Some Applications of Hyperbolic Geometry in String Perturbation Theory

by

Seyed Faroogh Moosavian

A thesis
presented to the University of Waterloo
in fulfillment of the
thesis requirement for the degree of
Doctor of Philosophy
in
Physics

Waterloo, Ontario, Canada, 2019

Examining Committee Membership

The following served on the Examining Committee for this thesis. The decision of the Examining Committee is by majority vote.

External Examiner:	Duiliu-Emanuel Diaconescu, Rutgers University
Supervisor(s):	Davide Gaiotto, Perimeter Institute Jaume Gomis, Perimeter Institute
Internal Member:	Niayesh Afshordi, University of Waterloo
Internal-External Member:	Ruxandra Moraru, University of Waterloo
Other Member(s):	Kevin Costello, Perimeter Institute

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Abstract

In this thesis, we explore some applications of recent developments in the hyperbolic geometry of Riemann surfaces and moduli spaces thereof in string theory [1, 2, 3].

First we show how a proper decomposition of the moduli space of hyperbolic surfaces can be achieved using the hyperbolic parameters. The decomposition is appropriate to define off-shell amplitudes in bosonic-string, heterotic-string and type-II superstring theories. Since the off-shell amplitudes in bosonic-string theory are dependent on the choice of local coordinates around the punctures, we associate local coordinates around the punctures in various regions of the moduli space. The next ingredient to define the off-shell amplitudes is to provide a method to integrate the off-shell string measure over the moduli space of hyperbolic surfaces. We next show how the integrals appearing in the definition of bosonic-string, heterotic-string and type-II superstring amplitudes can be computed by lifting them to appropriate covering spaces of the moduli space. In heterotic-string and type-II superstring theories, we also need to provide a proper distribution of picture-changing operators. We provide such a distribution. Finally, we illustrate the whole construction in few examples. We then describe the construction of a consistent string field theory using the tools from hyperbolic geometry.

Acknowledgements

There were many people who made this thesis possible. First and foremost, I would like to thank my supervisor, Davide Gaiotto. Working with Davide was very challenging, mainly due to his perfectionism, but at the same time very rewarding for me. He taught me how to avoid asking random questions, which I was used to in the beginning of my Ph.D. studies, and instead try to understand what is worth pursuing and then boldly address those questions. I especially appreciate his attitude for giving me the freedom for exploring projects I personally found interesting. I am also grateful that Davide gave me the opportunity to work on some of his deep and beautiful ideas on the interplay between vertex operator algebras and supersymmetric gauge theory.

Secondly, I would like to thank Roji Pius with whom I explored the project consisting the material of this thesis. Roji played the role of my second supervisor in the first two years of my Ph.D. studies. I appreciate his patience and also thank him for innumerable discussions and explanations.

A special thanks goes to Kevin Costello for giving me the opportunity to work on some of his beautiful ideas on holography, integrable systems, and string field theory.

I am also grateful to the other members of my graduate committee, my co-supervisor Jaume Gomis, and Niayesh Afshordi whose questions were always challenging my basic knowledge of physics.

A very special acknowledgment goes to Ashoke Sen who patiently answered my questions about various aspects of string theory.

I am very grateful to Laurent Freidel who patiently and enthusiastically explained many subtle issues of The Problem of Quantum Gravity to me, and for all he taught me about this fascinating subject.

I also would like to thank my collaborators William Donnelly, Nafiz Ishtiaque, Heidar Moradi, Miroslav Rapčák, Antony Speranza, Apoorv Tiwari, and Yehao Zhou who taught me a lot about various aspects of gravity, holography, vertex algebra, condensed-matter physics, and mathematics.

So many other friends have been supportive with their encouraging words. I am deeply indebted to Farbod Kamiab, Mansour Karami, Mehdi Saravani, Anton van Niekerk and Yasaman Yazdi.

Finally, I must mention the Perimeter Institute. The years I spent there are among the best in my whole life. I am truly grateful to the Perimeter Institute for giving me the opportunity to being part of its amazing scientific community and also for providing such a wonderful work environment.

Dedication

To my Mother

Table of Contents

List of Tables

List of Figures

Chapter 1

Introduction

Quantum field theory is the best-understood framework for describing microscopic phenomena. Not only it provides a conceptual and theoretical framework for describing the Nature, the main power of quantum field theory, as a framework, is that it can be used to explicitly compute quantities that can be measured in real-world experiments. This computational strength is formulated in terms of a set of relatively simple rules. For instance, we compute the quantities of interest, mainly scattering amplitudes, as a perturbative expansion in the coupling parameters of the theory by evaluating Feynman diagrams and using Feynman rules. These rules can be derived from the Lagrangian of the theory. Using Feynman rules associated to a Lagrangian, one can express any amplitudes in a quantum field theory with arbitrary external states and loops as explicit integrals. Depending on the complexity of resulting expressions, simple or sophisticated methods are devised to compute them. Therefore, we can consider Feynman prescription as a calculable formulation of the perturbative quantum field theory. Any framework which claims to be more fundamental than quantum field theory first and foremost must be able to reproduce results of quantum field theory in some limit. Furthermore, it must be able to provide theoretical framework for explanation of theoretical questions that quantum field theory in its current formulations or some possible unification of it with general relativity are not able to resolve. Finally, this more fundamental framework must provide a way to compute physically-relevant quantities very explicitly just like in quantum field theories.

Superstring theory is an approach to the problem of quantization of gravity and its unification with the rest of fundamental forces of nature. In this approach, one replaces local interactions of quantum field theory with interactions of strings. Most of the insights about the structure of string theory comes from Polyakov's formulation of string perturbation theory [4, 5]. The absence of gauge and gravitational anomalies together with stability of the vacuum leaves five consistent theories for superstrings in $10d$

1. TYPE-I THEORY: This is a theory with $\mathcal{N} = (1,1)_2$ supersymmetry on the worldsheet, where the subscript 2 denotes the two dimensions of the worldsheet. This theory contains both open- and closed-strings with unoriented worldsheet [6];

2. TYPE-II THEORY: This theory has two variants: the non-chiral type-IIA theory, or the chiral type-IIB theory both with $\mathcal{N} = (1,1)_2$ supersymmetry on the worldsheet. The theory contains only closed strings with oriented worldsheet and it does not require an open-string sector for its consistency [7]. However, one can introduce D-branes into the story which in turn introduces open-string sector in the theory. The open- and closed-strings interact via open-closed interactions in the bulk [8];

3. HETEROTIC THEORY: This theory has $\mathcal{N} = (1,0)_2$ supersymmetry on the worldsheet and the only possible space-time gauge groups are $\mathtt{Spin}(32)/\mathbb{Z}_2$ or $\mathtt{E}_8 \times \mathtt{E}_8$ [9, 10, 11] or broken $\mathtt{SO}(16) \times \mathtt{SO}(16)$ [12]. It is believed that heterotic string can only be closed. However, $\mathtt{Spin}(32)/\mathbb{Z}_2$ heterotic string can have endpoints and admits a consistent boundary condition [13]. We do not consider this exotic case in this thesis;

In the conventional formulation of string theory, scattering amplitudes are obtained by evaluating the contribution from string diagrams, which are the string-theory analogue of Feynman diagrams. Unlike quantum field theories, the rules for evaluating string diagrams are not derived from a Lagrangian and it is given by Polyakov prescription [4, 5]. By exploring the conformal structure of string theory, powerful covariant methods to compute amplitudes in perturbative string theory were introduced in [14, 15]. However, this picture, which give rise to the so-called on-shell amplitudes, is not satisfying due to various type of divergences. There has been a considerable progress in recent years in resolving the issues associated to divergences in string theory [16, 17, 18, 19, 20, 21, 22, 23, 24, 25]. We can now safely say that perturbative string theory is a very well-developed formalism that provides a well-defined formal procedure for computing amplitudes, at least in the case of closed bosonic-string and closed superstring theories. We should emphasize that these developments are developed based on two complementary pictures of string dynamics, i.e. 1) formulation of off-shell amplitudes which provides the worldsheet perspective, and 2) string field theory which provides the space-time perspective. Also, most of these developments have been done in the so-called picture-changing formulation of string perturbation theory and string field theory [15], as we will explain below. Although two-loop computations have only been done in the so-called supergeometry formulation [26, 27, 28, 29, 30, 31, 32], the issues regarding divergences is barely studied especially the divergences associated to massive states [16].

Despite this firm theoretical ground, the practical computations in perturbative string theory is not still feasible. Although the path integral definition of scattering amplitudes in string theory reduces to a finite-dimensional integral, which is a huge simplification, the complicated structure of the resulting finite-dimensional space, the moduli space of Riemann surfaces, makes the practical computation almost hopeless. If we wish to ever get any testable prediction from string theory, one should come up with an efficient way to explicitly compute these integrals. In principle, there are two ways to compute the stringy effects

1. **Holography:** holography is a relation between a quantum field theory in d dimensions and a gravity theory in $d+1$ dimensions. The most celebrated realization of holography is The AdS/CFT Correspondence where the relevant quantum field theory is the $\mathcal{N}=4$ supersymmetric version of Yang-Mills theory (SYM) with gauge group $\mathtt{SU}(N)$ and the relevant (quantum) gravity theory is the type-IIB superstring theory on $\mathtt{AdS}^5 \times \mathbb{S}^5$ [33]. According to the assertion of The Correspondence, these two theories are considered to be dynamically equivalent which means that one can extract any information of one theory from the other one, i.e. they describe the same physics using completely different degrees of freedom. Free parameters of SYM theory g_{YM}, the Yang-Mills coupling, and N are mapped to free parameters in the string theory side, i.e. g_s, the string coupling, and L/l_s, where L is the radius of curvature of space-time and l_s is the string length, via

$$g_{YM}^2 = 2\pi g_s, \qquad 2\lambda \equiv 2N g_{YM}^2 = \left(\frac{L}{l_s}\right)^4. \tag{1.0.1}$$

   λ is called the 't Hooft coupling. Using these identification, one can consider various forms of AdS/CFT correspondence. The result is summarized in table 1.1. Therefore, The Mild Form of the AdS/CFT Correspondence can be used to extract perturbative stringy effects by studying $\mathcal{N}=4$ SYM theory in large-N limit with fixed 't Hooft coupling. One can thus explore quantum-gravitational effects by developing tools to study a quantum field theory.

   Although this method is very effective and practical, especially in large-N and large-λ limit, the other limits are not very well-explored [34, 35, 36]. The Correspondence also depends on some miraculous cancellations in the decoupling limit that led to the seminal paper of Maldacena [33]. On the other hand, establishing the duality completely provides a complete formulation of a quantum gravity theory in asymptotically-AdS space-times which is not The Universe we live in. A more systematic method is thus required to explore the stringy effects on physical phenomena. It seems reasonable to expect that if string theory is more fundamental than quantum field theories in describing physical phenomena, it should be described in its own terms not in terms of some supposedly-equivalent theory.

AdS/CFT Correspondence	$\mathcal{N}=4$ SYM theory side	type-IIB theory side
Weak Form	$N \longrightarrow \infty$, large λ	$g_s \longrightarrow 0, \quad \left(\frac{l_s}{L}\right)^2 \longrightarrow 0.$ (classical supergravity)
Mild Form	$N \longrightarrow \infty$, arbitrary fixed λ	$g_s \longrightarrow 0, \quad \left(\frac{l_s}{L}\right)^2 \neq 0.$ (perturbative string theory)
Strong Form	arbitrary N and λ	$g_s \neq 0, \quad \left(\frac{l_s}{L}\right)^2 \neq 0.$ (nonperturbative string theory)

Table 1.1: The various forms of AdS/CFT correspondence and the relevant limits in quantum field theory and string theory sides. The Mild Form of The AdS/CFT Correspondence can be used to extract perturbative stringy information by studying $\mathcal{N}=4$ SYM theory.

2. **Direct Approach:** Primary objects of study in string theory are scattering amplitudes. In this approach, one is studying scattering amplitudes directly, i.e. by trying to perform integrals over the moduli space of Riemann surface explicitly. The genus-$\mathfrak{g}$ contribution to an scattering process in string theory have the following schematic form

$$\mathcal{A}_{\mathfrak{g}} = \int_{\mathcal{M}_{\mathfrak{g}}} \mathbf{\Omega}(\mathbf{m}), \tag{1.0.2}$$

where $\mathbf{\Omega}$ is a form on $\mathcal{M}_{\mathfrak{g}}$, the moduli space of genus-$\mathfrak{g}$ Riemann surfaces, parametrized by a coordinate system $\mathbf{m}$. Regarding the fact that $\mathcal{M}_{\mathfrak{g}}$ does not have a simple geometry, once $\mathbf{\Omega}$ is constructed explicitly in term of $\mathbf{m}$, one needs to come-up with a procedure to compute these integrals explicitly.

In this thesis, we pursue the second approach. *It must be emphasized here that the method described in this thesis leads to practical computations if and only if the form* $\mathbf{\Omega}$ *in* (1.0.2) *can be computed explicitly in terms of a specific set of coordinates* $\mathbf{m}$ *on the moduli space, or more precisely, the so-called Teichmüller space of Riemann surfaces.* We will introduce these coordinates below. A method for computing $\mathbf{\Omega}$ on a Riemann surface is the gluing of pairs of pants. However, this method is not very effective for generic amplitudes. There should exist a systematic method to compute $\mathbf{\Omega}$ on any genus-$\mathfrak{g}$ surface in terms of these coordinates, a method similar to the one used in [37] to compute $\mathbf{\Omega}$ in terms of the period matrix of the surface.

1.1 Off-Shell Amplitudes in String Theory

In the presence of mass renormalization and dynamical shift of the vacuum, the conventional definition of scattering amplitudes in string theory breaks down [38, 39, 40, 41]. To deal with these issues, one needs to suitably generalize the definition of amplitudes in string theory. It turns out that a suitable generalization of the Polyakov prescription is provided by the so-called off-shell amplitudes [20]. Off-shell amplitudes in string theory are defined by relaxing the tree-level on-shell condition on external states. A state satisfying tree-level on-shell condition is represented by a vertex operator having conformal dimension $(0, 0)$. As such, an off-shell state is represented by a vertex operator having arbitrary conformal dimension. As a result, off-shell amplitudes will depend on the chosen set of local coordinates around marked points on Riemann surfaces. On the other hand, off-shell superstring amplitudes are dependent on the choice of distribution of picture-changing operators. An off-shell amplitude is schematically looks like

$$\mathcal{A}_{\mathfrak{g};\mathfrak{n}} = \int_{\mathcal{S}_{\mathfrak{g};\mathfrak{n}}} \mathbf{\Omega}_{\mathfrak{g},\mathfrak{n}}, \tag{1.1.1}$$

where $\mathcal{S}$ contains the information of genus-$\mathfrak{g}$ surface with $\mathfrak{n}$ marked points, the choice of local coordinates and/or the choice of picture-changing operators. These dependence are undesirable feature of the formalism since it implies that the result of computation is dependent on the chosen local coordinates and/or distribution of picture-changing operators. However, it turns out that none of the physical quantities extracted from off-shell amplitudes depend on these choices as long as they satisfy a special condition, namely **The Gluing-Compatibility Condition** [17, 18, 20]. A gluing-compatible set of local coordinates and/or a gluing-compatible choice of distribution of PCOs respects the gluing of Riemann surfaces. Therefore, if one is interested in taming infrared divergences in bosonic-string and superstring theories, it is essential to find an explicit choice of gluing-compatible set of local coordinates and/or distribution of picture-changing operators. We postpone a more detailed discussion of off-shell amplitudes to the main body of the thesis.

1.2 String Field Theory

The Polyakov prescription for computing scattering amplitudes in string theory is a formal definition. However, as we have mentioned, this prescription is very incomplete. On the other hand, many of formal properties of the string perturbation theory like unitarity, background independence etc seems to be obscure in this formulation. Therefore, it is necessary to find a machinery that such formal properties can be understood and also provides a more systematic framework for computing scattering amplitudes. String field theory is a quantum field theory that is constructed in such a way that its Feynman diagrams computes S-matrix elements. One advantage of this framework is that, unlike the ad hoc definition of string perturbation theory series, all the rich toolbox of quantum field theory is available within string field theory. Although "the physical content of the theory is buried under mountains of computationally-inaccessible data"[1], however regarding recent developments, it seems indispensable to use string field theory to provide a full definition of string perturbation theory. Some theoretical applications of string field theory are [43, 42]

- **formal properties of string perturbation theory:** The formal properties of string perturbation theory is difficult to establish using the worldsheet perspective. However, string field theory can be used to address these issues. For example unitarity [44], analyticity and crossing symmetry of amplitudes [45], and background independence [46, 47, 48] can be established using string field theory for closed-superstring field theories. It can also be shown that amplitudes satisfy cutting rules [49].

- **studying the non-perturbative regime:** Since string field theory is based on a Lagrangian, it has the potential to open the door towards the non-perturbative regime of the theory. This possibility has been explored in the case of open-string field theory [50, 51, 52, 53]. However, the case of closed-string field theory has not been explored much due to the complicated structure of corresponding Lagrangian [54].

- **low-energy dynamics of the theory:** string field theory can be used for the first-principle construction of the effective action describing the dynamics below certain mass scale [55].

[1]This sentence has been borrowed from [42].

- **treatment of divergences:** string field theory can be used to address problems associated to divergences like mass renormalization or the vacuum-shift issue. These issues have been settled using 1PI effective actions of closed-superstring field theories in an important recent paper [23].

- **string theory in RR backgrounds:** Since string field theory can be formulated in any background, it has been used recently to analyze string spectrum in the RR background flux [56].

String field theory is thus a very powerful theoretical ground. However, similar to the string perturbation theory, it is not very well-suited to compute amplitudes explicitly. The main reason again goes back to the appearance of moduli space of Riemann surfaces.

The complicated gauge structure of closed-string field theory demands the use of the most sophisticated available machinery for the quantization of gauge systems, i.e. the Batalian-Vilkovisky (BV) formalism [57, 58]. The perturbative solution of BV quantum master equation for the closed bosonic-string field theory and closed-superstring field theories have been constructed [59, 24]. The theory has only one independent parameter, the closed-string coupling. The interaction strengths of elementary interactions are expressed as integrals over the distinct two dimensional worldsheets describing elementary interactions of closed strings. The collection of worldsheets describing elementary interactions of closed strings are known as string vertices. The interaction term associated to $\mathcal{V}_{g,n}$, a genus-g string vertex with n marked points, takes the following form

$$\int_{\mathcal{V}_{g,n}} \mathbf{\Omega}, \tag{1.2.1}$$

where $\mathbf{\Omega}$ is a form on the moduli space. This term shows that we need to integrate over a region in the moduli space corresponding to the string vertex to be able to compute interaction terms of string field theory. The explicit evaluation of these integrals requires

1. a convenient choice of parametrization of the moduli space of Riemann surfaces. This parametrization must be accompanied by a condition that specifies regions of moduli spaces corresponding to string vertices;

2. an explicit procedure for constructing string measure in terms of the chosen coordinates of moduli space;

3. a consistent choice of local coordinates around marked points and/or a consistent choice of picture-changing operators on Riemann surfaces belonging to string vertices.

4. an explicit procedure for integrating string measures over the region corresponding to the string vertex.

A consistent set of string vertices provides a decomposition of the moduli space of Riemann surfaces. String field theory must be formulated in such a way that it gives a construction of the moduli space, i.e. a region corresponding to string vertex together with Feynman diagrams of the theory must provide a single cover of the moduli space. The main challenge in constructing string field theory is thus to find such a decomposition. String vertices that provide such a decomposition of the moduli space can be constructed using Riemann surfaces endowed with minimal-area metrics [59]. There exists an alternative construction of string vertices using Riemann surfaces endowed with the constant-curvature metric [2, 3]. It is also possible to construct vertices of effective actions of closed-superstring field theories using hyperbolic geometry [60]. This alternative approach is based on ideas presented in this thesis [1].

1.3 Moduli Space of Riemann Surfaces

Regarding what we have explained so far, we conclude that in order to obtain a practical prescription for evaluating amplitudes in string theory, whether using off-shell formulation of the theory or using string field theory, we need to deal with the moduli space of genus-g Riemann surfaces with n punctures and integration over it. g plays the role of loop-counting parameter and n is the number of external incoming and outgoing states. Unfortunately there is no simple description of the moduli space. This is mainly due to the following fact. Characterizing the space of Riemann surfaces requires varying the complex structure by infinitesimal deformation of the metric on the surface. However, infinitesimal deformations only lead us to the space $\mathcal{T}_{g,n}$, the Teichmüller space of Riemann surfaces, instead of the moduli space. The Teichmüller space is the space of inequivalent Riemann surfaces in the following sense [61]. Fix a Riemann surface $\mathcal{R}$. Consider a pair $(\mathcal{S}, \phi)$, where $\mathcal{S}$ is a surface and $\phi : \mathcal{R} \longrightarrow \mathcal{S}$ is an orientation-preserving diffeomorphism. Two such pairs $(\mathcal{S}_1, \phi_1)$ and $(\mathcal{S}_2, \phi_2)$ are equivalent if $\phi_2 \circ \phi_1^{-1} : \mathcal{S}_1 \longrightarrow \mathcal{S}_2$ is homotopy-equivalent to a biholomorphism between $\mathcal{S}_1$ and $\mathcal{S}_2$. It is easy to see that this relation is an equivalence relation. We denote the equivalence classes by $[\mathcal{S}, \phi]$. The set of all such equivalence classes is the Teichmüller space of $\mathcal{R}$ and it is denoted by $\mathcal{T}(\mathcal{R})$.

It turns out that we need to deal with more elaborate spaces in string theory, the moduli spaces. They can be obtained from the corresponding Teichmüller spaces. Let $\mathrm{Mod}(\mathcal{R})$ be the set of homotopy-equivalence class of orientation-preserving diffeomorphisms $f : \mathcal{R} \longrightarrow \mathcal{R}$. Such diffeomorphisms form a group, the so-called mapping-class group (MCG) of the surface. An element $[f] \in \mathrm{Mod}(\mathcal{R})$ acts on $\mathcal{T}(\mathcal{R})$ as follows

$$[f]([\mathcal{S}, \phi]) = [\mathcal{S}, \phi \circ f^{-1}], \qquad [\mathcal{S}, \phi] \in \mathcal{T}(\mathcal{R}). \tag{1.3.1}$$

The moduli space $\mathcal{M}(\mathcal{R})$ is defined by the quotient of the Teichmüller space by the action of mapping-class group

$$\mathcal{M}(\mathcal{R}) \equiv \frac{\mathcal{T}(\mathcal{R})}{\mathrm{Mod}(\mathcal{R})}. \tag{1.3.2}$$

In the following we use the notation $\mathcal{T}(\mathcal{R})$, $\mathrm{Mod}(\mathcal{R})$, and $\mathcal{M}(\mathcal{R})$ or, in the case we want to emphasize the topological type of the surface, $\mathcal{T}_{\mathfrak{g},\mathfrak{n}}$, $\mathrm{Mod}_{\mathfrak{g},\mathfrak{n}}$, and $\mathcal{M}_{\mathfrak{g},\mathfrak{n}}$ for the Teichmüller space, the mapping-class group, and the moduli space, respectively.

Geometry and topology of the moduli space is complicated because the action of mapping-class group on Teichmüller space is complicated, and as a result, finding explicit fundamental domains for the action of mapping-class group on the Teichmüller space is very challenging. This makes explicit integration over the moduli space very difficult. More concretely,

1. Mapping-class groups act properly-discontinuously on the corresponding Teichmüller space but the fundamental domain of the action is not known explicitly, i.e. the fundamental interval of coordinates which specify the fundamental domain of the action inside the Teichmüller space for some set of coordinates;

2. The moduli space is not a manifold. It is a connected orbifold [62]. The set of fixed points of the action of the mapping-class group is not dense and is finite;

3. The explicit mapping-class-group-invariant volume form on the moduli space is not known for some arbitrary coordinate system.

To the best of author's knowledge, the first explicit computation of an integral over the moduli space of Riemann surfaces is due to Wolpert [63]. He computed the Weil-Petersson (WP) volume of $\mathcal{M}_{1,1}$, the moduli space of once-punctured tori. The result is

$$\mathrm{Vol}_{\mathbf{WP}}(\mathcal{M}_{1,1}) = \int_{\mathcal{M}_{1,1}} 1 = \frac{\pi^2}{6}. \tag{1.3.3}$$

Another computation of this sort was done by Penner [64]. He described a method to compute the integral of a top-degree differential form over the moduli space of Riemann surfaces. He applied his method to compute the WP volume of $\mathcal{M}_{1,2}$, the moduli space of twice-punctured tori

$$\mathrm{Vol}_{\mathbf{WP}}(\mathcal{M}_{1,2}) = \int_{\mathcal{M}_{1,2}} 1 = \frac{\pi^4}{8}. \tag{1.3.4}$$

Zograf has obtained a recursive formula for WP volumes of the moduli space of punctured spheres [65]. Manin and Zograf obtained generating functions for WP volumes of the moduli space of punctured Riemann surfaces [66]. The systematic generalization of all these results, i.e. the explicit computation of WP volumes of $\mathcal{M}_{\mathfrak{g},\mathfrak{n}}(\mathbf{L})$, the moduli space of genus-$\mathfrak{g}$ surfaces with $\mathfrak{n}$ borders of fixed lengths $\mathbf{L} \equiv (L_1, \cdots, L_n)$, subject to the condition $2\mathfrak{g} + \mathfrak{n} \geq 3$, is due to Mirzakhani in her seminal work [67]. This outstanding progress is based on the idea that an integral over the moduli space of Riemann surfaces can be lifted to integrals over a covering space of the moduli space. These covering spaces are obtained by the quotient of the corresponding Teichmüller space by a subgroup of the mapping-class group. A set of global coordinates on the Teichmüller space is the so-called Fenchel-Nielsen (FN) coordinates [68]. We can thus hope that these coordinates could be used to formulate string perturbation theory. This is one of the main points explored in this thesis. More precisely, we describe a practical procedure for performing computations in perturbative string theory.

1.4 Objectives of the Thesis

In this thesis, we explore the formulation of on-shell and off-shell string perturbation theory using hyperbolic geometry. As we have explained above, the genus-$\mathfrak{g}$ contribution to the off-shell string amplitudes involving $\mathfrak{n}$ external states[2] is given by

$$\mathcal{A}_{\mathfrak{g}} = \int_{\mathcal{S}_{\mathfrak{g},\mathfrak{n}}} \mathbf{\Omega}, \tag{1.4.1}$$

where $\mathcal{S}$ is a section of a fiber bundle whose fibers involve the information about possible choice of local coordinates around marked point and, in the case of heterotic-string and superstring theories, involve the information about the possible choice of locations of

[2]For the case of heterotic-string theory, $\mathfrak{n} = \mathfrak{n}_{\mathrm{NS}} + \mathfrak{n}_{\mathrm{R}}$.

picture-changing operators (PCOs). Therefore, we need to specify the following data to be able to compute (off-shell) amplitudes like (1.4.1) explicitly

1. Local coordinates around the marked points must be specified. To be able to deal with divergences, it is known that the choice of local coordinates must be gluing-compatible [17, 18].

2. Locations of PCOs should be specified. As in the previous case, this choice of PCOs must be gluing-compatible [17, 18];

3. The form $\boldsymbol{\Omega}$ should be computed in terms of a specific coordinates on the moduli space;

4. The integral of $\boldsymbol{\Omega}$ should be done.

In this thesis, we consider Riemann surfaces equipped with hyperbolic metric with constant curvature -1. As far as we aware, there is no alternative method that provides all necessary ingredients for computing off-shell amplitudes. We briefly explain some of these alternative approaches in appendix 3.A. Main objectives of this thesis are as follows

1. We first argue that local coordinates induced from the hyperbolic metric is not gluing-compatible. This means that hyperbolic metric is not suitable for formulating off-shell amplitudes in string theory.

2. We then argue that this apparent conflict with the requirement of gluing-compatibility can be resolved by resorting to The Uniformization Theorem and doing a rescaling which makes the metric on the family of plumbed surfaces hyperbolic [69]. The idea is to use the so-called curvature-correction equation [70, 71, 72, 73].

3. We then provide a gluing-compatible choice of local coordinates around the marked points.

4. We next provide a distribution of PCOs on Riemann surfaces.

5. Finally, we turn to integration over the moduli space based on the work of Mirzakhani [67]. In this approach, one needs to use Fenchel-Nielsen coordinates for the associated Teichmüller space.

These steps provides a prescription for computing amplitudes in string theory, whether on-shell or off-shell. We again emphasize that *this prescription can be used for practical computations if and only if the form* $\boldsymbol{\Omega}$ *can be computed explicitly in terms of Fenchel-Nielsen coordinates.*

Chapter 2

Off-Shell Amplitudes in Bosonic-String and Superstring Theories

In this chapter, we shall review the general construction of off-shell amplitudes in bosonic-string and superstring theories [20].

2.1 Off-Shell Bosonic-String Amplitudes

In this section, we review the construction of off-shell amplitudes in bosonic-string theory. On-shell amplitudes in bosonic-string theory is obtained by integrating an appropriate $6g - 6 + 2n$ real dimensional differential form over the moduli space of Riemann surfaces denoted by $\mathcal{M}_{g,n}$. This differential form on $\mathcal{M}_{g,n}$ is constructed by computing the correlator of the vertex operators with conformal dimension $(0, 0)$. These operators correspond to the external states satisfying the tree-level on-shell condition. However, generic states of string theory undergo mass renormalization. States having masses different from that at tree level are mapped to vertex operators having conformal dimension different from $(0, 0)$. Therefore, on-shell amplitudes defined using vertex operators having conformal dimension $(0, 0)$ do not compute S-matrix elements beyond tree level for generic states in string theory. This forces us to consider the off-shell amplitudes constructed using vertex operators of arbitrary conformal dimensions.

2.1.1 The World-Sheet Theory

The bosonic-string theory is formulated in terms of a conformal field theory (CFT) defined on a Riemann surface. This conformal field theory has two sectors: the matter sector and the ghost sector. The matter CFT has central charge $(26, 26)$ and the central charge of CFT of reparametrization ghosts is $(-26, -26)$. The ghost system is composed of the anti-commuting fields $b, c, \bar{b}, \bar{c}$. The total CFT has central charge $(0, 0)$. The Hilbert space of CFT is denoted by $\mathcal{H}$. We denote by $\mathcal{H}_0$ a subspace of $\mathcal{H}$ consists of states $|\Psi\rangle$

$$(b_0 - \bar{b}_0)|\Psi\rangle = 0, \qquad (L_0 - \bar{L}_0)|\Psi\rangle = 0, \tag{2.1.1}$$

where L_n and $\overline{L}_n$ denote the total Virasoro generators in the holomorphic and anti-holomorphic sectors of the world-sheet theory [74]. A complete set of states $\{\phi_i\} \in \mathcal{H}_0$ satisfies the following orthonormality and completeness relation

$$\langle\phi_i^c|\phi_j\rangle = \delta_{ij}, \qquad \sum_i |\phi_i\rangle\langle\phi_i^c| = \mathbf{1}, \tag{2.1.2}$$

where $\langle\phi_i^c|$ is the BPZ-conjugate of the state $|\phi_i\rangle$. Physical states that appear as external states in S-matrix computation belong to the subspace $\mathcal{H}_1$ of $\mathcal{H}_0$ consists of states $|\psi\rangle \in \mathcal{H}_0$ satisfying following additional constrains

1. They satisfy the Siegel-gauge condition $(b_0 + \bar{b}_0)|\Psi\rangle = 0$;

2. The ghost number of $|\Psi\rangle$ is 2.

2.1.2 Off-Shell Amplitudes

On-shell amplitudes are defined using the unintegrated vertex operators of conformal dimension $(1, 1)$, and as such the associated integration measures do not depend on the choice of local coordinates around the marked points at which vertex operators are inserted. This means that the integration measure of an on-shell amplitude is a genuine differential form on $\mathcal{M}_{\mathfrak{g},\mathfrak{n}}$. One can then integrate this integrand over $\mathcal{M}_{\mathfrak{g},\mathfrak{n}}$ to get on-shell amplitudes. However, off-shell amplitudes defined using vertex operators of arbitrary conformal dimension depend on the choice of local coordinates around marked points. Therefore, we can not consider the integration measure of an off-shell amplitude as a genuine differential form on $\mathcal{M}_{\mathfrak{g},\mathfrak{n}}$. Instead, we need to think of them as differential forms on a section of a larger space $\mathcal{P}_{\mathfrak{g},\mathfrak{n}}$. This space is defined as a fiber bundle over $\mathcal{M}_{\mathfrak{g},\mathfrak{n}}$. The fiber direction of $\mathcal{P}_{g,n}$

corresponds to possible choices of local coordinates around the marked points of a genus-$\mathtt{g}$ Riemann surface with $\mathtt{n}$ marked points $\mathcal{R}_{\mathtt{g},\mathtt{n}} \in \mathcal{M}_{\mathtt{g},\mathtt{n}}$

$$\mathcal{P}_{\mathtt{g},\mathtt{n}} \longrightarrow \mathcal{M}_{\mathtt{g},\mathtt{n}}. \tag{2.1.3}$$

If we restrict ourselves to states belong to $\mathcal{H}_0$, we can consider the string measure to be defined on a section of space $\widehat{\mathcal{P}}_{\mathtt{g},\mathtt{n}}$. This space is smaller compared to $\mathcal{P}_{\mathtt{g},\mathtt{n}}$[1]. We can understand $\widehat{\mathcal{P}}_{\mathtt{g},\mathtt{n}}$ as a base space of the fiber bundle $\mathcal{P}_{g,n}$ with the fiber direction corresponds to a choice of local coordinates up to phase rotation

$$\mathcal{P}_{\mathtt{g},\mathtt{n}} \longrightarrow \widehat{\mathcal{P}}_{\mathtt{g},\mathtt{n}} \longrightarrow \mathcal{M}_{\mathtt{g},\mathtt{n}}. \tag{2.1.4}$$

We can thus say that a genus-$\mathtt{g}$ contribution to a bosonic-string amplitude involving $\mathtt{n}$ external states and characterized by the integrand $\mathbf{\Omega}$ can be written obtained as

$$\mathcal{A}_{\mathtt{g}} = \int_{\mathcal{S}_{\mathtt{g},\mathtt{n}}} \mathbf{\Omega}, \qquad \mathcal{S}_{\mathtt{g},\mathtt{n}} \in \Gamma(\widehat{\mathcal{P}}_{\mathtt{g},\mathtt{n}}), \tag{2.1.5}$$

where $\Gamma(\widehat{\mathcal{P}}_{\mathtt{g},\mathtt{n}})$ denotes the space of sections of $\widehat{\mathcal{P}}_{\mathtt{g},\mathtt{n}}$. Once a choice of local coordinates has been made, we can express them in terms of the moduli parameters, and integrate the resulting expression over the moduli space $\mathcal{M}_{\mathtt{g},\mathtt{n}}$.

Tangent Vectors of $\mathcal{P}_{\mathtt{g},\mathtt{n}}$

In order to construct a differential form on any space, we need to first study its tangent space at a generic point. Since we are interested in constructing a differential form on a section of $\mathcal{P}_{\mathtt{g},\mathtt{n}}$, we need to study the tangent space of $\mathcal{P}_{\mathtt{g},\mathtt{n}}$ associated with deformations of the punctured Riemann surface and/or the choice of local coordinates around the punctures. A tool that turns out to be useful is the so-called Schiffer variation [75, 20].

To elucidate the idea of Schiffer variation consider a Riemann surface $\mathcal{R} \in \mathcal{P}_{\mathtt{g},\mathtt{n}}$. This means that $\mathcal{R}$ is a genus-$\mathtt{g}$ Riemann surface with $\mathtt{n}$ punctures and a specific choice of local coordinates around its marked points. A typical Riemann surface equipped with a choice of local coordinates is shown in figure 2.1. We shall denote the local coordinate around the i^{th} marked points by w_i and the disc around i^{th} puncture, defined by the equation $|w_i| < 1$,

[1] We mode out the phases of local coordinates. Thus we essentially consider the quotient of $\mathcal{P}_{\mathtt{g},\mathtt{n}}$ by the phase of local coordinates. The resulting space, $\widehat{\mathcal{P}}_{\mathtt{g},\mathtt{n}}$, is thus smaller that $\mathcal{P}_{\mathtt{g},\mathtt{n}}$.

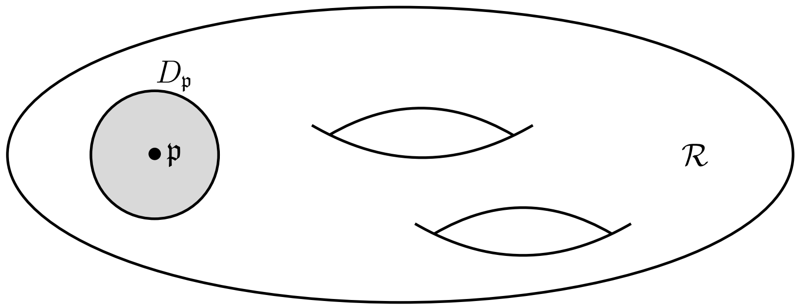


Figure 2.1: A genus-2 surface with local coordinate z around the marked point $\mathfrak{p}$ defined by the equation $|z - z(\mathfrak{p})| = 1$. We take the local coordinate such that $z(\mathfrak{p}) = 0$. Therefore, the disk $D_{\mathfrak{p}}$ is defined by $|z| = 1$.

by D_i for $i = 1, \cdots, \mathfrak{n}$. Consider a pair-of-pants decomposition of $\mathcal{R} - \sum_i D_i$ by choosing $3\mathfrak{g} - 3 + \mathfrak{n}$ homotopically non-trivial disjoint curves on it. This gives $2\mathfrak{g} - 2 + \mathfrak{n}$ pairs of pants denoted by $P_i,\ i = 1, \cdots, 2\mathfrak{g} - 2 + \mathfrak{n}$. We denote the coordinate inside P_i by z_i. Assume that i^{th} disc D_i shares its boundary $|w_i| = 1$ with the j^{th} pair of pants P_j, and also the k^{th} pair of pants P_k shares a boundary with the m^{th} pair of pants P_m. Then,

- On $D_i \cap P_j$, there is a transition function

$$z_j = f_i(w_i). \tag{2.1.6}$$

  f_j can have singularities elsewhere.

- On $P_k \cap P_m$, there is a transition function

$$z_k = f_{km}(z_m). \tag{2.1.7}$$

  f_{km} can have singularities elsewhere.

The Schiffer variation generates all deformations of $\mathcal{P}_{\mathfrak{g},\mathfrak{n}}$ by varying the transition function associated with discs around the punctures $f_i(w_i),\ i = 1, \cdots, n$ and by keeping all other transition functions $f_{km}(z_m)$ fixed [75]. We can generate such variations by keeping the coordinates z_k inside the pair of pants $P_k,\ k = 1, \cdots, 2\mathfrak{g} - 2 + \mathfrak{n}$ fixed and then changing the coordinates inside discs D_i from $w_i \to w_i^{\epsilon}$ for $i = 1, \cdots, \mathfrak{n}$. This change of coordinates deforms the transition function associated with the disc D_i around the i^{th} puncture as follows

$$f_i^{\epsilon}(w_i) = f_i(w_i) - \epsilon v^{(i)}(z_j), \qquad v^{(i)}(z_j) \equiv f_i'(w_i) v^{(i)}(w_i). \tag{2.1.8}$$

We have assumed that the boundary of D_i is shared with P_j. The form of $v^{(i)}(w_i)$ can be obtained from the fact that $f_i^{\epsilon}(w_i^{\epsilon}) = z_k = f_i(w_i)$. Then the tangent vector of $\mathcal{P}_{\mathfrak{g},\mathfrak{n}}$ is given by

$$\vec{v} = (v^{(1)}, \cdots, v^{(n)}). \tag{2.1.9}$$

The behavior of $v^{(i)}$ on $\mathcal{R}$ determines the type of deformation it induces on $\mathcal{P}_{\mathfrak{g},\mathfrak{n}}$. The proofs of following statements are given in section 7 of [59]

- $\vec{v}$ **is a null vector:** If it is holomorphic everywhere except possibly at marked points.
- $\vec{v}$ **deforms the local coordinates around the puncture:** if it is holomorphic inside $D_i,\ i = 1, \cdots, \mathfrak{n}$, vanishes at the puncture, and it does not homomorphically extend into $\mathcal{R} - \sum_i D_i$.
- $\vec{v}$ **moves the puncture:** if it is holomorphic inside $D_i,\ i = 1, \cdots, \mathfrak{n}$, it is non-vanishing at the marked point, and also it does not homomorphically extend into $\mathcal{R} - \sum_i D_i$.
- $\vec{v}$ **deforms the moduli of** $\mathcal{R}$**:** if it has poles at one or more punctures and further it can not be homomorphically extended into $\mathcal{R} - \sum_i D_i$. A set of $3\mathfrak{g} - 3$ of such vector fields with poles of order $1, \cdots, 3\mathfrak{g} - 3$ at any of punctures generate the complete set of deformations of $\mathcal{M}_{\mathfrak{g}}$.

Differential Forms on $\widehat{\mathcal{P}}_{\mathfrak{g},\mathfrak{n}}$

Consider p tangent vectors $V_1, \cdots, V_p$ of $\mathcal{P}_{\mathfrak{g},\mathfrak{n}}$ and let $\vec{v}_1, \cdots, \vec{v}_p$ be the corresponding n-tuple vector fields. We can construct an operator-valued p-form $\mathcal{B}_p$. The contraction of $\mathcal{B}_p$ tangent vectors $V_1, \cdots, V_p$ is given by

$$\mathcal{B}_p[V_1, \cdots, V_p] \equiv \mathfrak{b}(\vec{v}_1) \cdots \mathfrak{b}(\vec{v}_p), \tag{2.1.10}$$

where $\mathfrak{b}(\vec{v})$ is defined as

$$\mathfrak{b}(\vec{v}) \equiv \sum_{a=1}^{\mathfrak{n}} \oint \frac{dw_a}{2\pi \mathfrak{i}} v^{(a)}(w_a) b^{(a)}(w_a) + \sum_{a=1}^{\mathfrak{n}} \oint \frac{d\bar{w}_a}{2\pi \mathfrak{i}} \bar{v}^{(a)}(\bar{w}_a) \bar{b}^{(a)}(\bar{w}_a). \tag{2.1.11}$$

$b, \bar{b}$ denote the anti-ghost fields. Using $\mathcal{B}_p$, we can define a p-form on $\mathcal{P}_{\mathfrak{g},\mathfrak{n}}$

$$\mathbf{\Omega}_p(|\Phi\rangle) \equiv (2\pi \mathfrak{i})^{-(3\mathfrak{g}-3+\mathfrak{n})} \langle \mathcal{R} | \mathcal{B}_p | \Phi \rangle. \tag{2.1.12}$$

Here, $|\Phi\rangle$ is some element of $\mathcal{H}^{\otimes n}$ with ghost number

$$G_{\#} = p + 6 - 6\mathfrak{g}. \tag{2.1.13}$$

$\langle\mathcal{R}|$ is the surface-state associated with the surface $\mathcal{R}$. $\langle\mathcal{R}|$ describes the state that is created on the boundaries of D_i by performing a functional integral over fields of CFT on $\mathcal{R} - \sum_i D_i$. It is clear that $\mathbf{\Omega}_p^{(g,n)}(|\Phi\rangle)$ is a p-form on $\mathcal{P}_{g,n}$. Remember that a p-form on a space generates a number when contracted with p tangent vectors of this space and this number is anti-symmetric under the exchange of any pair of tangent vectors. Since the anti-ghost fields $b, \bar{b}$ are anti-commuting $\mathbf{\Omega}_p^{(g,n)}(|\Phi\rangle)$ also has this property. It is thus clear that $\mathbf{\Omega}_p^{(g,n)}(|\Phi\rangle)$ is a p-form on $\mathcal{P}_{\mathfrak{g},\mathfrak{n}}$.

However, we are interested to construct differential forms on $\widehat{\mathcal{P}}_{\mathfrak{g},\mathfrak{n}}$. To construct a p-form on $\widehat{\mathcal{P}}_{\mathfrak{g},\mathfrak{n}}$, we need to just impose a restriction on the state $|\Phi\rangle$. Instead of allowing $|\Phi\rangle$ to be any state belongs to $\mathcal{H}^{\otimes n}$ restrict it to be an element of $\mathcal{H}_0^{\otimes n}$. We can check this claim by showing that if any tangent vector of $\mathcal{P}_{\mathfrak{g},\mathfrak{n}}$ is chosen from the set of tangent vectors which change the phase of local coordinates, the result of the contraction of the form with such a tangent vector vanishes. To see this, note that such a vector has components of the form $v^{(i)}(w_i) = w_i$ and $\bar{v}^{(i)}(\bar{w}_i) = -\bar{w}_i$. It is clear that such vectors do not change $\mathbf{\Omega}_p^{(\mathfrak{g},\mathfrak{n})}(|\Phi\rangle)$ if $|\Phi\rangle \in \mathcal{H}_0$. This is because for $\vec{v} = (0, \cdots, w_i, \cdots, 0)$, $b(\vec{v}) = b_0^{(i)} - \bar{b}_0^{(i)}$ and $T(\vec{v}) = L_0^{(i)} - \bar{L}_0^{(i)}$, where superscripts denote that the modes b_0, L_0 and their complex conjugates are defined with respect to the local coordinates inside the i^{th} puncture, i.e. it acts on the Hilbert space associated to the i^{th} puncture. Here

$$T(\vec{v}) \equiv \sum_{a=1}^{\mathfrak{n}} \oint \frac{dw_a}{2\pi \mathfrak{i}} v^{(a)}(w_a) T^{(a)}(w_a) + \sum_{a=1}^{\mathfrak{n}} \oint \frac{d\bar{w}_i}{2\pi \mathfrak{i}} \bar{v}^{(a)}(\bar{w}_a) \bar{T}^{(a)}(\bar{w}_a). \tag{2.1.14}$$

Since $(b_0 - \bar{b}_0)|\Phi\rangle = (L_0 - \bar{L}_0)|\Phi\rangle = 0$, the change in $\Omega_p^{(\mathfrak{g},\mathfrak{n})}(|\Phi\rangle)$ due to the phase rotation of local coordinates vanishes [20].

Differential Forms on Sections of $\widehat{\mathcal{P}}_{\mathfrak{g},\mathfrak{n}}$

We shall now discuss a general construction of tangent vectors on a section of $\widehat{\mathcal{P}}_{g,n}$, where we allow both f_i and f_{km} to vary [76]. Since we are interested in a section that corresponds to choosing a specific local coordinates around punctures, we need to just worry about tangent vectors that deform the intrinsic moduli of the surface. In this case, the tangent vector will have $3\mathfrak{g}-3+2\mathfrak{n}$ number of components associated with $3\mathfrak{g}-3+\mathfrak{n}$ homotopically non-equivalent curves that are used for the pair of pant decomposition of $\mathcal{R}$ and $\mathfrak{n}$ circles bounding the discs $D_i,\ i = 1, \cdots, \mathfrak{n}$ around $\mathfrak{n}$ punctures. These $3\mathfrak{g} - 3 + 2\mathfrak{n}$ closed curves on $\mathcal{R}$ gives $2\mathfrak{g} - 2 + 2\mathfrak{n}$ coordinate patches. Let us denote the local coordinate on the m^{th} patch by z_m, and the real coordinates of the moduli space of genus-$\mathfrak{g}$ Riemann surface with $\mathfrak{n}$ punctures by $(t^{(1)}, \cdots, t^{(6\mathfrak{g}-6+2\mathfrak{n})})$. Assume that coordinate patches m and n have a non-empty intersection and contour C_{mn} runs between them. The change in transition function that relates patches m and n under a change in the moduli $t^{(k)}$ is given by [76]

$$\left.\frac{\partial z_m}{\partial t^{(k)}}\right|_{z_n} = v^{z_m}_{km} - \left.\frac{\partial z_m}{\partial z_n}\right|_t v^{z_n}_{kn} = v^{z_m}_{km} - v^{z_m}_{kn}, \tag{2.1.15}$$

where $v^{z_m}_{km} = \frac{dz_m}{dt^{(k)}}$. The $(3\mathfrak{g}-3+2\mathfrak{n})$-tuple vector field $\vec{v}_k = \left(\cdots, \left.\frac{\partial z_m}{\partial t^{(k)}}\right|_{z_n}, \cdots\right)$ corresponds to varying the moduli $t^{(k)}$ and

$$\mathfrak{b}(\vec{v}_k) = \sum_{(mn)} \oint_{C_{mn}} \left(dz_m \left.\frac{\partial z_m}{\partial t^k}\right|_{z_n} b_{z_m z_m} - d\bar{z}_m \left.\frac{\partial \bar{z}_m}{\partial t^k}\right|_{z_n} b_{\bar{z}_m \bar{z}_m} \right), \tag{2.1.16}$$

where the summation over (mn) denotes the summation over $3\mathfrak{g} - 3 + 2\mathfrak{n}$ closed curves on $\mathcal{R}$ which gives $2\mathfrak{g} - 2 + 2\mathfrak{n}$ coordinate patches. Using (2.1.15), we have

$$b(\vec{v}_k) = \sum_{m=1}^{2\mathfrak{g}-2+2\mathfrak{n}} \oint_{C_m} \left(dz_m v^{z_m}_{km} b_{z_m z_m} - d\bar{z}_m v^{\bar{z}_m}_{km} b_{\bar{z}_m \bar{z}_m} \right). \tag{2.1.17}$$

Using The Stokes Theorem, we get

$$\mathfrak{b}(\vec{v}_k) = \int_{\mathcal{R}} d^2 z \left(b_{zz} \mu^{z}_{k\bar{z}} + b_{\bar{z}\bar{z}} \mu^{\bar{z}}_{kz} \right). \tag{2.1.18}$$

Here μ_k denotes the Beltrami differential associated with the moduli $t^{(k)}$. The variation of the surface associated with the Beltrami differential can be related to vector fields as

$$\mu^{\bar{z}_m}_{kz_m} = \partial_{z_m} v^{\bar{z}_m}_{km}, \qquad \mu^{z_m}_{k\bar{z}_m} = \partial_{\bar{z}_m} v^{z_m}_{km}. \tag{2.1.19}$$

2.1.3 Gluing-Compatible Integration Cycles

Off-shell amplitudes relevant to the computation of S-matrix elements in bosonic-string theory are constructed by integrating differential forms on $\widehat{\mathcal{P}}_{\mathfrak{g},\mathfrak{n}}$. Such forms are built using state $|\Phi\rangle$ belongs to $\mathcal{H}_1^{\otimes n}$. In particular the ghost number of $|\Phi\rangle$ is $2\mathfrak{n}$. So the rank of relevant differential form is $p = 6\mathfrak{g} - 6 + 2\mathfrak{n}$, as it should be since it matches with the dimension of moduli space of genus-$\mathfrak{g}$ Riemann surfaces with $\mathfrak{n}$ punctures. However, we stress that although the rank of $\mathbf{\Omega}_p^{(\mathfrak{g},\mathfrak{n})}(|\Phi\rangle)$ matches with the dimension of the moduli space $\mathcal{M}_{\mathfrak{g},\mathfrak{n}}$, we can not regard it as a genuine top form on $\mathcal{M}_{\mathfrak{g},\mathfrak{n}}$. The reason is that it depends on the choice of local coordinates around the punctures and it is non-zero even if the tangent vectors $\vec{v}$ generate deformations of local coordinate without varying the surface $\mathcal{R}$. We thus need to integrate $\mathbf{\Omega}_p^{(\mathfrak{g},\mathfrak{n})}(|\Phi\rangle)$ over a section of the fiber bundle $\widehat{\mathcal{P}}_{\mathfrak{g},\mathfrak{n}}$ whose base space is $\mathcal{M}_{\mathfrak{g},\mathfrak{n}}$.

A necessary requirement for off-shell amplitudes is that physical quantities that can be extracted from off-shell amplitudes like the renormalized masses and S-matrix elements must be independent of the choice of the section of $\widehat{\mathcal{P}}_{\mathfrak{g},\mathfrak{n}}$ [20]. To ensure this requirement, we need to impose a condition on the choice of this section. To describe this condition, known as **The Gluing-Compatibility Requirement** introduced in [17, 18], consider two Riemann surfaces $\mathcal{R}_1$ and $\mathcal{R}_2$. $\mathcal{R}_1$ is a genus $\mathfrak{g}_1$ surface with $\mathfrak{n}_1$ punctures and $\mathcal{R}_2$ is a genus $\mathfrak{g}_2$ surface with $\mathfrak{n}_2$ punctures. Denote the collection of pairs of pants on $\mathcal{R}_1$ by $\{P_k^{(1)}\}$ and disks around punctures by $D_1^{(1)}, \cdots, D_{\mathfrak{n}_1}^{(1)}$. Similarly denote the collection of pairs of pants on $\mathcal{R}_2$ by $\{P_k^{(2)}\}$ and disks around punctures by $D_1^{(2)}, \cdots, D_{\mathfrak{n}_2}^{(2)}$. We can glue disks $D_i^{(1)}$ and $D_j^{(2)}$ using the plumbing fixture relation:

$$w_i^{(1)} w_j^{(2)} = e^{-s+\mathrm{i}\theta}, \qquad 0 \leq s < \infty, \qquad 0 \leq \theta < 2\pi. \tag{2.1.20}$$

This will produce another surface $\mathcal{R}$ having genus $\mathfrak{g} = \mathfrak{g}_1 + \mathfrak{g}_2$ and $\mathfrak{n} = \mathfrak{n}_1 + \mathfrak{n}_2 - 2$ punctures. Surfaces that can be constructed this way belong to the region near points at infinity of $\mathcal{M}_{\mathfrak{g},\mathfrak{n}}$. This part of $\mathcal{M}_{\mathfrak{g},\mathfrak{n}}$ can be parametrized by moduli of $\mathcal{M}_{\mathfrak{g}_1,\mathfrak{n}_1}$, the moduli of $\mathcal{M}_{\mathfrak{g}_2,\mathfrak{n}_2}$ and (s,θ). The gluing-compatibility condition requires that the section in fiber bundle $\widehat{\mathcal{P}}_{\mathfrak{g},\mathfrak{n}}$ over this region of moduli space be chosen such that the relationship between coordinates of $\{P_k^{(1)}\}$ and $D_1^{(1)}, \cdots, \cancel{D_i^{(1)}}, \cdots, D_{\mathfrak{n}_1}^{(1)}$ depends only on the moduli of $\mathcal{M}_{\mathfrak{g}_1,\mathfrak{n}_1}$ and not on the moduli of $\mathcal{M}_{\mathfrak{g}_2,\mathfrak{n}_2}$ and (s,θ). Similarly the relation between coordinates of $\{P_k^{(2)}\}$ and $D_1^{(2)}, \cdots, \cancel{D_j^{(2)}}, \cdots, D_{\mathfrak{n}_2}^{(2)}$ depends only on the moduli of $\mathcal{M}_{\mathfrak{g}_2,\mathfrak{n}_2}$ and not on the moduli of $\mathcal{M}_{\mathfrak{g}_1,n_1}$ and (s,θ). Also the dependence of these relations on the moduli of $\mathcal{M}_{\mathfrak{g}_l,\mathfrak{n}_l}$ must be the one induced from the choice of of the section $\widehat{\mathcal{P}}_{g_l,n_l}$ for $l = 1, 2$. Therefore, we can say that the requirement of gluing-compatibility is equivalent of the following two conditions [17, 18]

- The choice of local coordinates around marked points of the two component surfaces must be such that the relations between the local coordinates on each piece are independent of the local coordinates of the other piece and the parameters (s, θ) used for plumbing fixture of the two pieces in (2.1.20);
- The choice of local coordinate in each of the two pieces is induced from the choice of an appropriate section $\widehat{\mathcal{P}}_{\mathfrak{g}_i,\mathfrak{n}_i},\ i = 1, 2$ over the moduli space of each piece. In other words, on the regions of the moduli space that we can use (2.1.20) to construct the surface, we have:

$$\widehat{\mathcal{P}}_{\mathfrak{g},\mathfrak{n}} = \widehat{\mathcal{P}}_{\mathfrak{g}_1,\mathfrak{n}_1} \bigoplus \widehat{\mathcal{P}}_{\mathfrak{g}_2,\mathfrak{n}_2} \bigoplus \widehat{\mathcal{P}}(s, \theta). \tag{2.1.21}$$

  In this relation, $\widehat{\mathcal{P}}(s, \theta)$ is the space of possible local coordinates on the plumbing tube.

2.2 Off-Shell Superstring Amplitudes

We now turn to the general construction of the off-shell amplitudes in superstring theory [20]. For concreteness, we shall discuss the heterotic-string theory whose holomorphic sector is similar to a superstring theory and whose antiholomorphic sector is similar to a bosonic-string theory compactified on a 16 dimensional integer, even, self-dual lattice [9, 10, 11]. The generalization of this construction to type-II superstring theories is straightforward.

2.2.1 The World-Sheet Theory

The worldsheet theory of the heterotic-string theory contains the matter field theory with central charge $(26, 15)$, and the ghost system of total central charge $(-26, -15)$ containing anti-commuting fields $b, c, \bar{b}, \bar{c}$ and commuting β and γ ghosts. Most of complications in RNS formulation of the superstring theory stems from the curious properties of the $\beta\gamma$ system. Therefore, we shall discuss the $\beta\gamma$ system and its representations in some detail.

The $\beta\gamma$ system is a commuting fermionic system. The mode expansion of its fields is given by

$$\beta(z) = \sum_{n \in \mathbb{Z}+a} \beta_n z^{-n-\frac{3}{2}}, \qquad \gamma(z) = \sum_{n \in \mathbb{Z}+a} \gamma_n z^{-n+\frac{1}{2}}, \tag{2.2.1}$$

where

$$a = \begin{cases} \frac{1}{2}, & \text{for NS sector,} \\ 0, & \text{for R sector.} \end{cases} \tag{2.2.2}$$

The modes β_n and γ_n satisfy the following commutation relation

$$[\gamma_m, \beta_n] = \delta_{n,-m}. \tag{2.2.3}$$

There are infinite number of inequivalent representations of this algebra. These inequivalent representations can be constructed using the raising operators by acting on infinite number of vacuum states $|q\rangle$, where q is the ghost charge of the vacuum state $|q\rangle$, is integer or half integer. The ghost charge is the eigenvalue of the ghost charge operator given by

$$\mathcal{Q}_{\text{gh}} = \sum_n \beta_n \gamma_{-n}. \tag{2.2.4}$$

The number q is called *the Bose-sea level* or *the picture number* of the representation. These sets of vacua are inequivalent because unlike in the case of the degenerate ground states of the bc system, here we can not go from the vacuum with one value of q to another vacuum with a different value for q by acting with finite number of oscillators [15].

Let us denote the operators which can increase or decrease q by $\delta(\gamma_m)$ and $\delta(\beta_n)$. The action of these operators on the q-vacua are given by

$$\begin{aligned} \delta(\beta_{-q-\frac{3}{2}})|q\rangle &= |q+1\rangle, \\ \delta(\gamma_{q+\frac{1}{2}})|q\rangle &= |q-1\rangle. \end{aligned} \tag{2.2.5}$$

Similarly, we can define operators, spin fields, Σ_+ and Σ_- mapping states in the R-sector to states in the NS-sector or vice versa. The ghost charge for the NS-sector is an integer and for the R-sector is a half-integer. They are also defined by their action on the q-vacua

$$\Sigma_+(0)|0\rangle = \left|\frac{1}{2}\right\rangle, \quad \Sigma_-(0)|0\rangle = \left|-\frac{1}{2}\right\rangle. \tag{2.2.6}$$

Therefore, each of the states in the Hilbert space of superstring theory has an infinite number of inequivalent representation based on the q-vacua that we use for building the tower of states. From the operator-state correspondence, we know that there exist a vertex operator associated to each state in the Hilbert space. Therefore, there are infinite number of inequivalent vertex operators for any specific state and we distinguish them by associating a *picture number* which indicates the q-vacua used for constructing the state.

Things become more transparent if we represent $\beta\gamma$ system using a free scalar ϕ and a pair of free chiral fermionic fields ξ and η of conformal weight $(0,0)$ and $(1,0)$, respectively, known as the *bosonization* of the $\beta\gamma$ system [15]. The action for the combined system is

$$S[\phi,\xi,\eta] = \frac{1}{2\pi}\int_{\mathcal{R}}\left(\partial_z\phi(z)\partial_{\bar{z}}\phi(z) - \frac{1}{2}R\phi(z)\right) + \frac{1}{\pi}\int_{\mathcal{R}}\eta(z)\partial_{\bar{z}}\xi(z). \tag{2.2.7}$$

The stress-energy tensor is given by

$$T_{\beta,\gamma}(z) = T_\phi + T_{\eta,\xi}, \tag{2.2.8}$$

where

$$T_{\eta,\xi}(z) \equiv -\eta(z)\partial\xi(z), \qquad T_\phi(z) \equiv -\frac{1}{2}\partial\phi(z)\partial\phi(z) - \partial^2\phi(z). \tag{2.2.9}$$

The bosonization prescription reads

$$\beta(z) = \partial_z\xi(z)e^{-\phi(z)}, \qquad \gamma(z) = \eta(z)e^{\phi(z)}. \tag{2.2.10}$$

Reversing this prescription, we get the following identifications [77]

$$\begin{aligned}&\xi(z) = \mathrm{H}(\beta(z)), \qquad \eta(z) = \partial_z\gamma(z)\delta(\gamma(z)),\\ &e^{\phi(z)} = \delta(\beta(z)), \qquad e^{-\phi(z)} = \delta(\gamma(z)),\end{aligned} \tag{2.2.11}$$

where H denotes the Heaviside step function. Therefore, the complete set of bosonization relations are:

$$\begin{aligned}&\beta(z) = \partial_z\xi(z)e^{-\phi(z)}, \qquad \delta(\beta(z)) = e^{\phi(z)},\\ &\gamma(z) = \eta(z)e^{\phi(z)}, \qquad \delta(\gamma(z)) = e^{-\phi(z)}.\end{aligned} \tag{2.2.12}$$

The q-vacua in this representation is defined as

$$\begin{aligned}&\phi_n|q\rangle = \eta_n|q\rangle = \xi_m|q\rangle = 0, \qquad n \geq -1, \quad m \geq 0,\\ &\phi_0|q\rangle = q|q\rangle.\end{aligned} \tag{2.2.13}$$

We also have

$$\Sigma_{\pm}(z) = e^{\pm\frac{1}{2}\phi(z)}. \tag{2.2.14}$$

It is important to note that only the derivative of ξ-field is present in the identification (2.2.12) between the two systems. This means that the Hilbert space $\mathcal{H}_{\xi\eta\phi}$ of the (ξ, η, ϕ) has more states than the Hilbert space $\mathcal{H}_{\beta\gamma}$ of the β-γ system. The precise equivalence is the following [15]

$$\mathcal{H}_{\beta\gamma} \equiv \{|\psi\rangle \in \mathcal{H}_{\xi\eta\phi}|\ \eta_0|\psi\rangle = 0\}\,. \tag{2.2.15}$$

We saw that it is possible to construct the operators which can take us from one q-vacua to another. This means that it should be possible to change the Bose-sea charge of a vertex operator and this procedure is known as **The Picture-Changing Operation**. This operation is done using the picture-changing operator (PCO) $\chi(z)$ defined as follows

$$\begin{aligned}\chi(z) &\equiv \{\mathcal{Q}_B, \zeta(z)\} = \oint \frac{dw}{2\pi\mathrm{i}}\, j_B(w)\zeta(z),\\ j_B(z) &\equiv c(z)\left(T_m(z) + T_{\beta,\gamma}(z)\right) + \gamma(z)T_F(z) + b(z)c(z)\partial c(z) - \frac{1}{4}\gamma^2(z)b(z).\end{aligned} \tag{2.2.16}$$

Here $\mathcal{Q}_B$ denotes the worldsheet BRST charge, j_B denotes the BRST current and $T_F(z)$ denote the superpartner of the matter stress tensor $T_m(z)$. The picture-changing operator is BRST-invariant, and a dimension-zero primary operator with the picture number one. Therefore, although the picture-changing operator looks like an exact operator, it acts non-trivially on states.

We end our discussion on pictures by mentioning that we need to introduce enough number of PCOs on the worldsheet to make sure that the total picture number is zero to get a sensible superstring amplitude [20]. On a genus-$\mathfrak{g}$ surface with $\mathfrak{n}_{\text{NS}}$ NS marked point in picture number q_{NS} and $\mathfrak{n}_{\text{R}}$ R marked points in picture number q_{NS}, the number of PCOs is given by

$$\#\text{ of PCOs} = 2\mathfrak{g} - 2 + (-q_{\text{NS}})\mathfrak{n}_{\text{NS}} + (-q_{\text{R}})\mathfrak{n}_{\text{R}}. \tag{2.2.17}$$

The canonical picture numbers for states in the NS sector is -1 and for states in the R sector is $-\frac{1}{2}$. It is not very clear how to do the superstring perturbation theory with positive picture numbers [16].

Let us denote the total Hilbert space of the worldsheet theory by $\mathcal{H} = \mathcal{H}^{\text{NS}} \oplus \mathcal{H}^{\text{R}}$, where $\mathcal{H}^{\text{NS}}$ denotes the Hilbert space of the Nevue-Schwarz (NS) sector and $\mathcal{H}^{\text{R}}$ denotes the Hilbert space of the Ramond (R) sector. We denote by $\mathcal{H}_0 \equiv \mathcal{H}_0^{\text{NS}} \oplus \mathcal{H}_0^{\text{R}}$, a subspace of $\mathcal{H}$

defined as all the states $|\Psi\rangle$ satisfying

$$(b_0 - \bar{b}_0)|\Psi\rangle = 0, \qquad (L_0 - \bar{L}_0)|\Psi\rangle = 0, \qquad \eta_0|\Psi\rangle = 0. \tag{2.2.18}$$

Picture numbers of states in these spaces are chosen as follows:

$$\begin{aligned} &\text{picture number of } |\Psi\rangle = -1, \qquad |\Psi\rangle \in \mathcal{H}_0^{\mathrm{NS}}, \\ &\text{picture number of } |\Psi\rangle = -\frac{1}{2}, \qquad |\Psi\rangle \in \mathcal{H}_0^{\mathrm{R}}. \end{aligned} \tag{2.2.19}$$

The physical states that appear as external states in the S-matrix computation belong to the subspace $\mathcal{H}_1$ of $\mathcal{H}_0$ satisfying extra conditions

$$(b_0 + \bar{b}_0)|\Psi\rangle = 0, \qquad \text{ghost number}(|\Psi\rangle) = 2. \tag{2.2.20}$$

2.2.2 Off-Shell Amplitudes

The construction of off-shell amplitudes in the superstring theory is similar to the construction of off-shell amplitudes in the bosonic-string theory. For instance, we need to choose a gluing-compatible local coordinates around marked points. However, there are additional complications that we need to address. These complications are arising from the following fact. In order to construct the genus-$\mathfrak{g}$ contribution to amplitudes in the heterotic-string theory with $\mathfrak{n}_{\mathrm{NS}}$ NS marked points and $2\mathfrak{n}_{\mathrm{R}}$ R marked points, we need to insert $2\mathfrak{g} - 2 + \mathfrak{n}_{\mathrm{NS}} + \mathfrak{n}_{\mathrm{R}}$ PCOs on genus-$\mathfrak{g}$ surfaces with $\mathfrak{n} = \mathfrak{n}_{\mathrm{NS}} + 2\mathfrak{n}_{\mathrm{R}}$ marked points. Inserting these operators on the Riemann surface introduces the following additional issues compared to the bosonic-string theory

- The distribution of PCOs should be gluing-compatible, i.e. it should be compatible with sewing of surfaces.
- The existence of PCOs introduces divergences into integrands of amplitudes. These divergences are called *spurious singularities* and they have three origins 1) the collision of PCOs with each other, 2) the collision of PCOs with marked points, 3) singularities coming from correlation functions of $\beta\gamma$ system. As a result, in order to define superstring amplitudes we need to define the superstring measure and the integration cycle carefully by avoiding the occurrence of these spurious singularities.

Therefore, just like in the bosonic-string theory, the integration measure of off-shell amplitudes in the superstring theory is not a genuine differential form on the moduli space.

Instead, we need to think of integration measures of off-shell superstring amplitudes involving $\mathfrak{n}_{\rm NS}$ NS marked points and $2\mathfrak{n}_{\rm R}$ R marked points as a differential form defined on a section of a larger space $\mathcal{P}_{\mathfrak{g},\mathfrak{n}_{\rm NS},2\mathfrak{n}_{\rm R}}$, defined as a fiber bundle over $\mathcal{M}_{\mathfrak{g},\mathfrak{n}_{\rm NS},2\mathfrak{n}_{\rm R}}$. The fiber direction corresponds to different choices of local coordinates around punctures and positions of $2\mathfrak{g}-2+\mathfrak{n}_{\rm NS}+\mathfrak{n}_{\rm R}$ PCOs. If we restrict ourselves to states belonging to the Hilbert space $\mathcal{H}_0$, then we can consider the differential form of our interest as defined on a section of a space $\widetilde{\mathcal{P}}_{\mathfrak{g};\mathfrak{n}_{\rm NS},\mathfrak{n}_{\rm R}}$, where phases of local coordinates are forgotten. The rest of the construction is similar to the case of bosonic-string theory [20]. The genus-$\mathfrak{g}$ contribution to the heterotic-string theory with $\mathfrak{n}_{\rm NS}$ external states and $\mathfrak{n}_{\rm R}$ external states is given by

$$\mathcal{A}_{\mathfrak{g}}=\int_{\widetilde{\mathcal{S}}_{\mathfrak{g};\mathfrak{n}_{\rm NS},\mathfrak{n}_{\rm R}}}\widetilde{\boldsymbol{\Omega}}_d,\qquad \widetilde{\mathcal{S}}\in\Gamma(\widetilde{\mathcal{P}}_{\mathfrak{g};\mathfrak{n}_{\rm NS},\mathfrak{n}_{\rm R}}),\tag{2.2.21}$$

where $d\equiv 6\mathfrak{g}-6+2\mathfrak{n}_{\rm NS}+2\mathfrak{n}_{\rm R}$ is the dimension of $\mathcal{M}_{\mathfrak{g};\mathfrak{n}_{\rm NS},\mathfrak{n}_{\rm R}}$.

2.2.3 Gluing-Compatible Integration Cycles

In the superstring theory, the gluing-compatibility of integration cycle refers to choosing local coordinates and PCOs distribution that respect the plumbing-fixture construction. The definition of the gluing-compatible choice of the local coordinates is the same as described in subsection 2.1.3 for the bosonic-string theory. We shall briefly discuss the meaning of the gluing-compatibility requirement on the distribution of PCOs.

Consider the situation where a genus-$\mathfrak{g}$ Riemann surface with $(\mathfrak{n}_{\rm NS},2\mathfrak{n}_{\rm R})$ marked points degenerates into surfaces with signatures $(\mathfrak{g}_i;\mathfrak{n}^i_{\rm NS},2\mathfrak{n}^i_{\rm R})$ for $i=1,2$. There are two possible types of degenerations:

- The degeneration where an NS-sector state propagates along the tube connecting the two component surfaces. We have

$$\mathfrak{g}=\mathfrak{g}_1+\mathfrak{g}_2,\qquad \mathfrak{n}_{\rm NS}=\mathfrak{n}^1_{\rm NS}+\mathfrak{n}^2_{\rm NS}-2,\qquad \mathfrak{n}_{\rm R}=\mathfrak{n}^1_{\rm R}+\mathfrak{n}^2_{\rm R}.\tag{2.2.22}$$

  We should make sure that the component surfaces have appropriate number of PCOs. It turns out that it is always possible to do this.

- The degeneration where an R-sector state propagates along the tube connecting the two component surfaces. We have

$$\mathfrak{g}=\mathfrak{g}_1+\mathfrak{g}_2,\qquad \mathfrak{n}_{\rm NS}=\mathfrak{n}^1_{\rm NS}+\mathfrak{n}^2_{\rm NS},\qquad \mathfrak{n}_{\rm R}=\mathfrak{n}^1_{\rm R}+\mathfrak{n}^2_{\rm R}-1.\tag{2.2.23}$$

We again should make sure that the component surfaces have appropriate number of PCOs. It turns out that one needs to put a PCO on the tube connecting the two surfaces

$$\chi_0 = \oint \frac{dz}{2\pi\mathrm{i}} \frac{\chi(z)}{z}. \tag{2.2.24}$$

There is a reason for distributing the extra PCO on a homotopically non-trivial cycle on the plumbing tube, instead of placing it at a point on the tube. If we put the extra PCO on a point in the plumbing tube, then it is not possible to interpret the integration over the moduli of the plumbing tube with the R-sector state propagating through it as the propagator of the R-sector states. This is due to the fact that the reparametrization ghost field mode b_0 do not commute with the PCO. However, if we work within the Hilbert space $\mathcal{H}_0$, then the PCO distributed over a cycle commute with b_0, L_0 and $\bar{b}_0, \bar{L}_0$. This means that smearing the PCO on a cycle in the plumbing tube allows us to interpret the integration over the moduli of the plumbing tube with the R-sector state propagating through it as the propagator in the R sector [20].

Appendix

2.A On-Shell Amplitudes

There are two ways to formulate the superstring perturbation theory for the RNS superstring

1. The natural setting for doing RNS superstring is supergeometry, i.e. *super-Riemann surfaces and supermoduli spaces thereof*. This formulation and the concept of super-Riemann surface was introduced by Friedan [14]. This formulation is natural in the sense that a supersymmetric theory is naturally formulated on a supermanifold rather than an ordinary manifold. A super-Riemann surfaces is the superspace on which the RNS superstring theory lives.

2. The RNS superstring can be formulated using ordinary Riemann surfaces and moduli spaces thereof using the so-called *picture-changing formalism*. This formalism introduced by Friedan, Shenker, and Martinec [15].

The relation between two formulations was first found in [77], and then elaborated in [25]. In the following, we work within the second formalism. Extensive reviews of the first approach can be found in [78, 79, 80, 16].

Here we provide a very heuristic explanation of path integral of string perturbation theory. A systematic treatment can be found in chapter 5 of [76]. A modern treatment is given in sections 2 and 3 of [16].

In path integral formulation of the dynamics of point-like objects (particles), one sums over all the paths between initial and final configurations. This provides a first-quantized description of particle dynamics. Similarly, one can consider path integral definition of the dynamics of one-dimensional objects (strings). To define string scattering amplitude, the wisdom of S-matrix theory is that string states are coming from far past (asymptotic incoming states), interact, and go to the far future (asymptotic outgoing states). During this process, the one-dimensional strings sweeps a surface called worldsheet of string propagation. However, due to a symmetry of the world-sheet, i.e. the conformal (gauge) symmetry[2], one can map these asymptotic states to some points on an intermediate surface, i.e. one has an intermediate surface together with several of the so-called *vertex operators* describing the asymptotic states inserted on some marked points on the surface. It can also have a number of handles. Therefore, the stringy processes can be described by the path integral over all such surfaces as the summation over all one-dimensional worldline of particles gives quantization of particle motion. For asymptotic incoming states $\{I\}$ and asymptotic outgoing states $\{O\}$, the string correlation function, i.e. scattering amplitude, can be written schematically as

$$\mathcal{A}_{\mathbf{String}} = \int_{\{I\},\{O\}} \mathcal{D}(\mathcal{R})\ \mathcal{A}_{\mathbf{CFT}}[\mathcal{R}], \tag{2.A.1}$$

where $\mathcal{R}$ denotes a possible worldsheet, $\mathcal{A}_{\mathbf{CFT}}[\mathcal{R}]$ denotes the correlation function of a CFT, describing the worldsheet theory, on $\mathcal{R}$, and $\mathcal{D}(\mathcal{R})$ denotes the formal summation over all possible worldsheets between the incoming states $\{I\}$ and outgoing states $\{O\}$. For special case of zero incoming and outgoing state, we get the string partition function, i.e. the vacuum amplitude. The possible worldsheet can be different in two ways

1. **Topological Equivalence**: one can distinguish two topologically-distinct oriented surfaces by *the number of handles* and *number of boundaries*. In closed-string theory[3], the number of boundaries are fixed by the number of asymptotic states, and therefore, distinct topologies are classified by the number of handles of the surface, i.e. its genus.

2. **Conformal Equivalence**: One of the advantages of string theory is that space-time processes can be described by an S-matrix computed in a $2d$ quantum field theory. In the naive path integral approach, any process in the space-time involving quantum gravitational effects can be computed upon integration over the space of all Lorentzian metrics on a four-dimensional manifold modulo the action of the diffeo-

[2]There is no need to work with gauge-fixed version of the Polyakov action. Any $2d$ (super)CFT with appropriate central charges can be used to define string perturbation theory or string field theory [81, 82, 83].

[3]In open- and open-closed-string theories, the number of boundaries can change. For example a disk can develop a hole that adds a boundary.

morphism group[4] by knowing the UV-completion of general relativity. This space is infinite-dimensional and is difficult to handle. In string theory, after Wick rotating the $2d$ coordinates, we are dealing with a $2d$ conformally-invariant quantum field theory which is only sensitive to the conformal class of the metric of the surface on which the $2d$ theory is defined. Not all two topologically-equivalent Riemann surfaces can be mapped to each other by a conformal transformation. The ones that can be mapped to each other by such transformations form elements of an equivalence class and are called *conformally-equivalent*. Therefore, the computation of above integrals involves the integration over the space of distinct conformal classes of metrics on a Riemann surface.

Using these facts, the above integral can be written as

$$\mathcal{A}_{\mathbf{String}} = \sum_{\mathfrak{g}=0}^{\infty} \int_{\overline{\mathcal{M}}_{\mathfrak{g}}} \mathcal{A}_{\mathbf{CFT}}[\mathcal{R}_{\mathfrak{g}}], \tag{2.A.2}$$

where $\mathcal{R}_{\mathfrak{g}}$ is a genus-$\mathfrak{g}$ surface, and $\mathcal{A}_{\mathbf{CFT}}[\mathcal{R}_{\mathfrak{g}}]$ is the CFT correlation function on $\mathcal{R}_{\mathfrak{g}}$. $\overline{\mathcal{M}}_{\mathfrak{g}}$ denotes the space of distinct conformal classes on a genus-$\mathfrak{g}$ surface. The vertex operators in this expression are *integrated vertex operators*, i.e. the vertex operator integrated over the whole surface. Amplitudes in string theory can be written either as an integral over $\overline{\mathcal{M}}_{\mathfrak{g}}$ using integrated vertex operators or *equivalently* as an integral over $\overline{\mathcal{M}}_{\mathfrak{g},\mathfrak{n}}$, the space of distinct conformal classes on a genus-$\mathfrak{g}$ surface with $\mathfrak{n}$ marked points, using *unintegrated vertex operators*

$$\mathcal{A}_{\mathbf{String}} = \sum_{\mathfrak{g}=0}^{\infty} \int_{\overline{\mathcal{M}}_{\mathfrak{g},\mathfrak{n}}} \mathcal{A}_{\mathbf{CFT}}[\mathcal{R}_{\mathfrak{g},\mathfrak{n}}]. \tag{2.A.3}$$

With some modification, this expression also works for superstring theory. In superstring theory, $\mathfrak{n} = \mathfrak{n}_{\text{NS}} + \mathfrak{n}_{\text{R}}$ correspond to $\mathfrak{n}_{\text{NS}}$ states from the NS sector and $\mathfrak{n}_{\text{R}}$ states from the R sector. Also, $\mathcal{A}_{\mathbf{CFT}}$ should be interpreted as a correlation function in a superconformal field theory. Therefore, we need to deal with integrals over $\overline{\mathcal{M}}_{\mathfrak{g},\mathfrak{n}}$ in string perturbation theory[5].

Two metrics g_{ab} and g'_{ab} are said to be *conformally-equivalent* or *in the same conformal*

[4] This is correct if one ignores the topology-changing processes.

[5] Although one usually prefers to work with integrated vertex operators for practical purposes, however, due to the subtleties coming from the infinities of the moduli space, we rely on the unintegrated form of the vertex operator which always works. For illustration of this point see figures 1 and 2 in [16].

class if they are related by

$$g'_{ab} = e^{2\omega} g_{ab}, \qquad\qquad a, b = 1, 2, \tag{2.A.4}$$

for some function $\omega \equiv \omega(x^1, x^2)$, where x^1 and x^2 are coordinates on the worldsheet. To handle the conformal classes of metrics on a Riemann surface, it turns out to be useful to deal with *complex structures* instead. There are two equivalent ways to think about complex structure which are essentially the same. The first is the usual viewpoint of the theory of manifolds. The exceptionality of $2d$ is that any metric g on a 2d manifold is conformally-flat, i.e. it can be written as

$$ds^2 = g_{ab}\, dx^a \otimes dx^b = f(x^1, x^2) \left(dx^1 \otimes dx^1 + dx^2 \otimes dx^2 \right), \tag{2.A.5}$$

where $f(x^1, x^2)$ is a real and positive function on the local chart (x^1, x^2). Therefore, the transition functions between two local charts are provided by orientation-preserving[6] conformal diffeomorphisms or biholomorphisms, i.e. we have a complex structure defined by the local charts and the biholomorphisms between them. By definition, this complex structure depends only on the conformal class of the metric. From the second point of view, we use the fact that there is a one-to-one correspondence between metrics and complex structures in $2d$ [78]. Having a metric g_{ab} in local coordinates (x^1, x^2), a mixed tensor can be defined as

$$J_a{}^b \equiv \sqrt{g}\, \varepsilon_{ac}\, g^{cb}, \qquad a, b, c = 1, 2, \tag{2.A.6}$$

where ε is the totally-antisymmetric tensor in two dimensions defined by $\varepsilon_{12} = -\varepsilon_{21} = 1$. It satisfies

$$J_a{}^c J_c{}^b = -\delta_a{}^b, \qquad\qquad \boldsymbol{\nabla}_c J_a{}^b = 0, \tag{2.A.7}$$

and therefore, defines a complex structure. It is clear from (2.A.6) that two conformally-equivalent metrics define the same complex structure. This fact has an important implication, namely, the integration over distinct conformal classes of metrics reduces to the integration over the space of all complex structures modulo *infinitesimal diffeomorphisms*, i.e. *the Teichmüller space*. This space is finite-dimensional and the continuous parameters specifying a conformal class or a complex structure are called *Teichmüller parameters*. We emphasize that on a generic Riemann surface two arbitrary metrics cannot be related by conformal transformation. The reason is that in a local complex coordinate $z \equiv x^1 + \mathrm{i}x^2$, a (conformal) diffeomorphism is given by a holomorphic function $z \longrightarrow f(z)$. However, a general infinitesimal transformation is of the form $z \longrightarrow f(z, \bar{z})$ and therefore changes a metric to another metric which is not in the same conformal class. As a result, it defines

[6] In this thesis, we are only interested in oriented surfaces.

another complex structure. Such transformations are called *quasi-conformal transformations*.

There is a twist in the story: the theory must be invariant under the action of $\mathrm{Mod}_{\mathfrak{g},\mathfrak{n}}$, *the mapping-class group of the surface*. The reason is that there are two types of coordinate transformations $(x^1, x^2) \to 2(\widetilde{x}^1 = f^1(x^1, x^2), \widetilde{x}^2 = f^2(x^1, x^2))$, 1) the infinitesimal coordinate transformations, i.e. those that can be reached from the identity transformation, and 2) the global or large coordinate transformations that can not be reached by successive infinitesimal transformations from the identity. The second type of coordinate transformations form a group called the mapping-class group of the surface, and its elements are called *mapping-classes*. Any genus-zero surface with at least four marked points and all higher-genus surfaces have non-trivial mapping-class groups. These are residual gauge-redundancies of the theory and must be divide-out in the path integral. From another point of view, the invariance under large diffeomorphism is essential for the unitarity of the theory. In the light-cone gauge, the Teichmüller parameters of the worldsheet are related to the physical light-cone time, and also there is a one-to-one correspondence between the S-matrix poles and a surface with a fixed-set of Teichmüller parameters, i.e. there is a unique set of Teichmüller parameters for each S-matrix pole. Using the fact that in the light-cone gauge, the perturbation theory is manifestly unitary, one deduce that unitarity of the theory demands the invariance under large diffeomorphisms [84]. Using this fact, the integrals of string perturbation theory reduce to a fundamental region of the action of the mapping-class group on the Teichmüller space. This fundamental region is called *the moduli space of Riemann surfaces*. Technically, one has to consider the compactification of moduli space by adding the so-called Riemann surfaces with nodes to the uncompactified moduli space. This region give rises to divergences in string theory [16].

There is a further twist in the story for superstring theories. These theories contain fermions which are fields on the worldsheet. To define these fermions, one need to choose a *spin structure*, i.e. specify the behavior of fermions under parallel transport along non-trivial 1-cycles of the surface. On a genus-$\mathfrak{g}$ surface, there are $2^{2\mathfrak{g}}$ choices of spin structure. The summation over all spin structures is important for at least three reasons

1. Each choice of boundary condition for fermions gives rise to a sector in the Hilbert space. To get the contribution from all sectors, one thus need to sum over all spin structures;

2. One has to impose GSO projection [85]. Some of its significance are 1) it ensures that all of the multiplets are supersymmetric, 2) it removes the tachyon from the

spectrum of the theory. In the path integral formulation of theory, the summation over all possible spin structures imposes GSO projection on the spectrum [86].

3. To integrate over the moduli space, we need to construct a modular-invariant top-degree form on the moduli space $\overline{\mathcal{M}}_{g,n}$. One can construct this form for a specific choice of spin structure. However, a generic element of the mapping-class group of the surface changes the spin structure. Therefore, to get a modular-invariant form, one must some over all possible choices of spin structure. By abuse of notation, we denote the (compactified) moduli space of Riemann surfaces with $\overline{\mathcal{M}}_{g,n}$ where the summation over spin structures is understood. The modular invariance is also important from another point of view. There is no UV divergences in string theory. The reason is as follows [87]. The propagator in open- or closed-string theory can be written as integration over the proper time s, or the so-called *Schwinger proper time*, that strings propagates. The $s \longrightarrow 0$ limit normally corresponds to UV divergences in a quantum field theory. However, since string theory is modular invariant, a worldsheet in which the proper time $s \longrightarrow 0$ is equivalent to a worldsheet in which all the proper times are away from zero. Therefore, modular invariance is the origin of UV-finiteness of string theory. This is intimately related to the fact that the Deligne-Mumford compactification of the moduli space of Riemann surfaces can be constructed by adding divisors containing Riemann surfaces with only simple nodal singularities [16].

4. It can be shown that the divergence due to dilaton tadpole in type-II-superstring theory cancels only after summation over all spin structures [88]. This shows the finiteness of the type-II-superstring perturbation theory.

Putting all the ingredients together, we can write on-shell amplitudes as

$$\boxed{\mathcal{A}_{\mathbf{String}} = \sum_{g=0}^{\infty} \int_{\overline{\mathcal{M}}_{g,n}} \sum_{s=0}^{2^{2g}} \mathcal{A}_{\mathbf{SCFT}}[\mathcal{R}_{g,n;s}],} \tag{2.A.8}$$

where $\mathcal{R}_{g,n;s}$ is a genus-g Riemann surface with n marked points and spin structure s. One can construct $\mathcal{A}_{\mathbf{SCFT}}[\mathcal{R}_{g,n;s}]$ and sum over all spin structures. This gives $\mathcal{A}_{\mathbf{CFT}}[\mathcal{R}_{g,n}]$ that can be integrated over $\overline{\mathcal{M}}_{g,n}$. It can be shown that the string measure (the integrand of (2.A.8)) reproduces the correct measure on the moduli space [84, 81, 82], and therefore, can be integrated over it. This series is thus *the definition of superstring perturbation theory.*

The bosonic-string theory contains Tachyons and also involves non-vanishing massless tadpoles. These properties shows themselves as IR singularities. Due to all these reasons, the definition of S-matrix in the bosonic-string theory is purely formal [89, 16]. We can however do the following procedure to define superstring amplitudes[7]

1) Choose a set of vertex operators associated to external states. These are representatives of the cohomology classes of $\mathcal{Q}$, the BRST operator of the theory. Due to the fact that all BRST-exact states $A = \{\mathcal{Q}, \cdots\}$ decouple from the S-matrix computations, any set of representatives of cohomology classes give the same result [81, 82];

2) Construct $\mathcal{A}_{\mathbf{SCFT}}[\mathcal{R}_{\mathfrak{g},\mathfrak{n};\mathfrak{s}}]$ using the vertex operators. After summing over all spin structures, it becomes a representative of the top-degree de-Rahm cohomology class of the moduli space $\overline{\mathcal{M}}_{\mathfrak{g},\mathfrak{n}}$, where $\mathfrak{n} = \mathfrak{n}_{\text{NS}} + \mathfrak{n}_{\text{R}}$ [81, 82];

3) Compute $\sum_{\mathfrak{s}} \mathcal{A}_{\mathbf{SCFT}}[\mathcal{R}_{\mathfrak{g},\mathfrak{n};\mathfrak{s}}]$ in terms of a set of coordinate system on $\overline{\mathcal{M}}_{\mathfrak{g},\mathfrak{n}}$;

4) Since $\sum_{\mathfrak{s}} \mathcal{A}_{\mathbf{SCFT}}[\mathcal{R}_{\mathfrak{g},\mathfrak{n};\mathfrak{s}}]$ is a top-degree form on the moduli space, and by construction is invariant under the mapping-class group of $\mathcal{R}_{\mathfrak{g},\mathfrak{n}}$, one can integrate it over $\overline{\mathcal{M}}_{\mathfrak{g},\mathfrak{n}}$. The result is the scattering amplitude of the relevant process.

This is the on-shell prescription for computing scattering amplitudes in string theory.

2.B The Need for Off-Shell Amplitudes

In this section, we explain why the prescription given in the previous section is incomplete in general. This leads us to the necessity of defining off-shell amplitudes.

2.B.1 Divergences in String Theory

In the quantum field theories of point-like particles, interactions happens at points in space-time. To do computation, one thus need to integrate over all points in space-time. In particular two interaction points can become arbitrarily close to each other. This is the source of UV divergences in quantum field theory. Of course quantum field theory of point-like particles can involve long distance IR divergences as well. However, a string

[7] For the corresponding procedure using supergeometry formulation of superstring perturbation theory see [90, 91].

is an extended object, and therefore, naturally puts cut-off on the integrals appearing in the computations. This means that *there is no UV divergences in string theory*[8]. The only source of divergence is therefore IR divergences. It is easier to see the dominant contributions in this regime using Nambu-Goto action. In this formulation, the string action is written as the area of the world-sheet

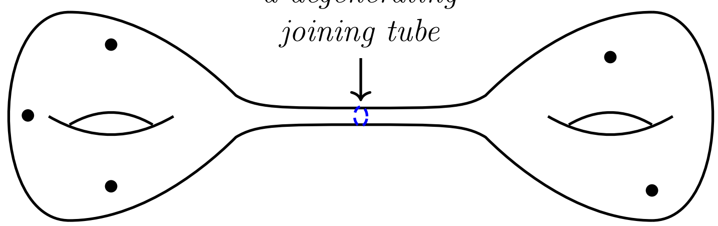
$$S_{\mathbf{NG}} = T \int_{\mathcal{R}} d^2\sigma \, \sqrt{-\gamma(\sigma,\tau)}, \tag{2.B.1}$$

were γ is the pull-back of the metric $G_{\mu\nu}$ on the space-time M, defined by the maps $X^\mu : \mathcal{R} \longrightarrow M$. In long distances, a string stretches across the space-time, and therefore, the dominant contribution according to the above action comes from those worldsheet configurations that minimize the area, i.e. *world-sheets with long tubes or conformally-equivalently tubes with vanishing circumference*. In the complete degeneration, i.e. when the circumference is exactly zero and a point called a *node*[9] formed, the neighborhood of the node is described by two disks joined at a single point, the node. It turns out that the deformation of a degenerating surface near the degeneration locus is totally independent of the rest of the surface. In particular, the nodes are always disjoint, i.e. there is no notion of approaching nodes because it is conformally-equivalent to a case where the nodes are well-separated. This is also clear from The Keen's Collar Lemma which we state in the next chapter. Therefore, adding Riemann surfaces with nodes to $\mathcal{M}_{\mathfrak{g};\mathfrak{n}_{\mathrm{NS}},\mathfrak{n}_{\mathrm{R}}}$ gives a compact space, the compactification of the moduli space of Riemann surfaces, denoted by $\mathcal{M}_{\mathfrak{g};\mathfrak{n}_{\mathrm{NS}},\mathfrak{n}_{\mathrm{R}}}$.

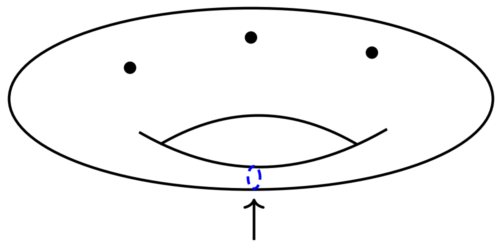
Regarding these facts, there are two types of degenerations

1. *Separating-type degenerations*: Such degenerations happen when a tube connecting two component surfaces, i.e. a cycle which is homologous to zero cycle in the underlying unmarked genus-$\mathfrak{g}$ surface, degenerates. In this case, there are two separate component surfaces joined at a point in the limit of complete degeneration. If the original surface is a genus-$\mathfrak{g}$ surface with $\mathfrak{n}$ marked points[10], the resulting component surfaces are genus-$\mathfrak{g}_i$, with $\mathfrak{g}_1 + \mathfrak{g}_2 = \mathfrak{g}$ surfaces with $\mathfrak{n}_i + 1$ marked points, including the node, where $\mathfrak{n}_1 + \mathfrak{n}_2 = \mathfrak{n} - 2$. The total number of parameters after the complete degeneration is $3\mathfrak{g}_1 - 3 + \mathfrak{n}_1 + 1 + 3\mathfrak{g}_2 - 3 + n_2 + 1 = 3\mathfrak{g} - 3 + 2\mathfrak{n} - 1$. Therefore, adding

[8]More precisely, the removal of UV divergences is related to the modular invariance of the theory.
[9]In the math literature, it is also called an *ordinary double point*.
[10]To avoid cluttering, we avoid to mention the type of the marked points.



(a) A degenerating joining tube. The small circles represent marked points. Such degenerations are called *separating-type degenerations* because after the complete degeneration, the resulting surface consists of two separate subsurfaces joined at a single point.



(b) A degenerating handle. The small circles represent marked points. Such degenerations are called *nonseparating-type degenerations* because after the complete degeneration, the resulting surface remains a single surface.

Figure 2.B.1: The two possible degenerations of a surface.

these type of degenerate surfaces amount to adding a loci of complex-codimention one to $\mathcal{M}_{\mathfrak{g};\mathfrak{n}_{\mathrm{NS}},\mathfrak{n}_{\mathrm{R}}}$. An example of such diagrams is sketched in figure 2.B.1a.

2. *Non-separating-type degenerations*: Such degenerations happen when one of the non-trivial cycles of the underlying genus-$\mathfrak{g}$ surface degenerates. In this case, there is a single surface two of whose points are joined together in the limit of complete degeneration. If the original surface is a genus-$\mathfrak{g}$ surface with $\mathfrak{n}$ marked points, the resulting surface is a genus-$(\mathfrak{g}-1)$ surface with $\mathfrak{n}+2$ marked points. The number of parameters can be counted as follows: $3(\mathfrak{g}-1)-3+\mathfrak{n}+2 = 3\mathfrak{g}-3+\mathfrak{n}-1$. Again, adding these type of degenerate surfaces amount to adding a loci of complex-codimention one to $\mathcal{M}_{\mathfrak{g};\mathfrak{n}_{\mathrm{NS}},\mathfrak{n}_{\mathrm{R}}}$. An example of such diagrams is sketched in figure 2.B.1b.

The fact that in both types of degenerations *the compactification divisor* $\boldsymbol{\mathfrak{D}} \equiv \overline{\mathcal{M}}_{\mathfrak{g},\mathfrak{n}} - \mathcal{M}_{\mathfrak{g};\mathfrak{n}_{\mathrm{NS}},\mathfrak{n}_{\mathrm{R}}}$ is a complex-codimention one locus shows that $\overline{\mathcal{M}}_{\mathfrak{g},\mathfrak{n}}$ *does not have a boundary* *and the surfaces in $\mathfrak{D}$ should be thought of as points at infinity*[11]. The situation is similar to the one-point compactification of the complex plane by adding the point ∞ which is a complex codimension-one submanifold. The resulting manifold, the Riemann sphere, does not have a boundary.

2.B.2 Mass Renormalization

As we have seen above, the conventional formulation of scattering amplitudes in string theory is based on a prescription given by Polyakov [4, 5]. Consider $\mathfrak{n}$ states represented by vertex operators $\mathcal{V}_i(a_i; p_i)$ with momentum p_i and quantum numbers a_i, then the scattering amplitude is schematically can be written as

$$\mathcal{A}(p_1, a_1; \cdots; p_{\mathfrak{n}}, a_{\mathfrak{n}}) = \sum_{\mathfrak{g}=0}^{\infty} \int \mathcal{D}(\text{moduli}) \underbrace{\left\langle (\text{ghost insertions}) \prod_{i=1}^{\mathfrak{n}} \mathcal{V}_i(p_i; a_i) \right\rangle_{\mathcal{R}_{\mathfrak{g}}}}_{\text{a correlation function on } \mathcal{R}_{\mathfrak{g}}} . \tag{2.B.2}$$

In a conformally-invariant formalism vertex operators have conformal dimension $(0, 0)$. Equivalently in the BRST formulation, the vertex operators belong to the cohomology of the BRST operator. In turn, this demands that physical states should satisfy classical mass-shell condition. We therefore conclude that the Polyakov prescription can be used to give correct scattering amplitudes for 1) tree-level scatterings, and 2) BPS and massless states which are protected from renormalization in the perturbation theory.

However, generic states in string theory undergo mass renormalization due to loop corrections [38, 41, 40, 39]. For example, it is shown in [38] that the unitarity of the theory demands the shift in the momentum of vertex operators in higher loops which corresponds to mass renormalization. Similar shift must be taken into account to cancel a BRST anomaly, a contribution comes from points at infinity of moduli space, generated in the computation of on-shell two-point function in the one-loop order [40]. Hence in order to compute physical S-matrix elements, the external momenta should be shifted to their renormalized values. This means that the Polyakov prescription breaks down for computing the scattering amplitudes of massive states beyond tree level. A typical string-theory worldsheet associated to the mass renormalization is given in figure 2.B.2.

To overcome this challenge there are two ways to proceed

[11]In contrast, the moduli spaces of surfaces with boundary do have boundaries.

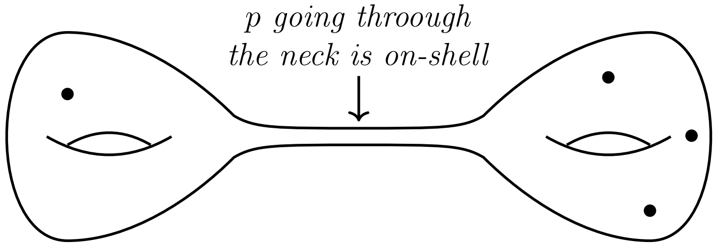


Figure 2.B.2: A typical diagram associated with the mass renormalization in string theory. In such diagrams, there is one of component surfaces of the degenerating surface that there is only one external states on it. In an on-shell formulation, where all of the external states satisfy the tree-level mass condition $p_i^2 = m_{i,0}$, for tree-level masses $m_{i,0}$, the momentum p going through the joining neck is on-shell.

1. Many states can be generated as single-particle intermediate states in some scattering process in which all external states are massless and/or BPS states. One can then find the renormalized masses of such states by examining poles of the S-matrix of such processes. However all massive states are not produced like this. Some examples are as follows [17, 18]
   - Consider the compactification of bosonic-string theory on S^1. An example of a state in this theory is a massive state which has non-zero winding number. Such state cannot be produced in the scattering of massless states which do not carry winding numbers. Pairs of such states can be produced in the intermediate channel. Such states produce a cut in the S-matrix of massless states. One can determine renormalized masses by examining endpoints of the cut. However, this is hard in general.
   - $\mathtt{SO}(32)$ heterotic-string theory contains massive states belonging to the spinor representation of $\mathtt{SO}(32)$ [17]. They cannot appear as single-particle intermediate states in the scattering of massless external states which are all in the adjoint or singlet representation of $\mathtt{SO}(32)$. As such, the renormalized masses of such states cannot be computed by examining the S-matrix of some scattering process of massless and/or BPS states.
2. The alternative option is to compute renormalized masses directly. This requires the computation of off-shell amplitudes in string theory, i.e. scattering amplitudes of states whose momenta are not satisfying the classical mass-shell condition. There are

various subtleties for extracting renormalized masses using the off-shell amplitudes such as finding a proper definition of the analog of the off-shell Green's function in string theory and mixing between physical and unphysical states [17, 18]. We briefly explain some of these below.

Consider a string theory amplitude corresponding to the scattering of $\mathfrak{n}$ external states representing particles carrying momenta $p_1, \cdots, p_{\mathfrak{n}}$ and other discrete quantum numbers $a_1, \cdots, a_{\mathfrak{n}}$ with tree level masses $m_{a_1}, \cdots, m_{a_{\mathfrak{n}}}$. The momenta p_i are required to satisfy the tree level on-shell condition $p_i^2 = -m_{a_i}^2$. The conventional formulation of string theory yields the result for what in a quantum field theory can be called *the truncated Green's function on classical mass shell*:

$$
\begin{aligned}
R^{(n)}(p_1, a_1; \cdots; p_{\mathfrak{n}}, a_{\mathfrak{n}}) &\equiv \lim_{p_i^2 \to -m_{a_i}^2} F^{(\mathfrak{n})}(p_1, a_1; \cdots; p_{\mathfrak{n}}, a_{\mathfrak{n}}), \\
F^{(\mathfrak{n})}(p_1, a_1; \cdots; p_{\mathfrak{n}}, a_{\mathfrak{n}}) &\equiv G^{(\mathfrak{n})}(p_1, a_1; \cdots; p_{\mathfrak{n}}, a_{\mathfrak{n}}) \prod_{i=1}^{\mathfrak{n}} (p_i^2 + m_{a_i}^2).
\end{aligned} \tag{2.B.3}
$$

where $G^{(\mathfrak{n})}(p_1, a_1; \cdots; p_{\mathfrak{n}}, a_{\mathfrak{n}})$ is the momentum space Green's function in the quantum field theory. This expression is significantly different from the S-matrix elements in quantum field theory. To define the S-matrix elements in quantum field theory, we need to first consider the two point function $G_{ab}^{(2)}(p, p')$ for all set of fields whose tree level masses are all equal to m described by the matrix

$$
G_{ab}^{(2)}(p, p') = (2\pi)^{D+1} \delta^{(D+1)}(p + p')\, Z^{1/2}(p)_{ac} (p^2 + M_p^2)^{-1}_{cd} (Z^{1/2}(-p))^T_{db}, \tag{2.B.4}
$$

where M_p^2 is the mass-squared matrix and $Z^{1/2}(k)$ is the wave-function renormalization matrix, the latter being free from poles near $k^2 + m^2 \simeq 0$. The sum over c, d are restricted to states which have the same tree level mass m as the states labelled by the indices a, b. $D + 1$ is the total number of non-compact space-time dimensions. We can diagonalize M_p^2 and absorb the diagonalizing matrices into the wave-function renormalization factor $Z^{1/2}(p)$ to express M_p^2 as a diagonal matrix. These eigenvalues, which we shall denote by $m_{a;\mathbf{p}}^2$, are the squares of the physical masses. Then the S-matrix elements are defined by [17, 18]

$$
S^{(\mathfrak{n})}(p_1, a_1; \cdots p_{\mathfrak{n}}, a_{\mathfrak{n}}) = \lim_{p_i^2 \to -m_{a_i,\mathbf{p}}^2} G^{(\mathfrak{n})}(p_1, b_1; \cdots; p_{\mathfrak{n}}, b_{\mathfrak{n}}) \prod_{i=1}^{\mathfrak{n}} \left[Z_i^{-1/2}(p_i)_{a_i, b_i} (p_i^2 + m_{a_i,\mathbf{p}}^2) \right]. \tag{2.B.5}
$$

$m_{a_i,\mathbf{p}}$ is the physical mass of the i-th particle, defined as the location of the pole as a function of $-p^2$ in the untruncated two-point Green's function $G^{(2)}$, and $Z_i(p_i)_{a_i,b_i}$ is the residue at this pole. At the tree level $Z = 1$, $M_p^2 = m^2\, I$ and hence $R^{(\mathfrak{n})}$ defined in (2.B.3) and $S^{(\mathfrak{n})}$ defined in (2.B.5) agree. In general however $R^{(\mathfrak{n})}$ and $S^{(\mathfrak{n})}$ are different. While $S^{(\mathfrak{n})}$ is the physically-relevant quantity, conventional formulation of string theory directly computes $R^{(\mathfrak{n})}$. This is a serious trouble for states whose masses are renormalized. For example consider the case that the mass of an external state with the quantum number a_i and the tree level mass m_{a_i} is being renormalized due to the loop effects. Then radiative corrections introduce series of $\frac{1}{p_i^2+m_{a_i}^2}$. However, as we have explained above, the Polyakov prescription demands that $p_i^2 + m_{a_i}^2 = 0$. Therefore, the resulting on-shell amplitude will be ill-defined.

Regarding these facts, the mass renormalization in string theory is one of the main motivations to define off-shell amplitudes is bosonic-string and superstring theories.

2.B.3 Dynamical Shift of the Vacuum

Demanding the Weyl-invariance at the quantum level imposes stringent constraints on the dimension and geometry of the background space-time through which an string is propagating. For the bosonic-string theory, the sum of dimension of compact and non-compact dimension must be 26 and that of the superstring theory must be 10. Also the background spacetimes that avoid the Weyl anomaly must satisfy a set of classical equations [83]. Below we shall discuss a situation in which quantum corrections modify the background.

Consider an $\mathcal{N} = 1$ supersymmetric compactification of string theory down to 3+1 dimensions, where we have $\mathtt{U}(1)$ gauge fields with Fayet-Iliopoulos (FI) terms generated at one loop [92, 93, 94, 95]. It is possible to ensure that only one gauge field has FI term by choosing suitable linear combination of these gauge fields. Typically there are also massless scalars ϕ_i charged under this $\mathtt{U}(1)$ gauge field. The FI term generates a term in the potential of the form

$$\frac{1}{g_s^2}\left(\sum_i q_i \phi_i^* \phi_i - C\, g_s^2\right)^2, \tag{2.B.6}$$

where q_i is the charge carried by ϕ_i, C is a numerical constant that determines the coefficient of the FI term and g_s is the string coupling constant. C could be positive or negative and q_is for different fields could have different signs. If we expand the potential in powers of ϕ_i around the perturbative vacuum $\phi_i = 0$, it is clear that some of these

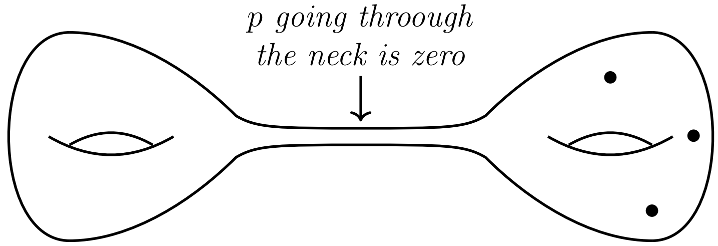


Figure 2.B.3: A typical diagram associated with a massless tadpole. In such diagrams, there is no external state on one of the component surfaces of the degenerating surface. The momentum conservation forces the momentum p going through the joining neck to be zero. When the state propagating through the neck is associated with a massless particle, this is an on-shell condition on p.

scalars can become tachyonic. The form of the effective potential suggests that the correct procedure to compute physical quantities is to shift the corresponding fields so that we have a new vacuum where $\sum_i q_i \langle \phi_i^* \rangle \langle \phi_i \rangle = C\, g_s^2$, and quantize string theory around this new background. However since classically the $C\, g_s^2$ term is absent from the potential (2.B.6), this new vacuum is not a solution to the classical equations of motion. As a result on-shell methods given by the Polyakov prescription is not suitable for carrying out a systematic perturbation expansion around this new background [19]. This is another reason for introducing off-shell methods in superstring theory.

In conventional quantum field theory the 1PI effective action can be used to deal with issues regarding the mass renormalization and vacuum shift. 1PI effective action by definition is the generating function of off-shell 1PI amplitudes, and is free from all infrared divergences associated with mass renormalization or massless tadpoles. On the other hand, infrared divergences associated with internal lines in the loop going on-shell are tamed by the usual $i\varepsilon$ prescription. Given the 1PI effective action, we are supposed to first find its local extremum, and then find plane wave solutions to the classical linearized equations of motion derived from the action. If these occur at momenta p then the values of $-p^2$ give the renormalized squared masses. Once we have determined them, the tree level S-matrix computed from the 1PI action gives us the full renormalized S-matrix of the original quantum field theory. The 1PI approach is indeed can be introduced in string theory to deal with problems of mass renormalization and vacuum shift in the theory. The 1PI effective actions for heterotic and type-II superstring theories in the picture-changing formalism have been constructed recently [21, 22]. These theories were used to deal with issues of mass renormalization and vacuum shift in an important recent paper [23].

Chapter 3

Hyperbolic Geometry and String Theory

In this chapter, we shall explicitly construct off-shell amplitudes in bosonic-string theory using hyperbolic Riemann surfaces following the general construction described in chapter 2. By explicit construction of off-shell amplitudes we mean a gluing-compatible choice of local coordinates and/or distribution of PCOs, and a prescription for integration over the moduli space. In the last section, we explain the applications of hyperbolic geometry to string field theory.

3.1 Sewing of Hyperbolic Surfaces

A Riemann surface is called hyperbolic if it is equipped with a hyperbolic metric, i.e. a metric with constant curvature -1 everywhere on the surface. One important advantage of using hyperbolic metric is that Riemann surfaces with nodes obtained by the plumbing fixture of hyperbolic surfaces are again hyperbolic. This suggests that if we choose local coordinates around the punctures as the one induced from the hyperbolic metric on the surface, at least on the complete degeneration limit, where the plumbing parameter vanishes and a node is developed, this choice will match with local coordinates induced from the component surfaces. This is an essential constraint satisfied by a gluing-compatible section of $\widehat{\mathcal{P}}_{\mathfrak{g},\mathfrak{n}}$. There are two ways to construct hyperbolic metric by gluing two surfaces

1. We choose two cusps[1] and glue them by a gluing relation. As we argue below, the

[1]We use the work cusp to denote a marked point or puncture on a hyperbolic surface. In what follows,

resulting family of surfaces, depending on a number of complex sewing parameters, is not hyperbolic.

2. We cut neighborhood of two cusps such that the resulting boundaries have the same length. Gluing these boundaries generates a hyperbolic surfaces by construction.

In this thesis, we pursue the first approach. As we will argue below, away from the degeneration locus, the hyperbolic metric is not gluing-compatible. We then see that this statement is too naive, and the hyperbolic metric can indeed be made gluing-compatible.

3.1.1 Incompatibility of the Hyperbolic Metric with Sewing

A hyperbolic surface can be represented as a quotient of the upper half-plane $\mathbb{H}$ by a Fuchsian group. A puncture on a hyperbolic surface corresponds to the fixed point of a parabolic element of the Fuchsian group acting on the upper half-plane $\mathbb{H}$. For a puncture $\mathfrak{p}$, there is a natural local coordinate w with $w(\mathfrak{p}) = 0$ and the hyperbolic metric around the puncture is locally given by[2]

$$ds^2 = \left(\left|\frac{dw}{w \ln |w|}\right|\right)^2 . \tag{3.1.1}$$

Let z be the coordinate on upper half-plane. Then, the local coordinate around a puncture is given as

- For a puncture corresponding to a parabolic element whose fixed point is located at infinity, the local coordinate is given by

$$w = \exp\left(2\pi \mathtt{i}\, z\right) . \tag{3.1.2}$$

As required this choice of local coordinate is invariant under the translation, $z \to z+1$, which represents the action of the generator of corresponding parabolic element. In terms of coordinate z, the metric around the puncture takes the form

$$ds^2 = \frac{dz d\bar{z}}{(\mathtt{Im} z)^2}, \tag{3.1.3}$$

which is the Poincaré metric for the upper half-plane $\mathbb{H}$, as it should be.

we use these words interchangeably.

[2]For derivation of this result, see around equation (3.B.6) in appendix 3.B.

- For a puncture corresponding to a parabolic element whose fixed point is located at $x \in \mathbb{R}$, the local coordinate is given by

$$w = \exp\left(-\frac{2\pi\mathrm{i}}{z-x}\right). \tag{3.1.4}$$

In terms of coordinate z, the metric around the puncture takes the form of the Poincaré metric for the upper half-plane $\mathbb{H}$, as it should be.

As it is clear from the form of the local coordinate around the puncture w given by (3.1.1), it is unique modulo a phase factor. This means that it is perfectly compatible with the requirement of local coordinates around marked points in string theory.

Before using the proposed choice of local coordinates for constructing off-shell amplitudes, it is important to ensure that it satisfies the gluing-compatibility requirement. We should first answer the following question: *What are the local coordinates induced around punctures on the family of surfaces obtained via the plumbing of two hyperbolic surfaces?* We thus need to find the metric on this family of surfaces to determine whether it is hyperbolic or not.

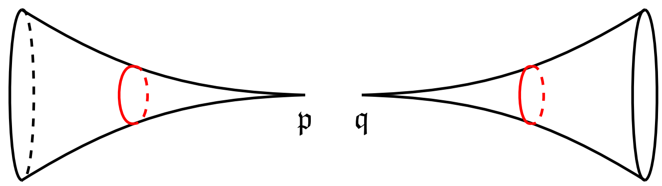
Plumbing of two Hyperbolic Surfaces

In this section, we describe the construction of a family of Riemann surfaces parametrized by a set of complex parameters. A degenerating family of surfaces constructed by considering two cusps, one on each surface, and glue them by a specific relation [96, 97]. We describe in detail this construction since it is essential for later discussion. We discuss the case of genus-$\mathfrak{g}$ surfaces. The construction can be readily generalize to a surface with any number of cusps.

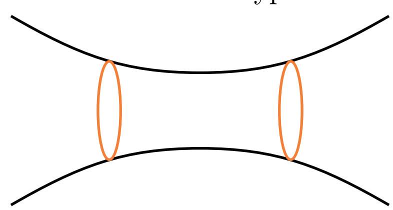
Consider $\overline{\mathcal{M}}_{\mathfrak{g}}$, the Deligne-Mumford compactification of the moduli space of genus-$\mathfrak{g}$ Riemann surfaces [62]. Let us denote the compactification divisor of $\overline{\mathcal{M}}_{\mathfrak{g}}$ by $\boldsymbol{\mathfrak{D}} \equiv \overline{\mathcal{M}}_{\mathfrak{g}} - \mathcal{M}_{\mathfrak{g}}$. A point of $\boldsymbol{\mathfrak{D}}$ represents a surface $\mathcal{R}$ with nodes. By definition, the neighborhood of a node $\mathfrak{n}$ of $\mathcal{R}$ is isomorphic to

$$U \equiv \left\{w^{(1)}w^{(2)} = 0 |\ |w^{(1)}|, |w^{(2)}| < \epsilon\right\}, \tag{3.1.5}$$

where $w^{(1)}$ and $w^{(2)}$ are local coordinates around the two sides of $\mathfrak{n}$. To move away from the compactification divisor, i.e. to open the node, let us consider the following family of



(a) The plumbing-fixture identification of two cusps on two disconnected hyperbolic surfaces.



(b) The resulting tube connecting the two disconnected surfaces.

Figure 3.1: The plumbing of two hyperbolic cusps $\mathfrak{p}$ and $\mathfrak{q}$.

surfaces fibered over a disk with complex coordinate t for some $\epsilon \ll 1$

$$\left\{ w^{(1)} w^{(2)} = t | \; |w^{(1)}|, |w^{(2)}| < \epsilon, |t| < \epsilon \right\} . \tag{3.1.6}$$

We can identify U with the fiber at $t = 0$. A deformation of $\mathcal{R} \in \overline{\mathfrak{M}}_{\mathfrak{g}}$ which opens the node is given by varying the parameter t. Consider a surface $\mathcal{R}_0 \in \boldsymbol{\mathfrak{D}} \subset \overline{\mathfrak{M}}_g$ with m nodes denoted by $\mathfrak{n}_1, \cdots, \mathfrak{n}_m$. For the node $\mathfrak{n}_i$, the punctures $\mathfrak{p}_i$ and $\mathfrak{q}_i$ of $\mathcal{R}_0 - \{\mathfrak{n}_1, \cdots, \mathfrak{n}_m\}$ are paired. Let

$$U_i^{(1)} = \left\{ |w_i^{(1)}| < 1 \right\}, \qquad U_i^{(2)} = \left\{ |w_i^{(2)}| < 1 \right\}, \qquad i = 1, \cdots, m, \tag{3.1.7}$$

be disjoint neighborhoods of punctures $\mathfrak{p}_i$ and $\mathfrak{q}_i$, respectively. Here, $w_i^{(1)}$ and $w_i^{(2)}$ with $w_i^{(1)}(\mathfrak{p}_i) = 0$ and $w_i^{(2)}(\mathfrak{q}_i) = 0$ are local coordinates around the two sides of the node $\mathfrak{n}_i$. Consider an open set $\mathcal{V} \subset \mathcal{R}_0$ disjoint from the set $U_i^{(1)}, U_i^{(2)}$ which support the Beltrami differentials $\{\mu_a\}$. Beltrami differentials span the tangent space of the Teichmüller space of $\mathcal{R}_0 - \{\mathfrak{n}_1, \cdots, \mathfrak{n}_m\}$. The dimension of this space is $3\mathfrak{g} - 3 - m$. Given

$$\boldsymbol{s} \equiv (s_1, \cdots, s_{3\mathfrak{g}-3-m}) \in \mathbb{C}^{3\mathfrak{g}-3-m}, \tag{3.1.8}$$

for a neighborhood of the origin, the sum $\mu(\boldsymbol{s}) = \sum_j s_j \mu_j$ is a solution $\omega^{\mu(s)}$ of the Beltrami

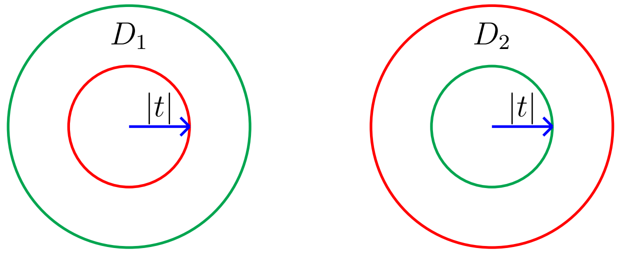


Figure 3.2: Two annuli with inner radius $|t|$ and outer radius 1 obtained by removing a disc of radius $|t|$ from D_1 and D_2, where t is a complex parameter. The resulting annuli are glued by identifying $w^{(1)}$ with $\frac{t}{w^{(2)}}$.

equation. Assume that the surface $\omega^{\mu(s)}(\mathcal{R}_0) = \mathcal{R}_s$ is a quasiconformal deformation of $\mathcal{R}_0$. Then, we shall parametrize the opening of nodes as follows. The map $\omega^{\mu(s)}$ is conformal on U_i^1 and U_i^2 and therefore $w_i^{(1)}$ and $w_i^{(2)}$ serve as local coordinates for $\omega^{\mu(s)}(U_i^1)$, $\omega^{\mu(s)}(U_i^2) \subset \omega^{\mu(s)}(\mathcal{R}_0)$. Given

$$\boldsymbol{t} \equiv (t_1, \cdots, t_m) \in \mathbb{C}^m, \qquad |t_i| < 1. \tag{3.1.9}$$

We construct the family of surface $\mathcal{R}_{\boldsymbol{t},\boldsymbol{s}}$, parametrized by $\boldsymbol{s}$ and $\boldsymbol{t}$ as follows. We first remove the discs $\{0 \leq |w_i^{(1)}| \leq |t_i|\}$ and $\{0 \leq |w_i^{(2)}| \leq |t_i|\}$ from $\mathcal{R}_{\boldsymbol{s}}$, and then attach $\{|t_i| < |w_i^{(1)}| < 1\}$ to $\{|t_i| < |w_i^{(2)}| < 1\}$ by identifying $w_i^{(1)}$ and $\frac{t_i}{w_i^{(2)}}$. This provides a local coordinate near $\boldsymbol{\mathfrak{D}}$ [96]. This construction has been illustrated in figure 3.2.

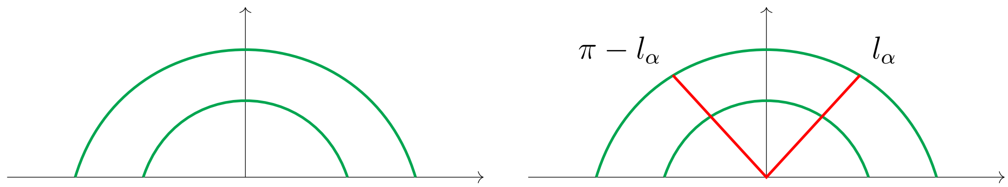
We now turn to the plumbing of two hyperbolic surfaces. For a geodesic α on the hyperbolic surface $\mathcal{R}$ of length l_α, a neighborhood with area $2l_\alpha \cot \frac{l_\alpha}{2}$ is called the collar around α. The standard collar around the geodesic α is the collection of points p whose hyperbolic distance from α is less than $w(\alpha)$ given by [98, 97]

$$\sinh w(\alpha) \sinh \frac{l_\alpha}{2} = 1. \tag{3.1.10}$$

This follows from the Keen's Collar Lemma that we will state below. The collar can be described as a quotient of the upper half-plane $\mathbb{H}$. To describe the resulting space, we consider the transformation $z \to e^{l_\alpha} z$ which is represented by the following matrix in $\mathtt{PSL}(2, \mathbb{R})$

$$M = \begin{pmatrix} \exp\left(\frac{l_\alpha}{2}\right) & 0 \\ 0 & \exp\left(-\frac{l_\alpha}{2}\right) \end{pmatrix}. \tag{3.1.11}$$

It is clear from this form that it generates a cyclic subgroup of $\mathtt{PSL}(2, \mathbb{R})$. We shall denote



(a) The fundamental domain of a the action of subgroup of $\mathtt{PSL}(2,\mathbb{R})$ generated by M given in (3.1.11). The result is the hyperbolic annulus.

(b) The fundamental domain for generating a standar collar is the region bounded by the strip and the wedge. The group which acting is still Γ_α.

Figure 3.3: The construction of the hyperbolic annulus and the standard collar.

this subgroup by Γ_α. The fundamental domain is given by a strip in $\mathbb{H}$ and it is shown in figure 3.3. We can then construct

- **The hyperbolic annulus:** If we quotient $\mathbb{H}$ with $z \to e^{l_\alpha} z$ relation, we identify the two sides of the strip. This fundamental domain is shown in figure 3.3a. This gives a cylinder which is topologically an annulus and it has an induced hyperbolic structure from $\mathbb{H}$. The simple closed geodesic of this annulus has hyperbolic length l_α.
- **The standard collar:** By definition, the standard collar is given by the Γ_α-quotient of the region bounded by the wedge $\{l_\alpha < \arg z < \pi - l_\alpha\}$ and the strip. The fundamental domain is shown in figure 3.3b.

The degeneration of an annulus can be described as follows. Consider two cusps $\mathfrak{c}_1$ and $\mathfrak{c}_2$ whose local coordinates are denoted by w_1 and w_2. These local coordinates define two disks D_1 and D_2. Consider the plumbing-fixture locus

$$\mathcal{F}_t = \{w_1 w_2 = t \mid |w_1|, |w_2|, |t| < 1\}\,. \tag{3.1.12}$$

It is a complex manifold fibered over the disk $D = \{|t| < 1\}$. One can describe the hyperbolic metric on the resulting family of annuli explicitly. There are two different situation

- **the $t = 0$ fiber:** The fiber is singular. D_1 and D_2 are joined in a single point. In terms of local coordinates w_1 and w_2 around the two cusps $\mathfrak{c}_1$ and $\mathfrak{c}_2$, each of the punctures disks $D_{\bullet 1}$ and $D_{\bullet 2}$

$$ds_0^2 = \left(\frac{|dw_i|}{|w_i| \ln |w_i|}\right)^2, \qquad i = 1, 2. \tag{3.1.13}$$

- $t \neq 0$ **fibers:** These are annuli. We can equip them with the hyperbolic metric,

$$ds_t^2 = \left(\frac{\pi}{\ln|t|}\csc\left(\pi\frac{\ln|w_i|}{\ln|t|}\right)\left|\frac{dw_i}{w_i}\right|\right)^2. \tag{3.1.14}$$

  For derivation of this result see around equation (3.B.9) in appendix 3.B. For small $|t|$, we have the following expansion of the hyperbolic metric on the punctured disc

$$ds_t^2 = \left(1 + \frac{1}{3}\Theta^2 + \frac{1}{15}\Theta^4 + \cdots\right) ds_0^2, \tag{3.1.15}$$

  where $\Theta \equiv \pi\frac{\ln|w_i|}{\ln|t|}$.

The length of the core geodesic is given by [96]

$$l(t) = -\frac{2\pi^2}{\ln|t|}. \tag{3.1.16}$$

It is known that there exists a positive constant c_* such that if the length l of a geodesic γ on a hyperbolic surface $\mathcal{R}$ is less than or equal to c_*, then the standard collar embeds about γ [99, 100, 96]. This constant c_* is known as the *collar constant*. We shall call a geodesic whose length is at most c_* a *short geodesics*. As it is clear from (3.1.16), this length is under control. Therefore, whenever the length of a simple geodesic along which the cut-and-paste construction can be done becomes less than the collar constant, a wide collar is formed around this simple closed geodesic. The width of this collar is given by the Keen's Collar Lemma [98]

Lemma 3.1 (Keen's Collar Lemma)**.** *Around a simple closed geodesic γ on a hyperbolic surface $\mathcal{R}$, there is always an embedded hyperbolic cylinder, called a collar, of width*

$$w(\gamma) = \operatorname{arcsinh}\left(\frac{1}{\sinh\left(\frac{l_\gamma}{2}\right)}\right). \tag{3.1.17}$$

Furthermore, assuming that cutting along simple closed curves $\{\gamma_1, \cdots, \gamma_{3\mathfrak{g}-3+\mathfrak{n}}\}$ produces a pairs of pants decomposition of $\mathcal{R}$, the collars around γ_as are all disjoint.

The second part of the lemma has an important consequence: *The fundamental result on the degeneration of 2d hyperbolic metrics is that the deformation localizes into collar* *neighborhoods about short geodesics, i.e. geodesics γ with length $l_\gamma \leq c_*$* [96]. This means that the geometry of collar is completely determined by the length of its core geodesics, and also it is independent of the rest of surface.

The disc $\{|t| < 1\}$ can be thought of as the moduli space for the hyperbolic annulus. The Weil-Petersson (WP) metric on it is given by [96]

$$ds^2_{\mathbf{WP}} = -\frac{2\pi^3}{|t|^2(\ln|t|)^3}|dt|^2. \tag{3.1.18}$$

This formula can be put into a form that is known as The Wolpert's Magic Formula [101]. We can identify Fenchel-Nielsen coordinates (ℓ, τ) for the moduli space as follows:

$$\ell \equiv -\frac{2\pi^2}{\ln|t|}, \qquad \frac{2\pi\tau}{\ell} \equiv \arg t = \mathtt{Im}(\ln t). \tag{3.1.19}$$

Using these relation, (3.1.18) can be written as $d\ell \wedge d\tau$.

This construction can be generalized to the case of a hyperbolic surface with m disjoint short geodesics. The collar neighborhood of each of the short geodesics can be interpreted as a plumbing collar. Given an $\epsilon > 0$, we have the following estimate for the length l_i of the simple closed geodesic of the t_i^{th} annulus

$$\left|\frac{2\pi^2}{l_i} - \ln\frac{1}{|t_i|}\right| < \epsilon, \quad i = 1, \cdots, m. \tag{3.1.20}$$

Therefore, we see that *the degeneration of a hyperbolic metric is associated to the formation of wide collars about short geodesics.*

Let us summarize what we have explained so far. Based on this, we make a conclusion that is important. For simplicity, we consider the case where a single degeneration happens, and denote the resulting surface by $\mathcal{R}_0$. The generalization to more degenerations are straightforward. Deformations that open the node generates a family of surfaces parametrized by a single complex parameter t, and we denote the family by $\mathcal{R}_t$. We consider two disconnected surfaces $\mathcal{R}_1$ and $\mathcal{R}_2$, each of which is equipped with the hyperbolic metric. We consider cusps $\mathfrak{c}_1$ and $\mathfrak{c}_2$ with local coordinates w_1 and w_2 and glue them via the plumbing relation

$$w_1 w_2 = t, \qquad 0 \leq |t| \leq 1. \tag{3.1.21}$$

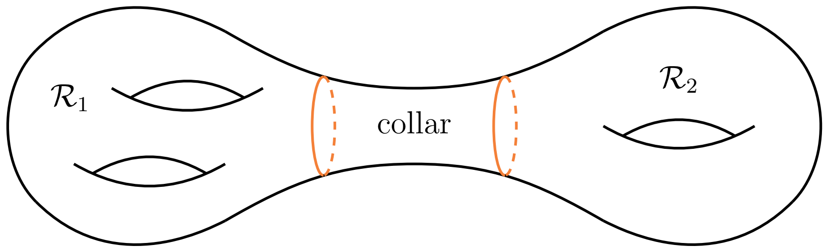


Figure 3.4: $\mathcal{R}_t$ for a particular value of t. For simplicity in the illustration, we have assumed that surfaces $\mathcal{R}_1$ and $\mathcal{R}_2$ do not have any other punctures.

$\mathcal{R}_t$ for each t is the following combination

$$\mathcal{R}_t = (\mathcal{R}_1 - D_1)\#(\text{collar})\#(\mathcal{R}_2 - D_2), \qquad \forall t, \tag{3.1.22}$$

where D_i are small disks around cusps, and # denotes the connected sum of components. For a particular value of t, $\mathcal{R}_t$ is illustrated in figure 3.4. We can then take the following metric on $\mathcal{R}_t$

- The metric for $t = 0$ fiber is given by (3.1.13).
- The metric for $t \neq 0$ fiber is given by (3.1.14).
- Away from the collar, the metric is taken to be the hyperbolic metrics on $\mathcal{R}_1$ and $\mathcal{R}_2$.
- At the collar boundaries, we consider an interpolation between the above metrics. We describe an appropriate interpolation in the next section.

These choices will give rise to a *grafted metric* $ds^2_{\mathbf{g}}$ on $\mathcal{R}_t$. The question is the following: *is the grafted metric hyperbolic?* To find the answer to this question, we need to compute $\mathbf{C_g}$, the curvature of grafted metric. It turns out that [73]

$$\mathbf{C_g} = -1 - \mathcal{O}\left((\ln|t|)^{-2}\right). \tag{3.1.23}$$

This result shows that except at $t = 0$, where $\mathbf{C_g} = -1$, for all $t \neq 0$ the grafted metric is not hyperbolic. Since the metric is hyperbolic away from boundaries of the collar, the only deviation can happen at the collar boundaries where the interpolation leads to a deviation from the hyperbolicity. We can thus state the two main conclusions of this section as follows

1. Sewing of two surfaces equipped with the hyperbolic metric does not give rise to a family of surfaces whose metrics are not hyperbolic except at $t = 0$. This proves that the hyperbolic metric is not compatible with sewing. This result generalizes when there are more sewing between two surfaces or there are sewing of more surfaces.

2. To construct off-shell amplitudes in bosonic-string theory, we need to choose a gluing-compatible choice of local coordinates around the cusps, i.e. the choice of local coordinates must be compatible with gluing [17, 18]. Since the hyperbolic metric is not compatible with the gluing, as it is clear from (3.1.23), we conclude that we cannot use the hyperbolic metric to construct off-shell amplitudes in bosonic-string or superstring theories.

3.1.2 The Hyperbolic Metric on the Family of Sewn Surfaces

In this section, we explain that the two conclusions of the last section were too naive. We indeed find that

1. Although the sewing of two hyperbolic surfaces does not automatically give rises to a hyperbolic surface, we can correct the metric systematically such that it becomes hyperbolic.

2. Since we can make the metric on the family of sewn surfaces hyperbolic, we can indeed use the hyperbolic metric to construct off-shell amplitudes in bosonic-string or superstring theories.

It is indeed possible to construct a hyperbolic metric on the family of sewn surfaces. The main result that helps us to find the hyperbolic metric is found in [73]. It is possible to find an expansion of the hyperbolic metric $ds^2_{\mathbf{h}}$ on $\mathcal{R}_t$ in terms of the grafted metric by doing a rescaling

$$ds^2_{\mathbf{h}} = \Sigma(t) ds^2_{\mathbf{g}}, \tag{3.1.24}$$

where $\Sigma(t)$ is a scaling factor given by [73]

$$\Sigma(t) \equiv 1 - \frac{1}{2}\left(\frac{\pi}{\ln|t|}\right)^2 (\mathcal{D}_{\mathbf{h}} - 2)^{-1}(\Lambda(w_1) + \Lambda(w_2)) + \mathcal{O}\left(\ln|t|^{-4}\right), \tag{3.1.25}$$

where $\mathcal{D}_{\mathbf{h}}$ is the Laplace-Beltrami operator computed in $ds^2_{\mathbf{h}}$ metric, and

$$\Lambda(w_i) \equiv \frac{\partial}{\partial a}\left(a^4 \frac{\partial \eta}{\partial a}\right), \qquad a \equiv \ln|w_i|, \quad i = 1, 2. \tag{3.1.26}$$

and η is a unit step function with step at the collar boundary. The existence of hyperbolic metric in (3.1.24) is guaranteed by The Uniformization Theorem. The main tool to deal with the problem is **The Curvature-Correction Equation** [70, 71, 72, 73]. Consider a compact Riemann surface with metric ds^2 with curvature $\mathbf{C}$. Then $\exp(2f)ds^2$ is a conformally-equivalent metric with curvature $\mathbf{C}'$. f satisfies the following equation known as the curvature-correction equation

$$\mathcal{D}f - \mathbf{C} = -\exp(2f)\mathbf{C}', \tag{3.1.27}$$

where $\mathcal{D}$ is the Laplace-Beltrami operator computed using the ds^2 metric. For curvature functions $\mathbf{C} < 0$ and $\mathbf{C}' < 0$, (3.1.27) has a unique solution [70, 72].

We are interested in the case that $\mathbf{C}' \equiv -1$ and $\mathbf{C} \equiv \mathbf{C}_{\mathbf{g}}$, where $\mathbf{C}_{\mathbf{g}}$ is the curvature of the grafted metric given by (3.1.23). Then, the curvature-correction equation can be written as

$$\mathcal{D}_{\mathbf{g}}f - \exp(2f) = \mathbf{C}_{\mathbf{g}}. \tag{3.1.28}$$

To proceed, we need to specify the grafted metric precisely. We consider a surface $\mathcal{R}_\bullet$ with a single node $\mathfrak{n}$. The generalization for an arbitrary surface with more than one node is straightforward and can be found in [69]. Assume b_* is a positive constant less than one. For $|t| < b_*^4$, we remove from $\widehat{\mathcal{R}} \equiv \mathcal{R}_\bullet - \{\mathfrak{n}\}$, a disconnected surface with a pair of punctures $\mathfrak{p}_1$ and $\mathfrak{p}_2$, the punctured disks $\{0 < |w^{(1)}| \leq |t|/b_*\}$ about $\mathfrak{p}_1$ and $\{0 < |w^{(2)}| \leq |t|/b_*\}$ about $\mathfrak{p}_2$ to obtain a surface $\mathcal{R}_{t/b_*}$. For $t \neq 0$, we can form an identification space $\mathcal{R}_t$, the family of sewn surfaces, by identifying the annulus $\{|t|/b_* < |w^{(1)}| < b_*\} \subset \mathcal{R}_{t/b_*}$ with the annulus $\{|t|/b_* < |w^{(2)}| < b_*\} \subset \mathcal{R}_{t/b_*}$ by the rule $w^{(1)}w^{(2)} = t$. To get the grafted metric, we should combine two different metrics

1. The metric ds^2 on $\mathcal{R}$ which defines the metric away from the plumbing collar.

2. To define the other metric, we first define:

$$\mathcal{F}_t \equiv \{(z, w, t)| \ zw = t, \quad |z|, |w|, |t| < 1\}. \tag{3.1.29}$$

The metric of this family is denoted by ds^2_t. The metric on the family of surfaces $\mathcal{F}_t$ is given by (3.1.13) for $t = 0$, and by (3.1.14) for $t \neq 0$. This metric can be restricted

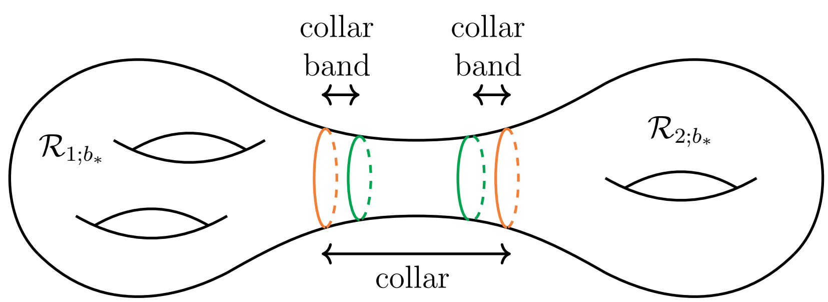


Figure 3.5: The construction of the grafted metric on the family of sewn surfaces $\mathcal{R}_t$.

to the following plumbing family considered above

$$\widetilde{\mathcal{F}}_{b_*} = \left\{(w^{(1)}, w^{(2)}, t)|\ w^{(1)}w^{(2)} = t,\ |t| < b_*^4\right\}, \tag{3.1.30}$$

The second metric that is used to define the grafted metric is ds_t^2 restricted to $\widetilde{\mathcal{F}}_{b_*}$.

We also need to define an interpolating metric around the annular regions $|w^{(1)}| = b_*$ and $|w^{(2)}| = b_*$. We first define a function η by

$$\eta(a) \equiv \begin{cases} 1, & a \le a_0 < 0, \\ 0, & a \ge 0. \end{cases} \tag{3.1.31}$$

We further restrict t to satisfy $e^{2a_0}b_*^2 \ge |t|$. $\{e^{a_0}b_* \le |w^{(1)}| \le b_*\}$ and $\{e^{a_0}b_* \le |w^{(2)}| \le b_*\}$ are called the collar band and are included in the collar. This region is illustrated in figure 3.5. We denote local coordinates around cusps of $\mathcal{R}_{t/b_*}$ and the fiber coordinate of the $\widetilde{\mathcal{F}}_{b_*}$ by ζ, the smooth grafted metric is then defined as follows

- We remove the disks $D_{i,b_*} \equiv \{0 \le |w^{(1)}| \le b_*\}$ from $\widehat{\mathcal{R}}$. The result is two disconnected surfaces $\mathcal{R}_{1;b_*}$ and $\mathcal{R}_{2;b_*}$. By abuse of notation, we use the restriction of metrics of $\widehat{\mathcal{R}}$ to $\mathcal{R}_{1;b_*}$ and $\mathcal{R}_{2;b_*}$ by ds^2. In this region and away from the collar, we take the grafted metric to be $ds^2_{\mathbf{g}} = ds^2$.
- In the region complement to the collar band inside the collar, the grafted metric is taken to be $ds^2_{\mathbf{g}} = ds_t^2$.
- In the collar band, the grafted metric is taken to be $ds^2_{\mathbf{g}} = ds^2_{\boldsymbol{i}}$, the interpolation

metric, which is given by

$$ds^2_{\boldsymbol{i}} \equiv (ds^2)^{1-\eta}(ds^2_t)^{\eta}, \qquad \eta \equiv \eta\left(\ln\left(\frac{|w^{(i)}|}{b_*}\right)\right), \quad i = 1, 2. \tag{3.1.32}$$

$ds^2_{\boldsymbol{i}}$ matches with the metric in the corresponding region. This is easy to check. At $|\xi| = b_*$, $\eta = \eta(0) = 0$, and $ds^2_{\boldsymbol{i}} = ds^2$. At $|\xi| = e^{a_0}b_*$, $\eta = \eta(a_0) = 1$, and $ds^2_{\boldsymbol{i}} = ds^2_t$. Therefore, the interpolation metric is an interpolation between the metric on $\widehat{\mathcal{R}} - D_{1,b_*} - D_{2,b_*}$ and the metric on the collar.

We now equipped with the required tools to construct the hyperbolic metric on $\mathcal{R}_t$ in terms of the grafted metric $ds^2_{\mathbf{g}}$. A prominent role is played by the Eisenstein series $E(z,s)$ for $s = 2$. The Eisenstein series can be defined as follows. Consider a surface uniformized by a Fuchsian group Γ. We consider the following subgroup of Γ

$$\Gamma_\infty \equiv \left\{\pm\begin{pmatrix}1 & n\\ 0 & 1\end{pmatrix} : n \in \mathbb{Z}\right\} = \left\langle\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\right\rangle. \tag{3.1.33}$$

This last notation means that it is a cyclic group generated by the transformation $z \to z+1$ for $z \in \mathbb{H}$. This subgroup is the stabilizer of a cusp at infinity. $E(z;s)$ associated to the cusp at infinity, is defined by the following Eisenstein-Maass series:

$$E(z;s) \equiv \sum_{\gamma\in\Gamma/\Gamma_\infty} (\mathtt{Im}(\gamma\cdot z))^s, \tag{3.1.34}$$

where z is the coordinate on $\mathbb{H}$ and

$$\gamma = \begin{pmatrix}a & b\\ c & d\end{pmatrix}, \quad \longrightarrow \quad \gamma\cdot z \equiv \frac{az+b}{cz+d}. \tag{3.1.35}$$

This series converges absolutely for all of the values of s with $\mathtt{Re}(s) > 1$ and uniformly in z on compact subsets of $\mathbb{H}$ [102]. One can show that $E(z,s)$ can be extended onto the entire complex s-plane by analytic continuation [102]. The case we are interested in corresponds to $s = 2$

$$E(z;2) = \sum_{\gamma\in\Gamma/\Gamma_\infty} (\mathtt{Im}(\gamma\cdot z))^2. \tag{3.1.36}$$

The function $(\mathrm{Im}z)^2$ on $\mathbb{H}$ is an eigenfunction of the hyperbolic Laplacian with eigenvalue 2. This formula shows that there is an Eisenstein series associated to each cusp. For a cusp represented by (3.1.36), the quotient space $\{\mathtt{Im}(z) > 1\}/\Gamma_\infty$ embeds in $\mathbb{H}/\Gamma$ and determines a cusp region. Cusp regions for distinct cusp are disjoint [69]. The Eisenstein series, transformed for a cusp represented at infinity, has the expansion:

$$E(z;2) = \mathtt{Im}(z)^2 + \widehat{e}(z), \tag{3.1.37}$$

where $\widehat{e}(z)$ bounded as $\mathcal{O}((\mathtt{Im}(z))^{-1})$ for large values of $\mathtt{Im}(z)$ [69].

A special truncation of the Eisenstein series is specified by the parameters t and b_* and the function η. The special truncation $E^{\#}$ of the Eisenstein series is given by a modification in the cusp regions. To modify the cusp region, we divide the cusp region into two parts and define a modification $E^{\#}$ of the $E(z;2)$. These two regions and a suitable definition of the modification $E^{\#}$ in each region are specified as follows [69]

- For the cusp $\mathfrak{c}$ defining the series where $E(z;2) = (\mathrm{Im} z)^2 + \hat{e}(z)$, for $\mathtt{Im}(z) > 1$ and $\chi \equiv 1 - \eta$, we define

$$E^{\#}(z;2) \equiv \chi(-2\pi\,\mathtt{Im}(z) - \ln b_*)(\mathtt{Im}(z))^2 + \chi\left(-2\pi\,\mathtt{Im}(z) + \ln\left(\frac{b_*}{|t|}\right) + a_0\right)\widehat{e}(z). \tag{3.1.38}$$

- For a remaining cusp represented at infinity and for $\mathfrak{Im}(z) > 1$, we define

$$E^{\#}(z;2) = \chi\left(-2\pi\,\mathtt{Im}(z) + \ln\left(\frac{b_*}{|t|}\right) + a_0\right)E(z;2). \tag{3.1.39}$$

- For the components of the surface not containing $\mathfrak{c}$, $E^{\#}$ is taken to be zero.

For cusp coordinates $w^{(1)}, w^{(2)}$ of $\mathcal{R}$, the punctured discs $\{0 < |w^{(1)}| \leq |t|/b_*\}$, $\{0 < |w^{(2)}| \leq |t|/b_*\}$ are removed and the annuli $\{|t|/b_* < |w^{(1)}| \leq b_*\}$, $\{|t|/b_* < |w^{(2)}| \leq b_*\}$ are identified by the rule $w^{(1)}w^{(2)} = t$ to form a collar. For $w^{(1)} = e^{2\pi \mathrm{i} z}$, $z \in \mathbb{H}$, the identified annulus is covered by $\{\ln(|t|/b_*) < -2\pi\mathtt{Im}(z) < \ln(b_*)\}$. This is easy to see

$$\left.\begin{aligned}
&\ln(w^{(1)}) = 2\pi i z = 2\pi(-\mathtt{Im}(z) + \mathrm{i}\,\mathtt{Re}(z)),\\
&\ln(w^{(1)}) = \ln(|w^{(1)}|) + 2\pi\mathrm{i}\arg(w^{(1)}),\\
&\ln(|t|/b_*) < \ln(w^{(1)}) \leq \ln b_*,
\end{aligned}\right\} \implies \ln(|t|/b_*) < -2\pi\mathtt{Im}(z) \leq \ln(b_*). \tag{3.1.40}$$

We also have:

- *The primary collar band* $\{b_* e^{a_0} \leq |w^{(1)}| \leq \ln b_*\}$ is covered by the strip $\{\ln b_* + a_0 \leq -2\pi \mathtt{Im}(z) \leq \ln b_*\}$.
- *The secondary collar band* $\{|t|/b_* \leq |w^{(1)}| \leq |t|e^{-a_0}/b_*\}$ is covered by the strip $\ln(|t|/b_*) \leq -2\pi \mathfrak{Im}(z) \leq \ln(|t|/b_*) - a_0\}$

The extended $E^{\#}$ has support in the $w^{(1)}, w^{(2)}$ cusp regions contained in $\{|w^{(1)}| \geq |t|/b_*\} \cup \{|w^{(2)}| \geq |t|/b_*\}$. To define the expansion of the hyperbolic metric in terms of the grafted metric, we need a counterpart interpolation for the Eisenstein series. It is called the melding $E^{\dagger}$ on $\mathcal{R}_t$ of the Eisenstein series. It is a smooth function associated to any pair of cusps sewn to form a collar. The melding is defined as follows

- On the overlap $\{|t|/b_* < |w^{(1)}| < b_*\} \cap \{|t|/b_* < |w^{(2)}| < b_*\}$, it is given by the sum of $E^{\#}$ for pairs of cusp regions

$$E^{\dagger}\left(w^{(1)}\right) \equiv E^{\#}\left(w^{(1)}; 2\right) + E^{\#}\left(w^{(2)}; 2\right) = E^{\#}\left(w^{(1)}; 2\right) + E^{\#}\left(\frac{w^{(1)}}{t}; 2\right). \quad (3.1.41)$$

- On the complement of the identified annuli $\{|t|/b_* < |w^{(1)}| < b_*\}$ and $\{|t|/b_* < |w^{(2)}| < b_*\}$, it is given by the value of $E^{\#}$.

As we mentioned earlier, $(\mathtt{Im}(z))^2$ is the eigenfunction of the hyperbolic Laplacian with eigenvalue 2. The contribution of the truncation of the Eisenstein series to the hyperbolic metric can be determined by analysing the quantity $(\mathcal{D}_{\mathbf{g}} - 2)E^{\dagger}$ on the collar, where $\mathcal{D}_{\mathbf{g}}$ is the Laplacian computed using the grafted metric. In the complement of the collar, the grafted metric is the hyperbolic metric and $E^{\dagger} = E$ and $(\mathcal{D}_{\mathbf{g}} - 2)E^{\dagger}$ becomes $(\mathcal{D} - 2)E$ for $\mathcal{D} \equiv y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$. By the definition of E given in (3.1.36), this quantity is zero. However, This quantity is non-zero on the collar and can be used to determine the contribution to the hyperbolic metric on the collar from the grafted metric defined on the collar band. It can be shown that [73, 69]:

$$(\mathcal{D}_{\mathbf{g}} - 2)E^{\dagger}(\zeta) = -\frac{1}{4\pi}\Lambda + \mathcal{O}\left((-\ln|t|)^{-1}\right), \quad (3.1.42)$$

and Λ is given by (3.1.26). One can use this result to find an expansion for the hyperbolic metric on $\mathcal{R}_t$ in terms of $ds^2_{\mathbf{g}}$. We state the result for a surface formed by a pair of punctures. The result can be generalized to the case where the surface constructed by sewing m pairs of punctures. Assume that $|t| < b_*^4$ is the plumbing-fixture parameter. To find expansion of the hyperbolic metric on $\mathcal{R}_t$, one should use the curvature-correction equation (3.1.28). This approach asserts that the hyperbolic metric $ds^2_{\mathbf{h}}$ on the degenerate surface $\mathcal{R}_t$ has the following expansion [73, 69]:

$$ds^2_{\mathbf{h}} = \left[1 + 2(\mathcal{D}_{\mathbf{g}} - 2)^{-1}(1 + \mathbf{C}_{\mathbf{g}}) + \mathcal{O}\left(\|1 + \mathbf{C}_{\text{graft}}\|\right)^2\right] ds^2_{\mathbf{g}}. \tag{3.1.43}$$

In which $\|\cdot\|$ is an appropriate norm and $\mathbf{C}_{\mathbf{g}}$ is the curvature of the grafted metric and is given by [73]

$$\mathbf{C}_{\mathbf{g}} = -1 - \frac{\epsilon^2}{6}\Lambda + \mathcal{O}(\epsilon^4), \qquad \epsilon \equiv \frac{\pi}{\ln|t|}. \tag{3.1.44}$$

Using this expansion, the hyperbolic metric $ds^2_{\mathbf{h}}$ on $\mathcal{R}_t$ in terms of the grafted metric $ds^2_{\mathbf{g}}$ is given by the following expansion (Theorem 4 of [69])

Theorem 3.1. *Given a choice of $b_* < 1$, a cut-off function η and for small t, the hyperbolic metric $ds^2_{\mathbf{h}}$ of $\mathcal{R}_t$ obtained by sewing a pair of punctures, as explained above, has the following expansion*

$$ds^2_{\mathbf{h}} = \left\{1 + \frac{4\pi^4}{3}(\ln|t|)^{-2}(E_1^\dagger + E_2^\dagger) + \mathcal{O}\left((\ln|t|)^{-3}\right)\right\} ds^2_{\mathbf{g}}. \tag{3.1.45}$$

The functions $E_1^\dagger$ and $E_2^\dagger$ are the melding of Eisenstein series $E(\cdot;2)$ associated to the pair of cusps sewn to form the collar.

This metric can be written in terms of the length of the core geodesics on the collar computed in the metric ds^2_t given in (3.1.14). It is given by $l = -\frac{2\pi^2}{\ln|t|} + \mathcal{O}\left((\ln|t|)^{-2}\right)$. The metric can then be written as

$$ds^2_{\mathbf{h}} = \left\{1 + \frac{l^2}{3}(E_1^\dagger + E_2^\dagger) + \mathcal{O}\left(l^3\right)\right\} ds^2_{\mathbf{g}}. \tag{3.1.46}$$

Using this metric, we can compute the lengths of two sets of geodesics in the degenerating surface $\mathcal{R}_t$ [69]

- **The length of the collar core geodesic**
  The length of the core geodesic is given by:

$$l_{\mathbf{h}} = -\frac{2\pi}{\ln|t|} + \mathcal{O}\left((-\ln|t|)^{-4}\right) = l + \mathcal{O}\left(l^4\right). \tag{3.1.47}$$

- **The length of closed geodesics away from the collars**
  For the length of a closed geodesic α disjoint from the plumbing collars

$$l_{\alpha;\mathbf{h}}\left(\{l\}\right)=l_{\alpha;\mathbf{h}}\left(\{0\}\right)+\frac{l^2}{6}\int_{\alpha}ds\,\left(E_1^{\dagger}+E_2^{\dagger}\right)+\mathcal{O}\left(l^3\right). \tag{3.1.48}$$

  In this formula, $l_{\alpha,\mathbf{h}}\left(\{l\}\right)$ is the length of α when the length of core geodesic, computed in the ds_t^2 metric, is l and $l_{\alpha;\mathbf{h}}\left(0\right)$ denotes the length of α on $\mathcal{R}_{t=0}$. Away from the collars, $E_1^{\dagger}(z,2)=E_1(z,2)$, so we can write:

$$l_{\alpha;\mathbf{h}}\left(\{l\}\right)=l_{\alpha;\mathbf{h}}\left(\{0\}\right)+\frac{l^2}{6}\int_{\alpha}ds\left(E_1+E_2\right)+\mathcal{O}\left(l^3\right). \tag{3.1.49}$$

The result (3.1.45) shows that the metric on the sewn family of surfaces $\mathcal{R}_t$, which we naively concluded that is not hyperbolic, can be made hyperbolic. The generalization to family of surfaces involving m collars is as follows. Assuming that $\boldsymbol{t}\equiv(t_1,\cdots,t_m)\in\mathbb{C}$, we consider a family of surfaces $\mathcal{R}_{\boldsymbol{t}}$, parametrized by $\boldsymbol{t}$, containing m collars. Then, we have the following [69]

Corollary 3.1. *Given a choice of $b_*<1$, a cut-off function η and for small t_i, the hyperbolic metric $ds^2_{\mathbf{h}}$ of $\mathcal{R}_{\boldsymbol{t}}$ obtained by sewing m pairs of punctures, as explained above, has the following expansion*

$$ds^2_{\mathbf{h}}=\left\{1+\frac{4\pi^4}{3}\sum_{a=1}^{m}(\ln|t_a|)^{-2}(E_{a,1}^{\dagger}+E_{a,2}^{\dagger})+\mathcal{O}\left(\sum_{a=1}^{m}(\ln|t_a|)^{-3}\right)\right\}ds^2_{\mathbf{g}}. \tag{3.1.50}$$

$E_{a,1}^{\dagger}$ and $E_{a,2}^{\dagger}$ are the melding of Eisenstein series $E(\cdot;2)$ associated to the pair of cusps plumbed to form the a^{th} collar. We can also write the metric in terms of lengths of core geodesics computed in ds_t^2

$$ds^2_{\mathbf{h}}=\left\{1+\sum_{a=1}^{m}\frac{l_a^2}{3}(E_{a,1}^{\dagger}+E_{a,2}^{\dagger})+\mathcal{O}\left(\sum_{a=1}^{m}l_a^3\right)\right\}ds^2_{\mathbf{g}}. \tag{3.1.51}$$

The length of core geodesics is given by [73]

$$l_{a;\mathbf{h}}(\boldsymbol{t})=-\frac{2\pi^2}{\ln|t_a|}+\mathcal{O}\left(\frac{1}{\left(\ln|t_a|\right)^2}\sum_{b=1}^{m}\frac{1}{\left(\ln|t_b|\right)^2}\right)\equiv l_{a;\mathbf{h}}^{(1)}+l_{a;\mathbf{h}}^{(2)},\qquad a=1,\cdots,m. \tag{3.1.52}$$

The length of a closed geodesic α disjoint from the plumbing collars is also given by

$$l_{\alpha;\mathbf{h}}(\{l_i\}) = l_{\alpha;\mathbf{h}}(\{0\}) + \sum_{a=1}^{m} \frac{l_a^2}{6} \int_{\alpha} ds \left(E_{a,1}^{\dagger} + E_{a,2}^{\dagger}\right) + \mathcal{O}\left(\sum_{a=1}^{m} l_a^3\right). \tag{3.1.53}$$

Using these results, we can state the two main conclusions of this section as follows:

1. The plumbing of two surfaces equipped with the hyperbolic metric does not give rise to a family of surfaces with hyperbolic metrics except at $t = 0$. However, one can find a hyperbolic metric on the resulting surface using the grafted metric $ds^2_{\mathbf{g}}$. The result is $ds^2_{\mathbf{h}}$ and it is given by the expansion (3.1.50).

2. To construct off-shell amplitudes in bosonic-string theory, we need to choose a gluing-compatible choice of local coordinates around marked points, i.e. the choice of local coordinates must be compatible with the gluing [17, 18]. Using the result of this section, we found metric on the family of plumbed surfaces can be made hyperbolic using the expansion (3.1.50). Therefore, we conclude that the hyperbolic metric is gluing-compatible, and as such, can be used to construct off-shell amplitudes in bosonic-string or superstring theories.

3.2 A Gluing-Compatible Section of $\widehat{\mathcal{P}}_{\mathfrak{g},\mathfrak{n}}$

In this section, we construct a gluing-compatible integration cycle using hyperbolic geometry. To construct off-shell amplitudes, we need to find the 1PI decomposition of the moduli space [20]. The fundamental result that we need to use is a criteria for the degeneration of hyperbolic structure. As we stated above, the result is as follows: the deformation of a degenerating hyperbolic surface localizes into collar neighborhoods about short geodesics. We can thus use this result to give a 1PI decomposition of the moduli space.

Define $\boldsymbol{t} \equiv (t_1, \cdots, t_m) \in \mathbb{C}$, we consider a family of surfaces $\mathcal{R}_{\boldsymbol{t}}$, parametrized by $\boldsymbol{t}$, containing m collars. Such a family can be constructed by gluing m pairs of punctures $\mathfrak{p}_a$ and $\mathfrak{p}_b$ using

$$w_a^{(1)} w_b^{(2)} = t_a, \qquad 0 \leq |t_a| \leq f_i(c_*), \tag{3.2.1}$$

where $w_i^{(a)}$ is the local coordinate around $\mathfrak{p}_a$, and $f_i(c_*)$ is a function of the collar constant c_*. The reason that the upper tail of the plumbing-fixture relation is a function of c_* is that the collar constant provides a first-order approximation to the length of core geodesics.

The main parameters that enters the construction are lengths of core geodesics inside collars computed in the metric $ds^2_{\mathbf{h}}$. Let us denote these lengths as $l_{a;\mathbf{h}}$. If the hyperbolic metric on $ds^2_{\mathbf{h}}$ has the expansion $ds^2_{\mathbf{h}} = \Sigma(\boldsymbol{t})ds^2_{\mathbf{g}}$, then the solution of the curvature-correction equation has the following expansion

$$\Sigma(\boldsymbol{t}) = \sum_{n=0}^{\infty}\sum_{a=1}^{m} l^n_{a;\mathbf{h}}(\boldsymbol{t})\Sigma_{a;n}, \tag{3.2.2}$$

For example, we found in the previous section that

$$\Sigma_{a;0} = \frac{1}{m}, \qquad \Sigma_{a;1} = 0, \qquad \Sigma_{a;2} = \frac{E^{\dagger}_{a,1} + E^{\dagger}_{a,2}}{3}, \qquad \forall a = 1, \cdots, m. \tag{3.2.3}$$

The decomposition of moduli space in terms of lengths of core geodesics is done by computing $l_{a,\mathbf{h}}$ at the upper tail of plumbing parameters $\boldsymbol{t}$, i.e. $l_{a,\mathbf{h}}(f_a(c_*))$. To proceed further, we define two classes of simple closed geodesics on the surface

- **separating-type simple closed geodesics:** a simple closed geodesic γ on a hyperbolic surface $\mathcal{R}$ is called separating-type if cutting $\mathcal{R}$ along γ generates two disconnected surfaces $\mathcal{R}_1$ and $\mathcal{R}_2$.

- **nonseparating-type simple closed geodesics:** a simple closed geodesic γ on a hyperbolic surface $\mathcal{R}$ is called nonseparating-type cutting $\mathcal{R}$ along γ generates a connected surface $\mathcal{R}'$.

We can then define 1PI and 1PR regions of the moduli space as follows

Definition 3.1 (The 1PR Region)**.** *Assume that we solve curvature-correction equation and computed the length of the core geodesics to be $l_{a,\mathbf{h}}$. Consider the set of all hyperbolic surfaces containing at least one separating-type simple closed geodesic whose length is less than or equal to $l_{a,\mathbf{h}}$. We call the region of moduli space containing such surfaces as the 1PR region of moduli space.*

Regarding this definition, we can define the 1PI region of the moduli space as follows

Definition 3.2 (The 1PI Region)**.** *We call the region of the moduli space containing surfaces not belonging to the 1PR region as the 1PI region of moduli space.*

Having these definitions at hand, we can propose the following gluing-compatible integration cycle for the off-shell bosonic-string amplitudes. Consider the genus-$\mathfrak{g}$ contribution to an off-shell bosonic-string amplitude containing $\mathfrak{n}$ external states

$$\mathcal{A}_{\mathrm{g}}^{\mathrm{b}} = \int_{\mathcal{S}_{\mathrm{g},\mathfrak{n}}} \mathbf{\Omega}. \tag{3.2.4}$$

Also, assume that we have the solution $ds^2_{\mathbf{h}}$ to the curvature-correction equation and the length of the a^{th} core geodesics computed in $ds^2_{\mathbf{h}}$ is $l_{a;\mathbf{h}}$. We want to choose a gluing-compatible integration cycle $\mathcal{S}_{\mathbf{g},\mathfrak{n}}$. We follow the following procedure

1. We define 1PI and 1PR regions of $\mathcal{M}_{\mathbf{g},\mathfrak{n}}$ using the value of $l_{a;\mathbf{h}}$ computed at the upper limit of the plumbing parameters $t_1, \cdots, t_m$.

2. On the 1PI region of the moduli space, surfaces are equipped with hyperbolic metrics. We use those metrics to define local coordinates around the marked points. The hyperbolic metric near punctures with local coordinates w_a takes the following form

$$\left(\left|\frac{dw_a}{w_a \ln |w_a|}\right|\right)^2, \qquad a = 1, \cdots, \mathfrak{n}. \tag{3.2.5}$$

3. On the 1PR region of the moduli space, we equip surfaces with metrics $ds^2_{\mathbf{h}}(\boldsymbol{t})$. We use this metric to define local coordinates around punctures. The metric satisfies $ds^2_{\mathbf{h}} = \Sigma(\boldsymbol{t}) ds^2_{\mathbf{g}}$. If we denote local coordinates around punctures in the 1PR region by $\widetilde{w}_a$, they satisfy

$$\left(\left|\frac{d\widetilde{w}_a}{\widetilde{w}_a \ln |\widetilde{w}_a|}\right|\right)^2 = \Sigma(\boldsymbol{t}) ds^2_{\mathbf{g}}, \qquad a = 1, \cdots, \mathfrak{n}. \tag{3.2.6}$$

4. Since the metric on the 1PI and 1PI region do not match, we need to introduce a region at the boundary of 1PI region and introduce a family of hyperbolic metric $ds^2_{\mathbf{h},\epsilon}$ parametrized by an infinitesimal parameter $\delta \ll 1$ such that

$$ds^2_{\mathbf{h};\epsilon}(\boldsymbol{l};\epsilon) = \Sigma(\boldsymbol{t};\epsilon) ds^2_{\mathbf{g}} = \Sigma(\boldsymbol{l}_{\mathbf{h}};\epsilon) ds^2_{\mathbf{g}}, \qquad 0 \leq \epsilon \leq \delta, \tag{3.2.7}$$

where $\boldsymbol{l}_{\mathbf{h}} \equiv (l_{1,\mathbf{h}}, \cdots, l_{m,\mathbf{h}})$. $ds^2_{\mathbf{h};\epsilon}(\boldsymbol{l};\epsilon)$ provides an interpolation between ds^2 in the 1PI region and $ds^2_{\mathbf{h}}$ in the 1PR region. If we denote the local coordinates around the punctures in $ds^2_{\mathbf{h};\epsilon}$ metric by $\widehat{w}_{a;\epsilon}$, they must satisfy

$$\left(\left|\frac{d\widehat{w}_{a,\epsilon}}{\widehat{w}_{a,\epsilon}\ln|\widehat{w}_{a,\epsilon}|}\right|\right)^2 = \Sigma(\boldsymbol{t};\epsilon)ds^2_{\mathbf{g}}, \qquad a=1,\cdots,\mathfrak{n}. \tag{3.2.8}$$

Let us summarize what we have describe, 1) we have given a choice of local coordinates around the punctures all over the moduli space, and 2) this choice of local coordinates is unique up to a phase rotation. We should now answer an important question. In constructing the hyperbolic metric on the family $\mathcal{R}_{\boldsymbol{t}}$, we have introduced various parameters like c_*, η, ϵ, a_0, etc. The question is the following: *if we fix the parameters for constructing the hyperbolic metric on the family of surfaces $\mathcal{R}_{\boldsymbol{t}}$, and then compute the off-shell amplitudes using this hyperbolic metric, to which extent the result is dependent on the choice of these parameters?* It turns out that as far as we are sticking to use a gluing-compatible choice of local coordinates around the punctures, the physical quantities of interest like renormalized masses and S-matrix elements are independent of the choice of spurious data that is used to construct the gluing-compatible local coordinates [17, 18, 20].

Let us apply this proposal for the explicit $ds^2_{\mathbf{h}}$ we described in section 3.1.2. Consider a surface with m collars. The length of core geodesics are c_*. The family of sewn surfaces is described by

$$\left\{ w_i^{(1)}w_i^{(2)} = t \,\middle|\, 0 \le |t| \le e^{-\frac{2\pi^2}{c_*}} \right\}. \tag{3.2.9}$$

In order to define local coordinates in the 1PI region, we divide it into subregions. Let us denote the subregion in the 1PI region consists of surfaces with m simple closed geodesics of length between c_* and $(1+\epsilon)c_*$, where ϵ is an infinitesimal parameter, by $\mathbf{R}_m$. For surfaces belong to the subregion $\mathbf{R}_0$, the local coordinate around the punctures is given by (3.2.5). For surfaces belong to the region $\mathbf{R}_m$ with $m \neq 0$, the local coordinate $\widehat{w}_{a,\epsilon}$ around the a^{th} puncture satisfies

$$\left(\left|\frac{d\widehat{w}_{a,\epsilon}}{\widehat{w}_{a,\epsilon}\ln|\widehat{w}_{a,\epsilon}|}\right|\right)^2 = ds^2_{\mathbf{g}}\left(1+\sum_{b=1}^{m} g(l_b)\left(E^{\dagger}_{b,1}+E^{\dagger}_{b,2}\right)\right). \tag{3.2.10}$$

$g(x)$ is a smooth function such that $g(c_*) = \frac{c_*^2}{3}$ and $g((1+\epsilon)c_*) = 0$. The 1PR region consists of surfaces with m simple closed geodesics of length $0 \le l_a < c_*,\ a=1,\cdots,m$.

Denoting the local coordinate around the a^{th} puncture by $\widetilde{w}_a$, it satisfies

$$\left(\left|\frac{d\widetilde{w}_a}{\widetilde{w}_a \ln|\widetilde{w}_a|}\right|\right)^2 = ds^2_{\mathbf{g}}\left(1+\sum_{b=1}^{m}\frac{l_b^2}{3}\left(E^{\dagger}_{b,1}+E^{\dagger}_{b,2}\right)\right). \tag{3.2.11}$$

Since we are equipped with a gluing-compatible choice of local coordinates around the punctures, we can now move to the next task in the construction of off-shell amplitudes, i.e. the integration of string measure over the moduli space of hyperbolic surfaces. We postpone the discussion of a gluing-compatible choice of PCOs distribution to section 3.4.

3.3 Integration over the Moduli Space of Hyperbolic Surfaces

In this section, we explain how to integrate over the moduli space. Throughout this section, we work with genus-$\mathfrak{g}$ surfaces with $\mathfrak{n}$ boundaries having fixed lengths $\mathbf{L} \equiv (L_1, \cdots, L_{\mathfrak{n}})$. The Teichmüller space of such surfaces is denoted by $\mathcal{T}_{\mathfrak{g},\mathfrak{n}}(\mathbf{L})$, and their moduli space is denoted by $\mathcal{M}_{\mathfrak{g},\mathfrak{n}}(\mathbf{L})$. A surface with marked points is a special case of a surface with boundaries where the length of all boundaries goes to zero. We also denote the mapping-class group of such a surface by $\text{Mod}_{\mathfrak{g},\mathfrak{n}}(\mathbf{L})$, and the mapping class group of a generic surface $\mathcal{R}$ by $\text{Mod}(\mathcal{R})$.

The moduli space $\mathcal{M}_{\mathfrak{g},n}$ can be understood as the quotient of the Teichmüller space $\mathcal{T}_{\mathfrak{g},\mathfrak{n}}$ with the action of mapping-class groups. However, in the generic case, an explicit fundamental region for the action of mapping-class groups is not known. This is due to the fact that the form of the action of mapping-class groups on the coordinate on the Teichmüller space is either complicated or unknown [103, 104]. This is one of the main reasons that integration of a form over the moduli space of hyperbolic surfaces is not a straightforward operation. In this section, we discuss a way to bypass this difficulty using the prescription for performing the integration over the moduli space introduced by Mirzakhani [67]. Her method is basically based on unwinding an integral over the moduli space using an identity and then lift the result to a covering space obtained by the quotient of the associated Teichmüller space with a subgroup of mapping-class group of the surface. She uses a specific global parametrization of the Teichmüller space called Fenchel-Nielsen coordinates denoted by $(\boldsymbol{\ell}, \boldsymbol{\tau}) = (\ell_1, \cdots, \ell_{3g-3+n}, \tau_1, \cdots, \tau_{3\mathfrak{g}-3+\mathfrak{n}})$. ℓ_is and τ_is are length-twists coordinates of a set of curves which gives a pair-of-pants decomposition of a genus-$\mathfrak{g}$ surface with $\mathfrak{n}$ boundaries. An example of such a pants decomposition is shown in figure 3.1. In an important

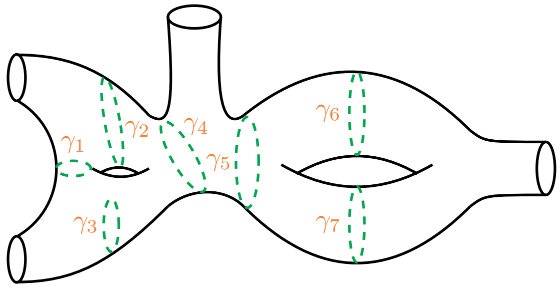


Figure 3.1: A pants decomposition of a genus-2 surface with four boundaries having fixed lengths L_1, L_2, L_3, L_4. We need 7 simple closed curves to get 6 pairs of pants. The length and twist coordinates of $\gamma_1, \cdots, \gamma_7$, $(\ell_{\gamma_1}, \tau_{\gamma_1}; \cdots; \ell_{\gamma_7}, \tau_{\gamma_7})$, provide a global coordinate on $\mathcal{T}_{2,4}(L_1, L_2, L_3, L_4)$.

paper [101], Wolpert discovered the following remarkable facts which underlie Mirzakhani's seminal paper

1. The 2-form
$$\omega_{\mathbf{WP}} = \sum_{a=1}^{3\mathfrak{g}-3+\mathfrak{n}} d\ell_a \wedge d\tau_a, \tag{3.3.1}$$
on $\mathcal{T}_{\mathfrak{g},\mathfrak{n}}$ does not depend on the choice of pants decomposition.

2. The interval in which Fenchel-Nielsen coordinates take value is
$$0 \le \ell_a < \infty, \qquad -\infty < \tau_a < \infty, \qquad a = 1, \cdots, 3\mathfrak{g} - 3 + \mathfrak{n}. \tag{3.3.2}$$
The Teichmüller space is a cell of dimension $6\mathfrak{g} - 6 + 2\mathfrak{n}$.

3. $\omega_{\mathbf{WP}}$ is the Kähler form of the Weil-Petersson metric.

4. $\omega_{\mathbf{WP}}$ is invariant under the action of mapping-class group of surfaces with signature $(\mathfrak{g}, \mathfrak{n})$.

5. $\omega_{\mathbf{WP}}$ extends to $\overline{\mathcal{M}}_{\mathfrak{g},\mathfrak{n}}$ [105].

The Weil-Petersson volume-form on $\mathcal{M}_{\mathfrak{g},\mathfrak{n}}$ is defined as
$$dV_{\mathbf{WP}}(\mathcal{M}_{\mathfrak{g},\mathfrak{n}}) = \omega^{3\mathfrak{g}-3+\mathfrak{n}}. \tag{3.3.3}$$

This formula is also applicable to hyperbolic surfaces with signature $(\mathfrak{g}, \mathfrak{n}; \mathbf{L})$, where $\mathbf{L}$ denotes the fixed length of geodesic boundary components [67]. The Weil-Petersson volume the moduli space of such surfaces is thus given by

$$V_{\mathbf{WP}}(\mathcal{M}_{\mathfrak{g},\mathfrak{n}}(\mathbf{L})) = \int_{\mathcal{M}_{\mathfrak{g},\mathfrak{n}}(\mathbf{L})} \omega^{3\mathfrak{g}-3+\mathfrak{n}}. \tag{3.3.4}$$

This is the simplest integral over the moduli space.

3.3.1 A Warm-Up Example: The Weil-Petersson Volume of $\mathcal{M}_{1,1}$

In order to demonstrate the non-triviality of even the simplest integration over the moduli space, namely the computation of the volume of the moduli space, let us discuss the volume calculation of the moduli space of once-punctured tori [63]. Suppose that the hyperbolic transformations $A, B \in \mathtt{PSL}(2;\mathbb{R})$, with $ABA^{-1}B^{-1}$ parabolic i.e. $(\mathrm{tr}\,(ABA^{-1}B^{-1}))^2 = 4$, freely generate the Fuchsian group $\Gamma \subset \mathtt{PSL}(2;\mathbb{R})$ which uniformizes a once-punctured torus. Assume that the repelling fixed-point of A is 0 and its attracting fixed-point is ∞. The attractive fixed-point of B is 1. It is not difficult to see that

$$\mathrm{tr}\left(ABA^{-1}B^{-1}\right) = -2. \tag{3.3.5}$$

The quantities $x = \mathrm{tr}(A)$, $y = \mathrm{tr}(B)$ and $z = \mathrm{tr}(AB)$ uniquely characterize the above description of Γ. Using (3.3.5), we obtain the following unique relation satisfied by the triple (x, y, z)

$$x^2 + y^2 + z^2 = xyz. \tag{3.3.6}$$

According to [106, 107], $\mathcal{T}_{1,1}$ is the following sublocus

$$x^2 + y^2 + z^2 = xyz, \qquad\qquad x, y, z > 2. \tag{3.3.7}$$

Let us introduce a different coordinate tuple (a, b, c) where $a \equiv x/yz$, $b \equiv y/xz$ and $c \equiv z/xy$. $\mathcal{T}_{1,1}$ can now be described by the following sublocus

$$a + b + c = 1, \qquad\qquad a, b, c > 0. \tag{3.3.8}$$

This sublocus is the equation of a simplex. By studying the action of $\mathrm{Mod}_{1,1}$, the mapping-class group of once-punctured tori, on the Fuchsian group Γ, it is possible to show that the following domain inside $\mathcal{T}_{1,1}$

$$\Delta = \left\{ (a, b, c) \in \mathcal{T}_{1,1} \;\middle|\; a, b, c \leq \frac{1}{2} \right\}, \tag{3.3.9}$$

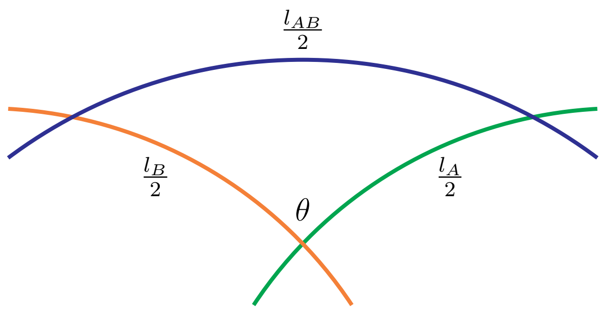


Figure 3.2: The geodesics corresponding to hyperbolic elements A, B and AB of the group Γ form a hyperbolic triangle, where A and B generate Γ, the Fuchsian group uniformizing once-punctured tori.

is a union of three copies of a fundamental domain $\mathcal{M}_{1,1}$, the moduli space of once-punctured tori, for the action of $\mathrm{Mod}_{1,1}$ [63].

Consider the Weil-Petersson Kähler form $\omega_{\mathbf{WP}}$ on $\mathcal{T}(\mathcal{R})$, the $2m$ dimensional Teichmüller space of $\mathcal{R}$. It is a classical result that free homotopy classes $\gamma_1, \cdots, \gamma_{2m}$ can be chosen such that lengths ℓ_{γ_a}, $1 \leq a \leq 2m$ provide local real coordinates for $\mathcal{T}(\mathcal{R})$ near $\mathcal{R}$ [63]. Then ω is given by

$$\omega = \sum_{a<b} M^{-1}_{ab} d\ell_a \wedge d\ell_b, \tag{3.3.10}$$

where M^{-1} is the inverse of a matrix whose components are $M_{ab} \equiv \tau_{\gamma_a} \ell_{\gamma_b}$. By definition, if Γ represents a point of $\mathcal{T}$ with generators A, B then

$$\begin{aligned} x &= \mathrm{tr}(A) = 2\cosh\left(\frac{\ell_A}{2}\right), \\ y &= \mathrm{tr}(B) = 2\cosh\left(\frac{\ell_B}{2}\right), \\ z &= \mathrm{tr}(AB) = 2\mathrm{cosh}\left(\frac{l_{AB}}{2}\right), \end{aligned} \tag{3.3.11}$$

where l_A denotes the length of geodesic corresponding to A, l_B denotes the length of geodesic corresponding to B, and l_{AB} denotes the length of geodesic corresponding to AB. We thus have

$$\omega = (t_A \ell_B)^{-1} \, d\ell_A \wedge d\ell_B = \frac{d\ell_A \wedge d\ell_B}{\cos\theta}, \tag{3.3.12}$$

where θ is measured from A to B. The geodesics correspond to A, B and AB forms a hyperbolic triangle. They have shown in figure 3.2. From the laws of cosines for the hyperbolic triangle, we have

$$\cos\theta = \frac{\cosh\left(\frac{l_A}{2}\right)\cosh\left(\frac{l_B}{2}\right) - \cosh\left(\frac{l_{AB}}{2}\right)}{\sinh\left(\frac{l_A}{2}\right)\sinh\left(\frac{l_B}{2}\right)}, \tag{3.3.13}$$

Using (3.3.11), we get

$$\omega = \frac{4dx \wedge dy}{xy - 2z}. \tag{3.3.14}$$

In terms of the variables a, b, c, the Kähler form is given by

$$\omega = \frac{da \wedge db}{ab(1-a-b)}. \tag{3.3.15}$$

Expressing everything in terms of a, b, and c, we are ready to perform the integration over the moduli space $\mathcal{M}_{1,1}$. Using the explicit integration domain (3.3.9)

$$\begin{aligned} V_{1,1;\mathbf{WP}} &= \int_{\mathcal{M}_{1,1}} \omega \\ &= \frac{1}{3}\int_{\Delta} \omega \\ &= \frac{1}{3}\int_0^{\frac{1}{2}} \int_{\frac{1}{2}-b}^{\frac{1}{2}} \frac{da \wedge db}{ab(1-a-b)} \\ &= -\frac{1}{3}\int_0^{\frac{1}{2}} db\, \frac{\ln(1-2b)}{b(1-b)}. \end{aligned} \tag{3.3.16}$$

We define a new variable $v = 1 - 2b$, we get

$$V_{1,1;\mathbf{WP}} = -\frac{4}{3}\int_0^1 dv\, \frac{\ln(v)}{1-v^2} = \frac{\pi^2}{6}. \tag{3.3.17}$$

This calculation was possible since the fundamental domain Δ for $\mathrm{Mod}_{1,1}$ was explicitly constructed. In general, it is very difficult to construct this domain. As such, the integration over the moduli space is a difficult task. We thus need to have a more efficient method.

3.3.2 The General Integration Procedure

In this section, we describe an integration procedure to compute integrals over the moduli space of genus-$\mathfrak{g}$ surfaces with $\mathfrak{n}$ boundaries having fixed lengths $\mathbf{L} = (L_1, \cdots, L_{\mathfrak{n}})$.

Consider the space $\mathcal{M}$ with a covering space $\mathcal{N}$. The covering map is given by

$$\pi : \mathcal{N} \longrightarrow \mathcal{M}. \tag{3.3.18}$$

If $dV_{\mathcal{M}}$ is a volume form for $\mathcal{M}$, then its pullback gives the volume form on $\mathcal{N}$

$$dV_{\mathcal{N}} \equiv \pi^{-1} * (dV_{\mathcal{M}}). \tag{3.3.19}$$

Assume that f is a smooth function defined in the space $\mathcal{N}$. Then the push forward of the function f at a point x in the space $\mathcal{M}$, which is denoted by $\pi_* f(x)$, can be obtained by the summation over the values of the function f at all points in the fiber of the point x in $\mathcal{N}$

$$(\pi_* f)(x) \equiv \sum_{y \in \pi^{-1}\{x\}} f(y). \tag{3.3.20}$$

This relation defines a smooth function on $\mathcal{M}$. As a result, the integral of this pull-backed function over $\mathcal{M}$ can be lifted to the covering space $\mathcal{N}$ as follows

$$\int_{\mathcal{M}} dV_{\mathcal{M}} \ (\pi_* f) = \int_{\mathcal{N}} dV_{\mathcal{N}} \ f. \tag{3.3.21}$$

For illustration, we consider a simple example, the integration over $\mathbb{S}^1$ as an integration over $\mathbb{R}$. Consider the real line $\mathbb{R} = (-\infty, \infty)$ as the covering space of circle $\mathbb{S}^1 = [0, 1)$. We denote the covering map by $\pi : \mathbb{R} \longrightarrow \mathbb{S}^1$. Assume that $f(x)$ is a function living in $\mathbb{S}^1$, i.e. $f(x + k) = f(x), \ k \in \mathbb{Z}$. We can then convert the integration over $\mathbb{S}^1$ into an integration over $\mathbb{R}$ with the help of the identity

$$\sum_{k=-\infty}^{\infty} \frac{\sin^2 (\pi[x - k])}{\pi^2 (x - k)^2} = 1. \tag{3.3.22}$$

The steps are as follows

$$\int_0^1 dx \ f(x) = \int_0^1 dx \ \left(\sum_{k=-\infty}^{\infty} \frac{\sin^2 (\pi[x - k])}{\pi^2 (x - k)^2} \right) f(x)$$

$$
\begin{aligned}
&= \int_0^1 dx \sum_{k=-\infty}^{\infty} \left(\frac{\sin^2 (\pi[x-k])}{\pi^2 (x-k)^2} f(x-k) \right) \\
&= \sum_{k=-\infty}^{\infty} \int_0^1 dx \, \frac{\sin^2 (\pi[x-k])}{\pi^2 (x-k)^2} f(x-k) \\
&= \int_{-\infty}^{\infty} dx \, \frac{\sin^2 (\pi x)}{\pi^2 x^2} f(x).
\end{aligned} \tag{3.3.23}
$$

In the last step, we absorbed the summation over k and then converted the integration over $\mathbb{S}^1$ to the integration over $\mathbb{R}$. As an example consider $f(x) = 1$. Using (3.3.23), we get

$$
\int_0^1 dx = \int_{-\infty}^{\infty} dx \, \frac{\sin^2 (\pi x)}{\pi^2 x^2} = 1. \tag{3.3.24}
$$

The second equality can be easily checked using Mathematica

Integrate $\left[\frac{\mathbf{Sin}[\pi x]^2}{\pi^2 x^2}, \{x, -\infty, \infty\}\right]$.

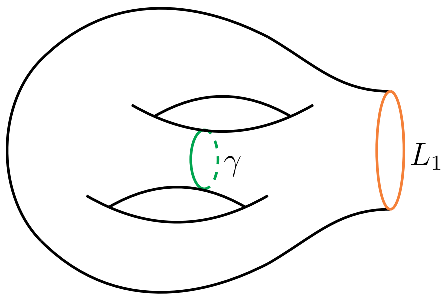
1.

What we have explained is the essence of Mirzakhani's procedure for integration over the moduli space. If we have an appropriate covering space of the moduli space, then the integration of a function defined in the moduli space can be performed by expressing the function as a push-forward of a function in that covering space of the moduli space using the covering map. Therefore, if we want to do explicit computations, we need two ingredients, 1) we need a convenient covering map, and 2) we need to construct the covering map explicitly. In the remaining part of this section, we shall explain these two ingredients. Based on this, we then explain how to perform integration over the moduli space.

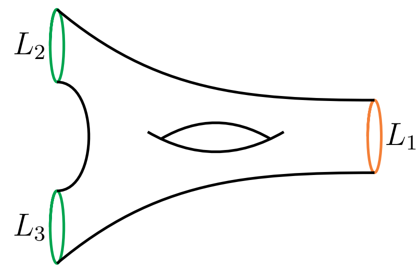
Let $\mathcal{R}$ be a genus-$\mathfrak{g}$ hyperbolic surface with $\mathfrak{n}$ boundary components. Consider a multi-curve of the following form on $\mathcal{R}$

$$
\gamma = \sum_{a=1}^{k} c_a \gamma_a. \tag{3.3.25}
$$

where $c_1, \cdots, c_k$ are real weights and $\gamma_1, \cdots, \gamma_k$ are disjoint non-homotopic simple closed geodesics on $\mathcal{R}$. Let us define the subgroup $\text{Stab}(\gamma)$ of $\text{Mod}_{\mathfrak{g},\mathfrak{n}}(\mathbf{L})$. Consider a set $\mathcal{S}$ of homotopy classes of simple closed geodesics on $\mathcal{R}$. We define the following subgroup of



(a) A genus-2 surface $\mathcal{R}$ with a single boundary is cut along the curve γ.



(b) Cutting $\mathcal{R}$ along γ generates a genus-1 surface with 3 boundaries having fixed lengths L_1, and $L_2 = L_3 = \ell_\gamma(\mathcal{R})$, where $\ell_\gamma(\mathcal{R})$ is the hyperbolic length of γ in $\mathcal{R}$.

Figure 3.3: The cutting of a surface along a simple closed curve. The situation for more general cutting is the same.

$\text{Mod}_{\mathfrak{g},\mathfrak{n}}(\mathbf{L})$

$$\text{Stab}(\mathcal{S}) \equiv \{\mathfrak{m} \in \text{Mod}_{\mathfrak{g},\mathfrak{n}}(\mathbf{L})|\ \mathfrak{m} \cdot \mathcal{S} = \mathcal{S}\} \subset \text{Mod}_{\mathfrak{g},\mathfrak{n}}(\mathbf{L}), \tag{3.3.26}$$

The elements of $\text{Stab}(\gamma)$ may permute components of the multi-curve with equal weights. The subgroup of $\text{Stab}(\gamma_a)$ that preserves the orientation of γ_a is denoted by $\text{Stab}_0(\gamma_a)$. The symmetry group of γ is then defined as follows

$$\text{Sym}(\gamma) \equiv \frac{\text{Stab}(\gamma)}{\cap_{a=1}^k \text{Stab}_0(\gamma_a)}. \tag{3.3.27}$$

Elements of this finite symmetry group possibly permute and reverse the orientation of the component geodesics in γ. We now consider cutting the surface along γ and denote the resulting surface by $\mathcal{R}(\gamma)$. An example of this is illustrated in figure 3.3.

Cutting along each component γ_a in γ generates two new boundaries. $\mathcal{R}(\gamma)$ may contain disconnected components as well. Suppose that the geodesic γ_a has length ℓ_a. We denote the product of Teichmüller spaces of the component surfaces in $\mathcal{R}(\gamma)$ by $\mathcal{T}\left(\mathcal{R}(\gamma); \boldsymbol{\ell}\right)$, $\boldsymbol{\ell} \equiv (\ell_1, \cdots, \ell_k)$, the mapping-class group of components by $\text{Mod}\mathcal{R}(\gamma)$, and the corresponding product of moduli spaces by $\mathcal{T}\left(\mathcal{R}(\gamma); \boldsymbol{\ell}\right)/\text{Mod}\left(\mathcal{R}(\gamma)\right)$.

We need to find an appropriate covering space associated to the multicurve γ. We define this covering space of $\mathcal{M}_{\mathfrak{g},\mathfrak{n}}(\mathbf{L})$ as follows

$$\mathcal{M}^\gamma_{\mathfrak{g},\mathfrak{n}}(\mathbf{L}) \equiv \{(\mathcal{R}, \eta)|\ \mathcal{R} \in \mathcal{M}_{\mathfrak{g},\mathfrak{n}}(\mathbf{L}), \quad \eta \in \mathcal{O}_\gamma\}, \tag{3.3.28}$$

where $\mathcal{O}_\gamma$ is the set of homotopy classes of all images of γ under the mapping-class group

$$\mathcal{O}_\gamma \equiv \{[\alpha]|\ \alpha \in \mathrm{Mod}_{\mathfrak{g},\mathfrak{n}}(\mathbf{L}) \cdot \gamma\}\,. \tag{3.3.29}$$

Covering spaces relevant for the integration procedure are of this kind. We thus analyze the structure of this covering space in some detail. Denoting the Dehn twist along the curve γ is denoted by ϕ_γ, the group G_γ, defined by

$$G_\gamma \equiv \bigcap_{a=1}^{k} \mathrm{Stab}(\gamma_a) \subset \mathrm{Mod}_{\mathfrak{g},\mathfrak{n}}(\mathbf{L}), \tag{3.3.30}$$

is generated by 1) the set of Dehn twists ϕ_{γ_a}, $a = 1, \cdots, k$, and 2) the elements of $\mathrm{Mod}\,(\mathcal{R}(\gamma))$. $\mathcal{M}^\gamma_{\mathfrak{g},\mathfrak{n}}(\mathbf{L})$ is then given by

$$\mathcal{M}^\gamma_{\mathfrak{g},\mathfrak{n}}(\mathbf{L}) \equiv \frac{\mathcal{T}_{\mathfrak{g},\mathfrak{n}}(\mathbf{L})}{G_\gamma}. \tag{3.3.31}$$

Moreover, $\mathcal{T}_{\mathfrak{g},\mathfrak{n}}(\mathbf{L})$ admits the following factorization [67]

$$\mathcal{T}(\mathcal{R}) = \prod_{a=1}^{s} \mathcal{T}(\mathcal{R}^a(\gamma)) \times \prod_{\gamma_a} \mathbb{R}_{>0} \times \mathbb{R}, \tag{3.3.32}$$

where s is the number of disconnected components and $\mathcal{R}^a(\gamma)$ is the surface obtained by cutting $\mathcal{R}$ along γ_a. We then have [67]

$$\mathcal{M}^\gamma_{\mathfrak{g},\mathfrak{n}}(\mathbf{L}) = \prod_{a=1}^{s} \mathcal{T}(\mathcal{R}^a(\gamma))/\mathrm{Mod}(\mathcal{R}^a(\gamma)) \times \prod_{\gamma_a} (\mathbb{R}_{>0} \times \mathbb{R})/\mathrm{Dehn}_*(\gamma_a). \tag{3.3.33}$$

In the generic case, the group $\mathrm{Dehn}_*(\gamma_i)$ is generated by the simple twist. $\mathrm{Dehn}_*(\gamma_a)$ acts only on the variables τ_a with the fundamental domain $0 \leq \tau_{\gamma_a} \leq \ell_{\gamma_a}$. However, if the curve γ_a separates a torus with a single boundary, then $\mathrm{Dehn}_*(\gamma_a)$ is generated by the half twist. The reason is that any one-holed torus comes with an elliptic involution. In this case $\mathrm{Stab}_0(\gamma)$ contains a half-twist. A Dehn half-twist acts on the associated Fenchel-Nielsen coordinates by

$$(\ell_a, \tau_a) \to (\ell_a, \tau_a + \ell_a/2). \tag{3.3.34}$$

As such, the fundamental interval for the twist parameter along γ_a is $0 \leq \tau_a \leq \ell_a/2$.

Finally, the volume form on $\mathcal{T}(\mathcal{R})$ is decomposed as

$$dV = \prod_{a=1}^{s} dV(\mathcal{R}^a(\gamma)) \times \prod_{\gamma_a} d\ell_{\gamma_a} \wedge d\tau_{\gamma_a}. \tag{3.3.35}$$

Equipped with these results, we can then perform the integration over the covering space $\mathcal{M}^{\gamma}_{\mathfrak{g},\mathfrak{n}}(\mathbf{L})$. Denoting the moduli space of component surfaces by $\mathcal{M}(\mathcal{R}^a(\gamma)) = \mathcal{T}(\mathcal{R}^a(\gamma))/\mathrm{Mod}(\mathcal{R}^a(\gamma))$, we can do the integration by first integrating over $\prod_{a=1}^{s} \mathcal{M}(\mathcal{R}^a(\gamma))$ for fixed values of lengths of the curves γ_a. Then, we can perform the integration over $d\tau_{\gamma_j}$ followed by the $d\ell_{\gamma_j}$ integration for $j = 1, \cdots s$.

To be able to lift the integration over the moduli space to an integration over the covering space $\mathcal{M}^{\gamma}_{\mathfrak{g},\mathfrak{n}}(\mathbf{L})$, we need the second ingredients we mentioned above, i.e. we need a covering map. Assume that there exists an identity of the following form

$$\sum_{[\alpha]\in \mathrm{Mod}_{\mathfrak{g},\mathfrak{n}}\cdot[\gamma]} f\left(\ell_\alpha(\mathcal{R})\right) = \text{constant}, \tag{3.3.36}$$

where f is a real function of the hyperbolic length $\ell_\alpha(\mathcal{R})$ of curve α, and is suitably small at infinity. γ is a multicurve of the form (3.3.25). For simplicity, we set the constant in right-hand side to be 1. This identity can be compared with the identity (3.3.22) which we used to lift the integral over $\mathbb{S}^1$ to $\mathbb{R}$. We assume that cutting along the curves $\gamma_a,\ a = 1, \cdots, k$ generates s disconnected bordered hyperbolic surfaces.

Consider a form $\boldsymbol{\Omega}(\boldsymbol{\ell}, \boldsymbol{\tau}; \mathbf{L})$ on the moduli space $\mathcal{M}_{\mathfrak{g},\mathfrak{n}}(\mathbf{L})$. It satisfies the following identity

$$\boldsymbol{\Omega}(\mathfrak{m}\cdot\boldsymbol{\ell}, \mathfrak{m}\cdot\boldsymbol{\tau}; \mathbf{L}) = \boldsymbol{\Omega}(\boldsymbol{\ell}, \boldsymbol{\tau}; \mathbf{L}), \qquad \forall \mathfrak{m} \in \mathrm{Mod}_{\mathfrak{g},\mathfrak{n}}(\mathbf{L}). \tag{3.3.37}$$

We would like to integrate $\boldsymbol{\Omega}(\boldsymbol{\ell}, \boldsymbol{\tau}; \mathbf{L})$ over $\mathcal{M}_{\mathfrak{g},\mathfrak{n}}(\mathbf{L})$. We have

$$\begin{aligned}\mathcal{I}_{\mathcal{M}_{\mathfrak{g},\mathfrak{n}}(\mathbf{L})} \equiv \int_{\mathcal{M}_{\mathfrak{g},\mathfrak{n}}(\mathbf{L})} \boldsymbol{\Omega}(\boldsymbol{\ell}, \boldsymbol{\tau}; \mathbf{L}) &= \sum_{[\alpha]\in \mathrm{Mod}_{\mathfrak{g},\mathfrak{n}}(\mathbf{L})\cdot[\gamma]} \int_{\mathcal{M}_{\mathfrak{g},\mathfrak{n}}(\mathbf{L})} f\left(\sum_{a=1}^{k} c_a \ell_{\mathfrak{m}\cdot\gamma_a}\right) \boldsymbol{\Omega}(\boldsymbol{\ell}, \boldsymbol{\tau}; \mathbf{L}) \\ &= \sum_{\mathfrak{m}\in \mathrm{Mod}_{\mathfrak{g},\mathfrak{n}}(\mathbf{L})/\mathrm{Stab}(\gamma)} \int_{\mathcal{M}_{\mathfrak{g},\mathfrak{n}}(\mathbf{L})} f\left(\sum_{a=1}^{k} c_a \ell_{\mathfrak{m}\cdot\gamma_a}\right) \boldsymbol{\Omega}(\mathfrak{m}\cdot\boldsymbol{\ell}, \mathfrak{m}\cdot\boldsymbol{\tau}; \mathbf{L}).\end{aligned} \tag{3.3.38}$$

To proceed further, we note that

$$\sum_{\mathfrak{m}\in\mathrm{Mod}_{\mathfrak{g},\mathfrak{n}}(\mathbf{L})/\mathrm{Stab}(\gamma)} f\left(\sum_{a=1}^{k} c_a \ell_{\mathfrak{m}\cdot\gamma_a}\right),$$

is invariant under the finite group $\mathrm{Stab}(\gamma)/\cap_{a=1}^{k}\mathrm{Stab}(\gamma_a)$. Using this fact, we have

$$\begin{aligned}\sum_{\mathfrak{m}\in\mathrm{Mod}_{\mathfrak{g},\mathfrak{n}}(\mathbf{L})/\mathrm{Stab}(\gamma)} f\left(\sum_{a=1}^{k} c_a \ell_{\mathfrak{m}\cdot\gamma_a}\right) &= \frac{1}{|\mathrm{Sym}(\gamma)|}\sum_{\mathfrak{m}\in\mathrm{Mod}_{\mathfrak{g},\mathfrak{n}}(\mathbf{L})/\mathrm{Stab}(\gamma)}\ \sum_{\mathrm{Stab}(\gamma)/\cap_{a=1}^{k}\mathrm{Stab}(\gamma_a)} f\left(\sum_{a=1}^{k} c_a \ell_{\mathfrak{m}\cdot\gamma_a}\right)\\ &= \frac{1}{|\mathrm{Sym}(\gamma)|}\sum_{\mathfrak{m}\in\mathrm{Mod}_{\mathfrak{g},\mathfrak{n}}(\mathbf{L})/\cap_{a=1}^{k}\mathrm{Stab}(\gamma_a)} f\left(\sum_{a=1}^{k} c_a \ell_{\mathfrak{m}\cdot\gamma_a}\right).\end{aligned} \tag{3.3.39}$$

Substitute this in (3.3.38), we get

$$\begin{aligned}\mathcal{I}_{\mathcal{M}_{\mathfrak{g},\mathfrak{n}}(\mathbf{L})} &= \frac{1}{|\mathrm{Sym}(\gamma)|}\int_{\mathcal{M}_{\mathfrak{g},\mathfrak{n}}(\mathbf{L})}\sum_{\mathfrak{m}\in\mathrm{Mod}_{\mathfrak{g},\mathfrak{n}}/\cap_{a=1}^{k}\mathrm{Stab}(\gamma_a)} f\left(\sum_{a=1}^{k} c_a \ell_{\mathfrak{m}\cdot\gamma_a}\right)\boldsymbol{\Omega}(\mathfrak{m}\cdot\boldsymbol{\ell},\mathfrak{m}\cdot\boldsymbol{\tau};\mathbf{L})\\ &= \frac{1}{|\mathrm{Sym}(\gamma)|}\sum_{\mathfrak{m}\in\mathrm{Mod}_{\mathfrak{g},\mathfrak{n}}(\mathbf{L})/\cap_{a=1}^{k}\mathrm{Stab}(\gamma_a)}\int_{\mathcal{M}_{\mathfrak{g},\mathfrak{n}}(\mathbf{L})} f\left(\sum_{a=1}^{k} c_a \ell_{\mathfrak{m}\cdot\gamma_a}\right)\boldsymbol{\Omega}(\mathfrak{m}\cdot\boldsymbol{\ell},\mathfrak{m}\cdot\boldsymbol{\tau};\mathbf{L})\\ &= \frac{1}{|\mathrm{Sym}(\gamma)|}\int_{\mathcal{M}^{\gamma}_{\mathfrak{g},\mathfrak{n}}(\mathbf{L})} f(\ell_\gamma)\boldsymbol{\Omega}(\boldsymbol{\ell},\boldsymbol{\tau};\mathbf{L}).\end{aligned} \tag{3.3.40}$$

Finally, we use (3.3.33) to get

$$\mathcal{I}_{\mathcal{M}_{\mathfrak{g},\mathfrak{n}}(\mathbf{L})} = \frac{1}{|\mathrm{Sym}(\gamma)|}\int_{\mathcal{M}(\gamma)} dV(\gamma)\prod_{a=1}^{s}\int_{\mathcal{M}(\mathcal{R}^a(\gamma))} f(\ell_\gamma)\boldsymbol{\Omega}(\boldsymbol{\ell},\boldsymbol{\tau};\mathbf{L}). \tag{3.3.41}$$

In this relation, $\mathcal{M}(\gamma)$ denotes the fundamental interval for integration over length-twist coordinates (ℓ_a,τ_a) associated to $\gamma_1,\cdots,\gamma_k$

$$0\le \ell_a<\infty,\qquad 0\le\tau_a\le 2^{-M_{\gamma_a}}\ell_a,\qquad a=1,\cdots,k, \tag{3.3.42}$$

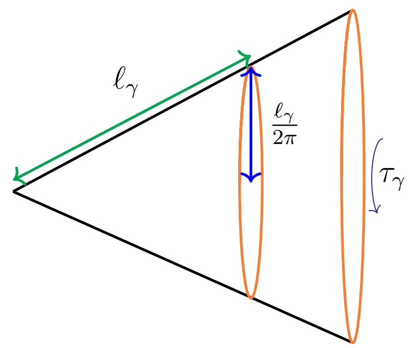


Figure 3.4: The domain of integration for Fenchel-Nielsen length ℓ_γ and twist τ_γ coordinates associated to a curve γ.

and

$$\int_{\mathcal{M}(\gamma)} dV(\gamma) \equiv \prod_{a=1}^{k} \int_0^\infty \int_0^{2^{-M_a}\ell_a} d\ell_a d\tau_a. \tag{3.3.43}$$

In these relations, $M_{\gamma_a} = 1$ if γ_a bounds a one-holed torus, and is zero otherwise. The reason for this extra factor, as we have explained above, is the existence of non-trivial automorphism group for one-holed tori. For all other cases, i.e. $(\mathfrak{g}, \mathfrak{n}) \neq (1,1)$, a generic point in $\mathcal{M}_{\mathfrak{g},\mathfrak{n}}(\mathbf{L})$ does not have any non-trivial automorphism that fixes the boundary components [67]. We can then repeat this for the integration over $\mathcal{M}(\mathcal{R}^a(\gamma))$ and continue the same process until we end up with an integration over a set of infinite cones similar to (3.3.42). An example of such a cone is illustrated in figure 3.4. The final integrand would be a product of some number of function similar to f appearing in (3.3.36) times $\boldsymbol{\Omega}(\boldsymbol{\ell}, \boldsymbol{\tau}; \mathbf{L})$. The fundamental interval of integration is

$$0 \leq \ell_a < \infty, \qquad 0 \leq \tau_a \leq 2^{-M_{\gamma_a}}\ell_a, \qquad a = 1, \cdots, 3\mathfrak{g} - 3 + \mathfrak{n}. \tag{3.3.44}$$

To be able to do the explicit integration, we need to specify the function which appears in (3.3.36). Some of these identities have been explained in appendix 3.C.

3.4 A Gluing-Compatible Section of $\widetilde{\mathcal{P}}_{\mathfrak{g};\mathfrak{n}_{\mathbf{NS}},\mathfrak{n}_{\mathbf{R}}}$

In this section, we describe a gluing-compatible section of $\widetilde{\mathcal{P}}_{\mathfrak{g};\mathfrak{n}_{\mathrm{NS}},\mathfrak{n}_{\mathrm{R}}}$.

Let us remind a few facts about off-shell superstring measure. The genus-$\mathfrak{g}$ contribution to a generic off-shell heterotic-string amplitude with $\mathfrak{n}_{\rm NS}$ NS external states and $\mathfrak{n}_{\rm R}$ R external states is given by [20]

$$\mathcal{A}^{\mathbf{h}}_{\mathfrak{g};\mathfrak{n}_{\rm NS},\mathfrak{n}_{\rm R}} = \int_{\widetilde{\mathcal{S}}_{\mathfrak{g};\mathfrak{n}_{\rm NS},\mathfrak{n}_{\rm R}}} \mathbf{\Omega}^{\mathbf{h}}_{\mathfrak{g};\mathfrak{n}_{\rm NS},\mathfrak{n}_{\rm R}}, \qquad \widetilde{\mathcal{S}}_{\mathfrak{g};\mathfrak{n}_{\rm NS},\mathfrak{n}_{\rm R}} \in \Gamma\left(\widetilde{\mathcal{P}}_{\mathfrak{g};\mathfrak{n}_{\rm NS},\mathfrak{n}_{\rm R}}\right), \tag{3.4.1}$$

where $\mathbf{\Omega}^{\mathbf{h}}_{\mathfrak{g};\mathfrak{n}_{\rm NS},\mathfrak{n}_{\rm R}}$, a $(6\mathfrak{g}-6+2\mathfrak{n}_{\rm NS}+2\mathfrak{n}_{\rm R})$-differential form, is given by

$$\mathbf{\Omega}^{\mathbf{h}}_{\mathfrak{g};\mathfrak{n}_{\rm NS},\mathfrak{n}_{\rm R}} \equiv \langle \mathcal{R}|\mathcal{B}_{6\mathfrak{g}-6+2\mathfrak{n}_{\rm NS}+2\mathfrak{n}_{\rm R}}|\Phi\rangle, \qquad \Phi \in \mathcal{H}_1^{\mathrm{NS}\otimes\mathfrak{n}_{\rm NS}} \otimes \mathcal{H}_1^{\mathrm{R}\otimes\mathfrak{n}_{\rm R}}. \tag{3.4.2}$$

To define $\mathcal{B}_{6\mathfrak{g}-6+2\mathfrak{n}_{\rm NS}+2\mathfrak{n}_{\rm R}}$, we first define the following p-form

$$\mathcal{K}^{(p)} \equiv \left[\mathbb{X}(z_1)\wedge\cdots\wedge\mathbb{X}(z_K)\right]^{(p)}, \qquad K \equiv 2\mathfrak{g}-2+\mathfrak{n}_{\rm NS}+\frac{1}{2}\mathfrak{n}_{\rm R}, \tag{3.4.3}$$

where $\mathbb{X}(z_a) \equiv \mathcal{X}(z_a) - d_{z_a}\xi(z_a)$, and the superscript (p) shows that we are taking the p-form part of the expression. Using this, $\mathcal{B}_{6\mathfrak{g}-6+2\mathfrak{n}_{\rm NS}+2\mathfrak{n}_{\rm R}}$ can be written as

$$\mathcal{B}_{6\mathfrak{g}-6+2\mathfrak{n}_{\rm NS}+2\mathfrak{n}_{\rm R}} \equiv \sum_{a=0}^{2\mathfrak{g}-2+\mathfrak{n}_{\rm NS}+\frac{1}{2}\mathfrak{n}_{\rm R}} \mathcal{K}^{(p)} \wedge \mathcal{B}_{6\mathfrak{g}-6+2\mathfrak{n}_{\rm NS}+2\mathfrak{n}_{\rm R}-p}. \tag{3.4.4}$$

The distribution of PCOs must have the following properties

1. It must be gluing-compatible;
2. It should be invariant under the mapping-class group of the surface.
3. It must avoid unphysical singularities.

The general method for avoiding unphysical singularities is the vertical integration procedure [20, 25, 108]. We explain this procedure in appendix 3.D.

To distribute PCOs which are invariant under the mapping-class group of the surface, we use the following facts

1. The number of PCOs is always less than or equal to the number of pairs of pants

$$\#\mathrm{PCOs} \leq 2\mathfrak{g}-2+\mathfrak{n}_{\rm NS}+\mathfrak{n}_{\rm R}. \tag{3.4.5}$$

# of curves	type of boundary	type of external marked point(s)	# of PCOs
1 curve	one NS boundary	two NS marked points	one
	one R boundary	one NS and one R marked points	zero
	one NS boundary	two R marked points	zero
2 curves	two NS boundaries	one NS marked point	one
	one NS and one R boundaries	one R marked point	zero
	two R boundaries	one NS marked point	zero

Table 3.1: Removing a pairs of pants from a genus-$\mathfrak{g}$ surface with $\mathfrak{n}_{\text{NS}}$ NS marked points and $\mathfrak{n}_{\text{R}}$ R marked points. The first column shows the number of required curves. The second column specifies the type of boundary it removes. The third column specifies the type of bounded external marked points, and the last column denotes the number of PCOs on such a separated pair of pants.

This means that we can put PCOs on different pairs of pants. One of the unphysical singularities in string integrand happens when two PCOs collide. By putting PCOs on different pairs of pants, it is ensured that such unphysical singularities will not occur.

2. As we have explained in section 3.3, there exists an identity of the following form on any hyperbolic surface with geodesic boundary components $\mathcal{R}$

$$\sum_{[\alpha]\in \text{Mod}_{\mathfrak{g},\mathfrak{n}}(\mathbf{L})\cdot\gamma} f(\ell_\alpha(\mathcal{R})) = 1, \tag{3.4.6}$$

where γ is a multicurve on the surface, and $\text{Mod}_{\mathfrak{g},\mathfrak{n}}(\mathbf{L})$ denotes the mapping-class group of the surface.

We begin by explaining the possible types of degeneration of $\mathcal{R}$, a genus-$\mathfrak{g}$ surface with $\mathfrak{n}_{\text{NS}}$ NS marked points and $\mathfrak{n}_{\text{R}}$ R marked points. To remove a pair of pants from $\mathcal{R}$, there are two ways, 1) we can either choose a curve on $\mathcal{R}$ and two marked points, or 2) we can choose two curves on $\mathcal{R}$ and a single marked point. In general there are six possible cases which are summarized in table 3.1.

Accordingly, generic boundary components, some of them might be the external marked points, of a pair of pants in a pants decomposition of a surface can be one of the following types 1) three NS boundary components which requires a single PCO, 2) one NS and two

R boundary components which requires no PCO. We use the following prescription for the possible cases

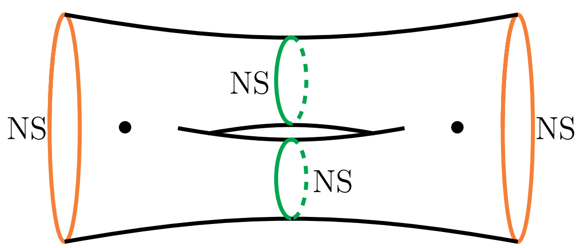
- When the associated pair of pants contains two external NS marked points, we can put a PCO on such a pair of pants. The only possibility for the other boundary component is an NS-type boundary.
- When the associated pair of pants contains two external R marked points, we do not put PCO on such a pair of pants. The only possibility for the other boundary component is an NS-type boundary.
- When the associated pair of pants contains a single external NS marked points, we can put a PCO on such a pair of pants. However, there is complication here. There are two possible boundary components: 1) two NS-type boundary components; we do not move the PCO when the surface develops short geodesics, and 2) two R-type boundary components; we use vertical integration to distribute the PCO on the core geodesic of annulus around the short geodesics.
- When the associated pair of pants contains a single external R marked points, we can put a PCO on such a pair of pants. The only possibility for other boundary components is two NS-type boundaries.

To illustrate these rule, let us consider an example. Consider a genus-1 surface with two NS boundaries. Such surfaces contributes to the two-point one-loop amplitudes of two NS-sector states. In this case, we need 2 PCOs. Possible degenerations of such a surface is illustrated in figure 3.1. Instead of punctures, we considered NS and R boundary components.

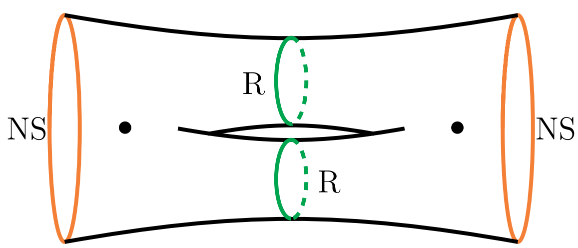
Let us discuss the possible pants decompositions separately. In figure 3.1a, the resulting surfaces are two pairs of pants each with three NS marked points. Such pairs of pants require a single PCO. According to above rules, we put a PCO on each of these pairs of pants. When the surface develops short geodesics, we do not need to move PCOs.

In figure 3.1b, the resulting surfaces are two pairs of pants with one NS boundary and two R boundaries, each of which contain a PCO. Once the surface develops short geodesics, we need to move PCOs to plumbing tubes using the vertical integration.

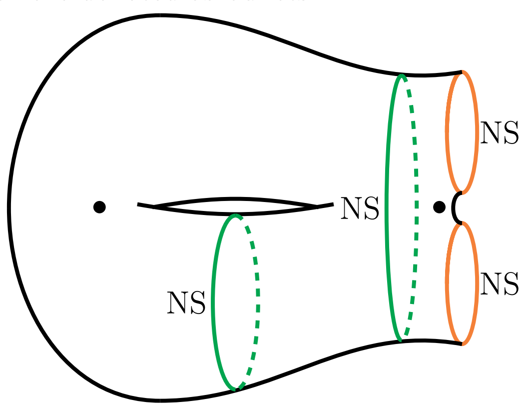
In figure 3.1c, the resulting surfaces are two pairs of pants with three NS boundaries, each of which contain a single PCO. Once the surface develop short geodesics, we do not need to move PCOs.



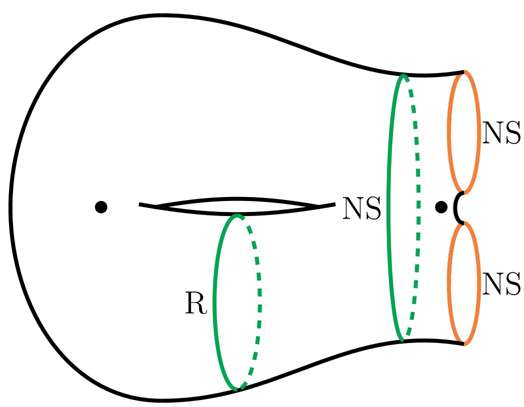
(a) A possible pants decomposition of a genus-1 surface with 2 NS boundaries. We need 2 PCOs which are denoted as bullets.



(b) A possible pants decomposition of a genus-1 surface with 2 NS boundaries. We need 2 PCOs which are denoted as bullets.

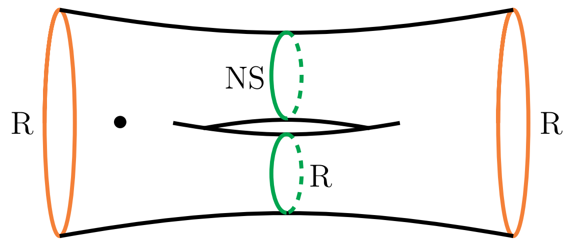


(c) A possible pants decomposition of a genus-1 surface with 2 NS boundaries. We need 2 PCOs which is denoted as bullets.

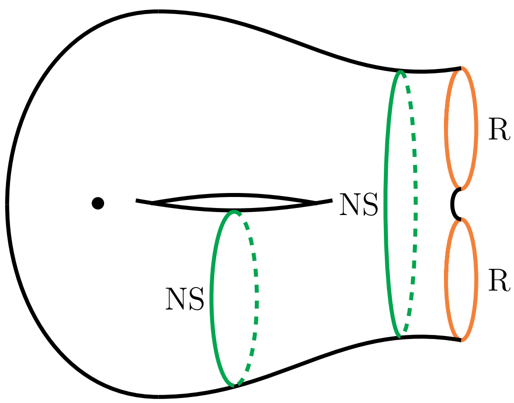


(d) A possible pants decomposition of a genus-1 surface with 2 NS boundaries. We need 2 PCOs which is denoted as bullets.

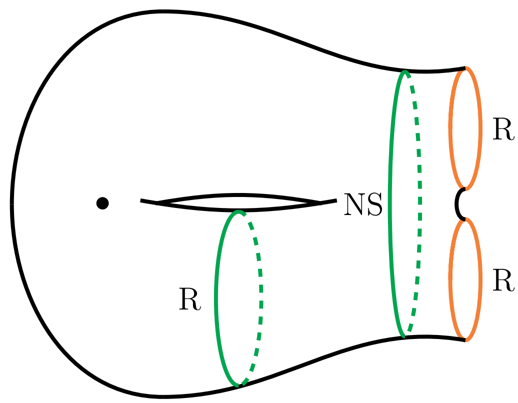
Figure 3.1: Possible degenerations of a genus-1 surface with two NS boundaries.



(a) A possible pants decomposition of a genus-1 surface with 2 R boundaries. We need 1 PCO which is denoted as a bullet.



(b) A possible pants decomposition of a genus-1 surface with 2 R boundaries. We need a PCO which is denoted as a bullet.



(c) A possible pants decomposition of a genus-1 surface with 2 R boundaries. We need a PCO which is denoted as a bullet.

Figure 3.2: Possible degenerations of a genus-1 surface with two R boundaries.

In figure 3.1d, the resulting surface are two pairs of pants. One of them has three NS boundaries, and the other one has one NS boundary and two R boundaries. Once the surface develops short geodesics, we do not need to move the PCO on the pair of pants with three NS boundaries. However, we need to move the PCO on the pair of pants with two R boundaries to the plumbing tube using the vertical integration.

This example illustrates the consistency of the rules given above. For comparison, let us consider a genus-1 surface with two R boundary components. Such a surface require one PCO. We can put it on any of the pairs of pants.

In figure 3.2a, the resulting surfaces are two pairs of pants with one NS boundary and two R boundaries. Once the surface develops short geodesics, we need to move the PCO on the plumbing tube corresponding to R degeneration using the vertical integration.

In figure 3.2b, the resulting surfaces are two pairs of pants, one with two R boundaries and one NS boundaries, and the other one with three NS boundaries. According to our rule, it is necessary to put the PCO on the pair of pants with three NS boundaries. Once the surface develops short geodesics, we do not need to move the PCO. This example also shows that why we have chosen the rule that avoid to put a PCO on a pair of pants containing two external R marked points. Note that if we put the PCO on the pair of pants containing two external R boundaries, we end up with a pair of pants with one NS boundary and two R boundaries. Such a pair of pants does not require a PCO. We thus can not achieve gluing-compatibility.

In figure 3.2c, the resulting surfaces are two pairs of pants with one NS boundaries and two R boundaries. According two our rule, it is necessary to put the PCO on the pairs of pants containing internal R boundaries. Once the surface develops short geodesics, we need to move the PCO to the plumbing tube using the vertical integration. Note that if we put the PCO on the pair of pants containing two external R boundaries, we end up with a pair of pants with one NS boundary and two external R boundaries. Such a pair of pants does not require a PCO. We thus can not achieve gluing-compatibility. On the other hand, once the surface develops an R degeneration, we cannot produce the correct propagator in the R sector without putting PCO on the plumbing tube [20].

These examples show that PCOs-distribution rules mentioned above are consistent. We need to emphasize two points, 1) the PCO inserted on a pair of pants can be put on one of the unique geodesics orthogonal to its boundaries, 2) these geodesics is determined by the hyperbolic metric on the surface. Therefore, we should use the hyperbolic metrics, described in section 3.2, in the respective regions of the moduli space to determine these geodesics.

We now consider the second issue, i.e. the fact that the PCOs distribution must be invariant under the mapping-class group of surface. For describing such a distribution, we consider the following expression

$$\sum_{\mathfrak{m}\in \mathrm{Mod}_{\mathfrak{g};\mathfrak{n}_{\mathrm{NS}},\mathfrak{n}_{\mathrm{R}}}} f(\ell_{\mathfrak{m}\cdot\gamma}(\mathcal{R}))\mathcal{K}^{(p)}(\mathfrak{m}\cdot z_1,\cdots,\mathfrak{m}\cdot z_K), \tag{3.4.7}$$

where γ is a multicurve on the surface. $\mathfrak{m}\cdot z_a$ denotes the action of the mapping-class $\mathfrak{m}$ on z_a, the location of insertion of a^{th} PCO. As we have explained in the previous section, the integral of $\langle\mathcal{R}|\mathcal{B}_{6\mathfrak{g}-6+2\mathfrak{n}_{\mathrm{NS}}+2\mathfrak{n}_{\mathrm{R}}}|\Phi\rangle$ can be written as

$$\mathcal{A}^{\mathbf{h}}_{\mathfrak{g};\mathfrak{n}_{\mathrm{NS}},\mathfrak{n}_{\mathrm{R}}} = \frac{1}{|\mathrm{Sym}(\gamma)|}\int_{\mathcal{M}(\gamma)} dV(\gamma)\prod_{a=1}^{s}\int_{\mathcal{M}(\mathcal{R}^a(\gamma))} f(\ell_\gamma)\langle\mathcal{R}|\mathcal{B}_{6\mathfrak{g}-6+2\mathfrak{n}_{\mathrm{NS}}+2\mathfrak{n}_{\mathrm{R}}}|\Phi\rangle. \tag{3.4.8}$$

To proceed further, we need to have an MCG-invariant PCOs distribution but the integrand does not involve such an expression. Therefore, (3.4.7) is not a complete expression. The complete expression should contain the summation over all elements of mapping-class groups of all the resulting surfaces obtained by cutting the surface along multicurves that gives a pair of pants decomposition. We thus have the following procedure

1. Let consider cutting $\mathcal{R}$ along a multicurve γ_1, containing k_1 curves, produces a surface with s_1 disconnected components $\mathcal{R}^1(\gamma_1), \cdots, \mathcal{R}^{s_1}(\gamma_2)$. We need to integrate over moduli space of these surfaces. Assume that we want to integrate over the moduli space of $\mathcal{R}^{a_1}(\gamma_1)$. We need a multicurve $\gamma_2^{a_1}$, containing $k_2^{a_1}$ curves. Assume that cutting $\mathcal{R}^{a_1}(\gamma_1)$ along these curves produces a disconnected surface with $s_2^{a_1}$ disconnected surfaces $\mathcal{R}^{a_1,1}(\gamma_1;\gamma_2^{a_1}), \cdots, \mathcal{R}^{a_1,s_2^{a_1}}(\gamma_1;\gamma_2^{a_1})$. We need to integrate over the moduli space of such surfaces. Assume that we want to integrate over the moduli space of $\mathcal{R}^{a_1,a_2}(\gamma_1;\gamma_2^{a_1})$. We need a multicurve $\gamma_3^{a_1,a_2}$, containing $k_3^{a_1,a_2}$ curves. Assume that cutting along these curves produces a surface with $s_3^{a_1,a_2}$ disconnected components $\mathcal{R}^{a_1,a_2,1}(\gamma_1;\gamma_2^{a_1};\gamma_3^{a_1,a_2}), \cdots, \mathcal{R}^{a_1,a_2,s_3^{a_1,a_2}}(\gamma_1;\gamma_2^{a_1};\gamma_3^{a_1,a_2})$. These process should be continued until we furnish all of the associated Fenchel-Nielsen coordinates.

2. Assume that the surfaces generated in the previous steps are denoted by $\mathcal{R}_0 \equiv \mathcal{R}, \mathcal{R}_1, \cdots, \mathcal{R}_N$, for some N. A mapping-class-group-invariant PCOs distribution can be then written as

$$\left(\prod_{a=0}^{N} \sum_{\mathfrak{m}_a \in \mathrm{Mod}(\mathcal{R}_a)} f_a(\ell_{\mathfrak{m}_a\cdot\gamma_a}(\mathcal{R}_a))\right) \mathcal{K}^{(p)}(\mathfrak{m}\cdot z_1, \cdots, \mathfrak{m}\cdot z_K), \tag{3.4.9}$$

   where we have defined $\mathfrak{m} \equiv \mathfrak{m}_0 \cdots \mathfrak{m}_N$.

As an example, consider the figure 3.1c. For such a surface, the possible PCOs distribution is given by

$$\sum_{\mathfrak{m}_0 \in \mathrm{Mod}_{1;2,0}} \sum_{\mathfrak{m}_1 \in \mathrm{Mod}_{1;1,0}} f_0(\ell_{\mathfrak{m}_0\cdot\gamma_0}(\mathcal{R}_{1;2,0})) f_1(\ell_{\mathfrak{m}_1\cdot\gamma_1}(\mathcal{R}_{1;1,0}))\mathcal{K}^{(p)}(\mathfrak{m}_0\mathfrak{m}_1\cdot z_1, \mathfrak{m}_0\mathfrak{m}_1\cdot z_2), \tag{3.4.10}$$

where γ_0 is the curve that together with two NS boundaries separates a pair of pants form the surface, and γ_1 is the curve turns a one-holed torus to a pair of pants. This ends our discussion of a gluing-compatible section of $\widetilde{\mathcal{P}}_{\mathfrak{g};\mathfrak{n}_{\mathrm{NS}},\mathfrak{n}_{\mathrm{R}}}$.

3.5 Applications in String Perturbation Theory

Integration over the moduli space is an essential ingredients of computation of amplitudes in string perturbation theory. We have seen that the integration over the moduli space can be done using the method explained in the previous section. The two quantities of interest in string theory are partition functions (i.e. vacuum amplitudes) and scattering amplitudes. Accordingly, there are two types of Riemann surfaces that one has to deal with:

- To compute vacuum amplitudes, one must integrate an appropriate form $\mathbf{\Omega}$ of degree $6\mathfrak{g}-6$ over the moduli space of genus-$\mathfrak{g}$ hyperbolic surfaces without any puncture;
- To compute scattering amplitudes of a scattering process involving $\mathfrak{n}$ external states (or $\mathfrak{n}_{\text{NS}}$ states from the NS sector and $\mathfrak{n}_{\text{R}}$ states from the R sector in superstring theory, for which $\mathfrak{n}=\mathfrak{n}_{\text{NS}}+\mathfrak{n}_{\text{R}}$), one has to integrate an appropriate form $\mathbf{\Omega}$ of degree $6\mathfrak{g}-6+2\mathfrak{n}$ over the moduli space of genus-$\mathfrak{g}$ hyperbolic surfaces with $\mathfrak{n}$ punctures (with $\mathfrak{n}_{\text{NS}}$ NS punctures and $\mathfrak{n}_{\text{R}}$ R punctures in the case of superstring theory);

As we have explained in appendix 3.C, there are various identities for hyperbolic surfaces of each of the above type. There is an identity, known as **The Luo-Tan Identity**[3] [109], that can be used for both types of surfaces. However, the Luo-Tan identity has a very complicated form and it is not useful for practical computations. In principle, we can consider the following procedure to compute amplitudes in string theory

- **Computation of Partition Functions (i.e. Vacuum Amplitudes)**
  To do the integration over the moduli space of genus-$\mathfrak{g}$ hyperbolic surfaces without puncture, one begins with an appropriate identity for such surfaces, namely **The Luo-Tan Identity** for borderless surfaces. By inserting this identity, the integration over the moduli space reduces to a product of integration over the moduli spaces of surfaces obtained by cutting along the curves appearing in the statement of the Luo-Tan Identity times a number of cones. To be able to continue the integration, one requires an identity for surfaces with borders. One can use a simpler identity, known as **The Mirzakhani-McShane Identity** or **The Generalized McShane Identity** [67];
- **Computation of Scattering Amplitudes**
  To do the integration over the moduli space of genus-$\mathfrak{g}$ hyperbolic surfaces with $\mathfrak{n}$

[3] We would like to thank G. McShane for introducing this paper.

cusps, one begins with an appropriate identity for such surfaces. This identity is known as **The McShane Identity** for hyperbolic punctured surfaces [110, 111]. By inserting this identity, the integration over the moduli space reduces to a product of integration over the moduli spaces of surfaces obtained by cutting along the curves appearing in the statement of The McShane Identity times a number of cones. To be able to continue the process of integration, we require an identity for surfaces with a mixture of borders and punctures. We can then use The Mirzakhani-McShane Identity or its various special cases where the length of some of boundaries tends to zero to do this integration.

We can use this prescription to compute any amplitudes in string perturbation theory. The only missing ingredients is to compute the form $\boldsymbol{\Omega}$ that appears as the integrand in the computation of string amplitudes in terms of Fenchel-Nielsen coordinates.

The curves which give a pair of pants decomposition of a surface provide various ways that a surface can degenerate. These curves are either separating-type or non-separating type. To deal with divergences in string theory, we need to impose the condition of gluing-compatibility on all separating-type degenerations [17, 18].

We also emphasize that the computation of on-shell amplitudes of external states which do not undergo mass renormalization do not require a choice of local coordinates around marked points or a choice of locations of PCOs. In these cases, the integration over the moduli space we explained above can be used to compute any amplitudes involving only such states.

We thus have all ingredients for the construction of on-shell/off-shell amplitudes in string perturbation theory, namely

1. We have an explicit gluing-compatible choice of local coordinates around the marked points;

2. We have an explicit gluing-compatible distribution of picture-changing operators;

3. We have an explicit integration procedure over the moduli space of hyperbolic surfaces.

We illustrate the general construction in the some examples.

3.5.1 Tree-Level Four-Point Amplitudes in Bosonic-String Theory

As the first example, let us consider tree level string amplitude with four off-shell external states in the bosonic-string theory. We first consider the computation of the Weil-Petersson volume of the moduli space. We consider a more general situation where the surface has four boundaries having fixed lengths $\mathbf{L} = (L_1, L_2, L_3, L_4)$. The Mirzakhani-McShane identity for these surfaces is given by:

$$\sum_{a=2}^{4} \sum_{\gamma \in \mathcal{F}_{1,a}} \mathcal{D}_2(L_1, L_a, \ell_\gamma) = L_1, \tag{3.5.1}$$

where the function $\mathcal{D}_2$ is defined in (3.C.25). The curves that bound the boundary with length L_1 and other boundaries are illustrated in figure 3.1.

Using this identity, the volume can be computed as follows:

$$\begin{aligned}
\mathrm{Vol}_{\mathbf{WP}}(\mathcal{M}_{0,4}(\mathbf{L})) &= \frac{1}{L_1} \sum_{a=2}^{4} \int_{\mathcal{M}_{0,4}(\mathbf{L})} \sum_{\gamma \in \mathcal{F}_{1,a}} \mathcal{D}_2(L_1, L_a, \ell_\gamma) \\
&= \frac{1}{L_1} \sum_{a=2}^{4} \int_{\mathcal{M}_{0,4}^{\gamma_a}(\mathbf{L})} d\ell_{\gamma_a} d\tau_{\gamma_a} \mathcal{D}_2(L_1, L_a, \ell_{\gamma_a}) \\
&= \frac{1}{L_1} \sum_{a=2}^{4} \int_0^{\infty} d\ell_{\gamma_a} \int_0^{\ell_{\gamma_a}} d\tau_{\gamma_a} \mathcal{D}(L_1, L_a, \ell_{\gamma_a}) \\
&= 2\pi^2 + \frac{1}{2}\left(L_1^2 + L_2^2 + L_3^2 + L_4^2\right).
\end{aligned} \tag{3.5.2}$$

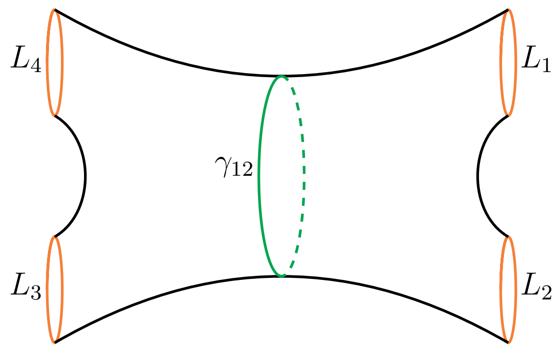
In particular, the Weil-Petersson volume of the moduli space of four-punctured spheres is $2\pi^2$.

We now turn to the computation of amplitudes with four external states. Any such amplitude can be written as

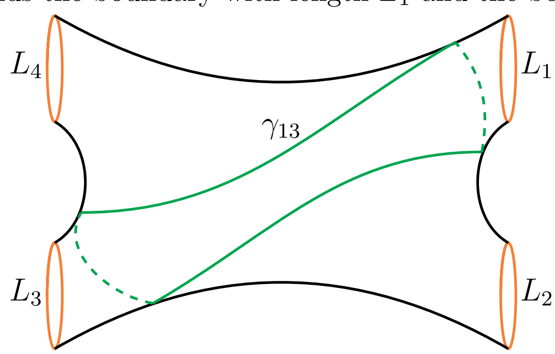
$$\mathcal{A}_{0,4}^{\mathbf{b}} = \int_{\mathcal{M}_{0,4}} \mathbf{\Omega}_{0,4}, \tag{3.5.3}$$

where $\mathbf{\Omega}$ is given by

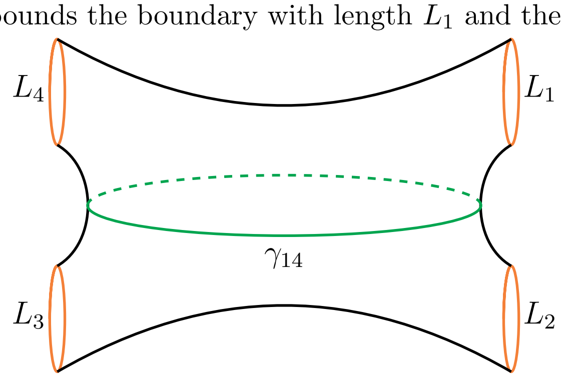
$$\mathbf{\Omega} = \langle \mathcal{R}_{0,4} | \mathcal{B} | \Phi \rangle. \tag{3.5.4}$$



(a) The curve γ_{12} bounds the boundary with length L_1 and the boundary with length L_2.



(b) The curve $\gamma_{13} \in \mathcal{F}_{13}$ bounds the boundary with length L_1 and the boundary with length L_3.



(c) The curve γ_{14} bounds the boundary with length L_1 and the boundary with length L_4.

Figure 3.1: The curves appearing in the statement of Mirzakhani-McShane Identity for spheres with four boundary components.

$\Phi \equiv \Phi_1\Phi_2\Phi_3\Phi_4$ represents the external states. One should be able to write $\mathbf{\Omega}$ in terms of Fenchel-Nielsen coordinates as

$$\mathbf{\Omega}_{0,4} = \Omega_{0,4}(\ell,\tau)d\ell \wedge d\tau. \tag{3.5.5}$$

To be able to perform the integration over the moduli space, we need the form of Mirzakhani-Mcshane Identity for fourth-punctured spheres. It is given by (3.5.1). If we denote the cusps by $\mathfrak{c}_1, \mathfrak{c}_2, \mathfrak{c}_3, \mathfrak{c}_4$, the different pairs of pants decomposition correspond to the following grouping of cusps

$$\begin{aligned}
&\mathcal{P}_1 : \{(\mathfrak{c}_1,\mathfrak{c}_2),(\mathfrak{c}_3,\mathfrak{c}_4)\} &&\longrightarrow &&\text{cutting along } \gamma_2,\\
&\mathcal{P}_2 : \{(\mathfrak{c}_1,\mathfrak{c}_3),(\mathfrak{c}_2,\mathfrak{c}_4)\} &&\longrightarrow &&\text{cutting along } \gamma_3,\\
&\mathcal{P}_3 : \{(\mathfrak{c}_1,\mathfrak{c}_4),(\mathfrak{c}_2,\mathfrak{c}_3)\} &&\longrightarrow &&\text{cutting along } \gamma_4.
\end{aligned} \tag{3.5.6}$$

Therefore, there are three terms in the decomposition of tree-level 4-point bosonic-string amplitudes

$$\begin{aligned}
\mathcal{A}^{\mathbf{b}}_{0,4} &= \sum_{a=2}^{4} \int_{\mathcal{M}^{\gamma_a}_{0,4}} d\ell_{\gamma_a} d\tau_{\gamma_a} \frac{2\Omega^a_{0,4}(\ell_{\gamma_a},\tau_{\gamma_a})}{1+\exp\left(\frac{\ell_{\gamma_a}}{2}\right)}\\
&= \sum_{a=2}^{4} \int_0^\infty d\ell_{\gamma_a} \int_0^{\ell_{\gamma_a}} d\tau_{\gamma_a} \frac{2\Omega^a_{0,4}(\ell_{\gamma_a},\tau_{\gamma_a})}{1+\exp\left(\frac{\ell_{\gamma_a}}{2}\right)}.
\end{aligned} \tag{3.5.7}$$

To proceed, we consider two separate cases

1. **On-Shell Amplitudes:** For the cases that Φ_is represent states that do not undergo mass renormalization, one can construct $\Omega^a_{0,4}$ in terms of Fenchel-Nielsen coordinates and integrate (3.5.7) to get the associated on-shell amplitude.

2. **Off-Shell Amplitudes:** For off-shell amplitudes, we need to consider additional complications. As it is clear from figure 3.2, all curves γ_1, γ_2, and γ_3 are separating-type simple closed curves. We thus need to impose the condition of gluing-compatibility on all separating-type curves. This means that we need to evaluate $\mathbf{\Omega}_{0,4}$ in different regions of moduli space using hyperbolic metrics in those regions. The local coordinates around the marked points are given by (3.2.5), (3.2.11), and (3.2.10) in the

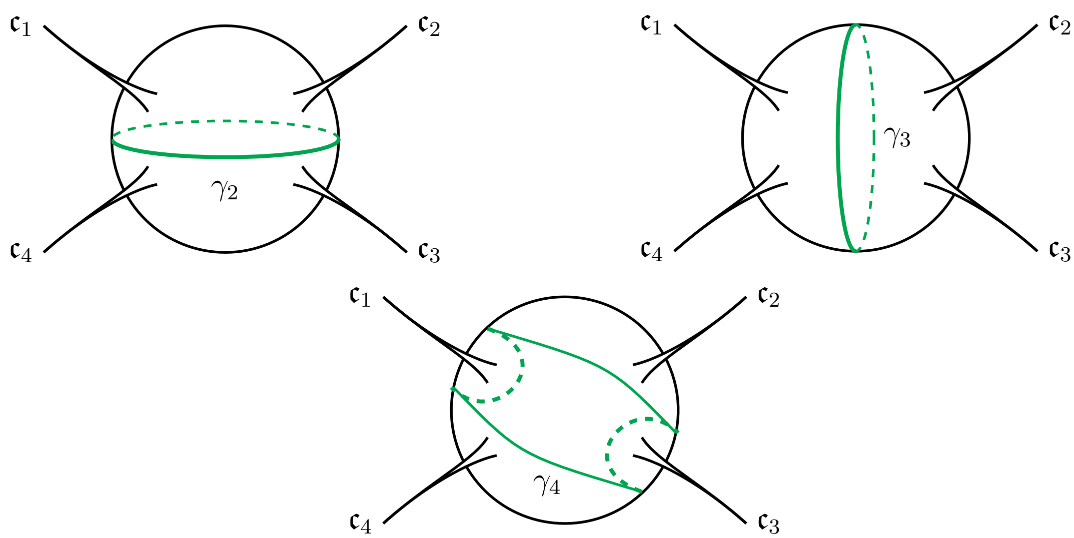


Figure 3.2: Spheres with four punctures with hyperbolic metric. External vertex operators are inserted at cusps $\mathfrak{c}_a$, $a = 1, 2, 3, 4$.

respective region.

$$\mathcal{A}^{\mathbf{b}}_{0,4} = \sum_{a=2}^{4} \Bigg\{ \int_0^{c_*} d\ell_{\gamma_a} \int_0^{\ell_{\gamma_a}} d\tau_{\gamma_a} \frac{2\Omega^{a,1}_{0,4}(\ell_{\gamma_a}, \tau_{\gamma_a})}{1 + \exp\left(\frac{\ell_{\gamma_a}}{2}\right)} + \int_{c_*}^{(1+\epsilon)c_*} d\ell_{\gamma_a} \int_0^{\ell_{\gamma_a}} d\tau_{\gamma_a} \frac{2\Omega^{a,2}_{0,4}(\ell_{\gamma_a}, \tau_{\gamma_a})}{1 + \exp\left(\frac{\ell_{\gamma_k}}{2}\right)}$$
$$+ \int_{(1+\epsilon)c_*}^{\infty} d\ell_{\gamma_a} \int_0^{\ell_{\gamma_a}} d\tau_{\gamma_a} \frac{2\Omega^{a,3}_{0,4}(\ell_{\gamma_a}, \tau_{\gamma_a})}{1 + \exp\left(\frac{\ell_{\gamma_k}}{2}\right)} \Bigg\}. \tag{3.5.8}$$

This concludes the discussion of tree-level 4-point amplitudes in bosonic-string theory.

3.5.2 Tadpole Amplitudes in Bosonic-String Theory

We next consider tadpole diagrams in bosonic-string theory. We again start with the computation of Weil-Petersson volume of $\mathcal{M}_{1,1}(L)$, the moduli space of tori with a boundary of length L. This surface is illustrated in figure 3.3. To simplify the computations, we first multiply the volume by the length L of the boundary. Following the integration procedure,

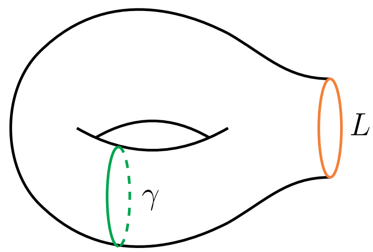


Figure 3.3: A torus with a boundary of length L. Cutting the surface along γ give a pair of pants.

we get

$$\begin{aligned} L \cdot \mathrm{Vol}_{\mathbf{WP}}(\mathcal{M}_{1,1}(L)) &= \int_{\mathcal{M}_{1,1}(L)} \sum_{\gamma_1,\gamma_2\in\mathcal{F}} \mathcal{D}_1(L_1, L_{\gamma_1}, L_{\gamma_2}) = \int_{\mathcal{M}_{1,1}(L)} \sum_{\gamma\in\mathcal{F}} \mathcal{D}_1(L_1, L_\gamma, L_\gamma) \\ &= \int_{\mathcal{M}^{\gamma}_{1,1}(L)} \mathcal{D}_1(L_1, L_\gamma, L_\gamma) = \int_0^\infty d\ell_\gamma \int_0^{\ell_\gamma} \mathcal{D}_1(L_1, L_\gamma, L_\gamma) \\ &= 2\int_0^\infty d\ell\, \ell \ln\left(\frac{\exp \ell + \exp\left(\frac{L}{2}\right)}{\exp \ell + \exp\left(-\frac{L}{2}\right)}\right) = \frac{\pi^2 L}{6} + \frac{L^3}{24}, \end{aligned} \tag{3.5.9}$$

where $\mathcal{D}_1$ is given by (3.C.23). We thus get

$$\mathrm{Vol}_{\mathbf{WP}}(\mathcal{M}_{1,1}(L)) = \frac{\pi^2}{6} + \frac{L^3}{24}. \tag{3.5.10}$$

In particular, (3.3.17) is reproduced in the limit $L \longrightarrow 0$.

We now turn to the computation of tadpole amplitudes. A generic tadpole amplitude in the bosonic-string theory, whose diagram is illustrated in figure 3.4, has the following form

$$\mathcal{A}^{\mathbf{b}}_{1,1} = \int_{\mathcal{M}_{1,1}} \mathbf{\Omega}_{1,1}, \tag{3.5.11}$$

where

$$\mathbf{\Omega}_{1,1} = \langle \mathcal{R}_{1,1} | \mathcal{B} | \Phi \rangle = \Omega_{1,1}(\ell,\tau) d\ell \wedge d\tau. \tag{3.5.12}$$

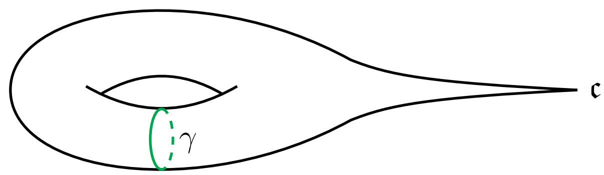


Figure 3.4: A tadpole diagram in bosonic-string theory. The external vertex operator is inserted on the cusp $\mathfrak{c}$.

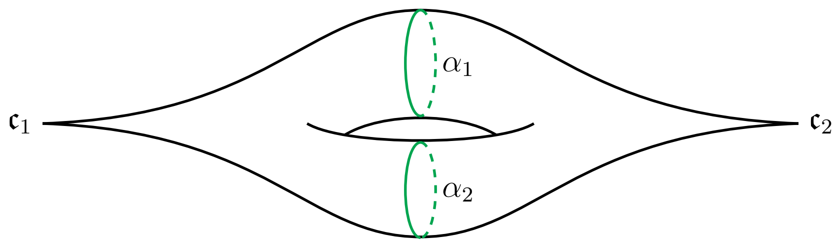
We proceed as follows

$$\begin{aligned}
\mathcal{A}^{\mathbf{b}}_{1,1} &= \int_{\mathcal{M}_{1,1}} \mathbf{\Omega}_{1,1} = \sum_{\gamma\in\mathcal{F}_1}\int_{\mathcal{M}_{1,1}} \frac{2\mathbf{\Omega}_{1,1}(\ell_\gamma,\tau_\gamma)}{1+\exp\ell_\gamma} \\
&= \int_{\mathcal{M}^{\gamma}_{1,1}} \frac{2\mathbf{\Omega}_{1,1}(\ell_\gamma,\tau_\gamma)}{1+\exp\ell_\gamma} \\
&= \int_0^\infty d\ell_\gamma \int_0^{\ell_\gamma} d\tau_\gamma\, \frac{2\Omega_{1,1}(\ell_\gamma,\tau_\gamma)}{1+\exp\ell_\gamma}.
\end{aligned} \tag{3.5.13}$$

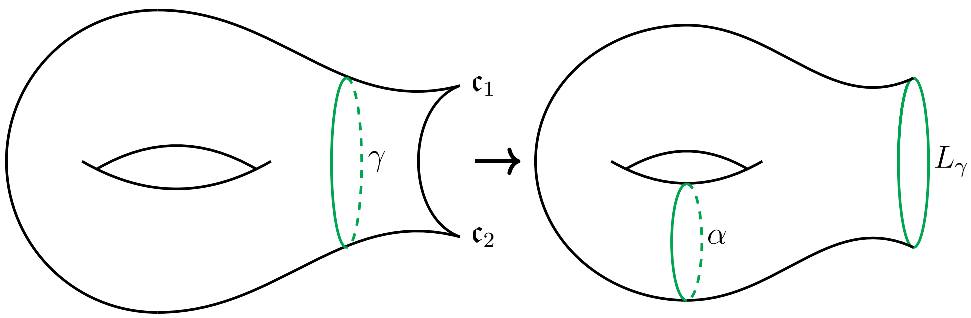
This is the final result for on-shell tadpole amplitude that the associated state does not undergo mass renormalization. However, we have one extra complication for off-shell tadpole amplitude. As it is clear from the figure 3.4, the curve γ is a non-separating-type simple closed curve. Therefore, the tadpole diagram belongs to 1PI region of the moduli space for all values of ℓ_γ. For an off-shell tadpole amplitude, we thus have

$$\begin{aligned}
\mathcal{A}^{\mathbf{b}}_{1,1} &= \int_0^\infty d\ell_\gamma \int_0^{\ell_\gamma} d\tau_\gamma \frac{2\Omega_{1,1}(\ell_\gamma,\tau_\gamma)}{1+\exp\ell_\gamma} \\
&= \int_0^{c_*} d\ell_\gamma \int_0^{\ell_\gamma} d\tau_\gamma \frac{2\Omega^1_{1,1}(\ell_\gamma,\tau_\gamma)}{1+\exp\ell_\gamma} + \int_{c_*}^\infty d\ell_\gamma \int_0^{\ell_\gamma} d\tau_\gamma \frac{2\Omega^2_{1,1}(\ell_\gamma,\tau_\gamma)}{1+\exp\ell_\gamma},
\end{aligned} \tag{3.5.14}$$

where the local coordinate around the cusp in the respective region of the moduli space is given by (3.2.5) and (3.2.11).



(a) The curves α_1 and α_2 together with the cusp $\mathfrak{c}_1$ removes a pair of pants from the surface. These curves appear in the first term of The McShane Identity for hyperbolic punctures surface (3.C.28).



(b) The curve γ together with cusps $\mathfrak{c}_1$ and $\mathfrak{c}_2$ separates a pairs of pants from the surface. It is the curve that appears in the second term of The McShane Identity for hyperbolic punctures surface (3.C.28). The resulting surface is a one-holed torus with a boundary of length L_γ. Any simple closed curve which is nonhomotopic to the resulting boundary gives a pair of pants. One of such curves is the curve α.

Figure 3.5: The possible pants decomposition of a torus with two cusps.

3.5.3 One-Loop Two-Point Amplitudes in Bosonic-String Theory

We next consider one-loop two-point amplitudes in bosonic-string theory. Let us compute the Weil-Petersson volume of $\mathcal{M}_{1,2}$, i.e. the moduli space of twice-punctured tori. We use the following form of The McShane Identity for hyperbolic punctured surface

$$\sum_{\{\alpha_1,\alpha_2\}\in\mathcal{F}_1} \frac{2}{1+\exp\left(\frac{\ell_{\alpha_1}(\mathcal{R}_{1,2})+\ell_{\alpha_2}(\mathcal{R}_{1,2})}{2}\right)} + \sum_{\gamma\in\mathcal{F}_{1,2}} \frac{2}{1+\exp\left(\frac{\ell_\gamma(\mathcal{R}_{1.2})}{2}\right)} = 1, \tag{3.5.15}$$

where curves α_1, α_2 are shown in figure 3.5a, and the curve γ is shown in figure 3.5b on a twice-cusped tori $\mathcal{R}_{1,2}$. We proceed as follows

$$
\begin{aligned}
\mathrm{Vol}_{\mathbf{WP}}(\mathcal{M}_{1,2}) &= \int_{\mathcal{M}_{1,2}} \sum_{\{\alpha_1,\alpha_2\}\in\mathcal{F}_1} \frac{2}{1+\exp\left(\frac{\ell_{\alpha_1}(\mathcal{R}_{1,2})+\ell_{\alpha_2}(\mathcal{R}_{1,2})}{2}\right)} + \sum_{\gamma\in\mathcal{F}_{1,2}} \frac{2}{1+\exp\left(\frac{\ell_\gamma(\mathcal{R}_{1.2})}{2}\right)} \\
&= \int_{\mathcal{M}_{1,1}^{\{\alpha_1,\alpha_2\}}} \frac{1}{1+\exp\left(\frac{\ell_{\alpha_1}(\mathcal{R}_{1,2})+\ell_{\alpha_2}(\mathcal{R}_{1,2})}{2}\right)} + \int_{\mathcal{M}_{1,1}^{\gamma}} \frac{2}{1+\exp\left(\frac{\ell_\gamma(\mathcal{R}_{1.2})}{2}\right)} \\
&= \int_0^\infty d\ell_{\alpha_1} \int_0^\infty d\ell_{\alpha_2} \frac{\ell_{\alpha_1}\ell_{\alpha_2}}{1+\exp\left(\frac{\ell_{\alpha_1}+\ell_{\alpha_2}}{2}\right)} + \int_0^\infty d\ell_\gamma \int_0^{\ell_\gamma} d\tau_\gamma \frac{2}{1+\exp\left(\frac{\ell_\gamma}{2}\right)} \int_{\mathcal{M}_{1,1}(\ell_\gamma)} 1.
\end{aligned}
$$

We know that

$$
\int_{\mathcal{M}_{1,1}(\ell_\gamma)} 1 = \mathrm{Vol}_{\mathbf{WP}}(\mathcal{M}_{1,1}(\ell_\gamma)) = \frac{\pi^2}{6} + \frac{\ell_\gamma^2}{24}. \tag{3.5.16}
$$

Using this, we get

$$
\mathrm{Vol}_{\mathbf{WP}}(\mathcal{M}_{1,2}) = \int_0^\infty d\ell_{\alpha_1} \int_0^\infty d\ell_{\alpha_2} \frac{\ell_{\alpha_1}\ell_{\alpha_2}}{1+\exp\left(\frac{\ell_{\alpha_1}+\ell_{\alpha_2}}{2}\right)} + \int_0^\infty d\ell_\gamma \int_0^{\frac{\ell_\gamma}{2}} d\tau_\gamma \frac{2\left(\frac{\pi^2}{6}+\frac{\ell_\gamma^2}{24}\right)}{1+\exp\left(\frac{\ell_\gamma}{2}\right)}.
$$

These integral can be computed using Mathematica.

$\mathbf{Integrate}\left[\frac{xy}{1+\mathrm{Exp}\left[\frac{x+y}{2}\right]}, \{x,0,\infty\}, \{y,0,\infty\}\right].$

$\frac{7\pi^4}{45}.$

$\mathbf{Integrate}\left[\frac{x\left(\frac{\pi^2}{6}+\frac{x^2}{24}\right)}{1+\mathrm{Exp}\left[\frac{x}{2}\right]}, \{x,0,\infty\}\right].$

$\frac{17\pi^4}{180}.$

We thus finally get

$$
\mathrm{Vol}_{\mathbf{WP}}(\mathcal{M}_{1,2}) = \frac{7\pi^4}{45} + \frac{17\pi^4}{180} = \frac{\pi^4}{4}. \tag{3.5.17}
$$

This result is in apparent conflict with (1.3.4). However, this conflict is related to different normalizations used in the two approaches. A more-involved but similar computation shows that the Weil-Petersson volume of the moduli space of tori with two boundary components having fixed lengths $\mathbf{L} = \{L_1, L_2\}$ is given by

$$\mathrm{Vol}_{\mathbf{WP}}(\mathcal{M}_{1,2}(\mathbf{L})) = \frac{(4\pi^2 + L_1^2 + L_2^2)\,(12\pi^2 + L_1^2 + L_2^2)}{192}. \tag{3.5.18}$$

We now turn to the computation of amplitudes. Any one-loop two-point amplitude can be written as

$$\mathcal{A}^{\mathbf{b}}_{1,2} = \int_{\mathcal{M}_{1,1}} \mathbf{\Omega}_{1,2}. \tag{3.5.19}$$

Using the above identity, we have

$$\begin{aligned}\mathcal{A}^{\mathbf{b}}_{1,2} &= \int_{\mathcal{M}_{1,1}^{(\alpha_1,\alpha_2)}} \frac{2\mathbf{\Omega}_{1,2}}{1+\exp\left(\frac{\ell_{\alpha_1}+\ell_{\alpha_2}}{2}\right)} + \frac{\mathbf{\Omega}_{1,2}}{1+\exp\left(\frac{\ell_\gamma}{2}\right)} \\ &= \int_0^\infty d\ell_{\alpha_1} \int_0^{\ell_{\alpha_1}} d\tau_{\alpha_1} \int_0^\infty d\ell_{\alpha_2} \int_0^{\ell_{\alpha_2}} d\tau_{\alpha_2} \frac{2\,\Omega_{1,2}(\ell_{\alpha_1},\tau_{\alpha_1};\ell_{\alpha_2},\tau_{\alpha_2})}{1+\exp\left(\frac{\ell_{\alpha_1}+\ell_{\alpha_2}}{2}\right)} \\ &+ \int_0^\infty d\ell_\gamma \int_0^{\frac{\ell_\gamma}{2}} d\tau_\gamma \int_{\mathcal{M}_{1,1}(\ell_\gamma)} \frac{\Omega_{1,2}(\ell_\gamma,\tau_\gamma;\ell_\alpha,\tau_\alpha)}{1+\exp\left(\frac{\ell_\gamma}{2}\right)}.\end{aligned} \tag{3.5.20}$$

To proceed further, we need to insert the identity associated to the one-holed tori which is given by

$$\sum_{\alpha\in\mathcal{F}_\gamma} \frac{\mathcal{D}_1(\ell_\gamma, \ell_\alpha(\mathcal{R}_{1,1}(\ell_\gamma)), \ell_\alpha(\mathcal{R}_{1,1}(\ell_\gamma)))}{\ell_\gamma} = 1. \tag{3.5.21}$$

Using this identity, the final expression can be written as

$$\begin{aligned}\mathcal{A}^{\mathbf{b}}_{1,2} &= \int_0^\infty d\ell_{\alpha_1} \int_0^{\ell_{\alpha_1}} d\tau_{\alpha_1} \int_0^\infty d\ell_{\alpha_2} \int_0^{\ell_{\alpha_2}} d\tau_{\alpha_2} \frac{2\,\Omega_{1,2}(\ell_{\alpha_1},\tau_{\alpha_1};\ell_{\alpha_2},\tau_{\alpha_2})}{1+\exp\left(\frac{\ell_{\alpha_1}+\ell_{\alpha_2}}{2}\right)} \\ &+ \int_0^\infty d\ell_\gamma \int_0^{\frac{\ell_\gamma}{2}} d\tau_\gamma \int_0^\infty d\ell_\alpha \int_0^{\ell_\alpha} d\tau_\alpha \frac{\mathcal{D}_1(\ell_\gamma, \ell_\alpha(\mathcal{R}_{1,1}(\ell_\gamma)), \ell_\alpha(\mathcal{R}_{1,1}(\ell_\gamma)))}{\ell_\gamma} \frac{\Omega_{1,2}(\ell_\gamma,\tau_\gamma;\ell_\alpha,\tau_\alpha)}{1+\exp\left(\frac{\ell_\gamma}{2}\right)}.\end{aligned} \tag{3.5.22}$$

This is the final expression for any one-loop two-point on-shell amplitude in the bosonic-string theory. However, there are some complications for off-shell amplitudes. Let us denote the first term in (3.5.22) by $\mathcal{A}^{\mathbf{b};(1)}_{1,2}$ and the second term by $\mathcal{A}^{\mathbf{b};(2)}_{1,2}$. We then have

$$\begin{aligned}\mathcal{A}^{\mathbf{b};(1)}_{1,2} &= \int_0^{c_*} d\ell_{\alpha_1}\int_0^{\ell_{\alpha_1}} d\tau_{\alpha_1}\left\{\int_0^{c^*} d\ell_{\alpha_2}\int_0^{\ell_{\alpha_2}} d\tau_{\alpha_2} + \int_{c_*}^{\infty} d\ell_{\alpha_2}\int_0^{\ell_{\alpha_2}} d\tau_{\alpha_2}\right\}\frac{2\,\Omega_{1,2}(\ell_{\alpha_1},\tau_{\alpha_1};\ell_{\alpha_2},\tau_{\alpha_2})}{1+\exp\left(\frac{\ell_{\alpha_1}+\ell_{\alpha_2}}{2}\right)}\\ &+ \int_{c_*}^{\infty} d\ell_{\alpha_1}\int_0^{\ell_{\alpha_1}} d\tau_{\alpha_1}\left\{\int_0^{c^*} d\ell_{\alpha_2}\int_0^{\ell_{\alpha_2}} d\tau_{\alpha_2} + \int_{c_*}^{\infty} d\ell_{\alpha_2}\int_0^{\ell_{\alpha_2}} d\tau_{\alpha_2}\right\}\frac{2\,\Omega_{1,2}(\ell_{\alpha_1},\tau_{\alpha_1};\ell_{\alpha_2},\tau_{\alpha_2})}{1+\exp\left(\frac{\ell_{\alpha_1}+\ell_{\alpha_2}}{2}\right)}.\end{aligned}$$

Similary, we have

$$\begin{aligned}\mathcal{A}^{\mathbf{b};(2)}_{1,2} &= \int_0^{c_*} d\ell_{\gamma}\int_0^{\frac{\ell_\gamma}{2}} d\tau_{\gamma}\left\{\int_0^{c_*} d\ell_{\alpha}\int_0^{\ell_{\alpha}} d\tau_{\alpha} + \int_{c_*}^{\infty} d\ell_{\alpha}\int_0^{\ell_{\alpha}} d\tau_{\alpha}\right\}\frac{\mathcal{D}_1(\ell_\gamma,\ell_\alpha(\mathcal{R}_{1,1}(\ell_\gamma)),\ell_\alpha(\mathcal{R}_{1,1}(\ell_\gamma)))}{\ell_\gamma}\frac{\Omega_{1,2}(\ell_\gamma,\tau_\gamma;\ell_\alpha,\tau_\alpha)}{1+\exp\left(\frac{\ell_\gamma}{2}\right)}\\ &+ \int_{c_*}^{(1+\epsilon)c_*} d\ell_{\gamma}\int_0^{\frac{\ell_\gamma}{2}} d\tau_{\gamma}\left\{\int_0^{c_*} d\ell_{\alpha}\int_0^{\ell_{\alpha}} d\tau_{\alpha} + \int_{c_*}^{\infty} d\ell_{\alpha}\int_0^{\ell_{\alpha}} d\tau_{\alpha}\right\}\frac{\mathcal{D}_1(\ell_\gamma,\ell_\alpha(\mathcal{R}_{1,1}(\ell_\gamma)),\ell_\alpha(\mathcal{R}_{1,1}(\ell_\gamma)))}{\ell_\gamma}\frac{\Omega_{1,2}(\ell_\gamma,\tau_\gamma;\ell_\alpha,\tau_\alpha)}{1+\exp\left(\frac{\ell_\gamma}{2}\right)}\\ &+ \int_{(1+\epsilon)c_*}^{\infty} d\ell_{\gamma}\int_0^{\frac{\ell_\gamma}{2}} d\tau_{\gamma}\left\{\int_0^{c_*} d\ell_{\alpha}\int_0^{\ell_{\alpha}} d\tau_{\alpha} + \int_{c_*}^{\infty} d\ell_{\alpha}\int_0^{\ell_{\alpha}} d\tau_{\alpha}\right\}\frac{\mathcal{D}_1(\ell_\gamma,\ell_\alpha(\mathcal{R}_{1,1}(\ell_\gamma)),\ell_\alpha(\mathcal{R}_{1,1}(\ell_\gamma)))}{\ell_\gamma}\frac{\Omega_{1,2}(\ell_\gamma,\tau_\gamma;\ell_\alpha,\tau_\alpha)}{1+\exp\left(\frac{\ell_\gamma}{2}\right)}.\end{aligned}$$

The final expression is the sum of the above two expressions

$$\mathcal{A}^{\mathbf{b}}_{1,2} = \mathcal{A}^{\mathbf{b};(1)}_{1,2} + \mathcal{A}^{\mathbf{b};(2)}_{1,2}. \tag{3.5.23}$$

These examples clearly show the power of using hyperbolic geometry in computing the amplitudes in string perturbation theory. The only missing ingredient is a method to compute $\mathbf{\Omega}$ in terms of Fenchel-Nielsen coordinates. Once such a method is developed, any amplitude whether on-shell or off-shell can be computed very explicitly using the method explained in this thesis.

3.6 Applications in String Field Theory

In this section, we describe the construction of a consistent string field theory using tools from hyperbolic geometry following [2, 3]. Since all the basic ingredients is already discussed in previous sections, we just emphasize the new pieces. For the sake of discussion, we restrict ourselves to the case of bosonic-string field theory [59]. The case of superstring field theory based on the picture-changing formalism, formulated in [24], can be dealt using similar considerations.

By a consistent string field theory, we mean a description of the string vertices and Feynman diagrams such that the regions corresponding to these pieces provide a single cover of the moduli space. This is ensured if the string vertices $\mathcal{V}_{\mathfrak{g},\mathfrak{n}}$ satisfy the BV quantum master equation

$$\partial\mathcal{V}_{\mathfrak{g},\mathfrak{n}}+\Delta\mathcal{V}_{\mathfrak{g}-1,\mathfrak{n}+2}+\frac{1}{2}\sum_{\substack{\mathfrak{g}_1+\mathfrak{g}_2=\mathfrak{g}\\ \mathfrak{n}_1+\mathfrak{n}_2=\mathfrak{n}+2}}\{\mathcal{V}_{\mathfrak{g}_1,\mathfrak{n}_1},\mathcal{V}_{\mathfrak{g}_2,\mathfrak{n}_2}\}=0, \tag{3.6.1}$$

where ∂ is the operation of taking the boundary, Δ is the operation of gluing two punctures on the same surface, and $\{\cdot,\cdot\}$ is the operation of gluing of two punctures on two disconnected surface[4]. The BV quantum master equation (3.6.1) tells us that the surfaces living at the boundary of string vertex, i.e. surface belong to $\partial\mathcal{V}_{\mathfrak{g},\mathfrak{n}}$, can be constructed by gluing of punctures on a single surface or on two disconnected surface. This is essentially the condition of gluing compatibility that we imposed on the decomposition of moduli space into 1PI and 1PR regions. Actually the basic insights of the formulation of off-shell amplitudes by 1PI decomposition of the moduli space is inspired from string field theory. However, there is a subtle difference. String field theory decomposes the moduli space into string vertices and Feynman diagrams. In this sense, all surface obtained by gluing of punctures on a single surface or two disconnected surfaces must be included in the regions of the moduli space corresponding to the Feynman diagrams of the string field theory. We can then define the following string-field-theory decomposition of the moduli space as follows

Definition 3.3 (The Feynman-Diagrams Region)**.** *Assume that we solve curvature-correction equation and computed the length of the core geodesics to be $l_{a,\mathbf{h}}$ for $a=1,\cdots,m$. Consider the set of all hyperbolic surfaces containing at least one simple closed geodesic, either separating-type or nonseparating-type, whose length is less than or equal to $l_{a,\mathbf{h}}$. We call the region of moduli space containing such surfaces the Feynman-diagrams region of moduli space.*

We note that the first-order approximation to $l_{a,\mathbf{h}}$ is given by

$$l_{a,\mathbf{h}}=c_*+\mathcal{O}(c_*^4). \tag{3.6.2}$$

[4] In the case of heterotic-string and type-II superstring theories, $\Delta=\Delta_{\mathrm{NS}}+\Delta_{\mathrm{R}}$, where Δ_{NS} is the operation of gluing of two NS punctures and Δ_{R} is the operation of gluing of two R punctures on the same surface. Similar comments hold for the operation of gluing of punctures on two disconnected surfaces $\{\cdot,\cdot\}$, i.e. $\{\cdot,\cdot\}=\{\cdot,\cdot\}_{\mathrm{NS}}+\{\cdot,\cdot\}_{\mathrm{R}}$. Note that an NS puncture and an R puncture cannot be glued together.

Regarding this definition, we can define the region of the moduli space corresponding to the string vertex as follows

Definition 3.4 (The String-Vertex Region). *We call the region of the moduli space containing surfaces not belonging to the Feynman-diagram region as the string-vertex region of moduli space.*

Note that if we can solve the curvature-correction equation exactly, we can obtain the exact hyperbolic metric on the family of plumbed surfaces. We can then construct a string field theory in exactly the same manner that we constructed off-shell amplitudes. We proceed as follows. We equip surfaces belong to the string-vertex region with the hyperbolic metric, and the local coordinates around the punctures are induced from it. We equip the surfaces belong to the Feynman-diagrams region of the moduli space with the hyperbolic metric obtained by solving the curvature-correction equation, and the local coordinates around the punctures are induced from it. Finally we equip the surfaces belong to the region between the Feynman-diagrams region and the string-vertex region with an interpolating hyperbolic metric, and the local coordinates around the punctures are induced from it. This prescription provides a concrete realization of the BV quantum master equation (3.6.1), and thus leads to a consistent string field theory. Regarding (3.6.2) and the approximate solution to the curvature-correction equation given by (3.1.50), we can construct an approximate gauge-invariant but explicit bosonic-string field theory using hyperbolic geometry.

The next ingredients is the explicit expression for interaction vertices of the bosonic-string field theory. An interaction vertex in string field theory is the integration of a convenient form over the string-vertex region of moduli space

$$\mathcal{I}_{\mathfrak{g},\mathfrak{n}} \equiv \int_{\mathcal{V}_{\mathfrak{g},\mathfrak{n}}} \mathbf{\Omega}(\mathbf{m}; z_1(\mathbf{m}), \cdots, z_{\mathfrak{n}}(\mathbf{m})), \tag{3.6.3}$$

where $z_a(\mathbf{m})$s are the local coordinates around the punctures depending on $6\mathfrak{g} - 6 + 2\mathfrak{n}$ moduli parameters $\mathbf{m} \equiv \{m_1, \cdots, m_{6\mathfrak{g}-6+2\mathfrak{n}}\}$. Since a surface $\mathcal{R} \in \mathcal{V}_{\mathfrak{g},\mathfrak{n}}$ is equipped with local coordinates around the punctures, the above integral should be understood as the pullback of a form $\mathbf{\Omega}(\mathbf{m}; z_1, \cdots, z_{\mathfrak{n}})$ defined on $\widehat{\mathcal{P}}_{\mathfrak{g},\mathfrak{n}}$ to the moduli space using a section $s : \mathcal{M}_{\mathfrak{g},\mathfrak{n}} \longrightarrow \widehat{\mathcal{P}}_{\mathfrak{g},\mathfrak{n}}$

$$\mathbf{\Omega}(\mathbf{m}; z_1(\mathbf{m}), \cdots, z_{\mathfrak{n}}(\mathbf{m})) = s^*(\mathbf{\Omega}(\mathbf{m}; z_1, \cdots, z_{\mathfrak{n}})). \tag{3.6.4}$$

Let us denote the vertex operator corresponding to the basis-state $|\Phi_s\rangle$ by $\mathcal{Y}(\Phi_s)$, and we define $|\mathcal{Y}_s, p\rangle \equiv \mathcal{Y}(\Phi_s)|\mathbf{1}, p\rangle$. The string field entering in the BV quantum master action can be expressed as

$$|\Psi\rangle = \sum_{\mathbb{G}(\Phi_s)\leq 2}{}' \sum_p \psi^s(p)\mathcal{Y}(\Phi_s)|\mathbf{1}, p\rangle + \sum_{\mathbb{G}(\Phi_s)\leq 2}{}' \sum_p \psi^*_s(p)\mathcal{Y}(\widetilde{\Phi}_s)|\mathbf{1}, p\rangle, \tag{3.6.5}$$

where $|\mathbf{1}, p\rangle$ denotes the $\mathrm{SL}(2,\mathbb{C})$-invariant family of vacua for the worldsheet CFT for the closed bosonic-string theory, parameterized by p, $\mathbb{G}(\Phi_s)$ is the ghost number of the basis-state Φ_s, and $\psi^s(p)$ and $\psi^*_s(p)$ are the target-space fields and antifields respectively. The state $|\widetilde{\Phi}\rangle$ is defined by $|\widetilde{\Phi}\rangle \equiv b_0^-|\Phi_s^c\rangle$, where $\langle \Phi_r^c|\Phi_s\rangle = \delta_{rs}$ [59]. The prime over the summation sign reminds us that the sum is only over those states that are annihilated by L_0^-. Using this expression, the BV quantum master action in terms of the target space fields and the target space antifields is given by

$$\begin{aligned} S(\Psi) = &\frac{1}{2g_s^2} \sum_{\substack{\mathbb{G}(\Phi_{s_1})\leq 2 \\ \mathbb{G}(\Phi_{s_2})\leq 2}}{}' \sum_{\substack{\phi^{s_1}\in\mathcal{S} \\ \phi^{s_2}\in\mathcal{S}}} \sum_{p_1,p_2} \phi^{s_1}(p_1)\mathbf{P}_{s_1s_2}(p_1,p_2)\,\phi^{s_2}(p_2) \\ &+ \sum_{\substack{\mathfrak{g}\geq 0 \\ \mathfrak{n}\geq 1}} \frac{\hbar^{\mathfrak{g}} g_s^{2\mathfrak{g}-2+\mathfrak{n}}}{\mathfrak{n}!} \sum_{\substack{\mathbb{G}(\Phi_{s_1})\leq 2 \\ \vdots \\ \mathbb{G}(\Phi_{s_\mathfrak{n}})\leq 2}}{}' \sum_{\substack{\phi^{s_1}\in\mathcal{S} \\ \vdots \\ \phi^{s_\mathfrak{n}}\in\mathcal{S}}} \sum_{p_1,\cdots,p_\mathfrak{n}} \mathbf{V}^{\mathfrak{g},\mathfrak{n}}_{s_1\cdots s_\mathfrak{n}}(p_1,\cdots,p_\mathfrak{n})\,\phi^{s_1}(p_1)\cdots\phi^{s_\mathfrak{n}}(p_\mathfrak{n}), \end{aligned} \tag{3.6.6}$$

where $\mathcal{S} = \{\psi^s, \psi^*_s\}$ is the set of all fields and antifields of the closed bosonic-string field theory spectrum, and g_s is the string coupling constant. $\mathbf{P}_{s_1s_2}(p_1,p_2)$, the inverse of the propagator, is given by

$$\mathbf{P}_{s_1s_2}(p_1,p_2) \equiv \left\langle \mathcal{Y}_{s_1}, p_1 \left| c_0^- \mathcal{Q}_B \right| \mathcal{Y}_{s_2}, p_2 \right\rangle, \tag{3.6.7}$$

where $\mathcal{Q}_B$ is the BRST operator of the theory, and

$$c_0^- \equiv \frac{1}{2}(c_0 - \bar{c}_0). \tag{3.6.8}$$

$\mathbf{V}^{\mathfrak{g},\mathfrak{n}}_{s_1\cdots s_\mathfrak{n}}(p_1,\cdots,p_\mathfrak{n})$, the $\mathfrak{g}$-loop interaction vertex of $\mathfrak{n}$ target spacetime fields/antifields

$\{\phi^{s_1}(p_1), \cdots, \phi^{s_n}(p_n)\}$, is given by

$$\mathbf{V}^{\mathfrak{g},\mathfrak{n}}_{s_1\cdots s_\mathfrak{n}}(p_1, \cdots, p_\mathfrak{n}) \equiv \int_{\mathcal{V}_{\mathfrak{g},\mathfrak{n}}} \mathbf{\Omega}\left(\mathbf{m}; |\mathcal{Y}_{s_1}, p_1\rangle, \cdots, |\mathcal{Y}_{s_\mathfrak{n}}, p_\mathfrak{n}\rangle\right). \tag{3.6.9}$$

Here, the ath string field is inserted at the ath puncture using the local coordinates $z_a(\mathbf{m})$. The state $|\mathcal{Y}_{s_a}, p_a\rangle$ is annihilated by both $b_0^{(a)-}$ and $L_0^{(a)-}$, where these operators act on the state-space of the ath puncture. To be able to do computation in the bosonic-string field theory, we need to explicitly evaluate the BV quantum master action. The explicit evaluation of the BV quantum master action requires the explicit evaluation of $\mathbf{V}^{\mathfrak{g},\mathfrak{n}}_{s_1\cdots s_\mathfrak{n}}(p_1, \cdots, p_\mathfrak{n})$. The explicit evaluation requires

1. A convenient choice of parametrization of the Teichmuüller space and the conditions on them that specify the string-vertex region of the moduli space inside the Teichmüller space.

2. A choice of local coordinates around the punctures on surfaces belong to the string-vertex region.

3. An explicit procedure for constructing the off-shell string measure in terms of the chosen coordinates of the moduli space in terms of parameters $\mathbf{m}$.

4. Finally, a procedure for integrating the off-shell string measure over the string-vertex region inside the moduli space.

Regarding what we have explained in 3.3.2, and the definition of the string-vertex region given above, (3.6.9) can be explicitly computed once we compute $\mathbf{\Omega}\left(\mathbf{m}; |\mathcal{Y}_{s_1}, p_1\rangle, \cdots, |\mathcal{Y}_{s_\mathfrak{n}}, p_\mathfrak{n}\rangle\right)$ in terms of the Fenchel-Nielsen coordinates. The region of integration is given by

$$l_{a,\mathbf{h}} \leq \ell_a < \infty, \qquad 0 \leq \tau_a \leq 2^{-M_{\gamma_a}}\ell_a, \qquad a = 1, \cdots, 3\mathfrak{g} - 3 + \mathfrak{n}. \tag{3.6.10}$$

Note that the first-order approximation to $l_{a,\mathbf{h}}$ is given by (3.6.2). As an example consider the four-leg genus-0 string vertex $\mathbf{V}^{0,4}_{s_1\cdots s_4}(p_1, \cdots, p_4)$. Considerations similar to those led to (3.5.7) can be used to compute this vertex. The result is

$$\mathbf{V}^{0,4}_{s_1\cdots s_4}(p_1, \cdots, p_4) = \sum_{a=2}^{4} \int_{l_{a,\mathbf{h}}}^{\infty} d\ell_{\gamma_a} \int_0^{\ell_{\gamma_a}} d\tau_{\gamma_a} \frac{2\mathbf{\Omega}\left(\ell_{\gamma_a}, \tau_{\gamma_a}; |\mathcal{Y}_{s_1}, p_1\rangle, \cdots, |\mathcal{Y}_{s_4}, p_4\rangle\right)}{1 + \exp\left(\frac{\ell_{\gamma_a}}{2}\right)}. \tag{3.6.11}$$

Using (3.6.2), this can be written explicitly as

$$\mathbf{V}^{0,4}_{s_1\cdots s_4}(p_1,\cdots,p_4) = \sum_{a=2}^{4}\int_{c_*}^{\infty} d\ell_{\gamma_a}\int_0^{\ell_{\gamma_a}} d\tau_{\gamma_a}\frac{2\mathbf{\Omega}\left(\ell_{\gamma_a},\tau_{\gamma_a};|\mathcal{Y}_{s_1},p_1\rangle,\cdots,|\mathcal{Y}_{s_4},p_4\rangle\right)}{1+\exp\left(\frac{\ell_{\gamma_a}}{2}\right)}. \tag{3.6.12}$$

Alternatively, we can keep the region of integration as (3.3.44) by extending the form $\mathbf{\Omega}\left(\mathbf{m};|\mathcal{Y}_{s_1},p_1\rangle,\cdots,|\mathcal{Y}_{s_\mathrm{n}},p_\mathrm{n}\rangle\right)$ by zero to the whole moduli space.

Once interaction vertices $\mathbf{V}^{g,\mathrm{n}}_{s_1\cdots s_\mathrm{n}}(p_1,\cdots,p_\mathrm{n})$ is computed explicitly by the prescription explained, (3.6.6) is the complete action for the bosonic-string field theory. This action resembles the action of a quantum field theory with infinite number of interaction terms. We can use the usual methods of quantum field theory to compute the physically-interesting quantities. Similar comments hold for the heterotic-string and type-II superstring field theories except that the definition of string vertices involves a choice of locations of PCOs. This concludes our discussion of string field theory. More details can be found in [2, 3].

Appendix

3.A Parametrizations of the Space of Riemann Surfaces

There are various ways to characterize a Riemann surface of a given topological type $(\mathfrak{g}, \mathfrak{n})$, i.e. to describe a point in the parametrization space of such surfaces. In this appendix, we briefly explain some of these parametrizations.

3.A.1 The Period Matrix

The period matrix for a genus-$\mathfrak{g}$ surface $\mathcal{R}$ is a natural generalization of the complex parameter of the torus. Using the fact that there are $\mathfrak{g}$ holomorphic and $\mathfrak{g}$ anti-holomorphic Abelian differential forms ω_a and $\overline{\omega}_a$ on a compact genus-$\mathfrak{g}$ Riemann surfaces, one can choose a basis for such forms such that:

$$\begin{aligned} \oint_{\alpha_a} \omega_b &= \delta_{ab}, \qquad\qquad a, b = 1, \cdots, \mathfrak{g} \\ \oint_{\beta_a} \omega_b &= \tau_{ab}, \qquad\qquad a, b = 1, \cdots, \mathfrak{g}. \end{aligned} \tag{3.A.1}$$

Where the set $(\alpha_1, \cdots, \alpha_{\mathfrak{g}}; \beta_1, \cdots, \beta_{\mathfrak{g}})$ is a basis for first homology group $H_1(\mathcal{R}, \mathbb{Z})$, with the following properties:

$$\langle \alpha_a, \beta_b \rangle = \delta_{ab} \qquad \langle \alpha_a, \alpha_b \rangle = 0 \qquad \langle \beta_a, \beta_b \rangle = 0 \tag{3.A.2}$$

and $\langle \cdot, \cdot \rangle : H_1(\mathcal{R}, \mathbb{Z}) \times H_1(\mathcal{R}, \mathbb{Z}) \longrightarrow \mathbb{Z}$ is the intersection pairing on the first homology group. $\boldsymbol{\tau} \equiv [\tau_{ab}]$ is called the *period matrix* of the surface $\mathcal{R}$. $\boldsymbol{\tau}$ is symmetric (which can be seen using Riemann's bilinear relations), and so it has $\frac{1}{2}\mathfrak{g}(\mathfrak{g}+1)$ independent elements, and its imaginary part is positive definite. There are pros and cons of using period matrix for characterization of Riemann surfaces of a given topological type. The good properties for such a parametrization of the moduli space are as follows:

1. The period matrix can be defined explicitly for any Riemann surface;

2. The theta function, in terms of which one can compute one-loop amplitudes, can be generalized for genus-$\mathfrak{g}$ surfaces and is defined in terms of period matrix of the surface. As we mentioned above, the correlation function on a fixed Riemann surface $\mathcal{R}$ is known in terms of well-defined quantities on $\mathcal{R}$, namely theta-function and prime form [37]. This means that one can in principle do the integration over the moduli space.

3. Modular invariance of the correlation functions on a fixed surface can be checked explicitly;

4. A degenerating surface $\mathcal{R}(t)$ near the points at infinity of $\overline{\mathcal{M}}_{\mathfrak{g},\mathfrak{n}}$ can be constructed by gluing two surfaces $\mathcal{R}_1$ and $\mathcal{R}_2$ using a specific gluing procedure. t is a complex parameter whose absolute value is related to the length of the cylinder connecting the two component surfaces. The period matrix of the degenerating surface $\mathcal{R}$ can be written in terms of the component surfaces as [112, 61]:

$$\boldsymbol{\tau}_{\mathcal{R}(t)} = \left(\begin{array}{c|c} \boldsymbol{\tau}_{\mathcal{R}_1} & 0 \\ \hline 0 & \boldsymbol{\tau}_{\mathcal{R}_2} \end{array}\right) + \mathcal{O}(t), \qquad 0 \leq |t| \leq 1, \tag{3.A.3}$$

   and the corrections can be systematically computed;

Due to these pros, period matrix is the favorite parametrization of the moduli space for string theorists. However, there are disadvantages as well:

1. The number of complex moduli parameter of a genus-$\mathfrak{g}$ Riemann surface is $3\mathfrak{g}-3+$ # of CKVs. The difference between the number of parameters is:

$$\frac{1}{2}\mathfrak{g}(\mathfrak{g}+1) - (3\mathfrak{g}-3) - \#\text{ of CKVs} = \frac{1}{2}(\mathfrak{g}-2)(\mathfrak{g}-3) - \#\text{ of CKVs}. \tag{3.A.4}$$

   For $\mathfrak{g} \geq 2$, # of CKVs $= 0$, and there is one complex CKV for $\mathfrak{g}=1$. Therefore, the only cases that the number of complex moduli parameter matches with the number of independent components of the corresponding period matrices are $\mathfrak{g} = 1, 2, 3$. There are redundancies for the description of moduli space in terms of period matrix for $\mathfrak{g} \geq 4$;

2. The explicit integration region is not known for $\mathfrak{g} \geq 2$;

3. As far as we know, the explicit MCG-invariant volume form has not been constructed for genus $\mathfrak{g} \geq 2$;

3.A.2 Schottky Groups

We now turn to the construction of families of Riemann surfaces by quotient of certain covering space by the so-called Schottky group.

The group $\mathtt{PSL}(2, \mathbb{C})$ is defined as matrices M in $\mathtt{GL}(2, \mathbb{C})$ such that

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathtt{GL}(2, \mathbb{C}), \qquad ad - bc = 1, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \sim \begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix}. \tag{3.A.5}$$

This group acts on Riemann sphere $\mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$ as follows

$$z \mapsto M(z) = \frac{az + b}{cz + d}. \tag{3.A.6}$$

An element of $\mathtt{PSL}(2, \mathbb{C})$ is called hyperbolic (or loxodromic) if it has two fixed points, one attractive and one repulsive. If $z_{\mathbf{at}}$ and $z_{\mathbf{re}}$ are attractive and repulsive fixed points, then a hyperbolic element M can be defined as follows

$$\frac{M(z) - z_{\mathbf{at}}}{M(z) - z_{\mathbf{re}}} = \kappa_M \frac{z - z_{\mathbf{at}}}{x - z_{\mathbf{re}}}, \qquad 0 < |k_M| < 1, \tag{3.A.7}$$

where κ_M is called the multiplier of M. This relation shows that any hyperbolic element of $\mathtt{PSL}(2, \mathbb{C})$ is parametrized by three parameters, $z_{\mathbf{at}}$, $z_{\mathbf{re}}$, and κ_M. We then have the following definitions.

Suppose that there are $2\mathfrak{g}$ circles $\mathcal{C}_a$ and $\mathcal{C}'_a$ for $a = 1, \cdots, \mathfrak{g}$ such that 1) there are $\mathfrak{g}$ hyperbolic elements $M_a \in \mathtt{PSL}(2, \mathbb{C})$ such that $M_a(\mathcal{C}_a) = \mathcal{C}'_a$, and 2) these circles bound a connected region $\mathcal{F}$ whose boundary is given by $\partial\mathcal{F} = \sum_{a=1}^{\mathfrak{g}} (\mathcal{C}'_a - \mathcal{C}_a)$.

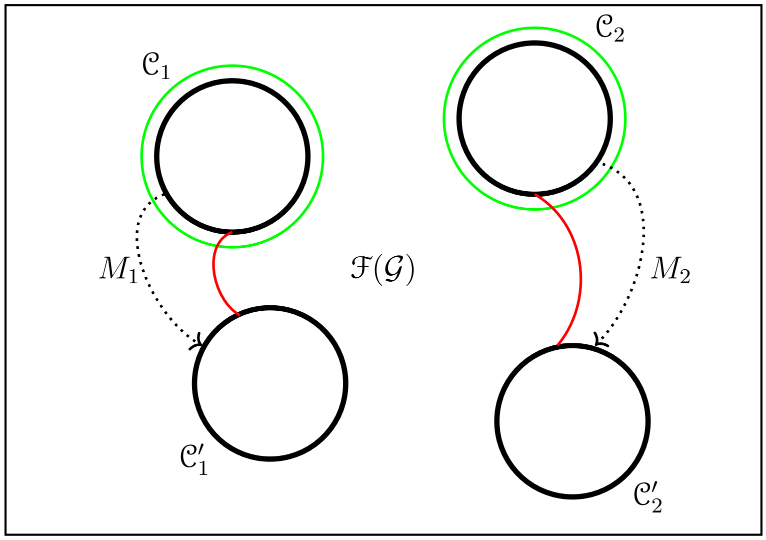


Figure 3.A.1: The construction of a genus-2 Riemann surface using the quotient of a fundamental domain $\overline{\mathcal{F}}(\mathcal{G})$ by a genus-2 Schottky group $\mathcal{G} \equiv \langle M_1, M_2 \rangle$ freely-generated by $\{M_1, M_2\}$. The dotted line show how $\mathcal{C}_a$ is mapped to $\mathcal{C}'_a$ for $a = 1, 2$. The green and red lines denoted the standard 1-cycles of a genus-2 Riemann surface.

Definition 3.5. *The group $\mathcal{G}$ freely-generated by $\mathfrak{g}$ hyperbolic elements M_a is called a genus-$\mathfrak{g}$ Schottky group.*

$\mathcal{F}$ is a fundamental domain for the action of $\mathcal{G}$ whose closure, which is obtained by including the boundary circles $\mathcal{C}_a$ and $\mathcal{C}'_a$, is denoted by $\overline{\mathcal{F}}$. If $\Lambda(\mathcal{F}) \subset \mathbb{CP}^1$ is the limit set of $\mathcal{G}$, i.e. the set of points in Riemann sphere that are not equivalent to a point in $\overline{\mathcal{F}}$, then a genus-$\mathfrak{g}$ Riemann surface $\mathcal{R}_\mathfrak{g}$ can be formed using the following quotient

$$\mathcal{R}_\mathfrak{g} \simeq \frac{\mathbb{CP}^1 - \Lambda(\mathcal{G})}{\mathcal{G}}, \tag{3.A.8}$$

Since all points in $\mathbb{CP}^1 - \Lambda(\mathcal{G})$ are equivalent to a point in $\overline{\mathcal{F}}$, $\mathcal{R}_\mathfrak{g}$ can be constructed by taking $\overline{\mathcal{F}}$ and identify the boundary circles $z \sim M_a(z)$ for $z \in \mathcal{C}_a$ and $M_a(z) \in \mathcal{C}'_a$. Any genus-$\mathfrak{g}$ Riemann surface can be constructed using a Schottky group [113]. The construction of a genus-2 Riemann surface using a Schottky group is illustrated in figure 3.A.1.

If we choose a specific set of $\mathfrak{g}$ generators, the Schottky group is called *a marked genus-$\mathfrak{g}$ Schottky group*. We can then parametrize Schottky groups by $3\mathfrak{g}$ parameters $z_{\mathbf{at},a}$, $z_{\mathbf{re},a}$, and

κ_{M_a}, for $a = 1, \cdots, \mathfrak{g}$, the so-called *Schottky parameters*. However, this parametrization is redundant since two Schottky groups which are conjugate in $\mathtt{PSL}(2,\mathbb{C})$ will give rise to the same genus-$\mathfrak{g}$ Riemann surfaces. The redundancy can be taken care of by fixing the locations of three fixed points. We are thus left with $3\mathfrak{g}-3$ parameters which describe the deformation space of genus-$\mathfrak{g}$ Schottky groups which is called Schottky space [113].

One of the main reasons that the Schottky groups were used in the early literature of string perturbation theory is the relation between Schottky parameters and the gluing of Riemann surfaces [114, 115, 116, 117]. The construction of a genus-$\mathfrak{g}$ Riemann surface from sphere requires $\mathfrak{g}$ gluings and in fact they can be identified with the $\mathfrak{g}$ hyperbolic generators of a marked genus-$\mathfrak{g}$ Schottky group $\mathcal{G} = \langle M_1, \cdots, M_{\mathfrak{g}} \rangle$. The construction is recursive. To construct a genus-$\mathfrak{g}$ Riemann surface $\mathcal{R}_g$, we can start with a genus-$(\mathfrak{g}-1)$ Riemann surface $\mathcal{R}_{\mathfrak{g}-1}$, and we consider two marked points $z_{\mathbf{at},\mathfrak{g}}$ and $z_{\mathbf{re},\mathfrak{g}}$ on it. $z_{\mathbf{at},\mathfrak{g}}$ and $z_{\mathbf{re},\mathfrak{g}}$ are located inside the fundamental domain of $\mathcal{G}_{\mathfrak{g}-1}$, the Schottky group associated with $\mathcal{R}_{\mathfrak{g}-1}$. If we denote the local coordinates around $z_{\mathbf{at},\mathfrak{g}}$ and $z_{\mathbf{re},\mathfrak{g}}$ by z_1 and z_2 respectively, $z_{\mathbf{at},\mathfrak{g}}$ and $z_{\mathbf{re},\mathfrak{g}}$ are located at $z_1 = 0$ and $z_2 = 0$, respectively. One can glue these marked point using the following gluing relation

$$z_1 z_2 = -\kappa_{M_{\mathfrak{g}}}. \tag{3.A.9}$$

The gluing is done by first removing disks $D_i = \{|z_i| < |\kappa_{M_{\mathfrak{g}}}|\}$, for $i = 1, 2$, around the marked points $z_{\mathbf{at},\mathfrak{g}}$ and $z_{\mathbf{re},\mathfrak{g}}$, and then identifying the points $|k_{M_{\mathfrak{g}}}| < |z_i| < 1$ using (3.A.9). The explicit construction of a genus-1 surface by starting from Riemann sphere is given in Appendix B of [118]. Therefore, multipliers of Schottky generators $M_1, \cdots, M_{\mathfrak{g}}$ can be thought of as gluing parameters to produce handles of a genus-$\mathfrak{g}$ surface $\mathcal{R}_{\mathfrak{g}}$. Therefore, it is very easy to describe the nonseparating-type degenerations in the Schottky parameterization, we simply take $\kappa_{M_a} \longrightarrow 0$. However, the description of separating-type degenerations using the Schottky parametrization is more involved, and can be found in section B.1.2 of [118].

We are now mentioning some advantage and disadvantages of the Schottky parametrization of Riemann surfaces. The main advantage of using the Schottky parameter is that the factorization property of amplitudes due to the gluing relations and the Schottky parameters, as has been briefly explained above. Due to this property, the unitarity of the amplitudes by factorization into different channels can be explicitly shown. However, since the action of mapping-class group on Schottky parameters is not known, the modular invariance is not easy to check. Also, as far as we are aware, the explicit integration regions for Schottky parameters has not been considered in the literature.

3.A.3 The Minimal-Area Metric

The minimal-area metric has been the main tool in the construction of covariant string field theory [119, 120, 121, 59, 122, 123]. Consider a Riemann surface $\mathcal{R}$, and a set $\Gamma = \{[\gamma_a]\}$ of finite or infinite base-point free homotopy classes of nontrivial closed curves on it. We denote the representative of each classes as $\widehat{\gamma}_a$ and for each homotopy class $[\gamma_a]$, we choose a constant $c_a \geq 0$. A metric ρ on $\mathcal{R}$ is called admissible if for any curve $\gamma'_a \sim \widehat{\gamma}_a$ satisfies the following inequality

$$\int_{\gamma'_a} \rho |dz| \geq c_a, \tag{3.A.10}$$

and a runs over all homotopy classes. A metric ρ is called a minimal-area metric if in addition to (3.A.10) also minimizes the area of $\mathcal{R}$. The minimal-area problem relevant for closed-string field theory can then be state as follows [124, 125]: *Under the condition that all nontrivial closed curves on $\mathcal{R}$ are longer than or equal to 2π, any string diagram is represented by a surface $\mathcal{R}$ equipped with the minimal-area metric.* The length condition should be imposed on all homotopy classes of curves on $\mathcal{R}$. There are similar minimal-area problems for open- and open-closed-string field theory [120, 121]. It can be shown that if a surface $\mathcal{R}$ is constructed by gluing together flat cylinders with circumference 2π and no closed curve on $\mathcal{R}$ is shorter than 2π, then $\mathcal{R}$ has the minimal-area metric [59].

Although minimal-area metric is suitable to formally satisfy the properties required by a string field theory, the explicit construction for most surfaces is lacking. On the other hand, the integration region for minimal-area parameters is not known.

3.A.4 The Constant-Curvature Hyperbolic Metric

The Euler characteristic of a genus-$\mathfrak{g}$ Riemann surface $\mathcal{R}$ with $\mathfrak{n}$ marked points is given by:

$$\boldsymbol{\chi}(\mathcal{R}) = 2 - 2\mathfrak{g} - \mathfrak{n}. \tag{3.A.11}$$

This quantity is negative for:

$$2\mathfrak{g} + \mathfrak{n} \geq 3. \tag{3.A.12}$$

If a surface satisfies this relation, then according to Gauss-Bonnet theorem, it can admit a negative constant curvature metric which can be taken to be -1. This is the content of **The Uniformization Theorem** of Poincaré and Koebe, according to which, there exists a unique hyperbolic metric for every conformal class of a Riemann surface. If $\mathbf{m}$ denotes the parameters that characterize a conformal structure on the surface:

$$\mathcal{R}(\mathbf{m}) \cong \mathcal{R}(\boldsymbol{h}_{\mathbf{m}}), \tag{3.A.13}$$

where $\boldsymbol{h}_{\mathbf{m}}$ is the parameters that characterize the corresponding hyperbolic metric. Let's consider the cases that do not satisfy condition (3.A.12)

1. **Sphere with zero, one, or two marked points**: In these cases, the volumes of the (residual) CKVs are infinite, and the amplitudes simply vanish[5].

2. **Torus with no marked point**: Torus has vanishing curvature and admits flat metric. It is well-known that the one-loop vacuum amplitude vanishes due to the space-time supersymmetry[6].

The condition (3.A.12) is thus satisfied for all the interesting and non-trivial cases.

The recent developments in hyperbolic geometry has provided the required tools to construct off-shell amplitudes in string theory [67, 69]. The integration over the moduli space requires the use of a specific set of coordinates for the corresponding Teichmüller space of the moduli space known as the Fenchel-Nielsen length and twist coordinates. As far as we know, the first appearance of these coordinates in the string theory literature goes back to [129]. The use of these coordinates has several advantages:

1. The explicit integration region can be constructed [67]. This is done by lifting the integration over the moduli space to an appropriate covering space;

2. The explicit volume form on the Teichmüller space can be constructed [101]. This volume form is given by the Weil-Petersson (WP) symplectic form via **The Wolpert's Magic Formula**:

$$\omega_{\mathbf{WP}} = \sum_{a=1}^{3g-3+n} d\ell_a \wedge d\tau_a, \tag{3.A.14}$$

   and the WP volume form is given by

$$dV_{\mathbf{WP}}(\mathcal{M}_{g,n}) \equiv (\omega_{\mathbf{WP}})^{3g-3+n}. \tag{3.A.15}$$

[5]Regarding the two-point amplitudes on sphere, see the interesting observation of [126].

[6]For a detailed computation of one-loop superstring vacuum amplitude see section 8 of [127] or section 7.6 of [128]. The two-loop computations are done in [26, 27, 28, 29, 30, 31, 32].

3. The WP volume form extends to the boundary of the Teichmüller space and it can be written as (3.A.14) [105, 96];
4. The WP volume form is invariant under the action of MCG [101];
5. The explicit hyperbolic metric on a degenerating surface near the boundary of the compactified moduli space can be constructed explicitly [73, 69];

There are few other parametrization of the space of Riemann surfaces like the Fuchsian parameterization, the Ribbon graphs, light-cone parameters, etc. Since none of these parameterization play any essential role in this thesis, we do not explain them here.

3.B Hyperbolic Metrics near a Puncture and on an Annulus

In this appendix, we derive the hyperbolic metric near a puncture and on the annulus.

The metric on a planar domain can be always written as

$$ds^2 = \rho^2(z,\bar{z})dzd\bar{z} = (\rho(z,\bar{z})|dz|)^2, \tag{3.B.1}$$

where $\rho(z,\bar{z})$ is a positive continuous function called *the density of metric*. A plane domain $\mathcal{H}$ with at least two boundary points is called a *hyperbolic domain*, and as such, the universal cover of a hyperbolic domain is the unit disk $\mathbb{D}$, i.e. we have the following

$$\pi : \mathbb{D} \longrightarrow \mathcal{H}. \tag{3.B.2}$$

One of the important aspects of The Uniformization Theorem is that the metric on an arbitrary Riemann surface can be defined using the metric on its covering space. On the other hand, the density of hyperbolic metric on the unit disk is given by

$$\rho_{\mathbb{D}}(z,\bar{z}) = \frac{1}{1-z\bar{z}}. \tag{3.B.3}$$

We can thus use π to push-forward $\rho_{\mathbb{D}}(z,\bar{z})$ to $\rho_{\mathcal{H}}(z,\bar{z})$ on the hyperbolic domain $\mathcal{H}$. For a point $x \in \mathbb{D}$ such that $\pi(x) = z$, we have [130]

$$\rho_{\mathcal{H}}(z) = \frac{\rho_{\mathbb{D}}(x)}{\left|\frac{d\pi(x)}{dx}\right|}. \tag{3.B.4}$$

One can show that this is a well-defined density on $\mathcal{H}$. It also shows that π is an infinitesimal hyperbolic isometry, and that $\mathcal{H}$ equipped with the density $\rho_{\mathcal{H}}$ is a complete metric space.

Using (3.B.4) and the Riemann map $f : \mathbb{D} \longrightarrow \mathbb{H}$ for $f = \mathtt{i}\frac{1+z}{1-z}$, the density of hyperbolic metric on upper half-plane $\mathbb{H}$ is given by

$$\rho_{\mathbb{H}}(z, \bar{z}) = \frac{1}{2\mathtt{Im}(z)}. \tag{3.B.5}$$

We can now consider our cases of interest. We first derive the hyperbolic metric near a puncture. A model for a hyperbolic surface near a puncture is the punctured disk $\mathbb{D}_\bullet$. The universal covering map $\pi : \mathbb{H} \longrightarrow \mathbb{D}_\bullet$ is given $\pi(z) = \exp(\mathtt{i}z) \equiv w$. Using (3.B.4) and (3.B.5), we have

$$\rho_{\mathbb{D}_\bullet}(w) = \frac{\rho_{\mathbb{H}}}{|\pi'(z)|} = \frac{1}{2|w| \ln \frac{1}{|w|}}. \tag{3.B.6}$$

This is the form of hyperbolic metric near a puncture.

To derive the density of hyperbolic metric on an annulus, consider the strip $S = \{w|\ 0 < \mathtt{Im}(w) < L\}$. The universal covering map between $\mathbb{H}$ and S is given by

$$f(z) = \frac{L \ln z}{\pi}. \tag{3.B.7}$$

Using this relation, we can compute the density of hyperbolic metric on strip using (3.B.4)

$$\rho_S(w) = \frac{\pi}{2L \sin\left(\frac{\pi}{L}\mathtt{Im}(w)\right)}. \tag{3.B.8}$$

The map $\exp(\mathtt{i}w)$ maps S to an annuli $\mathbb{A} = \{t|e^{-L} < |t| < 1\}$. Proceeding as before, we get the density of hyperbolic metric on an annulus to be

$$\rho_{\mathbb{A}}(t) = \frac{\pi}{2|t| \ln L} \csc\left(\pi \frac{\ln |t|}{\ln L}\right), \tag{3.B.9}$$

which is the result we were looking for.

3.C Identities for Hyperbolic Surfaces

In this appendix, we collect some identities on hyperbolic surfaces. These identities are the statement of the fact that the sum over the orbit of mapping-class group of the values of a function, which depends on lengths of simple closed curves on a hyperbolic surfaces, is a constant. We first give a brief general introduction to the identities on a hyperbolic surface and then explain some of these identities in some detail.

3.C.1 General Introduction

The study of hyperbolic geometry of surfaces has the advantage that there are many explicit results which are simply lacking in other methods of studying surfaces. On the other hand, many of these results rely on the study of very basic objects, namely simple closed curves on the surface. For example the mapping-class group of a surface can be determined by the action of its elements on the homotopy classes of simple closed curve on the surface, just like a linear transformation acting on a vector space can be determined through its action on basis vectors [131]. Also, there are results for hyperbolic surfaces that do not exist for Euclidean surfaces. Three classic examples of these results are

1. The Briman-Series result that the set of all simple closed geodesics on a hyperbolic surface has Hausdorff dimension zero. This means that most points of a hyperbolic surface do not lie on one of such geodesics [132];
2. The Anosov result that the geodesic flow on a hyperbolic surface (and more generally higher-dimensional hyperbolic manifolds) is ergodic. This means that if one choose an arbitrary point on a hyperbolic surface and in addition choose an arbitrary direction and generates geodesics paths, one can reach arbitrarily close to all points on the surface [133];
3. The existence of hyperbolic identities on hyperbolic surfaces where the sum of values of a function of hyperbolic length of (open or closed) curves is equal to a constant. The proof of the existence of such identities relies one the first two results.

The general idea behind finding identities on a hyperbolic surface is as follows. Let σ be a measure on a hyperbolic surface $\mathcal{R}$ and choose a finite-measure set $\mathcal{R}' \subset \mathcal{R}$. One can show that $\mathcal{R}'$ can be decomposed as:

$$\mathcal{R}' = \left(\bigcup_a \mathcal{R}_a \right) \cup \mathcal{R}'', \tag{3.C.1}$$

where $\{\mathcal{R}_a\}$ is a set of countable disjoint finite-measure subsets of $\mathcal{R}'$ and $\mathcal{R}''$ is a set of measure zero[7]. Then, schematically the identity takes the following form

$$\sigma(\mathcal{R}') = \sum_a \sigma(\mathcal{R}_a) + \sigma(\mathcal{R}'') = \sum_a \sigma(\mathcal{R}_a). \tag{3.C.2}$$

By computing the measures of the subsets in $\{\mathcal{R}_a\}$, one gets an identity for the hyperbolic surface $\mathcal{R}$. For example by choosing $\mathcal{R}' = \mathcal{T}_{\mathbb{I}}(\mathcal{R})$ (the unit tangent bundle of $\mathcal{R}$) or $\mathcal{R}' = \partial\mathcal{R}$, one can get the Luo-Tan identity and the generalized McShane identity, respectively. In the following, we introduce some of these identities [134].

3.C.2 Luo-Tan Identities

The Luo-Tan identity is an identity for hyperbolic surfaces with borders and cusps. It looks like the general form (3.3.36). To elucidate and state the Luo-Tan identity, we have to introduce two concepts which appear in (3.3.36): *the geometric objects over which we sum* and *the explicit form of the function f*. As we see, the first concept is given by *two classes of embedded subsurfaces namely properly-embedded geometric pairs of pants and quasi-embedded geometric pairs of pants*. Let $\mathcal{R}$ be a hyperbolic surface. Then a pair of pants $P \subset \mathcal{R}$ is called

- *Geometric* if the boundaries of P are geodesics;
- *Proper-embedded* if the inclusion map $\iota : P \hookrightarrow \mathcal{R}$ is injective and all of its boundaries are mapped onto different geodesics;
- *Quasi-embedded* is the inclusion map $\iota : P \hookrightarrow \mathcal{R}$ is injective, all of its boundaries are mapped onto geodesics, but two of its boundaries are mapped to the same geodesic;

The explicit form of the function f is given by the *Roger's dilogarithm function*. The Luo-Tan identity can be schematically written as the following form

$$\sum_{P_{\mathbf{0}}} f_{\mathbf{0}}(P_{\mathbf{0}}) + \sum_{P_{\mathbf{1}}} f_{\mathbf{1}}(P_{\mathbf{1}}) + \sum_{P_{\mathbf{2}}} f_{\mathbf{2}}(P_{\mathbf{2}}) + \sum_{P} f(P) = \text{constant}, \tag{3.C.3}$$

where

[7] To prove that $\mathcal{R}''$ has measure zero, one must resort to the results of Birman-Series or Anosov mentioned above.

- The first term is the sum over all properly-embedded geometric spheres with three holes $P_{\mathbf{0}}$ such that $\partial P_{\mathbf{0}} \cup \partial \mathcal{R} = \emptyset$;
- The second term is the sum over all properly-embedded geometric spheres with three holes $P_{\mathbf{1}}$ such that one of its boundaries is a boundary of $\mathcal{R}$;
- The third term is the sum over all properly-embedded geometric spheres with three holes $P_{\mathbf{2}}$ such that two of its boundaries are boundaries of $\mathcal{R}$;
- The fourth term is the sum over all quasi-embedded geometric spheres with three holes P;

The explicit forms of $f_{\mathbf{0}}$, $f_{\mathbf{1}}$, $f_{\mathbf{2}}$, and f will be given bellow. To write the explicit form of these functions, we need to introduce the Roger's dilogarithm function. For $|z| < 1$, the dilogarithm function Li_2 is defined by the following Taylor series:

$$\mathrm{Li}_2(z) \equiv \sum_{n=1}^{\infty} \frac{z^n}{n^2}. \tag{3.C.4}$$

For $z \in \mathbb{R}$ and $z \leq 1$, $\mathrm{Li}_2(z)$ can be written as follows

$$\mathrm{Li}_2(z) = -\int_0^z dz' \, \frac{\ln(1-z')}{z'}. \tag{3.C.5}$$

In the following, we only need dilogarithm function whose argument is real. We can then define the Roger's dilogarithm function

$$\begin{aligned} \mathcal{L}(z) &\equiv \mathrm{Li}_2(z) + \frac{1}{2} \ln|z| \ln(1-z) \\ &= -\frac{1}{2} \int_0^z dz' \left(\frac{\ln(1-z')}{z'} + \frac{\ln(z')}{1-z'} \right), \qquad 0 \leq z \leq 1. \end{aligned} \tag{3.C.6}$$

This function has the following properties

1. $\mathcal{L}'(z) = -\frac{1}{2}\left(\frac{\ln(1-z)}{z} + \frac{\ln(z)}{1-z}\right)$;
2. $\mathcal{L}(0) = 0$;
3. $\mathcal{L}(z) + \mathcal{L}(1-z) = \frac{\pi^2}{6}$;

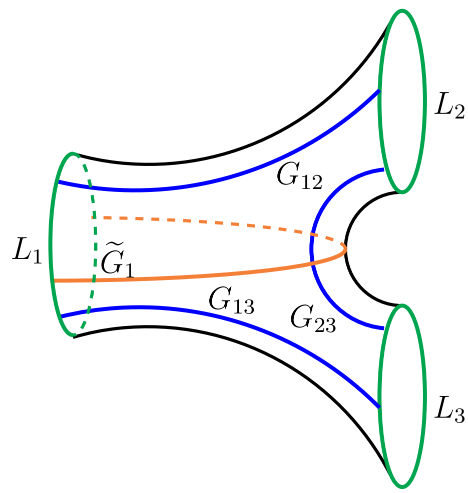


Figure 3.C.1: Geodesics that are connecting boundaries of a hyperbolic pair of pants.

4. *The pentagon relation*: the function (3.C.6) satisfies the following fundamental identity:

$$\mathcal{L}(x)+\mathcal{L}(y)+\mathcal{L}(1-xy)+\mathcal{L}\left(\frac{1-x}{1-xy}\right)+\mathcal{L}\left(\frac{1-y}{1-xy}\right)=\frac{\pi^2}{2}, \qquad x,y\in[0,1], \quad xy\neq 1. \tag{3.C.7}$$

We also need the so-called *Lasso function* $\mathrm{La}(x,y)$

$$\mathrm{La}(x,y)\equiv\mathcal{L}(y)+\mathcal{L}\left(\frac{1-y}{1-xy}\right)-\mathcal{L}\left(\frac{1-x}{1-xy}\right), \qquad x,y\in[0,1]\,. \tag{3.C.8}$$

Equipped with these definitions, we now move on to describe the terms in (3.C.3).

Properly-Embedded Geometric Pairs of Pants

The first term of (3.C.3) is a summation over all properly-embedded pairs of pants. The function f is a function of some geodesics on each of these pairs of pants. We shall discuss these geodesics and the explicit form of the function f separately:

- **Length Invariants of a Properly-Embedded Pair of Pants**
  Let $\mathcal{R}_{0,3}$ be a pair of pants, and let $\beta_i,\ i=1,2,3$ denote its boundary. There are two

types of orthogonal geodesic between the boundaries: *the orthogonal geodesic from a boundary to another boundary* and *the orthogonal geodesic from a boundary to itself*. These orthogonal geodesics are unique. We denote the orthogonal geodesic from the boundary β_i to boundary β_j by G_{ij}, its hyperbolic length by l_{G_i}, the orthogonal geodesic from the boundary β_i to itself by $\widetilde{G}_i$, and its hyperbolic length by $l_{\widetilde{G}_i}$. These geodesics for a hyperbolic pair of pants is shown in figure (3.C.1). We also denote the hyperbolic length of the boundary β_i by ℓ_{β_i}. We can relate the length of the types of orthogonal geodesics to the lengths of the boundaries by the following two identities:

$$
\begin{aligned}
\cosh(l_{G_{ij}}) &= \frac{\cosh\left(\frac{\ell_{\beta_i}}{2}\right) + \cosh\left(\frac{\ell_{\beta_j}}{2}\right)\cosh\left(\frac{\ell_{\beta_k}}{2}\right)}{\sinh\left(\frac{\ell_{\beta_j}}{2}\right)\sinh\left(\frac{\ell_{\beta_k}}{2}\right)}, && \{i,j,k\} = \{1,2,3\}, \\
\cosh(l_{\widetilde{G}_i}) &= \sinh\left(\frac{\ell_{\beta_j}}{2}\right)\sinh\left(\frac{l_{G_k}}{2}\right), && \{i,j,k\} = \{1,2,3\}. \quad (3.C.9)
\end{aligned}
$$

Using these two identities and $\cosh^2(x) - \sinh^2(x) = 1$, one can express all $l_{\widetilde{G}_i}$ and $l_{G_{ij}}$ in terms of ℓ_{β_1}, ℓ_{β_2}, and ℓ_{β_3}, the lengths of boundary components.

- **The Covering Maps f_0, f_1, and f_2**
  The function f_0, f_1, and f_2 in (3.C.3) depends on the lengths of three types of geodesics on $\mathcal{R}_{0,3}$: orthogonal geodesics from one boundary to the other, orthogonal geodesics from one boundary to itself, and lengths of boundary components. Using (3.C.9), these functions can be written entirely in terms of ℓ_{β_1}, ℓ_{β_2}, and ℓ_{β_3}, lengths of boundary components of $\mathcal{R}_{0,3}$. We can now express the functions. Let's define the following two parameters:

$$
\begin{aligned}
x_i &\equiv e^{-\ell_{\beta_i}}, \\
y_k &\equiv \tanh^2\left(\frac{l_{G_i}}{2}\right), \qquad k \neq i,j. \qquad (3.C.10)
\end{aligned}
$$

There is a useful relation between them:

$$
\frac{x_i(1-y_j)^2}{(1-x_i)^2 y_j} = \frac{1}{\cosh^2\left(\frac{l_{\widetilde{G}_k}}{2}\right)}. \qquad (3.C.11)
$$

Then functions appearing in (3.C.3) can be written as

$$
\begin{aligned}
f_{\mathbf{0}}(P) &\equiv 4\pi^2 - 8\left\{\sum_{i=1}^{3}\left[\mathcal{L}\left(\frac{1}{\cosh^2(l_{G_i}/2)}\right)+\mathcal{L}\left(\frac{1}{\cosh^2(l_{\widetilde{G}_i}/2)}\right)\right]+\sum_{i\neq j}\mathrm{La}\left(\ell_{\beta_i}, l_{G_i}\right)\right\}\\
&= 4\sum_{i\neq j}\left[2\mathcal{L}\left(\frac{1-x_i}{1-x_iy_i}\right)-2\mathcal{L}\left(\frac{1-y_j}{1-x_iy_i}\right)-\mathcal{L}\left(y_j\right)-\mathcal{L}\left(\frac{x_i(1-y_j)^2}{(1-x_i)^2y_j}\right)\right].
\end{aligned}
\tag{3.C.12}
$$

If $\partial P\cup\partial\mathcal{R}=L_1$, where L_1 is one of the boundaries of $\mathcal{R}$, then

$$
f_{\mathbf{1}}(P)\equiv f_{\mathbf{0}}(P)+8\left\{\mathcal{L}\left(\frac{1}{\cosh^2\left(\frac{l_{\widetilde{G}_1}}{2}\right)}\right)+\mathrm{La}(\ell_{\beta_2}, l_{G_3})+\mathrm{La}(\ell_{\beta_3}, l_{G_2})\right\}. \tag{3.C.13}
$$

And finally if $\partial P\cup\partial\mathcal{R}=L_1\cup L_2$, where L_1 and L_2 are two of the boundaries of $\mathcal{R}$, then

$$
\begin{aligned}
f_{\mathbf{2}}(P) \equiv f_{\mathbf{0}}(P) &+ 8\left\{\mathcal{L}\left(\frac{1}{\cosh^2\left(\frac{l_{\widetilde{G}_1}}{2}\right)}\right)+\mathcal{L}\left(\frac{1}{\cosh^2\left(\frac{l_{\widetilde{G}_1}}{2}\right)}\right)+\mathcal{L}\left(\frac{1}{\cosh^2\left(\frac{l_{G_3}}{2}\right)}\right)\right\}\\
&+ 8\left(\mathrm{La}(\ell_{\beta_1}, l_{G_3})+\mathrm{La}(\ell_{\beta_3}, l_{G_1})+\mathrm{La}(\ell_{\beta_2}, l_{G_3})+\mathrm{La}(\ell_{\beta_3}, l_{G_2})\right).
\end{aligned}
\tag{3.C.14}
$$

Quasi-Embedded Geometric Pairs of Pants

The function f in (3.C.3) is a summation over all quasi-embedded geometric pairs of pants. Such pants are in one-to-one correspondence with properly-embedded one-bordered tori because cutting such a tori along any simple closed geodesic which is not parallel to a boundary component gives a quasi-embedded geometric pair of pants. The function f is a function of some geodesics on one-bordered tori. We shall discuss these geodesics and the explicit form of the function f separately:

- **Length Invariants of a Quasi-Embedded Geometric Pair of Pants**
  Let $\mathcal{R}_{1,1}(L)$ be a torus with one hole whose boundary has fixed length L. If we cut this surface along non-boundary parallel simple closed geodesic α, we get a pair of pants $\mathcal{R}_{0,3}$ in which two of the boundaries have the same length. An example of such a pair of pants is illustrated in figure 3.C.2. We denote the border of $\mathcal{R}_{1,1}(L)$ by β, and the borders resulting from cutting along α by $\beta^{\pm}_{\alpha}$. On the resulting pair of

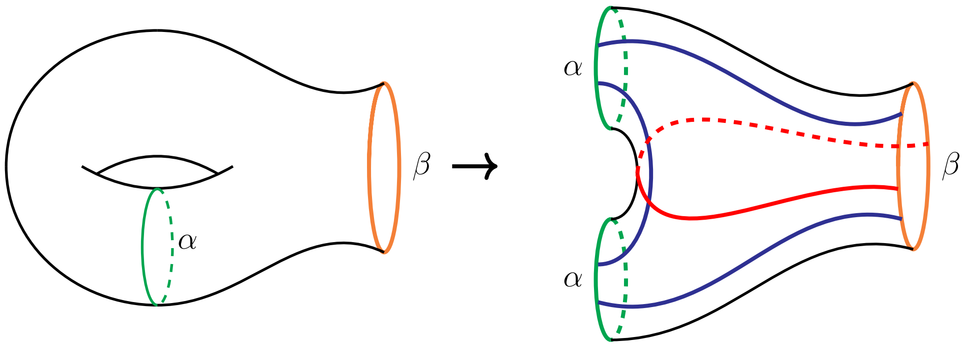


Figure 3.C.2: Cutting along the curve α gives a quasi-embedded geometric pair of pants. Geodesics orthogonal to boundaries are shown with blue and red line.

pants, we can again define the same orthogonal geodesics to the boundaries, whose lengths can be written in terms of the lengths of the borders of $\mathcal{R}_{0,3}$. The function f in (3.C.3) depends on the lengths of three types of geodesics on $\mathcal{R}_{0,3}$: orthogonal geodesics from one boundary to the other, orthogonal geodesics from one boundary to itself, and the length of the borders. Using (3.C.9), f can be written entirely in terms of the lengths of the borders of $\mathcal{R}_{0,3}$. The important point to notice is that *the pair of pants obtained by cutting a one-bordered torus along a curve not homotopic to the border is not a properly-embedded pair of pants but a quasi-embedded geometric pair of pants.*

- **The Covering Map** f

  Using the definitions of Roger's dilogarithm function (3.C.6) and the Lasso function (3.C.8), we have:

$$f(P) \equiv 4\pi^2 - 8\sum_{\alpha}\left[\mathcal{L}\left(\frac{1}{\cosh^2(l_{\widetilde{G}_\alpha}/2)}\right) + 2\mathrm{La}(\ell_\alpha, l_{G_\alpha})\right], \tag{3.C.15}$$

  where

  † ℓ_α is the hyperbolic length of the curve α, a non-boundary parallel simple closed geodesic along which we cut $\mathcal{R}_{1,1}(L)$ to get a quasi-embedded pair of pants;

  † l_{G_α} is the hyperbolic length of the orthogonal geodesics from $\beta^{\pm}_{\alpha}$ to β;

  † $l_{\widetilde{G}_\alpha}$ is the hyperbolic length of the orthogonal geodesics from β to itself;

Using (3.C.9), l_{G_α} and $l_{\widetilde{G}_\alpha}$ can be written entirely in terms of ℓ_α and ℓ_β, the hyperbolic length of the borders of $\mathcal{R}_{0,3}(\ell_\alpha, \ell_\alpha, \ell_\beta = L)$, the pair of pants obtained by cutting a one-holed tori along a simple closed curve α that is not homotopic to β. Therefore, everything can be written in terms of the hyperbolic length of the borders of the relevant pair of pants.

The Luo-Tan Identity for Bordered/Cusped Hyperbolic Surfaces

Equipped with these definitions, we are now in a position to state the Luo-Tan identity for bordered/cusped hyperbolic surfaces [109]:

Theorem 3.2. *Let $\mathcal{R}$ be a genus-$\mathfrak{g}$ hyperbolic Riemann surface with $\mathfrak{n}$ borders, i.e. $2\mathfrak{g}+\mathfrak{n} \geq 3$. There exist functions $f_{\mathbf{0}}$, $f_{\mathbf{1}}$, $f_{\mathbf{2}}$, and f, given by (3.C.12), (3.C.13), (3.C.14), and (3.C.15) respectively, involving the dilogarithm of lengths of simple closed geodesics, such that*

$$\sum_{P_{\mathbf{0}}} f_{\mathbf{0}}(P_{\mathbf{0}}) + \sum_{P_{\mathbf{1}}} f_{\mathbf{1}}(P_{\mathbf{1}}) + \sum_{P_{\mathbf{2}}} f_{\mathbf{2}}(P_{\mathbf{2}}) + \sum_{P} f(P) = 4\pi^2(2g+n-2), \tag{3.C.16}$$

where in the first three terms the sum runs over all properly-embedded geometric pairs of pants which have zero, one, or two shared boundary component with $\mathcal{R}$. The fourth term is the sum over all quasi-embedded geometric pairs of pants. Furthermore, if the length of $\mathfrak{m} \leq \mathfrak{n}$ of the borders becomes zero, we get an identity for a genus-$\mathfrak{g}$ surface with $\mathfrak{m}$ cusps and $\mathfrak{n}-\mathfrak{m}$ borders. We note that all the functions which appear in (3.C.16) are functions of only the lengths of boundary components of the associated pairs of pants.

Another way to state this summation is as follows: the first three summations are over all curves that remove a properly-embedded pairs of pants from $\mathcal{R}$. These curves may or may not be related through the action of mapping-class group. The second summation is over all curves that remove a properly-embedded one-bordered tori from $\mathcal{R}$. Again, these curves may or may not be related through the action of the mapping-class group.

The Luo-Tan Identity for Borderless/Uncusped Hyperbolic Surfaces

The Luo-Tan identity can be used for surfaces without any boundary component (i.e. without borders or cusps)[8]. The Luo-Tan identity for such surfaces can be stated as follows [109]

[8] A general hyperbolic surface can have borders, cusps, and cone points. For string theory, we only need to consider borders and cusps. There are hyperbolic identities for surfaces with cone points as well.

Theorem 3.3. *Let $\mathcal{R}$ be a genus-$\mathfrak{g}$ hyperbolic surface, i.e. $\mathfrak{g} \geq 2$. There exist functions $f_{\mathbf{0}}$ and f, given by (3.C.12) and (3.C.15) respectively, involving the dilogarithm of the lengths of simple closed geodesics, such that*

$$\sum_{P_{\mathbf{0}}} f_{\mathbf{0}}(P_{\mathbf{0}}) + \sum_{P} f(P) = 8\pi^2(g-1), \tag{3.C.17}$$

where in the first term the sum runs over all properly-embedded geometric pairs of pants embedded in $\mathcal{R}$, and the second term is the sum over all quasi-embedded geometric pairs of pants. We note that all functions appearing in (3.C.17) are functions of only the lengths of boundary components of the associated pairs of pants.

3.C.3 McShane-Type Identities

As it is explained above, we can obtain hyperbolic identities by choosing a finite-measure subset of the surface. Different choices would give rise to different identities on the same surfaces. One can choose one of the boundaries as finite-measure subset and try to prove identities like (3.C.2). Such a process will give rise to *McShane-type identities*. Here we shall explain the basic idea behind these identities.

The McShane Identity for Cusped Surfaces

The first identities of this sort was obtained by McShane for a once-punctured tori [110]:

Theorem 3.4. *Let $\mathcal{R}$ be any hyperbolic torus with a single cusp. Then, the following identity holds*

$$\sum_{\gamma} \frac{1}{1+e^{\ell_\alpha(\mathcal{R})}} = \frac{1}{2}, \tag{3.C.18}$$

where the sum is over all simple closed geodesics α, and $\ell_\alpha(\mathcal{R})$ is the hyperbolic length of α in $\mathcal{R}$.

The generalization of this identity to all cusped hyperbolic surfaces was obtained by McShane himself [111]

Theorem 3.5. *Let $\mathcal{R}$ be any genus-$\mathfrak{g}$ hyperbolic surface with $\mathfrak{n} \neq 0$ cusps. Then, the following identity holds:*

$$\sum_{\{\alpha_1,\alpha_2\}} \frac{1}{1+\exp\left(\frac{\ell_{\alpha_1}(\mathcal{R})+\ell_{\alpha_2}(\mathcal{R})}{2}\right)} = \frac{1}{2}, \tag{3.C.19}$$

where $\{\alpha_1, \alpha_2\}$ are a pair of curves which together with a fixed cusp bound a pair of pants. We need to consider all cusps in this formula.

To implement Mirzakhani's integration trick, we need to consider the generalization of this identity for surfaces which have a mixture of boundary components and cusps. A cusp is the limit of a border when the length of the border goes to zero. Therefore, the most general form of the McShane identity is for hyperbolic surfaces with geodesic boundary components.

The Mirzakhani-McShane Identity for Bordered/Cusped Hyperbolic Surfaces

To introduce The Mirzakhani-McShane Identity, we consider a surface $\mathcal{R}$ with boundary components $(b_1, \cdots, b_\mathfrak{n})$ having lengths $(L_1, \cdots, L_\mathfrak{n})$. The basic idea of the proof of The Mirzakhani-McShane Identity is the analysis of relation (3.C.2). One can base the identity on an arbitrary boundary component. This means that, one takes $\mathcal{R}' = L_i$ for $i = 1, \cdots, \mathfrak{n}$. To be specific, we base the identity on the boundary component b_1. This means that in (3.C.2)

$$\sigma(\mathcal{R}') = \sigma(b_1) = L_1. \tag{3.C.20}$$

Now we need to consider the sum over $\sigma(\mathcal{R}_a)$'s in the other side of the identity. The subsets $\mathcal{R}_a$ are taken to be all pairs of pants P that is bounded by two curves γ_1 and γ_2 and b_1. The relation (3.C.2) takes the following form

$$L_1 = \sum_P \sigma(P). \tag{3.C.21}$$

To establish the identity, we need to compute $\sigma(P)$. We consider a pair of pants with boundaries β_1, β_2, and β_3 with lengths l_1, l_2, and l_3. Let $p_\pm^{\beta_2}$ and $p_\pm^{\beta_3}$ be the point of intersection of simple geodesics which start at β_1 and spiraling towards β_2 and β_3. The $\pm$ means two directions of spiraling. We consider the order of the points to be $(p_+^{\beta_2}, p_+^{\beta_3}, p_-^{\beta_3}, p_-^{\beta_2})$. These points divide the boundary β_1 into four intervals [67]

- Each of the intervals $[p_+^{\beta_2}, p_+^{\beta_3}]$ and $[p_-^{\beta_3}, p_-^{\beta_2}]$ contains a unique point which is the end point of the unique simple orthogeodesics[9] from β_1 to itself;

[9] An othogeodesic to a border is a geodesic which is orthogonal to another border or the border itself.

- The interval $[p_-^{\beta_2}, p_+^{\beta_2}]$ contains a unique point which is the end-point of the unique simple orthogeodesic from β_1 to β_2;
- The interval $[p_+^{\beta_3}, p_-^{\beta_3}]$ contains a unique point which is the end-point of the unique simple orthogeodesic from β_1 to β_3;

Assume that $\beta_1 \equiv b_1$, one of the borders of the surface $\mathcal{R}$. One can compute $\sigma(P)$ in (3.C.21). It turns out that one can consider the following two cases [67]

- β_2 and β_3 are two interior curves on the surface $\mathcal{R}$ (i.e. they are not borders of $\mathcal{R}$). We can consider the following cases

$$\sigma(P) = 2l_{[p_+^{\beta_2}, p_+^{\beta_3}]}(\mathcal{R}) \equiv \mathcal{D}_1(l_{\beta_1}, l_{\beta_2}, l_{\beta_3}) = \mathcal{D}_1(L_1, l_2, l_3), \tag{3.C.22}$$

where $l_{[p_+^{\beta_2}, p_+^{\beta_3}]}(\mathcal{R})$ is the hyperbolic length of the interval $[p_+^{\beta_2}, p_+^{\beta_3}]$ in $\mathcal{R}$, and

$$\mathcal{D}_1(x, y, z) \equiv 2\ln\left(\frac{e^{\frac{x}{2}} + e^{\frac{y+z}{2}}}{e^{-\frac{x}{2}} + e^{\frac{y+z}{2}}}\right). \tag{3.C.23}$$

- β_2 is an interior curve and $\beta_3 \equiv b_i$, one of the borders of $\mathcal{R}$. Then

$$\sigma(P) = 2l_{\beta_3 \perp \beta_1}(\mathcal{R}) \equiv \mathcal{D}_2(l_{\beta_1}, l_{\beta_2}, l_{\beta_3}) = \mathcal{D}_2(L_1, L_i, l_3), \tag{3.C.24}$$

where $l_{\beta_3 \perp \beta_1}(\mathcal{R})$ is the hyperbolic length of the projection of β_3 to β_1 in $\mathcal{R}$, and

$$\mathcal{D}_2(x, y, z) \equiv x - \ln\left(\frac{\cosh\left(\frac{y}{2}\right) + \cosh\left(\frac{x+z}{2}\right)}{\cosh\left(\frac{y}{2}\right) + \cosh\left(\frac{x-z}{2}\right)}\right). \tag{3.C.25}$$

We can now state The Mirzakhani-McShane Identity for bordered hyperbolic surfaces (Theorem 4.2 of [67])

Theorem 3.6. *Let $\mathcal{R}$ be any genus-$\mathfrak{g}$ hyperbolic surface with $\mathfrak{n} \neq 0$ geodesic boundary components $\{b_1, \cdots, b_{\mathfrak{n}}\}$. For such a surface, the following identity holds*

$$\sum_{\{\alpha_1, \alpha_2\} \in \mathcal{F}_1} \mathcal{D}_1\left(L_1, \ell_{\alpha_1}(\mathcal{R}), \ell_{\alpha_2}(\mathcal{R})\right) + \sum_{i=2}^{\mathfrak{n}} \sum_{\gamma \in \mathcal{F}_{1,i}} \mathcal{D}_2\left(L_1, L_i, \ell_\gamma(\mathcal{R})\right) = L_1, \tag{3.C.26}$$

where $\mathcal{F}_i$ is the set of isotopy class of all pair of curves $\{\alpha_1, \alpha_2\}$ that together with the boundary component b_i bound a pair of pants, and $\mathcal{F}_{ij}$ is the set of isotopy class of all curves γ that together with the boundary components b_i and b_j bound a pair of pants.

A special case of (3.C.26) is the case that the length of one of the borders goes to zero, i.e. we have a surface with a single cusp and $\mathfrak{n}-1$ borders of lengths $(L_2, \cdots, L_\mathfrak{n})$:

$$\sum_{\{\alpha_1,\alpha_2\}\in\mathcal{F}_1} \frac{1}{1+\exp\left(\frac{\ell_{\alpha_1}(\mathcal{R})+\ell_{\alpha_2}(\mathcal{R})}{2}\right)} + \frac{1}{2}\sum_{i=2}^{\mathfrak{n}}\sum_{\gamma\in\mathcal{F}_{1,i}} \frac{1}{1+\exp\left(\frac{\ell_\gamma(\mathcal{R})+L_i}{2}\right)} + \frac{1}{1+\exp\left(\frac{\ell_\gamma(\mathcal{R})-L_i}{2}\right)} = \frac{1}{2}. \tag{3.C.27}$$

A further restriction is the case that all of the boundary components have zero length, i.e. we have a surface with $\mathfrak{n}$ cusps. Sending $(L_2, \cdots, L_\mathfrak{n}) \longrightarrow (0, \cdots, 0)$ in (3.C.27), we get

$$\sum_{\{\alpha_1,\alpha_2\}\in\mathcal{F}_1} \frac{1}{1+e^{\frac{\ell_{\alpha_1}(\mathcal{R})+\ell_{\alpha_2}(\mathcal{R})}{2}}} + \sum_{i=2}^{\mathfrak{n}}\sum_{\gamma\in\mathcal{F}_{1,i}} \frac{1}{1+e^{\frac{\ell_\gamma(\mathcal{R})}{2}}} = \frac{1}{2}. \tag{3.C.28}$$

This is the Mcshane identity for hyperbolic surfaces with cusps which we have stated in (3.C.19).

The McShane Identity for Genus Two Hyperbolic Surfaces

In the case of genus-two surfaces, there exist simpler identities. Here we mention one of such identities due to McShane [135]. To state the identity, we need to define the notion of a *hyperelliptic involution*. A hyperelliptic involution of a genus-$\mathfrak{g}$ surface is a conformal automorphism of order two[10] which has exactly $2\mathfrak{g}+2$ fixed points. If such an automorphism exists, it is unique [136]. A surface which has such an automorphism is called a *hyperelliptic surface*. If $\mathcal{R}$ is a genus-2 hyperbolic surface, it is known that it has a hyperelliptic involution J which admits six fixed points called *Weierstrass points* [136]. Therefore, any genus-2 surface is hyperelliptic. There are two facts:

- Every non-separating simple closed geodesic (i.e. a simple closed geodesics that does not separate $\mathcal{R}$ into disconnected pieces) passes through exactly two of Weierstrass points;

[10]The *order of an authormoshim* $J : \mathcal{R} \longrightarrow \mathcal{R}$ of a surface $\mathcal{R}$ is the smallest positive number n such that $J^n = \mathbb{I}$, where $\mathbb{I}$ is the identity map.

- Every separating simple closed geodesic separates $\mathcal{R}$ into two one-holed torus each of which contains exactly three of the Weierstrass points;

If p is a Weierstrass point of $\mathcal{R}$, the *Weierstrass class of* p is the set of all simple closed geodesics that pass through p. The *dual Weierstrass class of* p, denoted by $\mathcal{W}_p$, is the collection of pairs (α, β) in $\mathcal{R} - \{p\}$, where α is a non-separating simple closed geodesics, β is a separating simple closed geodesics, α and β are disjoint, and α and p are in the same component of $\mathcal{R} - \beta$. One can state the following result [135]

Theorem 3.7. *Let $\mathcal{R}$ be a genus-2 borderless hyperbolic surface, and $\mathcal{W}_p$ be a fixed dual Weierstrass class. For such a surface, the following identity holds*

$$\sum_{(\alpha,\beta)\in\mathcal{W}_p} \tanh^{-1}\left(\frac{2\cosh\left(\frac{\ell_\alpha}{2}-\frac{\ell_\beta}{4}\right)}{\sinh(\ell_\alpha)+\sinh\left(\frac{\ell_\beta}{2}\right)}\right) = \frac{\pi}{2}. \tag{3.C.29}$$

This completes our discussion of identities on hyperbolic surfaces.

3.D The Vertical Integration Procedure

In this section, we describe a method to deal with spurious singularities arising in the integrands of superstring theories. This method is first suggested in [20] and then further elaborated in [25].

3.D.1 Spurious Singularities

A typical integrand of superstring theories has the following schematic form

$$\mathbf{\Omega}_d \equiv \left\langle \mathcal{R}|\mathcal{K}^{(p)} \wedge \mathcal{B}_{d-p}|\Phi\right\rangle, \tag{3.D.1}$$

where Φ is a state in the tensor product of Hilbert space of external states, and $d \equiv 6\mathfrak{g} - 6 + 2\mathfrak{n}_{\rm NS} + 2\mathfrak{n}_{\rm R}$. There are however some complications due to the presence of the PCOs. There is no global choice for locations of PCOs on a surface. The reason is that the integration over the odd coordinates of the supermoduli space requires a gauge choice for the gravitino field. However, there is no global gauge choice. Hence, we have to integrate over the odd coordinates of the supermoduli space locally, by a local choice of gauge.

This procedure introduces PCOs [77]. From the point of view of the moduli space, the distribution of the PCOs is local. Therefore, a complete prescription for the computation of scattering amplitudes in superstring theory in the picture-changing formalism involves a procedure for gluing local description of PCOs distributions. One the other hand, as we move in the moduli space, the locations of PCOs, which are dependent on moduli parameters, change. Regarding this, three phenomenon can happen as we move in the moduli space

- **two PCOs collide**
  As we explain bellow, the distribution of PCOs on a surface, is a function of the moduli parameters. If we denote coordinates of the moduli space by $\mathbf{m}$, and the moduli-dependence of two PCOs $\mathbb{X}(z_a)$ and $\mathbb{X}(z_b)$ by $z_a = f_a(\mathbf{m})$ and $z_b = f_b(\mathbf{m})$, these PCOs collide whenever $f_a(\mathbf{m}) - f_b(\mathbf{m}) = 0$. The result is an spurious singularity in the superstring integrand.

- **a PCO collides with an external vertex operator**
  Locations of marked points are moduli parameters of the surface. A PCO can collide with an external marked point as we move in the moduli space. The result is again an spurious singularity in the superstring integrand.

- **genuine singularities**
  The PCO formalism forces us to consider the bosonization of the (β, γ) ghost system. After bosonization, the equivalent system can be described by the set of fields (η, ξ, ϕ). It turns out that the correlation of these fields can be written in terms of the Riemann theta function and prime forms [37]

$$\left\langle \prod_{i=1}^{n+1} \xi(x_i) \prod_{j=1}^{n} \eta(y_i) \prod_{k=1}^{m} e^{q_k} \phi(z_k) \right\rangle_{\alpha,\beta} \tag{3.D.2}$$

$$= \frac{\prod\limits_{j=1}^{n} \Theta[^{\alpha}_{\beta}](-\vec{y}_j + \sum \vec{x} - \sum \vec{y} + \sum q\vec{z} - 2\vec{\Delta})}{\prod_{j=1}^{n+1} \Theta[^{\alpha}_{\beta}](-\vec{x}_j + \sum \vec{x} - \sum \vec{y} + \sum q\vec{z} - 2\vec{\Delta})} \frac{\prod_{i_1<i_2} E(x_{i_1}, x_{i_2}) \prod_{j_1<j_2} E(y_{j_1}, y_{j_2})}{\prod_{i<j} E(x_i, y_j) \prod_{k<l} E(z_k, z_l)^{q_k q_l} \prod_k \sigma(z_k)^{2q_k}}, \tag{3.D.3}$$

  where

$$\sum_{k=1}^{m} q_k = 2\mathrm{g} - 2. \tag{3.D.4}$$

$\Theta[^\alpha_\beta]$ is the theta function with characteristic (α, β) for genus-$\mathfrak{g}$ Riemann surfaces. The characteristic (α, β) corresponds to a specific choice of spin structure, $E(x, y)$ is the prime form and $\sigma(z)$ is a $\frac{\mathfrak{g}}{2}$ differential representing the conformal anomaly of the ghost system. $\sum \vec{x}, \sum \vec{y}$ and $\sum q\vec{z}$ denote respectively $\sum_{i=1}^{n+1} \vec{x}_i, \sum_{j=1}^{n} \vec{y}_j$ and $\sum_{k=1}^{m} q_k\vec{z}_k$ with

$$x_i \equiv \int_{P_0}^{x} \omega_i, \tag{3.D.5}$$

where ω_i are the Abelian differentials and P_0 is an arbitrary point on the surface. $\vec{\Delta}$ is the Riemann class vector characterizing the divisor of the zeroes of the theta function. The components of the Riemann class vector are given by

$$\Delta_a \equiv \mathrm{i}\pi + \frac{\tau_a}{2} - \frac{1}{2\pi\mathrm{i}} \sum_{b\neq a} \int_{A_b} \omega_b(P) \int_{P_0}^{P} \omega_a, \tag{3.D.6}$$

where $\boldsymbol{\tau}$ denotes the period matrix and A_b denote the b^{th} A-cycle on the surface. The dependence of the Riemann class on P_0 will cancel the dependence of $\vec{x}$ on P_0 to make the theta function independent of P_0. There are two kinds of singularities that appear in these correlation functions

1. The prime form $E(x, y)$ has a simple zero at $x = y$. Therefore, the prime forms appearing in the denominator introduces poles in correlation functions. These poles corresponds to the collision of the operators.
2. It is known that on genus-$\mathfrak{g}$ Riemann surface the Theta function vanishes. Therefore, the factor $\prod_{j=1}^{n+1} \Theta[^\alpha_\beta](-\vec{x}_j + \sum \vec{x} - \sum \vec{y} + \sum q\vec{z} - 2\vec{\Delta})$ in the denominator also introduces poles in correlation functions.

Using the identifications (2.2.12) and operator product expansions of η, ξ fields, it is possible to see that this denominator factor becomes independent of the locations of βs and γs [137, 138]. As a result, the poles associated with theta functions do note depend on the locations of insertion of βs and γs. Note that vertex operators and the BRST charge are constructed using the β and γ fields, but the picture changing operator contains $\partial\xi, \eta$ and $e^{q\phi}$ factors that can not be expressed as polynomials of β and γ fields. This suggest that the picture changing operators are the source of poles in the correlation function that corresponds to the vanishing of Theta functions appearing in the denominator. Also the locations of these poles are functions of the locations of PCOs in the correlation function. These quantities appear in the denominator of the resulting expression. Therefore, if these expressions vanish, we have a singularity in the integrand. The singularities coming from the vanishing of the prime forms are physical singularities that can be avoided. However, the singularities that coming from the vanishing of the Riemann theta function is a genuine singularity which does not have any physical origin.

These singularities are collectively known as *spurious singularities*. In all these cases, the spurious singularity specifies a complex codimension one locus on the Riemann surface. The Riemann surface has complex dimension one. Therefore, the locations on the Riemann surface which introduce spurious singularity into the integrand is a finite set of points. If we choose a point in $\widetilde{\mathcal{P}}_{\mathfrak{g};\mathfrak{n}_{\mathrm{NS}},\mathfrak{n}_{\mathrm{R}}}$, we have to remove these points from each fibers of $\widetilde{\mathcal{P}}_{\mathfrak{g};\mathfrak{n}_{\mathrm{NS}},\mathfrak{n}_{\mathrm{R}}}$ in such a way that spurious singularities can be avoided. By considering what we have said so far, the correct prescription for the computation of scattering amplitudes must be accompanied by a prescription for gluing local data about the distribution of PCOs which avoid spurious singularities. The data about the locations of PCOs is contained in $\widetilde{\mathcal{P}}_{\mathfrak{g};\mathfrak{n}_{\mathrm{NS}},\mathfrak{n}_{\mathrm{R}}}$. A point in this space contains three pieces of information:

1. A genus-$\mathfrak{g}$ Riemann surface with $\mathfrak{n}_{\mathrm{NS}}+\mathfrak{n}_{\mathrm{R}}$ marked point. This is a point $\mathbf{m}\in\mathcal{M}_{\mathfrak{g};\mathfrak{n}_{\mathrm{NS}},\mathfrak{n}_{\mathrm{R}}}$. Let us denote this Riemann surface by $\mathcal{R}(\mathbf{m})$;

2. A choice of $\mathfrak{n}_{\mathrm{NS}}+\mathfrak{n}_{\mathrm{R}}$ local coordinates around the marked points of $\mathcal{R}(\mathbf{m})$;

3. A choice of points of $\mathcal{R}(\mathbf{m})$ corresponding to the locations of insertion of K PCOs. We have to remove *the bad points* from the Riemann surface on which we should not insert any PCO to avoid spurious singularities. The space obtained by removing the bad points from each fibers of $\widetilde{\mathcal{P}}_{\mathfrak{g};\mathfrak{n}_{\mathrm{NS}},\mathfrak{n}_{\mathrm{R}}}$ is denoted by $\widetilde{\mathcal{P}}^{\bullet}_{\mathfrak{g};\mathfrak{n}_{\mathrm{NS}},\mathfrak{n}_{\mathrm{R}}}$. We note that $\widetilde{\mathcal{P}}^{\bullet}_{\mathfrak{g};\mathfrak{n}_{\mathrm{NS}},\mathfrak{n}_{\mathrm{R}}}$ is not a smooth fiber bundle over $\mathcal{M}_{\mathfrak{g};\mathfrak{n}_{\mathrm{NS}},\mathfrak{n}_{\mathrm{R}}}$ since fibers are not smooth due to the elimination of bad points.

In the following, we shall explain a systematic procedure called **The Vertical Integration** for finding a spurious-pole-free integration cycle.

3.D.2 A Warm-Up Example: A Single PCO

To see the general construction of the integration cycle that avoids spurious singularities, let us consider the case that there is only one PCO. For simplicity, we further assume

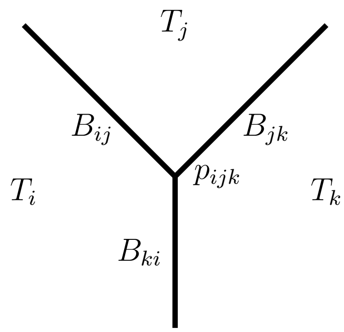


Figure 3.D.1: The triangulation of the two-dimensional moduli space.

that we have a complex-one-dimensional moduli space. We denote a local coordinate in $\widetilde{\mathcal{P}}^{\bullet}_{\mathfrak{g};\mathfrak{n}_{\rm NS},\mathfrak{n}_{\rm R}}$ by $(\mathbf{m}, a)$. Hence, the expression (3.D.1) takes the following form:

$$\mathbf{\Omega} = \langle\mathcal{R}|\left(\mathcal{X}(a) - \partial_a\xi(a)da\right)\wedge\mathcal{B}|\Phi\rangle\,, \tag{3.D.7}$$

$\mathbf{\Omega}$ is defined on a section $s : \mathcal{M}_{\mathfrak{g};\mathfrak{n}_{\rm NS},\mathfrak{n}_{\rm R}} \longrightarrow \widetilde{\mathcal{P}}^{\bullet}_{\mathfrak{g};\mathfrak{n}_{\rm NS},\mathfrak{n}_{\rm R}}$, the scattering amplitude $\mathcal{A}_{\mathfrak{g};\mathfrak{n}_{\rm NS},\mathfrak{n}_{\rm R}}$ is defined by the pull-back of this form to $\mathcal{M}_{\mathfrak{g};\mathfrak{n}_{\rm NS},\mathfrak{n}_{\rm R}}$ under s

$$\mathcal{A}_{\mathfrak{g};\mathfrak{n}_{\rm NS},\mathfrak{n}_{\rm R}} = \int_{\mathcal{M}_{\mathfrak{g};\mathfrak{n}_{\rm NS},\mathfrak{n}_{\rm R}}} s^*\mathbf{\Omega} = \int_{\mathcal{M}_{\mathfrak{g};\mathfrak{n}_{\rm NS},\mathfrak{n}_{\rm R}}} \mathbf{\Omega}(\mathbf{m}, s(\mathbf{m})). \tag{3.D.8}$$

In which $s(\mathbf{m}) = a$. In general, there is no global section. The general idea for dealing with spurious singularities is to find a fine tiling of $\mathcal{M}_{\mathfrak{g};\mathfrak{n}_{\rm NS},\mathfrak{n}_{\rm R}}$ and define a local section for each of the tiles in the tiling. In the simple case we are considering, the fine tiling is given by a triangulation of the moduli space. As is clear from the figure 3.D.1, there are three contributions to the scattering from different regions of the triangulated moduli space: *the triangles T_is, common boundaries of the triangles B_{ij}s and the common vertex of the triangles p_{ijk}s.* Hence, we need four pieces of information:

1) A triangulation of the moduli space. We denote a triangulation by $T \equiv \bigcup_{i=1}^{\#\text{ of triangles}} T_i$, in which T_is are triangles of the triangulation;

2) A set of local sections of $\widetilde{\mathcal{P}}^{\bullet}_{\mathfrak{g};\mathfrak{n}_{\rm NS},\mathfrak{n}_{\rm R}}$ defined on the triangles. The local section of $\widetilde{\mathcal{P}}^{\bullet}_{\mathfrak{g};\mathfrak{n}_{\rm NS},\mathfrak{n}_{\rm R}}$ defined on T_a is given by the following map

$$s_a : T_a \longrightarrow \widetilde{\mathcal{P}}^{\bullet}_{\mathfrak{g};\mathfrak{n}_{\rm NS},\mathfrak{n}_{\rm R}}, \qquad a = 1, \cdots, \#\text{ of triangles}. \tag{3.D.9}$$

These sections avoid spurious singularities because it is a section of $\widetilde{\mathcal{P}}^{\bullet}_{\mathfrak{g};\mathfrak{n}_{\rm NS},\mathfrak{n}_{\rm R}}$. The contribution from T_a to scattering amplitudes is given by

$$\mathcal{A}^{(a)}_{\mathfrak{g};\mathfrak{n}_{\rm NS},\mathfrak{n}_{\rm R}} \equiv \int_{T_a} s_a^*\mathbf{\Omega} = \int_{T_a} \mathbf{\Omega}(\mathbf{m}, s_a(\mathbf{m})), \qquad a = 1, \cdots, \#\text{ of triangles.} \tag{3.D.10}$$

3) In general, the contributions from the triangle, T_a and T_b, which have common boundary B_{ab}, does not agree on B_{ab}. The reason is clear: local sections s_a and s_b do not agree on B_{ab}. Therefore, we need to consider appropriate correction factor from B_{ab} to the scattering amplitude. We denote correction factors from various common boundaries of the triangles by $\mathcal{A}^{(ab)}_{\mathfrak{g};\mathfrak{n}_{\rm NS},\mathfrak{n}_{\rm R}}, \quad a, b = 1, \cdots, \#$ of triangles.

4) In general the contribution from the boundaries B_{ab}, B_{bc} and B_{ca} do not agree on the intersection of these boundaries because the corresponding sections do not agree there. Since the moduli space is 2-dimensional, $B_{ab}\cap B_{bc}\cap B_{ca}$ is a point p_{abc}. We need to consider an appropriate correction factor from p_{abc} to scattering amplitudes. We denote the correction factors from various common points of various boundaries of triangles by $\mathcal{A}^{(abc)}_{\mathfrak{g};\mathfrak{n}_{\rm NS},\mathfrak{n}_{\rm R}}, \quad a, b, c = 1, \cdots, \#$ of triangles. In general, there can be more that three triangles that share the same vertex. We denote the correction factors from the common vertex of n triangles by $\mathcal{A}^{(a_1\cdots a_n)}_{\mathfrak{g};\mathfrak{n}_{\rm NS},\mathfrak{n}_{\rm R}}, \quad a_1, \cdots, a_n = 1, \cdots, \#$ of triangles.

Considering all these contributions, scattering amplitudes are given by

$$\mathcal{A}_{\mathfrak{g};\mathfrak{n}_{\rm NS},\mathfrak{n}_{\rm R}} = \sum_{a=1}^{\#\text{ of triangles}} \mathcal{A}^{(a)}_{\mathfrak{g};\mathfrak{n}_{\rm NS},\mathfrak{n}_{\rm R}} \pm \frac{1}{2!} \sum_{\substack{a_1,a_2=1\\ a_1\neq a_2}}^{\#\text{ of triangles}} \mathcal{A}^{(a_1a_2)}_{\mathfrak{g};\mathfrak{n}_{\rm NS},\mathfrak{n}_{\rm R}} \pm \sum_{n\geq 3} \frac{1}{n!} \sum_{\substack{a_1,\cdots a_n=1\\ a_1\neq\cdots\neq a_n}}^{\#\text{ of triangles}} \mathcal{A}^{(a_1\cdots a_n)}_{\mathfrak{g};\mathfrak{n}_{\rm NS},\mathfrak{n}_{\rm R}}, \tag{3.D.11}$$

in which $\mathcal{A}^{(a)}_{\mathfrak{g};\mathfrak{n}_{\rm NS},\mathfrak{n}_{\rm R}}$ is given by (3.D.10). The $\pm$ sign is fixed by specifying the orientations of the B_{ab}'s. To find the full amplitude $\mathcal{A}_{\mathfrak{g},\mathfrak{n}_{\rm NS},\mathfrak{n}_{\rm R}}$, we need to find suitable expressions for $\mathcal{A}^{(a_1a_2)}_{\mathfrak{g};\mathfrak{n}_{\rm NS},\mathfrak{n}_{\rm R}}$ and all $\mathcal{A}^{(a_1,\cdots,a_n)}_{\mathfrak{g};\mathfrak{n}_{\rm NS},\mathfrak{n}_{\rm R}}$ for $n\geq 3$. The determination of these expressions is done by the method of vertical integration. We explore each piece separately.

- **Determining $\mathcal{A}^{(a_1a_2)}_{\mathfrak{g};\mathfrak{n}_{\rm NS},\mathfrak{n}_{\rm R}}$**
  The expression for $\mathcal{A}^{(a_1a_2)}_{\mathfrak{g};\mathfrak{n}_{\rm NS},\mathfrak{n}_{\rm R}}$ can be obtained by choosing a vertical segment $\mathcal{V}_{a_1a_2}$ over $B_{a_1a_2}$. To construct this segment, we choose a point $\mathbf{m}\in B_{a_1a_2}$ and a curve $P_{a_1a_2}(\mathbf{m}, v)$ that connects $s_{a_1}(\mathbf{m})$ and $s_{a_2}(\mathbf{m})$ in $\mathcal{R}(\mathbf{m})\in\widetilde{\mathcal{P}}^{\bullet}_{\mathfrak{g};\mathfrak{n}_{\rm NS},\mathfrak{n}_{\rm R}}$. The parameter

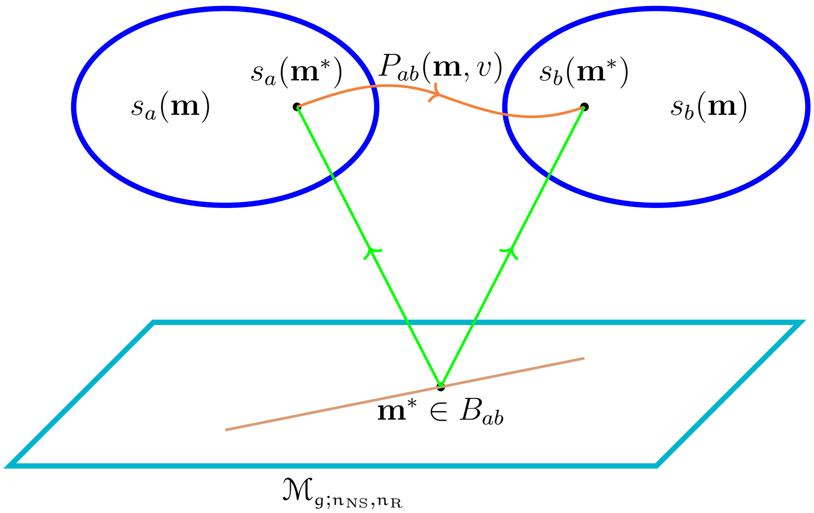


Figure 3.D.2: The vertical segment for a two dimensional moduli space. $s_a(\mathbf{m})$ and $s_b(\mathbf{m})$ are sections over T_a and T_b. The definition of a vertical segment involves a choice of the curve $P_{ab}(\mathbf{m}, v)$ in $\mathcal{R}(\mathbf{m}^*)$ that connects sections $s_a(\mathbf{m}^*)$ and $s_b(\mathbf{m}^*)$ over $\mathbf{m}^* \in B_{ab}$. For a fixed $\mathbf{m}^* \in B_{ab}$, as the parameter v changes over the interval $[0, 1]$, the curve connects the two sections $s_a(\mathbf{m}^*)$ and $s_b(\mathbf{m}^*)$ over $\mathbf{m}^*$.

$v \in [0, 1]$ labels a position along the curve

$$P_{a_1a_2}(\mathbf{m}, v) : [0, 1] \longrightarrow \widetilde{\mathcal{P}}^{\bullet}_{\mathrm{g};\mathrm{n}_{\mathrm{NS}},\mathrm{n}_{\mathrm{R}}}. \tag{3.D.12}$$

The vertical segment can be parametrized as follows

$$\mathcal{V}_{a_1a_2} \equiv \{(\mathbf{m}, v) \,|\, \mathbf{m} \in B_{a_1a_2},\ v \in [0, 1]\}\,. \tag{3.D.13}$$

To get the correction factor from the boundary $B_{a_1a_2}$, $\boldsymbol{\Omega}$ should be integrated over the path $P_{a_1a_2}(\mathbf{m}, v)$ that connects the sections $s_{a_1}(\mathbf{m})$ and $s_{a_2}(\mathbf{m})$ over the $B_{a_1a_2}$ instead of the sections themselves. This is illustrated in figure 3.D.2. A point on $P_{a_1a_2}(\mathbf{m}, v)$ is given by a value of the parameter v. Therefore, the form is dependent on $(\mathbf{m}, v)$. From (3.D.7), we can first integrate over v:

$$\mathcal{A}^{(ab)}_{\mathrm{g};\mathrm{n}_{\mathrm{NS}},\mathrm{n}_{\mathrm{R}}} = \int_{\mathcal{V}_{ab}} \Omega(\mathbf{m}, v) = \int_{\mathbf{m}\in B_{ab}} \int_{v\in[0,1]} \langle \mathcal{R}|\left(\mathcal{X}(\mathbf{m}, v) - \partial_v \xi(\mathbf{m}, v) dv\right) \wedge \mathcal{B}|\Phi\rangle$$

$$= \int_{\mathbf{m}\in B_{ab}} \underbrace{\langle \mathcal{R}|[\xi(s_a(\mathbf{m})) - \xi(s_b(\mathbf{m}))] \wedge \mathcal{B}|\Phi\rangle}_{\text{a } d-1 \text{ form}} . \tag{3.D.14}$$

- **Determining $\mathcal{A}^{(a_1\cdots a_n)}_{\mathrm{g};\mathfrak{n}_{\mathrm{NS}},\mathfrak{n}_{\mathrm{R}}}$ for $n \geq 3$**
  Let's consider the simplest case, namely $n = 3$. The contribution $\mathcal{A}^{(abc)}_{\mathrm{g};\mathfrak{n}_{\mathrm{NS}},\mathfrak{n}_{\mathrm{R}}}$ is present only if the vertical segments $\mathcal{V}_{ab}$ over B_{ab}, $\mathcal{V}_{bc}$ over B_{bc} and $\mathcal{V}_{ca}$ over B_{ca} do not match over p_{abc}. However, the formula (3.D.14) is independent of the choice of path $P_{ab}(\mathbf{m}, v)$ and hence the choice of vertical segment. Therefore, we can always choose the curves $P_{ab}(\mathbf{m}, v)$, $P_{bc}(\mathbf{m}, v)$ and $P_{ca}(\mathbf{m}, v)$ such that vertical segments $\mathcal{V}_{ab}$ over B_{ab}, $\mathcal{V}_{bc}$ over B_{bc} and $\mathcal{V}_{ca}$ over B_{ca} match over the triple intersection point p_{abc}. Therefore, there is no contribution from p_{abc}. Similarly, all $\mathcal{A}^{(a_1\cdots a_n)}_{\mathrm{g};\mathfrak{n}_{\mathrm{NS}},\mathfrak{n}_{\mathrm{R}}}$ can make to vanish for $n \geq 4$ [25].

Putting everything together, we conclude that the final expression for a scattering process involving $\mathfrak{n}_{\mathrm{NS}}$ external NS states and $\mathfrak{n}_{\mathrm{R}}$ R external states which has a (real) two-dimensional moduli space and needs only one PCO insertion is given by:

$$\mathcal{A}_{\mathrm{g};\mathfrak{n}_{\mathrm{NS}},\mathfrak{n}_{\mathrm{R}}} = \sum_{a=1}^{\#\text{ of triangles}} \mathcal{A}^{(a)}_{\mathrm{g};\mathfrak{n}_{\mathrm{NS}},\mathfrak{n}_{\mathrm{R}}} \pm \frac{1}{2!} \sum_{\substack{a,b=1\\ a\neq b}}^{\#\text{ of triangles}} \mathcal{A}^{(ab)}_{\mathrm{g};\mathfrak{n}_{\mathrm{NS}},\mathfrak{n}_{\mathrm{R}}}, \tag{3.D.15}$$

where $\mathcal{A}^{(a)}_{\mathrm{g};\mathfrak{n}_{\mathrm{NS}},\mathfrak{n}_{\mathrm{R}}}$ and $\mathcal{A}^{(ab)}_{\mathrm{g};\mathfrak{n}_{\mathrm{NS}},\mathfrak{n}_{\mathrm{R}}}$ are given by (3.D.10) and (3.D.14), respectively. Once, we fixed the orientations of the B_{ab}s, the $\pm$ sign in (3.D.15) will be fixed and we get a unique answer for the final amplitude.

This procedure is difficult to generalize to the higher-dimensional moduli spaces. The two main difficulties are

1. A fine tiling of the moduli space requires an appropriate notion of triangulation, as in the two dimensional case. The analog of a triangle for the higher-dimensional case is the notion of simplex. However, simplexes can meet on higher-dimensional subspaces of the moduli space. It turns out that the contribution for these common subspaces can not be made to vanish, unlike the case of two-dimensional moduli space. Therefore, we need to include the corrections from these subspaces as well [25].

2. If we want to specify a path between sections, say sections s_a and s_b, over a subspace C of the moduli space that two or more triangles meet, we have to move one PCO at

a time from s_a to s_b over a specified path. Therefore, there would be an ambiguity from the order of moving of PCOs from s_a to s_b over C. This implies that the process of integration is dependent on the choice of vertical segment over C because different order of moving of PCOs gives another vertical segment. However, there should not be an ambiguity in the definition of amplitudes.

In the next section, we explain a systematic procedure which resolves these issues.

3.D.3 The General Vertical Integration Procedure

The procedure described above for the case of one PCO can be suitably generalized for the higher-dimensional moduli spaces with a systematic way to *fill the gaps* for all subspaces on which tiles of the tiling of the moduli space meet. The fiber of $\widetilde{\mathcal{P}}_{\mathfrak{g};\mathfrak{n}_{\mathrm{NS}},\mathfrak{n}_{\mathrm{R}}}$ over the point $\mathbf{m}$ is

$$\Xi(\mathbf{m}) \equiv \underbrace{\mathcal{R}(\mathbf{m}) \times \cdots \times \mathcal{R}(\mathbf{m})}_{K \text{ factors}}. \tag{3.D.16}$$

We denote local coordinates by

$$(\mathbf{m}; a_1, \cdots, a_K) \in \mathcal{M}_{\mathfrak{g};\mathfrak{n}_{\mathrm{NS}},\mathfrak{n}_{\mathrm{R}}} \times \Xi(\mathbf{m}). \tag{3.D.17}$$

In which $\mathbf{m} \in \mathcal{M}_{\mathfrak{g};\mathfrak{n}_{\mathrm{NS}},\mathfrak{n}_{\mathrm{R}}}$ and a_is denote a choice of the locations of K PCOs in the i^{th} factor of $\Xi(\mathbf{m})$ in (3.D.16)

$$a_i \equiv (z_{1,i}, \cdots, z_{K,i}), \qquad z_{1,i}, \cdots, z_{K,i} \in \mathcal{R}(\mathbf{m}). \tag{3.D.18}$$

A local coordinate in $\widetilde{\mathcal{P}}^{\bullet}_{\mathfrak{g};\mathfrak{n}_{\mathrm{NS}},\mathfrak{n}_{\mathrm{R}}}$ is obtained by removing bad points from each fibers of $\widetilde{\mathcal{P}}_{\mathfrak{g};\mathfrak{n}_{\mathrm{NS}},\mathfrak{n}_{\mathrm{R}}}$ to avoid spurious singularities. If we denote the Riemann surface obtained by removing the bad points by $\mathcal{R}^{\bullet}(\mathbf{m})$, we can define

$$\Xi^{\bullet}(\mathbf{m}) \equiv \underbrace{\mathcal{R}^{\bullet}(\mathbf{m}) \times \cdots \times \mathcal{R}^{\bullet}(\mathbf{m})}_{K \text{ factors}}. \tag{3.D.19}$$

Therefore, a set of local coordinate on $\widetilde{\mathcal{P}}^{\bullet}_{\mathfrak{g};\mathfrak{n}_{\mathrm{NS}},\mathfrak{n}_{\mathrm{R}}}$ is given by

$$(\mathbf{m}; a_1^{\bullet}, \cdots, a_K^{\bullet}) \in \mathcal{M}_{\mathfrak{g};\mathfrak{n}_{\mathrm{NS}},\mathfrak{n}_{\mathrm{R}}} \times \Xi^{\bullet}(\mathbf{m}), \tag{3.D.20}$$

In which $\mathbf{m} \in \mathcal{M}_{g;n_{\rm NS},n_{\rm R}}$ and $a_i^\bullet$s denote a choice of the locations of K PCOs in the $i^{\rm th}$ factor of $\Xi^\bullet(\mathbf{m})$ in (3.D.19) that avoids spurious singularities. It is given by

$$a_i^\bullet \equiv (z_{1,i}, \cdots, z_{K,i}), \qquad z_{1,i}, \cdots, z_{K,i} \in \mathcal{R}^\bullet(\mathbf{m}). \tag{3.D.21}$$

As we noted above $\widetilde{\mathcal{P}}^\bullet_{g;n_{\rm NS},n_{\rm R}}$ is not a fiber bundle over $\mathcal{M}_{g;n_{\rm NS},n_{\rm R}}$. Therefore, the projection $\pi : \widetilde{\mathcal{P}}^\bullet_{g;n_{\rm NS},n_{\rm R}} \longrightarrow \mathcal{M}_{g;n_{\rm NS},n_{\rm R}}$ can not give rise to a global section of $\widetilde{\mathcal{P}}^\bullet_{g;n_{\rm NS},n_{\rm R}}$. However, we do not need a global section. We first tile the moduli space by appropriate tiles and then define local sections for each of these tiles. In general, the tiles can meet on subspaces of codimension $1 \leq k \leq d$[11]. A systematic procedure for computing string amplitudes must give a local section for each tile and also correction factors that fill the gaps between sections over all codimension $1 \leq k \leq d$ subspaces, on which the tiles can meet. Therefore, the first task is to specify an appropriate geometric notion of the tiles we are going to use for tiling of the moduli space.

The Notion of Dual Triangulation

An appropriate tiling of moduli spaces should provide a proper control over the subspaces on which two or more tiles of the tiling can meet. By definition, a *dual triangulation* Υ *of an n-dimensional manifold* is given by gluing together the n-dimensional polyhedra along their boundary faces. The faces of an n-dimensional polyhedra have codimensions $1 \leq k \leq n$. The gluing should be in such a way that every codimension-k face of a polyhedra in Υ is contained in exactly $k+1$ polyhedra in Υ. The later property of a dual triangulation gives better control over the number of polyhedra which have a common codimension-k face. Therefore, it would be easier to find the correction factors from these faces. Hence, we assume that we have a dual triangulation Υ of the moduli space. A typical example of dual triangulation for the two-dimensional moduli spaces is illustrated in figure 3.D.3.

We denote by $\mathcal{M}_k^{\alpha_0,\cdots,\alpha_k}$, a codimension-$k$ face with $0 \leq k \leq d$ of a dual triangulation Υ of $\mathcal{M}_{g;n_{\rm NS},n_{\rm R}}$ which is shared by $k+1$ codimension-zero faces $\mathcal{M}_0^{\alpha_0}, \cdots, \mathcal{M}_0^{\alpha_k}$, i.e. the polyhedra of the dual triangulation. If $\mathcal{M}_{g;n_{\rm NS},n_{\rm R}}$ has an orientation, it induces an orientation for each polyhedron. The orientation on each codimension-k face $\mathcal{M}_k^{\alpha_0,\cdots,\alpha_k}$ can be taken to be:

$$\partial\mathcal{M}_k^{\alpha_0,\cdots,\alpha_k} = -\sum_{\substack{\alpha_{k+1} \\ \alpha_{k+1}\neq\alpha_0 \\ \cdots \\ \alpha_{k+1}\neq\alpha_k}} \mathcal{M}_{k+1}^{\alpha_0,\cdots,\alpha_k,\alpha_{k+1}}. \tag{3.D.22}$$

[11] Once we introduced an appropriate notion of tiling, we mention an improved upper bound.

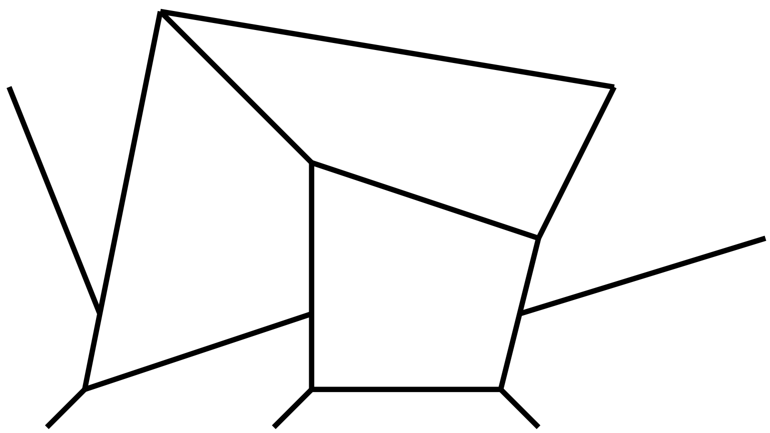


Figure 3.D.3: The dual triangulation of the two-dimensional moduli space. As it is clear, each codimension-one face (i.e. an edge) is shared by two polyhedra and each codimension-two face (i.e. a vertex) is shared by three polyhedra. This is in accordance with the rule that in a dual triangulation codimension-k faces are contained in $k+1$ polyhedra.

Under this choice, it is clear that the orientation changes sign under $\alpha_i \longleftrightarrow \alpha_j$ for any pair (i,j). We also consider the sections defined on the codimension-zero faces:

$$\begin{aligned} &s_i : \mathcal{M}_0^{\alpha_i} \longrightarrow \widetilde{\mathcal{P}}_{\mathfrak{g};\mathfrak{n}_{\mathrm{NS}},\mathfrak{n}_{\mathrm{R}}}, \qquad i = 0,\cdots,\#\text{ of polyhedra}, \\ &s_i(\mathbf{m}) \equiv (\mathbf{m}; a_i) = (\mathbf{m}; z_{1,i},\cdots,z_{K,i}), \qquad \mathbf{m} \in \mathcal{M}_0^{\alpha_i}. \end{aligned} \tag{3.D.23}$$

Each of these sections determine a configuration of PCOs in Ξ.

A general scattering amplitude in superstring theory is given by an integration over the moduli space. Therefore, if we tile the moduli space by dual triangulation, the scattering amplitude is given by an appropriate sum of the contributions from codimension-k faces with $0 \le k \le d$. If we denote the contribution from the codimension-k face $\mathcal{M}_k^{\alpha_0,\cdots,\alpha_k}$ by $\mathcal{A}_k^{\alpha_0,\cdots,\alpha_k}$, the full scattering amplitude is given by

$$\mathcal{A}_{\mathfrak{g};\mathfrak{n}_{\mathrm{NS}},\mathfrak{n}_{\mathrm{R}}} = \sum_{k=0}^{d} \sum_{\{\alpha_0,\cdots,\alpha_k\}} (\pm)\mathcal{A}_k^{\alpha_0,\cdots,\alpha_k}. \tag{3.D.24}$$

The factor $\pm$ determines the sign of the contribution from codimension-k faces. We will fix this factor later. To find the contributions from codimension k faces $\mathcal{M}_k^{\alpha_0,\cdots,\alpha_k}$ which is shared by co-dimension zero faces $\mathcal{M}_0^{\alpha_0},\cdots,\mathcal{M}_0^{\alpha_k}$, we have to find a condition on sections s_is. Hence, we assume that $a_0,\cdots,a_k$ stands for $(k+1)$ possible PCOs arrangements in

$\mathcal{R}(\mathbf{m})$. Each a_i stands for a K-tuple $(z_{1,i}, \cdots, z_{K,i})$ with $z_{j,i} \in \mathcal{R}(\mathbf{m}),\ j = 1, \cdots, K$. Any of coordinates z_i in a K-tuple $(z_1, \cdots, z_K)$, which specifies a location for one of the PCOs, can take values in one of the $z_{j,i},\ j = 1, \cdots, K,\ i = 0, \cdots, k$. Therefore there are $(k+1)^K$ possible PCOs arrangement. The condition on sections s_i is that they must avoid spurious singularities for $\mathbf{m} \in \mathcal{M}_k^{\alpha_0, \cdots, \alpha_k},\ k = 0, \cdots, d$. Using (3.D.23)[12]

$$\begin{aligned} &s_i : \mathcal{M}_0^{\alpha_i} \longrightarrow \widetilde{\mathcal{P}}^{\bullet}_{\mathfrak{g};\mathfrak{n}_{\rm NS},\mathfrak{n}_{\rm R}}, \qquad i = 0, \cdots, \#\text{ of polyhedra}, \\ &s_i(\mathbf{m}) \equiv (m; a_i) = (m; z_{1,i}, \cdots, z_{K,i}) \in \mathcal{M}_k^{\alpha_0, \cdots, \alpha_k} \times \Xi^{\bullet}. \end{aligned} \tag{3.D.25}$$

In other words, we have the following local expression

$$(m; s_0(\mathbf{m}), \cdots, s_k(\mathbf{m})) \in \mathcal{M}_k^{\alpha_0, \cdots, \alpha_k} \times \underbrace{\widetilde{\mathcal{P}}^{\bullet}_{\mathfrak{g};\mathfrak{n}_{\rm NS},\mathfrak{n}_{\rm R}} \times \cdots \times \widetilde{\mathcal{P}}^{\bullet}_{\mathfrak{g};\mathfrak{n}_{\rm NS},\mathfrak{n}_{\rm R}}}_{k \text{ factors}}. \tag{3.D.26}$$

It can be shown that for sufficiently fine dual triangulation such a choice of sections exists [25]. The general prescription for the vertical integration consists of finding contributions from all codimension-k faces $\mathcal{M}_k^{\alpha_0, \cdots, \alpha_k}$ to scattering amplitudes. We consider the cases $k = 0$ and $k \neq 0$ separately.

Contributions from Codimension-Zero Faces $\mathcal{M}_0^{\alpha_0}$

Consider a dual triangulation of moduli space with codimension-zero polyhedra $\{\mathcal{M}_0^{\alpha_0}\}$. The contribution of the polyhedron $\mathcal{M}_0^{\alpha_0}$ is given by

$$\mathcal{A}_0^{\alpha_0} = \int_{\mathcal{M}_0^{\alpha_0}} s_0^* \mathbf{\Omega} = \int_{\mathcal{M}_0^{\alpha_0}} \mathbf{\Omega}(\mathbf{m}, s_0(\mathbf{m})). \tag{3.D.27}$$

Therefore, we pull-back the form $\mathbf{\Omega}$ to $\mathcal{M}_0^{\alpha_0}$ using s_0 and then integrate it over $\mathcal{M}_0^{\alpha_0}$. The section s_0 satisfies the condition (3.D.26) and as such they avoid spurious singularities by construction.

Contributions from Codimension $k \neq 0$ Faces $\mathcal{M}_k^{\alpha_0, \cdots, \alpha_k}$

To find the contribution from all codimension $k \neq 0$ faces $\mathcal{M}_k^{\alpha_0, \cdots, \alpha_k}$, we need to explain the basic logic behind the vertical integration. The vertical integration states that we can avoid

[12] By abuse of notation, we use s_i for sections of $\widetilde{\mathcal{P}}^{\bullet}_{\mathfrak{g};\mathfrak{n}_{\rm NS},\mathfrak{n}_{\rm R}}$.

spurious poles by finding a spurious-pole free integration cycle. To find such an integration cycle in $\widetilde{\mathcal{P}}^{\bullet}_{\mathfrak{g};\mathfrak{n}_{\rm NS},\mathfrak{n}_{\rm R}}$, for a given codimension-$k \neq 0$ face $\mathcal{M}_k^{\alpha_0,\cdots,\alpha_k}$, we have to find a k-dimensional subspace, i.e. a path, $P_{\alpha_0,\cdots,\alpha_k}$ between PCOs locations in $\mathcal{M}_0^{\alpha_0},\cdots,\mathcal{M}_0^{\alpha_k}$ that share $\mathcal{M}_k^{\alpha_0,\cdots,\alpha_k}$. PCOs locations are defined in $\Xi(\mathbf{m})$, as described in (3.D.16). Therefore, $P_{\alpha_0,\cdots,\alpha_k}$ is a path in $\Xi(m)$. The vertical integration for a codimension $k \neq 0$ face $\mathcal{M}_k^{\alpha_0,\cdots,\alpha_k}$ involves the following steps

1. Finding an appropriate path $P_{\alpha_0,\cdots,\alpha_k}$ in $\Xi(\mathbf{m})$ that connects the locations of PCOs in $\mathcal{M}_0^{\alpha_0},\cdots,\mathcal{M}_0^{\alpha_k}$ over a point $\mathbf{m} \in \mathcal{M}_k^{\alpha_0,\cdots,\alpha_k},\ k \neq 0$;
2. Finding the form that needs to be integrated over codimension $k \neq 0$ faces $\mathcal{M}_k^{\alpha_0,\cdots,\alpha_k}$ by integrating the form $\mathbf{\Omega}$ over the path $P_{\alpha_0,\cdots,\alpha_k}$. The path is k-dimensional and we vary the locations of PCOs on it. On the other hand, the integration only depends on endpoints of the path. Therefore, we get an $(n-k)$-form. We denote it by $\mathbf{\Omega}_{n-k}^{\alpha_0,\cdots,\alpha_k}$.
3. Integrating $\mathbf{\Omega}_{n-k}^{\alpha_0,\cdots,\alpha_k}$ over $\mathcal{M}_k^{\alpha_0,\cdots,\alpha_k}$ to find $\mathcal{A}_k^{\alpha_0,\cdots,\alpha_k}$ in (3.D.24).

Therefore, the first task is to find an appropriate way to construct a path $P_{\alpha_0,\cdots,\alpha_k}$ inside $\Xi(\mathbf{m})$ for $\mathbf{m} \in \mathcal{M}_k^{\alpha_0,\cdots,\alpha_k}$. The basic idea is that for K PCOs and a codimension $k \neq 0$ face $\mathcal{M}_k^{\alpha_0,\cdots,\alpha_k}$, we choose a path on a lattice in $\mathbb{R}^K$ that avoids spurious singularities and then map this back into a path in $\Xi(\mathbf{m})$ for $\mathbf{m} \in \mathcal{M}_k^{\alpha_0,\cdots,\alpha_k}$. We shall describe the way to construct these paths.

Consider the K-dimensional Euclidean space $\mathbb{R}^K$. Given a codimension $k \neq 0$ face $\mathcal{M}_k^{\alpha_0,\cdots,\alpha_k}$ which is shared by k codimension-zero faces $\mathcal{M}_0^{\alpha_0},\cdots,\mathcal{M}_0^{\alpha_k}$, the $(k+1)^K$ possible PCOs arrangements which is determined by the sections $s_0,\cdots,s_k$ are points on a lattice in $\mathbb{R}^k$. The coordinate of this point is given by the index of sections s_i, i.e. the coordinates of these point is a K-tuple $(i_1,\cdots,i_K)$ with the rule that if i_j is determined by the section s_l, then $i_j = l$.

As an example, consider codimension-one faces $\mathcal{M}_1^{\alpha_0,\alpha_1}$ with 2 PCOs. There are 2^2 PCOs arrangements. It is a codimension-1 face, therefore in a dual triangulation, it is shared with two codimension-zero faces $\mathcal{M}_0^{\alpha_0}$ and $\mathcal{M}_0^{\alpha_1}$. Regarding the condition (3.D.26), appropriate sections are defined as follows

$$
\begin{aligned}
&s_i : \mathcal{M}_0^{\alpha_i} \longrightarrow \widetilde{\mathcal{P}}^{\bullet}_{\mathfrak{g};\mathfrak{n}_{\rm NS},\mathfrak{n}_{\rm R}}, \qquad i = 0,1,\\
&s_i(\mathbf{m}) = (\mathbf{m}; z_{1,i}, z_{2,i}), \qquad \mathbf{m} \in \mathcal{M}_1^{\alpha_0,\alpha_1}.
\end{aligned} \tag{3.D.28}
$$

The possible PCOs arrangement is given by a 2-tuple (i_1,i_2) in $\mathbb{R}^2$. Possible paths are illustrated in figure 3.D.4. We assign $i_j = 0$, if it is determined by s_0 and $i_j = 1$, if it is

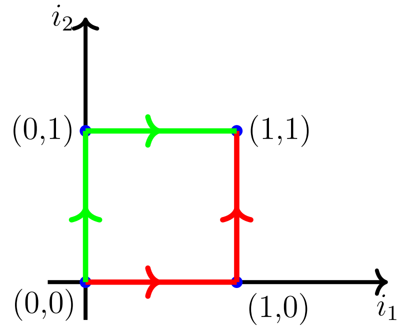


Figure 3.D.4: The Euclidean space $\mathbb{R}^2$ corresponds to a codimension-1 face $\mathcal{M}_1^{\alpha_0,\alpha_1}$ with two PCOs. The red path and the green path are the possible one-dimensional paths for moving PCOs for a point $\mathbf{m} \in \mathcal{M}_1^{\alpha_0,\alpha_1}$.

determined by s_1. The possible arrangements of PCOs are thus given by the four 2-tuples, 1) $(0,0)$, the locations of both PCOs are determined by s_0, 2) $(0,1)$, the location of the first PCO is determined by s_0 and the location of the second one is determined by s_1, 3) $(1,0)$, the location the first PCO is detemined by s_1, and the location of the second one is determined by s_0, and finally 4) $(1,1)$, the locations of both PCOs are determined by s_1.

Returning to the general case, we can determine $(k-1)$-dimensional subspaces $Q_{\alpha_0,\cdots,\alpha_{k-1}}$ of $\mathbb{R}^K$ that that are paths for the moving of PCOs. Each of these $(k-1)$-dimensional subspaces is composed of a union of hypercubes (the multi-dimensional generalization of a rectangle). Vertices of any of these hypercubes label locations of PCOs by the rule mentioned above. On each hypercube only $k-1$ of coordinates of $\mathbb{R}^k$ vary. We choose a k-dimensional subspace $Q_{\alpha_0,\cdots,\alpha_k}$ of $\mathbb{R}^K$ whose boundary consists of $(k-1)$-dimensional subspaces $Q_{\alpha_0,\cdots,\alpha_{k-1}}$

$$\partial Q_{\alpha_0,\cdots,\alpha_k} = -\sum_{i=0}^{k}(-1)^{k-i} Q_{\alpha_0,\cdots,\alpha_{i-1},\alpha_{i+1},\cdots,\alpha_k}. \tag{3.D.29}$$

$Q_{\alpha_0,\cdots,\alpha_k}$ is chosen to be a union of k-dimensional hypercubes. Vertices of these hypercubes are given by k-tuples of integers according to the rule mentioned above. Along these hypercubes only k of the coordinates of $\mathbb{R}^K$ vary. $Q_{\alpha_0,\cdots,\alpha_k}$ is chosen to be antisymmetric under $\alpha_i \longleftrightarrow \alpha_j$ for any pair (i,j). Once a $Q_{\alpha_0,\cdots,\alpha_k}$ with the above-mentioned properties was constructed, a k-dimensional subspace $P_{\alpha_0,\cdots,\alpha_k}(m)$ of $\Xi(\mathbf{m})$ for $\mathbf{m} \in \mathcal{M}_k^{\alpha_0,\cdots,\alpha_k}$ can be constructed. $Q_{\alpha_0,\cdots,\alpha_k}$ is the union of k-dimensional hypercubes $\mathbb{H}^k_a$

$$Q_{\alpha_0,\cdots,\alpha_k} = \bigcup_a \mathbb{H}^k_a. \tag{3.D.30}$$

We thus only need to know how to construct a k-dimensional hypercube-shaped regions in $\Xi(\mathbf{m})$ corresponding to a k-dimensional hypercube in $\mathbb{R}^K$. If we know how to construct these k-dimensional hypercube-shaped regions in $\Xi(\mathbf{m})$, then $P_{\alpha_0,\cdots,\alpha_k}(m)$ would be the union of them. The construction of these k-dimensional hypercube-shaped regions in $\Xi(\mathbf{m})$ is as follows. One replaces $\Xi(\mathbf{m})$ by its universal cover $\widetilde{\Xi}(\mathbf{m})$

$$\widetilde{\Xi}(\mathbf{m}) \equiv \underbrace{\widetilde{\mathcal{R}}(\mathbf{m}) \times \cdots \times \widetilde{\mathcal{R}}(\mathbf{m})}_{K \text{ factors}}, \tag{3.D.31}$$

where $\widetilde{\mathcal{R}}(\mathbf{m})$ is the universal cover of $\mathcal{R}(\mathbf{m})$. Each point in $\mathcal{R}(m)$ is represented by an infinite number of points in $\widetilde{\mathcal{R}}(m)$. We choose arbitrary $(k+1)^K$ points in $\widetilde{\Xi}(\mathbf{m})$ corresponding to $(k+1)^K$ locations of PCOs in $\Xi(\mathbf{m})$. Given a k-dimensional hypercube $\mathbb{H}^k$ in $\mathbb{R}^K$, we define the following map

$$\begin{gathered}\phi : \mathbb{R}^K \longrightarrow \widetilde{\Xi}(\mathbf{m}) \\ \phi(i_1, \cdots, i_K) = (z_{1,i_1}, \cdots, z_{1,i_K})\end{gathered} \tag{3.D.32}$$

To construct the k-dimensional hypercube-shaped subspace in $\widetilde{\Xi}(\mathbf{m})$, we consider a p-dimensional face (with $p \leq k$) of $\mathbb{H}^k_a$. We construct the k-dimensional subspace of $\widetilde{\Xi}(\mathbf{m})$ by starting from $p = 1$ up to $p = k$. Along a p-dimensional face only the coordinates $i_1, \cdots, i_p$ change and the other $k - p$ coordinates of $\mathbb{H}^k_a$ are fixed. We associate a p-dimensional subspace of $\widetilde{\Xi}(\mathbf{m})$, along which only $z_{i_1}, \cdots, z_{i_p}$ vary. The boundary of this p-dimensional subspace is fixed by the choice of the $(p-1)$-dimensional subspaces in the previous step, i.e. when we construct $(p-1)$-dimensional subspaces. However, the way coordinates $z_{i_1}, \cdots, z_{i_p}$ change in the interior of this p-dimensional subspace is completely arbitrary. We continue this process up to $p = k$ [13]. Proceeding in this way, we can construct the k-dimensional hypercube-shaped subspace of $\widetilde{\Xi}(\mathbf{m})$ by mapping the whole of $\mathbb{H}^k_a$. If we call the k-dimensional subspaces of $\widetilde{\Xi}(\mathbf{m})$ that constructed in this way by $\widetilde{\mathbb{H}}^k_a$, then the k-dimensional subspace of $\widetilde{\Xi}(\mathbf{m})$ or its projection in $\Xi(\mathbf{m})$, which we are looking for, is the

[13]This process is similar to the construction of the image of a triangle in a space by first specifying the location of the vertices mapping it into the space and then map the edges to the space.

union of these subspaces

$$P_{\alpha_0,\cdots,\alpha_k} \equiv \bigcup_a \widetilde{\mathbb{H}}^k_a. \tag{3.D.33}$$

Relation (3.D.29) induces the following relation

$$\partial P_{\alpha_0,\cdots,\alpha_k} \simeq -\sum_{i=0}^{k} P_{\alpha_0,\cdots,\alpha_{i-1},\alpha_{i+1},\cdots,\alpha_k}, \tag{3.D.34}$$

where $\simeq$ meant that the boundary of $P_{\alpha_0,\cdots,\alpha_k}$ is a collection of $(k-1)$-dimensional subspaces of $\Xi(\mathbf{m})$ whose corner points are exactly the corner points of $(k-1)$-dimensional subspaces on the right hand side. This process has to be done for every $k \neq 0$. Another issue is the upper bound on k i.e. to which value of k we need to do this process. In general $0 \leq k \leq d$. However, we have the relation $K \leq d$[14]. Therefore, the upper bound on k is given by:

$$k \leq K. \tag{3.D.35}$$

Assuming the condition (3.D.26), an integration cycle in $\widetilde{\mathcal{P}}_{\mathfrak{g};\mathfrak{n}_{\rm NS},\mathfrak{n}_{\rm R}}$ which avoids spurious singularities can be constructed as follows

1. Consider a dual triangulation $\Upsilon = \{\mathcal{M}_0^{\alpha_i}\}$, consists of codimension-zero faces (i.e. polyhedra of the dual triangulation) $\mathcal{M}_0^{\alpha_i}$, and the corresponding sections $\{s_i\}$ defined by
$$s_i : \mathcal{M}_0^{\alpha_i} \longrightarrow \widetilde{\mathcal{P}}^{\bullet}_{\mathfrak{g};\mathfrak{n}_{\rm NS},\mathfrak{n}_{\rm R}}. \tag{3.D.36}$$
These sections give subspaces $\{\mathcal{V}_{\alpha_i}\}$ in $\widetilde{\mathcal{P}}^{\bullet}_{\mathfrak{g};\mathfrak{n}_{\rm NS},\mathfrak{n}_{\rm R}}$;

2. In general, the sections s_i and s_j do not match on the boundary $\mathcal{M}_1^{\alpha_i,\alpha_j}$ of $\mathcal{M}_0^{\alpha_i}$ and $\mathcal{M}_0^{\alpha_i}$. We fill this gap by finding a path P_{α_i,α_j} fibered over $\mathcal{M}_1^{\alpha_i,\alpha_j}$. This gives a subspace $\mathcal{V}_{\alpha_i,\alpha_j}$ in $\widetilde{\mathcal{P}}^{\bullet}_{\mathfrak{g};\mathfrak{n}_{\rm NS},\mathfrak{n}_{\rm R}}$. Doing this for all the pairs (i,j), we get the collection of subspaces $\left\{\mathcal{V}_{\alpha_i,\alpha_j}\right\}$;

3. The $\left\{P_{\alpha_i,\alpha_j}\right\}$ are also defined over the codimension-two faces $\left\{\mathcal{M}_2^{\alpha_i,\alpha_j,\alpha_k}\right\}$ which form the boundaries of codimension-one faces $\left\{\mathcal{M}_1^{\alpha_i,\alpha_j}\right\}$. In general, the paths $P_{\alpha_i,\alpha_j}, P_{\alpha_j,\alpha_k}$ and P_{α_k,α_i} enclose a non-zero subspace in $\Xi(\mathbf{m})$. Therefore the subspaces $\mathcal{V}_{\alpha_i,\alpha_j}, \mathcal{V}_{\alpha_j,\alpha_k}$ and $\mathcal{V}_{\alpha_k,\alpha_i}$ do not meet on $\mathcal{M}_2^{\alpha_i,\alpha_j,\alpha_k}$. We fill this gap by finding a path $P_{\alpha_i,\alpha_j,\alpha_k}$

[14] With the picture number we associate to the states in NS sector and R sector, the bound is usually satisfied.

fibered over $\mathcal{M}_2^{\alpha_i,\alpha_j,\alpha_k}$. This gives a subspace $\mathcal{V}_{\alpha_i,\alpha_j,\alpha_k}$ in $\widetilde{\mathcal{P}}^{\bullet}_{\mathrm{g};\mathfrak{n}_{\mathrm{NS}},\mathfrak{n}_{\mathrm{R}}}$. Doing this for all the triples (i,j,k), we get the collection of subspaces $\left\{\mathcal{V}_{\alpha_i,\alpha_j,\alpha_k}\right\}$

4. Similarly, we fill the gap for higher codimension-k faces up to $k=K$. In the k^{th} step, we fiber the path $P_{\alpha_{i_0},\cdots,\alpha_{i_k}}$ over the codimension-k face $\mathcal{M}_k^{\alpha_{i_0},\cdots,\alpha_{i_k}}$ which determine the subspace $\mathcal{V}_{\alpha_{i_0},\cdots,\alpha_{i_k}}$ of $\widetilde{\mathcal{P}}^{\bullet}_{\mathrm{g};\mathfrak{n}_{\mathrm{NS}},\mathfrak{n}_{\mathrm{R}}}$. Doing this for all $(k+1)$-tuple $(i_0,\cdots,i_k)$ determines the subspaces $\left\{\mathcal{V}_{\alpha_{i_0},\cdots,\alpha_{i_k}}\right\}$.

Then we can define the integration cycle in $\widetilde{\mathcal{P}}^{\bullet}_{\mathrm{g};\mathfrak{n}_{\mathrm{NS}},\mathfrak{n}_{\mathrm{R}}}$ as a formal sum of subspaces $\{\mathcal{V}_{\alpha_0,\cdots,\alpha_k}\}$

$$\begin{array}{c}\text{continuous integration}\\ \text{cycle in } \widetilde{\mathcal{P}}^{\bullet}_{\mathrm{g};\mathfrak{n}_{\mathrm{NS}},\mathfrak{n}_{\mathrm{R}}}\end{array} \equiv \sum_{k=1}^{K} \bigcup_{\{\alpha_0,\cdots,\alpha_k\}} \mathcal{V}_{\alpha_0,\cdots,\alpha_k}. \tag{3.D.37}$$

Subspaces $\mathcal{V}_{\alpha_0,\cdots,\alpha_k}$ for $k\geq 1$ are called *vertical segments*. Each vertical segment comes with its own correction factor. Our next task is to obtain the correction factors from a vertical segment $\mathcal{V}_{\alpha_0,\cdots,\alpha_k}$ for $k\geq 1$. The correction factor for a codimension-k face $\mathcal{M}_k^{\alpha_0,\cdots,\alpha_0}$ is the integration of the form $\mathbf{\Omega}$ over the vertical segment $\mathcal{V}_{\alpha_0,\cdots,\alpha_k}$ defined over $\mathcal{M}_k^{\alpha_0,\cdots,\alpha_0}$ (i.e. the information about the arrangement of the PCO's) and then integrating the result over $\mathcal{M}_k^{\alpha_0,\cdots,\alpha_0}$ (i.e. the information about the moduli space). However, the integration over the vertical segment $\mathcal{V}_{\alpha_0,\cdots,\alpha_k}$ is nothing but the integration over k-dimensional subspaces $P_{\alpha_0,\cdots,\alpha_k}$ of $\Xi(\mathbf{m})$. The reason is that $\mathcal{V}_{\alpha_0,\cdots,\alpha_k}$ is constructed by fibering $P_{\alpha_0,\cdots,\alpha_k}$ over $\mathcal{M}_k^{\alpha_0,\cdots,\alpha_0}$. Therefore, the contribution from the codimension $k\neq 0$ face $\mathcal{M}_k^{\alpha_0,\cdots,\alpha_k}$ is given by

$$\mathcal{A}_k^{\alpha_0,\cdots,\alpha_k} = \int_{\mathcal{M}_k^{\alpha_0,\cdots,\alpha_0}} \int_{\mathcal{V}_{\alpha_0,\cdots,\alpha_k}} \mathbf{\Omega} = \int_{\mathcal{M}_k^{\alpha_0,\cdots,\alpha_0}} \int_{P_{\alpha_0,\cdots,\alpha_k}} \mathbf{\Omega}. \tag{3.D.38}$$

This is the generalization of the result with one PCO which is given by (3.D.14). The integration over $P_{\alpha_0,\cdots,\alpha_k}$ is the generalization of integration over the path $P_{ij}(\mathbf{m},v)$ i.e. the integration over the variable v. To proceed, we use a property of $P_{\alpha_0,\cdots,\alpha_k}$. Along $P_{\alpha_0,\cdots,\alpha_k}$, the location of only k PCOs $z_{i_1},\cdots,z_{i_k}$ change while the locations of remaining $K-k$ PCOs are fixed. If the initial and final values of the $z_{i_j},\ j=1,\cdots,k$ along $P_{\alpha_0,\cdots,\alpha_k}$ are $z_{i_j}^{(\mathbf{in})}$ and $z_{i_j}^{(\mathbf{fi})}$ respectively, we can define the following $d-k$ form

$$\mathbf{\Omega}_{d-k}^{\alpha_0,\cdots,\alpha_k} \equiv \int_{P_{\alpha_0,\cdots,\alpha_k}} \mathbf{\Omega}. \tag{3.D.39}$$

By a generalization of (3.D.14), we have

$$\boldsymbol{\Omega}_{d-k}^{\alpha_0,\cdots,\alpha_k}=\pm\left\langle\prod_{j=1}^{k}\left[\xi\left(z_{i_j}^{(\mathbf{fi})}\right)-\xi\left(z_{i_j}^{(\mathbf{in})}\right)\right]\prod_{\substack{p=1\\p\neq i_1,\cdots,i_k}}^{K}\left(\mathcal{X}\left(z_p\right)-\partial_{z_p}\xi\left(z_p\right)dz_p\right)\wedge\mathcal{B}\right\rangle. \quad (3.D.40)$$

The $\pm$ is fixed by the choice of the orientation of $Q_{\alpha_0,\cdots,\alpha_k}$ defined (3.D.29) which in turn induces an orientation for $P_{\alpha_0,\cdots,\alpha_k}$ given in (3.D.34). Therefore, we get the following contribution from the codimension $k\neq 0$ face $\mathcal{M}_{\alpha_0,\cdots,\alpha_k}$ to scattering amplitudes

$$\mathcal{A}_k^{\alpha_0,\cdots,\alpha_k}=\int_{\mathcal{M}_k^{\alpha_0,\cdots,\alpha_k}}\int_{\mathcal{V}_{\alpha_0,\cdots,\alpha_k}}\boldsymbol{\Omega}=\int_{\mathcal{M}_k^{\alpha_0,\cdots,\alpha_k}}\boldsymbol{\Omega}_{d-k}^{\alpha_0,\cdots,\alpha_k}. \quad (3.D.41)$$

All sections satisfy the condition (3.D.25) which means that $\left(z_{i_j}^{(\mathbf{in})},z_{i_j}^{(\mathbf{fi})}\right)$, $j=1,\cdots,k$ and z_p, $p=k+1,\cdots,K$ take value in a section of $\widetilde{\mathcal{P}}^{\bullet}_{\mathrm{g};\mathrm{n}_{\mathrm{NS}},\mathrm{n}_{\mathrm{R}}}$. They thus avoid spurious singularities by construction.

The Full Amplitude

The full amplitude, associated to a dual triangulation Υ of the moduli space, consists of the contribution from codimension zero faces $\mathcal{M}_0^{\alpha_0}$, given in (3.D.27), and the contribution from codimension $k\neq 0$ faces $\mathcal{M}_k^{\alpha_0,\cdots,\alpha_k}$, given in (3.D.41) [25]

$$\mathcal{A}_{\mathrm{g};\mathrm{n}_{\mathrm{NS}},\mathrm{n}_{\mathrm{R}}}=\sum_{k=0}^{K}(-1)^{\frac{k(k+1)}{2}}\sum_{\{\alpha_0,\cdots,\alpha_k\}}\int_{\mathcal{M}_0^{\alpha_0,\cdots,\alpha_k}}\boldsymbol{\Omega}_{d-k}^{\alpha_0,\cdots,\alpha_k}. \quad (3.D.42)$$

The above expression can be shown to be [25]

- **independent of the choice of vertical segments $\{\mathcal{V}_{\alpha_0,\cdots,\alpha_k}\}$.**
  The factor $(-1)^{\frac{k(k+1)}{2}}$ in (3.D.42) comes from the details of the proof of the independence from the choice of vertical segments.

- **gauge invariant**
  The gauge invariance means that if all the external states are BRST invariant, and at least one of them is BRST-trivial, the amplitude vanishes.

- **equivalent to the results from supergeometry formulation**
  The scattering amplitudes in superstring perturbation theory can naturally be understood as integral of an appropriate form over a cycle $\Gamma \in \widehat{\mathcal{M}}_l \times \widehat{\mathcal{M}}_r$, in which $\widehat{\mathcal{M}}_l$ and $\widehat{\mathcal{M}}_r$ parametrize the holomorphic and anti-holomorphic complex structure of the string worldsheet considered as a super-Riemann surface [79, 80]. It can be shown that the amplitude obtained this way is equivalent to the result given by (3.D.42).

3.D.4 Vertical Integration and Superstring Amplitudes

The vertical integration is used to avoid spurious singularities in the superstring amplitudes. On the other hand, we have seen in section 3.3 that the integration over moduli space of hyperbolic surface involves various term coming from integration over the moduli space and all moduli spaces associated to surfaces generated by cutting the original surface along some multicurve. Therefore, we have to implement the vertical integration for each term in such a decomposition. This can be achieved by finding a fine dual triangulation of the moduli space. According to the vertical integration prescription, we need to compute the contribution of all codimension $k \leq K$ faces. For each k, we can extend the definition of its integrand by zero to the whole moduli space. In this way, we have a form over the whole moduli space. We can then proceed as before for the computation of the resulting integral over the moduli space.

This concludes our discussion of the vertical-integration prescription.

Chapter 4

Future Directions

In this thesis, we gave a prescription to compute any on-shell or off-shell scattering amplitudes in the bosonic-string and superstring theories. It seems difficult to compute these integrals analytically. However, if the string integrand can be constructed in terms of Fenchel-Nielsen coordinates, it should be possible to explore the resulting expressions numerically. This would provide a framework for explicit computations in bosonic-string and superstring theories.

Here, there are some problems which are not yet satisfactorily solved. We conclude by mentioning some of these problems

- The most interesting direction is to compute $\mathbf{\Omega}$ in terms of Fenchel-Nielsen coordinates. Some of directions for solving these problems are as follows

  1. These correlation functions are known in terms of well-defined objects like theta functions, prime forms, etc [37]. One can express these objects in terms of Poincaré series [139]. These Poincaré series are dependent on generators of the uniformizing Fuchsian group. On the other hand, the generators of any finitely-generated Fuchsian group can be expressed in terms of Fenchel-Nielsen coordinates [140]. In this way, one obtain an expression in terms of Fenchel-Nielsen coordinates that can be integrated to get the desired amplitudes. The appearance of infinite series makes the resulting expression very convoluted and it is cumbersome to handle the integration.
  2. The gluing of pairs of pants or conformal blocks [141] is another method to get correlation functions in terms of Fenchel-Nielsen coordinates.

The basic question is how to do conformal field theory on a hyperbolic surface. In other words, what are the intrinsic objects similar to theta functions and prime forms for a hyperbolic surface that can be expressed naturally in terms of Fenchel-Nielsen coordinates?

Let us give an example. The chiral partition function in superstring theory is given by a combination of the determinant of the operator $\bar{\partial}_j$, the Cauchy-Riemann operator acting on the space of $(j,0)$-differentials on a genus-$\mathfrak{g}$ surface. For the gauge-fixed RNS action, the relevant values of j are $j=0$ for the X^μ fields, $j=+\frac{1}{2}$ for the ψ^μ fields, $j=-\frac{1}{2}$ for the $\beta\gamma$ system, and $j=-1$ for the bc system, i.e.

$$\mathcal{Z}_{\mathfrak{g}}(\mathbf{m})=\sum_{\mathfrak{s}=0}^{2^{2\mathfrak{g}}}\frac{\det\bar{\partial}_{-1}}{\left(\det\bar{\partial}_0\right)^{\frac{d}{2}}}\left[\frac{\left(\det\bar{\partial}_{\frac{1}{2}}\right)^{\frac{d}{2}}}{\det\bar{\partial}_{-\frac{1}{2}}}\right]_{\mathfrak{s}}, \tag{4.0.1}$$

where we sum over all spin structures for the contribution of fields having half-integer spin and $\mathbf{m}$ denotes the moduli parameters [37]. The precise question is that how these determinants can be computed in terms of the Fenchel-Nielsen coordinates.

- As we have explained in appendix 3.D, a class of spurious singularities is characterized by zeros of theta functions which are specific loci in the moduli space. The vertical integration is a systematic way to avoid any type of spurious singularities. However, the actual implementation of this procedure requires the study of zeros of theta function. Therefore, another question of importance for computation of superstring amplitudes is as follows: How can we specify the zeros of theta functions in terms of Fenchel-Nielsen coordinates? This question can be addressed more easily if we can understand how to do conformal field theory on a hyperbolic surface and construct the correlation functions in terms of Fenchel-Nielsen coordinates.

- It is known that amplitudes in the two-dimensional topological gravity obey certain recursion relations [142]. It would be interesting to see whether such recursion relations exist for amplitudes in the full superstring theories or some limit thereof.

References

[1] S. F. Moosavian and R. Pius, *Hyperbolic Geometry of Superstring Perturbation Theory*, *arXiv e-prints* (2017) [1703.10563].

[2] S. F. Moosavian and R. Pius, *Hyperbolic Geometry and Closed Bosonic-String Field Theory I: The String Vertices via Hyperbolic Riemann Surfaces*, *arXiv e-prints* (2017) [1706.07366].

[3] S. F. Moosavian and R. Pius, *Hyperbolic Geometry and Closed Bosonic-String Field Theory II: The Rules for Evaluating the Quantum BV Master Action*, *arXiv e-prints* (2017) [1708.04977].

[4] A. M. Polyakov, *Quantum Geometry of Bosonic Strings*, *Phys Lett.* **B103** (1981) 207.

[5] A. M. Polyakov, *Quantum Geometry of Fermionic Strings*, *Phys Lett.* **B103** (1981) 211.

[6] M. B. Green and J. H. Schwarz, *Anomaly-Cancellations in Supersymmetric $D = 10$ Gauge Theory and Superstring Theory*, *Phys. Lett.* **B149** (1984) 117.

[7] M. B. Green and J. H. Schwarz, *Supersymmetrical String Theories*, *Phys. Lett.* **B109** (1982) 444.

[8] J. Polchinski, *Dirichlet Branes and Ramond-Ramond Charges*, *Phys. Rev. Lett.* **75** (1995) 4724 [hep-th/9510017].

[9] D. J. Gross, J. A. Harvey, E. J. Martinec and R. Rohm, *Heterotic String*, *Phys. Rev. Lett.* **54** (1985) 502.

[10] D. J. Gross, J. A. Harvey, E. J. Martinec and R. Rohm, *Heterotic String Theory I. The Free Heterotic String*, *Nucl. Phys.* **B256** (1985) 253.

[11] D. J. Gross, J. A. Harvey, E. J. Martinec and R. Rohm, *Heterotic String Theory II. The Interacting Heterotic String*, Nucl. Phys **B267** (1986) 75.

[12] L. J. Dixon and J. A. Harvey, *String Theories in Ten Dimensions without Spacetime Supersymmetry*, Nucl. Phys. **B274** (1986) 93.

[13] J. Polchinski, *Open Heterotic Strings*, J. High Energ. Phys. **2006** (2006) 082 [hep-th/0510033].

[14] D. Friedan, *Notes on String Theory and Two-Dimensional Conformal Field Theory*, *Unified String Theories* (1986) 162.

[15] D. Friedan, E. J. Martinec and S. Shenker, *Conformal Invariance, Supersymmetry and String Theory*, Nucl. Phys. **B271** (1986) 93.

[16] E. Witten, *Superstring Perturbation Theory Revisited*, *ArXiv e-prints* (2012) [1209.5461].

[17] R. Pius, A. Rudra and A. Sen, *Mass Renormalization in String Theory: Special States*, J. High Energ. Phys. **2014** (2014) [arXiv:1311.1257].

[18] R. Pius, A. Rudra and A. Sen, *Mass Renormalization in String Theory: General States*, J. High Energ. Phys. **2014** (2014) [arXiv:1401.7014].

[19] R. Pius, A. Rudra and A. Sen, *String Perturbation Theory around Dynamically-Shifted Vacuum*, J. High Energ. Phys. **2014** (2014) 70 [1404.6254].

[20] A. Sen, *Off-Shell Amplitudes in Superstring Theory*, Fortsch. Phys. **63** (2015) 149 [1408.0571].

[21] A. Sen, *Gauge Invariant 1PI Effective Action for Superstring Field Theory*, J. High Energ. Phys. **2015** (2015) 22 [1411.7478].

[22] A. Sen, *Gauge Invariant 1PI Effective Superstring Field Theory: Inclusion of the Ramond Sector*, J. High Energ. Phys. **2015** (2015) 25 [1501.00988].

[23] A. Sen, *Supersymmetry Restoration in Superstring Perturbation Theory*, J. High Energ. Phys. **12** (2015) 75 [1508.02481].

[24] A. Sen, *BV Master Action for Heterotic and Type-II String Field Theories*, J. High Energ. Phys. **2016** (2016) 87 [1508.05387].

[25] A. Sen and E. Witten, *Filling the Gaps with PCOs*, J. High Energ. Phys. **2015** (2015) [1504.00609].

[26] E. D'Hoker and D. H. Phong, *Two-Loop Superstrings I. Main Formulas*, Phys. Lett. **B529** (2002) 241 [hep-th/0110247].

[27] E. D'Hoker and D. H. Phong, *Two-Loop Superstrings II. The Chiral Measure on Moduli Space*, Nucl. Phys. **B636** (2002) 3 [hep-th/0110283].

[28] E. D'Hoker and D. H. Phong, *Two-Loop Superstrings III. Slice-Independence and Absence of Ambiguities*, Nucl. Phys. **B636** (2002) 61 [hep-th/0111016].

[29] E. D'Hoker and D. H. Phong, *Two-Loop Superstrings IV. The Cosmological Constant and Modular Forms*, Nucl. Phys. **B639** (2002) 129 [hep-th/0111040].

[30] E. D'Hoker and D. H. Phong, *Two-Loop Superstrings V. Gauge-Slice Independence of the N-Point Function*, Nucl. Phys. **B715** (2005) 91 [hep-th/0501196].

[31] E. D'Hoker and D. H. Phong, *Two-Loop Superstrings VI. Nonrenormalization Theorems and the 4-Point Function*, Nucl. Phys. **B715** (2005) 3 [hep-th/0501197].

[32] E. D'Hoker and D. H. Phong, *Two-Loop Superstrings VII. Cohomology of Chiral Amplitudes*, Nucl. Phys. **B804** (2008) 421 [0711.4314].

[33] J. M. Maldacena, *The Large N Limit of Superconformal Field Theories and Supergravity*, *Adv. Theor. Math. Phys* **2** (1998) 231 [hep-th/9711200].

[34] J. M. Maldacena and H. Ooguri, *Strings in AdS_3 and the $SL(2,\mathbb{R})$ WZW Model. I: The Spectrum*, J. Math. Phys. **42** (2001) 2929 [hep-th/0001053].

[35] J. M. Maldacena, H. Ooguri and J. Son, *Strings in AdS_3 and the $SL(2,\mathbb{R})$ WZW Model. II: Euclidean Black Hole*, J. Math. Phys. **42** (2001) 2961 [hep-th/0005183].

[36] J. M. Maldacena and H. Ooguri, *Strings in AdS_3 and the $SL(2,\mathbb{R})$ WZW Model. III Correlation Functions*, Phys. Rev. **D65** (2002) [hep-th/0111180].

[37] E. Verlinde and H. Verlinde, *Chiral Bosonization, Determinants and the String Partition Function*, Nucl. Phys. **B288** (1987) 357.

[38] S. Weinberg, *Radiative Corrections in String Theories*, in *Proc. Division of Particles and Fields, Eugene, Oregon, 1985*, World Scientific, 1986.

[39] N. Seiberg, *Anomalous Dimensions and Mass Renormalization in String Theory*, *Phys. Lett.* **B187** (1987) 56.

[40] A. Sen, *Mass Renormalization and Cancellation of BRST Anomaly in String Theories*, *Nucl. Phys.* **B304** (1988) 403.

[41] A. Berera, *All-Order Mass Renormalization of String Theory*, *Phys. Rev.* **D49** (1994) 6674.

[42] T. Erler, *Four Lectures on Closed-String Field Theory*, *arXiv e-prints* (2019) [1905.06785].

[43] C. de Lacroix, H. Erbin, S. P. Kashyap, A. Sen and M. Verma, *Closed Superstring Field Theory and its Applications*, *Int. J. Mod. Phys.* **A32** (2017) 1730021 [1703.06410].

[44] A. Sen, *Unitarity of Superstring Field Theory*, *J. High Energ. Phys.* **12** (2016) 115 [1607.08244].

[45] C. de Lacroix, H. Erbin and A. Sen, *Analyticity and Crossing Symmetry of Superstring Loop Amplitudes*, *arXiv e-prints* (2018) [1810.07197].

[46] A. Sen and B. Zwiebach, *A Proof of Local Background-Independence of Classical Closed-String Field Theory*, *Nucl. Phys.* **B414** (1994) 649 [hep-th/9307088].

[47] A. Sen and B. Zwiebach, *Quantum Background-Independence of Closed-String Field Theory*, *Nucl. Phys.* **B423** (1994) 580 [hep-th/9311009].

[48] A. Sen, *Background-Independence of Closed-Superstring Field Theory*, *J. High Energ. Phys.* **2** (2018) 155 [1711.08468].

[49] R. Pius and A. Sen, *Cutkosky Rules for Superstring Field Theory*, *Journal of High Energy Physics* **10** (2016) 24 [1604.01783].

[50] A. Sen, *Universality of the Tachyon Potential*, *J. High Energ. Phys.* **12** (1999) 027 [hep-th/9911116].

[51] A. Sen, *Tachyon Condensation on the Brane-Antibrane System*, *J. High Energ. Phys.* **8** (1998) 012 [hep-th/9805170].

[52] M. Schnabl, *Analytic Solution for Tachyon Condensation in Open-String Field Theory*, *Adv. Theor. Math. Phys.* **10** (2006) 433 [arXiv:hep-th/0511286].

[53] T. Erler, *Analytic Solution for Tachyon Condensation in Berkovits' Open-Superstring Field Theory*, Journal of High Energy Physics **11** (2013) 7 [1308.4400].

[54] H. Yang and B. Zwiebach, *A closed-string tachyon vacuum?*, J. High Energ. Phys. **2005** (2005) 054 [arXiv:hep-th/0506077].

[55] A. Sen, *Wilsonian Effective Action of Superstring Theory*, J. High Energ. Phys. **2017** (2017) [arXiv:1609.00459].

[56] M. Cho, S. Collier and X. Yin, *Strings in Ramond-Ramond Backgrounds from the Neveu-Schwarz-Ramond Formalism*, *arXiv e-prints* (2018) [1811.00032].

[57] I. A. Batalin and G. A. Vilkovisky, *Gauge Algebra and Quantization*, Phys. Lett. **B102** (1981) 27.

[58] I. A. Batalin and G. A. Vilkovisky, *Quantization of Gauge Theories with Linearly-Dependent Generators*, Phys. Rev. **D28** (1983) 2567.

[59] B. Zwiebach, *Closed-String Field Theory: Quantum Action and the BV Master Equation*, Nucl. Phys. **B390** (1993) 33 [hep-th/9206084].

[60] S. F. Moosavian, *Construction of Effective Actions of Closed-Superstring Field Theories Using Hyperbolic Geometry*, *in Preparation* (2019) .

[61] Y. Imayoshi and M. Taniguchi, *An Introduction to Teichmüller Spaces*. Springer Japan, 1992.

[62] P. Deligne and D. Mumford, *The Irreducibility of the Space of Curves of Given Genus*, Publications Mathématiques de lÍHÉS **36** (1969) 75.

[63] S. A. Wolpert, *On the Kähler Form of the Moduli Space of Once-Punctured Tori*, Comment. Math. Helvetici **58** (1983) 246.

[64] R. C. Penner, *Weil-Petersson Volumes*, J. Differential Geom. **35** (1992) 559.

[65] P. Zograf, *The Weil-Petersson Volume of the Moduli Space of Punctured Spheres*, in *Mapping Class Groups and Moduli Spaces of Riemann Surfaces*, pp. 367–372. American Mathematical Society, 1993.

[66] Y. I. Manin and P. Zograf, *Invertible Cohomological Field Theories and Weil-Petersson Volumes*, Annales de l'institut Fourier **50** (2000) 519 [math/9902051].

[67] M. Mirzakhani, *Simple Geodesics and Weil-Petersson Volumes of Moduli Spaces of Bordered Riemann Surfaces*, *Inventiones Mathematicae* **167** (2006) 179.

[68] W. Fenchel and J. Nielsen, *Discontinuous Groups of Isometries in the Hyperbolic Plane*, de Gruyter Studies in Mathematics. De Gruyter, 2002.

[69] K. Obitsu and S. A. Wolpert, *Grafting Hyperbolic Metrics and Eisenstein Series*, *Mathematische Annalen* **341** (2008) 685 [arXiv:0704.3169].

[70] M. S. Berger, *Riemannian Structures of Prescribed Gaussian Curvature for Compact 2-Manifolds*, *J. Differential Geom.* **5** (1971) 325.

[71] J. Kazdan, *Prescribing the Curvature of a Riemannian Manifold*. American Mathematical Society, 1985.

[72] J. L. Kazdan and F. W. Warner, *Curvature Functions for Compact 2-Manifolds*, *The Annals of Mathematics* **99** (1974) 14.

[73] S. A. Wolpert, *The Hyperbolic Metric and the Geometry of the Universal Curve*, *J. Differential Geom.* **31** (1990) 417.

[74] P. Nelson, *Covariant Insertion of General Vertex Operators*, *Phys. Rev. Lett.* **62** (1989) 993.

[75] M. Schiffer and D. C. Spencer, *Functionals of Finite Riemann Surfaces*. Princeton University Press, 2015.

[76] J. Polchinski, *String Theory, Vol. 1 - An Introduction to the Bosonic String*. Cambridge University Press, 1998.

[77] E. Verlinde and H. Verlinde, *Multiloop Calculations in Covariant Superstring Theory*, *Phys. Lett.* **B192** (B1987) 95.

[78] E. D'Hoker and D. H. Phong, *The Geometry of String Perturbation Theory*, *Rev. Mod. Phys.* **60** (1988) 917.

[79] E. Witten, *Notes On Supermanifolds and Integration*, *ArXiv e-prints* (2012) [1209.2199].

[80] E. Witten, *Notes On Super Riemann Surfaces and Their Moduli*, *ArXiv e-prints* (2012) [1209.2459].

[81] L. Alvarez-Gaumé, C. Gomez, G. W. Moore and C. Vafa, *Strings in the Operator Formalism*, *Nucl. Phys.* **B303** (1988) 455.

[82] L. Alvarez-Gaumé, P. Nelson, C. Gomez, G. Sierra and C. Vafa, *Fermionic Strings in the Operator Formalism*, *Nucl. Phys.* **B311** (1988) 333.

[83] C. G. Callan, D. Friedan, E. J. Martinec and M. J. Perry, *Strings in Background Fields*, *Nucl. Phys.* **B262** (1985) 593.

[84] S. B. Giddings and E. Martinec, *Conformal Geometry and String Field Theory*, *Nucl. Phys.* **B278** (1986) 91.

[85] F. Gliozzi, J. Scherk and D. Olive, *Supergravity and the Spinor Dual Model*, *Phys. Lett.* **B65** (1976) 282.

[86] N. Seiberg and E. Witten, *Spin Structures in String Theory*, *Nucl. Phys.* **B276** (1986) 272.

[87] J. A. Shapiro, *Loop Graph in the Dual-Tub Model*, *Phys. Rev.* **D5** (1972) 1945.

[88] S. Mandelstam, *The n-Loop String Amplitude. Explicit Formulas, Finiteness and Absence of Ambiguities*, *Phys. Lett.* **B277** (1992) 82.

[89] D. J. Gross and V. Periwal, *String Perturbation Theory Diverges*, *Phys. Rev. Lett.* **60** (1988) 2105.

[90] A. Belopolsky, *De Rham Cohomology of the Supermanifolds and Superstring BRST Cohomology*, *Phys. Lett.* **B403** (1997) 47 [hep-th/9609220].

[91] A. Belopolsky, *New Geometrical Approach to Superstrings*, *ArXiv e-prints* (1997) [hep-th/9703183].

[92] J. J. Atick, L. J. Dixon and A. Sen, *String Calculation of Fayet-Iliopoulos D-Terms in Arbitrary Supersymmetric Compactifications*, *Nucl. Phys.* **B292** (1987) 109.

[93] M. Dine, I. Ichinose and N. Seiberg, *F-Terms and D-Terms in String Theory*, *Nucl. Phys.* **B293** (1987) 253.

[94] M. Dine, N. Seiberg and E. Witten, *Fayet-Iliopoulos Terms in String Theory*, *Nucl. Phys.* **B289** (1987) 589.

[95] M. B. Green and N. Seiberg, *Contact Interactions in Superstring Theory*, *Nucl. Phys.* **B299** (1988) 559.

[96] S. A. Wolpert, *Asymptotics of the Spectrum and the Selberg Zeta-Function on the Space of Riemann Surfaces*, Comm. Math. Phys. **112** (1987) 283.

[97] S. A. Wolpert, *Families of Riemann Surfaces and Weil-Petersson Geometry*. American Mathematical Society, 2010.

[98] L. Keen, *Collars on Riemann Surface*, in *Discontinuous Groups and Riemann Surfaces (AM-79)* (L. Greenberg, ed.), pp. 263–268. Princeton University Press, dec, 1974.

[99] J. P. Matelski, *A Compactness Theorem for Fuchsian Groups of the Second Kind*, Duke Math. J. **43** (1976) 829.

[100] B. Randol, *Cylinders in Riemann Surfaces*, Commentarii Mathematici Helvetici **54** (1979) 1.

[101] S. A. Wolpert, *On the Weil-Petersson Geometry of the Moduli Space of Curves*, American Journal of Mathematics **107** (1985) 969.

[102] A. B. Venkov, *Spectral Theory of Automorphic Functions and its Applications*. Springer Netherlands, 1990.

[103] A. Hatcher and W. Thurston, *A Presentation for the Mapping-Class Group of a Closed Orientable Surface*, Topology **19** (1980) 221.

[104] A. Hatcher, P. Lochak and L. Schneps, *On the Teichmüller Tower of Mapping-Class Groups*, J. reine angew. Math. (Crelles Journal) **2000** (2000) 1.

[105] H. Masur, *The Extension of the Weil-Petersson Metric to the Boundary of Teichmüller Space*, Duke Math. J. **43** (1976) 623.

[106] F. Klein and R. Fricke, *Lectures on the Theory of Automorphic Functions*. Higher Education, 2017.

[107] L. Keen, *On Fundamental Domains and the Teichmüller Modular Group*, in *Contributions to Analysis*, pp. 185–194. Elsevier, 1974.

[108] T. Erler and S. Konopka, *Vertical Integration from the Large Hilbert Space*, J. High Energ. Phys. **12** (2017) 112 [1710.07232].

[109] F. Luo and S. P. Tan, *A Dilogarithm Identity on Moduli Spaces of Curves*, J. Differential Geom. **97** (2014) 255 [1102.2133].

[110] G. McShane, *A Remarkable Identity for Lengths of Curves*, Ph.D. thesis, University of Warwick, 1991.

[111] G. McShane, *Simple Geodesics and a Series Constant over Teichmüller Space*, Invent. Math. **132** (1998) 607.

[112] J. D. Fay, *Theta Functions on Riemann Surfaces*, Springer **352** (2006) .

[113] L. R. Ford, *Automorphic Functions*. AMS Chelsea Publishing, 1951.

[114] C. Lovelace, *M-Loop Generalized Veneziano Formula*, Phys. Lett. **B32** (1970) 703.

[115] M. Kaku and L. Yu, *The General Multi-Loop Veneziano Amplitude*, Phys. Lett. **B33** (1970) 166.

[116] V. Alessandrini, *A General Approach to Dual Multi-Loop Diagrams*, Nuovo Cimento A Serie **2** (1971) 321.

[117] V. Alessandrini and D. Amati, *Properties of Dual Multi-Loop Amplitudes*, Nuovo Cimento A Serie **4** (1971) 793.

[118] S. Playle, *Deforming Super-Riemann Surfaces with Gravitinos and Super-Schottky Groups*, J. High Energ. Phys. **12** (2016) 35 [1510.06749].

[119] B. Zwiebach, *Quantum Closed-Strings from Minimal Area*, Mod. Phys. Lett. **A05** (1990) 2753.

[120] B. Zwiebach, *Quantum Open-String Theory with Manifest Closed-String Factorization*, Phys. Lett. **B256** (1991) 22.

[121] B. Zwiebach, *Minimal-Area Problems and Quantum Open Strings*, Comm. Math. Phys. **141** (1991) 577.

[122] M. Headrick and B. Zwiebach, *Convex Programs for Minimal-Area Problems*, *arXiv e-prints* (2018) [1806.00449].

[123] M. Headrick and B. Zwiebach, *Minimal-Area Metrics on the Swiss Cross and Punctured Torus*, *arXiv e-prints* (2018) [1806.00450].

[124] B. Zwiebach, *Consistency of Closed-String Polyhedra from Minimal Area*, Phys. Lett. **B241** (1990) 343.

[125] B. Zwiebach, *How Covariant Closed-String Theory Solves a Minimal-Area Problem*, *Comm. Math. Phys.* **136** (1991) 83.

[126] H. Erbin, J. Maldacena and D. Skliros, *Two-Point String Amplitudes*, *arXiv e-prints* (2019) [1906.06051].

[127] M. E. Peskin, *Introduction to String and Superstring Theory II*, *From the Planck Scale to the Weak Scale: Toward a Theory of the Universe* **2** (1987) .

[128] E. Kiritsis, *String Theory in a Nutshell*. Princeton University Press, 2007.

[129] E. D'Hoker and D. H. Phong, *Length-Twist Parameters in String Path Integrals*, *Phys. Rev. Lett.* **56** (1986) 912.

[130] L. Keen and N. Lakic, *Hyperbolic Geometry from a Local Viewpoint*, London Mathematical Society Student Texts. Cambridge University Press, 2007.

[131] B. Farb and D. Margalit, *A Primer on Mapping Class Groups*, *Princeton University Press* **49** (2011) .

[132] J. S. Birman and C. Series, *Geodesics with Bounded Intersection Number on Surfaces are Sparsely Distributed*, *Topology* **24** (1985) 217.

[133] D. V. Anosov, *Geodesic Flows on Closed Riemannian Manifolds of Negative Curvature*, *Proc. Steklov Inst. Math.* **90** (1967) 3.

[134] M. Bridgeman and S. P. Tan, *Identities on Hyperbolic Manifolds*, in *Handbook of Teichmüller Theory, Volume V*, pp. 19–53. European Mathematical Society Publishing House, 2016.

[135] G. McShane, *Simple Geodesics on Surfaces of Genus 2*, *Annales-Academiae Scientiarum Fennicae Mathematica* **31** (2006) 31.

[136] A. Haas and P. Susskind, *The Geometry of the Hyperelliptic Involution in Genus Two*, *Proc. Amer. Math. Soc.* **105** (1989) 159.

[137] O. Lechtenfeld, *Superconformal Ghost Correlations on Riemann Surfaces*, *Phys. Lett.* **B232** (1989) 193.

[138] A. Morozov, *A Straightforward Proof of Lechtenfeld's Formula for the $\beta\gamma$ Correlator*, *Phys. Lett.* **B234** (1990) 15.

[139] J. D. Fay, *Fourier Coefficients of the Resolvent for a Fuchsian Group*, *J. reine angew. Math. (Crelles Journal)* **1977** (1977) .

[140] B. Maskit, *Matrices for Fenchel-Nielsen Coordinates*, *Annales Academiæ Scientiarum Fennicæ Mathematica* **26** (2001) 267.

[141] J. Teschner, *Nonrational Conformal Field Theory*, in *New Trends in Mathematical Physics*, pp. 697–739. Springer Netherlands, 2009. arXiv:0803.0919.

[142] E. Witten, *Two-Dimensional Gravity and Intersection Theory on Moduli Space*, *Surveys Diff. Geom.* **1** (1990) 243.

[143] A. Belopolsky, *Picture Changing Operators in Supergeometry and Superstring Theory*, *ArXiv e-prints* (1997) [hep-th/9706033].

[144] J. H. Hubbard, *Teichmüller Theory And Applications To Geometry, Topology, And Dynamics*. Matrix Editions, 2006.