Some Stochastic Optimization Problems in Reinsurance and Insurance

by

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I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

Insurance, which hedges against the risk of a contingent loss, is an indispensable risk management tool for both institutions and individuals. Reinsurance, namely, a form of insurance accessible to insurers, helps limit the liability of an insurer on certain set of risks and protect against catastrophic events, while various insurance products are available for individuals to cover uncertain losses from almost every aspect of their daily life. This thesis focuses on dynamically controlling the utilities of decision makers by imposing various controls, including reinsurance for insurers, and life annuity and term life insurance for individuals, either analytically or numerically.

Utilizing (re)insurance to attain certain objectives has long been a central focus in the actuarial science literature. This thesis aims at making contributions in the existing literature by applying models that are more in line with reality, both in regard to the underlying dynamic models and control variables.

In Chapter 3, we study the optimal reinsurance-investment strategy for dynamic contagion claims. Such a claim process no longer possesses the stationary and independent increment property, and can capture contagion due to endogenous (self-exciting) and exogenous (externally-exciting) factors. Adopting the time-consistent mean-variance criterion, we analytically solve for the equilibrium strategies and analyze the impact of some contagion factors on the resulting optimal reinsurance strategies.

Chapter 4 models the basic surplus process as a spectrally negative Lévy process, and focuses on the partial information of the unobservable stock return rate to look into the optimal reinsurance-investment problem under the time-consistent mean-variance criterion. Analytical solutions are obtained by solving an extended HJB equation, and hedging demand due to partial information is carefully studied.

Chapter 5 is devoted to the study of the optimal allocation of life annuity, term life insurance and consumption for an individual under a general force of mortality. In our setup, an individual's decision of life annuity, term life insurance and consumption are allowed to depend on the current wealth, existing life annuity and existing term life insurance, and realistic lump-sum purchases are considered. Assuming a CRRA preference, a penalty method is applied to numerically solve for the optimal allocations of wealth in life annuity, term life insurance and consumption.

To ensure that the thesis flows smoothly, Chapter 1 introduces the background literature and main motivations of this thesis. Chapter 2 is devoted to mathematical preliminaries for the latter chapters. Finally, Chapter 6 concludes the thesis with potential directions for future research.

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Dedication

To my parents, Quan Cao and Shuirong Li.

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Chapter 1

Introduction

1.1 Background and Motivation

Insurance is a form of risk management, primarily used to hedge against the risk of a contingent loss. Under an insurance contract, a person or entity (the insured) pay certain fees, known as the premium, to the insurer in exchange for the insurer's promise to compensate the insured in the event of a covered loss. As a way to transfer risks, insurance is commonly used by financial institutions and individuals to achieve certain objectives. Typical examples include auto insurance, life insurance, casualty insurance, liability insurance and many others . The insurer may hedge its own risk by taking out *reinsurance*, whereby another insurer agrees to carry some of the risks.

The roots of reinsurance can be traced back to the 14th century when it was used for marine and fire insurance according to the Reinsurance Association of America. Since then, it has grown to cover every aspect of the modern insurance market. Two basic types of reinsurance treaties are proportional reinsurance and excess-of-loss reinsurance. As the name indicates, a proportional reinsurance treaty is one for which the reinsurer bears a pre-determined proportion of the underlying claims, while under an excess-of-loss policy, reinsurance kicks in only when the claim exceeds a certain limit. In addition to these two basic forms, there exist a wide variety of reinsurance contracts in practice which are often tailored to the particular situation. Using reinsurance, an insurer can limit its liability on a specific risk, protect against catastrophic events, reduce the reserve level and increase its capacity.

Early work of optimal reinsurance goes back to Borch [17] for risk minimization and Arrow [4] for expected utility maximization. Indeed, in a static environment, Borch [17] demonstrated that the excess-of-loss reinsurance is the best contract if the insurer measures risks by variance and the reinsurer prices risks by the expected value premium principle. Arrow [4] also showed that the excess-of-loss reinsurance is an optimal one if the insurer is an expected utility maximizer under the assumption of the expected value premium principle. These fundamental results have been generalized in a number of interesting and important directions. Just to name a few, for risk minimization, Cai et al. [22] extended Borch's result under the Value-at-Risk (VaR) and Conditional tail expectation (CTE) in a class of increasing convex ceded loss functions, Zhuang et al. [111] and Cheung et al. [27] investigated the optimal reinsurance to minimize distortion risk measures and coherent risk measure, respectively. For expected utility maximization, Young [105] elaborated Arrow's result taking into account Wang's premium principle, while Xu et al. [102] and Ghossoub [49] considered rank-dependent expected utility. The problem of optimal reinsurance is also studied in a dynamic environment. Along this strand of literature, various optimization criteria are investigated (see Promislow and Young [88] and Schmidli [93] for minimizing the probability of ruin, Bai and Guo [7] and Zhao et al. [109] for maximizing expected utility, Li et al. [67] and Landriault et al. [66] for maximizing time-consistent mean-variance criterion) over different reinsurance treaties (see e.g.,Højgaard and Taksar [58], Bai and Guo [7], Meng et al. [78], Gu et al. [51] and Schmidli [92] for proportional reinsurance, Tapiero and Zuckerman [99], Asmussen et al. [5], Zhao et al. [109], Liang and Young [71], Chunxiang et al. [28] and Moore and Young [82] for excess-of-loss reinsurance, Liang and Guo [73] and Zou and Cadenillas [112] for general reinsurance treaties).

Within most of the literature on dynamic optimal reinsurance, the basic underlying claim process is assumed to follow the classic Cramér-Lundberg model (see e.g., Dickson and Waters [40], Hipp and Plum [57] and Schmidli [92]), a linear Brownian motion (see e.g., Zhang et al. [108], Bai and Guo [7], Promislow and Young [88]) or more generally a spectrally negative Lévy process (see Li et al. [68] and Li et al. [67]), whose intrinsic feature is stationary and independent increments, an assumption which is often challenged or seriously violated in a large number of insurance contexts, see Seal et al. [94] and Beard [12]. For instance, the clustering effect due to *exogenous (externally-excited)* factors, such as earthquakes, flood, and hurricanes violates the stationary increments property, and can be more adequately captured adopting a Cox process (see Cox [29]). In insurance contexts, many researchers, e.g., Björk and Grandell [15], Embrechts et al. [41], Schmidli [91], and Albrecher and Asmussen [2] have suggested using a Cox process to model the claim arrival dynamics. On the other hand, clustering due to *endogenous (self-excited)* factors, such as contagious diseases and aggressive driving habits violates the independent increment property, and can be addressed using a Hawkes process (see Hawkes [55]). For applications, Aït-Sahalia and Hurd [1] considers an infinite time investment and consumption problem with mutually-exciting asset price dynamics for a utility maximizer. Buccioli et al. [21] investigates the optimal portfolio selection to minimize expected shortfall when asset prices embed self-exciting jumps. We refer interested readers to Hawkes [56] and Bacry et al. [6] for a comprehensive survey on Hawkes processes and their applications in finance. In the actuarial science literature, Stabile and Torrisi [97] studied the ruin problem of an insurance risk model adopting the Hawkes process. Delong and Gerrard [39] obtains the pre-commitment investment strategy for a mean-variance insurer with a Cox claim arrival process. Despite the capability in modelling clustering effects, the literature on optimal reinsurance problems with contagious claim arrivals is quite limited. **The first objective of this thesis is to contribute to this line of research by taking into account claim arrivals with both self-exciting and externally-exciting factors**.

When certain models are assumed for the underlying basic surplus process and the financial market, insurers are assumed to know perfectly the financial market and the claim process, including the underlying dynamics and model parameters, none of which are directly observable in practice. Such model risk plays an important role in the decision making process, and is mainly studied within two frameworks in the literature: ambiguity and partial information. Ambiguity describes the situation of unknown probability. In this framework, the decision maker has multiple prospectives and each prospective is associated with a known probability measure. In light of ambiguity aversion and risk aversion, a decision maker typically tries to maximize the robust utility function of the form:

$$\inf_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}^{\mathbb{Q}}\big[U(X) + h(\mathbb{Q})\big], \tag{1.1.1}$$

where X is a random payoff, $\mathcal{Q}$ is a set of probability measures, U is a utility function and $h(\cdot) : \mathcal{Q} \to [0, \infty]$ is a function that penalizes the deviation from some reference measure $\mathbb{P} \in \mathcal{Q}$. Zheng et al. [110] and Gu et al. [50] study the robust investment-reinsurance problem under (1.1.1). Zeng et al. [107] investigates the robust investment-reinsurance problem for a mean-variance insurer, while Li et al. [67] proposes a α-maxmin mean-variance criterion which differentiates between the level of ambiguity aversion and ambiguity.

Another method to characterize the model risk is through partial information. At any time point, the decision maker estimates the unobservable process and makes decisions based on the accumulated observable information up to that time point. For example, suppose the stock return rate process μ_t is the unobservable learning object, and the available observed information is the filtration $\mathcal{F}_t^S$ generated by the stock price. The investor uses filtering technique to characterize the dynamic of the following learning process

$$\hat{\mu}_t = \mathbb{E}\big[\mu_t | \mathcal{F}_t^S\big], \tag{1.1.2}$$

by which the original problem with partial information can be reduced to an equivalent one with complete information with an additional state variable.

The role of learning the unobserved information is critical. An investor is called myopic if he or she ignores the future learning of partial information, computes the optimal strategy with complete information and substitutes the unobserved part with its estimator directly. An investor is called non-myopic if he or she employs dynamic learning in the process of searching for an optimal strategy. The difference between the non-myopic strategy and myopic strategy is defined as the hedging demand induced by model uncertainty. Cvitanić et al. [32] and Honda [59] showed that the hedging component induced by learning about the stock return rate can be a substantial part of the demand, especially for long horizon investors. Brennan [19], Rieder and Bäuerle [90] and Longo and Mainini [77] explore the effect of learning for a CRRA investor, and they show that the direction of the hedging demand depends on whether the investor is more or less risk tolerant than a logarithmic investor.

In a Markovian setting, the dynamics of the learning process can be characterized by the Zakai equation, which is an infinite-dimension SDE, driven by the so-called innovation process. Pham [86] provides a survey of the methods involved in portfolio optimization with partial information, which covers the three finite-dimension cases for modelling the unobservable return, namely, Bayesian, linear-Gaussian and hidden finite-state Markov chain. These three cases are extensively studied for the optimal investment problem, see Karatzas and Zhao [64], Longo and Mainini [77] for the Bayesian case, Li et al. [69] for the linear-Gaussian case, and Rieder and Bäuerle [90], Shen and Siu [95] Bäuerle and Rieder [9], Jeanblanc et al. [63] and Callegaro et al. [23] for the hidden finite-state Markov chain framework. In actuarial contexts, Meng et al. [78], Zhang et al. [108] and Liang and Song [74] consider the partial information of stock return for the reinsurance-investment problem on the investment part. Moreover, Liang and Bayraktar [72] studies the optimal reinsurance-investment problem considering the partial information of the underlying claim process. **The second objective of this thesis is to study the optimal reinsurance-investment strategy with Bayesian learning of the unobservable stock return rate under the time-consistent mean-variance criterion**.

Optimal insurance from an individual's point of view is also of great importance in managing an individual's risk exposure. Directly related to an individual's uncertain lifetime are the so-called longevity risk, namely, the risk of running out of savings and falling into poverty before dying, and mortality risk, i.e., the risk of the loss of family income due to the breadwinner's premature death. In exchange for an initial lump-sum premium, a life annuity is a financial contract between an individual and an insurer that pays out a periodic amount for as long as the annuitant is alive, and a term life insurance provides certain death benefit at the individual's premature death when in effect. Therefore, life annuities and term life insurance can be used to manage one's longevity risk and mortality risk.

Yaari [103] is a starting point for modern research on the demand for life insurance and life annuity, who introduced an optimal consumption problem for an individual with uncertain time of death within the setup of a pure deterministic investment environment, and showed that assuming actuarially fair annuity prices and no bequest motives, expected utility maximizers should put all of their assets in actuarial notes, that is, they should annuitize all of their wealth. Then in a discrete-time setup, Hakansson [52] analyzed the problem of optimal investment, consumption and life insurance with CRRA utility, and find conditions under which zero insurance is optimal. Fischer [45] used a similar discrete-time model, where only two assets -a bond and an insurance asset are available, to examine the comparative statics and dynamics of the insurance demand functions. In a continuous-time setting, Richard [89] combined the model from Merton [79] and the insurance literature to include consumption, investment, life insurance and annuity decisions for an investor with a known distribution of lifetime, to maximize the expected utility from inter-temporal consumption and bequest motive. Along the strand of literature on optimal annuitization, Milevsky and Young [81] first incorporates life and pension annuity products into the portfolio choice literature and focuses on the optimal annuitization strategy for a retiree in so-called *all or nothing* and *anything anytime* annuity market, respectively. Explicit optimal annuitization strategies for a retiree were given in Wang and Young [101] under power utility and Liang and Young [70] under exponential utility. By incorporating a non-tradeble labor income risk, Horneff et al. [60] and Chai et al. [25] include both working life and retirement in their analysis. Following Richard [89], most studies about life annuity and insurance demands simplify the decision by including an instantaneous term annuity or insurance contract in continuous-time setting (see Pliska and Ye [87], Huang and Milevsky [61], Bayraktar et al. [11] and references therein) or a series a renewable one-year term life insurance in a discrete time setting (see Chen et al. [26]).

Several questions arise in regard to modelling the life annuity and term life insurance in the existing body of literature. A single control variable, namely the premium rate, is commonly used to characterize both life annuity and life insurance decision(see e.g., Pliska and Ye [87], Huang and Milevsky [61], Bayraktar et al. [11]): a positive premium rate indicates a positive amount of life insurance while a negative one represents a life annuity whose premium is due at death. First, this model doesn't take existing life annuity or life insurance into account as state variables, and the two decisions solely depend on an investor's current wealth as a result. Second, in practice life annuities and term life insurance can only be bought or surrendered at realistic lump-sum costs. Third, simultaneous holdings of life annuity and term life insurance are not allowed, while in practice, a substantial number of the families that own annuities also have life insurance policies according to Brown [20]. Fourth, the individual is commonly assumed to have a constant force of mortality when life annuities or term life insurance are allowed with lump-sum purchase, except for Milevsky and Young [81], where optimal life annuity without surrendering feature for an infinite time horizon is studied. The second issue is addressed separately by Bayraktar and Young [10] and Hambel et al. [54] for life insurance, and Milevsky and Young [81], Wang and Young [101] and Liang and Young [70] for life annuity, but not both. **The study of allocations of life annuity, term life insurance and consumption addressing the above four issues serves as the third objective of this thesis.**

The third objective is also motivated by empirical findings widely documented in life annuity and insurance market: (a) *Annuity puzzle*: Yaari [103] showed that, in a perfect market setting, expected utility maximizers with no bequest motive should annuitize their entire wealth, which was further confirmed by Davidoff et al. [37] under more general assumptions. Empirical studies find, however, that only a small portion of private wealth is used to purchase annuities. This discrepancy between theoretical findings and empirical observations is referred to as the annuity puzzle. (b) *Adverse selection* in annuity market: people with a higher level of longevity risk purchase more life annuity, see Finkelstein and Poterba [44]. (c) *Advantageous selection*: those who have more insurance are lower risk, observed in life and long-term care insurance market, see Cawley and Philipson [24] for life insurance, Finkelstein and McGarry [43] for longterm care insurance and Cutler et al. [31] for a comprehensive review. (d) Insufficient life insurance among the working-aged, see Bernheim et al. [13, 14]. (e) Excessive life insurance among the elderly, see Brown [20] and Cutler and Zeckhauser [30]. We seek to see if these empirical findings can be realized under certain model and parameter setup.

1.2 Structure of the Thesis

Motivated by all the above reasons, this thesis is a collection of three research projects and is organized as follows: Chapter 2 is devoted to introducing the core mathematical preliminaries for the latter chapters. Chapters 3 and 4 consider optimal reinsurance-investment problem with contagion claims and partial information of stock return rate, respectively. Chapter 5 examines how an individual should allocate wealth in life annuity, term life insurance and consumption. In Chapter 6, we end the thesis by providing potential directions for future work. The methodology and the main results of each problem is demonstrated as follows.

In Chapter 3, we study the optimal reinsurance-investment problem applying a dynamic contagion claim model introduced by Dassios and Zhao [35], which allows for self-exciting and externally-exciting clustering effect for the claim arrivals, and includes the well-known Cox process with shot noise intensity and the Hawkes process as special cases. For tractability, we assume that the insurer's risk preference is the time-consistent mean-variance criterion. By utilizing an extended HJB equation approach, a closed-form expression is obtained for the equilibrium reinsurance-investment strategy. An excess-of-loss reinsurance type is shown to be optimal even in the presence of self-exciting and externally-exciting contagion claims, and the strategy depends on both the claim size and claim arrivals assumptions. Further, we show that the self-exciting effect is of a more dangerous nature than the externally-exciting effect as the former requires more risk management controls than the latter. In addition, we find that the reinsurance strategy does not always become more conservative (i.e., transferring more risk to the reinsurer) when the claim arrivals are contagious. Indeed, the insurer can be better off retaining more risk if the claim severity is relatively light-tailed.

In Chapter 4, we investigate optimal reinsurance-investment problem with Bayesian learning of the unobservable stock return rate under the time-consistent mean-variance criterion, where the stock's expected return is modelled as a constant random variable, and the surplus process is modelled as a spectrally negative Lévy process. By reducing the problem to an equivalent one with complete information, and solving an extended HJB equation, we obtain the explicit equilibrium investment-reinsurance strategy. Moreover, comparison between the equilibrium strategy and its myopic counterpart is carefully investigated. We show that the optimal investment strategy with Bayesian learning acts on both the first and second order moment of the posterior distribution of the market price of risk, while its myopic counterpart can only act on the posterior mean.

In Chapter 5, we consider the optimal allocation of life annuity, term life insurance and consumption for an individual with CRRA preference, who seeks to maximize the expected utility from life time discounted consumption, bequest motive and terminal wealth upon survival. Life annuity and term life insurance allow for surrender feature, and are modelled using singular and impulse control. Under a general force of mortality, the problem boils down to finding the solution of a variational inequality with gradient constraints. We numerically solve for the optimal buying and surrendering boundaries respectively for life annuities and term life insurance using penalty methods, and analyze the impact of subjective force of mortality, pricing rate, wealth return rate, wealth volatility, risk preference, safety loading factor, penalty rate factor, tax on legacy on one's willingness to hold life annuities and term life insurance. Discussions are provided on widely documented empirical findings in regard to life annuity and life insurance along our numerical examples, including annuity puzzle, adverse selection in life annuity markets, advantageous selection in life insurance markets, insufficient life insurance among the working-aged, and excessive life insurance among the elderly. These empirical findings are possible in our model under certain parameter settings.

Note that both models and methodologies vary across these three research problems. In Chapter 3, the underlying basic claim process has contagious features, which no longer possesses the stationary and independent increment property, while in Chapter 4, a spectrally negative Lévy process is applied. In Chapters 3 and 4, the reinsurance-investment problem is solved under the time-consistent mean-variance criteria within a deterministic finite horizon, and the optimal strategy is of an equilibrium type, while in Chapter 5, we work in a expected utility maximization framework with a random time horizon. In terms of control variables, in Chapters 3 and 4, control variables are assumed to be absolutely continuous in time with finite rate, while in Chapter 5 this restriction is removed. In regard to methodology, in Chapters 3 and 4, the goal is to solve for explicit solutions of the extended-HJB equations, while in Chapter 5, optimal strategies are found via numerical solutions of variational inequalities.

Finally, it is important to note that each of the Chapters 3-5 corresponds to a research project, which was written independently of each other. Although efforts have been made to keep the notation as consistent as possible, some inconsistencies may remain. The reader is therefore invited to treat each chapter separately from a notational standpoint.

Chapter 2

Mathematical Preliminaries

2.1 Itô's formula and Stochastic Differential Equations

Consider a stochastic process $X = \{X_t\}_{t \geq 0}$ defined on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ satisfying the usual conditions of completeness and right continuity. Throughout the chapter, we assume the state space of X is $\mathbb{R}$.

First we introduce the definition of semimartingales, which form the largest class of processes with respect to which the Itô integral can be defined.

Definition 2.1.1 (Semimartingale)**.** *A semimartingale is a càdlàg adapted process X having a decomposition in the form:*

$$X_t = X_0 + M_t + A_t,$$

where X_0 is finite and $\mathcal{F}_0$-measurable, $M_0 = A_0 = 0$, M is a càdlàg local martingale, and A is an adapted process with finite variation.

Theorem 2.1.2 (Itô's Formula)**.** *Let X be a semimartingale and let ϕ be a $\mathcal{C}^2(\mathbb{R})$ real* *function. Then $\phi(X)$ is again a semimartingale, and the following formula holds:*

$$\phi(X_t) - \phi(X_0) = \int_{0+}^{t} \phi'(X_{s-})dX_s + \frac{1}{2}\int_{0+}^{t} \phi''(X_{s-})d[X,X]_s^c + \sum_{0<s\le t}\{\phi(X_s) - \phi(X_{s-}) - \phi'(X_{s-})\Delta X_s\},$$

where ϕ' (ϕ'' resp.) is the first (second resp.) order derivative of ϕ with respect to X, $[X,X]_s^c$ is the continuous part of the quadratic variation process of X and $\Delta X_t = X_t - X_{t-}$.

For the remainder of this chapter, we introduce the solution of a stochastic differential equation (SDE) and consider the formulation of some stochastic optimization problems based on a one-dimensional diffusion process driven by a one-dimensional Brownian motion, simply as preliminary background. More general dynamics, for instance, a general jump-diffusion process (Chapter 3) and a spectrally negative Lévy process (Chapter 4) will be studied in the later chapters. We refer interested readers to Øksendal and Sulem [84] for stochastic control problems on jump-diffusion processes and Fleming and Soner [46] on Markov processes.

Given deterministic measurable functions $b(s,x)$, $\sigma(s,x) : [t,+\infty)\times\mathbb{R}\to\mathbb{R}$, consider the SDE valued in $\mathbb{R}$:

$$\begin{cases} dX_s = b(s,X_s)ds + \sigma(s,X_s)dW_s, \ s>t, \\ X_t = \zeta. \end{cases} \tag{2.1.1}$$

where $\{W_t\}_{t\ge 0}$ is a one-dimensional Brownian motion with respect to $\{\mathcal{F}_t\}_{t\ge 0}$ and ζ is $\mathcal{F}_t$-measurable.

Definition 2.1.3 (Strong solution of a SDE)**.** *An $\{\mathcal{F}_s\}_{s\ge t}$-adapted continuous process X is called a strong solution of (2.1.1) if $X_t = \zeta$, $\mathbb{P}$-a.s.,*

$$\int_t^s |b(u,X_u)|du + \int_t^s |\sigma(u,X_u)|^2 du < \infty, \ \forall s \ge t, \mathbb{P}\text{-}a.s.,$$

and

$$X_s = X_t + \int_t^s b(u, X_u)du + \int_t^s \sigma(u, X_u)dW_u, \ \ s \geq t, \mathbb{P}\text{-}a.s.$$

A sufficient condition for the existence and uniqueness of a strong solution to the SDE (2.1.1) is given in the following theorem (Oksendal [83] Theorem 5.2.1 on page 66).

Theorem 2.1.4. *Let $b, \sigma : [t, +\infty) \times \mathbb{R} \to \mathbb{R}$ be measurable functions satisfying*

(1) $|b(s, x) - b(s, y)| + |\sigma(s, x) - \sigma(s, y)| \leq K|x - y|$ for all $x, y \in \mathbb{R}, s \geq t$.

(2) $|b(s, x)| + |\sigma(s, x)| \leq K(1 + |x|)$ for all $x \in \mathbb{R}, s \geq t$.

Given a $\mathcal{F}_t$-measurable random variable ζ with $\mathbb{E}[|\zeta|^2] < \infty$, there exists a unique continuous solution X to the SDE (2.1.1), and the solution is square integrable.

2.2 Formulation of Some Stochastic Control Problems

2.2.1 Standard Stochastic Control Problems

A diffusion process $X = \{X_t\}_{t \geq 0}$ under a control $\alpha = \{\alpha_t\}_{t \geq 0}$ can be described by a SDE valued in $\mathbb{R}$:

$$dX_s = b(s, X_s, \alpha_s)ds + \sigma(s, X_s, \alpha_s)dW_s. \tag{2.2.1}$$

The control $\alpha = \{\alpha_t\}_{0 \leq t \leq T}$ is a progressively measurable process, valued in $A \subset \mathbb{R}$. When this controlled SDE admits a unique strong solution starting from x at $s = t$, we then denote by $\{X_s^{t,x}, t \leq s \leq T\}$ this solution with a.s. continuous paths. For a constant control a and a function $\eta \in \mathcal{C}^{1,2}(\mathbb{R}_+ \times \mathbb{R})$, define the infinitesimal generator of the controlled diffusion process by

$$\mathcal{L}^a \eta = b(t, x, a)\eta_x + \frac{1}{2}\sigma^2(t, x, a)\eta_{xx},$$

where η_x (η_{xx} resp.) is the first (second resp.) order partial derivative of η with respect to x.

We fix a finite time horizon $T > 0$ with $0 < T < \infty$. Let $f : [0, T] \times \mathbb{R} \times A \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be two measurable functions satisfying:

(i) g is lower-bounded or

(ii) $|g(x)| \leq C(1+|x|^2),\ \forall x \in \mathbb{R}$ for some constant C independent of x.

For $(t,x) \in [0,T]\times\mathbb{R}$, denote by $\mathcal{A}(t,x)$ the set of controls α such that $\mathbb{E}\Big[\int_t^T |f(s, X_s^{t,x}, \alpha_s)|ds\Big] < \infty$.

Definition 2.2.1 (Value function)**.** *For all $(t,x) \in [0,T]\times\mathbb{R}$ and $\alpha \in \mathcal{A}(t,x)$, the gain function is defined as:*

$$J(t,x;\alpha) = \mathbb{E}\Big[\int_t^T f(s, X_s^{t,x}, \alpha_s)ds + g(X_T^{t,x})\Big], \tag{2.2.2}$$

and the associated value function is

$$v(t,x) = \sup_{\alpha\in\mathcal{A}(t,x)} J(t,x;\alpha). \tag{2.2.3}$$

Given an initial condition $(t,x) \in [0,T)\times\mathbb{R}$, we say that $\hat{\alpha} \in \mathcal{A}(t,x)$ is an optimal control if $v(t,x) = J(t,x;\hat{\alpha})$. A control process α in the form $\alpha_s = a(s, X_s^{t,x})$ for some measurable function a from $[t,T]\times\mathbb{R}$ into A, is called a Markovian control. In the sequel, we are interested in Markovian controls.

Theorem 2.2.2 (HJB equation)**.** *Suppose functions b, σ, f and g are uniformly continuous, and there exists a constant $L>0$ such that for $\varphi(t,x,a) = b(t,x,a), \sigma(t,x,a), f(t,x,a), g(x)$, $\forall t \in [0,T], x, y \in \mathbb{R},\ a \in A$,*

$$|\varphi(t,x,a) - \varphi(t,y,a)| \leq L|x-y| \ \text{and}\ |\varphi(t,0,a)| \leq L.$$

If the value function $v \in \mathcal{C}^{1,2}([0,T]\times\mathbb{R})$, then v is a solution of the following terminal value problem of a second-order PDE, called the Hamilton-Jacobi-Bellman equation (HJB equation, for short):

$$\begin{cases} -\dfrac{\partial v}{\partial t}(t,x) - H(t,x,v_x,v_{xx}) = 0,\ (t,x) \in [0,T)\times\mathbb{R}, \\ v(T,x) = g(x),\ x \in \mathbb{R}, \end{cases} \tag{2.2.4}$$

where for $(t, x, p, M) \in [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$,

$$H(t, x, p, M) = \sup_{a \in A} \Big[b(t, x, a)p + \frac{1}{2}\sigma^2(t, x, a)M + f(t, x, a) \Big]. \tag{2.2.5}$$

H is called the Hamiltonian of the associated control problem.

See Yong and Zhou [104] (Proposition 3.5 on page 182) for a formal derivation. In the following, we briefly discuss how the solutions to the HJB equation might help us in finding an optimal control.

Theorem 2.2.3 (Verification theorem)**.** *Let* $w \in \mathcal{C}^{1,2}([0, T] \times \mathbb{R}))$ *be a solution to (2.2.4) satisfying a quadratic growth condition:*

$$|w(t, x)| \leq C(1 + |x|^2), \ \forall (t, x) \in [0, T] \times \mathbb{R}. \tag{2.2.6}$$

If there exists a measurable function $\alpha^*(t, x) : [0, T] \times \mathbb{R} \to A$ *such that*

$$\begin{aligned} -\frac{\partial w}{\partial t}(t, s) - \sup_{a \in A} \Big[\mathcal{L}^a w(t, x) + f(t, x, a) \Big] &= -\frac{\partial w}{\partial t} - \mathcal{L}^{\alpha^*(t,x)} w(t, x) - f(t, x, \alpha^*(t, x)) \\ &= 0, \end{aligned} \tag{2.2.7}$$

and the SDE

$$dX_s = b(s, X_s, \alpha^*(s, X_s))ds + \sigma(s, X_s, \alpha^*(s, X_s))dW_s$$

admits a unique strong solution, denoted by $X_s^{*(t,x)}$ *starting at* $X_t = x$*, and the process* $\{\alpha^*(s, X_s^{*(t,x)}), t \leq s \leq T\} \in \mathcal{A}(t, x)$*. Then*

$$w = v \ \text{on} \ [0, T] \times \mathbb{R},$$

and α^* *is an optimal Markovian control.*

In Chapters 3 and 4, we will look for explicit solutions of time-inconsistent stochastic control problems using verification argument, in the same spirit as the one described above. This technique requires that the HJB equation admit classical solutions, meaning that the solutions be smooth enough, which isn't true in general, unfortunately. See for instance, example 2.3 on page 163 in Yong and Zhou [104]. To find a rigorous assertion similar in nature to Theorem 2.2.2 but without restrictive assumptions, viscosity solutions were introduced by Crandall and Lions in the early 1980s, whose key feature is to replace the conventional derivatives by the (set-valued) super/subdifferentials while maintaining the uniqueness of solutions under very mild conditions, see Fleming and Soner [46] for a detailed discussion.

For standard stochastic control problems, the displacement of the state due to control effort is differentiable in time. However, the state may be affected drastically due to possible "singular behaviour" of a control at a certain time point. A *singular control* can better model such situations. In the following, we briefly introduce an optimization problem with mixed type of controls: an absolutely continuous control and a singular control. A similar formulation, but within a finite time horizon will be carried out in Chapter 5.

2.2.2 Singular Control Problems

Let $\kappa : \mathbb{R} \to \mathbb{R}$ and $\theta : \mathbb{R} \to \mathbb{R}$ be given continuous functions. With slightly abuse of notations, let $X_t = X_t^{\alpha,\xi} \in \mathbb{R}$ be described by:

$$dX_t = b(X_t, \alpha_t)dt + \sigma(X_t, \alpha_t)dW_t + \kappa(X_{t-})d\xi_t,\ X_0 = x \in \mathbb{R}. \tag{2.2.8}$$

Here $\{\xi_t\}_{t\geq 0}$ valued in $\mathbb{R}$ is an adapted càdlàg finite variation process with increasing components and $\xi(0-) = 0$. Since $d\xi_t$ is allowed to be singular with respect to Lebesgue measure dt, we call ξ a singular control. The process $\{\alpha_t\}_{t\geq 0}$ is an adapted càdlàg process with values in $A \subset \mathbb{R}$, and we call $\alpha_t dt$ the absolutely continuous control.

Definition 2.2.4 (Value function)**.** *The gain functional $J(x;\alpha,\xi)$ is of the form*

$$J(x;\alpha,\xi) = \mathbb{E}\Big[\int_0^{\tau_S} f(X_t,\alpha_t)dt + g(X_{\tau_S})\mathbb{I}_{\tau_S<\infty} + \int_0^{\tau_S} \theta(X_{t-})d\xi_t\Big], \tag{2.2.9}$$

where $f : \mathbb{R} \times A \to \mathbb{R}$, $g : \mathbb{R} \to \mathbb{R}$ and $\theta : \mathbb{R} \to \mathbb{R}$ are given continuous functions and $\tau_S = \inf\{t > 0 : X_t^{\alpha,\xi} \notin S\} \leq \infty$ is the time of bankruptcy, where $S \subset \mathbb{R}$ is a given solvency *set with $S \subset \bar{S}^0$, where S^0 is the interior of S and $\bar{S}^0$ its closure. The problem is to find the value function*

$$v(x) = \sup_{(\alpha,\xi)\in\mathcal{A}} J(x;\alpha,\xi) = J(x;\alpha^*,\xi^*),$$

where $\mathcal{A}$ is a family of admissible controls (α,ξ) such that a unique strong solution X of (2.2.8) exists and $\mathbb{E}\Big[\int_0^{\tau_S}|f(X_t,u(t)|dt + |g(X_{\tau_S})|\mathbb{I}_{\tau_S<\infty} + \int_0^{\tau_S}|\theta(X_{t-})|d\xi_t\Big] < \infty$.

Let $\mathcal{L}^\alpha$ be the infinitesimal generator when we apply a constant control $\alpha \in A$ and $d\xi = 0$, i.e.,

$$\mathcal{L}^\alpha\eta = b(x,\alpha)\eta'(x) + \frac{1}{2}\sigma^2(x,\alpha)\eta''(x). \tag{2.2.10}$$

Next we introduce a verification theorem for the above optimization problem (Øksendal and Sulem [84] Theorem 8.2 on page 229).

Theorem 2.2.5 (Verification theorem)**.**

(a) Suppose there exists a function $w \in \mathcal{C}^2(S^0)\cap\mathcal{C}(\mathbb{R})$ such that

- *(i) $\mathcal{L}^a w(x) + f(x,a) \le 0$ for all constant $a \in A$ and $x \in S$.*
- *(ii) $\kappa(x)w'(x) + \theta(x) \le 0$ for all $x \in S$.*
- *(iii) $\mathbb{E}\Big[\int_0^{\tau_S}(\sigma(X_s,a)w'(X_s))^2 ds\Big] < \infty$ for all $(\alpha,\xi)\in\mathcal{A}$.*
- *(iv) $w(x) = g(x)$ for all $x \notin S$.*
- *(v) $\{w^-(X_\tau)\}_{\tau\le\tau_S}$ is uniformly integrable for all $(\alpha,\xi)\in\mathcal{A}$, $x\in S$.*

Then $w(x) \ge v(x)$ for all $x \in S$.

(b) Define the non-transaction region D by

$$D = \{x \in S : \kappa(x)w'(x) + \theta(x) < 0\}.$$

Suppose, in addition to (i)-(v) above, that for all $x \in \bar{D}$ there exists $\alpha^* = \alpha^*(x)$ such that *(i)* $\mathcal{L}^{\alpha^*(x)}w(x)+f(x,\alpha^*(x))=0$. *Moreover, suppose there exists* ξ^* *such that* $(\alpha^*,\xi^*)\in\mathcal{A}$ *and*

(ii) $X_t^{\alpha^*,\xi^*}\in\bar{D}$ *for all* t.

(iii) $(\kappa(X_{t-})w'(X_{t-})+\theta(X_{t-}))d\xi^{*(c)}=0$ *for all* t, *where* $\xi_t^{(c)}$ *is the continuous part of* ξ_t.

(iv) $\Delta w(X_{t_n})+\theta(X_{t_n-})\Delta\xi^*_{t_n}=0$ *for all jumping times* t_n *of* ξ^*.

(v) $\lim_{R\to\infty}\mathbb{E}[w(X_{T_R}^{\alpha^*,\xi^*})]=\mathbb{E}[g(X_{\tau_S}^{\alpha^*,\xi^*})\mathbb{I}_{\tau_S<\infty}]$, *where* $T_R=\min(\tau_S,R)$ *for* $R<\infty$.

Then $w(y)=v(y)$ *and* (α^*,ξ^*) *is an optimal control.*

In Chapter 5, we will consider a finite-horizon problem where the decisions of life annuity and term life insurance will be formulated using singular control and the consumption will be modelled as an absolutely continuous control. Instead of looking for the explicit solution with verification argument, which unlikely exists, we seek to numerically solve for the non-transaction region based on a time-dependent variational inequality. Also note that when the state starts from outside the non-transaction region, an impulse control will be exercised to move instantaneously to its boundary, and we refer interested readers to Øksendal and Sulem [84] for formulations of impulse control problems for jump-diffusion states.

Chapter 3

Optimal Reinsurance-Investment Strategy for a Dynamic Contagion Claim Model

3.1 Introduction

Optimal reinsurance-investment problem is one of the core research problems in actuarial science. Indeed, purchasing reinsurance can protect insurers against adverse claim experience, while investment further allows insurers to diversify their risks and enjoy a higher rate of return on the insurance portfolio's cash flows. Deeply entrenched in the comprehensive body of literature on this research topic, the goal often consists in solving for the optimal reinsurance arrangement and investment decision to achieve a clearly defined objective (e.g., minimizing ruin probability or maximizing expected utility).

In the existing literature on this topic, the underlying surplus process (before adopting a joint reinsurance and investment strategy) is commonly assumed to follow a compound Poisson, a linear Brownian motion, or more generally a spectrally negative Lévy process (e.g., Schmidli [93], Liu and Yang [76], Promislow and Young [88], Bai and Guo [7], Zeng and Li [106], and Li et al. [68]). In this Lévy framework, it is assumed that the claim arrivals have independent and stationary increments, an assumption which is often challenged or seriously violated in a large number of insurance contexts (e.g., catastrophic risks); see Seal et al. [94] and Beard [12]. More specifically, insurance claims are known to have various degrees of contagion and such *clustering* feature cannot be captured by a Lévy model.

Clustering due to *exogenous (externally-excited)* factors, such as earthquakes, flood, and hurricanes, might be captured using a Cox process which was introduced by Cox [29]. In insurance contexts, many researchers have suggested using a Cox process to model the claim arrival dynamics including Björk and Grandell [15], Embrechts et al. [41], Schmidli [91], and Albrecher and Asmussen [2]. The jump intensity of a Cox process not only depends on time but is also allowed to be a stochastic process. On the other hand, clustering due to *endogenous (self-excited)* factors, such as aggressive driving habits and poor health conditions, can be characterized using a Hawkes process (see Hawkes [55]). The self-exciting property of Hawkes processes means that the occurrence of any event increases the likelihood of future such events. Stabile and Torrisi [97] studied the ruin problem of an insurance risk model modelled by the Hawkes process. We also refer the readers to Hawkes [56] and Bacry et al. [6] for a comprehensive survey on Hawkes processes and their applications in finance. Recently, Dassios and Zhao [35] introduced a dynamic contagion process by generalizing both the Cox process with shot noise intensity and the Hawkes process. This process includes both self-excited and externally-excited jumps and is extremely versatile for modelling purposes, allowing for a wide variety of features in the claim arrival dynamics (such as the frequency, magnitude of the impact, and the decay with time) to be captured. We refer the reader to Dassios and Zhao [35] for more on this, as well as an analysis of a ruin problem in infinite time horizon in Dassios and Zhao [36].

Claim data generated from a Markov-modulated Poisson process can also *resemble* the feature of contagion and Liang and Bayraktar [72] apply this model to study the problem of optimal proportional reinsurance. As the state transits from one with lower intensity to another with higher intensity, the arrival of claims can be observed to be more exciting. However, there are no external events or dependent mechanism triggering such phenomenon. Thus a dynamic contagion process is more appropriate to model claim arrival with external triggering events and dependent occurrence explicitly. Models where there can be dependence between the claim arrival and the claim size, see e.g., Boudreault et al. [18] and Albrecher and Teugels [3], are also of great importance and worthy to work on for future research.

In this chapter, we propose to study the optimal reinsurance-investment problem in the framework of the claim contagion model introduced by Dassios and Zhao [35]. To mitigate the insurance risk, the insurer determines the optimal reinsurance arrangement. In contrast to most of the relevant literature on this topic, we do not limit the type of reinsurance to be of proportional or excess-of-loss form. We find later that the excess-of-loss reinsurance treaty is indeed optimal. The insurer is also allowed to participate in a financial market consisting of a risk-free bond with fixed risk-free rate and a risky stock whose price follows a geometric Brownian motion. The objective is to maximize the insurer's expected terminal utility with the utility function chosen to be the time-consistent mean-variance criterion. Following the seminal work by Basak and Chabakauri [8] and Björk et al. [16], this risk preference has become very popular in recent years; we refer the reader to Li et al. [67] and Landriault et al. [66] for a more detailed discussion of its applications in insurance and finance. The main advantage of this time-consistent mean-variance criterion is that the form of the corresponding value function is very simple and hence, more likely to yield explicit solutions.

It is worth pointing out that, the literature is rather scarce on optimal reinsurance and investment problems beyond the traditional Lévy framework. Delong and Gerrard [39] is a notable exception for which this chapter significantly differs on the following grounds. First, our dynamic contagion process can capture both (endogenous) self-exciting and (exogenous) externally-exciting factors while Delong and Gerrard [39] uses the diffusion type Cox process which only has the exogenous factor. Second, Delong and Gerrard [39] solves for the optimal investment problem when the stock price is driven by a Lévy process, while in our work we solve for both the investment and reinsurance problem. Third, Delong and Gerrard [39] uses the pre-commitment mean-variance criterion, but we use the time-consistent mean-variance criterion. As pointed out by many researchers, time consistency is a basic requirement of rational decision making (e.g., Strotz [98]).

The joint equilibrium reinsurance-investment strategy[1] is obtained in closed form by solving the associated extended Hamilton-Jacobi-Bellman equation. Next, we summarize our main findings and implications:

- Unlike in the Lévy setup, the insurer's optimal reinsurance strategy is shown to take both the claim arrival rate and the claim size distribution into account.
- The excess-of-loss reinsurance type is shown to be optimal even in the presence of a self-exciting and externally-exciting claim contagion effect, a finding in line with Li et al. [68] in the standard Lévy risk model framework.
- The externally-exciting effect is shown be well hedged by adjusting only the premium rate, while the self-exciting effect shall to be mitigated by adjusting both the premium rate and reinsurance strategy. In other words, the self-exciting effect is of a more dangerous nature because its control requires more risk management tools.
- The optimal reinsurance strategy does not always become more conservative (i.e., transferring more risk to the reinsurer) when the claim arrivals become contagious. In fact, if the claim severity is relatively light-tailed, the insurer is shown to be better off by retaining more risk. This is because more insurance premium can be collected as the expected premium principle is adopted.

The remainder of this chapter is organized as follows. Section 2 formally introduces our problem including the dynamic contagion process, controlled surplus process, and the objective function. Section 3 presents the main results of this chapter. In Section 4, several numerical studies are carried to determine the impact of the model's parameters on the optimal reinsurance strategy. All technical proofs can be found in Appendix.

[1] The optimal solution under the time-consistent mean-variance criterion is called an equilibrium solution because time inconsistency is addressed through a non-cooperative game.

3.2 Problem Formulation

Let $(\Omega, \mathcal{F}, \mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a filtered probability space satisfying the usual conditions of completeness and right continuity. We consider a fixed time horizon $T > 0$. In what follows, all the processes are assumed to live in this space.

3.2.1 Dynamic contagion process

For completeness, we first recall the definition of the dynamic contagion process from Dassios and Zhao [35].

Definition 3.2.1. *A dynamic contagion process $\{N_t\}_{t\geq 0}$ is a point process defined as $N_t = \sum_{k\geq 1} \mathbb{I}_{\{T_k \leq t\}}$, where the stochastic intensity process $\lambda_t = \lim_{\varepsilon \downarrow 0} \frac{\mathbb{E}[N_{t+\varepsilon} - N_t | \mathcal{F}_t]}{\varepsilon}$ is defined as*

$$\lambda_t = \beta + (\lambda_0 - \beta)e^{-\alpha t} + \sum_{i\geq 1} Z_i e^{-\alpha(t - T_i^{(1)})} \mathbb{I}_{\{T_i^{(1)} \leq t\}} + \sum_{k\geq 1} R_k e^{-\alpha(t - T_k)} \mathbb{I}_{\{T_k \leq t\}}, \tag{3.2.1}$$

where

- $\beta \geq 0$ *is the constant reversion level,*
- $\lambda_0 > 0$ *is the initial value of* λ_t*,*
- $\alpha > 0$ *is the constant rate of exponential decay,*
- *The externally-excited jumps $\{Z_i\}_{i\geq 1}$ form a sequence of iid (independent and identically distributed) nonnegative random variables whose arrival times $\{T_i^{(1)}\}_{i\geq 1}$ are those of an independent Poisson process $\{M_t\}_{t\geq 0}$ with constant intensity $\rho > 0$,*
- *The self-excited jumps $\{R_k\}_{k\geq 1}$ is a sequence of iid nonnegative random variables with arrival times $\{T_k\}_{k\geq 1}$.*
- $\{Z_i\}_{i\geq 1}$, $\{T_i^{(1)}\}_{i\geq 1}$ *and* $\{R_k\}_{k\geq 1}$ *are assumed to be independent of each other.*

As mentioned above, the dynamic contagion process $\{N_t\}_{t\geq 0}$ is a generalization of the Hawkes process and the Cox process with shot noise intensity. It offers a great deal of versatility for modelling purposes as the clustering effect of claims can be captured in two possible ways. On the one hand, it is self-exciting. An arrival in the contagion process $\{N_t\}_{t\geq 0}$ increases the jump intensity by an instantaneous amount of R_k (for the k-th arrival of $\{N_t\}_{t\geq 0}$). On the other hand, it is externally-exciting. The jump of an external Poisson process $\{M_t\}_{t\geq 0}$ increases the jump intensity of the contagion process $\{N_t\}_{t\geq 0}$ by an instantaneous amount of Z_k (for the k-th arrival of $\{M_t\}_{t\geq 0}$). In addition, the jump intensity decays exponentially between two consecutive jumps, and thus the pair $(N_t, \lambda_t)_{t\geq 0}$ is a Markov processes. Specifically, the intensity λ_t defined in (3.2.1) can be rewritten in a Markovian form as

$$d\lambda_t = \alpha(\beta - \lambda_t)dt + ZdM_t + RdN_t. \tag{3.2.2}$$

The objective is to study the optimal reinsurance-investment problem for this risk process. In the following, we denote by $m_R = \mathbb{E}[R]$, $m_Z = \mathbb{E}[Z]$, $n_R = \mathbb{E}[R^2]$, and $n_Z = \mathbb{E}[Z^2]$. We use $v_Z(\cdot)$ and $v_R(\cdot)$ to represent the probability measure of Z and R, respectively. Moreover, we assume that $\alpha > m_R$, which is the stationary condition for the intensity process (see Dassios and Zhao [36] for more detailed discussion on this condition).

3.2.2 Controlled surplus process

Suppose that an insurer's aggregate claim process is modelled as a compound dynamic contagion process $\{C_t\}_{t\geq 0}$ defined as

$$C_t = \begin{cases} \sum_{i=1}^{N_t} Y_i, & N_t > 0, \\ 0, & N_t = 0, \end{cases} \tag{3.2.3}$$

where $\{Y_i\}_{i\geq 1}$ is a sequence of iid nonnegative random variables representing the claim severity with mean $m_Y = \mathbb{E}[Y]$, second moment $n_Y = \mathbb{E}[Y^2]$, survival function $S_Y(\cdot)$, and probability measure $v_Y(\cdot)$. The claim arrival process $\{N_t\}_{t\geq 0}$, independent of $\{Y_i\}_{i\geq 1}$, is as defined in Definition 3.2.1.

Note that $m_Y \int_0^t \lambda(u)du$ is the compensator of $\{C_t\}_{t\geq 0}$ and thus $\{C_t - m_Y \int_0^t \lambda(u)du\}_{t\geq 0}$ is a martingale, then for any $0 \leq s < t$, $\mathbb{E}[C_t - m_Y \int_0^t \lambda(u)du|\mathcal{F}_s] = C_s - m_Y \int_0^s \lambda(u)du$, or equivalently,

$$\mathbb{E}[C_t - C_s|\mathcal{F}_s] = m_Y \int_s^t \lambda(u)du.$$

Then the insurance premium is collected using the expected value principle, and the insurer's surplus process $\{U_t\}_{t\geq 0}$ (without investment and reinsurance) follows

$$dU_t = (1+\theta)\mathbb{E}[dC_t|\mathcal{F}_t] - dC_t = (1+\theta)m_Y\lambda_t dt - Y dN_t, \tag{3.2.4}$$

where $\theta > 0$ is the insurer's risk loading factor. As in Delong and Gerrard [39], we note that the premium rate is proportional to the time-dependent and stochastic jump intensity λ_t. We want to point out that a trusted third party can be hired to estimate and collect the premium, provided certain model calibration techniques can be applied. Kirchner [65] and Lim et al. [75] discuss the simulation, model calibration and estimation procedures for multivariate Hawkes processes, which include the dynamic contagion process as a special case.

The insurer can mitigate the insurance risk by entering into a reinsurance arrangement. For an instantaneous loss of size y occurring at time t, the reinsurance strategy adopted by the insurer can be represented by the retention function $l(t,y) : [0,T] \times (0,+\infty) \to [0,y]$, and later we show that the optimal strategy would indeed be a feedback control. The remaining part of the risk $y - l(t,y)$ will be undertaken by a reinsurer with risk loading factor η with $\eta > \theta$. Under the reinsurance strategy $\{l(t,y)\}_{t\in[0,T],y>0}$, the surplus process evolves as

$$\begin{aligned} dR_t^l &= dU_t + (Y - l(t,Y))\, dN_t - (1+\eta)\mathbb{E}\left[(Y - l(t,Y))\, dN_t|\mathcal{F}_t\right] \\ &= (1+\theta)\lambda_t \int_0^{+\infty} y v_Y(dy)dt - l(t,Y)dN_t - (1+\eta)\lambda_t \int_0^{+\infty} (y - l(t,y))\, v_Y(dy)dt \\ &= \lambda_t \int_0^{+\infty} ((\theta-\eta)y + (1+\eta)l(t,y))\, v_Y(dy)dt - l(t,Y)dN_t. \end{aligned} \tag{3.2.5}$$

We further assume that the insurer can invest in the financial market consisting of a risk-free bond with constant risk-free rate $r \geq 0$ and a risky stock $\{S_t\}_{t\geq 0}$, whose dynamic is assumed to follow a geometric Brownian motion

$$dS_t = \mu S_t dt + \sigma S_t dW_t,$$

where $\mu > 0$ is the stock's return rate, σ is the stock's volatility and $\{W_t\}_{t\geq 0}$ is a standard Brownian motion, independent of the original surplus process $\{U_t\}_{t\geq 0}$[2]. Let π_t denote the dollar amount invested in the stock at time t. Under the reinsurance-investment strategy $u := (l(t), \pi_t)_{t\in[0,T]}$, the controlled surplus process $\{X_t^u\}_{t\in[0,T]}$ follows the dynamic

$$\begin{aligned} dX_t^u &= \frac{\pi_t}{S_t} dS_t + r(X_t^u - \pi_t)dt + dR_t^l \\ &= \left((\mu - r)\pi_t + rX_t^u + \lambda_t \int_0^{+\infty} ((\theta-\eta)y + (1+\eta)l(t,y))\, v_Y(dy) \right) dt \\ &\quad + \sigma \pi_t dW_t - l(t,Y)dN_t. \end{aligned} \tag{3.2.6}$$

We then denote by $\{(X_s^{u,t,x,\lambda}, \lambda_s^{t,\lambda})\}_{t\leq s\leq T}$ the solution of SDEs (3.2.6) and (3.2.2) starting from (x, λ) at time t under strategy u. A reinsurance-investment strategy $u = \{l_t, \pi_t\}_{t\in[0,T]}$ is called an admissible strategy if both l and π are $\mathbb{F}$-progressively measurable such that

$$\int_0^T \int_0^{+\infty} l^2(t,y) v_Y(\mathrm{d}y)\mathrm{d}t + \int_0^T \pi_t^2 \mathrm{d}t < +\infty \text{ a.s.}$$

The set of all admissible strategies is denoted by Π.

3.2.3 Objective function

The insurer's reward function is assumed to be of the mean-variance form given by

[2]It is standard to incorporate dependence between $\{S_t\}_{t\geq 0}$ and $\{U_t\}_{t\geq 0}$ by adding a Brownian motion to $\{U_t\}_{t\geq 0}$ which is correlated with the one in $\{S_t\}_{t\geq 0}$. The only consequence is an additional hedging term in the optimal investment strategy, while the optimal reinsurance strategy remains unchanged (e.g., Li et al. [67] and Li et al. [68]). The interested readers may refer to Li et al. [68] to see how this interdependence may affect the equilibrium investment strategy. We assume independence here to focus on the impact of the claims dynamic contagion effect on the optimal reinsurance strategy .

$$J^u(t,x,\lambda) := \mathbb{E}[X_T^{u,t,x,\lambda}] - \frac{\gamma}{2}\mathrm{Var}[X_T^{u,t,x,\lambda}],$$

where $\gamma > 0$ is the insurer's risk aversion parameter. Our objective is thus to maximize the total reward at maturity T over the set of admissible strategies Π, i.e.,

$$\max_{u\in\Pi} J^u(t,x,\lambda). \tag{3.2.7}$$

Due to the presence of the variance part, the dynamic mean-variance criterion (3.2.7) has the well-known issue of time-inconsistency, that is, the dynamic programming principle fails. We follow one of the main approaches to handle this problem by treating the decision-making process as a non-cooperative game against all strategies implemented by future players (see, e.g., Björk et al. [16] for the general theory and Landriault et al. [66] for a particular application). The solutions of this game problem are called *equilibrium strategies*, which are defined as follows.

Definition 3.2.2. *Let $u^* = (l^*, \pi^*) \in \Pi$ be an admissible strategy. For any $(t,x,\lambda) \in [0,T]\times\mathbb{R}\times(0,+\infty)$, we define a perturbed strategy u^ε as*

$$u^\varepsilon(s,y) = \begin{cases} u, & s\in[t,t+\varepsilon),\ y>0, \\ (l^*(s,y),\pi^*_s), & s\in[t+\varepsilon,T],\ y>0, \end{cases}$$

where $u = (l,\pi) \in [0,y]\times\mathbb{R}$ and $\varepsilon > 0$. Suppose that, for any $(t,x,\lambda) \in [0,T]\times\mathbb{R}\times(0,+\infty)$ and $u = (l,\pi) \in [0,y]\times\mathbb{R}$ we have

$$\liminf_{\varepsilon\to 0} \frac{J^{u^*}(t,x,\lambda) - J^{u^\varepsilon}(t,x,\lambda)}{\varepsilon} \geq 0.$$

Then u^* is called an equilibrium reinsurance-investment strategy for (3.2.7) and $J^{u^*}(t,x,\lambda)$ is the corresponding value function.

3.3 Main Results

We denote by $C^{1,2,1}([0,T] \times \mathbb{R} \times (0,+\infty))$ the space of functions which are first-order continuously differentiable in $t \in [0,T]$, second-order continuously differentiable in $x \in \mathbb{R}$, and first-order continuously differentiable in $\lambda \in (0,+\infty)$. For any $\phi(t,x,\lambda) \in C^{1,2,1}([0,T] \times \mathbb{R} \times (0,+\infty))$, by (5.5.1) and (3.2.2), the infinitesimal generator of X^u is given by

$$
\begin{aligned}
&\mathcal{A}^u\phi(t,x,\lambda)\\
&= \left(rx + (\mu - r)\pi + \lambda \int_0^{+\infty} \left((\theta-\eta)y + (1+\eta)l(t,y)\right) v_Y(dy) \right) \phi_x(t,x,\lambda)\\
&\quad + \frac{1}{2}\sigma^2\pi^2\phi_{xx}(t,x,\lambda) + \phi_t(t,x,\lambda) + \alpha(\beta-\lambda)\phi_\lambda(t,x,\lambda)\\
&\quad + \rho \int_0^{+\infty} \left(\phi(t,x,\lambda+z) - \phi(t,x,\lambda)\right) v_Z(dz)\\
&\quad + \lambda \int_0^{+\infty}\int_0^{+\infty} \left(\phi(t,x-l(t,y),\lambda+r) - \phi(t,x,\lambda)\right) v_Y(dy)v_R(dr),
\end{aligned}
$$

provided that $\int_0^{+\infty} |\phi(t,x,\lambda+z)|v_Z(dz) < +\infty$ and $\int_0^{+\infty}\int_0^{+\infty} |\phi(t,x-l(t,y),\lambda+r)|v_Y(dy)v_R(dr) < +\infty$.

Next, we provide a verification theorem for an equilibrium reinsurance-investment strategy. This is a special case of Theorem 5.2 in Björk et al. [16]. A sketch of proof is provided in Appendix 3.6.

Theorem 3.3.1. *Suppose there exist functions $V(t,x,\lambda)$, $g(t,x,\lambda) \in C^{1,2,1}([0,T] \times \mathbb{R} \times [0,+\infty))$ satisfying a quadratic growth condition in x and the following conditions hold:*

1. *For all $(t,x,\lambda) \in [0,T] \times \mathbb{R} \times (0,+\infty)$,*

$$
\sup_{u=(l(t,y),\pi)\in[0,y]\times\mathbb{R}} \left\{ \mathcal{A}^u V(t,x,\lambda) - \frac{\gamma}{2}\mathcal{A}^u g^2(t,x,\lambda) + \gamma g(t,x,\lambda)\mathcal{A}^u g(t,x,\lambda) \right\} = 0. \tag{3.3.1}
$$

We denote by $u^* = (l^*, \pi^*)$ the strategy that attains the supremum in (3.3.1).

2. For all $(t,x,\lambda)\in[0,T]\times\mathbb{R}\times(0,+\infty)$,

$$\mathcal{A}^{u^*}g(t,x,\lambda)=0. \tag{3.3.2}$$

3. For all $(x,\lambda)\in\mathbb{R}\times[0,+\infty)$,

$$V(T,x,\lambda)=g(T,x,\lambda)=x. \tag{3.3.3}$$

Then $u^*=(l^*,\pi^*)$ *is an equilibrium investment-reinsurance strategy for objective (3.2.7). Moreover,* $V(t,x,\lambda)=J^{u^*}(t,x,\lambda)$ *and* $g(t,x,\lambda)=\mathbb{E}[X_T^{u^*,t,x,\lambda}]$.

In the following theorem, an equilibrium reinsurance-investment strategy and the corresponding value function are formally stated.

Theorem 3.3.2. *An equilibrium reinsurance-investment strategy* $u^*=(\pi^*,l^*)$ *for objective (3.2.7) is given by*

$$l^*(t,y)=y\wedge RL^*(t)\ \text{and}\ \pi_t^*=\frac{\mu-r}{\gamma\sigma^2}e^{-r(T-t)}, \tag{3.3.4}$$

where the retention limit $RL^*(t)=\dfrac{[\eta+\gamma m_R k(t)]^+}{\gamma}e^{-r(T-t)}$*, and the function* $\{k(t)\}_{t\in[0,T]}$ *is the unique solution of the ordinary differential equation (ODE)*

$$\begin{cases} k'(t)+(m_R-\alpha)k(t)+e^{r(T-t)}\left((\theta-\eta)m_Y+\eta\int_0^{\frac{1}{\gamma}[\eta+\gamma m_R k(t)]^+e^{-r(T-t)}}S_Y(y)dy\right)=0,\\ k(T)=0. \end{cases} \tag{3.3.5}$$

Moreover, the equilibrium value function is given by

$$V(t,x,\lambda)=J^{u^*}(t,x,\lambda)=e^{r(T-t)}x+A(t)+K(t)\lambda,$$

where

$$A(t)=\int_t^T\left((\alpha\beta+\rho m_Z)K(s)+\frac{(\mu-r)^2}{2\gamma\sigma^2}-\frac{\gamma\rho}{2}k^2(s)n_Z\right)ds, \tag{3.3.6}$$

and $K(t)$ is the unique solution of the ODE

$$\begin{cases} K'(t) + (m_R - \alpha)K(t) + e^{r(T-t)}(\theta - \eta)m_Y + G(t) - \frac{\gamma}{2}k^2(t)n_R = 0, \\ K(T) = 0. \end{cases} \tag{3.3.7}$$

with

$$\begin{aligned} G(t) \;\; &= \;\; \int_0^{+\infty} \Bigg(-\frac{\gamma}{2} e^{2r(T-t)} \left(y \wedge \frac{[\eta + \gamma m_R k(t)]^+}{\gamma} e^{-r(T-t)} \right)^2 \\ &\quad + (\eta + \gamma m_R k(t)) e^{r(T-t)} \left(y \wedge \frac{[\eta + \gamma m_R k(t)]^+}{\gamma} e^{-r(T-t)} \right) \Bigg) v_Y(dy). \end{aligned}$$

In addition,

$$g(t, x, \lambda) = \mathbb{E}[X_T^{u^*,t,x,\lambda}] = e^{r(T-t)}x + \int_t^T \left(\frac{(\mu - r)^2}{\gamma\sigma^2} + k(s)(\alpha\beta + \rho m_Z) \right) ds + k(t)\lambda. \tag{3.3.8}$$

To investigate the impact of contagion claims, we shall compare our equilibrium reinsurance-investment strategy (l^*, π^*) in (3.3.4) with Li et al. [68]. The setting of Li et al. [68] is the same as the one in this chapter except for the aggregate claim process which is modelled by a Lévy process which has no self-exciting or externally-exciting effect. We find that the equilibrium investment strategy π^* is identical to the one in Li et al. [68], a conclusion which can be explained by the independence between the aggregate claim process and the stock price dynamic. The equilibrium reinsurance strategy in Li et al. [68] is given by

$$\tilde{l}(t, y) = y \wedge \widetilde{RL}(t), \tag{3.3.9}$$

where

$$\widetilde{RL}(t) = \frac{\eta}{\gamma} e^{-r(T-t)}.$$

First of all, we see from (3.3.4) and (3.3.9) that both equilibrium reinsurance strategies are of excess-of-loss form under the time-consistent mean-variance criterion.

Second, for the equilibrium reinsurance strategy $\tilde{l}$ in Li et al. [68], the retention limit

$\widetilde{RL}(t)$ is independent of the claim severity Y (which may be viewed as a possible shortcoming in actuarial practice). Our results show that, in the presence of claim contagion, the equilibrium reinsurance strategy l^* does rely on the distribution of the claim severity Y (through the function $k(t)$).

Third, when $m_R = 0$ (i.e., there is no self-exciting effect as $R = 0$), we have $l^* = \tilde{l}$. Further, the equilibrium reinsurance strategy l^* does not depend on the externally-exciting jumps (e.g., the distribution of Z and the Poisson intensity ρ). Note that both the insurer and reinsurer charge premium according to the expected value principle and as such, a change in the claim arrival intensity λ_t is offset by the corresponding change in insurance/reinsurance premium. Hence, one concludes from the form of l^* that externally-exciting effect can be perfectly hedged by adjusting the premium rate while the self-exciting effect needs to be mitigated by adjusting both the premium rate and the reinsurance strategy. In other words, the self-exciting effect is of a more dangerous nature than the externally-exciting effect, as the former requires more risk management tools.

The sign of $k(t)$ plays an important role in the reinsurance strategy l^*. If $k(t) \geq (\leq) 0$, the retention limits are such that $RL^*(t) \geq (\leq) \widetilde{RL}(t)$, which implies that the insurer retains more (less) risk when the claims are contagious. The following proposition provides the sufficient and necessary condition to determine the sign of $k(t)$ when $r = 0$.

Proposition 3.3.3. *Suppose that $r = 0$.*

(1) If $\frac{\mathbb{E}[[Y-\frac{\eta}{\gamma}]^+]}{\mathbb{E}[Y]} \leq \frac{\theta}{\eta}$, we have $k(t) \geq 0$ for all $t \in [0, T]$. Further, $k(t)$ and $RL^*(t)$ are decreasing in α and increasing in m_R.

(2) If $\frac{\mathbb{E}[[Y-\frac{\eta}{\gamma}]^+]}{\mathbb{E}[Y]} \geq \frac{\theta}{\eta}$, we have $k(t) \leq 0$ for all $t \in [0, T]$. Further, $k(t)$ and $RL^*(t)$ are increasing in α and decreasing in m_R.

Commonly used measures to indicate a heavily-tailed distribution include nonexistence of certain moments, a decreasing hazard rate, to name a few. The mean excess loss ratio $\frac{\mathbb{E}[[Y-\frac{\eta}{\gamma}]^+]}{\mathbb{E}[Y]} \in [0, 1]$ can also be viewed as a measure of tail risk of Y. Larger values of this ratio indicate a heavier tail for Y. Part (1) of Proposition 3.3.3 implies that, if the claim size Y is light-tail, the insurer will retain more risk due to the contagion effect. This is the case as the insurer collects more premium income to undertake this relatively low risk. The latter part very much agrees with intuition that the insurer retains more risk if the claim arrival intensity decays slower (α is smaller) or the self-exciting effect is more significant (m_R is larger). The converse implication is true for part (2).

In summary, Proposition 3.3.3 shows that for $r = 0$ the insurer's reinsurance strategy becomes more sophisticated when the claims contagion effect is considered. More precisely, the insurer's preference is strongly dependent on the tail heaviness of the generic claim size Y. We expect the implications to hold for sufficiently small r (the formal proof goes beyond the scope of this chapter). Nonetheless, we refer the reader to Section 4.1 for some numerical examples supporting this claim.

In the next proposition, we further analyze the behaviour of the optimal retention limit in terms of some other model parameters.

Proposition 3.3.4. *(1) The optimal reinsurance strategy $RL^*(t)$ and $k(t)$ are decreasing in γ.*

(2) The optimal reinsurance strategy $RL^*(t)$ and $k(t)$ are increasing in θ.

(3) When the risk-free rate $r = 0$, both $k(t)$ and $RL^*(t)$ are monotone in t. Specifically, if $\frac{\mathbb{E}[[Y-\frac{\eta}{\gamma}]^+]}{\mathbb{E}[Y]} \leq (\geq) \frac{\theta}{\eta}$ and $r = 0$, then $k(t) \downarrow (\uparrow) 0$ and $RL^*(t) \downarrow (\uparrow) \widetilde{RL}(\cdot)$ as $t \to T$.

Part (1) of Proposition 3.3.4 implies that all else being equal, a more risk-averse insurer retains less risk than its less risk-averse counterpart. This is consistent with the strategy $\tilde{l}(t, y)$ in (3.3.9). For part (2), we observe that a larger insurer's loading factor θ leads the insurer to take on more risk, while for $\tilde{l}(t, y)$ no such incentives exist. Part (3) of Proposition 3.3.4 implies such adjustment will diminish as time approaches to maturity because the impact of claims contagion reduces over time.

3.4 Numerical examples

Throughout this section, we assume that the generic claim size Y follows a gamma distribution with density function

$$\upsilon_Y(dy) = \frac{1}{b^a \Gamma(a)} y^{a-1} e^{-\frac{y}{b}} dy,$$

where a is the so-called shape parameter and b is the scale parameter. Also throughout, we make use of the well-known finite difference method to evaluate the function k satisfying the ODE (3.5) in the retention limit RL^*.

Unless otherwise stated, we assume the following default values of model parameters: $a = 10$, $b = 3$, $\alpha = 3$, $m_R = 1$, $n_R = 2$, $m_Z = 1$, $n_Z = 2$, $\theta = 0.3$, $\eta = 0.4$, $\gamma = 0.05$, $\rho = 2$, $\sigma = 0.3$, $\mu = 0.05$, $r = 0.01$ and $T = 5$.

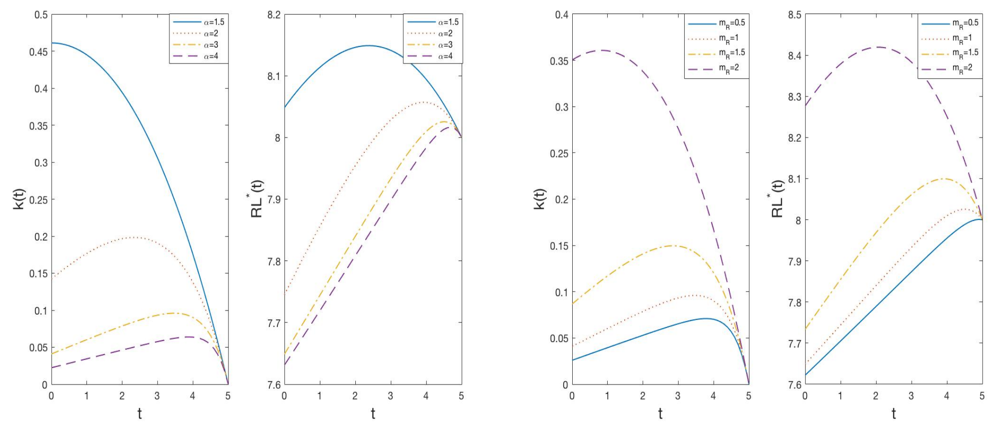
3.4.1 Proposition 3.3.3 for small r

First, we show via numerical examples that the conclusions of Proposition 3.3.3 for $r = 0$ also holds for small risk-free rates $r > 0$. Figure 3.1 depicts the function $k(t)$ and retention limit $RL^*(t)$ as a function of t under the default parameter set for which $\frac{\mathbb{E}[[Y-\frac{\eta}{\gamma}]^+]}{\mathbb{E}[Y]} \leq \frac{\theta}{\eta}$. In Figure 3.2, we redo the same exercise under the same parameter setting except that $b = 4$ for which $\frac{\mathbb{E}[[Y-\frac{\eta}{\gamma}]^+]}{\mathbb{E}[Y]} \geq \frac{\theta}{\eta}$. As we can see, the results are consistent with those in Proposition 3.3.3.

3.4.2 Impact of model parameters on $RL^*(t)$

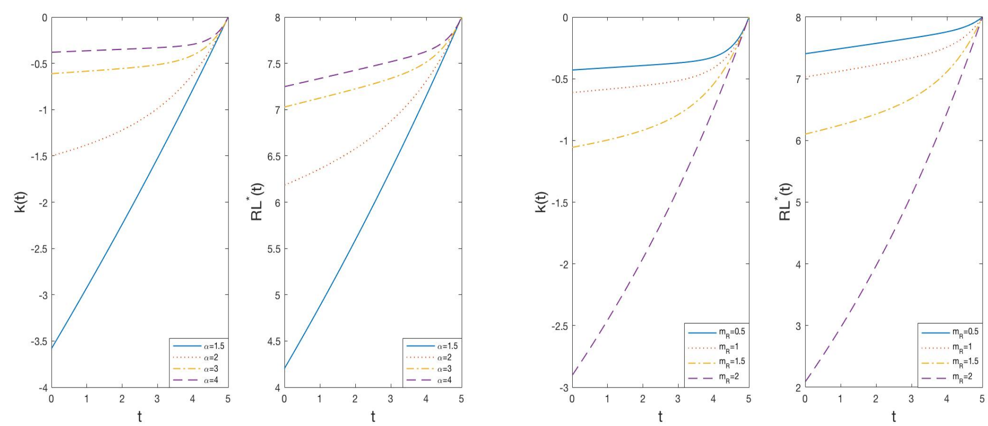
The impact of most model parameters were analytically studied in Propositions 3.3.3 and 3.3.4. In this section, we conduct a sensitivity analysis of the equilibrium retention limit $RL^*(t)$ in terms of the risk-free rate r and the reinsurer's premium rate η.

Figure 3.3 depicts the retention limit $RL^*(t)$ as a function of the risk-free rate r. As expected, we observe that the insurer is incentivized to take on less risk as r increases.



(a) Impact of α (b) Impact of m_R

Figure 3.1: Impact of α and m_R on $k(t)$ and $RL^*(t)$ when $\frac{\mathbb{E}[[Y-\frac{\eta}{\gamma}]^+]}{\mathbb{E}[Y]} \leq \frac{\theta}{\eta}$



(a) Impact of α (b) Impact of m_R

Figure 3.2: Impact of α and m_R on $k(t)$ and $RL^*(t)$ when $\frac{\mathbb{E}[[Y-\frac{\eta}{\gamma}]^+]}{\mathbb{E}[Y]} \geq \frac{\theta}{\eta}$

Note that this relation was also shown to hold in the Lévy framework (see Li et al. [68]). It can be rationalized as follows: the increased return from the risk-free bond allows the insurer to increase the reinsurance coverage by lowering the retention limit. As expected, the sensitivity of the retention limit $RL^*(t)$ with respect to the risk-free rate r is more

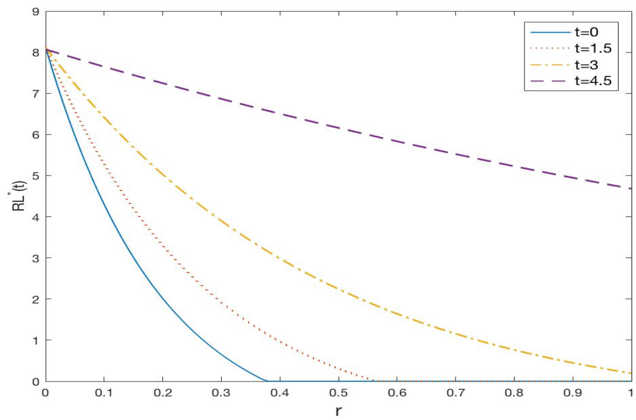
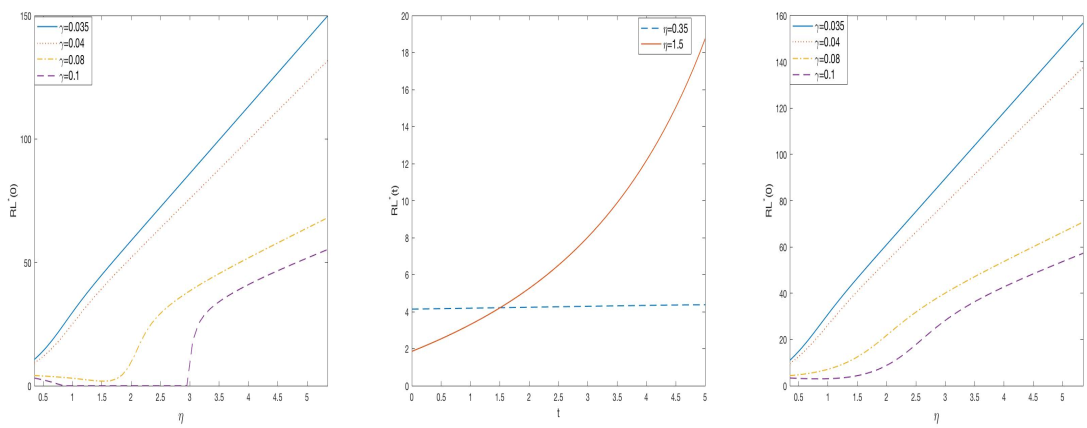


Figure 3.3: Effect of risk-free rate r on retention limit $RL^*(t)$

pronounced for smaller t.



(a) $RL^*(0)$, $T = 5$ (b) $RL^*(t)$, $\gamma = 0.08$, $T = 5$ (c) $RL^*(0)$, $T = 1$

Figure 3.4: Impact of η on the retention limit RL^*

Figure 3.4a depicts the retention limit $RL^*(0)$ as a function of the reinsurer's risk loading factor η. As expected, we note that $RL^*(t) \longrightarrow +\infty$ as $\eta \to +\infty$ which means that the insurer retains the entire risk if reinsurance is extremely expensive. Moreover, it is very interesting to see from Figure 3.4a (for the cases $\gamma = 0.08$ and $\gamma = 0.1$) that the retention limit $RL^*(0)$ is not necessarily monotonically increasing in η. In other words, the insurer may buy more reinsurance at time 0 when the reinsurer's loading factor is higher. To shed more light into this phenomenon, we display in Figure 3.4b the equilibrium retention limit $RL^*(t)$ as a function of t for two reinsurer's loading factors, namely $\eta = 0.35$ and $\eta = 1.5$. We observe that when the cost of reinsurance increases from $\eta = 0.35$ to $\eta = 1.5$, the insurer first purchases more reinsurance but beyond a certain time point, the optimal strategy dictates that less reinsurance should be purchased. By the nature of the equilibrium strategies, one shall compare the strategy trajectories as a whole rather than performing a point-wise comparison of them. This is due to the fact that the equilibrium strategies are *dynamic*. If we venture into the comparison of the two equilibrium strategies displayed in Figure 3.4b, the behaviour observed in the time-0 reinsurance strategy can be largely explained by *the presence of self-exciting claims*. Indeed, with no self-exciting effect, we know from Theorem 3.3.2 that the equilibrium retention limit increases in η for all t. As alluded above, when the self-exciting contagion risk is included, the equilibrium retention limit is not necessarily increasing in η at each time point. The aforementioned time-0 phenomenon is also related to *the length of time horizon* T, that is, it is more likely to occur with longer time horizon because the impact of self-exciting claims is more significant. Indeed, in Figure 3.4c, we observe that for the shorter time horizon $T = 1$, the monotonicity of the retention limit with respect to η is observed.

3.5 Conclusion

In this chapter, an optimal reinsurance-investment problem for a dynamic contagion process is considered. The claim arrival process is versatile, allowing for self-exciting and externally-exciting clustering behaviour in the process claim arrivals. Under the time-consistent mean-variance criterion, we obtain the explicit equilibrium reinsurance-investment strategy. Our main conclusion is that unlike the result in the Lévy framework, the insurer should take both the claim size distribution and the claim arrival rate into consideration.

Most notably, we find that an excess-of-loss type of reinsurance is optimal even in the presence of self-exciting and externally-exciting effect of claim contagion. Second, the self-exciting contagion risk is more dangerous in nature than the externally-exciting risk, since more advanced risk hedging tools are necessary. Third, the insurer's attitude towards claim risk depends on the tail heaviness of the claim size distribution. When the claim size is light tail distributed, the insurer can be better off retaining more risk.

3.6 Appendix

3.6.1 Proof of Theorem 3.3.1

We start by showing that for a function $f(t,x,\lambda) \in C^{1,2,1}\big([0,T]\times\mathbb{R}\times[0,\infty)\big)$ satisfying a quadratic growth condition in x, i.e., there exists a constant $C>0$ such that

$$|f(t,x,\lambda)| \leq C(1+|x|^2),$$

$\forall (t,x,\lambda) \in [0,T]\times\mathbb{R}\times[0,\infty)$ and any admissible strategy u, we have

$$\mathbb{E}\big[f(T, X_T^{u,t,x,\lambda}, \lambda_T^{t,\lambda})\big] = f(t,x,\lambda) + \mathbb{E}\Big[\int_t^T \mathcal{A}^u f(s, X_s^{u,t,x,\lambda}, \lambda_s^{t,\lambda})ds\Big].$$

To see this, for any stopping time τ valued in $[t,\infty)$, by Itô's formula,

$$\begin{aligned}
& f(T\wedge\tau, X_{T\wedge\tau}^{u,t,x,\lambda}, \lambda_{T\wedge\tau}^{t,\lambda}) \\
&= f(t,x,\lambda) + \int_t^{T\wedge\tau} \mathcal{A}^u f(s, X_s^{u,t,x,\lambda}, \lambda_s^{t,\lambda})ds + \int_t^{T\wedge\tau} \sigma\pi f_x(s, X_s^{u,t,x,\lambda}, \lambda_s^{t,\lambda})dW_s \\
&\quad + \int_t^{T\wedge\tau}\int_0^{+\infty} \Big(f(s, X_s^{u,t,x,\lambda}, \lambda_s^{t,\lambda}+z) - f(t,x,\lambda)\Big)\upsilon_Z(dz)d\tilde{M}_s \\
&\quad + \int_t^{T\wedge\tau}\int_0^{+\infty}\int_0^{+\infty} \Big(f(s, X_s^{u,t,x,\lambda} - l(s,y), \lambda_s^{t,\lambda}+r) - f(t,x,\lambda)\Big)\upsilon_Y(dy)\upsilon_R(dr)d\tilde{N}_s,
\end{aligned}$$

where $\tilde{M}_s = M_t - \rho t$ and $\tilde{N}_t = N_t - \int_0^t \lambda_s ds$ are the compensated process of M and N, respectively. We choose $\tau = \tau_n = \inf\{w \geq t : \int_t^w f_x^2(s, X_s^{u,t,x,\lambda}, \lambda_s^{t,\lambda})ds > n\}$, then $\lim_{n\to\infty}\tau_n = +\infty$ and the stopped process $\{\int_t^{w\wedge\tau_n} f_x(s, X_s^{u,t,x,\lambda}, \lambda_s^{t,\lambda})dW_s\}_{t\leq w\leq T}$ is a martingale. By taking expectation, we get

$$\mathbb{E}\Big[f(T\wedge\tau_n, X^{u,t,x,\lambda}_{T\wedge\tau_n}, \lambda^{t,\lambda}_{T\wedge\tau_n})\Big] = f(t,x,\lambda) + \mathbb{E}\Big[\int_t^{T\wedge\tau_n} \mathcal{A}^u f(s, X^{u,t,x,\lambda}_s, \lambda^{t,\lambda}_s)ds\Big].$$

Since f satisfies the quadratic growth condition, we have

$$|f(T\wedge\tau_n, X^{u,t,x,\lambda}_{T\wedge\tau_n}, \lambda^{t,\lambda}_{T\wedge\tau_n})| \le C(1 + \sup_{w\in[t,T]} |X^{u,t,x,\lambda}_w|^2),$$

and the righthand side is integrable, see Pham [85] (Theorem 1.3.15 on page 23) for instance. We can then apply the dominated convergence theorem and let $n\to\infty$, then we have

$$\mathbb{E}\big[f(T, X^{u,t,x,\lambda}_T, \lambda^{t,\lambda}_T)\big] = f(t,x,\lambda) + \mathbb{E}\Big[\int_t^T \mathcal{A}^u f(s, X^{u,t,x,\lambda}_s, \lambda^{t,\lambda}_s)ds\Big].$$

Therefore for functions V, g and g^2, we have

$$\mathbb{E}\big[V(T, X^{u^*,t,x,\lambda}_T, \lambda^{t,\lambda}_T)\big] = V(t,x,\lambda) + \mathbb{E}\Big[\int_t^T \mathcal{A}^{u^*} V(s, X^{u^*,t,x,\lambda}_s, \lambda^{t,\lambda}_s)ds\Big], \tag{3.6.1}$$

$$\mathbb{E}\big[g(T, X^{u^*,t,x,\lambda}_T, \lambda^{t,\lambda}_T)\big] = g(t,x,\lambda) + \mathbb{E}\Big[\int_t^T \mathcal{A}^{u^*} g(s, X^{u^*,t,x,\lambda}_s, \lambda^{t,\lambda}_s)ds\Big], \tag{3.6.2}$$

$$\mathbb{E}\big[g^2(T, X^{u^*,t,x,\lambda}_T, \lambda^{t,\lambda}_T)\big] = g^2(t,x,\lambda) + \mathbb{E}\Big[\int_t^T \mathcal{A}^{u^*} g^2(s, X^{u^*,t,x,\lambda}_s, \lambda^{t,\lambda}_s)ds\Big]. \tag{3.6.3}$$

Then we show that V is the value function corresponding to strategy u^*, and $g(t,x,\lambda) = \mathbb{E}\big[X^{u^*,t,x,\lambda}_T\big]$.

From (3.3.2), (3.3.3) and (3.6.2), we can see that

$$g(t,x,\lambda) = \mathbb{E}\big[X^{u^*,t,x,\lambda}_T\big].$$

Together with the terminal condition (3.3.3), it follows that

$$\mathbb{E}\Big[\int_t^T \mathcal{A}^{u^*} g^2(s, X_s^{u^*,t,x,\lambda}, \lambda_s^{t,\lambda})ds\Big] = \mathbb{E}\big[(X_T^{u^*t,x,\lambda})^2\big] - \Big(\mathbb{E}\big[X_T^{u^*,t,x,\lambda}\big]\Big)^2 = \mathrm{Var}\big[X_T^{u^*,t,x,\lambda}\big]. \tag{3.6.4}$$

From (3.3.1) and (3.3.2), we have

$$\mathcal{A}^{u^*} V(t,x,\lambda) - \frac{\gamma}{2}\mathcal{A}^{u^*} g^2(t,x,\lambda) = 0. \tag{3.6.5}$$

Now by (3.6.1) and the terminal condition (3.3.3),

$$\begin{aligned} V(t,x,\lambda) &= \mathbb{E}\big[X_T^{u^*t,x,\lambda}\big] - \mathbb{E}\Big[\int_t^T \mathcal{A}^{u^*} V(s, X_s^{u^*,t,x,\lambda}, \lambda_s^{t,\lambda})ds\Big] \\ &= \mathbb{E}\big[X_T^{u^*,t,x,\lambda}\big] - \frac{\gamma}{2}\mathbb{E}\Big[\int_t^T \mathcal{A}^{u^*} g^2(s, X_s^{u^*,t,x,\lambda}, \lambda_s^{t,\lambda})ds\Big] \\ &= \mathbb{E}\big[X_T^{u^*,t,x,\lambda}\big] - \frac{\gamma}{2}\mathrm{Var}\big[X_T^{u^*,t,x,\lambda}\big] \\ &= J^{u^*}(t,x,\lambda), \end{aligned} \tag{3.6.6}$$

where the second equality is due to (3.6.5) and the third equality is due to (3.6.4).

We now go on to show that u^* is indeed an equilibrium strategy. For any $u = (l,\pi) \in [0,y] \times \mathbb{R}$, $\varepsilon > 0$ we define a strategy u^ε as follows:

$$u^\varepsilon(s,y) = \begin{cases} u, & s \in [t, t+\varepsilon),\ \ y > 0, \\ (l^*(s,y), \pi_s^*), & s \in [t+\varepsilon, T],\ \ y > 0. \end{cases}$$

From the definition of u^ε, we have

$$J^{u^\varepsilon}(t+\varepsilon, X_{t+\varepsilon}^{u^\varepsilon,t,x,\lambda}, \lambda_{t+\varepsilon}^{t,\lambda}) = J^{u^*}(t+\varepsilon, X_{t+\varepsilon}^{u^\varepsilon,t,x,\lambda}, \lambda_{t+\varepsilon}^{t,\lambda}) = V(t+\varepsilon, X_{t+\varepsilon}^{u^\varepsilon,t,x,\lambda}, \lambda_{t+\varepsilon}^{t,\lambda}),$$

$$\mathbb{E}\big[X_T^{u^\varepsilon,t+\varepsilon,X_{t+\varepsilon}^{u^\varepsilon,t,x,\lambda},\lambda_{t+\varepsilon}^{t,\lambda}}\big] = \mathbb{E}\big[X_T^{u^*,t+\varepsilon,X_{t+\varepsilon}^{u^\varepsilon,t,x,\lambda},\lambda_{t+\varepsilon}^{t,\lambda}}\big] = g(t+\varepsilon, X_{t+\varepsilon}^{u^\varepsilon,t,x,\lambda}, \lambda_{t+\varepsilon}^{t,\lambda}).$$

Then we analyze the relationship between objective functions under u^ε and u^*, by noting that

$$
\begin{aligned}
&J^{u^\varepsilon}(t,x,\lambda)\\
&=\mathbb{E}\big[X_T^{u^\varepsilon,t,x,\lambda}-\frac{\gamma}{2}(X_T^{u^\varepsilon,t,x,\lambda})^2\big]+\frac{\gamma}{2}\Big(\mathbb{E}\big[X_T^{u^\varepsilon,t,x,\lambda}\big]\Big)^2\\
&=\mathbb{E}\Big[\mathbb{E}\big[X_T^{u^\varepsilon t,x,\lambda}-\frac{\gamma}{2}(X_T^{u^\varepsilon,t,x,\lambda})^2\big|X_{t+\varepsilon}^{u^\varepsilon,t,x,\lambda},\lambda_{t+\varepsilon}^{t,\lambda}\big]+\frac{\gamma}{2}\Big(\mathbb{E}\big[X_T^{u^\varepsilon t,x,\lambda}\big|X_{t+\varepsilon}^{u^\varepsilon,t,x,\lambda},\lambda_{t+\varepsilon}^{t,\lambda}\big]\Big)^2\Big]\\
&-\frac{\gamma}{2}\mathbb{E}\Big[\Big(\mathbb{E}\big[X_T^{u^\varepsilon,t,x,\lambda}\big|X_{t+\varepsilon}^{u^\varepsilon,t,x,\lambda},\lambda_{t+\varepsilon}^{t,\lambda}\big]\Big)^2\Big]+\frac{\gamma}{2}\Big(\mathbb{E}\Big[\mathbb{E}\big[X_T^{u^\varepsilon,t,x,\lambda}\big|X_{t+\varepsilon}^{u^\varepsilon,t,x,\lambda},\lambda_{t+\varepsilon}^{t,\lambda}\big]\Big]\Big)^2\\
&=\mathbb{E}\big[J^{u^\varepsilon}(t+\varepsilon,X_{t+\varepsilon}^{u^\varepsilon,t,x,\lambda},\lambda_{t+\varepsilon}^{t,\lambda})\big]+\frac{\gamma}{2}\Big(\mathbb{E}\big[g(t+\varepsilon,X_{t+\varepsilon}^{u^\varepsilon,t,x,\lambda},\lambda_{t+\varepsilon}^{t,\lambda})\big]\Big)^2-\frac{\gamma}{2}\mathbb{E}\big[g^2(t+\varepsilon,X_{t+\varepsilon}^{u^\varepsilon,t,x,\lambda},\lambda_{t+\varepsilon}^{t,\lambda})\big]\\
&=\mathbb{E}\big[V(t+\varepsilon,X_{t+\varepsilon}^{u^\varepsilon,t,x,\lambda},\lambda_{t+\varepsilon}^{t,\lambda})\big]+\frac{\gamma}{2}g^2(t,x,\lambda)-\frac{\gamma}{2}\mathbb{E}\big[g^2(t+\varepsilon,X_{t+\varepsilon}^{u^\varepsilon t,x,\lambda},\lambda_{t+\varepsilon}^{t,\lambda})\big].
\end{aligned}
\tag{3.6.7}
$$

From (3.3.1), for the given u, we have

$$\mathcal{A}^uV(t,x,\lambda)-\frac{\gamma}{2}\mathcal{A}^ug^2(t,x,\lambda)+\gamma g(t,x,\lambda)\mathcal{A}^ug(t,x,\lambda)\le 0,$$

From the definition of the infinitesimal generator, a discretized version of the above inequality is

$$
\begin{aligned}
\mathbb{E}\big[V(t+\varepsilon,X_{t+\varepsilon}^{u^\varepsilon,t,x,\lambda},\lambda_{t+\varepsilon}^{t,\lambda})\big]-V(t,x,\lambda)-\frac{\gamma}{2}\Big(\mathbb{E}\big[g^2(t+\varepsilon,X_{t+\varepsilon}^{u^\varepsilon,t,x,\lambda},\lambda_{t+\varepsilon}^{t,\lambda})\big]-g^2(t,x,\lambda)\Big)\\
+\gamma g(t,x,\lambda)\Big(\mathbb{E}\big[g(t+\varepsilon,X_{t+\varepsilon}^{u^\varepsilon},\lambda_{t+\varepsilon})\big]-g(t,x,\lambda)\Big)\le o(\varepsilon),
\end{aligned}
$$

or

$$V(t,x,\lambda)\ge\mathbb{E}\big[V(t+\varepsilon,X_{t+\varepsilon}^{u^\varepsilon},\lambda_{t+\varepsilon})\big]+\frac{\gamma}{2}g^2(t,x,\lambda)-\frac{\gamma}{2}\mathbb{E}\big[g^2(t+\varepsilon,X_{t+\varepsilon}^{u^\varepsilon},\lambda_{t+\varepsilon})\big]+o(\varepsilon).\tag{3.6.8}$$

Compare (3.6.7) and (3.6.8), and notice that $V(t,x,\lambda)=J^{u^*}(t,x,\lambda)$, we obtain,

$$J^{u^*}(t,x,\lambda)-J^{u^\varepsilon}(t,x,\lambda)\ge o(\varepsilon).$$

The proof is now complete.

3.6.2 Proof of Theorem 3.3.2

We first derive the equilibrium strategy $u^* = (l^*, \pi^*)$. With some calculations, the extended HJB equation (3.3.1) becomes

$$\begin{aligned}
0 &= \sup_{u=(l(t,y),\pi)\in[0,y]\times\mathbb{R}} \left\{ \mathcal{A}^u V(t,x,\lambda) - \frac{\gamma}{2}\mathcal{A}^u g^2(t,x,\lambda) + \gamma g(t,x,\lambda)\mathcal{A}^u g(t,x,\lambda) \right\} \\
&= V_t(t,x,\lambda) + (rx + (\theta-\eta)m_Y\lambda)\, V_x(t,x,\lambda) + \alpha(\beta-\lambda)V_\lambda(t,x,\lambda) - \frac{\gamma(\rho+\lambda)}{2} g^2(t,x,\lambda) \\
&\quad + \rho\int_0^{+\infty} \Big(V(t,x,\lambda+z) - V(t,x,\lambda) + \gamma g(t,x,\lambda)g(t,x,\lambda+z) - \frac{\gamma}{2}g^2(t,x,\lambda+z) \Big)\, v_Z(dz) \\
&\quad + \sup_{\pi\in\mathbb{R}} \left\{ (\mu-r)\pi V_x(t,x,\lambda) + \frac{1}{2}\sigma^2\pi^2 \left(V_{xx}(t,x,\lambda) - \gamma g_x^2(t,x,\lambda) \right) \right\} \\
&\quad + \lambda \sup_{l(t,y)\in[0,y]} \Bigg\{ (1+\eta)V_x(t,x,\lambda) \int_0^{+\infty} l(t,y) v_Y(dy) \\
&\quad + \int_0^{+\infty}\int_0^{+\infty} \left(V(t,x-l(t,y),\lambda+u) - V(t,x,\lambda) \right) v_Y(dy) v_R(du) \\
&\quad + \int_0^{+\infty}\int_0^{+\infty} \Big(\gamma g(t,x,\lambda) g(t,x-l(t,y),\lambda+u) - \frac{\gamma}{2} g^2(t,x-l(t,y),\lambda+u) \Big)\, v_Y(dy) v_R(du) \Bigg\}.
\end{aligned}$$

We then consider the following ansatzes

$$V(t,x,\lambda) = e^{r(T-t)}x + A(t) + K(t)\lambda \text{ and } g(t,x,\lambda) = e^{r(T-t)}x + a(t) + k(t)\lambda,$$

where the functions $A(t), K(t), a(t), k(t)$ are to be determined. From equation (3.3.3), we know that

$$V(T,x,\lambda) = x + A(T) + K(T)\lambda = x \text{ and } g(T,x,\lambda) = x + a(T) + k(T)\lambda = x$$

for all $(x,\lambda) \in \mathbb{R}\times(0,+\infty)$. We deduce that $A(T) = K(T) = a(T) = k(T) = 0$. Using the above forms of $V(t,x,\lambda)$ and $g(t,x,\lambda)$, the above two maximization problems with respect to π and l becomes

$$\begin{aligned} & \sup_{\pi\in\mathbb{R}} \left\{ (\mu-r)\pi V_x(t,x,\lambda) + \frac{1}{2}\sigma^2\pi^2 \left(V_{xx}(t,x,\lambda) - \gamma g_x^2(t,x,\lambda) \right) \right\} \\ = & \sup_{\pi\in\mathbb{R}} \left\{ -\frac{\gamma\sigma^2}{2} e^{2r(T-t)} \left(\pi - \frac{\mu-r}{\gamma\sigma^2} e^{-r(T-t)} \right)^2 \right\} + \frac{(\mu-r)^2}{2\gamma\sigma^2}, \end{aligned} \tag{3.6.9}$$

and

$$\begin{aligned} & \sup_{l(t,y)\in[0,y]} \left\{ (1+\eta)V_x(t,x,\lambda) \int_0^{+\infty} l(t,y) v_Y(dy) \right. \\ & + \int_0^{+\infty}\int_0^{+\infty} \left(V(t,x-l(t,y),\lambda+u) - V(t,x,\lambda) \right) v_Y(dy) v_R(du) \\ & \left. + \int_0^{+\infty}\int_0^{+\infty} \Big(\gamma g(t,x,\lambda) g(t,x-l(t,y),\lambda+u) - \frac{\gamma}{2} g^2(t,x-l(t,y),\lambda+u) \Big) v_Y(dy) v_R(du) \right\} \\ = & \sup_{l(t,y)\in[0,y]} \left\{ \int_0^{+\infty} \Big(-\frac{\gamma}{2} e^{2r(T-t)} l^2(t,y) + (\eta+\gamma m_R k(t)) e^{r(T-t)} l(t,y) \Big) v_Y(dy) \right\} \\ & + m_R K(t) + \frac{\gamma}{2} g^2(t,x,\lambda) - \frac{\gamma}{2} n_R k^2(t). \end{aligned} \tag{3.6.10}$$

From (3.6.9), we obtain

$$\pi_t^* = \frac{\mu-r}{\gamma\sigma^2} e^{-r(T-t)}.$$

For the problem (3.6.10), note that the quadratic function

$$f(l) = -\frac{\gamma}{2} e^{2r(T-t)} l^2 + (\eta+\gamma m_R k(t)) e^{r(T-t)} l$$

attains the maximum in the region $l \in [0,y]$ at

$$l^*(t,y) = y \wedge \frac{[\eta+\gamma m_R k(t)]^+}{\gamma} e^{-r(T-t)}.$$

Using the same ansatz $g(t,x,\lambda) = e^{r(T-t)}x + a(t) + k(t)\lambda$ and the form of $u^* = (l^*, \pi^*)$, equation (3.3.2) becomes

$$
\begin{aligned}
0 &= \mathcal{A}^{u^*} g(t,x,\lambda) \\
&= \lambda \left[k'(t) + (m_R - \alpha)k(t) + e^{r(T-t)} \left((\theta-\eta)m_Y + \eta \int_0^{\frac{1}{\gamma}[\eta+\gamma m_R k(t)]^+ e^{-r(T-t)}} S_Y(y)dy \right) \right] \\
&\quad + a'(t) + \frac{(\mu-r)^2}{\gamma\sigma^2} + k(t)(\alpha\beta + \rho m_Z),
\end{aligned}
$$

which holds for all $(t,x,\lambda) \in [0,T] \times \mathbb{R} \times (0,+\infty)$. It implies that

$$
\begin{cases}
k'(t) + (m_R - \alpha)k(t) + e^{r(T-t)} \left((\theta-\eta)m_Y + \eta \int_0^{\frac{1}{\gamma}[\eta+\gamma m_R k(t)]^+ e^{-r(T-t)}} S_Y(y)dy \right) = 0, \\
a'(t) + \frac{(\mu-r)^2}{\gamma\sigma^2} + k(t)(\alpha\beta + \rho m_Z) = 0.
\end{cases}
$$

By the boundary condition $a(T) = k(T) = 0$, we deduce that

$$
a(t) = \int_t^T \left(\frac{(\mu-r)^2}{\gamma\sigma^2} + k(s)(\alpha\beta + \rho m_Z) \right) ds,
$$

and $k(t)$ satisfies the ODE

$$
\begin{cases}
k'(t) + (m_R - \alpha)k(t) + e^{r(T-t)} \left((\theta-\eta)m_Y + \eta \int_0^{\frac{1}{\gamma}[\eta+\gamma m_R k(t)]^+ e^{-r(T-t)}} S_Y(y)dy \right) = 0, \\
k(T) = 0.
\end{cases}
$$

Note that the above ODE satisfies the uniform Lipschitz condition in the sense that its generator

$$
f(t,x) = (\alpha - m_R)x - e^{r(T-t)} \left((\theta-\eta)m_Y + \eta \int_0^{\frac{1}{\gamma}[\eta+\gamma m_R x]^+ e^{-r(T-t)}} S_Y(y)dy \right)
$$

satisfies

$$\begin{aligned}|f(t,x)-f(t,z)| &\le (\alpha-m_R)\,|x-z| + e^{r(T-t)}\eta\left|\int_{\frac{1}{\gamma}[\eta+\gamma m_R z]^+e^{-r(T-t)}}^{\frac{1}{\gamma}[\eta+\gamma m_R x]^+e^{-r(T-t)}} S_Y(y)dy\right| \\ &\le (\alpha-m_R)\,|x-z| + \frac{\eta}{\gamma}\left|[\eta+\gamma m_R x]^+ - [\eta+\gamma m_R z]^+\right| \\ &\le (\alpha-m_R)\,|x-z| + \frac{\eta}{\gamma}\gamma m_R|x-z| \\ &= (\alpha-m_R+\eta m_R)|x-z|.\end{aligned}$$

Therefore, ODE (3.3.5) has a unique solution.

It remains to derive the form of the value function $V(t,x,\lambda)$. From the form of the equilibrium strategy $u^*=(l^*,\pi^*)$, the extended HJB equation (3.3.1) becomes

$$\begin{aligned}0 &= \mathcal{A}^{u^*}V(t,x,\lambda)-\frac{\gamma}{2}\mathcal{A}^{u^*}g^2(t,x,\lambda)+\gamma g(t,x,\lambda)\mathcal{A}^{u^*}g(t,x,\lambda) \\ &= \lambda\left(K'(t)+(m_R-\alpha)K(t)+e^{r(T-t)}(\theta-\eta)m_Y+G(t)-\frac{\gamma}{2}k^2(t)n_R\right) \\ &\quad +A'(t)+(\alpha\beta+\rho m_Z)K(t)+\frac{(\mu-r)^2}{2\gamma\sigma^2}-\frac{\gamma\rho}{2}k^2(t)n_Z,\end{aligned}$$

which holds for all $(t,x,\lambda)\in[0,T]\times\mathbb{R}\times(0,+\infty)$, where

$$\begin{aligned}G(t) &= \int_0^{+\infty}\left(-\frac{\gamma}{2}e^{2r(T-t)}\left(y\wedge\frac{[\eta+\gamma m_R k(t)]^+}{\gamma}e^{-r(T-t)}\right)^2\right. \\ &\quad\left.+(\eta+\gamma m_R k(t))e^{r(T-t)}\left(y\wedge\frac{[\eta+\gamma m_R k(t)]^+}{\gamma}e^{-r(T-t)}\right)\right)v_Y(dy).\end{aligned}$$

It follows that

$$\begin{cases}K'(t)+(m_R-\alpha)K(t)+e^{r(T-t)}(\theta-\eta)m_Y+G(t)-\dfrac{\gamma}{2}k^2(t)n_R=0,\\ A'(t)+(\alpha\beta+\rho m_Z)K(t)+\dfrac{(\mu-r)^2}{2\gamma\sigma^2}-\dfrac{\gamma\rho}{2}k^2(t)n_Z=0.\end{cases}$$

By the boundary condition $A(T) = K(T) = 0$, we deduce that

$$A(t) = \int_t^T \left((\alpha\beta + \rho m_Z)K(s) + \frac{(\mu - r)^2}{2\gamma\sigma^2} - \frac{\gamma\rho}{2} k^2(s) n_Z \right) ds,$$

and $K(t)$ is the unique solution of the following ODE

$$\begin{cases} K'(t) + (m_R - \alpha)K(t) + e^{r(T-t)}(\theta - \eta)m_Y + G(t) - \frac{\gamma}{2} k^2(t) n_R = 0. \\ K(T) = 0. \end{cases}$$

The uniqueness is due to the uniform Lipschitz condition of its generator. This ends the proof.

3.6.3 Proof of Proposition 3.3.3

We only prove part (1) as the other part is completely symmetric to it. We first transform equation (3.3.5) to be forward in time. Define $\tilde{k}(t) = k(T-t)$ for $t \in [0, T]$, which satisfies

$$\begin{cases} \tilde{k}'(t) = f(\tilde{k}(t)), \\ \tilde{k}(0) = 0, \end{cases} \tag{3.6.11}$$

with generator

$$f(x) = (m_R - \alpha)x + (\theta - \eta)m_Y + \eta \int_0^{\frac{1}{\gamma}[\eta + \gamma m_R x]^+} S_Y(y)dy.$$

Note that when $\frac{\mathbb{E}[[Y - \frac{\eta}{\gamma}]^+]}{\mathbb{E}[Y]} = \frac{\theta}{\eta}$ or equivalently $\int_0^{\frac{\eta}{\gamma}} S_Y(y)dy = (1 - \dfrac{\theta}{\eta})m_Y$, $\tilde{k}(t) \equiv 0$ is the solution to (3.6.11). When $\frac{\mathbb{E}[[Y - \frac{\eta}{\gamma}]^+]}{\mathbb{E}[Y]} \leq \frac{\theta}{\eta}$ or equivalently $\int_0^{\frac{\eta}{\gamma}} S_Y(y)dy \geq (1 - \dfrac{\theta}{\eta})m_Y$, it follows that

$$f(0) = (\theta - \eta)m_Y + \eta \int_0^{\frac{\eta}{\gamma}} S_Y(y)dy \geq 0.$$

By the comparison principle (e.g., Terrell [100, lemma E.4]), we deduce that $\tilde{k}(t) \geq 0$ for all $t \in [0, T]$. Equivalently, we have $k(t) \geq 0$ for all $t \in [0, T]$.

Next we show the monotonicity of $k(t)$ with respect to α and m_R. The monotonicity of $RL^*(t)$ follows immediately. Let $p(t) = \frac{\partial k(t)}{\partial \alpha}$ and $q(t) = \frac{\partial k(t)}{\partial m_R}$ for $t \in [0, T]$. Differentiating equation (3.3.5) with respect to α yields

$$\begin{aligned} p'(t) &= \frac{\partial}{\partial \alpha}\left((\alpha - m_R)k(t) - (\theta - \eta)m_Y - \eta \int_0^{\frac{1}{\gamma}[\eta + \gamma m_R k(t)]^+} S_Y(y)dy\right) \\ &= k(t) + (\alpha - m_R)p(t) - \eta m_R S_Y\left(\frac{\eta}{\gamma} + m_R k(t)\right) 1_{\{\eta + \gamma m_R k(t) > 0\}} p(t) \\ &= \left[\alpha - m_R - \eta m_R S_Y\left(\frac{\eta}{\gamma} + m_R k(t)\right) 1_{\{\eta + \gamma m_R k(t) > 0\}}\right] p(t) + k(t), \end{aligned}$$

with boundary condition $p(T) = 0$ as $k(T) = 0$. It follows that

$$p(t) = -\int_t^T k(u) \exp\left\{-\int_t^u \left[\alpha - m_R - \eta m_R S_Y\left(\frac{\eta}{\gamma} + m_R k(s)\right) 1_{\{\eta + \gamma m_R k(s) > 0\}}\right] ds\right\} du.$$

Since $k(t)$ is nonnegative for all $t \in [0, T]$, we deduce that $p(t) \leq 0$ for all $t \in [0, T]$. Thus, $k(t)$ is decreasing in α. By the same argument, one can show that

$$\begin{aligned} q(t) &= \int_t^T \left[1 + \eta S_Y\left(\frac{\eta}{\gamma} + m_R k(u)\right) 1_{\{\eta + \gamma m_R k(u) > 0\}}\right] k(u) \times \\ &\quad \exp\left\{-\int_t^u \left[\alpha - m_R - \eta m_R S_Y\left(\frac{\eta}{\gamma} + m_R k(s)\right) 1_{\{\eta + \gamma m_R k(s) > 0\}}\right] ds\right\} du \\ &\geq 0, \end{aligned}$$

which implies that $k(t)$ is increasing in m_R.

3.6.4 Proof of Proposition 3.3.4

(1) We denote $m(t) := \frac{\partial k(t)}{\partial \gamma}$. Differentiating equation (3.3.5) with respect to γ yields

$$
\begin{aligned}
m'(t) &= (\alpha - m_R)m(t) - \eta S_Y\left(\frac{1}{\gamma}e^{-r(T-t)}[\eta+\gamma m_R k(t)]^+\right)\left(-\frac{\eta}{\gamma^2}+m_R m(t)\right)1_{\{\eta+\gamma m_R k(t)>0\}} \\
&= \left(\alpha - m_R - \eta m_R S_Y\left(\frac{1}{\gamma}e^{-r(T-t)}[\eta+\gamma m_R k(t)]^+\right)1_{\{\eta+\gamma m_R k(t)>0\}}\right)m(t) \\
&\quad + \frac{\eta^2}{\gamma^2}S_Y\left(\frac{1}{\gamma}e^{-r(T-t)}[\eta+\gamma m_R k(t)]^+\right)1_{\{\eta+\gamma m_R k(t)>0\}},
\end{aligned}
$$

together with the boundary condition $m(T) = 0$. It follows that

$$
\begin{aligned}
m(t) &= -\int_t^T \frac{\eta^2}{\gamma^2}S_Y\left(\frac{1}{\gamma}e^{-r(T-u)}[\eta+\gamma m_R k(u)]^+\right)1_{\{\eta+\gamma m_R k(u)>0\}} \times \\
&\quad \exp\left\{-\int_t^u \left(\alpha - m_R - \eta m_R S_Y\left(\frac{1}{\gamma}e^{-r(T-s)}[\eta+\gamma m_R k(s)]^+\right)1_{\{\eta+\gamma m_R k(s)>0\}}\right)ds\right\}du \\
&\leq 0.
\end{aligned}
$$

We then deduce that both $k(t)$ and $RL^*(t)$ are decreasing in γ.

(2) By the same argument as in part (1), by letting $n(t) := \frac{\partial k(t)}{\partial \theta}$, we can show that

$$
\begin{aligned}
n(t) &= \int_t^T m_Y e^{r(T-u)} \times \\
&\quad \exp\left\{-\int_t^u \left(\alpha - m_R - \eta m_R S_Y\left(\frac{1}{\gamma}e^{-r(T-s)}[\eta+\gamma m_R k(s)]^+\right)e^{-r(T-s)}1_{\{\eta+\gamma m_R k(s)>0\}}\right)ds\right\}du \\
&\geq 0.
\end{aligned}
$$

Therefore, it is immediate that both $k(t)$ and $RL^*(t)$ are increasing in θ.

(3) Note that equation (3.3.5) is an autonomous ordinary differential equation, and hence the function $k(t)$ is monotone in t (see, e.g., Lemma 1.7 in Hale and Koçak [53]). Since $k(T) = 0$, the remaining immediately follows from Proposition 3.3.3.

Chapter 4

Optimal Reinsurance-Investment Strategy with Bayesian Learning

4.1 Introduction

This chapter studies the optimal reinsurance-investment strategy with Bayesian learning for a mean-variance insurer. The stock return rate $\boldsymbol{\mu}$ is modelled as an unobservable constant random variable (not time-dependent). The insurer utilizes observed stock prices to learn about $\boldsymbol{\mu}$, and then uses the learning results and reinsurance to maximize the mean-variance objective function. As new stock prices are observed, learning results and investment strategies will be updated accordingly. The main contribution of this work is as follows:

First, an explicit form of optimal reinsurance-investment strategy up to a solution of a linear parabolic PDE is obtained. Zhang et al. [108] considers the optimal investment and proportional reinsurance problem for an exponential utility maximizer. Liang and Song [74] studies the same mean-variance problem, where reinsurance is constrained to be a proportional one and the stock return rate is modelled as a finite-state hidden Markov chain. Optimal strategies with Bayesian learning of the stock return rate for a mean-variance insurer is not yet studied to the best of our knowledge.

Second, the hedging demand due to the uncertainty of the stock return rate is analyzed qualitatively. An investor is called myopic if he or she refrains from future learning of partial information, and acts as if the learning result is certain going forward. The difference between the optimal strategy and its myopic counterpart is called hedging demand, a term introduced by Merton et al. [80], representing the demand the insurer needs to protect himself against unfavourable shifts in the stock return.

Our work shows that: (1) The optimal investment strategy degenerates to the myopic one when the stock return rate is a deterministic constant. (2) The hedging demand diminishes at the end of the time horizon. (3) If the learning result of the market price of risk always lies outside of a deterministic band and the optimal investment strategy and its myopic counterpart are of the same sign, then the optimal investment strategy is always more conservative. (4) Whether the two strategies are of the same sign depends on the magnitude of the posterior variance of the market price of risk. From our numerical analysis, when the posterior variance of the market price of risk is relatively large, the optimal investment strategy will take a short position even when the market price of risk is positive. This makes economic sense, since when the posterior mean of the market price of risk is positive, close to 0 and with a large posterior variance, there is a high probability that it will go below zero. The optimal investment strategy can capture this effect and take a short position accordingly, while the myopic one will take a long position as long as the posterior mean of market price of risk is positive. Therefore, our optimal investment strategy is better in the sense that it can act on both the first and second moments of the posterior distribution. (5) From our numerical analysis, when the distribution of market price of risk is not too scattered, the longer the time horizon, the more conservative our optimal investment strategy would be.

The remainder of this chapter is organized as follows. In section 4.2 the mathematical model of reinsurance-investment problem is formulated, and the objective function is presented. In section 4.3 we reduce the partially observable problem to an equivalent one with complete observation. In section 4.4, the main results for the explicit optimal reinsurance-investment strategy and the value function are given, and the hedging demand is analyzed qualitatively. In section 4.5, some numerical examples are provided. All proofs are postponed to Appendix 4.6.

4.2 Problem formulation

In this section we introduce our models for the insurance risk, the financial market and the objective function. Consider a fixed time horizon $T > 0$, and a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. In what follows, we use $\mathbb{E}$ to denote the expectation with respect to $\mathbb{P}$. Suppose the probability space is large enough to accommodate two uncertainties from the insurance risk and the financial market.

4.2.1 Surplus process for the insurance company

Consider an insurer's basic surplus process characterized by a spectrally negative Lévy process defined on $(\Omega, \mathcal{F}, \mathbb{P})$ with dynamics

$$dU_t = cdt + \sigma_1 dB_t - \int_0^\infty zN(dz, dt), \quad U_0 > 0, \tag{4.2.1}$$

where $c > 0$ is the insurance premium rate, $B = \{B_t\}_{0 \le t \le T}$ is a standard Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P})$ which represents a perturbation of insurance surplus with a volatility $\sigma_1 > 0$, and $N(dz, dt)$ is a Poisson random measure, independent of B, representing the number of insurance claims of size$(z, z + dz)$ within the time period $(t, t + dt)$. We denote the compensated measure of $N(dz, dt)$ by $\tilde{N}(dz, dt) = N(dz, dt) - v(dz)dt$, in which v is a Lévy measure such that $\int_0^\infty z^2 v(dz) < \infty$, and $v(dz)$ represents the expected number of insurance claims of size $(z, z + dz)$ within a unit time interval.

The insurance premium rate is computed under the expected value principle, that is, $c = (1 + \theta) \int_0^\infty zv(dz)$, in which $\theta > 0$ is the so-called safety loading of the insurer. At any time $t \in [0, T]$, the insurer facing a claim size z can manage the claim risk by purchasing a reinsurance policy with retention level $l(t, z)$, that is, $l(t, z)$ will be covered by the insurer and the remaining part $z - l(t, z)$ will be paid by the reinsurer. Certain reinsurance premium needs to be paid to transfer the risk, and we assume such reinsurance premium is also collected via expected value principal with the safety loading factor η. More specifically, the cost of transferring $z-l(t,z)$ to the reinsurer is at a rate of $(1+\eta)\int_0^\infty \big[z-l(t,z)\big]v(dz)$. Then the controlled surplus process under the reinsurance strategy l follows the dynamic:

$$\begin{aligned} dR_t^l &= dU_t - (1+\eta)\int_0^\infty [z-l(t,z)]v(dz)dt + \int_0^\infty [z-l(t,z)]N(dz,dt) \\ &= \int_0^\infty [(\theta-\eta)z + (1+\eta)l(t,z)]v(dz)dt + \sigma_1 dB_t - \int_0^\infty l(t,z)N(dz,dt) \\ &= \int_0^\infty [(\theta-\eta)z + \eta l(t,z)]v(dz)dt + \sigma_1 dB_t - \int_0^\infty l(t,z)\tilde{N}(dz,dt). \end{aligned} \tag{4.2.2}$$

We assume that the reinsurance is more expensive than the insurance, i.e., $\eta > \theta$ to exclude trivial results, and the retention level is nonnegative and no larger than the total claim size, i.e., $0 \leq l(t,z) \leq z$, for any $t \in [0,T]$, $z \geq 0$.

4.2.2 The financial market

Now assume that the insurer can further invest the collected premium in a financial market consisting of a risk-free bond with a constant interest rate $r > 0$ and a risky stock, whose price $\{S_t\}_{t\geq 0}$ is governed by

$$dS_t = S_t(\boldsymbol{\mu} dt + \sigma_2 dW_t), \qquad S_0 > 0, \tag{4.2.3}$$

where $\{W_t\}_{0\geq 0}$ is a standard Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P})$ with respect to its natural filtration $\mathbb{F}^W := \{\mathcal{F}_t^W\}_{t\geq 0}$, $\sigma_2 > 0$ (a known constant) is the volatility of the stock, and $\boldsymbol{\mu}$ is a random variable, which we use to characterize the uncertainty of the stock's return rate. We further assume that B, $N(\cdot,\cdot)$ and $\boldsymbol{\mu}$ are mutually independent. Moreover, B and W are correlated with a constant correlation coefficient ρ, that is, $\mathbb{E}\big[dW_t dB_t\big] = \rho dt$. In the Bayesian framework, we denote the prior distribution of $\boldsymbol{\mu}$ under $\mathbb{P}$ as follows:

$$\chi(A) = \mathbb{P}[\boldsymbol{\mu} \in A], \qquad A \in \mathscr{B}(\mathbb{R})$$

and assume that $\mathbb{E}[\boldsymbol{\mu}] < \infty$. $\mathscr{B}(\mathbb{R})$ denotes the Borel σ-algebra of $\mathbb{R}$.

A reinsurance-investment strategy is described by a pair process $u = \{(\pi_t, l_t)\}_{t\in[0,T]}$, where π_t represents the amount of money invested in stock at time t. Corresponding to a strategy u, the insurer's surplus process X^u follows dynamic:

$$\begin{aligned} dX_t^u &= \pi_t \frac{dS_t}{S_t} + r[X_t^u - \pi_t]dt + dR_t^l \\ &= \left[(\boldsymbol{\mu} - r)\pi_t + rX_t^u + \int_0^\infty \big[(\theta - \eta)z + \eta l(t,z)\big]v(dz)\right]dt \\ &\qquad\qquad + \sigma_1 dB_t + \pi_t\sigma_2 dW_t - \int_0^\infty l(t,z)\tilde{N}(dz,dt). \end{aligned} \tag{4.2.4}$$

Different from a model with complete observation, we assume that we cannot observe the stock return rate $\boldsymbol{\mu}$ nor the driven Brownian motion W directly, and we can only observe the stock price process $\{S_t\}_{t\geq 0}$, whose natural filtration is denoted by $\mathbb{F}^S := \{\mathcal{F}_t^S\}_{t\geq 0}$. We denote by $\mathbb{H} := \{\mathcal{H}_t\}_{t\geq 0}$ where $\mathcal{H}_t = \mathcal{F}_t^{S,B,N} = \mathcal{F}_t^S \vee \mathcal{F}_t^B \vee \mathcal{F}_t^N$, which is the smallest σ-algebra generated by $\mathcal{F}_t^S$, $\mathcal{F}_t^B$ and $\mathcal{F}_t^N$. $\mathbb{H}$ collects the information that the insurer actually has access to. Next, we define our admissible strategy set.

Definition 4.2.1. *(Admissible Strategy). A strategy $u = \{(\pi_t, l_t)\}_{t\in[0,T]}$ is called admissible if it satisfies the following conditions:*

(1) u is $\mathbb{H}$-progressively measurable;

(2) For all $t \in [0,T]$ and $z \geq 0$, $0 \leq l(t,z) \leq z$;

(3) $\mathbb{E}\big[\int_0^T \pi_t^2 dt\big] < \infty$.

We write Π as the set of all admissible strategies.

4.2.3 Objective function

For a controlled surplus process X^u with initial state $X_t = x$, the objective function is defined as

$$J^u(t,x) = \mathbb{E}[X_T^u|\mathcal{H}_t] - \frac{\gamma}{2}\text{Var}[X_T^u|\mathcal{H}_t], \qquad (t,x) \in [0,T) \times \mathbb{R}, \tag{4.2.5}$$

where $\gamma > 0$ is the insurer's risk aversion.

Note that due to the variance term, $J^u(t,x)$ is time-inconsistent in the sense that it doesn't admit the Bellman optimality principle. Following Björk et al. [16], we attach the problem by viewing it within a game theoretic framework, and look for Nash subgame equilibrium points. Moreover, since the wealth process (4.2.4) contains unobservable quantities $\boldsymbol{\mu}$ and W, we have a time-inconsistent stochastic control problem with partial information.

Definition 4.2.2. *(Equilibrium reinsurance-investment strategies). Consider an admissible strategy $u^* = \{(\pi_t^*, l_t^*)\}_{t\in[0,T]} \in \Pi$. For any $\varepsilon > 0$, $t \in [0,T]$, define a perturbed strategy*

$$u^\varepsilon(s,z) = \begin{cases} u, & t \le s \le t+\varepsilon, \\ (\pi_s^*, l^*(s,z)), & t+\varepsilon \le s \le T, \end{cases} \tag{4.2.6}$$

where $u = (\pi, l) \in \mathbb{R} \times [0,z]$. If

$$\liminf_{\varepsilon \downarrow 0} \frac{J^{u^*}(t,x) - J^{u^\varepsilon}(t,x)}{\varepsilon} \ge 0,$$

for any initial state $(t,x) \in [0,T] \times \mathbb{R}$ and $\varepsilon > 0$. Then u^* is called an equilibrium reinsurance-investment strategy and $J^{u^*}(t,x)$ is the associated equilibrium value function.

4.3 Posterior predictive distribution of $\boldsymbol{\mu}$

Since the stock return rate $\boldsymbol{\mu}$ is unobservable, the insurer needs to learn about $\boldsymbol{\mu}$ based on its prior distribution χ and the observable stock price, i.e, the observation filtration $\mathbb{F}^S$. Pham [86] provides a survey of the methods involved in portfolio optimization with partial information, which covers 3 cases for modelling the unobservable return rate, namely, Bayesian, linear-Gaussian and finite state Markov chain. Karatzas and Zhao [64] focuses on Bayesian adaptive portfolio to maximize expected terminal utility. We follow their methods in the Bayesian framework to derive the dynamic of the posterior estimator:

$$\hat{\boldsymbol{\mu}}_t = \mathbb{E}[\boldsymbol{\mu} | \mathcal{F}_t^S]. \tag{4.3.1}$$

We define

$$Y_t = W_t + \frac{\boldsymbol{\mu} - r}{\sigma_2} t, \quad t \in [0, T]. \tag{4.3.2}$$

Denote by $\mathbb{F}^Y = \{\mathcal{F}_t^Y\}_{t \geq 0}$ the filtration generated by the process Y, and by $\mathbb{G} = \{\mathcal{G}_t\}_{t \geq 0}$ the auxiliary, enlarged filtration $\mathcal{G}_t^{\boldsymbol{\mu}, W} = \sigma(\boldsymbol{\mu}, W_s : 0 \leq s \leq t)$ generated by the Brownian motion W and the random variable $\boldsymbol{\mu}$. Then $\mathcal{F}_t^Y \subseteq \mathcal{G}_t$ for every $t \in [0, T]$.

The following two lemmas are cited from Karatzas and Zhao [64](Lemma 2.1 and Lemma 2.2 on page 634).

Lemma 4.3.1. *W is a $(\mathbb{G}, \mathbb{P})$-Brownian motion, and the following exponential process*

$$K_t = \exp\{-\frac{\boldsymbol{\mu} - r}{\sigma_2} W_t - \frac{(\boldsymbol{\mu} - r)^2}{2\sigma_2^2} t\}, \quad 0 \leq t \leq T \tag{4.3.3}$$

is a $(\mathbb{G}, \mathbb{P})$-martingale.

We then define a new probability $\mathbb{Q}$ equivalent to $\mathbb{P}$ on $\mathcal{G}_T$ as follows,

$$\mathbb{Q}(A) = \mathbb{E}[K_T \cdot \mathbb{I}_A], \quad A \in \mathcal{G}_T. \tag{4.3.4}$$

For any $t \in [0, T]$, define

$$M_t = K_t^{-1} = \exp\{\frac{\boldsymbol{\mu} - r}{\sigma_2} Y_t - \frac{(\boldsymbol{\mu} - r)^2}{2\sigma_2^2} t\}. \tag{4.3.5}$$

Lemma 4.3.2.

(1) $\{M_t\}_{t \in [0,T]}$ is a $(\mathbb{G}, \mathbb{Q})$-martingale;

(2) $\{Y_t\}_{0 \in [0,T]}$ is a $(\mathbb{G}, \mathbb{Q})$ (thus also with respect to $\mathbb{F}^Y$)-standard Brownian motion and is independent of the random variable $\boldsymbol{\mu}$ under $\mathbb{Q}$;

(3) $\boldsymbol{\mu}$ has the same distribution under $\mathbb{Q}$, i.e., we have

$$\mathbb{P}[\boldsymbol{\mu} \in A] = \mathbb{Q}[\boldsymbol{\mu} \in A] = \chi(A), \qquad \forall A \in \mathscr{B}(\mathbb{R}).$$

By Itô's lemma, we have

$$d\ln S_t = \sigma_2 dY_t + rdt - \frac{\sigma_2^2}{2}dt, \tag{4.3.6}$$

which means the filtration generated by Y coincides with the filtration generated by the stock price S. In other words, $\mathbb{F}^Y = \mathbb{F}^S$. In the following, we will use the above results to compute the posterior mean $\hat{\boldsymbol{\mu}}_t$ and reduce the problem with partial observation to one with complete observation.

$$\begin{aligned}
\hat{\boldsymbol{\mu}}_t &= \mathbb{E}[\boldsymbol{\mu}|\mathcal{F}_t^S] \\
&= \frac{\mathbb{E}^{\mathbb{Q}}[\boldsymbol{\mu}\cdot M_T|\mathcal{F}_t^Y]}{\mathbb{E}^{\mathbb{Q}}[M_T|\mathcal{F}_t^Y]} \\
&= \frac{\mathbb{E}^{\mathbb{Q}}[\boldsymbol{\mu}\cdot M_t|\mathcal{F}_t^Y]}{\mathbb{E}^{\mathbb{Q}}[M_t|\mathcal{F}_t^Y]} \\
&= \frac{\mathbb{E}^{\mathbb{Q}}\big[\boldsymbol{\mu}\exp\{\frac{\boldsymbol{\mu}-r}{\sigma_2}Y_t - \frac{(\boldsymbol{\mu}-r)^2}{2\sigma_2^2}t\}|\mathcal{F}_t^Y\big]}{\mathbb{E}^{\mathbb{Q}}[\exp\{\frac{\boldsymbol{\mu}-r}{\sigma_2}Y_t - \frac{(\boldsymbol{\mu}-r)^2}{2\sigma_2^2}t\}|\mathcal{F}_t^Y]},
\end{aligned} \tag{4.3.7}$$

where the third line is due to $\mathbb{E}^{\mathbb{Q}}[M_T|\mathcal{F}_t^Y] = \mathbb{E}^{\mathbb{Q}}[\mathbb{E}^{\mathbb{Q}}(M_T|\mathcal{G}_t)|\mathcal{F}_t^Y] = \mathbb{E}^{\mathbb{Q}}[M_t|\mathcal{F}_t^Y]$, and $\mathbb{E}^{\mathbb{Q}}[\boldsymbol{\mu}M_T|\mathcal{F}_t^Y] = \mathbb{E}^{\mathbb{Q}}[\mathbb{E}^{\mathbb{Q}}(\boldsymbol{\mu}M_T|\mathcal{G}_t)|\mathcal{F}_t^Y] = \mathbb{E}^{\mathbb{Q}}[\boldsymbol{\mu}M_t|\mathcal{F}_t^Y]$, since $M(\cdot)$ is a $(\mathbb{G},\mathbb{Q})$ martingale by lemma 4.3.2. Moreover, since $Y(\cdot)$ and $\boldsymbol{\mu}$ are independent under $\mathbb{Q}$, we have

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}}[\exp\{\frac{\boldsymbol{\mu}-r}{\sigma_2}Y_t - \frac{(\boldsymbol{\mu}-r)^2}{2\sigma_2^2}t\}|\mathcal{F}_t^Y] &= \int_{\mathbb{R}} \exp\left\{\frac{x-r}{\sigma_2}y - \frac{(x-r)^2}{2\sigma_2^2}t\right\}\chi(dx)\Big|_{y=Y_t} \\
&=: F(t,y)\Big|_{y=Y_t}.
\end{aligned} \tag{4.3.8}$$

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}}\big[\boldsymbol{\mu}\exp\{\frac{\boldsymbol{\mu}-r}{\sigma_2}Y_t - \frac{(\boldsymbol{\mu}-r)^2}{2\sigma_2^2}t\}|\mathcal{F}_t^Y\big] &= \int_{\mathbb{R}} x\exp\left\{\frac{x-r}{\sigma_2}y - \frac{(x-r)^2}{2\sigma_2^2}t\right\}\chi(dx)\Big|_{y=Y_t} \\
&= \big(rF(t,y) + \sigma_2\frac{\partial F}{\partial y}(t,y)\big)\Big|_{y=Y_t}.
\end{aligned} \tag{4.3.9}$$

Substitute (4.3.8) and (4.3.9) into (4.3.7), we have the following lemma for the learning of $\boldsymbol{\mu}$ at time t:

Lemma 4.3.3.

$$\hat{\boldsymbol{\mu}}_t = r + \sigma_2 G(t, Y_t), \quad t \in (0, T]. \tag{4.3.10}$$

where $G(t,y) = \dfrac{1}{F(t,y)} \dfrac{\partial F}{\partial y}(t,y), (t,y) \in (0,T] \times \mathbb{R}$. *When* $t = 0$, $\hat{\boldsymbol{\mu}}_0$ *is just the prior mean of* $\boldsymbol{\mu}$.

Modelling the unobservable expected return rate in a Bayesian framework, Longo and Mainini [77] investigated the optimal investment problem for a CRRA investor. In a same modelling and learning framework, Zhang et al. [108] derived the optimal investment and reinsurance strategy which maximizes the exponential utility, where the reinsurance was restricted to a proportional one, basic surplus process was approximated as a diffusion process, and no comparison to the myopic strategy was analyzed.

Lemma 4.3.4. *Define*

$$\hat{W}_t := Y_t - \int_0^t \frac{\hat{\boldsymbol{\mu}}(s) - r}{\sigma_2} ds = Y_t - \int_0^t G(s, Y_s) ds, \; t \in [0, T]. \tag{4.3.11}$$

$\{\hat{W}\}_{t \in [0,T]}$ *is a* $(\mathbb{F}^S, \mathbb{P})$*-standard Brownian motion.*

Now we are ready to convert the problem with partial observation into one with complete observation. From

$$Y_t = W_t + \frac{\boldsymbol{\mu} - r}{\sigma_2} t = \hat{W}_t + \int_0^t \frac{\hat{\boldsymbol{\mu}}_s - r}{\sigma_2} ds, \tag{4.3.12}$$

substitute $dW_t = d\hat{W}_t + \dfrac{\hat{\boldsymbol{\mu}}_t}{\sigma_2} dt - \dfrac{\boldsymbol{\mu}}{\sigma_2} dt$ into wealth process (4.2.4), we have the complete observation process:

$$
\begin{cases} dX_t^u = \Big[\sigma_2\pi_t G(t,Y_t) + rX_t^u + \int_0^\infty [(\theta-\eta)z + \eta l(t,z)]v(dz)\Big]dt \\ \qquad\qquad\qquad\qquad + \sigma_1 dB_t + \pi_t\sigma_2 d\hat{W}_t - \int_0^\infty l(t,z)\tilde{N}(dz,dt), \\ dY_t = G\big(t,Y_t\big)dt + d\hat{W}_t. \end{cases} \tag{4.3.13}
$$

Since the insurer can observe the information set $\mathcal{H}_t$, each quantity in the above dynamics are observable to the insurer. The original problem now becomes the following time-inconsistent stochastic control problem with complete observation:

$$
J^u(t,x,y) = \mathbb{E}[X_T^{u,t,x,y}] - \frac{\gamma}{2}\mathrm{Var}[X_T^{u,t,x,y}], \qquad (t,x,y) \in [0,T) \times \mathbb{R} \times \mathbb{R}, \tag{4.3.14}
$$

where $\{(X_s^{u,t,x,y}, Y_s^{t,y})\}_{t\in[0,T]}$ denotes the solution of (4.3.13) starting from (x,y) at time point t.

4.4 Main results

For ease of notation, denote by $C^{1,2,2}([0,T]\times\mathbb{R}\times\mathbb{R})$ the space of functions which is first-order continuously differentiable in $t \in [0,T]$ and second-order continuously differentiable in $x \in \mathbb{R}$ and $y \in \mathbb{R}$. From (4.3.13), for any $\phi(t,x,y) \in C^{1,2,2}([0,T] \times \mathbb{R} \times \mathbb{R})$, the infinitesimal generator of X_t^u is given by

$$
\begin{aligned} \mathcal{A}^u\phi(t,x,y) = \phi_t(t,x,y) + \Big[\sigma_2\pi G(t,y) + rx + \int_0^\infty \big[(\theta-\eta)z + (\eta+1)l(t,z)\big]v(dz)\Big]\phi_x(t,x,y) \\ + \frac{1}{2}\big(\sigma_1^2 + \sigma_2^2\pi^2 + 2\pi\sigma_1\sigma_2\rho\big)\phi_{xx}(t,x,y) + G(t,y)\phi_y(t,x,y) + \frac{1}{2}\phi_{yy}(t,x,y) \\ + \big(\sigma_1\rho + \pi\sigma_2\big)\phi_{xy}(t,x,y) + \int_0^\infty \Big[\phi(t,x-l(t,z),y) - \phi(t,x,y)\Big]v(dz) \end{aligned} \tag{4.4.1}
$$

The following Verification Theorem is similar to Theorem 3.3.1 in Chapter 3, which is proved in Appendix 3.6, we thus omit the proof here.

Theorem 4.4.1. *(Verification Theorem). Suppose there exist $V(t,x,y)$ and $g(t,x,y) \in C^{1,2,2}([0,T]\times\mathbb{R}\times\mathbb{R})$ satisfying the following conditions:*

(1) For all $(t,x,y) \in [0,T]\times\mathbb{R}\times\mathbb{R}$,

$$\sup_{u\in\Pi}\Big\{\mathcal{A}^u V(t,x,y) - \frac{\gamma}{2}\mathcal{A}^u g^2(t,x,y) + \gamma g(t,x,y)\mathcal{A}^u g(t,x,y)\Big\} = 0; \tag{4.4.2}$$

Denote the strategy that attains the supremum by $u^* = (\pi^*, l^*)$.

(2) For all $(t,x,y) \in [0,T]\times\mathbb{R}\times\mathbb{R}$,

$$\mathcal{A}^{u^*} g(t,x,y) = 0; \tag{4.4.3}$$

(3) For all $(x,y) \in \mathbb{R}\times\mathbb{R}$,

$$V(T,x,y) = g(T,x,y) = x. \tag{4.4.4}$$

Then u^* is an equilibrium strategy, $V(t,x,y) = J^{u^*}(t,x,y)$ is the corresponding equilibrium value function, and $g(t,x,y) = \mathbb{E}[X_T^{u^*,t,x,y}]$ is the expectation of terminal surplus under the optimal strategy u^*.

Theorem 4.4.2. *The equilibrium strategy $u^* = \big(\pi_t^*, l^*(t,z)\big)$ is given by*

$$\begin{cases} \pi^*(t,y) = \big(\dfrac{\hat{\boldsymbol{\mu}}_t - r}{\gamma\sigma_2^2} - \dfrac{k_y(t,y)}{\sigma_2}\big)e^{-r(T-t)} - \dfrac{\sigma_1}{\sigma_2}\rho, \\ l^*(t,z) = z \wedge \dfrac{\eta}{\gamma}e^{-r(T-t)}, \end{cases} \tag{4.4.5}$$

for which $\{(k(t,y)\}_{(t,y)\in[0,T]\times\mathbb{R}}$ is the solution of the following linear parabolic PDE:

$$\begin{cases} k_t(t,y) + \dfrac{1}{2}k_{yy}(t,y) + \dfrac{G^2(t,y)}{\gamma} - e^{r(T-t)}\sigma_1\rho G(t,y) \\ \qquad\qquad + e^{r(T-t)}\displaystyle\int_0^\infty \Big[(\theta-\eta)z + \eta l^*(t,z)\Big]v(dz) = 0, \\ k(T,y) = 0. \end{cases} \tag{4.4.6}$$

The equilibrium value function is characterized by

$$J^{u^*}(t,x,y) = V(t,x,y) = e^{r(T-t)}x + K(t,y),$$

where $\{K(t,y)\}_{(t,y)\in[0,T]\times\mathbb{R}}$ *is the solution of*

$$\begin{cases} K_t(t,y) + \frac{1}{2}K_{yy}(t,y) + G(t,y)K_y(t,y) - k_y(t,y)G(t,y) \\ \qquad\qquad + \frac{G^2(t,y)}{2\gamma} + \frac{\gamma\sigma_1^2}{2}e^{2r(T-t)}(\rho^2-1) - \sigma_1\rho G(t,y)e^{r(T-t)} \\ \qquad\qquad\qquad + e^{r(T-t)}\int_0^\infty \Big[-\frac{\gamma}{2}(l^*)^2(t,z)e^{r(T-t)} + (\theta-\eta)z + \eta l^*(t,z)\Big]v(dz) = 0, \\ K(T,y) = 0. \end{cases} \tag{4.4.7}$$

Moreover, the expectation of the terminal surplus is characterized by

$$g(t,x,y) = e^{r(T-t)}x + k(t,y).$$

First, the optimal reinsurance strategy l^* takes an excess-of-loss reinsurance type, with a time-dependent retention limit $\frac{\eta}{\gamma}e^{-r(T-t)}$. Therefore, under the time-consistent mean-variance criterion, an excess-of-loss reinsurance treaty is better than a proportional one. As reinsurance becomes more expensive, which is measured by η, the insurer will retain more of the claim risk to himself and purchase less reinsurance. When the insurer becomes more risk averse, as measured by γ, he or she will undertake less insurance risk and transfer more to the reinsurer. Also, when the interest rate r increases, it is more costly to borrow money, and more insurance risk would be retained by the insurer. Moreover, our optimal reinsurance strategy l^* is the same with that of Li et al. [68] , since we both assume that the basic surplus process is completely observable.

Second, the optimal investment strategy π^* depends on the state variable y through the learning result $\hat{\boldsymbol{\mu}}_t$ and function k. Recall that the state process Y represents the dynamic of the learning result of $\boldsymbol{\mu}$. Therefore, as more stock prices are observed, both the learning result and the optimal investment strategy will be updated accordingly.

Note from the result of Li et al. [68], when $\boldsymbol{\mu}$ is a deterministic constant μ, the optimal investment strategy is

$$\tilde{\pi}_t^* = \frac{\mu - r}{\gamma\sigma_2^2} e^{-r(T-t)} - \rho\frac{\sigma_1}{\sigma_2}. \tag{4.4.8}$$

When uncertainty about the stock return rate is considered, if the insurer first solves the optimal investment strategy treating the return rate as a known constant, and then replace the return rate in the strategy by its estimator, then the insurer would follow the investment strategy:

$$\hat{\pi}_t = \frac{\hat{\boldsymbol{\mu}}_t - r}{\gamma\sigma_2^2} e^{-r(T-t)} - \rho\frac{\sigma_1}{\sigma_2}. \tag{4.4.9}$$

Such an insurer is called myopic, since the dynamic of the learning result of $\boldsymbol{\mu}$ is not incorporated in the decision making process. Having observed $Y_t = y$ at time t, the insurer behaves as if the estimator $\hat{\boldsymbol{\mu}}_t$ were certain and constant from t onwards. However, by comparing strategy (4.4.5) and (4.4.9), we can see that the uncertainty about the stock return rate will actually change the form of the investment strategy, and an additional hedging term is necessary for optimal purchasing behaviour. The difference between π^* and $\hat{\pi}$ is defined as the hedging demand induced by parameter uncertainty, see Merton et al. [80]. The term k_y in (4.4.5) arises from the incentive of the insurer to hedge against unfavourable realizations of the unknown parameter $\boldsymbol{\mu}$. Cvitanić et al. [32] showed that the hedging component induced by learning about the expected return rate can be a substantial part of the demand, especially for long time horizons.

The direction of the hedging demand may be positive or negative. The following theorem demonstrates the properties of the optimal investment strategies from this perspective.

Theorem 4.4.3.

(1) When $\boldsymbol{\mu}$ is a known constant, the optimal strategy π^* will degenerate to the one with complete observation, i.e. if $\boldsymbol{\mu}$ is a known constant μ, then

$$\pi^*(t) = \hat{\pi}(t), \quad \forall t \in [0, T]. \tag{4.4.10}$$

(2) At time T the optimal strategy is equal to the myopic counterpart, i.e.

$$\lim_{t \to T} \pi^*(t, y) = \hat{\pi}(T, y) \tag{4.4.11}$$

for all $y \in \mathbb{R}$.

(3) If

$$\frac{2}{\gamma} G(t, y) \leq e^{r(T-t)} \sigma_1 \rho \quad \forall t \in [0, T], y \in \mathbb{R}, \tag{4.4.12}$$

then

$$\pi^*(t, y) \geq \hat{\pi}(t, y) \quad \forall t \in [0, T], y \in \mathbb{R}.$$

And if

$$\frac{2}{\gamma} G(t, y) \geq e^{r(T-t)} \sigma_1 \rho \quad \forall t \in [0, T], y \in \mathbb{R}, \tag{4.4.13}$$

then

$$\pi^*(t, y) \leq \hat{\pi}(t, y) \quad \forall t \in [0, T], y \in \mathbb{R}.$$

Function G is defined in equation (4.3.10).

Part (2) of Theorem 4.4.3 says that at the end of the time horizon, the insurer will act as an myopic investor, since the investment is coming to an end and no further stock prices would be observed.

To understand part (3) of Theorem 4.4.3, let's consider the special case when $\rho = 0$, i.e., the insurance risk and financial market are independent. When $G(t, y) \geq 0$ for all $(t, y) \in [0, T] \times \mathbb{R}$, then from equation (4.4.9), $\hat{\pi}_t \geq 0$, and a myopic investor will take a long position. At this time, if the optimal investment strategy π^* is also positive, then Theorem 4.4.3 tells us that π^* will buy less and is more conservative than the myopic investor. The case when $G(t, y) \leq 0$ can be analyzed similarly. However, π^* and $\hat{\pi}$ are not necessarily of the same sign, it depends on the stock volatility, time horizon and prior distribution of $\boldsymbol{\mu}$ as can be seen from later numerical examples.

The study of the hedging demand for uncertain model parameters can be found in Brennan [19] and Longo and Mainini [77], where they both look for the optimal portfolio maximizing the expected power utility, while the former modelled $\boldsymbol{\mu}$ as a random variable with a Normal priori, and the latter modelled $\boldsymbol{\mu}$ as a general integrable random variable. Different from our result, in the context of maximizing the expected terminal power utility, the hedging demand depends on the sign of the risk tolerance parameter. Assuming the market price of risk is constant in sign, when the risk tolerance is positive, the optimal strategy would be more aggressive, and when the risk tolerance is negative, the associated optimal strategy would be more conservative. Therefore, the direction of the hedging demand depends on whether the investor is more or less risk tolerant than a logarithmic investor in their setting. While for a time-consistent mean-variance insurer, the sign of the hedging demand is not fixed and depends on other model parameters, even if the market price of risk is constant in sign.

4.5 Numerical Analysis

In this section, we assume that the basic surplus process follows the dynamic

$$dU_t = cdt + \sigma_1 dB_t - V_t,$$

where $\sigma_1 = 0.3$, $\{V_t\}_{t\in[0,T]}$ is a compound Poisson process with jump rate $\lambda = 1$ and severity distribution $\Gamma(\alpha, \beta)$, where $\alpha = 0.5$ and $\beta = 0.7$. Note that $\{U_t\}_{t\geq 0}$ is a special case of the spectrally negative Lévy process defined in (4.2.1) with Lévy measure given by

$$v(dz) = \lambda \frac{\beta^\alpha z^{\alpha-1} e^{-\beta z}}{\Gamma(\alpha)} dz.$$

The default values for other parameters are: $r = 0.03, \sigma_2 = 0.03, \gamma = 0.5, T = 5, \theta = 0.3, \rho = 0.2$, and $\eta = 0.6$.

Example 4.5.1. *When $\boldsymbol{\mu} \sim N(a, b^2)$, for any $(t, y) \in [0, T] \times \mathbb{R}$, we can get an explicit expression of $G(t, y)$ as follows,*

$$G(t, y) = \frac{b^2 y - \sigma_2(r - a)}{b^2 t + \sigma_2^2}.$$

When $a = 0.3$, $b = 1$, we can see from figure 4.1 that π^* doesn't always lie under $\hat{\pi}$ or above $\hat{\pi}$. Actually, function $G(t, \cdot)$ is a linear function of y for any fixed t, therefore when condition (4.4.12) or condition (4.4.13) doesn't hold, part (3) of Theorem (4.4.3) is not necessarily true.

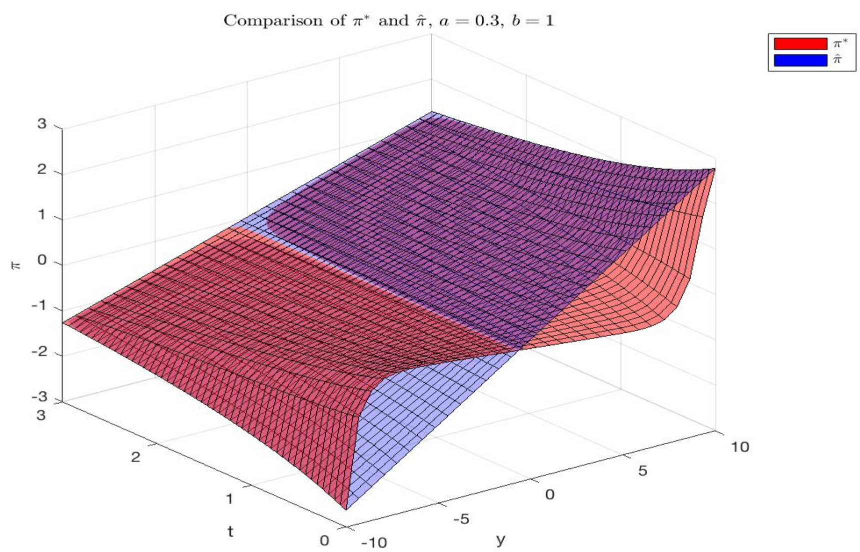


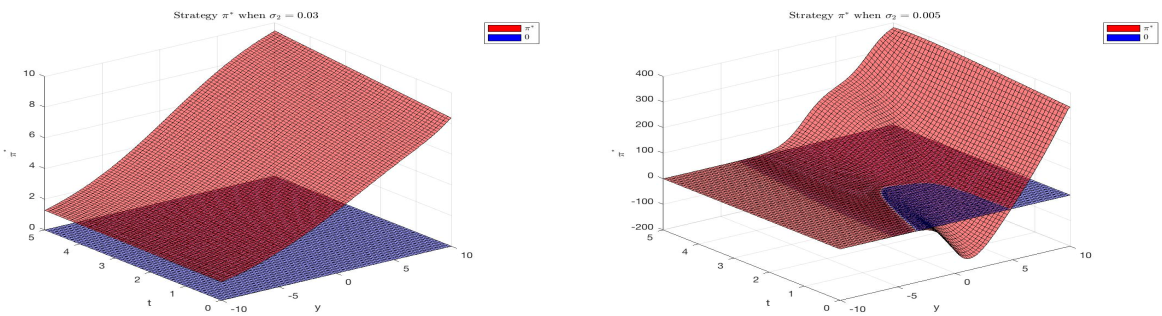
Figure 4.1: Comparison of π^* and $\hat{\pi}$

In the following, we study the case when $\boldsymbol{\mu}$ is discrete random variable with finite possible realizations and analyze the impact of the market price of risk $\frac{\boldsymbol{\mu} - r}{\sigma_2}$ and time horizon T on the optimal investment strategy π^*. For simplicity, assume from now on that $\rho = 0$, i.e. the stock dynamic and the surplus process are independent. Moreover, assume that $\boldsymbol{\mu}$ has 3 possible values μ_1, μ_2 and μ_3 with corresponding probability p_1, p_2 and p_3.

Example 4.5.2. *Impact of prior distribution of $\frac{\boldsymbol{\mu} - r}{\sigma_2}$ on π^*.*
Assume $\mu_1 = 0.04$, $\mu_2 = 0.035$, $\mu_3 = 0.03$, $p_1 = 0.3$, $p_2 = 0.4$, $p_3 = 0.3$. Since $r = 0.03$, the market price of risk $\frac{\boldsymbol{\mu} - r}{\sigma_2} \geq 0$. Therefore, we have $\hat{\pi} \geq 0$ and $\hat{\pi} \geq \pi^*$. However, π^* is not necessarily positive.

(1) Impact of stock volatility σ_2.

From figure [4.2](), when $\sigma_2 = 0.03$ we have $\pi^* \geq 0$. When $\sigma_2 = 0.005$, there exists a region where $\pi^* < 0$. As shown in figure 4.3, as σ_2 decreases, the prior variance of market price of risk increases, and the minimum value of $\pi^*(t, y)$ in the simulation region decreases and extends below zero when σ_2 is too small. More specifically, σ_2 affects on π^* through $G_y(\cdot, \cdot)$, which is the post variance of market price of risk from equation (4.6.6). Figure 4.4a and figure 4.4b show that π^* is negative when G_y is too big. This makes economic sense: when there is too much uncertainty, the optimal investment strategy may hold a short position even when the market price of risk is positive.



(a) $\sigma_2 = 0.03$ (b) $\sigma_2 = 0.005$

Figure 4.2: Impact of σ_2 on π^*

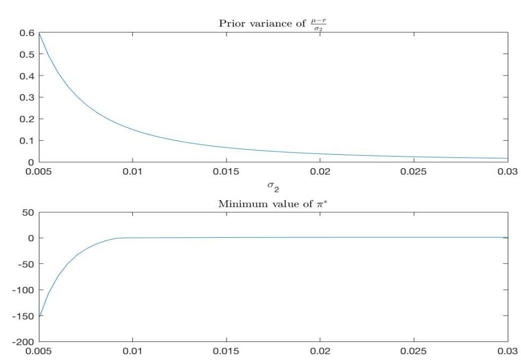
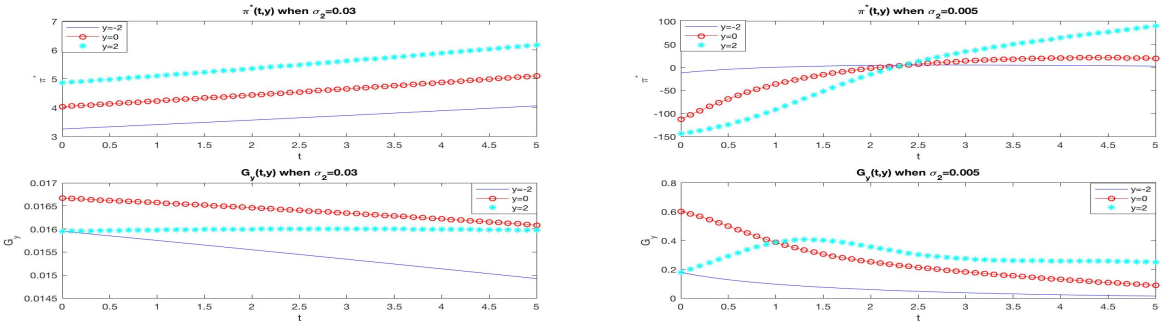


Figure 4.3: Minimum value of $\pi^*(t, y)$ w.r.t σ_2

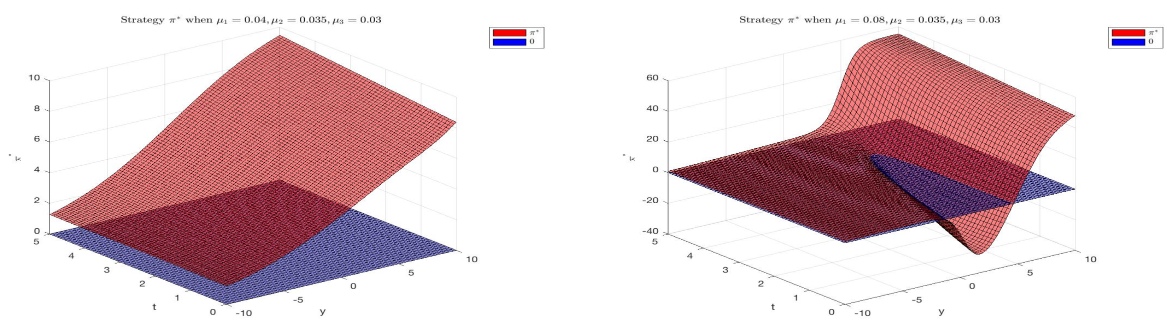


(a) $\sigma_2 = 0.03$ (b) $\sigma_2 = 0.005$

Figure 4.4: Effect of σ_2 on G_y

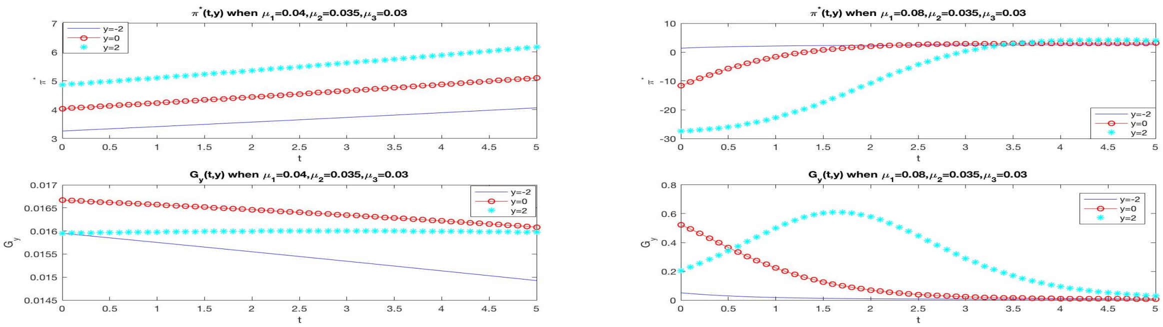
(2) Impact of prior distribution of $\boldsymbol{\mu}$.

With $\sigma_2 = 0.03$ *fixed, figure 4.5 compares the optimal investment strategies with different* μ_1*. With* $\mu_2, \mu_3, p_1, p_2, p_3$ *fixed, increase in* μ_1 *increases the prior variance of* $\boldsymbol{\mu}$ *from* $1.5 * 10^{-5}$ *to* $4.7 * 10^{-4}$*, thus increasing the post variance* G_y*, and* π^* *changes to a short position accordingly, as can be seen from figure 4.6a and figure 4.6b.*



(a) $\mu_1 = 0.04$ (b) $\mu_1 = 0.08$

Figure 4.5: Impact of μ_1 on π^*



(a) $\mu_1 = 0.04$ (b) $\mu_1 = 0.08$

Figure 4.6: Impact of μ_1 on G_y

Example 4.5.3. *Impact of* T *on* π^*.
The impact of time horizon T *also depends on* σ_2 *and the prior distribution of* $\boldsymbol{\mu}$. *When* $\mu_1 = 0.04$, $\mu_2 = 0.035$, $\mu_3 = 0.03$ *and* $\sigma_2 = 0.03$, *figure 4.7 depicts the 3 optimal investment strategies on the same region when* $T = 5$, $T = 15$ *and* $T = 25$ *respectively. It is shown that in this setting, all the strategies take a long position, and the longer the time horizon, the smaller amount is invested in stock. In this case, the optimal investment strategy is more conservative with a longer investment period.*

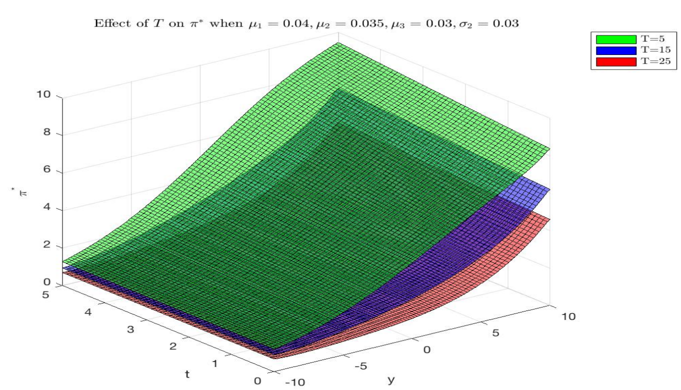
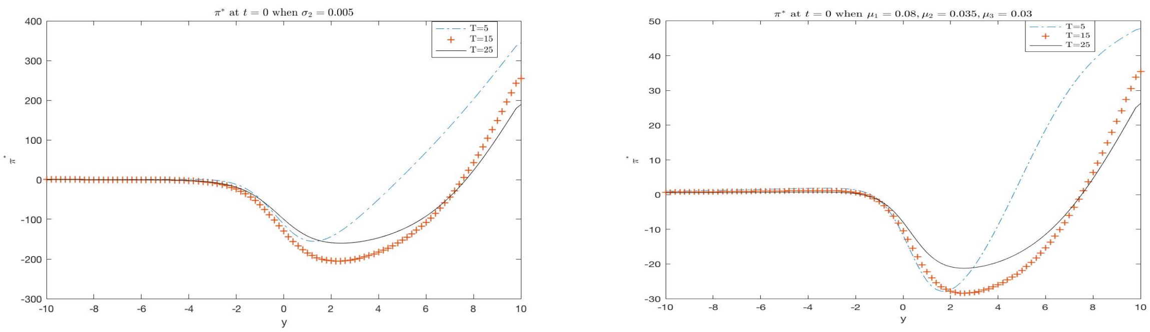


Figure 4.7: Impact of T on π^* for less uncertain market price of risk

However, when there is more uncertainty with the market price of risk, things will be more complicated as can be seen in figure 4.8, where either σ_2 *is decreased or* $\boldsymbol{\mu}$ *is more* *scattered. In both cases, when all 3 strategies are positive, the longer T is, the smaller amount of money is invested in stock. However, when all 3 strategies are negative, there is no monotonicity between investment horizon and equilibrium strategy.*



(a) $\sigma_2 = 0.005$ (b) $\mu_1 = 0.08$

Figure 4.8: Impact of T on π^* for more uncertain market price of risk

4.6 Appendix

4.6.1 Proof of Theorem 4.4.2

Proof. We now conjecture that $V(t,x,y)$ and $g(t,x,y)$ are separable in the surplus x and make the Ansatz

$$\begin{cases} V(t,x,y) = e^{r(T-t)}x + K(t,y), & K(T,y) = 0, \forall y \in \mathbb{R} \\ g(t,x,y) = e^{r(T-t)}x + k(t,y), & k(T,y) = 0, \forall y \in \mathbb{R} \end{cases} \tag{4.6.1}$$

for some deterministic functions k and K of t and y.

Then for all $(t,x,y) \in [0,T] \times \mathbb{R} \times \mathbb{R}$, we have

$$
\begin{aligned}
&V_t(t,x,y) = -re^{r(T-t)}x + K_t(t,y), V_x(t,x,y) = e^{r(T-t)},\\
&V_y(t,x,y) = K_y(t,y), V_{yy}(t,x,y) = K_{yy}(t,y),\\
&V_{xx}(t,x,y) = V_{xy}(t,x,y) = 0\\
&V(x-l(t,z),t,y) - V(t,x,y) = -l(t,z)e^{r(T-t)}.
\end{aligned}
$$

The same applies to function g if we replace V with g and K with k. Therefore, for any admissible strategy $u = (\pi, l)$, we have

$$
\begin{aligned}
&g(t,x,y)\mathcal{A}^u g(t,x,y) - \frac{1}{2}\mathcal{A}^u g^2(t,x,y)\\
&= -\frac{1}{2}\big(\sigma_1^2 + \sigma_2^2\pi^2 + 2\pi\sigma_1\sigma_2\rho\big)e^{2r(T-t)} - \frac{1}{2}\big(k_y(t,y)\big)^2\\
&\qquad\qquad - \big(\sigma_1\rho + \pi\sigma_2\big)e^{r(T-t)}k_y(t,y) - \frac{1}{2}\int_0^\infty l^2(t,z)e^{2r(T-t)}v(dz).
\end{aligned}
$$

Then it is easy to check that

$$
\begin{aligned}
&\sup_{u\in\Pi}\Big\{\mathcal{A}^u V(t,x,y) - \frac{\gamma}{2}\mathcal{A}^u g^2(t,x,y) + \gamma g(t,x,y)\mathcal{A}^u g(t,x,y)\Big\}\\
&= K_t + G(t,y)K_y + \frac{1}{2}K_{yy} - \frac{\sigma_1^2}{2}\gamma e^{2r(T-t)} - \frac{\gamma}{2}(k_y)^2 - \gamma\sigma_1\rho e^{r(T-t)}k_y\\
&\quad + \sup_\pi\Big\{-\frac{\gamma}{2}\sigma_2^2 e^{2r(T-t)}\pi^2 + e^{r(T-t)}\pi\big[\sigma_2 G(t,y) - \sigma_1\sigma_2\rho\gamma e^{r(T-t)} - \gamma\sigma_2 k_y\big]\Big\}\\
&\quad\quad + \sup_l\Big\{e^{r(T-t)}\int_0^\infty\big[-\frac{\gamma}{2}l^2(t,z)e^{r(T-t)} + (\theta-\eta)z + \eta l(t,z)\big]v(dz)\Big\}\\
&= K_t + G(t,y)K_y + \frac{1}{2}K_{yy} - \frac{\sigma_1^2}{2}\gamma e^{2r(T-t)} - \frac{\gamma}{2}(k_y)^2 - \gamma\sigma_1\rho e^{r(T-t)}k_y\\
&\quad + \sup_\pi\Big\{-\frac{\gamma}{2}\sigma_2^2 e^{2r(T-t)}\big[\pi - \frac{\beta(t,y)e^{-r(T-t)}}{\gamma\sigma_2^2}\big]^2 + \frac{\beta^2(t,y)}{2\gamma\sigma_2^2}\Big\}\\
&\quad\quad + \sup_l\Big\{e^{r(T-t)}\int_0^\infty\big[-\frac{\gamma}{2}l^2(t,z)e^{r(T-t)} + (\theta-\eta)z + \eta l(t,z)\big]v(dz)\Big\},
\end{aligned}
$$

where $\beta(t,y) = \sigma_2 G(t,y) - \sigma_1\sigma_2\rho\gamma e^{r(T-t)} - \gamma\sigma_2 k_y$.

Therefore,

$$\pi^*(t,y) = \frac{\beta(t,y)e^{-r(T-t)}}{\gamma\sigma_2^2} = (\frac{\hat{\boldsymbol{\mu}}_t - r}{\gamma\sigma_2^2} - \frac{k_y(t,y)}{\sigma_2})e^{-r(T-t)} - \frac{\sigma_1}{\sigma_2}\rho,$$

and

$$l^*(t,z) = \frac{\eta}{\gamma}e^{-r(T-t)} \wedge z$$

attain the last two supremum respectively. Note that the equilibrium strategy doesn't depend on function $K(\cdot,\cdot)$.

By plugging in $u^* = (\pi^*, l^*)$ and equation (4.4.3), equation (4.6.1) we have

$$\begin{aligned}
&\mathcal{A}^{u^*}g(t,x,y) \\
&= k_t(t,y) + \frac{1}{2}k_{yy}(t,y) + \frac{G(t,y)^2}{\gamma} - e^{r(T-t)}G(t,y)\sigma_1\rho \\
&\qquad\qquad + e^{r(T-t)}\int_0^\infty \Big[(\theta-\eta)z + \eta l^*(t,z)\Big]v(dz) = 0,
\end{aligned}$$

and from equation (4.4.4) we have $k(T,y) = 0, \forall y \in \mathbb{R}$.

By plugging in $u^* = (\pi^*, l^*)$ and equation (4.4.2) we have

$$\begin{aligned}
&\sup_{u\in\Pi}\Big\{\mathcal{A}^u V(t,x,y) - \frac{\gamma}{2}\mathcal{A}^u g^2(t,x,y) + \gamma g(t,x,y)\mathcal{A}^u g(t,x,y)\Big\} \\
&= K_t(t,y) + \frac{1}{2}K_{yy}(t,y) + G(t,y)K_y(t,y) - k_y(t,y)G(t,y) \\
&\quad + \frac{G^2(t,y)}{2\gamma} + \frac{\gamma\sigma_1^2}{2}e^{2r(T-t)}(\rho^2-1) - \sigma_1\rho G(t,y)e^{r(T-t)} \\
&\quad\quad + e^{r(T-t)}\int_0^\infty \Big[-\frac{\gamma}{2}(l^*)^2(t,z)e^{r(T-t)} + (\theta-\eta)z + \eta l^*(t,z)\Big]v(dz) = 0,
\end{aligned}$$

and $K(T,y) = 0, \forall y \in \mathbb{R}$.

From Theorem (4.4.1), $u^* = (\pi^*, l^*)$ is the equilibrium strategy, and $\{V(t,x,y) : 0 \leq t \leq T\}$ is the associated equilibrium value function. □

4.6.2 Proof of Theorem 4.4.3

Proof. Since

$$\pi^*(t,y) - \hat{\pi}(t,y) = -\frac{k_y(t,y)}{\sigma_2} e^{-r(T-t)} \tag{4.6.2}$$

We only need to analyze the function k_y. Note that function k is the solution of PDE (4.4.6), from Feynman-Kac formula,

$$\begin{aligned} k(t,y) &= \mathbb{E}^{\mathcal{Q}}\Big[\int_t^T \Big(\frac{G^2(s,Y_s)}{\gamma} - e^{r(T-s)}\sigma_1\rho G(s,Y_s) \\ &\qquad + e^{r(T-s)}\int_0^\infty \big[(\theta-\eta)z + \eta l^*(s,z)\big]v(dz)\Big)ds\Big|Y_t = y\Big] \\ &= \mathbb{E}^{\mathcal{Q}}\Big[\int_t^T \Big(\frac{G^2(s,y+Y_{s-t})}{\gamma} - e^{r(T-s)}\sigma_1\rho G(s,y+Y_{s-t}) \\ &\qquad + e^{r(T-s)}\int_0^\infty \big[(\theta-\eta)z + \eta l^*(s,z)\big]v(dz)\Big)ds\Big]. \end{aligned} \tag{4.6.3}$$

Here note that Y is a standard Brownian motion under $\mathcal{Q}$.

Therefore we have

$$k_y(t,y) = \mathbb{E}^{\mathcal{Q}}\Big[\int_t^T \Big(\frac{2G(s,y+Y_{s-t})}{\gamma} - e^{r(T-s)}\sigma_1\rho\Big)G_y(s,y+Y_{s-t})ds\Big]. \tag{4.6.4}$$

From the definition of function G,

$$\begin{aligned} G(t,y) &= \frac{F_y(t,y)}{F(t,y)} \\ &= \frac{\int_{\mathbb{R}} \frac{\vartheta - r}{\sigma_2}\exp\{\frac{\vartheta-r}{\sigma_2}y - \frac{(\vartheta-r)^2}{2\sigma_2^2}t\}\chi(\vartheta)d\vartheta}{\int_{\mathbb{R}} \exp\{\frac{\vartheta-r}{\sigma_2}y - \frac{(\vartheta-r)^2}{2\sigma_2^2}t\}\chi(\vartheta)d\vartheta} \\ &= \int_{\mathbb{R}} \frac{\vartheta-r}{\sigma_2}D(t,y,\vartheta)d\vartheta, \end{aligned} \tag{4.6.5}$$

where $D(t,y,\vartheta) := \dfrac{\exp\{\frac{\vartheta - r}{\sigma_2}y - \frac{(\vartheta - r)^2}{2\sigma_2^2}t\}\chi(\vartheta)}{\int_{\mathbb{R}} \exp\{\frac{\vartheta - r}{\sigma_2}y - \frac{(\vartheta - r)^2}{2\sigma_2^2}t\}\chi(\vartheta)d\vartheta}$.

Note that for fixed t and y, $D(t,y,\cdot)$ can be seen as a probability density function for random variable $\mathbf{X}(t,y)$.

Moreover, after simple algebra, we have

$$\begin{aligned} G_y(t,y) &= \int_{\mathbb{R}} \Big(\frac{\vartheta - r}{\sigma_2}\Big)^2 D(t,y,\vartheta)d\vartheta - \Big(\int_{\mathbb{R}} \frac{\vartheta - r}{\sigma_2} D(t,y,\vartheta)d\vartheta\Big)^2 \\ &= Var\Big(\frac{\mathbf{X}(t,y) - r}{\sigma_2}\Big) \geq 0, \end{aligned} \tag{4.6.6}$$

which completes the proof for part (2).

Particularly, when $\boldsymbol{\mu}$ is a known constant μ, then

$$G(t,y) = \frac{\mu - r}{\sigma_2}, \quad G_y(t,y) = 0, \quad \hat{\mu}(t) = \mu. \tag{4.6.7}$$

Then $k_y = 0$, and equation (4.4.10) follows. □

Chapter 5

Optimal Allocation of Life Annuity, Term Life Insurance and Consumption under General Force of Mortality

5.1 Introduction

Life annuity and term life insurance are two important types of financial instruments used by individuals to manage the risks related to their uncertain future lifetimes. In exchange for an initial lump-sum premium, a life annuity is a financial contract between an individual and an insurer that pays out a periodic amount as long as the annuitant is alive, while a term life insurance provides certain death benefit at the individual's premature death when in effect. As such, life annuities can be used to hedge longevity risk, namely, the risk of running out of savings and falling into poverty before dying. On the other hand, term life insurance can help protect against the loss of family income due to the breadwinner's premature death.

This chapter focuses on the optimal life annuity, term life insurance and consumption strategies for an individual facing a stochastic time of death, to maximize the expected CRRA utility from lifetime discounted consumption, bequest motive and terminal wealth upon survival. A starting point for the modern research on the demand for life insurance and life annuity, is the seminal paper by Yaari [103] who introduced an optimal consumption problem for an individual with an uncertain time of death within the setup of a purely deterministic investment environment. Hakansson [52] and Fischer [45] extended Yaari's model to a discrete time setup with uncertainty including risky assets. Then in a continuous-time setup, Richard [89] extended the model from Merton [79] to include consumption, investment, life insurance rules for an investor with a known distribution of lifetime, to maximize the expected utility from inter-temporal consumption and bequest motive. Note that in the aforementioned literature, a single control variable, either the actuarial note as in Yaari [103] or the premium rate as in Hakansson [52] and Richard [89], is used to characterize the purchasing decisions for both products.

Along the strand of literature on optimal annuitization when no life insurance is available, Milevsky and Young [81] incorporates life and pension annuity products into the portfolio selection literature. In an all or nothing market, where the individual is required to annuitize all her wealth at retirement, they solve for the optimal age to retire. In an anything anytime market, where the individual has the flexibility to purchase any amount of life annuity at any time, they utilize singular control and solve for the optimal annuity to wealth ratio. Although a general force of mortality is employed, the problem is not stationary within the infinite time horizon setup. Assuming a constant force of mortality, Wang and Young [101] and Liang and Young [70] obtained the explicit optimal annuitization strategy under power utility and exponential utility, respectively. By incorporating a non-tradeble labor income risk, Horneff et al. [60] and Chai et al. [25] include both working life and retirement in their analysis.

Following Richard [89], most studies about life insurance demands simplify the insurance decision by including an instantaneous term life insurance contract in a continuous-time setting (see Pliska and Ye [87], Huang and Milevsky [61], Bayraktar et al. [11] and references therein) or a series a renewable one-year term life insurance in a discrete time setting (see Chen et al. [26]). More realistic lump-sum purchases for life insurance are studied in Bayraktar and Young [10] and Hambel et al. [54].

The work of this chapter is different from the existing literature on demand for life annuities and term life insurance on the following points. First, we allow for the simultaneous holdings of life annuity and term life insurance as state variables. In the existing literature, decisions of life annuities and term life insurance are typically modelled using a *single control variable*, namely, the premium rate, where the controls only depend on the current wealth. Moreover, when the premium rate is positive, it represents a positive amount of life insurance, while when the premium rate is negative, it represents a positive amount of life annuity, where annuitants receive an annuity income when alive and premium is due at death, under which simultaneous holdings of two products is not possible. However, the existing amount of life annuity and term life insurance should play an important role in one's willingness to purchase more of these products. In practice, a substantial number of the families that own annuities also have life insurance policies according to Brown [20]. Second, life annuities and term life insurance can only be bought or surrendered at realistic lump-sum costs. Formally, we model the insurance decisions as singular and impulse controls. Such settings are applied for life insurance (see Bayraktar and Young [10] and Hambel et al. [54]) and life annuities (see Milevsky and Young [81], Wang and Young [101] and Liang and Young [70]) separately, but not simultaneously. Third, our model allows for a general force of mortality. Under singular and impulse control setup, only Milevsky and Young [81] considered a general force of mortality setup to the best of our knowledge. However, they considered life annuity purchasing without surrender feature, and didn't take into account a term life insurance component. Moreover, since they solved for an infinite-horizon problem, the model is not stationary when mortality beliefs are time-dependent. Most of the other work in this setting employs a constant force of mortality, which means the individual's lifetime follows an exponential distribution.

Mathematically, our problem boils down to solving a variational inequality with gradient constraints. Assuming the individual has a CRRA preference, we reduce the problem by one dimension and seek for the numerical solution using penalty methods. In the PDE theory, penalty approximations have been widely used to show the existence of solution to variational inequalities, see e.g., Friedman and Spruck [48]. This method has also been widely applied in mathematical finance, see Forsyth and Vetzal [47] for pricing American vanilla options and Dai and Zhong [33] for portfolio selection with proportional transaction costs, while it's rarely seen in actuarial content.

We establish some results that are consistent with intuition. Namely, the individual should allocate more wealth in life annuity if (1) He or she is wealthier or has less existing life annuity; (2) He or she has a longer expected life time than the pricing group; (3) Less is taxed on legacy; (4) The pricing rate of life annuity is higher; (5) The wealth process has a lower return rate; (6) The wealth process is more volatile. The individual is optimal to allocate more wealth in term life insurance if (1) He or she is wealthier or has less existing death benefit (2) He or she has a shorter expected lifetime than the pricing group; (3) More is taxed on legacy; (4) The pricing rate of the term life insurance is higher; (5) The wealth process has a lower return rate; (6) The wealth process is more volatile; (7) The individual is more risk averse. For both products, the individual should trade less frequently for a higher safety loading factor or surrender penalty rate.

We also find some interesting and unforeseen results on the impact of risk attitudes on one's willingness to annuitize. We find that the pattern depends on the level of interest rate. When the interest rate is low, a more risk averse individual should allocate a higher proportion of wealth in life annuity, a common results also found in other work in the literature, see Wang and Young [101], Milevsky and Young [81]. But when the interest rate is relatively high, instead of acting on the size of annuity income, risk attitudes directly affect the trading frequency as more risk averse individual should trade even more frequently. This finding is inconsistent with the problem of portfolio selection, and it may root in the difference between a stock and a life annuity.

An extensive discussion on several widely documented empirical findings in regard to life annuity and term life insurance is also included along with our numerical examples. Among others, this include the *annuity puzzle* and the *adverse selection* effect in life annuity markets, the *advantageous selection* effect in life insurance markets, the insufficient life insurance among the working-aged, and the excessive life insurance among the elderly, which will be reviewed in the later sections of this chapter. These empirical findings are possible in our model under certain parameter settings.

To have a clear picture of the impact of model parameters and state variables, we first look into two special cases which are new on their own. In Section 5.3, we study the demand for life annuity when term life insurance is not available. Compared to this special case, Milevsky and Young [81] and Liang and Young [70] neglect the surrender feature; Wang and Young [101] incorporates the surrender feature but assume a constant force of mortality and all three papers are working under an infinite time horizon. In Section 5.4, we investigate the case when only term life insurance is available. The closest paper to this special case utilizing singular control is Bayraktar and Young [10], but their major concern is optimal life insurance for a household with two wage earners, and a constant force of mortality is also assumed.

The rest of this chapter is organized as follows. In Section 5.2 we present the wealth process for the individual, and introduce the life annuity and term life insurance. In Section 5.3, we study the optimization problem when only the life annuity is available. In Section 5.4, we discuss the alternative case when only the term life insurance is available. In Section 5.5, we consider the general problem when the individual has access to both products. Section 5.6 concludes this chapter.

5.2 Wealth process, life annuity and term life insurance markets

Let $(\Omega, \mathcal{F}, \mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a filtered probability space, and all the processes in the following live in this space. Let $T > 0$ be the fixed time horizon.

Consider an individual with future lifetime described by the random variable τ_x, where x is the age of the individual at time 0. Under a consumption policy $\{c_t\}_{t\in[0,T]}$, the individual's wealth process satisfies

$$dW_t = \Big[\mu W_t - c_t\Big]dt + \sigma W_t dB_t, \tag{5.2.1}$$

where $\mu > 0$, $\sigma > 0$ and $\{B_t\}_{t\geq 0}$ is a standard Brownian motion[1].

[1]Note that this model can be considered as incorporating a constant proportion investment strategy π

We let ${}_tp_x^S$ denote the subjective probability that an individual aged x believes he or she will survive to age $x+t$. It is defined via the subjective force of mortality function, λ_{x+t}^S (deterministic), by

$$ {}_tp_x^S = \exp\Big(-\int_0^t \lambda_{x+s}^S ds \Big), \quad t > 0. $$

We have a similar formula for the objective probability of survival, ${}_tp_x^O$, in terms of the objective force of mortality function, λ_{x+t}^O, which can be used to characterize the average force of mortality of the pricing group. Then we can determine the price for a life annuity and a term life insurance as follows. The actuarially fair price of a life annuity that pays \$1 per year continuously to an individual aged x at the time of purchase is $\bar{a}_x$, given by

$$ \bar{a}_x = \int_0^\infty e^{-rt} {}_tp_x dt. \tag{5.2.2} $$

The actuarially fair cost of a \$1 death benefit, payable at τ_x if death occurred before T purchased at age x is $\bar{A}^1_{x:\overline{T}|}$, given by:

$$ \bar{A}^1_{x:\overline{T}|} := \mathbb{E}[e^{-r\tau_x}\mathbb{I}_{\tau_x<T}] = \int_0^T e^{-rt} {}_tp_x \lambda_{x+t} dt. \tag{5.2.3} $$

In terms of notation, if we use the subjective force of mortality to calculate the survival probabilities, then we write τ_x^S, $\bar{a}_x^S$ and $\bar{A}^1_{x:\overline{T}|}{}^S$, while if we use the objective (pricing) force of mortality to calculate the survival probabilities, then we write τ_x^O, $\bar{a}_x^O$ and $\bar{A}^1_{x:\overline{T}|}{}^O$. For the life annuity and term life insurance, we suppose that both premiums are payable when the contract is issued, thus $\bar{a}_x$ and $\bar{A}^1_{x:\overline{T}|}$ are both lump-sum premiums.

The individual wants to maximize the expected utility from a lifetime consumption up until T, the utility from a bequest motive if a premature death occurred before T, and the utility from the terminal wealth at T upon survival. Due to the model complexity, we will

in the stock market. Suppose the stock price follows a geometric Brownian motion with a drift rate of μ' and volatility σ': $dS_t = \mu' S_t dt + \sigma' S_t dB_t$, then the wealth process with a constant proportion π in stock has dynamic: $dW_t = \big[\big(\mu'\pi + r(1-\pi)\big)W_t - c_t\big]dt + \sigma'\pi W_t dB_t$. Augmenting the model to include a dynamic investment component would raise more complexity and could be studied in future research.

demonstrate our findings in three separate models to have a better grasp of the impact of different model parameters on the optimal strategies. We separately solve for the optimal annuitization strategy and the optimal term life insurance strategy in Sections 5.3 and 5.4, respectively. Note that these two problems are new on their own. For each special case, compared to literature on optimal annuity (or life insurance) utilizing singular control, we further allow for surrendering behaviour, assume a general force of mortality, and work under a finite horizon. In section 5.5 we study the general case when both products are available in the market.

5.3 Optimal annuity purchasing

In this section, assume that the individual can only consume and buy/surrender life annuity, without access to term life insurance. Let N_t^B denote the cumulative amount of life annuity income purchased on or before time t, and N_t^S be the cumulative amount of life annuity income surrendered on or before time t. Then $N_t = N_t^B - N_t^S$ represents the cumulative net amount of immediate life annuity income at time t. The wealth and annuity dynamics of the individual for $t < \min(\tau_x^S, T)$ are given by

$$\begin{cases} dW_t = \Big[\mu W_{t-} - c_{t-} + N_{t-}\Big]dt + \sigma W_{t-} dB_t - (1+l_A)\bar{a}^O_{x+t} dN_t^B + (1-p_A)\bar{a}^O_{x+t} dN_t^S, \\ dN_t = dN_t^B - dN_t^S, \end{cases} \tag{5.3.1}$$

with $W_0 = w \geq 0$ and $N_0 = N \geq 0$, where $l_A \geq 0$ is the safety loading factor of life annuity, $0 \leq p_A \leq 1$ is the surrender penalty rate and $\bar{a}^O_{x+t}$ is defined in (5.2.2). The surrender value of \$1 of annuity income is $(1-p_A)\bar{a}^O_{x+t}$. That is to say, the individual receives $(1-p_A)\bar{a}_{x+t}$ from the issuer by surrendering \$1 of annuity income. A more realistic setup is to allow p_A to change in time, but here we assume the surrender penalty rate is constant for simplicity. Since we allow the life annuity strategy to jump due to lump-sum purchases, we write the subscript $t-$ instead of t to denote the values of the corresponding process before any such jump.

If there is a premature death. i.e., $\tau_x < T$, the legacy would be left with kW_{τ_x}, where

$k \in [0,1]$ measures the after-tax proportion of wealth inherited. If $\tau_x > T$, the terminal wealth of the individual at T is $W_T + \bar{a}_{x+T} N_T$, where we include the present fair value of the remaining life annuity income in the terminal wealth. Note that in Pliska and Ye [87] such credit was omitted, since their life annuity is of an instantaneous term with the use of premium rate as the control variable. However, in our setup, it is more appropriate to include the remaining lifetime income over T, or the incentive to purchase life annuity cannot be captured sufficiently.

The consumption and life annuity strategies $\{c_t, N_t^B, N_t^S\}_{t\in[0,T]}$ are said to be admissible if

(i) $\{c_t\}_{t\in[0,T]}$, $\{N_t^B\}_{t\in[0,T]}$ and $\{N_t^S\}_{t\in[0,T]}$ are adapted to the filtration $\mathbb{F}$.

(ii) The controls $c_t \geq 0$, $N_t^B \geq 0$ and $N_t^S \geq 0$ a.s. for all $t \in [0,T]$.

(iii) $\int_0^t c_s ds < \infty$ a.s. for all $t \in [0,T]$.

(iv) $\{N_t^B\}_{t\in[0,T]}$ and $\{N_t^S\}_{t\in[0,T]}$ are nondecreasing in t, $W_t \geq 0$ and $N_t \geq 0$ a.s. for all $t \in [0,T]$.

5.3.1 Objective function

Fix a time point t, initial states $W_t = w, N_t = N$, and policy $\{\mathbf{c}, \mathbf{N}^B, \mathbf{N}^S\} := \{c_s, N_s^B, N_s^S\}_{s\in[t,T]}$, the objective function for the individual (who is now aged $x+t$) is given by

$$J^{(N)}(w,N,t;\mathbf{c},\mathbf{N}^B,\mathbf{N}^S) := \mathbb{E}\Big[\int_t^{\tau_x^S\wedge T} e^{-r(s-t)}u_1(c_s)ds + e^{-r(\tau_x^S-t)}u_2(kW_{\tau_x^S-})\mathbb{I}_{\tau_x^S\leq T} + e^{-r(T-t)}u_3(W_T+\bar{a}_{x+T}N_T)\mathbb{I}_{\tau_x^S>T}\Big|\tau_x^S>t\Big],$$

for which $u_1(\cdot)$, $u_2(\cdot)$ and $u_3(\cdot)$ are utilities of consumption, bequest motive and terminal wealth upon survival, respectively. In what follows, we assume that

$$u_1(x) = u_2(x) = u_3(x) = \frac{x^\gamma}{\gamma}, \quad \gamma < 1, \gamma \neq 0, \tag{5.3.2}$$

i.e., the individual is assumed to have a constant relative risk aversion of

$$\bar{\gamma} := 1 - \gamma. \tag{5.3.3}$$

Such utility is common in this line of research, see Milevsky and Young [81] and Pliska and Ye [87]. Since the Brownian motion $\{B_t\}_{t\in[0,T]}$ characterizes the randomness from the financial market or more generally, the underlying economy, we can assume it is independent from the lifetime random variable τ_x^S and then rewrite $J^{(N)}$ as follows:

$$\begin{aligned}
&J^{(N)}(w, N, t; \mathbf{c}, \mathbf{N}^B, \mathbf{N}^S) \\
&= \mathbb{E}\Big[\int_t^T e^{-r(s-t)}{}_s p_x^S \Big(u_1(c_s) + \lambda_{x+s}^S u_2(kW_{s-})\Big) ds + {}_T p_x^S e^{-r(T-t)} u_3(W_T + \bar{a}_{x+T}^O N_T)\Big] \\
&= \mathbb{E}\Big[\int_t^T e^{-\int_t^s (r+\lambda_{x+u}^S)du} \Big(u_1(c_s) + \lambda_{x+s}^S u_2(kW_{s-})\Big) ds + e^{-\int_t^T (\lambda_{x+s}^S + r)ds} u_3(W_T + \bar{a}_{x+T}^O N_T)\Big].
\end{aligned}$$

The individual seeks to maximize such expected utility from discounted consumption, bequest motive and terminal wealth upon survival, over all admissible strategies $\{\mathbf{c}, \mathbf{N}^B, \mathbf{N}^S\}$, and the value function is given by

$$U^{(N)}(w, N, t) = \sup_{\mathbf{c}, \mathbf{N}^B, \mathbf{N}^S} J^{(N)}(w, N, t; \mathbf{c}, \mathbf{N}^B, \mathbf{N}^S). \tag{5.3.4}$$

By standard stochastic control theory (see for instance, Shreve et al. [96] and Fleming and Soner [46]), $U^{(N)}$ solves the following variational inequality:

$$\begin{cases}
\max\Big\{U_t^{(N)} + \max\limits_{c\geq 0} \mathcal{L}_0^c U^{(N)}, U_N^{(N)} - (1+l_A)\bar{a}_{x+t}^O U_w^{(N)}, (1-p_A)\bar{a}_{x+t}^O U_w^{(N)} - U_N^{(N)}\Big\} = 0, \\
\qquad\qquad (w, N, t) \in \mathbb{R}_+ \times \mathbb{R}_+ \times [0, T), \\
U^{(N)}(w, N, T) = u_3(w + \bar{a}_{x+T}^O N), \quad (w, N) \in \mathbb{R}_+ \times \mathbb{R}_+,
\end{cases} \tag{5.3.5}$$

with

$$\mathcal{L}_0^c U = -(\lambda_{x+t}^S + r)U + \Big(\mu w - c + N\Big)U_w + \frac{1}{2}\sigma^2 w^2 U_{ww} + u_1(c) + \lambda_{x+t}^S u_2(kw).$$

5.3.2 Change of variables

The value function $U^{(N)}$ defined in (5.3.4) is homogeneous of degree γ with respect to both the wealth w and the annuity N due to the homogeneity property of the CRRA utility function (see Davis and Norman [38] for a proof). Specifically, for a constant $\alpha > 0$, $U^{(N)}(\alpha w, \alpha N, t) = \alpha^\gamma U^{(N)}(w, N, t)$. Define $y = \frac{N}{w+N}$ as the proportion of the current annuity income in total wealth and $V^{(N)}(y,t) := U^{(N)}(1-y, y, t)$, then

$$U^{(N)}(w, N, t) = (w+N)^\gamma U^{(N)}(1-y, y, t) = (w+N)^\gamma V^{(N)}(y,t).$$

By doing so, the original problem is simplified into a one-dimensional problem. Additionally, it is easy to check that $V^{(N)}$ satisfies

$$\begin{cases} \max\{V_t^{(N)} + \max\limits_{\tilde{c}\geq 0} \mathcal{L}_1^{\tilde{c}} V^{(N)}, \tilde{\mathcal{B}}_N V^{(N)}, \tilde{\mathcal{S}}_N V^{(N)}\} = 0, \ (y,t) \in [0,1] \times [0,T), \\ V^{(N)}(y,T) = \dfrac{(1 - y + \bar{a}_{x+T} y)^\gamma}{\gamma}, \ y \in [0,1], \end{cases} \tag{5.3.6}$$

where

$$\begin{aligned} \tilde{c} &= \frac{c}{w+N}, \\ \tilde{\mathcal{B}}_N V &= \Big[1 - (1+l_A)\bar{a}_{x+t}^O\Big]\gamma V + \Big[1 - y + y(1+l_A)\bar{a}_{x+t}^O\Big] V_y, \\ \tilde{\mathcal{S}}_N V &= \Big[(1-p_A)\bar{a}_{x+t}^O - 1\Big]\gamma V - \Big[(1-p_A)\bar{a}_{x+t}^O y + 1 - y\Big] V_y, \end{aligned} \tag{5.3.7}$$

and

$$\begin{aligned} \mathcal{L}_1^{\tilde{c}} V = & -(\lambda_{x+t}^S + r)V + \Big[\mu(1-y) - \tilde{c} + y\Big](\gamma V - y V_y) \\ & + \frac{\sigma^2}{2}(1-y)^2\Big[\gamma(\gamma-1)V + 2(1-\gamma) y V_y + y^2 V_{yy}\Big] + \frac{\tilde{c}^\gamma}{\gamma} + \lambda_{x+t}^S \frac{(k(1-y))^\gamma}{\gamma}. \end{aligned} \tag{5.3.8}$$

To avoid numerical oscillation due to term $\gamma V^{(N)}$ in the gradient constraints, we further make the following transformation similar as in Dai and Zhong [33],

$$\phi^{(N)}(y,t) = \frac{\log(\gamma V^{(N)}(y,t))}{\gamma}.$$

It follows that $\phi^{(N)}$ satisfies

$$\begin{cases} \max\{\phi_t^{(N)} + \max_{\tilde{c}\geq 0} \mathcal{L}^{\tilde{c}}\phi^{(N)}, \mathcal{B}_N\phi^{(N)}, \mathcal{S}_N\phi^{(N)}\} = 0,\ (y,t)\in[0,1]\times[0,T), \\ \phi^{(N)}(y,T) = \log(1-y+\bar{a}_{x+T}y),\ y\in[0,1], \end{cases} \tag{5.3.9}$$

where

$$\begin{aligned} \mathcal{B}_N\phi &= 1-(1+l_A)\bar{a}^O_{x+t} + \Big[1-y+y(1+l_A)\bar{a}^O_{x+t}\Big]\phi_y, \\ \mathcal{S}_N\phi &= (1-p_A)\bar{a}^O_{x+t} - 1 - \Big[(1-p_A)\bar{a}^O_{x+t}y + 1 - y\Big]\phi_y, \end{aligned} \tag{5.3.10}$$

and

$$\begin{aligned} \mathcal{L}^{\tilde{c}}\phi = {} & \frac{\sigma^2(1-y)^2}{2}\Big[y^2\phi_{yy} + \gamma y^2\phi_y^2\Big] + y\phi_y\Big[\sigma^2(1-y)^2(1-\gamma) - \big(\mu(1-y)-\tilde{c}+y\big)\Big] \\ & - \frac{\lambda^S_{x+t}+r}{\gamma} + \mu(1-y) - \tilde{c} + y + \frac{\sigma^2(1-y)^2}{2}(\gamma-1) + e^{-\gamma\phi}\Big[\frac{\tilde{c}^\gamma}{\gamma} + \lambda^S_{x+t}\frac{(k(1-y))^\gamma}{\gamma}\Big]. \end{aligned} \tag{5.3.11}$$

We define

$$BR_N = \{(y,t)\in[0,1]\times[0,T) : \mathcal{B}_N\phi^{(N)} = 0\},$$

$$SR_N = \{(y,t)\in[0,1]\times[0,T) : \mathcal{S}_N\phi^{(N)} = 0\},$$

$$NTR_N = [0,1]\times[0,T)\setminus\big(BR_N\cup SR_N\big),$$

where BR_N, SR_N and NTR_N represent the buy region, surrender region and non-transaction region of the life annuity, respectively.

We solve (5.3.9) using the penalty approximation:

$$\begin{cases} \phi_t^{(N)} + \max_{\tilde{c}} \mathcal{L}^{\tilde{c}}\phi^{(N)} + K\big(\mathcal{B}_N\phi^{(N)}\big)^+ + K\big(\mathcal{S}_N\phi^{(N)}\big)^+ = 0,\ (y,t)\in[0,1]\times[0,T), \\ \phi^{(N)}(y,T) = \log(1-y+\bar{a}_{x+T}y),\ y\in[0,1], \end{cases} \tag{5.3.12}$$

where K is a positive constant. (5.3.12) is expected to converge to (5.3.9) as K goes to infinity. We further impose the following boundary conditions: $\mathcal{B}_N\phi^{(N)} = 0$ at $y = 0$, and $\mathcal{S}_N\phi^{(N)} = 0$ at $y = 1$. The boundary conditions imply buying life annuity at $y = 0$ and surrendering life annuity at $y = 1$. We apply finite difference discretization, and upwind scheme for first order terms especially. Then Newton iteration is applied for nonlinear terms.

5.3.3 Numerical results

For the numerical study, we consider a Gompertz force of mortality, which is common in the actuarial literature for annuity pricing. It is written as

$$\lambda^S_{x+t} = a_s e^{b_s(x+t)} \quad \text{and} \quad \lambda^O_{x+t} = a_o e^{b_o(x+t)}.$$

a_s	b_s	a_o	b_o	age	k	γ
$2.1457 * 10^{-5}$	0.09524	$2.1457 * 10^{-5}$	0.09524	35	1	-1
T	r	μ	σ	l_A	p_A	
30	0.02	0.06	0.3	0.1	0.3	

Table 5.1: Default model parameters

Table 5.1 lists the default parameter values. Mortality parameter values a_s, b_s, a_o, b_o are from Milevsky and Young [81], which are fitted to the individual annuity mortality 2000 (basic) table (male) with projection scale G. As demonstrated in Footnote 1, this set of financial parameters can be interpreted as a constant investment strategy with 50% of wealth allocated to a stock with a 10% return rate, with the remaining 50% in a bank account with an interest rate of 2%.

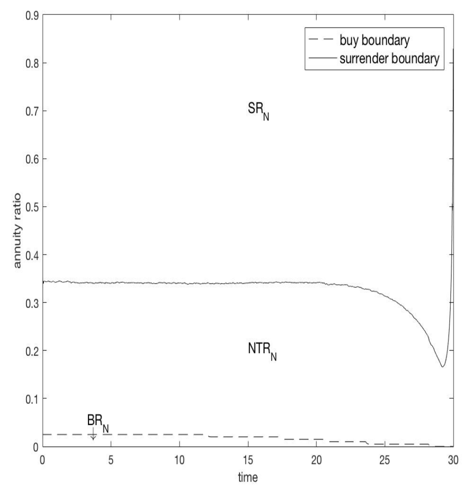
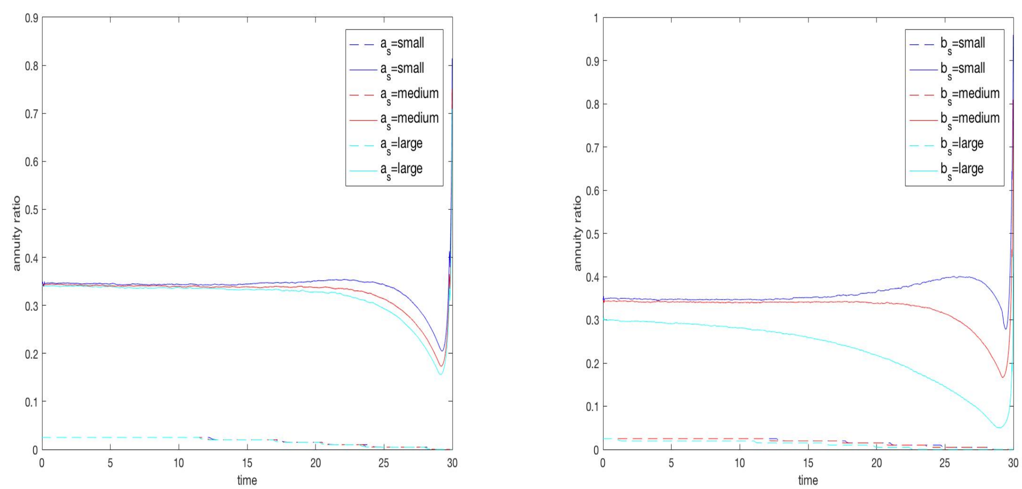


Figure 5.1: Optimal annuitization strategy under default parameters

Figure 5.1 depicts the optimal annuitization strategy as a function of time and the existing life annuity ratio under default parameters. The *dashed line is the life annuity buy boundary*, below which is the buy region BN_R. When the existing life annuity ratio $y = \frac{N}{w+N}$ lies below the buying boundary, the individual should buy an additional amount of life annuity ΔN immediately (impulse control) so that the updated life annuity ratio $\frac{N+\Delta N}{w-(1+l_A)\bar{a}^O_{x+t}\Delta N+N+\Delta N}$ moves up to the buy boundary. The *solid line is the life annuity surrender boundary*, above which is the surrender region SN_R. When the existing life annuity ratio lies above the surrender boundary, an amount of life annuity ΔN should be surrendered immediately (impulse control) so that the updated annuity ratio $\frac{N-\Delta N}{w+(1-p_A)\bar{a}^O_{x+t}\Delta N+N-\Delta N}$ moves down to the surrender boundary. Between the buy boundary and surrender boundary is the life annuity non-transaction region NTR_N. In the interior of NTR_N, the individual shouldn't purchase or surrender any life annuity and just consume continuously. At the boundary of NTR_N, the individual exercises singular control to buy or surrender life annuity to stay in NTR_N. Therefore, all else being equal, the actual amount of life annuity one should hold increases in wealth and decreases in the existing amount of life annuity.

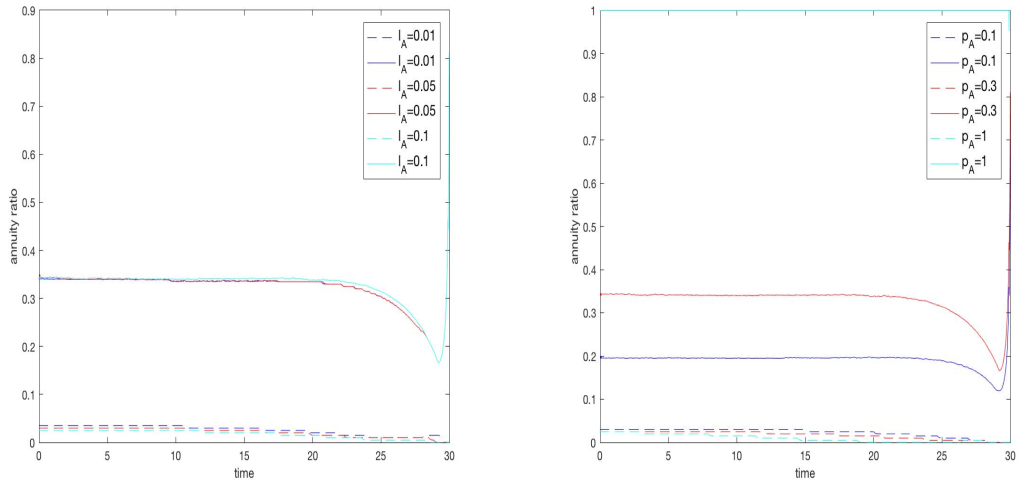
Figure 5.2 examines the impact of the subjective force of mortality on the annuitization strategy. In both figures, *small*, *medium* and *large* represent scenarios where either one of the parameter values of the subjective force of mortality is 50% lower, the same and 50% higher (resp.) than the default parameter setting. As expected, the graph shows that a larger value of a_s (or b_s) leads to a lower buy boundary and surrender boundary. Namely, the individual will keep a smaller proportion of wealth in life annuity if he or she believes his or her expected lifetime is smaller than that of the pricing group, and hold more life annuity if a longer than average lifetime is presumed. This is consistent with the standard *adverse selection* observed in annuity market, that is, people with a higher level of longevity risk purchase more lifetime annuity, see Finkelstein and Poterba [44].



(a) a_s (b) b_s

Figure 5.2: Impact of subjective force of mortality a_s and b_s

Figure 5.3 illustrates the impact of transaction costs, i.e., the loading factor l_A and surrender penalty rate p_A. In Figure 5.3a, when the loading factor l_A increases, the buy boundary decreases and the surrender boundary increases, leading to a larger non-transaction region. Similarly, the non-transaction region expands as the surrender penalty rate p_A increases. Particularly, when $p_A = 1$, we observe that it is never optimal for the individual to surrender. The intuition is that as transaction costs increase, the individual tends to decrease the trading frequency to save on the significant transaction costs.



(a) l_A (b) p_A

Figure 5.3: Impact of safety loading factor l_A and surrender penalty rate p_A

In Figure 5.4 we plot the optimal buy and surrender boundaries with varying after-tax effect k. Larger values of k lead to higher buy and surrender boundaries, which indicates a higher proportion of wealth in life annuity. With a smaller tax effect, more legacy can be inherited to cover bequest motive, hence, there is more incentive to invest wealth in life annuity.

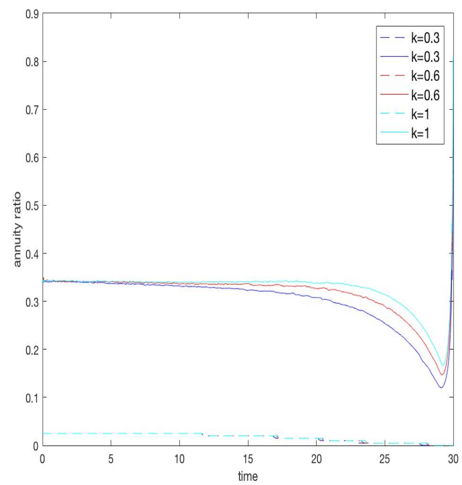
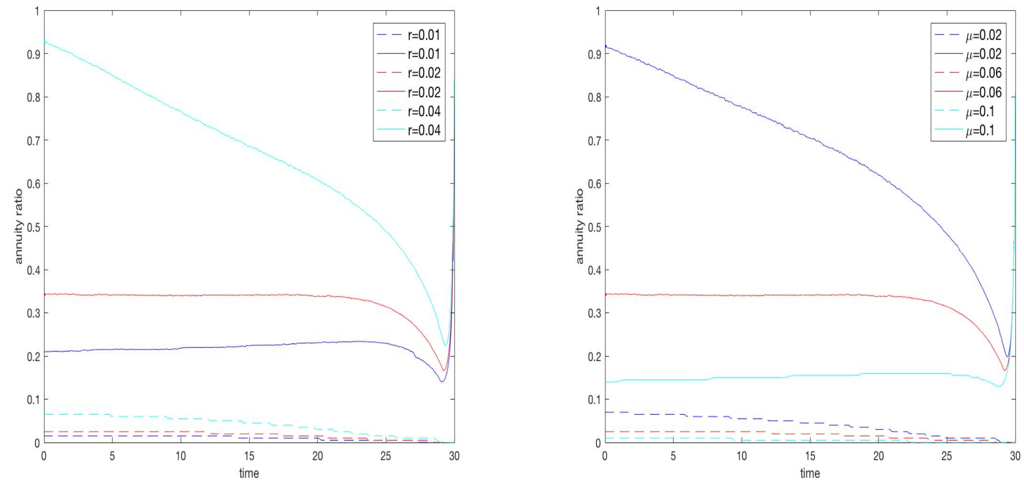


Figure 5.4: Impact of after-tax proportion k

Figure 5.5 investigates the effect of the interest rate r (used to price annuity) and the wealth growth rate μ on the annuitization strategy. When r increases, both buy and surrender boundaries increase and more will be held in life annuity. The reverse effect holds when μ increases, where the individual is better off enjoying a higher growth rate in the *bank account*[2]. This is consistent with intuition as r and μ measure the growth rate of the two "investment products", namely, life annuity and *bank account*, respectively. Particularly, when the interest rate r is low, it is optimal for the individual to allocate a small fraction of wealth to life annuities.

Yaari [103] showed that, in a perfect market setting, expected utility maximizers with no bequest motive should annuitize their entire wealth. This result was further confirmed by Davidoff et al. [37] under more general assumptions. Empirical studies find, however, that only a small portion of private wealth is used to purchase annuities. This discrepancy between theoretical findings and empirical observations is referred to as the *annuity puzzle*. As can be seen in Figure 5.5a, the buy boundary is close to zero when the interest rate is around 2%, thus the low interest environment might be one possible explanation for the low annuity demand.

[2]By *bank account* we mean the account where wealth grows following the wealth process in (5.2.1).



(a) r (b) μ

Figure 5.5: Impact of interest rate r and wealth growth rate μ

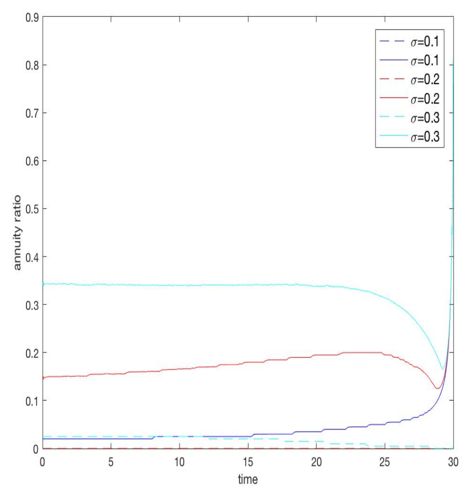
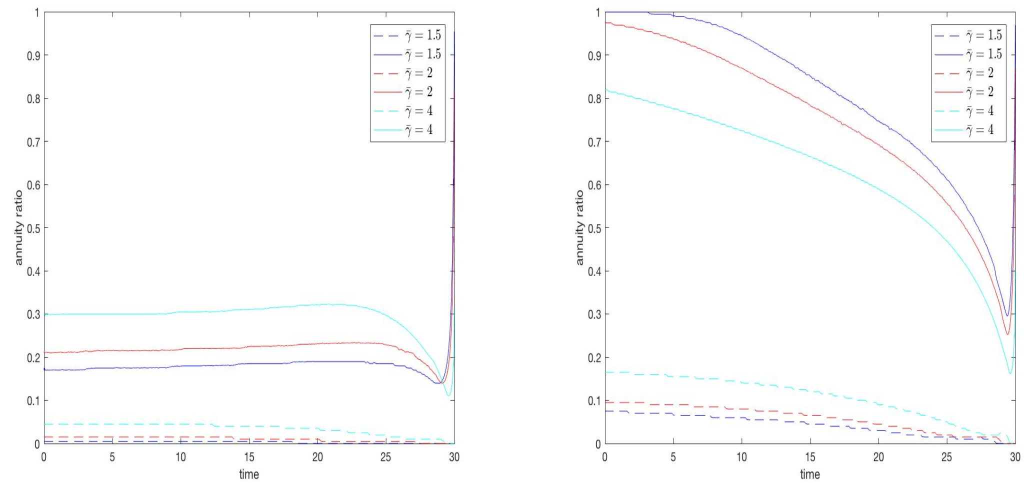


Figure 5.6: Impact of wealth volatility σ

Figure 5.6 depicts the buy and surrender boundaries for various wealth volatility σ. We observe that both the buy boundary and surrender boundary increase in σ. As expected, this implies that it is optimal to allocate a larger fraction of wealth in annuity when there is more uncertainty in wealth.



(a) $r = 0.01$

(b) $r = 0.05$

Figure 5.7: Impact of risk aversion $\bar{\gamma}$ under various interest rates

In Figure 5.7, we study the impact of the risk aversion parameter $\bar{\gamma}$ (recall that $\bar{\gamma} = 1-\gamma$) on the annuitization behaviour, which is more interesting and less obvious than for the other parameters. It turns out that the pattern for annuitization behaviour with different risk attitudes depends on the interest rate r. [3]

When the interest rate r is low as in Figure 5.7a, except when close to maturity, we observe that both the buy boundary and the surrender boundary increase in the risk aversion parameter $\bar{\gamma}$, which implies that it is optimal for a more risk averse individual to keep a larger fraction of wealth in life annuity. This reflects the life annuity's feature of hedging against longevity risk, when the interest rate is relatively low.

[3]We also run the numerical analysis updating r and μ simultaneously: (1) Assume a stock return rate of $\mu' = 10\%$, and fix a 50% investment in stock, then the wealth growth rate is $\mu = 0.5r + 0.5\mu'$. (2) Change r, and μ is determined using a Kelly's strategy $\pi = \frac{\mu'-r}{\sigma}$. The results for the 3 methods are consistent: If the individual is more risk averse, he or she should hold more life annuity if r is small, and should trade more frequently if r is large.

When the interest rate r is relatively high, the impact of the risk aversion parameter $\bar{\gamma}$ takes a different course. From Figure 5.7b, when the individual is more risk averse as $\bar{\gamma}$ increases, the buy boundary increases while the surrender boundary decreases, in other words, the non-transaction region for life annuity shrinks. This indicates for high level of interest rates, rather than working on the magnitude of annuity income, the risk preference affects the annuity strategy through the trading frequency. A more risk averse individual should trade more frequently. This phenomenon is inline with intuition in the sense that a more risk averse individual is more devoted to confining himself or herself to the specific optimal annuitization strategy and is willing to pay more transaction costs to achieve this goal. While a less risk averse individual cares less about fixating on the exact strategy and is fine trading less frequently to adjust the strategy. As a comparison to the portfolio selection problem, Fellner and Maciejovsky [42] finds that individual risk attitude is systematically related to market behaviour: the higher the degree of risk aversion, the lower the observed market activity. This may root in the difference between a stock and a life annuity.

5.3.4 Summary of findings on demand for life annuity

In this section, we consider the problem of maximizing utility from consumption, bequest motive and wealth upon survival when only life annuity is available. By numerically solving the optimal purchasing and surrendering boundaries via penalty methods, we establish results that are consistent with intuition, as well as find some interesting and unforeseen results. Namely, all else being equal, the individual should allocate *more wealth to life annuity* if

- He or she is wealthier;
- He or she has less existing life annuity;
- He or she has a longer expected lifetime than the pricing group;
- Less will be taxed on legacy;

- The pricing rate of the life annuity is higher;
- The wealth process has a lower return rate;
- The wealth process is more volatile.

The individual should trade less frequently for more expensive transaction costs (as measured by the safety loading factor l_A or the surrender penalty rate p_A).

We also find some interesting and somewhat unexpected results of the impact of risk attitudes on one's willingness to annuitize. We find that the pattern depends on the level of interest rate r. When the interest rate is low, a more risk averse individual should allocate a higher proportion of wealth in life annuity, a common result also found in some relevant work in the literature, see Wang and Young [101], Milevsky and Young [81]. But when the interest rate is relatively high, instead of acting on the size of annuity income, risk attitudes directly affect the trading frequency, and more risk averse individuals should trade more frequently.

Moreover, we find that the low interest rate environment could be one possible reason for the well-known *annuity puzzle*.

5.4 Optimal term life insurance purchasing

In this section, we consider the problem of optimal term life insurance purchasing and consumption, when access to life annuity is not available. Let D_t^B and D_t^S denote the cumulative amount of term life insurance benefit purchased and surrendered, respectively, on or before time t. Then $D_t = D_t^B - D_t^S$ is the net cumulative amount of term life insurance benefit at time t. When $t < \min(\tau_x^S, T)$, the wealth and term life insurance dynamics are given by

$$\begin{cases} dW_t = \Big[\mu W_{t-} - c_{t-}\Big]dt + \sigma W_{t-} dB_t - (1+l_I)\bar{A}_{x+t:\overline{T-t}|}^{\,1}{}^{O} dD_t^B + (1-p_I)\bar{A}_{x+t:\overline{T-t}|}^{\,1}{}^{O} dD_t^S, \\ dD_t = dD_t^B - dD_t^S, \end{cases} \tag{5.4.1}$$

with $W_0 = w \geq 0$, $D_0 = D \geq 0$, where $l_I \geq 0$ and $p_I \in [0,1]$ are the safety loading factor and surrender penalty rate for the term life insurance, respectively, and the price function $\bar{A}^{1}_{x+t:\overline{T-t|}}{}^{O}$ is given in (5.2.3). Remind that the term life insurance is effective until T. Similarly to the life annuity case, by surrendering \$1 of death benefit, the individual receives $(1-p_I)\bar{A}^{1}_{x+t:\overline{T-t|}}{}^{O}$ from the term life insurance issuer. The dynamics contain jumps due to lump-sum purchases of term life insurance, and $t-$ denotes the values of the associated process right before any such jump.

If death occurred before T, the term life insurance would be in effect and the legacy would be $kW_{\tau_x^S-} + D_{\tau_x^S-}$, where k is still the after-tax proportion on legacy. Note that most life insurance death benefit are not counted as taxable income.

The consumption and term life insurance strategies $\{c_t, D_t^B, D_t^S\}_{t\in[0,T]}$ are said to be admissible if

(i) $\{c_t\}_{t\in[0,T]}$, $\{D_t^B\}_{t\in[0,T]}$ and $\{D_t^S\}_{t\in[0,T]}$ are adapted to the filtration $\mathbb{F}$.

(ii) The controls $c_t \geq 0$, $D_t^B \geq 0$ and $D_t^S \geq 0$ a.s. for all $t \in [0,T]$.

(iii) $\int_0^t c_s ds < \infty$ a.s. for all $t \in [0,T]$.

(iv) $\{D_t^B\}_{t\in[0,T]}$ and $\{D_t^S\}_{t\in[0,T]}$ are nondecreasing in t, $W_t \geq 0$ and $D_t \geq 0$ a.s. for all $t \in [0,T]$.

5.4.1 Objective function

Fix a time point t ($t < T$), given initial wealth $W_t = w$, initial death benefit $D_t = D$ and policy $\{\mathbf{c}, \mathbf{D}^B, \mathbf{D}^S\} := \{c_s, D_s^B, D_s^S\}_{s\in[t,T)}$, the objective function for the individual (aged $x+t$ now) is given by

$$
\begin{aligned}
&J^{(D)}(w, D, t; \mathbf{c}, \mathbf{D}^B, \mathbf{D}^S) \\
&:= \mathbb{E}\Big[\int_t^{\tau_x^S\wedge T} e^{-r(s-t)}u_1(c_s)ds + e^{-r(\tau_x^S-t)}u_2(kW_{\tau_x^S-} + D_{\tau_x^S-})\mathbb{I}_{\tau_x^S\leq T} + e^{-r(T-t)}u_3(W_T)\mathbb{I}_{\tau_x^S>T}\Big|\tau_x^S > t\Big], \\
&= \mathbb{E}\Big[\int_t^{T} e^{-\int_t^s(r+\lambda_{x+u}^S)du}\Big(u_1(c_s) + \lambda_{x+s}^S u_2(kW_{s-} + D_s)\Big)ds + e^{-\int_t^T(\lambda_{x+s}^S+r)ds}u_3(W_T)\Big],
\end{aligned}
$$

with the same utility functions defined in (5.3.2), and τ_x^S is also assumed to be independent from the Brownian motion $\{B_t\}_{t\geq 0}$. The value function for this individual is defined for $(w, D, t) \in \mathbb{R}_+ \times \mathbb{R}_+ \times [0, T)$ by

$$U^{(D)}(w, D, t) = \sup_{\mathbf{c}, \mathbf{D}^B, \mathbf{D}^S} J^{(D)}(w, D, t; \mathbf{c}, \mathbf{D}^B, \mathbf{D}^S). \tag{5.4.2}$$

Again, from standard stochastic control theory, $U^{(D)}$ solves the following variational inequality:

$$\begin{cases} \max\Big\{U_t^{(D)} + \max\limits_{c\geq 0} \mathcal{L}_0^c U^{(D)}, U_D^{(D)} - (1 + l_I)\bar{A}^{\;\;\;\;O}_{\overset{1}{x+t}:\overline{T-t|}} U_w^{(D)}, (1 - p_I)\bar{A}^{\;\;\;\;O}_{\overset{1}{x+t}:\overline{T-t|}} U_w^{(D)} - U_D^{(D)}\Big\} = 0, \\ \qquad\qquad (w, D, t) \in \mathbb{R}_+ \times \mathbb{R}_+ \times [0, T), \\ U^{(D)}(w, D, T) = u_3(w), \quad (w, D) \in \mathbb{R}_+ \times \mathbb{R}_+, \end{cases} \tag{5.4.3}$$

in which

$$\mathcal{L}_0^c U = -(\lambda_{x+t}^S + r)U + \Big(\mu w - c\Big)U_w + \frac{1}{2}\sigma^2 w^2 U_{ww} + u_1(c) + \lambda_{x+t}^S u_2(kw + D).$$

We define $z = \frac{D}{w+D}$ as the proportion of death benefit in total wealth and $V^{(D)}(z, t) = U^{(D)}(1 - z, z, t)$ to reduce (5.4.3) into a one-dimensional problem. With the new variable z, we define

$$\phi^{(D)}(z, t) = \frac{\log(\gamma V^{(D)}(z, t))}{\gamma}.$$

Then we can check that $\phi^{(D)}$ satisfies

$$\begin{cases} \max\{\phi_t^{(D)} + \max\limits_{\tilde{c}\geq 0} \mathcal{L}^{\tilde{c}}\phi^{(D)}, \mathcal{B}_D\phi^{(D)}, \mathcal{S}_D\phi^{(D)}\} = 0, \;\; (z, t) \in [0, 1] \times [0, T), \\ \phi^{(D)}(z, T) = \log(1 - z), \;\; z \in [0, 1], \end{cases} \tag{5.4.4}$$

where

$$
\begin{aligned}
&\tilde{c} = \frac{c}{w + D}, \\
&\mathcal{B}_D\phi = 1 - (1 + l_I)\bar{A}^{\,1}_{x+t:\overline{T-t|}}{}^{O} + \Big[1 - z + (1 + l_I)\bar{A}^{\,1}_{x+t:\overline{T-t|}}{}^{O} z\Big]\phi_z, \\
&\mathcal{S}_D\phi = (1 - p_I)\bar{A}^{\,1}_{x+t:\overline{T-t|}}{}^{O} - 1 - \Big[1 - z + (1 - p_I)\bar{A}^{\,1}_{x+t:\overline{T-t|}}{}^{O} z\Big]\phi_z,
\end{aligned}
\tag{5.4.5}
$$

and

$$
\begin{aligned}
\mathcal{L}^{\tilde{c}}\phi = &\frac{\sigma^2}{2}(1 - z)^2\Big[z^2\phi_{zz} + \gamma z^2\phi_z^2\Big] + z\phi_z\Big[\sigma^2(1 - z)^2(1 - \gamma) - \mu(1 - z) + \tilde{c}\Big] \\
&- \frac{\lambda^S_{x+t} + r}{\gamma} + \mu(1 - z) - \tilde{c} + (\gamma - 1)\frac{\sigma^2}{2}(1 - z)^2 \\
&+ e^{-\gamma\phi}\Big[\frac{\tilde{c}^\gamma}{\gamma} + \lambda^S_{x+t}\frac{(k(1 - z) + z)^\gamma}{\gamma}\Big].
\end{aligned}
\tag{5.4.6}
$$

The buy region, surrender region and non-transaction region for the term life insurance are defined as

$$
\begin{aligned}
BR_D &= \{(z, t) \in [0, 1] \times [0, T) : \mathcal{B}_D\phi^{(D)} = 0\}, \\
SR_D &= \{(z, t) \in [0, 1] \times [0, T) : \mathcal{S}_D\phi^{(D)} = 0\}, \\
NTR_D &= [0, 1] \times [0, T) \setminus \big(BR_D \cup SR_D\big).
\end{aligned}
$$

5.4.2 Numerical results

We still apply Gompertz force of mortality as in Section 5.3 and Table 5.2 lists the default parameter values. Note that some of the parameter values in Table 5.2 are adjusted to be different from the ones in Table 5.1, to gain a clearer view of the sensitivity analysis.

a_s	b_s	a_o	b_o	age	k	γ
$2.1457 * 10^{-5}$	0.09524	$2.1457 * 10^{-5}$	0.09524	35	1	-2
T	r	μ	σ	l_I	p_I	
30	0.02	0.06	0.2	0.05	0.5	

Table 5.2: Default model parameters

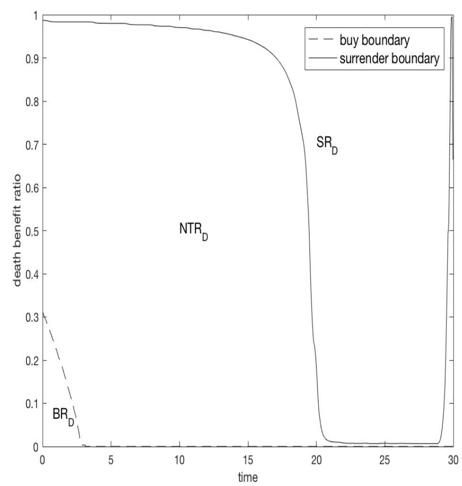
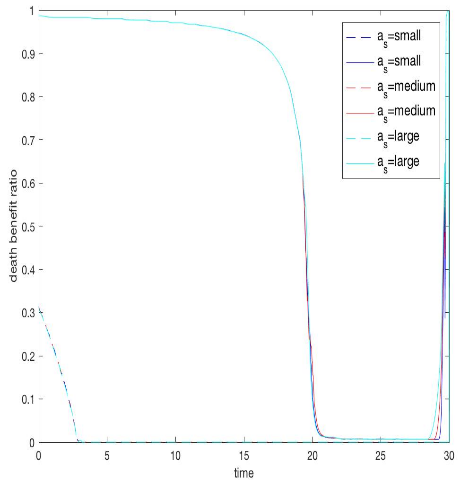


Figure 5.8: Optimal term life insurance strategy under default parameters

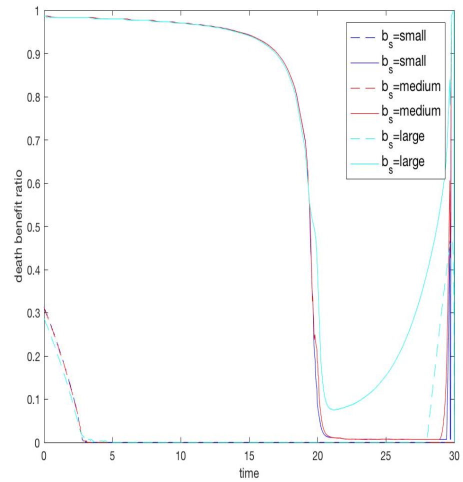
Figure 5.8 depicts the optimal term life insurance strategy under default parameters. Similar to Section 5.3, the dashed line and solid line are term life insurance buy boundary and surrender boundary, respectively. When the existing death benefit lies below the buy boundary (above the surrender boundary), a finite amount of death benefit will be purchased (surrendered) immediately to move up (down) to the buy (surrender boundary). Within the non-transaction region NTR_D, the individual should stay with what he or she has and only consume. Similar as before, the amount of term life insurance one should hold increases in wealth and decreases in the existing amount of death benefit.

In terms of the evolution of demand in time: we observe that one should allocate a positive fraction of wealth in term life insurance at the beginning of the time period, since we start with an individual with no insurance at time 0 and it's natural for the individual to likely purchase some coverage early on. Then it is followed by a large non-transaction region for a certain length of time, where one should stay with what he or she has and stop from purchasing or surrendering any term life insurance. After that, it is followed by a period where surrender activity picks up, and the surrender amount mainly depends on the mortality risk, risk preference, after-tax effect, wealth growth rate and wealth volatility (will be discussed later on). When close to maturity, the wealth composition stay fixed and one should refrain from trading any more.

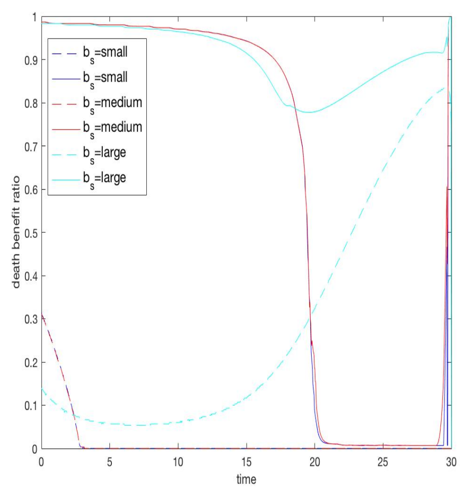
Figure 5.9 illustrates the impact of the individual's subjective force of mortality on his willingness to purchase term life insurance. In these figures, *small*, *medium* and *larger* represent scenarios where either one of the parameter values of the subjective force of mortality is 50% lower, the same and 50% higher (resp.) than the default parameter setting, unless otherwise specified in the figure captions. From Figure 5.9a, both boundaries are mostly insensitive to a_s, except when close to maturity, where it is optimal for people with higher mortality risk (large a_s) to surrender less and hold more term life insurance. b_s has a more obvious effect on the boundaries. In Figures 5.9b and 5.9c, we observe that, over most of the time horizons, a larger value of b_s leads to higher buy and surrender boundaries, namely, the individual should hold more term life insurance with the increase in the subjective mortality risk. However, note that there are time intervals over which such relation no longer holds. For example, when the subjective mortality risk is significantly higher than the objective one (as in Figure 5.9c), people with higher mortality risk purchase less when young, and the shape of the buy and surrender region are remarkably different.



(a) a_s



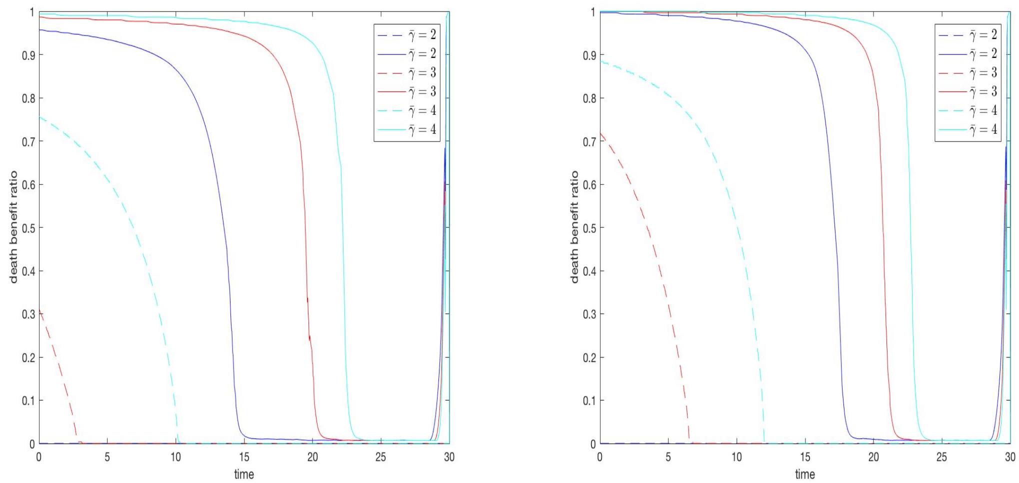
(b) b_s = large is 50% higher



(c) b_s = large is 100% higher

Figure 5.9: Impact of subjective force of mortality a_s and b_s

In Figure 5.10 we analyze the impact of the individual's risk aversion parameter $\bar{\gamma}$ (recall that $\bar{\gamma} = 1 - \gamma$). When the individual is more risk averse as $\bar{\gamma}$ increases, both buy and surrender boundaries increase. This implies that, all else being equal, more risk averse people should hold more term life insurance. Also note that such trend does not depend on the level of interest rate r, which is different from the annuitization behaviour[4].



(a) $r = 0.02$ (b) $r = 0.05$

Figure 5.10: Impact of risk preference $\bar{\gamma}$ under various interest rates

After studying the impact of the subjective force of mortality and risk preference, we want to discuss a phenomenon, commonly known as *advantageous selection*-where lower risk individuals (i.e., individuals with lower mortality risks) have more insurance, see Cawley and Philipson [24] for life insurance, Finkelstein and McGarry [43] for long-term care insurance and Cutler et al. [31] for a comprehensive review. In Figure 5.9c, we have seen people with lower mortality risk purchase more term life insurance when young, even for a fixed level a risk aversion. Next, in Figure 5.11, we compare the buy and surrender boundaries for two individuals. The red lines correspond to an individual with higher mortality risk (higher values of a_s and b_s than their counterparts for the objective mortality) and lower risk aversion (larger γ), and the blue lines correspond to another individual with lower mortality risk (smaller values of a_s and b_s than their counterparts for the objective

[4]As in the numerical analysis for optimal annuitization, we also test the effect of $\bar{\gamma}$ incorporating an investment strategy: either with a constant proportion strategy or a Kelly's strategy, see Footnote 5.3.3 for details. It turns out that more risk averse individuals should keep more term life insurance for both small and large interest rate r.

mortality) and higher risk aversion (or smaller γ). Over the first 20 years (i.e., from time 0 to time 20 in Figure 5.11), the individual with lower mortality risk and higher risk aversion allocates more wealth in term life insurance than the one with higher mortality risk and lower risk aversion. As studied in Cutler et al. [31], in life insurance market, heterogeneous risk aversion can help explain why people with lower mortality rates have more insurance. From their empirical study, individuals who don't engage in what are commonly thought of as risky behaviours or who take measures to reduce risk are systematically more likely to hold life insurance products, but these same individuals tend to have lower expected claims, leading the lower risk to have more coverage. Using our model, the phenomenon of *advantageous selection* for life insurance can be realized. It also provides support to the explanations in Cutler et al. [31] pertaining to the life-cycle optimal control.

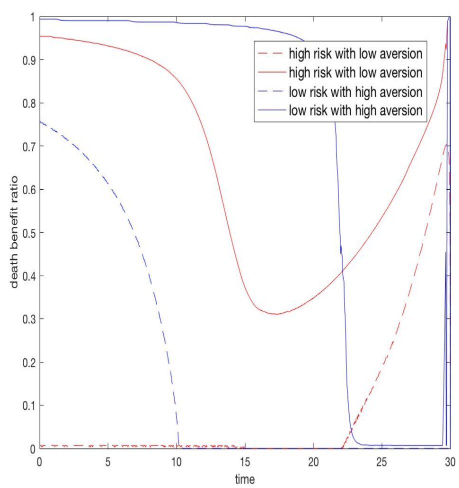
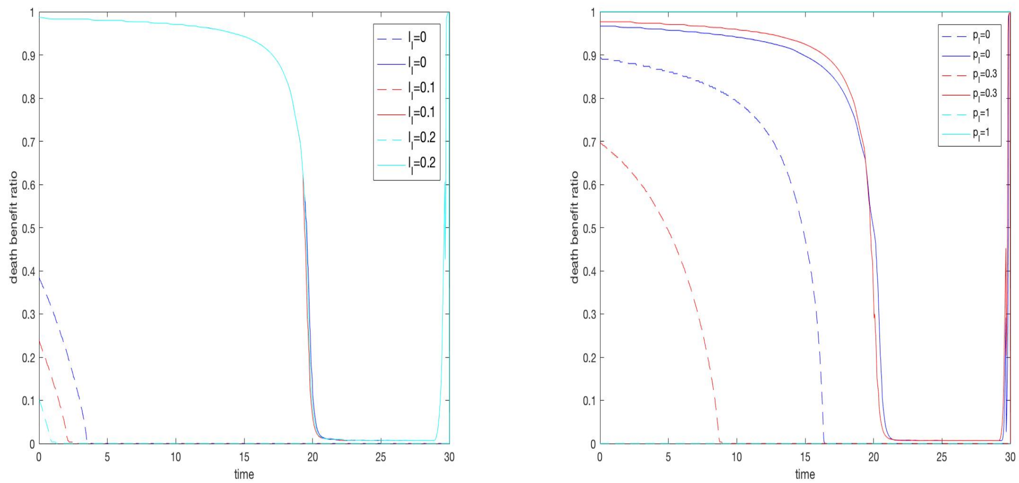


Figure 5.11: Illustration of advantageous selection

Next, Figure 5.12 depicts the impact of transaction costs, including the loading factor l_I and the surrender penalty rate p_I. From Figure 5.12a, the surrender boundary is minimally impacted by the change in l_I, while the buy boundary decreases in l_I, which means less term life insurance should be purchased when the safety loading increases. Equivalently, the non-transaction region expands as l_I increases. The same is true for the surrender penalty rate p_I as can be seen from Figure 5.12b. Worthy of mention is the case when

$p_I = 1$ where it is never optimal to surrender. Therefore, the safety loading and surrender penalty rate act on the non-transaction region, and people should trade less frequently when transaction costs increase.



(a) l_I (b) p_I

Figure 5.12: Impact of safety loading factor l_I and surrender penalty rate p_I

Figure 5.13 examines the effect of the after-tax rate k. When less legacy is taxed (i.e., as k increases), both the buy boundary and the surrender boundary decrease. Hence, all else being equal, it is optimal for the individual to allocate less wealth in term life insurance, because more legacy can be inherited and the bequest motive is better managed. Note that when the tax rate is extremely high on legacy, e.g. when $k = 0$, the individual will allocate a positive fraction of wealth in term life insurance over the entire time period.

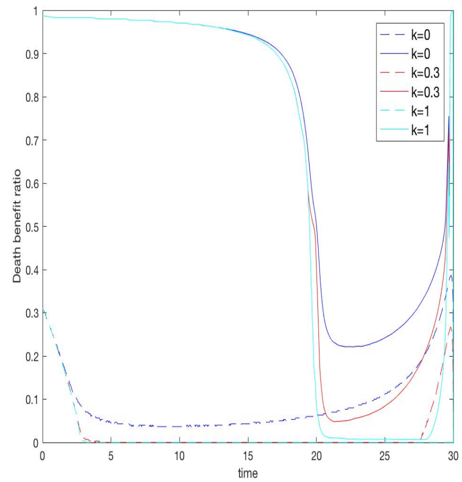
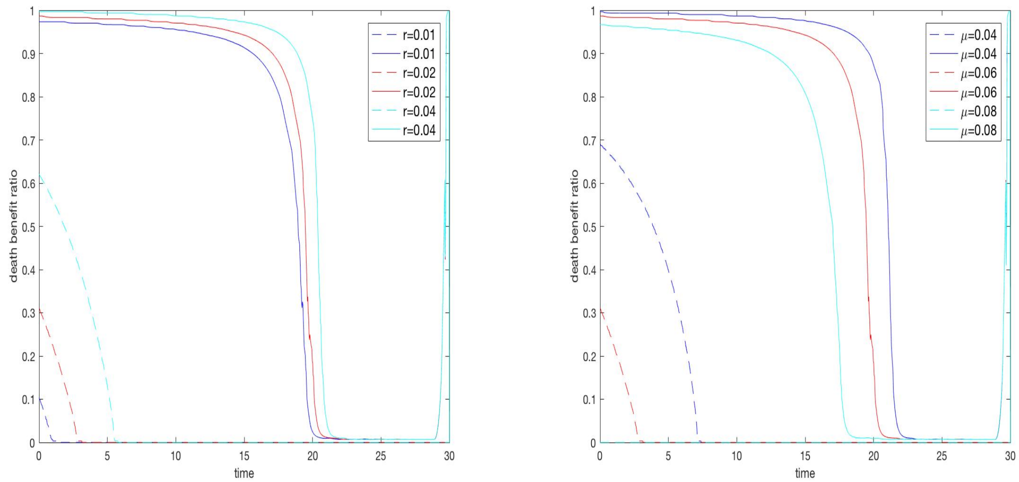


Figure 5.13: Impact of after-tax proportion k

Figure 5.14a depicts the impact of the interest (pricing) rate r. Both the buy boundary and the surrender boundary increase in r. That is to say, the individual should keep a larger fraction of wealth in term life insurance for a higher r. The intuition is that for the same amount of death benefit, the premium is lower with a higher value of r.

Figure 5.14b illustrates how the wealth accumulation rate μ affects the term life insurance buy/surrender behaviour. We observe that both the buy boundary and surrender boundary decrease in μ, which leads to a smaller fraction of wealth in term life insurance. Note that wealth can be inherited as a legacy to hedge the risk of a premature death. As μ increases, the accumulation of wealth is preferred over the alternative of investing in term life insurance.



(a) r (b) μ

Figure 5.14: Impact of interest rate r and stock return rate μ

Figure 5.15 shows the impact of the wealth volatility σ. When the wealth is more volatile (i.e., as σ increases), both the buy boundary and the surrender boundary increase, and consequently, it is optimal for the individual to keep more wealth in term life insurance. Note that the two boundaries are more sensitive to changes in σ (in comparison to changes in the interest rate r and the wealth growth rate μ).

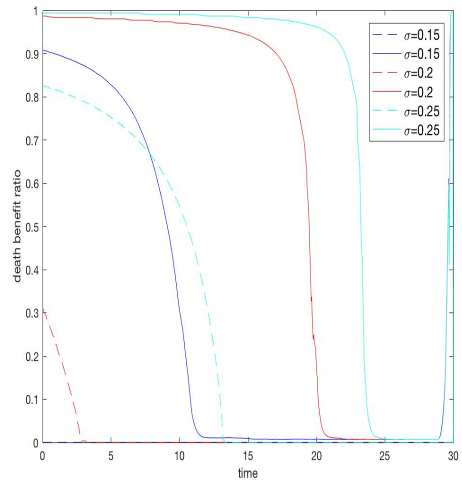


Figure 5.15: Impact of wealth volatility σ

At this moment, we want to discuss two empirical puzzles observed in life insurance market. One is insufficient life insurance among the working-aged, see Bernheim et al. [14, 13]. This observation is possible for most of our scenarios, since the buy boundary easily stays at zero after an initial short time period in most of our figures. Especially, expensive safety loading and surrender penalty rate (Figure 5.12), low level of interest rate and high wealth growth rate (Figure 5.14) can push the buy boundary to be zero over the entire time horizon. The other empirical observation is excessive life insurance among the elderly, see Brown [20] and Cutler and Zeckhauser [30]. From our result, high level of subjective force of mortality (Figure 5.9c) and high tax rate (Figure 5.13) can both trigger a positive buy boundary in late time periods. Hurd and McGarry [62] finds that the subjective force of mortality increases in age due to unanticipated change in health and with the death of a parent. As such, higher than average subjective force of mortality might help explain the excessive holdings of life insurance among the elderly. Although these empirical observations are from life insurance market and we study term life insurance, we argue that life insurance can be approximated by term life insurance when T is long enough.

5.4.3 Summary of findings on demand for term life insurance

In this section, we study the problem of maximizing utility from consumption, bequest motive and terminal wealth upon survival when only term life insurance is available. By applying a penalty method to the free boundary problem, we obtain a numerical solution to the optimal buy and surrender region, and an analysis of the corresponding results allows us to reach the following conclusions: All else being equal, it is optimal for an individual to allocate *more wealth in term life insurance* if

- He or she is wealthier;
- He or she has less existing death benefit;
- He or she has a shorter expected lifetime compared to the objective pricing group;
- More is taxed on legacy;
- The pricing rate of the term life insurance is higher;
- The wealth process has a lower return rate;
- The wealth process is more volatile;
- The individual is more risk averse.

An individual should trade less frequently when the transaction costs are more expensive (higher l_I or p_I).

Moreover, some empirical findings in life insurance market including *advantageous selection*, insufficient life insurance among the working-aged and excessive life insurance among the elderly, can be realized and partially explained using our model.

5.5 General case

In this section, we consider the general case when both life annuity and term life insurance are available in the market. N_t^B, N_t^S, D_t^B and D_t^S are defined as in the previous two sections. The controlled wealth process W_t, after considering consumption, and the purchase and surrender of life annuity and term life insurance, follows the dynamic

$$\begin{cases} dW_t = \Big[\mu W_{t-} - c_{t-} + N_{t-}\Big]dt + \sigma W_{t-}dB_t - (1+l_I)\bar{A}^{\;\;O}_{\substack{1\\x+t:\overline{T-t}|}} dD_t^B - (1+l_A)\bar{a}^O_{x+t}dN_t^B \\ \qquad\qquad + (1-p_I)\bar{A}^{\;\;O}_{\substack{1\\x+t:\overline{T-t}|}} dD_t^S + (1-p_A)\bar{a}^O_{x+t}dN_t^S, \quad t < \min(\tau_x^S, T), \\ dD_t = dD_t^B - dD_t^S, \\ dN_t = dN_t^B - dN_t^S, \end{cases} \tag{5.5.1}$$

with $W_0 = w \geq 0$, $D_0 = D \geq 0$ and $N_0 = N \geq 0$.

For a death time $\tau_x^S \leq T$, the total legacy is $D_{\tau_x^S-} + kW_{\tau_x^S-}$, where k is the after-tax proportion. If $\tau_x^S > T$, the term life insurance expires and the terminal wealth of the individual at T is $W_T + \bar{a}_{x+T}N_T$. Admissible strategies are defined similarly as in Sections 5.3 and 5.4.

5.5.1 Objective function

At each time point $t \in [0, T)$, with initial wealth, life annuity and term life benefit to be w, N and D, respectively, the individual seeks to maximize the expected utility of the discounted lifetime consumption up to T, the bequest motive for a premature death, and the terminal wealth upon survival to T, over all admissible strategies $\{\mathbf{c}, \mathbf{N}^B, \mathbf{N}^S, \mathbf{D}^B, \mathbf{D}^S\}$:

$$
\begin{aligned}
&J(w,N,D,t;\mathbf{c},\mathbf{N}^B,\mathbf{N}^S,\mathbf{D}^B,\mathbf{D}^S)\\
&:=\mathbb{E}\Big[\int_t^{\tau_x^S\wedge T} e^{-r(s-t)}u_1(c_s)ds+e^{-r(\tau_x^S-t)}u_2(D_{\tau_x^S}+kW_{\tau_x^S})\mathbb{I}_{\tau_x^S\le T}\\
&\qquad\qquad\qquad\qquad + e^{-r(T-t)}u_3(W_T+\bar{a}^O_{x+T}N_T)\mathbb{I}_{\tau_x^S>T}\Big|\tau_x^S>t\Big]\\
&=\mathbb{E}\Big[\int_t^T e^{-r(s-t)}{}_sp_x^S\Big(u_1(c_s)+\lambda^S_{x+s}u_2(D_{s-}+kW_{s-})\Big)ds\\
&\qquad\qquad\qquad\qquad + {}_Tp_x^S e^{-r(T-t)}u_3(W_T+\bar{a}^O_{x+T}N_T)\Big|\tau_x^S>t\Big]\\
&=\mathbb{E}\Big[\int_t^T e^{-\int_t^s(r+\lambda^S_{x+u})du}\Big(u_1(c_s)+\lambda^S_{x+s}u_2(D_{s-}+kW_{s-})\Big)ds+e^{-\int_t^T(\lambda^S_{x+s}+r)ds}u_3(W_T+\bar{a}^O_{x+T}N_T)\Big],
\end{aligned}
\tag{5.5.2}
$$

where $u_1(\cdot)$, $u_2(\cdot)$ and $u_3(\cdot)$ are defined in (5.3.2). Still, we assume that τ_x^S and the Brownian motion $\{B_t\}_{t\ge0}$ are independent.

The value function U, defined by

$$
U(w,N,D,t)=\sup_{\mathbf{c},\mathbf{N}^B,\mathbf{N}^S,\mathbf{D}^B,\mathbf{D}^S} J(w,N,D,t;\mathbf{c},\mathbf{N}^B,\mathbf{N}^S,\mathbf{D}^B,\mathbf{D}^S), \tag{5.5.3}
$$

solves the HJB equation

$$
\begin{cases}
\max\Big\{U_t+\max\limits_{c\ge0}\mathcal{L}_0^cU, U_D-(1+l_I)\bar{A}^{\ 1\ \ O}_{x+t:\overline{T-t|}}U_w, U_N-(1+l_A)\bar{a}^O_{x+t}U_w,\\
\qquad (1-p_I)\bar{A}^{\ 1\ \ O}_{x+t:\overline{T-t|}}U_w-U_D, (1-p_A)\bar{a}^O_{x+t}U_w-U_N\Big\}=0,\quad (w,N,D,t)\in\mathbb{R}^3_+\times[0,T),\\
U(w,D,N,T)=u_3(w+\bar{a}^O_{x+T}N),\quad (w,N,D)\in\mathbb{R}^3_+,
\end{cases}
\tag{5.5.4}
$$

where

$$
\mathcal{L}_0^cU=-(\lambda^S_{x+t}+r)U+\Big(\mu w-c+N\Big)U_w+\frac{1}{2}\sigma^2w^2U_{ww}+u_1(c)+\lambda^S_{x+t}u_2(D+kw).
$$

Due to the homogeneity property of the CRRA utility function, i.e., $U(\alpha w,\alpha N,\alpha D,t)=$

$\alpha^\gamma U(w,N,D,t)$ for $\alpha>0$, we define

$$y=\frac{N}{w+D+N},\qquad z=\frac{D}{w+D+N},\qquad V(y,z,t)=U(1-y-z,y,z,t).$$

Then

$$U(w,N,D,t)=(w+N+D)^\gamma V(y,z,t),$$

and the problem is simplified to the quantity V which has one fewer state variable (than U). With new variables y and z, define

$$\phi(y,z,t)=\frac{\log(\gamma V(y,z,t))}{\gamma},$$

then the function ϕ satisfies

$$\begin{cases}\max\{\phi_t+\max\limits_{\tilde{c}\ge 0}\mathcal{L}^{\tilde{c}}\phi,\mathcal{B}_D\phi,\mathcal{B}_N\phi,\mathcal{S}_D\phi,\mathcal{S}_N\phi\}=0,\\ \phi(y,z,T)=\log(1-y-z+\bar{a}^O_{x+T}y),\end{cases}\tag{5.5.5}$$

where

$$\tilde{c}=\frac{c}{w+N+D},$$
$$\mathcal{B}_D\phi=1-(1+l_I)\bar{A}^{1}_{x+t:\overline{T-t}|}{}^O+(1-z+(1+l_I)\bar{A}^{1}_{x+t:\overline{T-t}|}{}^O z)\phi_z+((1+l_I)\bar{A}^{1}_{x+t:\overline{T-t}|}{}^O-1)y\phi_y,$$
$$\mathcal{B}_N\phi=1-(1+l_A)\bar{a}^O_{x+t}+((1+l_A)\bar{a}^O_{x+t}-1)z\phi_z+(1-y+(1+l_A)\bar{a}^O_{x+t}y)\phi_y,$$
$$\mathcal{S}_D\phi=(1-p_I)\bar{A}^{1}_{x+t:\overline{T-t}|}{}^O-1-\big[(1-p_I)\bar{A}^{1}_{x+t:\overline{T-t}|}{}^O z+1-z\big]\phi_z-\big[(1-p_I)\bar{A}^{1}_{x+t:\overline{T-t}|}{}^O-1\big]y\phi_y,$$
$$\mathcal{S}_N\phi=(1-p_A)\bar{a}^O_{x+t}-1-\big[(1-p_A)\bar{a}^O_{x+t}-1\big]z\phi_z-\big[(1-p_A)\bar{a}^O_{x+t}y+1-y\big]\phi_y,$$

and

$$\mathcal{L}^{\tilde{c}}\phi = \frac{1}{2}\sigma^2(1-y-z)^2\Big[z^2(\phi_{zz}+\gamma\phi_z^2)+y^2(\phi_{yy}+\gamma\phi_y^2)+2yz(\phi_{yz}+\gamma\phi_z\phi_y)\Big]$$
$$+(z\phi_z+y\phi_y)\Big[\sigma^2(1-\gamma)(1-y-z)^2-\big(\mu-\mu z+(1-\mu)y-\tilde{c}\big)\Big]$$
$$-\frac{(\lambda_{x+t}^S+r)}{\gamma}+\big(\mu-\mu z+(1-\mu)y-\tilde{c}\big)+\frac{1}{2}\sigma^2(1-y-z)^2(\gamma-1) \tag{5.5.6}$$
$$+e^{-\gamma\phi}\Big[\frac{\tilde{c}^\gamma}{\gamma}+\lambda_{x+t}^S\frac{(z+k(1-z-y))^\gamma}{\gamma}\Big].$$

Buy, surrender and non-transaction regions for life annuity and term life insurance are defined similarly as before, except that now the space is two-dimensional in terms of the state variables y and z. Below is a hypothetical example of a time snapshot for each region.

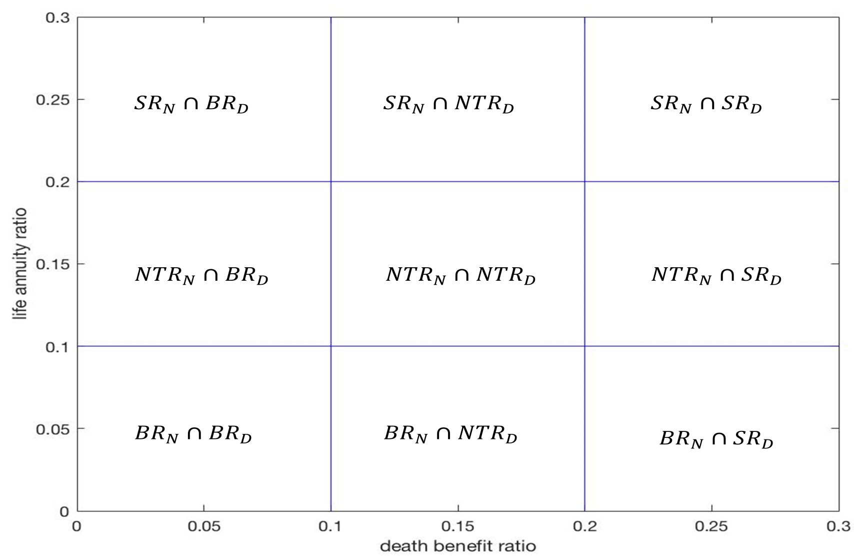


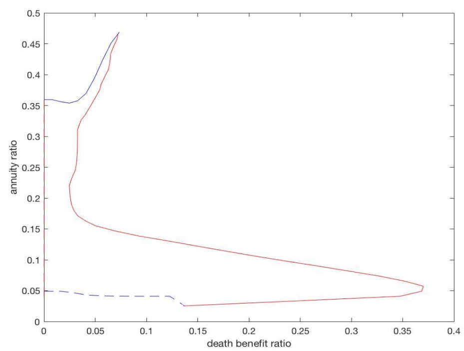
Figure 5.16: A hypothetical example of regions

5.5.2 Numerical results

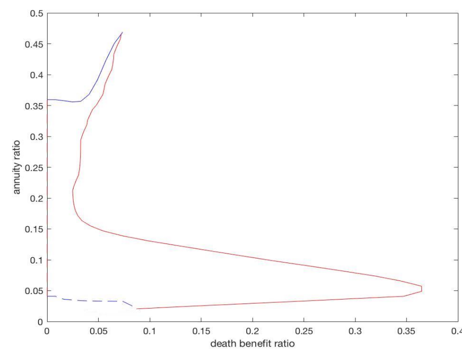
As for the previous two sections, we consider a Gompertz mortality law and Table 5.3 lists the default parameter values.

a_s	b_s	a_o	b_o	age	k	γ	p_I
$2.1457 * 10^{-5}$	0.09524	$2.1457 * 10^{-5}$	0.09524	35	1	-2	0.3
T	r	μ	σ	l_A	l_I	p_A	
30	0.02	0.06	0.3	0.05	0.05	0.3	

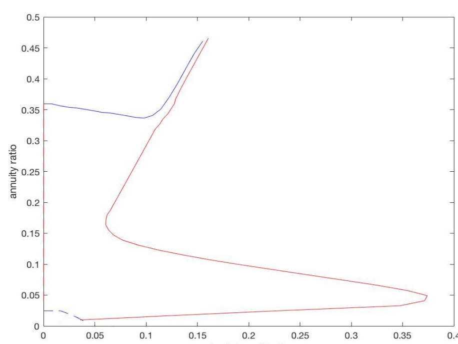
Table 5.3: Default parameter values



(a) $t = 0$



(b) $t = 15$



(c) $t = 21$

Figure 5.17: Time snapshots of life annuity and term life insurance region

In Figure 5.17, we plot the non-transaction region, i.e., $NTR_N \cap NRR_D$ in Figure 5.16. As can be seen, both the optimal buy (blue dashed line) and surrender (blue solid line) boundaries for life annuities are function of the death benefit ratio. Similarly, the optimal buy (red dashed line) and surrender (red solid line) boundaries for term life insurance are also function of the annuity ratio. We observe that no term life insurance is purchased. Actually, we find that there is no noticeable difference for term life insurance when we change the values of the after-tax proportion k, and the individual only purchases term life insurance when it is close to maturity. We also observe that the term life insurance surrender boundary is high for low values of life annuity ratio. As for the life annuity, we see that it is optimal to allocate a positive fraction of wealth in life annuity. The intuition behind this result is that the income from the life annuity position immediately increases the size of wealth which will be inherited as a result of a death prior to time T. As in the previous sections, the non-transaction region expands as time approaches the maturity.

5.6 Conclusion

In this chapter, by modelling the life annuity decision and term life insurance decision using singular and impulse control separately, we consider the problem of maximizing the expected utility from discounted lifetime consumption, bequest motive and terminal wealth upon survival. Applying a penalty method, we solve for the optimal buy and surrender boundaries for both products. The main takeaways are as follows:

1. All else being equal, the individual should allocate *more wealth in life annuity* if
    - He or she is wealthier;
    - He or she has less existing life annuity;
    - He or she has a longer expected lifetime than the pricing group;
    - Less is taxed on legacy;
    - The pricing rate for life annuities is higher;
    - The wealth process has a lower return rate;

- The wealth process is more volatile.

2. All else being equal, the individual is optimal to allocate *more wealth in term life insurance* if

   - He or she is wealthier;
   - He or she has less existing death benefit;
   - He or she has a shorter expected lifetime than the pricing group;
   - More is taxed on legacy;
   - The pricing rate for term life insurance is higher;
   - The wealth process has a lower return rate;
   - The wealth process is more volatile;
   - The individual is more risk averse.

3. For both products, the individual should trade less frequently for a higher safety loading factor or surrender penalty rate.

4. The individual may not allocate more wealth in life annuity if he or she is more risk averse. This depends on the level of interest rate.

Chapter 6

Conclusion and Future works

The main contribution of this thesis is that it has employed models more in line with reality, in regard to both the underlying dynamics and control variables, to the study of stochastic control problems of interest in insurance contexts.

More specifically, in Chapter 3, instead of adopting the commonly used spectrally negative Lévy process (SNLP), the underlying basic surplus process for the insurer is allowed to have contagious features, which allows for contagion due to endogenous (self-exciting) and exogenous (externally-exciting) factors. We generalized the result within the SNLP framework that an excess-of-loss reinsurance treaty is optimal under the time-consistent mean-variance criterion, and demonstrated new findings that contagion risk does play an important role in the reinsurance strategy through the tail heaviness of the claim size distribution. In Chapter 4, we recognized the partial information of the stock return rate, and explicitly analyzed the hedging demand due to such partial information with Bayesian learning under time-consistent mean-variance criterion. In Chapter 5, in addition to the current wealth, which is the sole state variable in most of the existing literature, we further took into account existing life annuity and term life insurance as state variables. Moreover, life annuity and term life insurance can only be purchased or surrendered at realistic lump-sum costs. Under a general force of mortality, optimal non-transaction regions have been solved numerically.

This work can be further extended in the following possible directions:

First, in Chapter 3, for tractability, both the insurance and reinsurance premium rates are assumed to be time-dependent and stochastic, which may not be allowed by reinsurance contract or regulations in practice. One way to address this issue is by using a constant rate under the expectation principle, where one associated issue is to demonstrate the existence of an analytical solution. On the other hand, the optimal reinsurance strategy does not depend on parameters of the externally-exciting effect, namely, it does not act on catastrophic events. One possible reason is the assumption of stochastic premium rate which fully hedges the catastrophic risk as pointed out in Chapter 3, but other utility preferences are also worthy of investigation.

Second, the work in Chapter 5 can be further extended in the following possible directions: (1) *Incorporate labor income in the wealth process*: Since a major function of term life insurance is to protect against the loss of the breadwinner's income in case of a premature death, incorporating labor income is more reasonable. Actually, as demonstrated in Richard [89] and Pliska and Ye [87], the demand for life insurance is directly affected by one's human capital, which is the discounted value of future income. One difficulty arising from the labor income is that the value function is no longer homogeneous in the state variables, and numerical solution with three spatial states are generally challenging. (2) *Allow for investment in risky assets*: Chen et al. [26] showed that individuals should make asset allocation decisions and life insurance decisions jointly.

The third direction is on framing the problem in an equilibrium setting. This thesis only focuses on one party's interest when studying the optimal problem, while " an agreement which is quite attractive to one party may not be acceptable to its counterparty", as demonstrated by Borch in 1960s. In an equilibrium setting, premiums are not taken as given, but rather set based on interactions between sellers and buyers. Especially, the empirical findings from life annuity and insurance market are the result of multiple participants' interaction, including individuals and insurance companies. After understanding individual's behaviour when prices are given, it will definitely be worth studying whether a market equilibrium exists and if so how to find it.

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