Applications of Operator Systems in Dynamics, Correlation Sets, and Quantum Graphs

by

Se-Jin Kim

A thesis
presented to the University of Waterloo
in fulfillment of the
thesis requirement for the degree of
Doctor of Philosophy
in
Pure Mathematics

Waterloo, Ontario, Canada, 2020

Examining Committee Membership

The following served on the Examining Committee for this thesis. The decision of the Examining Committee is by majority vote.

External Examiner: Elias Katsoulis
Professor, Dept. of Mathematics, East Carolina University

Supervisor(s): Kenneth Davidson
Professor, Dept. of Pure Mathematics, University of Waterloo
Matthew Kennedy
Professor, Dept. of Pure Mathematics, University of Waterloo

Internal Member: Vern Paulsen
Professor, Dept. of Pure Mathematics, University of Waterloo

Internal Member: Nico Spronk
Professor, Dept. of Pure Mathematics, University of Waterloo

Internal-External Member: Achim Kempf
Professor, Dept. of Applied Mathematics, University of Waterloo

Author's Declaration

This thesis consists of material all of which I authored or co-authored: see Statement of Contributions included in the thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Statement of Contributions

Parts of Chapter 2 as well as Chapter 3 is joint work with Samuel Harris. Chapters 5 and 6 is joint work with Vern Paulsen and Chris Schafhauser, and Chapter 7 is joint work with Arthur Mehta. All other chapters are the author's own. Contributions to these papers are equally split amongst the authors. The authors are listed alphabetically, as is common practice in pure mathematics.

Abstract

The recent works of Kalantar-Kennedy, Katsoulis-Ramsey, Ozawa, and Dykema-Paulsen have demonstrated that many problems in the theory of operator algebras and quantum information can be approached by looking at various subspaces of bounded operators on a Hilbert space. This thesis is a compilation of papers written by the author with various coauthors that apply the theory of operator systems to expand on some of these results. This thesis is split into two parts.

In Part I, we start by expanding on the theory of crossed product of operator algebras of Katsoulis and Ramsey. We first develop an analogous crossed product of operator systems. We then reduce two open problems on the uniqueness of universal crossed product operator algebras into one of operator systems and show that it has answers in the negative. In the final chapter of Part I, we generalize results of Kakariadis, Dor On-Salmon, and Katsoulis-Ramsey to characterize which tensor algebras of C*-correspondences admit hyperrigidity.

In Part II, we look at synchronous correlation sets, introduced by Dykema-Paulsen as a symmetric form of Tsirelson's quantum correlation sets. These sets have the distinct advantage that there is a nice C*-algebraic characterization that we present in Chapter 6. We show that the correlation sets coming from the tensor models on finite and infinite dimensional Hilbert spaces cannot be distinguished by synchronous correlation sets and that one can distinguish this set from the correlation sets which arise as limits of correlation sets arising from finite dimensional tensor models. Beyond this, we show that Tsirelson's problem is equivalent its synchronous analogue, expanding on a result of Dykema-Paulsen.

We end the thesis by looking at generalizations of graphs by the ways of operator subspaces of the space of matrices. We construct an analogue of the graph complement and show its robustness by deriving various generalizations of known graph inequalities.

Acknowledgements

I would firstly like to thank my advisors Kenneth Davidson and Matthew Kennedy for taking time out of almost every week of the past four years[1] to teach me the value of a good idea.

I would also like to thank my coauthors Samuel Harris, Arthur Mehta, Christopher Schafhauser, and Vern Paulsen, as well as the graduate students in Analysis at the University of Waterloo for many interesting and illuminating discussions.

This degree would have been a far more difficult pursuit had I not been in the capable hands of the staff at the Pure Mathematics Department. I would like to thank Lis D'Alessio, Jackie Hilts, Nancy Maloney, Pavlina Penk for their excellent work in this regard. As well, I would like to thank the Graduate Chairs Barbara Csima, Spiro Karigiannis, and Nico Spronk for all the work that they have put in into making the graduate student experience a welcoming one.

Finally, my PhD studies is financially supported by the Natural Sciences and Engineering Research Council of Canada.

[1] For Ken Davidson, this would be every week for the past five years.

Table of Contents

Chapter 1

Introduction

Aside from examples of C*-algebras as algebras generated by concrete operators on a Hilbert space, examples of C*-algebras come from constructions associated to objects from other areas of mathematics. For example, the group C*-algebra construction associates to each group G a C*-algebra $C^*(G)$ that captures its representation theory. More generally, the crossed product construction gives us a C*-algebra $A \rtimes G$ associated to any invertible C*-dynamical system (A, G). On the more combinatorial side, there are C*-algebras associated to directed graphs. More generally, there are the Cuntz-Pimsner algebras $\mathcal{O}_X$ associated to a C*-correspondence X. The interplay between the objects and their relation to algebraic properties of the associated C*-algebras allow for a rich source of examples. See [13] for more detail on these constructions and many others.

There are two natural generalizations of C*-algebras: there are non-self adjoint operator algebras, which are normed closed subalgebras of $B(H)$ that is not necessarily closed under the involution *, as well as operator systems, which are unital and *-closed but fail to be closed under multiplication. Part I of this thesis will focus on operator systems and their interplay between C*-algebras and non-self adjoint operator algebras. This project is fueled by the work of Hamana [35, 36] as well as recent work of Kalantar-Kennedy [42], Kawabe [54], and Davidson-Kennedy [20] that show that even in the C*-setting, understanding operator systems can give us deeper understanding of the structure of C*-algebras. I would also be remiss if I did not point out that much of the recent work on nuclear C*-algebras relies heavily on the machinery of operator systems and that, in the sense of continuous logic, the work of Goldbring-Sinclair show that C*-algebras as a subclass of the category of operator systems are first-order axiomatizable [33].

On the side of operator algebras, the work of Katsoulis-Ramsey [47, 46, 44] demonstrate that understanding C*-algebras associated to C*-correspondences may be the way of resolving the Hao-Ng isomorphism problem [37]. The Hao-Ng isomorphism problem asks the following: if G is a locally compact group acting non-degenerately on a C*-correspondence X, is it the case that

$$\mathcal{O}_X \rtimes G = \mathcal{O}_{X \rtimes G} \ ?$$

That is, is it the case that the Cuntz-Pimsner algebra construction and the crossed product construction commute? It is named after Hao and Ng as they show that it is indeed the case that these constructions commute in the case when the group acting on the C*-correspondence is amenable. Katsoulis and Ramsey make much headway in this direction, giving us the best known results on when the Hao-Ng isomorphism problem holds. They do this by first reducing the Hao-Ng problem down to the analogous problem about Tensor algebras $\mathcal{T}_X^+$ associated to a C*-correspondence X, which are non-self adjoint operator subalgebras of $\mathcal{O}_X$. In Chapter 3, in joint work with Sam Harris, we answer two questions of Katsoulis and Ramsey, who pose to what extent we can make this reduction in the general case of arbitrary operator algebras. We show that such a reduction is not possible in general and, in fact, that such a reduction is tied directly to nuclearity of the algebra $C^*(G)$. We do this by first defining a crossed product construction for operator systems in analogy to the work of crossed products of operator algebras of Katsoulis and Ramsey. After deriving some basic properties about these crossed products, we reduce the problem of Katsoulis and Ramsey to one about crossed product operator systems. We then show this cannot hold due to the existence of operator systems known as nuclearity detectors, discovered by Kavruk in [50].

Katsoulis and Ramsey then proceed to tackle the monumental task of constructing a notion of crossed product for non-selfadjoint operator algebras. As it turns out, once all the basic results about crossed product operator algebras are established, they are able to establish that

$$\mathcal{T}_X^+ \rtimes_r G = \mathcal{T}_{X \rtimes_r G}^+$$

for any locally compact G acting on X. As well, Katsoulis is able to establish that

$$\mathcal{T}_X^+ \rtimes G = \mathcal{T}_{X \rtimes G}^+$$

in the case of discrete G so long as the tensor algebra $\mathcal{T}_X^+$ admits a property called hyperrigidity. Chapter 4 is dedicated to characterizing exactly when the tensor algebra $\mathcal{T}_X^+$ is hyperrigid. This generalizes results of Kakariadis [41] and Dor On-Salomon [21], who establish an exact characterization in the case of directed graphs, as well as Katsoulis-Ramsey [46], who derive one direction of our characterization.

Part II is focused on applications of C*-algebras and operator systems to Tsirelson's correlation sets and quantum graphs. Tsirelson's correlation sets arise from the following set-up: suppose that one has two isolated labs run by Alice and Bob respectively. The two labs play a cooperative game run by a referee Charlie. Alice and Bob are allowed to come up with a strategy for winning this game beforehand but must not have any communication once the game starts. Beyond this, Alice and Bob's labs are allowed to share any number of entangled quantum states with one another. Tsirelson's correlation sets are matrices of probabilities that describe, given a strategy by Alice and Bob as well as some input by Charlie to each lab, what the probabilty is of winning with some given output by Alice and Bob. Since Alice and Bob share entangled states, their correlation sets may depend on the model of quantum system that we take. The goal of Tsirelson's correlation sets is to disguish different models of quantum systems by separating these correlation sets.

Indeed, to show that one model of quantum systems is distinct from another, it is enough to show that one has a winning stategy for the game assuming one model but cannot always win the game assuming another. Interest in separating physical models of quantum phenomena can be traced back as far as 1935, with the conception of the Einstein-Poldosky-Rosen paradox [26]. Bohm gives a formulation of the EPR paradox as follows: suppose that we have an entangled pair of particles whose total spin is zero (we say that such a pair is in a spin singlet state). Denote by their spins the vectors σ_1 and σ_2 respectively. If the two particles are separated, and measurement of the spin of one particle along a unit vector v is measured in a lab to have $\langle \sigma_1, v\rangle = 1$, then we must know that $\langle \sigma_2, v\rangle = -1$. It seems then that the only way we could have gained information about the state of the other particle faster than information about the particle can reach our lab is because there were additional hidden variables that the two particles knew before being separated.

In his monumental paper [8], John Bell shows that no hidden variables can exist in our system. His original argument is surprisingly elegant and so we sketch it here. Suppose that some additional parameters for our system exist. Let us call this collection of hidden parameters λ. We will do two experiments, denoted by experiment A and experiment B. Experiments A and B are going to measure one of two particles, denoted by particles 1 and 2, in a singlet spin state. In experiment A, we will measure particle 1 along a unit vector a and in experiment B, we will measure particle 2 along a unit vector b. We may denote the outcome of such an experiment by functions $A(a, \lambda) \in \{-1, +1\}$ and $B(b, \lambda) \in \{-1, +1\}$. Fix a probability measure μ on the set Λ of hidden parameters. The expectation value of the product $A(a,\lambda)\cdot B(b,\lambda)$ among these hidden parameters is then given by

$$P(a,b) := \int_\Lambda A(a,\lambda)B(b,\lambda)d\mu(\lambda)\ .$$

We choose μ so that this expectation value agrees with the expectation value of the product $\langle\sigma_1, a\rangle\langle\sigma_2, b\rangle$ according to the quantum model, which is $-\langle a, b\rangle$. If the quantum model agrees with experiments and hidden parameters exist, then such a μ must exist.

Since our particles are in a spin singlet state, we must have the identity

$$A(a,\lambda) = -B(a,\lambda)$$

for all unit vectors a and almost every λ. In particular, $P(a,b)$ may be rewritten as

$$P(a,b) = -\int_\Lambda A(a,\lambda)A(b,\lambda)d\mu(\lambda)\ .$$

Using the fact that for all b and λ, $A(b,\lambda)^2 = 1$, for all unit vectors a, b, c,

$$\begin{aligned}|P(a,b) - P(a,c)| &\leq \int_\Lambda |A(a,\lambda)(A(b,\lambda) - A(c,\lambda))|d\mu(\lambda)\\ &= \int_\Lambda |A(a,\lambda)A(b,\lambda)|(1 - A(b,\lambda)A(c,\lambda))d\mu(\lambda)\\ &\leq \int_\Lambda 1 - A(b,\lambda)A(c,\lambda)d\mu(\lambda) = 1 + P(b,c)\ .\end{aligned}$$

The above inequality is referred to as *Bell's inequality*. If $P(a,b) = -\langle a, b\rangle$ for all a, b, we have the inequality

$$|\langle a, b - c\rangle| \leq 1 + \langle b, c\rangle$$

for all unit vectors a, b, c. Substituing $b = -c$ and $a = b$ then leads to a contradiction. Thus, we may conclude that the quantum model for pairs of particles in a singlet quantum state is distinct from the classical model, even assuming additional parameters.

The biggest problem about correlation sets is Tsirelson's problem [76, 77]. This problem asks whether the correlation sets arising from the commuting model for mixed quantum systems, called C_{qc}, can be approximated by those models which assume that mixing two systems associated with Hilbert spaces H_A and H_B arises from their tensor product, called C_{qs}. That is, must it be the case that $C_{qc} = \overline{C_{qs}}$? Indeed, it seems that due to the work of Ji et al. that these correlation sets *are* distinct [39].

In Chapter 6, in joint work with Chris Schafhauser and Vern Paulsen, we look at a class of cooperative games called synchronous games, introduced by Dykema-Paulsen [24], and their associated correlation sets. We show that each correlation set can be distinguished using traces on a C*-algebra, the trace depending on the choice of model. Indeed, extending a result of Ozawa [63] and Dykema-Paulsen, we show that Tsirelson's problem is equivalent to distingushing the correlation sets arising from synchrous correlations. Modifying a construction of Solfstra [74], we also show that there is a synchronous game that distinguish between C_{qs} and $\overline{C_{qs}}$.

Finally, in Chapter 7, we look at a generalization of graphs by operator subspaces of $n \times n$-matrices that are *-closed. This was done originally by Duan, Severini, and Winter to analyze subspaces of matrices associated to a quantum channel [23]. By thinking of these subspaces as generalizations of graphs, they are able to generalize graph parameters to invariants of operator spaces. Indeed, in their paper they generalize the notion of an independence number α and Lovasz theta ϑ to subspaces of matrices. In joint work with Arthur Metha, we generalize the notion of a graph complement and chromatic number χ to these operator spaces. In doing so, we are able to establish a generalization of Lovasz's Sandwich Theorem, which states that

$$\alpha(G) \leq \vartheta(G) \leq \chi(G^c)$$

for any graph G. This is a special case of the work of Stalhke, who also establish a generalization with respect to subspaces for which the trace of each element is zero [75].

Part I

Dynamics on Operator Systems

Chapter 2

Preliminaries

In this chapter, we introduce the preliminary material and notation needed for Part I of this thesis. Since Part I is dedicated to operator systems, their crossed products, and the associated tensor algebras, we start with a discussion of operator systems and their C*-covers. In particular, we focus on describing the C*-envelope of an operator system through maximal dilations as developed by Dritschel and McCollough. Beyond this, we describe Arveson's notion of hyperrigidity, which is only defined for separable operator systems, and show that his definition is robust enough to extend to non-separable operator systems. Many of our arguments involving hyperrigidity and C*-covers will use dilations, and so we will take as the definition of hyperrigidity the unique extension property for all representations.

The next section is dedicated to crossed products of operator algebras as developed by Katsoulis and Ramsey. Our goal is to describe enough of the theory to explain why crossed products of operator algebras are tensor products under a trivial action. This will be used in Chapter 3 to give an answer to a problem of Katsoulis and Ramsey. This ties in to the subsequent section on tensor products of operator systems. Many of the results in this section will be used to describe the behaviour of crossed products of operator systems in the case where the action is trivial. Unlike the case of operator algebras, there are many natural tensor products that arise for operator systems. Indeed, we will analyze exactly which tensor products arise from crossed products of operator systems under the trivial action in the next chapter.

Unlike the case of C*-algebras we will see in Proposition 3.1.26 that even for abelian groups, G-equivariant quotient maps on operator systems will not induce a quotient map on their associated crossed product. The section on finite-dimensional operator systems and quotient maps is a brief review on the definition of quotients in the category of operator systems.

Finally, we review C*-correspondences and their associated tensor algebras. These results will be used heavily in Chapter 4. We end with a section on notation that is to be used in Part I of this thesis. Although we try to stay consistent in notation between the two sections, one minor change will be made in Part II which conflicts with the notation of Part I: operator algebras will be denoted by $\mathcal{A}$ and $\mathcal{B}$ in Part I, whereas $\mathcal{A}$ and $\mathcal{B}$ will be reserved specifically for a C*-algebra in Part II.

2.1 Operator systems, C*-envelopes, and maximal dilations

An operator system $\mathcal{S}$ is a subspace of a unital C*-algebra $\mathcal{C}$ for which $1_{\mathcal{C}} \in \mathcal{S}$ and $\mathcal{S}^* = \mathcal{S}$. The class of operator systems has an abstract axiomatization [14]. We will only say a word about the abstract characterization: to axiomatize operator systems it is enough to keep track of the involution *, the cone $M_n(\mathcal{S})_+$ of positive operators on $M_n(\mathcal{S}) \subseteq M_n(\mathcal{C})$, and the unit $1_{M_n(\mathcal{C})} \in M_n(\mathcal{S})$. The appropriate morphisms for operator systems are unital completely positive maps, for which we use the shorthand ucp, and the appropriate embeddings for operator systems are unital complete order embeddings, that is, maps $\varphi : \mathcal{S} \to \mathcal{T}$ for which φ and $\varphi^{-1} : \varphi(\mathcal{S}) \to \mathcal{S}$ are ucp.

Let $\mathcal{S}$ be an operator system. A C*-*cover* for $\mathcal{S}$ is a pair $(\mathcal{C}, \rho)$, where $\mathcal{C}$ is a C*-algebra and $\rho : \mathcal{S} \hookrightarrow \mathcal{C}$ is a unital complete order isomorphism such that $C^*(\rho(\mathcal{S})) = \mathcal{C}$. If $(\mathcal{C}, \rho)$ is a C*-cover for $\mathcal{S}$ and $\mathcal{I}$ is an ideal in $\mathcal{C}$, we say that $\mathcal{I}$ is a *boundary ideal* if the restriction of the canonical quotient map $q : \mathcal{C} \to \mathcal{C}/\mathcal{I}$ to $\mathcal{S}$ is a complete order embedding [2]. The *Shilov ideal* $\mathcal{J}_{\mathcal{S}}$ corresponding to the C*-cover $(\mathcal{C}, \rho)$ is the maximal boundary ideal; that is, whenever $\mathcal{I}$ is a boundary ideal for $\mathcal{S}$ in $(\mathcal{C}, \rho)$, we have $\mathcal{I} \subseteq \mathcal{J}_{\mathcal{S}}$. M. Hamana showed that the Shilov ideal for $\mathcal{S}$ in $(\mathcal{C}, \rho)$ always exists [35]. Moreover, in [35], it is shown that every operator system $\mathcal{S}$ admits a unique C*-cover $(C^*_{\text{env}}(\mathcal{S}), \iota)$, called the C*-*envelope* of $\mathcal{S}$, satisfying the following universal property: whenever $(\mathcal{C}, \rho)$ is a C*-cover of $\mathcal{S}$, there is a unique $*$-epimorphism $\pi : \mathcal{C} \to C^*_{\text{env}}(\mathcal{S})$ for which the diagram

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "\pi"] & & C^*_{\text{env}}(\mathcal{S}) \\ & \mathcal{S} \arrow[ul, "\rho"] \arrow[ur, "\iota"'] & \end{tikzcd}$$

commutes. In this setting, the Shilov ideal $\mathcal{J}_{\mathcal{S}}$ is given precisely by the kernel of the map π [35]. In other words, $\mathcal{C}/\mathcal{J}_{\mathcal{S}} \simeq C^*_{env}(\mathcal{S})$. The proof in [35] that a C*-envelope always exists uses the injective envelope of an operator system. Although this construction is useful in many respects, the construction of the C*-envelope on which we wish to concentrate in this thesis is the construction given by maximal dilations. Given a unital completely positive map $\varphi : \mathcal{S} \to B(H)$, we say that a representation $\rho : \mathcal{S} \to B(K)$ is a *dilation* of φ if there is an isometry $V : H \hookrightarrow K$ for which $V\varphi(x) = \rho(x)V$ for all $x \in \mathcal{S}$. We will always assume without loss of generality that $H \subseteq K$. In this way, we may always set $K = H \oplus H^{\perp}$ and represent $\rho(x)$ as the block 2×2-matrix

$$\rho(x) = \begin{bmatrix} \varphi(x) & a_x \\ c_x & b_x \end{bmatrix}$$

for some $a_x \in B(H^{\perp}, H)$, $b_x \in B(H^{\perp})$ and $c_x \in B(H, H^{\perp})$. Note that $c_x = a_{x^*}$. The compression to the $(2,2)$-corner of $\rho(x)$ is also a ucp map $\varphi' : \mathcal{S} \to B(H^{\perp})$. We say that the dilation ρ of φ is *trivial* if $\rho = \varphi \oplus \varphi'$; that is, $a_x = 0$ for all $x \in \mathcal{S}$. A ucp map $\varphi : \mathcal{S} \to B(H)$ is *maximal* if the only dilations of φ are trivial dilations. For an operator system $\mathcal{S}$ and a ucp map $\varphi : \mathcal{S} \to B(H)$, we say that φ has the *unique extension property* if there is a unique ucp extension of φ to $C^*_{\text{env}}(\mathcal{S})$ and the unique extension is a $*$-homomorphism. Dritschel and McCollough (see [3, Theorem 2.5] and [22]) show that a ucp map $\varphi : \mathcal{S} \to B(H)$ is maximal if and only if φ satisfies the unique extension property. In an unpublished work of Arveson, it is shown that every representation of an operator system has a maximal dilation [5, Theorem 1.3] and that if $\varphi : \mathcal{S} \to B(H)$ is maximal, then $\varphi(\mathcal{S})$ necessarily generates the C*-envelope of $\mathcal{S}$ [5, Corollary 3.3]. When it is convenient, we will always assume that our operator system $\mathcal{S}$ lies as a subspace of the C*-envelope $C^*_e(\mathcal{S})$.

An operator subsystem $\mathcal{S}$ of a C*-algebra $\mathcal{C}$ is said to be hyperrigid in $\mathcal{C}$ if we have the following unique extension property: whenever $\pi : C^*(\mathcal{S}) \to B(H)$ is a *-homomorphism and whenever $\varphi : C^*(\mathcal{S}) \to B(H)$ is a unital completely positive map extending the unital completely positive map $\pi|_{\mathcal{S}}$ then we must have $\varphi = \pi$. Hyperrigid operator systems give us a strong relation between operator systems and their C*-envelope. For example, if $\mathcal{S}$ is hyperrigid in $\mathcal{C}$ then we must have $C^*(\mathcal{S}) \simeq C^*_e(\mathcal{S})$. We say that $\mathcal{S}$ is hyperrigid if $\mathcal{S}$ is hyperrigid in $C^*_e(\mathcal{S})$. The above definition of hyperrigidity is not the original one. In [4, Definition 1.1], a subspace (that is not necessarily *-closed or unital) $\mathcal{S} \subseteq \mathcal{C}$ is said to be hyperrigid if whenever we have a faithful embedding $\mathcal{C} \subseteq B(H)$ and whenever $\varphi_n : B(H) \to B(H)$ is a sequence of completely contractive and completely positive maps, we have the implication

$$\lim_{n\to\infty} \|\varphi_n(x) - x\| = 0 \text{ for all } x \in \mathcal{S} \text{ implies } \lim_{n\to\infty} \|\varphi_n(a) - a\| = 0 \text{ for all } a \in \mathcal{C}\ .$$

In [4, Theorem 2.1], Arveson proves that these two definitions are equivalent in the separable case. The density character of a topological space X is the smallest cardinal κ for which there is a subset $E \subseteq X$ of size κ that is dense in X. Arveson's proof will go through verbatim when we replace all instances of *separable* with *density character at most* κ for any infinite cardinal κ. For completeness, we provide a sketch of the proof here.

Theorem 2.1.1. *Let $\mathcal{S}$ be an operator system of density character at most κ and let $\mathcal{C}$ be a C*-algebra generated by $\mathcal{S}$. The following are equivalent:*

1. *$\mathcal{S}$ is hyperrigid.*

2. *If $\pi : \mathcal{C} \hookrightarrow B(H)$ is a faithful representation for some Hilbert space H and $(\varphi_\lambda : B(H) \to B(H))_{\lambda\in\Lambda}$ is a net of ucp maps for which*

$$\lim_{\lambda\to\Lambda} \varphi_\lambda(\pi(s)) = \pi(s)$$

in norm for all $s \in \mathcal{S}$ then

$$\lim_{\lambda\to\Lambda} \varphi_\lambda(\pi(a)) = \pi(a)$$

in norm for all $a \in \mathcal{C}$.

3. *For every representation $\pi : \mathcal{C} \to B(H)$, where H is a Hilbert space of density character at most κ, and for every sequence $\varphi_n : \mathcal{C} \to B(H)$ of ucp maps which satisfy*

$$\lim_{n\to\infty} \|\varphi_n(s) - \pi(s)\| = 0$$

for all $s \in \mathcal{S}$, we must have

$$\lim_{n\to\infty} \|\varphi_n(a) - \pi(a)\| = 0$$

for all $a \in \mathcal{C}$.

4. *For every representation $\pi : \mathcal{C} \to B(H)$, where H is a Hilbert space of density character at most κ, π has the unique extension property.*

5. *For every unital C*-algebra $\mathcal{D}$, for every unital *-homomorphism $\theta : \mathcal{C} \to \mathcal{D}$, and for every ucp map $\varphi : \mathcal{D} \to \mathcal{D}$, if*

$$\varphi(\theta(s)) = \theta(s)$$

for all $s \in \mathcal{S}$ then

$$\varphi(\theta(a)) = \theta(a)$$

for all $a \in \mathcal{C}$.

Proof. Statement (1) is a special case of statement (2) and Arveson's proof that statement (1) is equivalent to statements (3), (4), and (5) go through vebatim in our case so we only need to verify that statement (5) implies statement (2).

For (5) implies (2), fix a net $(\varphi_\lambda : \lambda \in \Lambda)$ of ucp maps $\varphi_\lambda : B(H) \to B(H)$ for which φ_λ point-norm converges to the identity on $\mathcal{S}$. Assume without loss of generality that $\mathcal{C}$ is a C*-subalgebra of $B(H)$. Let $\mathcal{D} = B(H)$ and consider the asymptotic sequence algebra

$$\mathcal{D}_\Lambda := \ell^\infty(\Lambda, \mathcal{D})/c_\Lambda(\mathcal{D})$$

where $c_\Lambda(\mathcal{D})$ is the ideal of sequences $(b_\lambda)_\lambda$ for which $\lim_\lambda b_\lambda = 0$. Define the map

$$\varphi : \mathcal{D}_\Lambda \to \mathcal{D}_\Lambda : (b_\lambda) + c_\Lambda(\mathcal{D}) \mapsto (\varphi_\lambda(b_\lambda)) + c_\Lambda(\mathcal{D}) \ .$$

This map is well-defined since whenever $\lim_\lambda b_\lambda = 0$, as φ_λ are all contractions, $\lim_\lambda \varphi_\lambda(b_\lambda) = 0$. The map φ is completely positive since whenever $x \in M_n(\mathcal{D}_\Lambda)$ is positive, there is some $(b_\lambda) \in M_n(\ell^\infty(\Lambda, \mathcal{D}))$ positive such that $x = (b_\lambda) + M_n(c_\Lambda(\mathcal{D}))$. Let $\theta : \mathcal{C} \to \mathcal{D}_\Lambda$ be the diagonal embedding $a \mapsto (a, a, a, \ldots) + c_\Lambda(\mathcal{D})$. Since $\varphi(\theta(s)) = \theta(s)$ for all $s \in \mathcal{S}$, we have $\varphi(\theta(a)) = \theta(a)$ for all $a \in \mathcal{C}$. That is, $(\varphi_\lambda(a) - a)_\lambda$ belongs in the ideal $c_\Lambda(\mathcal{D})$. Thus, $\varphi_\lambda(a)$ norm converges to a for all $a \in \mathcal{C}$. □

Statement (2) in the next Corollary is what we will take as the definition of hyperrigidity in this Thesis.

Corollary 2.1.2. *Let $\mathcal{S}$ be an operator system generating a C*-algebra $\mathcal{C}$. The following are equivalent:*

1. *$\mathcal{S}$ is hyperrigid.*
2. *For every representation $\pi : \mathcal{C} \to B(H)$, π has the unique extension property.*

Proof. Let μ denote the density character of $\mathcal{S}$. For (1) implies (2), take any representation $\pi : \mathcal{C} \to B(H)$. Say H has density character λ. Apply the above Theorem with $\kappa = \lambda + \mu$.

The direction (2) implies (1) is a special case of (4) implies (1) of the above Theorem. □

If $\mathcal{S}$ is *-closed but non-unital, so long as $\mathcal{S}$ contains an approximate unit of $\mathcal{C}$, it follows from [72, Proposition 3.6] that $\mathcal{S}$ is hyperrigid in $C^*(\mathcal{S})$ if and only if $\mathcal{S}^+ := \mathcal{S} + \mathbb{C}1$ in the unitization $C^*(\mathcal{S})^+$ is hyperrigid. For a unital operator algebra $\mathcal{A}$, we say that $\mathcal{A}$ is hyperrigid in a C*-cover $\mathcal{D}$ if for all representations π of $\mathcal{D}$, there is a unique unital and completely contractive extension of $\pi|_{\mathcal{A}}$ to $\mathcal{D}$.

Lemma 2.1.3. *If $\mathcal{A}$ is a unital operator algebra with C^*-cover $\mathcal{D}$, and $\mathcal{S} = \mathcal{A} + \mathcal{A}^*$, then $\mathcal{A}$ is hyperrigid in $\mathcal{D}$ if and only if $\mathcal{S}$ is hyperrigid in $\mathcal{D}$.*

Proof. If $\mathcal{A}$ is hyperrigid in $\mathcal{D}$ and $\pi : \mathcal{D} \to B(H)$ is a unital $*$-homomorphism, then any ucp extension $\phi : \mathcal{D} \to B(H)$ of $\pi|_{\mathcal{S}} : \mathcal{S} \to B(H)$ is necessarily a unital, completely contractive extension of $\pi|_{\mathcal{A}}$, so that $\phi = \pi$. Hence, $\mathcal{S}$ is hyperrigid in $\mathcal{D}$. Similarly, if $\mathcal{S}$ is hyperrigid in $\mathcal{D}$ and $\pi : \mathcal{D} \to B(H)$ is a unital $*$-homomorphism, then any unital completely contractive extension $\psi : \mathcal{D} \to B(H)$ of $\pi|_{\mathcal{A}} : \mathcal{A} \to B(H)$ is ucp and satisfies $\psi(a + a^*) = \pi(a) + \pi(a)^*$ for all $a \in \mathcal{A}$. Thus, ψ is a ucp extension of $\pi|_{\mathcal{S}}$, so that $\psi = \pi$, which establishes the converse direction. □

In particular, the conditions in Theorem 2.1.1 are equivalent to hpyerrigidity when we replace all instances of $\mathcal{S}$ with $\mathcal{A}$.

A representation $\pi : C^*(\mathcal{S}) \to B(H)$ is said to be boundary if π is irreducible and π admits the unique extension property. Arveson's hyperrigidity conjecture asserts that if all irreducible representations are boundary then the operator system $\mathcal{S}$ must be hyperrigid in $\mathcal{C}$. Very little is known about the hyperrigidity conjecture. For more information on operator systems, see [64]. See [4] for the formulation of the hyperrigidity conjecture and more details on the above results.

2.2 Crossed products of operator algebras

Let $\mathcal{A}$ be an operator algebra. That is, $\mathcal{A}$ is a norm closed subalgebra of $B(H)$ with an approximate unit and morphisms given by completely contractive algebra homomorphisms. For this thesis, we will assume that all operator algebras are approximately unital and not necessarily closed under *. We will always require that representations of $\mathcal{A}$ be non-degenerate. An *automorphism* on $\mathcal{A}$ is a completely isometric isomorphism $\phi : \mathcal{A} \to \mathcal{A}$. Note that if $\mathcal{A}$ is unital, then any automorphism on $\mathcal{A}$ is automatically unital. An *operator algebra dynamical system* is a triple $(\mathcal{A}, G, \alpha)$, where $\mathcal{A}$ is an approximately unital operator algebra, G is a locally compact group, and $\alpha : G \to \mathrm{Aut}(\mathcal{A})$ is a strongly continuous group homomorphism into the group $\operatorname{Aut}(\mathcal{A})$ of automorphisms on $\mathcal{A}$. In [47], crossed products for operator algebras are introduced. Katsoulis and Ramsey define these as operator subalgebras of a C*-algebraic crossed product. To do this, they first define an *α-admissible* C*-*cover* of a dynamical system $(\mathcal{A}, G, \alpha)$ to be any C*-dynamical system $(\mathcal{C}, G, \tilde{\alpha})$ with a completely isometric homomorphism $\rho : \mathcal{A} \hookrightarrow \mathcal{C}$ such that, for all $s \in G$, the diagram

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\widetilde{\alpha}_s"] & \mathcal{C} \\ \mathcal{A} \arrow[u, "\rho"] \arrow[r, "\alpha_s"] & \mathcal{A} \arrow[u, "\rho"] \end{tikzcd}$$

commutes. It is shown in [47] that, given a dynamical system $(\mathcal{A}, G, \alpha)$, both the C*-envelope and universal C*-algebra of $\mathcal{A}$ admit an α-admissible C*-cover, with action denoted by the symbol α.

Definition 2.2.1. Let $(\mathcal{A}, G, \alpha)$ be an operator algebraic dynamical system and let $(\mathcal{C}, G, \alpha)$ be an α-admissible C*-cover with embedding $\rho : \mathcal{A} \hookrightarrow \mathcal{C}$.

1. The *reduced crossed product* $\mathcal{A} \rtimes_{\alpha}^{\lambda} G$ is the norm closure of $C_c(G, \rho(\mathcal{A}))$ in $\mathcal{C} \rtimes_{\alpha,\lambda} G$.
2. The *full crossed product relative to* $\mathcal{C}$, denoted by $\mathcal{A} \rtimes_{\mathcal{C},\alpha} G$, is the norm closure of $C_c(G, \rho(\mathcal{A}))$ in $\mathcal{C} \rtimes_{\alpha} G$.
3. The *full crossed product* $\mathcal{A} \rtimes_{\alpha} G$ is the full crossed product relative to $C^*_{\max}(\mathcal{A})$ (see [10] for the definition of $C^*_{\max}(\mathcal{A})$).

The reduced crossed product is independent of the choice of admissible C*-cover: if $(\mathcal{C}, G, \alpha)$ is any admissible C*-cover of $(\mathcal{A}, G, \alpha)$ with embedding $\rho : \mathcal{A} \hookrightarrow \mathcal{C}$, then the map

$$\varphi : C_c(G, \mathcal{A}) \subseteq C^*_{\text{env}}(\mathcal{A}) \rtimes_{\lambda} G \to C_c(G, \rho(A)) \subseteq \mathcal{C} \rtimes_{\lambda} G : f \mapsto \rho \circ f$$

extends to a completely isometric isomorphism of operator algebraic crossed products [47]. Theorem 3.3.2 will show that the analogous result for full relative crossed products need not hold in general. As in the case of C*-algebras, the full crossed product $\mathcal{A} \rtimes_{\alpha} G$ of an operator algebra satisfies the following universal property: if $(\pi, u) : (\mathcal{A}, G) \to B(H)$ is a covariant pair, in the sense that π is a completely contractive homomorphism on $\mathcal{A}$ and u is a homomorphism on G with $\pi(\alpha_s(a)) = u_s \pi(a) u_s^*$ for all $s \in G$ and $a \in \mathcal{A}$, then there is a canonical completely contractive homomorphism

$$\pi \rtimes u : \mathcal{A} \rtimes_{\alpha} G \to B(H)$$

extending π and u. This map is called the *integrated form* of (π, u).

When the group action is trivial, the resulting crossed product structures become operator algebra tensor products. We briefly recall the definition of these tensor products. We refer the reader to [10] for more information on operator algebra tensor products.

Definition 2.2.2. Let $\mathcal{A} \subseteq B(H)$ and $\mathcal{B} \subseteq B(K)$ be approximately unital operator algebras. The *minimal tensor product* of $\mathcal{A}$ and $\mathcal{B}$, denoted by $\mathcal{A} \otimes_{\min} \mathcal{B}$, is the completion of $\mathcal{A} \otimes \mathcal{B}$ with respect to the norm inherited from $B(H \otimes K)$.

Note that matrix norms for $\mathcal{A} \otimes_{\min} \mathcal{B}$ are also inherited from $B(H \otimes K)$. The definition of the minimal tensor product does not depend on the choice of embeddings.

Definition 2.2.3. For approximately unital operator algebras $\mathcal{A}$ and $\mathcal{B}$, the *maximal tensor product* of $\mathcal{A}$ and $\mathcal{B}$, denoted by $\mathcal{A} \otimes_{\max} \mathcal{B}$, is the completion of $\mathcal{A} \otimes \mathcal{B}$ with respect to the norm on $\mathcal{A} \otimes \mathcal{B}$ given by

$$\|x\|_{\max} = \sup\{\|\pi \cdot \rho(x)\|_{\mathcal{B}(\mathcal{H})}\},$$

where the supremum is taken over all Hilbert spaces $\mathcal{H}$ and all completely contractive representations $\pi : \mathcal{A} \to B(H)$ and $\rho : \mathcal{B} \to B(H)$ with commuting ranges.

We note that matrix norms for $\mathcal{A} \otimes_{\max} \mathcal{B}$ are defined similarly. In the supremum, the maps π and ρ can be assumed to be non-degenerate [10, 6.1.11]. Moreover, if $\mathcal{B}$ is a C*-algebra, then $\mathcal{A} \otimes_{\max} \mathcal{B}$ is completely isometrically contained in the C*-algebraic tensor product $C^*_{\max}(\mathcal{A}) \otimes_{\max} \mathcal{B}$ [10, 6.1.9].

Example 2.2.4. Let $\mathcal{A}$ be an approximately unital operator algebra and let G be a locally compact group. Let $\mathrm{id} : G \to \mathrm{Aut}(\mathcal{A})$ be the trivial action on $\mathcal{A}$; that is, $\mathrm{id}_s = \mathrm{id}_{\mathcal{A}}$ for all $s \in G$. We have natural isomorphisms

$$\begin{aligned} \mathcal{A} \rtimes^{\lambda}_{\mathrm{id}} G &\simeq \mathcal{A} \otimes_{\min} C^*_{\lambda}(G) : a\lambda_s \mapsto a \otimes \lambda_s, \text{ and} \\ \mathcal{A} \rtimes_{\mathrm{id}} G &\simeq \mathcal{A} \otimes_{\max} C^*(G) : au_s \mapsto a \otimes u_s \ . \end{aligned}$$

In the reduced case, we have the natural isomorphism

$$C^*_{\mathrm{env}}(\mathcal{A}) \rtimes_{\mathrm{id},\lambda} G = C^*_{\mathrm{env}}(\mathcal{A}) \otimes_{\min} C^*_{\lambda}(G)$$

since the left regular representation is exactly the representation of the minimal tensor product. In the full case, we have the natural isomorphism

$$C^*_{\max}(\mathcal{A}) \rtimes_{\mathrm{id}} G = C^*_{\max}(\mathcal{A}) \otimes_{\max} C^*(G)$$

since the universal property for both algebras are identical. Restricting these isomorphisms to $C_c(G, \mathcal{A})$ yields the result. For more details on these isomorphisms, see [79, Lemma 2.73 and Corollary 7.17].

2.3 Operator system tensor products

We briefly recall some facts about operator system tensor products here. More information can be found in [52].

An *operator system tensor product* τ is a map that sends a pair of operator systems $(\mathcal{S}, \mathcal{T})$ to an operator system $\mathcal{S} \otimes_\tau \mathcal{T}$ such that

1. if $X = (X_{ij}) \in M_n(\mathcal{S})_+$ and $Y = (Y_{k\ell}) \in M_m(\mathcal{T})_+$, then $X \otimes Y := (X_{ij} \otimes Y_{k\ell}) \in M_{nm}(\mathcal{S} \otimes_\tau \mathcal{T})_+$; and
2. if $\phi : \mathcal{S} \to M_n$ and $\psi : \mathcal{T} \to M_m$ are unital completely positive (ucp) maps, then $\phi \otimes \psi : \mathcal{S} \otimes_\tau \mathcal{T} \to M_{nm}$ is ucp.

An operator system tensor product τ is said to be *symmetric* if, for every pair of operator systems $(\mathcal{S}, \mathcal{T})$, the flip map $\mathcal{S} \otimes \mathcal{T} \to \mathcal{T} \otimes \mathcal{S}$ induces a complete order isomorphism $\mathcal{S} \otimes_\tau \mathcal{T} \to \mathcal{T} \otimes_\tau \mathcal{S}$.

We will be working with four main operator system tensor products:

Definition 2.3.1. The *minimal tensor product* of $\mathcal{S}$ and $\mathcal{T}$, denoted by $\mathcal{S} \otimes_{\min} \mathcal{T}$, is defined such that $X \in M_n(\mathcal{S} \otimes_{\min} \mathcal{T})$ is positive if and only if $(\phi \otimes \psi)^{(n)}(X) \in M_{kmn}^+$ for every pair of ucp maps $\phi : \mathcal{S} \to M_k$ and $\psi : \mathcal{T} \to M_m$.

A fact that will be used throughout the paper is that the minimal tensor product is injective; i.e., whenever $\mathcal{S}_1, \mathcal{S}_2, \mathcal{T}_1, \mathcal{T}_2$ are operator systems with unital complete order embeddings $\iota : \mathcal{S}_1 \subseteq \mathcal{S}_2$ and $\kappa : \mathcal{T}_1 \subseteq \mathcal{T}_2$, the tensor product map $\iota \otimes \kappa : \mathcal{S}_1 \otimes_{\min} \mathcal{T}_1 \to \mathcal{S}_2 \otimes_{\min} \mathcal{T}_2$ is a complete order embedding [52, Theorem 4.6]. In particular, if $\mathcal{S}$ is an operator subsystem of $B(H)$ and $\mathcal{T}$ is an operator subsystem of $B(K)$, then $\mathcal{S} \otimes_{\min} \mathcal{T}$ is completely order isomorphic to the image of $\mathcal{S} \otimes \mathcal{T}$ in $B(H \otimes K)$ [52, Theorem 4.4].

For two linear maps $\phi : \mathcal{S} \to B(H)$ and $\psi : \mathcal{T} \to B(H)$ with commuting ranges (i.e. $\phi(s)\psi(t) = \psi(t)\phi(s)$ for all $s \in \mathcal{S}$ and $t \in \mathcal{T}$), we let $\phi \cdot \psi : \mathcal{S} \otimes \mathcal{T} \to B(H)$ be defined on the vector space tensor product by $\phi \cdot \psi(s \otimes t) = \phi(s)\psi(t)$.

Definition 2.3.2. The *commuting tensor product* of $\mathcal{S}$ and $\mathcal{T}$, denoted by $\mathcal{S} \otimes_c \mathcal{T}$, is defined such that $X \in M_n(\mathcal{S} \otimes_c \mathcal{T})$ is positive if and only if $(\phi \cdot \psi)^{(n)}(X) \in M_n(B(H))_+$ for every pair of ucp maps $\phi : \mathcal{S} \to B(H)$ and $\psi : \mathcal{T} \to B(H)$ with commuting ranges.

We note that $\mathcal{S} \otimes_c \mathcal{T}$ is completely order isomorphic to the inclusion $\mathcal{S} \otimes \mathcal{T} \subseteq C_u^*(\mathcal{S}) \otimes_{\max} C_u^*(\mathcal{T})$ [52, Theorem 6.4].

Definition 2.3.3. The *maximal tensor product* of $\mathcal{S}$ and $\mathcal{T}$, denoted by $\mathcal{S} \otimes_{\max} \mathcal{T}$, is defined such that $X \in M_n(\mathcal{S} \otimes_{\max} \mathcal{T})$ is positive if and only if, for every $\varepsilon > 0$, there are $S_\varepsilon \in M_{k(\varepsilon)}(\mathcal{S})_+$, $T_\varepsilon \in M_{m(\varepsilon)}(\mathcal{T})_+$ and a linear map $A_\varepsilon : \mathbb{C}^n \to \mathbb{C}^{k(\varepsilon)} \otimes \mathbb{C}^{m(\varepsilon)}$ such that

$$X + \varepsilon 1 = A_\varepsilon^*(S_\varepsilon \otimes T_\varepsilon)A_\varepsilon.$$

Definition 2.3.4. The *essential tensor product* of $\mathcal{S}$ and $\mathcal{T}$, denoted by $\mathcal{S} \otimes_{\text{ess}} \mathcal{T}$, is the operator system structure on $\mathcal{S} \otimes \mathcal{T}$ inherited from the inclusion $\mathcal{S} \otimes \mathcal{T} \subseteq C^*_{\text{env}}(\mathcal{S}) \otimes_{\max} C^*_{\text{env}}(\mathcal{T})$.

For any two operator system tensor products α and β, we write $\alpha \leq \beta$ if, for all operator systems $\mathcal{S}$ and $\mathcal{T}$, the identity map $\text{id} : \mathcal{S} \otimes_\beta \mathcal{T} \to \mathcal{S} \otimes_\alpha \mathcal{T}$ is ucp. An operator system $\mathcal{S}$ is said to be (α, β)-*nuclear* if for every operator system $\mathcal{T}$, the identity map $\text{id} : \mathcal{S} \otimes_\alpha \mathcal{T} \to \mathcal{S} \otimes_\beta \mathcal{T}$ is a complete order isomorphism. For example, every unital C*-algebra is $(c, \max)$-nuclear [52, Theorem 6.7].

2.4 Finite-dimensional operator system quotients and duals

In general, the dual space of an operator system can always be made into a matrix-ordered $*$-vector space [14, Lemma 4.2, Lemma 4.3] as follows: if $\mathcal{S}$ is an operator system with Banach space dual $\mathcal{S}^d$, and $f = (f_{ij}) \in M_n(\mathcal{S}^d)$, then we define $f^* = (f_{ji}^*)$, where $f_{ij}^*(s) := \overline{f_{ij}(s^*)}$ for all $1 \leq i, j \leq n$ and $s \in \mathcal{S}$. We say that a self-adjoint element $f = (f_{ij}) \in M_n(\mathcal{S}^d)$ is positive if the associated map $F : \mathcal{S} \to M_n$ given by $F(s) = (f_{ij}(s))$ is completely positive. With this structure, $\mathcal{S}^d$ becomes a matrix-ordered $*$-vector space. If $\mathcal{S}$ is not finite-dimensional, then $\mathcal{S}^d$ may not have an order unit, and hence may not be an operator system. However, if $\mathcal{S}$ is finite-dimensional, then $\mathcal{S}^d$ is an operator system, and any faithful state on $\mathcal{S}^d$ will be an order unit for $\mathcal{S}^d$ [14].

The theory of operator system quotients is rather new and not well understood. If $\phi : \mathcal{S} \to \mathcal{T}$ is a surjective ucp map between operator systems, then we may endow the quotient vector space $\mathcal{S}/\ker(\phi)$ with an operator system structure [53]. For $s \in \mathcal{S}$, we write $\dot{s}$ to denote its image in $\mathcal{S}/\ker(\phi)$. We say that $\dot{X} = (\dot{X}_{ij}) \in M_n(\mathcal{S}/\ker(\phi))$ is positive if, for every $\varepsilon > 0$, there is $Y_\varepsilon \in M_n(\mathcal{S})_+$ such that $\dot{Y}_\varepsilon = \dot{X} + \varepsilon \dot{I}_n$, where I_n denotes the $n \times n$ identity matrix in $M_n(\mathcal{S})$. Note that whenever $\phi : \mathcal{S} \to \mathcal{T}$ is a surjective ucp map, the induced map $\dot{\phi} : \mathcal{S}/\ker(\phi) \to \mathcal{T}$ is ucp [53, Proposition 3.6]. This leads to the following definition.

Definition 2.4.1. A surjective ucp map $\phi : \mathcal{S} \to \mathcal{T}$ between operator systems is said to be a *complete quotient map* if the induced map $\dot{\phi} : \mathcal{S}/\ker(\phi) \to \mathcal{T}$ is a complete order isomorphism.

For our purposes, it will be helpful to translate between complete quotient maps and complete order embeddings, via the Banach space adjoint. Recall that whenever $\phi : \mathcal{S} \to \mathcal{T}$ is a ucp map between finite-dimensional operator systems, we may define a ucp map $\phi^d : \mathcal{T}^d \to \mathcal{S}^d$ by $[\phi^d(f)](s) = f(\phi(s))$. Then a ucp map $\phi : \mathcal{S} \to \mathcal{T}$ between finite-dimensional operator systems is a complete quotient map if and only if $\phi^d : \mathcal{T}^d \to \mathcal{S}^d$ is a complete order embedding [28, Proposition 1.8].

2.5 C*-correspondences and the tensor algebra $\mathcal{T}_X^+$

A Hilbert module is a generalization of Hilbert space with the scalar coefficients replaced by a fixed C*-algebra. That is, a Hilbert $\mathcal{C}$-module is a pair $(\mathcal{C}, X)$, where $\mathcal{C}$ is a C*-algebra and X is a right $\mathcal{C}$-module with a $\mathcal{C}$-valued inner product

$$\langle \cdot, \cdot \rangle : X \times X \to \mathcal{C}$$

satisfying the following axioms:

1. For fixed $x \in X$, the map $\langle x, \cdot \rangle : X \to X$ is $\mathcal{C}$-linear.
2. For any $x, y \in X$, $\langle x, y \rangle^* = \langle y, x \rangle$.
3. For every $x \in X$, $\langle x, x \rangle \geq 0$. As well, $x = 0$ if and only if $\langle x, x \rangle = 0$.
4. The space X is complete with respect to the 2-norm $\|x\| := \sqrt{\| \langle x, x \rangle \|}$.

Given a Hilbert $\mathcal{C}$-module, one can define the C*-algebra $\mathcal{L}(X)$ of bounded $\mathcal{C}$-linear maps from X to itself with norm given by the supremum norm. A C*-correspondence is a Hilbert $\mathcal{C}$-module $(\mathcal{C}, X)$ along with a *-homomorphism

$$\lambda : \mathcal{C} \to \mathcal{L}(X)\ .$$

Whenever convenient, we will denote this action by left multiplication: $\lambda(a)x = a \cdot x$.

Let $(\mathcal{C}, X)$ be a C*-correspondence and let $\mathcal{D}$ be a C*-algebra. We say that a pair of maps $(\pi^0, \pi^1) : (\mathcal{C}, X) \to \mathcal{D}$ is a Toeplitz pair if

1. $\pi^0 : \mathcal{C} \to \mathcal{D}$ is a *-homomorphism,
2. $\pi^1 : X \to \mathcal{D}$ is a linear map,
3. For any $a \in \mathcal{C}$ and $x \in X$ we have $\pi^0(a)\pi^1(x) = \pi^1(a \cdot x)$, and
4. For any x and y in X we have $\pi^0(\langle x, y\rangle) = \pi^1(x)^*\pi^1(y)$.

Given a Toeplitz pair (π^0, π^1), we can show that

$$(\pi^1(x)\pi^0(a) - \pi^1(x \cdot a))^*(\pi^1(x)\pi^0(a) - \pi^1(x \cdot a)) = 0 \ .$$

Because of this, we always have $\pi^1(x)\pi^0(a) = \pi^1(x \cdot a)$ for any $x \in X$ and $a \in \mathcal{C}$. A Toeplitz pair can also be thought of as a morphism from the C*-correspondence $(\mathcal{C}, X)$ into the C*-correspondence $(\mathcal{D}, \mathcal{D})$ where left and right action is given by multiplication and the inner product is given by $\langle x, y\rangle = x^*y$. There is always a maximal C*-algebra associated to C*-correspondences called the Toeplitz-Pimsner algebra $\mathcal{T}_X$. This C*-algebra is maximal in the following sense: there is always a Toeplitz pair

$$\begin{aligned} \kappa^0 &: \mathcal{C} \to \mathcal{T}_X \\ \kappa^1 &: X \to \mathcal{T}_X \end{aligned}$$

into $\mathcal{T}_X$ and whenever $(\pi^0, \pi^1) : (\mathcal{C}, X) \to \mathcal{D}$ is a Toeplitz pair then there is a *-homomorphism

$$\pi^0 \times \pi^1 : \mathcal{T}_X \to \mathcal{D}$$

for which the diagram

$$\begin{tikzcd} \mathcal{T}_X \arrow[r, "\pi^0\times\pi^1"] & \mathcal{D} \\ (\mathcal{C}, X) \arrow[u, "(\kappa^0,\kappa^1)"] \arrow[ur, "(\pi^0,\pi^1)"'] & \end{tikzcd}$$

commutes. The Toeplitz-Pimsner algebra always contains a canonical norm closed non-selfadjoint operator algebra $\mathcal{T}_X^+$ called the Tensor algebra. This algebra is described as the non-selfadjoint operator algebra generated by $\kappa^0(\mathcal{C})$ and $\kappa^1(X)$ in $\mathcal{T}_X$.

The Toeplitz-Pimsner algebra $\mathcal{T}_X$ always admits a canonical continuous $\mathbb{T}$-action γ called the gauge action. Using the universal property of $\mathcal{T}_X$, it is enough to define γ as an action on $(\mathcal{C}, X)$: for $z \in \mathbb{T}$,

$$\begin{aligned}\gamma_z^0 &: \mathcal{C} \to \mathcal{C} : a \mapsto a \\ \gamma_z^1 &: X \to X : x \mapsto z \cdot x\end{aligned}$$

will give us the action.

Although the Toeplitz-Pimsner algebra $\mathcal{T}_X$ is a canonical algebra associated to $(\mathcal{C}, X)$, it is often too big for our purposes as the gauge-invariant uniqueness theorem for graph algebras will not generalize to $\mathcal{T}_X$.

Example 2.5.1. Here we show that the gauge-invariant uniqueness theorem will not generalize to $\mathcal{T}_X$. That is, we show that there is a C*-correspondence $(\mathcal{C}, X)$ and a Toeplitz pair $(\pi^0, \pi^1) : (\mathcal{C}, X) \to \mathcal{D}$ for which $\mathcal{D}$ admits a gauge action but the *-homomorphism $\pi^0 \times \pi^1$ is not injective.

For our C*-correspondence, we take the correspondence $(\mathbb{C}, \mathbb{C})$ associated to the C*-algebra $\mathbb{C}$. The algebra $\mathcal{T}_{\mathbb{C}}$ is the universal C*-algebra generated by a single isometry. To see this, suppose that $C^*(V)$ is the universal C*-algebra generated by a single isometry. Define the pair of maps

$$(\kappa^0, \kappa^1) : (\mathbb{C}, \mathbb{C}) \to C^*(V)$$

for which $\kappa^0(1) = 1$ and for which $\kappa^1(1) = V$. Since V is an isometry, for every $a, x, y \in \mathbb{C}$, we have the relations

$$\kappa^0(\langle x, y\rangle) = \bar{x}y = \bar{x}V^*yV = \left\langle \kappa^1(x), \kappa^1(y)\right\rangle \text{ and } \kappa^1(ax) = axV = \kappa^0(a)\kappa^1(x) .$$

It follows that (κ^0, κ^1) is a Toeplitz pair. Next we claim that whenever there is a Toeplitz pair (π^0, π^1) from $(\mathbb{C}, \mathbb{C})$ into a C*-algebra $\mathcal{D}$, that $\pi^1(1)$ is an isometry. This follows from the identity $1 = \pi^0(\langle 1, 1\rangle) = \langle \pi^1(1), \pi^1(1)\rangle = \pi^1(1)^*\pi^1(1)$. The universal property of $\mathcal{T}_{\mathbb{C}}$ follows since $C^*(V)$ is the universal C*-algebra generated by an isometry. On the other hand, there is always a gauge-invariant Toeplitz pair from $(\mathbb{C}, \mathbb{C})$ into $C^*(\mathbb{Z})$ by mapping 1 to the canonical unitary u associated to $1 \in \mathbb{Z}$. To see this, we define the Toeplitz pair (π^0, π^1) from $(\mathbb{C}, \mathbb{C})$ into $C^*(\mathbb{Z})$ by $\pi^0(1) = 1$ and $\pi^1(1) = u$. The above calculation shows that (π^0, π^1) is indeed a Toeplitz pair. As well, there is a $\mathbb{T}$-action $\gamma : \mathbb{T} \curvearrowright C^*(\mathbb{Z})$ given by the extending the group homomorphism

$$U_z : \mathbb{Z} \to U(C^*(\mathbb{Z})) : 1 \mapsto zu$$

for all $z \in \mathbb{T}$. That this action is continuous follows from an application of triangle inequality on *-polynomials generated by u. Therefore, there is always a gauge-invariant *-homomorphism $\pi^0 \times \pi^1 : \mathcal{T}_{\mathbb{C}} \to C^*(\mathbb{Z})$ but, while the gauge-invariant uniqueness theorem would state that such a map should be injective, this map is not.

The remedy for the failure of the gauge-invariant uniqueness theorem is to restrict our class of representations.

Fix a C*-correspondence $(\mathcal{C}, X)$. The compact operators $\mathcal{K}(X)$ is the C*-subalgebra of the space $\mathcal{L}(X)$ of adjointable right-$\mathcal{C}$-linear operators on X spanned by the rank one operators $x \langle y, \cdot \rangle$ for $x, y \in \mathcal{X}$. Given a Toeplitz pair $(\pi^0, \pi^1) : (\mathcal{C}, X) \to \mathcal{D}$, there is always a *-homomorphism

$$\varphi_\pi : \mathcal{K}(X) \to \mathcal{D} : x \langle y, \cdot \rangle \mapsto \pi^1(x)\pi^1(y)^* \ .$$

The Katsura ideal $\mathcal{J}_X$ associated to $(\mathcal{C}, X)$ consists of elements $a \in \mathcal{C}$ for which $\lambda(a) \in \mathcal{K}(X)$ and for which $ab = 0$ whenever b belongs to the kernel of λ. A Toeplitz pair $(\pi^0, \pi^1) : (\mathcal{C}, X) \to \mathcal{D}$ is said to be covariant if for any element $a \in \mathcal{J}_X$, we have the identity

$$\pi^0(a) = \varphi_\pi(\lambda(a)) \ .$$

The appropriate choice of C*-algebra is the universal C*-algebra associated to covariant Toeplitz pairs. This algebra is called the Cuntz-Pimsner algebra $\mathcal{O}_X$. We will let

$$\begin{aligned} \iota^0 &: \mathcal{C} \to \mathcal{O}_X \\ \iota^1 &: X \to \mathcal{O}_X \end{aligned}$$

be the canonical covariant Toeplitz pair. Since the gauge action $(\gamma^0, \gamma^1) : \mathbb{T} \curvearrowright (\mathcal{C}, X)$ is covariant, $\mathcal{O}_X$ has a gauge action as well. As well, there is a canonical quotient map $\mathcal{T}_X \to \mathcal{O}_X$. We also have the gauge invariant uniqueness theorem [49, Theorem 6.4].

Theorem 2.5.2 (Gauge-invariant uniqueness theorem)**.** *Suppose that there is a covariant Toeplitz pair $(\pi^0, \pi^1) : (\mathcal{C}, X) \to \mathcal{D}$ with π^0 injective and suppose that there is a gauge action $\mathbb{T} \curvearrowright C^*(\pi^0, \pi^1)$ for which the Toeplitz pair (π^0, π^1) is $\mathbb{T}$-equivariant. The *-homomorphism*

$$\pi^0 \times \pi^1 : \mathcal{O}_X \to \mathcal{D}$$

is necessarily injective.

Example 2.5.3. Let $E = (E^0, E^1, s, r)$ be a topological graph. That is, E^0, E^1 are locally compact topological spaces, and $s, r : E^1 \to E^0$ are continuous maps. They are called graphs as we are thinking of E^0 as the space of vertices and E^1 as the space of edges. The maps s and r determine the source and the range of an edge. Therefore, these graphs are necessarily directed. We will also assume that s is a local homeomorphism: for every point $e \in E^1$, there is an open neighbourhood U of e such that s forms a homeomorphism of U onto its range. Define a C*-correspondence $X(E)$ over the C*-algebra $C_0(E^0)$ as the completion of $C_c(E^1)$ with left and right actions given by

$$\begin{aligned} f \cdot g &: e \mapsto f(e)g(s(e)) \text{ and} \\ g \cdot f &: e \mapsto g(r(e))f(e) \end{aligned}$$

for any $f \in C_c(E^1)$ and $g \in C_0(E^0)$ and with inner product given by

$$\langle f, h \rangle : x \in E^0 \mapsto \sum_{e \in E^1 : s(e) = x} \overline{f(e)} h(e)$$

for any $f, h \in C_c(E^1)$. The graph C*-algebra $C^*(E)$ is the Cuntz-Pimsner algebra $\mathcal{O}_{X(E)}$. This construction of C*-correspondences associated to topological graphs are introduced by Katsura in [48].

A result of Katouslis and Kribs shows that the tensor algebra $\mathcal{T}_X^+$ always sits completely isometrically as a subset of $\mathcal{O}_X$ [45, Lemma 3.5]. Moreover, they show that $\mathcal{O}_X$ is the C*-envelope of $\mathcal{T}_X^+$ [45, Theorem 3.7].

Definition 2.5.4. Let $(\mathcal{C}, X)$ be a C*-correspondence. We define the operator space $S(\mathcal{C}, X)$ as the *-closed operator subspace of $\mathcal{O}_X$ generated by X and $\mathcal{C}$.

An elementary argument shows that $S(\mathcal{C}, X)$ sits completely isometrically in both $\mathcal{T}_X$ and $\mathcal{O}_X$.

2.6 Notational conventions

In Part I of this thesis all non-selfadjoint operator algebras will be denoted by the script letters $\mathcal{A}$ and $\mathcal{B}$. Our groups will be denoted by the letter G and are assumed to be discrete unless otherwise stated. Unless otherwise specified, all C*-algebras will be denoted by the letters $\mathcal{C}$ and $\mathcal{D}$. Operator systems will generally be denoted by the letters $\mathcal{S}$ and $\mathcal{T}$. The order unit of an operator system $\mathcal{S}$ will be denoted by the unit 1 or $1_{\mathcal{S}}$ if it needs to be specified, with the exception of the order unit of the matrix algebra $M_n(\mathbb{C})$, which will be denoted by I_n.

Chapter 3

Crossed products of operator systems

In this chapter, which is joint work with Samuel Harris, we introduce the notion of a crossed product of an operator system. Much like C* and operator algebraic crossed products, crossed products of operator systems form a functor from the category of G-operator systems, where G is a discrete group, into the category of operator systems. Motivation for this work comes from two fronts: firstly, the work of Hamana, Kalantar, Kennedy, and many others demonstrate that G-operator systems are interesting objects in their own right and they can provide new insight even in the C*-category. Secondly, the work of Katsoulis and Ramsey demonstrates that looking at crossed products for subobjects of C*-algebras provide new insight and new results in the isomorphism problem of Hao and Ng.

Our construction of the crossed products mimick the construction of Katsoulis and Ramsey, who construct crossed products on operator algebras by giving a concrete description as subalgebras of an ambient C*-cover. As mentioned in the preliminaries, although crossed products of operator systems are functorial, they do not preserve G-equivariant quotient maps. Beyond this, the universal C*-cover of an operator system crossed product need not be the universal crossed product of some C*-cover. These differences mean that crossed products of operator systems are not just a straight-forward generalization of crossed products of operator algebras.

Finally, we finish this chapter by solving two problems of Katsoulis and Ramsey that ask whether universal crossed products of operator algebras are independent of the ambient C*-cover. We resolve this problem by first encoding the problem to one about operator system crossed products then appealing to Kavruk's nuclearity detectors to show that their problem is answered in the negative. The counter-example algebra is extremely tame: it is a five dimensional subalgebra of M_6 and, using the fact that the hyperrigidity conjecture holds when the spectrum is at most countable, we show that it is hyperrigid.

3.1 Reduced Crossed Products

Let G be a discrete group. We will assume that we are working with a set of generators $\mathfrak{g}$ for G such that $\mathfrak{g}^{-1} = \mathfrak{g}$ and $e \in \mathfrak{g}$.

Definition 3.1.1. If $\mathcal{S}$ is an operator system, then $\mathrm{Aut}(\mathcal{S})$ is the group of unital complete order isomorphisms $\phi : \mathcal{S} \to \mathcal{S}$. An *(operator system) dynamical system* is a 4-tuple $(\mathcal{S}, G, \mathfrak{g}, \alpha)$, where $\mathcal{S}$ is an operator system, G is a group with generating set $\mathfrak{g}$, and $\alpha : G \to \mathrm{Aut}(\mathcal{S})$ is a group homomorphism.

Let $(\mathcal{S}, G, \mathfrak{g}, \alpha)$ be a dynamical system, and let $\rho : \mathcal{S} \to \mathcal{C}$ be a complete order embedding for which $C^*(\rho(\mathcal{S})) = \mathcal{C}$. The C*-cover $\mathcal{C}$ is said to be α-*admissible* if there is a group action $\overline{\alpha} : G \to \mathrm{Aut}(\mathcal{C})$ for which the diagram

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\overline{\alpha}_g} & \mathcal{C} \\ {\scriptstyle\rho}\big\uparrow & & {\scriptstyle\rho}\big\uparrow \\ \mathcal{S} & \xrightarrow{\alpha_g} & \mathcal{S} \end{array}$$

commutes for all $g \in G$. We denote such an α-admissible C*-cover by the triple $(\mathcal{C}, \rho, \overline{\alpha})$.

Given an α-admissible C*-cover $(\mathcal{C}, \rho, \overline{\alpha})$ of a dynamical system $(\mathcal{S}, G, \mathfrak{g}, \alpha)$, the *reduced crossed product relative to* $\mathfrak{g}$ is defined as the operator subsystem of the reduced crossed product C*-algebra $\mathcal{C} \rtimes_{\overline{\alpha},\lambda} G$ given by

$$\mathcal{S} \rtimes_{\alpha,\lambda}^{(\mathcal{C},\rho)} \mathfrak{g} := \mathrm{span}\{\rho(a)\lambda_g : a \in \mathcal{S}, g \in \mathfrak{g}\} \subseteq \mathcal{C} \rtimes_{\overline{\alpha},\lambda} G \ .$$

Finally, given two operator system dynamical systems $(\mathcal{S}, G, \mathfrak{g}, \alpha)$ and $(\mathcal{T}, G, \mathfrak{h}, \beta)$, we say that a ucp map $\varphi : \mathcal{S} \to \mathcal{T}$ is G-*equivariant* if for every $g \in G$ and $s \in \mathcal{S}$, we have

$$\beta_g(\varphi(s)) = \varphi(\alpha_g(s)).$$

Remark 3.1.2. If G is a discrete group with generating sets $\mathfrak{g}$ and $\mathfrak{h}$ and $\mathfrak{g} \subseteq \mathfrak{h}$ then for any G-action $\alpha : G \to \mathrm{Aut}(\mathcal{S})$ and any admissible C*-cover $(\mathcal{C}, \rho, \alpha)$ we have the complete order embedding

$$\mathcal{S} \rtimes_{\alpha,\lambda}^{(\mathcal{C},\rho)} \mathfrak{g} \subseteq \mathcal{S} \rtimes_{\alpha,\lambda}^{(\mathcal{C},\rho)} \mathfrak{h}, \ .$$

given by the canonical inclusion. Thus, although different choices of generating sets will yield different crossed products in general, the choice of generators is not essential in the structure of the crossed product.

Remark 3.1.3. Recall that if $\mathcal{A}$ is a unital operator algebra with C*-cover $(\mathcal{C}, \rho)$ and $\phi : \rho(\mathcal{A}) \to B(H)$ is a linear map, then ϕ is unital and completely contractive if and only if the map $\widetilde{\phi} : \rho(\mathcal{A}) + \rho(\mathcal{A})^* \to B(H)$ given by

$$\widetilde{\phi}(\rho(a) + \rho(b)^*) = \phi(\rho(a)) + \phi(\rho(b))^*$$

is ucp [2]. In particular, if $\alpha \in \operatorname{Aut}(\rho(\mathcal{A}))$, then it readily follows that $\widetilde{\alpha} \in \operatorname{Aut}(\rho(\mathcal{A}) + \rho(\mathcal{A})^*)$. Suppose that G is a discrete group. For a group action $\alpha : G \to \operatorname{Aut}(\mathcal{A})$ and an α-admissible C*-cover $(\mathcal{C}, \rho, \alpha)$, there is an associated group action $\widetilde{\alpha} : G \to \operatorname{Aut}(\rho(\mathcal{A}) + \rho(\mathcal{A})^*)$ given by the assignment $g \mapsto \widetilde{\alpha}_g$. In fact, any α-admissible C*-cover $(\mathcal{C}, \rho, \alpha)$ for $(\mathcal{A}, G, \alpha)$ is also $\widetilde{\alpha}$-admissible for $(\rho(\mathcal{A}) + \rho(\mathcal{A})^*, G, \widetilde{\alpha}, \mathfrak{g})$. Let $\widetilde{\rho} : \rho(\mathcal{A}) + \rho(\mathcal{A})^* \to \mathcal{C}$ be the canonical inclusion. Then for reduced crossed products, setting $\mathfrak{g} = G$, we have the identity

$$(\rho(\mathcal{A}) + \rho(\mathcal{A})^*) \rtimes_{\alpha,\lambda}^{(\mathcal{C},\widetilde{\rho})} G = (\mathcal{A} \rtimes_{\alpha,\lambda}^{(\mathcal{C},\rho)} G) + (\mathcal{A} \rtimes_{\alpha,\lambda}^{(\mathcal{C},\rho)} G)^* \subseteq \mathcal{C} \rtimes_{\alpha,\lambda} G\ .$$

This means that there is a bijective correspondence between unital completely positive maps on the reduced crossed product $(\rho(\mathcal{A}) + \rho(\mathcal{A})^*) \rtimes_{\alpha,\lambda}^{(\mathcal{C},\widetilde{\rho})} G$ and unital completely contractive maps on $\mathcal{A} \rtimes_{\alpha,\lambda}^{(\mathcal{C},\rho)} G$. In this way, any reduced crossed product of a unital operator algebra by a discrete group is contained completely isometrically in an associated operator system reduced crossed product.

We would like an abstract notion of the reduced crossed product. Indeed, we shall show that the reduced crossed product is independent of its admissible C*-cover. Until that fact is established, we will always make reference to the C*-cover in question when discussing (relative) reduced crossed products.

Example 3.1.4. Let G be a group. Consider the trivial action of G on $\mathbb{C}$; i.e., $\alpha_g(1) = 1$ for all $g \in G$. In this case, we simply recover the reduced group operator system corresponding to the generating set $\mathfrak{g}$. That is to say,

$$\mathbb{C} \rtimes_{\mathrm{id},\lambda}^{C^*_\lambda(G)} \mathfrak{g} = \mathcal{S}_\lambda(\mathfrak{g}),\ .$$

where $\mathcal{S}_\lambda(\mathfrak{g}) = \operatorname{span}\{\lambda_g : g \in \mathfrak{g}\} \subseteq C^*_\lambda(G)$ [27].

Proposition 3.1.5. *Suppose that $(\mathcal{S}, G, \mathfrak{g}, id)$ is the trivial dynamical system; i.e., $\alpha_g = id_{\mathcal{S}}$ for all $g \in G$. Then the map*

$$\Psi : \mathcal{S} \rtimes_{\mathrm{id},\lambda} \mathfrak{g} \to \mathcal{S} \otimes_{\min} \mathcal{S}_\lambda(\mathfrak{g})$$
$$s\alpha_g \mapsto s \otimes \lambda_g$$

is a complete order isomorphism.

Proof. From the theory of C*-algebras, there is an isomorphism $\Phi : C^*_{\mathrm{env}}(\mathcal{S}) \rtimes_{id,\lambda} G \to C^*_{\mathrm{env}}(\mathcal{S}) \otimes_{\min} C^*_\lambda(G)$ which sends generators to generators [79, Lemma 2.73]. This restricts to an isomorphism $\Psi : \mathcal{S} \rtimes_{id,\lambda} \mathfrak{g} \to \mathrm{span}\{a \otimes \lambda_g : a \in \mathcal{S}, g \in \mathfrak{g}\}$ which sends generators to generators. By [52, Corollary 4.10], the latter operator system is precisely $\mathcal{S} \otimes_{\min} \mathcal{S}_\lambda(\mathfrak{g})$. □

Let $\mathcal{C}$ be a C*-algebra and $\mathcal{S}$ be an operator system contained in $\mathcal{C}$. We say that $\mathcal{S}$ *contains enough unitaries* in $\mathcal{C}$ if the set of elements in $\mathcal{S}$ which are unitary in $\mathcal{C}$ generate $\mathcal{C}$ as a C*-algebra. This property of operator systems was first considered in [53]. A result of Kavruk [50, Proposition 5.6] states that if $\mathcal{S} \subseteq \mathcal{C}$ is an operator subsystem of a C*-cover $\mathcal{C}$ for which $\mathcal{S}$ contains enough unitaries, then $\mathcal{C}$ is the C*-envelope of $\mathcal{S}$. In particular, $C^*_{\mathrm{env}}(\mathbb{C} \rtimes^{C^*_\lambda(G)}_{\mathrm{id},\lambda} \mathfrak{g}) = C^*_\lambda(G)$.

Before working more with C*-envelopes corresponding to dynamical systems, we first show that for any dynamical system, the group action can be extended to a group action on the C*-envelope.

Proposition 3.1.6. *Suppose that $(\mathcal{S}, G, \mathfrak{g}, \alpha)$ is a dynamical system. Suppose that the pair $(C^*_{env}(\mathcal{S}), \iota)$ is the C*-envelope of $\mathcal{S}$, where $\iota : \mathcal{S} \hookrightarrow C^*_{env}(\mathcal{S})$ is the canonical complete order embedding. Then there exists a G-action $\overline{\alpha}$ on $C^*_{env}(\mathcal{S})$ which makes $(C^*_{env}(\mathcal{S}), \iota, \overline{\alpha})$ an α-admissible C*-cover of $\mathcal{S}$.*

Proof. Let $g \in G$. Since $\iota \circ \alpha_g$ is a complete order embedding of $\mathcal{S}$ into $C^*_{\mathrm{env}}(\mathcal{S})$, by the universal property of C*-envelopes, there is a unique surjective $*$-homomorphism $\overline{\alpha}_g : C^*_{\mathrm{env}}(\mathcal{S}) \to C^*_{\mathrm{env}}(\mathcal{S})$ for which the diagram

$$\begin{tikzcd} C^*_{\mathrm{env}}(\mathcal{S}) \arrow[r, "\overline{\alpha}_g"] & C^*_{\mathrm{env}}(\mathcal{S}) \\ \mathcal{S} \arrow[u, "\iota"] \arrow[r, "\alpha_g"] & \mathcal{S} \arrow[u, "\iota"'] \end{tikzcd}$$

commutes. For every $g, h \in G$, the map $\overline{\alpha}_{gh} \circ (\overline{\alpha}_g \circ \overline{\alpha}_h)^{-1}$ restricts to the identity on $\mathcal{S}$. By uniqueness, we must have $\overline{\alpha}_{gh} \circ (\overline{\alpha}_g \circ \overline{\alpha}_h)^{-1} = \mathrm{id}_{C^*_{\mathrm{env}}(\mathcal{S})}$, so that $\overline{\alpha}_{gh} = \overline{\alpha}_g \circ \overline{\alpha}_h$. Evidently we have $\overline{\alpha}_e = \mathrm{id}_{C^*_{\mathrm{env}}(\mathcal{S})}$. In particular, $\overline{\alpha}_g \circ \overline{\alpha}_{g^{-1}} = \overline{\alpha}_{g^{-1}} \circ \overline{\alpha}_g = \mathrm{id}_{C^*_{\mathrm{env}}(\mathcal{S})}$, so each $\overline{\alpha}_g$ is an automorphism. Then $\overline{\alpha} : G \curvearrowright C^*_{\mathrm{env}}(\mathcal{S})$ is a group action which is admissible with respect to the dynamical system. □

Definition 3.1.7. Let $(\mathcal{S}, G, \mathfrak{g}, \alpha)$ be a dynamical system. Define the *reduced crossed product* $\mathcal{S} \rtimes_{\alpha,\lambda} \mathfrak{g}$ to be the reduced crossed product $\mathcal{S} \rtimes_{\alpha,\lambda}^{(C^*_{env}(\mathcal{S}),\iota)} \mathfrak{g}$ relative to $(C^*_{env}(\mathcal{S}), \iota)$.

3.1.1 The C*-envelope of a reduced crossed product

The goal of this section is to prove the identity

$$C^*_{\mathrm{env}}(\mathcal{S} \rtimes_{\alpha,\lambda} \mathfrak{g}) = C^*_{\mathrm{env}}(\mathcal{S}) \rtimes_{\alpha,\lambda} G$$

for any dynamical system $(\mathcal{S}, G, \mathfrak{g}, \alpha)$.

The next Lemma contains some useful facts relating to hyperrigidity.

Lemma 3.1.8. *Let $\mathcal{S}$ be an operator system.*

1. *If α is an automorphism on $\mathcal{S}$ and $\pi : \mathcal{S} \to B(H)$ is a representation with the unique extension property, then $\pi \circ \alpha$ has the unique extension property.*
2. *If $\pi_i : \mathcal{S} \to B(H_i)$ is a maximal representation for each $i \in I$, then $\bigoplus_i \pi_i$ is also maximal.*

Proof. The fact that (1) holds is by the definition of the unique extension property. The proof of (2) is due to Arveson [4, Proposition 4.4]. □

Let $\mathcal{S}$ be an operator system in a C*-algebra $\mathcal{C}$. Recall that $\mathcal{S}$ is hyperrigid in $\mathcal{C}$ if whenever $\pi : \mathcal{C} \to B(H)$ is a $*$-representation, the map $\pi|_{\mathcal{S}}$ satisfies the unique extension property. The following Theorem is helpful for our purposes.

Theorem 3.1.9. *If $\mathcal{S}$ is an operator system in a unital C*-algebra $\mathcal{C}$ and $\mathcal{S}$ is hyperrigid in $\mathcal{C}$, then $\mathcal{C} = C^*_{env}(\mathcal{S})$.*

Proof. If $\mathcal{S} \subseteq \mathcal{C}$ is hyperrigid, then any faithful representation π of $\mathcal{C}$ is such that $\pi_{\mathcal{S}}$ has the unique extension property. Thus, $\pi_{\mathcal{S}}$ is maximal on $\mathcal{S}$. By a theorem of Dritschel and McCollough [22, Theorem 1.1], since maximal representations generate C*-envelopes, $C^*(\pi|_c alS)$ is isomorphic to $C^*_{\mathrm{env}}(\mathcal{S})$. Evidently $\mathcal{C} \simeq C^*(\pi|_{\mathcal{S}})$, so this proves that $\mathcal{C}$ is the C*-envelope of $\mathcal{S}$. □

Recall that an operator subsystem $\mathcal{S}$ of a unital C*-algebra $\mathcal{C}$ *contains enough unitaries* if the the set of elements in $\mathcal{S}$ that are unitary in $\mathcal{C}$ generates $\mathcal{C}$ as a C*-algebra. The next result is folklore.

Lemma 3.1.10. *Suppose that $\mathcal{C}$ is a unital C*-algebra and that $\mathcal{S} \subseteq \mathcal{C}$ is an operator system which contains enough unitaries in $\mathcal{C}$. Then the operator system $\mathcal{S}$ is hyperrigid.*

Proof. Let $\pi : \mathcal{C} \to B(H)$ be a $*$-representation, and suppose that the ucp map $\pi|_{\mathcal{S}}$ is not maximal. Let $\psi : \mathcal{S} \to B(K)$ be a maximal dilation of $\pi|_{\mathcal{S}}$. Since ψ is maximal, ψ induces a $*$-homomorphism on $C^*(\mathcal{S}) = \mathcal{C}$, which we will also denote by ψ. For $s \in \mathcal{S}$, we may decompose the operator $\psi(s)$ with respect to $\mathcal{K} = H \oplus H^{\perp}$ as

$$\psi(s) = \begin{bmatrix} \pi(s) & a_s \\ b_s & \chi(s) \end{bmatrix}$$

for operators $a_s \in B(H^{\perp}, H)$, $b_s \in B(H, H^{\perp})$ and $\chi(s) \in B(H^{\perp})$. Let $u \in \mathcal{S}$ be unitary in $\mathcal{C}$. Since $\psi(u)$ must also be unitary, we know that the (1,1)-corner of the operators $\psi(u)\psi(u^*)$ and $\psi(u^*)\psi(u)$ must be the identity. A calculation shows that $(\psi(u)\psi(u^*))_{1,1} = I_H + a_u a_u^*$ and $(\psi(u^*)\psi(u))_{1,1} = I_H + b_u^* b_u$. Thus, both a_u and b_u must be zero, so that $\psi(u) = \pi(u) \oplus \chi(u)$. Therefore, if $u, v \in \mathcal{S}$ are unitaries in $\mathcal{C}$, then using the fact that ψ is a $*$-homomorphism,

$$\begin{bmatrix} \pi(u)\pi(v) & 0 \\ 0 & \chi(u)\chi(v) \end{bmatrix} = \psi(u)\psi(v) = \psi(uv) = \begin{bmatrix} \pi(uv) & a_{uv} \\ b_{uv} & \chi(uv) \end{bmatrix}.$$

Hence, a_{uv} and $b_{uv} = 0$. It easily follows that for any elements $u_1, ..., u_n \in \mathcal{S}$ that are unitary in $\mathcal{C}$, we have

$$\psi(u_1 \cdots u_n) = \begin{bmatrix} \pi(u_1 \cdots u_n) & 0 \\ 0 & \chi(u_1 \cdots u_n) \end{bmatrix}.$$

Since $\mathcal{C}$ is generated by unitaries in $\mathcal{S}$, $\mathcal{C}$ is the span of elements of the form $u_1 \cdots u_n$ for elements $u_1, ..., u_n$ of $\mathcal{S}$ that are unitary in $\mathcal{C}$. It follows that ψ decomposes as $\pi \oplus \chi$ for some ucp map χ on $\mathcal{C}$. Restricting to $\mathcal{S}$, this proves that $\pi|_{\mathcal{S}}$ is maximal, which is a contradiction. $\square$

This gives an alternate proof of Kavruk's result on C*-envelopes [50, Proposition 5.6].

Corollary 3.1.11. *Suppose that $\mathcal{C}$ is a unital C*-algebra and $\mathcal{S} \subseteq \mathcal{C}$ is an operator system that contains enough unitaries in $\mathcal{C}$. Then $\mathcal{C}$ is the C*-envelope of $\mathcal{S}$.*

Proof. By Lemma 3.1.10, $\mathcal{S}$ is hyperrigid. By Theorem 3.1.9, $\mathcal{C} = C^*_{\text{env}}(\mathcal{S})$. □

In order to show that $C^*_{\text{env}}(\mathcal{S} \rtimes_{\alpha,\lambda} \mathfrak{g}) = C^*_{\text{env}}(\mathcal{S}) \rtimes_{\alpha} G$, we require the following lemma.

Lemma 3.1.12. *Suppose that $(\mathcal{S}, G, \mathfrak{g}, \alpha)$ is a dynamical system. Let $\pi : C^*_{env}(\mathcal{S}) \hookrightarrow B(H)$ be a faithful representation that is maximal on $\mathcal{S}$. Let $(\overline{\pi}, \lambda_H)$ be the covariant extension of π to $H \otimes \ell^2(G)$. That is, for any $a \in A$, $h \in H$, and $s, g \in G$,*

$$\overline{\pi} : A \to B(H \otimes \ell^2(G)) : \overline{\pi}(a)(h \otimes \delta_g) = \pi(a)h \otimes \delta_g \text{ and}$$
$$\lambda_H : G \to U(H \otimes \ell^2(G)) : \lambda_{H,s}(h \otimes \delta_g) = h \otimes \delta_{sg} \; .$$

The integrated form $\overline{\pi} \rtimes \lambda_H : C^*_{env}(\mathcal{S}) \rtimes_{\alpha,\lambda} G \to B(H \otimes \ell^2(G))$ is maximal on $\mathcal{S} \rtimes_{\alpha,\lambda} \mathfrak{g}$.

Proof. Since $\overline{\pi} = \bigoplus_{g \in G} \pi \circ \alpha_g$, Lemma 3.1.8 shows that $\overline{\pi}$ has the unique extension property on $\mathcal{S}$. We claim that $\overline{\pi} \rtimes \lambda_H$ has the unique extension property on $\mathcal{S} \rtimes_{\alpha,\lambda} \mathfrak{g}$. If this were true, then $\overline{\pi} \rtimes \lambda_H$ would be maximal on $\mathcal{S} \rtimes_{\alpha,\lambda} \mathfrak{g}$. Thus, it remains to show that $\overline{\pi} \rtimes \lambda_H$ has the unique extension property. Suppose that $\rho : C^*_{\text{env}}(\mathcal{S}) \rtimes_{\alpha,\lambda} G \to B(H)$ is a ucp extension of $\overline{\pi} \rtimes \lambda_H|_{\mathcal{S} \rtimes_{\alpha,\lambda} \mathfrak{g}}$. We observe that, by maximality of $\overline{\pi}$, we must have $\rho|_{C^*_{\text{env}}(\mathcal{S})} = \overline{\pi} \rtimes \lambda_H|_{C^*_{\text{env}}(\mathcal{S})} = \pi$. Recall that $\mathcal{S}_\lambda(\mathfrak{g}) = \text{span}\{\lambda_g : g \in \mathfrak{g}\}$. By Lemma 3.1.10, $\mathcal{S}_\lambda(\mathfrak{g})$ is hyperrigid, since it contains enough unitaries in its C*-envelope. Since $\mathcal{S}_\lambda(\mathfrak{g})$ is hypperrigid, we get the identity $\rho|_{C^*(\mathcal{S}_\lambda(\mathfrak{g}))} = \overline{\pi} \rtimes \lambda_H|_{C^*(\mathcal{S}_\lambda(\mathfrak{g}))}$. Thus, $C^*(\mathcal{S}_\lambda(\mathfrak{g}))$ is in the multiplicative domain of ρ. Now, let $g \in G$. Since $\lambda_g \in C^*(\mathcal{S}_\lambda(\mathfrak{g}))$, for $a \in C^*_{\text{env}}(\mathcal{S})$ we obtain the identity

$$\rho(a\lambda_g) = \rho(a)\rho(\lambda_g) = (\overline{\pi} \rtimes \lambda_H(a))(\overline{\pi} \rtimes \lambda_H(\lambda_g)) = \overline{\pi} \rtimes \lambda_H(a\lambda_g) \; .$$

This proves that $\rho = \overline{\pi} \rtimes \lambda_H$, so that $\overline{\pi} \rtimes \lambda_H$ has the unique extension property. □

We are now in a position to prove the desired result on C*-envelopes of reduced crossed products.

Theorem 3.1.13. *Suppose that $(\mathcal{S}, G, \mathfrak{g}, \alpha)$ is a dynamical system. Then there is a canonical isomorphism*

$$C^*_{env}(\mathcal{S} \rtimes_{\alpha,\lambda} \mathfrak{g}) \simeq C^*_{env}(\mathcal{S}) \rtimes_{\alpha,\lambda} G \; .$$

Proof. Let $\pi : C^*_{\text{env}}(\mathcal{S}) \hookrightarrow B(H)$ be a maximal representation, and let $\overline{\pi} \rtimes \lambda_H$ be the associated integrated form of the covariant extension $(\overline{\pi}, \lambda_H)$ of π. By Lemma 3.1.12,

$\overline{\pi} \rtimes \lambda_H$ is maximal on $\mathcal{S} \rtimes_{\alpha,\lambda} \mathfrak{g}$. Thus, the C*-algebra generated by $(\overline{\pi} \rtimes \lambda_H)(\mathcal{S} \rtimes_{\alpha,\lambda} \mathfrak{g})$ is the C*-envelope of $\mathcal{S} \rtimes_{\alpha,\lambda} \mathfrak{g}$. Thus, we obtain the isomorphism

$$\overline{\pi} \rtimes \lambda_H : C^*_{\text{env}}(\mathcal{S}) \rtimes_{\alpha,\lambda} G \to C^*_{\text{env}}(\mathcal{S} \rtimes_{\alpha,\lambda} \mathfrak{g}) \ .$$

which completes the proof. □

Hyperrigidity is also preserved by the reduced crossed product.

Corollary 3.1.14. *Suppose that $(\mathcal{S}, G, \mathfrak{g}, \alpha)$ is a dynamical system. If $\mathcal{S}$ is hyperrigid in $C^*_{env}(\mathcal{S})$, then $\mathcal{S} \rtimes_{\alpha,\lambda} \mathfrak{g}$ is hyperrigid in $C^*_{env}(\mathcal{S} \rtimes_{\alpha,\lambda} \mathfrak{g})$.*

Proof. By Theorem 3.1.13, we have $C^*_{\text{env}}(\mathcal{S} \rtimes_{\alpha,\lambda} \mathfrak{g}) = C^*_{\text{env}}(\mathcal{S}) \rtimes_{\alpha,\lambda} G$. Suppose that $\rho : C^*_{\text{env}}(\mathcal{S}) \rtimes_{\alpha,\lambda} G \to B(H)$ is a unital $*$-homomorphism and let $\varphi : C^*_{\text{env}}(\mathcal{S}) \rtimes_{\alpha,\lambda} G \to B(H)$ be a ucp extension of $\rho|_{\mathcal{S} \rtimes_{\alpha,\lambda} \mathfrak{g}}$. Since $\mathcal{S}$ is hyperrigid in $C^*_{\text{env}}(\mathcal{S})$, $\varphi|_{C^*_{\text{env}}(\mathcal{S})}$ satisfies the unique extension property on $\mathcal{S}$. On the other hand, $\mathcal{S}_\lambda(\mathfrak{g})$ is hyperrigid in $C^*_\lambda(G)$ by Lemma 3.1.10. Thus, φ agrees with ρ when restricted to the copy of $C^*_\lambda(G)$ in $C^*_{\text{env}}(\mathcal{S}) \rtimes_{\alpha,\lambda} G$. In particular, $C^*_{\text{env}}(\mathcal{S})$ and the copy of $C^*_\lambda(G)$ are contained in the multiplicative domain of φ. As these two algebras generate $C^*_{\text{env}}(\mathcal{S}) \rtimes_{\alpha,\lambda} G$ as a C*-algebra, it follows that $\rho = \varphi$. Therefore, ρ is maximal on $\mathcal{S} \rtimes_{\alpha,\lambda} \mathfrak{g}$. Since ρ was an arbitrary representation of $C^*_{\text{env}}(\mathcal{S}) \rtimes_{\alpha,\lambda} G$, it follows that $\mathcal{S} \rtimes_{\alpha,\lambda} \mathfrak{g}$ is hyperrigid in $C^*_{\text{env}}(\mathcal{S} \rtimes_{\alpha,\lambda} \mathfrak{g})$. □

Example 3.1.15. Consider the commutative C*-algebra $C(\mathbb{T})$ with generator $u : \mathbb{T} \to \mathbb{C}$ given by $u(z) = z$. Fix $\theta \in [0, 1]$ and define the action $\alpha : \mathbb{Z} \curvearrowright C(\mathbb{T})$ by the automorphism

$$\alpha : u \mapsto e^{2\pi i\theta} u \ .$$

Define $\mathcal{S}_{\mathbb{T}} := \text{span}\{1, u, u^*\}$. Observe that α restricts to an action on $\mathcal{S}_{\mathbb{T}}$. Set $\mathfrak{g} = \{1, 0, -1\} \subseteq \mathbb{Z}$. The crossed product $\mathcal{S}_{\mathbb{T}} \rtimes_{\alpha,\lambda} \mathfrak{g}$ has C*-envelope $C(\mathbb{T}) \rtimes_\alpha \mathbb{Z}$. Therefore, all rotation algebras are C*-envelopes of finite-dimesional operator systems.

Example 3.1.16. We consider a generalization of Example 3.1.15. Let $n \geq 1$, and let $U(n)$ act on the Cuntz algebra $\mathcal{O}_n$ via the mapping

$$\alpha_g : s_i \mapsto \sum_{j=1}^{n} g_{ji} s_j$$

where $s_1, \ldots, s_n$ are the isometries generating $\mathcal{O}_n$ and $g = (g_{ij})$ is the matrix representation of the element g of $U(n)$ with respect to the canonical basis. For a subgroup G of $U(n)$, we say that an action $G \curvearrowright \mathcal{O}_n$ is quasi-free if G acts by α (see [57]). Let $\mathfrak{g} \subseteq U(n)$ be a finite symmetric subset containing the identity. Let $G = \langle \mathfrak{g} \rangle$. Set $\mathcal{S}_n = \text{span}\{s_1, \ldots, s_n, 1, s_1^*, \ldots, s_n^*\}$. If $G \curvearrowright_\alpha \mathcal{O}_n$ is a quasi-free action, then G restricts to an action on $\mathcal{S}_n$. The system $\mathcal{S}_n \rtimes_{\alpha,\lambda} \mathfrak{g}$ has C*-envelope $\mathcal{O}_n \rtimes_{\alpha,\lambda} G$.

3.1.2 An abstract characterization of reduced crossed products

We now move towards showing that the reduced crossed product does not depend on the choice of C*-cover for the operator system. Recall the following characterization of positivity for reduced C*-algebraic crossed products (see [13, Corollary 4.1.6]):

Proposition 3.1.17. *Suppose that $(\mathcal{C}, G, \alpha)$ is a C*-dynamical system. An element $x = \sum_{g\in G} a_g\lambda_g \in C_c(G, \mathcal{C})$ is positive if and only if for any finite set $\{g_1, \ldots, g_n\} \subseteq G$, the matrix*

$$\begin{bmatrix} \quad \alpha_{g_i}^{-1}(a_{g_ig_j^{-1}}) \quad \end{bmatrix}_{i,j=1}^{n}$$

is positive in $M_n(\mathcal{C})$.

The following is a well known result. For a proof, see [79, Lemma 7.16].

Proposition 3.1.18. *Let $(\mathcal{C}, G, \alpha)$ be a C*-dynamical system and let $n \geq 1$. We have the isomorphism*

$$M_n(\mathcal{C} \rtimes_{\alpha,\lambda} G) \simeq M_n(\mathcal{C}) \rtimes_{\alpha^{(n)},\lambda} G\ .$$

The next corollary immediately follows from Proposition 3.1.18.

Corollary 3.1.19. *Let $(\mathcal{S}, G, \mathfrak{g}, \alpha)$ be a dynamical system, and let (A, ρ) be an α-admissible C*-cover. For $n \geq 1$, we have a complete order isomorphism*

$$M_n(\mathcal{S} \rtimes_{\alpha,\lambda}^{(\mathcal{C},\rho)} \mathfrak{g}) \simeq M_n(\mathcal{S}) \rtimes_{\alpha^{(n)},\lambda}^{(M_n(\mathcal{C}),\rho^{(n)})} \mathfrak{g}\ .$$

Therefore, we have the following characterization of reduced crossed products.

Proposition 3.1.20. *Let $(\mathcal{S}, G, \mathfrak{g}, \alpha)$ be a dynamical system and let $(\mathcal{C}, \rho)$ be an α-admissible C*-cover. For $n \geq 1$, the positive cones $C_n := M_n(\mathcal{S} \rtimes_{\alpha,\lambda}^{(\mathcal{C},\rho)} \mathfrak{g})_+$ are given by the following rule: an element $x = \sum_{g\in\mathfrak{g}} x_g\lambda_g \in M_n(\mathcal{S} \rtimes_{\alpha,\lambda}^{(\mathcal{C},\rho)} \mathfrak{g})$ is in C_n if and only if, for every finite subset F of G, the matrix*

$$\begin{bmatrix} \quad \alpha_{g^{-1}}^{(n)}(\rho(x_{gh^{-1}})) \quad \end{bmatrix}_{g,h\in F}$$

is positive in $M_F(M_n(\rho(\mathcal{S})))$.

We now prove that the reduced crossed product is independent of the C*-cover. The proof is essentially the same as the proof of the analogous result for the operator algebras [47, Lemma 3.11].

Lemma 3.1.21. *Suppose that $\rho : \mathcal{S} \hookrightarrow \mathcal{C}$ is a complete order embedding of an operator system $\mathcal{S}$ into a C*-cover $\mathcal{C}$. Let $\mathcal{J}_\mathcal{S}$ be the Shilov ideal of $\mathcal{S}$ in $\mathcal{C}$. If $\alpha : \mathcal{C} \to \mathcal{C}$ is an automorphism such that $\alpha(\rho(\mathcal{S})) = \rho(\mathcal{S})$, then $\alpha(\mathcal{J}_\mathcal{S}) = \mathcal{J}_\mathcal{S}$.*

Proof. Let $n \geq 1$ and $x \in M_n(\mathcal{S})$. A calculation shows that

$$\|\rho^{(n)}(x) + M_n(\alpha(\mathcal{J}_\mathcal{S}))\| = \|(\alpha^{-1})^{(n)}(\rho^{(n)}(x)) + M_n(\mathcal{J}_\mathcal{S})\| = \|(\alpha^{-1})^{(n)}(\rho^{(n)}(x))\| = \|x\| ,$$

by definition of $\mathcal{J}_\mathcal{S}$. Therefore, $\mathcal{S} \to \mathcal{C}/\alpha(\mathcal{J}_\mathcal{S}) : x \mapsto \rho(x) + \alpha(\mathcal{J}_\mathcal{S})$ is a complete order isometry. Thus, $\alpha(\mathcal{J}_\mathcal{S})$ is a boundary ideal for $\mathcal{S}$ in $(\mathcal{C}, \rho)$. Since the Shilov ideal is maximal amongst boundary ideals, $\alpha(\mathcal{J}_\mathcal{S}) \subseteq \mathcal{J}_\mathcal{S}$. Since α is an automorphism, applying the same argument for α^{-1} shows that $\mathcal{J}_\mathcal{S} \subseteq \alpha(\mathcal{J}_\mathcal{S})$. Thus, $\alpha(\mathcal{J}_\mathcal{S}) = \mathcal{J}_\mathcal{S}$. □

Lemma 3.1.22. *Suppose that $(\mathcal{S}, G, \mathfrak{g}, \alpha)$ is a dynamical system and suppose that $(\mathcal{C}, \rho, \alpha)$ is an α-admissible C*-cover. Then the G-action α induces a G-action on $\mathcal{C}/\mathcal{J}_\mathcal{S}$ via $g \mapsto \dot{\alpha}_g$, where $\dot{\alpha}_g(x + \mathcal{J}_\mathcal{S}) = \alpha_g(x) + \mathcal{J}_\mathcal{S}$. Moreover, if $q : \mathcal{C} \to \mathcal{C}/\mathcal{J}_\mathcal{S}$ is the canonical quotient map, then $(\mathcal{C}/\mathcal{J}_\mathcal{S}, q \circ \rho, \dot{\alpha})$ is an α-admissible C*-cover for $(\mathcal{S}, G, \mathfrak{g}, \alpha)$.*

Proof. Let $g \in G$. Since $(\mathcal{C}, \rho, \alpha)$ is an α-admissible C*-cover, $\alpha_g(\rho(\mathcal{S})) = \rho(\mathcal{S})$. By Lemma 3.1.21, $\alpha_g(\mathcal{J}_\mathcal{S}) = \mathcal{J}_\mathcal{S}$. Hence, the map $\dot{\alpha}_g$ as defined above is a well-defined unital $*$-homomorphism. It is easy to check that the assignment $g \mapsto \dot{\alpha}_g$ induces a group action $\dot{\alpha}$ of G on $\mathcal{C}/\mathcal{J}_\mathcal{S}$.

To see that $(\mathcal{C}/\mathcal{J}_\mathcal{S}, q \circ \rho, \dot{\alpha})$ is an α-admissible C*-cover for $(\mathcal{S}, G, \mathfrak{g}, \alpha)$, let $x \in \mathcal{S}$ and let $g \in G$. We have the identity

$$\begin{aligned}\dot{\alpha}_g(q \circ \rho(x)) &= \dot{\alpha}_g(\rho(x) + \mathcal{J}_\mathcal{S}) = (\alpha_g \circ \rho)(x) + \mathcal{J}_\mathcal{S} \\ &= \rho(\alpha_g(x)) + \mathcal{J}_\mathcal{S} = q \circ \rho(\alpha_g(x)),\end{aligned}$$

thus proving that $(\mathcal{C}/\mathcal{J}_\mathcal{S}, q \circ \rho, \dot{\alpha})$ is α-admissible. □

Theorem 3.1.23. *Suppose that $(\mathcal{S}, G, \mathfrak{g}, \alpha)$ is a dynamical system and suppose that $(\mathcal{C}, \rho, \alpha)$ is an α-admissible C*-cover. If $\mathcal{J}_\mathcal{S}$ is the Shilov boundary of $\mathcal{S}$ in $\mathcal{C}$, then*

$$\mathcal{S} \rtimes_{\alpha,\lambda}^{(\mathcal{C}/\mathcal{J}_\mathcal{S}, q\circ\rho)} \mathfrak{g} \simeq \mathcal{S} \rtimes_{\alpha,\lambda}^{(\mathcal{C},\rho)} \mathfrak{g}$$

canonically. In particular, the reduced crossed product does not depend on the choice of C-cover.*

Proof. By Lemma 3.1.22, the reduced crossed product $\mathcal{S} \rtimes_{\alpha,\lambda}^{(\mathcal{C}/\mathcal{J}_{\mathcal{S}}, q\circ\rho)} \mathfrak{g}$ is well-defined. It remains to prove that the map

$$\begin{aligned} \Phi : \mathcal{S} \rtimes_{\alpha,\lambda}^{(\mathcal{C},\rho)} \mathfrak{g} &\to \mathcal{S} \rtimes_{\alpha,\lambda}^{(\mathcal{C}/\mathcal{J}_{\mathcal{S}}, q\circ\rho)} \mathfrak{g} \\ x\lambda_g &\mapsto (x + \mathcal{J}_{\mathcal{S}})\lambda_g \end{aligned}$$

is a complete order isomorphism. This map is unital and completely positive since it arises as the restriction of a *-homomorphism

$$\begin{aligned} \mathcal{C} \rtimes_{\alpha,\lambda} G &\to \mathcal{C}/\mathcal{J}_{\mathcal{S}} \rtimes_{\dot{\alpha},\lambda} G \\ x\lambda_s &\mapsto (x + \mathcal{J}_{\mathcal{S}})\lambda_s \ . \end{aligned}$$

Conversely, for $X \in M_n(\mathcal{S} \rtimes_{\alpha,\lambda}^{(\mathcal{C},\rho)} \mathfrak{g})$, suppose that $\Phi(X) \in M_n(\mathcal{S} \rtimes_{\alpha,\lambda}^{(\mathcal{C}/\mathcal{J}_{\mathcal{S}}, q\circ\rho)} \mathfrak{g})$ is positive. By Proposition 3.1.20, X is positive if and only if for every finite subset F of G, the matrix

$$\left[\quad \dot{\alpha}_{g^{-1}}^{(n)}(q \circ \rho^{(n)}(X_{gh^{-1}})) \quad \right]_{g,h\in F}$$

is positive in $M_F(M_n(q \circ \rho(\mathcal{S})))$. $\dot{\alpha}_g \circ (q \circ \rho) = (q \circ \rho) \circ \alpha_g$ and $q \circ \rho : \mathcal{S} \to \mathcal{C}/\mathcal{J}_{\mathcal{S}}$ is a complete order embedding, we see that the matrix

$$\left[\quad \alpha_{g^{-1}}^{(n)}(X_{gh^{-1}}) \quad \right]_{g,h\in F}$$

is positive in $M_F(M_n(\mathcal{S}))$. Since $\alpha_g \circ \rho = \rho \circ \alpha_g$, the matrix

$$\left[\quad \alpha_{g^{-1}}^{(n)}(\rho^{(n)}(X_{gh^{-1}})) \quad \right]_{g,h\in F}$$

is positive in $M_F(M_n(\rho(\mathcal{S}))$. Applying Proposition 3.1.20 again, we see that X is positive in $M_n(\mathcal{S} \rtimes_{\alpha,\lambda}^{(\mathcal{C},\rho)} \mathfrak{g})$, establishing the complete order isomorphism.

□

We close this section with a short discussion on G-equivariant ucp maps. First, we have the following result ([13, Exercise 4.1.4]):

Proposition 3.1.24. *Suppose that* $(\mathcal{S}, G, \mathfrak{g}, \alpha)$ *and* $(\mathcal{T}, G, \mathfrak{g}, \beta)$ *are dynamical systems and suppose that* $\varphi : \mathcal{S} \to \mathcal{T}$ *is a* G*-equivariant ucp map. The map*

$$\widetilde{\varphi} : \mathcal{S} \rtimes_{\alpha,\lambda} \mathfrak{g} \to \mathcal{T} \rtimes_{\beta,\lambda} \mathfrak{g} : a\lambda_g \mapsto \varphi(a)\lambda_g$$

is ucp. If the map φ *is a complete order embedding, then the map* $\tilde{\varphi}$ *is also a complete order embedding.*

Proof. It is clear that $\widetilde{\varphi}$ is unital. Since the amplifications $\varphi^{(n)} : M_n(\mathcal{S}) \to M_n(\mathcal{T})$ are G-equivariant, it suffices to show that $\widetilde{\varphi}$ is positive. If $x = \sum_{g\in\mathfrak{g}} x_g\lambda_g$ is positive in $\mathcal{S} \rtimes_{\alpha,\lambda} \mathfrak{g}$, then for each finite $F \subseteq G$ the matrix

$$P := \left[\quad \alpha_{g^{-1}}(x_{gh^{-1}}) \quad \right]_{g,h\in F} .$$

is positive in $M_F(\mathcal{S})$. Since φ is ucp, $\varphi^{(F)}(P) \geq 0$ in $\mathcal{T}$. By G-equivariance, this means that the matrix

$$\left[\quad \beta_{g^{-1}}(\varphi(x_{gh^{-1}})) \quad \right]_{g,h\in F}$$

is positive in $M_F(\mathcal{T})$. This occurs if and only if the element $\sum_{g\in\mathfrak{g}} \varphi(x_g)\lambda_g$ is positive in $\mathcal{T} \rtimes_{\beta,\lambda} \mathfrak{g}$. A similar argument shows that $\widetilde{\phi}$ is a complete order embedding whenever ϕ is a complete order embedding. $\square$

In the case of C*-algebras, a G-equivariant quotient map between two C*-algebras produces a quotient map on the reduced crossed product. This fails in the case of operator systems. For example, let $\mathfrak{z} := \{1, 0, -1\} \subseteq \mathbb{Z}$. Let $E_{00}, E_{01}, E_{10}, E_{11}$ enumerate the canonical system of matrix units for M_2. Define

$$\varphi : M_2 \to \mathcal{S}(\mathfrak{z}) : E_{ij} \mapsto \frac{1}{n} u_i u_j^* .$$

It was shown in [28, Theorem 2.4] that φ is a complete quotient map. However, the following holds.

Proposition 3.1.25. *[27, Proposition 3.10] The map* $\varphi \otimes \varphi : M_2 \otimes_{\min} M_2 \to \mathcal{S}(\mathfrak{z}) \otimes_{\min} \mathcal{S}(\mathfrak{z})$ *is not a complete quotient map.*

Proposition 3.1.26. *Let $\mathfrak{z} = \{1, 0, -1\} \subseteq \mathbb{Z}$. There is a G-equivariant complete quotient map*

$$\varphi : (M_2, \mathrm{id}, \mathfrak{z}, \mathbb{Z}) \to (S(\mathfrak{z}), \mathrm{id}, \mathfrak{z}, \mathbb{Z})$$

which does not induce a complete quotient map on the reduced crossed product.

Proof. Let $\varphi : M_2 \to \mathcal{S}(\mathfrak{z})$ be the complete quotient map as above, and suppose that the induced ucp map

$$\varphi \rtimes_{\mathrm{id}} \mathfrak{z} : M_2 \rtimes_{\mathrm{id},\lambda} \mathfrak{z} \to S(\mathfrak{z}) \rtimes_{\mathrm{id},\lambda} \mathfrak{z}$$

is a complete quotient map. Observe that, since the $\mathbb{Z}$-action is trivial, under the canonical isomorphisms we have $M_2 \rtimes_{\mathrm{id},\lambda} \mathfrak{z} = M_2 \otimes_{\min} \mathcal{S}(\mathfrak{z})$ and $\mathcal{S}(\mathfrak{z}) \rtimes_{\mathrm{id},\lambda} \mathfrak{z} = \mathcal{S}(\mathfrak{z}) \otimes_{\min} \mathcal{S}(\mathfrak{z})$. In this way, we can identify $\varphi \rtimes_{\mathrm{id}} \mathfrak{z} = \varphi \otimes_{\min} \mathrm{id}_{S(\mathfrak{z})}$. If $\varphi \rtimes_{\mathrm{id}} \mathfrak{z}$ were a complete quotient map, then by an amplification, $\mathrm{id}_{M_2} \otimes_{\min} \varphi$ would also be a complete quotient map. This would imply that $\varphi \otimes_{\min} \varphi = (\mathrm{id}_{M_2} \otimes_{\min} \varphi) \circ (\varphi \otimes_{\min} \mathrm{id}_{S(\mathfrak{z})})$ is a complete quotient map, contradicting Proposition 3.1.25. Hence, $\phi \rtimes_{\mathrm{id}} \mathfrak{z}$ is not a complete quotient map.

□

3.2 Full Crossed Products

In this section we turn to the theory of full crossed products, motivated by the approach for operator algebras in [47]. In general, there are many choices for a relative full crossed product for operator systems. We will focus on those regarding the smallest C*-cover of an operator system (the C*-envelope) and the largest C*-cover (the universal C*-algebra of an operator system).

Definition 3.2.1. Suppose that $(\mathcal{S}, G, \mathfrak{g}, \alpha)$ is a dynamical system. If $(\mathcal{C}, \rho, \alpha)$ is an α-admissible C*-cover of $(\mathcal{S}, G, \mathfrak{g}, \alpha)$, then define the *full crossed product relative to* $\mathcal{C}$ to be the subsystem

$$\mathcal{S} \rtimes_{\alpha}^{\mathcal{C}} \mathfrak{g} := \mathrm{span}\{a u_g : a \in \mathcal{S}, g \in \mathfrak{g}\} \subseteq \mathcal{C} \rtimes_{\alpha} G\ .$$

The *full enveloping crossed product* of $(\mathcal{S}, G, \mathfrak{g}, \alpha)$ is the crossed product

$$\mathcal{S} \rtimes_{\alpha,\mathrm{env}} \mathfrak{g} := \mathcal{S} \rtimes_{\alpha}^{C^*_{\mathrm{env}}(\mathcal{S})} \mathfrak{g}.$$

Remark 3.2.2. The analogue of Remark 3.1.2 holds for relative full crossed products as well. Whenever $(\mathcal{S}, G, \mathfrak{g}, \alpha)$ is an operator system dynamical system, $(\mathcal{C}, \rho)$ is an α-admissible C*-cover and $\mathfrak{h}$ is another generating set for G with $\mathfrak{g} \subseteq \mathfrak{h}$, then there is a canonical complete order embedding

$$\mathcal{S} \rtimes_{\alpha}^{(\mathcal{C},\rho)} \mathfrak{g} \hookrightarrow \mathcal{S} \rtimes_{\alpha}^{(\mathcal{C},\rho)} \mathfrak{h}.$$

Remark 3.2.3. Remark 3.1.3 also applies to relative full crossed products. That is to say, if $\mathcal{A}$ is a unital operator algebra, G is a discrete group and $(\mathcal{A}, G, \alpha)$ is an operator algebraic dynamical system with α-admissible C*-cover $(\mathcal{C}, \rho, \alpha)$, then

$$(\rho(\mathcal{A}) + \rho(\mathcal{A})^*) \rtimes_{\widetilde{\alpha}}^{(\mathcal{C},\widetilde{\rho})} G = (\mathcal{A} \rtimes_{\alpha}^{(\mathcal{C},\rho)} G) + (\mathcal{A} \rtimes_{\alpha}^{(\mathcal{C},\rho)} G)^* \subseteq \mathcal{C} \rtimes_{\alpha} G.$$

To define the relative full crossed product with respect to $C_u^*(\mathcal{S})$, we need the following proposition.

Proposition 3.2.4. *Suppose that $(\mathcal{S}, G, \mathfrak{g}, \alpha)$ is a dynamical system. Then there is a unique G-action $\overline{\alpha}$ on $C_u^*(\mathcal{S})$ which extends the action on $\mathcal{S}$. Moreover, if $j : \mathcal{S} \to C_u^*(\mathcal{S})$ is the canonical complete order embedding, then $(C_u^*(\mathcal{S}), j, \overline{\alpha})$ is an α-admissible C*-cover for $(\mathcal{S}, G, \mathfrak{g}, \alpha)$.*

Proof. Suppose that $g \in G$. By the universal property of $C_u^*(\mathcal{S})$, there is a unique $*$-homomorphism $\overline{\alpha}_g$ on $C_u^*(\mathcal{S})$ for which the diagram

$$\begin{array}{ccc} C_u^*(\mathcal{S}) & \xrightarrow{\overline{\alpha}_g} & C_u^*(\mathcal{S}) \\ {\scriptstyle j}\uparrow & & {\scriptstyle j}\uparrow \\ \mathcal{S} & \xrightarrow{\alpha_g} & \mathcal{S} \end{array}$$

commutes. It is not hard to check that $\overline{\alpha}$ defines a G-action on $C_u^*(\mathcal{S})$. $\square$

For an operator system dynamical system $(\mathcal{S}, G, \mathfrak{g}, \alpha)$, we will often denote the associated G-action on $C_u^*(\mathcal{S})$ by the same letter α.

Definition 3.2.5. Suppose that $(\mathcal{S}, G, \mathfrak{g}, \alpha)$ is a dynamical system. Define the *full crossed product* to be the subsystem

$$\mathcal{S} \rtimes_{\alpha} \mathfrak{g} := \operatorname{span}\{a u_g : a \in \mathcal{S}, g \in \mathfrak{g}\} \subseteq C_u^*(\mathcal{S}) \rtimes_{\alpha} G\ .$$

The following fact gives us a universality for the full crossed products of operator systems.

Proposition 3.2.6. *Let $(\mathcal{S}, G, \mathfrak{g}, \alpha)$ be a dynamical system, and let $(\mathcal{C}, \iota)$ be any α-admissible C^*-cover for $(\mathcal{S}, G, \mathfrak{g}, \alpha)$. Then there is a unique surjective ucp map $\iota \rtimes \alpha : \mathcal{S} \rtimes_\alpha \mathfrak{g} \to \mathcal{S} \rtimes_\alpha^{\mathcal{C}} \mathfrak{g}$ such that $\iota \rtimes \alpha(au_g) = \iota(a)u_g$ for all $a \in \mathcal{S}$ and $g \in \mathfrak{g}$.*

Proof. Note that the map $\iota : \mathcal{S} \to \mathcal{C}$ is G-equivariant with respect to α. By the universal property of $C_u^*(\mathcal{S})$, there is a unique unital $*$-homomorphism map $\Phi : C_u^*(\mathcal{S}) \to \mathcal{C}$ such that $\Phi_{|\mathcal{S}} = \iota$. It is easy to see that Φ is still G-equivariant, so we obtain a unital $*$-homomorphism $\Phi \rtimes \alpha : C_u^*(\mathcal{S}) \rtimes_\alpha G \to \mathcal{C} \rtimes_\alpha G$. Restricting to $\mathcal{S} \rtimes_\alpha \mathfrak{g}$ yields the desired map. □

Recall from [27] that for a group G with generating set $\mathfrak{g}$,

$$\mathcal{S}(\mathfrak{g}) := \operatorname{span}\{u_g : g \in \mathfrak{g}\} \subseteq C^*(G) \ .$$

There are two difficulties in working with full crossed products. The first is that surjective ucp maps between operator systems are not, in general, quotient maps of operator systems. This problem arises even in low dimensions, such as in Proposition 3.1.25. The other key difficulty can be seen by considering any dynamical system $(\mathcal{S}, G, \mathfrak{g}, \alpha)$ equipped with the trivial action $\alpha = \mathrm{id}$. Proposition 3.2.12 below shows that $\mathcal{S} \rtimes_{\mathrm{id,env}} \mathfrak{g} \simeq \mathcal{S} \otimes_{\mathrm{ess}} \mathcal{S}(\mathfrak{g})$, while Proposition 3.1.5 shows that $\mathcal{S} \rtimes_{\mathrm{id},\lambda} \mathfrak{g} = \mathcal{S} \otimes_{\min} \mathcal{S}_\lambda(\mathfrak{g})$. On the other hand, the tensor product structures arising from $\mathcal{S} \rtimes_{\mathrm{id}} \mathfrak{g}$ are not as well understood.

Proposition 3.2.7. *Let $(\mathcal{S}, G, \mathfrak{g}, \mathrm{id})$ be a dynamical system with the trivial action. Then $\mathcal{S} \rtimes_{\mathrm{id}} \mathfrak{g}$ is completely order isomorphic to the inclusion of the subspace $\mathcal{S} \otimes \mathcal{S}(\mathfrak{g}) \subseteq C_u^*(\mathcal{S}) \otimes_{\max} C^*(G)$.*

Proof. We note that $C_u^*(\mathcal{S}) \rtimes_{\mathrm{id}} G$ is canonically isomorphic to $C_u^*(\mathcal{S}) \otimes_{\max} C^*(G)$, and that this isomorphism maps $\mathcal{S} \rtimes_\alpha \mathfrak{g}$ onto $\mathcal{S} \otimes \mathcal{S}(\mathfrak{g})$, which completes the proof. □

One could define a *universal-enveloping* tensor product of operator systems $\mathcal{S}$ and $\mathcal{T}$ to be the operator system structure $\mathcal{S} \otimes_{ue} \mathcal{T}$ arising from the inclusion $\mathcal{S} \otimes \mathcal{T} \subseteq C_u^*(\mathcal{S}) \otimes_{\max} C_{\mathrm{env}}^*(\mathcal{T})$. In this way, for any trivial dynamical system $(\mathcal{S}, G, \mathfrak{g}, \mathrm{id})$, we have $\mathcal{S} \rtimes_{\mathrm{id}} \mathfrak{g} \simeq \mathcal{S} \otimes_{ue} \mathcal{S}(\mathfrak{g})$. However, the properties of this tensor product are unclear. If $\pi_{\mathcal{S}} : C_u^*(\mathcal{S}) \to C_{\mathrm{env}}^*(\mathcal{S})$ and $\pi_{\mathcal{T}} : C_u^*(\mathcal{T}) \to C_{\mathrm{env}}^*(\mathcal{T})$ are the canonical quotient maps, then we obtain the sequence of ucp maps

$$\begin{array}{ccccc} C_u^*(\mathcal{S}) \otimes_{\max} C_u^*(\mathcal{T}) & \xrightarrow{\mathrm{id} \otimes \pi_{\mathcal{T}}} & C_u^*(\mathcal{S}) \otimes_{\max} C_{\mathrm{env}}^*(\mathcal{T}) & \xrightarrow{\pi_{\mathcal{S}} \otimes \mathrm{id}} & C_{\mathrm{env}}^*(\mathcal{S}) \otimes_{\max} C_{\mathrm{env}}^*(\mathcal{T}) \\ \uparrow & & \uparrow & & \uparrow \\ \mathcal{S} \otimes_c \mathcal{T} & \longrightarrow & \mathcal{S} \otimes_{ue} \mathcal{T} & \longrightarrow & \mathcal{S} \otimes_{\mathrm{ess}} \mathcal{T}. \end{array}$$

In particular, we have ess $\leq ue \leq c$. Similarly, one can define the *enveloping-universal* tensor product of operator systems $\mathcal{S}$ and $\mathcal{T}$ to be the operator system structure $\mathcal{S} \otimes_{eu} \mathcal{T}$ arising from the inclusion $\mathcal{S} \otimes \mathcal{T} \subseteq C_{\mathrm{env}}^*(\mathcal{S}) \otimes_{\max} C_u^*(\mathcal{T})$. Clearly, the flip map $\mathcal{S} \otimes \mathcal{T} \to \mathcal{T} \otimes \mathcal{S}$ induces a complete order isomorphism $\mathcal{S} \otimes_{ue} \mathcal{T} \to \mathcal{T} \otimes_{eu} \mathcal{S}$. On the other hand, we can at least distinguish ue from c. To this end, we need a slight generalization of a result of Kavruk [50, Corollary 5.8]. The proof is almost identical to [28, Proposition 3.6]; we include it for completeness.

Proposition 3.2.8. *Let $\mathcal{S}$ be an operator system, and let $\mathcal{T} \subseteq \mathcal{C}$ be an operator system that contains enough unitaries in a unital C*-algebra $\mathcal{C}$. If $\mathcal{S} \otimes_{\min} \mathcal{T} = \mathcal{S} \otimes_c \mathcal{T}$, then $\mathcal{S} \otimes_{\min} \mathcal{C} = \mathcal{S} \otimes_{\max} \mathcal{C}$.*

Proof. As every unital C*-algebra is $(c, \max)$-nuclear [52, Theorem 6.7], we need only show that $\mathcal{S} \otimes_{\min} \mathcal{C} = \mathcal{S} \otimes_c \mathcal{C}$. Let $X \in M_k(\mathcal{S} \otimes_{\min} \mathcal{C})$ be positive, and let $\phi : \mathcal{S} \to B(H)$ and $\psi : \mathcal{C} \to B(H)$ be ucp maps with commuting ranges. We consider a minimal Stinespring representation $\psi = V^* \pi(\cdot) V$ of ψ on some Hilbert space $\mathcal{H}_\pi$. We apply the commutant lifting theorem [2, Theorem 1.3.1] to obtain a unital $*$-homomorphism $\rho : \psi(\mathcal{C})' \to \mathcal{B}(H_\pi)$ such that $V^* \rho(a) = a V^*$ for all $a \in \psi(\mathcal{C})'$. The fact that $\phi(\mathcal{S}) \subseteq \psi(\mathcal{C})'$ implies that $\gamma := \rho \circ \phi : \mathcal{S} \to \mathcal{B}(H_\pi)$ is ucp and its range commutes with the range of π. Since the restriction of $\gamma \cdot \pi$ to $\mathcal{S} \otimes_c \mathcal{T}$ is ucp and $\mathcal{S} \otimes_{\min} \mathcal{T} = \mathcal{S} \otimes_c \mathcal{T}$, it follows that $\gamma \cdot \pi$ is ucp on $\mathcal{S} \otimes_{\min} \mathcal{T}$. We extend $\gamma \cdot \pi$ by Arveson's extension theorem [2] to a ucp map $\eta : C_{\mathrm{env}}^*(\mathcal{S}) \otimes_{\min} \mathcal{C} \to \mathcal{B}(H_\pi)$. Let $\{u_\alpha\}_{\alpha \in A}$ be a collection of unitaries in $\mathcal{T}$ that generate $\mathcal{C}$ as a C*-algebra. Then for each $\alpha \in A$, we have

$$\eta(1 \otimes u_\alpha) = \gamma \cdot \pi(1 \otimes u_\alpha) = \pi(u_\alpha),$$

which is unitary. Then each u_α is in the multiplicative domain $\mathcal{M}_\eta$ of η, from which it follows that $1 \otimes \mathcal{C} \subseteq \mathcal{M}_\eta$. Therefore, for $s \in \mathcal{S}$ and $b \in \mathcal{C}$, we obtain

$$\eta(s \otimes b) = \eta(s \otimes 1)\eta(1 \otimes b) = \gamma(s)\pi(b).$$

In particular, it follows that

$$\phi \cdot \psi(s \otimes b) = \phi(s)\psi(b) = \phi(s) V^* \pi(b) V = V^* \rho(\phi(s)) \pi(b) V = V^* \gamma \cdot \pi(s \otimes b) V.$$

Therefore, $\phi \cdot \psi = V^* \eta(\cdot)|_{\mathcal{S} \otimes_{\min} \mathcal{C}} V$ is ucp, so that $\phi \cdot \psi(X) \in M_k(B(H))_+$. Hence, X is positive in $M_k(\mathcal{S} \otimes_c \mathcal{C})$ as well, which completes the proof. $\square$

To show that $ue \neq c$, we consider the operator system

$$\mathcal{W}_{3,2} := \left\{ \begin{bmatrix} a & b & 0 & 0 & 0 & 0 \\ b & a & 0 & 0 & 0 & 0 \\ 0 & 0 & a & c & 0 & 0 \\ 0 & 0 & c & a & 0 & 0 \\ 0 & 0 & 0 & 0 & a & d \\ 0 & 0 & 0 & 0 & d & a \end{bmatrix} : a, b, c, d \in \mathbb{C} \right\} \subseteq M_6(\mathbb{C}).$$

If $\mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2$ is the free product of three copies of $\mathbb{Z}_2$ and h_i is the generator of the i-th copy of $\mathbb{Z}_2$, and $NC(3)$ is the operator subsystem of $C^*(*_3\mathbb{Z}_2)$ spanned by h_1, h_2, h_3, then the dual operator system $NC(3)^d$ is unitally completely order isomorphic to $\mathcal{W}_{3,2}$ [27, Proposition 5.13]. Moreover, $\mathcal{W}_{3,2}$ is a nuclearity detector [51, Theorem 0.3], and since $\mathcal{W}_{3,2}$ has a finite-dimensional C*-cover, it follows that $C^*_{env}(\mathcal{W}_{3,2})$ is nuclear. By [34, Proposition 4.2], $\mathcal{W}_{3,2}$ is $(\min, \mathrm{ess})$-nuclear.

Proposition 3.2.9. *Let G be a discrete group with generating set $\mathfrak{g}$. If G is not amenable, then $\mathcal{W}_{3,2} \otimes_{ue} \mathcal{S}(\mathfrak{g}) \neq \mathcal{W}_{3,2} \otimes_c \mathcal{S}(\mathfrak{g})$.*

Proof. Suppose that $\mathcal{W}_{3,2} \otimes_{ue} \mathcal{S}(\mathfrak{g}) = \mathcal{W}_{3,2} \otimes_c \mathcal{S}(\mathfrak{g})$. Note that the commuting tensor product is symmetric [52, Theorem 6.3]. Thus, if $\mathcal{W}_{3,2} \otimes_{ue} \mathcal{S}(\mathfrak{g})$ is not completely order isomorphic to $\mathcal{S}(\mathfrak{g}) \otimes_{ue} \mathcal{W}_{3,2}$ via the flip map, then we are done. Suppose that this flip map is a complete order isomorphism with respect to the ue-tensor product. Then $\mathcal{W}_{3,2} \otimes_{ue} \mathcal{S}(\mathfrak{g}) \simeq \mathcal{S}(\mathfrak{g}) \otimes_{ue} \mathcal{W}_{3,2}$. Since $C^*_{\mathrm{env}}(\mathcal{W}_{3,2})$ is C*-nuclear and $\mathcal{S}(\mathfrak{g}) \otimes_{ue} \mathcal{W}_{3,2} \subseteq C^*_u(G) \otimes_{\max} C^*_{\mathrm{env}}(\mathcal{W}_{3,2}) = C^*_u(G) \otimes_{\min} C^*_{\mathrm{env}}(\mathcal{W}_{3,2})$, we see that $\mathcal{S}(\mathfrak{g}) \otimes_{\min} \mathcal{W}_{3,2} = \mathcal{S}(\mathfrak{g}) \otimes_{ue} \mathcal{W}_{3,2}$. In particular, $\mathcal{S}(\mathfrak{g}) \otimes_{\mathrm{ess}} \mathcal{W}_{3,2} = \mathcal{S}(\mathfrak{g}) \otimes_{ue} \mathcal{W}_{3,2}$. Since ess is also symmetric, applying the flip map, we have that

$$\mathcal{W}_{3,2} \otimes_{\mathrm{ess}} \mathcal{S}(\mathfrak{g}) = \mathcal{W}_{3,2} \otimes_{ue} \mathcal{S}(\mathfrak{g}).$$

Since $\mathcal{W}_{3,2}$ is $(\min, \mathrm{ess})$-nuclear, it follows that $\mathcal{W}_{3,2} \otimes_{\min} \mathcal{S}(\mathfrak{g}) = \mathcal{W}_{3,2} \otimes_c \mathcal{S}(\mathfrak{g})$. By Proposition 3.2.8, we have $\mathcal{W}_{3,2} \otimes_{\min} C^*(G) = \mathcal{W}_{3,2} \otimes_{\max} C^*(G)$, which implies that $C^*(G)$ is nuclear [51, Theorem 0.3], which is a contradiction since G is not amenable. Hence, $\mathcal{W}_{3,2} \otimes_{ue} \mathcal{S}(\mathfrak{g}) \neq \mathcal{W}_{3,2} \otimes_c \mathcal{S}(\mathfrak{g})$. ☐

Remark 3.2.10. In the case of operator algebraic dynamical systems $(\mathcal{A}, G, \alpha)$, Katsoulis and Ramsey proved [47, Theorem 4.1] that

$$C^*_u(\mathcal{A} \rtimes_\alpha G) = C^*_u(\mathcal{A}) \rtimes_\alpha G\ .$$

It is known that in general $C^*_u(\mathcal{S}(\mathfrak{g}))$ does not coincide with $C^*(G)$ (see [27]), so such a theorem is not expected to hold for operator system dynamical systems. In fact, $C^*_u(\mathcal{S}(\mathfrak{g}))$

and $C^*(G)$ fail to coincide in the case where $G = \mathbb{Z}$ and $\mathfrak{g} = \{-1, 0, 1\}$ [27]. Moreover, since we are interested in C*-envelopes, we will focus on properties of the full enveloping crossed product, rather than the full crossed product.

Like the reduced crossed product, the full enveloping crossed product preserves hyperrigidity. The proof is exactly the same as in Theorem 3.1.13, so we omit it. For a proof in the operator algebraic case, see [44, Theorem 2.7].

Theorem 3.2.11. *Suppose that $(\mathcal{S}, G, \mathfrak{g}, \alpha)$ is a dynamical system. Suppose that $\mathcal{S}$ is hyperrigid. Then the full enveloping crossed product is hyperrigid. In particular, $C^*_{env}(\mathcal{S} \rtimes_{\alpha,env} \mathfrak{g}) \simeq C^*_{env}(\mathcal{S}) \rtimes_\alpha G$.*

We now give the tensor product description of the full enveloping crossed product with respect to a trivial action.

Proposition 3.2.12. *Suppose that $(\mathcal{S}, G, \mathfrak{g}, id)$ is a trivial dynamical system. We have the isomorphism*

$$\mathcal{S} \rtimes_{id,env} \mathfrak{g} \simeq \mathcal{S} \otimes_{ess} \mathcal{S}(\mathfrak{g}) \ .$$

Proof. We know that

$$C^*_{\text{env}}(\mathcal{S}) \rtimes_{id} G \simeq C^*_{\text{env}}(\mathcal{S}) \otimes_{\max} C^*(G) \ .$$

This induces an isomorphism

$$\mathcal{S} \rtimes_{id,\text{env}} \mathfrak{g} \simeq \text{span}\{a \otimes u_g \in C^*_{\text{env}}(\mathcal{S}) \otimes_{\max} C^*(G) : a \in \mathcal{S}, g \in \mathfrak{g}\} \ .$$

By definition, the span on the right hand side is $\mathcal{S} \otimes_{\text{ess}} \mathcal{S}(\mathfrak{g})$. ∎

For amenable groups, there is no difference between the reduced and the full enveloping crossed products.

Proposition 3.2.13. *Suppose that $(\mathcal{S}, G, \mathfrak{g}, \alpha)$ is a dynamical system with G amenable. Then $\mathcal{S} \rtimes_{\alpha,\lambda} \mathfrak{g} = \mathcal{S} \rtimes_{\alpha,env} \mathfrak{g}$.*

Proof. Since G is amenable, we have the isomorphism $C^*_{\text{env}}(\mathcal{S}) \rtimes_{\alpha,\lambda} G = C^*_{\text{env}}(\mathcal{S}) \rtimes_\alpha G$ sending generators to generators. By restricting this isomorphism to $\mathcal{S} \rtimes_{\alpha,\lambda} \mathfrak{g}$, we get the identity $\mathcal{S} \rtimes_{\alpha,\lambda} \mathfrak{g} = \mathcal{S} \rtimes_{\alpha,\text{env}} \mathfrak{g}$. ∎

3.3 Two Problems of Katsoulis and Ramsey

If X is an operator space in $B(\mathcal{H})$, then define the associated non self-adjoint operator algebra $\mathcal{U}(X)$ (see [10, Sections 2.2.10-2.2.11] for more on this algebra) as the subalgebra of $B(H^2) \simeq M_2(B(H))$ given by

$$\mathcal{U}(X) := \left\{ \begin{bmatrix} \lambda & x \\ 0 & \lambda \end{bmatrix} : \lambda \in \mathbb{C},\, x \in X \right\} .$$

Note that the algebra $\mathcal{U}(X)$ does not depend on the representation chosen for X. The goal of this section is to prove the following two theorems:

Theorem 3.3.1. *Suppose that G is a locally compact group such that $C^*_\lambda(G)$ admits a tracial state. Let $\mathcal{A} := \mathcal{U}(\mathcal{W}_{3,2})$ be the operator subalgebra of $M_2\left(\bigoplus_{k=1}^3 M_2(\mathbb{C})\right)$ endowed with the trivial G-action $\mathrm{id} : G \curvearrowright \mathcal{A}$. The following are equivalent:*

1. $\mathcal{A} \rtimes_{C^*_{env}(\mathcal{A}),\mathrm{id}} G = \mathcal{A} \rtimes_{C^*_u(\mathcal{A}),\mathrm{id}} G$.
2. *The group G is amenable.*

Theorem 3.3.1 provides a counterexample to Problem 2 in [47] for a large class of locally compact groups. Indeed, by [55], $C^*_\lambda(G)$ admits a tracial state if and only if G admits an open amenable normal subgroup. The operator algebra $\mathcal{A} = \mathcal{U}(\mathcal{W}_{3,2})$ is surprisingly tame. Because $\mathcal{A}$ is constructed from Kavruk's nuclearity detector, which is four-dimensional, the operator algebra we obtain satisfies $\dim(\mathcal{A}) = 5$. Moreover, $\mathcal{A}$ is hyperrigid in its C^*-envelope (see Theorem 3.4.10).

The counterexample of Theorem 3.3.1 allows us to give a counterexample to Problem 1 of [47].

Theorem 3.3.2. *Let $\mathcal{A} = \mathcal{U}(\mathcal{W}_{3,2})$. If G is a discrete group, then the following are equivalent:*

1. *We have the identity $C^*_{env}(\mathcal{A}) \rtimes_{\mathrm{id}} G = C^*_{env}(\mathcal{A} \rtimes_{\mathrm{id}} G)$.*
2. *The group G is amenable.*

Before proving Theorems 3.3.1 and 3.3.2, we need some facts about $\mathcal{U}(X)$. Recall that an operator system $\mathcal{S}$ *detects nuclearity* (or is a *nuclearity detector*) if, whenever $\mathcal{D}$ is a unital C*-algebra and

$$\mathcal{S} \otimes_{\min} \mathcal{D} = \mathcal{S} \otimes_c \mathcal{D},$$

then $\mathcal{D}$ is a nuclear C*-algebra [51]. Our first goal is to show that if $\mathcal{S}$ is a nuclearity detector, then $\mathcal{U}(\mathcal{S})$ is a nuclearity detector for C*-algebras; that is, if $\mathcal{U}(\mathcal{S}) \otimes_{\min} \mathcal{D} = \mathcal{U}(\mathcal{S}) \otimes_{\max} \mathcal{D}$ then $\mathcal{D}$ is a nuclear C*-algebra.

We first wish to interpret the relevant tensor products of an operator system $\mathcal{S}$ with a unital C*-algebra as an operator space. We know that the norm arising from the operator system $\mathcal{S} \otimes_{\min} \mathcal{D}$ agrees with the minimal operator space tensor norm [52, Corollary 4.9]. For the norm arising from the commuting tensor product, we have the following result.

Lemma 3.3.3. *Let $\mathcal{S}$ be an operator system; let $\mathcal{D}$ be a unital C*-algebra; and let $d \in \mathbb{N}$. For any element $Z \in M_d(\mathcal{S} \otimes_c \mathcal{D}) \simeq \mathcal{S} \otimes_c M_d(\mathcal{D})$, we have*

$$\|Z\|_{M_d(\mathcal{S}\otimes_c\mathcal{D})} = \sup_{\theta,\pi} \|\theta \cdot \pi(Z)\|$$

where the supremum is taken over all pairs of commuting maps

$$\theta : \mathcal{S} \to B(H) \text{ and}$$
$$\pi : M_d(\mathcal{D}) \to B(H),$$

where θ is ucp and π is a unital $*$-homomorphism.

Proof. Since $M_d(\mathcal{D})$ is itself a unital C*-algebra, by replacing $M_d(\mathcal{D})$ with $\mathcal{D}$ if necessary, we may assume that $d = 1$. We know that the order structure of $\mathcal{S} \otimes_c \mathcal{D}$ is inherited from the order structure of $C_u^*(\mathcal{S}) \otimes_{\max} \mathcal{D}$. From this fact, it follows that the norm structure of $\mathcal{S} \otimes_c \mathcal{D}$ is also inherited from the norm structure of $C_u^*(\mathcal{S}) \otimes_{\max} \mathcal{D}$. Given an element $z \in \mathcal{S} \otimes_c \mathcal{D}$, we have

$$\|z\|_c = \sup_{\theta',\pi'} \|\theta' \cdot \pi'(z)\|,$$

where the supremum is taken over all unital $*$-homomorphisms $\theta' : C_u^*(\mathcal{S}) \to B(H)$ and $\pi' : \mathcal{D} \to B(H)$, with commuting ranges. Since every unital $*$-homomorphism $\theta : C_u^*(\mathcal{S}) \to B(H)$ restricts to a ucp map on $\mathcal{S}$, we obtain the inequality

$$\|z\|_c \leq \sup_{\theta,\pi} \|\theta \cdot \pi(z)\|,$$

where the supremum is taken over all ucp maps $\theta : \mathcal{S} \to B(H)$ and unital $*$-homomorphisms $\pi : \mathcal{D} \to B(H)$ with commuting ranges. Conversely, if $\theta : \mathcal{S} \to B(H)$ is ucp and $\pi : \mathcal{D} \to B(H)$ is a unital $*$-homomorphism whose range commutes with the range of θ, then by the universal property of $C_u^*(\mathcal{S})$, there is a unital $*$-homomorphism $\theta' : C_u^*(\mathcal{S}) \to B(H)$ such that $\theta'|_{\mathcal{S}} = \theta$. Since the range of θ commutes with the range of π and and $C_u^*(\mathcal{S})$ is generated as a C*-algebra by $\mathcal{S}$, we have that θ' and π have commuting ranges. Moreover, $\|\theta' \cdot \pi(z)\| = \|\theta \cdot \pi(z)\|$. Therefore, the reverse inequality holds, and the result follows. $\square$

The next lemma is the operator system analogue of a result of Blecher and Duncan [9, Lemma 6.3].

Lemma 3.3.4. *Let $\mathcal{S}$ be an operator system and let $\mathcal{D}$ be a unital C*-algebra. If $\mathcal{U}(\mathcal{S}) \otimes_{\min} \mathcal{D} = \mathcal{U}(\mathcal{S}) \otimes_{\max} \mathcal{D}$, then $\mathcal{S} \otimes_{\min} \mathcal{D} = \mathcal{S} \otimes_c \mathcal{D}$. In particular, if $\mathcal{S}$ is a nuclearity detector, then $\mathcal{U}(\mathcal{S})$ is a nuclearity detector.*

Proof. Suppose that $\mathcal{U}(\mathcal{S}) \otimes_{\min} \mathcal{D} = \mathcal{U}(\mathcal{S}) \otimes_{\max} \mathcal{D}$, and let $z \in \mathcal{S} \otimes \mathcal{D}$. Since we always have $\|z\|_c \geq \|z\|_{\min}$, it suffices to show that $\|z\|_c \leq \|z\|_{\min}$. We may write

$$z = \sum_{i=1}^{k} z_i \otimes d_i,$$

where $z_i \in \mathcal{S}$ and $d_i \in \mathcal{D}$. Let $\theta : \mathcal{S} \to B(H)$ be ucp and $\pi : \mathcal{D} \to B(H)$ be a unital $*$-homomorphism whose range commutes with the range of θ. We define

$$\theta' : \mathcal{U}(\mathcal{S}) \to B(\mathcal{H}^2) : \begin{bmatrix} \lambda & x \\ 0 & \lambda \end{bmatrix} \mapsto \begin{bmatrix} \lambda & \theta(x) \\ 0 & \lambda \end{bmatrix}.$$

It follows from [64, Lemma 8.1] that θ' is a unital completely contractive map. A calculation also shows that θ' is a homomorphism. The amplification $\pi \oplus \pi : \mathcal{D} \to B(H^2)$ is a unital $*$-homomorphism such that θ' and $\pi \oplus \pi$ have commuting ranges. In particular, by definition of the maximal operator algebra tensor norm,

$$\begin{aligned} \|\theta \cdot \pi(z)\| &= \left\| \theta' \cdot (\pi \oplus \pi) \left(\sum_i \begin{bmatrix} 0 & z_i \\ 0 & 0 \end{bmatrix} \otimes d_i \right) \right\| \\ &\leq \left\| \sum_i \begin{bmatrix} 0 & z_i \\ 0 & 0 \end{bmatrix} \otimes d_i \right\|_{\max} \\ &= \left\| \sum_i \begin{bmatrix} 0 & z_i \\ 0 & 0 \end{bmatrix} \otimes d_i \right\|_{\min} \\ &= \|z\|_{\min} \end{aligned}$$

Since this is for arbitrary pairs (θ, π), it follows that $\|z\|_c \leq \|z\|_{\min}$. A similar argument shows that $\|Z\|_c = \|Z\|_{\min}$ for every $Z \in M_d(\mathcal{S} \otimes \mathcal{D})$ and $d \geq 1$. Thus, $\mathcal{S} \otimes_{\min} \mathcal{D} = \mathcal{S} \otimes_c \mathcal{D}$.

If $\mathcal{S}$ is a nuclearity detector and $\mathcal{D}$ is a unital C*-algebra such that $\mathcal{U}(\mathcal{S}) \otimes_{\min} \mathcal{D} = \mathcal{U}(\mathcal{S}) \otimes_{\max} \mathcal{D}$, then the above proof shows that $\mathcal{S} \otimes_{\min} \mathcal{D} = \mathcal{S} \otimes_c \mathcal{D}$. Since $\mathcal{S}$ is a nuclearity detector, $\mathcal{D}$ must be a nuclear C*-algebra. □

Recall that

$$\mathcal{W}_{3,2} := \left\{ \begin{bmatrix} a & b & 0 & 0 & 0 & 0 \\ b & a & 0 & 0 & 0 & 0 \\ 0 & 0 & a & c & 0 & 0 \\ 0 & 0 & c & a & 0 & 0 \\ 0 & 0 & 0 & 0 & a & d \\ 0 & 0 & 0 & 0 & d & a \end{bmatrix} : a, b, c, d \in \mathbb{C} \right\} \subseteq M_6(\mathbb{C}).$$

Note that $C^*_{\text{env}}(\mathcal{W}_{3,2})$ is nuclear. Indeed, if we conjugate $\mathcal{W}_{3,2}$ by the unitary matrix $\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{\oplus 3}$, we have the isomorphism

$$\mathcal{W}_{3,2} \simeq \{\operatorname{diag}(a+b, a-b, a+c, a-c, a+d, a-d) : a, b, c, d \in \mathbb{C}\}.$$

Therefore, the C*-envelope of $\mathcal{W}_{3,2}$ must be a quotient of $\mathbb{C}^6$. In particular, $\mathcal{U}(\mathcal{W}_{3,2})$ has a nuclear C*-envelope as well. On the other hand, since $\mathcal{W}_{3,2}$ is a nuclearity detector [51, Theorem 0.3], Lemma 3.3.4 shows that $\mathcal{U}(\mathcal{W}_{3,2})$ is a nuclearity detector. Unlike the situation for operator systems, the operator algebra $\mathcal{U}(\mathcal{W}_{3,2})$ also detects nuclearity for non-unital C*-algebras. To show this fact, we first need the following fact.

Proposition 3.3.5. *Let τ be the minimal or maximal operator algebra tensor product. Let $\mathcal{A}$ be a unital operator algebra and let $\mathcal{D}$ be a C*-algebra. If $\mathcal{D}^+$ is the minimal unitization of $\mathcal{D}$ and $\pi : \mathcal{D}^+ \to \mathcal{D}^+/\mathcal{D} \simeq \mathbb{C}$ is the canonical quotient map, then the sequence*

$$0 \longrightarrow \mathcal{A} \otimes_\tau \mathcal{D} \longrightarrow \mathcal{A} \otimes_\tau \mathcal{D}^+ \xrightarrow{\mathrm{id}_{\mathcal{A}} \otimes \pi} \mathcal{A} \longrightarrow 0$$

is exact. In other words, $\mathcal{A} \otimes_\tau \mathcal{D} \subseteq \mathcal{A} \otimes_\tau \mathcal{D}^+$ completely isometrically and $\mathrm{id}_{\mathcal{A}} \otimes \pi$ is a complete quotient map of $\mathcal{A} \otimes_\tau \mathcal{D}^+$ onto $\mathcal{A}$.

Proof. The inclusion $\mathcal{A} \otimes_{\min} \mathcal{D} \subseteq \mathcal{A} \otimes_{\min} \mathcal{D}^+$ holds by definition of the minimal tensor product. The fact that $\mathcal{A} \otimes_{\max} \mathcal{D} \subseteq \mathcal{A} \otimes_{\max} \mathcal{D}^+$ completely isometrically follows by [9, Proposition 2.5].

The tensor product map $\mathrm{id}_{\mathcal{A}} \otimes \pi : \mathcal{A} \otimes_\tau \mathcal{D}^+ \to \mathcal{A} \otimes_\tau \mathbb{C} = \mathcal{A}$ is completely contractive. On the other hand, the map $\varphi : \mathcal{A} \to \mathcal{A} \otimes_\tau \mathcal{D}^+$ given by $\varphi(a) = a \otimes 1_{\mathcal{D}^+}$ is a completely contractive unital homomorphism. Hence, φ is a completely contractive splitting of $\mathrm{id}_{\mathcal{A}} \otimes \pi$. It follows that $\mathrm{id}_{\mathcal{A}} \otimes \pi$ is a complete quotient map. $\square$

Lemma 3.3.6. *Let $\mathcal{A}$ be a unital operator algebra; let $\mathcal{D}$ be a non-unital C^*-algebra, and let $\mathcal{D}^+$ be its minimal unitization. Then $\mathcal{A} \otimes_{\min} \mathcal{D} = \mathcal{A} \otimes_{\max} \mathcal{D}$ if and only if $\mathcal{A} \otimes_{\min} \mathcal{D}^+ = \mathcal{A} \otimes_{\max} \mathcal{D}^+$.*

Proof. If $\mathcal{A} \otimes_{\min} \mathcal{D}^+ = \mathcal{A} \otimes_{\max} \mathcal{D}^+$, then applying Proposition 3.3.5 shows that $\mathcal{A} \otimes_\tau \mathcal{D} \subseteq \mathcal{A} \otimes_\tau \mathcal{D}^+$ completely isometrically for $\tau \in \{\min, \max\}$. It immediately follows that $\mathcal{A} \otimes_{\min} \mathcal{D} = \mathcal{A} \otimes_{\max} \mathcal{D}$.

Conversely, suppose that $\mathcal{A} \otimes_{\min} \mathcal{D} = \mathcal{A} \otimes_{\max} \mathcal{D}$. Using the canonical maps, the following diagram commutes:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{A} \otimes_{\max} \mathcal{D} & \longrightarrow & \mathcal{A} \otimes_{\max} \mathcal{D}^+ & \longrightarrow & \mathcal{A} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{A} \otimes_{\min} \mathcal{D} & \longrightarrow & \mathcal{A} \otimes_{\min} \mathcal{D}^+ & \longrightarrow & \mathcal{A} & \longrightarrow & 0
\end{array}$$

By Proposition 3.3.5, the rows are exact. By the Five-Lemma for operator algebras [11, Lemma 3.2], since the outer two vertical arrows are complete isometries, the middle arrow is a complete isometry. Thus, $\mathcal{A} \otimes_{\min} \mathcal{D}^+ = \mathcal{A} \otimes_{\max} \mathcal{D}^+$. □

Lemma 3.3.7. *Suppose that $\mathcal{A}$ is a unital operator algebra for which for every unital C^*-algebra $\mathcal{C}$, $\mathcal{A} \otimes_{\min} \mathcal{C} = \mathcal{A} \otimes_{\max} \mathcal{C}$ if and only if $\mathcal{C}$ is a nuclear C^*-algebra. Then for every non-unital C^*-algebra $\mathcal{D}$, $\mathcal{A} \otimes_{\min} \mathcal{D} = \mathcal{A} \otimes_{\max} \mathcal{D}$ if and only if $\mathcal{D}$ is a nuclear C^*-algebra.*

Proof. Let $\mathcal{D}$ be a non-unital C*-algebra. Let $\mathcal{D}^+$ be its minimal unitization. We know that $\mathcal{D}$ is nuclear if and only if $\mathcal{D}^+$ is nuclear. Thus, $\mathcal{D}$ is nuclear if and only if $\mathcal{A} \otimes_{\min} \mathcal{D}^+ = \mathcal{A} \otimes_{\max} \mathcal{D}^+$. By Lemma 3.3.6, the latter condition is equivalent to having $\mathcal{A} \otimes_{\min} \mathcal{D} = \mathcal{A} \otimes_{\max} \mathcal{D}$, as desired. □

We are now in a position to prove Theorem 3.3.1, which gives a counterexample to the second problem of Katsoulis and Ramsey.

Proof of Theorem 3.3.1. The implication (2) $\to$ (1) holds by [47, Theorem 3.14]. Conversely, suppose we have the identity $\mathcal{A} \rtimes_{C^*_{env}(\mathcal{A}),\mathrm{id}} G = \mathcal{A} \rtimes_{C^*_u(\mathcal{A}),\mathrm{id}} G$. Since $C^*_{\mathrm{env}}(\mathcal{A})$ is nuclear,

$$C^*_{env}(\mathcal{A}) \otimes_{\min} C^*(G) = C^*_{env}(\mathcal{A}) \otimes_{\max} C^*(G) = C^*_{\mathrm{env}}(\mathcal{A}) \rtimes_{\mathrm{id}} G\,.$$

Thus, $\mathcal{A} \rtimes_{C^*_{\text{env}}(\mathcal{A}),\text{id}} G$ is completely isometrically isomorphic to the completion of $\mathcal{A} \otimes C^*(G)$ in $C^*_{\text{env}}(\mathcal{A}) \otimes_{\min} C^*(G)$. We conclude that

$$\mathcal{A} \rtimes_{C^*_{\text{env}}(\mathcal{A},\text{id})} G = \mathcal{A} \otimes_{\min} C^*(G).$$

On the other hand, by Example 2.2.4, we have

$$\mathcal{A} \rtimes_{C^*_u(\mathcal{A}),\text{id}} G = \mathcal{A} \otimes_{\max} C^*(G).$$

Then we have the identity

$$\mathcal{A} \otimes_{\min} C^*(G) = \mathcal{A} \rtimes_{C^*_{env}(\mathcal{A}),\text{id}} G = \mathcal{A} \rtimes_{C^*_u(\mathcal{A}),\text{id}} G = \mathcal{A} \otimes_{max} C^*(G)\ .$$

Since $\mathcal{A}$ is a nuclearity detector, $C^*(G)$ is nuclear. In particular, since nuclearity of C*-algebras passes to quotients [14], it follows that $C^*_\lambda(G)$, which is a quotient of $C^*(G)$, must be nuclear. By [61,] (see also [29, Corollary 3.3]), G is amenable if and only if $C^*_\lambda(G)$ is nuclear and $C^*_\lambda(G)$ admits a tracial state. Therefore, G is amenable. □

Theorem 3.3.1 allows for a proof of Theorem 3.3.2, which gives a counterexample to the first problem of Katsoulis and Ramsey.

Proof of Theorem 3.3.2. The proof that (2) implies (1) was done by Katsoulis [44, Theorem 2.5]. For the converse, if $C^*_{env}(\mathcal{A}) \rtimes_{\text{id}} G = C^*_{env}(\mathcal{A} \rtimes_{\text{id}} G)$, then $\mathcal{A} \rtimes_{C^*_u(\mathcal{A}),\text{id}} G$ embeds faithfully and canonically into $C^*_{env}(\mathcal{A}) \rtimes_{\text{id}} G$. Thus, $\mathcal{A} \rtimes_{C^*_{env}(\mathcal{A}),\text{id}} G = \mathcal{A} \rtimes_{C^*_u(\mathcal{A}),\text{id}} G$. By Theorem 3.3.1, this implies that G is amenable. □

3.4 Hyperrigidity and $\mathcal{U}(\mathcal{W}_{3,2})$

In this final section, we show that (min, ess)-nuclearity is preserved under the full enveloping crossed product whenever the operator system is hyperrigid. We also show that $\mathcal{U}(\mathcal{W}_{3,2})$ is hyperrigid, which shows that the equation $C^*_{\text{env}}(\mathcal{A} \rtimes_\alpha G) = C^*_{\text{env}}(\mathcal{A}) \rtimes_\alpha G$ can fail even in the case of a hyperrigid operator algebra.

Let $\mathcal{S} \subseteq \mathcal{C}$ and $\mathcal{T} \subseteq \mathcal{D}$ be hyperrigid operator subsystems of unital C*-algebras $\mathcal{C}$ and $\mathcal{D}$, respectively. By injectivity of the minimal tensor product, $\mathcal{S} \otimes_{\min} \mathcal{T} \subseteq \mathcal{C} \otimes_{\min} \mathcal{D}$. Moreover, by Theorem 3.1.9, $C^*_{\text{env}}(\mathcal{S}) = \mathcal{C}$ and $C^*_{\text{env}}(\mathcal{T}) = \mathcal{D}$. In particular, by definition of the essential tensor product, we have that $\mathcal{S} \otimes_{\text{ess}} \mathcal{T} \subseteq \mathcal{C} \otimes_{\max} \mathcal{D}$. In fact, more is true.

Lemma 3.4.1. *Suppose that $\mathcal{S} \subseteq \mathcal{C}$ and $\mathcal{T} \subseteq \mathcal{D}$ are hyperrigid operator subsystems of unital C*-algebras $\mathcal{C}$ and $\mathcal{D}$ respectively. Then $\mathcal{S} \otimes_{\min} \mathcal{T} \subseteq \mathcal{C} \otimes_{\min} \mathcal{D}$ and $\mathcal{S} \otimes_{ess} \mathcal{T} \subseteq \mathcal{C} \otimes_{\max} \mathcal{D}$ are hyperrigid.*

Proof. We prove the result for the minimal tensor product; the proof for the essential tensor product is the same. Let $\iota_{\mathcal{C}} : \mathcal{C} \to \mathcal{C} \otimes_{\min} \mathcal{D} : a \mapsto a \otimes 1$ and $\iota_{\mathcal{D}} : \mathcal{D} \to \mathcal{C} \otimes_{\min} \mathcal{D} : b \mapsto 1 \otimes b$ be the canonical *-homomorphisms. Suppose that $\pi : \mathcal{C} \otimes_{\min} \mathcal{D} \to B(\mathcal{H})$ is a $*$-representation and suppose that $\rho : \mathcal{C} \otimes_{\min} \mathcal{D} \to B(\mathcal{H})$ is a ucp map extending the map $\pi|_{\mathcal{S} \otimes_{\min} \mathcal{T}}$. Since $\rho \circ \iota_{\mathcal{C}}$ agrees with $\pi \circ \iota_{\mathcal{C}}$ on $\mathcal{S}$, by hyperrigidity of $\mathcal{S}$, $\rho \circ \iota_{\mathcal{C}} = \pi \circ \iota_{\mathcal{C}}$. Similarly, $\rho \circ \iota_{\mathcal{D}} = \pi \circ \iota_{\mathcal{D}}$. For any $a \in \mathcal{C}$ and $b \in \mathcal{D}$, since $\iota_{\mathcal{D}}(b)$ is in the multiplicative domain of ρ,

$$\rho(a \otimes b) = \rho(\iota_{\mathcal{C}}(a)\iota_{\mathcal{D}}(b)) = \rho(\iota_{\mathcal{C}}(a))\rho(\iota_{\mathcal{D}}(b)) = \pi(a \otimes b) \ .$$

Extending by linearity and continuity shows that $\rho = \pi$. □

The following proposition is a generalization of a result of Gupta and Luthra [34, Theorem 4.3].

Proposition 3.4.2. *Suppose that $\mathcal{S} \subseteq \mathcal{C}$ is a hyperrigid operator system. Then $\mathcal{S}$ is* (min, *ess*)*-nuclear if and only if $\mathcal{C}$ is nuclear.*

Proof. If $\mathcal{C}$ is nuclear, then by [34, Proposition 4.2], $\mathcal{S}$ is $(\min, \mathrm{ess})$-nuclear. Conversely, suppose that $\mathcal{S}$ is $(\min, \mathrm{ess})$-nuclear, and let $\mathcal{D}$ be any unital C*-algebra. First, let us show that $C^*_{\mathrm{env}}(\mathcal{S} \otimes_{\min} \mathcal{D}) = \mathcal{C} \otimes_{\min} \mathcal{D}$ and $C^*_{\mathrm{env}}(\mathcal{S} \otimes_{\mathrm{ess}} \mathcal{D}) = \mathcal{C} \otimes_{\max} \mathcal{D}$. By Theorem 3.1.9, it suffices to show that $\mathcal{S} \otimes_{\min} \mathcal{D}$ and $\mathcal{S} \otimes_{\mathrm{ess}} \mathcal{D}$ are hyperrigid in their C*-covers. For the minimal case, suppose that

$$\pi : \mathcal{C} \otimes_{\mathrm{C^*\text{-}min}} \mathcal{D} \to B(H)$$

is a unital $*$-homomorphism and suppose that φ is the restriction of π to $\mathcal{S} \otimes_{\min} \mathcal{D}$. If ρ is any ucp extension of φ to $\mathcal{C} \otimes_{\min} \mathcal{D}$, then the ucp map $\rho|_{\mathcal{C}}$ is a ucp extension of $\varphi|_{\mathcal{S}}$. By hyperrigidity of $\mathcal{S}$, $\rho|_{\mathcal{C}} = \pi$. As well, $\rho|_{\mathcal{D}} = \varphi|_{\mathcal{D}} = \pi|_{\mathcal{D}}$. Thus, $\mathcal{C} \otimes 1$ and $1 \otimes \mathcal{D}$ are in the multiplicative domain of ρ. Therefore, for all $c \in \mathcal{C}$ and $d \in \mathcal{D}$,

$$\rho(c \otimes d) = \rho(c \otimes 1)\rho(1 \otimes d) = \pi(c \otimes 1)\pi(1 \otimes d) = \pi(c \otimes d).$$

Extending by linearity and continuity, we have $\rho = \pi$ on $\mathcal{C} \otimes_{\min} \mathcal{D}$. The proof for the maximal tensor product is the same, replacing all tensors with the appropriate type. Hence, $\mathcal{S} \otimes_{\min} \mathcal{D}$ is hyperrigid in $\mathcal{C} \otimes_{\min} \mathcal{D}$, and $\mathcal{S} \otimes_{\mathrm{ess}} \mathcal{D}$ is hyperrigid in $\mathcal{C} \otimes_{\max} \mathcal{D}$.

Since $\mathcal{S} \otimes_{\min} \mathcal{D} = \mathcal{S} \otimes_{\text{ess}} \mathcal{D}$, and both operator systems are hyperrigid, we get

$$\mathcal{C} \otimes_{\min} \mathcal{D} = C^*_{\text{env}}(\mathcal{S} \otimes_{\min} \mathcal{D}) = C^*_{\text{env}}(\mathcal{S} \otimes_{\text{ess}} \mathcal{D}) = \mathcal{C} \otimes_{\max} \mathcal{D} \,.$$

As $\mathcal{D}$ was an arbitrary unital C*-algebra, it follows that $\mathcal{C}$ is nuclear. $\square$

Corollary 3.4.3. *Suppose that $(\mathcal{S}, G, \mathfrak{g}, \alpha)$ is a dynamical system with $\mathcal{S}$ hyperrigid in $C^*_{env}(\mathcal{S})$ and G amenable. Then $\mathcal{S}$ is $(\min, ess)$-nuclear if and only if $\mathcal{S} \rtimes_{\alpha, env} \mathfrak{g}$ is $(\min, ess)$-nuclear.*

Proof. By Proposition 3.4.2, $\mathcal{S}$ is (min,ess)-nuclear if and only if $C^*_{\text{env}}(\mathcal{S})$ is nuclear. Since G is discrete and amenable, $C^*_{\text{env}}(\mathcal{S})$ is nuclear if and only if $C^*_{\text{env}}(\mathcal{S}) \rtimes_\alpha G$ is nuclear [13, Theorem 4.2.6]. Using Theorem 3.2.11, $\mathcal{S} \rtimes_{\alpha,\text{env}} \mathfrak{g}$ is hyperrigid in $C^*_{\text{env}}(\mathcal{S}) \rtimes_\alpha G$. Applying Proposition 3.4.2 again, $C^*_{\text{env}}(\mathcal{S}) \rtimes_\alpha G$ is nuclear if and only if $\mathcal{S} \rtimes_{\alpha,\text{env}} \mathfrak{g}$ is $(\min, \text{ess})$-nuclear. $\square$

We now will work towards showing that $\mathcal{U}(\mathcal{W}_{3,2})$ is hyperrigid in its C*-envelope. We begin with the following helpful fact about $C^*_{\text{env}}(\mathcal{U}(\mathcal{S}))$.

Proposition 3.4.4. *Let $\mathcal{S}$ be an operator system with C*-cover $\mathcal{D}$. Then $M_2(\mathcal{D})$ is a C*-cover for $\mathcal{U}(\mathcal{S})$. Moreover, $C^*_{env}(\mathcal{U}(\mathcal{S})) = M_2(C^*_{env}(\mathcal{S}))$.*

Proof. Let $\phi : \mathcal{S} \to \mathcal{D}$ be a complete order isomorphism, where $\mathcal{D}$ is a unital C*-algebra such that $\mathcal{D} = C^*(\phi(\mathcal{S}))$. Then there is an associated unital, completely isometric homomorphism $\psi : \mathcal{U}(\mathcal{S}) \to M_2(\mathcal{D})$ such that

$$\psi\left(\begin{bmatrix} \lambda & x \\ 0 & \lambda \end{bmatrix}\right) = \begin{bmatrix} \lambda & \phi(x) \\ 0 & \lambda \end{bmatrix}$$

for all $\lambda \in \mathbb{C}$ and for all $x \in \mathcal{S}$. The matrix units E_{12}, E_{21} are in $C^*(\psi(\mathcal{U}(\mathcal{S})))$ since ϕ is unital. Hence, every element of the form $E_{ii} \otimes \phi(x)$, for $x \in S$, is in $C^*(\psi(\mathcal{U}(\mathcal{S})))$. Thus, $E_{ii} \otimes a \in C^*(\psi(\mathcal{U}(\mathcal{S})))$ for all $a \in \mathcal{D}$. Using the matrix units E_{12} and E_{21}, we see that $C^*(\psi(\mathcal{U}(\mathcal{S}))) = M_2(\mathcal{D})$. Thus, $M_2(\mathcal{D})$ is a C*-cover for $\mathcal{U}(\mathcal{S})$ whenever $\mathcal{D}$ is a C*-cover for $\mathcal{S}$.

Lastly, we must show that $C^*_{\text{env}}(\mathcal{U}(\mathcal{S})) = M_2\left(C^*_{\text{env}}(\mathcal{S})\right)$. To this end, we let $\rho : M_2\left(C^*_{\text{env}}(\mathcal{S})\right) \to C^*_{\text{env}}(\mathcal{U}(\mathcal{S}))$ be a surjective, unital $*$-homomorphism that preserves the copy of $\mathcal{U}(\mathcal{S})$. Let γ be the restriction of ρ to the subalgebra $M_2 \otimes 1_{C^*_{\text{env}}(\mathcal{S})}$. Since γ is a unital $*$-homomorphism and M_2 is simple, γ must be injective. In particular, if $\mathcal{D} = \rho(I_2 \otimes C^*_{\text{env}}(\mathcal{S}))$, then M_2 and $\mathcal{D}$ are commuting unital C*-subalgebras of $C^*_{\text{env}}(\mathcal{U}(\mathcal{S}))$

that generate the whole algebra. Let η be the restriction of ρ to $I_2 \otimes (C^*_{\text{env}}(\mathcal{S}))$. Since ρ is a completely isometric homomorphism on $\mathcal{U}(\mathcal{S})$, a standard canonical shuffle argument [64, p. 97] shows that ρ must be a complete isometry when restricted to $E_{12} \otimes \mathcal{S}$. As ρ is multiplicative, we see that η is a complete isometry on $I_2 \otimes \mathcal{S}$. Thus, $\eta|_{I_2 \otimes \mathcal{S}}$ maps $\mathcal{S}$ completely order isomorphically into $\mathcal{D}$. Since $I_2 \otimes \mathcal{S}$ generates $I_2 \otimes C^*_{\text{env}}(\mathcal{S})$ as a C*-algebra, the image of $I_2 \otimes \mathcal{S}$ under η generates $\mathcal{D}$. By the definition of the C*-envelope, there is a unique, surjective unital $*$-homomorphism $\delta : \mathcal{D} \to C^*_{\text{env}}(\mathcal{S})$ such that $\delta(\eta(I_2 \otimes s)) = s$ for all $s \in \mathcal{S}$. The map $\eta : C^*_{\text{env}}(\mathcal{S}) = I_2 \otimes C^*_{\text{env}}(\mathcal{S}) \to \mathcal{D}$ also fixes the copy of $\mathcal{S}$. Hence, $\delta \circ \eta$ and $\eta \circ \delta$ are the identity when restricted to $\mathcal{S}$. As $C^*_{\text{env}}(\mathcal{S})$ and $\mathcal{D}$ are generated by their respective copies of $\mathcal{S}$, it follows that δ and η are inverses of each other. Therefore, $\mathcal{D} \simeq C^*_{\text{env}}(\mathcal{S})$. We conclude that $C^*_{\text{env}}(\mathcal{U}(\mathcal{S})) = M_2(C^*_{\text{env}}(\mathcal{S}))$. $\square$

The following shows that hyperrigidity of $\mathcal{U}(S)$ passes to $\mathcal{S}$.

Proposition 3.4.5. *Let $\mathcal{D}$ be a C*-cover of S. If $\mathcal{U}(S)$ is hyperrigid in $M_2(\mathcal{D})$, then S is hyperrigid in $\mathcal{D}$.*

Proof. Assume without loss of generality that $\mathcal{S} \subseteq \mathcal{D}$. Suppose that $\pi : \mathcal{D} \to B(H)$ is a unital $*$-homomorphism with restriction $\rho = \pi|_{\mathcal{S}} : \mathcal{S} \to B(H)$ to $\mathcal{S}$. Let $\varphi : \mathcal{D} \to B(H)$ be any ucp extension of ρ. Then $\pi^{(2)} : M_2(\mathcal{D}) \to B(H^2)$ is a unital $*$-homomorphism. Moreover, if $\eta = (\pi^{(2)})|_{\mathcal{U}(\mathcal{S})}$, then $\varphi^{(2)} : M_2(\mathcal{D}) \to B(H^2)$ is a ucp extension of η, since $\phi|_{\mathcal{S}} = \rho$. By hyperrigidity of $\mathcal{U}(\mathcal{S})$, we have $\varphi^{(2)} = \pi^{(2)}$. In particular, $\phi = \pi$, as desired. $\square$

Corollary 3.4.6. *Let $\mathcal{S}$ be an operator system with C*-cover $\mathcal{D}$. If $\mathcal{U}(\mathcal{S})$ is hyperrigid in $M_2(\mathcal{D})$, then $C^*_{env}(\mathcal{S}) = \mathcal{D}$ and $C^*_{env}(\mathcal{U}(\mathcal{S})) = M_2(\mathcal{D})$.*

Proof. By Proposition 3.4.5, since $\mathcal{U}(\mathcal{S})$ is hyperrigid in $M_2(\mathcal{D})$, $\mathcal{S}$ is hyperrigid in $\mathcal{D}$. By Theorem 3.1.9, $C^*_{\text{env}}(\mathcal{S}) = \mathcal{D}$. The fact that $C^*_{\text{env}}(\mathcal{U}(\mathcal{S})) = M_2(\mathcal{D})$ follows by Proposition 3.4.4. $\square$

Let $\mathcal{S}$ be an operator subsystem of a C*-algebra $\mathcal{D}$. Suppose that any irreducible representation $\pi : \mathcal{D} \to B(H)$ of $\mathcal{D}$ is maximal on $\mathcal{S}$. In general, it is unknown whether this implies $\mathcal{S}$ is hyperrigid in $\mathcal{D}$. That is, it is not known if this implies that all representations of $\mathcal{D}$, irreducible or not, have the unique extension property. In our case however, this obstruction is not a concern.

Proposition 3.4.7. *Let $\mathcal{S}$ be an operator system with finite-dimensional C*-cover $\mathcal{F}$. If every irreducible representation of $\mathcal{F}$ is maximal on $\mathcal{S}$, then S is hyperrigid in $\mathcal{F}$.*

Proof. Since $\mathcal{F}$ is finite-dimensional, it has finite spectrum; i.e., there are only finitely many irreducible representations of $\mathcal{F}$ up to unitary equivalence. The desired result then follows by a theorem of Arveson [4, Theorem 5.1]. $\square$

For notational convenience, we let

$$\mathcal{B} = \left\{ \begin{bmatrix} a & b \\ b & a \end{bmatrix} : a, b \in \mathbb{C} \right\}.$$

Note that $\mathcal{B} \simeq \mathbb{C}^2$ as C*-algebras; however, the given presentation of $\mathcal{B}$ will be most useful for our purposes. We have the following lemma.

Lemma 3.4.8. *Let $S \subseteq \bigoplus_{k=1}^3 M_2(\mathcal{B})$ be the operator system defined by*

$$\mathcal{S} := \left\{ \bigoplus_{k=1}^{3} \left[\begin{array}{cc|cc} a_k & & b_k & c_k \\ & a_k & c_k & b_k \\ \hline d_k & f_k & a_k & \\ f_k & d_k & & a_k \end{array} \right] : a_k = a_\ell,\, b_k = b_\ell,\, d_k = d_\ell,\, \textit{for all } 1 \leq k, \ell \leq 3 \right\}$$

Then $\mathcal{S}$ is hyperrigid in its C-cover $M_2(\mathcal{B}) \oplus M_2(\mathcal{B}) \oplus M_2(\mathcal{B})$.*

Proof. Let $\pi : \bigoplus_{k=1}^3 M_2(\mathcal{B}) = (\bigoplus_{k=1}^3 M_2) \otimes \mathcal{B} \to B(H)$ be an irreducible representation. Then up to unitary equivalence, we may assume that $H = K \otimes L$ and $\pi = \rho \otimes \sigma$, where $\rho : M_2 \oplus M_2 \oplus M_2 \to B(K)$ and $\sigma : \mathcal{B} \to B(L)$ are $*$-homomorphisms. Since $\rho \otimes \sigma$ is irreducible and $\sigma(\mathcal{B})$ is abelian, we must have $\dim(\sigma(\mathcal{B})) = 1$ and $L = \mathbb{C}$. Therefore, we may identify $H = K$ and $\sigma\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right) = \omega I_H$, where $\omega \in \{1, -1\}$. Since π is irreducible, it must be surjective. Hence, ρ is surjective. Thus, H is finite-dimensional and we may write $B(H) = M_D$ for some $D \in \mathbb{N}$. Let $\rho_k : M_2 \to M_D$ be the restriction of ρ to the k-th summand of M_2 in $M_2 \oplus M_2 \oplus M_2$. Each ρ_k is a $*$-homomorphism, so since M_2 is simple, ρ_k is either injective or the zero map. If at two of the ρ_k's were injective (say, ρ_1 and ρ_2) and $T_1 \oplus T_2 \in M_2 \oplus M_2$ were such that $\rho(T_1 \oplus T_2 \oplus 0) = 0$, then we would have

$$0 = \rho(T_1 \oplus T_2 \oplus 0)^* \rho(T_1 \oplus T_2 \oplus 0) = \rho_1(T_1^* T_1) + \rho_2(T_2^* T_2).$$

By injectivity of each ρ_k, we must have $T_1 = T_2 = 0$, so that $\rho_{|M_2 \oplus M_2 \oplus 0}$ is injective. But then this restriction would be an isomorphism of $M_2 \oplus M_2$ onto M_D, with the latter being simple, which is a contradiction. Similarly, it is impossible to have all three ρ_k's injective. Moreover, since $I_D = \rho(I_2 \oplus I_2 \oplus I_2) = \sum_{k=1}^3 \rho_k(I_2)$, we must have that exactly one ρ_k is non-zero. Therefore, we may assume without loss of generality that ρ_1 is non-zero (and hence an isomorphism), while $\rho_2 = \rho_3 = 0$. In particular, we may assume that $D = 2$.

Up to unitary conjugation, we may assume that $\rho_1 = \mathrm{id}_{M_2}$. Let ϕ be the restriction of π to S, and let $\psi : \bigoplus_{k=1}^3 M_2(\mathcal{B}) \to M_2$ be any ucp extension of ϕ. Let $\psi = V^*\Pi(\cdot)V$ be a minimal Stinespring representation of ψ on some Hilbert space H_Π. We consider $\bigoplus_{k=1}^3 M_2(\mathcal{B})$ as a subalgebra of $\bigoplus_{k=1}^3 M_4$.

Let $X = E_{14} + E_{23} \in S$. We note that $\psi(X) = \pi(X) = \omega E_{12} \in M_2$. With respect to the decomposition $H_\Pi = \mathrm{ran}\ (V) \oplus \mathrm{ran}\ (V)^\perp$, for $i, j, k, \ell \in \Lambda$, we have

$$\Pi(X) = \begin{bmatrix} \phi(X) & A \\ B & C \end{bmatrix} = \begin{bmatrix} \omega E_{12} & A \\ B & C \end{bmatrix},$$

for some operators A, B, C. Noting that $X^*X + XX^* = \sum_{i=1}^4 E_{ii}$, we have

$$\Pi\left(\sum_{i=1}^4 E_{ii}\right) = \Pi(X^*X + XX^*) = \Pi(X)^*\Pi(X) + \Pi(X)\Pi(X)^* = \begin{bmatrix} I + B^*B + AA^* & * \\ * & * \end{bmatrix}.$$

Thus, $\phi\left(\sum_{i=1}^4 E_{ii}\right) = I + B^*B + AA^* \leq \phi(I_{12}) = I_2$. This forces $B^*B + AA^* = 0$, so that $A = 0$ and $B = 0$. Considering the $(1,1)$-block, it follows that $\psi(X^*X) = E_{22} = \psi(X)^*\psi(X)$ and $\psi(XX^*) = E_{11} = \psi(X)\psi(X)^*$. Thus, X lies in the multiplicative domain $\mathcal{M}_\psi$ of ψ. Moreover, $\psi(E_{ii}) = 0$ for all $5 \leq i \leq 12$. It readily follows that $\psi_{|0 \oplus M_2(\mathcal{B}) \oplus M_2(\mathcal{B})} = 0$. Replacing X with $Y = E_{13} + E_{24}$, it is easy to see that $Y \in \mathcal{M}_\psi$ as well, while $\psi(Y) = E_{12}$. Let $W = E_{12} + E_{21}$ and $Z = E_{34} + E_{43}$. Then the first copy of $M_2(\mathcal{B})$ is generated as a C*-algebra by the four elements X, Y, Z, W. Since we have $W^*W = E_{11} + E_{22}$ and $Z^*Z = E_{33} + E_{44}$, we need only show that $\psi(W) = \pi(Z)$ and $\psi(Z) = \pi(Z)$. If this assertion holds, then W and Z would lie in $\mathcal{M}_\psi$, from which it would follow that $M_2(\mathcal{B}) \oplus 0 \oplus 0 \subseteq \mathcal{M}_\psi$ and $\pi = \psi$. Since $W = XY^*$, we may write

$$\psi(W) = \psi(XY^*) = \psi(X)\psi(Y)^* = \omega E_{12}E_{21} = E_{11}.$$

Similarly, since $Z = X^*Y$, we have

$$\psi(Z) = \psi(X^*Y) = \psi(X)^*\psi(Y) = \omega E_{21}E_{12} = \omega E_{22}.$$

It readily follows that $\psi(Z) = \pi(Z)$ and $\pi(W) = \psi(W)$, while $Z, W \in \mathcal{M}_\psi$. Therefore, $\psi = \pi$. Applying Proposition 3.4.5 completes the proof. $\square$

Lemma 3.4.9. *The operator sub-algebra of $\bigoplus_{k=1}^3 M_2(\mathcal{B})$ given by*

$$\mathcal{A} := \left\{ \bigoplus_{k=1}^{3} \left[\begin{array}{cc|cc} a_k & & b_k & c_k \\ & a_k & c_k & b_k \\ \hline & & a_k & \\ & & & a_k \end{array}\right] : a_k = a_\ell,\, b_k = b_\ell,\, \forall 1 \le k, \ell \le 3 \right\}$$

is hyperrigid in $M_2(\mathcal{B}) \oplus M_2(\mathcal{B}) \oplus M_2(\mathcal{B})$.

Proof. The operator system $\mathcal{S}$ in Lemma 3.4.8 is precisely $\mathcal{A} + \mathcal{A}^*$ in $\bigoplus_{k=1}^3 M_2(\mathcal{B})$. The desired result follows by Lemma 2.1.3. $\square$

We can now obtain hyperrigidity of the nuclearity detectors $\mathcal{W}_{3,2}$ and $\mathcal{U}(\mathcal{W}_{3,2})$.

Theorem 3.4.10. *The operator algebra $\mathcal{U}(\mathcal{W}_{3,2})$ is hyperrigid in $M_2\left(\bigoplus_{k=1}^3 \mathcal{B}\right)$, and $\mathcal{W}_{3,2}$ is hyperrigid in $\bigoplus_{k=1}^3 \mathcal{B}$.*

Proof. Since $\mathcal{W}_{3,2}$ is the set of all matrices in $\mathcal{B} \oplus \mathcal{B} \oplus \mathcal{B}$ of the form

$$\bigoplus_{k=1}^{3} \begin{bmatrix} a_k & b_k \\ b_k & a_k \end{bmatrix},$$

where $a_k = a_\ell$ for all $1 \le k, \ell \le 3$, it follows that $C^*(\mathcal{W}_{3,2}) \subseteq \oplus_{k=1}^3 \mathcal{B}$. On the other hand, for each k, the element V_k defined by $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ in the k-th summand and 0 in the other summands is an element of $\mathcal{W}_{3,2}$, and $V_k^2 = V_k^* V_k$ is I_2 in the k-th summand and 0 otherwise. It follows that $C^*(\mathcal{W}_{3,2}) = \oplus_{k=1}^3 \mathcal{B}$. Using Proposition 3.4.4, $M_2(\oplus_{k=1}^3 \mathcal{B})$ is a C*-cover of $\mathcal{U}(\mathcal{W}_{3,2})$.

To establish hyperrigidity of $\mathcal{U}(\mathcal{W}_{3,2})$ in $M_2\left(\bigoplus_{k=1}^3 \mathcal{B}\right)$, let π be the restriction of the canonical shuffle $M_2(M_6) \simeq M_6(M_2)$ to $M_2\left(\bigoplus_{k=1}^3 \mathcal{B}\right)$. Then π is a $*$-isomorphism from $M_2\left(\bigoplus_{k=1}^3 \mathcal{B}\right)$ onto $\oplus_{k=1}^3 M_2(\mathcal{B})$ that sends $\mathcal{U}(\mathcal{W}_{3,2})$ onto the operator algebra $\mathcal{A}$ in Lemma 3.4.9. Applying Lemma 3.1.8, since $\mathcal{A}$ is hyperrigid in $\oplus_{k=1}^3 M_2(\mathcal{B})$, $\mathcal{U}(\mathcal{W}_{3,2})$ is hyperrigid in $M_2\left(\bigoplus_{k=1}^3 \mathcal{B}\right)$. The analogous claim for $\mathcal{W}_{3,2}$ follows by Proposition 3.4.5. The claim about C*-envelopes immediately follows by Corollary 3.4.6. $\square$

Chapter 4

Characterizing hyperrigidity for C*-correspondences

This chapter is concerned with the Hao-Ng isomorphism problem, as stated in the introduction. Recall that the isomorphism problem asks: if $(\mathcal{C}, X)$ is a non-degenerate C*-correspondence and G is a locally compact group acting continuously on $(\mathcal{C}, X)$ then is it the case that we have the identity

$$\mathcal{O}_X \rtimes G = \mathcal{O}_{X \rtimes G} \ ?$$

At its core, the Hao-Ng isomorphism problem is asking whether the functor which maps a C*-correspondence $(\mathcal{C}, X)$ to its Cuntz-Pimsner algebra $\mathcal{O}_X$ is closed under crossed products. Because of this, the isomorphism problem is fundamental in the understanding of the dynamics of Cuntz-Pimsner algebras. Our goal is to establish an intrinsic characterization of hyperrigidity for C*-correspondences.

We say that a C*-correspondence $(\mathcal{C}, X)$ is hyperrigid if the operator space

$$S(\mathcal{C}, X) := \mathrm{span}\{x + a + y^* : x, y \in X, a \in \mathcal{C}\} \subseteq \mathcal{O}_X$$

has the following extension property: given a representation $\pi : \mathcal{O}_X \to B(H)$, if $\varphi : \mathcal{O}_X \to B(H)$ is a completely positive and completely contractive map which agrees with π on $S(\mathcal{C}, X)$ then φ must agree with π on $\mathcal{O}_X$. In [46], Katsoulis and Ramsey establish:

1. If $(\mathcal{C}, X)$ is a hyperrigid C*-correspondence and G is a locally compact group that acts on $(\mathcal{C}, X)$ then we have the identity

$$\mathcal{O}_X \rtimes G = \mathcal{O}_{X \hat{\rtimes} G}$$

where $(\mathcal{C}\widehat{\rtimes}G, X\widehat{\rtimes}G)$ is the completion of the pair $(C_c(G,\mathcal{C}), C_c(G,X))$ in $\mathcal{O}_X \rtimes G$. In particular, for hyperrigid C*-correspondences, the crossed product $\mathcal{O}_X \rtimes G$ is a Cuntz-Pimsner algebra.

2. If $(\mathcal{C}, X)$ is a hyperrigid C*-correspondence and G is a locally compact group that acts on $(\mathcal{C}, X)$ then we have the identity

$$\mathcal{O}_X \rtimes_r G = \mathcal{O}_{X \rtimes_r G} \ .$$

It is an open question whether $(\mathcal{C}\widehat{\rtimes}G, X\widehat{\rtimes}G)$ is the same as $(\mathcal{C} \rtimes G, X \rtimes G)$.

We denote by $\lambda : \mathcal{C} \to \mathcal{L}(X)$ the left action of $\mathcal{C}$ on X. Recall that the Katsura ideal $\mathcal{J}_X$ of a C*-correpsondence $(\mathcal{C}, X)$ is the ideal

$$\mathcal{J}_X := \{a \in \mathcal{C} : \lambda(a) \in \mathcal{K}(X) \text{ and } ax = 0 \text{ for all } x \in \ker \lambda\} \ .$$

The main Theorem of this chapter is the following.

Theorem 4.0.1. *Let $(\mathcal{C}, X)$ be a C*-correspondence. The following are equivalent:*

1. *The C*-correspondence $(\mathcal{C}, X)$ is hyperrigid.*
2. *We have the identity $\mathcal{J}_X \cdot X = X$.*

This extends a result of Kakariadis [41, Theorem 3.3] and Dor-On and Salomon [21, Theorem 3.5] who establish the equivalence for C*-correspondences associated to discrete graphs and a result of Katsoulis and Ramsey who prove the forward direction of our Theorem [46, Theorem 3.1]. Finally, we use this result to give an exact characterization for when the C*-correspondence associated to a topological graph is hyperrigid when the range map r is open.

4.1 Hyperrigidity of operator spaces $S(\mathcal{C}, X)$

In [46, Theorem 3.1], Katsoulis and Ramsey show that to achieve hyperrigidity of a C*-correspondence X it is sufficient for the left action of $\mathcal{J}_X$ to act non-degenerately on X. Our main result shows that this condition is in fact equivalent to hyperrigidity of X. The following two definitions are in [69].

Definition 4.1.1. Let $(\mathcal{C}, X)$ be a Hilbert $\mathcal{C}$-module. We treat the multiplier algebra $M(\mathcal{C})$ as the C*-algebra $\mathcal{L}(\mathcal{C})$. The Hilbert $M(\mathcal{C})$-module $M(X)$ is defined as follows: As a linear space, $M(X)$ consists of the space of adjointable right-$\mathcal{C}$-linear maps from $\mathcal{C}$ into X. That is, $M(X) = \mathcal{L}(\mathcal{C}, X)$. The right action is given by composition and the inner product is given by $\langle x, y\rangle := x^* \circ y$.

If $x \in X$ and $y \in M(X)$ then $\langle y, x\rangle \in \mathcal{C}$ and if $a \in \mathcal{C}$ then $y \cdot a \in X$. If $(\mathcal{C}, X)$ is a C*-correspondence and $a \in \mathcal{C}$ is such that $\lambda(a) \in K(X)$ then for any $x \in M(X)$, we have $a \cdot x \in X$. In particular, if $a \in \mathcal{J}_X$ then $a \cdot x \in X$.

Definition 4.1.2. Let $(\mathcal{C}, X)$ be a Hilbert $\mathcal{C}$-module. We say that X is countably generated over $\mathcal{C}$ if there exists a sequence $(x_n)_{n\geq 1}$ in $M(X)$ for which the $\mathcal{C}$-linear span of $(x_n)_n$ is dense in X. A standard normalized frame for $(\mathcal{C}, X)$ is a sequence $(x_n)_{n\geq 1}$ in $M(X)$ for which for every $x \in X$ we have the identity

$$\langle x, x\rangle = \sum_{n\geq 1} \langle x, x_n\rangle \langle x_n, x\rangle \ .$$

By [69, Corollary 3.3], whenever X is countably generated over $\mathcal{C}$, a standard normalized frame for X exists.

The reconstruction formula [69, Theorem 3.4] states that a sequence $(x_n)_{n\geq 1}$ is a standard normalized frame if and only if we have the identity

$$x = \sum_{n\geq 1} x_n \langle x_n, x\rangle$$

for every $x \in X$.

Let $(\mathcal{C}, X)$ be a C*-correspondence. In the following Lemma, we consider the set $\mathcal{M}$ of countably generated right $\mathcal{C}$-submodules of X. This set is directed under inclusion and it is cofinal in X.

Lemma 4.1.3. *Suppose that $(\mathcal{C}, X)$ is a C*-correspondence. Let $\mathcal{M}$ denote the space of all countably generated right $\mathcal{C}$-submodules of X. For each $Y \in \mathcal{M}$, let $(x_n(Y))_{n\geq 1}$ denote a standard normalized frame for Y. Let*

$$e_n(Y) := \sum_{k=1}^{n} x_k(Y) \langle x_k(Y), \cdot\rangle \ .$$

The set $(e_n(Y))_{(n,Y)\in\mathbb{N}\times\mathcal{M}}$ is an approximate unit for $\mathcal{K}(X)$ in the following sense: if $T \in \mathcal{K}(X)$ then we have the identity

$$\lim_{Y\to\infty} \lim_{n\to\infty} e_n(Y) \cdot T = T \ .$$

Proof. Let $T \in \mathcal{K}(X)$. Let $\epsilon > 0$. Suppose that $y_1, \ldots, y_n, z_1, \ldots, z_n \in X$ is such that

$$\|T - \sum_{k=1}^{n} y_k \langle z_k, \cdot \rangle \| < \epsilon \; .$$

Let $S = \sum_k y_k \langle z_k, \cdot \rangle$. Consider any $Y \in \mathcal{M}$ for which y_k, z_k belong to Y for all k. For any $x \in X$,

$$\sum_k y_k \langle z_k, x \rangle \in Y \; .$$

By the reconstruction formula, we know that

$$y_k = \sum_{n \geq 1} x_n(Y) \langle x_n(Y), y_k \rangle = \lim_{n \to \infty} e_n(Y)(y_k) \; .$$

for all k. This means in particular, that for n large enough,

$$\|e_n(Y)S - S\| < \epsilon \; .$$

Therefore,

$$\|T - e_n(Y)T\| \leq 2\|T - S\| + \|S - e_n(Y)S\| < 3\epsilon \; .$$

This proves that $e_n(Y)$ is an approximate unit for $\mathcal{K}(X)$. $\square$

The following Lemma provides a quantitative variant of [64, Theorem 3.18].

Lemma 4.1.4. *Let $\mathcal{C}$ be a C^*-algebra. Fix $m, n \geq 1$. Suppose that $\varphi : \mathcal{C} \to B(H)$ is a completely positive and contractive map for which for some $\epsilon > 0$ and $a \in M_{m,n}(\mathcal{C})$, we have the bound*

$$\|\varphi(aa^*) - \varphi(a)\varphi(a)^*\| < \epsilon \; .$$

It is then the case that for any $b \in M_{m,n}(\mathcal{C})$, we have the estimate

$$\|\varphi(ab) - \varphi(a)\varphi(b)\| < \sqrt{\epsilon}\|b\| \; .$$

Proof. For a positive $p \in M_{2m}(\mathcal{C})$, let

$$P(p) := \left[\begin{array}{c|c} I_n & a^* \;\; b^* \\ \hline \begin{matrix} a \\ b \end{matrix} & p \end{array}\right] .$$

The same argument as in [64, Lemma 3.1] shows that the matrix $P(M)$ is positive if and only if we have the bound

$$\begin{bmatrix} a \\ b \end{bmatrix} \begin{bmatrix} a^* & b^* \end{bmatrix} \leq p \ .$$

Taking

$$p = \begin{bmatrix} aa^* & ab^* \\ ba^* & bb^* \end{bmatrix} ,$$

we can conclude $P(p)$ is positive in this case. Since φ is contractive and completely positive, applying the $(2n+2)$-amplification of the unitization of φ onto $P(p)$, we get the bound

$$\begin{bmatrix} \varphi(a) \\ \varphi(b) \end{bmatrix} \begin{bmatrix} \varphi(a)^* & \varphi(b)^* \end{bmatrix} \leq \varphi(p) \ .$$

That is, the matrix

$$\begin{bmatrix} \varphi(aa^*) - \varphi(a)\varphi(a)^* & \varphi(ab^*) - \varphi(a)\varphi(b^*) \\ \varphi(ba^*) - \varphi(b)\varphi(a)^* & \varphi(bb^*) - \varphi(b)\varphi(b)^* \end{bmatrix}$$

is positive. Since the $(1,1)$ corner of this matrix is at most ϵ, we get positivity of the matrix

$$\begin{bmatrix} \epsilon I_2 & \varphi(ab^*) - \varphi(a)\varphi(b^*) \\ \varphi(ba^*) - \varphi(b)\varphi(a)^* & \varphi(bb^*) - \varphi(b)\varphi(b)^* \end{bmatrix} .$$

In particular, we have the bound

$$\begin{aligned} \|\varphi(ab^*) - \varphi(a)\varphi(b)^*\|^2 &\leq \epsilon \|\varphi(bb^*) - \varphi(b)\varphi(b)^*\| \\ &\leq \epsilon \|bb^*\| \ , \end{aligned} \tag{4.1}$$

where the final inequality follows from the fact that

$$0 \leq \varphi(bb^*) - \varphi(b)\varphi(b)^* \leq \varphi(bb^*) \leq \|bb^*\| 1 \ .$$

□

Theorem 4.1.5. *Let $(\mathcal{C}, X)$ be a C^*-correspondence. The following are equivalent:*

1. *The left action of $\mathcal{J}_X$ on X is non-degenerate.*

2. *$S(\mathcal{C}, X)$ is hyperrigid.*

Proof. First assume that $\mathcal{J}_X$ acts on X non-degenerately. We denote by (i^0, i^1) the canonical covariant pair

$$(i^0, i^1) : (\mathcal{C}, X) \to \mathcal{O}_X \ .$$

Fix any *-homomorphism $\pi : \mathcal{O}_X \to B(H)$ and suppose that $\varphi : \mathcal{O}_X \to B(H)$ is any cpcc-extension of $\pi|_{S(\mathcal{C},X)}$. By a multiplicative domain argument and by non-degeneracy of $\mathcal{J}_X$ acting on X, it suffices to show that for any $a \in \mathcal{C}$ and $x \in X$, we have

$$\varphi(\iota^1(a \cdot x)\iota^1(a \cdot x)^*) = \varphi(\iota^1(a \cdot x))\varphi(\iota^1(a \cdot x))^* \ .$$

Let $\mathcal{M}$ and $x_n(Y), e_n(Y)$ be as in Lemma 4.1.3. Let

$$\phi_\iota : \mathcal{K}(X) \to \mathcal{O}_X : x \langle y, \cdot \rangle \mapsto \iota^1(x)\iota^1(y)^* \ .$$

For any $a \in \mathcal{J}_X$, since $\lambda(a)$ is a compact operator, we have

$$\iota^0(aa^*) = \phi_\iota\left(\lim_{Y\to\infty} \lim_{n\to\infty} \lambda(a) \cdot e_n(Y)\lambda(a)^*\right) = \lim_{Y\to\infty} \lim_{n\to\infty} \sum_{k<n} \iota^1(a \cdot x_k(Y))\iota^1(a \cdot x_k(Y))^* \ .$$

By the Schwarz inequality,

$$\begin{aligned}
\varphi(\iota^0(aa^*)) &= \lim_Y \lim_n \sum_{k<n} \varphi(\iota^1(a \cdot x_k(Y))\iota^1(a \cdot x_k(Y))^*) \\
&\geq \lim_Y \lim_n \sum_{k<n} \varphi(\iota^1(a \cdot x_k(Y)))\varphi(\iota^1(a \cdot x_k(Y)))^* \\
&= \lim_Y \lim_n \sum_{k<n} \pi(\iota^1(a \cdot x_k(Y)))\pi(\iota^1(a \cdot x_k(Y)))^* \\
&= \pi(\iota^0(aa^*)) = \varphi(\iota^0(aa^*)) \ .
\end{aligned}$$

From this, we have the identity

$$\lim_Y \lim_n \sum_{k<n} \varphi(\iota^1(a \cdot x_k(Y))\iota^1(a \cdot x_k(Y))^*) = \lim_Y \lim_n \sum_{k<n} \pi(\iota^1(a \cdot x_k(Y))\iota^1(a \cdot x_k(Y))^*) \ .$$

By the reconstruction formula, for any $x \in X$ and for any $Y \in \mathcal{M}$ with $x \in Y$, we have for all $a \in \mathcal{J}_X$,

$$a \cdot x = \sum_{n\geq 1} a \cdot x_n(Y) \langle x_n(Y), x \rangle \ .$$

Let $\epsilon > 0$. Fix any $Y \in \mathcal{M}$ for which we have the bound

$$0 \leq \sum_{n\geq 1} \varphi(\iota^1(a\cdot x_n(Y))\iota^1(a\cdot x_n(Y))^*) - \varphi(\iota^1(a\cdot x_n(Y)))\varphi(\iota^1(a\cdot x_n(Y)))^* \leq \epsilon 1 \ .$$

Let $\alpha_n = \iota^1(a\cdot x_n(Y))$ and let $\beta_n = \iota^1(a\cdot x\,\langle x, x_n(Y)\rangle)$. Observe that

$$\iota^1(a\cdot x)\iota^1(a\cdot x)^* = \sum_{n\geq 1} \alpha_n\beta_n^* \ .$$

Consider for fixed $n \geq 1$ the $1\times n$-matrices $A_n = (\alpha_1, \cdots, \alpha_n)$ and $B_n = (\beta_1, \cdots, \beta_n)$. A calculation shows that

$$\begin{aligned}
\|B_nB_n^*\| &= \left\| \sum_{k\leq n} \iota^1(a\cdot x)\iota^0(\langle x, x_k(Y)\rangle\,\langle x_k(Y), x\rangle)\iota^1(a\cdot x)^* \right\| \\
&= \left\| \iota^1(a\cdot x)\ \iota^0\left(\sum_{k\leq n}\langle x, x_k(Y)\rangle\,\langle x_k(Y), x\rangle\right)\iota^1(a\cdot x)^* \right\| \\
&\leq \|a\cdot x\|^2 \left\| \sum_{k\leq n}\langle x, x_k(Y)\rangle\,\langle x_k(Y), x\rangle \right\| \ .
\end{aligned}$$

Since the sequence $x_n(Y)$ is a standard normalized frame, we have the inequality

$$\|B_nB_n^*\| \leq \|a\cdot x\|^2\|x\|^2$$

for any n. As well, a calculation shows that

$$\begin{aligned}
\lim_{n\to\infty} \varphi(A_nB_n^*) - \varphi(A_n)\varphi(B_n)^* &= \lim_{n\to\infty}\sum_{k\leq n} \varphi(\iota^1(a\cdot x_n(Y)\,\langle x_n(Y), x\rangle)\iota^1(a\cdot x)^*) \\
&\quad - \varphi(\iota^1(a\cdot x_n(Y)\,\langle x_n(Y), x\rangle))\varphi(\iota^1(a\cdot x))^* \\
&= \varphi(\iota^1(a\cdot x)\iota^1(a\cdot x)^*) - \varphi(\iota^1(a\cdot x))\varphi(\iota^1(a\cdot x))^* \ .
\end{aligned}$$

The above calculation with Lemma 4.1.4 give us the bound

$$\begin{aligned}
\|\varphi(\iota^1(a\cdot x)\iota^1(a\cdot x)^*) - \varphi(\iota^1(a\cdot x))\varphi(\iota^1(a\cdot x))^*\|^2 &= \lim_{n\to\infty} \|\varphi(A_nB_n^*) - \varphi(A_n)\varphi(B_n)^*\|^2 \\
&\leq \epsilon\|a\cdot x\|^2\|x\|^2 \ .
\end{aligned}$$

Since this identity is independent of the choice of Y and ϵ, we may conclude that for any $a \in \mathcal{J}_X$ and for any $x \in X$, the element $\iota^1(a\cdot x)$ belongs to the multiplicative domain of φ, showing hyperrigidity.

For the converse, assume that $\mathcal{J}_X$ does not act on X non-degenerately. Fix a faithful covariant representation $(\pi^0, \pi^1) : (\mathcal{C}, X) \to B(H)$. Let $N \subseteq \mathcal{J}_{X,+}$ form a contractive approximate unit for $\mathcal{J}_X$ under the ordering induced by the positive operators. Define operators $P = \lim_{a \in N} \pi^0(a)$ and $Q = 1 - P$ where the limit is taken in the strong operator topology on $B(H)$. For any isometry $V \in B(\mathcal{K})$, let $(\tau^0, \tau^1_V) : (\mathcal{C}, X) \to B(H \otimes \mathcal{K})$ be the following pair of maps

$$\begin{aligned}
&\tau^0 : A \to B(H \otimes \mathcal{K}) : a \mapsto \pi^0(a) \otimes I \\
&\tau^1_V : X \to B(H \otimes \mathcal{K}) : x \mapsto P\pi^1(x) \otimes I + Q\pi^1(x) \otimes V \ .
\end{aligned}$$

It is immediate that τ^0 is a *-homomorphism and that τ^1_V is linear. For any $a \in \mathcal{C}$ and $x \in X$, first observe that since P is the projection which generates the ideal $\pi^0(\mathcal{J}_X)$ in $\pi^0(\mathcal{C})$, that P commutes with $\pi^0(a)$. Thus,

$$\begin{aligned}
\tau^0(a)\tau^1_V(x) &= (\pi^0(a) \otimes I)(P\pi^1(x) \otimes I + Q\pi^1(x) \otimes V) \\
&= P(\pi^0(a)\pi^1(x)) \otimes I + Q\pi^0(a)\pi^1(x) \otimes V \\
&= P\pi^1(a \cdot x) \otimes I + Q\pi^1(a \cdot x) \otimes V = \tau^1(a \cdot x) \ .
\end{aligned}$$

As well, for $x, y \in X$, we have

$$\begin{aligned}
\tau^1_V(x)^*\tau^1_V(x) &= (P\pi^1(x) \otimes I + Q\pi^1(x) \otimes V)^*(P\pi^1(y) \otimes I + Q\pi^1(y) \otimes V) \\
&= \pi^1(x)^*P\pi^1(y) \otimes I + \pi^1(x)^*Q\pi^1(y) \otimes I \\
&= \pi^1(x)^*(P + Q)\pi^1(y) \otimes I = \pi^0(\langle x, y\rangle) \otimes I \\
&= \tau^0(\langle x, y\rangle) \ .
\end{aligned}$$

This is therefore a Toeplitz representation for $(\mathcal{C}, X)$. To see that this representation is covariant, let $a \in \mathcal{J}_X$. Since $\lambda(a) \in \mathcal{K}(X)$, for $\epsilon > 0$, let $x_1, \ldots, x_n, y_1, \ldots, y_n \in X$ such that for any contraction $z \in X$, we have

$$\left\| a \cdot z - \sum_{k=1}^{n} x_k \langle y_k, z\rangle \right\| < \epsilon \ .$$

For any $b \in N$, we have

$$\left\| bab \cdot z - \sum_{k=1}^{n} b \cdot x_k \langle b \cdot y_k, z\rangle \right\| < \epsilon \ .$$

In particular, $\lambda(bab)$ is within ϵ of the compact operator $\sum_k bx_k \langle by_k, \cdot\rangle$. Let

$$\varphi_V : \mathcal{K}(X) \to B(H) : x_i \langle y_i, \cdot\rangle \mapsto \tau^1_V(x_i)\tau^1_V(y_i)^* \ .$$

A calculation shows

$$\begin{aligned}
&\varphi_V\left(\sum\nolimits_k bx_k\langle by_k,\cdot\rangle\right) = \sum\nolimits_k \tau^1(bx_k)\tau^1(by_k)^* \\
&= \sum\nolimits_k (P\pi^1(bx_k)\otimes I + Q\pi^1(bx_k)\otimes V)(P\pi^1(by_k)\otimes I + Q\pi^1(by_k)\otimes V)^* \\
&= \sum\nolimits_k (P\pi^1(bx_k)\otimes I)(P\pi^1(by_k)\otimes I)^* \\
&= \sum\nolimits_k (\pi^1(bx_k)\otimes I)(\pi^1(by_k)\otimes I)^* \\
&= \left(\sum\nolimits_k \pi^1(bx_k)\pi^1(by_k)^*\right)\otimes I\ .
\end{aligned}$$

For any $b \in N$,

$$\begin{aligned}
\left\|\varphi_V(\lambda(bab)) - \pi^0(bab)\otimes I\right\| \le& \left\|\varphi_V(\lambda(bab)) - \varphi_V\left(\sum\nolimits_k bx_k\langle by_k,\cdot\rangle\right)\right\| \\
&+ \left\|\varphi_V\left(\sum\nolimits_k bx_k\langle by_k,\cdot\rangle\right) - \pi^0(bab)\otimes I\right\| \\
<& 2\epsilon\ .
\end{aligned}$$

Since this is true for arbitrary $\epsilon > 0$, we conclude that $\varphi_V(\lambda(bab)) = \tau^0(bab)$ for all $b \in N$. Since N is an approximate unit for $\mathcal{J}_X$ and $a \in \mathcal{J}_X$, we have $\varphi_V(\lambda(a)) = \tau^0(a)$.

Let us fix the unilateral shift $V \in B(\ell^2(\mathbb{Z}_+))$ and the bilateral shift $U \in B(\ell^2(\mathbb{Z}))$. Let $\Phi : B(\ell^2(\mathbb{Z})) \to B(\ell^2(\mathbb{Z}_+))$ be the ucp map given by restriction. The diagram

$$\begin{tikzcd}
 & B(H\otimes\ell^2(\mathbb{Z})) \arrow[d, "\Phi"] \\
\mathcal{O}_X \arrow[ur, "\tau^0\times\tau^1_U"] \arrow[r, "\tau^0\times\tau^1_V"] & B(H\otimes\ell^2(\mathbb{Z}_+))
\end{tikzcd}$$

commutes. So long as we can show $Q\pi^1(X) \neq 0$, we are done, since $\Phi\circ(\tau^0\times\tau^1_U) \neq \tau^0\times\tau^1_V$ but agree on $S(\mathcal{C}, X)$. Suppose that $Q\pi^1(X) = 0$ in order to derive a contradiction. Since $P+Q = I$, this means that $P\pi^1(x) = \pi^1(x)$ for every $x \in X$. If $\mathcal{J}_X$ acts on X degenerately, then by taking a subnet if necessary, there is some $\epsilon > 0$ and some $x \in X$ so that for every $b \in N$, there is some unit vector $h_b \in H$ for which we have the identity

$$\left\langle(\pi^1(x)^*\pi^1(x) - \pi^1(x)^*\pi^0(b)\pi^1(x))h_b, h_b\right\rangle \ge \epsilon\ .$$

If $a \ge b$ in N then we have the identity

$$\begin{aligned}
\left\langle(\pi^1(x)^*\pi^1(x) - \pi^1(x)^*\pi^0(b)\pi^1(x))h_a, h_a\right\rangle \ge& \left\langle(\pi^1(x)^*\pi^1(x) - \pi^1(x)^*\pi^0(a)\pi^1(x))h_a, h_a\right\rangle \\
\ge& \epsilon\ .
\end{aligned}$$

If we could replace the net $(h_b)_{b\in N}$ with a fixed vector $h_b = h \in H$ for all b then we may conclude from the above inequality that $P\pi^1(x) \neq \pi^1(x)$ and we would have our contradiction.

In order to guarantee that a vector $h \in H$ as above exists, we need to fix a specific faithful representation. Take any non-principal ultrafilter $\mathcal{U}$ over N containing the set

$$S := \{\{a \in N : a \geq b\} : b \in N\} .$$

Such an ultrafilter exists since S has the finite intersection property. Consider the covariant pair

$$(\overline{\pi}^0, \overline{\pi}^1) : (\mathcal{C}, X) \to B(H^{\mathcal{U}})$$

so that $\overline{\pi}^0(a)(\lim_{\mathcal{U}} k_b) = \lim_{\mathcal{U}} \pi^0(a) \cdot k_b$ and $\overline{\pi}^1(x)(\lim_{\mathcal{U}} k_b) = \lim_{\mathcal{U}} \pi^1(x) \cdot k_b$. Replacing (π^0, π^1) with $(\overline{\pi}^0, \overline{\pi}^1)$ and taking $h = \lim_{\mathcal{U}} h_b$ will do. $\square$

As an application, we will characterize the topological graphs with range map r open for which the associated space $S(C_0(E^0), X(E))$ is hyperrigid. This generalizes a result of Kakariadis [41, Theorem 3.3] and Dor-On and Salomon [21, Theorem 3.5] that give a characterization for E discrete. First a bit of notation: let $E^0_{\text{fin.}}$ be the open subset of E^0 for which we have the identity

$$C_0(E^0_{\text{fin.}}) = \lambda^{-1}(\mathcal{K}(X(E))) .$$

The kernel of λ consists of those elements $f \in C_0(E^0)$ for which $f|_{r(E^1)} = 0$. Thus,

$$\ker \lambda = C_0(E^0 \setminus \overline{r(E^1)}) .$$

This implies that $\mathcal{J}_{X(E)} = C_0(E^0_{\text{fin.}} \cap \operatorname{int}(\overline{r(E^1)}))$. Let $Y = \operatorname{int}(\overline{r(E^1)})$. Assume that $E^0_{\text{fin.}} \cap Y$ is dense in Y. I claim that $\mathcal{J}_{X(E)}X(E) = X(E)$. Let φ_i be an approximate unit for $C_0(E^0_{\text{fin.}} \cap Y)$. For any $f \in C_c(E^1)$, I claim that $\varphi_i \cdot f$ converges to f. Consider the positive function $\mathcal{F}_i = \langle f - \varphi_i \cdot f, f - \varphi_i \cdot f\rangle$. Observe that as f is compactly supported that all $\mathcal{F}_i$ are supported on a compact set K. As well, $\mathcal{F}_i(x)$ is a decreasing net for all $x \in E^0$. By the uniform limit theorem, the function

$$\mathcal{F} : E^0 \to \mathbb{C} : x \mapsto \lim_{i\to\infty} \mathcal{F}_i(x)$$

is continuous and compactly supported. We need to show that $\mathcal{F} = 0$. If not, there is some open set $U \subseteq E^0$ for which $\mathcal{F}|_U > 0$. If $x \in U$ then for any $e \in s^{-1}(x)$, $r(e) \notin E^0_{\text{fin.}}$.

That is, if $x \in r(s^{-1}(U))$ then $x \notin E^0_{\text{fin.}}$. Assume that r is open. That $r(s^{-1}(U))$ is an open subset of Y and that $r(s^{-1}(U)) \cap E^0_{\text{fin.}} = \varnothing$ is a contradiction of the density of $E^0_{\text{fin.}} \cap Y$ in Y. Thus we have $\mathcal{J}_{X(E)}X(E) = X(E)$.

If $E^0_{\text{fin.}} \cap Y$ is not dense in Y then there is some open subset U of Y so that $U \cap E^0_{\text{fin.}} = \varnothing$. Consider any non-zero function $f \in C_c(E^1)$ supported on $r^{-1}(U)$. If $\mathcal{J}_{X(E)}$ acts non-degenerately on $X(E)$, then by Cohen's factorization theorem, there is some $x \in X(E)$ and some $g \in \mathcal{J}_{X(E)}$ for which $g \cdot x = f$. Say $f_i \in C_c(E^1)$ for which $\lim_i f_i = x$. For any point $e \in E^1$, if $f(e) \neq 0$ then $r(e) \in U$. This implies that $g(r(e)) = 0$. For any i,

$$\langle g \cdot f_i, f \rangle : x \mapsto \sum_{e \in E^1 : s(e) = x} \overline{g(r(e))} f(e) \overline{f_i(e)} = 0 \ .$$

Thus we have $\langle f, f \rangle = \lim_i \langle g \cdot f_i, f \rangle = 0$ – a contradiction.

All this proves:

Theorem 4.1.6. *Let E be a topological graph and let r be open. The following are equivalent:*

1. *The space $S(C_0(E^0), X(E))$ is hyperrigid.*
2. *The set $E^0_{\text{fin.}}$ is dense in E^0.*

Proof. Let $Y = \operatorname{int}(\overline{r(E^1)})$. By the above argument, hyperrigidity of $S(C_0(E^0), X(E))$ is equivalent to density of $E^0_{\text{fin.}} \cap Y$ in Y. To finish the argument, suppose that $E^0_{\text{fin.}} \cap Y$ is dense in Y. If x is a point in $E^1 \setminus \overline{r(E^1)}$ then there is a non-negative function f supported outside of $\overline{r(E^1)}$ for which $f(x) = 1$. Since $\lambda(f) = 0$, we must conclude that $x \in E^0_{\text{fin.}}$. In particular, whenever U is an open set in E^0 for which $U \cap E^0_{\text{fin.}} = \varnothing$ then we must have $U \subseteq Y$. By our assumption, we must have $U = \varnothing$. □

Indeed, at the time that the contents of this chapter has been written, Katsoulis and Ramsey establish from different techniques than ours that the following Theorem is true for general topological graphs:

Theorem 4.1.7 (Katsoulis-Ramsey)**.** *Let E be a topological graph. The following are equivalent:*

1. *The space $S(C_0(E^0), X(E))$ is hyperrigid.*
2. *The map $r : E^1 \to E^0$ is proper and $r(E^1) \subseteq \operatorname{int}(\overline{r(E^1)})$.*

Part II

Synchonous Correlation Sets and Quantum Graphs

Chapter 5

Preliminaries

In this chapter we explain the concept of a correlation set. These correlation sets arise from different models of quantum systems where we have two isolated labs that share any number of entangled states. Correlation sets are matrices of probabilities that are associated to various models of quantum systems. Because correlation sets provide a description of some part of a physical system, we will begin by talking about the physical set-up that describe a correlation set. We will then formalize this concept using the language of C*-algebras. At the end, we will discuss certain quantum graph parameters which arise from modifying classical graph parameters such as the independence and chromatic number into a co-operative game and considering whether one can have a winning strategy for such games using different quantum systems. This, and the subsequent chapter is joint work with Vern Paulsen and Chris Schafhauser.

5.1 Correlation Sets and Graph Parameters

Suppose that Alice has n_A quantum experiments each with m_A outcomes and Bob has n_B quantum experiments each with m_B outcomes and that their combined labs are in some combined, possibly entangled, state. We let $p(a,b|x,y)$ denote the conditional probability that if Alice conducts experiment x and Bob conducts experiment y then they get outcomes a and b, respectively. The $n_A n_B m_A m_B$-tuple

$$\big(p(a,b|x,y)\big)_{1\le x\le n_A, 1\le y\le n_B, 1\le a\le m_A, 1\le b\le m_B}$$

of real numbers is, informally, called a *quantum correlation*. There are several different mathematical models that can be used to describe these values, denoted by the subscripts,

q, qs, qa and qc. The Tsirelson problems are concerned with whether or not these different mathematical models yield the same sets. Due to the work of [74] and the remarkable work of [39], it seems to be the case that most of these sets are distinct.

We now recall the formal definitions of these sets. First, the different mathematical models shall be denoted by a subscript t where t can be either q, qs, qa, or qc. We let $C_t(n_A, n_B, m_A, m_B) \subseteq \mathbb{R}^{n_A n_B m_A m_B}$ denote the set of all possible tuples $\big(p(a, b|x, y)\big)$ that can be obtained using the model t. We now describe each of these models.

When $t = q$, we have that $p(a, b|x, y) \in C_q(n_A, n_B, m_A, m_B)$ if and only if there exist finite dimensional Hilbert spaces H_A and H_B, orthogonal projections $E_{x,a} \in B(H_A)$, $1 \leq x \leq n_A, 1 \leq a \leq m_A$ satisfying $\sum_{a=1}^{m_A} E_{x,a} = I_{H_A}$, for all x, orthogonal projections $F_{y,b} \in B(H_B)$, $1 \leq y \leq n_B, 1 \leq b \leq m_B$ satisfying $\sum_{b=1}^{m_B} F_{y,b} = I_{H_B}$, for all y and a unit vector $\psi \in H_A \otimes H_B$ such that

$$p(a, b|x, y) = \langle E_{x,a} \otimes F_{y,b}\psi, \psi\rangle.$$

When $t = qs$, the set $C_{qs}(n_A, n_B, m_A, m_B)$ is defined similarly, except the condition that the Hilbert spaces H_A and H_B be finite dimensional is dropped.

It is known that the closure of the set $C_q(n_A, n_B, m_A, m_B)$ is equal to the closure of the set $C_{qs}(n_A, n_B, m_A, m_B)$, see [30, 40, 68], and we denote the closure by $C_{qa}(n_A, n_B, m_A, m_B)$

The set $C_{qc}(n_A, n_B, m_A, m_B)$ is defined by eliminating the tensor product and instead having a single Hilbert space H, a unit vector $\psi \in H$, together with orthogonal projections $E_{x,a}, F_{y,b} \in B(\mathcal{H})$ satisfying

1. $E_{x,a}F_{y,b} = F_{y,b}E_{x,a}$ for all a, b, x, y,
2. $\sum_{a=1}^{m_A} E_{x,a} = \sum_{b=1}^{m_B} F_{y,b} = I_H$ for all x, y, and
3. $p(a, b|x, y) = \langle E_{x,a}F_{y,b}\psi, \psi\rangle$ for all a, b, x, y.

In each of the cases, i.e., for $t \in \{q, qs, qa, qc\}$, when $n_A = n_B = n$ and $m_A = m_B = m$, we set $C_t(n, m) = C_t(n, n, m, m)$.

A correlation $\big(p(a, b|x, y)\big) \in C_t(n, m)$ is called *synchronous* provided that whenever $a \neq b$, $p(a, b|x, x) = 0$, for all $1 \leq x \leq n$. For each t, we write $C_t^s(n, m)$ for the subset of synchronous correlations. Characterizations of synchronous correlations in terms of traces are known for the cases $t = q, qc$. In Section 6.1, we give characterizations of synchronous correlations for the remaining cases, $t = qs, qa$.

By a *finite input-output game*, we mean a tuple $\mathcal{G} = (I_A, I_B, O_A, O_B, V)$ where I_A, I_B, O_A, O_B are finite sets, representing the inputs that Alice and Bob can receive and the outputs that they can produce, respectively, and a function

$$V : I_A \times I_B \times O_A \times O_B \to \{0, 1\}$$

called the *rule* or *predicate* function. Here $V(x, y, a, b) = 1$ means that if Alice and Bob receive $(x, y) \in I_A \times I_B$ and produce outputs $(a, b) \in O_A \times O_B$ then they win the game and if $V(x, y, a, b) = 0$, then they lose the game.

A game is called *synchronous* provided that $I_A = I_B$, $O_A = O_B$ and the function V satisfies $V(x, x, a, b) = 0$, for all x, and for all $a \neq b$.

Given a game, a correlation $\big(p(a, b|x, y)\big) \in C_t(|I_A|, |I_B|, |O_A|, |O_B|)$ is called a *perfect* or *winning* t-correlation, if the probability that it produces a losing output is 0, i.e., provided that whenever $V(x, y, a, b) = 0$, then $p(a, b|x, y) = 0$. When a game has a perfect t-correlation, then we say that the game possesses a perfect t-strategy. Note that if a game is synchronous, then any perfect correlation must be synchronous.

From the definition of the set $C_{qa}(n_A, n_B, m_A, m_B)$ it readily follows that a game possesses a perfect qa-strategy if and only if for every $\epsilon > 0$, there is a q-correlation $\big(p(a, b|x, y)\big)$ in $C_q(n_A, n_B, m_A, m_B)$ satisfying that whenever $V(x, y, a, b) = 0$, then we must have $p(a, b|x, y) < \epsilon$.

Every synchronous game $\mathcal{G}$ has a unital *-algebra $\mathcal{A}(\mathcal{G})$ [38] affiliated with it (possibly the zero algebra), defined by generators and relations. It has generators

$$\{E_{x,a} : 1 \leq x \leq n, 1 \leq a \leq m\}$$

and relations

1. $E_{x,a} = E_{x,a}^* = E_{x,a}^2$ for all a and x,
2. $\sum_{a=1}^m E_{x,a} = I$ for all x, and
3. for all a, b, x, and y, if $V(x, y, a, b) = 0$, then $E_{x,a}E_{y,b} = 0$.

One of the results of [38] is that a synchronous game $\mathcal{G}$ has a perfect q-strategy if and only if $\mathcal{A}(\mathcal{G})$ has a unital *-representation as operators on a non-zero, finite dimensional Hilbert space. Thus, a synchronous game $\mathcal{G}$ has a perfect q-strategy if and only if one can find projections $E_{x,a}$ on a finite dimensional Hilbert space satisfying the above relations for the given rule V. Similarly, $\mathcal{G}$ has a perfect qc-strategy if and only if $\mathcal{A}(\mathcal{G})$ has a unital *-representation into a C*-algebra with a trace. The results of Section 6.1 will show that $\mathcal{G}$ has a perfect qa-strategy if and only if $\mathcal{A}(\mathcal{G})$ approximately has unital *-representations on non-zero, finite-dimensional Hilbert spaces; more precisely, $\mathcal{A}(\mathcal{G})$ has a unital *-representation on $\mathcal{R}^{\mathcal{U}}$, the tracial ultrapower of the hyperfinite II_1-factor $\mathcal{R}$. For readers not familiar with this ultrapower construction, more details can be found in [13, Appendix A].

There are two families of synchronous games, both involving graphs, that we wish to recall.

Let $G = (V, E)$ be a finite undirected graph without loops. That is, V is a finite set of *vertices* and $E \subseteq V \times V$ denotes the set of *edges*. Thus, for each $v \in V$, $(v, v) \notin E$, and $(v, w) \in E \implies (w, v) \in E$, since it is undirected. We let K_n denote the complete graph on n vertices so that $(v, w) \in E$, for all $v \neq w$.

Given two graphs $G = (V(G), E(G))$ and $H = (V(H), E(H))$, a *graph homomorphism from G to H* is a function $f : V(G) \to V(H)$ satisfying

$$(v, w) \in E(G) \implies (f(v), f(w)) \in E(H).$$

We write $G \to H$ to indicate that there is a graph homomorphism from G to H.

Many graph parameters can be defined in terms of graph homomorphisms. For an integer c, let K_c denote the complete graph on c vertices. The *chromatic number* of G is

$$\chi(G) = \min\{c : G \to K_c\}.$$

The *clique number* of G is

$$\omega(G) = \max\{c : K_c \to G\},$$

and the *independence number* of G is

$$\alpha(G) = \omega(\overline{G}),$$

where $\overline{G} = (V, \overline{E})$ denotes the *complement* of G; i.e., the graph with the same vertex set but for $v \neq w$, $(v, w) \in \overline{E} \iff (v, w) \notin E$.

Given graphs G and H, the *graph homomorphism game from G to H* is the synchronous game with inputs $V(G)$, outputs $V(H)$ and rule function

$$V(v, w, x, y) = 0 \iff \big((v, w) \in E(G) \text{ and } (x, y) \notin E(H)\big) \text{ or } \big(v = w \text{ and } x \neq y\big).$$

For $t \in \{q, qs, qa, qc\}$, we write $G \xrightarrow{t} H$ to indicate that the graph homomorphism game from G to H has a perfect t-strategy.

In parallel with the above characterizations we set:

$$\chi_t(G) = \min\{c : G \xrightarrow{t} K_c\},\ \omega_t(G) = \max\{c : K_c \xrightarrow{t} G\},\ \alpha_t(G) = \omega_t(\overline{G}) \ .$$

It is not hard to verify that for complete graphs,

$$\chi(K_n) = \chi_q(K_n) = \chi_{qs}(K_n) = \chi_{qa}(K_n) = \chi_{qc}(K_n) = n$$

and that

$$\alpha(K_n) = \alpha_q(K_n) = \alpha_{qs}(K_n) = \alpha_{qa}(K_n) = \alpha_{qc}(K_n) = 1 \ .$$

Indeed, by [38], we have that

$$n = \chi(K_n) \geq \chi_q(K_n) \geq \chi_{qs}(K_n) \geq \chi_{qa}(K_n) \geq \chi_{qc}(K_n) = n \ .$$

To see the second set of equalities, note that if $K_c \xrightarrow{t} \overline{K}_n$ with $c > 1$, then we would have inputs $v \neq w$ and the perfect t-correlation for this game would satisfy $p(x, y|v, w) = 0$ for every x, y, contradicting $\sum_{x,y} p(x, y|v, w) = 1$.

Since for $t \in \{q, qs, qa, qc\}$, $\chi_t(K_n) = n$, we have that if there exists $K_n \xrightarrow{t} K_c$ then $n \leq c$. This in turn implies that

$$c = \omega(K_c) \leq \omega_q(K_c) \leq \omega_{qs}(K_c) \leq \omega_{qa}(K_c) \leq \omega_{qc}(K_c) \leq c \ ,$$

where the last inequality follows since $\omega_{qc}(K_c)$ is the largest n for which $K_n \xrightarrow{qc} K_c$. Thus, we see that for complete graphs, these quantum analogues all have the same values as their classical counterparts.

The second game that we shall need is the (G, H)-*isomorphism game* defined in [6]. This game is intended to capture the concept of two graphs being isomorphic. It is a synchronous game with input set and output set both equal to $V(G) \cup V(H)$ where we view the vertex sets as disjoint. We refer the reader to [6] for the rules of this game. For $t \in \{q, qs, qa, qc\}$ we write $G \cong_t H$ to indicate that there is a perfect t-strategy for the (G, H)-isomorphism game. In [6], they only introduced and studied the cases $t = q$ and $t = ns$ (which we have not introduced here).

However, we shall use the fact that since this is a synchronous game, it will have an affiliated *-algebra with generators and relations that can be used to characterize when perfect t-strategies exist. In fact, the generators and relations for the *-algebra of the game are precisely the relations (IQP_d) in [6]. We now recall the *-algebra $\mathcal{A}(\mathcal{G})$ corresponding to the (G, H)-isomorphism game $\mathcal{G}$. First we need some notation. Given vertices $g, g' \in V(G)$ and $h, h' \in V(H)$, write $\mathrm{rel}(g, g') = \mathrm{rel}(h, h')$ if any of the following hold:

1. $g = g'$ and $h = h'$;
2. $(g, g') \in E(G)$ and $(h, h') \in E(H)$;
3. $g \neq g'$, $(g, g') \notin E(G)$, $h \neq h'$, and $(h, h') \notin E(H)$,

while $\mathrm{rel}(g, g') \neq \mathrm{rel}(h, h')$ when all three fail to hold. The *-algebra $\mathcal{A}(\mathcal{G})$ is generated by elements

$$\{X_{g,h} : g \in V(G), h \in V(H)\}$$

subject to the relations

$$X_{g,h} = X_{g,h}^* = X_{g,h}^2, \quad \text{and} \quad \sum_{h' \in V(H)} X_{g,h'} = \sum_{g' \in V(G)} X_{g',h} = 1$$

for all $g \in V(G)$ and $h \in V(H)$ and

$$\mathrm{rel}(g, g') \neq \mathrm{rel}(h, h') \quad \Rightarrow \quad X_{g,h} X_{g',h'} = 0$$

for all $g, g' \in V(G)$ and $h, h' \in V(H)$.

5.2 Notation

Let $\mathbb{F}(n, m)$ denote the group freely generated by n elements of order m. That is, $\mathbb{F}(n, m) = (\mathbb{Z}/m\mathbb{Z})^{*n}$. Let $\mathrm{C}^*(\mathbb{F}(n, m))$ denote the universal group C*-algebra of $\mathbb{F}(n, m)$. For $x = 1, \ldots, n$, let u_x be the unitary in $\mathrm{C}^*(\mathbb{F}(n, m))$ corresponding to the xth generator of $\mathbb{F}(n, m)$. If ω_m denotes a primitive mth root of unity, the spectral values of u_x are ω_m^i for $i = 1, \ldots, m$. Let $e_{x,i}$ denote the spectral projection of u_x at the spectral value ω_m^i. Then $e_{x,i}$ is a projection for all x and i and $\sum_i e_{x,i} = 1$ for all x.

Conversely, given a C*-algebra $\mathcal{A}$ and projections $e_{x,i} \in \mathcal{A}$ for $1 \leq x \leq n$ and $1 \leq i \leq n$ such that $\sum_i e_{x,i} = 1$ for all x, the element $v_x = \sum_i \omega_m^i e_{x,i}$ is a unitary in $\mathcal{A}$ with order m. Hence there is a unique *-homomorphism $\mathrm{C}^*(\mathbb{F}(n, m)) \to \mathcal{A}$ determined by $u_i \mapsto v_i$. A straight forward calculation shows these constructions are inverses of each other and hence $\mathrm{C}^*(\mathbb{F}(n, m))$ is the universal C*-algebra generated by projections $e_{x,i}$ for $1 \leq x \leq n$ and $1 \leq j \leq m$ such that $\sum_i e_{x,i} = 1$ for all x.

Let M_n denote the vector space of $n \times n$ matrices over $\mathbb{C}$. This vector space can also be viewed as a Hilbert space using the inner product $\langle A, B \rangle := \mathrm{tr}(B^* A)$. By a submatricial operator system, we mean a linear subspace S of M_n for which the identity matrix I

belongs to S and for which S is closed under the adjoint map *. A submatricial traceless self-adjoint operator space is a linear subspace $\mathcal{J} \subseteq M_n$ for which $\mathcal{J}$ is closed under the adjoint operation * and for which given any $A \in \mathcal{J}$, the trace of A is zero.

We will also be using the following graph theory terminology. A graph $G = (V, E)$ is an ordered pair consisting of a vertex set V and edge set $E \subseteq V \times V$. Since we are working with undirected graphs we require that if $(i_1, i_2) \in E$ then $(i_2, i_1) \in E$. We say vertices i_1 and i_2 are *adjacent*, or connected by an edge, and write $i_1 \sim i_2$, whenever $(i_1, i_2) \in E$. An *independent set* of a graph G is a subset $v \subseteq V$ such that for any two distinct elements $i_1, i_2 \in v$ we have $i_1 \not\sim i_2$. For a graph with n vertices it will be standard to consider the vertex set to be $V = \{1, \ldots, n\}$, which we will denote by $[n]$.

Finally, we will always denote by $\mathcal{U}$ for an arbitrary free ultrafilter over $\mathbb{N}$.

Chapter 6

A synchronous game for binary constraint systems

This chapter is about synchronous correlation sets. We start this chapter by characterizing synchronous correlation sets in terms of tracial states on the algebra $C^*(\mathbb{F}(n,m))$ as introduced in the preliminaries. We show that while the sets C_q and C_{qs} cannot be distinguished by synchronous games, the case of whether C_{qa} and C_{qc} can be distinguished by a synchronous game is equivalent to Connes' embedding problem.

In Section 2, we describe a class of games called the synchronous BCS games. These are games based on trying to find a solution to the linear equation $Ax = b$ over the field $\mathbb{Z}/2$. The synchronous BCS game is a variation of the BCS game used by Slofstra to show that C_{qs} cannot be closed [74]. We make a modification of Slofstra's argument to reduce the problem of finding a winning strategy for this game to finding a representation for a group $\Gamma(A,b)$ associated to the game. Indeed, by Slofstra's argument, it follows that one can separate the set C_{qs} from the set C_{qa} by a synchronous game.

Finally, in Section 3, we show that for a graph $G_{A,b}$ associated to a synchronous BCS game, the synchronous BCS game has a winning t-strategy if and only if the associated t-independence number $\alpha_t(G_{A,b})$ is maximal. This, combined with the work in Section 2 demonstrates that the parameter α_{qa} is distinct from the parameter α_q.

6.1 Characterizations of Synchronous strategies

In [67] it was shown that synchronous quantum strategies arise from various families of traces. In particular, it was shown that $p(i,j|v,w) \in C_{qc}^s(n,m)$ if and only if there is a tracial state $\tau : \mathrm{C}^*(\mathbb{F}(n,m)) \to \mathbb{C}$ such that $p(i,j|v,w) = \tau(e_{v,i}e_{w,j})$ and $p(i,j|v,w) \in C_q^s(n,m)$ if and only if there was a tracial state as before such that in addition the GNS representation of $\big(\mathrm{C}^*(\mathbb{F}(n,m)),\tau\big)$ is finite dimensional. But at the time no characterization were given of the traces that arise from synchronous quantum spatial correlations or synchronous quantum approximate correlations. In this section we provide characterizations of those two types of traces.

Definition 6.1.1. Let $\mathcal{A} \subseteq B(H)$ be a C*-algebra. A tracial state τ on $\mathcal{A}$ is called *amenable* provided there is a state ρ on $B(H)$ such that $\rho|_{\mathcal{A}} = \tau$ and $\rho(uTu^*) = \rho(T)$ for all $T \in B(H)$ and all unitaries $u \in \mathcal{A}$.

Note that when $\mathcal{A} = C_\lambda^*(G)$ for a discrete group G, then amenability of τ_e is equivalent to amenability of G. One direction can be seen by taking a restriction of the state ρ extending τ_e on $B(\ell^2(G))$ to $\ell^\infty(G)$ in order to get a G-invariant state. By an application of Arveson's Extension Theorem, the amenability of τ is independent of the choice of faithful representation of $\mathcal{A}$. The following is [56, Proposition 3.2] (see also Theorem 6.2.7 in [13]). Here $\mathcal{R}$ denotes the hyperfinite II_1-factor, $\mathcal{U}$ is a free ultrafilter over the positive integers, and $\mathcal{R}^{\mathcal{U}}$ is the corresponding tracial ultrapower. See Appendix A in [13] for the relevant definitions.

Theorem 6.1.2. *Suppose $\mathcal{A}$ is a separable C*-algebra and τ is a tracial state on $\mathcal{A}$. The following are equivalent:*

1. *the tracial state τ is amenable;*
2. *there is a *-homomorphism $\phi : \mathcal{A} \to \mathcal{R}^{\mathcal{U}}$ with a completely positive, contractive lift $\mathcal{A} \to \ell^\infty(\mathcal{R})$ such that $\mathrm{tr} \circ \phi = \tau$;*
3. *there is a sequence of completely positive, contractive maps $\phi_k : A \to M_{d(k)}$ such that*
$$\|\phi_k(ab) - \phi_k(a)\phi_k(b)\|_2 \to 0 \quad \text{and} \quad \mathrm{tr}_{d(k)}(\phi_k(a)) \to \tau(a)$$
for all $a, b \in \mathcal{A}$;
4. *the linear functional $\varphi : \mathcal{A} \otimes \mathcal{A}^{op} \to \mathbb{C}$ defined by $\varphi(a \otimes b^{op}) = \tau(ab)$ is bounded with respect to the minimal tensor product;*

The trace in condition (3) is the normalized trace, i.e., for $A = (a_{i,j}) \in M_n$, $tr_n(A) = \frac{1}{n}\sum_{i=1}^{n} a_{i,i}$ and the norm in condition (3) is the normalized Hilbert-Schmidt norm, i.e., $\|A\|_2 = tr_n(A^*A)^{1/2}$.

In condition (4), note that if φ is bounded then for any $x = \sum_i a_i \otimes b_i^{op}$ we have that

$$\varphi(x^*x) = \sum_{i,j} \tau(a_j^* a_i b_i b_j^*) = \sum_{i,j} \tau((a_i b_i)(a_j b_j)^*) \geq 0.$$

Since $\varphi(1 \otimes 1) = 1$, we see that if φ is bounded, then φ is a state.

Recall that $\mathrm{C}^*(\mathbb{F}(n,m))$ is generated by a set of n unitaries, $u_v, 1 \leq v \leq n$, of order m and $e_{v,i}$ denotes the spectral projection of u_v corresponding to the spectral value ω_m^i where ω_m is a primitive mth root of unity.

Lemma 6.1.3. *There is a *-isomorphism $\gamma : \mathrm{C}^*(\mathbb{F}(n,m)) \to \mathrm{C}^*(\mathbb{F}(n,m))^{op}$ with $\gamma(u_v^j) = u_v^j, 1 \leq v \leq n$, $1 \leq j \leq m-1$. Moreover, $\gamma(e_{v,i}) = e_{v,i}$ for $1 \leq v \leq n$ and $1 \leq i \leq m$.*

Proof. The words of the form

$$u_{v_1}^{n_1} \cdots u_{v_K}^{n_K}$$

span a dense *-subalgebra of $\mathrm{C}^*(\mathbb{F}(n,m))$. If we set

$$\gamma(u_{v_1}^{n_1} \cdots u_{v_K}^{n_K}) = u_{v_K}^{n_K} \cdots u_{n_1}^{n_1},$$

and extend linearly, then it is easily checked that γ extends to the desired *-isomorphism. The second claim is a simple computation. $\square$

Lemma 6.1.4. *Suppose $n \geq 1$ and $p \in M_n$ is a positive contraction. If q denotes the spectral projection of p for the interval $[1/2, 1]$, then*

$$\|p - q\|_2 \leq 2\sqrt{2}\|p - p^2\|_2.$$

Proof. Define $p_0 = (1-q)p$ and $p_1 = qp$. Note that $\|p_i - p_i^2\|_2 \leq \|p - p^2\|_2$ for $i = 0, 1$. Since $0 \leq p_0 \leq \frac{1}{2}$, we have

$$p_0 - p_0^2 = p_0(1 - p_0) \geq \frac{1}{2}p_0$$

and hence $\|p_0\|_2 \leq 2\|p_0 - p_0^2\|_2$. Similarly, since $\frac{1}{2}q \leq p_1 \leq 1$, we have

$$p_1 - p_1^2 = p_1(1 - p_1) \geq \frac{1}{2}q(1 - p_1) = \frac{1}{2}(q - p_1)$$

and hence $\|p_1 - q\|_2 \leq 2\|p_1 - p_1^2\|_2$. Since p_0 and $p_1 - q$ are orthogonal, the result follows from the Pythagorean identity. $\square$

Lemma 6.1.5. *Given $\varepsilon > 0$ and an integer $m \geq 1$, there is a $\delta > 0$ such that for any integer $d \geq 1$, if $p_1, \ldots, p_m \in M_d$ are positive contractions with $\|p_i^2 - p_i\|_2 < \delta$ and $\|p_i p_j\|_2 < \delta$ for all $i, j = 1, \ldots, m$ with $i \neq j$, then there are mutually orthogonal projections $q_1, \ldots, q_m \in M_d$ such that $\|p_i - q_i\|_2 < \varepsilon$ for all $i = 1, \ldots, m$.*

If in the statement above, we further require $\|\sum_i p_i - 1\|_2 < \delta$, then we may arrange for $\sum_i q_i = 1$.

Proof. We prove the first statement by induction on m. When $m = 1$, this is immediate from Lemma 6.1.4. Assume the result holds for an integer $m \geq 1$. Fix $\varepsilon > 0$ and define $\varepsilon_0 = \varepsilon/(40m + 3)$. Let $\delta_0 > 0$ be the constant obtained by applying the induction hypothesis to m and ε_0 and define $\delta := \min\{\delta_0, \varepsilon_0\}$. Suppose $d \geq 1$ and $p_1, \ldots, p_{m+1} \in M_d$ are positive contractions as above. By the choice of δ, there are mutually orthogonal projections $q_1, \ldots, q_m \in M_d$ such that

$$\|p_i - q_i\|_2 < \varepsilon_0 < \varepsilon.$$

Since $\|p_i p_{m+1}\|_2 < \delta$ for all $i = 1, \ldots, m$, we have

$$\|q_i p_{m+1}\|_2 < \varepsilon_0 + \delta < 2\varepsilon_0.$$

Define $r = (1 - q_1 - \cdots q_m)$ and define $p = p_{m+1}$. Then

$$\|rpr - p\|_2 \leq 2\|rp - p\|_2 \leq 2\sum_{i=1}^{m} \|q_i p\|_2 < 4m\varepsilon_0.$$

Now, note that

$$\begin{aligned}\|(rpr)^2 - rpr\|_2 &\leq \|(rpr)^2 - p^2\|_2 + \|p^2 - p\|_2 + \|p - rpr\|_2 \\ &\leq 3\|rpr - p\|_2 + \|p^2 - p\|_2 < 12m\varepsilon_0 + \delta < (12m + 1)\varepsilon_0.\end{aligned}$$

By the previous lemma, if q_{m+1} denotes the spectral projection of rpr corresponding to the interval $[1/2, 1]$, then

$$\|q_{m+1} - rpr\|_2 < 2\sqrt{2}(12m + 1)\varepsilon_0.$$

Therefore,

$$\|q_{m+1} - p\|_2 < 2\sqrt{2}(12m + 1)\varepsilon_0 + 4m\varepsilon_0 < (40m + 3)\varepsilon_0 = \varepsilon.$$

Note that each of the projections $q_1, \ldots q_m$ are orthogonal to r by construction. As q_{m+1} is a spectral projections of rpr, we also have that each of the projections $q_1, \ldots, q_m$ is orthogonal to q_{m+1}. This completes the proof of the first part of the lemma.

To see the final sentence holds, fix $m \geq 1$ and $\varepsilon > 0$. Let $\varepsilon_0 = \varepsilon/(m+2)$ and let δ_0 be the constant given by applying the first part of the lemma to m and ε_0. Define $\delta = \min\{\varepsilon_0, \delta_0\}$. Suppose $d \geq 1$ and $p_1, \ldots, p_m \in M_d$ are projections such that

$$\|p_i - p_i^2\|_2 < \delta, \quad \|p_i p_j\|_2 < \delta, \quad \text{and} \quad \Big\| \sum_k p_k - 1 \Big\|_2 < \delta$$

for all $i, j = 1, \ldots, m$ with $i \neq j$. By the choice of δ, there are mutually orthogonal projections $q_1', q_2, q_3, \ldots q_m \in \mathbb{M}_d$ such that $\|p_1 - q_1'\|_2 < \varepsilon_0$ and $\|p_i - q_i\|_2 < \varepsilon_0$ for $i = 2, \ldots, m$. Now, define $q_1'' = 1 - q_1' - \sum_{i=2}^m q_i$ and note that $\|q_1''\|_2 < (m+1)\varepsilon_0$. To complete the proof, define $q_1 = q_1' + q_1''$. □

Theorem 6.1.6. *Fix integers $n, m \geq 1$. For $\big(p(i,j|v,w)\big) \in \mathbb{R}^{n^2m^2}$, the following are equivalent:*

1. $\big(p(i,j|v,w)\big) \in C_{qa}^s(n,m)$*;*
2. *there are synchronous correlations $\big(p_k(i,j|v,w)\big) \in C_q^s(n,m)$ with*
$$\lim_k p_k(i,j|v,w) = p(i,j|v,w)$$
for all i, j, v, w;
3. *there is an amenable trace τ on $\mathrm{C}^*(\mathbb{F}(n,m))$ such that*
$$\tau(e_{v,i}e_{w,j}) = p(i,j|v,w)$$
for all i, j, v, w; and
4. *there are projections $f_{v,i} \in \mathcal{R}^{\mathcal{U}}$ such that $\sum_i f_{v,i} = 1$ for all v and*
$$\mathrm{tr}(f_{v,i}f_{w,j}) = p(i,j|v,w)$$
for all i, j, v, w.

Proof. It is clear that (2) implies (1). To see (1) implies (3), assume that $\big(p(i,j|v,w)\big)$ is a correlation in $C_{qa}^s(n,m)$. There exist correlations $\big(p_k(i,j|v,w)\big)$ in $C_q(n,m)$ for $k \geq 1$ such that

$$\lim_k p_k(i,j|v,w) = p(i,j|v,w)$$

for all i, j, v, w. Each $\big(p_k(i,j,v,w)\big)$ has a representation on a tensor product of finite dimensional vector spaces $\mathbb{C}^{d_k} \otimes \mathbb{C}^{r_k}$ as

$$p_k(i,j|v,w) = \langle E^k_{v,i} \otimes F^k_{w,j} \psi_k, \psi_k \rangle \ ,$$

where the matrices $E^k_{v,i}, F^k_{w,j}$ are all orthogonal projections satisfying $\sum_i E^k_{v,i} = I_{d_k}$ and $\sum_j F^k_{w,j} = I_{r_k}$ and each ψ_k is a unit vector.

Thus there is a representation $\pi_k : C^*(\mathbb{F}(n,m)) \otimes C^*(\mathbb{F}(n,m))^{op} \to M_{d_k} \otimes M_{r_k}$ with $\pi_k(e_{v,i} \otimes e^{op}_{w,j}) = E^k_{v,i} \otimes F^k_{w,j}$. Setting $\varphi_k(a \otimes b^{op}) = \langle \pi_k(a \otimes b^{op})\psi_k, \psi_k\rangle$ defines a sequence of states φ_k on $C^*(\mathbb{F}(n,m) \otimes C^*(\mathbb{F}(n,m))^{op}$. Let φ be any weak*-limit point of $(\varphi_k)_k$ and note that $\varphi(e_{v,i} \otimes e^{op}_{w,j}) = p(i,j|v,w)$.

If we let $\pi : C^*(\mathbb{F}(n,m)) \otimes_{min} C^*(\mathbb{F}(n,m))^{op} \to B(H)$ and $\psi \in H$ be a GNS representation of this state, then it follows by [67, Theorem 5.5], that $\tau(a) = \langle \pi(a \otimes 1)\psi, \psi\rangle$ is a trace and that

$$\pi(a \otimes e_{w,j})\psi = \pi(ae_{w,j} \otimes 1)\psi.$$

Hence, $\pi(a \otimes be_{w,j})\psi = \pi(1 \otimes b)\pi(a \otimes e_{w,j})\psi = \pi(ae_{w,j} \otimes b)\psi$ and it follows that

$$\varphi(a \otimes b^{op}) = \langle \pi(a \otimes b^{op})\psi, \psi\rangle = \langle \pi(ab \otimes 1)\psi, \psi\rangle = \tau(ab)\ .$$

Thus, τ is an amenable trace by Theorem 6.1.2.

To see (3) implies (2), it suffices to show that if τ is an amenable trace on $\mathrm{C}^*(\mathbb{F}(n,m))$, then there is a sequence of traces τ_k on $\mathrm{C}^*(\mathbb{F}(n,m))$ which factor through a finite dimensional matrix algebra such that $\tau_k(a) \to \tau(a)$ for all $a \in \mathrm{C}^*(\mathbb{F}(n,m))$. Since τ is amenable, Theorem 6.1.2 yields a sequence of completely positive, unital maps $\phi_k : \mathrm{C}^*(\mathbb{F}(n,m)) \to M_{d(k)}$ such that

$$\|\phi_k(ab) - \phi_k(a)\phi_k(b)\|_2 \to 0 \quad \text{and} \quad \mathrm{tr}(\phi_k(a)) \to \tau(a)$$

for all $a, b \in \mathrm{C}^*(\mathbb{F}(n,m))$. By passing to a subsequence and applying Lemma 6.1.5, we may find projections $p^k_{v,i} \in M_{d(k)}$ such that $\sum_i p^k_{v,i} = 1$ for all v and k and such that $\|\phi_k(e_{v,i}) - p^k_{v,i}\|_2 \to 0$ for all v and i. There is a *-homomorphism $\phi'_k : \mathrm{C}^*(\mathbb{F}(n,m)) \to M_{d(k)}$ such that $\phi'_k(e_{v,i}) = p^k_{v,i}$ for all v, i, and k. Using that $\mathrm{C}^*(\mathbb{F}(n,m))$ is generated as a C*-algebra by the projections $e_{v,i}$, one can show

$$\|\phi_k(a) - \phi'_k(a)\|_2 \to 0$$

for all $a \in \mathrm{C}^*(\mathbb{F}(n,m))$. In particular,

$$\lim \mathrm{tr}(\phi'_k(a)) = \lim \mathrm{tr}(\phi_k(a)) = \tau(a)$$

for all $a \in \mathrm{C}^*(\mathbb{F}(n,m))$.

To see (3) implies (4), note that if τ is an amenable trace on $\mathrm{C}^*(\mathbb{F}(n,m))$, then there is a trace preserving *-homomorphism $\phi : \mathrm{C}^*(\mathbb{F}(n,m)) \to \mathcal{R}^{\mathcal{U}}$ by Theorem 6.1.2. Define $f_{v,i} = \phi(e_{v,i}) \in \mathcal{R}^{\mathcal{U}}$ for all v and i. Conversely, given $f_{v,i}$ as in (4), there is a *-homomorphism $\phi : \mathrm{C}^*(\mathbb{F}(n,m)) \to \mathcal{R}^{\mathcal{U}}$ such that $\phi(e_{v,i}) = f_{v,i}$ for all v and i. As $\mathrm{C}^*(\mathbb{F}(n,m))$ has the local lifting property, ϕ has a completely positive, unital lift $\mathrm{C}^*(\mathbb{F}(n,m)) \to \ell^\infty(\mathcal{R})$. By Theorem 6.1.2, the trace $\tau := \mathrm{tr} \circ \phi$ on $\mathrm{C}^*(\mathbb{F}(n,m))$ is amenable. □

Corollary 6.1.7. *Let $\mathcal{G} = (I, O, V)$ be a synchronous game. The following are equivalent:*

(i) $\mathcal{G}$ has a perfect qa-strategy,

*(ii) there is a unital *-representation of $\mathcal{A}(\mathcal{G})$ into $\mathcal{R}^{\mathcal{U}}$,*

(iii) there is an amenable trace τ on $\mathrm{C}^*(\mathbb{F}(n,m))$ such that $V(v,w,i,j) = 0$ implies $\tau(e_{v,i}e_{w,j}) = 0$ for all i, j, v, w.

Corollary 6.1.8. *The following are equivalent:*

(i) Connes' embedding conjecture has an affirmative answer,

(ii) for all n, m, $C^s_{qa}(n,m) = C^s_{qc}(n,m)$,

(iii) for all n, m, $C_{qa}(n,m) = C_{qc}(n,m)$.

Proof. The equivalence of (i) and (ii) in Theorem 6.1.6 answers [24, Problem 3.8]. In the remarks following Problem 3.8, [24] shows how a positive solution of the problem leads to the above result. □

Remark 6.1.9. The implication $(iii) \implies (i)$ in the above corollary is due to [63]. The equivalence of (i) and (ii) follows from [24, Theorem 3.7] and our solution of their synchronous approximation problem. Note that the implication (iii) implies (ii) is trivial, so we have a different proof of Ozawa's implication. Ozawa's proof uses Kirchberg's results showing the equivalence of Connes' embedding conjecture to the equality of the minimal and maximal tensor products of certain C*-algebras of free groups. The above proof uses the results of [24] which in turn used Kirchberg's results about the equivalence of Connes' embedding conjecture to finite approximability of traces, often referred as the matricial microstates conjecture. Finally, due to the work of [39], it seems to be that the above corollary has an answer in the negative.

We next turn our attention to the set of synchronous quantum spatial correlations. We prove the somewhat surprising result that any synchronous correlation that that can be obtained using a tensor product of possible infinite dimensional Hilbert spaces has a representation using only finite dimensional spaces.

Theorem 6.1.10. *Let $n, m \geq 1$. Then $C_q^s(n,m) = C_{qs}^s(n,m)$.*

Proof. By definition, $C_q^s(n,m) \subseteq C_{qs}^s(n,m)$, so we must prove that $C_{qs}^s(n,m) \subseteq C_q^s(n,m)$. Let $\big(p(i,j|v,w)\big) \in C_{qs}^s(n,m)$ be represented as

$$p(i,j|v,w) = \langle E_{v,i} \otimes F_{w,j}\psi, \psi\rangle$$

where $\{E_{v,i}, 1 \leq v \leq n, 1 \leq i \leq m\}$ are orthogonal projections on some Hilbert space H satisfying $\sum_i E_{v,i} = I_H$ for all v, $\{F_{w,j} : 1 \leq w \leq n, 1 \leq j \leq m\}$ are orthogonal projections on some Hilbert space K satisfying $\sum_j F_{w,j} = I_{\mathcal{K}}$ for all w, and $\psi \in H \otimes K$ is a unit vector.

Note that if we are given any other Hilbert space $\mathcal{G}$ and we set $F'_{w,1} = F_{w,1} \oplus I_{\mathcal{G}}$ and $F'_{w,j} = F_{w,j} \oplus 0$, then $p(i,j|v,w) = \langle (E_{v,i} \otimes F'_{w,j})\psi, \psi\rangle$. In this manner we see that there is no loss of generality in assuming that $dim(H) = dim(K)$, so we assume that these two Hilbert spaces have the same dimension.

Let $\sum_{k\in X} \alpha_k e_k \otimes f_k$ be the Schmidt decomposition of ψ so that X is a countable set and $\{e_k : k \in X\}$ and $\{f_k : k \in X\}$ are orthonormal sets in their respective Hilbert spaces. By setting sufficiently many α's equal to 0, and direct summing with additional Hilbert spaces as needed, we may assume that these sets are orthonormal bases for their respective spaces.

Let $\{r_l : l \in Y\} = \{\alpha_k : k \in X\}$ be an enumeration of the set of distinct non-zero α_k's (which is at most countable) with $r_1 > r_2 > \ldots$ and let $S_l = \{k : \alpha_k = r_l\}$. Let $\mathcal{E}_l = span\{e_k : k \in S_l\}$ and $\mathcal{F}_l = span\{f_k : k \in S_l\}$. Since the α_k's are square summable, each set S_l is finite and so each of these spaces is finite dimensional.

We claim that the spaces $\mathcal{E}_l$ are reducing subspaces for $\{E_{v,i}\}$ and that the spaces $\mathcal{F}_l$ are reducing for the set $\{F_{w,j}\}$

First, we complete the proof assuming the claim. Let $E^l_{v,i}$ denote the compression of $E_{v,i}$ to the space $\mathcal{E}_l$ and let $F^l_{w,j}$ denote the compression of $F_{w,j}$ to the space $\mathcal{F}_l$ so that these are orthogonal projections and $\sum_i E^l_{v,i} = I_{\mathcal{E}_l}$ for all v and $\sum_j F^l_{w,j} = I_{\mathcal{F}_l}$ for all w. Set $d_l = \dim(\mathcal{E}_l) = \dim(\mathcal{F}_l) = card(S_l)$ and let $\psi_l = \frac{1}{\sqrt{d_l}} \sum_{k\in S_l} e_k \otimes f_k \in \mathcal{E}_l \otimes \mathcal{F}_l$, which is a unit vector. Let $t_l = r_l^2 d_l$ so that $\sum_l t_l = \sum_k \alpha_k^2 = 1$ and set

$$p_l(i,j|v,w) = \langle E^l_{v,i} \otimes F^l_{w,j}\psi_l, \psi_l\rangle \in C_q(n,m)\ .$$

Note that

$$\sum_l t_l p_l(i,j|v,w) = p(i,j|v,w) \ ,$$

so that for $i \neq j$, $\sum_l t_l p_l(i,j|v,v) = 0$ from which it follows that $p(i,j|v,v) = 0$ for all l. Thus, each $p_l(i,j|v,w) \in C_q^s(n,m)$.

Since $C_q^s(n,m)$ is convex, by [19], $p(i,j|v,w) \in C_q^s(n,m)$. The key point here is that by [19] a convex set need not be closed to ensure that such a series remains in the set.

Thus, we need only establish that these spaces reduce the operators. Let $\omega = e^{2\pi i/m}$ be a primitive m-th root of unity and let $A_v = \sum_{i=1}^m \omega^i E_{v,i}$ and let $B_w = \sum_{j=1}^m \omega^j F_{w,j}$ so that these are unitaries of order m and the original projections are the spectral projections of these unitaries. Note that these unitaries generate the same C*-algebras as the projections so that the projections are reduced by these subspaces if and only if these unitaries are reduced by these subspaces.

First recall that the synchronous condition guarantees that $(E_{v,i} \otimes I)\psi = (I \otimes F_{v,i})\psi$ by [67, Theorem 5.5i] and hence, $(A_v \otimes I)\psi = (I \otimes B_v)\psi$.

Now $(A \otimes I)\psi = (I \otimes B)\psi$ implies

$$\alpha_j \langle Ae_j, e_i \rangle = \langle (A \otimes I)\psi, e_i \otimes f_j \rangle = \langle (I \otimes B)\psi, e_i \otimes f_j \rangle = \alpha_i \langle Bf_i, f_j \rangle \ .$$

Thus for $i \in S_1$, using that $\alpha_1 \geq \alpha_j$, we have

$$\alpha_1^2 \geq \sum_j \alpha_j^2 |\langle A_v e_j, e_i \rangle|^2 = \sum_j \alpha_i^2 |\langle B_v f_i, f_j \rangle|^2 = \alpha_1^2 \|B_v f_i\|^2 = \alpha_1^2 \ ,$$

and so we must have equality throughout. But equality implies that $\langle A_v e_j, e_i \rangle = 0$ for all $j \notin S_1$. Hence, $A_v^* e_i \in \mathcal{E}_1$ for all $i \in S_1$. This shows that A_v^* leaves $\mathcal{E}_1$ invariant. Hence, $A_v = \left(A_v^*\right)^{m-1}$ also leaves this space invariant and so $\mathcal{E}_1$ is a reducing subspace for every A_v and hence for the entire C*-algebra that they generate. A similar proof shows that $\mathcal{F}_1$ is reducing for every B_v.

Now it follows that for $i \in S_2$, we have that for $j \in S_1$, $\langle A_v e_j, e_i \rangle = 0$ and so,

$$r_2^2 \geq \sum_j \alpha_j^2 |\langle A_v e_j, e_i \rangle|^2 = \sum_j r_2^2 |\langle B_v f_i, f_j \rangle|^2 = r_2^2 \ .$$

Similar reasoning shows that $A_v^* e_i \in \mathcal{E}_2$ and consequently that $\mathcal{E}_2$ reduces these unitaries.

We have now done the first two cases and the complete proof follows by induction along these lines. $\square$

Corollary 6.1.11. *A synchronous game has a perfect qs-strategy if and only if it has a perfect q-strategy.*

6.2 Separating C_{qs}^s and C_{qa}^s

Suppose $Ax = b$ is an $m \times n$ linear system over $\mathbb{Z}/2$; that is, $A = (a_{i,j}) \in \mathbb{M}_{m,n}(\mathbb{Z}/2)$ and $b \in (\mathbb{Z}/2)^n$. Let $V_i = \{j \in \{1, \ldots, n\} : a_{i,j} \neq 0\}$ denote the variables which occur in the ith equation for $i = 1, \ldots, m$. It will be convenient to write the system multiplicative notation where we identify $\mathbb{Z}/2$ with $\{\pm 1\}$ and write the ith equation of the linear system as

$$\prod_{j \in V_i} x_j = (-1)^{b_i} \tag{6.1}$$

for $i = 1, \ldots, m$ where $x_j \in \{\pm 1\}$. We recall the definition of the solution group $\Gamma(A, b)$ associated to the system $Ax = b$. The idea is to interpret (6.1) as the relations of a group with generators $x_1, \ldots, x_n$ and a generator J used to place the role of -1. More precisely, we make the following definition.

Definition 6.2.1. Given an $m \times n$ linear system as above, let $\Gamma(A, b)$ denote the group generated by $u_1, \ldots, u_n, J$ with relations

1. $u_j^2 = J^2 = 1$ for $j = 1, \ldots, n$,
2. $u_j u_k = u_k u_j$ for $j, k \in V_i$ and $i = 1, \ldots, m$,
3. $u_j J = J u_j$ for $j = 1, \ldots, n$, and
4. $\prod_{j \in V_i} u_j = J^{b_i}$ for $i = 1, \ldots, m$.

We call $\Gamma(A, b)$ the *solution group* associated to the linear system $Ax = b$.

For $i = 1, \ldots, m$, let

$$S_i = \{x \in \{\pm 1\}^n : \prod_{j \in V_i} x_j = (-1)^{b_i} \text{ and } x_j = 1 \text{ for } j \notin V_i\}.$$

We associate a synchronous game to $Ax = b$ as follows:

Definition 6.2.2. Suppose $Ax = b$ is an $m \times n$ linear system over $\mathbb{Z}/2$ and $b \in (\mathbb{Z}/2)^n$. The synchronous BCS game associated to $Ax = b$, denoted $\mathrm{synBCS}(A, b)$, is given as follows:

1. the input set is $\mathcal{I} = \{1, \ldots, m\}$;

2. the output set is $\mathcal{O} = \{\pm 1\}^n$;
3. given input (i, j), Alice and Bob win on output (x, y) if $x \in S_i$, $y \in S_j$, and for all $k \in V_i \cap V_j$, $x_k = y_k$.

Let $\mathcal{A} \cong \mathrm{C}^*(\mathbb{F}(m, 2^n))$ denote the universal C*-algebra generated by projections $e_{i,x}$ for $i = 1, \ldots, m$ and $x \in \{\pm 1\}^n$ subject to the relations $\sum_x e_{i,x} = 1$ for all $i = 1, \ldots, m$. The following result gives a relationship between correlations in $C_{qc}^s(m, 2^n)$ and the structure of the group $\Gamma(A, b)$.

Theorem 6.2.3. *Suppose every column of $\mathcal{A}$ contains a non-zero entry. Then there is a surjective *-homomorphism $\pi : \mathcal{A} \to \mathrm{C}^*(\Gamma(A, b))/\langle J + 1\rangle$, where $\langle J + 1\rangle$ denotes the ideal generated by $J + 1$, given by*

$$\pi(e_{i,x}) = \begin{cases} \prod_{j \in V_i} \chi_{x_j}(u_j) & x \in S_i \\ 0 & x \notin S_i, \end{cases} \tag{6.2}$$

where $\chi_{x_j}(u_j)$ denotes the spectral projection of u_j at the point x_j.

Moreover, the map $\tau \mapsto \tau \circ \pi$ is a bijection from the set of tracial states on the algebra $\mathrm{C}^*(\Gamma(A, b))/\langle J + 1\rangle$ to the set of tracial states τ' on $\mathcal{A}$ satisfying $\tau'(e_{i,x} e_{j,y}) = 0$ whenever Alice and Bob lose on outputs (x, y) given inputs (i, j).

Proof. First we show that the formula for π given in (6.2) defines a *-homomorphism on $\mathcal{A}$. Note that since $\{u_j : j \in V_i\}$ is a set of commuting self-adjoint unitaries, $\pi(e_{i,x})$ is defined and is a projection for each i and x. Moreover, for $i = 1, \ldots, m$, in the algebra $\mathrm{C}^*(\Gamma(A, b))/\langle J + 1\rangle$,

$$(-1)^{b_i} = \prod_{j \in V_i} u_j = \prod_{j \in V_i} (\chi_{+1}(u_j) - \chi_{-1}(u_j)) = \sum_{x \in \{\pm 1\}^{V_i}} \prod_{j \in V_i} x_j \chi_{x_j}(u_j) \ .$$

Note that if $x \in \{\pm 1\}^{V_i}$ and $\prod_{j \in V_i} x_j \neq (-1)^{b_i}$, then

$$\prod_{j \in V_i} x_j \chi_{x_j}(u_j) = - \prod_{j \in V_i} u_j \chi_{x_j}(u_j) = - \prod_{j \in V_i} x_j \chi_{x_j}(u_j)$$

and hence $\prod_{j \in V_i} x_j \chi_{x_j}(u_j) = 0$. Combining these calculations, we have

$$(-1)^{b_i} = \sum_{x \in S_i} \prod_{j \in V_i} x_j \chi_{x_j}(u_j)$$

and hence

$$\sum_{x\in\{\pm1\}^n} \pi(e_{i,x}) = \sum_{x\in S_i}\prod_{j\in V_i} \chi_{x_j}(u_j) = (-1)^{b_i}\sum_{x\in S_i}\prod_{j\in V_i} x_j\chi_{x_j}(u_j) = 1\ .$$

Thus the desired *-homomorphism π exists.

To see π is surjective, fix $k \in \{1, \ldots, m\}$. As the kth column of A contains a non-zero entry, there is an $i \in \{1, \ldots, m\}$ such that $k \in V_i$. Note that

$$\begin{aligned} u_k &= (\chi_{+1}(u_k) - \chi_{-1}(u_k))\sum_{x\in S_i}\prod_{j\in V_i}\chi_{x_j}(u_j) \\ &= \sum_{x\in S_i, x_k=1}\prod_{j\in V_i}\chi_{x_j}(u_j) - \sum_{x\in S_i, x_k=-1}\prod_{j\in V_i}\chi_{x_j}(u_j) \\ &= \sum_{x\in S_i, x_k=1}\pi(e_{v,x}) - \sum_{x\in S_i, x_k=-1}\pi(e_{v,x}). \end{aligned}$$

As $\mathrm{C}^*(\Gamma(A,b))/\langle J+1\rangle$ is generated by $u_1, \ldots, u_m$, the result follows.

We next work to prove the claim about traces. As π is surjective, the induced map on traces is injective. To see surjectivity, let τ' be a trace on $\mathcal{A}$ such that $\tau'(e_{i,x}e_{j,y}) = 0$ if $x \notin S_i$, $y \notin S_j$, or there is a $k \in V_i \cap V_j$ such that $x_k \neq y_k$. Define

$$\mathcal{N} = \{a \in \mathcal{A} : \tau'(a^*a) = 0\}$$

and note that $\mathcal{N}$ is an ideal in $\mathcal{A}$. We first show

1. if $x \notin S_i$, then $e_{i,x} \in \mathcal{N}$,
2. if $x_k \neq y_k$ for some $k \in V_i \cap V_j$, then $e_{i,x}e_{j,y} \notin \mathcal{N}$, and
3. if $k \in V_i \cap V_j$, then $\displaystyle\sum_{x\in S_i} x_k e_{i,x} - \sum_{y\in S_j} y_k e_{i,x} \in \mathcal{N}$.

For (1), if $x \notin S_i$, then $\tau'(e_{i,x}^*e_{i,x}) = \tau'(e_{i,x}e_{i,x}) = 0$ by the assumptions on τ'. For (2), if $x_k \neq y_k$ for some $k \in V_i \cap V_j$, then

$$\tau'((e_{i,x}e_{j,y})^*(e_{i,x}e_{j,y})) = \tau'(e_{j,y}e_{i,x}e_{j,y}) = \tau'(e_{i,x}e_{j,y}) = 0$$

by the assumptions on τ'. For (3), fix $k \in V_i \cap V_j$. Then

$$\tau'(x_k y_k e_{i,x}e_{j,y}) = \begin{cases} \tau'(e_{i,x}e_{j,y}) & x_k = y_k \\ 0 & x_k \neq y_k \end{cases}$$

by (2) above. Also,

$$\sum_{x\in S_i} \tau'(e_{i,x}) = \sum_{x\in S_j} \tau'(e_{j,y}) = 1$$

by (1) above. Now,

$$\tau'\left(\left(\sum_{x\in S_i} x_k e_{i,x} - \sum_{y\in S_j} y_k e_{i,x}\right)^* \left(\sum_{x\in S_i} x_k e_{i,x} - \sum_{y\in S_j} y_k e_{i,x}\right)\right)$$
$$= \sum_{x\in S_i} \tau'(e_{i,x}) + \sum_{y\in S_j} \tau'(e_{j,y}) - 2 \sum_{x\in S_i, y\in S_j} \tau'(e_{i,x} e_{j,y}) = 0$$

which proves (3).

Fix $k \in \{1, \ldots, n\}$. Since the jth column of A is non-zero, there is an $i \in \{1, \ldots, m\}$ such that $k \in V_i$. Define $v_k \in \mathcal{A}/\mathcal{N}$ by

$$v_k = \sum_{x\in S_i} x_k e_{i,x}.$$

By condition (3) above, the v_k is independent of the choice of i. Note that v_k is a self-adjoint unitary in $\mathcal{A}/\mathcal{N}$ and if $k, \ell \in V_i$ for some $i = 1 \ldots m$, then $v_k v_\ell = v_\ell v_k$. Finally for $i = 1, \ldots, m$, since the projections $e_{i,x}$ are orthogonal, we have

$$\prod_{k\in V_i} v_k = \prod_{k\in V_i} \sum_{x\in S_i} x_k e_{i,x} = \sum_{x\in S_i} \left(\prod_{k\in V_i} x_k\right) e_{i,x} = (-1)^{b_i}.$$

It follows that there is a group homomorphism $\rho : \Gamma(A, b) \to U(\mathcal{A}/\mathcal{N})$ given by $\rho(u_k) = v_k$ and $\rho(J) = -1$. Now, ρ induces a *-homomorphism, still denoted ρ, from $\mathrm{C}^*(\Gamma(A, b))/\langle J + 1\rangle$ to $\mathcal{A}/\mathcal{N}$.

Let $q : \mathcal{A} \to \mathcal{A}/\mathcal{N}$ denote the quotient map. Since τ' vanishes on $\mathcal{N}$, there is a trace $\bar{\tau}'$ on $\mathcal{A}/\mathcal{N}$ such that $\bar{\tau}' \circ q = \tau'$. Define a trace τ on $\mathrm{C}^*(\Gamma(A, b))/\langle J + 1\rangle$ by $\tau = \bar{\tau}' \circ \rho$. By construction, $\rho(\pi(e_{i,x})) = q(e_{i,x})$ for all i and x and hence $\rho \circ \pi = q$. Now, $\tau \circ \pi = \bar{\tau}' \circ \rho \circ \pi = \bar{\tau}' \circ q = \tau'$. This completes the proof. □

Corollary 6.2.4. *Let $Ax = b$ be a linear system.*

1. synBCS(A, b) *has a perfect qc-strategy if and only if $J \neq 1$ in $\Gamma(A, b)$,*

2. synBCS(A, b) *has a perfect qa-strategy if and only if there is representation $\Gamma(A, b) \to \mathcal{R}^{\mathcal{U}}$ such that $\rho(J) \neq 1$, and*

3. synBCS(A,b) *has a perfect q-strategy if and only if there is a finite dimensional representation* $\rho : \Gamma(A,b) \to U(M_d)$ *such that* $\rho(J) \neq 1$.

Proof. We may assume no column of A is identically zero. Assume A is an $m \times n$ linear system.

We first prove (1). If synBCS(A,b) has a perfect qc-strategy $p(x,y|i,j) \in C_{qc}^s(m, 2^n)$, there is a trace τ on $\mathcal{A}$ such that

$$p(x,y|i,j) = \tau(e_{i,x}e_{j,y}) \quad \text{for all } i,j,x,y.$$

By Theorem 6.2.3, there is a trace τ' on $\mathrm{C}^*(\Gamma(A,b))/\langle J+1\rangle$ such that $\tau' \circ \pi = \tau$. In particular, $\mathrm{C}^*(\Gamma(A,b))/\langle J+1\rangle$ is non-zero. Hence $J+1 \neq 2$ in $\mathrm{C}^*(\Gamma(A,b))$ and $J \neq 1$ in $\Gamma(A,b)$.

Conversely, suppose $J \neq 1$ in $\Gamma(A,b)$. As J is central, $\langle J\rangle \cong \mathbb{Z}/2$ is a normal subgroup of $\Gamma(A,b)$. There is a conditional expectation $E : \mathrm{C}^*(\Gamma(A,b)) \to \mathrm{C}^*(\langle J\rangle) \cong \mathbb{C}^2$ determined by $E(s) = s$ for $s \in \{1, J\}$ and $E(s) = 0$ for $s \in \Gamma(A,b)\backslash\{1,J\}$. Let $\chi : \mathrm{C}^*(\langle J\rangle) \to \mathbb{C}$ be the character defined by $\chi(J) = -1$. Then $\chi \circ E$ is a trace on $\mathrm{C}^*(\Gamma(A,b))$. As $(\chi\circ E)(J+1) = 0$ and $J+1 \geq 0$, the trace $\chi\circ E$ vanishes on the ideal $\langle J+1\rangle \subseteq \mathrm{C}^*(\Gamma(A,b))$ and hence induces a trace τ on $\mathrm{C}^*(\Gamma(A,b))/\langle J+1\rangle$. Now, the trace $\tau\circ\pi$ on $\mathcal{A}$ is a trace where π is the surjection in Theorem 6.2.3. We define a qc-correlation by

$$p(x,y|i,j) = \tau(\pi(e_{i,x}e_{j,y})) \quad \text{for all } i,j,x,y.$$

By Theorem 6.2.3, $\big(p(x,y|i,j)\big)$ is a perfect qc-strategy.

For (2) and (3), we let $\mathcal{B}$ denote either $\mathcal{R}^{\mathcal{U}}$ or M_d. Suppose $\rho : \Gamma(A,b) \to U(\mathcal{B})$ is a group homomorphism such that $\rho(J) \neq 1$. Let q denote the spectral projection of $\rho(J)$ corresponding to the eigenvalue -1. As $J \neq 1$, we have $q \neq 0$. As J is central in $\Gamma(A,b)$, the projection q commutes with the image of ρ. Now, $q\rho(\cdot)$ is a unitary representation of $\Gamma(A,b)$ on $U(q\mathcal{B}q)$ and $q\rho(J) = -q$. When $\mathcal{B} = M_d$, $q\mathcal{B}q \cong M_{d'}$ for some $d' \geq 1$, and when $\mathcal{B} = \mathcal{R}^{\mathcal{U}}$, $q\mathcal{B}q \cong \mathcal{R}^{\mathcal{U}}$. Hence after replacing $\mathcal{B}$ with $q\mathcal{B}q$ and ρ with $q\rho(\cdot)$, we may assume $\rho(J) = -1$. Now ρ induces a *-homomorphism $\mathrm{C}^*(\Gamma(A,b)) \to \mathcal{B}$ vanishing on $J+1$ and hence induces a *-homomorphism

$$\mathcal{A} \xrightarrow{\pi} \mathrm{C}^*(\Gamma(A,b))/\langle J+1\rangle \to \mathcal{B}\,.$$

The trace on $\mathcal{B}$ defines a trace on $\mathcal{A}$ which in turn defines a winning q-strategy when $\mathcal{B}$ is finite dimensional and a winning qa-strategy when $\mathcal{B} = \mathcal{R}^{\mathcal{U}}$.

Now suppose synBCS(A, b) has a perfect qa-strategy. As in Theorem 6.2.3, there is a trace τ on $\mathrm{C}^*(\Gamma(A, b))/\langle J+1\rangle$ which factors through the trace on $\mathcal{R}^{\mathcal{U}}$. The GNS representation of τ induces a representation of $\mathrm{C}^*(\Gamma(A, b))/\langle J+1\rangle \to \mathcal{R}^{\mathcal{U}}$ which in turn induces a representation $\rho : \Gamma(A, b) \to \mathcal{R}^{\mathcal{U}}$ with $\rho(J) = -1$. Similarly, if synBCS(A, b) has a perfect q-strategy, one produces a representation of $\Gamma(A, b)$ in the same way using a finite dimensional algebra in place of $\mathcal{R}^{\mathcal{U}}$. □

The following result is in [74].

Theorem 6.2.5. *There is a linear system $Ax = b$ such that there is a representation $\rho : \Gamma(A, b) \to U(\mathcal{R}^{\mathcal{U}})$ such that $\rho(J) \neq 1$ but for every finite dimensional representation $\rho_0 : \Gamma(A, b) \to U(M_d)$, $\rho(J) = 1$.*

Combining Theorem 6.2.5 with Corollary 6.2.4 provides a synchronous game which has a perfect qa-strategy but no perfect q-strategy. Hence we have the following strengthening of Slofstra's result[74].

Corollary 6.2.6. *For sufficiently large m and n, we have $C_q^s(m, 2^n) = C_{qs}^s(m, 2^n) \neq C_{qa}^s(m, 2^n)$. In particular, for sufficiently large m, n, $C_q^s(m, 2^n) = C_{qs}^s(m, 2^n)$ is not closed.*

Remark 6.2.7. If each row of the matrix A appearing in the above result has only k non-zero entries, then one can deduce that $C_q^s(m, 2^k) = C_{qs}^s(m, 2^k)$ is not closed.

Remark 6.2.8. If $C_{qs}(m, 2^n)$ or $C_q(m, 2^n)$ was closed, then their subsets of synchronous elements would be closed. Since $C_q^s(m, 2^n) = C_{qs}^s(m, 2^n)$, the above result implies Slofstra's result[74] that $C_q(m, 2^n)$ and $C_{qs}(m, 2^n)$ are not closed, for sufficiently large m, n.

Remark 6.2.9. It is shown in [70] that $C_q^s(3, 2)$ is always closed but as of the writing of this thesis, it is still open if $C_q(3, 2)$ must also be closed. As well, it is shown in [18] that $C_q \neq C_{qs}$.

6.3 Separating quantum independence numbers

In this section we prove that there exists a graph G for which $\alpha_q(G) < \alpha_{qa}(G)$. Recall from the preliminaries that for $t \in \{q, qa, qc\}$, the independence number $\alpha_t(G)$ is the largest $c \geq 1$ for which the graph homomorphism game $K_c \to \overline{G}$ has a perfect t-strategy.

First let us recall from [6, Section 6] the graph $G_{A,b}$ defined for a linear system $Ax = b$ over $\mathbb{Z}/2$.

Definition 6.3.1. Suppose $Ax = b$ is an $m \times n$ linear system over $\mathbb{Z}/2$ and $b \in (\mathbb{Z}/2)^n$. Define a graph $G_{A,b}$ with the following data:

1. the vertices of $G_{A,b}$ are pairs (i, x) where $i \in \{1, \ldots, m\}$ and $x \in S_i$;
2. there is an edge between distinct vertices (i, x) and (j, y) if and only if there exists some $k \in V_i \cap V_j$ for which $x_k \neq y_k$; that is, x and y are inconsistent solutions.

Lemma 6.3.2. *Suppose $t \in \{q, qa, qc\}$. If G and H are finite graphs and $G \cong_t H$ then $\alpha_t(G) = \alpha_t(H)$.*

Proof. Let $V = V(G) \cup V(H)$. It suffices to show that if $G \cong_t H$, then whenever $\alpha_t(G) \geq c$, we also have $\alpha_t(H) \geq c$. As $\alpha_t(G) \geq c$, we have $K_c \xrightarrow{t} \overline{G}$. The C*-algebra of this synchronous game is a C*-algebra $\mathcal{A}$, with a tracial state $\tau_{\mathcal{A}}$ on A, and projections $e_{i,v} \in \mathcal{A}$ for $i = 1, \ldots, c$ and $v \in V(G)$ such that $\sum_v e_{i,v} = 1$ for all $i = 1, \ldots, c$ and $\tau(e_{i,v}e_{j,w}) = 0$ whenever $(v, w) \in E(G)$. If $t = q$, we may assume $\tau_{\mathcal{A}}$ factors through a finite dimensional algebra and if $t = qa$, we may assume $\tau_{\mathcal{A}}$ is amenable.

Similarly, since $G \cong_t H$, there is a C*-algebra $\mathcal{B}$, a tracial state $\tau_{\mathcal{B}}$ on $\mathcal{B}$, and projections $q_{v,w} \in \mathcal{B}$ for $v, w \in V$ such that $\sum_{w \in V} q_{v,w} = 1$ for all $v \in V$ and such that if $v, w \in V(G)$ and $x, y \in V(H)$ with $\mathrm{rel}(v, w) \neq \mathrm{rel}(x, y)$ then $\tau_B(q_{vx}q_{wy}) = 0$. (Note that there are other relations in the graph isomorphism game; these are the only ones we will need to use here.) Again we choose $\tau_{\mathcal{B}}$ to factor through a finite dimensional algebra if $t = q$ and we choose $\tau_{\mathcal{B}}$ to be amenable if $t = qa$.

For $i = 1, \ldots, c$ and $x \in V(H)$, define

$$f_{i,x} = \sum_{v \in V(G)} e_{i,v} \otimes q_{v,x} \in \mathcal{A} \otimes \mathcal{B}.$$

Then each $f_{i,x}$ is a projection and for all $i = 1, \ldots, c$, we have $\sum_x f_{i,x} = 1$. If $x, y \in V(H)$ and $(x, y) \in E(H)$, then

$$\tau_{\mathcal{A}} \otimes \tau_{\mathcal{B}}(f_{i,x}f_{j,y}) = \sum_{v,w \in V(G)} \tau_{\mathcal{A}}(e_{i,v}e_{j,w})\tau_{\mathcal{B}}(q_{v,x}q_{w,y}).$$

For $v, w \in V(G)$, if $(v, w) \in E(G)$, then $\tau_{\mathcal{A}}(e_{i,v}e_{j,w}) = 0$, and if $(v, w) \notin E(G)$, then $\tau_{\mathcal{B}}(f_{v,x}f_{w,y}) = 0$. Hence the projections $f_{i,x} \in \mathcal{A} \otimes \mathcal{B}$ and the trace $\tau_{\mathcal{A}} \otimes \tau_{\mathcal{B}}$ determine a perfect qc-strategy for the graph homomorphism game from K_c to $\overline{H}$. If τ_A and τ_B factor through finite dimensional algebras, so does $\tau_A \otimes \tau_B$. If τ_A and τ_B are amenable, so is $\tau_A \otimes \tau_B$. Hence in all cases, $\alpha_t(H) \geq c$. $\square$

It is shown in [62, Theorem 3.7] that for $t \in \{q, qa, qc\}$ and graphs G, H and K, if $G \xrightarrow{t} H$ and $H \xrightarrow{t} K$ then $G \xrightarrow{t} K$. This leads to the following:

Proposition 6.3.3. *If $t \in \{q, qa, qc\}$ and G is a finite graph, then $\alpha_t(G) \leq \chi_t(\overline{G})$.*

Proof. Suppose that $\alpha_t(G) = c$. By definition, there is a t-homomorphism $K_c \xrightarrow{t} \overline{G}$. If $\chi_t(\overline{G}) = d$ then there is a t-homomorphism, $\overline{G} \xrightarrow{t} K_d$. Since qa-homomorphisms are closed under composition, there is a t-homomorphism $K_c \xrightarrow{t} K_d$ which implies that $\chi_t(K_c) \leq d$. As noted in the preliminaries, $\chi_t(K_c) = c$ and hence $c \leq d$ as claimed. □

In the case $t = q$, the following result appears as Theorem 6.2 in [6]. Since the publication of this result, an error has been found the initial publication of this Theorem by Adina Goldberg and it is rectified and generalized in [12, 32]. The following proof is an amendment with ideas coming from their correction.

Theorem 6.3.4. *Suppose $t \in \{q, qa, qc\}$ and let $Ax = b$ be an $m \times n$ linear system. The following are equivalent:*

1. *the game* synBCS(A, b) *has a winning t-strategy;*
2. *$G_{A,b} \cong_t G_{A,0}$;*
3. *$\alpha_t(G_{A,b}) = m$.*

Proof. (1) ⇒ (2): Suppose that we have a winning t-strategy for the synBCS(A, b). Fix a C*-algebra $\mathcal{B}$, a faithful trace $\tau \in \mathcal{B}$, and projections $e_{i,x} \in \mathcal{B}$ for $i = 1, \ldots, m$ and $x \in \{\pm 1\}^n$ such that $\sum_x e_{i,x} = 1$ for all i, $e_{i,x} = 0$ if $x \notin S_i$, and $e_{i,x}e_{j,y} = 0$ if there is a $k \in V_i \cap V_j$ with $x_k \neq y_k$. If $t = q$, we assume $\mathcal{B}$ is finite dimensional and if $t = qa$, we assume $\mathcal{B} \subseteq \mathcal{R}^{\mathcal{U}}$. Let $\mathcal{G}$ be the isomorphism game for $(G_{A,b}, G_{A,0})$ and let $\mathcal{A}(\mathcal{G})$ denote the algebra associated to $\mathcal{G}$ as defined in the preliminaries. It suffices to construct a unital *-homomorphism $\pi : \mathcal{A}(\mathcal{G}) \to \mathcal{B}$.

Let $S_i^0 \subseteq \{\pm 1\}^n$ denote the set of solutions to the ith equation of the linear system $Ax = 0$ and let $S_i^1 \subseteq \{\pm 1\}^n$ denote the set of solutions to the ith equation of the linear system $Ax = b$. Given $x, y \in \{\pm 1\}^n$, let $xy \in \{\pm 1\}^n$ denote the pointwise product of x and y. Note that if $x \in S_i^1$ and $y \in S_i^0$, then $xy \in S_i^1$. Moreover, for $x \in S_i^1$, the map $S_i^0 \to S_i^1$ given by $y \mapsto xy$ is a bijection.

For $(i,x) \in V(G_{A,b})$ and $(j,y) \in V(G_{A,0})$, define

$$q_{(i,x),(j,y)} = \begin{cases} e_{i,xy} & i = j \\ 0 & i \neq j \end{cases}$$

and note that each $q_{(i,x),(j,y)}$ is a projection. For $(i,x) \in V(G_{A,b})$, we have

$$\sum_{(j,y)\in V(G_{A,0})} q_{(i,x),(j,y)} = \sum_{j=1}^{n} \sum_{y \in S_j^0} q_{(i,x),(j,y)} = \sum_{y\in S_i^0} e_{i,xy} = \sum_{z \in S_i^1} e_{i,z} = 1.$$

A similar computation shows that for all $(j,y) \in V(G_{A,0})$, we have

$$\sum_{(i,x)\in V(G_{A,b})} q_{(i,x),(j,y)} = 1.$$

We need to show that for all $(i,x), (i',x') \in V(G_{A,b})$ and $(j,y), (j',y') \in V(G_{A,0})$, the implication

$$q_{(i,x),(j,y)} q_{(i',x'),(j',y')} \neq 0 \quad \Rightarrow \quad \mathrm{rel}((i,x),(i',x')) = \mathrm{rel}((j,y),(j',y'))$$

holds. To this end, suppose $q_{(i,x),(j,y)} q_{(i',x'),(j',y')} \neq 0$. Then $i = j$, $i' = j'$, and $e_{i,xy} e_{i',x'y'} \neq 0$. We consider several cases.

Suppose first $i = i'$. Then we have $xy = x'y'$. If $x = x'$, then $y = y'$ and we have both $(i,x) = (i',x')$ and $(j,y) = (j',y')$ so the right hand side of the implication holds in the case. Conversely, if $x \neq x'$ and $y \neq y'$, then $(i,x) \neq (i',x')$ and $(j,y) \neq (j',y')$. Note also that since $i = i'$, x and x' are necessarily inconsistent solutions so that (i,x) and (i',x') are adjacent. Similar reasoning shows (j,y) and (j',y') are adjacent. Hence the right hand side of the implication holds.

Now assume $i \neq i'$ so that, in particular, $(i,x) \neq (i',x')$. If (i,x) and (i',x') are adjacent, there is a $k \in V_i \cap V_{i'}$ such that $x_k \neq x'_k$. On the other hand, as $e_{i,xy} e_{i',x'y'} \neq 0$, we know $x_k y_k = (xy)_k = (x'y')_k = x'_k y'_k$. Therefore, $y_k \neq y'_k$ so that (i,y) and (i',y') are adjacent. Finally, suppose (i,x) and (i',x') are not adjacent. Then $x_k = x'_k$ for all $i \in V_i \cap V_{i'}$. Again since $e_{i,xy} e_{i',x'y'} \neq 0$, we also know $x_k y_k = x'_k y'_k$ for all $k \in V_i \cap V_{i'}$ and therefore $y_k = y'_k$ for all $k \in V_i \cap V_{i'}$ so that (j,y) and (j',y') are not adjacent. This covers all cases.

Now, the projections $q_{(i,x),(j,y)} \in \mathcal{B}$ define a unital *-representation $\pi : \mathcal{A}(\mathcal{G}) \to \mathcal{B}$ and it follows that $G_{A,b} \cong_t G_{A,0}$.

(2) $\Rightarrow$ (3): Suppose that $\overline{G_{A,b}} \cong_t \overline{G_{A,0}}$. By Lemma 6.3.2, it suffices to show that $\alpha_t(G_{A,0}) = m$. The map $f : \overline{G_{A,0}} \to \{1, \dots, m\} : (i,x) \mapsto i$ is an m-colouring of $\overline{G_{A,0}}$. Indeed, suppose are (i,x) and (j,y) are distinct vertices in $\overline{G_{A,0}}$ with $f(i,x) = f(j,y)$. Then $i = j$ and hence $x \neq y$. That is, there is some $k \in V_i$ such that $x_k \neq y_k$ and thus there is no edge between (i,x) and (j,y) in $\overline{G_{A,0}}$.

For each $i = 1, \dots, m$, the vector $x_0 = (1, \dots, 1)$ is in $S_i \subseteq \{\pm 1\}^n$ for the system $Ax = 0$. Hence for $i, j = 1, \dots, m$, there is no edge between the vertices (i, x_0) and (j, x_0) in $\overline{G_{A,0}}$ and we have $\alpha(G_{A,0}) \geq m$. Now by the previous proposition,

$$m \geq \chi(\overline{G_{A,0}}) \geq \chi_t(\overline{G_{A,0}}) \geq \alpha_t(G_{A,0}) \geq \alpha(G_{A,0}) \geq m,$$

and $\alpha_t(G_{A,0}) = m$.

(3) $\Rightarrow$ (1): Suppose $\alpha_t(G_{A,b}) = m$. Then the graph homomorphism game from K_m to $\overline{G_{A,b}}$ has a perfect t-strategy. Fix a C*-algebra $\mathcal{A}$ with a faithful trace τ and projections $e_{i,k,x} \in \mathcal{A}$ for $i = 1, \dots, m$, $(k,x) \in V(G_{A,b})$ such that

1. $\displaystyle\sum_{k=1}^{m} \sum_{x \in S_k} e_{i,k,x} = 1$ for all $1 \leq i \leq m$, and
2. $\tau(e_{i,k,x} e_{\ell,j,y}) = 0$ if there is an edge between (k,x) and (j,y) in $G_{A,b}$.

If $t = q$, we may assume $\mathcal{A}$ is finite dimensional and if $t = qa$, we may assume $\mathcal{A} = \mathcal{R}^{\mathcal{U}}$.

Define for $i = 1, \dots, m$ and $j \in V_i$,

$$v_{i,j} = \sum_{k=1}^{m} \sum_{x \in S_i} x_j e_{k,i,x} \ .$$

The $v_{i,j}$ are self adjoint since it is a $\mathbb{R}$-linear combination of projections. Next we show that for all $i = 1, \dots m$ and for all $j, k \in V_i$, $v_{i,j}$ commutes with $v_{i,k}$. An expansion of the product gives us

$$\begin{aligned} v_{i,j} v_{i,k} &= \left(\sum_{s=1}^{m} \sum_{x \in S_i} x_j e_{s,i,x} \right) \left(\sum_{t=1}^{m} \sum_{y \in S_i} y_k e_{t,i,y} \right) \\ &= \sum_{s,t} \sum_{x,y} x_j y_k e_{s,i,x} e_{t,i,y} \ . \end{aligned}$$

Using the fact that τ is tracial, whenever $x \neq y$, $\tau((e_{s,i,x}e_{t,i,y})^*(e_{s,i,x}e_{t,i,y})) = \tau(e_{s,i,x}e_{t,i,y}) = 0$. By faithfulness of τ we must have $e_{s,i,x}e_{t,i,y} = 0$. By the same reasoning, when $s \neq t$, we must have $e_{s,i,x}e_{t,i,x} = 0$ since we have a synchronous game. All this lets us conclude that

$$v_{i,j}v_{i,k} = \sum_s \sum_x x_j x_k e_{s,i,x} = v_{i,k}v_{i,j}$$

and hence the operators $v_{i,j}$ and $v_{i,k}$ commute. Setting $j = k$ gives us

$$v_{i,j}^2 = \sum_s \sum_x x_k^2 e_{s,i,x} = \sum_s \sum_x e_{s,i,x} \ .$$

Since each $e_{s,i,x}$ are pairwise orthogonal, $v_{i,j}^2$ is a projection. Beyond this,

$$\begin{aligned}\sum_{i=1}^m v_{i,j}^2 &= \sum_{i,s} \sum_x e_{s,i,x} = \sum_s \left(\sum_i \sum_{x \in S_i} e_{s,i,x} \right) \\ &= m \cdot 1 \ .\end{aligned}$$

In particular, we must have $v_{i,j}^2 = 1$ for all i, j.

Similarly, for all i, k and $j \in V_i \cap V_k$, from the above analysis it follows that

$$\begin{aligned}\tau(v_{i,j}v_{k,j}) &= \sum_{p,q=1}^m \sum_{x \in S_i, y \in S_k} x_j y_j \tau(e_{p,i,x}e_{q,i,y}) \\ &= \sum_p \sum_{x \in S_i} \tau(e_{p,i,x}) = \tau(v_{i,j}^2) = 1 \ .\end{aligned}$$

Hence we have

$$\tau((v_{i,j} - v_{k,j})^*(v_{i,j} - v_{k,j})) = 2 - 2\tau(v_{i,j}v_{k,j}) = 0 \ .$$

From this it follows that $v_{i,j} = v_{k,j}$ as τ is faithful.

Given $j = 1, \ldots, n$, define $w_j = v_{i,j}$ if $j \in V_i$ for some $i = 1, \ldots, m$ and $w_j = 1$ otherwise. By the previous paragraph, since the choice of $v_{i,j}$ is independent of the given i, the operator w_j is well-defined. If $1 \leq j, k \leq n$ and there is an $i = 1, \ldots, m$ with $j, k \in V_i$, then $w_j = v_{i,j}$ and $w_k = v_{i,k}$ commute. Moreover, for each $i = 1, \ldots, m$,

$$\prod_{j \in V_i} w_j = \prod_{j \in V_i} v_{i,j} = \sum_{k=1}^m \sum_{x \in S_i} \prod_{j \in V_i} x_j e_{k,i,x} = (-1)^{b_i}.$$

Hence there is a representation $\rho : \Gamma(A, b) \to U(\mathcal{A})$ such that $\rho(u_i) = w_i$ and $\rho(J) = -1$ for all $i = 1, \ldots, n$. By Corollary 6.2.4, the game synBCS(A, b) has a perfect t-strategy which proves (1). $\square$

By Corollary 6.2.4 and Theorem 6.2.5 applied to the above Theorem, we get the following two Corollaries.

Corollary 6.3.5. *There exists a graph G for which $\alpha_{qa}(G) > \alpha_q(G)$.*

Corollary 6.3.6. *There exist graphs G and H for which $G \cong_{qa} H$ but $G \ncong_q H$.*

Chapter 7

Chromatic numbers for quantum graphs

Given a graph on n vertices one can associate two different subspaces of the $n \times n$ matrices that encode all of the information of the graph. This has motivated the generalization of several well known graph theoretic concepts to a larger class of objects.

In [23], Duan, Severini, and Winter describe a version of non-commutative graph theory whose underlying objects consist of *submatricial operator systems*. The aforementioned authors generalize the independence number and Lovász theta number to submatricial operator systems.

In [75], Stahlke works with a similar but distinct definition of a non-commutative graph. Instead of working with submatricial operator systems, Stahlke associates a subspace of matrices whose elements all have zero trace to a graph. Stahlke generalizes several classical graph theory concepts to these traceless subspaces including the chromatic number, clique number and notion of graph homomorphism.

Thus, there are two quite different subspaces of matrices to associate to graphs that lead to two different ways to create a non-commutative graph theory. In this chapter we discuss both the *submatricial operator system* and *submatricial traceless self-adjoint operator space* definitions of a non-commutative graph.

There is currently no notion of the complement of a non-commutative graph that generalizes the graph complement. By working with both of the above definitions we are able to generalize the complement of a graph using the orthogonal complement with respect to the Hilbert-Schmidt inner product. We conclude this section by reviewing the definition of several non-commutative graph parameters and show that some of these parameters can be approximated by evaluating classical graph parameters.

In [59] Lovász introduced his well known theta number of a graph, $\theta(G)$. Lovász shows that this number determines the following bounds on the independence number, $\alpha(G)$, and the chromatic number of the graph complement $\chi(\overline{G})$.

$$\alpha(G) \leq \theta(G) \leq \chi(\overline{G}).$$

These two inequalities are often referred to as the Lovász sandwich theorem. In [23], it is shown that that the independence number of a submatricial operator system is bounded above by its Lovász number. This provides the first inequality for a generalized Lovász sandwich theorem. In [75] Stahlke introduces a version of the chromactic number denoted χ_{St}, that generalizes the second inequality.

In section 7.2 we introduce new generalizations of the chromatic number, χ_0 and $\widehat{\chi}$, that provide lower and upper bounds on χ_{St}. Using $\widehat{\chi}$ we provide a simplified proof of a weaker sandwich inequality. The advantage is that we can answer a question posed by Stahlke by generalizing the equation $\chi(G)\omega(\overline{G}) \geq n$ to non-commutative graphs.

Given two graphs G and H the Cartesian product is the graph $G\Box H$ with vertex set $V(G) \times V(H)$ and edge relation given by $(v,a) \sim (w,b)$ if one of $v \sim_G w$ and $a = b$ or $v = w$ and $a \sim_H b$ holds. A Theorem of Sabidussi tell us $\chi(G\Box H) = \max\{\chi(G), \chi(H)\}$ for any G and H. We introduce a Cartesian product and establish a generalization of this result for *submatricial traceless self-adjoint operator spaces* in Section 7.3 . In section 7.3 we also establish a categorical product for submatricial traceless self-adjoint operator space and extend a Theorem of Hedetniemi to submatricial traceless self-adjoint operator spaces.

7.1 Non-commutative graphs

A non-commutative graph is sometimes viewed as any submatricial operator system S. Non-commutative graphs have also been described as any submatricial traceless self-adjoint operator space $\mathcal{J}$. In this section we review how one can view a classical graph as either of these objects without losing information about the graph itself. We also discuss several parameters for non-commutative graphs.

7.1.1 Non-commutative graphs as operator systems

Definition 7.1.1. Let $G = (V, E)$ be a graph with vertex set $[n]$. Define $S_G \subseteq M_n$ by

$$S_G := \text{span}\{E_{i,j} : (i, j) \in E \text{ or } i = j\}.$$

Observe that for any graph G, S_G will be a submatricial operator system. In [62], it is shown that graphs G and H are isomorphic if and only if S_G and S_H are isomorphic in the category of operator systems. We discuss this in more detail in 7.1.2.

Given a graph G, if vertices i, j are not adjacent, then $e_i e_j^* = E_{i,j}$ is orthogonal to the submatricial operator system S_G in the sense that for all $X \in S_G$, the Hilbert-Schmidt inner product $\langle E_{i,j}, X\rangle := \text{tr}(E_{i,j} \cdot X^*)$ is zero. Similarly if $\{i_1, \dots, i_k\}$ is an independent set of vertices in G then for any $j \neq k$ we have $e_{i_j} e_{i_k}^*$ is orthogonal to S_G. If $v = (v_1, \dots, v_k)$ is an orthonormal collection of vectors in $\mathbb{C}^n$ then v called an *independent set* for a submatricial operator system $S \subseteq M_n$ if for any $i \neq j$, $v_i v_j^*$ is orthogonal to S.

Definition 7.1.2. Let S be a submatricial operator system. We define the *independence number*, $\alpha(S)$, to be the largest integer k such that there exists an independent set for S of size k.

A graph $G = (V, E)$ has a k-colouring if and only if there exists a partition of V into k independent sets. In [65] Paulsen defines a natural generalization of the chromatic number to non-commutative graphs. We say a submatricial operator system $S \subseteq M_n$ has *k-colouring* if there exists an orthonormal basis for $\mathbb{C}^n$, $v = (v_1, \dots, v_n)$, such that v can be partitioned into k independent sets for S.

Definition 7.1.3. Let $S \subseteq M_n$ be a submatricial operator system. The *chromatic number*, $\chi(S)$, is the least $k \in \mathbb{N}$ such that S has a k-colouring.

For any submatricial operator system $S \subseteq M_n$ we have $\chi(S) \leq n$ since you can partition any basis of $\mathbb{C}^n$ into n independent sets. In Theorem 7.1.14 we show that both of the above parameters provide a generalization of the classical graph theory parameters, that is we show $\alpha(S_G) = \alpha(G)$ and $\chi(S_G) = \chi(G)$. This first equality is originally found in [23] and the second can be found in [65].

Example 7.1.4. Consider the submatricial operator system $S := \text{span}\{I, E_{i,j} : i \neq j\} \subseteq M_n$. Let u_1, u_2 be two orthonormal vectors and let i be an element of the support of u_1. Since $u_1^* u_2 = 0$ there must be an element $j \neq i$ of the support of u_2. Then $\langle u_1 u_2^*, E_{i,j}\rangle = u_1(i)\overline{u_2(j)} \neq 0$. Thus we see that $\alpha(S) = 1$. This also tell us that $\chi(S) = n$.

As in [23], given a graph G one can compute the Lovász theta number $\vartheta(G)$ as,

$$\vartheta(G) = \max\{\|I+T\| : I+T \geq 0,\, T_{i,j} = 0 \text{ for } i \sim j\}.$$

Here the supremum is taken over all $n \times n$ matrices and $I+T \geq 0$ indicates that $I+T$ is positive semidefinite.

The following inequality is due to Lovász. For a good self-contained review please see [58].

Theorem 7.1.5. *Let G be a graph and $\overline{G}$ be the graph complement of G. Then,*

$$\alpha(G) \leq \vartheta(G) \leq \chi(\overline{G}).$$

In order to obtain an generalization of 7.1.5 we need to identify the the appropriate generalization of a graph complement. Given a submatricial operator system $S \subseteq M_n$ we use the orthogonal complement $S^\perp$ to generalize the graph complement. Note that the orthogonal complement of a submatricial operator system is no longer a submatricial operator system since it will fail to contain the identity operator. In fact since $I \in S$ we will have $\mathrm{tr}(A) = \langle A, I\rangle = 0$ for every A element of $S^\perp$. In [75], Stahlke works with precisely these objects. We show that it is useful to consider both submatricial operator systems and submatricial traceless self-adjoint operator spaces to generalize the graph complement.

7.1.2 The complement of a non-commutative graph

In this section, we introduce the analogue of the notion of a graph complement for non-commutative graphs. Using this, we define a notion of clique number, independence number, and chromatic number.

Definition 7.1.6. Let G be a finite graph with vertex set $[n]$. The *traceless self-adjoint operator space associated to G* is the linear space

$$\mathcal{J}_G := \mathrm{span}\{E_{i,j} : i \sim j\} \subseteq M_n\ .$$

A *traceless non-commutative graph* is any submatricial traceless self-adjoint operator space.

Remark 7.1.7. The traceless self-adjoint operator space $\mathcal{J}_G$ is the traceless non-commutative graph S_G given in [75]. Given a finite graph G with vertex set $[n]$, we have the identity $\mathcal{J}_G^\perp = S_{\overline{G}}$. This identity in particular suggests that the graph complement of a non-commutative graph should be its orthogonal complement. In [75], Stalhke suggests that the graph complement of $\mathcal{J}_G$ should be $(\mathcal{J}_G + \mathbb{C}I)^\perp$. However, this notion of complement would mean that $\mathcal{J}_{\overline{G}} \neq (\mathcal{J}_G + \mathbb{C}I)^\perp$ for any graph with at least two vertices. We shall see that, so long as one is willing to pay the price of working with two different notions of a non-commutative graph, the orthogonal complement is the correct analogue of the graph complement.

Proposition 7.1.8. *The traceless self-adjoint operator subspaces of M_n are exactly the orthogonal complements of submatricial operator systems. That is, S is a submatricial operator system if and only if $S^\perp$ is a traceless self-adjoint operator space.*

Proof. If S is an operator subsystem of M_n then for any $X \in S^\perp$, $tr(X) = \langle X, I \rangle = 0$. As well, if $X \in S^\perp$, for any $Y \in S$, $\operatorname{tr}(XY) = \overline{\operatorname{tr}(Y^*X^*)} = 0$. This proves that $S^\perp$ is a traceless self-adjoint operator space. Conversely, if S is a traceless self-adjoint operator space, then $S^\perp$ contains I since for all $X \in S$, $\langle X, I \rangle = \operatorname{tr}(X^*I) = 0$. If $X \in S^\perp$ then $\langle X^*, Y \rangle = \operatorname{tr}(XY) = \overline{\operatorname{tr}(Y^*X^*)} = 0$. Therefore, $X^* \in S^\perp$. This proves that $S^\perp$ is an operator system. $\square$

Proposition 7.1.9. *If G is a graph with vertex set $[n]$ then $S_G^\perp = \mathcal{J}_{\overline{G}}$.*

Proof. Observe that for $i, j, k, l \in [n]$, $E_{ij} \in S_G^\perp$ if and only if for all $k \simeq_G l$, $\operatorname{tr}(E_{ij}E_{kl}) = 0$. This is only possible if $i \sim_{\overline{G}} j$. $\square$

It is a result of Paulsen and Ortiz [62, Proposition 3.1] that two graphs G and H of the same vertex set $[n]$ are isomorphic if and only if there is a $n \times n$ unitary matrix U for which $US_GU^* = S_H$.

Corollary 7.1.10. *Suppose that G and H are graphs with vertex set $[n]$. The graphs G and H are isomorphic if and only if there is an $n \times n$ unitary matrix U such that $U\mathcal{J}_GU^* = \mathcal{J}_H$.*

Proof. For any $n \times n$ unitary matrix U, $(US_GU^*)^\perp = U\mathcal{J}_{\overline{G}}U^*$. Since G and H are isomorphic if and only if their graph complement is, the result follows. $\square$

Remark 7.1.11. In [78] a *quantum graph* is defined as a reflexive, symmetric quantum relation on a $*$-subalgebra $\mathcal{M} \subseteq M_n$. In this framework a submatricial operator system S is indeed quantum graph when taking $\mathcal{M} = M_n$. This approach fails to provide a complement for a quantum graph since $S^\perp$ will fail to be a reflexive quantum relation on any $\mathcal{M} \subseteq M_n$ and hence will not be a quantum graph.

The notion of an independent set for an submatricial operator system was described solely in terms of an orthogonality relation. We can similarly say that an orthonormal collection of vectors $v = (v_1, \ldots, v_n)$ in $\mathbb{C}^n$ is an *independent set* for a submatricial traceless self-adjoint operator space $\mathcal{J} \subseteq M_n$ if for any $i \neq j$, $v_i v_j^*$ is orthogonal to $\mathcal{J}$. We say $\mathcal{J}$ has a k-coloring if there exists an orthonormal basis $v = (v_1, \ldots, v_n)$ of $\mathbb{C}^n$, that can be partitioned into k independent sets for $\mathcal{J}$.

Definition 7.1.12. Let $\mathcal{J} \subseteq M_n$ be a submatricial traceless self-adjoint operator space.

1. The independence number, $\alpha(\mathcal{J})$, is the largest $k \in \mathbb{N}$ such that there exists an independent set of size k for $\mathcal{J}$.
2. The chromatic number $\chi(\mathcal{J})$ is the least integer k such that $\mathcal{J}$ has k-colouring.

It is not hard to show that χ is monotonic and α is reverse monotonic under inclusion. This holds when considering these as parameters on submatricial operator systems as well as submatricial traceless self-adjoint operator spaces.

Next we show that if G is a graph, S_G and $\mathcal{J}_G$ have the same independence number and chromatic number. We start with a lemma. The following proof is in [65, Lemma 7.28]:

Lemma 7.1.13. *Let $v_1, \ldots, v_n$ be a basis for $\mathbb{C}^n$. There exists a permutation σ on $[n]$ so that for each i, the $\sigma(i)$th component of v_i is non-zero.*

Proof. Let $A = [a_{i,j}]$ denote the matrix with column i equal to v_i. Since we have a basis, $\det(A) \neq 0$. But

$$\det(A) = \sum_{\sigma \in Sym([n])} sgn(\sigma) a_{1,\sigma(1)} \cdots a_{n,\sigma(n)} .$$

There must therefore be some σ for which the product $a_{1,\sigma(1)} \cdots a_{n,\sigma(n)}$ is non-zero. This permutation works. □

It has been shown that $\alpha(G) = \alpha(S_G)$ and $\chi(G) = \chi(S_G)$ in [23] and [65] respectively. We are able to obtain the analogous results for submatricial self-adjoint operator spaces.

Theorem 7.1.14. *Let G be a graph on n vertices, we have $\alpha(G) = \alpha(S_G) = \alpha(\mathcal{J}_G)$ and $\chi(G) = \chi(S_G) = \chi(\mathcal{J}_G)$.*

Proof. The inclusion $\alpha(S_G) \leq \alpha(\mathcal{J}_G)$ follow from reverse monotonicity. If $i_1, \ldots, i_k$ are an independent set of vertices in the graph G then we have that the standard vectors $e_{i_1}, \ldots, e_{i_k}$ is an independent set for S_G so we get

$$\alpha(G) \leq \alpha(S_G) \leq \alpha(\mathcal{J}_G) .$$

Next suppose that $v_1, \ldots, v_k$ are an independent set for $\mathcal{J}_G$. Then since $v_1, \ldots, v_k$ is an linearly independent set of vectors we can find a permutaiton σ on $[n]$ so that $\langle v_i, e_{\sigma(i)} \rangle$ is non-zero for all i.

We note that if vertices $\sigma(j)$ and $\sigma(k)$ are adjacent in G then we have $E_{\sigma(j),\sigma(k)} \in \mathcal{J}_G$. But then $\langle v_j v_k^*, E_{\sigma(j),\sigma(k)} \rangle = \langle v_j, e_\sigma(j) \rangle \langle e_{\sigma(k)}, v_k \rangle \neq 0$ a contradiction. Thus $\sigma(1), \ldots, \sigma(k)$ are an independent set for the graph G so $\alpha(\mathcal{J}_G) \leq \alpha(G)$. The proof for χ follows the same argument. $\square$

Recall for a classical graph G the clique number, $\omega(G)$, satisfies that $\omega(G) = \alpha(\overline{G})$.

Definition 7.1.15. Let S be a submatricial operator system and let $\mathcal{J}$ be a submatricial traceless self-adjoint operator space.

1. Define the clique number, $\omega(S)$, to be the independence number of the submatricial traceless self-adjoint operator space $S^\perp$.
2. Define the clique number, $\omega(\mathcal{J})$, to be the independence number of the submatricial operator system $\mathcal{J}^\perp$.

It should be noted that the above definition $\omega(\mathcal{J})$ of a traceless submatricial operator space is first mentioned in [75]. We can use Theorem 7.1.14 to conclude that for any graph G we have $\omega(G) = \omega(S_G) = \omega(\mathcal{J}_G)$.

The next proposition shows that α, ω, and χ may be computed purely from the associated parameters for graphs. We can achieve this by associating a family of graphs to each submatricial traceless self-adjoint operator space or submatricial operator system.

Definition 7.1.16. Given a submatricial operator system $S \subseteq M_n$ and an orthonormal basis $v = (v_1, \ldots, v_n)$ we can construct two different graphs.

1. *The confusability graph* of v, with respect to S, denoted $H_v(S)$, is the graph on n vertices with $i \sim j$ if and only if $v_i v_j^* \in S$.

2. *The distinguishability graph* of v, with respect to S, denoted $G_v(S)$ is the graph on n vertices with $i \sim j$ if and only if $v_i v_j^* \perp S$.

We can also define the confusability and distinguishability graphs of an orthonormal basis $v = (v_1, \ldots, v_n)$ with respect to submatricial traceless self-adjoint operator spaces $\mathcal{J} \subseteq M_n$ in the same way. We would then have for $S = \mathcal{J}^\perp$, $G_v(S) = H_v(\mathcal{J})$ and $H_v(S) = G_v(\mathcal{J})$. When it is clear what the underlying system or submatricial traceless self-adjoint operator space is we simply write G_v and H_v.

Theorem 7.1.17. *Let $\mathcal{J}$ be a submatricial traceless self-adjoint operator space in M_n and let $\mathcal{B}$ denote the set of ordered orthonormal bases for $\mathbb{C}^n$. We have the identities*

$$\alpha(\mathcal{J}) = \sup_{v \in \mathcal{B}} \alpha(\overline{G_v}) \ ,$$
$$\chi(\mathcal{J}) = \inf_{v \in \mathcal{B}} \chi(\overline{G_v}) \ , \ \textit{and}$$
$$\omega(\mathcal{J}) = \sup_{v \in \mathcal{B}} \omega(H_v) \ .$$

The same identity holds if we replace $\mathcal{J}$ with a submatricial operator system in M_n.

Proof. Suppose $v_1, \ldots, v_c$ is a maximal independent set for $\mathcal{J}$, that is for $i \neq j$ we have $v_i v_j^* \perp \mathcal{J}$. We can extend this collection to an orthonormal basis $v = (v_1, \ldots, v_c, v_{c+1}, \ldots v_n)$. Note that the vertices $1, \ldots, c$ in the graph $\overline{G_v}$ are an independence set since for distinct $i, j \in [c]$ we have $v_i v_j^* \perp \mathcal{J}$. This gives $i \sim j$ in G_v. Thus there is no edge between i and j in $\overline{G_v}$. Therefore $\alpha(\overline{G_v}) \geq c$ so we have $\alpha(\mathcal{J}) \leq \sup_{v \in \mathcal{B}} \alpha(\overline{G_v})$. Conversely, for each $v \in \mathcal{B}$ if $i_1, \ldots, i_c$ are an independent set for $\overline{G_v}$ then $i_j \sim i_k$ in G_V. We then have $v_{i_1}, \ldots, v_{i_c}$ is an independent set for $\mathcal{J}$. This gives $\alpha(\mathcal{J}) \geq \alpha(\overline{G_v})$.

The proof of the second identity is similar. If $\chi(\mathcal{J}) = c$ then there exists orthonormal basis $v = (v_1, \ldots, v_n)$ and a partition $P_1, \ldots P_c$ of $[n]$ such that $v_i v_j^* \perp \mathcal{J}$ for distinct i and j in the same partition. Define a colouring f of $\overline{G_v}$ by having $f(i) = l$ if and only if $i \in P_l$. We see that for $i \neq j$ if we have $f(i) = f(j)$ then $v_i v_j^* \perp \mathcal{J}$ giving that $i \sim j$ in G_v so f is indeed a c colouring of $\overline{G_v}$. This gives $\chi(\mathcal{J}) \geq \inf_{v \in \mathcal{B}} \chi(\overline{G_v})$. Conversely, if f is any c colouring of $\overline{G_v}$ for some $v \in \mathcal{B}$ then we can obtain a c colouring of $\mathcal{J}$ by partitioning $[n]$ into sets $P_1, \ldots P_c$ where $i \in P_l$ if and only if $f(i) = l$. Then if distinct $i, j \in P_l$ we have $i \sim j$ in G_v so $v_i v_j^* \perp \mathcal{J}$.

Lastly, suppose $(v_1, \ldots, v_k)$ is a collection of orthonormal vectors such that for distinc i, j we have $v_i v_j^* \in \mathcal{J}$. We can extend this set to a orthonormal basis $v = (v_1, \ldots v_n)$

and we immediately get that the vertices $\{1, \ldots, k\}$ form a clique in H_v. For the other direction note for any basis $v = (v_1, \ldots v_n)$ H_v has a clique $i_1, \ldots, i_k$ then $v_1, \ldots v_k$ will satisfy $v_i v_j^* \in \mathcal{J}$ for distinct i and j. $\square$

We next extend the definition of Lovász' theta function, [59], to non-commutative graphs. This was first extended to submatricial operator spaces in [23]. We introduce a natural extension to submatricial operator systems as well.

Definition 7.1.18. Let S be a submatricial operator system and $\mathcal{J}$ be a submatricial traceless self-adjoint operator space. Define the theta number of a submatricial operator system, $\vartheta(S)$ and the complementary theta number of a submatricial traceless self-adjoint operator space, $\overline{\vartheta}$ as follows.

1. $\vartheta(S) = \sup\{\|I + T\| : T \in M_n,\ I + T \geq 0,\ T \perp S\}$.
2. $\overline{\vartheta}(\mathcal{J}) = \sup\{\|I + T\| : T \in M_n,\ I + T \geq 0, T \in \mathcal{J}\}$.

Observe that $\vartheta(S_G) = \vartheta(G)$ and $\overline{\vartheta}(\mathcal{J}_G) = \vartheta(\overline{G})$ for all graphs G.

Example 7.1.19. Recall the previously mentioned submatricial operator system $S := \mathrm{span}\{I, E_{i,j} : i \neq j\} \subseteq M_n$. We see that $\vartheta(S) = n$ since we can take T to be the diagonal matrix with $n - 1$ for the $1, 1$ entry and -1 for all other diagonal entries. We also see that if $v = (e_1, \ldots, e_n)$ is the standard basis for C^n then v is a clqiue for S and thus we have $\chi(S^\perp) = 1$. This shows that using the definition of the chromatic number from [65] we can not hope to generalize the Lovász sandwich theorem.

7.2 Non-commutative Lovász inequality

We see by the previous example that one needs a different generalization of the chromatic number in order to obtain a Lovász sandwich Theorem for non-commutative graphs. Here we introduce the strong and minimal chromatic number of a submatricial operator system and provide a generalization on Lovász theorem.

7.2.1 The strong chromatic number

Let $\mathcal{J} \subseteq M_n$ be a submatricial traceless self-adjoint operator space. A collection of orthonormal vectors $v = (v_1, \ldots, v_k)$ in $\mathbb{C}^n$ is called a *strong independent set for $\mathcal{J}$* if for any i, j,we have $v_i v_j^*$ is orthogonal to $\mathcal{J}$. We say that $\mathcal{J}$ has a *strong k-colouring* if there exist an orthonormal basis $v = (v_1, \ldots, v_n)$ of $\mathbb{C}^n$ that can be partitioned into k strong independent sets for $\mathcal{J}$. We will show in Corollary 7.2.8, $\widehat{\chi}(\mathcal{J}_G)$ agrees with the chromatic number of G, for any graph G.

Definition 7.2.1. Let $\mathcal{J} \subseteq M_n$ be a submatricial traceless self-adjoint operator space. The *strong chromatic number*, $\widehat{\chi}(\mathcal{J})$, is the least $k \in \mathbb{N}$ such that $\mathcal{J}$ has a strong k-colouring. If $\mathcal{J}$ has no strong-k colouring then we say $\widehat{\chi}(\mathcal{J}) = \infty$.

As with χ we have $\widehat{\chi}$ is monotonic with respect to inclusion.

Example 7.2.2. Suppose that $S = \mathbb{C}1 + \text{span}\{E_{i,j} : i \neq j\} \subseteq M_n$ and ζ is a nth root of unity. Define $v_k = (1, \zeta^k, \zeta^{2k}, \ldots, \zeta^{(n-1)k})$. Observe that the v_i are orthogonal and that $v_k v_k^*$ belongs to S for all k. Thus $S^\perp$ does have a strong-n colouring and we get $\widehat{\chi}(S^\perp) \leq n$.

Example 7.2.3. Consider the submatricial traceless self-adjoint operator space $\mathcal{J} = \mathbb{C}\Delta \subseteq M_n$ where $\Delta = \text{diag}(n-1, -1, -1, \ldots, -1)$. Observe that $\mathcal{J} \subseteq S^\perp$. By monotonicity, $\widehat{\chi}(\mathcal{J}) \leq \widehat{\chi}(S^\perp) \leq n$. It is known that $\overline{\vartheta}(\mathcal{J}) = n$ (see [62, Remark 4.3]). We show in Theorem 7.2.9 that $\widehat{\chi}$ is bounded below by $\overline{\vartheta}$ Thus we have $\widehat{\chi}(\mathcal{J}) = \widehat{\chi}(S^\perp) = n$.

In [75] Stahlke introduces a different chromatic number for submatricial traceless self-adjoint operator spaces.

Definition 7.2.4. Let $\mathcal{J}$ and $\mathcal{K}$ be submatricial traceless self-adjoint operator spaces in M_n and M_m respectively. We say that there is a *graph homomorphism from $\mathcal{J}$ to $\mathcal{K}$*, denoted $\mathcal{J} \to \mathcal{K}$, if there is a completely positive and trace preserving map $\mathcal{E} : M_n \to M_m$ with associated Kraus operators $E_1, \ldots, E_r$ for which $E_i \mathcal{J} E_j^* \subseteq \mathcal{K}$ for any i and j.

Stalhke's chromatic number of a submatricial traceless self-adjoint operator space $\mathcal{J}$, denoted $\chi_{St}(\mathcal{J})$, is the least integer c for which there is a graph homomorphism $\mathcal{J} \to \mathcal{J}_{K_c}$ if one exists. We set $\chi_{St}(\mathcal{J}) = \infty$ otherwise.

Observe that χ_{St} is monotonic under graph homomorphism by construction.

Theorem 7.2.5. *For any submatricial traceless self-adjoint operator space $\mathcal{J} \subseteq M_n$ we have $\widehat{\chi}(\mathcal{J}) \geq \chi_{St}(\mathcal{J})$.*

Proof. Suppose $\widehat{\chi}(\mathcal{J}) = r$. There exists a orthonormal basis $v_1, \ldots, v_n$ that can be partitioned into strong independent sets $P_1, \ldots, P_r$. By reordering the vectors, we may assume that whenever $v_i \in P_\ell$ and $v_j \in P_{\ell+1}$, that $i < j$. By conjugating by the unitary $U : v_i \mapsto e_i$, we get the inclusion $\bigoplus_{i=1}^r M_{|P_i|} \subseteq U\mathcal{J}^\perp U^* = (U\mathcal{J}U^*)^\perp$.

This then gives us $(\bigoplus_{i=1}^r M_{|P_i|})^\perp \supset (U\mathcal{J}U^*)$. We have that $\mathcal{J} \to U\mathcal{J}U^*$ by conjugating by the unitary U. Similarly we have $U\mathcal{J}U^* \to (\oplus_{i=1}^r M_{|P_i|})^\perp$ by inclusion. Since χ_{St} is monotonic with respect to homomorphisms we get $\chi_{St}(\mathcal{J}) \leq \chi_{St}((\oplus_{i=1}^r M_{|P_i|})^\perp) = \chi(\overline{G}) = r$ where G is the disjoint union of r complete graphs. □

Corollary 7.2.6. *If $\mathcal{J} \subseteq M_n$ is a submatricial traceless self-adjoint operator space for which for some basis $v = (v_1, \dots, v_n)$, the diagonals $v_i v_i^*$ are orthogonal to $\mathcal{J}$, then $\chi_{St}(\mathcal{J}) \leq n$.*

In [66] it was shown that $\alpha(S) = \alpha(M_d(S))$ for all submatricial operator systems S. The proof they give will also work to show $\alpha(\mathcal{J}) = \alpha(M_d(\mathcal{J}))$ and $\widehat{\chi}(\mathcal{J}) = \widehat{\chi}(M_d(\mathcal{J}))$ for submatricial traceless self-adjoint operator spaces $\mathcal{J}$ and $d \in \mathbb{N}$.

Recall that for $d, n \geq 1$, the partial trace map is

$$M_d \otimes M_n \to M_n : X \otimes Y \mapsto \mathrm{tr}(X)Y \ .$$

As is the case with χ we can approximate $\widehat{\chi}$ using the chromatic number for classical graphs.

Theorem 7.2.7. *Let $\mathcal{J} \subseteq M_n$ be a submatricial traceless self-adjoint operator space. Suppose that $\mathcal{B}_{\mathcal{J}}$ denotes the set of ordered orthonormal bases $v = (v_1, \dots, v_n)$ of $\mathbb{C}^n$ for which $v_i v_i^* \perp \mathcal{J}$ for all i. For each $v = (v_1, \dots, v_n)$ in $\mathcal{B}_{\mathcal{J}}$, define the graph G_v with vertices $[n]$ and edge relation given by $i \sim j$ if $v_i v_j^*$ is orthogonal to $\mathcal{J}$. Then,*

$$\widehat{\chi}(\mathcal{J}) = \inf_{v \in \mathcal{B}} \chi(\overline{G_v}) \ ,$$

whenever $\widehat{\chi}(\mathcal{J})$ is finite.

Proof. The proof is exactly as in Theorem 7.1.17. □

Corollary 7.2.8. *For any finite graph G, $\widehat{\chi}(\mathcal{J}_G) = \chi(G)$.*

Proof. By Theorem 7.2.7, $\widehat{\chi}(\mathcal{J}_G) \leq \chi(\overline{G_v})$ where $v = (e_1, \dots, e_n)$. The complement of the graph G_v is the graph G. This gets us the bound $\widehat{\chi}(\mathcal{J}_G) \leq \chi(G)$. As well, by Theorem 7.2.5, $\chi(G) = \chi_{St}(\mathcal{J}_G) \leq \widehat{\chi}(\mathcal{J}_G)$. □

Using the strong chromatic number we are easily able to generalize other graph inequalities that for now remain unanswered for χ_{St}. In [75] Stahlke asks if for all submatricial traceless self-adjoint operator spaces $\mathcal{J} \subseteq M_n$, one can show $\chi_{St}(\mathcal{J})\omega(\mathcal{J}^c) \geq n$, where $\mathcal{J}^c$ is the proposed complement $\mathcal{J}^c = (\mathcal{J} + \mathbb{C}I)^{\perp}$. The question is motivated by the simple graph inequality $\chi(G)\omega(\overline{G}) \geq n$. Indeed for $\mathcal{J} \subseteq M_n$ a submatricial traceless self-adjoint operator space if we suppose $\widehat{\chi}(\mathcal{J}) = k$ then we can find an orthonormal basis $v = (v_1, \ldots, v_n)$ and a partition of v into independent sets $P_1, \ldots, P_k$. By definition of $\omega(\mathcal{J}^{\perp})$ we know that $|P_i| \leq \omega(\mathcal{J}^{\perp})$ for $i = 1, \ldots, k$. Thus we have $n = \sum_i |P_i| \leq \sum_i \omega(\mathcal{J}^{\perp}) = \widehat{\chi}(\mathcal{J})\omega(\mathcal{J}^{\perp})$.

Using [75], one can establish that $\overline{\vartheta}(\mathcal{J}) \leq \chi_{St}(\mathcal{J})$ for any submatricial traceless self-adjoint operator space $\mathcal{J} \subseteq M_n$: if $c = \chi_{St}(\mathcal{J})$, then there is a graph homomorphism $\mathcal{J} \to \mathcal{J}_{K_c}$. In [75, Theorem 19], it is shown that $\overline{\vartheta}_n$ is monotonic under graph homomorphisms. We therefore get the inequality

$$\overline{\vartheta}_n(\mathcal{J}) \leq \overline{\vartheta}_n(\mathcal{J}_{K_c}) = \vartheta(\overline{K_c}) \leq \chi(K_c) = c\ .$$

We can now establish a Lovász sandwich Theorem for $\widehat{\chi}$.

Theorem 7.2.9. *Let S be a submatricial operator system. For any $d \geq 1$, we have the inequalities*

$$\alpha(S) \leq \vartheta(S) \leq \widehat{\chi}(S^{\perp})\ .$$

Proof. The inequality $\alpha(S) \leq \vartheta(S)$ is a result in [23, Lemma 7] so we will only prove the other inequality. Let $v = (v_1, \ldots, v_n)$ be an orthonormal basis that can be partitioned into k strong independent sets for $S^{\perp}$. Then consider the graph $G_v(S^{\perp})$ as defined in Theorem 7.1.17. We have $\widehat{\chi}(S^{\perp}) = \chi(\overline{G_v})$. There exists a unitary $U \in M_n$ such that we get the the inclusion $S \supset US_{G_v}U^*$. Since ϑ is reverse monotonic under inclusion and invariant under conjugation by a unitary, we establish the inequalities

$$\vartheta(S) \leq \vartheta(S_{G_v}) = \vartheta(G_v) \leq \chi(\overline{G_v}) = \widehat{\chi}(S^{\perp}).$$

□

Similarly we get the follow inequality for any submatricial traceless self-adjoint operator space $\mathcal{J}$.

$$\alpha(\mathcal{J}^{\perp}) \leq \overline{\vartheta}(\mathcal{J}) \leq \widehat{\chi}(\mathcal{J}).$$

It should be pointed out that using $\chi_{St}(\mathcal{J}) \leq \widehat{\chi}(\mathcal{J})$, as shown in Theorem 7.2.5 , and the fact that $\omega(\mathcal{J}) = \alpha(\mathcal{J}^{\perp})$, one can obtain the the above inequality as a corollary of Corollary 20 in [75] . In this sense the above can be considered as a simplified proof of a weaker result.

7.2.2 The minimal chromatic number

In this section, we construct a concrete example of a homomorphism monotone chromatic number. Using this concrete definition, we are able to establish analogues of two classic identities for chromatic numbers under graph products: Sabidussi's Theorem and Hedetniemi's inequality.

Definition 7.2.10. Let $\mathcal{J} \subseteq M_n$ be a submatricial traceless self-adjoint operator space. Define the minimal chromatic number of $\mathcal{J}$, denoted $\chi_0(\mathcal{J})$, to be the least integer c for which there exists a basis $v_1, \ldots, v_n$ of $\mathbb{C}^n$ and a partition $P_1, \ldots, P_c$ of $[n]$ for which whenever $i, j \in P_s$, we have the relation $v_i v_j^* \perp \mathcal{J}$.

We note thatχ_0 differs from $\widehat{\chi}$ since we no longer require that we are working with an orthonormal basis. This parameter agrees with the chromatic number for graphs. We define the competely bounded version of this parameter by $\chi_{0,cb}(\mathcal{J}) = \inf_d \chi_0(M_d(\mathcal{J}))$.

Proposition 7.2.11. *Let G be a finite graph. We have the relation $\chi(G) = \chi_0(\mathcal{J}_G)$.*

Proof. Since $\chi_0(\mathcal{J}_G) \leq \chi(G)$, it suffices to show that $\chi(G) \leq \chi_0(\mathcal{J}_G)$. For this proof, let c be minimal and let $v_1, \ldots, v_n$ be a basis in $\mathbb{C}^n$ for which there is a partition $P_1, \ldots, P_c$ of $[n]$ such that whenever i, j in P_s, $v_i v_j^* \perp \mathcal{J}_G$. We then have a permutation σ of $[n]$ for which $\langle v_i, e_{\sigma(i)} \rangle$ is non-zero. By conjugating $\mathcal{J}_G$ by the permutation matrix defined by σ, assume that $\sigma(i) = i$ for all i. Define the c-colouring $f : V(G) \to [c]$ By $f(i) = s$ for s such that $i \in P_s$. To see that this is a colouring, suppose not. There are then $i \sim j$ for which $i, j \in P_s$ for some s. By definition then we have, $E_{i,j}$ belongs to $\mathcal{J}_G$. We observe then,

$$\langle v_i v_j^*, E_{i,j} \rangle = \operatorname{tr}(v_j v_i^* e_i e_j^*) = \langle v_i, e_i \rangle \langle v_j, e_j \rangle \neq 0 \, .$$

This is contradicts the fact that $v_i v_j^* \in \mathcal{J}^\perp$. ☐

We recall the following result, which arises as a consequence of the Stinespring dilation Theorem (see [75, Definition 7]).

Lemma 7.2.12. *Let $\mathcal{J} \subseteq M_n$ and $\mathcal{K} \subseteq M_m$ be submatricial traceless self-adjoint operator spaces. There is a graph homomorphism $\mathcal{J} \to \mathcal{K}$ if and only if there is a $d \geq 1$ and an isometry $E : \mathbb{C}^n \to \mathbb{C}^m \otimes \mathbb{C}^d$ for which $E \mathcal{J} E^* \subseteq M_d(\mathcal{K})$.*

We use this equivalent characterization to show that $\chi_{0,cb}$ is monotonic under graph homomorphisms.

Theorem 7.2.13. *Let $\mathcal{J} \subseteq M_n$ and $\mathcal{K} \subseteq M_m$ be submatricial traceless self-adjoint operator spaces. If there is a graph homomorphism $\varphi : \mathcal{J} \to \mathcal{K}$ with d associated Kraus operators, then we have the inequality*

$$\chi_0(\mathcal{J}) \leq \chi_0(M_d \otimes \mathcal{K}) \; .$$

In particular, $\chi_{0,cb}(\mathcal{J}) \leq \chi_{0,cb}(\mathcal{K})$.

Proof. Suppose that $P_1, \ldots, P_c$ is a partition of the set $[d] \times [m]$ and $(w_i : i \in [d] \times [m])$ is a basis for which whenever i, j are in the same P_s, then $w_i w_j^* \perp M_d \otimes \mathcal{K}$. By lemma 7.2.12, there is an isometry E for which the map $\varphi : M_n \to M_d \otimes M_m : X \mapsto EXE^*$ sends $\mathcal{J}$ to $M_d \otimes \mathcal{K}$. Consider the set $\{E^* w_i : i \in [d \times m]\}$. This set spans $\mathbb{C}^n$. To see this, for any $v \in \mathbb{C}^n$, since $Ev \in \mathbb{C}^d \otimes \mathbb{C}^m$, there are some λ_i for which $Ev = \sum_i \lambda_i w_i$. Multiplying on the left by E^* tell us that v is spanned by the $E^* w_i$. If i, j belong to the same P_s, then for any $X \in \mathcal{J}$,

$$\langle E^* w_i (E^* w_j)^*, X \rangle = \left\langle w_i w_j^*, EXE^* \right\rangle = 0 \; .$$

For each $i \in [c]$, let $C_i = \{E^* w_j : j \in P_i\}$. We will define a sequence of linear subspaces $V_1, \ldots, V_c$ for which $\sum_{i=1}^c V_i = \mathbb{C}^n$ inductively. For the base case, set $V_1 = \operatorname{span} C_1$. For $i > 1$, let

$$V_i = \operatorname{span} \left\{ v \in \operatorname{span} C_i : v \notin \sum_{k<i} V_k \right\} \; .$$

By construction, for distinct i and j, the vectors the V_i are linearly independent in relation to the vectors of V_j and $\sum_i V_i = \mathbb{C}^n$. For each s, let $Q_s = \{v_{s,1}, \ldots, v_{s,d_s}\}$ be a basis in V_s, where $d_s = dim(V_s)$. Since each vector in Q_s is a linear combination of the vectors in C_s, we get that whenever, $i, j \in [d_s]$, given any $X \in \mathcal{J}$,

$$\left\langle v_{s,i} v_{s,j}^*, X \right\rangle = 0 \; .$$

The vectors $\{v_{s,i} : s \in [c], i \in [d_s]\}$ then form a basis for $\mathbb{C}^n$ and are partitioned by the sets $\{Q_s : s \in [c]\}$. This proves that $\chi_0(\mathcal{J}) \leq \chi_0(M_d \otimes \mathcal{K})$. If $r \geq 1$ and E is an isometry for which the map $\varphi : M_n \to M_d \otimes M_m : X \mapsto EXE^*$ sends $\mathcal{J}$ to $M_d(\mathcal{K})$, then the map

$$1 \otimes \varphi : M_r \otimes M_n \to M_{r+d} \otimes M_m : X \otimes Y \mapsto X \otimes \varphi(Y)$$

is a map implemented by conjugation by the isometry $1 \otimes E$. By lemma 7.2.12, $1 \otimes E$ is a graph homomorphism $M_r(\mathcal{J}) \to M_r(\mathcal{K})$. By the above proof, we get the bound $\chi_0(M_r(\mathcal{J})) \leq \chi_0(M_{r+d}(\mathcal{K})) \leq \chi_{0,cb}(\mathcal{K})$ for every $r \geq 1$. This establishes the inequality

$$\chi_{0,cb}(\mathcal{J}) \leq \chi_{0,cb}(\mathcal{K}) .$$

$\square$

Corollary 7.2.14. *Let $\mathcal{J}$ be a submatricial traceless self-adjoint operator space. We have the inequality*

$$\chi_{0,cb}(\mathcal{J}) \leq \chi_{St}(\mathcal{J}) .$$

Proof. We first show that $\chi_{0,d}(\mathcal{J}_G) = \chi(\mathcal{J}_G)$ for any $d \geq 1$ and any graph G. Let $G^{[d]}$ denote the graph on vertices $V(G) \times [d]$ for which $(v, i) \sim (w, j)$ if $v \sim w$ in G. The projection $G^{[d]} \to G : (v, i) \mapsto v$ and the inclusion $G \to G^{[d]} : v \mapsto (v, 1)$ are graph homomorphisms. We therefore get by monotonicity of χ that $\chi(G) = \chi(G^{[d]})$. On the other hand, we know that $\chi_0(\mathcal{J}_G) = \chi(G) = \chi(G^{[d]}) = \chi_0(M_d(\mathcal{J}_G)) = \chi_{0,d}(\mathcal{J}_G)$. In particular, for any $c \geq 1$, $\chi_{0,cb}(\mathcal{J}_{K_c}) = \chi(K_c) = c$. $\square$

Remark 7.2.15. We were unable to determine if $\chi_{0,cb} = \chi_0$. Nevertheless, by working with $\chi_{0,cb}$, we can deduce our inequality since we know it is a homomorphism monotone parameter.

7.3 Sabidussi's Theorem and Hedetniemi's conjecture

As an application of our new graph parameters, in this section, we generalize two results for chromatic numbers on graph products. For convenience we will let $\overline{\chi}(X) = \widehat{\chi}(X^\perp)$ for X a submatricial traceless self-adjoint operator space or a submatricial operator system.

Definition 7.3.1. Let G and H be finite graphs.

1. Define the categorical product of G and H to be the graph $G \times H$ with vertex set $V(G) \times V(H)$ and edge relation given by $(v, a) \sim (w, b)$ if $v \sim_G w$ and $a \sim_H b$.

2. Define the Cartesian product of G and H to be the graph $G \square H$ with vertex set $V(G) \times V(H)$ and edge relation given by $(v, a) \sim (w, b)$ if one of the following holds

   (a) $v \sim_G w$ and $a = b$ or

   (b) $v = w$ and $a \sim_H b$.

7.3.1 Sabidussi's Theorem

We generalize the Theorem of Sabidussi [71].

Theorem 7.3.2 (Sabidussi)**.** *Let G and H be finite graphs. We have the identity*

$$\chi(G \square H) = \max\{\chi(G), \chi(H)\} .$$

The first step in generalizing this Theorem is to generalize the cartesian product.

Definition 7.3.3. Let $\mathcal{J} \subseteq M_n$ and let $\mathcal{K} \subseteq M_m$ be submatricial traceless self-adjoint operator spaces. Let $v \subseteq \mathbb{C}^n$ and $w \subseteq \mathbb{C}^m$ be bases. Define the cartesian product of $\mathcal{J}$ and $\mathcal{K}$ relative to (v, w) as the submatricial traceless self-adjoint operator space

$$(\mathcal{J} \square \mathcal{K})_{v,w} = \mathcal{J} \otimes \mathcal{D}_w + \mathcal{D}_v \otimes \mathcal{K}$$

where for a basis $x = (x_1, \ldots, x_n)$, $\mathcal{D}_x = \text{span}\{x_i x_i^* : i \in [n]\}$.

In the case when $e = (e_1, \ldots, e_n)$ and $f = (e_1, \ldots, e_m)$, we define the cartesian product $\mathcal{J} \square \mathcal{K}$ to be $(\mathcal{J} \square \mathcal{K})_{e,f}$.

Lemma 7.3.4. *Let G and H be finite graphs with $[n] = V(G)$ and $[m] = V(H)$. We have the identity $\mathcal{J}_G \square \mathcal{J}_H = \mathcal{J}_{G \square H}$.*

Proof. Observe that $\mathcal{J}_G \otimes \mathcal{D}_m = \text{span}\{E_{v,w} \otimes E_{i,i} : v \sim_G w, i \in [m]\}$ and that $\mathcal{D}_n \otimes \mathcal{J}_H = \text{span}\{E_{i,i} \otimes E_{v,w} : i \in [n], v \sim_H w\}$. Combining these, we get that $E_{i,j} \otimes E_{k,l} \in \mathcal{J}_G \square \mathcal{J}_H$ if and only if $i \sim_G j$ and $k = l$ or $i = j$ and $k \sim_H l$. This is exactly what it means to be a member of $\mathcal{J}_{G \square H}$. $\square$

Lemma 7.3.5. *Suppose that $\mathcal{J} \subseteq M_n$ and $\mathcal{K} \subseteq M_m$ are submatricial traceless self-adjoint operator spaces. Suppose $v \subseteq \mathbb{C}^n$ and $w \subseteq \mathbb{C}^m$ are bases. There exist graph homomorphisms $\mathcal{J} \to \mathcal{J} \otimes \mathcal{D}_w$ and $\mathcal{K} \to \mathcal{D}_v \otimes \mathcal{K}$. In particular, there exist graph homomorphisms $\mathcal{J} \to (\mathcal{J} \square \mathcal{K})_{v,w}$ and $\mathcal{K} \to (\mathcal{J} \square \mathcal{K})_{v,w}$.*

Proof. Define $\varphi : M_n \to M_n \otimes M_m : X \mapsto \frac{1}{\|w_1\|^2} X \otimes w_1 w_1^*$. This map has Kraus operator $E : \mathbb{C}^n \to \mathbb{C}^n \otimes \mathbb{C}^m : v \mapsto v \otimes w_1 / \|w_1\|$. Since this Kraus operator is an isometry, we know that φ is cptp. As well, $\varphi(\mathcal{J}) = \mathcal{J} \otimes w_1 w_1^* \subseteq \mathcal{J} \otimes \mathcal{D}_w$. Similarly, $\mathcal{K} \to \mathcal{D}_v \otimes \mathcal{K}$. Since $\mathcal{J} \otimes \mathcal{D}_w \subseteq (\mathcal{J} \square \mathcal{K})_{v,w}$ and $\mathcal{D}_v \otimes \mathcal{K} \subseteq (\mathcal{J} \square \mathcal{K})_{v,w}$, we conclude that $\mathcal{J} \to (\mathcal{J} \square \mathcal{K})_{v,w}$ and $\mathcal{K} \to (\mathcal{J} \square \mathcal{K})_{v,w}$. $\square$

Theorem 7.3.6. *Let $\mathcal{J} \subseteq M_n$ and $\mathcal{K} \subseteq M_m$ be submatricial traceless self-adjoint operator spaces. Let $v \subseteq \mathbb{C}^n$ and $w \subseteq \mathbb{C}^m$ be bases. We have the inequality*

$$\max\{\chi_{0,cb}(\mathcal{J}), \chi_{0,cb}(\mathcal{K})\} \leq \chi_{0,cb}((\mathcal{J}\square\mathcal{K})_{v,w}) \ .$$

Proof. By lemma 7.3.5 and by Theorem 7.2.13, we get the inequalities

$$\chi_{0,cb}(\mathcal{J}) \leq \chi_{0,cb}((\mathcal{J}\square\mathcal{K})_{v,w}) \text{ and } \chi_{0,cb}(\mathcal{K}) \leq \chi_{0,cb}((\mathcal{J}\square\mathcal{K})_{v,w}) \ .$$

Our Theorem follows. □

The reverse inequality seems to require the existence of orthogonal bases which colour our submatricial traceless self-adjoint operator spaces. The proof mimicks the proof of Sabidussi's Theorem in [31].

Theorem 7.3.7. *Let $\mathcal{J} \subseteq M_n$ and $\mathcal{K} \subseteq M_m$ be submatricial traceless self-adjoint operator spaces. Let $c = \max\{\chi_0(\mathcal{J}), \chi_0(\mathcal{K})\}$. Suppose that orthonormal bases $v \subseteq \mathbb{C}^n$ and $w \subseteq \mathbb{C}^m$ exist for which we have maps $f : [n] \to [c]$ and $g : [m] \to [c]$ for which whenever $f(i) = f(j)$, $v_{f(i)}v^*_{f(j)} \perp \mathcal{J}$ and whenever $g(l) = g(k)$, we have $w_{g(l)}w^*_{g(k)} \perp \mathcal{K}$. We have the inequality*

$$\chi_0((\mathcal{J}\square\mathcal{K})_{v,w}) \leq \max\{\chi_0(\mathcal{J}), \chi_0(\mathcal{K})\} \ .$$

Proof. Let $c = \max\{\chi_0(\mathcal{J}), \chi_0(\mathcal{K})\}$. Suppose that v, w, f, and g are as above. Define $h : [n] \times [m] \to [c] : (i, j) \mapsto f(i) + g(j) \mod c$. I claim that whenever $h(i, j) = h(k, l)$, that $(v_i \otimes w_j)(v_k \otimes w_l)^*$ is orthogonal to $(\mathcal{J}\square\mathcal{K})_{v,w}$. The identity $h(i, j) = h(k, l)$ tell us $f(i) - f(k) \equiv g(j) - g(l) \mod c$. If $f(i) - f(k) \equiv 0 \mod c$ then we have nothing to check since this means that $f(i) = f(k)$ and $g(j) = g(l)$. Otherwise, $v_iv^*_k \perp v_sv^*_s$ for all s and $w_jw^*_l \perp w_sw^*_s$ for all s. This guarantees that $v_iv^*_k \otimes w_jw^*_l$ is orthogonal to $(\mathcal{J}\square\mathcal{K})_{v,w}$. □

Remark 7.3.8. The same proof as above will show us that for some orthonormal bases v and w,

$$\chi((\mathcal{J}\square\mathcal{K})_{v,w}) \leq \max\{\chi(\mathcal{J}), \chi(\mathcal{K})\} \text{ and}$$
$$\overline{\chi}((\mathcal{J}\square\mathcal{K})^{\perp}_{v,w}) \leq \max\{\overline{\chi}(\mathcal{J}^{\perp}), \overline{\chi}(\mathcal{K}^{\perp})\} \ .$$

We are now ready to state a generalized version of Sabidussi's theorem. In the following statement please recall that $\overline{\chi}(X) = \widehat{\chi}(X^{\perp})$.

Corollary 7.3.9 (Sabidussi's Theorem for submatricial traceless self-adjoint operator spaces)**.** *Suppose that $\mathcal{J} \subseteq M_n$ and $\mathcal{K} \subseteq M_m$ are submatricial traceless self-adjoint operator spaces. There exist orthonormal bases $v \subseteq \mathbb{C}^n$ and $w \subseteq \mathbb{C}^m$ for which we have the inequalities*

$$\max\{\chi_{0,cb}(\mathcal{J}), \chi_{0,cb}(\mathcal{K})\} \leq \chi_{0,cb}((\mathcal{J}\Box\mathcal{K})_{v,w}) \leq \overline{\chi}((\mathcal{J}\Box\mathcal{K})^{\perp}_{v,w}) \leq \max\{\overline{\chi}(\mathcal{J}^{\perp}), \overline{\chi}(\mathcal{K}^{\perp})\} \ .$$

Proof. By Remark 7.3.8, we get the inequality

$$\overline{\chi}((\mathcal{J}\Box\mathcal{K})^{\perp}_{v,w}) \leq \max\{\overline{\chi}(\mathcal{J}^{\perp}), \overline{\chi}(\mathcal{K}^{\perp})\} \ .$$

By Theorem 7.2.5 and Corollary 7.2.14, we get the inequality

$$\chi_{0,cb}((\mathcal{J}\Box\mathcal{K})_{v,w}) \leq \chi_{st}((\mathcal{J}\Box\mathcal{K})_{v,w}) \leq \overline{\chi}((\mathcal{J}\Box\mathcal{K})^{\perp}_{v,w}) \ .$$

Finally, by Theorem 7.3.6 we get the final inequality. □

7.3.2 Hedetniemi's inequality

The inequality we wish to generalize in this section is a Theorem of Hedetniemi.

Theorem 7.3.10 (Hedetniemi's inequality)**.** *Suppose that G and H are finite graphs. We have the inequality*

$$\chi(G \times H) \leq \min\{\chi(G), \chi(H)\}$$

This Theorem follows as a special case of the analogous result for $\chi_{0,cb}$, first we generalize the categorical product.

Proposition 7.3.11. *Let G and H be finite graphs. We have the identity*

$$\mathcal{J}_G \otimes \mathcal{J}_H = \mathcal{J}_{G\times H} \ .$$

Proof. Observe that

$$\begin{aligned} \mathcal{J}_G \otimes \mathcal{J}_H &= \operatorname{span}\{E_{i,j} \otimes E_{k,l} : i \sim_G j, k \sim_H l\} \\ &= \mathcal{J}_{G\times H} \ . \end{aligned}$$

□

We now get a generalization of Hedetniemi's inequality to $\chi_{0,cb}$.

Proposition 7.3.12. *Suppose that $\mathcal{J} \subseteq M_n$ and $\mathcal{K} \subseteq M_m$ are submatricial traceless self-adjoint operator spaces. We have the inequality*

$$\chi_{0,cb}(\mathcal{J} \otimes \mathcal{K}) \leq \min\{\chi_{0,cb}(\mathcal{J}), \chi_{0,cb}(\mathcal{K})\} .$$

Proof. The partial trace maps produce graph homomorphisms $\mathcal{J} \otimes \mathcal{K} \to \mathcal{K}$ and $\mathcal{J} \otimes \mathcal{K} \to \mathcal{J}$. By Theorem 7.2.13, we get the inequality. □

Remark 7.3.13. The long standing conjecture of Hedetneimi asked whether we get the identity

$$\chi(G \times H) = \min\{\chi(G), \chi(H)\}$$

for any finite graphs G and H. This was recently resolved in the negative by the remarkable work of Yaroslav Shitov [73].

References

[1] M. Argerami, S. Coskey, M. Kalantar, M. Kennedy, M. Lupini, and M. Sabok. The classification problem for finitely generated operator systems and spaces. *preprint (arXiv:1411.0512 [math.OA])*, 2014.

[2] W. Arveson. Subalgebras of C*-algebras. *Acta Mathematica*, 123:141–224, 1969.

[3] W. Arveson. The non-commutative Choquet boundary. *Journal of the American Mathematical Society*, 21(4):1065–1084, 2008.

[4] W. Arveson. The non-commutative Choquet boundary II: hyperrigidity. *Israel Journal of Mathematics*, 184(1):349–385, 2011.

[5] W. Arveson. Notes on the unique extension property. *Available at https://math.berkeley.edu/ arveson/Dvi/unExt.pdf*, preprint (2006).

[6] A. Atserias, L. Mancinska, D. Roberson, R. Sámal, S. Severini, and A. Varvitsiotis. Quantum and non-signalling graph isomorphisms. *Journal of Combinatorial Theory, Series B*, 136:289–328, 2019.

[7] E. Bedos, S. Kaliszewski, J. Quigg, and D. Robertson. A new look at crossed product correspondences and associated C*–algebras. *Journal of Mathematical Analysis and Applications*, 426(2):1080–1098, 2015.

[8] J. Bell. On the einstein podolsky rosen paradox. *Physics Physique Fizika*, 1(3):195, 1964.

[9] D. Blecher and B. Duncan. Nuclearity-related properties for nonselfadjoint algebras. *Journal of Operator Theory*, 65(1):47–70, 2011.

[10] D. Blecher and C. Le Merdy. *Operator algebras and their modules–an operator space approach*, volume 30 of *London Mathematical Society Monographs. New Series*. The Clarendon Press, Oxford University Press, Oxford.

[11] D. Blecher and M. Royce. Extensions of operator algebras I. *Journal of Mathematical Analysis and Applications*, 339(2):1451–1467, 2008.

[12] M. Brannan, A. Chirvasitu, K. Eifler, S. Harris, V. Paulsen, X. Su, and M. Wasilewski. Bigalois extensions and the graph isomorphism game. *Communications in Mathematical Physics*, pages 1–33, 2019.

[13] N. Brown and N. Ozawa. *C*-algebras and finite dimensional approximations*, volume 88 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, Rhode Island, 2008.

[14] M.-D. Choi and E. Effros. Injectivity and operator spaces. *Journal of Functional Analysis*, 24(2):156–209, 1977.

[15] M.-D. Choi and E. Effros. Nuclear C*-algebras and injectivity: the general case. *Indiana University Mathematics Journal*, 26(3):443–446, 1977.

[16] R. Cleve, L. Liu, and W. Slofstra. Perfect commuting-operator strategies for linear system games. *Journal of Mathematical Physics*, 58(1):012202, 2017.

[17] R. Cleve and R. Mittal. *Characterization of binary constraint system games*. International Colloquium on Automata, Languages, and Programming. Springer, 2014.

[18] A. Coladangelo and J. Stark. Unconditional separation of finite and infinite-dimensional quantum correlations. *arXiv preprint arXiv:1804.05116*, 2018.

[19] W. Cook and R. Webster. Caratheodory's theorem. *Canadian Mathematical Bulletin*, 15:888, 1972.

[20] K. Davidson and M. Kennedy. Noncommutative choquet theory. *arXiv preprint arXiv:1905.08436*, 2019.

[21] A. Dor-On and G. Salomon. Full Cuntz-Krieger dilations via non-commutative boundaries. *Journal of the London Mathematical Society*, 98(2):416–438, 2018.

[22] M. Dritschel and S. Mccullough. Boundary representations for families of representations of operator algebras and spaces. *Journal of Operator Theory*, 53(1):159–167, 2005.

[23] R. Duan, S. Severini, and A. Winter. Zero-error communication via quantum channels, noncommutative graphs, and a quantum Lovász number. *IEEE Transactions on Information Theory*, 59(2):1164–1174, 2013.

[24] K. Dykema and V. Paulsen. Synchronous correlation matrices and Connes' embedding conjecture. *Journal of Mathematical Physics*, 57:015215, 2016.

[25] K. Dykema, V. Paulsen, and J. Prakash. Non-closure of the set of quantum correlations via graphs. *Communications in Mathematical Physics*, 365(3):1125–1142, 2019.

[26] A. Einstein, B. Podolsky, and N. Rosen. Can quantum-mechanical description of physical reality be considered complete? *Physical review*, 47(10):777, 1935.

[27] D. Farenick, A. Kavruk, V. Paulsen, and I. Todorov. Operator systems from discrete groups. *Communications in Mathematical Physics*, 329(1):207–238, 2014.

[28] D. Farenick and V. Paulsen. Operator system quotients of matrix algebras and their tensor products. *Mathematica Scandinavica*, 111(2):210–243, 2012.

[29] B. Forrest, N. Spronk, and M. Wiersma. Existence of tracial states on reduced group C*-algebras. *preprint (arXiv:1706.05354 [math.OA])*, 2017.

[30] T. Fritz. Tsirelson's problem and Kirchberg's conjecture. *Reviews in Mathematical Physics*, 24(05):1250012, 2012.

[31] C. Godsil, D. Roberson, R. Šámal, and S. Severini. Sabidussi versus Hedetniemi for three variations of the chromatic number. *Combinatorica*, 36(4):395–415, 2016.

[32] A. Goldberg. Synchronous linear constraint system games. *arXiv preprint arXiv:2007.02782*, 2020.

[33] I. Goldbring and T. Sinclair. On the axiomatizability of C*-algebras as operator systems. *Glasgow Mathematical Journal*, 61(3):629–635, 2019.

[34] V. Gupta and P. Luthra. Operator system nuclearity via C*-envelopes. *Journal of the Australian Mathematical Society*, 101(3):356–375, 2016.

[35] M. Hamana. Injective envelopes of operator systems. *Publications of the Research Institute for Mathematical Sciences, Kyoto University*, 15(3):773–785, 1979.

[36] M. Hamana. Injective envelopes of C*-dynamical systems. *Tohoku Mathematical Journal (2)*, 37(4):463–487, 1985.

[37] G. Hao and C.-K. Ng. Crossed products of C*-correspondences by amenable group actions. *Journal of Mathematical Analysis and Applications*, 345(2):702–707, 2008.

[38] J. Helton, K. Meyer, V. Paulsen, and M. Satriano. Algebras, synchronous games, and chromatic numbers of graphs. *New York Journal of Mathematics*, 25:328–361, 2019.

[39] Z. Ji, A. Natarajan, T. Vidick, J. Wright, and H. Yuen. MIP*= RE. *arXiv preprint arXiv:2001.04383*, 2020.

[40] M. Junge, M. Navascues, C. Palazuelos, D. Perez-Garcia, V. Scholz, and R. Werner. Connes' embedding problem and Tsirelson's problem. *Journal of Mathematical Physics*, 52(1):012102, 2011.

[41] E. Kakariadis. The Dirichlet property for tensor algebras. *Bulletin of the London Mathematical Society*, 45(6):1119–1130, 2013.

[42] M. Kalantar and M. Kennedy. Boundaries of reduced C*-algebras of discrete groups. *Journal für die reine und angewandte Mathematik*, 2017(727):247–267, 2017.

[43] S. Kaliszewski, J. Quigg, and D. Robertson. Functoriality of Cuntz-Pimsner correspondence maps. *Journal of Mathematical Analysis and Applications*, 405(1):1—11, 2013.

[44] E. Katsoulis. C*-envelopes and the hao-ng isomorphism for discrete groups. *International Mathematics Research Notices*, 2017(18):5751–5768, 2017.

[45] E. Katsoulis and D. Kribs. Tensor algebras of C*-correspondences and their C*-envelopes. *Journal of Functional Analysis*, 234(1):226–233, 2006.

[46] E. Katsoulis and C. Ramsey. A non-selfadjoint approach to the Hao-Ng isomorphism problem. *International Mathematics Research Notices*, 2019.

[47] E. Katsoulis and C. Ramsey. Crossed products of operator algebras. *Memoirs of the American Mathematical Society*, in press.

[48] T. Katsura. A class of C*-algebras generalizing both graph algebras and homeomorphism C*-algebras I, fundamental results. *Transactions of the American Mathematical Society*, 356(11):4287–4322, 2004.

[49] T. Katsura. On C*-algebras associated with C*-correspondences. *Journal of Functional Analysis*, 217(2):366–401, 2004.

[50] A. Kavruk. Nuclearity related properties in operator systems. *Journal of Operator Theory*, 71(1):95–156, 2014.

[51] A. Kavruk. On a non-commutative analogue of a classical result of Namioka and Phelps. *Journal of Functional Analysis*, 269(10):3282–3303, 2015.

[52] A. Kavruk, V. Paulsen, I. Todorov, and M. Tomforde. Tensor products of operator systems. *Journal of Functional Analysis*, 261(2):267–299, 2011.

[53] A. Kavruk, V. Paulsen, I. Todorov, and M. Tomforde. Quotients, exactness, and nuclearity in the operator system category. *Advances in Mathematics*, 235:321–360, 2013.

[54] T. Kawabe. Uniformly recurrent subgroups and the ideal structure of reduced crossed products. *preprint (arXiv:1701.03413 [math.OA])*, 2017.

[55] M. Kennedy and S. Raum. Traces on reduced group c*-algebras. *Bulletin of the London Mathematical Society*, 49(6):988–990, 2017.

[56] E. Kirchberg. Discrete groups with Kazhdan's property T and factorization property are residually finite. *Mathematische Annalen*, 299:35–63, 1994.

[57] A. Kishimoto and A. Kumjian. Crossed products of Cuntz algebras by quasi-free automorphisms. 13:173–192, 1997.

[58] D. Knuth. The sandwich theorem. *Electronic Journal of Combinatorics: A1*, 1994.

[59] L. Lovász. On the Shannon capacity of a graph. *IEEE Transactions on Information Theory*, 25(1):1–7, 1979.

[60] L. Mančinska and D. Roberson. Quantum homomorphisms. *Journal of Combinatorial Theory, Series B*, 118:228–267, 2016.

[61] C.-K. Ng. Strictly amenable representations of reduced group c*-algebras. *International Mathematics Research Notices*, 2015(17):7853–7860, 2015.

[62] C. Ortiz and V. Paulsen. Lovász theta type norms and operator systems. *Linear Algebra and its Applications*, 477:128–147, 2015.

[63] N. Ozawa. About the Connes embedding conjecture: algebraic approaches. *Japanese Journal of Mathematics*, 8:147–183, 2013.

[64] V. Paulsen. *Completely bounded maps and operator algebras*, volume 78 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2002.

[65] V. Paulsen. Entanglement and non-locality. 2016. Available at http://www.math.uwaterloo.ca/~vpaulsen/EntanglementAndNonlocality_LectureNotes_7.pdf. Accessed 24-03-2017.

[66] V. Paulsen, R. Levene, and I. Todorov. Complexity and capacity bounds for quantum channels. *IEEE Transactions on Information Theory*, 64(10):6917–6928, 2018.

[67] V. Paulsen, S. Severini, D. Stahlke, I. Todorov, and A. Winter. Estimating quantum chromatic numbers. *Journal of Functional Analysis*, 270(6):2188–2222, 2016.

[68] V. Paulsen and I. Todorov. Quantum chromatic numbers via operator systems. *The Quarterly Journal of Mathematics*, 66(2):677–692, 2015.

[69] E. Raeburn and S. Thompson. Countably generated Hilbert modules, the Kasparov stabilisation theorem, and frames in Hilbert modules. *Proceedings of the American Mathematical Society*, 131(5):1557–1564, 2003.

[70] T. Russell. Geometry of the set of synchronous quantum correlations. *Journal of Mathematical Physics*, 61(5):052201, 2020.

[71] G. Sabidussi. Graphs with given group and given graph-theoretical properties, 1957.

[72] G. Salomon. Hyperrigid subsets of Cuntz-Krieger algebras and the property of rigidity at zero. *Journal of Operator Theory*, 81(1):61–79, 2019.

[73] Y. Shitov. Counterexamples to Hedetniemi's conjecture. *Annals of Mathematics*, 190(2):663–667, 2019.

[74] W. Slofstra. The set of quantum correlations is not closed. *Forum of Mathematics, Pi*, 7, 2019.

[75] D. Stahlke. Quantum zero-error source-channel coding and non-commutative graph theory. *IEEE Transactions on Information Theory*, 62(1):554–577, 2016.

[76] B. Tsirelson. Some results and problems on quantum Bell-type inequalities. *Hadronic Journal Supplement 8, 4, 329 - 345*, 1993.

[77] B. Tsirelson. Bell inequalities and operator algebras. *http://www.imaph.tu-bs.de/qi/problems/33.html*, 2006.

[78] N. Weaver. Quantum graphs as quantum relations. *https://arxiv.org/abs/1506.03892*, 2015.

[79] D. Williams. *Crossed products of C^*-algebras*, volume 134 of *Mathematical Surveys and Monographs.* American Mathematical Society, Providence, RI, 2007.