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Local observers on linear Lie groups with linear estimation error dynamics

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***Abstract*—This paper proposes local exponential observers for systems on linear Lie groups. We study two classes of systems. In the first class, the full state of the system evolves on a linear Lie group and is available for measurement. In the second class, only part of the system's state evolves on a linear Lie group and this portion of the state is available for measurement. In each case, we propose two different observer designs. We show that, depending on the observer chosen, local exponential stability of one of the two observation error dynamics, left- or right-invariant error dynamics, is obtained. For the first class of systems these results are developed by showing that the estimation error dynamics are differentially equivalent to a stable linear differential equation on a vector space. For the second class of system, the estimation error dynamics are almost linear. We illustrate these observer designs on an attitude estimation problem.**

I. Introduction

Observers for systems on Lie groups is an active area of research [1]. Interest in this research has been partially motivated by the problem of controlling mobile robots, and in particular, unmanned aerial vehicles (UAVs). Precise control of these systems requires accurate estimates of the orientation of a rigid-body using low cost on-board sensors [2]. Small autonomous robots usually undergo significant vibration and other disturbances, while being restricted to carrying only a basic light-weight sensor package. For this reason, high-frequency noise is often present in the sensor measurements of these robots. Nonlinear observers for systems on Lie groups are useful because, in certain cases, they can be used to filter out the sensor noise.

Recent work on full-state observers for systems on $\mathsf{SO}(3,\mathbb{R})$, describing rigid-body rotational kinematics, was done in [3]. The algorithms in [3] rely on a projection of the measurement error from the Lie group to its Lie algebra. The projected vector in the Lie algebra is then used to drive the observer to converge to the system trajectory. While this projection based approach does not work for systems on the general linear Lie group, $\mathsf{GL}(n,\mathbb{R})$, the work in [4] may contain ideas to extend these projection based observers to the general linear group.

For systems on $\mathsf{SO}(3,\mathbb{R})$ with partial state measurements, the paper [5] proposes an observer that uses measurements of the orientation and of the torque to estimate the angular velocity of the rigid-body. The papers [6], [7] propose globally exponentially convergent observers using partial state measurements. The work in [8] also uses partial state measurements in their observers. The paper [9] analyses the effect of noise on an attitude estimation observer. The authors of [10] propose observers for $\mathsf{SO}(n,\mathbb{R})$.

For systems on $\mathsf{SE}(3,\mathbb{R})$, describing rigid-body pose, full-state observers were proposed in [11], [12], [13]. For systems on $\mathsf{SL}(3,\mathbb{R})$, describing a homography transformation, partial-state observers were proposed in [14].

In this paper, we consider left-invariant systems on closed subgroups of the general linear Lie group, i.e., the group of all invertible, real $n \times n$ matrices. The output of the system is taken to be that portion of the state evolving on the linear Lie group. We first consider the case in which the entire state evolves on the Lie group. We call these Lie group full-state observers. Then we consider the case where the states evolving on the Lie group are only a subset of the systems entire state. We call these Lie group partial-state observers.

A recent breakthrough in observer design on the general linear Lie group was achieved in [15], where exponentially converging observers are proposed for left-invariant and right-invariant systems on arbitrary finite dimensional, connected Lie groups. The proposed exponential observer uses gradient-like driving terms, derived from cost functions of the Lie group measurement error. In this paper we propose an alternative to gradient-like observers. Our observers are noteworthy because they yield linear estimation error dynamics. A weakness of our result is that we only prove local exponential stability. The research in [16], [17] considers systems that evolve on a vector space, but are such that a certain Lie group action leaves the system equations unchanged. They show that, if the plant is invariant under the action of a Lie group, then a subset of its states can be redefined as evolving on this Lie group, at least locally.

A. Contributions

The contributions of this paper are as follows: 1) We design a full-state observer in Section III-A for left-invariant systems on any closed subgroup of the general linear Lie group. The resulting estimation error dynamics are linear, distinguishing it from other observers in the literature. 2) Section III-A proposes an exponential partial-state observer, for a larger class of system. This class of system has only a proper subset of its states evolving on a closed subgroup of the general linear group. 3) Section IV shows differential equivalence between differential equations on $\mathsf{GL}(n,\mathbb{R})$. 4) The effectiveness of the proposed observers is illustrated via simulation in Section VI. An extended version of this paper is available [18].

B. Notation and Preliminaries

The symbols I_n and 0_n denote the $n \times n$ identity matrix and $n \times n$ zero matrix respectively. If $A \in \mathbb{R}^{n\times n}$ then $A^\top$ denotes the transpose of A and $\mathrm{trace}(A)$ denotes its trace. We denote by $\mathsf{GL}(n,\mathbb{R})$ the general linear Lie group of all invertible $n \times n$ matrices with real entries. We denote by $\mathsf{M}(n,\mathbb{R})$ the algebra of all $n \times n$ matrices with real entries. The bilinear product that makes $\mathsf{M}(n,\mathbb{R})$ an algebra is the matrix commutator, i.e., given $A, B \in \mathsf{M}(n,\mathbb{R})$, the product of A and B is $[A,B] := AB - BA$. For matrices $A \in \mathsf{M}(n,\mathbb{R})$ and $X \in \mathsf{GL}(n,\mathbb{R})$, the adjoint map is $\mathrm{Ad}_X(A) := XAX^{-1}$.

For a vector $x \in \mathbb{R}^n$, $\|x\|$ denotes the Euclidean norm and for a matrix $A \in \mathbb{R}^{n\times n}$, the induced matrix norm is denoted $\|A\|$. Given a matrix $A \in \mathbb{R}^{n\times n}$ and a real scalar $r > 0$, define the open ball $B(A,r) := \left\{X \in \mathbb{R}^{n\times n} : \|A - X\| < r\right\}$.

Definition I.1. *A linear Lie group* G *is a closed subgroup of* $\mathsf{GL}(n,\mathbb{R})$.

For brevity, the term Lie group is used in place of linear Lie group throughout. The Lie algebra of a linear Lie group G is denoted by $\mathrm{Lie}(\mathsf{G})$.

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II. Problem Statements

We introduce the two problems studied in this paper. The first problem deals with kinematic systems on linear Lie groups while the second relates to dynamic systems on linear Lie groups.

A. Full state observers

Let $\mathsf{G} \subseteq \mathsf{GL}\,(n,\mathbb{R})$ be a linear Lie group. Consider the following system on G

$$\begin{aligned} \dot{X} &= Xu \\ Y &= X, \end{aligned} \tag{1}$$

where $u : \mathbb{R} \to \mathrm{Lie}(\mathsf{G})$ is the control input to the system, and $Y \in \mathsf{G}$ is the measured output of the system. System (1) is left-invariant. This means that, for any fixed matrix $A \in \mathsf{G}$, if we redefine the state as $Z := AX$, then the new state Z satisfies the same differential equation as X, i.e., $\dot{Z} = Zu$. We assume that the control signal u in (1) is such that the corresponding solution is unique and piecewise differentiable.

Assumption 1. *For any initial condition $X(0) \in \mathsf{G}$ the corresponding solution $X(t)$ is bounded*

$$(\forall\, X(0) \in \mathsf{G})\, (\exists\, B \geq 0)\, (\forall\, t \geq 0)\, \|X(t)\| \leq B.$$

Assumption 1 is automatically satisfied if the group G is compact, for example $\mathsf{G} = \mathsf{SO}\,(3,\mathbb{R})$.

Problem 1: *Given a left-invariant system (1) on a linear Lie group $\mathsf{G} \subseteq \mathsf{GL}\,(n,\mathbb{R})$ with input $u \in \mathrm{Lie}(\mathsf{G})$ such that Assumption 1 holds, design a state estimator with estimate $\hat{X} \in \mathsf{G}$, access to $Y \in \mathsf{G}$ and $u \in \mathrm{Lie}(\mathsf{G})$, such that, for $\hat{X}(0)$ sufficiently close to $X(0)$, $\hat{X}(t) \longrightarrow X(t)$ exponentially, as $t \to \infty$.*

The results of this paper can be extended to right-invariant systems on Lie groups. However, we restrict the discussion to left-invariant systems to avoid repetition and for clarity.

B. Partial state observers

Consider the following system

$$\begin{aligned} \dot{X} &= Xx_2 \\ \dot{x}_2 &= x_3 \\ &\vdots \\ \dot{x}_d &= u \\ Y &= X, \end{aligned} \tag{2}$$

where $X \in \mathsf{G}$ evolves on a linear Lie group and $x_i \in \mathrm{Lie}(\mathsf{G})$, $i \in \{2,\ldots,d\}$. The input to (2) is $u : \mathbb{R} \to \mathrm{Lie}(\mathsf{G})$ which we assume is such that the corresponding solution is unique and piecewise differentiable.

Assumption 2. *For any initial condition $X(0) \in \mathsf{G}$, $x_2(0), \ldots, x_d(0) \in \mathrm{Lie}(\mathsf{G})$, the corresponding solution $(X(t), x_2(t), \ldots, x_d(t))$ of (2) is such that $X(t)$ is bounded*

$$(\forall\, X(0) \in \mathsf{G})\, (\exists\, B \geq 0)\, (\forall\, t \geq 0)\, \|X(t)\| \leq B.$$

Problem 2: *Given the system (1), design a state estimator with estimate $\hat{X} \in \mathsf{G}$, $\hat{x}_i \in \mathrm{Lie}\,(\mathsf{G})$, $i \in \{2,\ldots,d\}$, access to the output $Y \in \mathsf{G}$ and the input $u \in \mathrm{Lie}(\mathsf{G})$, such that, under Assumption 2, if $\|\hat{X}(0) - X(0)\|$, $\|\hat{x}_2(0) - x_2(0)\|$, …, $\|\hat{x}_d(0) - x_d(0)\|$ are sufficiently small, then $\|\hat{X}(t) - X(t)\| \to 0$, $\|\hat{x}_2(t) - x_2(t)\| \to 0$, …, $\|\hat{x}_d(t) - x_d(t)\| \to 0$ exponentially, as $t \to \infty$.*

III. Proposed observers

In this section we propose various observers that solve Problems 1 and 2. The analysis of the observers is presented in Section V where, using the results of Section IV, we provide conditions under which the observers solve Problems 1 and 2.

A. Local full state observers

For system (1), we propose two different observers, which we call local Lie group Full State Observers (LFSOs). The first is the passive LFSO, given by

$$\dot{\hat{X}} = \hat{X}u - a_0\hat{X}\log(Y^{-1}\hat{X}). \tag{3}$$

The second is the direct LFSO, given by

$$\dot{\hat{X}} = YuY^{-1}\hat{X} - a_0\hat{X}\log(Y^{-1}\hat{X}). \tag{4}$$

In the above two observers, the constant $a_0 > 0$ is a design parameter that, as we will show, can be used to change the rate of observer convergence. Following the terminology of [15], we call the term $\alpha(\hat{X}, Y) := -a_0\hat{X}\log(Y^{-1}\hat{X})$, appearing in (3) and (4), the innovation term of the observer. It can be verified that the term α satisfies the definition, given in [15, Definition 15], of an innovation term.

Remark III.1. *It is computationally costly and inefficient to compute the matrix logarithm map using the series definitions. Various studies have looked at the problem of approximating this computation. In particular the work [19], [20], [21], [22] may be useful for implementing the observers proposed in this paper. While we do not pursue the notion of using approximations to the matrix logarithm to implement the observers, we do observe, in Section VI, that in the special case $\mathsf{G} = \mathsf{SO}\,(3,\mathbb{R})$ the logarithm can be computed efficiently.*

B. Local partial state observers

For system (2), we propose two different observers which we call local Lie group Partial State Observers (LPSOs). The first is the direct LPSO, given by

$$\begin{aligned} \dot{\hat{X}} &= Y\hat{x}_2Y^{-1}\hat{X} - a_{d-1}\hat{X}\log(Y^{-1}\hat{X}) \\ \dot{\hat{x}}_2 &= \hat{x}_3 - a_{d-2}\log(Y^{-1}\hat{X}) \\ &\vdots \\ \dot{\hat{x}}_{d-1} &= \hat{x}_d - a_1\log(Y^{-1}\hat{X}) \\ \dot{\hat{x}}_d &= u - a_0\log(Y^{-1}\hat{X}) \end{aligned} \tag{5}$$

and the second is the passive LPSO, given by

$$\begin{aligned} \dot{\hat{X}} &= \hat{X}\hat{x}_2 - a_{d-1}\hat{X}\log(Y^{-1}\hat{X}) \\ \dot{\hat{x}}_2 &= \hat{x}_3 - a_{d-2}\log(Y^{-1}\hat{X}) \\ &\vdots \\ \dot{\hat{x}}_{d-1} &= \hat{x}_d - a_1\log(Y^{-1}\hat{X}) \\ \dot{\hat{x}}_d &= u - a_0\log(Y^{-1}\hat{X}). \end{aligned} \tag{6}$$

In the above two observers, the constants $a_0, \ldots, a_{d-1} \in \mathbb{R}$ are design parameters, chosen such that the polynomial $p(s) = s^d + a_{d-1}s^{d-1} + \cdots + a_1 s + a_0$ is Hurwitz. These design parameters can be used to modify the rate of convergence of the estimation error.

IV. Differential Equations on Matrices

In this section we study the properties of a pair of differential equations on linear Lie groups that arrise in the analysis of the error dynamics associated with the observers proposed in Section III.

A. A Differential Equation on $\mathsf{GL}(n,\mathbb{R})$

Consider the differential equation evolving on $\mathsf{GL}(n,\mathbb{R})$ given by

$$\dot{E} = -a_0 E \log(E), \tag{7}$$

where $E \in \mathsf{GL}(n,\mathbb{R})$ and $a_0 \in \mathbb{R}$ is a positive constant. The equation (7) arises in the analysis of the error dynamics associated with the observers (3), (4). The crucial property of the differential equation (7) is that the matrices $\dot{E}$ and E commute, i.e., $E\dot{E} = \dot{E}E$. This property is a consequence of matrices E and $\log(E)$ commuting. Commutativity of $\dot{E}$ and E, combined with the product rule, gives us the following result.

Lemma IV.1. *Let $E : \mathbb{R} \to \mathsf{GL}(n,\mathbb{R})$ be a curve in $\mathsf{GL}(n,\mathbb{R})$, such that E and $\dot{E}$ commute. Then for all positive integers k*

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[(E-I_n)^k\right] = k\dot{E}(E-I_n)^{k-1} = k(E-I_n)^{k-1}\dot{E}.$$

Lemma IV.1 is the key reason why (7) is differentially equivalent to a linear differential equation. The change of coordinates that realizes this equivalence is the matrix logarithm map defined on $B(I_n,1) \subset \mathsf{GL}(n,\mathbb{R})$. The matrix logarithm map $\log : B(I_n,1) \to \mathsf{M}(n,\mathbb{R})$ is a diffeomorphism onto its image. Furthermore, the codomain of the log map is the set $\mathsf{M}(n,\mathbb{R})$, which is isomorphic to $\mathbb{R}^{n^2}$, as a vector space. Therefore, the log map is a local coordinate transformation on $\mathsf{GL}(n,\mathbb{R})$, defined on the ball $B(I_n,1)$.

We denote by $e \in \mathsf{M}(n,\mathbb{R})$ the log coordinates of the matrix $E \in B(I_n,1)$

$$e := \log(E). \tag{8}$$

To express the differential equation (7) in log coordinates we differentiate e with respect to time, making use of Lemma IV.1, to obtain $\dot{e} = \sum_{k=0}^{\infty}(-1)^k(E-I_n)^k\dot{E} = E^{-1}\dot{E} = -a_0E^{-1}E\log(E) = -a_0e$. The resulting differential equation is linear with n^2 eigenvalues located at $-a_0$. Thus, for any positive constant $a_0 > 0$, the point $e = 0_n$ is an exponentially stable equilibrium of $\dot{e} = -a_0e$. Since stability of an equilibrium is a coordinate independent property, the equilibrium point $E = I_n$ is also locally exponentially stable for (7). The above discussion proves the following.

Lemma IV.2. *On the set $E \in B(I_n,1)$, the vector field (7) is differentially equivalent to the vector field*

$$\dot{e} = -a_0 e. \tag{9}$$

If $a_0 > 0$ then $E = I_n$ is locally exponentially stable.

B. A Differential Equation on $\mathsf{GL}(n,\mathbb{R})$ and $\mathsf{M}(n,\mathbb{R})$

Consider the following differential equation, which is a natural extension of the differential equation (7),

$$\begin{aligned} \dot{E} &= e_2E - a_{d-1}E\log(E) \\ \dot{e}_2 &= e_3 - a_{d-2}\log(E) \\ &\vdots \\ \dot{e}_{d-1} &= e_d - a_1\log(E) \\ \dot{e}_d &= -a_0\log(E), \end{aligned} \tag{10}$$

where $E \in \mathsf{GL}(n,\mathbb{R})$, $e_i \in \mathsf{M}(n,\mathbb{R})$ for $i = 2,\ldots,d$ and $a_0,\ldots,a_{d-1} \in \mathbb{R}$ are constants such that the polynomial $p(s) = s^d + a_{d-1}s^{d-1} + \cdots + a_1s + a_0$ is Hurwitz. System (10) arises in the analysis of the error dynamics associated with the direct LPSO (5).

In general the matrices E and $\dot{E}$ in (10) do not commute. This is because E and e_2 are generally non-commuting matrices. The non-commutativity of E and $\dot{E}$ means that, defining $e_1 := \log(E)$, the expression for $\dot{e}_1$ is not as simple as was the case for equation (7) in Section IV-A. In particular, we do not obtain a closed-form expression for $\dot{e}_1$. Instead we have the following, weaker, result.

Proposition IV.3. *In the open neighbourhood $B(I_n,1) \times (\mathsf{M}(n,\mathbb{R}))^{d-1}$ the differential equation (10) is differentially equivalent to*

$$\begin{aligned} \dot{e}_1 &= e_2 - a_{d-1}e_1 + K(e_1,e_2) \\ \dot{e}_2 &= e_3 - a_{d-2}e_1 \\ &\vdots \\ \dot{e}_{d-1} &= e_d - a_1e_1 \\ \dot{e}_d &= -a_0e_1, \end{aligned}$$

where $e_1 := \log(E)$ and $K : \mathrm{Lie}(G) \times \mathrm{Lie}(G) \to \mathrm{Lie}(G)$ is a smooth function that vanishes if e_1 and e_2 commute

$$(\forall\, e_1,e_2 \in \mathsf{M}(n,\mathbb{R})\ [e_1,e_2] = 0_n)\ K(e_1,e_2) = 0_n.$$

Proof: Since $e_1 = \log(E)$ is a Taylor series in E, term by term differentiation yields that $\dot{e}_1$ only depends on E and $\dot{E}$. Furthermore, from (10), we know that $\dot{E}$ only depends on E and e_2. Thus, using $E = \exp(e_1)$, we have that $\dot{e}_1$ only depends on e_1 and e_2. Let $K(e_1,e_2) := \dot{e}_1 - e_2 + a_{d-1}e_1$.

Assume that e_1 and e_2 commute. This implies that $E = \exp(e_1)$ and e_2 also commute and this implies that E and $\dot{E}$ commute. Since $E\dot{E} = \dot{E}E$, we can repeat almost the same analysis that we used in Section IV-A, doing this we get $\dot{e}_1 = \dot{E}E^{-1} = e_2 - a_{d-1}e_1$. Therefore $K(e_1,e_2) = 0_n$ for any commuting e_1 and e_2. The expressions of $\dot{e}_i$ for $i = 2,\ldots,d$ are computed by substituting $\log(E) = e_1$ into (10). ■

Lemma IV.4. *If the constants $a_0,\ldots,a_{d-1} \in \mathbb{R}$ are chosen such that the polynomial $p(s) = s^d + a_{d-1}s^{d-1} + \cdots + a_1s + a_0$ is Hurwitz then the equilibrium point $(E,e_2,\ldots,e_d) = (I_n,0_n,\ldots,0_n)$ of the differential equation (10) is locally exponentially stable.*

Proof: Adapting the proof of [14, Theorem 3.1 (ii)], we show that (10) is locally exponentially stable at the equilibrium point, by showing that its linearization, around the equilibrium point $(I_n,0_n,\ldots,0_n)$, is exponentially stable.

In a neighbourhood of the equilibrium point $(I_n,0_n,\ldots,0_n)$ define $\delta E := E - I_n$, $\delta e_2 := e_2 - 0_n$, …, $\delta e_d := e_d - 0_n$. Using the series definition of the matrix logarithm we deduce that, near $\delta E = 0_n$, $\log(E) \approx \delta E$. Similarly, using $E = \delta E + I_n$, and dropping higher order terms in δE, we get

$$E\log(E) = (\delta E + I_n)\left((\delta E) - \frac{1}{2}(\delta E)^2 + \cdots\right) \approx \delta E.$$

Finally, near the equilibrium point $(I_n,0_n,\ldots,0_n)$, $e_2E = (\delta e_2)(\delta E + I_n) \approx \delta e_2$. Substituting these approximations into the differential equation (10), we get the linearization of (10) at $(I_n,0_n,\ldots,0_n)$ has system matrix

$$\begin{pmatrix} -a_{d-1}I_n & I_n & 0_n & \ldots & 0_n & 0_n \\ -a_{d-2}I_n & 0_n & I_n & \ldots & 0_n & 0_n \\ -a_{d-3}I_n & 0_n & 0_n & \ldots & 0_n & 0_n \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -a_1I_n & 0_n & 0_n & \ldots & 0_n & I_n \\ -a_0I_n & 0_n & 0_n & \ldots & 0_n & 0_n \end{pmatrix}.$$

The eigenvalues of the system matrix are located at the roots of the polynomial $p(s) = s^d + a_{d-1}s^{d-1} + \cdots + a_1s + a_0$, with multiplicity n, for each (possibly repeating) root of $p(s)$. Since all the eigenvalues have negative real parts, the linearization above is exponentially stable. Therefore $(E,e_2,\ldots,e_d) = (I_n,0_n,\ldots,0_n)$ is a locally exponentially stable equilibrium of (10). ■

V. Estimation Error Dynamics

In this section we analyse the stability of the estimation error for each of the observers proposed in Section III. We show that, under Assumptions 1 and 2, the estimates exponentially converge to the state of the system.

A. Estimation Error Functions

Following [15], we introduce two canonical choices of estimation error functions for left-invariant systems on Lie groups.

Definition V.1. *Given system (1) with $X \in \mathsf{G}$, and an observer with state estimate $\hat{X} \in \mathsf{G}$, the canonical left-invariant error, $E_l : \mathsf{G} \times \mathsf{G} \to \mathsf{G}$, is*

$$E_l(X, \hat{X}) := X^{-1}\hat{X}$$

and the canonical right-invariant error, $E_r : \mathsf{G} \times \mathsf{G} \to \mathsf{G}$, is

$$E_r(X, \hat{X}) := \hat{X}X^{-1}.$$

In Problems 1 and 2 we seek to design observers so that $\|\hat{X} - X\| \to 0$ exponentially. To characterize this property we rely on the following result.

Proposition V.2. *Suppose that $X : \mathbb{R} \to \mathsf{G}$ is uniformly bounded. If either $E_r \to I_n$ exponentially, or $E_l \to I_n$ exponentially, as $t \to \infty$, then $\hat{X} \to X$ exponentially, as $t \to \infty$.*

Proof: For any X, $\hat{X} \in \mathsf{G}$, using Definition V.1, the following identities hold $\hat{X} - X = X(E_l - I_n)$, $\hat{X} - X = (E_r - I_n)X$. Taking the norms of these identities, we obtain

$$\begin{aligned} \|\hat{X} - X\| &\leq \|X\|\|E_l - I_n\| \\ \|\hat{X} - X\| &\leq \|X\|\|E_r - I_n\|. \end{aligned} \tag{11}$$

Additionally, for any $X, \hat{X} \in G$, $E_l - I_n = X^{-1}(\hat{X} - X)$, $E_r - I_n = (\hat{X} - X)X^{-1}$, so that

$$\begin{aligned} \|E_l - I_n\| &\leq \|X^{-1}\|\|\hat{X} - X\| \\ \|E_r - I_n\| &\leq \|X^{-1}\|\|\hat{X} - X\|. \end{aligned} \tag{12}$$

By hypothesis, $\|X(t)\|$ is uniformly bounded, i.e., $(\exists K_1 > 0)(\forall t \geq 0)\|X(t)\| \leq K_1$. This implies that $X(t)$ evolves on the compact subset $\mathcal{G} = \{X \in \mathsf{GL}(n, \mathbb{R}) : \|X\| \leq K_1\}$. Since the matrix inverse map is continuous, the image of $\mathcal{G}$ under the matrix inverse map is also a compact subset of $\mathsf{GL}(n, \mathbb{R})$. Therefore, $\|X^{-1}(t)\|$ is also uniformly bounded, i.e., $(\exists K_2 > 0)(\forall t \geq 0)\|X^{-1}(t)\| \leq K_2$.

Now suppose that $\|E_r(t) - I_n\| \to 0$ exponentially, as $t \to \infty$, then by the definition of exponential stability, we have $(\exists \delta, m, \alpha > 0)$ $(\forall E_r(0) \in B(I_n, \delta))$ $(\forall t \geq 0)$ $\|E_r(t) - I_n\| < me^{-\alpha t}\|E_r(0) - I_n\|$. By the inequalities (11), and uniform boundedness of $\|X\|$, we have that $\|E_r - I_n\| < m \Rightarrow \|\hat{X} - X\| < K_1 m$. By the inequalities (12), and uniform boundedness of $\|X^{-1}\|$, we have that $\|\hat{X} - X\| < \frac{\delta}{K_2} \Rightarrow \|E_r - I_n\| < \delta$. Combining the above results, we have exponential convergence of $\|\hat{X} - X\| \to 0$,

$$(\exists \delta, m, \alpha > 0) \left(\|\hat{X}(0) - X(0)\| < \delta/K_2\right) \\ (\forall t \geq 0) \;\; \|\hat{X}(t) - X(t)\| < K_1 K_2 m e^{-\alpha t}\|\hat{X}(0) - X(0)\|.$$

The proof for E_l is identical. ∎

Definition V.3. *For any $E_l \in B(I_n, 1)$, the log left-invariant error, $e_l : \mathsf{G} \times \mathsf{G} \to \mathrm{Lie}(\mathsf{G})$, is*

$$e_l(X, \hat{X}) := \log(E_l(X, \hat{X})) = \log(X^{-1}\hat{X}).$$

For any $E_r \in B(I_n, 1)$, the log right-invariant error, $e_r : \mathsf{G} \times \mathsf{G} \to \mathrm{Lie}(\mathsf{G})$, is

$$e_r(X, \hat{X}) := \log(E_r(X, \hat{X})) = \log(\hat{X}X^{-1}).$$

The variables e_l and e_r are useful because they are vectors in $\mathrm{Lie}(\mathsf{G})$ and they allow us to convert a differential equation on a Lie group into a differential equation on a vector space. The disadvantage of e_l and e_r is that they are only defined for E_l, $E_r \in B(I_n, 1)$.

Lemma V.4. *If $E_l, E_r \in B(I_n, 1)$, then*

$$e_r = X e_l X^{-1}.$$

Proof: By direct calculation and using elementary properties of the exp and log maps, we obtain $e_r = \log(E_r) = \log(XE_lX^{-1}) = X\log(E_l)X^{-1} = Xe_lX^{-1}$. ∎

Finally, in the context of partial state observers, since x_i and $\hat{x}_i$, for $i = 2, \ldots, d$ are vectors in $\mathrm{Lie}(\mathsf{G})$, to quantify the error between x_i and $\hat{x}_i$, we can use subtraction of vectors

$$e_i := x_i - \hat{x}_i, \quad i = 2, \ldots, d. \tag{13}$$

Since x_i and $\hat{x}_i$ are elements of the vector space $\mathrm{Lie}(\mathsf{G})$, e_i is also an element of $\mathrm{Lie}(\mathsf{G})$.

B. Local full state observers

We first analyze the dynamics of the error functions E_l and E_r under the observers defined by (3) and (4). We assume that $\hat{X}$ is initialized sufficiently close to X, so that E_l, $E_r \in B(I_n, 1)$.

1) Passive Observer: When the passive observer (3) is used to estimate the state of (1) dynamics of the right-invariant error, E_r are

$$\dot{E}_r = -a_0 E_r \log(E_r). \tag{14}$$

The above differential equation is formally the same as equation (7). Therefore if $\hat{X}$ is sufficiently close to X so that $E_r \in B(I_n, 1)$ then, by Lemma IV.2, system (14) is differentially equivalent to

$$\dot{e}_r = -a_0 e_r, . \tag{15}$$

By choosing $a_0 > 0$, Lemma IV.2 states the equilibrium point $E_r = I_n$ is locally exponentially stable for system (14). This discussion, in light of Proposition V.2, proves the following solution to Problem (1).

Corollary V.5. *For $E_r(0) \in B(I_n, 1)$, the passive observer (3) exponentially stabilizes $E_r = I_n$. Furthermore, under Assumption 1, the passive observer solves Problem (1).*

The convergence of E_r to I_n does not rely on the trajectories of (1) being bounded. Next, we examine the dynamics of the left-invariant error, E_l, to see if Assumption 1 can be weakened. The dynamics of the left invariant error E_l under the passive observer (3) are

$$\dot{E}_l = -a_0 E_l \log(E_l) + \delta_P(u, E_l), \tag{16}$$

where $\delta_P(u, E_l) := E_l u - u E_l$ is a perturbation term that vanishes when $E_l = I_n$. Since the matrices E_l and $\dot{E}_l$ do not, in general, commute, Lemma IV.1 does not hold for (16).

Next we transform the error dynamics (16) into log coordinates. Recall, by Lemma V.4, if $\hat{X}$ is sufficiently close to X then $e_l = X^{-1}e_rX$. Therefore to transform the dynamics (16) into e_l coordinates, we differentiate this alternate expression for e_l

$$\dot{e}_l = -a_0 e_l + [e_l, u]. \tag{17}$$

The above system, rewritten $\dot{e}_l = -a_0 e_l + [e_l, u]$, is bilinear. If $a_0 < 0$, then by [23, Corollary 4], system (17) is integral-input to state stable (iISS). Specifically, see [23], there exist class-$\mathcal{K}_\infty$ functions α, γ and a class-$\mathcal{KL}$ function β such that for any $e_l(0) \in \mathsf{M}(n, \mathbb{R})$, and any input $u(\cdot)$, $\alpha(\|e_l(t)\|) \leq \beta(e_l(0), t) + \int_0^t \gamma(\|u(\tau)\|)\mathrm{d}\tau$. As a result, if $u(t) \to 0_n$ as $t \to \infty$, then $e_l(t) \to 0_n$ as $t \to \infty$. Furthermore, if $\int_0^\infty \gamma(\|u(t)\|)\mathrm{d}t < \infty$, then by [23, Proposition 6], $e_l(t) \to 0_n$ as $t \to \infty$. Niether of these properties allow us to weaken Assumption 1. First, because we have no guarantees that the control signal satisfies the above properties and second, System (16) is only differentially equivalent to (17) if $E_l \in B(I_n, 1)$ and the iISS property does not ensure that $e_l \in \log(B(I_n, 1))$.

By showing that the system (16) is diffeomorphic to the system (17), we have found an easy way to prove the following, non-obvious, result.

Corollary V.6. *Let $G \subseteq \mathsf{GL}(n, \mathbb{R})$ be a linear Lie group and consider the system*

$$\dot{E} = [E, u], \tag{18}$$

where $E \in \mathsf{G} \subseteq \mathsf{GL}(n, \mathbb{R})$ is the state and $u \in \mathrm{Lie}(G) \subseteq \mathsf{M}(n, \mathbb{R})$ is an admissible input signal. On the open set $B(I_n, 1) \cap \mathsf{G}$, system (18) is differentially equivalent to

$$\dot{e} = [e, u], \tag{19}$$

where $e = \log(E)$.

Proof: Rewrite (18) as a difference of two vector fields

$$\dot{E} = ([E, u] + E\log(E)) - (E\log(E))$$

and let $f(E, u) := [E, u] + E\log(E)$ and $g(E) := E\log(E)$. Since the system (16) transforms into the system (17), we know that the vector field $f(E, u)$ transforms into $[e, u] + e$. Also, since the dynamics (7) transform into the dynamics (9), we know that the vector field $g(E)$ transforms into e. This means that the vector field $f(E, u) - g(E)$ transforms into $[e, u] + e - e = [e, u]$. ■

The analysis of equation (18) is facilitated by taking the systemic view of "splitting" the equation (18) into a pair consisting of "system" (1), with state X, and "observer" (3), with state $\hat{X}$. The splitting is done as $E = X^{-1}\hat{X}$, and allows us to convert the differential equation (18) into log coordinates.

2) Direct Observer: When the direct observer (4) is used to estimate the state of (1) dynamics of the left-invariant error E_l are

$$\dot{E}_l = -a_0 E_l \log(E_l). \tag{20}$$

The above equation (20) is the same as the equation (7), if we identify E_l with E. This means that if $\hat{X}$ is sufficiently close to X so that $E_l \in B(I_n, 1)$, then by Lemma IV.2, system (20) in e_l-coordinates reads

$$\dot{e}_l = -a_0 e_l. \tag{21}$$

If $a_0 > 0$, Lemma IV.2 states that the equilibrium point $E_l = I_n$ is locally exponentially stable for the dynamics (20).

Corollary V.7. *For $E_l(0) \in B(I_n, 1)$, the direct observer (3) exponentially stabilizes $E_l = I_n$. Furthermore, under Assumption 1, the passive observer solves Problem (1).*

As before, we seek to weaken Assumption 1 and hence we examine the dynamics of the right-invariant error E_r, when the direct observer is used

$$\dot{E}_r = \delta_D(u, X, E_r) - a_0 E_r \log(E_r). \tag{22}$$

Here, $\delta_D(u, X, E_r) := XuX^{-1}E_r - E_r XuX^{-1}$ is a perturbation term that vanishes when $E_r = I_n$. The above equation (22) has the same problem that we encountered when trying to analyze equation (16). Namely, the matrices E_r and $\dot{E}_r$ do not commute in general, because E_r and $\delta_D(u, X, E_r)$ do not commute in general. Fortunately, we can transform equation (16) into log coordinates by once again differentiating the identity $e_r = Xe_lX^{-1}$. To be able to do this, it is sufficient that the conditions of Lemma V.4 are satisfied, i.e., that $E_l, E_r \in B(I_n, 1)$. Doing so, one obtains

$$\dot{e}_r = -a_0 e_r + \left[XuX^{-1}, e_r\right]. \tag{23}$$

The above system, rewritten $\dot{e}_r = -a_0 e_r + \left[XuX^{-1}, e_r\right]$, is a non-autonomous, bilinear system. Once again, we cannot weaken the requirement of Assumption 1 and rely on Proposition V.2 to ensure that $E_l \to 0_n$ as $t \to \infty$ is equivalent to $E_r \to 0_n$ as $t \to \infty$.

C. Local partial state observers

We now analyze the estimation error dynamics when using the observers proposed in Section III-B and defined by (6) and (5).

1) Direct Observer: When the direct observer (5) is applied to estimate the state of system (2) the dynamics of the right-invariant error E_l are

$$\begin{aligned} \dot{E}_l &= e_2 E_l - a_{d-1} E_l \log(E_l) \\ \dot{e}_2 &= e_3 - a_{d-2}\log(E_l) \\ &\vdots \\ \dot{e}_{d-1} &= e_d - a_1 \log(E_l) \\ \dot{e}_d &= -a_0 \log(E_l). \end{aligned} \tag{24}$$

The above differential equation is formally the same as equation (10), if we identify E with E_l. Application of Lemma IV.4 immediately yields the following solution to Problem 2.

Corollary V.8. *For $(E_l, e_2, \ldots, e_d) \in B(I_n, I) \times (\mathsf{M}(n, \mathbb{R}))$, the direct observer (5) exponentially stabilizes $(I_n, 0_n, \ldots, 0_n)$. Furthermore, under Assumption 2, the direct observer solves Problem 2.*

2) Passive Observer: When the passive observer (6) is employed to estimate the state of system (2) the dynamics of the right-invariant error E_r are given by

$$\begin{aligned} \dot{E}_r &= E_r \,\mathrm{Ad}_X(e_2) - a_{d-1} E_r \log(E_r) \\ \dot{e}_2 &= e_3 - a_{d-2}\log(E_r) \\ &\vdots \\ \dot{e}_{d-1} &= e_d - a_1 \log(E_r) \\ \dot{e}_d &= -a_0 \log(E_r). \end{aligned} \tag{25}$$

Lemma IV.4 cannot be used to deduce the stability of the equilibrium point $(E_r, e_2, \ldots, e_d) = (I_n, 0_n, \ldots, 0_n)$. Unfortunately, we are not able to prove the stability of these error dynamics. We conjecture that the passive LPSO is locally exponentially convergent if Assumption 2 holds. This conjecture is supported by simulation, where the passive LPSO performs better than the direct LPSO, when a large amount of measurement noise is present in Y.

VI. Example : Dynamic Rigid-Body Orientation

Consider a dynamic model of a rotating rigid body

$$\begin{aligned} \dot{R} &= R\omega \\ \dot{\omega} &= u \\ Y &= R \end{aligned} \tag{26}$$

where $R \in \mathsf{SO}(3, \mathbb{R})$, ω and u are skew-symmetric matrices. Here $Y = R$ is directly measured as well as the angular acceleration u. A similar model was discussed in [24, Example 2]. For system (26), the proposed direct observer is

$$\begin{aligned} \dot{\hat{R}} &= Y\hat{\omega}Y^{-1}\hat{R} - a_1\hat{R}\log(Y^{-1}\hat{R}) \\ \dot{\hat{\omega}} &= u - a_0\log(Y^{-1}\hat{R}) \end{aligned} \tag{27}$$

Fig. 1: $\|\hat{R} - R\|$ versus time with $\sigma = 0.4$.

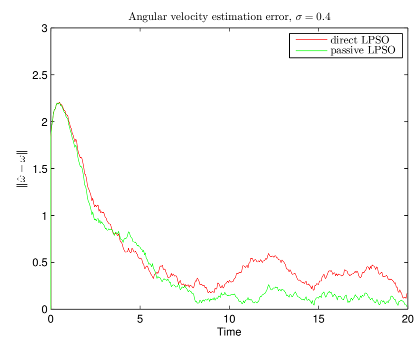

Fig. 2: $\|\hat{\omega} - \omega\|$ versus time with $\sigma = 0.4$.

and the passive LPSO is

$$\begin{aligned} \dot{\hat{R}} &= \hat{R}\hat{\omega} - a_1 \hat{R} \log(Y^{-1}\hat{R}) \\ \dot{\hat{\omega}} &= u - a_0 \log(Y^{-1}\hat{R}). \end{aligned} \tag{28}$$

We simulate the direct and the passive LPSOs, with increasing amounts of noise in the output. The initial conditions for the plant and the observer are chosen as

$$R(0) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad \omega(0) = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix},$$

$\hat{R}(0) = I_3$, $\hat{\omega}(0) = 0_3$. The angular acceleration input is chosen to be

$$u(t) = \begin{pmatrix} 0 & -2\sin(t) & \cos(t) \\ 2\sin(t) & 0 & -\sin(t) \\ -\cos(t) & \sin(t) & 0 \end{pmatrix}.$$

The observer gains are chosen as $a_0 = 1$, $a_1 = 2$. Noise is injected into the output via the random rotation matrix $N \in \mathsf{SO}(3, \mathbb{R})$, by setting $Y = RN$. The disturbance N is obtained as $\exp(n)$ where the elements of $n \in \mathrm{Lie}(\mathsf{SO}(3))$ are normally distributed, with zero-mean and standard deviation σ. The simulation results are shown in Figures 1, 2 with a large amount of measurement noise.

VII. Conclusions

We proposed observers for two classes of systems on linear Lie groups. The first class of system is one in which the entire state evolves on the general linear group and the entire state is measured. We have shown that if the systems state is bounded, then both the left- and right-invariant estimation errors are differentially equivalent to a stable LTI system and hence are locally exponentially stable. The second class of system is one in which only part of the state evolves on the general linear group and only this portion of the state is measured. We have shown that if the system's state is bounded, then the left and right estimation errors are locally exponentially stable using the direct observer. The passive observer was shown to work well in simulation in the presence of constant disturbances.

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