On Hopf Ore Extensions and Zariski Cancellation Problems

by

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Examining Committee Membership

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This thesis consists of material all of which I authored or co-authored: see Statement of Contributions included in the thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Statement of Contributions

Chapters 1 and 2 are mainly expository, although some of the exposition is taken from papers I have written or coauthored. I am the sole author of the material in Chapters 3 and 4; Chapters 5 and 6 are partially taken from joint work with Jason Bell, Maryam Hamidizadeh and Helbert Venegas.

Abstract

In this thesis, we investigate Ore extensions of Hopf algebras and the Zariski Cancellation problem for noncommutative rings. In particular, we improve upon the existing conditions for when $T = R[x; \sigma, \delta]$ is a Hopf Ore extension of a Hopf algebra R, and we give noncommutative analogues of a cancellation theorem of Abhyankar, Eakin, and Heinzer. In Chapter 3, we study the relationship between prime ideals of $T = R[x; \sigma, \delta]$ and their contractions under R. In Chapter 4, we look at when T is a Hopf algebra and by studying the coproduct of x, $\Delta(x)$, we provide a sequence of results that answers a question due to Panov; that is, given a Hopf algebra R, for which automorphisms σ and σ-derivations δ does the Ore extension $T = R[x; \sigma, \delta]$ have a Hopf algebra structure extending the given Hopf algebra structure on R? In Chapter 5, we consider the question of cancellation for finitely generated not-necessarily-commutative domains of Gelfand-Kirillov dimension one and show that such algebras are necessarily cancellative when the characteristic of the base field is zero. In particular, this recovers the cancellation result of Abhyankar, Eakin, and Heinzer in characteristic zero when one restricts to the commutative case. We also provide examples that show affine domains of Gelfand-Kirillov dimension one need not be cancellative when the base field has a positive characteristic, giving a counterexample to a conjecture of Tang et al. In Chapter 6, we prove a skew analogue of the result of Abhyankar-Eakin-Heinzer, in which one works with skew polynomial extensions as opposed to ordinary polynomial rings.

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Dedication

This is dedicated to my mother, who is the greatest Mom in the world. You always did your best to support me in pursuing educational opportunities. Thank you for giving me the time and space to do what I love.

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Chapter 1

Introduction

In this thesis, we investigate *Hopf Ore extensions* and *Zariski cancellation problems*. Let us begin by taking a look at the background of these two topics. An *Ore extension* is also called a *skew polynomial ring* and is defined as follows. Let k be a field and R be a k-algebra. Given a k-algebra endomorphism σ of R, we define a k-linear σ-derivation δ of R to be a k-linear map satisfying $\delta(ab) = \sigma(a)\delta(b) + \delta(a)b$ for $a, b \in R$. Then one can form an *Ore extension* $T = R[x, \sigma, \delta]$, which is the k-algebra generated by R and the indeterminate x subject to the relation $xr = \sigma(r)x + \delta(r)$ for all $r \in R$ [20]. There is currently great interest in the study of Ore extensions of algebras, which is largely due to the fact that many quantized algebras and their homomorphic images can be expressed in terms of (iterated) Ore extensions. New methods are developed to describe prime ideals in an Ore extension $T = R[x; \sigma, \delta]$ in [19, 25, 26]. In the case that R is commutative noetherian (and σ is an automorphism), a complete description of the prime ideals of T in terms of their contractions to R is given in [19]. In Chapter 3, we investigate the relationship between prime ideals of T and their contractions under R when R is a noetherian ring that satisfies a polynomial identity.

We are particularly interested in Ore extensions that have the additional property of being a Hopf algebra. A Hopf algebra $(R, m, \Delta, \mu, \epsilon)$ is an associative k-algebra that is also a coassociative k-coalgebra with the extra condition that Δ and ϵ are algebra maps, together with an antipode map S and some additional constraints about how the various maps interact [38]. We give further details in the next chapter.

Notably, Hopf algebras play an important role in many different areas of mathematics, including algebraic topology, group scheme theory, and group theory [2]. In [40], Panov considered the possible Hopf algebra structures on an Ore extension $T = R[x; \sigma, \delta]$ that extend the underlying Hopf structure on a Hopf algebra R. In Chapter 4, we will establish sufficient and necessary conditions on σ and δ to make T extend the Hopf algebra structure of R by improving the result of [11, theorem in §2.4]. Specifically, when R is noetherian and $R \otimes_k R$ is a domain, we show that after a suitable change of variables, we have $\Delta(x) = \beta^{-1} \otimes x + x \otimes 1 + w$, with $w \in R \otimes_k R$ and β a grouplike element of R. Since $T = R[x; \sigma, \delta]$ is a free R-module generated by $\{1, x, x^2, \ldots, \}$, it is of great significance to understand the nature of $\Delta(x)$ when studying the Hopf algebra structure of T.

Now let us shift to the second topic, i.e., the *Zariski cancellation problem* (ZCP). Kraft said in his 1995 survey [28] that "*there is no doubt that complex affine n-space $\mathbb{A}^n = \mathbb{A}^n_{\mathbb{C}}$ is one of the basic objects in algebraic geometry. It is therefore surprising how little is known about its geometry and its symmetries.*" Although there has been some remarkable progress in the last few years, many basic problems remain open. On a related note, the famous *Zariski cancellation problem* asks: is an affine variety X over an algebraically closed field k having the property that $X \times \mathbb{A}^1 \cong \mathbb{A}^{n+1}$ necessarily isomorphic to $\mathbb{A}^n$? The question is known to have an affirmative answer when $n = 1$ [1], and $n = 2$, with the characteristic zero case being done by Fujita [17] and Miyanishi-Sugie [37], and the positive characteristic case handled by Russell [43]. In positive characteristic, Gupta [21, 22] gave counterexamples to the Zariski cancellation problems in dimension at least three. Still, the problem remains open in dimension greater than two in the case that the base field has characteristic zero. Equivalently, the *Zariski cancellation problem* can be stated algebraically: if A is an *affine* (finitely generated) k-algebra such that $A[x] \cong k[x_1, \ldots, x_{n+1}]$, does it follow that A is isomorphic to $k[x_1, \ldots, x_n]$? More generally, we are interested in the question: when does $R[t] \cong S[t]$ imply that R and S are isomorphic as k-algebras for a some specific k-algebra R? If it is always the case that R is isomorphic to S whenever $R[t] \cong S[t]$, then R is called *cancellative*.

In [1], it has been shown that if R is commutative and has Krull dimension one, then R is cancellative. On another hand, many counterexamples were constructed in [14] when R has Krull dimension two. So it is natural to ask whether R is cancellative if R is noncommutative and has Gelfand-Kirillov dimension one. In Chapter 5, we look at noncommutative analogues of the result of Abhyankar, Eakin, and Heinzer. Their theorem, when one works in the category of commutative algebras, says that if A is a finitely generated algebra that is an integral domain of Krull dimension one, then A is strongly cancellative in the above sense. We consider a noncommutative analogue of this theorem, in which one considers finitely generated domains of Gelfand-Kirillov dimension one. When working with noncommutative algebras, it is generally preferable to work with Gelfand-Kirillov dimension rather than with the classical Krull dimension.

In light of the Zariski cancellation problem, it is then natural to ask when an algebra R is *skew cancellative*; that is, if $R[x;\sigma,\delta] \cong S[x;\sigma',\delta']$ when do we necessarily have $R \cong S$? In Chapter 6, we study the skew cancellativity of the two most important special cases of this construction; namely, the skew polynomial extensions of automorphism type, where $\delta = 0$; and skew polynomial extensions of derivation type, where σ is the identity. In the former case, where $\delta = 0$, it is customary to omit δ and write $R[x;\sigma]$; and in the latter case, where σ is the identity, it is customary to omit σ and write $R[x;\delta]$. We show that R is skew cancellative in the two cases just mentioned when the coefficient ring R is an affine commutative domain of Krull dimension one. We end this thesis by listing some relevant open questions in Chapter 7 that naturally arise from our investigations.

Chapter 2

Preliminaries

In this chapter, we summarize the notation and mathematical conventions used in this thesis. In addition, we will give detailed definitions of concepts that are involved in the later chapters. Throughout this thesis, we take k to be a field and all algebras are over k. Given two morphisms f, g, we denote $f \circ g$ the composition of f and g. A map δ is called a *derivation* of an algebra R if $\delta(ab) = a\delta(b) + \delta(a)b$ for all $a, b \in R$. A map σ is called an *endomorphism* of an algebra R if σ is a ring homomorphism sending R to R. In addition, if σ is a bijective endomorphism of R, then σ is called an automorphism of R. We will generally always assume that σ is an automorphism in this thesis.

2.1 Ore Extensions

2.1.1 Definition of an Ore extension

Let R be a k-algebra. An *Ore extension*, also called a *skew polynomial ring*, is a generalization of a polynomial extension of an algebra R in a variable x. In this setting, however, we no longer assume that the variable x commutes with the elements of R. If x commutes with

R, then the construction yields the trivial case, namely, a polynomial ring in the variable x with coefficient ring R. In general, in the algebra we construct, each element will be expressible uniquely in the form $\sum a_i x^i$ for some $a_i \in R$ and such that the degrees behave appropriately; i.e., for polynomials $f(x), g(x)$, we have $\deg(f(x)g(x)) \leq \deg f(x) + \deg g(x)$. In this case, it is required that $xa \in Rx + R$. In particular, $xa = \sigma(a)x + \delta(a)$ will satisfy the above requirement, where σ, δ are endomorphisms of the additive group R^+. Moreover, by looking at $x \cdot (ab)$ with $a, b \in R$, we notice that

$$x(ab) = \sigma(ab)x + \delta(ab)$$

and

$$(xa)b = \sigma(a)\sigma(b)x + \sigma(a)\delta(b) + \delta(a)b.$$

Hence, this implies that σ is an endomorphism of R and that

$$\delta(ab) = \sigma(a)\delta(b) + \delta(a)b.$$

We can state the precise definition of an *Ore extension* as follows.

Definition 2.1.1. Let R be a k-algebra with a k-algebra endmorphism σ and a k-linear σ-derivation δ of R, (i.e. $\delta : R \to R$ is a k-linear map with the property that $\delta(ab) = \sigma(a)\delta(b) + \delta(a)b$ for $a, b \in R$). A skew polynomial extension or Ore extension $T = R[x; \sigma, \delta]$ is the k-algebra generated by R and the indeterminate x, subject to the relations $xr - \sigma(r)x = \delta(r)$ for $r \in R$.

Thus T, as a set, is just the $R[x]$, but where the indeterminate x now *skew commutes* with elements of R. In the Ore extension $R[x; \sigma, \delta]$, if $\delta = 0$ this is written as $R[x; \sigma]$; and, if $\sigma = \mathrm{id}$, as $R[x; \delta]$. These give two special types of Ore extensions, which will be investigated in Chapter 6.

Remark 2.1.1. The algebra $T = R[x; \sigma, \delta]$ defined above has the universal property that if $\phi : R \longrightarrow S$ is a k-algebra homomorphism and y in an algebra S has the property that

$$y\phi(a) = \phi(\sigma(a))y + \phi(\delta(a))$$

for all a in R, then there exists a unique algebra homomorphism $\psi : R[x;\sigma,\delta] \longrightarrow S$ such that $\psi(x) = y$ and the diagram

$$\begin{tikzcd} R \arrow[rr] \arrow[dr, "\phi"] & & R[x;\sigma,\delta] \arrow[dl, "\psi"'] \\ & S & \end{tikzcd}$$

commutes.

Now we provide some examples of Ore extensions.

Example 2.1.2. Let $R[x]$ be the classic polynomial algebra over a ring R. Then $R[x]$ is the special Ore extension of R in which $\sigma \equiv \mathrm{id}_R$ and $\delta \equiv 0$. In particular, the polynomial algebra in $n \geq 2$ variables $R\left[x_1, \cdots, x_n\right]$ is also an example of Ore extension of $R\left[x_1, \cdots, x_{n-1}\right]$ with variable x_n.

Example 2.1.3. The quantum plane $k_q[x, y]$, with $q \in k\backslash\{0\}$, is an Ore extension of $R = k[x]$, in which σ is the algebra automorphism of R determined by $\sigma(x) = qx$ and $\delta \equiv 0$. In the notation of Ore extensions, we write $k[x][y;\sigma] = R[y;\sigma] = k_q[x, y]$, where $yx = \sigma(x)y = qxy$.

Example 2.1.4. A differential operator algebra $k[y][x;\delta]$ is an Ore extension in which $\sigma = \mathrm{id}_{k[y]}$ and δ is simply a derivation. For instance, if $\delta = \frac{d}{dy}$, then we have the relation $xy = yx + \delta(y) = yx + 1$ and $k[y][x;\delta]$ becomes the so called first Weyl algebra over the field k, $A_1(k)$.

Example 2.1.5. A quantum Weyl algebra $A_1^q(k)$ is an Ore extension of the form

$$k[y][x;\sigma,\delta],$$

with $q \in k\backslash\{0\}$, where σ is determined by $\sigma(y) = qy$ and δ is the unique σ-derivation satisfying $\delta(y) = 1$. The variables y and x satisfy the relation $xy = \sigma(y)x + \delta(y) = qyx + 1$.

2.1.2 Properties of Ore extensions

In this subsection, we list some properties of Ore extensions and some concepts relative to Ore extensions which will be involved in later chapters.

Definition 2.1.2. Let R be a ring, let a be in R, and let σ be an endomorphism of R. The rule $\delta_a(r) = ar - \sigma(r)a$ defines a σ-derivation δ_a on R, called the inner derivation induced by a. Any derivation of R that is not an inner derivation is called an outer derivation.

Definition 2.1.3. Let δ be a derivation and σ be an endomorphism on a ring R. A δ-ideal (resp. σ-ideal) of R is an ideal I of R such that $\delta(I) \subseteq I$ (resp. $\sigma(I) \subseteq I$). The ring R is called δ-simple (resp. σ-simple) if R is nonzero and the only δ-ideals (resp. σ-ideals) of R are (0) and R.

Theorem 2.1.6. [35, Theorem 2.9] *Let $T = R[x; \sigma, \delta]$.*

1. *If σ is injective and R is a domain, then T is a domain.*
2. *If σ is injective and R is a division ring, then T is a principal right ideal domain.*
3. *If σ is an automorphism and R is a prime ring, then T is a prime ring.*

Proof. See the proof in [35, Theorem 2.9]. □

Theorem 2.1.7. [35, Theorem 2.10] *Let R be a right noetherian ring, and T be an over ring generated by R and a variable x such that $Rx + R = xR + R$. Then T is right noetherian. In particular, if $T = R[x; \sigma, \delta]$ is an Ore extension and R is right noetherian, then T is right noetherian.*

Proof. See the proof in [35, Thereom 2.10]. □

2.2 Hopf Algebras

In this section, we introduce some basic information about Hopf algebras over a field k. Let us first give the definitions of algebras and coalgebras in the following subsection.

2.2.1 Algebras and Coalgebras

Definition 2.2.1. An algebra is a triple (A, m, μ) where A is a k-vector space and $m : A \otimes_k A \to A$ and $\mu : k \to A$ are k-linear maps that make the following diagrams commute:

$$\begin{tikzcd} A \otimes_k A \otimes_k A \arrow[r, "m \otimes \mathrm{id}"] \arrow[d, "\mathrm{id} \otimes m"] & A \otimes_k A \arrow[d, "m"] \\ A \otimes_k A \arrow[r, "m"] & A \end{tikzcd} \tag{2.1}$$

and

$$\begin{tikzcd} & A \otimes_k A \arrow[dd, "m"] & \\ A \otimes_k k \arrow[ur, "\mathrm{id} \otimes \mu"] \arrow[dr] & & k \otimes_k A \arrow[ul, "\mu \otimes \mathrm{id}"'] \arrow[dl] \\ & A & \end{tikzcd} . \tag{2.2}$$

The property of distributivity in A of *product* m (also called *multiplication*) and addition is captured in the definition of m as a map from the tensor product $A \otimes_k A$ to A. The isomorphisms $A \simeq k \otimes_k A$ and $A \simeq A \otimes_k k$ in the diagram (2.2) are the canonical ones. For instance, in $A \simeq k \otimes_k A, a \in A$ is mapped to $1 \otimes a$ and conversely, $\lambda \otimes a \in k \otimes_k A$ is mapped to λa. In general, we let ab or $a \cdot b$ denote the product of two elements a and b in an algebra.

The dual concept of a coalgebra arises naturally when we reverse all the arrows in the diagrams (2.1) and (2.2). Next we will introduce coalgebras and study some of their properties.

Definition 2.2.2. A coalgebra is a triple (C, Δ, ϵ) where C is a k-vector space and $\Delta : C \to C \otimes_k C$ and $\epsilon : C \to k$ are k-linear maps that make the following diagrams commute:

$$\begin{tikzcd} C \arrow[r, "\Delta"] \arrow[d, "\Delta"] & C \otimes_k C \arrow[d, "\mathrm{id} \otimes \Delta"] \\ C \otimes_k C \arrow[r, "\Delta \otimes \mathrm{id}"] & C \otimes_k C \otimes_k C \end{tikzcd} \tag{2.3}$$

and

$$\begin{tikzcd} & C \arrow[dl] \arrow[d, "\Delta"] \arrow[dr] & \\ C \otimes_k k & & k \otimes_k C \\ & C \otimes_k C \arrow[ul, "\mathrm{id} \otimes \epsilon"] \arrow[ur, "\epsilon \otimes \mathrm{id}"'] & \end{tikzcd} . \tag{2.4}$$

Many basic concepts of algebras find their analogues in coalgebra theory. Dually, we call Δ the *coproduct* (also call *comultiplication*) and call ϵ the *counit* map. For the notation of coproduct, we will use Sweedler's notation, which is named after Moss E. Sweedler who introduced it in his pioneering book [49]. It can be very useful to denote the coproduct of an element c in a coalgebra C by $\Delta(c) = \sum_{(c)} c_1 \otimes c_2$ or simply $\Delta(c) = \sum c_1 \otimes c_2$.

Remark 2.2.1. Using Sweedler's notation, the counit axiom in (2.2) says that for all $c \in C$,

$$c = \sum \epsilon(c_1) c_2 = \sum c_1 \epsilon(c_2) .$$

Moreover, we express the coassociativity above in this following formula. It simply says

$$(\mathrm{id} \otimes \triangle) \circ \Delta(c) = (\mathrm{id} \otimes \triangle) \left(\sum c_1 \otimes c_2\right) = \sum c_1 \otimes c_{21} \otimes c_{22}.$$

It should equal

$$(\Delta \otimes \mathrm{id}) \circ \Delta(c) = \sum c_{11} \otimes c_{12} \otimes c_2.$$

So, in this case we write both of the above simply as

$$\sum c_1 \otimes c_2 \otimes c_3. \tag{2.5}$$

Applying coassociativity to (2.5) we find that the three expressions

$$\sum \Delta(c_1) \otimes c_2 \otimes c_3, \sum c_1 \otimes \Delta(c_2) \otimes c_3 \text{ and } \sum c_1 \otimes c_2 \otimes \Delta(c_3)$$

are all equal in $C \otimes_k C \otimes_k C \otimes_k C$. Thus we write it as

$$\sum c_1 \otimes c_2 \otimes c_3 \otimes c_4.$$

By a similar way, we have $\Delta_n = (\mathrm{id} \otimes \Delta) \circ \Delta_{n-1} = (\Delta \otimes \mathrm{id}) \circ \Delta_{n-1}$ as an iterated application of Δ as above, so

$$\Delta_{n-1} : C \to C^{\otimes n}. \tag{2.6}$$

Note that commutative algebras are an important subclass of associative algebras. Analogously, there exists a dual concept for a coalgebra called cocommutativity. The coalgebra C is cocommutative if and only if $\Delta(c) = \sum c_2 \otimes c_1$ for all $c \in C$. Here we can use a diagram to express cocommutativity.

Definition 2.2.3. Let $\tau : C \otimes_k C \to C \otimes_k C$ be a k-linear map, called the flip, such that $\tau(a \otimes b) = b \otimes a$, for all $a, b \in C$. A coalgebra (C, Δ, ϵ) is called cocommutative if $\tau \circ \Delta = \Delta$, i.e., the following diagram commutes

$$\begin{tikzcd} C \otimes_k C \arrow[rr, "\tau"] & & C \otimes_k C \\ & C \arrow[ul, "\Delta"] \arrow[ur, "\Delta"'] & \end{tikzcd} . \tag{2.7}$$

We will give a number of examples of algebras and coalgebras in this subsection.

Example 2.2.2. It is easy to see that $\mathbb{C}[x]$ is an algebra and a coalgebra. We omit the checking of the algebra structure here. The coproduct on basis elements that determine the coproduct of $\mathbb{C}[x]$ is given by,

$$\Delta\left(x^n\right) = \sum_{i=0}^{n} \binom{n}{i} x^i \otimes x^{n-i}$$

extending linearly in $\mathbb{C}[x]$, where $\epsilon(1) = 1$, $\epsilon(x^n) = 0$, for $n \geq 1$ and $x^0 = 1$. It is not too difficult to see that this is coassociative. We can consider a small example, $\Delta\left(x^2\right) = 1 \otimes x^2 + 2x \otimes x + x^2 \otimes 1$. Notice that

$$\begin{aligned}
&(\mathrm{id} \otimes \Delta)\left(\Delta(x^2)\right) \\
=&(\mathrm{id} \otimes \Delta)(1 \otimes x^2 + 2x \otimes x + x^2 \otimes 1) \\
=&1 \otimes (1 \otimes x^2 + 2x \otimes x + x^2 \otimes 1) + 2x \otimes (1 \otimes x + x \otimes 1) + x^2 \otimes (1 \otimes 1) \\
=&1 \otimes 1 \otimes x^2 + 2 \otimes x \otimes x + 1 \otimes x^2 \otimes 1 + 2x \otimes 1 \otimes x + 2x \otimes x \otimes 1 + x^2 \otimes 1 \otimes 1
\end{aligned}$$

and

$$\begin{aligned}
&(\Delta \otimes \mathrm{id})\left(\Delta(x^2)\right) \\
=&(\Delta \otimes \mathrm{id})(1 \otimes x^2 + 2x \otimes x + x^2 \otimes 1) \\
=&(1 \otimes 1) \otimes x^2 + 2(1 \otimes x + x \otimes 1) \otimes x + \left(1 \otimes x^2 + 2x \otimes x + x^2 \otimes 1\right) \otimes 1 \\
=&1 \otimes 1 \otimes x^2 + 2 \otimes x \otimes x + 2x \otimes 1 \otimes x + 1 \otimes x^2 \otimes 1 + 2x \otimes x \otimes 1 + x^2 \otimes 1 \otimes 1.
\end{aligned}$$

So $(\mathrm{id} \otimes \Delta)\left(\Delta(x^2)\right) = (\Delta \otimes \mathrm{id})\left(\Delta(x^2)\right)$. Moreover, $(\mathrm{id} \otimes \epsilon)\left(\Delta(x^2)\right) = x^2 = (\epsilon \otimes \mathrm{id})\left(\Delta(x^2)\right)$.

Remark 2.2.3. In this above example, we also can define a different valid coproduct by

$$\Delta\left(x^n\right) = \sum_{i=0}^{n} x^i \otimes x^{n-i}$$

and extending linearly in $\mathbb{C}[x]$, and

$$\epsilon(x^n) = \begin{cases} 1, \text{ if } n = 0 \\ 0, \text{ if } n \neq 0. \end{cases}$$

Notice that

$$
\begin{aligned}
(\Delta \otimes \mathrm{id})\left(\Delta(x^n)\right) &= \sum_{i=0}^{n}\sum_{j=0}^{i} x^j \otimes x^{i-j} \otimes x^{n-i} \\
&= \sum_{i=0}^{n}\sum_{j=0}^{i} x^i \otimes x^j \otimes x^{n-i-j}
\end{aligned}
$$

and

$$
\begin{aligned}
(\mathrm{id} \otimes \Delta)\left(\Delta(x^n)\right) &= \sum_{i=0}^{n}\sum_{j=0}^{n-i} x^i \otimes x^j \otimes x^{n-i-j} \\
&= \sum_{i=0}^{n}\sum_{j=0}^{i} x^i \otimes x^j \otimes x^{n-i-j}.
\end{aligned}
$$

So coassociativity holds.

Example 2.2.4. Let $A = \{s_1, s_2, \ldots, s_k\}$ be a finite alphabet and let A^* denote the free monoid on the set A. Consider the vector space kA^* whose basis elements are all the elements of A^*. This is the free associative algebra on A. We define the product of basis elements to be simply concatenation and the unit element 1 to be the empty word. We define the coproduct on the basis elements in A via the rule

$$\Delta(s_n) = \sum_{i=0}^{n} s_i \otimes s_{n-i}$$

for each basis element $s_i \in A$, where we take $s_0 = 1$. For example,

$$\Delta(s_1) = 1 \otimes s_1 + s_1 \otimes 1,\ \Delta(s_2) = 1 \otimes s_2 + s_1 \otimes s_1 + s_2 \otimes 1, \ldots.$$

In this case, an order for the elements of A is required. If w is a word in A^*, say $w = s_1 s_2 \cdots s_m$, then we define

$$\Delta(w) = \Delta(s_1) \cdots \Delta(s_m)$$

and then extend linearly to kA^*. In fact, it is straightforward to show that,

$$\Delta\left(s_1^n\right) = \sum_{i=0}^{n} \binom{n}{i} s_1^i \otimes s_1^{n-i}.$$

We define

$$\epsilon(w) = \begin{cases} 1, \text{ if } w = 1 \\ 0, \text{ if } w \neq 1 \end{cases}$$

for all w in the algebra. Then the counit satisfies the relations,

$$\begin{cases} w = 1 \otimes w = (\epsilon \otimes \mathrm{id})\Delta(w) \\ w = w \otimes 1 = (\mathrm{id} \otimes \epsilon)\Delta(w). \end{cases}$$

A group algebra is another fundamental example of an object that has both an algebra structure and coalgebra structure.

Example 2.2.5. Let G be a group and let kG be the group algebra, where each element in kG is expressed as a sum $\sum \alpha_g g$, where $\alpha \in k$ and $g \in G$, and all but finitely many of the α_g are zero. The coproduct is defined by $\Delta : kG \to kG \otimes_k kG$ by $\Delta(g) = g \otimes g$ for $g \in G$, and $\epsilon(g) = 1$ for $g \in G$.

Definition 2.2.4. Let (C, Δ, ϵ) be a coalgebra. A subcoalgebra D of C is a vector subspace of C such that $\Delta(D) \subseteq D \otimes_k D$. A left coideal I of C is a vector subspace of C such that $\Delta(I) \subseteq I \otimes_k C$. A right coideal I of C is a vector subspace of C such that $\Delta(I) \subseteq C \otimes_k I$. A coideal I of C is a vector subspace of C such that $\Delta(I) \subseteq I \otimes_k C + C \otimes_k I$ and $\epsilon(I) = 0$.

We list algebra and coalgebra morphisms below and they will be used in constructing bialgebras.

Definition 2.2.5. Given algebras (A, m_A, μ_A) and (B, m_B, μ_B), an algebra morphism $f : A \to B$ is a k-linear map such that

$$f \circ m_A = m_B \circ (f \otimes f) \text{ and } f \circ \mu_A = \mu_B.$$

Definition 2.2.6. Given two coalgebras $(C, \Delta_C, \epsilon_C)$ and $(D, \Delta_D, \epsilon_D)$, a coalgebra morphism $\varphi : C \to D$ is a k-linear map such that

$$\Delta_D \circ \varphi = (\varphi \otimes \varphi) \circ \Delta_C, \text{ and } \epsilon_D \circ \varphi = \epsilon_C.$$

Example 2.2.6. Let $A = kG$ and $B = kH$ be two group algebras. Suppose ϕ is an algebra morphism and ψ is a coalgebra morphism from A to B. Then we have

$$\begin{cases} \phi(ab) = \phi(a)\phi(b) \\ \mu_B(1_B) = \phi(\mu_A(1_A)) \end{cases}$$

and

$$\begin{cases} \Delta_B(\psi(a)) = \sum \psi(a_1) \otimes \psi(a_2) \\ \epsilon_B(\psi(a)) = \epsilon_A(a) \end{cases}$$

for all $a, b \in A$.

2.2.2 Bialgebras and Convolutions

In the proceeding subsection, we notice that in Examples 2.2.2, 2.2.4, 2.2.5, (m, μ) and (Δ, ϵ) are compatible, namely, m and μ are colagebra morphisms and Δ and ϵ are algebra morphisms. In this situation, the resulting objects are called bialgebras.

Definition 2.2.7. A bialgebra H is a k -vector space $H = (H, m, u, \Delta, \epsilon)$, where (H, m, u) is an algebra; and (H, Δ, ϵ) is a coalgebra; and such that either (and hence both) of the following two conditions hold:

1. Δ and ϵ are algebra morphisms;

2. m and u are coalgebra morphisms.

We only require one condition in the definition of a bialgebra above to hold because of the following proposition.

Proposition 2.2.7. *Let $H = (H, m, u, \Delta, \epsilon)$ have both algebra structure and coalgebra structure. Then Δ and ϵ are algebra morphisms if and only if m and u are coalgebra morphisms.*

We need the information below to complete the proof the above proposition.

Given two k-algebras A and B, we can see that $A \otimes_k B$ is also a k-algebra by defining $(a \otimes b)(c \otimes d) = ac \otimes bd$ and extending linearly using distributivity. Expressed as a diagram this is:

$$A \otimes_k B \otimes_k A \otimes_k B \xrightarrow{\mathrm{id} \otimes \tau \otimes \mathrm{id}} A \otimes_k A \otimes_k B \otimes_k B \xrightarrow{m_A \otimes m_B} A \otimes_k B$$

where $\tau : B \otimes_k A \to A \otimes_k B$ is the flip: $\tau(b \otimes a) = a \otimes b$. The unit $u_{A \otimes_k B}$ of $A \otimes_k B$ is given by

$$k \cong k \otimes k \xrightarrow{\mu_A \otimes \mu_B} A \otimes_k B$$

Similarly, if C and D are coalgebras then so is $C \otimes D$ with $\Delta_{C \otimes D}$ given by

$$C \otimes_k D \xrightarrow{\Delta \otimes \Delta} C \otimes_k C \otimes_k D \otimes_k D \xrightarrow{\mathrm{id} \otimes \tau \otimes \mathrm{id}} C \otimes_k D \otimes_k C \otimes_k D$$

and counit

$$C \otimes_k D \xrightarrow{\epsilon_C \otimes \epsilon_D} k \otimes k \cong k$$

In particular, this applies when $A = B$ and when $C = D$. Now we give the detailed argument of the above proposition.

Proof. It suffices to see the following facts by the definitions of algebra morphisms and coalgebra morphisms.

1. Δ is an algebra morphism by (2.8) and (2.9)

2. ϵ is an algebra morphism by (2.10) and (2.11)

3. m is a coalgebra morphism by (2.8) and (2.10)

4. μ is a coalgebra morphism by (2.9) and (2.11).

$$\begin{tikzcd}
H \otimes_k H \arrow[r, "m"] \arrow[d, "\Delta \otimes \Delta"] & H \arrow[dd, "\Delta"] \\
H \otimes_k H \otimes_k H \otimes_k H \arrow[d, "\mathrm{id} \otimes \tau_{A,A} \otimes \mathrm{id}"] & \\
H \otimes_k H \otimes_k H \otimes_k H \arrow[r, "m_A \otimes m_B"] & H \otimes_k H
\end{tikzcd} \tag{2.8}$$

$$\begin{tikzcd}
k \arrow[r, "\mu"] \arrow[d] & H \arrow[d, "\Delta"] \\
k \otimes k \arrow[r, "\mu \otimes \mu"] & H \otimes_k H
\end{tikzcd} \tag{2.9}$$

$$\begin{tikzcd}
H \otimes_k H \arrow[r, "\epsilon \otimes \epsilon"] \arrow[d, "m"] & k \otimes k \arrow[d, "\epsilon"] \\
H \arrow[r] & H \otimes_k k
\end{tikzcd} \tag{2.10}$$

$$\begin{tikzcd}
& H \arrow[dr, "\epsilon"] & \\
k \arrow[ur, "\mu"] \arrow[rr, "\mathrm{id}"] & & k
\end{tikzcd} \tag{2.11}$$

$\square$

Definition 2.2.8. A morphism of bialgebras $f : A \to B$ is a k-linear map which is both an algebra and a coalgebra morphism.

Definition 2.2.9. If $f : A \to B$ is a bialgebra morphism, then $\ker f$ is called a biideal: this means that $\ker f$ is an ideal and a coideal (i.e. the kernel of a coalgebra morphism) of A.

Examples 2.2.2, 2.2.4 and 2.2.5 are bialgebras. A tensor algebra is also an example of bialgebra.

Example 2.2.8. Let V be a vector space and form $T(V)$, the tensor algebra,

$$T(V) = \bigoplus_{n\geq 0} \left(V^{\otimes n}\right).$$

The multiplication of pure tensors is given by tensoring and the rule is then obtained by extending linearly. Then $T(V)$ is a bialgebra if we define

$$\Delta(v) = v \otimes 1 + 1 \otimes v \in T(V) \otimes T(V)$$

and $\epsilon(v) = 0$ for $v \in V$.

Remark 2.2.9. Not every object which has both algebra and coalgebra structures is a bialgebra. A counterexample is given in Remark 2.2.3. Since $\Delta(x^2) = 1 \otimes x^2 + x \otimes x + x^2 \otimes 1$ and $\Delta(x)\Delta(x) = 1 \otimes x^2 + 2x \otimes x + x^2 \otimes 1$, $\Delta(x^2) \neq (\Delta(x))^2$.

2.2.3 Hopf algebras

Now we are ready to introduce Hopf algebras.

Definition 2.2.10. Let C be a k-coalgebra and A be a k-algebra and $f, g \in \mathrm{Hom}_k(C, A)$. The convolution of f and g is the linear map $f \star g := m \circ (f \otimes g) \circ \Delta : C \to A$, i.e., in Sweedler's notation

$$(f \star g)(c) = \sum f(c_1)\, g(c_2)$$

for all $c \in C$. The convolution product is the map $\star : \mathrm{Hom}_k(C, A) \times \mathrm{Hom}_k(C, A) \to \mathrm{Hom}_k(C, A)$ that sends a pair (f, g) to $f \star g$.

One particular case on which we will focus afterwards is when $(H, m, \mu, \Delta, \epsilon)$ is a bialgebra and the convolution product is considered between linear endomorphisms of H. We prove now some properties about the convolution product.

Proposition 2.2.10. *Let (C, Δ, ϵ) be a k-coalgebra and (A, m, μ) be a k-algebra. Then the convolution product $\star$ on $\mathrm{Hom}_k(C, A)$ is a bilinear and associative map. Moreover, $\mu \circ \epsilon \in \mathrm{Hom}_k(C, A)$ is the identity element. Therefore, $\mathrm{Hom}_k(C, A)$ is a monoid.*

Proof. Bilinearity follows from the fact that $f \star g := m \circ (f \otimes g) \circ \Delta$ and the fact that tensor products of maps are bilinear. Associativity can be obtained from the associative properties of m and the tensor product of maps, as well as from the coassociativity of Δ. Given $f, g, h \in \mathrm{Hom}_k(C, A)$ and $c \in C$, we write in Sweedler's notation

$$\begin{aligned}((f \star g) \star h)(c) &= \sum (f \star g)\,(c_1)\,h\,(c_2) \\ &= \sum f\,(c_1)\,g\,(c_2)\,h\,(c_3) \\ &= \sum f\,(c_1)\,(g \star h)\,(c_2) \\ &= (f \star (g \star h))(c)\end{aligned}$$

which proves that $(f \star g) \star h = f \star (g \star h)$. Hence, $\star$ is associative on $\mathrm{Hom}_k(C, A)$.

Next we will show $\mu \circ \epsilon \in \mathrm{Hom}_k(C, A)$ is the identity element. Let $f \in \mathrm{Hom}_k(C, A)$ and $c \in C$. Then we compute in Sweedler's notation

$$(f \star (\mu \circ \epsilon))(c) = \sum f\,(c_1)\,\mu\,(\epsilon\,(c_2)) = f\left(\sum c_1 \epsilon\,(c_2)\right) = f(c).$$

Similarly, we can prove that $(\mu \circ \epsilon) \star f = f$. $\square$

Definition 2.2.11. Let $H = (H, m, u, \Delta, \epsilon)$ be a bialgebra. An antipode of H is a morphism $S : H \to H$ such that

$$m\,(S \otimes \mathrm{id})\,\Delta = \mu\epsilon = m\,(\mathrm{id} \otimes S)\,\Delta$$

i.e. the following diagram commutes:

$$\begin{tikzcd}
 & H \otimes_k H \arrow[rr, "\mathrm{id} \otimes S"] & & H \otimes_k H \arrow[dr, "m"] & \\
H \arrow[ur, "\Delta"] \arrow[rr, "\epsilon"] \arrow[dr, "\Delta"'] & & k \arrow[rr, "\mu"] & & H \\
 & H \otimes_k H \arrow[rr, "S \otimes \mathrm{id}"] & & H \otimes_k H \arrow[ur, "m"'] &
\end{tikzcd} . \tag{2.12}$$

Remark 2.2.11. An antipode may not always exist in a bialgebra but when it does, it is unique by the uniqueness of inverses in the monoid $\mathrm{Hom}_k(H, H)$. An antipode of H is an anti-homomorphism.

Definition 2.2.12. A Hopf algebra H is a bialgebra with an antipode S.

Definition 2.2.13. Morphisms of Hopf algebras are just bialgebra maps preserving the antipode.

We can easily see that those examples of bialgebras in the above are Hopf algebras by endowing them with a proper antipode.

Example 2.2.12. Let $\mathbb{C}[x]$ be defined in Example 2.2.2. Then $\mathbb{C}[x]$ is a bialgebra. If we define the antipode S on $\mathbb{C}[x]$ given by $S(x^n) = (-1)^n x^n$ for $n > 0$, then $\mathbb{C}[x]$ is a Hopf algebra.

Example 2.2.13. Let kA^* be defined in Example 2.2.4. Then S is defined by $S(s_i) = -s_i$ for $s_i \in A$. Then kA^* is a Hopf algebra.

Example 2.2.14. In the group algebra kG, if we define $S(g) = g^{-1}$ for all $g \in G$, then kG is a Hopf algebra.

Definition 2.2.14. Let G be an algebraic variety over an algebraically closed field k, which also has the structure of a group, that is, the multiplication and inverse maps

$$m : G \times G \to G$$

$$(x, y) \mapsto xy$$

and

$$\tau : G \to G$$

$$x \mapsto x^{-1}$$

are morphisms of varieties. Then G is an algebraic group over k.

The *coordinate algebra* $\mathcal{O}(G)$ of an algebraic group G is the algebra of regular functions from G to k. So the identity element of $\mathcal{O}(G)$ is the constant function 1.

Proposition 2.2.15. *Let G be an algebraic group over k. Then $\mathcal{O}(G)$ is a Hopf algebra. We define*

$$\begin{aligned} \mu : k &\mapsto \mathcal{O}(G) \\ 1_k &\mapsto \mathrm{id}_{\mathcal{O}(G)}, \ \ \mathrm{id}_{\mathcal{O}(G)}(x) = 1_k \\ \Delta : \mathcal{O}(G) &\mapsto \mathcal{O}(G) \otimes_k \mathcal{O}(G) \cong \mathcal{O}(G \times G) \end{aligned}$$

and $\Delta(f)$ the function from $G \times G$ to k by

$$\Delta(f)((x, y)) = f(xy)$$

for $x, y \in G$; and $\varepsilon : \mathcal{O}(G) \to k$ is given by $f \mapsto f(1_G)$. Finally, let $S : \mathcal{O}(G) \to \mathcal{O}(G)$ be given by

$$(Sf)(x) = f\left(x^{-1}\right).$$

Proof. It is clear that $\mathcal{O}(G)$ is an algebra. Now we check the coalgebra structure and antipode.

$$\begin{aligned}
(\mathrm{id} \otimes \Delta)(\Delta(f))(x,y,z) &= \Delta(f)(x,yz) = f(xyz) \\
(\Delta \otimes \mathrm{id})(\Delta(f))(x,y,z) &= \Delta(f)(xy,z) = f(xyz) \\
(\mathrm{id} \otimes \epsilon)(\Delta(f))(x) &= \Delta(f)(x,1_G) = f(x1_G) = f(x) \\
(\mathrm{id} \otimes S)(\Delta(f))(x) &= (\Delta(f))(x,x^{-1}) = f(xx^{-1}) = f(1_G) = \epsilon(f) = \mu\epsilon(f)(x).
\end{aligned}$$

□

The following example will be involved in Chapter 4.

Example 2.2.16. Let G be the group of upper-triangular 3×3 unipotent complex matrices and let H be the coordinate ring of G. Then H is generated as a $\mathbb{C}$-algebra by the coordinate functions x,y,z, where evaluating x,y and z at an element of G corresponds to taking resp. the $(1,3)$-, $(1,2)$-, and $(2,3)$-entries of the element. Then $H = k[y,z][x]$ with coefficient Hopf algebra $R = k[y,z]$. The coproduct of $H = k[y,z][x]$ is determined by the products of $\Delta(x), \Delta(y)$ and $\Delta(z)$. Let

$$A = \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & d & e \\ 0 & 1 & f \\ 0 & 0 & 1 \end{bmatrix}. \quad \text{Then} \quad AB = \begin{bmatrix} 1 & a+d & e+af+b \\ 0 & 1 & c+f \\ 0 & 0 & 1 \end{bmatrix}.$$

By the definition in Proposition 2.2.15, we have

$$\begin{aligned}
\Delta(x)((A,B)) &= x(AB) = e + af + b = (x \otimes \mathrm{id}_{\mathcal{O}(G)} + \mathrm{id}_{\mathcal{O}(G)} \otimes x + y \otimes z)((A,B)) \\
\Delta(y)((A,B)) &= y(AB) = a + d = (y \otimes \mathrm{id}_{\mathcal{O}(G)} + \mathrm{id}_{\mathcal{O}(G)} \otimes y)((A,B)) \\
\Delta(z)((A,B)) &= z(AB) = c + f = (z \otimes \mathrm{id}_{\mathcal{O}(G)} + \mathrm{id}_{\mathcal{O}(G)} \otimes z)((A,B)).
\end{aligned}$$

Therefore, x is not primitive in H (see Definition 2.2.15).

Theorem 2.2.17. [44, Corollary 1.7] *The functor $G \mapsto \mathcal{O}(G)$ defines a contravariant equivalence of categories*

$$\left\{ \begin{array}{c} \textit{affine algebraic groups over } k \\ \textit{and their morphisms} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \textit{affine commutative semiprime} \\ \textit{Hopf } k\textit{-algebras and} \\ \textit{Hopf algebra morphisms} \end{array} \right\}.$$

Example 2.2.18. Let $\mathcal{G}$ be the set of unlabelled finite graphs and $H = \mathrm{Span}_k(\mathcal{G})$. Let us define the comultiplication $\Delta : H \to H \otimes_k H$ as follows. For a set $V_1 \subset V(\Gamma)$ of vertices of a graph Γ, let us denote by $G(V_1)$ the induced subgraph of Γ with the set of vertices V_1; i.e., V_1 is the set of vertices of $G(V_1)$, and $e \in E(\Gamma)$ is an edge in $G(V_1)$ if and only if both ends of e belong to V_1. We set

$$\Delta(\Gamma) = \sum_{V_1 \subseteq V(\Gamma)} G(V_1) \otimes_k G(V(\Gamma) \setminus V_1)).$$

We define the disjoint union of graphs as a multiplication. This comultiplication and multiplication can be extended by linearity to linear combinations of graphs. This makes the space H into a commutative algebra. Besides, we define $\mu(1) = \emptyset$, $\epsilon(\emptyset) = 1$ and $\epsilon(\Gamma) = 0$ for any nonempty graph Γ. It is easy to check that H is a bialgebra. Moreover, since H is graded and connected, S is unique and has the form of

$$S = \sum_{n \geq 0} (-1)^n m^{n-1} p^{\otimes n} \Delta^{n-1},$$

where p is the projection onto $\ker(\epsilon)$, and Δ^{n-1}, m^{n-1} are defined as in 2.6. Therefore, H is a Hopf algebra.

Example 2.2.19. The universal enveloping algebra $U(g)$ of Lie algebra g is a Hopf algebra. We define $\Delta(x) = x \otimes 1 + 1 \otimes x$ for every x in g. Notice that this rule is compatible with commutators and can therefore be uniquely extended to all elements of $U(g)$. Moreover, we define $\epsilon(x) = 0$ for all $x \neq 1$ in g (again, extended to $U(g)$) and $S(x) = -x$ all x in g). Clearly $U(g)$ is cocommutative. $U(g)$ is commutative if and only if g is abelian.

2.2.4 Some properties of Hopf algebras

We will use the following proposition in Chapter 4.

We define $\mathrm{Alg}_k(H, k)$ to be the set of algebra morphisms from H to k.

Corollary 2.2.20. *Let H be a bialgebra. The space $\mathrm{Alg}_k(H, k)$ is a monoid under the convolution product $\star$ with ϵ as the convolution identity element. Furthermore, if H is a Hopf algebra, with antipode S, then $\mathrm{Alg}_k(H, k)$ becomes a group, in which for every $\alpha \in \mathrm{Alg}_k(H, k)$, its convolution inverse is $\alpha \circ S$.*

Proof. As a direct consequence of Proposition 2.2.10, $\mathrm{Alg}_K(H, k)$ is a monoid. Assume that H is a Hopf algebra and S is the antipode. Given $h \in H$, we compute

$$(\alpha \star (\alpha \circ S))(h) = \sum \alpha\left(h_1\right) \alpha\left(S\left(h_2\right)\right) = \alpha\left(\sum h_1 S\left(h_2\right)\right) = \alpha(\epsilon(h)) = \epsilon(h)$$

since α is an algebra morphism and by applying the antipode property. Similarly, we prove that $(\alpha \circ S) \star \alpha = \epsilon$. □

So we denote $\mathrm{Alg}_k(H, k)$ the group of algebra automorphisms from H to k.

Proposition 2.2.21. [49, Proposition 4.01] *Let H be a Hopf algebra with antipode S. Then:*

1. *$S(gh) = S(h)S(g)$, for $g, h \in H$;*
2. *$S\left(1_H\right) = 1_H$;*
3. *$\Delta(S(h)) = \sum S\left(h_2\right) \otimes S\left(h_1\right)$, for $h \in H$;*
4. *$\epsilon(S(h)) = \epsilon(h)$, for $h \in H$.*

Proof. See the proof in [49, Proposition 4.01]. □

Definition 2.2.15. Let C be a coalgebra. An element $g \in C$ is called *grouplike* if $g \neq 0$ and $\Delta(g) = g \otimes g$. The set of grouplike elements of C is denoted $G(C)$. Let H be a bialgebra. An element $h \in H$ is called *primitive* if $\Delta(h) = 1 \otimes h + h \otimes 1$. The set of primitive elements of H is denoted $P(H)$.

Remark 2.2.22. If $g \in C$ is *grouplike*, then by the counit axiom we have $\epsilon(g)g = g$, from where it follows that $\epsilon(g) = 1$. Likewise, if $h \in H$ is *primitive*, then by the counit axiom it follows that $\epsilon(h) = 0$.

Definition 2.2.16. Let H be a Hopf algebra and $\alpha \in \mathrm{Alg}_k(H, k)$. The left winding automorphisms τ_α^ℓ is the algebra endomorphism $m \circ (\alpha \otimes I) \circ \Delta : H \to H$, i.e., in Sweedler's notation:

$$\tau_\alpha^\ell(h) = \sum \alpha\left(h_1\right) h_2$$

for all $h \in H$. Similarly, we can define the right winding automorphism τ_α^r as the map $m \circ (I \otimes \alpha) \circ \Delta : H \to H$, in Sweedler's notation:

$$\tau_\alpha^r(h) = \sum h_1 \alpha\left(h_2\right)$$

for all $h \in H$.

Now we introduce an important invariant of a coalgebra C, its *coradical*, which will be used in Chapter 4. A nonzero subcoalgebra of a coalgebra C is called *simple* if it does not have any nontrivial proper subcoalgebras.

Definition 2.2.17. Let C be a coalgebra. The coradical C_0 of C is the sum of the simple subcoalgebras of C. The coalgebra C is called connected if C_0 is trivial, i.e., $C_0 = k$.

In Chapter 4, we will apply the following proposition to a connected coalgebra C.

Proposition 2.2.23. [38, Theorem 5.2.2] *Let C be a coalgebra. Define inductively $C_n = \Delta^{-1}(C \otimes_k C_{n-1} + C_0 \otimes_k C)$ for $n \geq 1$. Then $\{C_n\}_{n \in \mathbb{N}}$ is a family of subcoalgebras of C, called the coradical filtration, that satisfies*

1. $C = \cup_{n \in \mathbb{N}} C_n$

2. $C_n \subseteq C_{n+1}$

3. $\Delta(C_n) \subseteq \sum_{i=0}^{n} C_i \otimes_k C_{n-i}$.

Proof. See [38, Theorem 5.2.2]. □

Remark 2.2.24. A family $\{A_i\}_{i \in \mathbb{N}}$ of subcoalgebras of a coalgebra that satisfies 1 to 3 above is called a coalgebra filtration.

2.2.5 Smash Products

In Chapter 4, we investigate cocommutative Hopf algebras. To understand the structure of cocommutative Hopf algebras, we will need the concept of nilpotent-by-finite groups and smash products.

Definition 2.2.18. A *polycyclic* group is a group G with a finite chain

$$1 = G_0 \triangleleft G_1 \triangleleft \cdots \triangleleft G_n = G,$$

where G_i is normal in G_{i+1}, and each factor G_{i+1}/G_i is cyclic.

Definition 2.2.19. A *polycyclic-by-finite* (resp. *nilpotent-by-finite*) group G is a group that has a polycyclic (resp. nilpotent) normal subgroup of finite index.

Remark 2.2.25. If G is a finitely generated and nilpotent-by-finite, then it is polycyclic-by-finite.

Before we give the definition of smash products, we introduce the concept of a *skew group ring*. Let R be a ring and G be a group that acts on R as automorphisms. We use both r^g and $g \cdot r$ to denote the action of $g \in G$ on $r \in R$. The skew group ring $R\#G$ is then the free right R-module with elements of G as a basis and with multiplication defined by $(rh)(sg) = \left(rs^h\right)(hg)$ for $g, h \in G, r, s \in R$. Thus each element of $R\#G$ has a unique expression as $\sum_{g\in G} r_g g$ with $r_g = 0$ for all but finitely many $g \in G$. For a k-Hopf algebra H, we have a somewhat analogous construction called the smash product as below.

Definition 2.2.20. Let H be a Hopf algebra. A k-algebra A is a left H-module algebra if

1. A is a left H-module via: $h \otimes a \longrightarrow h \cdot a$,
2. $h \cdot (ab) = \sum (h_1 \cdot a)(h_2 \cdot b)$,
3. $h1_A = \epsilon(h)1_A$

for all $h \in H$, $a, b \in A$.

Definition 2.2.21. Let H be a k-Hopf algebra and let A be a left H-module algebra. Then the smash product algebra $A\#H$ is defined as follows, for all $a, b \in A, h, g \in H$:

1. As a k-vector space, $A\#H = A \otimes_k H$ and we write $a\#h$ for the element $a \otimes h$
2. The multiplication is given by

$$(a\#h)(b\#g) = \sum a(h_1 \cdot b)\#h_2 g.$$

If $H = kG$, then $A\#kG = A\#G$, the skew group ring: multiplication is just $(ag)(bh) = a(g \cdot b)gh$, for all $a, b \in A, g, h \in G$.

2.3 Zariski Cancellation Problems

2.3.1 Preliminaries

A longstanding problem in affine algebraic geometry is the Zariski cancellation problem, which asks whether an affine variety X over an algebraically closed field k having the property that $X \times \mathbb{A}^1 \cong \mathbb{A}^{n+1}$ is necessarily isomorphic to $\mathbb{A}^n$. The question is known to have an affirmative answer when $n = 1$ [1], and $n = 2$, with the characteristic zero case being done by Fujita [17] and Miyanishi-Sugie [37], and the positive characteristic case handled by Russell [43]. In positive characteristic, Gupta [21, 22] gave counterexamples to the Zariski cancellation problem in dimension at least three, but the problem remains open in dimension greater than two in the case that the base field has characteristic zero. Ring theoretically, one can ask more generally:

Question 2.3.1. For R some specific k-algebra, when does $R[t] \cong S[t]$ imply that R and S are isomorphic as k-algebras?

In recent years, increased attention has been paid to the noncommutative analogue of the Zariski cancellation problem [7, 18, 31, 32, 50]. In this setting, a finitely generated k-algebra R that has the property that whenever $R[x] \cong S[x]$ implies $R \cong S$ where S is another finitely generated k-algebra is said to be *cancellative.* An algebra R is *strongly cancellative* if, for every $d \geq 1$, any isomorphism $R[x_1, \ldots, x_d] \cong S[x_1, \ldots, x_d]$ implies that R is isomorphic to S. It is known that many classes of noncommutative algebras are cancellative or strongly cancellative in the sense above. Notably, cancellation holds for algebras with trivial centre, for "noncommutative surfaces" that are not commutative, and many quantizations of coordinate rings of affine varieties (see the results in [7]).

We make use of the following definitions from [7, 31]. If B is a subring of a ring C and $f_1, \cdots, f_m$ are elements of C, then the subring generated by B and the set $\{f_1, \cdots, f_m\}$ is denoted by $B\{f_1, \cdots, f_m\}$.

Definition 2.3.1. Let A be an algebra.

1. We call A *cancellative* (resp. *strongly cancellative*) if for any algebra B, any isomorphism $A[t] \to B[t]$ (resp., for any $d \geq 1$, any isomorphism $A[t_1, \ldots, t_d] \to B[t_1, \ldots, t_d]$) implies that $A \cong B$.

2. We call A *retractable* (resp. *strongly retractable*) if for any algebra B, any isomorphism $\phi : A[t] \to B[t]$ (resp. for any $d \geq 1$, any isomorphism $\phi : A[t_1, \ldots, t_d] \to B[t_1, \ldots, t_d]$) implies that $\phi(A) = B$.

3. Let $Z(A)$ and $Z(B)$ denote the centres of A and B, respectively. We call A *Z-retractable* (resp. *strongly Z-retractable*) if for any algebra B, any algebra isomorphism $\phi : A[t] \to B[t]$ (resp. for any $d \geq 1$, any isomorphism $\phi : A[t_1, \ldots, t_d] \to B[t_1, \ldots, t_d]$), we necessarily have $\phi(Z(A)) = Z(B)$.

4. We call A *detectable* (resp. *strongly detectable*) if for any algebra B, any isomorphism $\phi : A[t] \to B[s]$ (resp. for any $d \geq 1$, any isomorphism $\phi : A[t_1, \ldots, t_d] \to B[s_1, \ldots, s_d]$), we necessarily have $s \in B\{\phi(t)\}$ (resp. $s_i \in B\{\phi(t_1), \ldots, \phi(t_d)\}$ for $i = 1, \ldots, d$).

2.3.2 Some useful tools

In this subsection, we provide the basic background on the Makar-Limanov invariant that was introduced by Makar-Limanov [34], who called the invariant AK, although it is now standard to use the terminology Makar-Limanov invariant and the notation ML.

We quickly recall the basic concepts involved in the definition of this invariant. These concepts can be found in [34, 7, 31].

Definition 2.3.2. Let k be a field and let A be a k-algebra.

1. We let $\mathrm{Der}(A)$ denote the collection of k-linear derivations of A.

2. A k-linear derivation δ of A is called locally nilpotent if for each $a \in A$ there exists a $n_a \in \mathbb{N}$ such that $\delta^n(a) = 0$ for all $n > n_a$.

3. We let $\mathrm{LND}(A) = \{\delta \in \mathrm{Der}(A) \mid \delta \text{ is a locally nilpotent derivation of } A\}$.

4. A *Hasse-Schmidt derivation* of A is a sequence of k-linear maps $\partial := \{\partial_n\}_{n\geq 0}$ such that:

$$\partial_0 = \mathrm{id}_A, \text{ and } \partial_n(ab) = \sum_{i=0}^{n} \partial_i(a)\partial_{n-i}(b)$$

for $a, b \in A$ and $n \geq 0$.

5. A Hasse-Schmidt derivation $\partial = \{\partial_n\}_{n\geq 0}$ is called *locally nilpotent* if for each $a \in A$ there exists an integer $N = N(a) \geq 0$ such that $\partial_n(a) = 0$ for all $n \geq N$ and the k-algebra homomorphism $A[t] \to A[t]$ given by $t \mapsto t$ and $a \mapsto \sum_{n\geq 0} \partial_n(a)t^n$ is a k-algebra isomorphism. If only the first condition holds then the map $A[t] \to A[t]$ is still an injective endomorphism but need not be onto; we will call Hasse-Schmidt derivations for which only the first condition holds (i.e., there exists an integer $N = N(a) \geq 0$ such that $\partial_n(a) = 0$ for all $n \geq N$) a *weakly locally nilpotent Hasse-Schmidt derivation*.

6. A Hasse-Schmidt derivation $\partial = \{\partial_n\}_{n\geq 0}$ is called *iterative* if $\partial_i \circ \partial_j = \binom{i+j}{i}\partial_{i+j}$ for all $i, j \geq 0$. The collection of Hasse-Schmidt derivations of an algebra A is denoted $\mathrm{Der}^H(A)$ and the collection of iterative Hasse-Schmidt derivations is denoted $\mathrm{Der}^I(A)$. The collection of locally nilpotent Hasse-Schmidt derivations (resp. iterative Hasse-Schmidt derivations, resp. weakly locally nilpotent Hasse-Schmidt derivations) of A is denoted $\mathrm{LND}^H(A)$ (resp. $\mathrm{LND}^I(A)$, resp. $\mathrm{LND}^{H'}(A)$).

7. Given $\partial = \{\partial_n\}_{n\geq 0}$ the kernel of ∂ is defined to be

$$\ker(\partial) = \bigcap_{n\geq 1} \ker(\partial_n).$$

8. The $*$-*Makar-Limanov invariant* of A is defined to be

$$\mathrm{ML}^*(A) = \bigcap_{\delta \in \mathrm{LND}^*(A)} \ker(\delta).$$

9. The $*$-*Makar-Limanov centre* of A is defined to be

$$\mathrm{ML}^*_Z(A) = \mathrm{ML}^*(A) \cap Z(A).$$

10. We say that A is LND^*-*rigid* (resp. *strongly* LND^*-*rigid*) if $\mathrm{ML}^*(A) = A$ (resp. $\mathrm{ML}^*(A[t_1, \ldots, t_d]) = A$, for any $d \geq 1$).

11. We say that A is LND^*_Z-*rigid* (resp. *strongly* LND^*_Z-*rigid*) if $\mathrm{ML}^*_Z(A)$ is equal to $Z(A)$ (resp. $\mathrm{ML}^*_Z(A[t_1, \ldots, t_d]) = Z(A)$, for any $d \geq 1$).

In items (7)–(10), $*$ is either blank, I, H, or H'.

Remark 2.3.2. Let k be a field and let A be a k-algebra. We recall some basic facts about derivations and Hasse-Schmidt derivations.

1. If $\partial = \{\partial_n\}_{n\geq 0}$ is a locally nilpotent Hasse-Schmidt derivation of A then by definition the map $G_{\partial,t} : A[t] \to A[t]$ defined by

$$a \mapsto \sum_{n=0}^{\infty} \partial_n(a)t^n, \text{ for all } a \in A, t \mapsto t \tag{2.13}$$

extends to a k-algebra automorphism of $A[t]$ and when $\partial = \{\partial_n\}_{n\geq 0}$ is a weakly locally nilpotent Hasse-Schmidt derivation then this map is an injective endomorphism.

2. Conversely, if one has a k-algebra automorphism (resp. endomorphism) $G : A[t] \to A[t]$ such that $G(t) = t$ and $G(a) - a \in tA[t]$ for $a \in A$, then for $a \in A$ we have

$$G(a) = \sum_{n=0}^{\infty} \partial_n(a)t^n,$$

and $\partial = \{\partial_n\}_{n\geq 0}$ is a locally nilpotent Hasse-Schmidt derivation (resp. weakly locally nilpotent Hasse-Schmidt derivation) of A (see [7, Lemma 2.2 (3)]).

3. If the characteristic of k is zero and $\delta : A \to A$ is a k-linear derivation, then the only iterative Hasse-Schmidt derivation $\partial = \{\partial_n\}_{n\geq 0}$ of A with $\partial_1 = \delta$ is given by

$$\partial_n = \frac{\delta^n}{n!} \tag{2.14}$$

for $n \geq 0$. This iterative Hasse-Schmidt derivation is called the *canonical Hasse-Schmidt derivation* associated to δ. If, moreover, δ is locally nilpotent, then by [7, Lemma 2.2(2)], the map $G_{\partial,t}$ defined in item (2.13) is an automorphism and $\partial = \{\partial_n\}_{n\geq 0}$ is a locally nilpotent iterative Hasse-Schmidt derivation, and conversely if $\partial = \{\partial_n\}_{n\geq 0}$ is locally nilpotent then so is δ. Thus locally nilpotent iterative Hasse-Schmidt derivations correspond naturally to locally nilpotent derivations in the characteristic zero case and so $\mathrm{ML}^I(A) = \mathrm{ML}(A)$ for algebras with characteristic zero base field.

4. [36, §1.1] If the characteristic of k is a positive integer p, then for an iterative derivation $\partial = \{\partial_n\}_{n\geq 0}$, ∂_n can be explicitly described as

$$\partial_n = \frac{(\partial_1)^{i_0}\,(\partial_p)^{i_1}\ldots(\partial_{p^r})^{i_r}}{(i_0)!\,(i_1)!\ldots(i_r)!},$$

where $n = i_0 + i_1 p + \cdots + i_r p^r$ is the base-p expansion of n. In this case, an iterative Hasse-Schmidt derivation ∂ is completely determined by $\partial_1, \partial_p, \partial_{p^2}, \ldots$.

5. Let T be the polynomial ring $A[t_1, \ldots, t_d]$ over a k-algebra A. We fix an integer $1 \leq i \leq d$. For each $n \geq 0$, we can define a divided power A-linear differential operator Δ_i^n as follows:

$$\Delta_i^n : t_1^{m_1}\cdots t_d^{m_d} \mapsto \begin{cases} \binom{m_i}{n} t_1^{m_1}\cdots t_i^{m_i-n}\cdots t_d^{m_d} & \text{if } m_i \geq n \\ 0 & \text{otherwise,} \end{cases} \tag{2.15}$$

where $\binom{m_i}{n}$ is defined in $\mathbb{Z}$ or in $\mathbb{Z}/(p)$. Then $\{\Delta_i^n\}_{n=0}^{\infty}$ is a locally nilpotent iterative Hasse-Schmidt derivation of T. We can also extend an element $\partial = \{\partial_n\}_{n\geq 0}$ in $\mathrm{LND}^{H'}(A)$ to an element of $\mathrm{LND}^{H'}(T)$ by declaring that $t_1, \ldots, t_d$ are in the kernel of $\partial = \{\partial_n\}_{n\geq 0}$; moreover, the extension is iterative if the original Hasse-Schmidt derivation is iterative, and it is in $\mathrm{LND}^H(T)$ if the original weakly locally nilpotent Hasse-Schmidt derivation is in $\mathrm{LND}^H(A)$. Combining this observation along with data from the maps Δ_i^n, we see

$$\mathrm{ML}^*(A[t_1, \ldots, t_d]) \subseteq \mathrm{ML}^*(A),$$

where $*$ is either I, H, or H'.

2.4 Gelfand-Kirillov dimension

In this section, we introduce two very important dimensions, *Krull dimension* and *Gelfand-Kirillov dimension*.

A useful invariant in commutative algebra is the Krull dimension, which is named after Wolfgang Krull. It is defined as the supremum of the lengths of all chains of prime ideals. For noncommutative rings there is an extension of Krull dimension due to Gabriel and Rentschler, which is defined as an ordinal given by the deviation of the poset of its left ideals (if it exists). We give some interesting facts about Krull dimension. A field k has Krull dimension 0; $k[x_1, \ldots, x_n]$ has Krull dimension n; a principal ideal domain that is not a field has Krull dimension 1. Krull dimension is also used in algebraic geometry. The dimension of the affine variety given by the zero set of a radical ideal I in a polynomial ring A is the Krull dimension of A/I.

However, for a noncommutative ring it is often more convenient to work with *Gelfand-Kirillov dimension*. For more information about Gelfand-Kirillov dimension, we refer the reader to the book of Krause and Lenagan [29] and we will state the definition in the next subsection. One well-known fact is that Krull dimension and Gelfand-Kirillov dimension coincide when one restricts one's focus to the class of finitely generated commutative algebras over a field.

2.4.1 Definition of Gelfand-Kirillov dimension

The following definition is from the textbook [29].

Definition 2.4.1. Let k be a field and let A be a finitely generated k-algebra. Choose a finite-dimensional k-subspace V of A, containing 1_A, such that A is generated as an algebra over k by V. Let V^n denote the span of $\{v_1 v_2 \cdots v_n \mid v_1, v_2, \cdots, v_n \in V\}$ for $n \geq 1$. There is an ascending chain of subspaces

$$k \subseteq V \subseteq V^2 \subseteq \cdots \subseteq V^n \subseteq \cdots \subseteq \bigcup_{n=0}^{\infty} V^n = A$$

with $\dim_k (V^n) < \infty$, for each $n \in \mathbb{N}$. The asymptotic behavior of the monotone increasing sequence $\{\dim_k (V^n)\}$ provides a useful invariant of the algebra A, known as the growth or Gelfand-Kirillov dimension of A, and defined by

$$\mathrm{GKdim}(A) = \overline{\lim_{n}} \frac{\log \dim_k (V^n)}{\log n}.$$

One of the first questions to consider is whether this definition depends on the vector subspace V chosen. It does not, as is proved in [29, Lemma 1.1]. The notion is extended to algebras A that are not finitely generated as follows:

$$\mathrm{GKdim}(A) = \sup\{\mathrm{GKdim}(B) : B \text{ is a finitely generated subalgebra of } A.\}$$

Algebras that are finite-dimensional (as vector spaces) have Gelfand-Kirillov dimension equal to 0. For integral domains (i.e., a commutative algebra without zero-divisors) that are finitely generated, the Gelfand-Kirillov dimension is equal to the transcendence degree, i.e., the maximum number of algebraically independent elements of the algebra. It is clear that $\mathrm{GKdim}(A) = 0$ if and only if A is locally finite-dimensional, meaning that every finitely generated k-subalgebra of A is finite-dimensional.

2.4.2 Gelfand-Kirillov dimension of an Ore extension

In this short subsection, we record two results that we will need in Chapter 5.

Proposition 2.4.1. [29, Lemma 3.4] *Let A be a k-algebra with k-derivation δ, and let $B = A[x;\delta]$. Then* $\mathrm{GKdim}(B) \geq \mathrm{GKdim}(A) + 1$.

Proof. See the proof in [29, Lemma 3.4]. $\square$

Proposition 2.4.2. [29, Proposition 3.5] *Let A be a k-algebra with k-derivation δ such that each finite-dimensional subspace of A is contained in a δ-stable finitely generated subalgebra of A. Then* $\mathrm{GKdim}(A[x;\delta]) = \mathrm{GKdim}(A) + 1$.

Proof. See the proof in [29, Proposition 3.5]. $\square$

Chapter 3

Prime ideals of $T = R[x; \sigma, \delta]$

3.1 Statement of results

Throughout this chapter, we always assume that R is a noetherian ring, and $T = R[x; \sigma, \delta]$ is an Ore extension as defined in Chapter 1 with σ an automorphism. When investigating the prime ideals of $T = R[x; \sigma, \delta]$, it is important to understand the actions of these maps on R. In particular when we can eliminate one of δ or σ and assume either $\delta = 0$ or $\sigma = \mathrm{id}$, then the analysis becomes more straightforward [26, 25]. We let $\mathrm{Spec}(R)$ denote the set of primes of R. Goodearl [19] exhibited a relationship between prime ideals of T and their contractions in the coefficient algebra R in the case when R is commutative and noetherian. The result is stated in the following theorem.

Theorem 3.1.1. [19, Theorem 3.1] *Let $T = R[x; \sigma, \delta]$, where R is a commutative noetherian ring and σ is an automorphism.*

1. *If P is a prime ideal of T and $I = P \cap R$, then one of the following cases must hold:*

   (a) I is a (σ, δ)-prime ideal of R. In this case, either

i. I is a σ-prime (σ,δ)-ideal of R, or

ii. I is a δ-prime (σ,δ)-ideal of R and R/I has a unique associated prime ideal, which contains $(1-\sigma)(R)$.

(b) I is a prime ideal of R and $\sigma(I) \neq I$.

2. Conversely, if I is any ideal of R satisfying (a) or (b), then $I = P \cap R$ for some prime ideal P of T. More specifically, in case (a), $IT \in \operatorname{Spec}(T)$ while in case (b), there exists a unique $P \in \operatorname{Spec}(T)$ such that $P \cap R = I$, and T/P is a commutative domain.

We recall that a (σ,δ)-ideal of R is simply an ideal that is invariant under the maps σ and δ. An ideal I is a (σ,δ)-prime ideal if whenever J and K are (σ,δ)-ideals with $JK \subseteq I$ we necessarily have either J or K is contained in I. The notions of σ- and δ-ideals and σ- and δ-prime ideals are defined analogously.

Theorem 3.1.1 can be roughly interpreted as follows: if R is commutative and noetherian and P is a prime ideal of $T = R[x;\sigma,\delta]$ then either T/P is commutative or $P \cap R$ is well-behaved under the maps σ and δ. Thus one can easily study the prime homomorphic images of T.

Since commutative rings are a special case of polynomial identity rings (PI rings, for short), that is rings that satisfy a nonzero identity in finitely many noncommuting variables, it is natural to consider whether Goodearl's results extend to this setting. We are unable to completely resolve this question and leave the complete resolution for future work.

We recall that a prime ideal of a ring is completely prime if the quotient ring formed by modding out by the ideal is a domain. Our main results in this direction are given in the theorem below.

Theorem 3.1.2. *Let R be a noetherian PI algebra and let σ and δ be resp. an automorphism and a σ-derivation of R. Then if P is a completely prime ideal of $T := R[x;\sigma,\delta]$*

then either T/P satisfies the same polynomial identity as R or $I := P \cap R$ is a σ-invariant and δ-invariant prime ideal of R.

The proof of this theorem will require some additional technology, and will be given in the next section.

3.2 Prime Ideals of $T = R[x; \sigma, \delta]$ and their contractions under R

To prove Theorem 3.1.2, we need to use noncommutative localization. We refer the reader to the book of Goodearl and Warfield [20] for further background.

If R is a ring and X is a right denominator set of R with $1 \in X$, then let RX^{-1} the ring of quotients of R with respect to X. Then there is a natural homomorphism ϕ from R to RX^{-1} given by $r \mapsto r1^{-1}$.

- For any right ideal A of R, we define A^e (the extension of A) to be $\phi(A)(RX^{-1})$.
- For any right ideal B of RX^{-1}, we define B^c (the contraction of B) to be $\phi^{-1}(B)$.
- Given $(a, s_1), (b, s_2) \in R \times X$, if there exists some $s \in X$ such that $(as_2 - bs_1)s = 0$, then we say (a, s_1) and (b, s_2) are equivalent and we write $as_1^{-1} \sim bs_2^{-1}$. Then as a set RX^{-1} is given by R$\times X/\sim$.

Then we can conclude the following proposition by the results in [30, Propsosition 10.32 and 10.33].

Proposition 3.2.1. *Given a ring R and a denominator set X, if A is a right ideal of R, then we have the following:*

1. *We define $A^e = \{as^{-1} \mid a \in A, s \in X\}$, and A^e is a right ideal of RX^{-1} if R is right noetherian.*

2. *If A is an ideal in R such that A^e is an ideal in RX^{-1}, then for any right ideal $A_1 \subseteq R$, $(A_1 A)^e = A_1^e A^e$.*

3. *$A^{ec} = \{r \in R \mid rs \in A$ for some $s \in X\}$ and so $A \subseteq A^{ec}$.*

4. *If B is a right ideal of RX^{-1}, then $B^{ce} = B$.*

5. *If R is right noetherian, there is a bijection between the set of primes of RX^{-1} and the set of primes of R which are disjoint from X, given by contraction and extension. Moreover, $P^{ec} = P$ for any prime ideal P of R.*

Proof. See the proof in [30, Propsosition 10.32 and 10.33]. □

We can now prove the main theorem of this chapter.

Proof of Theorem 3.1.2. Let $I = P \cap R$. Then if $\sigma(I) \not\subseteq I$, there is some $a \in I$ such that $\sigma(a) \notin I$. Moreover, since P is completely prime and R/I embeds in T/P, we see that I is a completely prime ideal of R. Thus $\sigma(a)$ is not a zero divisor mod I. Then we have $xa = \sigma(a)x + \delta(a) \in P$ and so $\sigma(a)x = -\delta(a) \pmod{P}$. Now we let X denote the set of nonzero divisors of T/P. Then Goldie's theorem (see [20]) gives that X is a right denominator set of T/P and that $\mathrm{Frac}(T/P) := (T/P)X^{-1}$ is a division ring. Similarly, we can construct $\mathrm{Frac}(R/I)$ and the universal property of localization gives $\mathrm{Frac}(R/I)$ embeds in $\mathrm{Frac}(T/P)$. Moreover, by construction $\sigma(a)$ is a unit in $\mathrm{Frac}(R/I)$ and so since $\sigma(a)x = -\delta(a)$ in $\mathrm{Frac}(T/P)$, we see that $x + P$ can be identified with the image of $-(\sigma(a))^{-1}\delta(a)$ in $\mathrm{Frac}(R/I)$. In particular, T/P is a subalgebra of $\mathrm{Frac}(R/I)$. Since R satisfies a polynomial identity, both R/I and $\mathrm{Frac}(R/I)$ satisfy the same identities as R (and possibly more) and since T/P is isomorphic to a subring of $\mathrm{Frac}(R/I)$, it satisfies the same identity as R. Thus we have obtained the result if $\sigma(I) \not\subseteq I$.

Thus we may assume that $\sigma(I) \subseteq I$. Then in this case if $a \in I$ then $xa = \sigma(a)x + \delta(a)$ and since $a, \sigma(a) \in I$, both xa and $\sigma(a)x$ are in P and so $\delta(a) \in P \cap R = I$. Thus I is a (σ, δ)-ideal. Moreover, we showed that it was a prime ideal and so we obtain the desired result.

$\square$

In future work, we would like to extend the result to a true analogue of Goodearl's theorem 3.1.2 for Ore extensions of PI rings and eliminate the completely prime hypothesis.

Chapter 4

Hopf Ore extensions

In Chapter 2 we introduced the notions of Ore extensions and Hopf algebras. We will now talk about Hopf Ore extensions, which were introduced by Panov [40]. Then we will look at work of Brown, O'Hagan, Zhang and Zhuang [11], which improved upon Panov's result and hence became the main reference in this particular topic. Throughout this chapter, we let k be a field, we let R be a Hopf algebra with antipode S, and we let $T = R[x; \sigma, \delta]$ be an Ore extension of R. We recall that T is the algebra generated by R and by x subject to the relations

$$xr = \sigma(r)x + \delta(r)$$

for all $r \in R$, where σ is an automorphism of R and δ is a σ-derivation. Every element in T can be written uniquely as $\sum_{i \in \mathbb{N}} r_i x^i$, with finitely many nonzero $r_i \in R$. This chapter is based on the content of [24]. The main result of this chapter is to establish the following theorems.

Theorem 4.0.1. *Let R be a noetherian Hopf k-algebra and let $T = R[x; \sigma, \delta]$ be an Ore extension of R. Suppose that $R \otimes_k R$ is a domain. Then T has a Hopf algebra structure extending that of R if and only if after a linear change of variables we have the following:*

1. *There exists a grouplike element β of R such that $\Delta(x) = \beta^{-1} \otimes x + x \otimes 1 + w$, and $S(x) = -\beta(x + \sum w_1 S(w_2))$ and $\beta \sum w_1 S(w_2) = \sum S(w_1) w_2$ with $w \in R \otimes_k R$;*

2. *There is a character $\chi : R \to k$ such that*

$$\sigma(r) = \tau^l_\chi(r) = \sum \chi(r_1) r_2 = \sum \beta^{-1} r_1 \beta \chi(r_2)$$

   for all $r \in R$;

3. *The σ-derivation δ satisfies the relation*

$$\Delta\delta(r) - \sum \delta(r_1) \otimes r_2 - \sum \beta^{-1} r_1 \otimes \delta(r_2) - w\Delta(r) - \Delta\sigma(r) w = 0$$

   and

$$w \otimes 1 + (\Delta \otimes I)(w) = \beta^{-1} \otimes w + (I \otimes \Delta) w.$$

It is shown that $R \otimes_k R$ is a domain when R satisfies certain conditions as follows.

Theorem 4.0.2. *Let k be an algebraically closed field of characteristic zero and let R be a noetherian cocommutative k-Hopf algebra of finite Gelfand-Kirillov dimension that is a domain. Then $R \otimes_k R$ is a domain. In particular, the results of Theorem 4.0.1 apply in this setting.*

4.1 Hopf Ore extensions and Panov's question

Panov united the two algebraic structures of Ore extensions and Hopf algebras in one algebraic object, more precisely, he raised the following question:

Question 4.1.1. Given a Hopf algebra R, for which automorphisms σ and σ-derivations δ does the Ore extension $T = R[x; \sigma, \delta]$ have a Hopf algebra structure extending the given Hopf algebra structure on R?

In general, since T is a free left R -module on basis $\{x^i : i \geq 0\}$, the general form of coproduct of x is

$$\Delta(x) = \sum_{i,j \geq 0} w_{ij} x^i \otimes x^j,$$

where each $w_{ij} \in R \otimes_k R$. To understand the Hopf algebra structure of T, it is necessary to establish the form of $\Delta(x)$. However, $\Delta(x)$ can be very complicated as described above. To simplify matters, Panov imposed a hypothesis on $\Delta(x)$ and defined the *Hopf Ore extension* as follow:

Definition 4.1.1. [40, Definition 1.0] Let R and $T = R[x; \sigma, \delta]$ be Hopf algebras over k. The Hopf algebra $T = R[x; \tau, \delta]$ is called a Hopf Ore extension if $\Delta(x) = x \otimes r_1 + r_2 \otimes x$ for some $r_1, r_2 \in R$ and R is a Hopf subalgebra of T.

Under this setting of the formula of $\Delta(x)$, Question 4.1.1 can be transformed as below:

Question 4.1.2. Given a Hopf algebra R, for which automorphisms σ and σ-derivations δ does the Ore extension $T = R[x; \sigma, \delta]$ become a Hopf Ore extension?

Moreover, if $\Delta(x) = x \otimes r_1 + r_2 \otimes x$, then Panov showed in [40] that $\Delta(x') = x' \otimes 1 + r' \otimes x'$, by replacing the generating element x by $x' = x r_1^{-1}$ in T, where $r' = r_2 r_1^{-1}$ is a grouplike element. Without loss of generality, we can assume that $\Delta(x) = x \otimes 1 + r \otimes x$ in definition 4.1.1. Then it follows from [40, Lemma 1.1] that

$$\epsilon(x) = 0$$
$$S(x) = -r^{-1} x$$

where $r^{-1} = S(r)$. This agrees with the Hopf structure on the classical polynomial algebra $K[x]$, where x is primitive. Since r is a grouplike element, we call x a skew primitive element. We recall Sweedler's notation, defined in Chapter 2 in which we write $\Delta(r) = \sum r_1 \otimes r_2$ for the coproduct of a general element $r \in R$. Then Panov showed the necessary and sufficiency condition of $T = R[x; \sigma, \delta]$ being a Hopf Ore extension and answered question 4.1.2 in the subsequent theorem.

Theorem 4.1.3. [40, Theorem 1.3] *The Hopf algebra $T = R[x;\sigma,\delta]$ is a Hopf Ore extension if and only if*

1. *there is a character $\chi : R \to k$ such that $\sigma(a) = \chi(a_1)\, a_2$ for any $a \in R$ (i.e., σ is a twisted automorphism of R;*

2. *the following relation holds: $\chi(a_1)\, a_2 = r a_1 r^{-1} \chi(a_2)$;*

3. *the σ-derivation δ satisfies the relation $\Delta\delta(a) = \delta(a_1) \otimes a_2 + r a_1 \otimes \delta(a_2)$.*

4.2 Generalization of Hopf Ore extensions

The additional hypothesis that the variable x of the extension T is skew primitive, typically, is not valid. In particular, Brown, O'Hagan, Zhang, and Zhuang (BOZZ) [11] gave a counterexample 2.2.16.

To deal with such examples, Brown et al. [11] relaxed Panov's hypothesis and studied skew polynomial extensions of Hopf algebras in which $\Delta(x)$ is of the form $s \otimes x + x \otimes t + v(x \otimes x) + w$, where $s, t \in R$ and $v, w \in R \otimes_k R$. In addition, Brown et al. [11] extended the definition of Hopf Ore extensions as follows:

Definition 4.2.1. [11, in §2.1] Let R be a Hopf k-algebra. A *Hopf Ore extension* (HOE) of R is a k-algebra T such that:

1. T is a Hopf k-algebra with Hopf subalgebra R;

2. there exists an algebra automorphism σ and a σ-derivation δ of R such that $T = R[x;\sigma,\delta]$;

3. there are $s, t \in R$ and $w, v \in R \otimes_k R$ such that

$$\Delta(x) = s \otimes x + x \otimes t + v(x \otimes x) + w. \tag{4.1}$$

The condition given by Equation (4.1) can be seen as imposing that x is not too "far" from being skew primitive. We observe that the "Hopf Ore extensions" originally defined by Panov are the Hopf Ore extensions (defined in 4.2.1) in which $v = w = 0$. It is natural to ask just how restrictive this condition Equation (4.1) is, namely:

Question 4.2.1. [11, in §2.1] Does the third condition in Definition 4.2.1 follow from the first two, after a change of the variable x?

Indeed, it is unclear in general whether an Ore extension of a Hopf algebra R that is itself a Hopf algebra should have an easy formula for $\Delta(x)$ described in the Definition 4.2.1. But understanding the form of $\Delta(x)$ is of significance in understanding the Hopf algebra structure of T.

4.3 Generalized Panov's theorem

In Definition 4.2.1, the hypothesis on $\Delta(x)$ in Definition 4.1.1 is extended to be

$$\Delta(x) = s \otimes x + x \otimes t + v(x \otimes x) + w.$$

In this section, the improved Panov's theorem is stated. Like Theorem 4.1.3, it establishes a criterion to assess when we can extend a Hopf algebra structure on R to an Ore extension $T = R[x; \sigma, \delta]$, that is, define on T a Hopf algebra structure compatible with the given structure on R.

Before we move forward to the improved theorem, we make observations that should be understood. The polynomial variable of a skew polynomial extension is far from uniquely determined. For if $T = R[x; \sigma, \delta]$ is a skew polynomial algebra and $\lambda \in k$, then a straightforward computation shows that $\delta_\lambda := \delta + \lambda(\mathrm{id} - \sigma)$ is another σ-derivation of R. Moreover, we can rewrite the Ore extension as

$$T = R\left[x + \lambda; \sigma, \delta_\lambda\right].$$

Moreover, given a unit of R, say for example b^{-1}, replacing x by $b^{-1}x$, and writing $ad(b^{-1})$ to denote conjugation by b^{-1}, then again, the Ore extension can be formed as

$$T = R\left[b^{-1}x; \mathrm{ad}\left(b^{-1}\right)\sigma, b^{-1}\delta\right].$$

In practice, b will be a grouplike element of a Hopf algebra when we apply this below, so this usage of the notation ad coincides with the standard Hopf notation ad_l, [38, page 33]. Therefore, without loss of generality, we are allowed to change the variable x as $x+\lambda$, $b^{-1}x$ or via a combination of the two $b^{-1}x+\lambda$ for the sake of convenience in the subsequent proofs.

After a suitable change of variable x and corresponding adjustments to σ, δ and w in the Ore extension $T = R[x;\sigma,\delta]$, the improved Panov's theorem is provided as follow.

Theorem 4.3.1. [11, Theorem in §2.4]

1. *Let R be a Hopf k-algebra and let $T = R[x;\sigma,\delta]$ be a HOE of R. Suppose that*

$$S(x) = \alpha x + \beta \text{ for } \alpha, \beta \in R \text{ with } \alpha \text{ a unit of } R. \tag{4.2}$$

Write $w = \sum w_1 \otimes w_2 \in R \otimes_k R$, with $\{w_1\}$ and $\{w_2\}$ chosen to be k-linearly independent subsets of R. Then the following hold.

   (a) *a, b are grouplike and $v = 0$.*

   (b) *After a change of the variable x and corresponding adjustments to σ, δ and w*

$$\epsilon(x) = 0 \tag{4.3}$$

   and

$$\Delta(x) = a \otimes x + x \otimes 1 + w. \tag{4.4}$$

   For the remainder of 1, we assume that (4.3) and (4.4) hold.

   (c) *$S(x) = -a^{-1}\left(x + \sum w_1 S\left(w_2\right)\right)$.*

(d) There is a character $\chi : R \to k$ such that

$$\sigma(r) = \sum \chi(r_1) r_2 = \sum a r_1 a^{-1} \chi(r_2), \tag{4.5}$$

for all $r \in R$. That is, σ is a left winding automorphism τ_χ^ℓ, and is the composition of the corresponding right winding automorphism with conjugation by a.

(e) The σ-derivation δ satisfies the relation

$$\Delta\delta(r) - \delta(r_1) \otimes r_2 - a r_1 \otimes \delta(r_2) = w\Delta(r) - \Delta\sigma(r) w \tag{4.6}$$

(f) The elements $\{w_1\}$ and $\{w_2\}$ of R satisfy the identities

$$S(w_1) w_2 = a^{-1} w_1 S(w_2) \tag{4.7}$$

and

$$w \otimes 1 + (\Delta \otimes \mathrm{id})(w) = a \otimes w + (\mathrm{id} \otimes \Delta)(w) \tag{4.8}$$

2. *Let R be a Hopf k-algebra. Suppose given $a \in G(R), w \in R \otimes_k R$, a k-algebra automorphism σ of R and a σ-derivation δ of R such that this data satisfies (4.5), (4.6), (4.7) and (4.8). Then the skew polynomial algebra $T = R[x; \sigma, \delta]$ admits a structure of Hopf algebra with R as a Hopf subalgebra, and with x satisfying* (4.2), (a), (b) *and (c) of (1). As a consequence, T is a HOE of R.*

4.4 Answer to Panov's original question

In this section, we will resolve Panov's original Question 4.1.1 affirmatively when R is noetherian and $R \otimes_k R$ is a domain. In practice, there are no known examples of Hopf algebras R that are domains for which $R \otimes_k R$ is not a domain as well, so one can think of this theorem, intuitively, as applying to noetherian domains.

Indeed, if we assume that $R \otimes_k R$ is a domain, then we can get a much simpler form of $\Delta(x)$ as described in the following lemma.

Lemma 4.4.1. [11, Lemma 1] *Let $T = R[x;\sigma,\delta]$ be a Hopf k-algebra with R a Hopf subalgebra. Suppose that $R \otimes_k R$ is a domain. Then*

$$\Delta(x) = s(1 \otimes x) + t(x \otimes 1) + v(x \otimes x) + w, \tag{4.9}$$

where s, t, v and $w \in R \otimes_k R$.

Comparing the form of $\Delta(x)$ in (4.9) and (4.1), it is natural to ask the following question:

Question 4.4.2. Can the conclusion (4.9) be replaced by (4.1) in the definition of HOE 4.2.1?

In fact, we observe that there are two examples which give positive answer to Question 4.4.2 described in the following two results. Brown, O'Hagan, Zhang, and Zhuang [11] showed that when R is a connected Hopf algebra, the answer to Question 4.4.2 is affirmative in the proposition below. Recall that R is called a connected Hopf algebra if its coradical is the base field k. If R is a connected Hopf k-algebra, then so is $R \otimes_k R$: for, if $\{R_i\}$ is the coradical filtration of R, then it is clear from the definition, [38, Theorem 5.2.2], that $A_n := \sum_{i=0}^{n} R_i \otimes_k R_{n-i}$ is a coalgebra filtration of $R \otimes_k R$, and hence by [38, Lemma 5.3.4] the coradical of $R \otimes_k R$ is contained in $A_0 = k$. Hence $R \otimes_k R$ is a domain by [38, Lemma 6.6]. Similarly, $R \otimes_k R \otimes_k R$ is a connected Hopf algebra domain. In this setting, [11] give the following result.

Proposition 4.4.3. [11, Proposition in §2.8] *Let k be algebraically closed of characteristic 0. Let R be a connected Hopf k-algebra and let $T = R[x;\sigma,\delta]$ be a Hopf algebra containing R as a Hopf subalgebra. Then*

$$\Delta(x) = 1 \otimes x + x \otimes 1 + w$$

for some $w \in R \otimes_k R$. As a consequence, T is a HOE of R and is a connected Hopf algebra.

Another recent example has been proved [6] in the theorem below.

Theorem 4.4.4. [6, Theorem C] *Suppose k is algebraically closed and R is a finitely generated commutative integral Hopf k-algebra. If an Ore extension $R[x;\sigma,\delta]$ admits a Hopf algebra structure extending that of R then, after a linear change of the variable x,*

$$\Delta(x) = a \otimes x + x \otimes b + w$$

for some $a, b \in R$, each of which is either 0 or grouplike, and some $w \in R \otimes_k R$. In particular, $R[x;\sigma,\delta]$ is a Hopf Ore extension of R.

Therefore, Question 4.2.1 seems likely to have a positive answer. In this section, in light of the above lemma, we will show that under the hypotheses that $R \otimes_k R$ is a domain and R is noetherian, then after a change of variables we have $\Delta(x) = \beta^{-1} \otimes x + x \otimes 1 + w$, with $w \in R \otimes_k R$ and β a grouplike element of R. More precisely, the hypothesis that R is noetherian ensures that the antipode is bijective [45], and allows us to use work of [11] to get that $S(x)$ has a linear form.

Suppose that T admits a Hopf algebra structure extending that of R. Recall that T is a free left R-module with basis $\{x^i \mid i \geq 0\}$ and $T \otimes_k T$ is a left $R \otimes_k R$-module with basis $\{x^i \otimes x^j : i, j \geq 0\}$. Thus we have that

$$\Delta(x) = \sum_{i,j} w_{i,j} x^i \otimes x^j,$$

with $w_{i,j} \in R \otimes_k R$. By Lemma 4.4.1, the hypothesis that $R \otimes_k R$ is a domain gives that

$$\Delta(x) = s(1 \otimes x) + t(x \otimes 1) + v(x \otimes x) + w, \tag{4.10}$$

with $s, t, v, w \in R \otimes_k R$. After a change of variables and corresponding adjustments to σ and δ, we may assume that $\epsilon(x) = 0$. For if $\epsilon(x) = c \neq 0 \in k$, then let $y = x - c$ and so

$$\epsilon(y) = 0,$$

and

$$yr = \sigma(r)y + \sigma(r)c + \delta(r) - cr.$$

Let $\delta'(r) = \delta(r) + \sigma(r)c - cr$. Then a straightforward computation shows that

$$\delta'(ab) = \sigma(a)\delta'(b) + \delta'(a)b,$$

whence δ' is a σ-derivation. Therefore, $R[x;\sigma,\delta] \cong R[y,\sigma,\delta']$.

In the next two Lemmas, we aim to answer question 4.4.2 and will show much more: after a change of variables we have $\Delta(x) = \beta^{-1} \otimes x + x \otimes 1 + w$, where β is a grouplike element of R. This is a significant step, as it shows that $\Delta(x)$ can be assumed to have a much simpler form, which gives an explicit Hopf algebra structure on the Ore extension T that is compatible with the Hopf structure on R.

To begin, we list the following facts which are useful in the proof of the subsequent lemmas. Using coassociativity of $\Delta : T \to T \otimes_k T$ and the form given in Equation (4.10) and then comparing the coefficients of all relevant terms

$$x \otimes x \otimes x, 1 \otimes x \otimes 1, 1 \otimes x \otimes x, 1 \otimes 1 \otimes x, x \otimes x \otimes 1$$

on both sides of the equation $(\mathrm{id} \otimes \Delta)\Delta(x) = (\Delta \otimes \mathrm{id})\Delta(x)$, we obtain the following equations:

$$(\mathrm{id} \otimes \Delta)(v) \cdot (1 \otimes v) = (\Delta \otimes \mathrm{id})(v) \cdot (v \otimes 1) \tag{4.11}$$

$$(\mathrm{id} \otimes \Delta)(s) \cdot (1 \otimes t) = (\Delta \otimes \mathrm{id})(t) \cdot (s \otimes 1) \tag{4.12}$$

$$(\mathrm{id} \otimes \Delta)(s) \cdot (1 \otimes v) = (\Delta \otimes \mathrm{id})(v) \cdot (s \otimes 1) \tag{4.13}$$

$$(\mathrm{id} \otimes \Delta)(s) \cdot (1 \otimes s) = (\Delta \otimes \mathrm{id})(s) + (\Delta \otimes \mathrm{id})(v) \cdot (w \otimes 1) \tag{4.14}$$

$$(\mathrm{id} \otimes \Delta)(v) \cdot (1 \otimes t) = (\Delta \otimes \mathrm{id})(t) \cdot (v \otimes 1). \tag{4.15}$$

We use these equations to derive additional useful equations. Note that we use Sweedler notation to make things more compact, that is, we simply write $f = \sum f_1 \otimes f_2$ in $R \otimes_k R$. We also note that we may always assume, in addition, that when we choose an expression for an element $\sum_{i=1}^{d} a_i \otimes b_i \in R \otimes_k R$, that $\{a_1, \ldots, a_d\}$ and $\{b_1, \ldots, b_d\}$ are k-linearly independent sets. We set

$$\alpha = (\mathrm{id} \otimes \epsilon)(s) = \sum s_1 \epsilon(s_2), \tag{4.16}$$

and

$$\beta = (\epsilon \otimes \mathrm{id})(t) = \sum \epsilon(t_1)(t_2). \tag{4.17}$$

Observe that applying $\mathrm{id} \otimes \epsilon \otimes \epsilon$ to Equation (4.13), we obtain on the left side

$$(\mathrm{id} \otimes \epsilon \otimes \epsilon)((\mathrm{id} \otimes \Delta)(s) \cdot (1 \otimes v)),$$

which is

$$\left(\sum s_1 \epsilon(s_2)\right) \cdot \left(\sum \epsilon(v_1)\epsilon(v_2)\right) = \alpha \cdot \left(\sum \epsilon(v_1)\epsilon(v_2)\right),$$

and on the right side, we obtain $(\mathrm{id} \otimes \epsilon \otimes \epsilon)((\Delta \otimes \mathrm{id})(v) \cdot (s \otimes 1))$, which is

$$(\sum v_1 \epsilon(v_2)) \cdot \left(\sum \epsilon(s_2) s_1\right) = \left(\sum v_1 \epsilon(v_2)\right) \cdot \alpha.$$

Thus we obtain the new equation

$$\alpha \cdot \left(\sum \epsilon(v_1)\epsilon(v_2)\right) = \left(\sum v_1 \epsilon(v_2)\right) \cdot \alpha. \tag{4.18}$$

We do not give the complete details of the following computations, as they can be done in a similar manner. We apply $\epsilon \otimes \epsilon \otimes \mathrm{id}$ to Equation (4.11) and we obtain

$$\left(\sum \epsilon(v_1) v_2\right) \cdot \left(\sum \epsilon(v_1) v_2\right) = \left(\sum \epsilon(v_1) v_2\right) \cdot \left(\sum \epsilon(v_1)\epsilon(v_2)\right). \tag{4.19}$$

By a result of Skryabin [45, Corollary 1], S is bijective on T and R so we must have $S(x) = ax + b$ with $a, b \in R$ and a a unit in R. Notice that

$$0 = \epsilon(x) = m \circ (S \otimes \mathrm{id}) \circ \Delta(x). \tag{4.20}$$

The coefficient of x^2 in the right side of Equation (4.20) is $\sum a\sigma(S(v_1)v_2)$, and the coefficient of x^2 on the left side of Equation (4.20) is 0. Since a is a unit and σ is an automorphism, we see that $\sum S(v_1)v_2 = 0$ and after the standard fact that $\epsilon \circ S = \epsilon$ then obtain that

$$\sum \epsilon(v_1)\epsilon(v_2) = 0. \tag{4.21}$$

Now we apply the $\mathrm{id} \otimes \epsilon \otimes \mathrm{id}$ to Equation (4.13). We obtain on the left side

$$(\mathrm{id} \otimes \epsilon \otimes \mathrm{id})((\mathrm{id} \otimes \Delta)(s) \cdot (1 \otimes v)) = \sum s_1 \otimes (\sum \epsilon(s_{21})\epsilon(v_1) \otimes s_{22}v_2) = \sum s_1 \otimes 1 \otimes s_2 \sum \epsilon(v_1)v_2$$

and on the right side

$$(\mathrm{id} \otimes \epsilon \otimes \mathrm{id})((\Delta \otimes \mathrm{id})(v) \cdot (s \otimes 1)) = \sum v_1 (\sum s_1 \epsilon(s_2)) \otimes 1 \otimes v_2.$$

By the multiplication map $m \otimes \mathrm{id} : R \otimes_k R \otimes_k R \to R \otimes_k R$, we can see that

$$s\left(1 \otimes (\sum \epsilon(v_1)v_2)\right) = v\left((\sum s_1 \epsilon(s_2)) \otimes 1\right). \tag{4.22}$$

Lemma 4.4.5. *Let R be a noetherian Hopf k-algebra and let $T = R[x; \sigma, \delta]$ admit a Hopf algebra structure with R a Hopf subalgebra. Suppose that $R \otimes_k R$ is a domain. Then after a change of the variable x" with the property that $\epsilon(x) = 0$ and corresponding adjustments to σ and δ, we can ensure that $v = 0$ in Equation (4.10); namely, that $\Delta(x) = s(1 \otimes x) + t(x \otimes 1) + w$, with $s, t, w \in R \otimes_k R$.*

Proof. Suppose $R \otimes_k R$ is a domain. By lemma 4.4.1, we have Equation (4.10)

$$\Delta(x) = s(1 \otimes x) + t(x \otimes 1) + v(x \otimes x) + w,$$

where $s, t, v, w \in R \otimes_k R$. Using the fact that $(\epsilon \otimes \mathrm{id}) \circ \Delta(x) = (\mathrm{id} \otimes \epsilon) \circ \Delta(x) = x$ and that $\epsilon(x) = 0$ in Equation (4.10) gives $1 = (\epsilon \otimes \mathrm{id})(s) = (\mathrm{id} \otimes \epsilon)(t)$; that is,

$$1 = \sum \epsilon(s_1)s_2 = \sum t_1 \epsilon(t_2). \tag{4.23}$$

Equations (4.16), (4.17) and (4.23) tell us that

$$1 = \epsilon(\alpha) = \epsilon(\beta),$$

so in particular α and β are nonzero. Thus, Equations (4.18) and (4.21) give

$$\sum \epsilon(v_1)\epsilon(v_2) = \sum v_1 \epsilon(v_2) = 0.$$

Further, Equation (4.19) tells that

$$\sum \epsilon(v_1)v_2 = \sum \epsilon(v_1)\epsilon(v_2) = \sum v_1\epsilon(v_2) = 0.$$

Thus by Equation (4.22), we see that $0 = v(\alpha \otimes 1)$.

Since $\alpha \otimes 1 \neq 0$, and $R \otimes_k R$ is a domain, we see that $v = 0$. Thus we have shown that $\Delta(x) = s(1 \otimes x) + t(x \otimes 1) + w$. □

Lemma 4.4.6. *Let R be a noetherian Hopf k-algebra and suppose that $T = R[x;\sigma,\delta]$ admits a Hopf algebra structure with R a Hopf subalgebra. Suppose that $R \otimes_k R$ is a domain and $\Delta(x) = s(1 \otimes x) + t(x \otimes 1) + w$, with $s, t, w \in R \otimes_k R$. Then after a change of the variable x, we can assume that $\Delta(x) = \beta^{-1} \otimes x + x \otimes 1 + w'$, where β is a grouplike element in R and $w' = \sum w'_1 \otimes w'_2 \in R \otimes_k R$. Moreover, $S(x) = -\beta(x + \sum w'_1 S(w'_2))$ and $\beta \cdot (\sum w'_1 S(w'_2)) = \sum S(w'_1)w'_2$.*

Proof. By the assumption that $\Delta(x)$ has the form of Equation (4.10) with $v = 0$, we get

$$(\mathrm{id} \otimes \Delta)(s) \cdot (1 \otimes s) = (\Delta \otimes \mathrm{id})(s) \tag{4.24}$$

from Equation (4.14). Applying $\mathrm{id} \otimes S \otimes \mathrm{id}$ to Equation (4.24), we obtain that

$$(\mathrm{id} \otimes S \otimes \mathrm{id})\left((\mathrm{id} \otimes \Delta)(s) \cdot (1 \otimes s)\right) = (\mathrm{id} \otimes S \otimes \mathrm{id})\left((\Delta \otimes \mathrm{id})(s)\right).$$

By the associativity of the product map, i.e., $m \circ (\mathrm{id} \otimes m) = m \circ (m \otimes \mathrm{id}) : R \otimes_k R \otimes_k R \to R$, we obtain on the left side

$$\begin{aligned} & m \circ (\mathrm{id} \otimes m) \circ ((\mathrm{id} \otimes S \otimes \mathrm{id})((\mathrm{id} \otimes \Delta)(s) \cdot (1 \otimes s))) \\ =& m \circ (\mathrm{id} \otimes m)(\sum s_1 \otimes (\sum S(s_1)S(s_{21}) \otimes s_{22}s_2)) \\ =& \sum s_1\epsilon(s_2)(\sum S(s_1)s_2) \\ =& \alpha(\sum S(s_1)s_2) \end{aligned}$$

and on the right side

$$m \circ (m \otimes \mathrm{id}) \circ (\mathrm{id} \otimes S \otimes \mathrm{id})((\Delta \otimes \mathrm{id})(s)) = m \circ (\epsilon \otimes \mathrm{id})(s) = \sum \epsilon(s_1)s_2 = 1.$$

Therefore, we have

$$\alpha(\sum S(s_1)s_2) = 1.$$

Since R is a domain and α is left invertible, α is invertible and $\alpha^{-1} = \sum S(s_1)s_2$. Applying $\mathrm{id} \otimes \epsilon \otimes \mathrm{id}$ to Equation (4.12), and using Equations (4.16) and (4.17), we see that

$$s(1 \otimes \beta) = t(\alpha \otimes 1). \tag{4.25}$$

Note that α is a unit, and thus

$$s(\alpha^{-1} \otimes \beta) = t. \tag{4.26}$$

Combining Equations (4.12) and (4.26), we have

$$(\mathrm{id} \otimes \Delta)(s) \cdot (1 \otimes s) \cdot (1 \otimes \alpha^{-1} \otimes \beta) = (\Delta \otimes \mathrm{id})(s) \cdot (\Delta(\alpha^{-1}) \otimes \beta) \cdot (s \otimes 1).$$

By Equation (4.24), we have

$$(\Delta \otimes \mathrm{id})(s) \cdot (1 \otimes \alpha^{-1} \otimes \beta) = (\Delta \otimes \mathrm{id})(s) \cdot (\Delta(\alpha^{-1}) \otimes \beta) \cdot (s \otimes 1). \tag{4.27}$$

Applying $(\mathrm{id} \otimes \mathrm{id} \otimes \epsilon)$ to Equation (4.27) and using the fact that $\epsilon(\beta) = 1$, it results that

$$\sum \epsilon(s_2)\Delta(s_1) \cdot (1 \otimes \alpha^{-1}) = \sum \epsilon(s_2)\Delta(s_1) \cdot \Delta(\alpha^{-1}) \cdot s. \tag{4.28}$$

Note again that $R \otimes_k R$ is a domain and $\alpha \neq 0$. Cancelling $\Delta(\alpha) = \sum \epsilon(s_2)\Delta(s_1)$ from both sides of Equation (4.28), we have

$$1 \otimes \alpha^{-1} = \Delta(\alpha^{-1}) \cdot s. \tag{4.29}$$

Then

$$\begin{aligned}
\Delta(\alpha^{-1}x) &= \Delta(\alpha^{-1}) \cdot \Delta(x) \\
&= \Delta(\alpha^{-1}) \cdot (s(1 \otimes x) + t(x \otimes 1) + w) \\
&= 1 \otimes \alpha^{-1}x + \Delta(\alpha^{-1}) \cdot t(x \otimes 1) + \Delta(\alpha^{-1})w \\
&= 1 \otimes \alpha^{-1}x + \Delta(\alpha^{-1}) \cdot t(\alpha \otimes 1)(\alpha^{-1}x \otimes 1) + \Delta(\alpha^{-1})w \\
&= 1 \otimes \alpha^{-1}x + \alpha^{-1}x \otimes \alpha^{-1}\beta + \Delta(\alpha^{-1})w \text{ (By Equations (4.25) and (4.29)).}
\end{aligned}$$

Replace x, β and w by $\alpha^{-1}x$, $\alpha^{-1}\beta$ and $\Delta(\alpha^{-1})w$, resp.. Then we have that

$$\Delta(x) = 1 \otimes x + x \otimes \beta + w. \tag{4.30}$$

Using the fact that $(\Delta \otimes \mathrm{id}) \circ \Delta(x) = (\mathrm{id} \otimes \Delta) \circ \Delta(x)$ along with Equation (4.30), if we compare the coefficients of $x \otimes 1 \otimes 1$, then we obtain the equation: $\Delta(\beta) = \beta \otimes \beta$. Hence β is a grouplike element and thus has inverse. Notice that

$$\Delta(x\beta^{-1}) = \Delta(x)\Delta(\beta^{-1}) = \beta^{-1} \otimes x\beta^{-1} + x\beta^{-1} \otimes 1 + w\Delta(\beta^{-1}).$$

To get a simpler form of $S(x)$ later, one can replace x by $x\beta^{-1}$ and after a change of variables, we can assume that

$$\Delta(x) = \beta^{-1} \otimes x + x \otimes 1 + w. \tag{4.31}$$

Using the identity that $m \circ (\mathrm{id} \otimes S) \circ \Delta(x) = m \circ (S \otimes \mathrm{id}) \circ \Delta(x) = \epsilon(x)$ and Equation (4.31), a direct computation shows that $S(x) = -\beta(x + \sum w_1 S(w_2))$ and $\beta \sum w_1 S(w_2) = \sum S(w_1) w_2$. □

As a consequence, we have the following corollary.

Corollary 4.4.7. *Let R be a noetherian Hopf k-algebra and suppose that $T = R[x; \sigma, \delta]$ admits a Hopf algebra structure extending that of R. Suppose that $R \otimes_k R$ is a domain. Then after a change of variables for the variable x, we have $\Delta(x) = \beta^{-1} \otimes x + x \otimes 1 + w$, where β is a grouplike element in R and $w = \sum w_1 \otimes w_2 \in R \otimes_k R$ and thus condition (iii) in Definition 4.2.1 follows from conditions* (1) *and* (2)*. In particular, the Question 4.2.1 has an affirmative answer under the above hypotheses.*

This corollary allows us to immediately obtain our main result.

Theorem 4.4.8. *Let R be a noetherian Hopf k-algebra and let $T = R[x; \sigma, \delta]$ be an Ore extension of R. Suppose that $R \otimes_k R$ is a domain. Then T has a Hopf algebra structure extending that of R if and only if after a linear change of variables we have the following:*

1. *There exists a grouplike element β of R such that $\Delta(x) = \beta^{-1} \otimes x + x \otimes 1 + w$, and $S(x) = -\beta(x + \sum w_1 S(w_2))$ and $\beta \sum w_1 S(w_2) = \sum S(w_1) w_2$ with $w \in R \otimes_k R$;*

2. *There is a character $\chi : R \to k$ such that*
$$\sigma(r) = \tau_\chi^l(r) = \sum \chi(r_1) r_2 = \sum \beta^{-1} r_1 \beta \chi(r_2)$$
for all $r \in R$;

3. *the σ-derivation δ satisfies the relation*
$$\Delta\delta(r) - \sum \delta(r_1) \otimes r_2 - \sum \beta^{-1} r_1 \otimes \delta(r_2) - w\Delta(r) - \Delta\sigma(r) w = 0$$
and
$$w \otimes 1 + (\Delta \otimes I)(w) = \beta^{-1} \otimes w + (I \otimes \Delta) w.$$

Proof. Suppose that $R \otimes_k R$ is a domain. Let $T = R[x; \sigma, \delta]$ be a Hopf algebra with a Hopf structure extending that of the Hopf algebra R. Then we have (1) follows from Lemmas 4.4.5 and 4.4.6.

The maps Δ, ϵ and S of T must preserve the relation $xr = \sigma(r)x + \delta(r)$. In particular, we have the following equations:

$$\begin{aligned}
&\Delta(x)\Delta(r) = \Delta(\sigma(r))\Delta(x) + \Delta(\delta(r));\\
&\epsilon(x)\epsilon(r) = \epsilon(\sigma(r))\epsilon(x) + \epsilon(\delta(r));\\
&S(r)S(x) = S(x)S(\sigma(r)) + S(\delta(r)).
\end{aligned}$$

Using arguments from [40, Theorem 1.3] and [11, Theorem, §2.4], we obtain (2) and (3).

Conversely, a similar argument to that used in [40, Theorem 1.3] and [11, Theorem, §2.4] shows that (1), (2) and (3) imply that T is a Hopf algebra with R as a Hopf subalgebra. $\square$

4.5 Cocommutative Hopf algebras

In light of Theorem 4.4.8, it becomes natural to ask when $R \otimes_k R$ is a domain. Obviously, the hypothesis that $R \otimes_k R$ is a domain plays a significant role in last section. However, it appears to be difficult to show that $R \otimes_k R$ is a domain when R is a Hopf algebra that is a domain. Rowen and Saltman [42] exhibit division k-algebras E and F, both finite-dimensional over their centres and each containing an algebraically closed field k of characteristic 0, such that $E \otimes_k F$ not a domain. Their construction is non-trivial and it does not obviously lend itself to produce a counterexample in the Hopf algebra case. In this section, we shall show that $R \otimes_k R$ is a domain when k is algebraically closed of characteristic zero and R is a noetherian cocommutative Hopf algebra of finite Gelfand-Kirillov dimension that is a domain. In this case, one has that R is isomorphic to the smash product of the enveloping algebra of a finite-dimensional Lie algebra $\mathcal{L}$ and a finitely generated nilpotent-by-finite group. The underlying Lie algebra $\mathcal{L}$ is generated by the primitive elements in R, and the nilpotent-by-finite group is just the group of grouplike elements of R, which acts on $\mathcal{L}$ via k-algebra automorphisms, giving the smash product structure. To complete the proof of Theorem 4.5.2, we will need a result describing when crossed products are domains. The proof of the following theorem can be found in the book of Passman.

We recall that a domain R is called left (right, resp.) *Ore domain* if and only if $Rr_1 \cap Rr_2 \neq (0)$ (resp. $r_1R \cap r_2R \neq (0)$), for all non-zero elements $r_1, r_2 \in R$. The domain R is called an Ore domain if R is both a left and right Ore domain. It is well-known that a domain of finite GK-dimension is an Ore domain.

Theorem 4.5.1. [41, Corollary 37.11] *Let R be an Ore domain and let let G be a group and suppose that G has a finite subnormal series*

$$\{1\} = G_0 \lhd G_1 \lhd \cdots \lhd G_n = G$$

with each quotient G_{i+1}/G_i locally polycyclic-by-finite. If G is torsion-free then $R\#G$ is an Ore domain. In particular if R is an Ore domain and G is a torsion-free polycyclic-by-finite group then the smash product $R\#G$ is a domain.

Using this result, we can give the proof of the following theorem.

Theorem 4.5.2. *Let k be an algebraically closed field of characteristic zero and let R be a noetherian cocommutative k-Hopf algebra of finite Gelfand-Kirillov dimension that is a domain. Then $R \otimes_k R$ is a domain. In particular, the results of Theorem 4.4.8 apply in this setting.*

Proof. By a refinement of a result of Kostant (see Bell and Leung [5, Proposition 2.1]), we have that $R \cong U(\mathcal{L}_0)\#kH$ where $\mathcal{L}_0$ is a finite-dimensional Lie algebra over k and H is a finitely generated nilpotent-by-finite group that acts on $\mathcal{L}_0$. Hence, we have $R \otimes_k R = U(\mathcal{L}_0 \oplus \mathcal{L}_0)\#k[H \times H]$. Let $\mathcal{L}$ denote the Lie algebra $\mathcal{L}_0 \oplus \mathcal{L}_0$ and let G denote $H \times H$. Then $R \otimes_k R = U(\mathcal{L})\#kG$, where G acts on $U(\mathcal{L})$ in the natural way induced from the action of H on $\mathcal{L}_0$.

Since R is a domain, H is torsion-free, and thus G is also torsion-free. Moreover, G is also finitely generated and nilpotent-by-finite, since H is. Since $\mathcal{L}$ is finite-dimensional, we have that $U(\mathcal{L})$ is an Ore domain; moreover G is a torsion-free polycyclic-by-finite group, and so we see that $R \otimes_k R$ is a domain from Theorem 4.5.1. $\square$

As a immediate consequence of 4.5.2, we get the following corollary.

Corollary 4.5.3. [24, Corollary 3.3] *Let k be an algebraically closed field k of characteristic zero and let R be a noetherian cocommutative Hopf algebra of finite Gelfand-Kirillov dimension over k which is a domain. Let $T = R[x; \sigma, \delta]$ be an Ore extension over R. Then T has a Hopf algebra structure extending that of R if and only if after a change of variables we have the following:*

1. *there exists a grouplike element* β *of* R *such that* $\Delta(x) = \beta^{-1} \otimes x + x \otimes 1 + w$, *and* $S(x) = -\beta(x + \sum w_1 S(w_2))$ *and* $\beta \sum w_1 S(w_2) = \sum S(w_1) w_2$ *with* $w \in R \otimes_k R$;

2. *there is a character* $\chi : R \to k$ *such that*

$$\sigma(r) = \tau_\chi^l(r) = \sum \chi(r_1) r_2 = \sum \beta^{-1} r_1 \beta \chi(r_2);$$

   for all $r \in R$.

3. *the* σ*-derivation* δ *satisfies the relation*

$$\Delta\delta(r) - \sum \delta(r_1) \otimes r_2 - \sum \beta^{-1} r_1 \otimes \delta(r_2) - w\Delta(r) - \Delta\sigma(r) w = 0$$

   and

$$w \otimes 1 + (\Delta \otimes I)(w) = \beta^{-1} \otimes w + (I \otimes \Delta) w.$$

Proof. Theorem 4.5.2 tells us that in this case $R \otimes_k R$ is a domain. Then the assertion immediately follows from Theorem 4.4.8. $\square$

Chapter 5

Zariski Cancellation Problems

Throughout this chapter, when A is a finitely generated k-algebra, we will simply say that A is an *affine algebra over* k, or simply an *affine algebra* when the base field is understood; we shall also let $Z(A)$ denote the centre of an algebra A. The goal of this chapter is to look at noncommutative analogues of the result of Abhyankar, Eakin, and Heinzer [1, Theorem 3.3 and Corollary 3.4]. Their theorem, when one works in the category of commutative algebras, says that if A is a finitely generated algebra that is an integral domain of Krull dimension one, then A is cancellative (see Definition 2.3.1). We consider a noncommutative analogue of this theorem, in which one considers finitely generated domains of Gelfand-Kirillov dimension one. When working with noncommutative algebras, it is generally preferable to work with Gelfand-Kirillov dimension rather than with the classical Krull dimension. The main result is as follows.

Theorem 5.0.1. *We have the following results for affine domains of Gelfand-Kirillov dimension one.*

1. *Let k be a field of characteristic zero and let A be an affine domain over k of Gelfand-Kirillov dimension one. Then A is cancellative.*

2. *Let p be prime. Then there exists a field k of characteristic p and an affine domain A over k of Gelfand-Kirillov dimension one that is not cancellative.*

In this section 5.1, we state some known results which are relevant to Theorem 5.0.1. Then we show some propositions which will be used in the proof of Theorem 5.0.1 in section 5.2. In addition, we prove a general result that suggests over "nice" base fields that cancellation should be controlled by the centre (see Proposition 5.2.7, Corollary 5.2.9, and Conjecture 5.2.10). In section 5.3, we prove Theorem 5.0.1 (a) and prove some positive results for domains of Gelfand-Kirillov dimension one over positive characteristic base fields. In Section 5.4, we construct the family of examples needed to establish Theorem 5.0.1 (b).

5.1 Some Known Results

In general, there are rings R for which the implication given in Question 2.3.1 does not hold. In fact, Danielewski [14] gave the following two families of examples of affine complex varieties as counterexamples.

1. Let $n \geq 1$ and let B_n be the coordinate ring of the surface $x^n y = z^2 - 1$ over $\mathbb{C}$. Then $B_i \neq B_j$ if $i \neq j$, but $B_i[s] \cong B_j[t]$ for all $i, j \geq 1$. Therefore, all the B_n 's are not cancellative.

2. $\mathbb{C}[x, y, z]/(p)$ is not isomorphic to $\mathbb{C}[x, y, z]/(q)$, while $(\mathbb{C}[x, y, z]/(p))[t]$ is isomorphic to $(\mathbb{C}[x, y, z]/(q))[t]$, where $p = xy - z^2 + 1; q = q(x, y, z) = x^2 y - z^2 + 1$.

It should be noted, however, that these non-cancellative examples all have dimension at least two, and if we restrict our attention to curves, cancellation holds: this is a result of Abhyankar, Eakin, and Heinzer [1] described in the theorem below.

Theorem 5.1.1. [1, Theorem 3.3] *Let A be an integral domain of transcendence degree one over a subfield k. Suppose that $A[x_1, \ldots, x_n] \cong B[y_1, \ldots, y_n]$ for some $n \geq 1$, and let k' denote the algebraic closure of k in A. If $A \ncong B$, then A and B are both polynomial rings over the field k'. Consequently, if A is not a polynomial ring, then A is strongly cancellative.*

Since Krull dimension and Gelfand-Kirillov dimension coincide for finitely generated commutative k-algebras, Theorem 5.0.1 specializes to the classical cancellation result of Abhyankar-Eakin-Heinzer in the case of characteristic zero base fields when one takes R to be commutative.

Part (2) of Theorem 5.0.1 gives a counterexample to the conjecture below when the base field has positive characteristic:

Conjecture 5.1.2. [50, Conjecture 0.3(1)] *Let A be a noetherian finitely generated prime algebra. If $Z(A)$ has Gelfand-Kirillov dimension less than or equal to one, then A is cancellative.*

While Theorem 5.0.1 (a) answers the following question of Lezama, Wang, and Zhang [31] in the case when the base field has characteristic zero in the domain case.

Question 5.1.3. [31, Question 0.5] Is every affine prime k-algebra of Gelfand-Kirillov dimension one cancellative?

We note that [1] show in fact prime affine commutative algebras of Gelfand-Kirillov dimension one are strongly cancellative and we do not know whether this conclusion holds in characteristic zero for Theorem 5.0.1 (a). We also point out that Lezama, Wang, and Zhang proved that for algebraically closed base fields k, affine prime k-algebras of Gelfand-Kirillov dimension one are cancellative in the theorem below:

Theorem 5.1.4. [31, Theorem 0.6] *Let k be algebraically closed. Then every affine prime k-algebra of Gelfand-Kirillov dimension one is cancellative.*

Notice that the algebraically closed property is needed, because the authors invoke Tsen's theorem at one point in their proof. Our example in section 5.4 shows that this application of Tsen's theorem is in some sense necessary to get their result in positive characteristic.

In characteristic zero, our Theorem 5.0.1 (a) is somewhat orthogonal to the result of [31], since domains of Gelfand-Kirillov dimension one over algebraically closed fields are commutative by an application of Tsen's theorem to a result of Small and Warfield [46] and hence the only part of Theorem 5.0.1 (a) covered by Theorem 5.1.4 is the commutative case, which was previously known from the result of Abhyankar-Eakin-Heinzer 5.1.1.

5.2 Preparation for the proof of theorem 5.0.1

To prove Theorem 5.0.1, we will need several propositions and lemmas. We will show a similar result to the lemma below in the case of left Goldie rings. In addition, we prove Proposition 5.2.7 and Corollary 5.2.9, which give further underpinning to the idea that the centre of an algebra plays a large role in whether the cancellation property holds for that algebra.

Lemma 5.2.1. [7, Lemma 3.2] *Let* $Y := \bigoplus_{i=0}^{\infty} Y_i$ *be an* $\mathbb{N}$*-graded domain. If* Z *is a subalgebra of* Y *containing* Y_0 *such that* $\mathrm{GKdim}(Z) = \mathrm{GKdim}(Y_0) < \infty$*, then* $Z = Y_0$.

We begin by proving a lemma, which is the counterpart of Lemma 5.2.1.

Lemma 5.2.2. *Let* $Y := \bigoplus_{i=0}^{\infty} Y_i$ *be an* $\mathbb{N}$*-graded* k*-algebra and suppose that* $Y_0 y Y_0$ *contains a regular element whenever* y *is a nonzero homogeneous element of* Y*. If* Z *is a subalgebra of* Y *containing* Y_0 *such that* $\mathrm{GKdim}(Z) = \mathrm{GKdim}(Y_0) < \infty$*, then* $Z = Y_0$.

Proof. Suppose that Z strictly contains Y_0 as a subalgebra. Since Y is a graded algebra, Z is an $\mathbb{N}$-filtered algebra with $X := F_0 Z$. By [29, Lemma 6.5], $\mathrm{GKdim}(Z) \geq \mathrm{GKdim}(\mathrm{gr}(Z))$, where $\mathrm{gr}(Z)$ is the associated graded ring of Z with respect to the filtration induced by the $\mathbb{N}$-grading on Y. Then $\mathrm{gr}(Z)$ is an $\mathbb{N}$-graded subalgebra of Y that strictly contains its degree 0 part, which is Y_0, and so $\mathrm{gr}(Z)$ contains some nonzero homogeneous element $y \in Y_d$ for some $d \geq 1$. Then it contains the Y_0-Y_0-bimodule $Y_0 y Y_0 \subseteq Y_d$. In particular, there is some regular homogeneous element $a \in Z$ of positive degree and so by considering the grading we have

$$Y_0 + Y_0 a + \cdots$$

is direct and is contained in $\mathrm{gr}(Z)$. From this one can easily show that

$$\mathrm{GKdim}(\mathrm{gr}(Z)) \geq \mathrm{GKdim}((\mathrm{gr}(Z))_0) + 1 \geq \mathrm{GKdim}(Y_0) + 1.$$

Combining these inequalities gives

$$\mathrm{GKdim}(Z) \geq \mathrm{GKdim}(Y_0) + 1,$$

a contradiction. Thus $Z = Y_0$. $\square$

We will use Lemma 5.2.2 in the case when A is a prime left Goldie algebra and $Y = A[t_1, \ldots, t_d]$, where we declare that elements of A have degree 0, and $t_1, \ldots, t_d$ are homogeneous of degree 1. Observe that if $p(t_1, \ldots, t_d)$ is a nonzero homogeneous polynomial of degree m in Y, then we can put a degree lexicographic order on the monomials in $t_1, \ldots, t_d$ by declaring that $t_1 > t_2 > \cdots > t_d$. Then we let $t_1^{i_1} \cdots t_d^{i_d}$ denote the degree lexicographically largest monomial that occurs in $p(t_1, \ldots, t_d)$ with nonzero coefficient and we let $a \in A$ denote this coefficient. Then since $A = Y_0$ is prime Goldie, $Y_0 a Y_0$ contains a regular element, and so $Y_0 p(t_1, \ldots, t_d) Y_0$ contains a nonzero homogeneous polynomial $q = q(t_1, \ldots, t_d)$ such that the degree lexicographically largest monomial that occurs in $q(t_1, \ldots, t_d)$ with nonzero coefficient has the property that this coefficient is regular; moreover, this monomial is again $t_1^{i_1} \cdots t_d^{i_d}$, and we let $c \in A$ denote this coefficient. We now claim that q must be regular. To see this, let h be a nonzero polynomial in Y. Then let $t_1^{j_1} \cdots t_d^{j_d}$ denote the degree lexicographically largest monomial that occurs in h with nonzero coefficient, and let $b \in A$ denote this coefficient. Then by construction the coefficient of $t_1^{i_1+j_1} \cdots t_d^{i_d+j_d}$ in qh is cb and since b is nonzero and c is regular, $qh \neq 0$; similarly, $hq \neq 0$ and so q is regular. In particular, Y satisfies the hypotheses of Lemma 5.2.2, in this case, which we will now apply in the following proposition.

Proposition 5.2.3. *Let A be a finitely generated prime left Goldie k-algebra of finite Gelfand-Kirillov dimension. Let $*$ be either blank, H, H' or I. When $*$ is blank we further assume k has characteristic zero.*

1. *If $\mathrm{ML}^*(A[t]) = A$, then A is retractable and so is cancellative.*
2. *If $\mathrm{ML}^*(A[t_1, \ldots, t_n]) = A$, then A is strongly-retractable and so is strongly cancellative.*
3. ([31, Lemma 2.6]) *Suppose $Z(A)$ is affine and $\mathrm{ML}_Z(A[t]) = Z(A)$ or $\mathrm{ML}_Z^H(A[t]) = Z(A)$ or $\mathrm{ML}_Z^{H'}(A[t]) = Z(A)$. Then A is Z-retractable.*
4. ([31, Lemma 2.6]) *Suppose $Z(A)$ is affine and A is strongly LND_Z^*-rigid where $*$ is either blank, H, or H'. Then A is strongly Z-retractable.*

Proof. The proof is identical to the proof given in [7, Theorem 3.3], with the one exception being that we invoke Lemma 5.2.2 with $Y = A[t_1, \ldots, t_d]$ (with elements of A having degree 0 and $t_1, \ldots, t_d$ having degree 1) as a replacement for [7, Lemma 3.2] used in [7, Lemma 3.3]. We point out that H' is not used in [31], but the argument in the H' case goes through in the same manner as it does for H. □

In Proposition 5.2.7 below, we give a result that is related to a conjecture of Makar-Limanov [33, p. 55], which has interesting implications in terms of cancellation. To prove this result, we need to invoke a result of [31] that requires that the algebras involved be strongly Hopfian. An algebra A is *strongly Hopfian* if whenever $d \geq 1$ and ϕ is a surjective endomorphism of $A[t_1, \ldots, t_d]$, ϕ is necessarily injective. The following proposition will be useful for proving the Hopfian property for certain algebras.

Proposition 5.2.4. [29, Proposition 3.15]) *Let I be an ideal of a k -algebra A, and assume that I contains a right regular element or a left regular element of A. Then*

$$\mathrm{GKdim}(A/I) + 1 \leq \mathrm{GKdim}(A).$$

By the above proposition, we can conclude the following corollary.

Corollary 5.2.5. *Let A be a prime Goldie algebra of finite Gelfand-Kirillov dimension. Then A is strongly Hopfian.*

Proof. To see this, we can assume that $A[t_1, \cdots, t_d]$ has epimorphism ϕ that is not injective with $d \geq 1$. Hence, $A[t_1, \cdots, t_d] \cong A[t_1, \cdots, t_d]/ker(\phi)$. Since A is prime Goldie, $A[t_1, \cdots, t_d]$ is prime Goldie and so $I := ker(\phi)$ has a a right regular element. It follows from Proposition 5.2.4 that $\mathrm{GKdim}(A[t_1, \cdots, t_d]/I) + 1 \leq \mathrm{GKdim}(A[t_1, \cdots, t_d])$. This contradicts that $A[t_1, \cdots, t_d] \cong A[t_1, \cdots, t_d]/ker(\phi)$. $\square$

We first need a basic result about vanishing of polynomials in noncommutative rings.

Remark 5.2.6. Let A be a prime ring and let $p(x) \in A[x]$ be a nonzero polynomial of degree d. If there are $d+1$ distinct central elements $z \in A$ such that $p(z) = 0$ then $p(x)$ is the zero polynomial.

Proof. Write $p(x) = a_0 + a_1 x + \cdots + a_d x^d$. Let Z denote the centre of A, which is an integral domain since A is prime. Suppose that there exist distinct $z_1, \ldots, z_{d+1} \in Z$ such that $p(z_i) = 0$ for $i = 1, \ldots, d+1$. Let M be the $(d+1) \times (d+1)$ matrix whose (i, j)-entry is z_j^{i-1}. Then considering A^{d+1} as a right $M_{d+1}(Z)$-module, we see $[a_0, a_1, \ldots, a_d]M = 0$. Then right-multiplying by the classical adjoint of M we obtain $a_i \det(M) = 0$ for $i = 0, \ldots, d$. Then M is a Vandermonde matrix and since Z is an integral domain and the z_i are pairwise distinct, $\det(M)$ is a nonzero central element of A, and hence is regular since A is prime. It follows that $a_0 = \cdots = a_d = 0$ and $p(x)$ is the zero polynomial. $\square$

Proposition 5.2.7. *Let A be a prime finitely generated k-algebra with infinite centre. Then $\mathrm{ML}^{H'}(A) = \mathrm{ML}^{H'}(A[x_1, x_2, \cdots, x_d])$. In particular, if, in addition, A is left Goldie, has finite Gelfand-Kirillov dimension, $Z(A)$ is affine, and either $\mathrm{ML}_Z^{H'}(A) = Z(A)$ or $\mathrm{ML}^{H'}(A) = A$, then A is strongly cancellative.*

Proof. Remark 2.3.2(5) gives that $\mathrm{ML}^{H'}(A[x_1, x_2, \ldots, x_d]) \subseteq \mathrm{ML}^{H'}(A) \subseteq A$ for all $d \geq 1$. It thus suffices to show that $\mathrm{ML}^{H'}(A) \subseteq \mathrm{ML}^{H'}(A[x_1, x_2, \ldots, x_d])$.

We show that $\mathrm{ML}^{H'}(A) \subseteq \mathrm{ML}^{H'}(A[x])$. Once we have proved this, it will immediately follow by induction that $\mathrm{ML}^{H'}(A) \subseteq \mathrm{ML}^{H'}(A[x_1, \ldots, x_d])$ and we will obtain the result. Let $\partial := \{\partial_n\}_{n\geq 0}$ be an element of $\mathrm{LND}^{H'}(A[x])$. As in Equation (2.13), we have an induced k-algebra homomorphism $\phi : A[t] \longrightarrow A[x][t]$, given by

$$\phi(a) = \sum_{n\geq 0} \partial_n(a) t^n \quad \text{for } a \in A, \quad \phi(t) = t.$$

In particular, if $a \in \mathrm{ML}^{H'}(A)$, then $\phi(a) = a + tp(x, t)$, for some polynomial $p(x, t) \in A[x, t]$. We now fix $z \in Z(A)$ and consider the map $e_z : A[x, t] \to A[t]$, defined by $e_z(g(x, t)) = g(z, t)$. Then the composition $\phi_z := e_z \circ \phi$ gives a homomorphism from $A[t]$ to $A[t]$ and by construction $\phi_z(a) \equiv a \pmod{(t)}$, and $\phi(t) = t$ and so this homomorphism is injective. Thus there are maps $\mu_j : A \to A$ with $\mu_0 = \mathrm{id}_A$ such that $\phi_z(a) = \sum_{j\geq 0} \mu_j(a) t^j$ for $a \in A$. In particular for $a \in A$, $\mu_n(a) = 0$ for n sufficiently large, and so (μ_n) is a weakly locally nilpotent Hasse-Schmidt derivation of A [7, Lemma 2.2(3)]. Thus for $a \in \mathrm{ML}^{H'}(A)$ we have $\mu_i(a) = 0$ for every $i \geq 1$; that is, for $i \geq 1$, $\partial_i(a)|_{x=z} = 0$ for every $z \in Z(A)$. Since $Z(A)$ is infinite and $\partial_n(a)$ is a polynomial in $A[x]$, Remark 5.2.6 gives that $\partial_n(a) = 0$ for $n \geq 1$ and hence $a \in \mathrm{ML}^{H'}(A[x])$. Thus $\mathrm{ML}^{H'}(A) \subseteq \mathrm{ML}^{H'}(A[x])$ as required.

Now suppose that $Z(A)$ is infinite and affine and that A is prime left Goldie and has finite Gelfand-Kirillov dimension. It follows that if $\mathrm{ML}_Z^{H'}(A) = Z(A)$ then from the above $\mathrm{ML}_Z^{H'}(A[x_1, \ldots, x_d]) = Z(A)$ and so A is strongly $\mathrm{LND}_Z^{H'}$-rigid and hence strongly Z-retractable by Proposition 5.2.3. Thus by [31, Lemma 3.2], A is strongly detectable, and so is strongly cancellative [31, Lemma 3.6], since A is strongly Hopfian by Corollary 5.2.5. On the other hand if $\mathrm{ML}^{H'}(A) = A$ then A is strongly $\mathrm{LND}^{H'}$-rigid and so by Proposition 5.2.3, A is strongly cancellative. $\square$

In analogy with terminology from algebraic geometry, given an algebraically closed field k and a finitely generated extension F of k, we will say that F is *uniruled* over k if there is a finitely generated field extension E of k with $\mathrm{trdeg}_k(E) = \mathrm{trdeg}_k(F) - 1$ and an injective k-algebra homomorphism $F \to E(t)$. The idea here is that F is the function field of a normal projective scheme X of finite type over k. Then the condition $F \subseteq E(t)$ says that there is a dominant rational map $Y \times \mathbb{P}^1 \dashrightarrow X$ for some variety Y with $\dim(Y) = \dim(X) - 1$. Similarly, if F is the function field of a normal projective scheme X of finite type over k, we define the *Kodaira dimension* of F to be the Kodaira dimension of X. Since Kodaira dimension is a birational invariant, this is well-defined. We refer the reader to Hartshorne [23] for background in algebraic geometry and on Kodaira dimension. If k has characteristic zero, a uniruled variety has Kodaira dimension $-\infty$ and the converse holds in dimensions one, two, and three; the main conjectures of the minimal model program imply that the converse should hold in higher dimensions, too.

Over uncountable fields, affine uniruled varieties have a pleasant characterization in terms of being covered by rational affine lines (see [27, 48]). We recall that if k is an algebraically closed field, then a *rational curve* over k is a curve that can be parametrized by rational functions in $k(x)$.

Proposition 5.2.8. *Let k be an uncountable algebraically closed field and let X be an irreducible affine variety over k of dimension at least one. Then following conditions are equivalent:*

1. *for every $x \in X$ there is a rational polynomial affine curve Y_x in X that passes through x;*

2. *there is a Zariski-dense open subset U of X, such that for every $x \in U$ there is a polynomial affine curve Y_x in X that passes through x;*

3. *X is uniruled; that is, there exists an affine variety Y with $\dim(Y) = \dim(X) - 1$ and a dominant morphism $Y \times \mathbb{A}^1 \to X$.*

Corollary 5.2.9. *Let k be an uncountable algebraically closed field, let A be a finitely generated prime left Goldie k-algebra of finite Gelfand-Kirillov dimension, and suppose that $Z(A)$ is affine. If A does not possess the strong cancellation property then $\mathrm{Frac}(Z(A))$ is uniruled. In particular, if k has characteristic zero and $\mathrm{Frac}(Z(A))$ has nonnegative Kodaira dimension then A is strongly cancellative.*

Proof. We claim that if there exists a weakly locally nilpotent Hasse-Schmidt derivation $\partial = \{\partial_n\}_{n\geq 0}$ such that $Z(A) \not\subseteq \mathrm{Ker}(\partial)$ then the field of fractions of $Z(A)$ is necessarily uniruled. To see this, suppose that $\partial = (\partial_n)_{n\geq 0}$ is a non-trivial weakly locally nilpotent Hasse-Schmidt derivation of $Z(A)$. Then by Remark 2.3.2 we have an injective $G_\partial : Z(A)[t] \to Z(A)[t]$ that sends t to t, and by assumption G_∂ is not the identity on $Z(A)$. Let $X = \mathrm{Spec}(Z(A))$, which is an affine scheme of finite type over k. Then the induced k-algebra homomorphism

$$Z(A) \to Z(A) \otimes_k k[t] = Z(A)[t]$$

from G_∂ yields a morphism $\Phi : \mathbb{A}^1 \times X \to X$ with $\Phi(0, x) = x$ for $x \in X$, and since $\partial = \{\partial_n\}_{n\geq 0}$ is non-trivial, there is some $x \in X$ such that $\Phi(\mathbb{A}^1 \times \{x\})$ is not a point. We claim there is a Zariski dense open set $U \subseteq X$ such that for $x \in U$ we have $\Phi(\mathbb{A}^1 \times \{x\}) = Y_x \subseteq X$ with Y_x birationally isomorphic to $\mathbb{P}^1$. To see this, notice that for each $x \in X$, Φ gives a map from $\mathbb{A}^1 \to Y_x$. Since $\mathbb{A}^1$ is one-dimensional and irreducible, Y_x is either a point or an irreducible curve. Moreover, if Y_x is a curve, then we have a dominant rational map $\mathbb{P}^1 \dashrightarrow Y_x$ and so Y_x is birationally isomorphic to $\mathbb{P}^1$ by Lüroth's theorem. So now let V denote the set of $x \in X$ such that Y_x is a point. Now there is at least one point $x \in X$

such that Y_x is infinite, so pick $p, q \in \mathbb{A}^1$ and $x_0 \in X$ such that $\Phi(p, x_0) \neq \Phi(q, x_0)$. Then $\Psi : X \to X \times X$ given by $x \mapsto (\Phi(p, x), \Phi(q, x))$ is a morphism and since the diagonal Δ is closed in $X \times X$, $Y := \Psi^{-1}(\Delta)$ is a closed subvariety of X and by assumption $x_0 \notin Y$ and so Y is proper. Thus $U := X \setminus Y$ has the property that $\Phi(p, x) \neq \Phi(q, x)$ for $x \in U$ and so Y_x is necessarily a rational curve for $x \in U$. Thus X has an open dense subset such that each k-point in U is covered by rational curves and so X is uniruled by Proposition 5.2.8. In particular, there is a dominant rational map from a variety of the form $Y \times \mathbb{P}^1$ to X, where $\dim(Y) = \dim(X) - 1$. Thus $\mathrm{Frac}(A) = k(X) \hookrightarrow k(Y \times \mathbb{P}^1) \cong k(Y)(t)$ and so F =Frac$(Z(A))$ is uniruled. Thus if F is not uniruled then $\mathrm{ML}^{H'}(Z(A)) = Z(A)$ and since an element of $\mathrm{LND}^{H'}(A)$ restricts to an element of $\mathrm{LND}^{H'}(Z(A))$, $\mathrm{ML}_Z^{H'}(A) = Z(A)$ and so A is strongly cancellative by Proposition 5.2.7. □

The above result shows under certain conditions that if the centre of an algebra is sufficiently "rigid" then the algebra is strongly cancellative. We conjecture that over "nice" base fields the centre completely determines cancellation. We make this precise with the following conjecture.

Conjecture 5.2.10. *Let k be an uncountable algebraically closed field of characteristic zero and let A be an affine noetherian domain over k. Suppose that $Z(A)$ is affine and cancellative (resp. strongly cancellative). Then A is cancellative (resp. strongly cancellative).*

5.3 Main Results

The following slice theorem is a powerful tool when working on the Zariski cancellation problem. Let A be a k-algebra with a derivation δ. Any element $x \in A$ with $\delta(x) = 1$ is a *slice* for A.

Theorem 5.3.1. [16, Theorem 1.26] *Suppose B is an integral domain and $\delta \in \mathrm{LND}(B)$ admits a slice $x \in B$, and let $A = \ker(\delta)$. Then:*

1. *$B = A[x]$ and $\delta = \frac{d}{dx}$*

2. *If B is affine, then A is affine.*

We shall prove a noncommutative analogue of this result. We make use of the fact that for a k-algebra A with a locally nilpotent derivation δ, the map δ restricts to a locally nilpotent derivation of the centre of A.

This following lemma is an extension of the slice theorem for a (not necessarily commutative) prime affine k-algebra.

Lemma 5.3.2. *(Noncommutative slice theorem) Let k be a field and let A be a k-algebra. Then the following statements hold.*

1. *Suppose that the characteristic of k is zero and $\delta \in \mathrm{LND}(A)$. If there exists $x \in Z(A)$ such that $\delta(x) = 1$, and if A_0 is the kernel of δ, then the sum $\sum_{i \geq 0} A_0 x^i$ is direct and $A = A_0[x]$.*

2. *Suppose that $\partial = \{\partial_n\}_{n \geq 0} \in \mathrm{LND}^I(A)$. If there exists $x \in Z(A)$ such that $\partial_1(x) = 1$ and $\partial_n(x) = 0$ for $n \geq 2$, and if A_0 is the kernel of ∂, then the sum $\sum_{i \geq 0} A_0 x^i$ is direct and $A = A_0[x]$.*

Before giving the proof of this result, we first make a basic remark.

Remark 5.3.3. Let k be a field, let A be a prime k-algebra, let $\partial = (\partial_n)_{n \geq 0} \in \mathrm{LND}^I(A)$, and let $B = \ker(\partial)$. Then the following hold:

1. if there is $x \in A$ and $m \geq 1$ such that $\partial_{m+i}(x) = 0$ for $i \geq 1$ and $\partial_m(x)$ is a regular element of A, then the sum

$$B + Bx + Bx^2 + \cdots$$

is direct;

2. if A is a field and A is algebraic over B then $A = B$;

3. if $\mathrm{GKdim}(A) < \infty$ and A is an affine domain and $B \neq A$ then $\mathrm{GKdim}(B) \leq \mathrm{GKdim}(A) - 1$;

4. if $\mathrm{GKdim}(A) = 1$ and A is an affine domain and $B \neq A$ then B is finite-dimensional.

Proof. Suppose there exist $x \in A$ and $m \geq 1$ such that $\partial_{m+i}(x) = 0$ for $i \geq 1$ and $\partial_m(x)$ is a regular element of A. By induction, we will show that $\partial_n(x^i) = 0$ for $im < n$ and $\partial_{ms}(x^s) = \partial_m(x)^s$. Suppose that $i = 2$ and $n > 2m$. Then

$$\begin{aligned}
\partial_n(x^2) &= \sum_{j=1}^{n} \partial_j(x)\partial_{n-j}(x) \\
&= \partial_1(x)\partial_{n-1}(x) + \cdots + \partial_m(x)\partial_{n-m}(x) \text{ (as } \partial_{m+i}(x) = 0 \text{ for } i \geq 1) \\
&= 0 \text{ (as } n-1 \geq \cdots \geq n-m > m), \\
\partial_{2m}(x^2) &= \sum_{j=1}^{2m} \partial_j(x)\partial_{2m-j}(x) \\
&= \partial_m(x)^2.
\end{aligned}$$

We assume that if $i = k$ and $km < n$, then $\partial_n(x^k) = 0$ and $\partial_{km}(x^k) = \partial_m(x)^k$. Then

$$\begin{aligned}
\partial_{n+m}(x^{k+1}) &= \sum_{j=1}^{n+m} \partial_j(x)\partial_{n+m-j}(x^k) \\
&= \partial_1(x)\partial_{n+m-1}(x^k) + \cdots + \partial_m(x)\partial_n(x^k) \\
&= 0, \\
\partial_{(k+1)m}(x^{k+1}) &= \sum_{j=1}^{(k+1)m} \partial_i(x)\partial_{(k+1)m-i}(x^k) \\
&= \partial_1(x)\partial_{(k+1)m-1}(x^k) + \cdots + \partial_m(x)\partial_{km}(x^k) \\
&= \partial_m(x)\partial_{km}(x^k) \\
&= \partial_m(x)^{k+1}.
\end{aligned}$$

Suppose that there is a non-trivial relation $b_0+b_1x+\cdots+b_sx^s=0$ with $b_0,\ldots,b_s\in B$ and b_s nonzero. Then applying ∂_{ms} to this dependence gives $b_s\partial_m(x)^s=0$, which is impossible as $b_s\neq 0$ and $\partial_m(x)$ is regular. This establishes (1). Next, to prove (2), observe that if A is a field, then B is a subfield of A. We have just shown that for $x\in A\setminus B$, the sum $B+Bx+\cdots$ is direct, and so if A is algebraic over B then we must have $A=B$.

We next prove (3). Suppose that A is a domain of finite Gelfand-Kirillov dimension and that $B\neq A$ and that

$$\mathrm{GKdim}(B)>\mathrm{GKdim}(A)-1.$$

Then there exists $\alpha>\mathrm{GKdim}(A)-1$ and a finite-dimensional k-vector subspace W of B that contains 1 and such that $\dim(W^n)\geq n^\alpha$ for n sufficiently large. Pick $x\in A\setminus B$. Then by (1), $B+Bx+Bx^2+\cdots$ is direct. Now let $V=W+kx$. Then $V^{2n}\supseteq W^n+W^nx+\cdots+W^nx^n$ and so $\dim(V^{2n})\geq(n+1)n^\alpha\geq n^{\alpha+1}$. Thus $\mathrm{GKdim}(A)\geq\alpha+1$, a contradiction. Thus we obtain (3).

We now prove (4). Suppose that A is an affine domain of Gelfand-Kirillov dimension one and that $B\neq A$. We claim that $\dim_k(B)<\infty$. Pick $z\in A\setminus B$. By part (1), the sum $B+Bz+Bz^2+\cdots$ is direct. Now suppose towards a contradiction that $\dim_k(B)$ is infinite and let V be a finite-dimensional subspace of A that contains 1 and z and which generates A as a k-algebra. Then since $\bigcup_{i\geq 0}V^i\supseteq B$, we have $W_n:=V^n\cap B$ has the property that $\dim(W_n)\to\infty$ as $n\to\infty$. Since A has Gelfand-Kirillov dimension one, by a result of Bergman (see the proof of [29, Theorem 2.5]) there is some positive constant C such that $\dim(V^n)\leq Cn$ for n sufficiently large. On the other hand, for each $d\geq 1$ we have

$$\dim(V^{n+d})\geq\dim(W_dV^n)\geq\dim(W_d+W_dz+\cdots+W_dz^n)=\dim(W_d)(n+1).$$

Thus $\dim(W_d)\leq C(n+d)/(n+1)$ for all n sufficiently large and so $\dim(W_d)\leq C$ for every $d\geq 1$, a contradiction. Thus B is finite-dimensional. □

Proof of Lemma 5.3.2. It suffices to prove part (2) by Remark 2.3.2. We let

$$A_0=\{a\in A\mid \partial_n(a)=0 \text{ for } n\geq 1\}.$$

We claim that $A = A_0[x]$. By Remark 5.3.3, $\sum A_0 x^i$ is direct. Thus A_0 and x generate a polynomial ring and $A \supseteq A_0[x]$. We next claim that $A \subseteq A_0[x]$. To see this, suppose that there exists some $a \in A \setminus A_0[x]$. Then there is some largest $m \geq 1$ such that $\partial_m(a) \neq 0$. Among all $a \in A \setminus A_0[x]$, we choose one with this m minimal. Since $\partial_i(\partial_m(a)) = \binom{i+m}{m}\partial_{i+m}(a) = 0$ for $i \geq 1$, $\partial_m(a)$ is in the kernel of ∂ and hence in A_0. Let $c = \partial_m(a) \in A_0$ and consider $a' = a - cx^m$. Observe that $\partial_j(a') = 0$ for $j > m$ and $\partial_m(a') = 0$ by construction. Thus by minimality of m, $a' \in A_0[x]$ and hence so is a, a contradiction. The result follows. $\square$

Proposition 5.3.4. *Let k be a field of characteristic zero and let A be a prime finitely generated k-algebra of finite Gelfand-Kirillov dimension, and suppose that $Z(A)$ is an affine domain of Gelfand-Kirillov dimension at most 1. Then one of the following alternatives must hold:*

1. $\mathrm{ML}_Z(A) = Z(A)$*; or*

2. *there is a prime k-subalgebra A_0 of A such that $A \cong A_0[t]$.*

Proof. If $\mathrm{ML}_Z(A) \neq Z(A)$, then there is some $\delta \in \mathrm{LND}(A)$ and some $z \in Z(A)$ such that $\delta(z) \neq 0$. We now pick the largest j such that $\delta^j(z) \neq 0$ and we replace z by $\delta^{j-1}(z)$. By construction, $\delta^i(z) = 0$ for $i \geq 2$ and $c := \delta(z) \neq 0$. Then $c \in A_0 \cap Z(A)$, where A_0 is defined in earlier lemmas. Now $A_0 \cap Z(A)$ is a subalgebra of $Z(A)$ and since $Z(A)$ has Krull dimension one and $A_0 \cap Z(A) \subsetneq Z(A)$, $A_0 \cap Z(A)$ is finite-dimensional by Remark 5.3.3 and thus is a field. Thus c is a unit and so if we replace z by $c^{-1}z$ then we have $\delta(z) = 1$ and we may invoke Lemma 5.3.2 to get that $A \cong A_0[t]$. Since A is prime, A_0 is necessarily prime too. $\square$

In general, the proof of Proposition 5.3.4 shows that if A is an affine prime k-algebra of finite Gelfand-Kirillov dimension over a field k of characteristic zero, then either $\mathrm{ML}_Z(A) = Z(A)$ or there is a prime subalgebra A_0 of A and some $c \in Z(A) \cap A_0$ such that $A[c^{-1}] \cong$

$A_0[c^{-1}][t]$. In the case, that $Z(A)$ is affine of Gelfand-Kirillov dimension one we are able to deduce that c is invertible in the proof, which gives us part (2) in the dichotomy occurring in Proposition 5.3.4.

It has been shown that if A is a prime finitely generated algebra of Gelfand-Kirillov dimension one then A is noetherian, satisfies a polynomial identity, and is a finitely generated module over its center [46]. We will use the above fact to prove the following theorem.

Theorem 5.3.5. *We have the following results for affine domains of Gelfand-Kirillov dimension one.*

1. *Let k be a field of characteristic zero and let A be an affine domain over k of Gelfand-Kirillov dimension one. Then A is cancellative.*

2. *Let p be prime. Then there exists a field k of characteristic p and an affine domain A of Gelfand-Kirillov dimension one that is not cancellative.*

Proof of Theorem 5.3.5 (1). (1) We recall that affine prime algebras of Gelfand Kirillov dimension one are noetherian and hence left Goldie [46]. If $\mathrm{ML}(A) = A$ then $\mathrm{ML}(A[x]) = \mathrm{ML}(A)$ by [7, Lemma 3.5] and so A is cancellative by Proposition 5.2.3. If on the other hand, $\mathrm{ML}(A) \neq A$, then there is a nonzero locally nilpotent derivation δ of A. Let A_0 denote the kernel of δ. Then by Remark 5.3.3 A_0 is finite-dimensional and since it is a domain, it is a division ring.

In particular, $Z(A) \not\subseteq A_0$, since $Z(A)$ has Gelfand-Kirillov dimension one [46]. We let $E = A_0 \cap Z(A)$. Then E is a commutative integral domain that is finite-dimensional over k and hence E is a field. Since δ is not identically zero on $Z(A)$ and is locally nilpotent on A, there exists some $z \in Z(A)$ such that $z \notin E$ and $c := \delta(z) \in E \setminus \{0\}$. As E is a field and is contained in the kernel of δ, $x := c^{-1}z \in Z(A)$ satisfies $\delta(x) = 1$ and so by the noncommutative slice theorem, we see $A \cong A_0[x]$. Then by the same analysis as above if $A[t] \cong B[t]$ then we necessarily have $\mathrm{ML}(B) \neq B$ and so $B \cong B_0[x]$ for some finite-dimensional division ring B_0. Since A_0 is a finite-dimensional division algebra, it follows from [31, Theorem 4.1] that A_0 is strongly cancellative and hence $A_0 \cong B_0$ and hence A is cancellative. Thus we obtain the result in this case.

(2) The counterexample is listed in the section § 5.4. $\square$

We next prove a result, which has rather technical hypotheses, although we believe the result is important in understanding cancellation in positive characteristic. Given an affine domain A over a field k, we say that k is *inseparably closed* in A if whenever F is a k-subalgebra of A that is a field, we have that F is separable over k. In particular, when k has characteristic zero, this always holds. Throughout this proof we make use of the so-called Lucas identity, which says that if p is prime and $0 \leq a, b < p$ and $A, B > 0$ then

$$\binom{pA+a}{pB+b} \equiv \binom{A}{B}\binom{a}{b} \pmod{p}.$$

Proposition 5.3.6. *Let k be a field and let A be an affine domain of Gelfand-Kirillov dimension one. Suppose that k is inseparably closed in A and that $\mathrm{ML}^I(A) \neq A$. Then A is strongly cancellative.*

Proof. When the characteristic of k is zero, this follows immediately from Theorem 5.3.5 (1). Thus we may assume that k has characteristic $p > 0$. Fix a non-trivial locally nilpotent iterative Hasse-Schmidt derivation $\partial = \{\partial_n\}_{n\geq 0}$ of A and let D denote the kernel of $\{\partial_n\}_{n\geq 0}$. Then by Remark 5.3.3, D is finite-dimensional and hence a finite-dimensional division ring over k. Given nonzero $a \in A$, we define $\nu(a) = \sup\{m \colon \partial_m(a) \neq 0\}$. Then we pick $a \in A \setminus D$ with $m := \nu(a)$ minimal among elements of $A \setminus D$. We claim that $\nu(a) = p^r$ for some $r \geq 0$. To see this, suppose that this is not the case. Then $m = \nu(a) = p^r s_0 + p^{r+1} s_1$ with $r \geq 0$, $1 \leq s_0 < p$, and $s_0 + ps_1 > 1$. If $s_1 \geq 1$, then observe that $b = \partial_{p^r s_0}(a)$ has the property that

$$\partial_{p^{r+1}s_1}(b) = \partial_{p^{r+1}s_1} \circ \partial_{p^r s_0}(a) = \binom{p^{r+1}s_1 + p^r s_0}{p^r s_0}\partial_m(a) = \binom{ps_1 + s_0}{s_0}\partial_m(a) \neq 0$$

and for $i > p^{r+1}s_1$, we have $\partial_i(b) \in k\partial_{m+i-p^{r+1}s}(a) = \{0\}$ and hence $\nu(b) = p^{r+1}s_1 < m$. If, on the other hand, $s_1 = 0$, we have $m = p^r s_0$ with $2 \le s_0 < p$. Then if we let $b = \partial_{p^r}(a)$, then as before we have $\partial_i(b) = 0$ for $i > p^r(s_0 - 1)$ and $\partial_{p^r(s_0-1)}(b) \neq 0$. It follows that $m = \nu(a)$ is necessarily of the form p^r for some $r \ge 0$. Let $\alpha = \partial_m(a)$. Then for $i \ge 1$,

$$\partial_i(\alpha) = \partial_i \circ \partial_m(a) = \binom{m+i}{i}\partial_{m+i}(a) = 0$$

and hence $\alpha \in D \setminus \{0\}$. Then by replacing a by $\alpha^{-1}a$, we may assume without loss of generality that $\alpha = 1$.

We next claim that $p^r = \nu(a)$ divides $\nu(b)$ for every $b \in A$. To see this, suppose this is not the case. Then there exists some $b \in A$ such that $\nu(b) = p^r s + q$ with $0 < q < p^r$. Consequently, there is some $i < r$ such that $q = p^i q'$ with $\gcd(q', p) = 1$ and $q' < p^{r-i}$. We let $b' = \partial_{p^r s}(b)$, and we have

$$\partial_q(b') = \partial_q \partial_{p^r s}(b) = \binom{p^r s + p^i q'}{p^i q'}\partial_{\nu(b)}(b) = \binom{p^{r-i}s + q'}{q'}\partial_{\nu(b)}(b) \neq 0.$$

Also for $i > q$ we have $\partial_i(b') \in k\partial_{m+i-q}(b) = \{0\}$ and so $0 < \nu(b') = q < m$, which contradicts minimality of m.

We now prove that $A \cong D[x]$. To see this, observe that for $\beta \in D$, $\partial_i([\beta, a]) = [\beta, \partial_i(a)] = 0$ for $i > m$ and since $\partial_m(a) = 1$, $\partial_m([\beta, a]) = 0$ for $\beta \in D$ and thus $\nu([\beta, a]) < m$ for all $\beta \in D$. By minimality of m, $\nu([\beta, a]) = 0$ for $\beta \in D$ and so $[D, a] \subseteq D$. Hence the map $\delta : D \to D$ given by $\delta(\beta) = [\beta, a]$ is a k-linear derivation of D. We claim that $A = D + Da + \cdots$. Since $D \subseteq A$ and $a \in A$, it suffices to show that A is contained in

$$\sum Da^i.$$

So suppose that this containment does not hold. Then there is some

$$b \in A \setminus (D + Da + Da^2 + \cdots).$$

Among all such b, pick one with $\nu(b)$ minimal. From the above we have $\nu(b) = p^r s = ms$ for some $s \ge 1$. Let $\gamma = \partial_{p^r s}(b) \in D$ and consider $b' := b - \gamma a^s$. By construction,

$\nu(b') < \nu(b)$ and so by minimality of $\nu(b)$, $b' \in D + Da + \cdots$, which then gives that b is too, a contradiction. It follows that $A = D + Da + \cdots$ and since A is infinite-dimensional and D is finite-dimensional, this sum is direct; moreover, $[a, \beta] = \delta(\beta)$ for $\beta \in D$, and so $A \cong D[x; \delta]$ with δ a k-linear derivation and k contained in $Z(D)$ and $[D : k] < \infty$. Now by assumption $Z(D)$ is separable over k and so δ vanishes on $Z(D)$ [10, Prop. 3, V. p. 128]. Thus δ is a $Z(D)$-linear derivation of D and by a straightforward application of the Skolem-Noether theorem it is thus inner [15, Theorem 3.22]. Hence by making a change of variables of the form $x' = x - c$ with suitably chosen $c \in D$, we have $A \cong D[x']$. But now D is strongly cancellative [31, Theorem 4.1] and thus A is strongly cancellative. $\square$

5.4 Examples

In this brief section, we give a family of examples that establish Theorem 5.3.5 (2).

Proof of Theorem 5.3.5 (2). Let p be a prime, and let $K = \mathbb{F}_p(x_1, \ldots, x_{p^2-1})$. We let $k = \mathbb{F}_p(x_1^p, \ldots, x_{p^2-1}^p)$ and we let δ be the k-linear derivation of K given by $\delta(x_i) = x_{i+1}$ for $i = 1, \ldots, p^2 - 1$, where we take $x_{p^2} = x_1$. Since k has characteristic $p > 0$, we have δ^{p^i} is a k-linear derivation for every $i \geq 0$, and since $\delta^{p^2}(x_i) = \delta(x_i) = x_{i+1}$ for $i = 1, \ldots, p^2 - 1$, $\delta^{p^{j+2}} = \delta^{p^j}$ for every $j \geq 0$. We let $\delta' := \delta^p$, which as we have just remarked is a k-linear derivation of K. We let $A = K[x; \delta]$ and we let $B = K[x'; \delta']$. Since $\mathrm{ad}_u^p = \mathrm{ad}_{u^p}$ for u in a ring of characteristic p, we have $z := x^{p^2} - x$ and $z' := (x')^{p^2} - x'$ are central by the above remarks. We claim that A and B have Gelfand-Kirillov dimension one, $A \not\cong B$, and $A[t] \cong B[t']$.

Since $[K : k] < \infty$, A and B are finitely generated k-algebras of Gelfand-Kirillov dimension 1 by the Proposition 2.4.2. We construct an isomorphism $\Phi : A[t] \to B[t']$ as follows. We define $\Phi(\alpha) = \alpha$ for $\alpha \in K$, $\Phi(x) = (x')^p + t'$ and $\Phi(t) = (x')^{p^2} - x' + (t')^p$. Then to show that Φ extends to a k-algebra homomorphism from $A[t]$ to $B[t']$, it suffices to show that

$$\delta(\alpha) = \Phi([x, \alpha]) = [\Phi(x), \alpha]$$

for $\alpha \in K$ and that $\Phi(t)$ is central. For $\alpha \in K$,

$$[\Phi(t), \alpha] = [(x')^{p^2} - (x'), \alpha] = (\delta')^{p^2}(\alpha) - \delta'(\alpha) = 0$$

and since $\Phi(t)$ also commutes with x', $\Phi(t)$ is central. To show that

$$\delta(\alpha) = \Phi([x, \alpha]) = [\Phi(x), \alpha]$$

for $\alpha \in K$, observe that $\Phi([x, \alpha]) = \Phi(\delta(\alpha)) = \delta(\alpha)$ and

$$[\Phi(x), \Phi(\alpha)] = [(x')^p + t', \alpha] = (\delta')^p(\alpha) = \delta^{p^2}(\alpha) = \delta(\alpha).$$

Thus Φ induces a homomorphism from $A[t]$ to $B[t']$. We claim that Φ is onto. We have $\Phi(z) = (z')^p + (t')^{p^2} - t'$ and $\Phi(t) = z' + (t')^p$. In particular,

$$\Phi(z - t^p) = (z')^p + (t')^{p^2} - t' - (z')^p - (t')^{p^2} = -t'$$

and so $\Phi(t + (z - t^p)^p) = z'$. Thus K, t' and z' are in the image of Φ. Since $\Phi(x) = (x')^p + t'$ we also have $(x')^p \in \mathrm{Im}(\Phi)$. Finally, observe that $z' = (x')^{p^2} - x'$ and since z' and $(x')^p$ are in the image of Φ, so is

$$x' = (x')^{p^2} - z' = ((x')^p)^p - z'.$$

Thus x', z' and K are in the image of Φ and so Φ is onto. Let I denote the kernel of Φ. Then since $\Phi : A[t] \to B[t]$ is onto, we have $A[t]/I \cong B[t]$. But $A[t]$ and $B[t]$ are both affine domains of Gelfand-Kirillov dimension two, and so I is necessarily zero by Proposition 5.2.4. Thus Φ is an isomorphism and so $A[t] \cong B[t]$.

Thus it only remains to show that $A \ncong B$ as k-algebras. To see this, suppose that $\Psi : A \to B$ is a k-algebra isomorphism. Then since the units group of A and B are both K^*, Ψ induces a k-algebra automorphism of K; furthermore, every $\alpha \in K$ satisfies $\alpha^p \in k$ and for $\beta \in k$ there is a unique $\alpha \in K$ such that $\alpha^p = \beta$. Since Ψ is the identity on k, Ψ

is the identity on K. Thus $\Psi(x) = p(x')$ for some $p(x') \in K[x'; \delta'] \setminus K$. Let $d \geq 1$ denote the degree of $p(x')$ as a polynomial in x'. If $d > 1$, it is straightforward to show that Ψ cannot be onto, as every element in the image of Ψ necessarily then has degree in x' equal to a multiple of d. Since $\Psi(x) \notin K$, we see $\Psi(x) = \alpha x' + \beta$ with $\alpha \in K^*$ and $\beta \in K$. Since Ψ is an isomorphism, for $\zeta \in K$ we have

$$\delta(\zeta) = \Psi(\delta(\zeta)) = \Psi([x, \zeta]) = [\Psi(x), \Psi(\zeta)] = [\alpha x' + \beta, \zeta] = \alpha[x', \zeta] = \alpha\delta'(\zeta).$$

But by construction $\delta(x_1) = x_2$ and $\delta'(x_1) = x_{p+1}$ and so $\alpha = x_2/x_{p+1}$. We also have $\delta(x_2) = x_3$ and $\delta'(x_2) = x_{p+2}$, and so $\alpha = x_3/x_{p+2}$, which gives $x_2 x_{p+2} = x_3 x_{p+1}$, where we take $x_{p+2} = x_1$ when $p = 2$. This is a contradiction. Thus $A \ncong B$. $\square$

Chapter 6

Skew Cancellativity

6.1 Preliminaries

We also prove a result in a different direction; namely, skew cancellativity. To describe this extension, we recall that given a ring R, an automorphism σ of R and a σ-derivation $\delta : R \to R$ of R (that is, δ satisfies $\delta(rs) = \sigma(r)\delta(s) + \delta(r)s$ for $r, s \in R$), one can define a *skew polynomial* extension $R[x; \sigma, \delta]$, which is just $R[x]$ as an additive abelian group and with multiplication given by $x \cdot r = \sigma(r)x + \delta(r)$ for $r \in R$, where we use the same multiplication rule for elements in R as before. The two most important special cases of this construction are the skew polynomial extensions of automorphism type, where $\delta = 0$; and skew polynomial extensions of derivation type, where σ is the identity. In light of the Zariski cancellation problem, it is then natural to ask when an algebra R is *skew cancellative*; that is, if $R[x; \sigma, \delta] \cong S[x; \sigma', \delta']$ when do we necessarily have $R \cong S$? We show that this holds in the two cases just mentioned when the coefficient ring R is an affine commutative domain of Krull dimension one and the characteristic of k is 0 in the derivation case.

Our main result in this chapter is Theorem 6.2.7. A special case of Theorem 6.2.7 was proved by Bergen [8] in the derivation case as follows.

Theorem 6.1.1. [8, Theorem] *Let R be an algebra over a field k of characteristic 0 such that the skew polynomial rings $R[t;\delta]$ and $k[x][y;d]$ are isomorphic, with derivations δ and d. If $d(x)$ has degree at least one, then $R \cong k[x]$.*

It would be interesting to give a "unification" of the two results occurring in Theorem 6.2.7 and prove that skew cancellation holds for general skew polynomial extensions, although this appears to be considerably more subtle than the cases we consider. The positive characteristic case for skew extensions of derivation type appears to have additional subtleties. In particular, the constructions given in §5.4 show that cancellation can behave strangely with skew extensions of derivation type in positive characteristic.

In this chapter, we consider skew cancellation and prove Theorem 6.2.7.

6.2 Skew Cancellativity

We now consider the case of when an isomorphism of skew polynomial extensions

$$R[x;\sigma,\delta] \cong S[x;\sigma';\delta']$$

implies that R and S are isomorphic. We consider the case when R and S are finitely generated commutative integral domains of Krull dimension one over a field. We observe that when σ, σ' are the identity maps and δ, δ' are zero, the question reduces to the classical cancellation problem for affine curves, answered by Abhyankar, Eakin, and Heinzer [1]. To prove Theorem 6.2.7, we must consider two types of extensions: skew extensions of automorphism type and skew extensions of derivation type. We first look at the automorphism type case, in which the analysis is more straightforward.

Lemma 6.2.1. *Let k be a field, let R be an affine commutative domain over k of Krull dimension one, and let σ be a k-algebra automorphism of R that is not the identity. If A*

is a commutative domain of Krull dimension one that is a homomorphic image of $R[x;\sigma]$ then either $A \cong R$ or $A \cong K[x]$ for some finite extension K of k; moreover R occurs as a homomorphic image of $R[x;\sigma]$ by modding out by (x).

Proof. We consider prime commutative homomorphic images of $T := R[x;\sigma]$ of Krull dimension one. Observe that if P is a prime ideal of T such that T/P is commutative, then since T/P is an integral domain and $R/(P \cap R)$ embeds in T/P, $R/(P \cap R)$ is also an integral domain. Since R is an integral domain of Krull dimension one, either $P \cap R = (0)$ or $P \cap R = I$, with I a maximal ideal of R. In the former case, observe that since $xr = \sigma(r)x \equiv x\sigma(r) \pmod P$, we have $x(r - \sigma(r)) \in P$. Moreover, since σ is not the identity and P is completely prime, we necessarily have $x \in P$. Thus T/P is a homomorphic image of $R[x;\sigma]/(x) \cong R$. Since T/P and R are both integral domains of Krull dimension one, we then have $T/P \cong R$ in this case. In the case where $P \cap R = I$, with I a maximal ideal of R, we claim that $I = I^\sigma$. To see this, suppose that this is not the case. Then since I is maximal, $I + \sigma(I) = R$. In particular, there are $a, b \in I$ such that $a + \sigma(b) = 1$. Then $ax, xb \in P$ and so $ax + xb \in P$. But $ax + xb = (a + \sigma(b))x = x$ and so $x \in P$. Thus T/P is a homomorphic image of R/I, which contradicts the assumption that T/P has Krull dimension one. Hence $I = \sigma(I)$. Then by the Nullstellensatz $K := R/I$ is a finite extension of k and σ induces a k-algebra automorphism of K. We next claim that σ is the identity on K; if not, there is some $\lambda \in K$ such that $\lambda \not\equiv \sigma(\lambda) \pmod P$. But since $[\lambda, x] = (\lambda - \sigma(\lambda))x \in P$ and since P is completely prime, we again have $x \in P$, which gives $T/P \cong K$, a contradiction. Thus σ induces the identity map on $R/I = K$ and so $T/IT \cong K[x]$. Since P contains IT, we then see that T/P is a homomorphic image of $K[x]$ and since T/P has Krull dimension one, we have $T/P \cong K[x]$. The result follows. □

Proposition 6.2.2. *Let k be a field and let R be an affine commutative domain over k of Krull dimension one. If $R[x;\sigma] \cong S[x;\sigma']$ then $R \cong S$.*

Proof. If σ is the identity then both $R[x;\sigma]$ and $S[x;\sigma']$ are commutative and so σ' is also the identity and the result follows from [1]. Hence we may assume that σ and σ' are not the identity maps on their respective domains. By Lemma 6.2.1, the set of isomorphism classes of prime commutative images of $R[x;\sigma]$ of Krull dimension one is contained in $\{K[x]\colon [K:k]<\infty\}\cup\{R\}$, with R occurring on the list. Similarly, the set of isomorphism classes of prime commutative images of $S[x;\sigma']$ of Krull dimension one is contained in $\{K[x]\colon [K:k]<\infty\}\cup\{S\}$, with S occurring on the list. It follows that either $R\cong S$ or $R\cong K[x]$ for some finite extension K of k. Similarly, either $S\cong R$ or $S\cong K'[x]$ for some finite extension K' of k. Thus we may assume without loss of generality that $R=K[t]$ and $S\cong K'[t]$ with K,K' finite extensions of k. Then the k-algebra isomorphism $R[x;\sigma]\to S[x;\sigma']$ restricts to an isomorphism of the unit groups. Since the unit groups of $R[x;\sigma]=K^*$ and the units group of $S[x;\sigma']$ is $(K')^*$, we see the isomorphism restricts to a k-algebra isomorphism between K and K'. Thus $K\cong K'$ and so $R\cong S$. □

We now prove a lemma, which is a straightforward extension of earlier work (see [33, Lemma 21], [7, Lemma 3.5]).

Lemma 6.2.3. *Let k be a field of characteristic zero, let A be a finitely generated Goldie domain over k, and let δ be a k-derivation of A. If $\mathrm{ML}(A)=A$ then $\mathrm{ML}(A[x;\delta])=\mathrm{ML}(A)$.*

Proof. Let μ_0 be a locally nilpotent derivation of A and suppose that μ_0 commute with δ. Then μ_0 extends to a locally nilpotent derivation of $A[x;\delta]$ by declaring that $\mu_0(x)=1$. Then the kernel of this extension of μ_0 is equal to $\ker(\mu_0|_A)$ and hence $\mathrm{ML}(A[x;\delta])\subseteq\mathrm{ML}(A)$. Now we show that the reverse containment holds. Let μ be a locally nilpotent derivation of $A[x;\delta]$. Suppose that μ is not identically zero on $\mathrm{ML}(A)$. Since A is finitely generated there is some largest $m\geq 0$ such that for $r\in A$ we have

$$\mu(r)=\partial(r)x^m+\text{lower degree terms},$$

with ∂ a derivation of A that is not identically zero on $\mathrm{ML}(A)$. If $m=0$ then ∂ is a locally nilpotent derivation of A and hence vanishes on $\mathrm{ML}(A)$, a contradiction. Thus we may assume that $m>0$. We now argue as in the three cases given in [7, Lemma 3.5].

Case I: Suppose that $\mu(x) \in A + Ax + \cdots + Ax^m$. So we have $\mu(x^i) \subseteq \sum_{n=0}^{i+m-1} Ax^n$ and $\mu(Ax^i) \subseteq \sum_{n=0}^{i+m} Ax^n$ for all i. Thus for every $r \in A$ we have

$$\mu^2(r) = \partial^2(r)x^{2m} + \text{ lower degree terms.}$$

More generally, we see that

$$\mu^j(r) = \partial^j(r)x^{mj} + \text{ lower degree terms.}$$

Thus ∂ is a locally nilpotent derivation and so $\partial(A) = 0$, contradicting the minimality of m. Thus $\mu(A) = 0$ in this case.

Case II: $\mu(x) = bx^{m+1}+$ lower degree terms for some $b \neq 0$ in A.
Applying μ to the equation $[x, r] = \delta(r)$, one sees that b commutes with every r in A and so b is in the center of A. Now we define a new derivation μ' of $A[x]$ (where the variable x now commutes) by declaring that $\mu'(r) = \partial(a)x^m$ for $r \in A$ and $\mu'(x) = bx^{m+1}$. Then we see that μ' sends Ax^i to Ax^{i+m} for every $i \geq 0$.

We can view μ' as an associated graded derivation of μ. Since μ is locally nilpotent, μ' is a locally nilpotent derivation of $A[x]$ [12, Lemma 4.11]. Since A is Goldie domain, A has a quotient algebra $Q(A)$. Applying Lemma 3.4[7] to the algebra $A[x], A[x]$ embeds in $E[y; \mu_1]$, where μ_1 is a derivation of E and $E = \{a \in Q(A) \mid \mu'(a) = 0\}$. Moreover, μ' extends to a locally nilpotent derivation of $E[y; \mu_1]$ by declaring that $\mu'(E) = 0$ and $\mu'(y) = 1$. Under this embedding $x = p(y)$ for some nonzero polynomial p. Let d denote the degree of this polynomial. Then bx^{m+1} gets sent to $q(y)p(y)^{m+1}$ for some nonzero polynomial $q(y)$. But since $\mu'(x)$ is nonzero, it has degree exactly $d - 1$ and so we have $(m + 1)d + \deg q(y) = d - 1$ which is impossible.

Case III: $\mu(x) = bx^i+$ lower degree terms for some $b \neq 0$ in A and some $i > m + 1$. In this case we see that, for each $n \geq 2$

$$\mu^n(x) = \left\{\prod_{s=1}^{n-1}((i-1)s+1)\right\} b^n x^{(i-1)n+1} + \text{ lower degree terms.}$$

so μ cannot be locally nilpotent, which contradicts the hypothesis. Combining these cases, we see that $\mu(A) = 0$. The result follows. □

The following result is due to Crachiola and Makar-Limanov [13, Lemma 2.3].

Remark 6.2.4. Let k be a field of characteristic zero and let R be an affine commutative domain of Krull dimension one. Then either $\mathrm{ML}(R) = R$ or $R \cong k'[t]$ for some finite field extension k' of k.

Proof. Suppose that $\mathrm{ML}(R) \neq R$. Then there is a locally nilpotent derivation δ of R that is not identically zero on R. In particular, the kernel of δ is a subalgebra R_0 of R. By Remark 5.3.3, R_0 is finite-dimensional and hence a finite extension k' of k. Then pick $x \in R$ such that $\delta(x) \neq 0$ and $\delta^2(x) = 0$. Then $\delta(x) \in (k')^*$ and so we may rescale and assume that $\delta(x) = 1$. Then by Lemma 5.3.2, $R \cong k'[x]$, as required. □

Corollary 6.2.5. *Let k be a field of characteristic zero, let R be a finitely generated k-algebra that is a commutative domain of Krull dimension one, and let δ be a k-linear derivation of R. Then either $R \cong k'[t]$ for some finite extension k' of k or* $\mathrm{ML}(R[x;\delta]) = R$.

Proposition 6.2.6. *Let k be a field of characteristic zero and let R and S be affine commutative domains over k of Krull dimension one. If δ and δ' are resp. k-linear derivations of R and S and $R[x;\delta] \cong S[x;\delta']$, then $R \cong S$.*

Proof. By Corollary 6.2.5, either $R \cong k'[t]$ for some finite extension k' of k or $\mathrm{ML}(R[x;\delta]) = R$ and same conclusion holds for S. If neither R nor S is isomorphic to $k'[x]$, with k' some finite extension of k, then $R = \mathrm{ML}(R[x;\delta]) \cong \mathrm{ML}(S[x;\delta']) = S$ and we get the result. If R is isomorphic to $k'[x]$ for some finite extension of k and S is not isomorphic to an algebra of this type, then $k' = \mathrm{ML}(R) \cong \mathrm{ML}(S) = S$, which is impossible. Thus we may assume that $R \cong k'[t]$ and $S \cong k''[t]$ where k' and k'' are finite extensions of k. But now the units group of $R[x;\delta]$ is $(k')^*$ and the units group of $S[x;\delta']$ is $(k'')^*$ and so the isomorphism from $R[x;\delta] \to S[x;\delta']$ restricts to an isomorphism between k' and k'' and so $R \cong S$ in this case. $\square$

We do not know whether Proposition 6.2.6 is true when the base field k has positive characteristic. We compare the examples from Theorem 5.3.5 (b) with the positive characteristic version of Proposition 6.2.6. In the positive characteristic version, there exists a field k and a finite extension K of k and k-linear derivations δ, δ' of K such that $K[t;\delta][x] \cong K[t;\delta'][x]$ but $K[t;\delta] \not\cong K[t;\delta']$. But we can extend δ and δ' to $K[x]$ by declaring that $\delta(x) = \delta'(x) = 0$ and we have

$$K[t;\delta][x] \cong K[x][t;\delta] \cong K[x][t;\delta'] \cong K[t;\delta'][x].$$

So the algebra $K[t;\delta][x] \cong K[x][t;\delta]$ is cancellative with respect to the variable t but not with respect to the variable x. Thus these examples do not give rise to counterexamples to the positive characteristic version of Proposition 6.2.6.

Theorem 6.2.7. *Let k be a field, let A and B be affine commutative integral domains of Krull dimension one, and let σ, σ' be k-algebra automorphisms of A and B resp. and let δ, δ' be k-linear derivations of A and B resp.. If $A[x;\sigma] \cong B[x';\sigma']$ then $A \cong B$. If, in addition, k has characteristic zero and if $A[x;\delta] \cong B[x';\delta']$ then $A \cong B$.*

Proof. This follows immediately from Propositions 6.2.2 and 6.2.6. $\square$

Chapter 7

Future Directions

7.1 Concluding Remarks

We note that in the paper [6], a version of Corollary 4.5.3 was proved for finitely generated commutative Hopf algebras that are domains over an algebraically closed field k. We also note that this follows immediately from the Theorem 4.4.8, as such an algebra R is of the form $\mathcal{O}(G)$ for G an irreducible affine algebraic group over k and so $R \otimes_k R$ is just $\mathcal{O}(G \times G)$, which is again a domain. In Corollary 4.5.3, we have the hypothesis that the ring has finite Gelfand-Kirillov dimension. Conjecturally, the result should hold for noetherian cocommutative Hopf algebras R over an algebraically closed field of characteristic zero, since such algebras are isomorphic to algebras of the form $U(L)\#G$; since R is faithfully flat over both $U(L)$ and the group algebra $k[G]$, if R is noetherian, then so must these two subalgebras. Conjecturally, enveloping algebras are noetherian if and only if L is finite-dimensional and $k[G]$ is noetherian if and only if G is polycyclic-by-finite. Hence Theorem 4.5.1 can be applied to give that $R \otimes_k R \cong U(L \oplus L)\#(G \times G)$ is a domain if R is a domain, since G is necessarily torsion-free. In light of this, we ask the following questions.

Question 7.1.1. Let R be a cocommutative noetherian Hopf algebra over an algebraically closed field k of characteristic zero. Is $R \otimes_k R$ a domain if R is a domain?

If the reader feels like being more ambitious, we raise the following question, which, combined with Theorem 4.4.8, would give an affirmative answer to Question 4.2.1 in the case when R is a domain over an algebraically closed field if it could be answered affirmatively.

Question 7.1.2. Let k be an algebraically closed field and let R be a noetherian Hopf algebra that is a domain. Is $R \otimes_k R$ a domain?

In another direction, recall that Böhm and Szlachányi [9] introduced weak Hopf algebras as a generalization of ordinary Hopf algebras. Roughly speaking, one can obtain a weak Hopf algebra by relaxing the axioms related to the unit and counit in the definition of an ordinary Hopf algebra. Weak Hopf algebras naturally appear in different areas of mathematics, such as functional analysis and representation theory (one can see the survey [39] for more details). But the general construction of weak Hopf algebras is not well-understood. This leads us to the following question.

Question 7.1.3. Given a weak Hopf algebra R, for which automorphisms σ and σ-derivations δ does the Ore extension $T = R[x, \sigma, \delta]$ have weak Hopf algebra structure extending the given weak Hopf algebra structure on R?

By work in [3], the classification of all finite-dimensional pointed Hopf algebra is well-understood. It is then natural to investigate infinite-dimensional k-Hopf algebras. Moreover, it is interesting to study these k-Hopf algebras that have Gelfand-Kirillov dimension one in the question below.

Question 7.1.4. Let k be an algebraically closed field of characteristic zero and H be a k-Hopf algebra of Gelfand-Kirillov dimension one. Which Hopf algebras H described as above can be expressed in terms of an (iterated) Ore extension of a finite dimensional k-Hopf algebra R?

Interesting problems for future work are to investigate the Zariski Cancellativity of an affine domain that has Gelfand-Kirillov dimension two or three, and to also work on the theory of weak Hopf algebras by extending the above results. More specifically, one can consider the following questions.

Question 7.1.5. Is a non-PI affine prime k-algebra A that has Gelfand-Kirillov dimension at most three cancellative?

For Question 7.1.5, one can approach it by considering the Gelfand-Kirillov dimension of the center $Z(A)$ of A. It follows from [47, Corollary 2] that $Z(A)$ has Gelfand-Kirillov dimension at most one. If $Z(A)$ has Gelfand-Kirillov dimension zero, then $Z(A)$ is a field. By the result [7, Proposition, 1.3], A is cancellative. So we may restrict our focus to the case when $Z(A)$ has Gelfand-Kirillov dimension one. In [4, Proposition 3.3], we have shown that such a domain A is either LND_Z-rigid or $A = A_0[t]$, where $A_0 = \ker(\delta)$ for some locally nilpotent derivation δ of A. This is close to proving the cancellativity of A.

A key feature of these examples in 5.4 is that they have centres that are not inseparably closed. It is natural to ask whether affine domains of Gelfand-Kirillov dimension one are cancellative when one adds the assumption that the base field is inseparably closed.

Question 7.1.6. Let k be a field of positive characteristic and let A be an affine domain of Gelfand-Kirillov dimension one with the property that k is inseparably closed in A. Is A cancellative?

If this question has a negative answer, a counterexample must be very constrained. By work of Lezama, Wang, and Zhang [31], if A is a counterexample we have $Z(A) \cong k'[x]$ for some finite extension k' of k, furthermore, A is Azumaya and the Brauer group of $k'[x]$ cannot be trivial. By Proposition 5.3.6, we have $\mathrm{ML}^I(A) = A$ and yet we must also have $\mathrm{ML}^{H'}(A) \neq A$ by Proposition 5.2.7.

When it comes to the skew cancellativity, we do not know whether cancellation holds for skew polynomial extensions of mixed type with coefficient rings being domains of Krull dimension one. We pose an unresolved question, which—if the answer were affirmative—would unify the two cases in Theorem 6.2.7 and would also extend Proposition 6.2.6 to base fields of positive characteristic. We will close by posing the following question.

Question 7.1.7. Let k be a field, let R be an affine commutative domain over k of Krull dimension one, and let σ and δ be respectively a k-algebra automorphism and a k-linear σ-derivation of R. Is R skew cancellative?

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