

“First Published in SIAM Journal on Control and Optimization in 2014, published by the Society for Industrial and Applied Mathematics (SIAM)” and the copyright notice as stated in the article itself (e.g., “Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.”)

TRANSVERSE FEEDBACK LINEARIZATION WITH PARTIAL INFORMATION FOR SINGLE-INPUT SYSTEMS*

CHRISTOPHER NIELSEN[†]

Abstract. This paper is motivated by the problem of asymptotically stabilizing invariant sets in the state space of control systems by means of output feedback. The sets considered are smooth embedded in submanifolds and the class of system is nonlinear, finite-dimensional, autonomous, deterministic, single-input and control-affine. Given an invariant set and a control system with fixed output, necessary and sufficient conditions are presented for feedback equivalence to a normal form that facilitates the design of output feedback controllers that stabilize the set using existing design techniques.

Key words. Transverse feedback linearization, feedback equivalence, output feedback, set stabilization, robust path following.

AMS subject classifications. 37N35, 93B10, 93B29, 93B27.

1. Introduction. In this paper the problem of asymptotically stabilizing sets using output feedback is investigated. Many control objectives can be accomplished by stabilizing an appropriate invariant set, often a submanifold, in the state space of a control system. This point of view is relevant in applications such as output regulation [6], [8], [13], synchronization [24], formation control problems for multi-agent systems [7], [14] and path following [9], [19]. Topological obstructions for submanifold stabilization using full-state feedback were characterized in [16]. Output feedback controllers are necessary whenever the state of the system is not available for feedback. This is common in applications where, due to economic or technological reasons, sensors cannot measure a system’s entire state.

The most natural approach to stabilizing sets using output feedback is to find an observable, i.e., available for feedback, function that yields a well-defined relative degree whose associated zero dynamics manifold coincides with the set to be stabilized. If such an observable function exists, then the set stabilization problem becomes an output stabilization problem which can be solved using classical and well-understood output feedback control design techniques [1], [2], [17], [27]. The main contribution of this paper are necessary and sufficient conditions for the existence of such an observable function (Theorem 5.3).

In the case of full-state feedback, i.e., the full information case, the above approach to set stabilization was studied in [21]. There we sought a coordinate and feedback transformation locally bringing the control system to a “normal form,” in which the system’s dynamics are decomposed into two cascade-connected subsystems. In our normal form, the driving system is linear, time-invariant, and controllable. It models the dynamics “transversal” to the target set in the sense that, in transformed coordinates, the target set corresponds to the origin of this linear system. We refer to the driving subsystem as the transversal subsystem. On the other hand, the restriction of the driven system to the target set represents the “tangential” motion of the control system on the set, and for this reason such restriction is referred to as the tangential subsystem. The process of bringing the original control system to the

*This work was supported by supported by the National Science and Engineering Research Council (NSERC) of Canada.

[†]Department of Electrical and Computer Engineering, University of Waterloo, Waterloo, ON, N2L 3G1 Canada. E-mail: cnelsen@uwaterloo.ca.

normal form just described is called local transverse feedback linearization (LTFL). This terminology originated with the work of Andrzej Banaszuk and John Hauser in [3].

1.1. Contributions. The contributions of the paper are the following. 1) The results in [20], [21] are extended to the partial information case in Theorem 5.3. In that work we assumed that the full state of the control system is available for feedback and that, in particular, the local transverse output was permitted to be a function of the entire state. In this paper we assume that the only information available for feedback is modeled by a fixed output. We refer to this as the partial information case. This work is complementary to the papers [2], [17], [18] [25], [27]. In those papers output feedback controllers are designed for systems in a given normal form. The main result of this work, motivated by set stabilization problems, provides necessary and sufficient conditions under which a system can be brought into the aforementioned normal form in which the control design techniques can be applied. 2) Sufficient conditions under which a global version of the problem can be solved are presented in Section 6.1. 3) In Section 6.2 we show how the results of this paper can be used to facilitate control design for systems affected by unmeasured disturbances. A preliminary version of this paper appeared in [22].

2. Motivating example. Consider a system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \end{bmatrix} = \begin{bmatrix} x_4 \\ -x_3 - x_2^3 \\ x_2 \\ 0 \\ x_1 \end{bmatrix} + \begin{bmatrix} x_1 \\ 0 \\ 0 \\ 1 \\ x_5 \end{bmatrix} u \quad (2.1)$$

with output

$$y = h(x) = \begin{bmatrix} x_4 \\ x_5 \end{bmatrix}. \quad (2.2)$$

We are interested in locally stabilizing the invariant set

$$\Gamma^* = \{x \in \mathbb{R}^5 : x_1 = x_4 = x_5 = 0\}.$$

As discussed in Section 1, the most direct approach to locally stabilizing this set using output feedback is to seek an observable function that yields, in a neighbourhood of a point $x_0 \in \Gamma^*$, a well-defined relative degree whose associated zero dynamics manifold coincides with Γ^* . If such a function exists, then zeroing the function locally solves the set stabilization problem (if the trajectories of the closed-loop system are bounded). Furthermore, if the function is observable, system (2.1) is feedback equivalent to a system that fits the framework of well-known and “standard” output feedback control design approaches [2], [27] that can zero the observable function.

Specifically, in this example we seek a function $\lambda : \mathbb{R}^5 \rightarrow \mathbb{R}$ that, in a neighbourhood of a point $x_0 \in \Gamma^*$, has the following properties.

1. The function λ yields a well-defined relative degree at $x_0 \in \Gamma^*$.
2. The zero dynamics manifold of System (2.1) with output λ coincides with the target set Γ^* in a neighbourhood of x_0 .
3. The function λ is observable. In other words, the function λ can be expressed as a composition $\lambda = \tilde{\lambda} \circ h$ of a sufficiently smooth function $\tilde{\lambda} : \mathbb{R}^2 \rightarrow \mathbb{R}$ with the output (2.2).

A natural first attempt to finding a function with the aforementioned properties is to check if any of the constraints that define Γ^* satisfy the conditions enumerated above. In this example neither of the functions x_1 and x_5 yield a well-defined relative degree at any point on Γ^* . Furthermore, x_1 is not observable. The constraint function x_4 does yield a well-defined relative degree and is observable, however the zero dynamics manifold associated to the output x_4 does not equal Γ^* . Hence making $x_4 \rightarrow 0$ does not ensure that the set Γ^* is locally attractive. These facts mean that it is not clear whether or not a function λ that satisfies the three conditions above exists and, therefore, it is not clear whether or not the above program can be carried out.

The main contribution of this paper is to provide, given a control system with fixed output and an invariant set, necessary and sufficient conditions for the existence of a function λ that satisfies the three conditions listed above. In this example the function

$$\lambda(x) = \tilde{\lambda} \circ h(x) = x_5 e^{-x_4} \quad (2.3)$$

meets the above criteria with $\tilde{\lambda}(y) = y_2 e^{-y_1}$. We now illustrate how this function facilitates output feedback stabilization of Γ^* .

Using the observable function (2.3) define the coordinate transformation

$$\begin{bmatrix} \eta_1 \\ \eta_2 \\ \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix} := \begin{bmatrix} x_2 \\ x_3 \\ x_5 e^{-x_4} \\ x_1 e^{-x_4} \\ x_4 e^{-x_4} \end{bmatrix}$$

which, by the inverse function theorem, is a diffeomorphism of a neighbourhood of any point $x \in \mathbb{R}^5$. The system in (η, ξ) -coordinates reads

$$\begin{bmatrix} \dot{\eta}_1 \\ \dot{\eta}_2 \\ \dot{\xi}_1 \\ \dot{\xi}_2 \\ \dot{\xi}_3 \end{bmatrix} = \begin{bmatrix} -\eta_2 - \eta_1^3 \\ \eta_1 \\ \xi_2 \\ \xi_3 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \phi(\eta, \xi) u$$

where $\phi(\eta, \xi) = (1 - x_4) e^{-x_4}|_{x=T^{-1}(\eta, \xi)}$ and where $\xi_1 = \lambda(x)$ is available for feedback. For this system there are various approaches one can take to stabilize the ξ -subsystem. For example, one can view the stabilization problem as the study of a system with unknown high frequency gain for which the techniques in [23] along with the switching strategy in [11] can be employed to stabilize Γ^* . Alternatively, using the results in [2], there exists a dynamic feedback that stabilizes Γ^* using only measurements of ξ_1 . The ‘‘high-gain’’ observer used in [2] takes the form

$$\dot{\hat{\xi}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \hat{\xi} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \phi_0(\hat{\xi}) u + \begin{bmatrix} \frac{\alpha_1}{\varepsilon} \\ \frac{\alpha_2}{\varepsilon^2} \\ \frac{\alpha_3}{\varepsilon^3} \end{bmatrix} (\xi_1 - \hat{\xi}_1)$$

where $\hat{\xi} \in \mathbb{R}^3$ is an estimate of ξ , $\phi_0(\hat{\xi}) = (1 - \hat{\xi}_3)$ is the nominal, inexact, model of $\phi(\eta, \xi)$, $\varepsilon > 0$ is a high-gain parameter and the constants α_i are chosen so that the polynomial $s^3 + \alpha_1 s^2 + \alpha_2 s + \alpha_3$ is Hurwitz. The control law is chosen as

$$u = \frac{1}{\phi_0(\hat{\xi})} \left(-k_1 \hat{\xi}_1 - k_2 \hat{\xi}_2 - k_3 \hat{\xi}_3 \right)$$

with $k_i > 0$, $i \in \{1, 2, 3\}$.

The applicability of the above approach to output feedback control design depends crucially on the existence of the observable function (2.3). Therefore, a key challenge in output feedback stabilization of invariant sets is finding such a function and, most importantly, in determining whether it exists or not. This paper completely solves the latter question for single-input systems.

3. Preliminaries. This section presents the notation used throughout the paper. Section 3.2 contains supporting material needed to prove the main result. Section 3.3 provides definitions for the concept of invariance used in this paper and the Lie derivative and Lie bracket.

3.1. Notation. Let $\text{col}(x_1 \dots, x_k) := [x_1 \ \dots \ x_n]^\top$ where \top denotes transpose. Let x and y be two column vectors, define $\text{col}(x, y) := [x^\top \ y^\top]^\top$. If $x \in \mathbb{R}^n$ then $\|x\|$ denotes the Euclidean norm. If \mathcal{V} and \mathcal{W} are subspaces of the finite-dimensional vector space \mathcal{X} , the notation $\mathcal{V} \oplus \mathcal{W}$ (internal direct sum) represents the subspace $\mathcal{V} + \mathcal{W}$ when \mathcal{V} and \mathcal{W} are independent.

If f is a scalar-valued function from an open set $U \subseteq \mathbb{R}^n$ into \mathbb{R} , and k times continuously differentiable for at every $x \in U$, then f is of differentiability class C^k on U , denoted $f \in C^k(U)$ or $f \in C^k$ when the domain of f is clear. If f is C^k for all k , then f is C^∞ or smooth. If $f : U \subseteq \mathbb{R}^n \rightarrow V \subseteq \mathbb{R}^m$ is a continuously differentiable map, then for each $x \in U$, the derivative of f at x , denoted df_x , is a linear map $df_x : \mathbb{R}^n \rightarrow \mathbb{R}^m$. Its matrix representation is the Jacobian matrix of f evaluated at x . If U is an open set of \mathbb{R}^n , let $\text{Diff}(U)$ denote the family of diffeomorphism with domain U .

For brevity, the term submanifold is used in place of embedded submanifold of \mathbb{R}^n . If M is a smooth manifold and $p \in M$, we denote by $T_p M$ the tangent space to M at p and by TM the tangent bundle of M . The cotangent space to M at p is denoted by $T_p^* M$ and the cotangent bundle is written as $T^* M$.

DEFINITION 3.1. *A smooth distribution D on a manifold M is an assignment to each $p \in M$ of a subspace $D(p) \subseteq T_p M$ which varies smoothly as a function of p . A point $p \in M$ is a regular point of the smooth distribution D if there exists a neighbourhood U containing p for which $\dim(D(q))$ is constant for all $q \in U$. In this case, D is said to be nonsingular on U . Similarly, a codistribution Ω on M assigns at each $p \in M$ a subspace $\Omega(p) \subseteq T_p^* M$.*

Given a smooth distribution D , we let $\text{inv}(D)$ be its involutive closure (the smallest involutive distribution containing D). The codistribution $\text{ann}(D)$ is the annihilator of D , i.e., an assignment to each $p \in M$ of a subspace $\text{ann}(D)(p) \subseteq T_p^* M$ with the property that if $\sigma \in \text{ann}(D)(p)$ and $\tau \in D(p)$, then $\sigma(\tau) = 0$.

3.2. Vector bundles. If D is a distribution defined on \mathbb{R}^n and N is a submanifold we at times consider objects like $TN + D$ and $TN \cap D$. These objects are examples of real vector bundles, more precisely, subbundles of $T\mathbb{R}^n|_N$. They are defined, for each $p \in N$, by $T_p N + D(p)$ and $T_p N \cap D(p)$, respectively. These subbundles, and the operations on them, can be defined formally using the framework of vector bundles [10, 15, 26].

DEFINITION 3.2. *A n -dimensional (real) vector bundle is a map*

$$\pi : E \rightarrow B$$

of manifolds E and B such that, for any $b \in B$, the inverse image $\pi^{-1}(b)$ has the structure of the n -dimensional vector space \mathbb{R}^n having the following property of local

triviality: For each $b \in B$, there exists a neighbourhood U of b in B and a diffeomorphism

$$h : \pi^{-1}(U) \rightarrow U \times \mathbb{R}^n$$

such that for every $b' \in U$ the assignment of $x \in \pi^{-1}(b')$ to $h(x) = (b', \hat{h}(x))$ is an isomorphism of $\pi^{-1}(b')$ to $\{b'\} \times \mathbb{R}^n$. The manifold E is called the total space, B is called the base space and the vector space $E_b := \pi^{-1}(b)$ is called the fibre over b .

All of the vector bundles encountered in this paper are finite-dimensional and real. We typically denote a vector bundle (π, E, B) by E alone. Given a bundle (π, E, B) , for each $b \in B$ we can replace the fibre $\pi^{-1}(b)$ with different vector spaces. In this paper we will only consider the simplest case, we replace each vector space $\pi^{-1}(b)$ with its dual space.

DEFINITION 3.3. Let $\xi = (\pi, E, B)$ be a vector bundle. The dual bundle to ξ , is $\xi^* = (\pi^*, E^*, B)$ where

$$E^* := \bigcup_{b \in B} (\pi^{-1}(b))^*,$$

and $\pi^* : E^* \rightarrow B$ is the natural projection $\pi^* : (\pi^{-1}(p))^* \mapsto p$.

When this construction is applied to the tangent bundle TM of a manifold M , the resulting bundle is the cotangent bundle T^*M of M .

DEFINITION 3.4. Let $\eta = (\pi_F, F, B)$ be a subbundle of the smooth vector bundle $\xi = (\pi, E, B)$. The annihilator $\text{ann}(\eta)$ of η , is the subbundle of ξ^* whose fibres are defined at each $b \in B$ by

$$\text{ann}(F_b) := \{e^* \in E_b^* : e^*(f) = 0, \forall f \in F_b\}.$$

Recall that, if \mathcal{X} is a finite dimensional vector space, then $(\mathcal{X}^*)^* =: \mathcal{X}^{**}$ is canonically isomorphic to \mathcal{X} . Using this fact, and applying Definition 3.4 twice to the vector bundle ξ , we obtain the following.

PROPOSITION 3.5. Let $\xi = (\pi, E, B)$ be a smooth vector bundle over B . Then

$$\text{ann}(\text{ann}(\xi)) = \xi.$$

Proposition 3.5 implies that, if η^* is a subbundle of the cotangent bundle T^*M , then $\text{ann}(\eta^*)$ is a subbundle of TM , the tangent bundle to M . The following results, needed in this paper, can be found in [10, 15, 26].

PROPOSITION 3.6. Let $\xi = (\pi, E, B)$, $\xi_1 = (\pi_{E_1}, E_1, B)$ and $\xi_2 = (\pi_{E_2}, E_2, B)$ be vector bundles such that $\xi_2 \subseteq \xi_1 \subseteq \xi$, then,

$$\text{ann}(\xi) \subseteq \text{ann}(\xi_1) \subseteq \text{ann}(\xi_2) \subseteq \xi^*.$$

PROPOSITION 3.7. Let $\xi_1 = (\pi_{E_1}, E_1, B)$, $\xi_2 = (\pi_{E_2}, E_2, B)$ be subbundles of the smooth vector bundle $\xi = (\pi, E, B)$. If $\xi_1 + \xi_2 = (\pi_F, F, B)$ is also a subbundle of ξ , then

$$\text{ann}(\xi_1 + \xi_2) = \text{ann}(\xi_1) \cap \text{ann}(\xi_2).$$

3.3. Invariant sets, Lie derivatives, Lie brackets. Denote the set of all C^∞ -vector fields on a smooth manifold M by $\mathbf{V}(M)$. Given $v \in \mathbf{V}(M)$ and a point $p \in M$, we denote the maximal integral curve, or flow, generated by the vector field v through the point p as $\phi_t^v(x)$.

DEFINITION 3.8. *A set $N \subset M$ is said to be invariant under $v \in \mathbf{V}(M)$ if*

$$(p \in N) \Rightarrow (\forall t \geq 0)(\phi_t^v(p) \in N).$$

The property of invariance in Definition 3.8 is sometimes called positive or forward invariance because N is invariant for $t \geq 0$. When N is a closed submanifold invariance for $t \geq 0$ is equivalent to invariance for $t \in \mathbb{R}$. If N is an n -dimensional submanifold of M expressed as $N = \{p \in M : \phi(p) = 0\}$, where $\phi(p) = \text{col}(\phi_1(p), \dots, \phi_{m-n}(p))$ is a smooth map $M \rightarrow \mathbb{R}^{m-n}$, and 0 is a regular value of ϕ , then there is a particularly simple criterion for invariance.

THEOREM 3.9. *Let $\phi : M \rightarrow \mathbb{R}^{m-n}$ be a smooth map, and 0 be a regular value of ϕ . Let $v \in \mathbf{V}(M)$, then, $N = \phi^{-1}(0)$ is invariant under v if, and only if,*

$$(\text{d}\phi_i)_p(v(p)) = 0$$

for all $i \in \{1, \dots, m-n\}$ and all $p \in \phi^{-1}(0)$.

Geometrically, the theorem asserts that N is invariant under v if and only if v is tangent to N , everywhere on N . The same is true for general closed submanifolds of M .

DEFINITION 3.10. *If $v \in \mathbf{V}(M)$ and $\lambda \in C^\infty(M)$ then the derivative of λ along v is a function $L_v\lambda : M \rightarrow \mathbb{R}$ defined by*

$$L_v\lambda(p) = \lim_{h \rightarrow 0} \frac{1}{h} [\lambda(\phi_h^v(p)) - \lambda(p)]$$

and called the Lie or directional derivative of λ along v at p . It is an element of $C^\infty(M)$.

DEFINITION 3.11. *If $f, g \in \mathbf{V}(M)$, then the Lie bracket of f and g is a vector field $[f, g] \in \mathbf{V}(M)$ defined by the relation*

$$(\forall \lambda \in C^\infty(M)) \quad L_{[f, g]}\lambda = L_f(L_g\lambda) - L_g(L_f\lambda).$$

Definitions 3.10 and 3.11 are implicit in that they do not directly indicate how to compute, respectively, the Lie derivative and Lie bracket. If $\lambda \in C^\infty(\mathbb{R}^n)$ and $v \in \mathbf{V}(\mathbb{R}^n)$ then $L_v\lambda(x)$ is computed as

$$L_v\lambda(x) = (\text{d}\lambda)_x(v(x)).$$

If $f, g \in \mathbf{V}(\mathbb{R}^n)$, the Lie bracket of f and g is computed as

$$[f, g](x) = \text{d}g_x(f(x)) - \text{d}f_x(g(x)),$$

where $\text{d}f_x, \text{d}g_x$ are the derivative maps of the vector functions $f, g : \mathbb{R}^n \rightarrow \mathbb{R}^n$. We use the following standard notation for iterated Lie derivatives and Lie brackets

$$\begin{aligned} L_g^0\lambda &:= \lambda, & L_g^k\lambda &:= L_g(L_g^{k-1}\lambda), \\ L_g L_f\lambda &:= L_g(L_f\lambda), \\ \text{ad}_f^0 g &:= g, & \text{ad}_f^k g &:= [f, \text{ad}_f^{k-1}g], \quad k \geq 1. \end{aligned}$$

4. Problem formulation. Consider a control system modeled by equations of the form

$$\dot{x} = f(x) + g(x)u. \quad (4.1)$$

Here $x \in \mathbb{R}^n$ is the state and $u \in \mathbb{R}$ is the control input. The vector fields f and $g : \mathbb{R}^n \rightarrow T\mathbb{R}^n$ are smooth (C^∞). Suppose that the state x is not available for feedback but, rather, the only available information is given by a smooth vector output

$$y = h(x), \quad h : \mathbb{R}^n \rightarrow \mathbb{R}^p. \quad (4.2)$$

We assume that the component functions $\text{col}(h_1(x), \dots, h_p(x))$ of the output $h(x)$ are linearly independent, i.e., we assume that dh_x has rank p for all $x \in \mathbb{R}^n$. Define the following distributions associated with control system (4.1)

$$\mathcal{G}_i := \text{span}\{ad_f^j g : 0 \leq j \leq i\}. \quad (4.3)$$

To the output (4.2) we associate the nonsingular, involutive, $(n - p)$ -dimensional distribution

$$\mathcal{W} := \text{ann}(\text{span}\{dh_1, \dots, dh_p\}). \quad (4.4)$$

Suppose that we are given a submanifold $\Gamma^* \subset \mathbb{R}^n$ of dimension $0 < n^* < n$ which is either invariant under the vector field $f(x)$ in (4.1) or controlled invariant, i.e., it can be made invariant by appropriate choice of smooth feedback.

DEFINITION 4.1. *A closed connected submanifold $N \subset \mathbb{R}^n$ is called controlled invariant for (4.1) if there exists a smooth feedback $\bar{u} : N \rightarrow \mathbb{R}$ making N an invariant set for the closed-loop system.*

In this paper, as in [21], we treat the controlled invariant set Γ^* as given data. Often, however, one is given a set $\Gamma \subset \mathbb{R}^n$, perhaps defined by virtual constraints or design goals, and then one must pare away pieces of Γ until all that remains is the maximal controlled invariant submanifold Γ^* contained in Γ . We now state the problem considered in this paper.

Local Transverse Feedback Linearization with Partial Information (LT-FLPI) Problem : Given a smooth single-input system (4.1) with smooth output (4.2), a closed, connected, embedded, n^* -dimensional controlled invariant submanifold $\Gamma^* \subset \mathbb{R}^n$ and a point $x_0 \in \Gamma^*$, find, if possible, a diffeomorphism $\Xi \in \text{Diff}(U)$

$$\begin{aligned} \Xi : U &\rightarrow \Xi(U) \subset (\Gamma^* \cap U) \times \mathbb{R}^{n-n^*} \\ x &\mapsto (\eta, \xi) \end{aligned} \quad (4.5)$$

where U is a neighbourhood of x_0 , such that

- (i) The restriction of Ξ to $\Gamma^* \cap U$ is

$$\Xi|_{\Gamma^* \cap U} : x \mapsto (\eta, 0).$$

- (ii) The dynamics of system (4.1) in (η, ξ) -coordinates reads

$$\begin{aligned} \dot{\eta} &= f_0(\eta, \xi) \\ \dot{\xi} &= A\xi + b(a_1(\eta, \xi) + a_2(\eta, \xi)u), \end{aligned} \quad (4.6)$$

where the pair (A, b) is in Brunovský normal form (one chain of integrators) and $a_2(\eta, \xi) \neq 0$ in $\Xi(U)$.

- (iii) The first component of ξ , denoted ξ_1 is observable, i.e., there exists a function $\tilde{\lambda} : h(U) \subseteq \mathbb{R}^p \rightarrow \mathbb{R}$ such that

$$\xi_1(x) = \tilde{\lambda} \circ h(x).$$

As illustrated in Section 2, solving LTFPLI is relevant for stabilizing the set Γ^* using output feedback. To understand this claim, suppose that LFTPLI is solvable at $x_0 \in \Gamma^*$. Let $\lambda(x) := \tilde{\lambda} \circ h(x)$. Using this function we partially define the diffeomorphism $\Xi(x)$ by letting $\xi := \text{col}(\lambda(x), L_f \lambda(x), \dots, L_f^{n-n^*-1} \lambda(x))$. Choose $n - n^*$ additional independent functions $\eta_i := \phi_i(x)$, $i \in \{1, \dots, n - n^*\}$, to complete the coordinate transformation $\Xi : U \rightarrow \mathbb{R}^{n^*} \times \mathbb{R}^{n-n^*}$, $x \mapsto (\eta, \xi)$. In the single-input case, since $\text{ann}(\text{span}\{g\})$ is spanned by exact differentials, the functions $\phi_i(x)$ can always be chosen (see [12]) so that their time derivative along the control system do not depend on u , i.e., so that for all $x \in U$ and all $i \in \{1, \dots, n^*\}$, $L_g \phi_i(x) = 0$. After applying this coordinate transformation, in (η, ξ) -coordinates the system is modeled by equation (4.6).

If the entire state x is available for feedback, as in the full information case, then the regular feedback transformation $u = -\frac{a_1(\eta, \xi)}{a_2(\eta, \xi)} + \frac{v}{a_2(\eta, \xi)}$ yields a system of the form

$$\begin{aligned} \dot{\eta} &= f_0(\eta, \xi) \\ \dot{\xi} &= A\xi + bv \end{aligned} \tag{4.7}$$

and we say that system (4.1) has been locally transversely feedback linearized with respect to the set Γ^* . In this case, stabilizing the subspace $\xi = 0$ in (η, ξ) -coordinates corresponds to stabilizing the set $\Gamma^* \cap U$ in original coordinates (if the trajectories of the closed-loop system are bounded). For this reason we call the ξ -subsystem of (4.6) the transverse dynamics of (4.1) with respect to Γ^* . Stabilizing $\xi = 0$ can be achieved easily using the auxiliary control input v since the pair (A, b) is controllable. On the target set, the system dynamics are governed by the ordinary differential equation

$$\dot{\eta} = f_0(\eta, 0). \tag{4.8}$$

For this reason the dynamics (4.8) are called the tangential dynamics of (4.1) with respect to Γ^* .

In the partial information case the state x is not available for feedback, the only available information is given by the output function (4.2). In this case (η, ξ) is not available for feedback, the feedback transformation above cannot be implemented, and it may be impossible to stabilize the ξ subsystem. In the partial information case the ξ -subsystem before feedback transformation is

$$\begin{aligned} \dot{\xi}_1 &= \xi_2 \\ &\dots \\ \dot{\xi}_{n-n^*-1} &= \xi_{n-n^*} \\ \dot{\xi}_{n-n^*} &= a_1(\eta, \xi) + a_2(\eta, \xi)u. \end{aligned} \tag{4.9}$$

Since $\xi_1 = \lambda(x) = \tilde{\lambda}(h(x))$, it is available for feedback. For system (4.6) with ξ_1 measured and $a_2(\eta, \xi)$ sign-definite, the results in [2], [17], [18] [25], [27] assert the existence of a dynamic feedback

$$\begin{aligned} \dot{\zeta} &= \varphi(\zeta, \xi_1) \\ u &= \rho(\zeta, \xi_1) \end{aligned}$$

capable of stabilizing the origin of (4.9). The selection of an appropriate output feedback design framework depends crucially on the properties of tangential system (4.8). In the simplest case, the results of [2] can be used whenever $\phi(\eta, \xi, u) := a_1(\eta, \xi) + a_2(\eta, \xi)u$ and $f_0(\eta, \xi)$ are locally Lipschitz with $\phi(0, 0, 0) = 0$, $f_0(0, 0) = 0$ and the tangential dynamics (4.8) are minimum phase. Alternatively, if practical stability is sought then the results of [27, Section 6] can be used, again provided the tangential subsystem is minimum phase.

In the cases when the tangential dynamics do not necessarily converge to zero but remain otherwise bounded, the results in [17] are relevant. In [17] a weak-minimum phase assumption is made on the tangential subsystem and the function $a_1(\eta, \xi)$ is not necessarily known. Furthermore, they require that the tangential dynamics $f_0(\eta, \xi)$ have the form $f_0(\eta, \xi_1)$. Sufficient conditions for this additional property to hold are given in Corollary 6.2. Finally, in cases where the tangential system has the form $f_0(\eta, \xi_1)$ and $\frac{\partial a_2(\eta, \xi)}{\partial \xi_i} \equiv 0$, $i \in \{2, \dots, n - n^*\}$ and $\frac{\partial a_2(\eta, \xi)}{\partial \eta} \equiv 0$, the results in [4], see also [1], are applicable.

Once a system is expressed in the normal form (4.6) there are many other output stabilization techniques one can consider. The survey [1] gives an excellent overview of the available techniques while also classifying them as direct or indirect approaches. Conceptually, the direct design approach is preferable because the state feedback stabilizing control law is known and therefore the estimation scheme can focus on estimating the control signal directly. On the other hand, the indirect approach is far more common in the research literature and in particular, the approach used in Section 2 is an example of “domination via a dominant model” [1]. Motivated by these observations, we seek conditions guaranteeing the existence of an observable transverse output function.

5. Main result. The next result, an obvious consequence of [20, Theorem 4.1] or [21, Theorem 3.1], shows that LFTLPI is solvable if and only if there exists a “virtual output” function yielding a well-defined relative degree.

THEOREM 5.1. *LFTLPI is solvable at $x_0 \in \Gamma^*$ if and only if there exists a smooth \mathbb{R} -valued function $\tilde{\lambda}$, defined on a neighbourhood of $h(x_0)$ in \mathbb{R}^p satisfying*

- (a) *for some neighbourhood U of $x_0 \in \mathbb{R}^n$, $\Gamma^* \cap U \subseteq \{x \in U : \tilde{\lambda} \circ h(x) = 0\}$, and*
- (b) *the system*

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y' &= \lambda(x) = \tilde{\lambda} \circ h(x) \end{aligned} \tag{5.1}$$

has relative degree $n - n^$ at x_0 .*

Moreover, if LFTLPI is solvable, then there exists a neighbourhood $V \subseteq U$ of x_0 such that, on V , a connected component \mathcal{Z}^ of the zero dynamics manifold of (5.1) coincides with $\Gamma^* : \mathcal{Z}^* \cap V = \Gamma^* \cap V$.*

The proof of Theorem 5.1 is omitted because it is almost identical to the proof of [20, Theorem 4.1].

DEFINITION 5.2. *Let $\tilde{\lambda}$ be a smooth \mathbb{R} -valued functions satisfying the conditions of Theorem 5.1. The map $\lambda(x) := \tilde{\lambda} \circ h(x)$ is called a local observable transverse output of (4.1), (4.2) with respect to Γ^* .*

The main result of this paper, presented next, gives necessary and sufficient conditions for the existence of an observable transverse output.

THEOREM 5.3. *Suppose that $\text{inv}(\mathcal{G}_{n-n^*-2} + \mathcal{W})$ is regular at $x_0 \in \Gamma^*$. Then LFTLPI is solvable at x_0 for system (4.1) if and only if*

- (a) $T_{x_0}\Gamma^* \oplus \mathcal{G}_{n-n^*-1}(x_0) = T_{x_0}\mathbb{R}^n$
(b) *there exists an open neighbourhood U of x_0 in \mathbb{R}^n such that, $(\forall x \in \Gamma^* \cap U)$,*

$$\dim(T_x\Gamma^* \oplus \mathcal{G}_{n-n^*-2}(x)) = \dim(T_x\Gamma^* \oplus \text{inv}(\mathcal{G}_{n-n^*-2} + \mathcal{W})(x)).$$

Proof. Suppose that LTFLPI is solvable at $x_0 \in \Gamma^*$. Condition (a) is coordinate and feedback invariant so it suffices to show that it holds for system (4.7). Let $V := \Xi(\Gamma^* \cap U)$. By the properties of the normal form (4.6), $\Xi(x_0) = \text{col}(p_0, 0)$. Hence in (η, ξ) -coordinates

$$T_{p_0}V + \mathcal{G}_{n-n^*-1}(\text{col}(p_0, 0)) = \text{Im} \left(\begin{bmatrix} I_{n^*} & \star & \star & \dots & \star \\ 0_{n-n^* \times n^*} & b & Ab & \dots & A^{n-n^*-1}b \end{bmatrix} \right).$$

Since (A, b) is a controllable pair it immediately follows that $T_pV + \mathcal{G}_{n-n^*-1}(\text{col}(p, 0)) = T_p\mathbb{R}^n$. Furthermore, since (4.1) is a single-input system and by the definition (4.3) of \mathcal{G}_i , $\dim(\mathcal{G}_{n-n^*-1}) \leq n - n^*$, and therefore the subspaces $T_{x_0}\Gamma^*$ and $\mathcal{G}_{n-n^*-1}(x_0)$ are independent which proves that condition (a) is necessary.

We are left to show that condition (b) is necessary. Since LTFLPI is solvable and $\xi_1(x) = \lambda(x) = \tilde{\lambda} \circ h(x)$, we have $\Gamma^* \cap U = \{x \in \mathbb{R}^n : \xi(x) = 0\} \subseteq \{x \in \mathbb{R}^n : \lambda(x) = 0\}$ so that, for all $x \in \Gamma^* \cap U$ and for any $v \in T_x\Gamma^*$, $L_v\lambda(x) = 0$. This implies that $d\lambda \in \text{ann}(T\Gamma^*)$. Furthermore, since $\lambda(x)$ yields a well-defined relative degree of $n - n^*$ at x_0 , for any x in an open neighbourhood of x_0 , without loss of generality U , $L_g\lambda(x) = L_{ad_f g}\lambda(x) = \dots = L_{ad_f^{n-n^*-2}g}\lambda(x) = 0$ and $L_{ad_f^{n-n^*-1}g}\lambda(x) \neq 0$. This means that, in a neighbourhood of x_0 , without loss of generality U ,

$$d\lambda \in \text{ann}(\mathcal{G}_{n-n^*-2}), \quad d\lambda \notin \text{ann}(\mathcal{G}_{n-n^*-1}).$$

By the chain rule, $d\lambda_x = d\tilde{\lambda}_{h(x)} \circ dh_x$, so that, for any vector field $w \in \mathcal{W}$, and all $x \in U$,

$$d\lambda_x(w(x)) = d\tilde{\lambda}_{h(x)} \circ dh_x(w(x)) = 0.$$

In other words, $d\lambda \in \text{ann}(\mathcal{W}) = \text{span}\{dh_1, \dots, dh_p\}$. This shows that, in U ,

$$\begin{aligned} d\lambda &\in \text{ann}(\mathcal{G}_{n-n^*-2}) \cap \text{ann}(\mathcal{W}) \\ &\Rightarrow \mathcal{G}_{n-n^*-2} + \mathcal{W} \subseteq \text{ann}(d\lambda). \end{aligned}$$

The distribution $\text{ann}(d\lambda)$ is involutive since its annihilator is spanned by smooth, exact one-forms. Therefore $\text{inv}(\mathcal{G}_{n-n^*-2} + \mathcal{W}) \subseteq \text{ann}(d\lambda)$ and by Proposition 3.6,

$$d\lambda \in \text{ann}(\text{inv}(\mathcal{G}_{n-n^*-2} + \mathcal{W})).$$

This shows that, on $\Gamma^* \cap U$, $d\lambda \in \text{ann}(T\Gamma^*) \cap \text{ann}(\text{inv}(\mathcal{G}_{n-n^*-2} + \mathcal{W}))$. Thus, by Proposition 3.7,

$$d\lambda \in \text{ann}(T\Gamma^* + \text{inv}(\mathcal{G}_{n-n^*-2} + \mathcal{W}))$$

which implies that on $\Gamma^* \cap U$,

$$\dim(\text{ann}(T\Gamma^* + \text{inv}(\mathcal{G}_{n-n^*-2} + \mathcal{W}))) \geq 1. \quad (5.2)$$

Therefore, by (5.2) and Proposition 3.5, at any point on $x \in \Gamma^* \cap U$

$$\dim(T_x\Gamma^* + \text{inv}(\mathcal{G}_{n-n^*-2} + \mathcal{W})(x)) < n.$$

We have already shown that condition (a) is necessary and therefore on $\Gamma^* \cap U$,

$$\dim(T_x \Gamma^* \oplus \mathcal{G}_{n-n^*-2}(x)) = n - 1. \quad (5.3)$$

Therefore, by (5.2) and (5.3), we have that for any point in $\Gamma^* \cap U$

$$\begin{aligned} n - 1 &= \dim(T_x \Gamma^* + \mathcal{G}_{n-n^*-2})(x) \\ &\leq \dim(T_x \Gamma^* + \mathcal{G}_{n-n^*-2}(x) + \mathcal{W}(x)) \\ &\leq \dim(T_x \Gamma^* + \text{inv}(\mathcal{G}_{n-n^*-2} + \mathcal{W})(x)) < n. \end{aligned}$$

which proves the necessity of condition (b).

We now turn to the proof of sufficiency. Conditions (a) and (b), and the regularity of $\text{inv}(\mathcal{G}_{n-n^*-2} + \mathcal{W})$ at x_0 imply that $T\Gamma^* \cap \text{inv}(\mathcal{G}_{n-n^*-2} + \mathcal{W})$ is a smooth nonsingular distribution near x_0 . Using an argument identical to that in the proof of [21, Lemma 4.5], it can be shown that, by taking U sufficiently small, there exists a smooth nonsingular distribution $\mathcal{G}^\parallel \subset \text{inv}(\mathcal{G}_{n-n^*-2} + \mathcal{W})$ on U enjoying the two properties

$$\begin{aligned} (\forall x \in U) \quad \text{inv}(\mathcal{G}_{n-n^*-2} + \mathcal{W})(x) &= \mathcal{G}^\parallel(x) \oplus \mathcal{G}_{n-n^*-2}(x), \\ (\forall x \in \Gamma^* \cap U) \quad \mathcal{G}^\parallel \Big|_{\Gamma^* \cap U}(x) &= T_x \Gamma^* \cap \text{inv}(\mathcal{G}_{n-n^*-2} + \mathcal{W})(x). \end{aligned}$$

Let $V := \Gamma^* \cap U$ and let w_1, \dots, w_μ be a set of local generators for \mathcal{G}^\parallel on U . Similarly, there exist $n^* - \mu$ vector fields $\{v_1, \dots, v_{n^*-\mu}\}$, $v_i : V \rightarrow TV$, such that, for all $x \in V$,

$$T_x V = \mathcal{G}^\parallel \Big|_V(x) \oplus \text{span}\{v_1, \dots, v_{n^*-\mu}\}(x).$$

Now we have a collection of n linearly independent vector fields,

$$\underbrace{\underbrace{\underbrace{v_1, \dots, v_{n^*-\mu}}_{TV/\mathcal{G}^\parallel|_V}, \underbrace{w_1, \dots, w_\mu}_{\mathcal{G}^\parallel}}_{TV}, \underbrace{g, \dots, ad_f^{n-n^*-2}g, ad_f^{n-n^*-1}g}_{\mathcal{G}_{n-n^*-2}}}_{T\mathbb{R}^n} \quad (5.4)$$

which we use to generate a coordinate transformation analogous to that used in Theorem [21, Theorem 3.2]. We work our way from left to right in the list (5.4) starting with the group of vector fields spanning $TV/\mathcal{G}^\parallel|_V$. Define the map $S_\emptyset := (s_1, \dots, s_{n^*-\mu}) \mapsto \Phi_{S_\emptyset}^\emptyset(x_0)$ as

$$\Phi_{S_\emptyset}^\emptyset(x_0) = \phi_{s_{n^*-\mu}}^{v_{n^*-\mu}} \circ \dots \circ \phi_{s_1}^{v_1}(x_0).$$

Next define $S^\parallel := (s_1^\parallel, \dots, s_\mu^\parallel) \mapsto \Phi_{S^\parallel}^\parallel(x)$, as

$$\Phi_{S^\parallel}^\parallel(x) := \phi_{s_\mu^\parallel}^{w_\mu} \circ \dots \circ \phi_{s_1^\parallel}^{w_1}(x),$$

and the map $S^\natural := (s_0^\natural, \dots, s_{n-n^*-2}^\natural) \mapsto \Phi_{S^\natural}^\natural(x)$, as

$$\Phi_{S^\natural}^\natural(x) := \phi_{s_0^\natural}^g \circ \dots \circ \phi_{s_{n-n^*-2}^\natural}^{ad_f^{n-n^*-2}g}(x).$$

Finally, let $s := (S_{\emptyset}, s_{n-n^*-1}^{\uparrow}, S^{\uparrow}, S^{\parallel}) \mapsto \Phi_s(x_0)$, with domain a neighbourhood U of $s = 0$, be defined as

$$\Phi_s(x_0) := \Phi_{S^{\parallel}}^{\parallel} \circ \Phi_{S^{\uparrow}}^{\uparrow} \circ \phi_{s_{n-n^*-1}^{\uparrow}}^{ad_f^{n-n^*-1}g} \circ \Phi_{S_{\emptyset}}^{\emptyset}(x_0). \quad (5.5)$$

Since the vector fields in the list (5.4) are linearly independent near x_0 , it follows from the inverse function theorem that there exists a neighbourhood U of $s = 0$ such that (5.5) is a diffeomorphism onto its image. Let

$$\lambda(x) = s_{n-n^*-1}^{\uparrow}(x). \quad (5.6)$$

We will not show that (5.6) yields a well-defined relative degree of $n - n^*$ at x_0 because the arguments are similar to the proof [21, Theorem 3.2]. Instead, we focus on showing that there exists a function $\tilde{\lambda}$ such that $\lambda = \tilde{\lambda}(h(x))$. The function (5.6) yields relative degree $n - n^*$ on near x_0 , so in particular

$$L_g \lambda(x) = L_{ad_f g} \lambda(x) = \cdots = L_{ad_f^{n-n^*-2} g} \lambda(x) = 0.$$

Furthermore, since $\Gamma^* \subset \lambda^{-1}(0)$, these facts imply that,

$$\begin{aligned} d\lambda \in \text{ann}(T\Gamma^*) \cap \text{ann}(\mathcal{G}_{n-n^*-2}) &= \text{ann}(T\Gamma^* + \mathcal{G}_{n-n^*-2}) \\ &= \text{ann}(T\Gamma^* + \text{inv}(\mathcal{G}_{n-n^*-2} + W)) \\ &\subseteq \text{ann}(T\Gamma^* + \mathcal{G}_{n-n^*-2} + W) \\ &= \text{ann}(T\Gamma^*) \cap \text{ann}(\mathcal{G}_{n-n^*-2} + W). \end{aligned}$$

Therefore,

$$\begin{aligned} d\lambda \in \text{ann}(\mathcal{G}_{n-n^*-2}) \cap \text{ann}(W) \\ = \text{ann}(\mathcal{G}_{n-n^*-2}) \cap \text{span}\{dh_1, \dots, dh_p\} \end{aligned}$$

so

$$d\lambda = \sum_{i=1}^p \sigma_i(x) dh_i(x)$$

which implies that $\lambda = \tilde{\lambda}(h(x))$ and $\sigma_i(x) = \frac{\partial \tilde{\lambda}}{\partial y_i} \Big|_{y=h(x)}$. \square

REMARK 5.4. *When the state x is available for feedback, i.e., $h(x) = x$, $\mathcal{W} = \{0\}$ and $\Gamma^* = \{x_0\}$ is an equilibrium point of the open-loop system, the conditions of Theorem 5.3 coincide with the necessary and sufficient conditions for solving the state-space exact feedback linearization problem [12, Theorem 4.2.3].*

REMARK 5.5. *The direct generalization of Theorem 5.3 to multi-input, multi-output systems gives sufficient, not necessary, conditions under which the MIMO version of LTFLPI is solvable. The MIMO proof of necessity is an open problem and the subject of future research.*

Example. We now return to the motivating example of Section 2 to illustrate the application of Theorem 5.3. In Section 2 we had a system of the form (4.1), (4.2) with $f(x) = \text{col}(x_4, -x_3 - x_2^3, x_2, 0, x_1)$, $g(x) = \text{col}(x_1, 0, 0, 1, x_5)$, $h(x) = \text{col}(x_4, x_5)$. The target set is given by $\Gamma^* = \{x \in \mathbb{R}^4 : x_1 = x_4 = x_5 = 0\}$. We now use Theorem 5.3 to justify the discussion from Section 2 and show that LTFLPI is solvable, for the given

system and set, in a neighbourhood of the origin. First note that $n^* = 1$ and that $(\forall x \in \Gamma^*) T_x \Gamma^* = \text{span}\{v_1, v_2\}(x) = \text{span}\{e_2, e_3\}$ where e_2 and e_3 are the second and third natural basis vectors for \mathbb{R}^5 . Furthermore

$$\mathcal{G}_2(x) = \text{span}\{g, ad_f g, ad_f^2 g\}(x) = \text{span}\left\{\begin{bmatrix} x_1 \\ 0 \\ 0 \\ 1 \\ x_5 \end{bmatrix}, \begin{bmatrix} x_4 - 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 - x_4 \end{bmatrix}\right\}.$$

Checking conditions of Theorem 5.3 we have that $T_0 \Gamma^* + \mathcal{G}_2(0) = T_0 \mathbb{R}^5 \simeq \mathbb{R}^5$ so condition (a) holds at $x_0 = 0$. In order to check condition (b) of Theorem 5.3 we first write $\mathcal{W} = \text{ann}(\text{span}\{dh_1, dh_2\}) = \text{span}\{w_1, w_2, w_3\}(x) = \text{span}\{e_2, e_3, e_1\}$. The distribution $\mathcal{G}_1 + \mathcal{W}$ is regular in a neighbourhood of the origin, it has dimension four is everywhere. Calculating the Lie brackets between the vector fields $g, ad_f g, w_1, w_2$ shows that, for all $x \in \mathbb{R}^5$, $\mathcal{G}_1 + \mathcal{W}(x) = \text{inv}(\mathcal{G}_1 + \mathcal{W})(x)$. This makes verifying that condition (b) of Theorem 5.3 holds easy to check. We conclude that there exists an observable function such that Theorem 5.1 holds and hence that LTFLPI is solvable.

In order to actually find the observable output, in this simple case, one can follow the semi-constructive procedure of Theorem 5.3. Construct the maps from the proof of Theorem 5.3 noting that, in a neighbourhood of x_0 , $T\Gamma^*/\mathcal{G}^\parallel = \{0\}$. We have

$$\Phi_{S^\parallel}^\parallel(x_0) = \phi_{s_2^\parallel}^{w_2} \circ \phi_{s_1^\parallel}^{w_1}(x_0) = \text{col}\left(x_0, s_1^\parallel + x_0, s_2^\parallel + x_0, x_0, x_0\right)$$

and $\Phi_{S_m}^\uparrow(x) = \phi_{s_0^\uparrow}^g \circ \phi_{s_1^\uparrow}^{ad_f g}(x)$. Hence the overall map (5.5) is given by $\Phi_s(x_0) = \Phi_{S^\parallel}^\parallel \circ \Phi_{S_m}^\uparrow \circ \phi_{s_2^\uparrow}^{ad_f^2 g}(x_0)$ where

$$\begin{aligned} s_0^\uparrow &\mapsto \phi_{s_0^\uparrow}^g(x) = \text{col}\left(e^{s_0^\uparrow} x_1, x_2, x_3, s_0^\uparrow + x_4, e^{s_0^\uparrow} x_5\right) \\ s_1^\uparrow &\mapsto \phi_{s_1^\uparrow}^{ad_f g}(x) = \text{col}\left((x_4 - 1)s_1^\uparrow + x_1, x_2, x_3, x_4, x_5\right) \\ s_2^\uparrow &\mapsto \phi_{s_2^\uparrow}^{ad_f^2 g}(x) = \text{col}\left(x_1, x_2, x_3, x_4, (1 - x_4)s_2^\uparrow + x_5\right). \end{aligned}$$

and therefore, with $x_0 = 0$,

$$\Phi_s(x_0) = \Phi_{S^\parallel}^\parallel \circ \Phi_{S_m}^\uparrow \circ \phi_{s_2^\uparrow}^{ad_f^2 g}(x_0) = \text{col}\left(-s_1^\uparrow e^{s_0^\uparrow}, s_1^\parallel, s_2^\parallel, s_0^\uparrow, s_2^\uparrow e^{s_0^\uparrow}\right).$$

Inverting this map yields

$$\text{col}\left(s_1^\parallel(x), s_2^\parallel(x), s_0^\uparrow(x), s_1^\uparrow(x), s_2^\uparrow(x)\right) = \text{col}\left(x_2, x_3, x_4, -x_1 e^{-x_4}, x_5 e^{-x_4}\right).$$

The observable transverse output is the function $\lambda(x) = s_2^\uparrow(x) = x_5 e^{-x_4}$. This function allows one to follow the design procedure in Section 2.

6. Extensions and applications. In this section we present extensions and applications of Theorem 5.3. In Section 6.1 we pose the global transverse feedback linearization problem with partial information in which, roughly speaking, one seeks a single coordinate transformation such that (4.1) is equivalent to the normal form (4.6) in a neighbourhood of the entire set Γ^* . We present sufficient conditions for the global problem to be solvable by restricting the geometry of the target set. In Section 6.2 we illustrate how these results can be applied to systems affected by disturbances that cannot be measured.

6.1. Global transverse feedback linearization with partial information.

The results of Section 5 are local, valid in a neighbourhood of a point on the target manifold Γ^* . In this section we seek a global solution. By global we mean a solution valid in a neighbourhood of Γ^* , not necessarily all of \mathbb{R}^n . The following is a global version of LTFLPI.

Global Transverse Feedback Linearization with Partial Information (GTFLPI) Problem : Given a smooth single-input system (4.1) with smooth output (4.2), a closed, connected, embedded, n^* -dimensional controlled invariant submanifold $\Gamma^* \subset \mathbb{R}^n$, find, if possible, a diffeomorphism $\Xi \in \text{Diff}(\mathcal{N})$

$$\begin{aligned} \Xi : \mathcal{N} &\rightarrow \Xi(\mathcal{N}) \subseteq \Gamma^* \times \mathbb{R}^{n-n^*} \\ x &\mapsto (\eta, \xi) \end{aligned} \quad (6.1)$$

where \mathcal{N} is a neighbourhood of Γ^* , such that

- (i) The restriction of Ξ to Γ^* is

$$\Xi|_{\Gamma^*} : x \mapsto (\eta, 0).$$

- (ii) The dynamics of system (4.1) in (η, ξ) -coordinates is given by (4.6) where the pair (A, b) is in Brunovský normal form (one chain of integrators) and $a_2(\eta, \xi) \neq 0$ in $\Xi(\mathcal{N})$.
- (iii) The first component of ξ , denoted ξ_1 is observable, i.e., there exists a function $\tilde{\lambda} : h(\mathcal{N}) \subseteq \mathbb{R}^p \rightarrow \mathbb{R}$ such that

$$\xi_1(x) = \tilde{\lambda} \circ h(x).$$

In order to solve GTFLPI we restrict the class of allowable target sets Γ^* .

ASSUMPTION 6.1. *The set Γ^* is diffeomorphic to a generalized cylinder, i.e., $\Gamma^* \simeq \mathbb{T}^k \times \mathbb{R}^{n^*-k}$, $k \in \{0, \dots, n^*\}$ where \mathbb{T}^k is the k -torus.*

Assumption 6.1 restricts the class of set considered in the global problem. However, there are many applications, most notably path following, where the set to be stabilized is a generalized cylinder [5]. This type of set arises when solving a path following problem for closed curves.

PROPOSITION 6.2. *Under Assumption 6.1, GTFLPI is solvable if*

- (a) $(\forall x \in \Gamma^*) T_x \Gamma \oplus \mathcal{G}_{n-n^*-1}(x) = T_x \mathbb{R}^n$
(b) *The distribution \mathcal{G}_{n-n^*-2} is non-singular and involutive in a neighbourhood of Γ^**
(c) *The distribution $\text{inv}(\mathcal{G}_{n-n^*-2} + \mathcal{W})$ is non-singular and, $(\forall x \in \Gamma^*)$,*

$$\dim(T_x \Gamma^* \oplus \mathcal{G}_{n-n^*-2}(x)) = \dim(T_x \Gamma^* \oplus \text{inv}(\mathcal{G}_{n-n^*-2} + \mathcal{W})(x)).$$

Proof. The proof of this result is based on the proof of [20, Theorem 4.4]. Assumption 6.1 and hypotheses (a) and (b) imply that the conditions of [20, Theorem 4.4] hold. This guarantees the existence of a diffeomorphism with properties (i) and (ii) in the GTFLPI problem. In particular it ensures the existence of a function $\lambda : \mathcal{N} \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ such that $\Gamma^* \subset \lambda^{-1}(0) = \{x \in \mathcal{N} : \lambda(x) = 0\}$ and which yields a uniform relative degree of $n - n^*$ over Γ^* .

Now, assume that (c) holds, then using similar arguments to those in Theorem 5.3 we have

$$\begin{aligned} d\lambda \in \text{ann}(T\Gamma^*) \cap \text{ann}(\mathcal{G}_{n-n^*-2}) &= \text{ann}(T\Gamma^* + \mathcal{G}_{n-n^*-2}) \\ &= \text{ann}(T\Gamma^* + \text{inv}(\mathcal{G}_{n-n^*-2} + W)) \\ &\subseteq \text{ann}(T\Gamma^* + \mathcal{G}_{n-n^*-2} + W) \\ &= \text{ann}(T\Gamma^*) \cap \text{ann}(\mathcal{G}_{n-n^*-2} + W). \end{aligned}$$

Therefore,

$$\begin{aligned} d\lambda &\in \text{ann}(\mathcal{G}_{n-n^*-2}) \cap \text{ann}(W) \\ &= \text{ann}(\mathcal{G}_{n-n^*-2}) \cap \text{span}\{dh_1, \dots, dh_p\} \end{aligned}$$

so

$$d\lambda = \sum_{i=1}^p \sigma_i(x) dh_i(x)$$

which implies that $\lambda = \tilde{\lambda}(h(x))$ with $\sigma_i(x) = \left. \frac{\partial \tilde{\lambda}}{\partial y_i} \right|_{y=h(x)}$ and therefore that $\xi_1(x) = \lambda(x)$ is observable. \square

The next result is partly motivated by the results in [17] as discussed in Section 4. Recall that once a system is represented in the normal form (4.6), the approach one uses to stabilize Γ^* using output feedback depends on the dynamics of the tangential subsystem (4.8). When the tangential dynamics do not necessarily converge to zero but remain bounded, one may use the results in [17]. However, in [17] the η -dynamics in (4.6) must only depend on the tangential states η and the observable transversal state ξ_1 . The next result gives sufficient conditions for this to be the case.

COROLLARY 6.3. *If, in addition to the hypothesis of Proposition 6.2,*

$$(d) \text{ For all } i, j \in \{0, \dots, n - n^* - 1\}, \left[ad_f^i g, ad_f^j g \right] = 0$$

then there exists a diffeomorphism solving the GTFLPI problem such that the dynamics of system (4.1) in (η, ξ) -coordinates is given by

$$\begin{aligned} \dot{\eta} &= f_0(\eta, \xi_1) \\ \dot{\xi} &= A\xi + b(a_1(\eta, \xi) + a_2(\eta, \xi)u), \end{aligned} \tag{6.2}$$

with (A, b) a controllable pair in Brunovský normal form.

Proof. By Proposition 6.2 there exists a real-valued function $\lambda : \mathcal{N} \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ with $\Gamma^* \subset \lambda^{-1}(0)$ and yields a uniform relative degree of $n - n^*$ over Γ^* . The remainder of the claim follows from the proof of [12, Proposition 9.1.1] and the assumption that (d) holds. In this case, since the vector fields $ad_f^i g$, $i \in \{1, \dots, n - n^* - 1\}$ are not assumed to be complete, we are only assured to obtain a diffeomorphism with the required properties in a neighbourhood of Γ^* . \square

6.2. Systems with unobservable disturbances. Theorem 5.3 finds applications on systems affected by disturbance signals that cannot be measured. Consider the single-input smooth control system

$$\begin{aligned} \dot{x} &= f(x, w) + g(x, w)u \\ \dot{w} &= s(w), \end{aligned} \tag{6.3}$$

with $x \in \mathbb{R}^n$, $w \in \mathbb{R}^k$ and control input $u \in \mathbb{R}$. The state w can be thought of as an unobservable disturbance generated by the dynamical system modeled by $\dot{w} = s(w)$. The variable x is assumed measured. Denote $\bar{n} := n + k$ and let $q := \text{col}(x, w)$. Re-writing (6.3) as

$$\dot{q} = F(q) + G(q)u \tag{6.4}$$

with $F(q) := \text{col}(f(x, w), s(w))$, $G(q) := \text{col}(g(x, w), 0)$ and with output

$$y = H(q), \quad H(q) = \begin{bmatrix} I_{n \times n} & 0_{n \times k} \end{bmatrix} q,$$

we immediately have the following corollary.

COROLLARY 6.4. *Suppose that $\Gamma^* \subset \mathbb{R}^{\bar{n}}$ is a closed, connected, embedded, n^* -dimensional controlled invariant submanifold for (6.3). Let $q_0 = (x_0, w_0) \in \Gamma^*$ and*

$$\mathcal{W} := \text{ann}(\text{span}\{dH_1, \dots, dH_n\}) = \text{span}\left\{\frac{\partial}{\partial w_1}, \dots, \frac{\partial}{\partial w_k}\right\}.$$

If the hypotheses of Theorem 5.3 hold, then there exists a diffeomorphism

$$\begin{aligned} \Xi : U &\longrightarrow \Xi(U) \\ q &\mapsto (\eta, \xi) \end{aligned}$$

where U is a neighbourhood of q_0 in $\mathbb{R}^{\bar{n}}$, such that system (6.3) is locally diffeomorphic to (4.6). Moreover, the set $\Gamma^ \cap U$ in (η, ξ) -coordinates is given by*

$$\Xi(\Gamma^* \cap U) = \{(\eta, \xi) \in \Xi(U) : \xi = 0\}.$$

Finally, the first component of ξ , denoted ξ_1 is observable.

As an application of these ideas, consider the path following problem in presence of disturbances. Path following problems can naturally be cast as set stabilization problems [20] and transverse feedback linearization has been effectively used to implement path following controllers [19]. When the control system is affected by unobservable disturbances, the results in this paper are more suitable to solving the path following problem.

Example. Consider a kinematic unicycle with unit translational velocity affected by an unobservable disturbance w described by the model

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} \cos(x_3) \\ \sin(x_3) \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} w. \quad (6.5)$$

Suppose that the disturbance w is generated by the exosystem

$$\begin{bmatrix} \dot{w}_1 \\ \dot{w}_2 \end{bmatrix} = \begin{bmatrix} w_2 \\ -w_1 \end{bmatrix}. \quad (6.6)$$

We assume that the disturbance satisfies, for all $t \in \mathbb{R}$, $\|w(t)\| < 1$. Our objective is to design a control law that makes the position (x_1, x_2) of the unicycle converge to the unit circle in the (x_1, x_2) -plane. The measured states are x_1, x_2, x_3 .

In the state space of (6.5), (6.6) we view the goal set as the maximal controlled invariant set contained in $\{(x, w) \in \mathbb{R}^3 \times \mathbb{R}^2 : x_1^2 + x_2^2 - 1 = 0\}$ which is given by

$$\Gamma^* = \{(x, w) \in \mathbb{R}^3 \times \mathbb{R}^2 : x_1^2 + x_2^2 - 1 = x_1 \cos(x_3) + x_2 \sin(x_3) + x_2 w_1 = 0\}. \quad (6.7)$$

Here, $n^* = \dim(\Gamma^*) = 3$. Writing $\Gamma^* = \gamma^{-1}(0)$ for the function $\gamma : \mathbb{R}^3 \times \mathbb{R}^2 \rightarrow \mathbb{R}^2$ defined by (6.7), one can verify that, under the assumption that $\|w\| < 1$, zero is a regular value of γ . The set (6.7) has two connected components, one corresponding to clockwise motion along the path, the other corresponding to counterclockwise motion. Since our results are local, a choice of $(x_0, w_0) \in \Gamma^*$ selects one of these two components.

To apply Corollary 6.4, set $q := \text{col}(x, w)$, and define $F(q) = \text{col}(\cos(x_3), \sin(x_3) + w_1, 0, w_2, -w_1)$, $G(q) = \text{col}(0, 0, 1, 0, 0)$, $H(q) = \text{col}(x_1, x_2, x_3)$. The question we ask is: can transverse feedback linearization be used to stabilize Γ^* using only y for feedback? The answer, of course, is yes. For, one can check that

the observable function $\lambda(q) = x_1^2 + x_2^2 - 1$ yields a well-defined relative degree of $2 = n - n^*$ at each $q \in \Gamma^*$. If we set $\xi_1(q) = \lambda(q)$, $\xi_2(q) = L_F \lambda(q)$ and let η_1, η_2, η_3 be any¹ three additional linearly independent functions, then, the local diffeomorphism $\Xi(q) = \text{col}(\eta(q), \xi(q))$ solves LTFLPI.

Next we confirm this observation using Corollary 6.4. In this case, with $q_0 = (x_0, w_0)$, conditions (a) and (b) of Theorem 5.3 become

- (a) $\dim(T_{q_0}\Gamma^* + \mathcal{G}_1(q_0)) = 5$
- (b) there exists an open neighbourhood U of q_0 in \mathbb{R}^5 such that

$$(\forall q \in \Gamma^* \cap U) \dim(T_q\Gamma^* + \mathcal{G}_0(q)) = \dim(T_q\Gamma^* + \text{inv}(\mathcal{G}_0 + \mathcal{W})(q)) = \text{constant},$$

where \mathcal{W} , in this case, is given by $\mathcal{W} = \text{ann}(\text{span}\{dH_1, dH_2, dH_3\}) = \text{span}\{e_4, e_5\}$ where e_4 and e_5 are the fourth and fifth natural basis vectors for \mathbb{R}^5 . At each $q \in \Gamma^*$ we have that

$$T_q\Gamma^* = \text{span} \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} x_2^2 \\ -x_1x_2 \\ 0 \\ \phi(q) \\ 0 \end{bmatrix}, \begin{bmatrix} -x_2\varphi(q) \\ x_1\varphi(q) \\ \phi(q) \\ 0 \\ 0 \end{bmatrix} \right\}$$

where $\phi(q) := x_1(\sin(x_3) + w_1) - x_2 \cos(x_3)$, $\varphi(q) := x_1 \sin(x_3) - x_2 \cos(x_3)$ are both non-zero on Γ^* if $\|w\| < 1$. Therefore

$$T_q\Gamma^* + \mathcal{G}_1(q) = \text{span} \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} x_2^2 \\ -x_1x_2 \\ 0 \\ \phi(q) \\ 0 \end{bmatrix}, \begin{bmatrix} -x_2\varphi(q) \\ x_1\varphi(q) \\ \phi(q) \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -\sin(x_3) \\ \cos(x_3) \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}.$$

The determinant of the matrix whose columns are the above basis vectors is given by $\phi(q)\varphi^2(q)$. If this determinant is non-zero then condition (a) holds. To this end, note that on the set (6.7)

$$\begin{aligned} & x_1 \cos(x_3) + x_2 \sin(x_3) + x_2 w_1 = 0 \\ \Rightarrow & x_2 (\sin(x_3) + w_1) = -x_1 \cos(x_3) \\ \Rightarrow & x_2^2 (\sin(x_3) + w_1)^2 = x_1^2 \cos^2(x_3) \\ \Rightarrow & x_2^2 (\sin(x_3) + w_1)^2 = (1 - x_2^2) \cos^2(x_3) \quad (\text{since } x_1^2 + x_2^2 - 1 = 0) \\ \Rightarrow & x_2 = \pm \frac{\cos(x_3)}{\sqrt{(\sin(x_3) + w_1)^2 + \cos^2(x_3)}}. \end{aligned}$$

Similarly, using $x_1^2 = 1 - x_2^2$,

$$x_1 = \mp \frac{\sin(x_3) + w_1}{\sqrt{(\sin(x_3) + w_1)^2 + \cos^2(x_3)}}.$$

¹If the functions η_1, η_2, η_3 are chosen so that $d\eta_1, d\eta_2, d\eta_3 \in \text{ann}(G(q))$, then we obtain the normal form (4.6). It is always possible to do this. If, on the other hand, $d\eta_1, d\eta_2, d\eta_3 \notin \text{ann}(G(q))$, then the control u will appear in the η subsystem.

Therefore, we have that

$$\phi(q) = \mp \frac{1 + 2w_1 \sin(x_3) + w_1^2}{\sqrt{(\sin(x_3) + w_1)^2 + \cos^2(x_3)}}, \quad \varphi(q) = \mp \frac{1 + w_1 \sin(x_3)}{\sqrt{(\sin(x_3) + w_1)^2 + \cos^2(x_3)}}$$

which, under the assumption that $\|w\| < 1$, are both non-zero.

To check condition (b) we first note that since \mathcal{G}_0 and W are constant distributions, $\mathcal{G}_0 + \mathcal{W} = \text{inv}(\mathcal{G}_0 + \mathcal{W})$. We have that, for any $q \in \Gamma^*$,

$$T_q \Gamma^* + \mathcal{G}_0(q) = \text{span} \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} x_2^2 \\ -x_1 x_2 \\ 0 \\ \phi(q) \\ 0 \end{bmatrix}, \begin{bmatrix} -x_2 \varphi(q) \\ x_1 \varphi(q) \\ \phi(q) \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\},$$

and

$$T_q \Gamma^* + \text{inv}(\mathcal{G}_0 + \mathcal{W})(q) = \text{span} \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} x_2^2 \\ -x_1 x_2 \\ 0 \\ \phi(q) \\ 0 \end{bmatrix}, \begin{bmatrix} -x_2 \varphi(q) \\ x_1 \varphi(q) \\ \phi(q) \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

Since, as already shown, when $\|w\| < 1$ the functions φ and ϕ are non-zero on Γ^* and since x_1 and x_2 are not equal to zero simultaneously, condition (b) is satisfied.

In (η, ξ) -coordinates the unicycle has the normal form (4.6). At this point a ‘‘high-gain’’ output feedback controller can be used to make the target set Γ^* attractive and have the unicycle traverse the desired path in the presence of unobservable disturbances.

7. Conclusions. In this paper we studied the problem of stabilizing a controlled invariant embedded submanifold in the state space of autonomous nonlinear control systems using output feedback. We studied the most natural approach to solving this problem : given a controlled invariant manifold (the target set), find an observable output function yielding a well-defined relative degree whose associated zero dynamics manifold locally coincides with the target set. We call this the local transverse feedback linearization problem with partial information. Necessary and sufficient conditions were presented under which this problem is solvable. We also presented a global solution to this problem in the case when the target set is a generalized cylinder. Finally we illustrated how this work may find applications in system affected by unobservable disturbances and to path following problems.

REFERENCES

- [1] V. ANDRIEU AND L. PRALY, *A unifying point of view on output feedback designs for global asymptotic stabilization*, *Automatica*, 45 (2009), pp. 1789–1798.
- [2] A. ATASSI AND H. KHALIL, *A separation principle for the stabilization of a class of nonlinear systems*, *IEEE Transactions on Automatic Control*, 44 (1999), pp. 1672–1687.
- [3] A. BANASZUK AND J. HAUSER, *Feedback linearization of transverse dynamics for periodic orbits*, *Systems and Control Letters*, 26 (1995), pp. 95–105.
- [4] C. BYRNES AND A. ISIDORI, *Asymptotic stabilization of minimum phase nonlinear systems*, *IEEE Transactions on Automatic Control*, 36 (1991), pp. 1122–1137.

- [5] L. CONSOLINI, M. MAGGIORE, C. NIELSEN, AND M. TOSQUES, *Path following for the PVTOL aircraft*, *Automatica*, 46 (2010), pp. 1284–1296.
- [6] E. DAVISON, *The robust control of a servomechanism problem for linear time-invariant multi-variable systems.*, *IEEE Transactions on Automatic Control*, 21 (1976), pp. 25–34.
- [7] M. EL-HAWWARY AND M. MAGGIORE, *Distributed circular formation stabilization for dynamic unicycles*, *IEEE Transactions on Automatic Control*, 58 (2013), pp. 149–162.
- [8] B. FRANCIS, *The linear multivariable regulator problem*, *SIAM Journal on Control and Optimization*, 15 (1977), pp. 486–505.
- [9] L. FREIDOVICH, A. ROBERTSSON, A. SHIRIAEV, AND R. JOHANSSON, *Periodic motions of the pendubot via virtual holonomic constraints: Theory and experiments*, *Automatica*, 44 (2008), pp. 785–791.
- [10] M. HIRSCH, *Differential Topology*, no. 33 in Graduate Texts in Mathematics, Springer - Verlag, New York, 1976.
- [11] A. ILCHMANN AND D. OWENS, *Threshold switching functions in high-gain adaptive control*, *IMA Journal of Mathematical Control and Information*, 8 (1991), pp. 409–429.
- [12] A. ISIDORI, *Nonlinear Control Systems*, Springer, New York, 3rd ed., 1995.
- [13] A. ISIDORI AND C. BYRNES, *Output regulation of nonlinear systems*, *IEEE Transactions on Automatic Control*, 25 (1990), pp. 131–140.
- [14] L. KRICK, M. BROUCKE, AND B. FRANCIS, *Stabilisation of infinitesimally rigid formations of multi-robot networks*, *International Journal of Control*, 82 (2009), pp. 423–439.
- [15] J. M. LEE, *Introduction to Smooth Manifolds*, Springer, New York, 2002.
- [16] A.-R. MANSOURI, *Topological obstructions to submanifold stabilization*, *IEEE Transactions on Automatic Control*, 55 (2010), pp. 1701–1703.
- [17] L. MARCONI, L. PRALY, AND A. ISIDORI, *Output stabilization via nonlinear Luenberger observers*, *SIAM Journal on Control and Optimization*, 45 (2007), pp. 2277–2298.
- [18] L. MARCONI, L. PRALY, AND A. ISIDORI, *Robust asymptotic stabilization of nonlinear systems with non-hyperbolic zero dynamics*, *IEEE Transactions on Automatic Control*, 55 (2010), pp. 907–921.
- [19] C. NIELSEN, C. FULFORD, AND M. MAGGIORE, *Path following using transverse feedback linearization: Application to a maglev positioning system*, *Automatica*, 46 (2010), pp. 585–590.
- [20] C. NIELSEN AND M. MAGGIORE, *Output stabilization and maneuver regulation: A geometric approach*, *Systems Control Letters*, 55 (2006), pp. 418–427.
- [21] ———, *On local transverse feedback linearization*, *SIAM J. Control and Optimization*, 47 (2008), pp. 2227–2250.
- [22] ———, *Local transverse feedback linearization with partial information for single-input systems*, in *International Conference of Numerical Analysis and Applied Mathematics (ICNAAM 2012)*, vol. 1479, Kos, Greece, September 2012, pp. 1454–1457.
- [23] R. D. NUSSBAUM, *Some remarks on a conjecture in parameter adaptive control*, *Systems & Control Letters*, 3 (1983), pp. 243 – 246.
- [24] A. POGROMSKY, G. SANTOBONI, AND H. NIJMEIJER, *Partial synchronization: from symmetry towards stability*, *Physica D. Nonlinear Phenomena*, 172 (2002), pp. 65–87.
- [25] F. D. PRISCOLI, A. ISIDORI, AND L. MARCONI, *A dissipativity-based approach to output regulation of non-minimum-phase systems*, *Systems & Control Letters*, 58 (2009), pp. 584 – 591.
- [26] M. SPIVAK, *A Comprehensive Introduction to Differential Geometry*, vol. 1, Publish or Perish, Houston, 3rd ed., 2005.
- [27] A. TEEL AND L. PRALY, *Tools for semiglobal stabilization by partial state and output feedback*, *SIAM Journal on Control and Optimization*, 33 (1995), pp. 1443–1488.