorphism from $X \setminus (\cup_{i,j} S_{ij})$ to $\mathbb{P}^2 \setminus (\cup_{i,j} \phi(S_{ij}))$. Assume $x \notin S_{ij}$. Then, to approximate $x \in X$ is equivalent to approximate $\phi(x)$ in $\mathbb{P}^2$, so in this case we can approximate x with any line through x and the approximation constant is 1. Assume $x \in S_{ij}$ for some i, j. Then a COBA for x with respect to T is S_{ij} with approximation constant $T \cdot S_{ij} = 0$. Hence, we consider two subcases of this case:

Case 1.1: $x \in S_{ij}$ for some i, j. Then, we summarize the results in the following table.

Divisor	COBA	α_x
L	S_{ij} or $C_k \sim L - E_k$	1
$L - E_i$	S_{ij}	0
$L - E_j$	S_{ij}	0
$L - E_k$ where $k \notin \{i, j\}$	$C_k \sim L - E_k$	0
T	S_{ij}	0

Let $D = a_0 L + a_i(L - E_i) + a_j(L - E_j) + a_k(L - E_k) + a_4 T \in \mathrm{Nef}(X)$. Assume that S_{ij} is a COBA for D, then $D \cdot S_{ij} \leq D \cdot (L - E_k)$, i.e. $a_0 + a_k \leq a_0 + a_i + a_j + a_4$. We divide the nef cone as $\mathrm{Nef}(X) = \Gamma_1 \cup \Gamma_2$, where $\Gamma_1 = \langle L, L - E_i, L - E_j, T, T + L - E_k, 2L - E_i - E_k, 2L - E_j - E_k \rangle$ and $\Gamma_2 = \langle L, L - E_k, 2L - E_i - E_k, 2L - E_j - E_k, T + L - E_k \rangle$. Notice that S_{ij} is a COBA for x with respect to any generator of Γ_1 that has intersection 0 with S_{ij}. Additionally, by Proposition 4.4, S_{ij} is also a COBA with respect to any generator of Γ_1 that has intersection 1 with S_{ij}. Thus, S_{ij} is a COBA for x with respect to any divisor in Γ_1. Similarly, $L - E_k$ is a COBA for any divisor in Γ_2. So we conclude:

Divisor	COBA	α_x
$b_0 L + b_1(L - E_i) + b_2(L - E_j) + b_3 T +$ $b_4(T + L - E_k) + b_5(2L - E_i - E_k) + b_6(2L - E_j - E_k)$	S_{ij}	$b_0 + b_4 + b_5 + b_6$
$c_0 L + c_1(L - E_k) + c_2(2L - E_i - E_k)$ $+ c_3(2L - E_j - E_k) + c_4(T + L - E_k)$	$L - E_k$	$c_0 + c_1 + c_2 + c_3 + c_4$

Case 1.2: $x \notin S_{ij}$ for all i, j. Then, we summarize the results in the following table.

Divisor	COBA	α_x
L	C_1 or C_2 or C_3	1
$L - E_1$	$C_1 \sim L - E_1$	0
$L - E_2$	$C_2 \sim L - E_2$	0
$L - E_3$	$C_3 \sim L - E_3$	0
T	C_1 or C_2 or C_3	1

In this case the nef cone is divided as $\mathrm{Nef}(X) = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$, where $\Gamma_i = \langle L, T, L - E_i, 2L - E_i - E_j, 2L - E_i - E_k \rangle$, where $\{i, j, k\} = \{1, 2, 3\}$. Notice that $L - E_i$ is a COBA for x with respect to any divisor in Γ_i because the intersection with a generator of Γ_i is 0 or 1. In the latter case, Proposition 4.4 proves that $L - E_i$ is a COBA. We summarize as follows.

Divisor	COBA	α_x
$b_0L + b_1T + b_2(L - E_i) + b_3(2L - E_i - E_j)$ $+b_4(2L - E_i - E_k)$	$C_i \sim L - E_i$	$b_0 + b_1 + b_3 + b_4$

Case 2: $x \in E_1(k) \cup E_2(k) \cup E_3(k)$, and $x \notin S_{ij}$ for $1 \le i < j \le 3$. Say, $x \in E_i(k)$ for some $i \in \{1, 2, 3\}$. In this case, Since E_i is contracted by L, E_i is a COBA for x with respect to L. On the other hand, since x is not in any of the S_{ij}, we see that E_i is also a COBA for T because E_i is a line passing through x. Notice that $C_i = \pi_i^{-1}(\pi_i(x)) \sim L - E_i$ is a line through x, so C_i is also a COBA for T. Lastly, let $C_m = \pi_m^{-1}(\pi_m(x))$, for $m \in \{1, 2, 3\}$, we know as before that C_m is a COBA for $L - E_m$, but notice that since $x \in E_i$ and $x \notin S_{ij} \cup S_{ik} \cup S_{jk}$, $C_j = C_k = E_i$. So we conclude the following table:

Divisor	COBA	α_x
L	E_i	0
$L - E_i$	$C_i \sim L - E_i$	0
$L - E_j$	E_i	0
$L - E_k$	E_i	0
T	E_i or C_i	1

Let $D = a_0L + a_i(L - E_i) + a_j(L - E_j) + a_k(L - E_k) + a_4T \in \mathrm{Nef}(X)$. Assume that E_i is a COBA for D, then $D \cdot E_i \le D \cdot (L - E_i)$, and we get $a_i \le a_0 + a_j + a_k$, which is the cone $\Gamma_1 = \langle L, T, L - E_j, L - E_k, 2L - E_i, 2L - E_i - E_j, 2L - E_i - E_k \rangle$. Notice that E_i is a COBA for all the generators in Γ_1. By analogous argument we find that $\Gamma_2 = \langle T, L - E_i, 2L - E_i, 2L - E_i - E_j, 2L - E_i - E_k \rangle$, and $L - E_i$ is a COBA for all generators as well. We summarize as follows:

Divisor	COBA	α_x
$b_0L + b_1T + b_2(L - E_j) + b_3(L - E_k)+$ $b_4(2L - E_i) + b_5(2L - E_i - E_j) + b_6(2L - E_i - E_k)$	E_i	$b_1 + b_4 + b_5 + b_6$
$b_0T + b_1(L - E_i) + b_2(2L - E_i)+$ $b_3(2L - E_i - E_j) + b_4(2L - E_i - E_k)$	$L - E_i$	$b_0 + b_2 + b_3 + b_4$

Case 3: $x \in \bigcup_{1\le i<j\le 3, m\in\{i,j\}}(E_m \cap S_{ij})$. Let $x \in E_i \cap S_{ij}$. In this case E_i is still a COBA for x with respect to L. Now, since the corresponding morphism of T contracts S_{ij} and $x \in S_{ij}$, we see that S_{ij} is a COBA for x with respect to T. We also know that a COBA for $L - E_i$ is inside of the fibre $\pi_i^{-1}(\pi_i(x))$, but since $\pi_i^{-1}(x) = E_j \cup S_{ij}$ and $x \in S_{ij} \setminus E_j$, S_{ij} is a COBA for $L - E_i$. Similarly, since the fibre $\pi_j^{-1}(\pi_j(x)) = E_i \cup S_{ij}$ and $x \in E_i \cap S_{ij}$, both E_i and S_{ij} are COBAs for x with respect to $L - E_j$. Finally, since E_i is an irreducible component of the fibre $\pi_k^{-1}(\pi_k(x)) = E_i \cup S_{ik}$ and $x \in E_i \setminus S_{ik}$, E_i is a COBA for x with respect to $L - E_k$.

Divisor	COBA	α_x
L	E_i	0
$L - E_i$	S_{ij}	0
$L - E_j$	E_i or S_{ij}	0
$L - E_k$	E_i	0
T	S_{ij}	0

Let $D = a_0L + a_i(L - E_i) + a_j(L - E_j) + a_k(L - E_k) + a_4T \in \mathrm{Nef}(X)$. Assume that E_i is a COBA for D, then $D \cdot E_i \le D \cdot S_{ij}$. As before we get $a_i + a_4 \le a_0 + a_k$, then $\mathrm{Nef}(X) = \Gamma_1 \cup \Gamma_2$, where $\Gamma_1 = \langle L, L - E_j, 2L - E_i, L + T, L - E_k + T, L - E_k, 2L - E_i - E_k\rangle$ and $\Gamma_2 = \langle L - E_j, 2L - E_i, L + T, L - E_k + T, T, L - E_i, 2L - E_i - E_k\rangle$. Notice that S_{ij} and E_i are both COBAs for the generators $2L - E_i$, $L + T$, $L - E_k + T$ and $2L - E_i - E_k$, thus E_i is a COBA for divisors in Γ_1 and S_{ij} is a COBA for divisors in Γ_2. We conclude:

Divisor	COBA	α_x
$b_0L + b_1(L - E_j) + b_2(2L - E_i) + b_3(L + T)+$ $b_4(L - E_k + T) + b_5(L - E_k) + b_6(2L - E_i - E_k)$	E_i	$b_2 + b_3 + b_4 + b_6$
$b_0T + b_1(L - E_j) + b_2(2L - E_i) + b_3(L + T)+$ $b_4(L - E_k + T) + b_5(L - E_i) + b_6(2L - E_i - E_k)$	S_{ij}	$b_2 + b_3 + b_4 + b_6$

We conclude from the examples developed in this chapter, that the conjecture 1.8 is true for blow up of the projective plane at one, two or three non-collinear points. The case of blowing up at two points is in the Appendix.

Chapter 5

Split rational surfaces of Picard rank at least 5

In this chapter we verify the following conjecture for surfaces obtained by blowing up smooth rational surfaces of Picard rank 4.

Conjecture 5.1. *Let X be an algebraic variety defined over k and D be any ample divisor on X. Let x be any k-rational point on X such that there is some rational curve on X, defined over k and passing through x. Then, there exists a curve of best approximation to x on X with respect to D.*

Recall that the Hirzebruch surface $\mathbb{F}_n$ is characterized by the two maps $\pi_0 : \mathbb{F}_n \longrightarrow \mathbb{P}^1$ and $\phi : \mathbb{F}_n \longrightarrow A$ where A is the cone in $\mathbb{P}^{n+1}$ over the rational normal curve in $\mathbb{P}^n$.

Since every time that we blow up a point in a surface X, the rank of the Picard group increases by one, we need to blow-up $\mathbb{F}_n$ three times to study the split rational surfaces X of Picard rank 5. We may assume that none of the blown up points lie on the $(-n)$-section on $\mathbb{F}_n$, since blowing up a point in that section would only increase the value of n by 1 (see Corollary 3.9). By Theorem 3.8, we can describe the process of blowing up the Hirzebruch surface $\mathbb{F}_n$ in the following commutative diagram:

$$
\begin{tikzcd}
 & & & & A \subset \mathbb{P}^{n+1} \\
X \arrow[r, "f_2"] \arrow[rrrru, bend left, "f"] \arrow[rrrrd, bend right, "\pi"'] & X_2 \arrow[r, "f_1"] & X_1 \arrow[r, "f_0"] & \mathbb{F}_n \arrow[ru, "\phi"] \arrow[rd, "\pi_0"] & \\
 & & & & \mathbb{P}^1
\end{tikzcd}
\tag{5.1}
$$

where $X_1 = Bl_1(\mathbb{F}_n)$ is the blow up of $\mathbb{F}_n$ at one point, $X_2 = Bl_1(X_1)$ is the blow up of X_1 at one point (i.e. $X_2 \cong Bl_2(\mathbb{F}_n)$ the blow up of $\mathbb{F}_n$ at two points), and f_i is the map of blowing down in steps $i = 0, 1, 2$. Notice that X_1 could be different depending on the blowup points. However, the Picard groups of all possible surfaces X_1 have the same rank.

Notation 5.2. *From now on, we use the following notation:*

- *$Pic(\mathbb{F}_n) = \mathbb{Z}F_n \oplus \mathbb{Z}S_n$, where $F_n \sim \pi_0^*(y)$ for any point $y \in \mathbb{P}^1$ and $nF_n + S_n = \phi^*(H)$ for any hyperplane section H in A, and S_n is the $(-n)$-section on $\mathbb{F}_n$.*

- *$Pic(X_1) = \mathbb{Z}S \oplus \mathbb{Z}E_1 \oplus \mathbb{Z}E_2$, where $S := f_0^*(S_n)$, E_1 is the strict transform of F_n under the map f_0, and E_2 is the exceptional divisor of the blow up map f_0. On X_1 we also will use the letter F for $F := f_0^*(F_n)$.*

- *$Pic(X_2) = \mathbb{Z}S \oplus \mathbb{Z}E_1 \oplus \mathbb{Z}E_2 \oplus \mathbb{Z}E_3$, where we use the same letter S for $S := f_1^*(S)$, and the same letters E_1 and E_2 for the strict transforms of E_1 and E_2 respectively, under the map f_1. Also, E_3 is the exceptional divisor corresponding to f_1, and we use the same letter F for $F := f_1^*(F)$.*

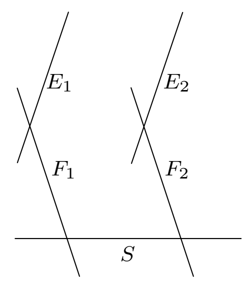
- *$Pic(X) = \mathbb{Z}S \oplus \mathbb{Z}E_1 \oplus \mathbb{Z}E_2 \oplus \mathbb{Z}E_3 \oplus \mathbb{Z}E_4$, where we use the same letter S for $S = f_2^*(S)$, and the letters E_i for the strict transform of E_i under f_2 for $i = 1, 2, 3$, E_4 is the exceptional divisor corresponding to f_2, and we use the same letter F for $F := f_2^*(F)$.*

The study of the conjecture on surfaces X_2 is given in [19, Theorem 3.17]. The proof in that paper was divided in three cases of X_2 that depend on the configuration of the reducible fibre of the map π. We summarize the results obtained in that paper since we use them for the study of surfaces of bigger Picard ranks.

Notation 5.3. *We use the following notation:*

Surfaces X_2 ***of type 1:*** *These are surfaces with two reducible fibres, each with two components that intersect transversely at one point. It was proven that* $Pic(X) = \mathbb{Z}S \oplus \mathbb{Z}E_1 \oplus \mathbb{Z}E_2 \oplus \mathbb{Z}E_3$ *with intersection theory given by:*

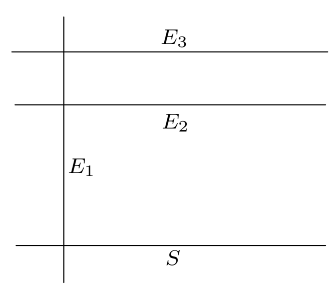
$\cdot$	F_1	E_1	E_2	S
F_1	-1	1	0	1
E_1	1	-1	0	0
E_2	0	0	-1	0
S	1	0	0	$-n$



Here $F \sim F_1 + E_1 \sim F_2 + E_2$*, and it was also proven that* $Eff(X) = Cone(S, F_1, E_1, F_2, E_2)$ *and* $Nef(X) = Cone(F, D_\alpha)$*, where* $\alpha \in \{0,1\}^2$ *and* $D_\alpha = S + nF - \alpha \cdot (E_1, E_2)$*. The surfaces* X *obtained by blowing up this type of surfaces are not considered in this chapter. We provide some ideas on how to develop these cases in the next chapter.*

Surfaces X_2 ***of type 2:*** *These are surfaces with one reducible fibre of the map* π *in diagram 5.1, with three components configured like a letter* F*. It was proven that* $Pic(X_2) = \mathbb{Z}S \oplus \mathbb{Z}E_1 \oplus \mathbb{Z}E_2 \oplus \mathbb{Z}E_3$ *with intersection table*

$\cdot$	E_1	E_2	E_3	S
E_1	-2	1	1	1
E_2	1	-1	0	0
E_3	1	0	-1	0
S	1	0	0	$-n$



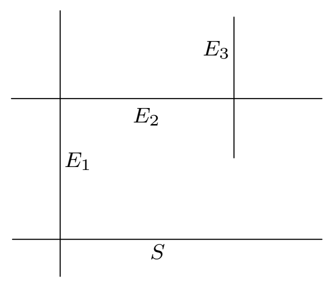
Here $F \sim E_1 + E_2 + E_3$*, and it was also proven that* $Eff(X) = Cone(E_1, E_2, E_3, S)$*, and*

Nef$(X) = $ *Cone*(F, D_1, D_2, D_3), *where*

$$\begin{aligned} D_1 &= nF + S, \\ D_2 &= D_1 - E_2, \\ D_3 &= D_1 - E_3. \end{aligned}$$

Surfaces X_2 ***of type 3:*** *These are surfaces with one reducible fibre of the map* π, *with three components configured like a letter H. In this case we only include the case where the reducible fibre has no multiple component. It was proven that* $Pic(X_2) = \mathbb{Z}S \oplus \mathbb{Z}E_1 \oplus \mathbb{Z}E_2 \oplus \mathbb{Z}E_3$ *with intersection table*

$\cdot$	E_1	E_2	E_3	S
E_1	-1	1	0	1
E_2	1	-2	1	0
E_3	0	1	-1	0
S	1	0	0	$-n$

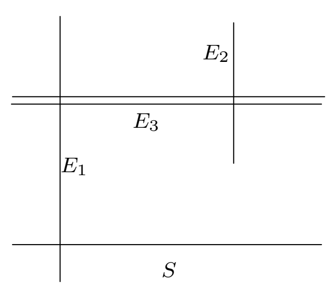


Here $F \sim E_1 + E_2 + E_3$, *Eff*$(X) = $ *Cone*(E_1, E_2, E_3, S), *and Nef*$(X) = $ *Cone*(F, D_1, D_2, D_3), *where*

$$\begin{aligned} D_1 &= nF + S, \\ D_2 &= D_1 - E_2 - E_3, \\ D_3 &= D_1 - E_2 - 2E_3. \end{aligned}$$

Surfaces X_2 ***of type 4:*** *These are surfaces of type* 3 *that have a double component. It was proven that* $Pic(X_2) = \mathbb{Z}S \oplus \mathbb{Z}E_1 \oplus \mathbb{Z}E_2 \oplus \mathbb{Z}E_3$ *with intersection table*

$\cdot$	E_1	E_2	E_3	S
E_1	-2	0	1	1
E_2	0	-2	1	0
E_3	1	1	-1	0
S	1	0	0	$-n$



Here $F \sim E_1+E_2+2E_3$, Eff$(X) =$ Cone(E_1, E_2, E_3, S), and Nef$(X) =$ Cone(F, D_1, D_2, D_3), where

$$\begin{array}{rcl} D_1 & = & nF + S, \\ D_2 & = & D_1 - E_2 - E_3, \\ D_3 & = & 2D_1 - E_2 - 2E_3. \end{array}$$

Remark 5.4. *In all cases it was proven that each of the generators of the nef cone was base point free. Thus, to study the case of blowing up a point in X_2, using the notation f_2 for that blow-down map in diagram 5.1, we see that $f_2^*(D_1)$, $f_2^*(D_2)$, and $f_2^*(D_3)$ are also base point free by Proposition 3.19. Additionally, by Theorem 3.5, they belong to the nef cone of X. We use the same letters D_1, D_2, and D_3 to denote $f_2^*(D_1)$, $f_2^*(D_2)$, and $f_2^*(D_3)$ respectively.*

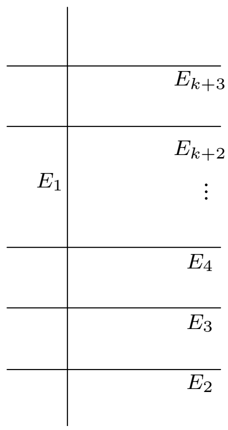
5.1 Blowing-up smooth points in the reducible fibre

In this section we prove Conjecture 5.1 for surfaces obtained by blowing up a surface X_2 at a smooth point of the fibre F. We find a complete description of the effective and the nef cone in these cases and we use the generators of the nef cone to find a COBA with respect to each generator of Nef(X).

Most of the proofs in this section seem very similar. However, in each case the divisors D_3 and D_4 were different, and in the cases in which they were base point free, the morphisms to projective space that they defined were also distinct. For this reason we kept our proofs as they are presented in this chapter.

Theorem 5.5. *Let X be the blow-up of the type 2 surface X_2 at k points all lying in $E_1 \setminus (S \cup E_2 \cup E_3)$, where $k, n \in \mathbb{N}$. Then Conjecture 5.1 is true for X.*

Proof. The reducible fibre of the map $\pi : X \longrightarrow \mathbb{P}^1$ is $F = \sum_{j=1}^{k+3} E_j$. Note that F is a comb with $k+2$ teeth.



Notice that $\text{Pic}(X) = \mathbb{Z}S \bigoplus_{j=1}^{k+3} \mathbb{Z}E_j$, where E_j is the exceptional divisor of the blow-up of X_2 at a point $p_j \in E_1 \setminus (S \cup E_2 \cup E_3)$, with the following intersections, where $i, j \in \{2, 3, \cdots, k+3\}$ and $i \neq j$:

$$\begin{aligned} E_1^2 &= -(k+2) \\ E_i^2 &= -1 \\ S^2 &= -n \\ E_1 \cdot E_i &= 1 \\ E_i \cdot E_j &= 0 \\ S \cdot E_1 &= 1 \\ S \cdot E_i &= 0 \end{aligned}$$

Claim 5.6. *The effective cone of X is generated by $E_1, \cdots, E_{k+3}$, and S. The nef cone of X is generated by the divisors F, $D_1 = nF + S$, and $D_i = nF + S - E_i$ for all $i \in \{2, 3, \cdots, k+3\}$.*

Proof. Let $\sigma = \text{Cone}(S, E_1, E_2, \cdots, E_{k+3})$. Since the generators of σ are all linearly independent over k, σ is a simplicial cone. So by Proposition 3.20, its dual cone is generated by divisors T_i where $T_i \cdot E_j = \delta_{ij}$, where δ is the Kronecker delta, and $i, j \in \{1, 2, \cdots, k+3\}$, $T_0 \cdot S = 1$, and $T_0 \cdot E_j = 0$ for all j.

We can verify that $D_i \cdot E_j = \delta_{ij}$, $F \cdot E_i = 0$, and $F \cdot S = 1$. Hence $T_i = D_i$ for all $i \in \{1, 2, \cdots, k+3\}$ and $T_0 = F$, which tells us that $\sigma^\vee = \text{Cone}(F, D_1, \cdots, D_{k+3})$. If we show that all the generators of $\sigma^\vee$ are base point free, we prove that $\sigma^\vee = \mathrm{Nef}(X)$.

Clearly F and D_1 are base point free, since they correspond to the morphisms π and f respectively in the diagram 5.1. Let $\phi_i : X \longrightarrow Y_i$ be the blow-down map of the exceptional divisors E_j where $j \in \{2, 3, \cdots, k+3\}$, where $j \neq i$, and let $\phi_{1,i} : Y_i \longrightarrow \mathbb{F}_{n-1}$ be the blow-down map of E_1.

Now we show that D_i is base point free for all $i \in \{2, 3 \cdots, k+3\}$. Let H be a hyperplane section of $A \subset \mathbb{P}^n$, where A is the cone over a rational normal curve in $\mathbb{P}^{n-1}$. Let $\phi : \mathbb{F}_{n-1} \longrightarrow A \subset \mathbb{P}^n$, the map in diagram 5.1. Then $\phi^*(H) \sim (n-1)F_{n-1} + S_{n-1}$. Then

$$\begin{aligned}
\phi^*(H) &\sim (n-1)F_{n-1} + S_{n-1} \\
\phi^*_{1,i}(\phi^*(H)) &\sim (n-1)(E_i + E_1) + (S + E_1) \\
\phi^*_i(\phi^*_{1,i}(\phi^*(H))) &\sim (n-1)(E_i + E_1 + \sum_{1<j\leq k,\ j\neq i} E_j) + (S + E_1 + \sum_{1<j\leq k,\ j\neq i} E_j) \\
\phi^*_i(\phi^*_{1,i}(\phi^*(H))) &\sim (n-1)F + S + F - E_i = nF + S - E_i = D_i.
\end{aligned}$$

Hence, since the E_i's and S are effective, and F and the D_i's are base point free, we see that

$$\begin{aligned}
\mathrm{Eff}(X) &= \mathrm{Cone}(S, E_1, E_2, \cdots, E_{k+3}), \\
\mathrm{Nef}(X) &= \mathrm{Cone}(F, D_1, \cdots, D_{k+3}).
\end{aligned}$$

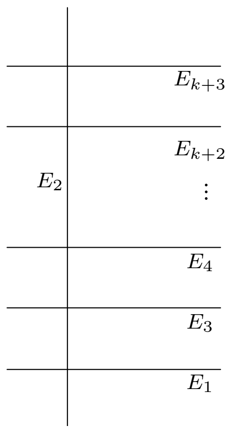
□

To finish the proof, we choose an arbitrary point $x \in X(k)$. We know that the only curves contracted by D_i are S and E_j where $j \neq i$. Hence, if x does not lie in the reducible fibre F, we let $C = \pi^*(\pi(x))$, then $C \sim F$, $C \cdot D_i = F \cdot D_i = 1$, and $C \cdot F = 0$. So by Proposition 2.15, C is a COBA for x with respect to any generator of the nef cone of X. Hence, for any $D \in \mathrm{Nef}(X)$, C is a COBA for x with respect to D and $\alpha_x(D) = D \cdot C$.

On the other hand, suppose that x belongs to the reducible fibre, i.e. x belongs to some exceptional curve E_i or S, and $D \in \mathrm{Nef}(X)$. We let $T = \{S, F, E_1, \cdots, E_k\}$. Then $\alpha_x(D) = \min_{\{R \in T:\ x \in R\}}\{D \cdot R\}$, and whichever curve in T that contains x and has minimal intersection with D is a COBA for x with respect to D. □

Theorem 5.7. *Let X be the blow-up of the type 3 surface X_2 at k points all lying in $E_2 \setminus (E_1 \cup E_3)$, where $k, n \in \mathbb{N}$, $n \geq 2$. Then Conjecture 5.1 is true for X.*

Proof. The reducible fibre of the map $\pi : X \longrightarrow \mathbb{P}^1$ is $F = \sum_{j=1}^{k+3} E_j$. Notice that F is a comb with $k+2$ teeth. We represent the fibre as:



Notice that $\mathrm{Pic}(X) = \mathbb{Z}S \bigoplus_{j=1}^{k+3} \mathbb{Z}E_j$, where E_j is the exceptional divisor of the blowup of X_2 at the point $p_j \in E_2 \setminus (E_1 \cup E_3)$, $j \in \mathbb{N}$, $4 \leq j \leq k+3$. We have the following intersection theory, where $i, j \in \{1, 3, 4, \cdots, k+3\}$ and $i \neq j$:

$$
\begin{aligned}
E_2^2 &= -(k+2) \\
E_i^2 &= -1 \\
S^2 &= -n \\
E_2 \cdot E_i &= 1 \\
E_i \cdot E_j &= 0 \\
S \cdot E_1 &= 1 \\
S \cdot E_2 = S \cdot E_i &= 0 \quad (\text{for } i \geq 3)
\end{aligned}
$$

Claim 5.8. *The effective cone of X is generated by $E_1, \cdots E_k$, and S. The nef cone of X is generated by the divisors F, $D_1 = nF + S$, $D_2 = D_1 - \sum_{i \geq 2} E_i$, and $D_i = D_2 - E_i$ for all $i \in \{3, \cdots, k+3\}$.*

Proof. Let $\sigma = \mathrm{Cone}(S, E_1, E_2 \cdots, E_{k+3})$. Since the generators of σ are linearly independent over k, σ is a simplicial cone. Hence, its dual $\sigma^\vee$ is generated by divisors

$T_0, T_1, \cdots, T_{k+3}$ of the dual space such that $T_n \cdot E_m = \delta_{nm}$, where $n, m \in \{0, 1, \cdots k+3\}$ where we have $E_0 := S$. But notice that $D_m \cdot E_m = 1$ for all $m \in \{0, 1, \cdots, k+3\}$ (where we have $D_0 := F$), and that $D_n \cdot E_m = 0$ for all n, m, $n \neq m$. Hence by Proposition 3.20, $\sigma^\vee = \mathrm{Cone}(F, D_1, D_2, \cdots, D_{k+3})$.

We know that F is base point free, corresponding to the morphism $\pi : X \longrightarrow \mathbb{P}^1$. Similarly D_1 is base point free, since it corresponds to the morphism f to a cone in the diagram 5.1. Also, D_2 is base point free, since it corresponds to a morphism to a cone in $\mathbb{P}^n$ that contracts all E_j $j \neq 2$ and S.

Now, for each $i \in \{3, \cdots, k+3\}$ we let $\phi_i : X \longrightarrow Y_i$ be the blow-down map with exceptional divisors E_j for $j \in \{1, 3, 4, \cdots k+3\}$ and $j \neq i$. Notice that $\phi_i(E_2)$ is a divisor, which we denote with the same letter E_2 in Y_i that has self intersection -1. So now we can contract it by letting $\phi_{2,i} : Y_i \longrightarrow \mathbb{F}_{n-2}$ be the blow-down map of E_2 on Y_i to $\mathbb{F}_{n-2}$. Recall that in $\mathbb{F}_{n-2}$ we have a morphism $\phi : \mathbb{F}_{n-2} \longrightarrow A \subset \mathbb{P}^{n-1}$ where A is a cone.

We let H be a hyperplane section in $\mathbb{P}^{n-1}$. Then

$$\begin{array}{rcl}
\phi^*(H) & \sim & (n-2)F_{n-2} + S_{n-2} \\
\phi^*_{2,i}(\phi^*(H)) & \sim & (n-2)(E_i + E_2) + (E_2 + S) \\
\phi^*_i(\phi^*_{2,i}(\phi^*(H))) & \sim & (n-2)\left(E_i + \displaystyle\sum_{\substack{j\neq i \\ 1 \leq j \leq k+3}} E_j\right) + \left(\displaystyle\sum_{\substack{j\neq i \\ 1 \leq j \leq k+3}} E_j + S + E_1\right) \\
& = & (n-2)F + F - E_i + S + E_1 = D_1 - \left(\displaystyle\sum_{2 \leq j \leq k+3} E_j\right) - E_i = D_i
\end{array}$$

Hence, we can conclude that

$$\begin{array}{rcl}
\mathrm{Eff}(X) & = & \mathrm{Cone}(S, E_1, E_2, \cdots, E_{k+3}) \\
\mathrm{Nef}(X) & = & \mathrm{Cone}(F, D_1, D_2, \cdots, D_{k+3})
\end{array}$$

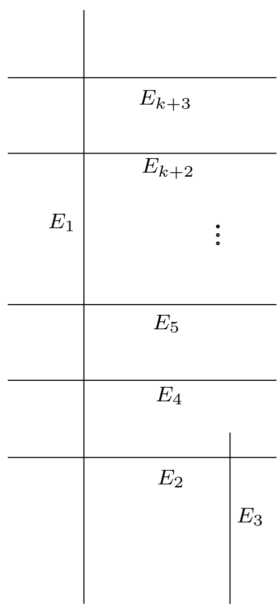
□

Finally, we consider a point $x \in X(k)$. Similarly to the earlier cases, since the product $F \cdot D_j = 1$ for all $j \in \{1, 2, 3, \cdots, k+3\}$, by Proposition 2.15, F is a COBA for x with respect to a divisor D unless that x belongs to an exceptional curve. We let $T = \{S, E_1, \cdots, E_{k+3}\}$.

If D belongs to the nef cone, $\alpha_x(D) = \min_{R \in T\colon x \in R}\{D \cdot F, D \cdot R\}$ and such a curve that provides the value of $\alpha_x(D)$ is a COBA for x with respect to D. $\square$

Theorem 5.9. *Let X be the blow-up of the type 3 surface X_2 at k points all lying in $E_1 \setminus (E_2 \cup S)$, where $k, n \in \mathbb{N}$. Then Conjecture 5.1 is true for X.*

Proof. The reducible fibre of the map $\pi : X \longrightarrow \mathbb{P}^1$ is $F = \sum_{j=1}^{k+3} E_j$. Note that F is a comb with $k+1$ teeth, but one of the teeth has been blown-up. We represent the fibre as:



Notice that $\mathrm{Pic}(X) = \mathbb{Z}S \bigoplus_{j=1}^{k+3} \mathbb{Z}E_j$, where E_j is the exceptional divisor of the blowup of X_2 at the point $p_j \in E_1 \setminus (E_2 \cup S)$, $j \in \mathbb{N}$, $4 \leq j \leq k+3$, with the following intersection theory, where $i, j \in \{3, 4, \cdots, k+3\}$ and $i \neq j$:

$$
\begin{aligned}
E_1^2 &= -(k+1) \\
E_2^2 &= -2 \\
E_i^2 &= -1 \\
S^2 &= -n \\
E_1 \cdot E_2 = E_1 \cdot E_i &= 1 \quad (\text{for } i \geq 4) \\
E_1 \cdot E_3 &= 0 \\
E_2 \cdot E_3 &= 1 \\
E_2 \cdot E_i &= 0 \quad (\text{for } i \geq 4) \\
E_i \cdot E_j &= 0 \\
S \cdot E_1 &= 1 \\
S \cdot E_2 = S \cdot E_i &= 0
\end{aligned}
$$

Claim 5.10. *The effective cone of X is generated by $E_1, \cdots E_k$, and S. The nef cone of X is generated by the divisors F, $D_1 = nF + S$, $D_2 = D_1 - E_2 - E_3$, $D_3 = D_1 - E_2 - 2E_3$, and $D_i = D_1 - E_i$ for all $i \in \{4, 5, \cdots, k+3\}$.*

Proof. Let $\sigma = \text{Cone}(S, E_1, E_2 \cdots, E_{k+3})$. Since the generators of σ are linearly independent over k, σ is a simplicial cone. Hence, its dual $\sigma^\vee$ is generated by divisors $T_0, T_1, \cdots, T_{k+3}$ of the dual space such that $T_n \cdot E_m = \delta_{nm}$, where $n, m \in \{0, 1, \cdots k+3\}$ and we have $E_0 := S$. But notice that $D_m \cdot E_m = 1$ for all $m \in \{0, 1, \cdots, k+3\}$ (where we have $D_0 := F$), and that $D_n \cdot E_m = 0$ for all n, m, $n \neq m$. Hence by Proposition 3.20, $\sigma^\vee = \text{Cone}(F, D_1, D_2, \cdots, D_{k+3})$.

As before, we know that F and D_1 are base point free.

To prove that D_2 and D_3 are base point free, consider the map $\phi_2 : X \longrightarrow X_2$ that blows down all E_j for $j \geq 4$. Now, recall that D_2 and D_3 in X_2 are base point free. Thus, in X, $D_2 = \phi_2^*(D_2)$ and $D_3 = \phi_2^*(D_3)$ which implies that D_2 and D_3 in X are also base point free.

Now, for each $i \in \{4, \cdots, k+3\}$ we let $\phi_i : X \longrightarrow Y_i$ be the blow-up map with exceptional divisors E_j for $j \in \{3, 4, \cdots, k+3\}$ and $j \neq i$. Notice that $\phi_i(E_2)$ is a divisor, which we denote with the same letter E_2, in Y_i that has self intersection -1. Similarly, $\phi_i(E_1)$ is a divisor, which we also call E_1 in Y_i that has self intersection -2. So now we can contract E_2 in Y_i by letting $\phi_{2,i} : Y_i \longrightarrow Y_2$ be the blow-down map of E_2 on Y_i. Notice that $E_1 := \phi_{2,i}(E_1)$ is a divisor in Y_2 with self intersection -1. Thus, we can contract it with a blow-down map $\phi_{1,i} : Y_2 \longrightarrow \mathbb{F}_{n-1}$. Recall that in $\mathbb{F}_{n-1}$ we have a morphism $\phi : \mathbb{F}_{n-1} \longrightarrow A \subset \mathbb{P}^n$ where A is a cone.

We let H be a hyperplane section in $\mathbb{P}^n$. Then

$$\begin{aligned}
\phi^*(H) &\sim (n-1)F_{n-1} + S_{n-1} \\
\phi^*_{1,i}(\phi^*(H)) &\sim (n-1)(E_i + E_1) + (E_1 + S) \\
\phi^*_{2,i}(\phi^*_{1,i}(\phi^*(H))) &\sim (n-1)(E_i + E_1 + E_2) + (E_1 + E_2 + S) \\
\phi^*_i(\phi^*_{2,i}(\phi^*_{1,i}(\phi^*(H)))) &= (n-1)\left(\sum_{1\leq j\leq k+3} E_j\right) + (E_1 + \sum_{\substack{j\neq i \\ 3<j\leq k+3}} E_j + E_2 + E_3 + S) \\
&= nF + S - E_i = D_1 - E_i = D_i
\end{aligned}$$

Hence, we can conclude that

$$\mathrm{Eff}(X) = \mathrm{Cone}(S, E_1, E_2, \cdots, E_{k+3})$$

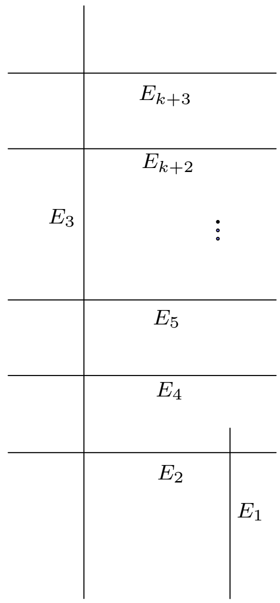
$$\mathrm{Nef}(X) = \mathrm{Cone}(F, D_1, D_2, \cdots, D_{k+3})$$

□

Finally, we consider a point $x \in X(k)$. Similarly to the earlier cases, since F has intersection 1 with each generator of the nef cone, by Proposition 2.15, F is a COBA for x with respect to a divisor D unless that x belongs to an exceptional curve. We let $T = \{S, E_1, \cdots, E_{k+3}\}$. If D belongs to the nef cone, then $\alpha_x(D) = \min_{R\in T:\ x\in R}\{D \cdot F, D \cdot R\}$. □

Theorem 5.11. *Let X be the blow-up of the type 3 surface X_2 at k points all lying in $E_3 \setminus E_2$, where $k, n \in \mathbb{N}$, $n \geq 3$. Then Conjecture 5.1 is true for X.*

Proof. The reducible fibre of the map $\pi : X \longrightarrow \mathbb{P}^1$ is $F = \sum_{j=1}^{k+3} E_j$. Notice that F is a comb with $k+1$ teeth, but one of the teeth has been blown-up. We represent the fibre as:



Notice that the fibre looks very similar to the fibre in Theorem 5.9, but the difference is that E_3 does not intersect S. So this is a different case. Also notice that $\mathrm{Pic}(X) = \mathbb{Z}S \bigoplus_{j=1}^{k+3} \mathbb{Z}E_j$, where E_j is the exceptional divisor of the blowup of X_2 at the point $p_j \in E_3 \setminus E_2$, $j \in \mathbb{N}$, $4 \leq j \leq k+3$, with the following intersection theory, where $i, j \in \{4, 5 \cdots, k+3\}$ and $i \neq j$:

$$\begin{aligned}
E_1^2 = E_i^2 &= -1 \\
E_2^2 &= -2 \\
E_3^2 &= -(k+1) \\
S^2 &= -n \\
E_1 \cdot E_2 = E_2 \cdot E_3 &= 1 \\
E_1 \cdot E_3 &= 0 \\
E_1 \cdot E_i = E_2 \cdot E_i &= 0 \\
E_3 \cdot E_i &= 1 \\
E_i \cdot E_j &= 0 \\
S \cdot E_1 &= 1 \\
S \cdot E_2 = S \cdot E_3 = S \cdot E_i &= 0
\end{aligned}$$

Claim 5.12. *The effective cone of X is generated by $E_1, \cdots, E_{k+3}$, and S. The nef cone* *of X is generated by the divisors F, $D_1 = nF + S$, $D_2 = (n-1)F + S + E_1$, $D_3 = (n-2)F + S + 2E_1 + E_2$ and $D_i = D_3 - E_i$ for all $i \in \{4, 5, \cdots, k+3\}$.*

Proof. Let $\sigma = \text{Cone}(S, E_1, E_2 \cdots, E_{k+3})$. Since the generators of σ are linearly independent over k, σ is a simplicial cone. Hence, its dual $\sigma^\vee$ is generated by divisors $T_0, T_1, \cdots, T_{k+3}$ of the dual space such that $T_n \cdot E_m = \delta_{nm}$, where $n, m \in \{0, 1, \cdots k+3\}$ and we have $E_0 := S$. But notice that $D_m \cdot E_m = 1$ for all $m \in \{0, 1, \cdots, k+3\}$ (where we have $D_0 := F$), and that $D_n E_m = 0$ for all n, m, $n \neq m$. Hence by Proposition 3.20, $\sigma^\vee = \text{Cone}(F, D_1, D_2, \cdots, D_{k+3})$.

As before, F and D_1 are base point free. To prove that D_2 and D_3 are base point free, consider the map $\phi_2 : X \longrightarrow X_2$ that is the blow-up of X_2 with exceptional divisors E_j for $j \geq 4$. Now, recall that D_2 and D_3 in X_2 are base point free. Now, in X,

$$\begin{aligned} \phi_2^*(D_2) &= \phi_2^*(D_1 - E_2 - E_3) = D_1 - E_2 - \left(\sum_{3 \leq j \leq k+3} E_j \right) = D_2, \\ \phi_2^*(D_3) &= \phi_2^*(D_1 - E_2 - 2E_3) = D_1 - E_2 - 2\left(\sum_{3 \leq j \leq k+3} E_j \right) = D_3. \end{aligned}$$

Hence, D_2 and D_3 in X are also base point free.

Now, for each $i \in \{4, \cdots, k+3\}$ we let $\phi_i : X \longrightarrow Y_i$ be the blow-up map with exceptional divisors E_j for $j \in \{1, 4, 5, \cdots, k+3\}$ and $j \neq i$. Notice that $\phi_i(E_2)$ is a divisor, which we denote with the same letter E_2, in Y_i that has self intersection -1. Similarly, $\phi_i(E_3)$ is a divisor, which we also call E_3 in Y_i that has self intersection -2. So now we can contract E_2 in Y_i by letting $\phi_{2,i} : Y_i \longrightarrow Y_2$ be the blow-up map of E_2 on Y_i. Notice that $E_3 := \phi_{2,i}(E_3)$ is a divisor in Y_2 with self intersection -1, thus, we can contract it with a blow-up map $\phi_{3,i} : Y_2 \longrightarrow \mathbb{F}_{n-3}$ (Notice that the target of the last map is $\mathbb{F}_{n-3}$ because we are contracting curves that intersect S, i.e. since $E_1 \subset X$ is contracted, $\phi_i(S) = S_{n-1}$. Since $E_2 \subset Y_i$ is contracted, $\phi_{2,i}(S_{n-1}) = S_{n-2}$, and since $E_3 \subset Y_2$ is contracted, $\phi_{3,i}(S_{n-2}) = S_{n-3}$). Recall that in $\mathbb{F}_{n-1}$ we have a morphism $\phi : \mathbb{F}_{n-1} \longrightarrow A \subset \mathbb{P}^n$ where A is a cone.

We let H be a hyperplane section in $\mathbb{P}^n$ that intersects A non-trivially. Then

$$\begin{aligned}
\phi^*(H) &\sim (n-3)F_{n-3} + S_{n-3} \\
\phi^*_{3,i}(\phi^*(H)) &\sim (n-3)(E_i + E_3) + (E_3 + S) \\
\phi^*_{2,i}(\phi^*_{3,i}(\phi^*(H))) &\sim (n-3)(E_i + E_3 + E_2) + (E_3 + E_2 + S + E_2) \\
\phi^*_i(\phi^*_{2,i}(\phi^*_{3,i}(\phi^*(H)))) &= (n-3)F + \left(\sum_{\substack{j\neq i \\ 3\leq j\leq k+3}} E_j \right) + 2(E_2+E_1) + S + E_1 \\
&= (n-2)F + E_2 + 2E_1 + S - E_i = D_3 - E_i = D_i.
\end{aligned}$$

Hence D_i is base point free for all i, and we can conclude that

$$\mathrm{Eff}(X) = \mathrm{Cone}(S, E_1, E_2, \cdots, E_{k+3})$$

$$\mathrm{Nef}(X) = \mathrm{Cone}(F, D_1, D_2, \cdots, D_{k+3})$$

□

Finally, we consider a point $x \in X(k)$. Similarly to the cases before, since F has intersection 1 with each generator of the nef cone, by Proposition 2.15, F is a COBA for x with respect to a divisor D unless that x belongs to an exceptional curve. We let $T = \{S, E_1, \cdots, E_{k+3}\}$. If D belongs to the nef cone, then $\alpha_x(D) = \min_{R\in T:\ x\in R}\{D\cdot F, D\cdot R\}$.

□

Theorem 5.13. *Let X_2 be a split rational surface of type 2. Let X be the blow up of X_2 at a point y in some exceptional curve and not on S, then Conjecture 5.1 is true for X.*

Proof. We consider two cases.

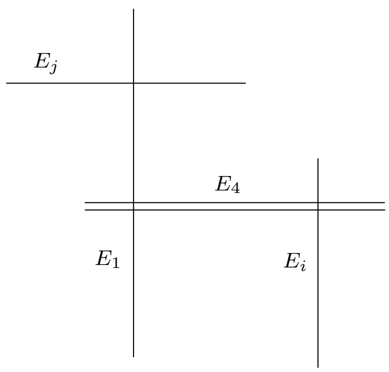
Case(1): y belongs to exactly one of E_2 or E_3. (Notice that the case of blowing up a point y in E_1 was covered in Theorem 5.5).

Case(2): $y = E_1 \cap E_2$ or $y = E_1 \cap E_3$.

We do not consider here the case of blowing-up the Hirzebruch surfaces $\mathbb{F}_0$ or $\mathbb{F}_1$, since $\mathbb{F}_0$ is the surface $\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{F}_1$ is isomporphic to the blowup of $\mathbb{P}^2$ at a point.

Case(1): Notice that this case was covered in Theorem 5.9. If for instance, y belongs to E_2 we can swap the notation of E_4 and E_3 in Theorem 5.9.

Case(2): Let $y = E_1 \cap E_i$, for $i \in \{2, 3\}$. We also let $j \in \{2, 3\}$ such that $j \neq i$. We represent the fibre of the map $\pi : X \longrightarrow \mathbb{P}^1$ as



In this case $F \sim E_1 + E_i + E_j + 2E_4$ and the intersection matrix is given by

$\cdot$	E_1	E_i	E_j	E_4	S
E_1	-3	0	1	1	1
E_i	0	-2	0	1	0
E_j	1	0	-1	0	0
E_4	1	1	0	-1	0
S	1	0	0	0	$-n$

Claim 5.14. *Eff*$(X) = \langle E_1, E_j, E_i, E_4, S \rangle$ *and Nef*$(X) = \langle F, D_1, D_j, D_i, D_4 \rangle$ *where*

$$\begin{aligned} D_1 &= nF + S \\ D_i &= D_1 - E_i - E_4 \\ D_j &= D_1 - E_j \\ D_4 &= 2(nF + S) - E_i - 2E_4 = D_1 + D_i - E_4 \end{aligned}$$

Proof. Let $\sigma = \text{Cone}(E_1, E_2, E_3, E_4, S)$. Using Matlab, we can verify that the dual cone of σ is $\sigma^\vee = \text{Cone}(F, D_1, D_i, D_j, D_4)$. Notice that S, E_1, E_j, E_i, and E_4 are all effective.

Therefore $\sigma \subset \mathrm{Eff}(X)$, and $\mathrm{Nef}(X) \subset \sigma^{\vee}$.

As we did in the previous case, use the same notation from diagram 5.1 and using the same letters for the generators D_1, D_i and D_j of the nef cone of X_2. We see that

$$\begin{aligned} f_2^*(D_1) &= D_1 \\ f_2^*(D_i) = f_2^*(nF + S - E_i) &= nF + S - E_i - E_4 = D_i \\ f_2^*(D_j) = f_2^*(nF + S - E_j) &= nF + S - E_j = D_j \end{aligned}$$

Hence, since D_1, D_i, and D_j in X_2 are base point free, then D_1, D_i, and D_j in X are also base point free.

Now we show that D_4 is base point free. We first consider the morphisms $\phi_1 : X \longrightarrow A_1 \subset \mathbb{P}^{n+1}$ and $\phi_i : X \longrightarrow A_2 \subset \mathbb{P}^n$ associated to D_1 and D_i respectively, where A_1 and A_2 are cones. Notice that ϕ_1 contracts E_i, E_j, E_4, and S, and ϕ_i contracts E_1, E_j, E_4, and S. Consider the composition $\psi : X \longrightarrow \mathbb{P}^N$ in the following commutative diagram:

$$\begin{tikzcd} X \arrow[d, "\phi_{1,i}"] \arrow[rrd, bend left, "\psi"] & & \\ A_1 \times A_2 \arrow[r, "\iota"] & \mathbb{P}^{n+1} \times \mathbb{P}^n \arrow[r, "\gamma"] & \mathbb{P}^N \end{tikzcd}$$

where $N = (n+2)(n+1) - 1$, ι is the inclusion of the Cartesian product of cones into the Cartesian product of projective spaces, γ is the Segre embedding, and for every point $p \in X$, $\phi_{1,i}(p) = (\phi_1(p), \phi_i(p))$. In fact $\phi_{1,i}^*(H) \sim D_1 + D_i$ for any hyperplane H in $\mathbb{P}^N$.

Since ϕ_1 and ϕ_i are isomorphisms away from the exceptional curves, we know that $\phi_{1,i}$ is also an isomorphism away from the exceptional curves. Now, let z_1 (and z_2) be the vertex of the cone A_1 (resp. A_2). Notice that $\phi_{1,i}(E_4) = (x, z_2)$ where x is a smooth point in the cone A_1, $\phi_{1,i}(E_i) = \{x\} \times l_1$, where l_1 is a line through the vertex z_2, $\phi_{1,i}(E_j) = (y, z_2)$ where y is a smooth point in A_1 and $y \neq x$, $\phi_{1,i}(E_1) = l_2 \times \{z_2\}$ where l_2 is a line passing through x and z_1, and finally, $\phi_{1,i}(S) = (z_1, z_2)$.

Hence, we see that $\phi_{1,i}^{-1}(x, z_2) = E_4$. So, to show that D_4 has no base points we make use of the morphism ψ. Let $p \in X \setminus E_4$, and let H be a hyperplane in $\mathbb{P}^N$ that passes through

$\psi(E_4)$ and not through $\psi(p)$. Then, $\psi^*(H) = \widetilde{C} + E_4$, where $\widetilde{C}$ is the strict transform of the curve C in $\mathbb{P}^{n+1} \times \mathbb{P}^n$ passing through $\iota(x, z_2)$ and corresponding to the hyperplane H. Thus, $\psi^*(H) - E_4 = \widetilde{C} + E_4 - E_4 \in |D_1 + D_i - E_4|$ and p does not belong to $\widetilde{C}$. Hence, p is not a base point for $D_1 + D_i - E_4$.

Now, if $p \in E_4$, then p corresponds to a tangent direction of a hyperplane section in $\mathbb{P}^N$ passing through the point $\psi(E_4)$. Let T be a hyperplane in $\mathbb{P}^N$, passing through $\psi(E_4)$ and missing the direction p. Then, $\psi^*(T)$ is a divisor in $|D_1 + D_i|$ that has a copy of E_4, thus $\psi^*(T) - E_4 \in |D_1 + D_i - E_4|$, and p is not a base point for $\psi^*(T) - E_4$. Hence $D_1 + D_i - E_4$ is base point free.

Finally we conclude that since all the generators of $\sigma^\vee$ are base point free, they are nef and the claim follows. □

We prove the conjecture for this case. Let $x \in X(k)$ be a rational point on X. Assume that x belongs to the reducible fibre F. Let C be an irreducible component of F containing x. If there is only one irreducible component C of F that contains x, then, since the intersection of C with any generator of $\mathrm{Nef}(X)$ is at most 1, by Proposition 2.15, C is a COBA for x with respect to any divisor $D \in \mathrm{Nef}(X)$, and $\alpha_x(D) = C \cdot D$. If x belongs to two irreducible components of F, say C_1 and C_2, then for any divisor $D \in \mathrm{Nef}(X)$ a COBA for x with respect to D is any of C_1 or C_2 that has the smallest intersection number with D, and $\alpha_x(D) = \min\{D \cdot C_1, D \cdot C_2\}$.

Assume that x is not in the reducible fibre F. Let $C = \pi^*(\pi(x))$ be the fibre of π that contains x. Recall that C is a smooth irreducible rational curve and $C \sim F$. Then, $C \cdot D_1 = C \cdot D_i = C \cdot D_j = 1$, and $C \cdot F = 0$. So, by Proposition 2.15, C is a COBA for any divisor $D \in \mathrm{Cone}(F, D_1, D_i, D_j)$. On the other hand, notice that $D_4 = 2D_i + E_i$ and since C is a COBA with respect to D_i and $C \cap E_i = \emptyset$, by Theorem 2.16, C is also a COBA to x with respect to D_4. Thus, C is a COBA with respect to any $D \in \mathrm{Nef}(X)$, and $\alpha_x(D) = C \cdot D$. Thus the conjecture is true in this case. □

Corollary 5.15. *The Conjecture 5.1 holds for X, where X is obtained by blowing up a smooth point in E_1 on X_2, where X_2 is of type 4.*

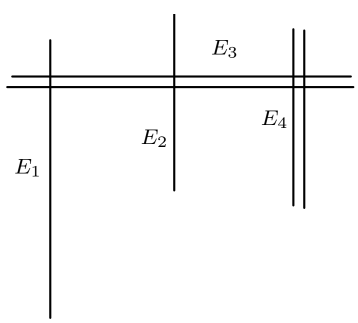
Proof. The intersection theory for X is given by the following table:

$\cdot$	E_1	E_2	E_3	E_4	S
E_1	-3	0	1	1	1
E_2	0	-2	1	0	0
E_3	1	1	-1	0	0
E_4	1	0	0	-1	0
S	1	0	0	0	$-n$

Notice that this case is the same case (2) in Theorem 5.13, where we change E_4 by E_3, $i = 2$, and $j = 4$. □

Theorem 5.16. *Let X_2 be a split rational surface of type 4. Let X be the blow-up of X_2 at a smooth point y of the fibre at the multiple component E_3. Then, Conjecture 5.1 is true for X.*

Proof. We represent the reducible fibre of the map $\pi : X \longrightarrow \mathbb{P}^1$ as the following:



In this case the reducible fibre on X is $F \sim E_1 + E_2 + 2E_3 + 2E_4$. The intersection matrix for X is:

$\cdot$	E_1	E_2	E_3	E_4	S
E_1	-2	0	1	0	1
E_2	0	-2	1	0	0
E_3	1	1	-2	1	0
E_4	0	0	1	-1	0
S	1	0	0	0	$-n$

So, we can see from the intersection table that all five divisors form a basis for $NS(X) \otimes \mathbb{R}$.

Claim 5.17. *Eff*$(X) = \langle E_1, E_2, E_3, E_4, S\rangle$ *and Nef*$(X) = \langle F, D_1, D_2, D_3, D_4\rangle$*, where:*

$$\begin{aligned} D_1 &= nF + S \\ D_2 &= D_1 - E_2 - E_3 - E_4 \\ D_3 &= 2D_1 - E_2 - 2E_3 - 2E_4 = 2D_2 + E_2 \\ D_4 &= D_3 - E_4 \end{aligned}$$

Proof. Let $\sigma = \mathrm{Cone}(E_1, E_2, E_3, E_4, S)$. Notice that E_i and S are effective divisors, $\sigma \subset \mathrm{Eff}(X)$. Using Matlab, we verify that the dual cone is indeed $\sigma^\vee = \mathrm{Cone}(F, D_1, D_2, D_3, D_4)$.

Let C be an irreducible curve in X. Then $C = a_0S + a_1E_1 + a_2E_2 + a_3E_3 + a_4E_4 \in \mathrm{NS}(X) \otimes \mathbb{R}$ for some $a_i \in \mathbb{R}$. Assume that C is different from E_i and S. Since C is an effective divisor, C intersects properly E_i and S, i.e. $C \cdot E_i \geq 0$ and $C \cdot S \geq 0$. So $C \in \sigma^\vee$. This tells us that $C = b_0F + b_1D_1 + b_2D_2 + b_3D_3 + b_4D_4$ for some $b_i \geq 0$. Then, C can be represented in two ways:

$$\begin{aligned} C &= a_0S + a_1E_1 + a_2E_2 + a_3E_3 + a_4E_4 = b_0F + b_1D_1 + b_2D_2 + b_3D_3 + b_4D_4 \\ &= b_0(E_1 + E_2 + 2E_3 + 2E_4) + b_1(nE_1 + nE_2 + 2nE_3 + 2nE_4 + S) + \\ &\quad b_2(nE_1 + (n-1)E_2 + (2n-1)E_3 + (2n-1)E_4 + S) + \\ &\quad b_3(2nE_1 + (2n-1)E_2 + (4n-2)E_3 + (4n-2)E_4 + 2S) + \\ &\quad b_4(2nE_1 + (2n-1)E_2 + (4n-2)E_3 + (4n-3)E_4 + 2S) \end{aligned}$$

Recall that $\mathrm{NS}(X)$ is a free group, so comparing coefficients for E_i and S in both expressions, we see that

$$\begin{aligned} a_0 &= b_1 + b_2 + 2b_3 + 2b_4 \\ a_1 &= b_0 + nb_1 + nb_2 + 2nb_3 + 2nb_4 \\ a_2 &= b_0 + nb_1 + (n-1)b_2 + (2n-1)b_3 + (2n-1)b_4 \\ a_3 &= 2b_0 + 2nb_1 + (2n-1)b_2 + (4n-2)b_3 + (4n-2)b_4 \\ a_4 &= 2b_0 + 2nb_1 + (2n-1)b_2 + (4n-2)b_3 + (4n-3)b_4 \end{aligned}$$

Notice that since $b_i \geq 0$ and $n \geq 1$, then $a_i \geq 0$. Then $C \in \sigma$. Hence $\sigma = \mathrm{Eff}(X)$ and $\sigma^\vee = \mathrm{Nef}(X)$ as desired. $\square$

We also can see that in fact F, D_1, D_2, and D_3 are all base point free. Clearly F and D_1 are base point free since they correspond respectively to the maps π and f in the diagram 5.1. Also, considering the morphism f_2 in 5.1, we see that

$$\begin{aligned} f_2^*(D_2) &= f_2^*(D_1 - E_2 - E_3) = D_1 - E_2 - E_3 - E_4 = D_2, \\ f_2^*(D_3) &= f_2^*(2D_1 - E_2 - 2E_3) = 2D_1 - E_2 - 2(E_3 + E_4) = D_3. \end{aligned}$$

Thus, D_2 and D_3 are base point free. Call ϕ_2 and ϕ_3 the morphisms associated to D_2 and D_3 respectively.

The morphism ϕ_2 is the blowdown map of the exceptional divisors E_4, E_3, and E_1 in that order. Thus, $\phi_2 : X \longrightarrow B \subset \mathbb{P}^n$, where B is a cone in $\mathbb{P}^n$.

Now we prove the conjecture for this case. Let $x \in X(k)$ be a rational point on X. We assume that x does not lie on S or any of the E_i's. Let C be an irreducible component of F passing through x. Recall that F corresponds to a map to $\mathbb{P}^1$ and that D_1 and D_2 correspond to maps to a cone in a projective space. Moreover,

$$\begin{aligned} D_1 \cdot D_1 &= n = \deg(f)\deg(\mathrm{Im}(f)) \\ D_2 \cdot D_2 &= n - 1 = \deg(\phi_2)\deg(\mathrm{Im}(\phi_2)) \end{aligned}$$

Since $\deg(\mathrm{Im}(f)) = n$ and $\deg(\mathrm{Im}(\phi_2)) = n - 1$, we see that $\deg(f) = \deg(\phi_2) = 1$. So, using the intersection table, we see that f contracts E_2, E_3, E_4, and S, and away from these divisors f is an isomorphism. Similarly, ϕ_2 contracts E_1, E_3, E_4, and S, and away from these divisors, ϕ_2 is an isomorphism.

Now, since x is not in any of the curves contracted by f or ϕ_2, by Proposition 2.15, we see that the approximation constant with respect to D_1 and D_2 is 1 and that C is a COBA with respect to these two divisors.

For D_3, recall that $D_3 = 2D_2 + E_2$. We already know that $C \sim F$ is a COBA with respect to D_2, E_2 is effective, and $F \cdot E_2 = 0$. By Theorem 2.16, we know that F is a COBA with respect to D_3.

Finally, we consider D_4. Let H be a hyperplane on the cone $A \subset \mathbb{P}^{n+1}$ passing through x' and P, where P is the point that we blow up to obtain X and x' is the point in $\mathbb{F}_n$

corresponding to the point $x \in X$ that we are approximating. Then, $2H$ is divisor on A vanishing with order 2 at both x' and P. Also $f^*(2H) = 2f^*(H) \sim 2(nF+S)$ on X. Then $2(nF+S) - 2(E_2+2E_3+2E_4)$ is effective, since $2H$ vanishes with order 2 at P, and $E_2+2E_3+2E_4$ is the exceptional divisor of f. We see that $D_4 = f^*(2H) - 2(E_2+2E_3+2E_4)+E_2+2E_3+E_4$. This means that $f^*(2H) - D_4 = E_2+2E_3+3E_4 \in \mathrm{Cone}(E_2, E_3, E_4)$.

Now, let $f_3 : BL_1(X) \longrightarrow X$ be the map of blowing up the point x that we are approximating, with exceptional divisor E. Since $f^*(2H)$ also vanishes with multiplicity 2 at x, we know that $f_3^*(f^*(2H)) - 2E$ is effective. Therefore,

$$f_3^*(D_4 + E_2 + 2E_3 + 3E_4) - 2E = f_3^*(D_4) + f_3^*(E_2) + 2f_3^*(E_3) + 3f_3^*(E_4) - 2E \geq 0.$$

But, since $x \notin E_2 \cup E_3 \cup E_4$, $E \cap f_3^*(E_i) = \emptyset$ for $i = 2,3,4$. This implies that $f_3^*(D_4) - 2E$ is effective. So by the Liouville-type Theorem 2.10, we know that outside of the base locus $B = f_3(B')$, where B' is the base locus of $f_3^*(D_4) - 2E$, $\alpha_x(D_4) \geq 2$. Notice also, that there are no curves in B passing through x, so by the Liouville-type theorem, we see that $\alpha_x(D_4) \geq 2$.

On the other hand, notice that C is a curve passing through x, $C \sim F$, and $F \cdot D_4 = 2$. Then, $\alpha_x(D_4) \leq 2$. This tells us that C is a COBA for x with respect to D_4. We conclude the analysis in the following table:

Divisor	COBA	α_x
F	$C \sim F$	0
D_1	$C \sim F$	1
D_2	$C \sim F$	1
D_3	$C \sim F$	2
D_4	$C \sim F$	2

Thus, for any divisor $D \in \mathrm{Nef}(X)$, C is a COBA for x with respect to D, and if we let $D = a_0F + a_1D_1 + a_2D_2 + a_3D_3 + a_4D_4$, for $a_i \in \mathbb{N} \cup \{0\}$, $\alpha_x(D) = D \cdot F = a_1 + a_2 + 2a_3 + 2a_4$.

Notice that this shows that for any divisor $D \in \mathrm{Nef}(X)$, and any $x \in X(k)$, F is a curve of best approximation to x with respect to D, unless that x belongs to an exceptional curve. If x belongs to some exceptional curve (i.e. S, E_1, E_2, E_3, or E_4), let $D \in \mathrm{Nef}(X)$, $x \in X(k)$, and let $T = \{S, E_1, E_2, E_3, E_4\}$. Since any curve in T has intersection 0 or 1 with any generator of the nef cone, then whichever curve that contains x that has minimal intersection with D will be a COBA for x with respect to D, and

$$\alpha_x(D) = \min_{\{R \in T:\ x \in R\}} \{D \cdot F,\ D \cdot R\}.$$

See the appendix for explicit calculations of α in the cases of this theorem. Although the theorem was proved here, there is an approach by dividing the nef cone into subcones of divisors that have the same curve of best approximation. This approach is the same as the one used in the previous chapter. □

Remark 5.18. *Up to this point we have proven that Conjecture 5.1 holds for every rational surface obtained by blowing up a smooth point in the reducible fibre of a surface X_2 of type 2, 3, and 4, except for a smooth point of the reducible fibre on E_2 of type 4. This last consideration will be made in Corollary 5.24.*

5.2 Blowing-up singular points in the reducible fibre

In this section we will work with cases where X is the blow-up of a surface X_2 at a singular point of the reducible fibre. We will study these cases in the following theorem:

Theorem 5.19. *Conjecture 5.1 holds for any split rational surface X of Picard rank five with only one reducible fibre of the map to $\mathbb{P}^1$.*

Proof. Any such rational surface is the blowup of a split rational surface of Picard rank 4. In our notation, it is the blowup of a surface X_2. The cases in which we blow-up a smooth point of the reducible fibre were covered in the previous theorems. Now we study the cases where we blow-up singular points in the reducible fibre of the map to $\mathbb{P}^1$ on X_2.

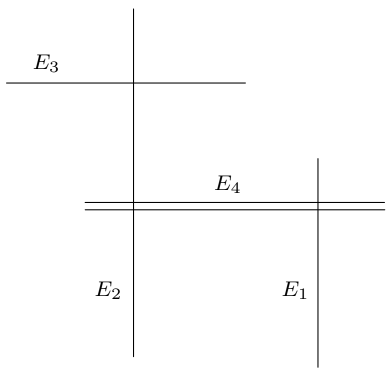
Case 1: X is the blow-up of a surface X_2 of type 2 at the point y, where $y = E_1 \cap E_2$ or $y = E_1 \cap E_3$.

This case was covered in Theorem 5.13, case (2).

Case 2: X is the blow-up of a surface X_2 of type 3 at a point y, where y is a singular point in the reducible fibre.

Case 2.1: $y = E_1 \cap E_2$ in X_2 of type 3.

In this case we represent the fibre of the map π as:



In this case the fibre of this map is $F \sim E_1 + E_2 + E_3 + 2E_4$ and the intersection matrix is given by:

$\cdot$	E_1	E_2	E_3	E_4	S
E_1	-2	0	0	1	1
E_2	0	-3	1	1	0
E_3	0	1	-1	0	0
E_4	1	1	0	-1	0
S	1	0	0	0	$-n$

Claim 5.20. *The effective cone of X is generated by E_1, E_2, E_3, E_4, and S. The nef cone of X is generated by the divisors F, $D_1 = nF + S$, $D_2 = D_1 - E_2 - E_3 - E_4$, $D_3 = D_2 - E_3$, and $D_4 = 2D_1 - E_2 - E_3 - 2E_4 = D_1 + D_2 - E_4$.*

Proof. Let $\sigma = \text{Cone}(S, E_1, E_2, E_3, E_4)$. As in previous cases, using the inner product of divisors in Matlab or Polymake one can verify that $\sigma^\vee = \text{Cone}(F, D_1, D_2, D_3, D_4)$. We want to prove that the generators of $\sigma^\vee$ are nef divisors.

We know that F and D_1 are base point free corresponding to the morphisms π and f in the diagram 5.1 respectively. On the other hand, in diagram 5.1, $f_2 : X \longrightarrow X_2$ is the blow-up map with exceptional divisor E_4. Notice that in X, $f_2^*(D_2) = f_2^*(D_1 - E_2 - E_3) =$

$D_1 - E_2 - E_4 - E_3 = D_2$ and $f_2^*(D_3) = f_2^*(D_2 - E_3) = D_2 - E_3 = D_3$. Hence, both D_2 and D_3 are base point free and thus, they are nef.

Finally, we study D_4. We first consider the morphisms $\phi_1 : X \longrightarrow A_1 \subset \mathbb{P}^{n+1}$ and $\phi_2 : X \longrightarrow A_2 \subset \mathbb{P}^n$ defined by D_1 and D_2 respectively, where A_1 and A_2 are cones. Recall that ϕ_1 contracts E_2, E_3, E_4, and S, and ϕ_2 contracts E_1, E_3, E_4, and S. Consider $\psi : X \longrightarrow \mathbb{P}^N$ be the composition:

$$\begin{tikzcd}
X \arrow[d, "\phi_{1,2}"] \arrow[rrd, bend left, "\psi"] & & \\
A_1 \times A_2 \arrow[r, "\iota"] & \mathbb{P}^{n+1} \times \mathbb{P}^n \arrow[r, "\gamma"] & \mathbb{P}^N
\end{tikzcd}$$

where $N = (n+2)(n+1) - 1$, ι is the inclusion of the cartesian product of cones into the cartesian product of projective spaces, γ is the Segre embedding, and for every point $p \in X$, $\phi_{1,2}(p) = (\phi_1(p), \phi_2(p))$. In fact $\phi_{1,2}^*(\mathcal{O}(1,1)) = D_1 + D_2$, where $\mathcal{O}(1,1)$ is the line bundle of hyperplane divisors in $\mathbb{P}^{n+1} \times \mathbb{P}^n$.

Since ϕ_1 and ϕ_2 are isomorphisms away from the exceptional curves, we know that $\phi_{1,2}$ is also an isomorphism away from the exceptional curves. Now, let z_1 (resp. z_2) be the vertex of the cone A_1 (resp. A_2). Notice that $\phi_{1,2}(E_4) = (x, z_2)$ where x is a smooth point in the cone A_1, $\phi_{1,2}(E_2) = \{x\} \times l_1$, where l_1 is a line through the vertex z_2, $\phi_{1,2}(E_3) = (x, y)$ where y is a smooth point in A_2 (notice that the line l_1 passes through y), $\phi_{1,2}(E_1) = l_2 \times \{z_2\}$ where l_2 is a line passing through x and z_1, and finally, $\phi_{1,2}(S) = (z_1, z_2)$.

Hence, we see that $\phi_{1,2}^{-1}(x, z_2) = E_4$. So, to show that D_4 has no base points we make use of the morphism ψ. Let $p \in X \setminus E_4$, and let H be a hyperplane in $\mathbb{P}^N$ that passes through $\psi(E_4)$ and not through $\psi(p)$. Then, $\psi^*(H) = \widetilde{C} + E_4$, where $\widetilde{C}$ is the strict transform of the curve C in $\mathbb{P}^{n+1} \times \mathbb{P}^n$ passing through $\iota(x, z_2)$ and corresponding to the hyperplane H. Thus, $\psi^*(H) - E_4 = \widetilde{C} + E_4 - E_4 \in |D_1 + D_2 - E_4|$ and so, p is not a base point for $D_1 + D_2 - E_4$.

Now, let $p \in E_4$. Then p corresponds to a tangent direction of a hyperplane section in $\mathbb{P}^N$ passing through the point $\psi(E_4)$. Let T be a hyperplane in $\mathbb{P}^N$, passing through $\psi(E_4)$ and missing the direction p. Then, $\psi^*(T)$ is a divisor in $|D_1 + D_2|$ that has a copy of E_4.

Thus p is not a base point for $\psi^*(T) - E_4$. Hence $D_4 = D_1 + D_2 - E_4$ is base point free.

We conclude that since all the generators of $\sigma^\vee$ are base point free, thus, they are nef. The claim follows. $\square$

Now, we find a COBA with respect to each generator of the nef cone. First, we study the COBAs with respect to D_4. Assume that $x \in X(k)$ is a point lying in some of E_1, E_2, E_3 or S. Then, since D_4 intersects any of these curves trivially, whichever contains x is a COBA for x with respect to D_4 and $\alpha_x(D_4) = 0$.

Assume that $x \in X(k)$ is a point not lying in any exceptional curve. Let $C = \pi^*(\pi(x))$. We know that $C \sim F$ and $F \cdot D_4 = 2$, so $\alpha_x(D_4) \leq 2$. Now, let $g : \widetilde{X} \longrightarrow X$ be the blow-up morphism of X at the point x with exceptional divisor E. We want to show that $g^*(D_4) - 2E$ is effective.

Recall from diagram 5.1, that D_1 is base point free with associated morphism f which is the composition $f = \phi \circ g_1$ where $g_1 := f_0 \circ f_1 \circ f_2 : X \longrightarrow \mathbb{F}_n$ and $\phi : \mathbb{F}_n \longrightarrow A \subset \mathbb{P}^{n+1}$. So, we have the following diagram:

$$\widetilde{X} \xrightarrow{\ g\ } X \xrightarrow{\ g_1\ } \mathbb{F}_n \xrightarrow{\ \phi\ } \mathbb{P}^{n+1}.$$

Let H be a hyperplane section on $\mathbb{P}^{n+1}$ passing through the points $\phi(g_1(x))$ and $\phi(g_1(E_4))$. Then $\phi^*(2H)$ is a curve in $\mathbb{F}_n$ that vanishes with order 2 at the points $g_1(x)$ and $g_1(E_4)$. Let us call this curve $M := \phi^*(2H)$. Thus, $g_1^*(M) - 2(E_2 + E_3 + 2E_4)$ is an effective divisor (recall that $E_2 + E_3 + 2E_4$ is the exceptional divisor corresponding to the morphism g_1). Moreover, $g_1^*(M) - 2(E_2 + E_3 + 2E_4) \sim D_4 - E_2 - E_3 - 2E_4$. Hence, $g_1^*(M) \sim D_4 + E_2 + E_3 + 2E_4$. Also, since $g_1^*(M)$ vanishes with order 2 at x, $g^*(g_1^*(M)) - 2E \geq 0$ is an effective divisor. Thus $g^*(D_4) + g^*(E_2) + g^*(E_3) + 2g^*(E_4) - 2E \geq 0$. But since $x \notin E_2 \cup E_3 \cup E_4$, $g^*(E_i) \cap E = \emptyset$ for all $i \in \{2, 3, 4\}$, so $g^*(D_4) - 2E$ is effective. Thus, by the Liouville-type Theorem 2.10, since outside the base locus $B = g(B')$ where B' is the base locus of $g^*(D_4) - 2E$, there are no curves in the base locus of B that pass through x. Then $\alpha_x(D_4) \geq 2$.

The previous analysis and the fact that $D_4 \cdot F = 2$ tells us that if $x \in X(k)$ and x does not lie in an exceptional curve, then $C \sim F$ is a COBA for x with respect to D_4.

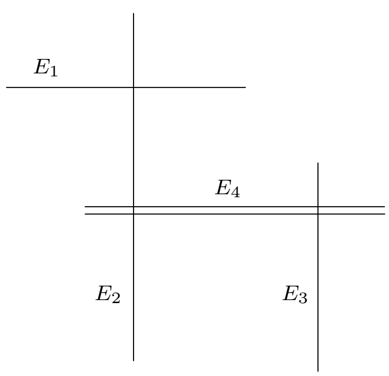
Finally, assume that $x \in E_4(k)$ is a point that belongs only to E_4 and not to any other exceptional curve. Then, since $E_4 \cdot D_4 = 1$, by Proposition 2.15, we know that E_4 is a COBA for x with respect to D_4 and that $\alpha_x(D_4) = 1$.

To finish this case, we study the intersections of F with the generators of the nef cone and we see that $F \cdot F = 0$, $F \cdot D_i = 1$ for all $i \in \{1, 2, 3\}$. Hence, by Proposition 2.15, given a point $x \in X(k)$, F is a COBA for x with respect to any of F, D_1, D_2, or D_3, unless x belongs to an exceptional curve that is contracted by some generator, in which case that exceptional curve would be a COBA.

Hence, we conclude that for any divisor $D \in \mathrm{Nef}(X)$, the component of F of minimal D-degree through x is always a COBA with respect to D, unless x lies in S, in which case whichever of S or $C = \pi^*(\pi(x))$ (if C is irreducible) that has minimal intersection with D is a COBA for x with respect to D, and $\alpha_x(D) = \min\{D \cdot C, D \cdot S\}$. So, the theorem follows.

Case 2.2: $y = E_2 \cap E_3$ on X_2, where X_2 is of type 3.

In this case we represent the fibre of the map π as:



The fibres of this map are $F \sim E_1 + E_2 + E_3 + 2E_4$ and the intersection matrix is given by:

$\cdot$	E_1	E_2	E_3	E_4	S
E_1	-1	1	0	0	1
E_2	1	-3	1	1	0
E_3	0	0	-2	1	0
E_4	0	1	1	-1	0
S	1	0	0	0	$-n$

Claim 5.21. *The effective cone of X is generated by E_1, E_2, E_3, E_4, and S. The nef cone of X is generated by the divisors F, $D_1 = nF + S$, $D_2 = D_1 - E_2 - E_3 - 2E_4$, $D_3 = D_2 - E_3 - E_4$, and $D_4 = D_2 + D_3 - E_4$.*

Proof. Let $\sigma = \text{Cone}(S, E_1, E_2, E_3, E_4)$. As in previous cases, using Matlab or Polymake we can verify that $\sigma^\vee = \text{Cone}(F, D_1, D_2, D_3, D_4)$. We want to prove that the generators of $\sigma^\vee$ are nef divisors.

We know that F and D_1 are base point free corresponding to the morphisms π and f in the diagram 5.1 respectively. On the other hand, in diagram 5.1, $f_2 : X \longrightarrow X_2$ is the blow-up map with exceptional divisor E_4. Notice that in X, $f_2^*(D_2) = f_2^*(D_1 - E_2 - E_3) = D_1 - E_2 - E_3 - 2E_4 = D_2$ and $f_2^*(D_3) = f_2^*(D_2 - E_3) = D_2 - E_3 - E_4 = D_3$. Hence, both D_2 and D_3 are base point free and thus, they are nef.

Now we study D_4. We first consider the morphisms $\phi_2 : X \longrightarrow A_2 \subset \mathbb{P}^n$ and $\phi_3 : X \longrightarrow A_3 \subset \mathbb{P}^{n-1}$ associated to D_2 and D_3 respectively, where A_2 and A_3 are cones. Recall that ϕ_2 contracts in order E_4, E_3, E_1, and S, and ϕ_3 contracts in order E_4, E_1, E_2, and S. Consider $\psi : X \longrightarrow \mathbb{P}^N$ be the composition:

$$\begin{tikzcd} X \arrow[d, "\phi_{2,3}"] \arrow[rr, bend left, "\psi"] & & \\ A_2 \times A_3 \arrow[r, "\iota"] & \mathbb{P}^n \times \mathbb{P}^{n-1} \arrow[r, "\gamma"] & \mathbb{P}^N \end{tikzcd}$$

where $N = (n+1)(n) - 1$, ι is the inclusion of the cartesian product of cones into the cartesian product of projective spaces, γ is the Segre embedding, and for every point $p \in X$, $\phi_{2,3}(p) = (\phi_2(p), \phi_3(p))$. In fact $\phi_{2,3}^*(\mathcal{O}(1,1)) = D_2 + D_3$.

Since ϕ_2 and ϕ_3 are isomorphisms away from the exceptional curves, we know that $\phi_{2,3}$ is also an isomorphism away from the exceptional curves. Now, let z_2 (resp. z_3) be the vertex of the cone A_2 (resp. A_3). Notice that $\phi_{2,3}(E_4) = (x, z_3)$ where x is a smooth point in the cone A_2, $\phi_{2,3}(E_2) = l_1 \times z_3$, where l_1 is a line through the vertex z_2 and through the point x, $\phi_{2,3}(E_3) = \{x\} \times l_2$ where l_2 is a line that passes through z_3, and $\phi_{2,3}(E_1) = \phi_{2,3}(S) = (z_2, z_3)$.

Hence, we see that $\phi_{2,3}^{-1}(x, z_3) = E_4$. So, to show that D_4 has no base points we make use of the morphism ψ. Let $p \in X \setminus E_4$, and let H be a hyperplane in $\mathbb{P}^N$ that passes through $\psi(E_4)$ and not through $\psi(p)$. Then, $\psi^*(H) = \widetilde{C} + E_4$, where $\widetilde{C}$ is the strict transform of the curve C in $\mathbb{P}^{n+1} \times \mathbb{P}^n$ passing through $\iota(x, z_3)$ and corresponding to the hyperplane H. Thus, $\psi^*(H) - E_4 = \widetilde{C} + E_4 - E_4 \in |D_2 + D_3 - E_4|$ and so, p is not a base point for $D_2 + D_3 - E_4$.

Now, let $p \in E_4$. Then p corresponds to a tangent direction of a hyperplane section in $\mathbb{P}^N$ passing through the point $\psi(E_4)$. Let T be a hyperplane in $\mathbb{P}^N$, passing through $\psi(E_4)$ and missing the direction p. Then, $\psi^*(T)$ is a divisor in $|D_2 + D_3|$ that has a copy of E_4. Thus p is not a base point for $\psi^*(T) - E_4$. Hence $D_4 = D_2 + D_3 - E_4$ is base point free.

We conclude that all the generators of $\sigma^\vee$ are base point free, and hence they are nef and the claim follows. $\square$

To finish this case, we study the intersections of F with the generators of the nef cone and we see that $F \cdot F = 0$, $F \cdot D_i = 1$ for all $i \in \{1, 2, 3\}$. Hence, by Proposition 2.15, given a point $x \in X(k)$, F is a COBA for x with respect to any of F D_1, D_2, or D_3, unless x belongs to an exceptional curve that is contracted by some generator, in which case that exceptional curve would be a COBA.

Finally we study the COBAs with respect to D_4. First assume that $x \in X(k)$ is a point lying in some of E_1, E_2, E_3 or S. Then, since D_4 intersects any of these curves trivially, whichever contains x is a COBA for x with respect to D_4 and $\alpha_x(D_4) = 0$.

Assume that $x \in X(k)$ is a point not lying in any exceptional curve. Let $C = \pi^*(\pi(x))$. We know that $C \sim F$ and $F \cdot D_4 = 2$, then $\alpha_x(D_4) \leq 2$. Now, let $g : \widetilde{X} \longrightarrow X$ be the blow-up morphism of X at the point x with exceptional divisor E. We want to show that $g^*(D_4) - 2E$ is effective.

Recall from diagram 5.1, that D_1 is base point free with associated morphism $f : X \longrightarrow A \subset \mathbb{P}^{n+1}$ where A is a cone. So, we have the following diagram:

$$\widetilde{X} \xrightarrow{g} X \xrightarrow{f} \mathbb{P}^{n+1}.$$

Let H be a hyperplane section on $\mathbb{P}^{n+1}$ passing through the points $f(x)$ and $f(E_4)$. Then $f^*(2H)$ is a curve in X that vanishes with order 2 at the point x and at E_4. Let us call this curve $M := f^*(2H)$. Notice that $M \sim 2(nF+S)$ and also $D_4 = 2(nF+S)-2E_2+3E_3-6E_4$. Thus, $M \sim D_4 + 2E_2 + 3E_3 + 6E_4$.

On the other hand, since M vanishes with order 2 at x, $g^*(M) - 2E$ is effective, which implies that $g^*(D_4) + 2g^*(E_2) + 3g^*(E_3) + 6g^*(E_4) - 2E$ is also effective. But notice that since x does not belong to any exceptional curve, $g^*(E_i) \cap E = \emptyset$ for all $i \in \{2, 3, 4\}$. This implies that $g^*(D_4) - 2E$ is effective. Hence, by the Liouville-type theorem 2.10, since outside the base locus $B = g(B')$ where B' is the base locus of $g^*(D_4) - 2E$, there are no curves in the base locus of B that pass through x. Then $\alpha_x(D_4) \geq 2$.

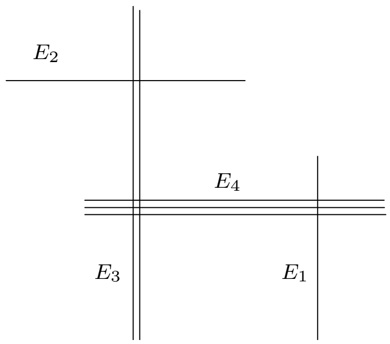
The previous analysis and the fact that $D_4 \cdot F = 2$ tells us that if $x \in X(k)$ and x does not lie in an exceptional curve, then F is a COBA for x with respect to D_4.

Finally, assume that $x \in E_4(k)$ is a point that belongs only to E_4 and not to any other exceptional curve. Then, since $E_4 \cdot D_4 = 1$, by Proposition 2.15, we know that E_4 is a COBA for x with respect to D_4 and that $\alpha_x(D_4) = 1$.

Hence, we conclude that for any divisor $D \in \mathrm{Nef}(X)$, the component of F through x is always a COBA with respect to D, unless x lies in S, in which case whichever of S or $C = \pi^*(\pi(x))$ (if C is irreducible) that has minimal intersection with D is a COBA for x with respect to D, and $\alpha_x(D) = \min\{D \cdot C, D \cdot S\}$. So, the theorem follows.

Case 3: X is the blow-up of a surface X_2 of type 4 at a point y, where y is a singular point of the reducible fibre.

Case 3.1: $y = E_1 \cap E_3$ on X_2, where X_2 is of type 4.
In this case the fibre F of the map $\pi : X \longrightarrow \mathbb{P}^1$ is $F \sim E_1 + E_2 + 2E_3 + 3E_4$ and we represent it as



As in cases before the Picard group of X is $\mathrm{Pic}(X) = \mathbb{Z}E_1 \oplus \mathbb{Z}E_2 \oplus \mathbb{Z}E_3 \oplus \mathbb{Z}E_4 \oplus \mathbb{Z}S$, with intersection theory given by:

$\cdot$	E_1	E_2	E_3	E_4	S
E_1	-3	0	0	1	1
E_2	0	-2	1	0	0
E_3	0	1	-2	1	0
E_4	1	0	1	-1	0
S	1	0	0	0	$-n$

Claim 5.22. *The effective cone of X is generated by E_1, E_2, E_3, E_4, and S. The nef cone of X is generated by the divisors F, $D_1 = nF + S$, $D_2 = D_1 - E_2 - E_3 - E_4$, $D_3 = 2D_1 - E_2 - 2E_3 - 2E_4$, and $D_4 = D_1 + D_2 - E_4$.*

Proof. Let $\sigma = \mathrm{Cone}(E_1, E_2, E_3, E_4, S)$. Using Matlab we can verify that the dual cone of σ is $\sigma^\vee = \mathrm{Cone}(F, D_1, D_2, D_3, D_4)$, and we know that $\sigma \subset \mathrm{Eff}(X)$. To show that σ is in fact the effective cone of X, we verify that all generators of the nef cone are base point free. We already know that F is base point free. Also, as before we have take the pullback of the generators of the nef cone of X_2 of type 4 and we have

$$
\begin{aligned}
f_2^*(D_1) &= D_1 \\
f_2^*(D_2) &= f_2^*(D_1 - E_2 - E_3) = D_1 - E_2 - E_3 - E_4 = D_2 \\
f_2^*(D_3) &= f_2^*(2D_1 - E_2 - 2E_3) = 2D_1 - E_2 - 2E_3 - 2E_4 = D_3
\end{aligned}
$$

Finally, notice that D_4 can be written as $D_4 = D_2 + D_3 + E_2 + E_3$. But, since D_2 and D_3 are base point free, this representation tells us that the possible base points of D_4 lie in E_2 or

E_3. Similarly, we can write D_4 as $D_4 = (3n-1)F + E_1 + 3S$, which tells us that the possible base points for D_4 should lie in E_1 or S. Notice that in this case $(E_2 \cup E_3) \cap (E_1 \cup S) = \emptyset$. Hence, D_4 is base point free, and the claim follows.

□

To prove the conjecture, first we check the intersection of the generators of the nef cone with the fibre F. The intersections are: $F \cdot F = 0$, $F \cdot D_1 = F \cdot D_2 = 1$, $F \cdot D_3 = 2$, and $F \cdot D_4 = 3$.

Let $x \in X(k)$ be a point. By Proposition 2.15, F is a COBA with respect to F, D_1, and D_2, unless x belongs to an exceptional curve, in which case the exceptional curve containing x and having the minimal intersection is a COBA with respect to all F, D_1, and D_2.

We study the curves of best approximation to x with respect to D_3. To do it we will use the blow-up morphism of x and use the Liouville-type theorem. Assume that x is not lying in any exceptional curve. Then, since $D_3 \cdot F = 2$, $\alpha_x(D_3) \leq 2$. Consider the following composition of maps, where we consider the maps as before (i.e. $\widetilde{X}$ is the blow-up of X at x, and D_1 corresponds to the morphism $f = \phi \circ g_1$):

$$\widetilde{X} \xrightarrow{g} X \xrightarrow{g_1} \mathbb{F}_n \xrightarrow{\phi} \mathbb{P}^{n+1}. \tag{5.2}$$

Let H be a hyperplane section on $\mathbb{P}^{n+1}$ passing through the points $\phi(g_1(x))$ and $\phi(g_1(E_4))$. Then $\phi^*(2H)$ is a curve in $\mathbb{F}_n$ that vanishes with order 2 at the points $g_1(x)$ and $g_1(E_4)$. Let us call this curve $M := \phi^*(2H)$. Thus, $g_1^*(M) - 2(E_2 + 2E_3 + 3E_4)$ is an effective divisor (recall that $E_2 + 2E_3 + 3E_4$ is the exceptional divisor corresponding to the morphism g_1). Moreover, $g_1^*(M) - 2(E_2 + 2E_3 + 3E_4) \sim D_3 - E_2 - 2E_3 - 4E_4$. So $g_1^*(M) \sim D_3 + E_2 + 2E_3 + 2E_4$. Also, since $g_1^*(M)$ vanishes with order 2 at x, $g^*(g_1^*(M)) - 2E \geq 0$ is an effective divisor. Thus $g^*(D_3) + g^*(E_2) + 2g^*(E_3) + 2g^*(E_4) - 2E \geq 0$. But since $x \notin E_2 \cup E_3 \cup E_4$, $g^*(E_i) \cap E = \emptyset$ for all $i \in \{2, 3, 4\}$, which implies that $g^*(D_3) - 2E$ is effective. Hence, by the Liouville-type theorem 2.10, the approximation constant outside the base locus $B = g(B')$ where B' is the base locus of $g^*(D_3) - 2E$, is at least 2. But, since there are no curves in the base locus of B that pass through x, we conclude that $\alpha_x(D_3) \geq 2$. Notice $F \cdot D_3 = 2$, then $\pi^*(\pi(x)) \sim F$ is a COBA for x with respect to D_3.

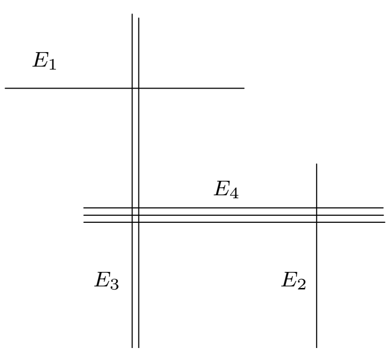
Notice that on the other hand, if x belongs to some exceptional curves then the exceptional curve with minimal intersection with D_3 will be a COBA for x with respect to D_3.

Similarly, we study the COBAs for D_4. We use the same diagram (5.2). Now, we call $M' := \phi^*(3H)$. Then $g_1^*(M') - 3(E_2 + 2E_3 + 3E_4)$ is an effective divisor, and moreover $g_1^*(M) \sim D_4 + 2E_2 + 3E_3 + 3E_4$, and since $g_1^*(M)$ vanishes with order 3 at x, $g^*(g_1^*(M)) - 3E \geq 0$ is an effective divisor. Thus $g^*(D_4) + 2g^*(E_2) + 3g^*(E_3) + 3g^*(E_4) - 3E \geq 0$, and similarly, since E does not intersect any exceptional curve, we see that $g^*(D_4) - 3E \geq 0$. Hence, by the Liouville-type theorem, outside the base locus $B = g(B')$ where B' is the base locus of $g^*(D_4) - 3E$, the approximation constant is at least 3. But since there are no curves in the base locus of B that pass through x, $\alpha_x(D_4) \geq 3$. Notice that $F \cdot D_4 = 3$, so $\pi^*(\pi(x)) \sim F$ is a COBA for x with respect to D_4. Which implies that $\pi^*(\pi(x)) \sim F$ is a COBA for x with respect to any divisor in the nef cone.

We conclude this case by noticing that if x belongs to any exceptional curve, since any exceptional curve has intersection 0 or 1 with any generator of the nef cone, whichever exceptional curve that has minimal intersection with a divisor $D \in \mathrm{Nef}(X)$, will be a COBA for x with respect to D.

Case 3.2: $y = E_2 \cap E_3$ on X_2, where X_2 is of type 4.

In this case the fibre F of the map $\pi : X \longrightarrow \mathbb{P}^1$ is $F \sim E_1 + E_2 + 2E_3 + 3E_4$ and we represent it as



As in cases before the Picard group of X is $\mathrm{Pic}(X) = \mathbb{Z}E_1 \oplus \mathbb{Z}E_2 \oplus \mathbb{Z}E_3 \oplus \mathbb{Z}E_4 \oplus \mathbb{Z}S$, with intersection theory given by:

$\cdot$	E_1	E_2	E_3	E_4	S
E_1	-2	0	1	0	1
E_2	0	-3	0	1	0
E_3	1	0	-2	1	0
E_4	0	1	1	-1	0
S	1	0	0	0	$-n$

Claim 5.23. *The effective cone of X is generated by E_1, E_2, E_3, E_4, and S and the nef cone of X is generated by F, D_1, $D_2 = D_1 - E_2 - E_3 - 2E_4$, $D_3 = 2D_1 - E_2 - 2E_3 - 3E_4$, and $D_4 = 3D_2 + E_2$.*

Proof. Let $\sigma = \text{Cone}(E_1, E_2, E_3, E_4, S)$. Using Matlab we can verify that the dual cone of σ is $\sigma^\vee = \text{Cone}(F, D_1, D_2, D_3, D_4)$, and we know that $\sigma \subset \text{Eff}(X)$. To show that σ is in fact the effective cone of X, we verify that all generators of the nef cone are indeed base point free. We already know that F is base point free. Also, as before we take the pullback of the generators of the nef cone of X_2 of type 4 and we have

$$\begin{aligned} f_2^*(D_1) &= D_1, \\ f_2^*(D_2) &= f_2^*(D_1 - E_2 - E_3) = D_1 - E_2 - E_3 - 2E_4 = D_2, \\ f_2^*(D_3) &= f_2^*(2D_1 - E_2 - 2E_3) = 2D_1 - E_2 - 2E_3 - 3E_4 = D_3. \end{aligned}$$

Finally, notice that D_4 is written in terms of D_2 and D_2 is base point free. Hence the only possible base point of D_4 should lie on E_2. Additionally, D_4 can be written as $D_4 = (3n-2)F + 2E_1 + E_3 + 3S$, which tells us that the possible base points of D_4 should lie on E_1, E_3, or on S. But since $(E_1 \cup E_3 \cup S) \cap E_2 = \emptyset$, we conclude that D_4 is base point free. Thus, the claim holds.

□

It is clear that for a point $x \in X(k)$ not lying in an exceptional curve, F is a COBA with respect to F, D_1, and D_2 (by Proposition 2.15). Otherwise, if x lies on some exceptional curve, the curve with minimal intersection with a divisor $D \in \text{Cone}(F, D_1, D_2)$ would be a COBA with respect to D.

To study the COBAs with respect to D_3, we first assume that x does not lie in any exceptional curve. We use the set up for the Liouville-type theorem as before, where $\widetilde{X}$ is the blow up of X at the point x.

$$\widetilde{X} \xrightarrow{g} X \xrightarrow{g_1} \mathbb{F}_n \xrightarrow{\phi} \mathbb{P}^{n+1}.$$

Let H be a hyperplane section on $\mathbb{P}^{n+1}$ passing through the points $\phi(g_1(x))$ and $\phi(g_1(E_4))$. Then $\phi^*(2H)$ is a curve in $\mathbb{F}_n$ that vanishes with order 2 at the points $g_1(x)$ and $g_1(E_4)$. Let us call this curve $M := \phi^*(2H)$. Then $g_1^*(M) \sim 2D_1 = D_3 + E_2 + 2E_3 + 3E_4$.

Also, since $g_1^*(M)$ vanishes with order 2 at x, $g^*(g_1^*(M)) - 2E$ is an effective divisor. Thus $g^*(D_3) + g^*(E_2) + 2g^*(E_3) + 3g^*(E_4) - 2E \geq 0$. But since $x \notin E_2 \cup E_3 \cup E_4$, $g^*(E_i) \cap E = \emptyset$ for all $i \in \{2, 3, 4\}$, $g^*(D_3) - 2E$ is effective. Hence, by the Liouville-type theorem 2.10, outside the base locus $B = g(B')$ where B' is the base locus of $g^*(D_3) - 2E$, the approximation constant is at least 2. But there are no curves in the base locus of B that pass through x. Thus $\alpha_x(D_3) \geq 2$. Notice that $F \cdot D_3 = 2$. So we conclude that F is a COBA for x with respect to D_3.

On the other hand, if x lies in some exceptional curves, then a curve with minimal intersection with D_3 that contains x is a COBA for x with respect to D_3.

Finally, we study COBAs with respect to D_4. If x lies on some exceptional curve, as we just described, there will be a COBA for x with respect to D_4. If x does not lie in any exceptional curve, then we know that F is a COBA for x with respect to D_2, and by Theorem 2.16, as $F \cap E_2 = \emptyset$, F is also a COBA for x with respect to D_4.

We conclude that for any divisor $D \in \mathrm{Nef}(X)$, the component of F through x is always a COBA with respect to D, unless x lies in S, in which case any of S or $C = \pi^*(\pi(x))$ (if C is irreducible) that has minimal intersection with D is a COBA for x with respect to D, and $\alpha_x(D) = \min\{D \cdot C, D \cdot S\}$. □

Corollary 5.24. *The Conjecture 5.1 is true for X, where X is the blowing up of a smooth point of the reducible fibre in E_2 on X_2, where X_2 is of type 4.*

Proof. The intersection theory for X is given by the following table:

$\cdot$	E_1	E_2	E_3	E_4	S
E_1	-2	0	1	0	1
E_2	0	-3	1	1	0
E_3	1	1	-1	0	0
E_4	0	1	0	-1	0
S	1	0	0	0	$-n$

Notice that this case is the same as the case 2.1 in Theorem 5.19, where we interchange E_4 and E_3. $\square$

Chapter 6

Further work

In this chapter, we present some possible directions in which the ideas in this thesis can be taken further.

Through this study it was important to completely describe the generators of the nef cone and the effective cone. It is also known that the number of generators of the effective cone grows pretty quickly. In fact, for $\mathbb{P}^2$ blown up at six rational points in general position, the nef cone has 99 generators. For the complete list of generators see [21]. It is also known that the nef cone of $\mathbb{P}^2$ blown up at 9 points in general position is not finitely generated. Thus, the methods used in this work may not be useful to study these cases. For more reference on the effective cones of blow-ups of projective space, see [8].

6.1 Blowing-up surfaces with multiple reducible fibres

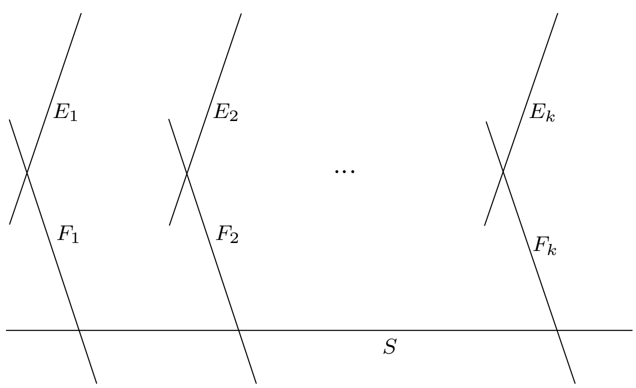
We consider an additional case in this section, that shows the techniques that can be used when blowing up a surface with multiple reducible fibres of the map to $\mathbb{P}^1$.

The surfaces described in the previous chapter are the result of blowing up a surface with only one reducible fibre of the map to $\mathbb{P}^1$. We still need to cover the cases in which the map to $\mathbb{P}^1$ has multiple reducible fibres. i.e. blowing up surfaces of type 1.

One of the main constraints on solving this problem is that the effective and nef cones are no longer simplicial and thus, the methods to prove that the effective cone is generated by the exceptional curves on X is more subtle.

The surfaces of type 1 can be presented in the following general form, in which we blow up a Hirzebruch surface $\mathbb{F}_n$ at $k < n$ points none of them lying in the $(-n)$-curve.

Surfaces X_k of type 1: We let $n \geq 2$. These type of surfaces are the blowup of a Hirzebruch surface $\mathbb{F}_n$ at $k < n$ points, say p_1, p_2, $\cdots$, p_k, no two of which lie in the same fibre of π. These surfaces have k reducible fibres, each with two components that intersect transversely at one point. (The case $n = 1$ is the blow-up of $\mathbb{P}^2$ at k points and the conjecture was proven for $k \leq 6$ in [21]). It is known that $\mathrm{Pic}(X) = \mathbb{Z}S \oplus \mathbb{Z}F \bigoplus_{i=1}^{k} \mathbb{Z}E_i$, where S is the $(-n)$-section, F is a fibre of π, and E_i is the exceptional curve of the blowup at the point p_i. The configuration of the reducible fibres is given by:



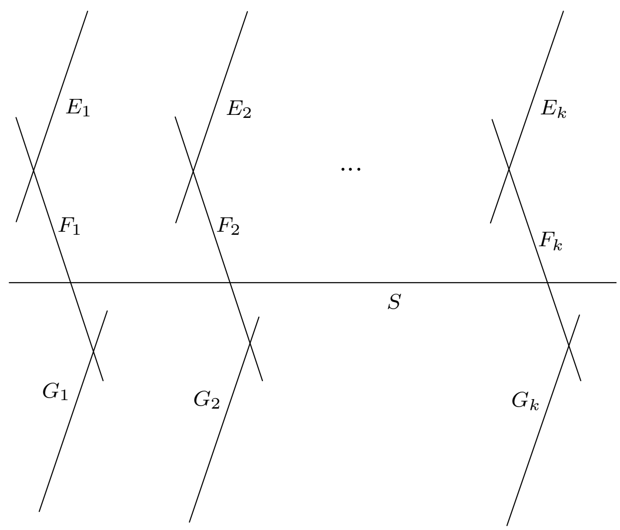
We have $F \sim F_i + E_i$ for all $i \in \{1, 2, \cdots, k\}$, with intersection theory given by:

$$
\begin{array}{rcl}
E_i^2 & = & -1 \quad \text{ for all } i \in \{1, 2, \cdots, k\} \\
S^2 & = & -n \\
F^2 & = & 0 \\
E_i \cdot E_j & = & 0 \quad \text{for all } i, j \in \{1, 2, \cdots, k\}, \ i \neq j \\
E_i \cdot S = E_i \cdot F & = & 0 \quad \text{for all } i \in \{1, 2, \cdots, k\} \\
F \cdot S & = & 1.
\end{array}
$$

It was proven in [19, Theorem 3.5] that $\mathrm{Eff}(X) = \mathrm{Cone}(S, F_i, E_i)$ for all $i \in \{1, 2, \cdots, k\}$, and $\mathrm{Nef}(X) = \mathrm{Cone}(F, D_\alpha)$, where $\alpha \in \{0, 1\}^k$ and $D_\alpha = S + nF - \alpha \cdot (E_1, \cdots, E_k)$.

Theorem 6.1. *Let X be the blow-up of the type 1 surface X_k at k points, each lying in $F_i \setminus (S \cup E_i)$, for $i \in \{1, 2, \cdots, k\}$ and $k \in \mathbb{N}$ and $k < n$. Then Conjecture 5.1 is true for X.*

Proof. The reducible fibre of the map $\pi : X \longrightarrow \mathbb{P}^1$ is $F \sim F_i + E_i + G_i$. The configuration of the reducible fibres is represented as follows:



For the proof we assume that X is the blow-up of X_1 at k points, each lying in a reducible fibre as is represented above, but this proof also works for X obtained by blowing up smooth points in some of the reducible fibres of X_1 and not necessarily all k fibres.

Notice that $\mathrm{Pic}(X) = \mathbb{Z}S \oplus \mathbb{Z}F_1 \bigoplus_{i=1}^{k} \mathbb{Z}E_i \bigoplus_{i=1}^{k} G_i$, where G_i is the exceptional divisor of the blow-up of X_k at the point $p_i \in F_i \setminus (S \cup E_i)$. With the following intersections, where $i, j \in \{1, 2, 3, \cdots, k\}$ and $i \neq j$:

$$
\begin{aligned}
E_i^2 &= -1 \\
G_i^2 &= -1 \\
S^2 &= -n \\
F_i^2 &= -2 \\
G_i \cdot G_j = E_i \cdot E_j = E_i \cdot G_j = E_i \cdot G_i &= 0 \\
F_i \cdot G_i = F_i \cdot E_i &= 1 \\
F_i \cdot G_j = F_i \cdot E_j = F_i \cdot F_j &= 0 \\
E_i \cdot S = G_i \cdot S &= 0 \\
F_i \cdot S &= 1.
\end{aligned}
$$

Claim 6.2. *The effective cone of X is generated by E_i, G_i, F_i, and S for all $i \in \{1, 2, \cdots, k\}$. The nef cone of X is generated by the divisors F, and $D_{\alpha,\beta}$, where $\alpha = (\alpha_1, \cdots, \alpha_k), \beta = (\beta_1, \cdots, \beta_k) \in \{0,1\}^k$, $\alpha_i + \beta_i \leq 1$ for all i, and $D_{\alpha,\beta} = nF + S - \alpha \cdot (E_1, \cdots, E_k) - \beta \cdot (G_1, \cdots, G_k)$.*

Proof. Let $\sigma = \mathrm{Cone}(S, G_i, E_i, F_i)$ and let $\tau = \mathrm{Cone}(F, D_1, D_{\alpha,\beta})$. First we prove that these cones are dual to one another.

Notation for the proof: For convenience we will use the notation

$$D_1 := nF + S = D_{\vec{0},\vec{0}}$$

$$I := \left\{ (\alpha, \beta) \in \{0,1\}^k \times \{0,1\}^k : \begin{array}{c} \alpha = (\alpha_1, \cdots, \alpha_k),\ \beta = (\beta_1, \cdots, \beta_k), \\ \text{and } \alpha_i + \beta_i \leq 1 \text{ for all } \ 1 \leq i \leq k \end{array} \right\}.$$

The intersection matrix of the generators of τ and σ is as follows:

$\cdot$	S	E_i	G_i	F_i
F	1	0	0	0
D_1	0	0	0	1
$D_{\alpha,\beta}$	0	α_i	β_i	$1 - \alpha_i - \beta_i \geq 0$

We see that since the intersections are non-negative, $\tau \subseteq \sigma^\vee$.

On the other hand, let $D \in \tau^\vee$. Then the intersection of D with each generator of τ is non-negative. More specifically let $D = a_0 S + \sum_{i=1}^k a_i E_i + \sum_{i=1}^k b_i G_i + b_0 F_1 \in \mathrm{NS}(X) \otimes \mathbb{R}$. Then,

$$
\begin{aligned}
D \cdot F &= a_0 \geq 0 && (6.1)\\
D \cdot D_1 &= b_0 \geq 0 && (6.2)\\
D \cdot D_{\alpha,\beta} &= \sum_{i=1}^k a_i \alpha_i + b_i \beta_i \geq 0 && (6.3)
\end{aligned}
$$

The inequality (6.3) holds for all $(\alpha, \beta) \in I$. Now, we want to show that D belongs to σ, which means that we need to find an expression of D of the form $A_0 S + \sum_{i=1}^k A_i E_i + \sum_{i=1}^k B_i G_i + \sum_{i=1}^k C_i F_i$ for some non-negative real values A_i, B_i, C_i. But recall that $F_i \sim F_1 + G_1 + E_1 - G_i - E_i$ for all i. Thus, we can rewrite the wanted expression of D as

$$
A_0 S + \left(A_1 + \sum_{i=2}^k C_i \right) E_1 + \sum_{i=2}^k (A_i - C_i) E_i + \left(B_1 + \sum_{i=2}^k C_i \right) G_1 + \sum_{i=2}^k (B_i - C_i) G_i + \left(\sum_{i=1}^k C_i \right) F_1.
$$

Hence, since $NS(X)$ is a free group, to show that D belongs to σ it is enough to find non-negative real numbers A_i, B_i, C_i such that

$$
\begin{aligned}
a_0 &= A_0 && (6.4)\\
b_0 &= C_1 + \sum_{i=2}^k C_i && (6.5)\\
a_1 &= A_1 + \sum_{i=2}^k C_i && (6.6)\\
b_1 &= B_1 + \sum_{i=2}^k C_i && (6.7)\\
a_i &= A_i - C_i \quad \text{for all} \quad 2 \leq i \leq k && (6.8)\\
b_i &= B_i - C_i \quad \text{for all} \quad 2 \leq i \leq k && (6.9)
\end{aligned}
$$

Clearly, by inequality (6.1), the equation (6.4) holds. Notice, from equation (6.3), that a_1 and b_1 are non-negative (by applying (6.3) with $\alpha = (1, 0 \cdots, 0), \beta = \vec{0}$, and $\alpha = \vec{0}$, $\beta = (1, 0 \cdots, 0)$ respectively). Thus, the previous system of equations can be reduced to find non-negative real numbers C_i that satisfy all the following inequalities:

$$b_0 - \sum_{i=2}^{k} C_i \geq 0 \tag{6.10}$$

$$a_1 - \sum_{i=2}^{k} C_i \geq 0 \tag{6.11}$$

$$b_1 - \sum_{i=2}^{k} C_i \geq 0 \tag{6.12}$$

$$a_i + C_i \geq 0 \text{ for all } 2 \leq i \leq k \tag{6.13}$$

$$b_i + C_i \geq 0 \text{ for all } 2 \leq i \leq k \tag{6.14}$$

Now, for each $i \in \{2, 3, \cdots, k\}$, let $C_i = -\min\{a_i, b_i, 0\}$. Notice that these values satisfy inequalities (6.13) and (6.14). Also, notice that we can write $\min\{a_i, b_i, 0\}$ as $\min\{a_i, b_i, 0\} = n_i a_i + m_i b_i$, for some $n_i, m_i \in 0, 1$ and $n_i + m_i \leq 1$. So, in particular inequality (6.10) follows from (6.3) with $\alpha = (0, n_2, \cdots, n_k)$ and $\beta = (0, m_2, \cdots, m_k)$. Similarly, inequality (6.11) follows from (6.3) with $\alpha = (1, n_2, \cdots, n_k)$ and $\beta = (0, m_1, \cdots, m_k)$, and inequality (6.12) follows with $\alpha = (0, n_2, \cdots, n_k)$ and $\beta = (1, m_1, \cdots, m_k)$. Thus, $D \in \sigma$, $\tau^\vee \subseteq \sigma$ and the cones τ, and σ are dual to one another.

Recall that F is base point free, corresponding to the map $\pi : X \longrightarrow \mathbb{P}^1$ (and it contracts all E_i, F_i, G_i).

For any $(\alpha, \beta) \in I$, define $r(\alpha, \beta) := |\{i : \alpha_i = 1\} \cup \{j : \beta_j = 1\}|$. Since $k \leq n$, $D_{\alpha,\beta}$ is base point free, corresponding to the morphism $\phi_{\alpha,\beta} : X \longrightarrow \mathbb{P}^{n-r(\alpha,\beta)+1}$ which contracts S, E_i for all i such that $\alpha_i = 0$, G_j for all j such that $\beta_j = 0$, and F_k for all k such that $\alpha_k = 1$ or $\beta_k = 1$.

Thus, since all the generators of $\sigma^\vee$ are base point free, they are nef, and so, the claim follows. $\square$

To finish the proof, we choose an arbitrary point $x \in X(k)$. We know that the only curves contracted by $D_{(\alpha,\beta)}$ are S and some of E_i's, G_i's, and F_i's. We see that if x does not belong to any of the reducible fibres F, let $C = \pi^*(\pi(x))$, then $C \sim F_i + E_i + G_i$, $C \cdot D_{(\alpha,\beta)} = 1$, and $C \cdot F = 0$. So by Proposition 2.15, C is a COBA for x with respect to any generator of the nef cone of X. Hence, for any $D \in \mathrm{Nef}(X)$, C is a COBA for x with respect to D and $\alpha_x(D) = D \cdot C$.

On the other hand, suppose x belongs to the reducible fibre, i.e. x belongs to some exceptional curve E_i, F_i, G_i or S. Notice that since $D_{(\alpha,\beta)}$ has intersection 0 or 1 with each of the generators of the effective cone, if x does not belong to a curve contracted by $D_{(\alpha,\beta)}$, but it belongs to some exceptional curve C which has intersection 1 with $D_{(\alpha,\beta)}$, we see that C is a COBA for x with respect to $D_{(\alpha,\beta)}$.

We let $T = \{S, F, E_i, F_i, G_i\}$ for all $i \in \{1, 2, \cdots, k\}$, and $D \in \mathrm{Nef}(X)$. Then $\alpha_x(D) = \min_{\{R \in T\colon\, x \in R\}}\{D \cdot R\}$, and whichever curve in T that contains x and has minimal intersection with D is a COBA for x with respect to D. $\square$

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APPENDICES

Appendix A

The case of $\mathbb{P}^2$ blown up at two points

Let $X = Bl_2\mathbb{P}^2$ be the blow up of $\mathbb{P}^2$ at two points. Without loss of generality, we can choose the blow up at the points $p_1 = [0, 0, 1]$, $p_2 = [0, 1, 0]$. Explicitly,

$$X = \{([x : y : z], [u : v], [s, t]) \in \mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1 : \ xv = yu, xt = zs\} \subseteq \mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1.$$

In this case there are three projections, $\pi : X \to \mathbb{P}^2 : ([x : y : z], [u : v], [s, t]) \to [x : y : z]$, $\pi_1 : X \to \mathbb{P}^1 : ([x : y : z], [u : v], [s, t]) \to [u : v]$, and $\pi_2 : X \to \mathbb{P}^1 : ([x : y : z], [u : v], [s, t]) \to [s : t]$. As in section 4.1, let $L = \pi^*(H)$ where H is a line in $\mathbb{P}^2$, and let $E_1 = \pi^{-1}(p_1)$ and $E_2 = \pi^{-1}(p_2)$ be the exceptional divisors. Notice that if we restrict $\pi : X \setminus \{E_1, E_2\} \longrightarrow \mathbb{P}^2 \setminus \{p_1, p_2\}$, we get an isomorphism. Hence, as in section 4.1, $\mathrm{Pic}(X \setminus \{E_1, E_2\}) \cong \mathrm{Pic}(\mathbb{P}^2 \setminus \{p_1, p_2\}) \cong \mathrm{Pic}(\mathbb{P}^2) \cong \mathbb{Z}L$. Now, using the exact sequences:

$$0 \longrightarrow \mathbb{Z}E_1 \longrightarrow \mathrm{Pic}(X) \longrightarrow \mathrm{Pic}(X \setminus \{E_1\}) \longrightarrow 0,$$

$$0 \longrightarrow \mathbb{Z}E_2 \longrightarrow \mathrm{Pic}(X \setminus \{E_1\}) \longrightarrow \mathrm{Pic}(X \setminus \{E_1, E_2\}) \longrightarrow 0,$$

we see that $\mathrm{Pic}(X) \cong \mathbb{Z}L \oplus \mathbb{Z}E_1 \oplus \mathbb{Z}E_2$. Also, by a similar argument as before $\pi_1^*(\mathcal{O}(1)) = L - E_1$ and $\pi_2^*(\mathcal{O}(1)) = L - E_2$, and the intersection theory on X is

$\cdot$	L	E_1	E_2
L	1	0	0
E_1	0	-1	0
E_2	0	0	-1

We see that the three generators are numerically independent, so $\mathrm{NS}(X) \otimes \mathbb{R} = \mathbb{R}L \oplus \mathbb{R}E_1 \oplus \mathbb{R}E_2$.

Now, let L' be the line joining the points p_1 and p_2. Consider the inverse image of the point $[0:1] \in \mathbb{P}^1$ under the projections π_1 and π_2.

$$\begin{aligned}
\pi_1^{-1}([0:1]) &= E_2 \cup \{([0:y:z],[0:1],[0:1]) : \ z \neq 0\} \\
\pi_2^{-1}([0:1]) &= E_1 \cup \{([0:y:z],[0:1],[0:1]) : \ y \neq 0\}
\end{aligned}$$

We let $S := \{([0:y:z],[0:1],[0:1]) : \ [y:z] \in \mathbb{P}^1\}$ be the strict transform of L'. Notice that $\pi(S) = L'$.

Claim A.1. *$L = S + E_1 + E_2$.*

Proof. Notice that S is a line in X, and $S \cdot L = S \cdot E_1 = S \cdot E_2 = 1$. Also, as a divisor $S \in \mathrm{Pic}(X)$, $S = a_0 L + a_1 E_1 + a_2 E_2$ for some $a_0, a_1, a_2 \in \mathbb{Z}$. Then, using the intersection pairing with S, we see that $a_0 = 1$ and $a_1 = a_2 = -1$, so the claim holds. $\square$

Now, let $\sigma = \mathrm{Cone}(S, E_1, E_2)$. Using the software polymake, we see that its dual is $\sigma^\vee = \mathrm{cone}(L, L - E_1, L - E_2)$. Notice that all the generators of $\sigma^\vee$ are base point free, so they are nef. This tells us that $\sigma = \mathrm{Eff}(X)$ and $\sigma^\vee = \mathrm{Nef}(X)$. Now, we find a COBA for each of the generators of $\mathrm{Nef}(X)$. Consider these cases:

Case 1: $x \in X(k) \setminus (E_1 \cup E_2)$. In this case, a COBA for x with respect to L is any line through x. Let C_i be the irreducible component of $\pi_i^{-1}(\pi_i(x))$ for $i = 1, 2$ that contains x. Since C_1 and C_2 are contained in the fibres of π_1 and π_2 respectively, we see that C_i is a COBA for $L - E_i$, $i = 1, 2$. Notice that if $x \in S$ then $C_1 = C_2 = S$. To find α with respect to each generator, we see that $C_i \cdot L = 1$ and $C_i \cdot (L - E_i) = 0$ for $i = 1, 2$ (regardless of $x \in S$ or $x \notin S$). We conclude

Divisor	COBA	α_x
L	C_1 or C_2	1
$L - E_1$	C_1	0
$L - E_2$	C_2	0

Now we study how to divide the nef cone into subcones where C_1 or C_2 are COBA. First, notice that if $x \in S$, then S is a COBA for all the generators of the nef cone, and by Theorem 2.14, S is a COBA for all divisors in $D \in \mathrm{Nef}(X)$. Also, if $D = a_0L + a_1(L-E_1) + a_2(L-E_2)$, then $\alpha_x(D) = D \cdot S = a_0$.

On the other hand, if $x \notin S$, let $D = a_0L + a_1(L - E_1) + a_2(L - E_2) \in \mathrm{Nef}(X)$ such that $C_1 \sim L - E_1$ is a COBA for D. Then $D \cdot C_1 \leq D \cdot C_2$, i.e. $a_0 + a_2 \leq a_0 + a_1$. We see that we can divide the nef cone as $\mathrm{Nef}(X) = \Gamma_1 \cup \Gamma_2$ where $\Gamma_1 = \langle L, L - E_1, 2L - E_1 - E_2\rangle$ and $\Gamma_2 \cup \langle L, L - E_2, 2L - E_1 - E_2\rangle$. By a similar argument as in Claim 4.3, we can verify that C_1 and C_2 are both COBA with respect to $2L - E_1 - E_2$. Hence C_i is a COBA for any divisor in Γ_i, $i = 1, 2$ and we conclude that:

Divisor	COBA	α_x
$a_0L + a_1(L - E_1) + a_2(2L - E_1 - E_2)$	C_1	$a_0 + a_2$
$a_0L + a_1(L - E_2) + a_2(2L - E_1 - E_2)$	C_2	$a_0 + a_2$

Case 2: $x \in E_1(k) \setminus S$ or $x \in E_2(k) \setminus S$. Assume, without loss of generality, that $x \in E_1$, then a COBA for x with respect to L is E_1. To approximate x with respect to $L - E_1$, notice that as before we can choose $C_1 = \pi_1^{-1}(\pi_1(x))$ as a COBA. To approximate x with respect to $L - E_2$, notice that we can pick the fibre $C_2 = \pi_1^{-1}(\pi_1(x)) = \pi_1^{-1}([0 : 1])$, but recall that $x \in C_2$, so in this case $C_2 = E_1$, and our table is:

Divisor	COBA	α_x
L	E_1	0
$L - E_1$	$C_1 \sim L - E_1$	0
$L - E_2$	E_1	0

Let $D = a_0L + a_1(L - E_1) + a_2(L - E_2) \in \mathrm{Nef}(X)$. Assume that E_1 is a COBA for D, then $D \cdot E_1 \leq D \cdot (L - E_1)$. In this case we get that $D \in \Gamma_1 = \langle L, L - E_2, 2L - E_1, 2L - E_1 - E_2\rangle$. We know that E_1 is a COBA for $L, L - E_2$. By a similar argument as in claim 4.3, we can see that E_1 and $L - E_1$ are both COBA for x with respect to $2L - E_1$ and $2L - E_1 - E_2$. On the other hand, if $L - E_1$ is a COBA for D, we see that $D \in \Gamma_2 = \langle L - E_1, 2L - E_1, 2L - E_1 - E_2\rangle$. Hence, we conclude with the table:

Divisor	COBA	α_x
$a_0L + a_1(L - E_2) + a_2(2L - E_1) + a_3(2L - E_1 - E_2)$	E_1	$a_2 + a_3$
$a_0(L - E_1) + a_1(2L - E_1) + a_2(2L - E_1 - E_2)$	$C_1 \sim L - E_1$	$a_1 + a_2$

Case 3: $x \in S(k) \cap E_1(k)$ or $x \in S(k) \cap E_2(k)$. Without loss of generality, assume that $x \in S(k) \cap E_1(k)$. We conclude that:

Divisor	COBA	α_x
L	E_1	0
$L - E_1$	S	0
$L - E_2$	S or E_1	0

Let $D = a_0L + a_1(L - E_1) + a_2(L - E_2) \in \mathrm{Nef}(X)$. Assume that E_1 is a COBA for D, then $D \cdot E_1 \leq D \cdot S$, i.e. $a_1 \leq a_0$. Similarly, if we assume that S is a COBA for D we get $a_1 \geq a_0$. Hence, we divide the nef cone into two regions, $\mathrm{Nef}(X) = \Gamma_1 \cup \Gamma_2$, where $\Gamma_1 = \langle L, 2L - E_1, L - E_2 \rangle$ and $\Gamma_2 = \langle 2L - E_1, L - E_1, L - E_2 \rangle$.

Claim A.2. *S and E_1 are both COBA for $2L - E_1$.*

Proof. This follows from Proposition 4.4. □

We see that E_1 is a COBA for any divisor in the subcone Γ_1 and S is a COBA for any divisor in the subcone Γ_2. We conclude:

Divisor	COBA	α_x
$a_0L + a_1(2L - E_1) + a_2(L - E_2)$	E_1	a_1
$a_0(L - E_1) + a_1(2L - E_1) + a_2(L - E_2)$	S	a_1

Appendix B

Explicit calculation of α

Although the complete proof of the conjecture in Theorem 5.16 was presented in the proof of that theorem, here there is an explicit calculation of the approximation constants that follow from the general case. But here we do an exhaustive study of the subcones of the nef cone.

Case 1. $x \in S(k) \setminus E_1$.

Let C be the irreducible component of the fibre of π through x. We know that $C \sim F$. Recall S is contracted by the morphisms corresponding to D_1, D_2, and D_3. Also, $S \cdot D_4 = 0$, and since D_4 is nef, $\alpha_x(D_4) \geq 0$, so S is a COBA for D_4. We conclude:

Divisor	COBA	α_x
F	F	0
D_1	S	0
D_2	S	0
D_3	S	0
D_4	S	0

Let $D = a_0F + a_1D_1 + a_2D_2 + a_3D_3 + a_4D_4 \in \text{Nef}(X)$. Assume that F is COBA with respect to D, then $a_1 + a_2 + 2a_3 + 2a_4 \leq a_0$, so we obtain the cone $\Gamma_1 = \langle F, F + D_1, F + D_2, F + 2D_3, F + 2D_4\rangle$. Similarly, if we assume that S is a COBA for D, we obtain the cone $\Gamma_2 = \langle D_1, D_2, D_3, D_4, F + D_1, F + D_2, 2F + D_3, 2F + D_4\rangle$. $\text{Nef}(X) = \Gamma_1 \cup \Gamma_2$.

We verify that F is a COBA for any divisor in Γ_1 and that S is COBA for any divisor in Γ_2.

S and F are both COBA with respect to $F + D_1$ and $F + D_2$: If C is an irreducible curve such that $C \cdot (F + D_1) = 0$, then $C \cdot F = C \cdot D_1 = 0$. So $C \in \{E_2, E_3, E_4\}$, but none of these curves contains x. Also $S \cdot (F + D_1) = F \cdot (F + D_1) = 1$, then by Proposition 2.15, S and F are COBA for x with respect to $F + D_1$. Similarly, the only irreducible curves that have intersection zero with $F + D_2$ are E_1, E_3, and E_4. But x does not belong to any E_i. Thus, since $S \cdot (F + D_2) = F \cdot (F + D_2) = 1$, by Proposition 2.15, S and F are also COBA for x with respect to $F + D_2$.

Claim B.1. *Both S and F are both COBA with respect to $2F + D_3$.*

Proof. Notice that since $x \in S \setminus E_1$, x does not belong to any curve C that is contracted by both F and D_3. Thus, since $S \cdot F = 1$, by Proposition 4.4, S is COBA for x with respect to $F + D_3$, which implies that S is also COBA for x with respect to $2F + D_3$ (again by Proposition 4.4).

The previous argument shows that $\alpha_x(2F + D) = \alpha_{x,S}((2F + D_3)|_S) = S \cdot (2F + D_3) = 2$. But also notice that the fibre $C = \pi^*(\pi(x)) \sim F$ is a smooth rational curve that contains x and $\alpha_{x,C}((2F + D_3)|_C) = C \cdot (2F + D_3) = 2 = \alpha_x(2F + D_3)$. Thus a sequence in C converging to x also attains the minimal value $\alpha_x(2F + D_3)$, so $C \sim F$ is also a COBA for x with respect to $2F + D_3$. □

A similar argument as in the previous claim shows that F and S are both COBA for x with respect to $2F + D_4$. Then, we can write the nef cone of X as $\mathrm{Nef}(X) = \Gamma_1 \cup \Gamma_2$, where

$$\Gamma_1 := \langle F, F + D_1, F + D_2, 2F + D_3, 2F + D_4 \rangle,$$

$$\Gamma_2 := \langle F + D_1, F + D_2, 2F + D_3, 2F + D_4, D_1, D_2, D_3, D_4 \rangle,$$

and we conclude the following table:

Divisor	COBA	α_x
$a_0F + a_1(D_1 + F) + a_2(F + D_2) +$ $a_3(2F + D_3) + a_4(2F + D_4)$	F	$a_1 + a_2 + 2a_3 + 2a_4$
$\sum_{i=1}^4 a_iD_i + b_1(F + D_1) + b_2(F + D_2) +$ $b_3(2F + D_3) + b_4(2F + D_4)$	S	$b_1 + b_2 + 2b_3 + 2b_4$

Case 2. x lies exactly on one E_i but not in S.

We know that E_i is COBA for x with respect to D_j, $j \neq i$ and F, since E_i is contracted by all of these and $x \in E_i$. The irreducible curves contracted by D_i are precisely E_j, $j \neq i$ and S, but x doesn't belong to any of these, thus by Proposition 2.15, E_i is COBA for x with respect to D_i.

Divisor	COBA	α_x
F	E_i	0
D_i	E_i	1
D_j, $j \neq i$	E_i	0

Since E_i is COBA for all the generators of the nef cone, by Theorem 2.14. E_i is COBA for x with respect to any divisor in Nef(X). Let $D \in \text{Nef}(X)$, say $D = a_0F + a_iD_i + \sum_{j\neq i} a_jD_j$, then $\alpha_x(D) = D \cdot E_i = a_i$.

Case 3. $x = E_1 \cap S$.

Divisor	COBA	α_x
F	E_1	0
D_1	S	0
D_2	S or E_1	0
D_3	S or E_1	0
D_4	S or E_1	0

We write the nef cone as $\text{Nef}(X) = \Gamma_S \cup \Gamma_{E_1}$, where

$$\Gamma_S = \langle D_1, D_2, D_3, D_4, F + D_1\rangle$$

$$\Gamma_{E_1} = \langle F, D_2, D_3, D_4, F + D_1\rangle$$

We saw in case 2 that S is a COBA for x with respect to any divisor in Γ_S. By Proposition 2.15, E_1 is also a COBA with respect to $(F + D_1)$, so E_1 is COBA for x with respect to any divisor in Γ_{E_1}. So we conclude

Divisor	COBA	α_x
$a_0(F + D_1) + \sum_{i=1}^4 a_iD_i$	S	a_0
$a_0F + a_1(F + D_1) + \sum_{i=2}^4 a_iD_i$	E_1	a_1

Case 4. $x = E_i \cap E_j$, where $(i, j) \in \{(1, 3), (2, 3), (3, 4)\}$.

Since $x = E_i \cap E_j$, we know that for any generator of the nef cone, we have E_i and E_j as candidates for COBA. We write the results as follows:

Divisor	COBA	α_x
F	E_i or E_j	0
D_i	E_j	0
D_j	E_i	0
D_k, $k \notin \{i, j\}$	E_i or E_j	0

We can write the nef cone as $\mathrm{Nef}(X) = \Gamma_i \cup \Gamma_j$, where

$$\Gamma_i = \langle F, D_k, D_j, D_i + D_j \rangle$$

$$\Gamma_j = \langle F, D_k, D_i, D_i + D_j \rangle$$

By Proposition 2.15, we know that E_i and E_j are both COBA for x with respect to $D_i + D_j$. So, by Theorem 2.14, E_i is COBA with respect to any divisor in Γ_i and E_j is COBA with respect to any divisor in Γ_j. We conclude the following table:

Divisor	COBA	α_x
$a_0F + a_jD_j + \sum_{k \neq i,j} a_kD_k + a_i(D_i + D_j)$	E_i	$a_j + a_i$
$a_0F + a_iD_i + \sum_{k \neq i,j} a_kD_k + a_j(D_i + D_j)$	E_j	$a_i + a_j$