Risk Modelling in Shariah Compliant Investment and Insurance Products

by

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Author's Declaration

This thesis consists of material all of which I authored or co-authored: see Statement of Contributions included in the thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Statement of Contributions

Dila Puspita was the sole author for Chapters 1, 2, 3, and 5 which were written under the supervision of Prof. Adam Kolkiewicz and Prof. Ken Seng Tan and were not written for publication.

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As lead author of this chapter, I was the primary author and responsible for contributing to conceptualizing study design, carrying out numerical simulations and analysis, and drafting and submitting manuscripts. My coauthors who are my supervisors provided guidance during each step of the research and provided feedback on draft manuscripts.

Abstract

The main objective of this study is to develop a risk model for Shariah (Islamic) compliant investment and insurance. Islamic finance is a part of Socially Responsible Investment (SRI) that incorporates religious ethics. In the first part, we propose a portfolio optimization model that complies with Shariah rules. Motivated by the Markowitz's mean variance model, this study proposes a new portfolio optimization model that takes into consideration both processes of purification and screening, which are key to constructing a Shariah-compliant portfolio. The model reflects the stochastic nature of purification and applies to both investment and dividend purification. Recognizing the importance of on-going screening and that assets that subsequently become non-compliance during the investment horizon could adversely affect the portfolio strategy, we impose probabilistic constraints to control the risk of compliance change. We conduct an extensive empirical study using a sample of Shariah-compliant public companies listed on the Indonesia Stock Exchange. We evaluate our proposed model by formulating the probabilistic constraints at both asset and portfolio levels, together with three different financial screening divisors that are broadly used by the international Shariah boards. The empirical results demonstrate that the effect of imposing non-compliance probabilistic constraints is highly sensitive to the adopted divisors and that these constraints can be an effective way of mitigating the risk of compliance change, thereby avoiding involuntary asset liquidation, and enhancing the performance of the resulting Shariah-compliant portfolio.

We extend the Shariah portfolio model to the multiperiod problem consisting of a risk-free asset and risky assets in the second part of this thesis. We assume geometric Brownian motion to capture the dynamic of purified asset return, and we apply two assumptions for the dynamic of the screening financial ratio. In the first one, we assume that the financial ratios are independent and follow beta distributions. The second assumption relaxes the independence assumption by allowing the financial ratios to follow the Beta-AR(p,q) model. We also take into account other Shariah constraints, namely no-short selling and no-leverage restrictions. We solve the multiperiod mean-variance Shariah portfolio problem using a pre-commitment approach and adopt the multi-stage strategy following by Stochastic Grid Bundling Method (SGBM) to find the optimal asset allocations. Finally, we apply the proposed algorithm to generate Shariah portfolio efficient frontiers for multiperiod time cases to see the impact of each Shariah variable on the performance of the portfolio. The empirical simulations show that the proposed probabilistic Shariah screening constraint is efficient in maintaining the sustainability of the asset in the portfolio.

In the third part of this study, we construct a new risk model representing the Hybrid-Takaful (Islamic Insurance) framework and develop a computational procedure to calculate

the associated ruin probability. The Hybrid-Takaful business model applies a Wakalah (agent-based) contract for underwriting activities and Mudharabah (profit sharing) contracts for investment activities. We consider the existence of the qard-hasan facility provided by the operator (shareholder) as a benevolent loan for participants' fund in case of a deficit. This facility is a no-interest loan that will be repaid if the business generates profit in the future. For better investment management, we propose a separate investment account of participants' fund. We implement several numerical examples to analyze the impact of several parameters on the Takaful business model. We also find that our proposed Takaful model has a better performance compare with the conventional counterpart in terms of the probability of ruin. At the end of this thesis, we describe the conclusions and possible future research topic for each component of this thesis.

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Dedication

To my family.

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Chapter 1

Introduction

1.1 Motivation

Islamic (Shariah) finance is the only financial system in the world today that is based on the teaching of a dominant religion (Hassan and Mahlknecht, 2011). The practice of Islamic finance follows the principles of the Quran, Holy Book of Islam, the Hadith, which are the teachings and sayings of Prophet Muhammad, and Ijtihad, comprising scholarly legal deductions.Two fundamental principles in Islamic finance are the prohibition of interest and the rejection of financial speculation with the noble purpose of social justice. However, religion is not a prerequisite for the participation in Islamic finance. In fact, the Islamic finance are surfacing not just among Muslim countries, but also in Australia and other western countries. In particular the report by Trowers and Hamlins (2019) indicates that the UK is the top western centre for Islamic finance. Based on the GIFR (2019) report, the Islamic finance industry has a positive annual growth of 6.58% during 2018 with total asset USD 2.6 trillion at the end of 2018 in across 75 countries from its three main sectors namely: Islamic banking, Shariah capital market, and Islamic insurance (Takaful). According to the ICD (2019) report, the global Islamic assets are projected to increase to US$ 3.472 trillion by 2024.

In recent years the concept of the Socially Responsible Investment (SRI) has received much attention among private and institutional investors. To construct a SRI portfolio, the investors do not just rely on the optimal tradeoff between risk and reward, as dictated by the modern portfolio theory of Markowitz (1952), but they also take into considerations other factors, depending on the definition of being "socially responsible". For example, an environment activist concerned with climate change will be excluding carbon intensive companies, such as coal mining companies, from their investment portfolio. The Shariah-compliant investment is another class of SRI, which complies with the principles of Shariah (Islamic rule).

Shariah-compliant investment implies that the investment strategy must adhere to the following four Shariah principles (see Abbes, 2017): (i) Haram Income (prohibit to invest in business that is unlawful from the perspective of Islam); (ii) Riba (prohibit to earn interest on lending); (iii) Maysir (must avoid speculative or unnecessary risk); and (iv) Gharar (prohibit to take excessive risk that might lead to loss). From a portfolio construction point of view, ensuring a portfolio that complies with the Shariah principles is equivalent to imposing additional constraints on the portfolio construction. Therefore, a Shariah-compliant investment is more restrictive and with a smaller feasible set of investable assets. For these reasons, there is a cost to the compliance, as typically reflected in lower expected return. For example, the studies conducted by Gasser et al. (2017) conclude a statistically significant decrease in expected returns for SRI. See also Nainggolan et al. (2015).

In practice, most Shariah boards agree that a portfolio that adheres to Shariah requires two processes, known as the screening process and the purification process. The screening process ensures that assets in the portfolio remain in compliance throughout the entire investment horizon. It involves pre-screening and on-going monitoring, which can be qualitative or quantitative (see Chapter 2.1.1). The purification process recognizes that, in today's complex business environment, it may be difficult to find a company that is 100% compliant with Shariah principles. For examples, part of a hotel's revenue may be from serving alcohol or part of a company's revenue may be attributed to interest income from the cash investment. These activities are considered non-halal and the resulting income is classified as non-permissible income. However, Islamic investors are still allowed to invest in these companies as long as these activities do not exceed a certain threshold and that the non-permissible income is "treated" appropriately. The act of cleansing and deducting the non-permissible income from the investors' returns is known as the purification process (Marzban, 2011; Elgari, 2000; Gamaleldin, 2015). As to be explained in Chapter 2.1.2, the method of purification can be based on dividend or investment.

Because of these complications, some extensions of Markowitz (1952) have been proposed recently for constructing a Shariah-compliant portfolio. Hazny et al. (2012) review Markowtiz's mean-variance model and conclude that most of the assumptions are consistent with Shariah. Derbali et al. (2017) and Hazny et al. (2020) extend one period Markowtiz's model by imposing short-selling constraint, a constant purification rate and zakat rate. Loosely speaking, zakat can be interpreted as a form of religious tax with value typically set at 2.5%. The Islamic investors are expected to pay zakat from their investment gains as a way of giving back to the society. By interpreting zakat as a form of purification, Masri (2017) incorporates zakat and obtains a Shariah-compliant portfolio via a goal programming method involving chance-constrained and a recourse approach.

The aforementioned literature represents some earlier attempts to construct a Shariah-compliant portfolio. Derigs and Marzban (2008) show that different opinions and inconsistencies exist among the Shariah scholars and their compliance strategies. While some approaches (such as Masri, 2017) do not make distinction between zakat and purification, Marzban (2011) argue that this practice is inconsistent with Sharia as it is based on different concept and that the investors cannot use the non-permissible income from their investment to pay for zakat. The assumption of a constant purification rate made in Derbali et al. (2017) may not be reasonable due to the volatility of the business activities and returns. A more serious limitation is that the existing literature appears to have focused only on asset pre-screening and overlooked the importance of the on-going screening. A portfolio manager may implement an investment strategy based on an initial set of Shariah-compliant assets. As time evolves, however, some of these assets may no longer be Shariah compliant. When this occurs, the non-compliant asset(s) will need to be liquidated and the portfolio will need to be re-balanced to maintain Shariah compliance. The involuntary liquidation may adversely distort the performance of the adopted investment strategy, and hence it is prudent to integrate the risk of non-compliance to the investment decision making.

Besides investment, insurance is an integral part of the financial plan as efforts to keep away from potential adversities whenever such things occur. However, most Islamic scholars agree that conventional insurance is not acceptable under Shariah (Islamic law) its interpretation with respect to some means and methods that are related to Gharar (uncertainty), Maisir (gambling), and Riba (interest-bearing) (Husain and Pasha, 2011). Takaful is an alternative innovative instrument that provides similar protection as the conventional insurance except that it complies with Shariah law. The word Takaful is from Arabic that means to take care of one's need (Yusof et al., 2011). General (non-life) Takaful was firstly established in 1979 while family (life) Takaful was introduced later in 1984 (Kassim et al., 2013). The Takaful industry is still growing at the rate of 19% in 2018, with total assets USD 46 billion from 324 Takaful operators in 47 countries, according to the IFDR (2018) report. Moreover, it is anticipated the Takaful will continue to grow, especially in the Muslim countries community. For example, in Indonesia, with 98% Muslim population, the Indonesian Health Social Security Organising Agency (BPJS) and the Employment Social Insurance Administration Organisation (BPJS Employment) are currently developing Shariah-based products to attract Muslim citizens (Bappenas, 2018).

The main difference between Takaful and commercial insurance lies in the contract design. In conventional insurance, the insurance company sells the contract with a promise to pay for the loss to the policyholder. However, this practice is forbidden in Islam, as it is not clear what is sold under the insurance product. The Takaful contract combines agency and profit/risk sharing in their business, instead. The role of Takaful companies is to manage the Takaful fund only, while the liability of any claims is borne by the Takaful fund, which is owned by Takaful participants. This contract's feature make Takaful quite similar to mutual insurance. However, the main difference is in the existence of the operator in Takaful insurance. In addition, to manage the Takaful business, the operator also provides capital. Hence, Takaful operator has some rights to a part of surplus from Takaful fund.

Similar to the conventional counterpart, in terms of protection, there are two types of Takaful business model, namely: general (non-life) Takaful and family (life) Takaful. Based on business models, the operation of Takaful can be structured as Wakalah (agent-based contract), Mudarabah (profit sharing), Hybrid (mixed), and Waqf model. Under the Wakalah (agency) contract, the role of Takaful operator is a wakeel (agent) that is paid by participants a predefined fee to manage the Takaful funds. The Wakalah fee is paid in advance as a percentage of contribution. After deducting the wakalah fee, the rest of the contributions are credited to the participants' funds, which are also called Tabaru funds. In the Mudarabah (profit sharing) contract, the operator and the participants should agree on a profit-sharing rate at the commencement of the contract. Under this contract, all participants' contribution is credited to Takaful fund without any deduction. A Hybrid contract applies the Wakalah contract for underwriting activities, while Mudarabah is adopted for investment activities. The latter model is the most dominant in the Takaful market (Tolefat and Asutay, 2013).

The growing trend of the Takaful market requires in-depth studies of its financial stability and actuarial modeling to make a better business decision. Ruin theory is fundamental study in actuarial science that analyses the dynamic evolution of the capital of insurance products driven by different sources of risk. One important problem in ruin theory is the estimation of the probability that surplus becomes negative at some point in the future, or often described as the ruin probability problem. A brief overview of some current research on ruin probability can be found in (Bulinskaya, 2017). Because Takaful products have different features when compared with its conventional counterpart, the two products have different types of risk leading to different risk models. However, until now there are no studies on risk for Takaful itself. This study contributes to the development of Takaful risk model.

In this study, we focus on the development of finite-time ruin probability for Takaful business, especially for a Hybrid model. In practice, this topic is helpful for Enterprise Risk Management to study the probability of becoming insolvent before 5 or 10 years in a steady regime, which can be used to assess whether the activity is sustainable in a steady regime (Gerber and Loisel, 2012). We incorporate qard-hasan facility (benevolent loan) in our Hybrid Takaful risk model (see Chapter 4.1), which is an essential element to maintain Takaful solvency requirement (Onagun, 2011; Rahim et al., 2017). The practical example of the qard-hasan facility is the situation where the Indonesian government may provide a qard-hasan facility through Baznas (Indonesia's national Zakat collection agency) to overcome the deficit of the Shariah-based products as a strategy to achieve the Sustainable Development Goals (SDGs) in Indonesia (Rehman, 2019). We present numerical simulations based on the constructed finite-time ruin probability to investigate the impact of some variables on the performance of Takaful business. Based on our numerical results, we find that some conditions under which the Takaful product is not only in line with Shariah rule but may also outperforms the conventional counterpart. This result answers the key concern related to the optimal structure of the Takaful model mentioned in the WTR (2016) report. According to this report, many shareholders expect to see profitability in line with conventional insurers, while participants expect to see a unique product that fully embraces the ideals principal of Takaful.

1.2 Research Objective

In this work, we study and proposed some mathematical models that comply with the Shariah rule for Islamic compliant portfolios and Takaful insurance.

First, we propose a variant of the Markowitz-based Shariah-compliant model with the following two distinctive features. The first feature assumes that the purification rates (both dividend and investment) are stochastic, hence relaxing the assumption of of Derbali et al. (2017) and Hazny et al. (2020). The second feature introduces probabilistic constraints to explicitly control the risk of non-compliance and to integrate them to the portfolio strategy. To the best of our knowledge, this is the first study that explicitly considers the non-compliance risk in the construction of Shariah-compliant portfolio.

Second, we extend the one-period Shariah problem to the multiperiod problem. Unlike conventional finance, the discussion of the multiperiod portfolio problem has never been mentioned before in Islamic finance. We incorporate the proposed purification and screening constraints in the multiperiod problem. We also integrate the existing risk-free asset in the Shariah portfolio and impose the leverage restriction on the portfolio. Finally, we follow the idea of Cong and Oosterlee (2016) to use the pre-commitment strategy and apply Stochastic Grid Bundling Method (SGBM) to solve the multiperiod portfolio problem.

Third, we propose a Takaful risk model that incorporates investment activities and qard-hasan (benevolent loan) facility, and derive a finite-time ruin probability formula to quantify the risk associated with Hybrid Takaful. We follow the idea of Kim and Drekic (2016) to construct a recursive formula to calculate ruin probability. We enhance the model by allowing an investment option with stochastic returns.

1.3 Thesis Outline

The thesis is organized as follows:

Chapter 2 discusses the proposed mathematical model for a Shariah compliant portfolio problem. We start the discussion from the practice of investment portfolio in Islamic capital market in Chapter 2.1. Following the construction of Markowitz-based Shariah compliant model in Chapter 2.2. Our extensive empirical studies, where we examine different type of purification methods and different quantitative screening methods involving different commonly adopted divisors, validate the importance of incorporating the non-compliance probabilistic constraints to the Markowitz-based Shariah-compliant model. See Chapter 2.3 for data and parameter estimations and Chapter 2.4 for the empirical analysis.

Chapter 3 provides the extension of the Shariah-compliant portfolio optimization problem with the purification and screening process for a multiperiod case. First, we construct the multiperiod Shariah portfolio problem and introduce several Shariah constraints in Chapter 3.1. Then, we explain the proposed method to solve the constructed Shariah problem in Chapter 3.2. Finally, Chapter 3.3 presents some numerical simulations to illustrate the impact of purification and screening on the optimal asset allocations and the portfolio performance.

Chapter 4 presents the formulation of Takaful risk model and the associated finite-time ruin probability. First we explain the practice of Hybrid Takaful and the definition of qard-hasan facility in Chapter 4.1. In Chapter 4.2, we introduce the surplus model and the associated mathematical variables. While the construction of the finite-time ruin probability is explained in Chapter 4.3. Chapter 4.4 presents some results of our numerical simulations.

Chapter 5 describes the conclusions and possible future research topic for each component of this thesis.

Chapter 2

A Single Period Markowitz-based Shariah Compliant Portfolio Model

This chapter provides the construction of the Shariah portfolio model in one period time setting with the following key contributions to the existing literature:

- We introduce two types of purification into the Shariah portfolio optimization problem: dividend and investment purification. We construct the purification model based on the current practice of the purification process in the Shariah capital market. We find that our proposed model is more realistic than the existing model proposed by Hazny et al. (2020) and Derbali et al. (2017).
- We propose non-compliance probabilistic constraints into the Shariah portfolio model. The current literature in the Shariah portfolio only considers a static screening process. However, in practice, the screening process is carried out throughout the investment period. Therefore, the probabilistic constraints are essential to maintain the sustainability compliance status of the portfolio.
- We study the sensitivity of several important variables to the performance of the Shariah portfolio. The study includes the impact of two types of purification, short selling restriction and screening process based on the rule issued by several prominent international Shariah boards.

2.1 Shariah-compliant portfolios

This section explains in greater detail two prominent processes in constructing Shariah-complaint portfolio, namely screening process and purification process.

2.1.1 Screening process

The screening process consists of two phases. The first screening phase relates to the pre-screening "portfolio selection" and the second screening phase involves the on-going monitoring. Let us focus on the first screening phase. While the screening criteria could vary from investor to investor, some plausible criteria include the types of securities (e.g. bonds versus stocks), sector consideration (e.g. high-tech stocks vs utilities stocks), securities' risk and return characteristics, etc. In addition to these screening considerations, a Shariah-compliant portfolio has the added constraint that the securities in the portfolio must adhere to Shariah. In practice, Islamic investors may have a Shariah board that prescribes the Islamic capital market's rule and issues a list of Shariah-compliant stocks. An example of Shariah board is the Accounting and Auditing Organization for Islamic Financial Institutions (AAOIFI) for Gulf Cooperation Council (GCC) clients. Shariah advisory council for Malaysian investors, and National Shariah board of Indonesian Ulema Council (DSN-MUI) for Indonesian investors.

The Shariah-compliant screening criteria, can be classified as either qualitative or quantitative. Qualitative screening examines the nature of the business and excludes stocks of companies with activities that are in direct conflict with Shariah. For example, conventional banks, insurance companies, and companies with dealings in pork, liquor, drugs, and weapons are excluded. Companies that pass the qualitative screening are then proceeded to quantitative screening, where some selected financial ratios are calculated and companies that exceed some preset thresholds are removed. Based on the notations defined in Table 2.1, below we list four commonly adopted quantitative financial screening ratios. For more information on the use of these ratios for quantitative screening, see Gamaleldin (2015).

1. Leverage Ratio $= \frac{TID}{TA \ or \ MC_{24} \ or \ MC_{36}}$;

2. Cash Ratio $= \frac{TC+IBS}{TA \ or \ MC_{24} \ or \ MC_{36}}$;

3. Liquidity Ratio $= \frac{AR+TC}{TA \ or \ MC_{24} \ or \ MC_{36}}$;

4. Non-permissible Income (NPI) Ratio $= \frac{NPI}{TR}$.

Table 2.1: *Notations.*

Notation	Definition	Notation	Definition
$AR:$	Account receivables	$TA:$	Total assets
$IBS:$	Interest-bearing securities	$TC:$	Total cash
$MC_{24}:$	24-month market capitalization	$TID:$	Total Interest-bearing debt
$MC_{36}:$	24-month market capitalization	$TR:$	Total revenue
$NPI:$	Non-permissible income		

The screening criteria based on Leverage Ratio and Cash Ratio are meant to restrict company's investment activities on riba, which is considered non-halal. Most Shariah boards use either 33% or 1/3 as the maximum threshold for these two ratios. A company with higher liquidity, in general, reflects positive financial condition, although this is not necessarily the case from the Shariah perspective. According to the Shariah's principle, returns should be gained from the illiquid assets only (Derigs and Marzban, 2008). For this reason, Shariah boards impose the liquidity screening condition with its ratio ranging from 33% to 55%. The final screening NPI Ratio recognizes that it may be difficult to find a company that is truly Shariah compliant. As long as the company's NPI Ratio is no more than, say 5%, the company is still deemed as Shariah-compliant.

Table 2.2 summarizes the quantitative screening financial ratios for Indonesian Shariah board (ISB) and four other Islamic indices, namely S&P Shariah Index, Dow Jones Islamic Market Index (DJIM), FTSE Shariah global equity Index series, and MSCI Islamic Index. Note that the screening thresholds and the divisors could vary depending on the index, hence demonstrating that there is no (strict) consensus among the Shariah boards.

Table 2.2: *Quantitative Shariah-Compliant Financial Ratios.*

Compliance	Cash Ratio	Leverage Ratio	Liquidity Ratio	NPI Ratio
ISB	$\frac{TID}{TA} < 45\%$	n.a.	n.a.	$\frac{NPI}{TR} < 10\%$
S&P	$\frac{TID}{MC_{36}} < 33\%$	$\frac{TC+IBS}{MC_{36}} < 33\%$	$\frac{AR}{MC_{36}} < 49\%$	$\frac{NPI}{TR} < 5\%$
DJIM	$\frac{TID}{MC_{24}} < 33\%$	$\frac{TC+IBS}{MC_{24}} < 33\%$	$\frac{AR}{MC_{24}} < 33\%$	$\frac{NPI}{TR} < 5\%$
FTSE	$\frac{TID}{TA} < 33.333\%$	$\frac{TC+IBS}{TA} < 33.333\%$	$\frac{AR+TC}{TA} < 50\%$	$\frac{NPI}{TR} < 5\%$
MSCI	$\frac{TID}{TA} < 33.333\%$	$\frac{TC+IBS}{TA} < 33.333\%$	$\frac{AR+TC}{TA} < 33.33\%$	$\frac{NPI}{TR} < 5\%$

Implementing the above qualitative and quantitative screening processes enables the Shariah board to produce a list of securities that complies with Shariah principles. This list, in turn, becomes a universe of feasible securities for the portfolio manager and investors to construct a Shariah-compliant portfolio. Once a Shariah-compliant portfolio is constructed after the above pre-screening process, it requires an on-going monitoring, and this corresponds to the second screening phase. The on-going monitoring is important as the quantitative screening financial ratios may fluctuate over the investment horizon due to business and market volatilities. Consequently, a security that is initially compliant may subsequently become non-compliant. When this occurs, the non-compliant security will need to be liquidated and the portfolio will need to be rebalanced to ensure compliance. The involuntary liquidation can be devastating to the investment return, thus highlighting the critical role of the on-going screening process. It is, therefore, prudent to select securities that minimize the risk of compliance change throughout the investment horizon.

This key insight motivates our proposed Shariah-compliant portfolio model, where we explicitly impose probabilistic constraints in order to control the risk of compliance change (see Subsection 2.2.3). In addition to the better investment perspective, this probabilistic constraint is also an important tool to maintain the sustainable Shariah compliance. As some Islamic investors want to invest their fund in a company that is conventionally compliant over time and not accidentally compliant over the current time period only (Marzban, 2008).

To the best of our knowledge, this is the first paper that explicitly considers the risk of non-compliance. To conclude this subsection, we note that the non-compliance screening process should be conducted continuously. While this is possible in theory, we assume that in practice the on-going monitoring is only conducted at the end of the investment horizon.

2.1.2 Purification process

The imposed thresholds on the financial screening ratios ensure that the NPI is kept at an acceptable level. The Islamic investors, however, are not allowed to accept any of the NPI. For this reason, the portion corresponding to the NPI needs to be removed from the earnings, and the act of cleansing and deducting the NPI from the investors' returns is known as the purification process (Marzban, 2011). There are two commonly adopted practices for purifying a portfolio, namely dividend and investment purifications. The dividend purification assumes that the dividend is a consequence of a company's business activities, and if part of the business activity is contaminated by non Shariah-compliant activities, then the portfolio's returns need to be cleansed. Hence the dividend received will be reduced by the percentage of the NPI.

While the dividend purification is widely accepted by most Shariah boards, including those of FTSE and S&P, Marzban (2011) raises some concerns. The author argues that the dividend purification is appropriate only if the company distributes all earnings in the form of dividend. In practice there are companies that tend not to distribute dividend but rather retain the earnings and reinvest in the companies for more growth opportunities. For these companies, dividend purification would result in zero deduction, even though part of the earnings may be non-halal. The investment purification rectifies this issue by deducting the NPI proportionally according to the number of shares owned by the investors relative to the outstanding shares of the company. This method is used in the GCC Shariah market.

Let p_i^s be the purification loss per share for asset i and purification method s for a given investment, where we use $s = 1$ for dividend purification and $s = 2$ for investment purification. Then, for a given investment horizon, p_i^s can be calculated as

$$p_i^s = \begin{cases} \dfrac{\text{NPI including interest}}{\text{Total income}} \times \text{ Dividend per share}, & \text{for } s = 1,\ i = 1, ..., N \\ \dfrac{\text{NPI including interest}}{\text{Total outstanding shares}}, & \text{for } s = 2,\ i = 1, ..., N. \end{cases} \tag{2.1}$$

These information are readily available from the company's financial statement. For example, the non-permissible income data is typically obtained from the reported interest income. Multiplying p_i^s by the number of shares owned by the investors becomes the purification loss for asset i. This will be the amount to be deducted from the investors' portfolio. Because of the purification process, the net return of the investors will be reduced by the magnitude of the purification loss. Hence, it is important for the investors to take this deduction into consideration when constructing a Shariah-complaint portfolio.

2.2 The proposed approach to Shariah-compliant portfolio

Subsection 2.2.1 first reviews the conventional mean-variance portfolio model of Markowitz (1952). Subsection 2.2.2 then discusses ways of incorporating purification process in constructing an investment portfolio. Finally, Subsection 2.2.3 formulates our proposed generalization of Markowitz model that incorporates both purification and screening processes.

2.2.1 Conventional Markowitz (1952) model

We begin by assuming that a portfolio manager is interested in constructing a Shariah-compliant portfolio from a feasible set of N risky assets that have passed the pre-screening. We assume that the current time is 0 and that the portfolio manager plans to hold the investment for T years, which could be a day, a month, etc.

Let $\boldsymbol{R} = (R_1, R_2, \ldots, R_N)'$ be the vector of total return random variables, with $R_i, i = 1, \ldots N$, representing the total return for asset i over the given investment horizon. By investing in these assets, the total investment return could arise from the assets' capital appreciations and/or dividends. To distinguish these two returns, we introduce the following two vectors $\boldsymbol{R}^C = (R_1^C, \ldots, R_N^C)'$ and $\boldsymbol{R}^D = (R_1^D, \ldots, R_N^D)'$ where R_i^C and R_i^D denote, respectively, the (stochastic) capital appreciation rate and the (stochastic) dividend yield for asset $i, i = 1, \ldots, N$. Hence $R_i = R_i^C + R_i^D$. We also define

$$\begin{aligned}
&\boldsymbol{\mu} = (\mu_1, \ldots, \mu_N)', && \boldsymbol{\mu}^C = (\mu_1^C, \ldots, \mu_N^C)', && \boldsymbol{\mu}^D = (\mu_1^D, \ldots, \mu_N^D)', \\
&\boldsymbol{\Sigma} = (\sigma_{ij})_{i,j=1,\ldots,N}, && \boldsymbol{\Sigma}^C = (\sigma_{ij}^C)_{i,j=1,\ldots,N}, && \boldsymbol{\Sigma}^D = (\sigma_{ij}^D)_{i,j=1,\ldots,N},
\end{aligned}$$

where $\mu_i = \mathrm{E}[R_i]$, $\mu_i^C = \mathrm{E}[R_i^C]$, $\mu_i^D = \mathrm{E}[R_i^D]$, $\sigma_{ij} = \mathrm{Cov}(R_i, R_j)$, $\sigma_{ij}^C = \mathrm{Cov}(R_i^C, R_j^C)$, and $\sigma_{ij}^D = \mathrm{Cov}(R_i^D, R_j^D)$, for $i, j = 1, \ldots, N$.

Suppose a portfolio is constructed with portfolio weight $\boldsymbol{x} = (x_1, \ldots, x_N)'$, where x_i denotes the percentage of the portfolio's total investment amount invested in the i-th asset, which implies that $\sum_{i=1}^N x_i = 1$. Then for any portfolio strategy $\boldsymbol{x}$, the portfolio rate of return $R_{\boldsymbol{x}}$ is given by

$$R_{\boldsymbol{x}} = \sum_{i=1}^N x_i R_i = \sum_{i=1}^N x_i (R_i^C + R_i^D). \tag{2.2}$$

The reward (portfolio expected return) and the riskiness (portfolio standard deviation or variance) of the investment strategy associated with $\boldsymbol{x}$ can be computed, respectively, as

$$\mu_{\boldsymbol{x}} = \mathrm{E}[R_{\boldsymbol{x}}] = \boldsymbol{\mu}'\boldsymbol{x}, \tag{2.3}$$

and

$$\sigma_{\boldsymbol{x}}^2 = \mathrm{Var}(R_{\boldsymbol{x}}) = \boldsymbol{x}'\boldsymbol{\Sigma}\boldsymbol{x}. \tag{2.4}$$

The mean-variance efficient portfolio of Markowitz (1952) solves the following minimization problem

$$\min_{\boldsymbol{x}} \sigma_{\boldsymbol{x}}^2, \qquad \text{subject to } \boldsymbol{x}'\boldsymbol{e} = 1 \text{ and } \mu_{\boldsymbol{x}} = \hat{\mu}, \tag{2.5}$$

where $\boldsymbol{e} = (1, \ldots, 1)'$ is a N-dimensional vector of ones and $\hat{\mu}$ is a given level of expected return. By solving the above optimization problem for all feasible values of $\hat{\mu}$, we can construct the classical "mean-variance efficient frontier" depicting the best possible trade-off pairs of risk and reward.

2.2.2 Incorporating purification

In this section, we focus on integrating the purification process with the conventional Markowitz model.

Definition 1. *Let $\boldsymbol{P}^s = (P_1^s, \ldots, P_N^s)'$ denote a vector of purification rates, with P_i^s representing the purification rate for asset i and purification method s. The purification rate P_i^s is defined as*

$$P_i^s = \frac{p_i^s}{S_i} \tag{2.6}$$

where p_i^s is the purification amount for each share of asset i based on the purification method s as defined by Equation (2.1), and S_i is the initial value of asset i, $i = 1, \ldots, N$.

As seen at time 0, p_i^s is a random variable, and so is P_i^s. Let $\boldsymbol{\psi}^s = (\psi_1^s, \ldots, \psi_N^s)'$ be the mean vector of $\boldsymbol{P}^s$, i.e. $\psi_i^s = \mathrm{E}[P_i^s]$, and $\boldsymbol{\Psi}^s = (\Psi_{ij}^s)_{i,j=1,\ldots,N}$ be the variance-covariance matrix of $\boldsymbol{P}^s$.

Let us denote by $R_{\boldsymbol{x}}^s$ the random rate of return of a Shariah-compliant portfolio with portfolio weight $\boldsymbol{x}$ and adjusted for purification based on method s. Then we have

$$R_{\boldsymbol{x}}^s = \sum_{i=1}^{N} x_i (R_i^C + R_i^D - P_i^s). \tag{2.7}$$

Comparing with the conventional portfolio construction (i.e. Equation (2.2)), the complication of a Shariah-compliant portfolio is the presence of the purification rate P_i^s. Setting $\boldsymbol{\mu}^s = \boldsymbol{\mu}^C + \boldsymbol{\mu}^D - \boldsymbol{\psi}^s$ and $\boldsymbol{\Sigma}^s = \boldsymbol{\Sigma}^C + \boldsymbol{\Sigma}^D + \boldsymbol{\Psi}^s - 2\mathrm{Cov}(\boldsymbol{R}^D, \boldsymbol{P}^s)$, the mean and variance of $R_{\boldsymbol{x}}^s$ become

$$\mathrm{E}[R_{\boldsymbol{x}}^s] = \boldsymbol{x}'\boldsymbol{\mu}^s, \tag{2.8}$$

$$\mathrm{Var}(R_{\boldsymbol{x}}^s) = \boldsymbol{x}'\boldsymbol{\Sigma}^s\boldsymbol{x}. \tag{2.9}$$

The variance representation (2.9) assumes that the capital appreciation rate is uncorrelated with the purification rate and that the dividend yield is possibly correlated with the purification rate. Based on (2.1), the loss due to dividend purification is a percentage of the dividend itself. Hence, there should be a positive correlation between dividend and dividend purification. We also assume that there is possibly a non-zero correlation between dividend and investment purification (see Timothy, 2012). Notice that the NPI is a part of the company performance. The assumption that the capital appreciation rate is not correlated with the purification rate is based on the study conducted by Mashoka (2013), which shows that the relationship between earning from financial statement and stock return for non-banking sector is weak.

A Markowitz-based Shariah-compliant portfolio is now readily constructed by solving (2.5) with $R_{\boldsymbol{x}}$ replaced by $R^s_{\boldsymbol{x}}$ and with the added short-selling constraint (i.e. $x_i \geq 0, i = 1, \ldots, N$), since it is prohibited in Shariah. We should mention that in the Markowitz-based Shariah-compliant portfolio models proposed by Hazny et al. (2020) and Derbali et al. (2017), the authors assume that for each asset the purification rate is constant. The assumption of constant purification rate constitutes the main difference between these approaches and our method. Since we do not make such an assumption, our approach to the construction of mean-variance Shariah portfolio is more general that those proposed by Hazny et al. (2020) and Derbali et al. (2017). To be more specific, in Hazny et al. (2020) and Derbali et al. (2017), the authors modify the random rate of return R_i for asset i to $\tilde{R}_i$ where $\tilde{R}_i = (1 - \tilde{\psi}_i)(1 - \zeta)R_i$ and $\tilde{\psi}_i$ is the pre-determined purification rate for asset i and ζ is the zakat rate.

2.2.3 Incorporating purification and screening

In this subsection, we describe our proposed method to the selection of Shariah-compliant portfolio. As alluded earlier, our proposed approach is motivated by the fact that any initially constructed Shariah-compliant portfolio is subject to an on-going monitoring. If at any future time any of the asset is no longer compliant, then the non-compliant asset will be liquidated. The involuntary liquidation can have an adverse impact on the performance of the investment. This suggests that when constructing a Shariah-compliant portfolio, the portfolio manager should be mindful of the potential involuntary liquidation, and therefore it will be prudent to keep this possibility as low as possible. This is exactly what our proposed Shariah-compliant portfolio aims to achieve.

In Subsection 2.1.1, we have presented four quantitative screening financial ratios: leverage ratio, cash ratio, liquidity ratio, and non-permissible income ratio. In a practical screening process, the definitions of the first three financial ratios depend on the choice of the divisor, which could be either the total asset value, or the 24-month market capitalization, or the 36-month market capitalization. Hence it is important to distinguish which divisor is used when dealing with the first three ratios. Suppose the random variable F_{ji}^k represents, at the end of the investment horizon, asset i's financial ratio j based on divisor k, where $i \in \{1, \dots, N\}; j \in \{1, 2, 3, 4\}; k \in \{1, 2, 3\}$. When $j = 4$, we have $F_{4i}^k = F_{4i}$. Suppose also that U_j^k denotes the threshold level for financial ratio j and divisor k. Note that $U_4^k = U_4$, and U_j^k is a known constant at time 0. Whenever the realized financial ratio j (based on divisor k) at the end of the investment period exceeds the threshold U_j^k, then asset i is no longer compliant and will need to be liquidated. Therefore, it is prudent to ensure that if asset i is a member of the Shariah-compliant portfolio, then the probability $\Pr(F_{ji}^k > U_j^k)$ should be kept as small as possible to reduce the chance of involuntary liquidation.

To control the probability of a compliance change, we use $\alpha \in (0, 1]$ to capture the maximum acceptable probability that F_{ji}^k exceeds U_j^k, i.e.

$$\Pr(F_{ji}^k \leq U_j^k) > 1 - \alpha \quad i = 1, .., N; \; j = 1, .., 4. \tag{2.10}$$

This probabilistic constraints can be incorporated into the Markowitz-based Shariah-compliant portfolio so that when α is set to be a small value, say 0.01, the resulting Shariah-compliant portfolio will have a small probability of facing involuntary liquidation.

In practice, implementing above probabilistic constraints can be a challenging problem as there are $4 \times N$ constraints. It is, therefore, useful to re-formulate these constraints via the following approach. Let z_{ji} and $z_i, i = 1, \dots, N, j = 1, \dots, 4$, be defined as follows:

$$z_{ji} = \begin{cases} 1, & \text{if } \Pr(F_{ji}^k < U_j^k) > 1 - \alpha \\ 0, & \text{else;} \end{cases} \tag{2.11}$$

$$z_i = \min\{z_{ji}\}_{j=1}^4. \tag{2.12}$$

Hence z_{ji} is an indicator function and z_i is a binary function that admits value of 1 if the probability that the asset i remains in compliance is at least equal to $1 - \alpha$, and 0 otherwise. Therefore, if asset i does not meet constraint (2.10), then we set the asset's weight to 0 through the value of z_i. With the above definitions, and together with the short-selling constraint, we have

$$0 \leq x_i \leq z_i, \qquad i = 1, \dots, N. \tag{2.13}$$

The above monitoring of non-compliance applies to each and every single asset within the Shariah-compliant portfolio. As argued in Derigs and Marzban (2009), the screening of financial ratio can also be applied at the portfolio level, in which case (2.10) revises to

$$\Pr\left(\sum_{i=1}^{N} x_i F_{ji}^k < U_j^k\right) > 1-\alpha, \qquad j = 1, ..., 4. \tag{2.14}$$

If the same threshold applies to both the asset level and the portfolio level, then it is easy to see that screening at the asset level is more stringent than at the portfolio level. In fact, it is easy to show that Equation (2.10) implies Equation (2.14).

Combining the above screening constraints (i.e. constraint (2.13) for the asset level screening or constraint (2.14) for the portfolio level screening) with the mean and variance representations given in (2.8) and (2.9), as well as the short-selling constraint, we are now ready to present our proposed Markowitz-based Shariah-compliant portfolio. This is formulated in the following two optimization problems:

Asset Level Screening:

$$\left.\begin{array}{ll} & \min\limits_{\boldsymbol{x}} \boldsymbol{x}'\boldsymbol{\Sigma}^s\boldsymbol{x} \\ \text{subject to} & \\ & \boldsymbol{x}'\boldsymbol{\mu}^s = \hat{\mu} \\ & \boldsymbol{x}'\boldsymbol{e} = 1 \\ & 0 \le x_i \le z_i, i = 1, \ldots, N. \end{array}\right\} \tag{2.15}$$

with z_i defined in (2.12).

Portfolio Level Screening:

$$\left.\begin{array}{ll} & \min\limits_{\boldsymbol{x}} \boldsymbol{x}'\boldsymbol{\Sigma}^s\boldsymbol{x} \\ \text{subject to} & \\ & \boldsymbol{x}'\boldsymbol{\mu}^s = \hat{\mu} \\ & \boldsymbol{x}'\boldsymbol{e} = 1 \\ & 0 \le x_i \le 1, i = 1, \ldots, N, \\ & \Pr\left(\sum_{i=1}^{N} x_i F_{ji}^k < U_j^k\right) > 1-\alpha, j \in \{1, \ldots, 4\}. \end{array}\right\} \tag{2.16}$$

We remark that the screening constraints (either at the asset level or portfolio level) are imposed explicitly to control the probability of non-compliance. This is achieved through the input parameter value α. This is new and a desirable feature of the model. By controlling the probability of non-compliance, we minimize the potential adverse impact on the performance of the Shariah-compliant portfolio, triggered by the involuntary liquidation. We describe the process to determine the probability of financial screening constraint in 2.3.2.

2.3 Data and estimation of the model parameters

This section describes the asset data that will be used to illustrate our proposed approach to constructing Shariah-compliant portfolio. We also discuss estimation methods of the model parameters. In particular, subsections 2.3.1 and 2.3.2 empirically estimate the parameter values that are needed for the purification process and the screening process.

The empirical data that we will use in our analysis is a subset of the Jakarta Islamic Index (JII). JII consists of 30 most liquid Islamic stocks that are listed on the Indonesian Stock Exchange. In our analysis, we have selected 6 stocks that are representative of their respective industry classification. This is shown in Table 2.3.

Table 2.3: List of 6 Selected Stocks and Their Industry Classification.

Ticker	Name	Industry Classification
TLKM	Telkom Indonesia	mobile telecommunications
UNVR	Unilever Indonesia	personal products
PGAS	Perusahaan Gas Negara	exploration&production
WIKA	Waskita Karya	heavy construction
KLBF	Kalbe Farma	Pharmaceuticals
ASII	Astra International	automotive&financial

We calculate the sample mean (μ_i^C) and sample variance-covariance matrix $\mathbf{\Sigma}^C$ of the quarterly asset's return of the 6 selected stocks from January 2008 to October 2018 that reported in Table 2.4. Similarly, the estimated average dividend yield (μ_i^D) and its standard deviation ($\sqrt{\sigma_{ii}^D} = \sigma_i^D$) are given in Table 2.5. Other descriptive statistics are provided in Appendix A. We assume that there is no correlation between each asset's dividend data.

Table 2.4: Sample Mean Vector And Sample Variance-Covariance Matrix.

Mean vector

	TLKM	UNVR	PGAS	WIKA	KLBF	ASII
	0.059	0.059	0.019	0.067	0.077	0.097

Variance-covariance matrix

Stock	TLKM	UNVR	PGAS	WIKA	KLBF	ASII
TLKM	0.0211	0.0067	0.0069	0.0106	0.013	0.0201
UNVR	0.0067	0.0161	0.0032	0.0049	0.0048	0.0091
PGAS	0.0069	0.0032	0.0334	0.0267	0.0188	0.0189
WIKA	0.0106	0.0049	0.0266	0.0666	0.0307	0.0196
KLBF	0.0131	0.0047	0.0187	0.0307	0.0424	0.0274
ASII	0.0201	0.0091	0.0189	0.0196	0.0274	0.0525

Table 2.5: Sample Mean and Sample Standard Deviation of Dividend Yield.

	TLKM	UNVR	PGAS	WIKA	KLBF	ASII
μ_i^D	0.069	0.008	0.009	0.006	0.006	0.023
σ_i^D	0.080	0.004	0.007	0.004	0.004	0.035

2.3.1 Purification Data

To get the purification data, for each company we retrieve the NPI and total income data from each company's quarterly financial statement from December 2013 to September 2018. While the total income data is readily available from the company's income statement, it is more complicated to gather the NPI data. The NPI is typically obtained from the company's interest income and other income from non-permissible activities. The interest income can be found in the company's income statement. However, to obtain the non-compliant activities income, we need to check in details the source of income, and disentangle core income from other income that is generated from non-compliant activities. As an example let us consider ASII, which is an automotive company. In addition to selling automotive and other automotive merchandize, ASII also offers insurance and interest-based financing services to their clients. The latter two financial services are considered non-permissible, and thus any profit generated from these services must be cleansed and deducted. Typically, the relevant data is also available from the company's financial statement, so that the purification rate can be estimated empirically. Figure 2.1 plots the dividend purification rate and the investment purification rate for the six selected stocks, while Table 2.6 tabulates their sample mean (ψ_i^s) and sample standard deviation ($\sqrt{\Psi_{ii}^s} = \Psi_i^s$). Other descriptive statistics of purification data are provided in Appendix A. Note that the rates based on the investment purification are consistently higher than that based on the dividend purification. We also plot constant purification rate in Figure 2.1 which follows a constant purification rate assumption in Derbali et al. (2017) and Hazny et al. (2020). For each company, we use a constant non permissible income percentage ($\tilde{\psi}_i$) as the average of non permissible data as given by Table 2.8. Furthermore, the purification rate is calculated by $\tilde{\psi}_i R_i$. Where R_i is the total return for asset i.

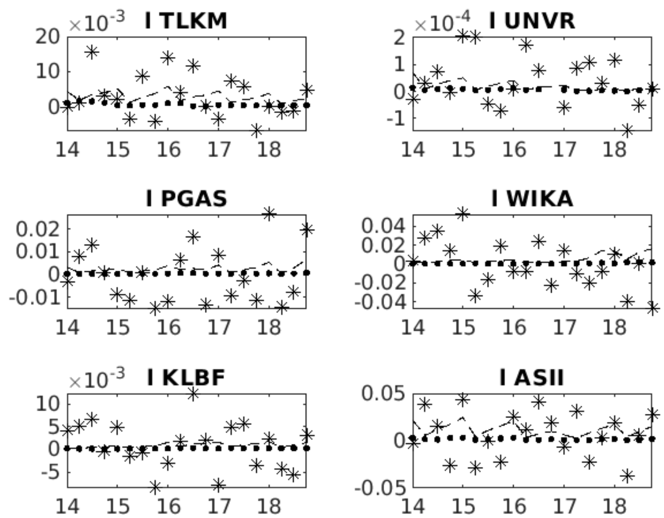


Figure 2.1: Comparison of dividend purification rate (dotted line), investment purification rate (dashed line), and constant purification rate (stars).

Table 2.6: Sample Mean and Sample Standard Deviation of The Purification Rates

	TLKM	UNVR	PGAS	WIKA	KLBF	ASII
Dividend purification method						
ψ_i^1	0.000158	0.000007	0.000691	0.000616	0.000188	0.007378
Ψ_i^1	0.000246	0.000008	0.001479	0.001335	0.000347	0.008234
Investment purification method						
ψ_i^2	0.002844	0.000021	0.002483	0.005245	0.000821	0.039644
Ψ_i^2	0.001392	0.000018	0.001564	0.004853	0.000471	0.021224

Table 2.7 displays the sample correlation ($\rho_{R_i^D, P_i^s}$) between dividend yield and purification rate for each asset i and purification method s. The sample correlation between dividend yield and dividend purification rate is higher than the sample correlation between dividend yield and investment purification rate. The strong dependence between dividend yield and dividend purification rate is straightforward to explain, as it can be inferred from Equation (2.1).

Table 2.7: Correlation Between Dividend Yield and Purification Rate ($\rho_{R_i^D, P_i^s}$)

Asset	TLKM	UNVR	PGAS	WIKA	KLBF	ASII
$\rho_{R_i^D, P_i^1}$	0.95	0.65	0.23	0.69	0.85	0.93
$\rho_{R_i^D, P_i^2}$	0.22	0.56	0.23	0.58	0.28	0.88

The sample mean and sample standard deviation of the NPI rate are given in Table 2.8.

Table 2.8: Sample Mean and Sample Standard Deviation of NPI Rate

Asset	TLKM	UNVR	PGAS	WIKA	KLBF	ASII
mean	0.0524	0.0011	0.0624	0.1681	0.0482	0.2362
standard deviation	0.0095	0.0005	0.0377	0.1108	0.0124	0.0405

2.3.2 Screening data and screening distributions

This subsection studies empirically the financial ratios that are used for quantitative screening. From Subsection 2.1.1, the four commonly adopted financial ratios require the following data: total interest-bearing debt (TID), total cash (C), interest-bearing securities (IBS), Account receivables (AR), non-permissible income (NPI), and total revenue (TR), along with three divisors data; i.e. total asset, 24-month, and 36-month market capitalization. Most of these data are readily available from the company's quarterly financial statement. The historical financial ratios covering 2013 to 2018 are depicted in Figure 2.2 for Leverage ratio (F_{1i}^k), Figure 2.3 for Cash ratio (F_{2i}^k), and Figure 2.4 for Liquidity ratio (F_{3i}^k).

From Figure 2.2, the leverage ratios for TLKM, UNVR, KLBF, and ASII with the total asset divisor are always higher than those based on the market capitalization. Furthermore, these ratios are always lower than the 33% threshold, indicating that these companies have maintained compliance status from the perspective of Shariah, irrespective of divisor. The

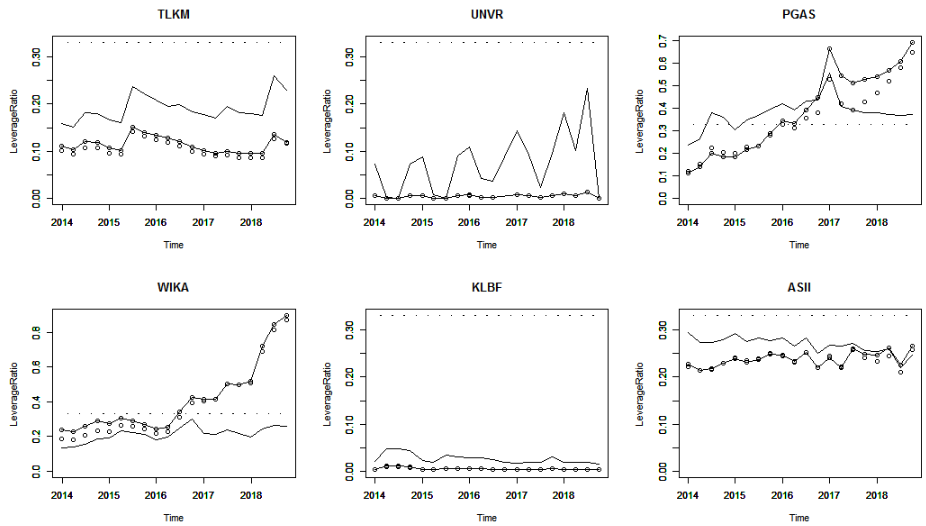


Figure 2.2: Leverage ratio (F_{1i}^k) based on the Total Asset divisor (continuous line), 24-month market capitalization (circles), and 36-month market capitalization (line and point), with 33% threshold level (dotted line).

same, however, does not apply to companies WIKA and PGAS. While both companies are Shariah compliant at the beginning of 2014, they revert to the non-compliance status subsequently, depending on the adopted divisor. More specifically, if the total asset were the divisor, then the leverage ratio of PGAS is mostly above the 33% threshold level starting mid 2014. If market capitalization were used as the divisor, PGAS would remain Shariah-compliant until 2016 and then becomes non-compliant thereafter. These empirical evidences demonstrate the sensitivity of the adopted divisor, as a company may or may not be Shariah compliant depending on the chosen divisor.

In comparison to the leverage ratio, a very similar empirical phenomenon can be observed for cash ratio as depicted in Figure 2.3. For a liquidity ratio, Figure 2.4 shows that there are more companies that are in violation of the Shariah compliance. In addition to WIKA and PGAS, there are many incidences where both UNVR and KLBF are non-compliant if the total asset were used as the divisor. This suggests that the liquidity ratio is a more stringent criterion.

We now turn to the NPI ratio. Among the six companies, ASII is the only company that has business activities that do no comply with Shariah principles. Figure 2.5 provides empirical values of the NPI, the total revenue, and the NPI ratio. Notice that the NPI data in the screening process is different from the one we need in the purification data.

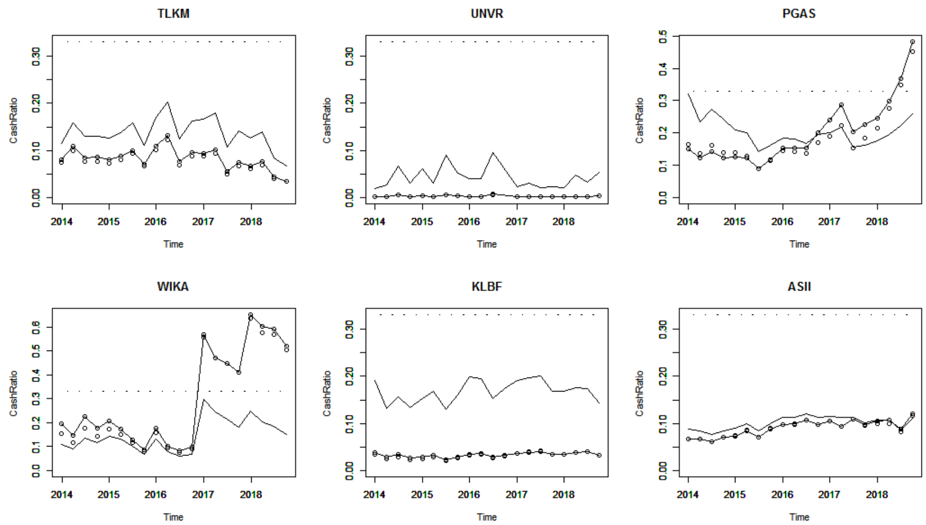


Figure 2.3: Cash ratio (F_{2i}^{k}) based on the Total Asset divisor (continuous line), 24-month market capitalization (circles), and 36-month market capitalization (line and point), with 33% threshold level (dotted line).

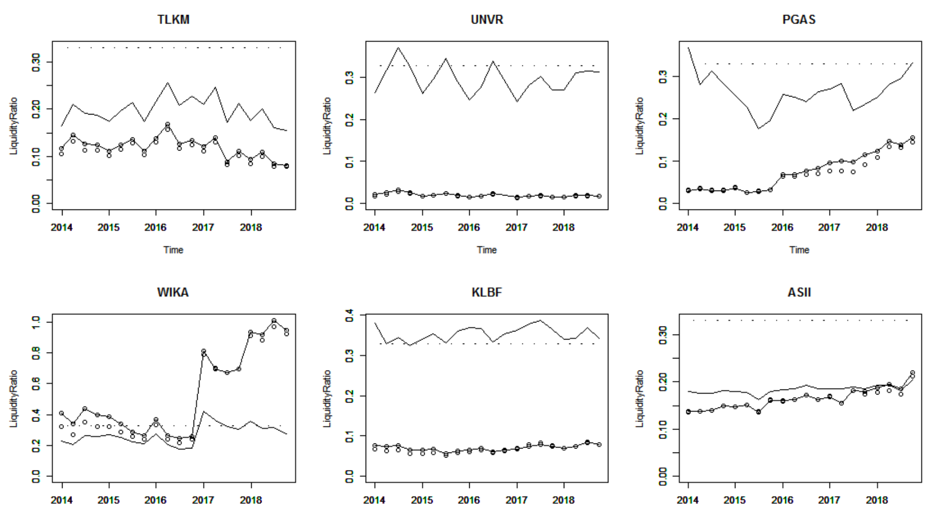


Figure 2.4: Liquidity ratio (F_{3i}^{k}) based on the Total Asset divisor (continuous line), 24-month market capitalization (circles), and 36-month market capitalization (line and point), with 33% threshold level (dotted line).

In the screening process, we focus only on income generated by non-halal activities, not including interest-bearing investment. ASII's NPI ratio is always lower than 5%, which means that ASII is deemed to be Shariah-compliant.

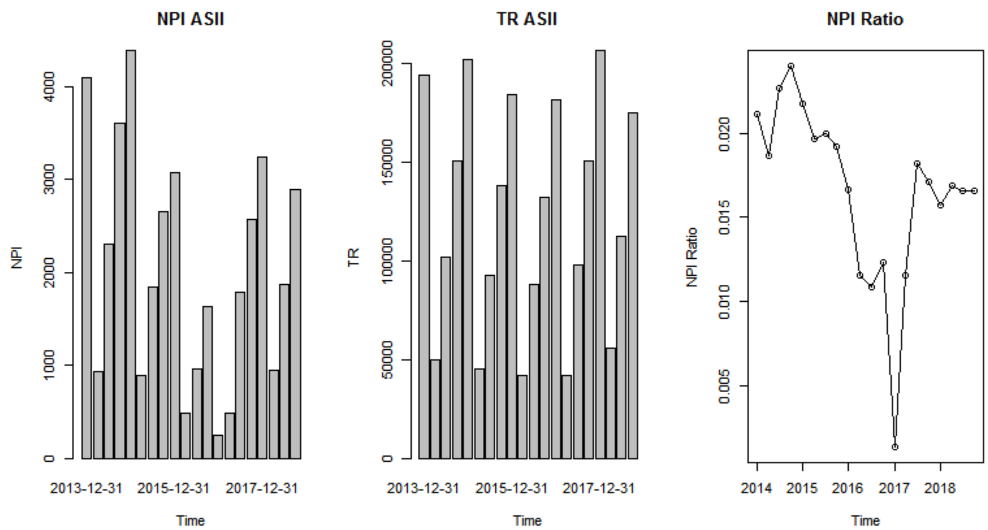


Figure 2.5: ASII's non-permissible income, total revenue and non-permission income ratio.

We model the risk of compliance change by fitting a probability distribution to the financial ratio random variable $F^k_{ji}, i = 1, \ldots, 6, j = 1, \ldots, 4, k = 1, 2, 3$. We first check the normality of each data by applying the Jarque Berra test from *tseries* package in R software. The p-values at the 5% significant level from J-B tests are reported in Table 2.9. Note that for F_{4i} there is only a reported value for ASII since this is the only company that has non-permissible income. Except for ASII's non-permissible ratio, the test indicates that the other financial ratios are consistent with the null hypothesis of the Jarque-Bera test (at 5% significance level) that is a joint hypothesis of the skewness being zero and the excess kurtosis being zero. Additional evidence based on QQ-plots (see B) also support the normality assumption on modeling most of financial ratios. Some financial ratios histogram indicates that the distribution is from a left-skewed distribution (for example, ASII cash ratio F^1_{16}) and a right-skewed distribution (for example, WIKA cash ratio F^2_{14}). However, in this Chapter, we use the normality assumption for the financial ratios (except for ASII NPI ratio) where the sample mean and sample standard deviation of F^k_{ji} are estimated in Table 2.10. The normality assumption has an advantage in the computation of the compliance probability at the portfolio level screening (constraint (2.10)) since the weighted sum of financial ratios is also normally distributed. Currently, we solve the Shariah portfolio problem in Subsection 2.4 by using LINGO optimization software which needs the analytical expression of the constraints. However, since our proposed model allows for a general distribution assumption, we can solve the problem with other financial ratio models by using another optimization method which is discussed in Chapter 3.

Table 2.9: p-values from Jarque-Berra normality test on the financial ratios

i	TLKM	UNVR	PGAS	WIKA	KLBF	ASII
F_{1i}^1	0.2693	0.3597	0.1745	0.9587	0.4632	0.1414
F_{1i}^2	0.3931	0.4891	0.3508	0.1217	0.4584	0.6299
F_{1i}^3	0.4199	0.5658	0.4943	0.0897	0.5427	0.6219
F_{2i}^1	0.9543	0.1377	0.2049	0.4764	0.6844	0.3944
F_{2i}^2	0.9983	0.1133	0.6463	0.2622	0.7585	0.5122
F_{2i}^3	0.9617	0.1022	0.5187	0.2961	0.5977	0.5319
F_{3i}^1	0.7016	0.7416	0.8627	0.6111	0.6183	0.6185
F_{3i}^2	0.9277	0.2094	0.6083	0.2612	0.6099	0.4199
F_{3i}^3	0.9809	0.1533	0.5710	0.3061	0.8174	0.4139
F_{4i}						0.0196

Table 2.10: Sample Means and Sample Standard Deviations (in Parenthesis) of F_{ji}^k.

i	TLKM	UNVR	PGAS	WIKA	KLBF	ASII
F_{1i}^1	0.191(0.029)	0.073(0.065)	0.380(0.065)	0.212(0.042)	0.023(0.006)	0.268(0.017)
F_{1i}^2	0.105(0.017)	0.004(0.004)	0.340(0.256)	0.383(0.208)	0.004(0.001)	0.235(0.014)
F_{1i}^3	0.114(0.017)	0.005(0.004)	0.277(0.217)	0.411(0.201)	0.004(0.001)	0.238(0.015)
F_{2i}^1	0.137(0.032)	0.042(0.023)	0.205(0.044)	0.148(0.067)	0.173(0.021)	0.101(0.013)
F_{2i}^2	0.076(0.021)	0.003(0.002)	0.163(0.134)	0.285(0.206)	0.033(0.006)	0.089(0.016)
F_{2i}^3	0.083(0.023)	0.003(0.002)	0.152(0.121)	0.303(0.202)	0.034(0.005)	0.090(0.017)
F_{3i}^1	0.198(0.028)	0.297(0.035)	0.263(0.044)	0.271(0.065)	0.358(0.017)	0.184(0.008)
F_{3i}^2	0.109(0.020)	0.019(0.004)	0.217(0.181)	0.498(0.281)	0.068(0.010)	0.162(0.019)
F_{3i}^3	0.119(0.022)	0.020(0.005)	0.198(0.161)	0.534(0.271)	0.070(0.007)	0.164(0.022)

Given the non-normality of the ASII's non-permissible income ratio F_{46}, additional statistical analysis has also been conducted. For example, we use the function descdist from fitdistr R package to gain some ideas about possible candidate distributions for F_{46}. The Cullen and Frey Graph is given in Figure 2.6. The kurtosis and squared skewness of our data is plotted as a blue point named "Observation." It appears that distributions such as Weibull and Gamma are plausible choices.

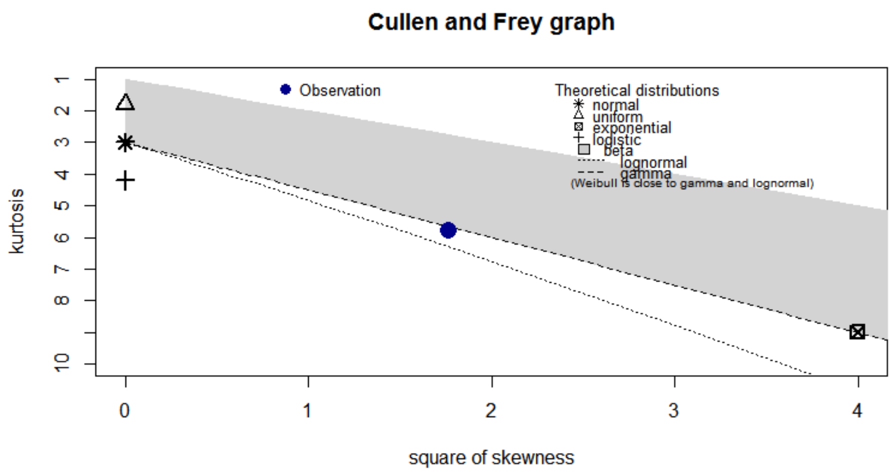


Figure 2.6: The Cullen-Frey graph for non-permissible income ratio of ASII F_{46}

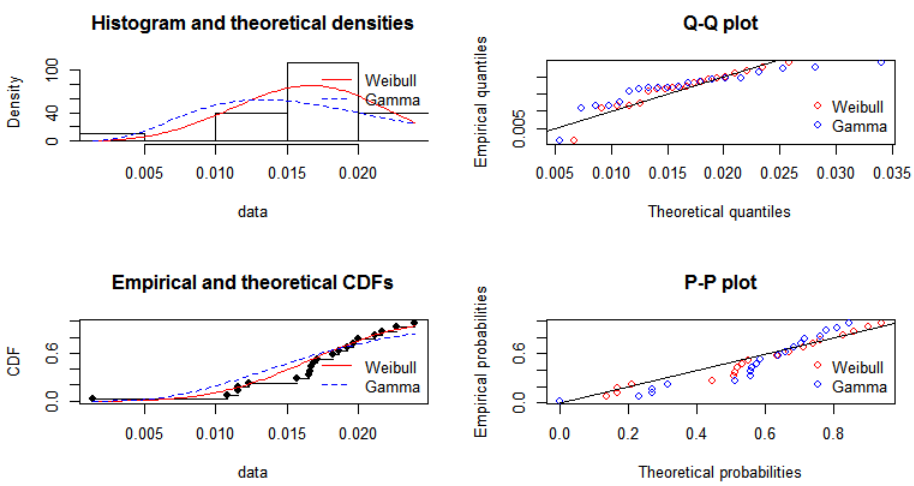


Figure 2.7: Densities, CDFs, Q-Q Plot, and P-P Plot Weibull, Gamma, and empirical distributions of Non-Permissible income ratio (F_{46})

Figure (2.7) plots the distribution and QQ-Plot of Weibull and Gamma distribution and compares to its empirical one. Based on the figure, especially the QQ-Plot, the Weibull distribution looks better than the Gamma distribution, especially at the tails. The choice of Weibull distribution is further supported by its smaller Akaike's Information Criterion (AIC) and Bayesian Information Criterion (BIC) values relative to those of the Gamma distribution (see Table 2.11). For these reasons, we conclude that Weibull with shape parameter 3.71 and scale parameter 0.18 is the best fitted distribution for ASII's non-permissible income ratio.

Table 2.11: AIC and BIC tests for non-permissible income ratio (F_{46})

	Weibull	Gamma
Akaike's Information Criterion (AIC)	-147.977	-138.007
Bayesian Information Criterion (BIC)	-145.985	-136.016

Armed with the above analysis, we now revisit the screening constraints based on the financial ratios. At the asset level compliance screening, $z_{ji}, i = 1, \ldots, 6$, defined in (2.11) become

$$z_{ji} = \begin{cases} 1, & \text{if} \quad U_j^k - \mu_{F_{ji}^k} - \sigma_{F_{ji}^k} \Phi^{-1}(1-\alpha), i \in \{1, \ldots, 6\}, \ j \in \{1, 2, 3\} \\ 0, & \text{else;} \end{cases}$$

$$z_{46} = \begin{cases} 1, & \text{if} \quad U_4 - F_{F_{46}}^{-1}(1-\alpha) > 0 \\ 0, & \text{else;} \end{cases}$$

where $(\hat{\mu}_{F_{ji}^k}, \hat{\sigma}_{F_{ji}^k})$ are the asset i's sample mean and sample standard deviation of financial ratio j with divisor k (as estimated in Table 2.10), Φ is the cumulative distribution function (CDF) of standard normal distribution and $F_{F_{46}}(x)$ is the CDF of the Weibull distribution with shape parameter $\kappa_{F_{46}} = 3.71$ and scale parameter $\lambda_{F_{46}} = 0.18$.

For the portfolio screening process, the probabilistic constraint given in (2.14) shows that it depends on a linear combination of random variables, i.e. $\sum_{i=1}^{N} x_i F_{ji}^k$. Since F_{ji}^k are normally distributed, we can use the fact that a sum of normally distributed random variables is also normally distributed (with appropriate mean and standard deviation adjustments). As for $j = 4$, we only need to worry about ASII, since it is the only company that generates NPI. In summary, we have

$$\begin{aligned}\sum_{i=1}^{6} x_i F_{ji}^{k} &\sim Normal\left(\sum_{i=1}^{6} x_i \mu_{F_{ji}^{k}}, \sum_{i=1}^{6} x_i \sigma_{F_{ji}^{k}}\right), \text{ for } j,k \in \{1,2,3\}, && (2.17)\\ \sum_{i=1}^{6} x_i F_{4i} &= x_6 F_{46} \sim \text{Weibull}(x_6 \lambda_{F_{46}}, \kappa_{F_{46}}), && (2.18)\end{aligned}$$

where $Normal(\mu, \sigma)$ denotes the normal distribution with mean μ and standard deviation σ, and $Weibull(\lambda, \kappa)$ denotes the Weibull distribution with scale parameter λ and shape parameter κ.

In our empirical studies, we apply LINGO software version 17 to solve the Shariah portfolio problems with asset and portfolio compliance strategies. LINGO is an optimization modeling software that may handle linear and non-linear problems with stochastic variables (see (Lindo Systems, 2021)). We set several values of the target return ($\hat{\mu_p}$) in Shariah portfolio problems (i.e. solving (2.15) and (2.16)) to develop portfolio's efficient frontier.

2.4 Empirical Analysis

Based on the parameters estimated in the preceding section, we now study our proposed Shariah-compliant portfolio by solving both optimization problems (2.15) and (2.16). We divide our empirical analysis into two subsections, with Subsection 2.4.1 examining the impact of purification and Subsection 2.4.2 evaluating the impact of both purification and compliance screening.

2.4.1 The impact of purification process

We can construct a Shariah-compliant portfolio without the screening constraints by solving the optimization problem (2.15) with $U_j^k = 1$ for all j and k. The required data on the purification process for all assets are reported in Tables 2.6 and 2.7. The resulting efficient frontiers, under both dividend purification and investment purification methods, are depicted in Figure 2.8. Along with these efficient frontiers, we also produce three other efficient frontiers based on conventional Markowitz model, conventional Markowitz model with short-selling constraint, and the Shariah-compliant portfolio of Derbali et al. (2017)

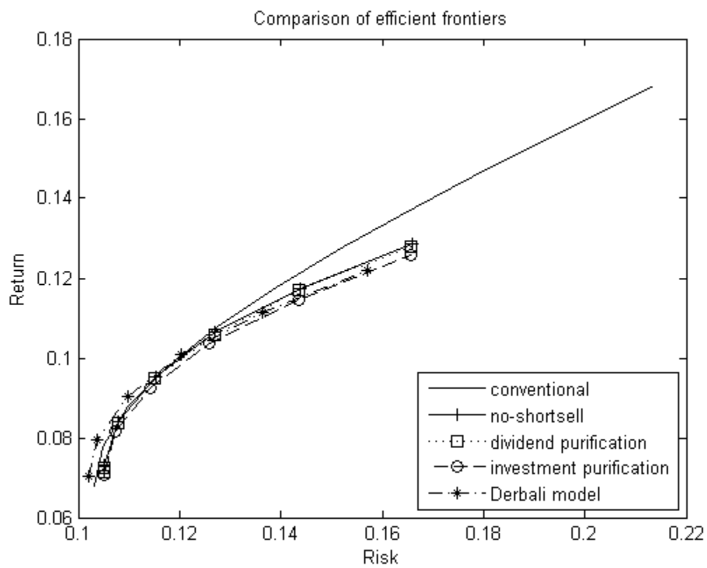


Figure 2.8: Comparison of efficient frontiers

with 0 zakat rate and NPI rates as given in Table 2.8. We exclude zakat rate from our calculation to focus the analysis on the purification process itself.

Key observations from Figure 2.8 are: (i) the presence of a purification process penalizes the risk and reward tradeoff, as depicted by the lower efficient frontiers for both dividend and investment purification; (ii) the dividend purification yields better performance than investment purification, which is consistent with Table 2.6 and Figure 2.1 showing that the estimated historical rates for the dividend purification method are lower than the corresponding investment purification rates; (iii) the efficient frontier from the Shariah-compliant portfolio of Derbali et al. (2017) and Hazny et al. (2020) presents some rather interesting phenomenon. For low return and low volatility portfolios, the model actually yields better risk and return tradeoff than the conventional Markowitz model. This results can be explained by the fact that the value of purification rate under Derbali et al. (2017) and Hazny et al. (2020) model, may produce negative values as shown by the empirical data in Figure (2.1). Furthermore, the negative value of purification rate will increase the total return. However, in reality a negative purification rate is never happen. As the investor's risk tolerance increases by accepting higher risk and higher return portfolio, the risk and reward tradeoff becomes progressively worse than the conventional Markowitz model. In fact, for high return and high volatility portfolios, the Derbali et al. (2017) model can also produce worse performance than our proposed Shariah-compliant portfolio with dividend purification method.

2.4.2 The impact of both purification and screening processes

Recall that our proposed Shariah-compliant portfolio model imposes probabilistic constraints to control the risk of compliance change via the parameter α. The objective of this subsection is to assess the impact of these constraints in addition to the purification process. By solving the proposed Shariah-compliant optimization problem (2.15) (for asset screening with $\alpha = 0.10$) and problem (2.16) (for portfolio screening with $\alpha = 0.01$), the respective efficient frontiers are depicted in upper and lower panels of Figure 2.9. Consistent with the recommendation by most Shariah boards, we consider financial thresholds $U_j^k = 0.33$ for $j = 1, 2, 3$ and $U_4^k = 0.05$, together with divisors $k = 1, 2, 3$ based on total asset, 24-month market capitalization, and 36-month market capitalization, respectively. The on-screening efficient frontier, i.e. without imposing the probabilistic constraints, is also plotted in Figure 2.9 to benchmark against the screening. This efficient frontier corresponds to the one in Figure 2.8 with dividend purification.

We now focus on the asset screening. This gives a more transparent way of assessing the impact of screening, since the probabilistic constraint is applied to each asset separately. By comparing the efficient frontiers in Figure 2.9, we can see that the effect of screening is more pronounced on the left part of the efficient frontier. Let us assess the impact of screening by comparing attainable minimum and maximum risk portfolios. Without screening, the optimal tradeoff for the maximum risk portfolio is $(0.1656, 0.1279)$, where the first and second coordinates denote the portfolio's standard deviation and expected return, respectively. This optimal portfolio is similarly obtained with screening, irrespective of the divisors. However, the minimum risk portfolios portray a vastly different picture. The minimum risk portfolio for no screening is $(0.1050, 0.0715)$, which is quite similar to the screening with divisors 2 and 3. If the adopted divisor is total asset, then the minimum risk portfolio becomes $(0.1615, 0.1261)$, which differs significantly from the other minimum risk portfolios (see Figure 2.10 which magnifies upper part of the efficient frontier of Figure 2.9). These comparisons indicate that the risk of compliance change is highly dependent on the screening divisor, and that the presence of probabilistic constraints could severely limit the optimal portfolios.

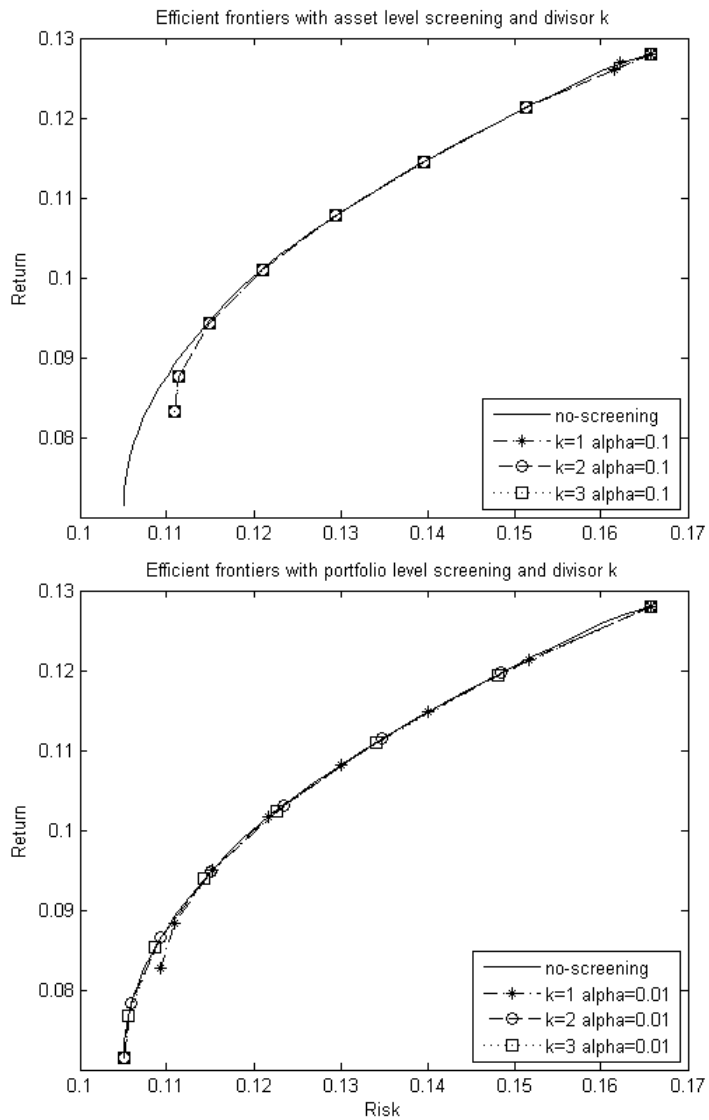


Figure 2.9: The impact of screening with different divisors. Upper panel gives the asset level screening (with $\alpha = 0.10$) and lower panel is for the portfolio screening (with $\alpha = 0.01$)

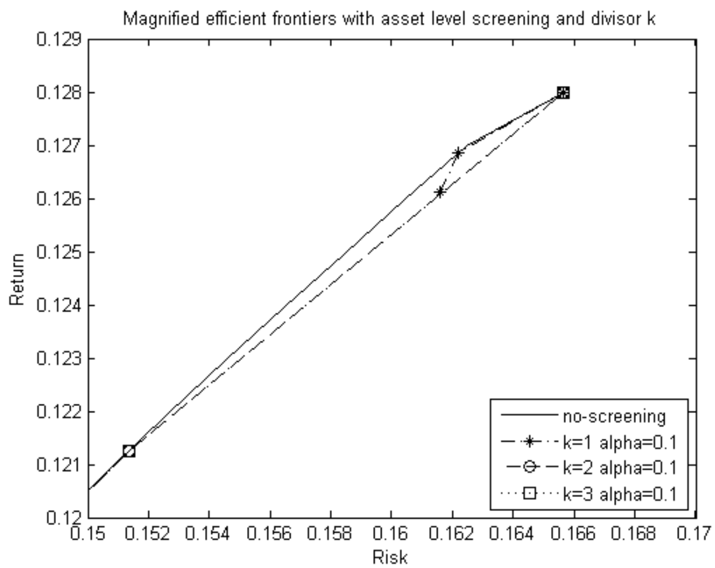


Figure 2.10: Magnified efficient frontier of upper panel of Figure 2.9 for portfolio risk over the range $(0.15, 0.17)$.

Table 2.12: Comparison of Asset Allocations for Minimum Risk Portfolio With and Without Screening.

Method	Risk	Return	TLKM	UNVR	PGAS	WIKA	KLBF	ASII
no screening	0.1050	0.0715	0.1933	0.5502	0.2117	0	0.0446	0
screening with divisor								
$k = 1$	0.1615	0.1261	0.8196	0	0	0	0	0.1803
$k = 2$	0.1108	0.0833	0.2230	0.6266	0	0	0.1502	0
$k = 3$	0.1108	0.0833	0.2230	0.6266	0	0	0.1502	0

A more illuminating insight can be gained from Table 2.12, which produces the asset allocation for the minimum risk portfolios. First, the results based on screening divisors 2 and 3 are identical. This should not be surprising since both divisors are based on market capitalization (24-month average vs 36-month average). Because of the averaging, both divisors tend to be quite similar. Second, and more importantly, is how the distribution of the asset allocation changes from no screening to screening. These changes are triggered by the non-compliance probabilistic constraints and hence highlight their importance. The changes can be understood by noticing that the probabilistic constraints are imposed to keep the risk of non-compliance at an acceptable level. Moreover, as soon as one of the financial ratios crosses the thresholds, then the asset will be deemed non-compliant. Hence the asset allocation will be re-distributed from assets with a high non-compliance risk to assets with a low non-compliance risk. To illustrate this point, let us consider the total asset divisor screening. Based on the empirical estimates of F_{ji}^k in Table 2.10, assets such as UNVR, PGAS, and KLBF exhibit a high non-compliance risk due to their high expected financial ratios (in at least one of them). For example, $\hat{\mu}_{F_{12}^1} = 0.297$ (liquidity ratio), $\hat{\mu}_{F_{13}^1} = 0.380$ (leverage ratio) and $\hat{\mu}_{F_{35}^1} = 0.358$ (liquidity ratio). Hence imposing the non-compliance probabilistic constraints may dramatically change the asset allocations from positive weights (with no screening) to zero, as confirmed in Table 2.12. The portfolio weights for screening with divisors 2 and 3 are also reduced to zero for PGAS. Similar justification applies by noticing their high leverage ratios, as can be inferred from either Table 2.10 or Figure 2.2, showing that most of the historical leverage ratios exceed 33% threshold.

It is also instructive to focus on UNVR. This asset has the highest investment weight under no screening and screening with divisors 2 and 3, while no investment under total asset screening divisor. By standardizing the reward-to-risk measure (defined as expected return divided by standard deviation), UNVR is found to have the highest ratio using either capital appreciation rate or dividend yield (see Tables 2.4 and 2.5). This implies that on its own, UNVR is a highly investable asset and thus explains its highest portfolio weight without screening. Under screening with divisors 2 and 3, UNVR is deemed to have very low non-compliance risk with an expected financial ratio not exceeding 2% (see Table 2.10). Hence even with these screenings, UNVR is still a desirable asset and thus accounts for its highest investment proportion. However, UNVR is highly non-compliant asset according to the total asset divisor criterion. t Therefore, we have to forgo UNVR to reduce the non-compliance risk and avoid involuntary liquidation under the screening constraint with total asset divisor. This, in part, explains the limited optimal portfolios as depicted in Figure 2.10.

So far we have focused on the screening at the asset level. For the portfolio level screening (lower panel of Figure 2.9), we similarly observe that the left part of the efficient frontier is affected the most. The magnitude of the impact, as compared to the asset level screening, appears to be less pronounced. This is to be expected since the portfolio screening is less restrictive than the asset screening. Consequently the performance of the portfolio screening should be better than the asset screening.

Our final set of comparison is to examine the impact of α. For brevity, we only consider the asset screening with total asset as the divisor. The resulting efficient frontiers are plotted in Figure 2.11 for $\alpha \in \{0.1, 0.3, 0.5\}$, together with the no screening efficient frontier.

Higher α implies less stringent conditions on the non-compliance risk, , and hence the resulting efficient frontiers converge to the non-screening efficient frontier, as shown in the figure.

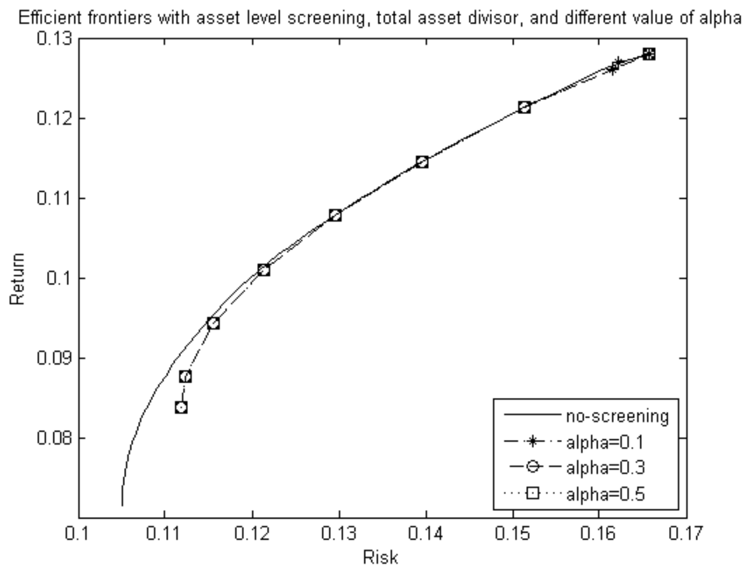


Figure 2.11: The impact of α on efficient frontier (asset level screening).

2.5 Conclusion

In this chapter, we discuss two important features in the Shariah compliant portfolio, namely purification and screening process. We have proposed an effective Shariah-compliant portfolio model that is more consistent with the current practice. It has the desirable features of incorporating both stochastic purification and non-compliance probabilistic constraints. The former addresses the constant purification rate that is commonly assumed in the existing literature, while the latter provides an effective way of controlling the non-compliance risk, which can have an adverse effect on the portfolio strategy. The conducted empirical studies affirm the importance of these features. In particular, they show that the effect of imposing non-compliance probabilistic constraints is highly sensitive to the adopted divisors and that these constraints can be an effective way of mitigating the risk of compliance change, thereby enhancing the effectiveness of the resulting Shariah-compliant portfolio.

Chapter 3

Multi-period Mean-Variance Shariah Compliant Portfolio Model

This chapter provides the extension of the mean-variance Shariah portfolio model to the multiperiod time setting. Some important contributions of this study to the existing literature include:

- To the best of our knowledge, our study of the Shariah portfolio optimization problem in a multiperiod time setting is the first study in the Islamic finance literature.
- In addition to dividend and investment purifications that have been discussed in Chapter 2, we propose a modification and an extension of Hazny et al. (2020) and Derbali et al. (2017) model. The proposed modified model is more realistic than the original one because it can avoid negative values of purification.
- In our study we adopt two assumptions about the dynamic of the financial screening ratios. In the first one, we assume that the financial ratios are independent and follow beta distributions. The second assumption relaxes the independence assumption by allowing the financial ratios to follow the Beta-AR(p,q) model.
- We also consider other possible constraints based on the current Shariah rules. They include a restriction on the short selling activity and a limitation of portfolio leverage facility.
- To solve the multiperiod Shariah portfolio problem, we follow the idea of a forward and backward algorithms proposed by Cong and Oosterlee (2016). While the authors

apply the algorithms in the context of conventional portfolio problems with bounded constraints, we use the algorithms to solve the Shariah-compliant portfolio problem with more complex constraints.

- We also perform several numerical calculations to study the impact of Shariah rules on the dynamic of optimal asset allocations and the portfolio's performance.

3.1 Problem formulation for multiperiod shariah portfolio with purification process

We assume that the financial market is defined on a probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0\leq t\leq T}, \mathcal{P})$ with a bounded time period $[0, T]$. The state space Ω is the set of all realizations of the financial market within the time horizon $[0, T]$. $\mathcal{F}$ is the sigma algebra of events at time T, i.e. $\mathcal{F} = \mathcal{F}_T$. The filtration $\{\mathcal{F}_t\}_{0\leq t\leq T}$ is generated by the price processes of the financial market and augmented with the null sets of $\mathcal{F}$. Probability measure $\mathcal{P}$ is defined on $\mathcal{F}$. In addition to the financial market information, in the Shariah portfolio we need the financial information from the companys' financial statement to calculate the purification rate and the financial screening ratios. Suppose that $\mathcal{G}_t$ is the smallest σ-algebra such that the purification rate and financial ratios are measurable at time t. We denote by $\mathcal{F}_t \vee \mathcal{G}_t$ the smallest sigma-algebra generated by the events in $\mathcal{F}_t$ and $\mathcal{G}_t$.

We assume that the portfolio consists of 1 risk-free asset and N risky assets. In the Shariah asset class, the riskless asset could be a fixed rate sukuk (Islamic bond) (Shazly and Tripathy, 2013) or other type of sukuk issued either by a government that has sound fiscal equilibrium or by a highly rated corporation (Krichene, 2013). In this multiperiod setting, we assume that the portfolio can be traded at discrete time opportunities $t = [0, \Delta t, \ldots, T - \Delta t]$, before terminal time T. If the total number of re-balancing opportunities before terminal time T is M, then $\Delta t = \frac{T}{M}$.

At each trading time t, an investor is seeking the best investment strategy that maximizes the expected value of the terminal wealth and minimizes the investment risk

$$v_t(W_t) = \max_{\{\mathbf{x}_s\}_{s=t}^{T-\Delta t}} \{E(W_T|\mathcal{F}_t \vee \mathcal{G}_t) - \lambda Var(W_T|\mathcal{F}_t \vee \mathcal{G}_t)\} \tag{3.1}$$

where W_t is the wealth at time t, $\mathbf{x}_t = [x_{1,t}, x_{2,t}, ..., x_{N,t}]$ is the asset allocations of investors' wealth in the risky assets over time interval $[t, t + \Delta t)$, and λ is a trade-off between maximizing the profit and minimizing the risk. It captures the risk aversion of the investor.

We denote by $R_0, R^c_{i,t}$, and $R^d_{i,t}$ the risk-free return, the capital appreciation rate, and the dividend yield of asset i over time interval $[t, t+\Delta t)$, respectively. For simplicity, in our model we assume the risk-free return R_0 to be constant. An important difference between the conventional and the Shariah portfolio is the application of cleansing or purification by which the portfolio return is deducted by non-permissible income percentage (Hashim et al., 2017). If we assume that $P^s_{i,t}$ is the purification rate per share of asset i at time t with purification type s, then following Definition 1 in Chapter 2.2.2, we have:

$$P^s_{i,t} = \frac{p^s_{i,t}}{S_{i,t-\Delta t}}, \text{ for } i = 1, \ldots, N, \; t = 0, \ldots, T, \text{ and } s \in \{1, 2\}, \tag{3.2}$$

where $S_{i,t}$ is the stock price of asset i at time t, and $p^s_{i,t}$ is the amount of purification per share for asset i at time t, with the purification type s. $p^s_{i,t}$ is calculated based on Equation (2.1) in Chapter 2.1.2. Under the assumption of self-financing trading strategy, we define the wealth dynamic of the Shariah compliant portfolio as:

$$W_{t+\Delta t} = W_t(\sum_{i=1}^{N} x_{i,t}(R^c_{i,t+\Delta t} + R^d_{i,t+\Delta t} - P^s_{i,t+\Delta t}) + (1 - \sum_{i=1}^{N} x_{i,t})R_0) \tag{3.3}$$

$$= W_t\left(\sum_{i=1}^{N} x_{i,t}\left(R^e_{i,t+\Delta t} + R^d_{i,t+\Delta t} - P^s_{i,t+\Delta t}\right) + R_0\right) \tag{3.4}$$

$$= W_t\left(\mathbf{x}'_t \mathbf{R}^{ps}_{t+\Delta t} + R_0\right), \; t = 0, \ldots, T - \Delta t, \; s \in \{1, 2\}, \tag{3.5}$$

with $R^e_{i,t}$ denoting the capital appreciation excess return, and $\mathbf{R}^{ps}_t = [R^{ps}_{1,t} \ldots R^{ps}_{N,t}]$ being the column vector of purified excess returns with purification type s. When $P^s_{i,t+\Delta t} = 0$ for $t = 0, \ldots, T$, and $i = 1, \ldots, N$, then the wealth dynamic of the Shariah compliant portfolio is equal as the wealth dynamic of the conventional portfolio.

In Subsection 3.2, we present a method of solving the problem of optimal portfolio selection through a pre-commitment strategy approach. Under this approach, we need to assume that the purified excess returns $\{\mathbf{R}^{ps}_t\}_{t=0}^{T-\Delta t}$ are statistically independent (Cong and Oosterlee, 2016). One popular model for $\mathbf{R}^{ps}_t$ is a geometric Brownian motion. Before explaining the proposed algorithm to solve the Shariah portfolio problem, in Subsection 3.1.1 we describe several constraints that are used in this research.

3.1.1 Incorporating Shariah Constraints

In Chapter 2.1.1 we have explained that in a Shariah-compliant portfolio, investors can invest their funds only in the subset of assets that complies with Shariah rules. There are two stages of screening activities in practice: the first one is the sector and activity-based screening, whereas the second one is the quantitative or financial screening (Hashim et al., 2017). In general, there are four types of financial ratios needed to be screened, namely: leverage ratio, cash ratio, liquidity ratio, and non-permissible income (NPI) ratio (Derigs and Marzban, 2008). These ratios should not exceed the permissible threshold to be considered as Shariah-compliant.

Similar with the notation in Chapter 2.2.3, we denote the 4 types of financial ratios of asset i at time t by $F^k_{ji,t} \in [0, 1]$, with $j = 1$ for cash ratio, $j = 2$ for leverage ratio, $j = 3$ for liquidity ratio, and $j = 4$ for NPI ratio. For each financial ratio j, there are 3 different definitions of financial ratio, which depend on the chosen divisor: total asset ($k = 1$), 24-month market capitalization ($k = 2$), or 36-month market capitalization ($k = 3$). The calculation of $F^k_{ji,t}$ is described in Chapter 2.1.1. For simplicity, in this Chapter, we only use the total asset divisor to define financial ratios. From now on, we will denote the financial ratio by $F_{ji,t}$, without the superscript k. It is possible to use our approach to incorporate other definitions of financial ratios.

In this study, we use two model assumptions for $F_{ji,t}$. In the first one, we use time independent assumption, while the second assumption relaxes the independence assumption.

1. Time independent model of financial ratio.

   For the time independent assumption, we assume that $F_{ji,t},\ t = 0, \Delta t, \ldots, T$, are independent and follow the beta distribution with parameters $\alpha^B_{ji} > 0$ and $\beta^B_{ji} > 0$. We choose a beta distribution because the value of $F_{ji,t}$ lies between zero and one. The beta probability density function is given by

$$f^B(F_{ji,t}; \alpha^B_{ji}, \beta^B_{ji}) = \frac{\Gamma(\alpha^B_{ji} + \beta^B_{ji})}{\Gamma(\alpha^B_{ji})\Gamma(\beta^B_{ji})} F_{ji,t}^{\alpha^B_{ji}-1} (1 - F_{ji,t})^{\beta^B_{ji}-1},\ 0 \leq F_{ji,t} \leq 1,\ t = 0, \ldots, T, \tag{3.6}$$

   where $\Gamma(.)$ is the gamma function. Under the above assumption, the mean and variance of $F_{ji,t}$ are respectively

$$E[F_{ji,t}] = \frac{\alpha^B_{ji}}{\alpha^B_{ji} + \beta^B_{ji}} \text{ and } Var(F_{ji,t}) = \frac{\alpha^B_{ji}\beta^B_{ji}}{(\alpha^B_{ji} + \beta^B_{ji})^2(\alpha^B_{ji} + \beta^B_{ji} + 1)}.$$

2. Time dependent model of financial ratio.

   For the time dependent assumption, we assume $F_{ji,t}$ to follow the Beta-ARMA(p,q) model, as introduced by Rocha and Neto (2009). Suppose that $\mathcal{G}_t$ is the smallest σ-algebra such that the variables $\{F_{ji,s}\}_{0\leq s\leq t}$ are measurable. Under the Beta-ARMA(p,q) model, we assume that the conditional density of $F_{ji,t}$ given $\mathcal{G}_{t-\Delta t}$ is given by:

$$f^A(F_{ji,t}|\mathcal{G}_{t-\Delta t}) = \frac{\Gamma(\delta_{ji})}{\Gamma(\pi_{ji,t}\delta_{ji})\Gamma((1-\pi_{ji,t})\delta_{ji})}(F_{ji,t})^{\pi_{ji,t}\delta_{ji}-1}(1-F_{ji,t})^{(1-\pi_{ji,t})\delta_{ji}-1}. \quad (3.7)$$

The conditional mean and variance of $F_{ji,t}$ are $\pi_{ji,t}$ and $\frac{\pi_{ji,t}(1-\pi_{ji,t})}{1+\delta_{ji}}$, respectively. From equation (3.7), we can see that $F_{ji,t}$ follows a beta distribution with parameter $\pi_{ji,t}\delta_{ji}$ and $(1-\pi_{ji,t})\delta_{ji}$. Furthermore, the parameter $\pi_{ji,t}$ in Equation (3.7) is defined as the ARMA(p,q) model:

$$g(\pi_{ji,t}) = \alpha_{ji}^A + \sum_{k=1}^{p} \beta_{ji}^{Ak} g(F_{ji,t-k\Delta t}) + \sum_{l=1}^{q} \theta_{ji}^{Al} \epsilon_{ji,t-l\Delta t} \quad (3.8)$$

with $E(\epsilon_{ji,t}) = 0$ and $Var(\epsilon_{ji,t}) = \frac{\pi_{ji,t}(1-\pi_{ji,t})}{1+\delta_{ji}}$. In Equation (3.8), we assume that $g(.)$ is a strictly monotonic and twicely differentiable link function that maps $(0,1)$ into $\mathbb{R}$. In our research we use a logit function as proposed by Ferrari and Neto (2004):

$$g(\pi) = \log(\frac{\pi}{1-\pi}). \quad (3.9)$$

After defining the dynamic of financial ratios, now we are ready to formulate some constraints for Shariah compliant portfolio.

Shariah screening constraint

As discussed in Section 2.1.1, a portfolio is considered Shariah compliant if all of the screening financial ratios $F_{ji,t}$ are less than their permissible boundaries U_j. Hence, at the beginning of each investment period $[t, t+\Delta t)$, $t = 0, \ldots, T-\Delta t$, we need to do the screening check either at the asset level or the portfolio level. Under the asset level screening process, if the financial ratio $F_{ji,t}$ is greater than the permissible boundary U_j, for some $j = \{1,2,3,4\}$, then at time t we need to exclude asset i from the portfolio. This procedure can be written mathematically as follows:

$$x_{i,t} \leq Y_{i,t}, \; i = 1,..,N, \; t = 0, \ldots, T-\Delta t, \quad (3.10)$$

where

$$Y_{i,t} = \min\{y_{ji,t}\}_{j=1}^{4} \tag{3.11}$$

$$y_{ji,t} = \begin{cases} u_i, & \text{if } F_{ji,t} < U_j \\ 0, & \text{else,} \end{cases} \tag{3.12}$$

and u_i is the ceiling constraint as explained in equation (3.18). If we apply the screening monitoring at the portfolio level as proposed by Derigs and Marzban (2009), then we add constraint (3.13)

$$\sum_{i=1}^{N} x_{i,t} F_{ji,t} < U_j, \; j = 1, \ldots, 4, \; t = 0, \ldots, T - \Delta t. \tag{3.13}$$

Shariah Sustainibility Constraint

Since the Islamic equity investment process is dynamic in nature, and new financial information and corporate news are published over time, the screening financial ratio results change over time. These events have an impact on Shariah compliance. An asset may turn from compliant to non-compliant, which results into a forced liquidation of the asset position from the portfolio, if included. For example, the company may become highly leveraged through rising capital in the form of debt as we have seen in Figure 2.2.

Marzban (2008, 2011) emphasize the importance of maintaining the status of compliance in the Shariah portfolio for two reasons:

1. Sustainable shariah compliance.
   Islamic investors are investing in Islamic products because they want the investment to comply with their beliefs and religious principles. This means that Islamic investors prefer Shariah-compliance over speculative return and risk. Thus, for some Islamic investors it is important that the company they are investing in is conventionally compliant over time and not accidentally compliant over the current time period only.

2. Better Fund Performance
   In a shariah portfolio, if the asset switches from being compliant to non-compliant, then the investors need to liquidate the assets. When the market is bearish, then the investors may experience a substantial loss. If the investors or fund managers can predict the possibility of the asset being liquidated, then they can mitigate the loss.

In Chapter 2.2.3 we have proposed a probabilistic constraint to measure the possibility of the Shariah portfolio to turn to be non-compliant in the future period. Suppose that $\alpha \in (0, 1]$ is the maximum acceptable probability that a financial ratio exceeds a permissible boundary over a future time interval. If we apply screening at the asset level, then an asset with a sustainable probability smaller than a preferred level $1 - \alpha$ can be eliminated prior to the optimization. Hence, the probabilistic constraint under the asset level screening assumption can be described as

$$0 \leq x_{i,t} \leq Z_{i,t+\Delta t}, \; i = 1, .., N, \tag{3.14}$$

where

$$Z_{i,t+\Delta t} = \min\{z_{ji,t+\Delta t}\}_{j=1}^{4} \tag{3.15}$$

$$z_{ji,t+\Delta t} = \begin{cases} u_i, & \text{if } \Pr(F_{ji,t+\Delta t} < U_j | F_{ji,t}) > 1 - \alpha \\ 0, & \text{else.} \end{cases} \tag{3.16}$$

If we monitor the screening process at the portfolio level, then we apply the following constraint

$$\Pr\left(\sum_{i=1}^{N} x_{i,t} F_{ji,t} < U_j\right) > 1 - \alpha, \; j \in \{1, \ldots, 4\}. \tag{3.17}$$

Floor and ceiling constraints

By using floor and ceiling constraints, fund managers are defining their preferences and binding requirements in terms of maximum (ceiling) and minimum (floor) permissible investment in each of the assets considered for investment. These restrictions can be formulated as follow:

$$l_i \leq x_{i,t} \leq u_i, \; i = 1, \ldots, N, \; t = 0, \ldots, T - \Delta t, \tag{3.18}$$

where l^i and u^i are the lower and the upper limits, respectively, of asset i.

Shortselling restriction

As discussed in Chapter 2, most Islamic scholars agree that shortselling is prohibited in Shariah capital market because muslim investors can not sell something what the investors do not own (Dusuki and Abozaid, 2008). Only the Shariah Advisory Council of the Securities Comission of Malaysia (SAC) went againts the majority by permitting short selling in Malaysia since 2006 (Sifat and Mohamad, 2016). If shortselling is prohibited, then we need to add the following constraint:

$$x_{i,t} \geq 0,\ i = 1, \ldots, N,\ t = 0, \ldots, T - \Delta t. \tag{3.19}$$

Portfolio leverage constraint

Let $x_{0,t} = 1 - \sum_{i=1}^{N} x_{i,t}$ be the allocation of the riskless asset at time t. The negative value of $x_{0,t}$ means that the investor has sold the risk-free asset to buy more of the risky asset. Krichene (2013) allows a negative value of $x_{0,t}$ in their constructed Shariah portfolio model. In practice, investors may borrow money from the Islamic money market by selling Sukuk or other Shariah compliant fixed income instruments. However, this facility is only applicable for investors who can issue money market instruments, like government Sukuk or corporate Sukuk.

In a conventional capital market, small or retail investors can use margin facility to leverage their portfolio. However, margin trading with interest is forbidden in Shariah capital market. Therefore, if we assume that there is no leveraging facility, then we need to impose a leverage constraint:

$$\sum_{i=1}^{N} x_{i,t} \leq 1,\ t = 0, \ldots, T - \Delta t. \tag{3.20}$$

3.2 Proposed method to solve the multiperiod shariah portfolio optimization problem

Recall that the objective of a multiperiod portfolio selection problem is finding a set of asset allocations $\mathbf{x}_t,\ t = 0, 1, .., T - \Delta t$, that maximize value function (3.1) subject to wealth restriction (3.5). This optimization problem belongs to a class of dynamic optimization problems (sometimes called dynamic programming), because it is based on a sequence of interrelated optimal decisions (Hillier and Lieberman, 2015). The dynamic programming approach is usually solved by a backward recursive procedure. Under this method, we need the separabality condition of the objective function (Stevanov, 2001).

Unfortunately, our objective function (3.1) is not a separable function due to the non-linearity of conditional variances. The expectation is a linear function because for all $s \leq t$, $E[E[W_T|\mathcal{F}_t \vee \mathcal{G}_t|\mathcal{F}_s \vee \mathcal{G}_s]] = E[W_T|\mathcal{F}_s \vee \mathcal{G}_s]]$, however the variance function does not satisfy the analogous condition $Var[Var[W_T|\mathcal{F}_t \vee \mathcal{G}_t]]|\mathcal{F}_s \vee \mathcal{G}_s]] \neq Var[W_T|\mathcal{F}_s \vee \mathcal{G}_s]]$. Hence, we need to transform the objective function into a tractable separable function. We use the same transformation as the one applied by Li and Ng (2000), Cong and Oosterlee (2016), and Wang and Forsyth (2010), namely:

$$J_t(W_t) := \min_{\{\mathbf{x_s}\}_{s=t}^{T-\Delta t}} E[(W_T - \frac{\gamma}{2})^2|\mathcal{F}_t \vee \mathcal{G}_t] \tag{3.21}$$

where $\gamma = \frac{1}{\lambda} + 2E_{\mathbf{x}^*}[W_T|\mathcal{F}_t \vee \mathcal{G}_t]$ for $t = 0, .., T - \Delta t$.

Li and Ng (2000) have proved that if the excess return $R_{t,i}^{ps}$ is statistically independent, then the solution of the LQ problem (3.21) is identical to the solution of the problem (3.1). We adopt this approach in our study, and from now on our objective function is the quadratic problem (3.21) subject to the budget constraint (3.5) and the Shariah constraints. In the literature, this approach is called a pre-commitment strategy (Basak and Chabakauri (2010),Cong and Oosterlee (2017b)). The advantage of this approach is that the Bellman dynamic programming principle is applicable to the objective function (3.21), and hence the backward recursive method can be applied to solve this optimization problem.

In this study, we implement backward dynamic programming through Monte Carlo based simulation proposed by Cong and Oosterlee (2016). The authors apply backward differential dynamic programming, and impose a multi-stage strategy solution as the initial solutions to the backward approach. Firstly, the multi stage strategy is performed in a forward recursive scheme to generate the initial values of $\{\mathbf{x}_t\}_{t=1}^{T-\Delta t}$. This step is known as the forward algorithm procedure. Next, the solution is updated through the dynamic programming in a backward recursive manner, which is called a backward algorithm procedure. We apply the bundling and regression technique in the backward algorithm. While Cong and Oosterlee (2016) apply the method to the conventional portfolio problem with a bounded or a box constraint, we adopt the method to the Shariah portfolio problem with more general constraints as described in Subsection 3.1.1. The next two subsections describe in detail the forward and backward algorithms that we have used in this research.

3.2.1 Forward Algorithm

The purpose of the forward algorithm is to find a sub-optimal solution, which we denote by $\{\tilde{\mathbf{x}}_t\}_{t=0}^{T-\Delta t}$. The idea of this method is to find the optimal asset allocations such that the wealth W_t achieves a certain target level at time t, for each $t \in [\Delta t, \dots, T]$. Then, at time t, we put all of the money in the risk-free asset so that the wealth at the terminal time is equal to the final target level $\frac{\gamma}{2}$. Therefore, to achieve the final target level, we set the intermediate target level as the final target's discounted value. This method is known as the multistage strategy (Cong and Oosterlee, 2016). Mathematically, we can represent the above strategy as follow:

$$\tilde{\mathbf{x}}_t := \arg\min_{\mathbf{x}_t} E\big[(W_t(\mathbf{x_t}'\mathbf{R}^{ps}_{t+\Delta t} + R_0) - \tau_{t+\Delta t})^2 | \mathcal{F}_t \vee \mathcal{G}_t\big],\ t = 0, \dots, T - \Delta t \tag{3.22}$$

where

$$\tau_t = \frac{\gamma}{2}(R_0)^{-(T-t)\Delta t}. \tag{3.23}$$

For a problem without any constraints, the optimal allocation in problem (3.22) can be calculated analytically as follow:

$$\tilde{\mathbf{x}}_t = \frac{(\tau_{t+\Delta t} - W_t R_0)}{W_t} \sum\nolimits^{-1} \mu^s \tag{3.24}$$

where

$$\mu^s = [E[R^{ps}_{1,t}] \cdots E[R^{ps}_{N,t}]]$$

and

$$\sum = \begin{bmatrix} Var(R^{ps}_{1,t}) & \cdots & Cov(R^{ps}_{1,t}, R^{ps}_{N,t}) \\ \vdots & \ddots & \vdots \\ Cov(R^{ps}_{N,t}, R^{ps}_{1,t}) & \cdots & Var(R^{ps}_{N,t}). \end{bmatrix}$$

The corresponding value function at time t is:

$$J_t(W_t) = (W_t R_0 - \tau_{t+\Delta t})^2 (1 - (\mu^s)^T \sum\nolimits^{-1} \mu^s). \tag{3.25}$$

To find the optimal allocation for a portfolio consisting of one risky asset and one risk-free asset with box constraints (for example, constraint (3.18)), we solve first the unconstrained problem using (3.24). Then, we penalize the solution on the bounded constraints by comparing the value functions (3.25). When solving a portfolio problems with 2 risky assets and 1 risk-free asset, this procedure is equivalent to solving five portfolio problems with 1 risky asset and 1 risk-free asset, and then choosing the set of allocations with the lowest value function. For problems with a larger number of risky assets and bounded constraints, we can adopt the same idea. To solve a problem with more complex constraints (for example, constraints (3.13), (3.17), and (3.20)), we use a Quadratic Programming method by means of a numerical approach. In this research we use *quadprog* function in Matlab.

The following is the procedure that we have used to find the optimal solution for the Shariah-compliant portfolio based on the multistage strategy algorithm.

- Step 1. Choose the final target level γ and specify a set of parameters and the distribution of the purified asset return $R^{ps}_{i,t}$ and the financial ratio $F_{ji,t}$. We assume that the following information is given:
  - initial wealth W_0
  - risk-free asset return $R_0 \geq 1$
  - the definition of the purification process, either dividend purification ($s = 1$) or investment purification ($s = 2$)
  - the screening permissible boundary $U_j,\ j = 1, \ldots, 4$
  - the set of Shariah constraints that is described in Subsection 3.1.1 based on either the chosen Shariah ruling sources or the investor preferences. For example, whether the leverage facility is permitted or not (constraint (3.20)), and the screening process at the asset level (constraint (3.10)) or at the portfolio level (constraint (3.13)).
- Step 2. Generate n paths of the purified excess return $R^{ps}_{i,t}$ for $i = 1, \ldots, N$ and $t = \Delta t, \ldots, T$. Also simulate the financial ratio $F^i_{j,t}$ for $i = 1, \ldots, N$, $j = 1, \ldots, 4$, and $t = 0, \ldots, T$.
- Step 3. Calculate the intermediate target values τ_t for $t = \Delta t, \ldots, T$ as described in (3.23).
- Step 4. Starting at the initial state, and for each path $k = 1, \ldots, n$, solve problem (3.22) subject to specified constraints to get optimal asset allocation $\tilde{\mathbf{x}}_t(k)$:

  $$\tilde{\mathbf{x}}_t(k) = \arg\min_{\mathbf{x}_t} E\big[(W_t(\mathbf{x_t}'\mathbf{R}^{ps}_{t+\Delta t}+R_0)-\tau_{t+\Delta t})^2|W_t = W_t(k), \mathcal{F}_t \vee \mathcal{G}_t\big],\ t = 0, \ldots, T-\Delta t.$$

  Once we get $\tilde{\mathbf{x}}_t(k)$, we can calculate $W_{t+\Delta t}(k) = W_t(k)\big(\tilde{\mathbf{x}}_t(k)\mathbf{R}^{ps}_{t+\Delta t}(k) + R_0\big)$. Then we use $W_{t+\Delta t}(k)$ to find $\tilde{\mathbf{x}}_{t+\Delta t}(k)$. Repeat this step in forward fashion, starting from the initial time until time $T - \Delta t$.

3.2.2 Backward Algorithm

The solution that we have found in Section 3.2.1 is the sub-optimal solution. To get the optimal asset allocation, we need to continue the process through a backward algorithm that will be discussed in this Subsection. We use the sub-optimal asset allocation $\tilde{\mathbf{x}}_t(k)$ and the corresponding wealth $W_{t+\Delta t}(k)$ for $t = 0, \ldots, T - \Delta t$ and $k = 1, \ldots, n$, that we have found through the forward algorithm in Section 3.2.1 as the initial guess of the optimal solution for backward algorithm.

Suppose that A is the admissible set for the asset allocation based on the specified constraints from Step 1 of the forward algorithm in the Subsection 3.2.1. Then we can rewrite the objective function (3.21) into the following recursive relation

$$J_t(W_t) = \min_{\mathbf{x_t} \in A} E[J_{t+\Delta t}(W_{t+\Delta t})|\mathcal{F}_t \vee \mathcal{G}_t], \tag{3.26}$$

$$= \min_{\mathbf{x_t} \in A} \{E[J_{t+\Delta t}(W_t(\mathbf{x_t}\mathbf{R}_t^{ps} + R_0))|\mathcal{F}_t \vee \mathcal{G}_t]\} \tag{3.27}$$

with $J_T(W_T) = (W_T - \frac{\gamma}{2})^2$. In general, this kind of problem needs to be solved using a backward recursive approach. In this research we adopt the backward programming method of Cong and Oosterlee (2016) to solve the portfolio problem (3.27). While Cong and Oosterlee (2016) is limited to the conventional portfolio problem with a bounded or box constraint problem, we apply the algorithm to the Shariah portfolio problem with more complex constraints as described in the Subsection 3.1.1.

A common approach to the problem described in (3.27) is to apply a simulation based approach by varying $\mathbf{x_t}$ around the initial solution $\tilde{\mathbf{x}}_t$. Firstly we need to construct an admissible control set $A_\eta = [\tilde{\mathbf{x}}_t - \eta, \tilde{\mathbf{x}}_t + \eta]$. Then we transform the recursive problem (3.27) into the following truncated problem:

$$J_t(W_t) = \min_{x_t \in A} \{E[J_{t+\Delta t}(W_{t+\Delta t})|\mathcal{F}_t \vee \mathcal{G}_t,\ W_{t+\Delta t} \in D_{t+\Delta t}\} \tag{3.28}$$

where

$$D_{t+\Delta t} := \{W_{t+\Delta t}|W_{t+\Delta t} = W_t.(\mathbf{x}_t\mathbf{R}_t^{ps} + R_0), \mathbf{x_t} \in A_\eta\}. \tag{3.29}$$

Using this transformation, we can solve the local optimization problem in the restricted domain $D_{t+\Delta t}$. We apply a Stochastic Grid Bundling Method (SGBM) to solve this truncated problem. The SGBM method was firstly introduced by Jain and Oosterlee (2015) to solve the Bermudan option pricing problem. This method combines the Monte Carlo path simulation and the dynamic programming techniques to determine optimal policies. To increase its computational efficiency, the authors propose regression and bundling techniques to approximate the continuation value function $J_t(W_t)$ at each time step.

The bundling technique is used to approximate domain (3.29) by:

$$\hat{D}_{t+\Delta t} = \{W_{t+\Delta t} | W_{t+\Delta t} = \hat{W}_t(\tilde{\mathbf{x}}_t.\mathbf{R}_t^{ps} + R_0),\ \hat{W}_t \in B_d\} \tag{3.30}$$

where $B_d = [W_t - d, W_t + d]$. By using this approximation, we vary W_t by considering the paths with the state are around (t, W_t). In order to solve the truncated problem (3.28) at state (t, W_t), we need to determine the value function $J_{t+\Delta t}(W_{t+\Delta t})$ conditional on $W_{t+\Delta t} \in \hat{D}_{t+\Delta t}$. In this research we employ the regress-later approach to approximate the value function. For all paths in the domain B_d, we regress $\{J_{t+\Delta t}(W_{t+\Delta t})\}$ on the polynomial up to order two formed by $\{W_{t+\Delta t}\}$ and obtain the parameters $\{a_p\}_{p=0}^2$. Suppose that $\mathbf{x}_t^{*b}$ is the optimal control of problem (3.28). The value function is then approximated by:

$$J_t^b(W_t) = E[J_{t+\Delta t}(W_{t+\Delta t}) | \mathcal{F}_t \vee \mathcal{G}_t, \mathbf{x}_t^{*b}] \approx E[\sum_{p=0}^{2} a_p W_{t+\Delta t}^p | \mathcal{F}_t \vee \mathcal{G}_t, \mathbf{x}_t^{*b}]. \tag{3.31}$$

The reason for using polynomials up to order two for the regression basis function is the fact that the value function is quadratic. In the local optimization problem, the truncated problem is considered a piecewise quadratic function. Hence, the polynomial up to order two should be sufficient (Fu et al., 2010).

After employing the forward algorithm described in Subsection 3.2.1, we employ the backward algorithm in the following steps:

- Step 1 Generate an initial guess of asset allocations $\tilde{\mathbf{x}}_t(k)$ and the corresponding wealth $W_{t+\Delta t}(k)$ for time $t = 0, \ldots, T - \Delta t$ and path $k = 1, \ldots, n$, and calculate the initial value function $J_T(W_T(k)) = (W_T(k) - \frac{\gamma}{2})^2$
- Step 2 Solve the objective problem (3.28) using the following bundling and regression technique:
    1. Bundle paths into B partitions, with each bundle having n_B paths so that the paths inside the bundles have a similar number of paths and similar wealth values at time t. Denote the bundles by $\{W_t^b(k)\}_{k=1}^{n_B}$.
    2. Determine $J_{t+\Delta t}^b = f_{t+\Delta t}^b(W_{t+\Delta t}^b)$ by regression using a polynomial up to order two.
    3. Find the minimum function of $E[f_{t+\Delta t}^b(W_{t+\Delta t}^b) | W_t = W_t^b(k), \mathcal{F}_t \vee \mathcal{G}_t]$ under the constrain $D_{t+\Delta t}$ to get the optimal asset allocations $\{\hat{\mathbf{x}}_t^b(k)\}_{k=1}^{n_B}$.

4. Since at time t we have $\{W_t^b(k)\}_{k=1}^{n_B}$ and the allocations $\{\hat{\mathbf{x}}_t^b(k)\}_{k=1}^{n_B}$, we can calculate the value function $\{\hat{J}_t^b(k)\}_{k=1}^{n_B}$ using regression as described by equation (3.31).

- Step 3 Update the initial guess of asset allocations through the following procedure:

  1. Calculate $\{\tilde{J}_t^b(k)\}_{k=1}^{n_B}$ using the old guess $\{\tilde{\mathbf{x}}_t^b(k)\}_{k=1}^{n_B}$ by following the same approach as in step 2.4
  2. If $\tilde{J}_t^b(k) > \hat{J}_t^b(k)$, we choose $\hat{\mathbf{x}}_t^b(k)$, otherwise we keep the old guess $\tilde{\mathbf{x}}_t^b(k)$ as the new optimal solution. We denote the updated allocations by $\{\mathbf{x}_t^{*b}(k)\}_{k=1}^{n_B}$

- Step 4 Use the updated allocations $\{\mathbf{x}_t^{*b}(k)\}_{k=1}^{n_B}$ to calculate the value function $\{J_t^b(k)\}_{k=1}^{n_B}$. Use these updated values to calculate the optimal allocations at $t - \Delta t$. We iterate this process in a backward fashion until the initial time to get $\{\mathbf{x}_t^{*b}(k)\}_{k=1}^{n_B}$ for $t = 0, \ldots, T - \Delta t$.

After one iteration of the backward algorithm, we will get one set of the updated asset allocation. Cong and Oosterlee (2016) apply Bellman operator in Equation (3.27) to prove that the backward recursive updating process is monotone in which $J_0^1(W_0) \geq J_0^2(W_0) \geq \cdots \geq J_0^l(W_0)$ and the value function will converge to $J_0^*(W_0) = \lim_{l\to\infty} J_0^l(W_0)$ with J_0^l being the value function at time 0 after l iterations of the backward procedures. Hence, we can perform the backward algorithm in several iterations to get a satisfactory result. In our research, we use the solution from the multi-stage strategy obtained in Subsection 3.2.1 as the initial guess for the first iteration. Then, we use the updated solution from the backward procedure obtained in Step 4 as the initial solution of the next iteration of the backward procedure.

3.3 Numerical Examples

In this section, we apply the proposed method described in Section 3.2 to generate the efficient frontiers of Shariah portfolios. An efficient frontier is generated by assigning different values of γ and solving the corresponding problem. As we obtain the optimal asset allocations for each γ, we can calculate the mean and standard deviation of the optimal portfolio's wealth at the terminal time denoted by $E[W_T]$ and $Std[W_T]$, respectively.

3.3.1 Conventional portfolio with 1 risky asset and 1 non-risky asset

Before we apply the optimization method to the Shariah portfolio problem, we check the performance of the proposed method in the context of a conventional portfolio consisting of 1 risky asset and 1 risk-free asset. In the conventional portfolio, there is no purification process. We denote the risky asset rate of return without purification as $R^{p0}_{1,t}$. In our simulation study we use the same input parameters as in Bielecki et al. (2005) :
$R_0 = 1.06, R^{p0}_{1,t} \sim N(\mu_1, \sigma_1^2)$ with $\mu_1 = 0.12, \sigma_1 = 0.15$, $T = 1$ year, and $W_0 = 1$. Zhou and Li (2000) provide the analytical solution (3.32) for this problem in a continuous time setting

$$E[W_T] = 0.0618 + 0.4165 Std[W_T]. \tag{3.32}$$

In addition to the unconstrained conventional portfolio, We also add a no-bankruptcy constraint to the conventional portfolio, i.e :

$$W_t(x_{1,t}R^{p0}_{1,t+\Delta t} + R_0) \geq 0, \; t = 0, \ldots, T - \Delta t. \tag{3.33}$$

We impose the following constraint to ensure that the no-bankruptcy condition (3.33) is valid with level of certainty $1 - 2\nu$:

$$\frac{-R_0}{R^{p0,1-\nu}_{1,t}} \leq x_{1,t} \leq \frac{-R_0}{R^{p0,\nu}_{1,t}}, \; t = 0, \ldots, T, \tag{3.34}$$

where $R^{p0,1-\nu}_{1,t}$ and $R^{p0,\nu}_{1,t}$ are the ν- and $1-\nu$- quantiles of the excess return $R^{p0}_{1,t}$, respectively. In this example we set $\nu = 10^{-8}$.

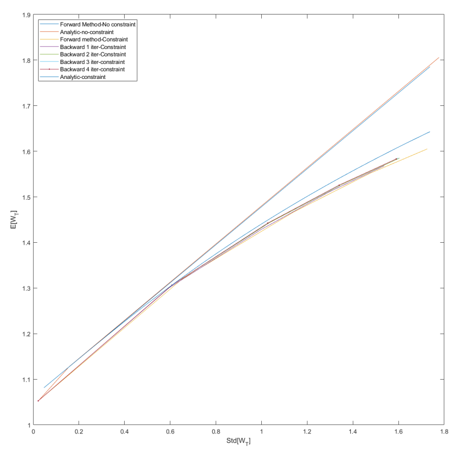
Bielecki et al. (2005) derived an analytical solution for continuous mean variance problem with a no-bankruptcy constraint, which is given by:

$$E[W_T] = \frac{e^{0.06}\eta N(\frac{\ln\eta + 0.14}{0.4}) - N(\frac{\ln\eta - 0.02}{0.4})}{\eta N(\frac{\ln\eta - 0.02}{0.4}) - e^{0.1} N(\frac{\ln\eta - 0.18}{0.4})} - 1, \tag{3.35}$$

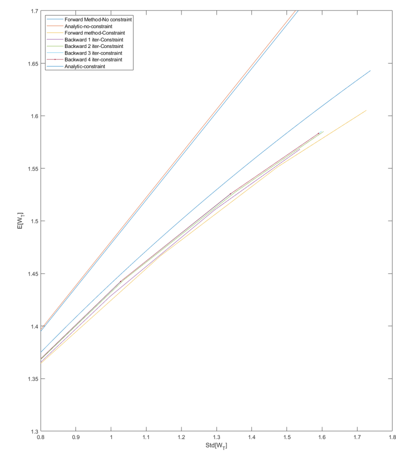
$$\begin{aligned} Std[W_T] = \Big[\Big(&\frac{\eta}{\eta N(\frac{\ln\eta+0.14}{0.4}) - e^{-0.06}N(\frac{\ln\eta-0.02}{0.4})} - 1\Big)(E[W_T]+1)^2 - \\ &\frac{1}{\eta N(\frac{\ln\eta+0.14}{0.4}) - e^{-0.06}N(\frac{\ln\eta-0.02}{0.4})}(E[W_T]+1)\Big]^{0.5} \end{aligned} \tag{3.36}$$

with $\eta \in [0, \infty)$.

We plot the analytical solution (3.32) for the non-constraint problem and solution (3.35) and (3.36) for no-bankruptcy problem in Figure 3.1. We also generate the efficient frontier using the assumption of a discrete time model, as proposed in this research, with $M = 32$ and sample size $n = 50000$. We provide the results from the forward and backward methods with up to four iterations.



(a) Simulation Results



(b) Zoom

Figure 3.1: Simulation results with no-constraint using the forward method M=32, and bankruptcy constraint using the forward method with M=32, and the backward method with M=32 and the number of iteration up to 4. The results are compared with the analytical solution (3.32) with no constraint, and analytical solution (3.35) with constraints.

For the unconstrained problem, the efficient frontier generated by the forward method is very close to the analytical solution. This result is consistent with the Cong and Oosterlee (2016) findings that for the non-constraint problem, the solution from the forward method is equivalent to the optimal asset allocation for the problem (3.21). Hence for the unconstrained problem the multistage strategy provides accurate approximations to the optimal solution. Note that the analytical solution is provided for continuous time rebalancing, while our simulation uses the assumption of discrete time model. Hence slightly different results between the simulation and analytical solution are reasonable.

For the constrained problem, the simulation result generated by the forward method underestimates the analytical result. After applying four iterations of the backward recursive method, the efficient frontier gets closer to the analytical solution. Cong and Oosterlee (2016) also apply 4 iterations of the backward method, which produce satisfactory results. In order to generate efficient frontiers for the Shariah portfolio problem that we consider in the next section, we utilize the forward and the backward methods with 4 iterations.

3.3.2 Shariah Compliant Portfolio with 1 Risky Asset and 1 Risk-free Asset

Incorporating purification and restriction to shortselling and leverage facility

In this part we provide some simulation results to examine the impact of Shariah variables, namely: no-shortsell, no-leverage, and purification on the performance of portfolio containing 1 risky asset and 1 risk free asset. In this simulation study, instead of defining the stochastic dynamics of the capital appreciation rate $R^c_{i,t}$, dividend yield $R^d_{i,t}$, and purification rate $P^s_{i,t}$ separately, as in Subsection 2.3.1, we define the dynamic of the excess purified return $R^{ps}_{i,t}$ itself. The problem with modeling the different components separately is that the independence assumption that is required for the validity of our algorithm is not satisfied. We can see from Figure 2.1 that the purification rate exhibits some temporal dependence. In each year, the data has an upward trend, with the lowest value in the first quarter and highest at the fourth quarter. This trend can be explained by the fact that the purification rate is a function of NPII (non permissible income including interest) (see Equation (2.1)). The quarterly NPII data is the accumulated data within the same year and hence this data is lowest at the beginning and highest at the end of the year. This suggests that the assumption that the purification rate is statistically independent is not realistic. However, the independence assumption is required in the mean-variance portfolio with pre-commitment approach (Cong and Oosterlee, 2016; Li and Ng, 2000). To overcome this problem, we model the dynamic of the purified return $R^{ps}_{i,t}$ itself, which can be justified by the fact that the independence assumption is acceptable for most of stock return data and usually the purification rate is of much smaller magnitude than the return itself. In our simulation study we assume that the risky asset follows a geometric Brownian motion with drift μ^{ps}_i and volatility σ^{ps}_i.

Figure 3.2 shows the empirical data of quarterly purified return of ASII.JK from 2013 to 2018 that we have used in this simulation. The dividend and investment purification are calculated by using Equation (3.2). Then the purified asset return is calculated by the following equation

$$R^{ps}_{i,t} = R^e_{i,t} + R^d_{i,t} - P^s_{i,t},\ \ s \in \{1, 2\}. \tag{3.37}$$

We may see from Figure 3.2 that return data with dividend purification is always higher than the return data with investment purification. This is because the investment purification rate is higher than the dividend purification rate data (see Figure 2.1). We also calculate the purified asset return based on the Shariah portfolio model proposed by Hazny et al. (2020) and Derbali et al. (2017) in Equation (3.38), in which

$$R_{i,t}^{p3} = (1 - \tilde{\psi}_i)(1 - \zeta)(R_{i,t}^e + R_{i,t}^d), \tag{3.38}$$

where $\tilde{\psi}_i$ is the constant purification percentage calculated by

$$\tilde{\psi}_i = \frac{NPII}{TI}. \tag{3.39}$$

In this formula, NPII and TI denote the non permissible income including interest and total income. We retrieve the NPII and TI historical data from the company's financial statement, and calculate the average NPII/TI to get the constant purification percentage $\tilde{\psi}_i$. In this simulation we set the zakat rate $\zeta = 0$. Based on the empirical data that we have retrieved from the ASII quarterly financial statements from 2013 to 2018, we find $\tilde{\psi}_1 = 0.23619$. As we can see in Figure 3.2, the purified return data with constant purification proposed by Hazny et al. (2020) and Derbali et al. (2017) have higher values than the return data without purification when the non purified return data is negative. With a zero zakat rate, we can write the amount of purification rate from (3.38) as $\tilde{\psi}_i(R_{i,t}^e + R_{i,t}^d)$, which will result in a negative value when $R_t^e + R_t^d$ is negative. However, in reality the amount of purification is never negative, so the purified asset return is always less than or equal to the non purified return. To deal with this problem, we modify the constant purified return by

$$R_{i,t}^{p4} = (1 - \tilde{\psi}_i)(1 - \zeta)\max(R_{i,t}^e + R_{i,t}^d, 0). \tag{3.40}$$

In equation (3.40), we assume that the purification will be processed only if investors get a positive return.

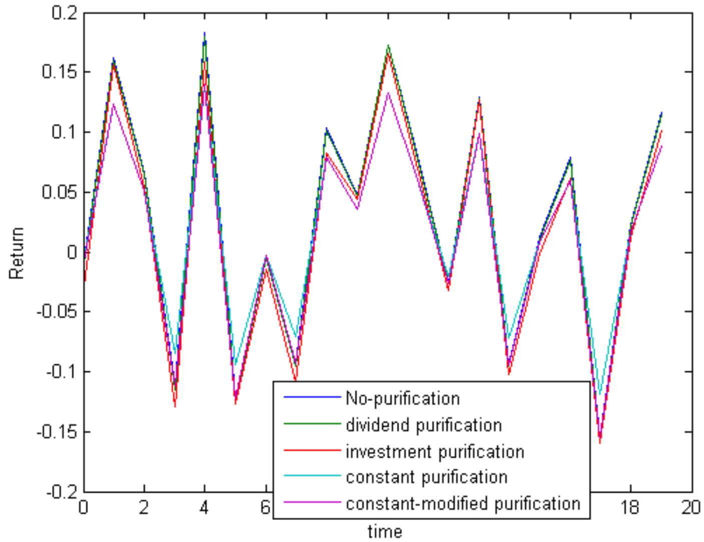


Figure 3.2: ASII empirical quarterly return data without purification and with several types purification from 2013-2018

We conduct a Jarque Berra test to check normality of returns and purified returns. The p-values at the 5% significant level from the J-B tests are reported in Table 3.1. We denoted ASII excess return data as $R_{1,t}^{p0}$, and the purified return data as $R_{1,t}^{ps}$, with $s = 1$ for dividend purification return, $s = 2$ for investment purification return, $s = 3$ for constant purification return, and $s = 4$ for modified constant return. Based on Table (3.1), the test indicates that ASII return and purified return data are consistent (at 5% significance level) with the null hypothesis of the skewness and kurtosis matching a normal distribution. Additional evidence based on histogram and QQ-plots (see Appendix C.1) also support the normality assumption on modeling these excess return and purified return data. The sample mean and sample standard deviation of ASII return and purified return data are in Table 3.2. Other input parameters needed in the simulation study are given in Table 3.3.

Table 3.1: p-values from Jarque-Berra normality test on the ASII return and purified return data

Variable	$R_{1,t}^{p0}$	$R_{1,t}^{p1}$	$R_{1,t}^{p2}$	$R_{1,t}^{p3}$	$R_{1,t}^{p4}$
p-value	0.3495	0.3452	0.3309	0.3495	0.2685

Table 3.2: Sample Means and Sample Standard Deviations of ASII return and purified return data

Variable	$R_{1,t}^{p0}$	$R_{1,t}^{p1}$	$R_{1,t}^{p2}$	$R_{1,t}^{p3}$	$R_{1,t}^{p4}$
μ_1^{ps}	0.02778	0.02605	0.01623	0.02122	0.01393
σ_1^{ps}	0.10372	0.10295	0.10384	0.07932	0.08997

Table 3.3: Input parameters for Shariah portfolio consisting 1 risky asset and 2 risky assets

Parameter	R_0	T	W_0	M	n
Value	1.01	5 (years)	1	20	50000

In this simulation we also assume no short selling and no leverage constraints, that is

$$x \in [0, 1]. \tag{3.41}$$

Figure 3.3 shows the efficient frontiers for the un-constrained conventional portfolio, constrained conventional portfolio, and constrained Shariah portfolio with 4 types of purification. Adding shortselling and leverage constraints to the conventional portfolio produce a lower efficient frontier. This result is to be expected because the optimal solution is restricted to the constraints (3.41). Incorporating a purification process to the constrained portfolio reduces the total return of the portfolio, and hence it produces a lower efficient frontier. The Shariah portfolio with the dividend type purification yields a better performance than the investment purification. This result is consistent with Figure 3.2 showing that the dividend purified return is always higher than the investment purified return. Because the dividend purification only occurs when the company issues dividend and only applies to the dividend, it is expected that Shariah portfolio with the dividend purification yields a better performance when compared with the investment purification.

The efficient frontier from the Shariah portfolio with constant purification proposed by Hazny et al. (2020) and Derbali et al. (2017) lies between the Shariah portfolio with dividend and investment types purification. The modified version of constant purification yields the worst performance among other type of purification. Both constant and modified constant purification are carried out regardless whether or not the dividend is issued, and it applies to the total return (capital gain and dividend). This process is similar to the investment purification. The difference is in the calculation methods: while the investment purification is the percentage of NPI to the total outsanding share (see Equation (2.1)), the constant purification is the percentage of NPI to TI (see Equation (3.39)). The negative values of purification rate from Hazny et al. (2020) and Derbali et al. (2017) contribute to a higher efficient frontier compared to the corresponding modification purification. Our proposed investment purification has a better performance than the constant modified purification, although the last type of purification only applies when the return is positive. Hence, if the purification should be applied to both capital appreciation and dividend yield, then our proposed investment purification is a better option for this case.

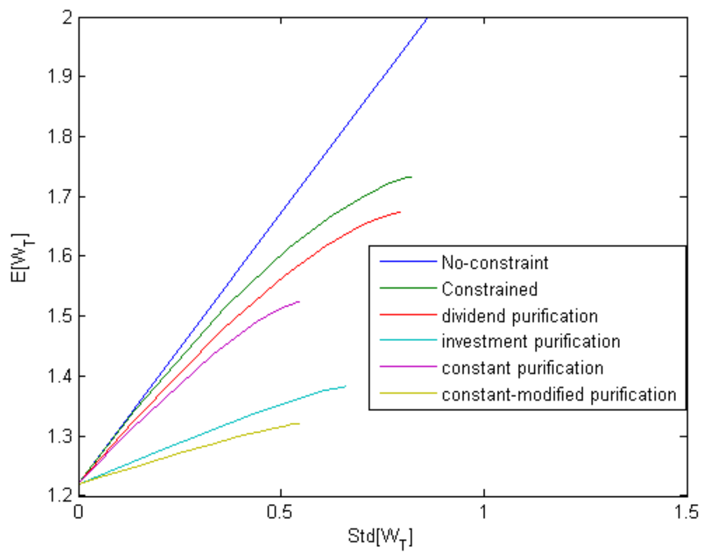


Figure 3.3: Efficient frontiers for conventional portfolio, conventional portfolio with constraint, and Shariah portfolio with four purification types without screening

Incorporating Purification, shortselling and leverage restrictions, and Shariah screening constraints

In this section, in addition to purification and constraint as formulated in (3.41), we incorporate screening constraints to construct a Shariah portfolio efficient frontier. We apply two different modeling assumptions of financial ratios, namely a beta distribution and a beta-AR(1) distribution. The parameters are estimated based on empirical data of the ASII financial ratios from 2013 to 2018 that we have retrieved from its quarterly financial statements. In our simulation, we use the Total Asset divisor to calculate financial ratios. The historical ASII financial ratios are depicted in Figure 3.4.

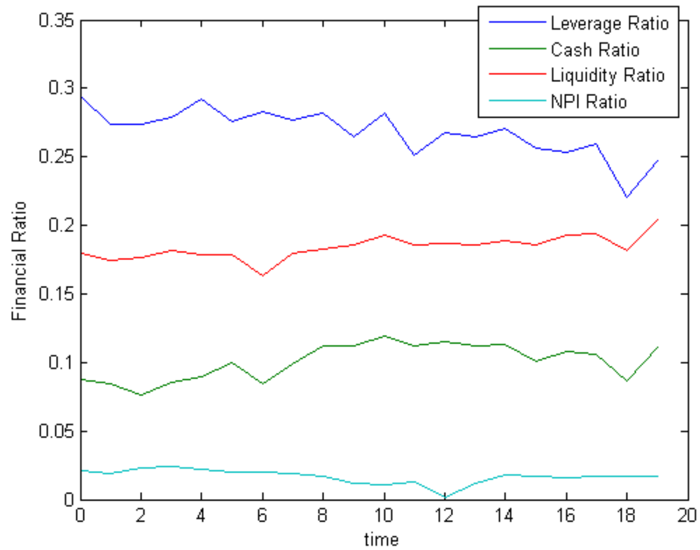


Figure 3.4: ASII empirical quarterly return data without purification and with several types purification from 2013-2018

For the Beta distribution assumption, we assume that $F_{j1,t} \sim \text{Beta}(\alpha^B_{j1}, \beta^B_{j1})$. We apply a *betafit* function and *adtest* function in Matlab software to estimate the parameters and calculate p-values from the Anderson-Darling test. The estimated parameters, and the corresponding p-values, are reported in Table 3.4. The Anderson-Darling test indicates that all ASII financial ratios are consistent with the null hypothesis (at 5% significance level) that the data follows a beta distribution with estimated parameter α^B_{j1} and β^B_{j1}.

Financial ratio	$F_{11,t}$	$F_{21,t}$	$F_{31,t}$	$F_{41,t}$
α^B_{j1}	179.6034	53.0245	402.9962	4.9817
β^B_{j1}	489.4	472.6	1787.7	295.2
p-value	0.8597	0.3776	0.8801	0.1245

Table 3.4: Beta distribution estimated parameters and p-values from ASII financial ratios data

For time-dependent model, we assume that the j-th financial ratio follows a beta-AR(1) model described in subsection 3.1.1 with estimated parameters $\alpha^A_{j1}, \beta^{A1}_{j1}$, and δ_{j1}. We apply a method of moments as described in Algorithm 1 of Appendix C.2 to be an initial guess of the maximum likelihood parameters of the beta-AR(1). Then, we use *fminsearch* function in Matlab to find the estimated parameters that maximize the conditional log-likelihood function:

$$\ell_{ji} = \sum_{t=2}^{S} \ell_t(\pi_{ji,t}, \delta_{ji}), \tag{3.42}$$

where S is the sample size, and $\ell_t(\pi_{ji,t}, \delta_{ji})$ is the logarithm of the likelihood function of beta-AR(1):

$$\begin{aligned}\ell_t(\pi_{ji,t}, \delta_{ji}) = &\log\Gamma(\delta_{ji}) - \log\Gamma(\delta_{ji}\pi_{ji,t}) - \log\Gamma(\delta_{ji}(1-\pi_{ji,t})) + (\pi_{ji,t}\delta_{ji} - 1)\log F_{ji,t} \\ &+ \big[(1-\pi_{ji,t})\delta_{ji} - 1\big]\log(1 - F_{ji,t}),\end{aligned}$$

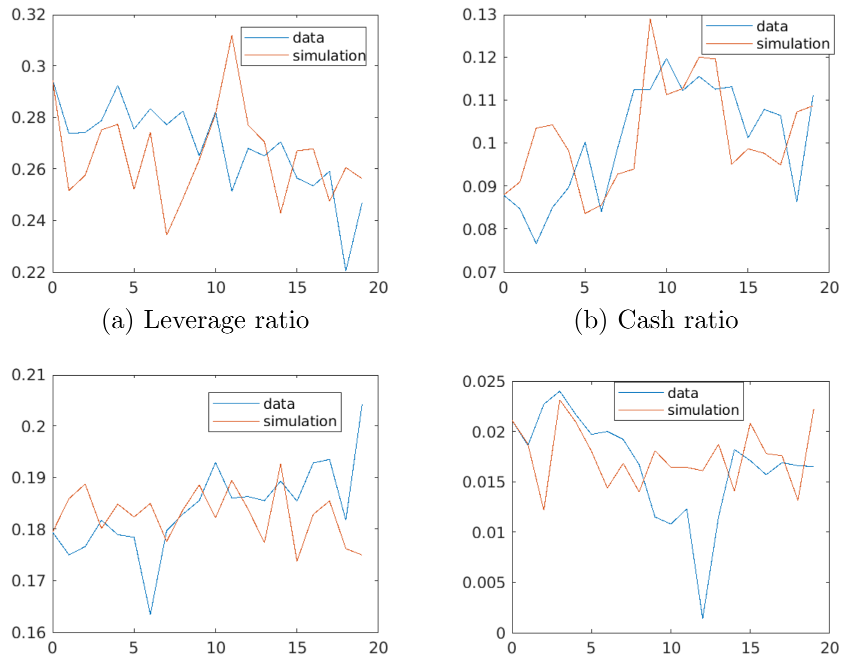
where

$$\pi_{ji,t} = \frac{\exp(\alpha^A_{ji} + \beta^{A1}_{ji} g(F_{ji,t-\Delta t}))}{1 - \exp(\alpha^A_{ji} + \beta^{A1}_{ji} g(F_{ji,t-\Delta t}))}.$$

Table 3.5 presents Beta-AR(1) estimated parameters for ASII financial ratios. Figure 3.5 depicts one path of the beta-AR(1) simulated financial ratios along with the historical data. These historical data are the same as those in Figures 3.4. These figures show qualitatively that the beta-AR(1) model captures reasonably well the dynamic of the data. The diagnostic in Appendix C.3 shows that the residuals scatter around a zero horizontal level have no trend. Furthermore, Figure C.7 does not show statistically significant evidence of nonzero autocorrelation in the residuals. Hence, we can conclude that beta-AR(1) is a good model for ASII financial ratios.

Financial ratio	α^A_{j1}	β^{A1}_{j1}	δ_{j1}
$F_{11,t}$	-0.5377	0.4729	942.3916
$F_{21,t}$	-0.7897	0.6329	898.9128
$F_{31,t}$	-1.2	0.2	4194
$F_{41,t}$	-2.098	0.4848	368.8105

Table 3.5: ASII Financial ratio beta-AR(1) estimated parameters



(a) Leverage ratio (b) Cash ratio

(c) Liquidity ratio (d) NPI ratio

Figure 3.5: Beta-AR ASII financial ratio

In our study we also use the following permissible boundary:

$$U_j = \begin{cases} 0.3, & \text{for } j = 1, 2, 3 \\ 0.1, & \text{for } j = 4. \end{cases} \tag{3.43}$$

The initial financial ratios are taken to be the values at the end of 2018, which gives $F_{11,0} = 0.2469$, $F_{21,0} = 0.111$, $F_{31,0} = 0.2042$, and $F_{41,0} = 0.0165$

In addition to constraint (3.41), we also incorporate the asset level screening constraint (3.10), portfolio level screening constraint (3.13), and both the asset level screening and the asset level probabilistic constraints ((3.10) and (3.14) with $\alpha = 0.05$) to solve the Shariah portfolio problem with dividend-type purification. The respective efficient frontiers are depicted in the upper panel in Figure 3.6 for the beta distribution model, and the lower panel, for beta-AR(1) model. The non-screening efficient frontier, i.e. without imposing the Shariah screening constraints, is also plotted in Figure 3.6 to benchmark against the case with screening. This efficient frontier corresponds to the dividend purification efficient frontier in Figure 3.3.

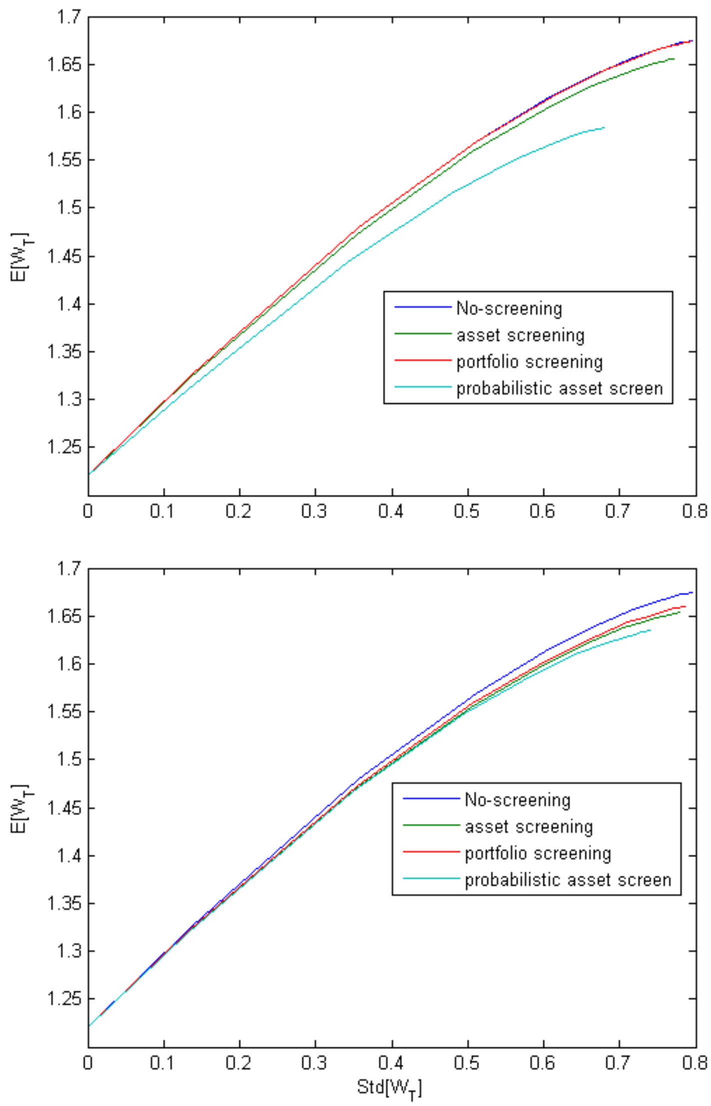
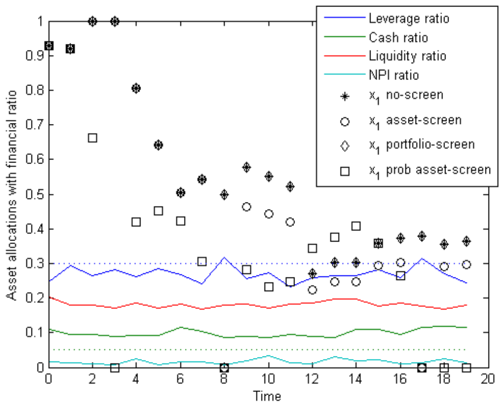
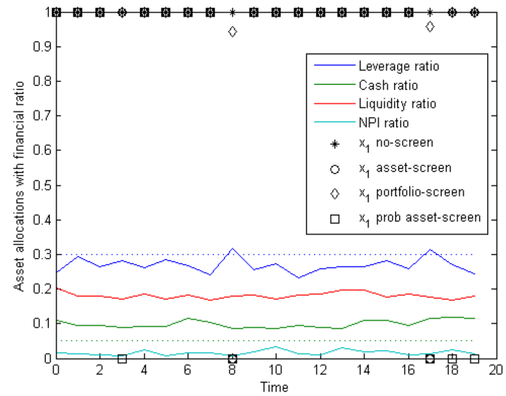


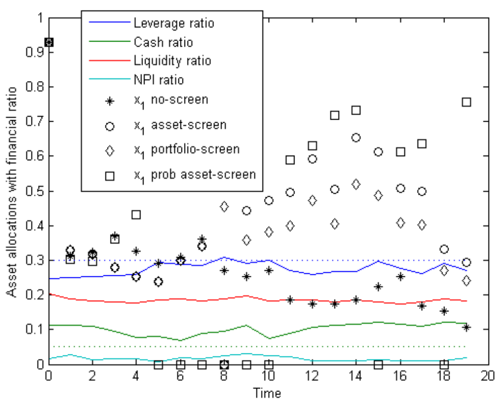
Figure 3.6: The impact of screening constraints on the multiperiod Shariah portfolio efficient frontier. Upper panel is based on the beta distribution model while the lower panel is based on the beta-AR(1) model.



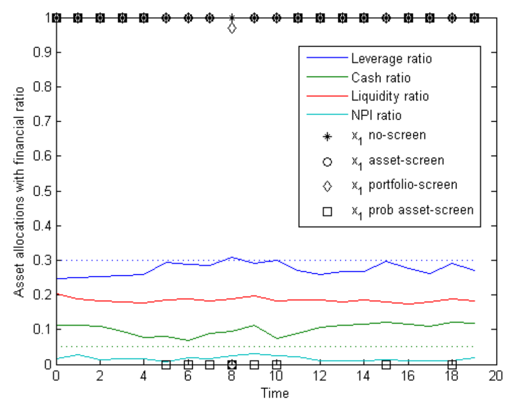
(a) Beta distribution with $\gamma = 4$



(b) Beta distribution with $\gamma = 18$



(c) Beta-AR(1) with $\gamma = 4$



(d) Beta-AR(1) with $\gamma = 18$

Figure 3.7: One sample asset allocations with screening constraints at $\gamma = 4$ and $\gamma = 18$ along with beta distribution and beta-AR(1) financial ratios

The impact of screening constraints in the multiperiod Shariah portfolio optimization problem are depicted in Figures 3.6 and 3.7, with the following key observations:

- Figure 3.6 shows that the impact of screening in the multiperiod Shariah portfolio with short-selling and leverage restrictions is consistent with the results found by Derigs and Marzban (2009), namely that the asset level screening has a better performance than the portfolio level screening. This result is reasonable since the asset level screening has a stricter rule than the portfolio level screening, in the sense that at the asset screening level once one of the financial ratios exceeds its permissible boundary, then the asset is excluded from the investment portfolio. Under the portfolio level screening, the risky asset is still admissible, however its asset allocation is reduced to maintain the requirement that the total amount of financial ratio is lower than the permissible boundary (i.e. $x_{i,t}F_{ji,t} < U_j$).

- Figure 3.7 shows the impact of screening constraints on the asset allocation distribution. The figure presents one sample of a set of financial ratios from the beta distribution and the beta-AR(1) model, along with the optimal asset allocations with two different inputs of γ. We set two different values of γ to assess the distribution of asset allocations in two different points of the efficient frontier. $\gamma = 4$ produces a low risk and low return portfolio (the right tail of the efficient frontiers), while $\gamma = 18$ produces a high risk and high return portfolio (the left tail of the efficient frontiers). Key observations from Figure 3.7 in terms of the performance of Shariah portfolio with the asset and portfolio level screening constraints are :

  (i) When the leverage ratio crosses the permissible boundary at times $t = 8$ and 18 for the beta distribution and at time $t = 8$ for the beta-AR(1) model, then the asset allocation is reduced to zero for the asset level screening constraints. This asset allocation is re-distributed to the non-risky asset that has a lower risk and lower return, hence it produces a lower risk and lower return portfolio

  (ii) In terms of the portfolio level screening, the asset allocation is slightly reduced, for example, from 100% without screening to 94.4% with screening at time $t = 8$ in the case of beta distribution model with $\gamma = 18$. Because the change of asset allocations is not significant in terms of the portfolio level screening, the efficient frontier of the Shariah portfolio with the portfolio level screening constraint is closer to the Shariah portfolio without screening constraint.

- Imposing probabilistic asset screening in addition to the screening constraint leads to a significant change in the asset allocation, from positive weights (with no screening or screening constraint only) to zero, as confirmed in Figure 3.3. The dramatic change of the asset allocation produces the lowest efficient frontiers in Figure 3.3. The change of the efficient frontier for the Beta distribution model is even more pronounced. Even though the leverage ratio sample from the Beta-AR(1) model only crosses the permissible boundary 1 time, these values are consistently high and very close to the permissible boundary during time period 5 to 10, and at time $t = 15, 18$. Hence, to keep the risk of non-compliance at an acceptable level, the asset allocations are reduced to zero during these time periods. This probabilistic constraint is effective in maintaining the sustainability of the portfolio in terms of Shariah compliance.

3.3.3 Shariah Compliant Portfolio with 2 Risky Assets and 1 Risk-free Asset

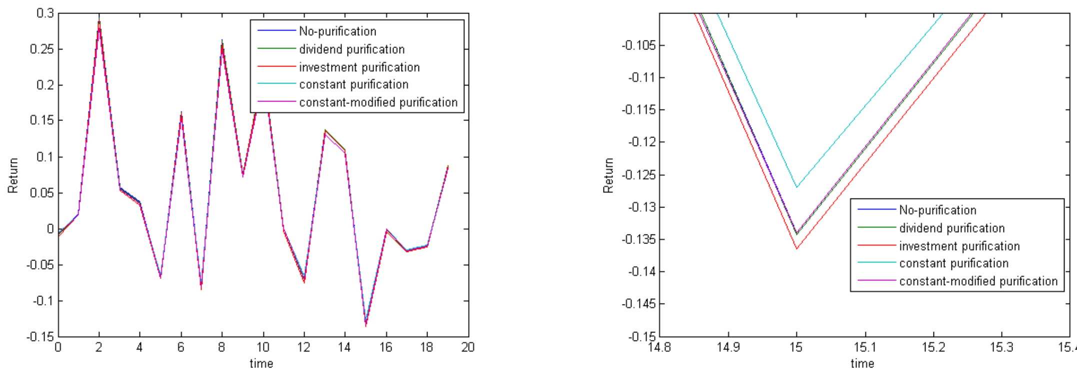
In this subsection, we expand the problem by solving the optimal Shariah portfolio consisting of 2 risky assets and 1 risk-free asset. We have estimated the parameters of the risky assets' purified returns and financial ratios from ASII.JK and TLKM.JK quarterly historical data spanning the time period from 2013 to 2018.

Incorporating purification and restriction to shortselling and leverage facility

We use the same input parameters for ASII purified return data $R_{1,t}^{ps}$ as those presented in Table 3.2. Now we present the estimated parameters for TLKM return and purified return data. We use the same approach for calculating 4 types of purifications for ASII as in subsection 3.3.2. In our discussion, we use the following notation of the TLKM return data at time t:

- $R_{2,t}^{p0}$ denotes the excess capital appreciation plus dividend yield.
- $R_{2,t}^{p1}$ denotes purified excess return with dividend type purification.
- $R_{2,t}^{p2}$ denotes purified excess return with investment type purification.
- $R_{2,t}^{p3}$ denotes purified excess return with constant purification model.
- $R_{2,t}^{p4}$ denotes purified excess return with modified constant purification model.

To calculate the constant purification and modified constant purification, we use a constant NPI percentage $\tilde{\psi}_2 = 0.0524$, which we have calculated as the average of the NPI percentage from 2013 to 2018. Figure 3.8 presents the TLKM historical returns from 2013 to 2018. When a non-purified return is negative, the constant purified return is higher than the return without purification. The investment purified return is always lower than the dividend purified return, since the deduction from the investment purification is always higher than the dividend purification, as depicted in Figure 2.1.



(a) $R_{2,t}^{ps}$ quarterly data from 2013 to 2018

(b) magnified $R_{2,t}^{ps}$ data

Figure 3.8: TLKM empirical quarterly return data without purification and with several types purification from 2013 to 2018

We assume that for each s, $s = 0, 1, 2, 3, 4$, the returns $R_{2,t}^{ps}$, $t = \Delta t, 2\Delta t, \ldots, T$, are independent and follow a normal distribution with mean μ_2^{ps} and volatility σ_2^{ps}. The estimated means and standard deviations, along with the p-values from the J-B tests, are presented in Table 3.6. The results in Table 3.6 indicate that TLKM return and purified returns data are consistent with the null hypothesis of the Jarque-Bera test (at 5% significance level) the skewness being zero and the excess kurtosis being zero. Additional evidence based on histograms and QQ-plots (see Appendix C.1) also support the normality assumption in modeling these excess return and purified return data.

Table 3.6: Sample Means, Sample Standard Deviations, and p-values of TLKM return and purified return data

Variable	$R_{2,t}^{p0}$	$R_{2,t}^{p1}$	$R_{2,t}^{p2}$	$R_{2,t}^{p3}$	$R_{2,t}^{p4}$
μ_2^{ps}	0.05184	0.04951	0.05245	0.04971	0.04861
σ_2^{ps}	0.11716	0.11707	0.11731	0.11115	0.11253
p-value	0.311	0.3174	0.3124	0.3124	0.3614

In addition to mean and volatility, we also estimate correlations between ASII and TLKM return and purified returns data, which are given in Table 3.7. Other input parameters required to find the solution of the Shariah portfolio problem consisting of 2 risky assets and 1 risky asset are the same as in Table 3.3.

Table 3.7: Sample correlations ASII and TLKM return and purified return data

$\rho(R^{p0}_{1,t}, R^{p0}_{2,t})$	$\rho(R^{p1}_{1,t}, R^{p1}_{2,t})$	$\rho(R^{p2}_{1,t}, R^{p2}_{2,t})$	$\rho(R^{p3}_{1,t}, R^{p3}_{2,t})$	$\rho(R^{p4}_{1,t}, R^{p4}_{2,t})$
0.43623	0.43696	0.43523	0.43623	0.43113

In this study, we consider 2 different cases. In Case 1, we impose a no-short selling restriction, and the upper limit constraint for each asset is 1. The portfolio constraint for Case 1 are given in (3.44). In Case 2, we add a no-leverage constraint in addition to the no-short selling constraint, which are presented in (3.45).

Case 1 constraint:

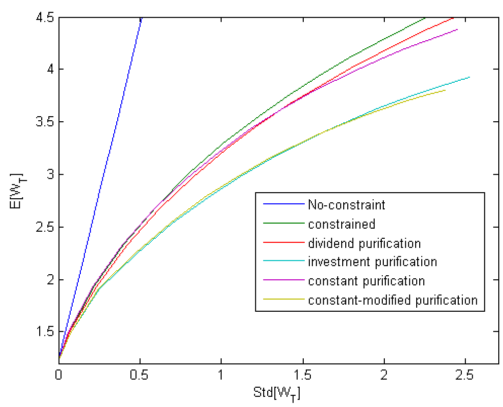
$$0 \leq x_{i,t} \leq 1, \text{ for } i = 1, 2, \text{ and } t = 0, 1, \ldots T - \Delta t. \tag{3.44}$$

Case 2 constraint:

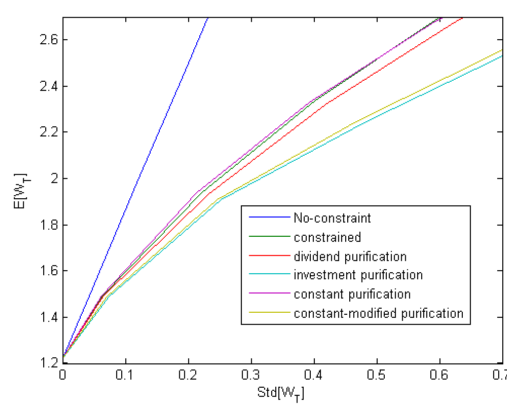
$$\left.\begin{array}{l} 0 \leq x_{i,t} \leq 1, \text{ for } i = 1, 2, \text{ and } t = 0, 1, \ldots T - \Delta t \\ \sum_{i=1}^{N} x_{i,t} \leq 1, \text{ for } t = 0, 1, \ldots T - \Delta t. \end{array}\right\} \tag{3.45}$$

Figure 3.9 shows efficient frontiers for the un-constrained conventional portfolio, the conventional portfolio with constraint (3.44), and the Shariah portfolios with constraint (3.44). As we can see from the graph, adding shortselling constraints to the conventional portfolio produces a lower efficient frontier. This result is expected because the optimal asset allocations for the risky assets are restricted to the region in constraint (3.44). The presence of a purification process penalizes the risk and reward tradeoff, as depicted by the lower efficient frontiers for both dividend and investment purification. The Shariah portfolio with a dividend type purification yields better performance than the investment purification. This result is consistent with Figures 3.2 and 3.8, which shows that the dividend purified return is always higher than the investment purified return. Because the dividend purification only occurs when the company issues dividend, and only applies to the dividend, it is expected that Shariah portfolio with dividend purification will yield a better performance compared to the investment purification.

The efficient frontier from the Shariah portfolio with constant purification proposed by Hazny et al. (2020) and Derbali et al. (2017) lies between the Shariah portfolio with dividend and the Shariah portfolio with the investment type purification for high risk portfolio, however its values outperform the conventional counterpart for the low risk portfolio. This phenomenon happen due to the negative values of purification rate under the constant purification model. The modified version of the constant purification yields the worst performance among the other types of purification for high risk portfolio, both constant and modified constant purifications are carried out regardless whether or not the dividend is being issued, and it applies to the total return (capital gain and dividend). This process is similar to the investment purification. The difference is in the calculation method: while the investment purification is the percentage of NPI to the total outstanding share (see Equation (2.1)), the constant purification is the percentage of NPI to TI (see Equation (3.38)). The negative values of purification rate from Hazny et al. (2020) and Derbali et al. (2017) contribute to a higher efficient frontier compared to the corresponding modification purification. Our proposed investment purification (Equation (2.1)) shows better performance than the constant modified purification for high risk portfolio, although the last type of purification only applies when the return is positive. Hence, if the purification should be applied to both the capital appreciation and the dividend yield, then our proposed investment purification is a better option for this case.



(a) Efficient frontiers for purified return



(b) Magnified efficient frontiers

Figure 3.9: Efficient frontiers of portfolio consisting of 2 risky assets and 1 risk-free asset for conventional portfolio, constrained conventional portfolio, and Shariah portfolio with 4 types purifications and with Case 1 constraint.

To see the impact of the leverage constraint on the portfolio, we generate the efficient frontier for the Shariah portfolio with dividend purification and Case 2 (no-short sell and no-leverage) constraints. The result is depicted in Figure 3.10. To compare results, in the same graph we also present the efficient frontier of the Shariah portfolio with the dividend purification and Case 1 (no-short sell) constraint. The presence of the leverage constraint penalizes the risk and return of the portfolio, and the decline is more pronounced for the high risk and high return portfolios.

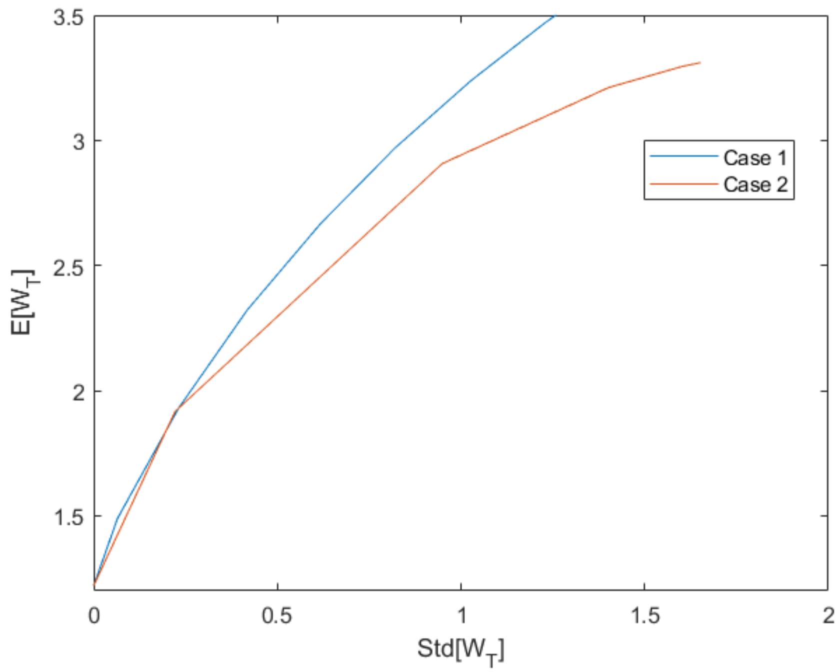


Figure 3.10: Efficient frontiers of Shariah portfolio with dividend purification consisting of 2 risky assets and 1 risky asset for Case 1 and Case 2 constraints.

Incorporating Purification, shortselling and leverage restrictions, and Shariah screening constraints

In this subsection, we add several screening constraints to Case 1 and Case 2 constraints. In Figure 3.11 we present historical data from 2013 to 2018 of the quarterly financial ratio screening for TLKM. The NPI ratio of TLKM is zero between 2013 to 2018, hence we only provide three other financial ratio data.

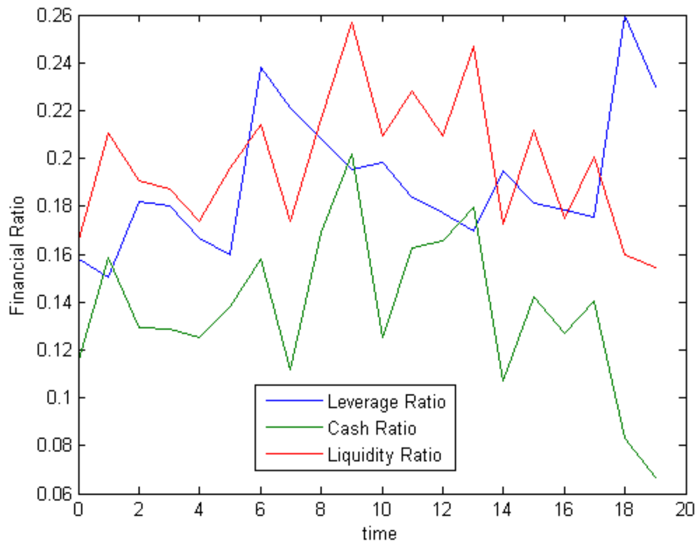
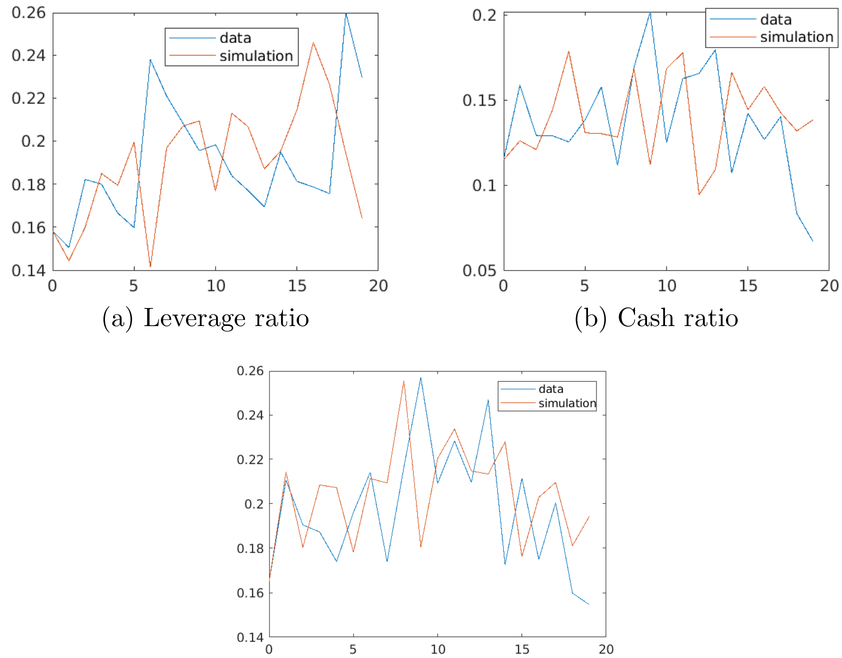


Figure 3.11: TLKM empirical quarterly financial ratio data from 2013-2018

In the portfolio with 2 risky assets and 1 risky-free asset, we only model the dynamic of financial ratios using the beta-AR(1) model. The estimated parameters for ASII are given in Table 3.5, while the estimated parameters for TLKM are presented in Table 3.8. Figure 3.12 depicts one simulated path of TLKM financial ratio using the beta-AR(1) model. The analysis of residuals from the fitted model in Appendix C.3 suggests that the residuals have nearly the properties of white noise. Hence, we can conclude that beta-AR(1) with the estimated parameters in Table 3.5 are reasonably close to the true values of TLKM financial ratios.

Financial ratio	α^{A}_{j2}	β^{A1}_{j2}	δ_{j2}
$F_{12,t}$	-0.8185	0.4214	259.8788
$F_{22,t}$	-1.2978	0.2933	113.107
$F_{32,t}$	-1.2149	0.1258	224.6933

Table 3.8: TLKM Financial ratio beta-AR(1) estimated parameters

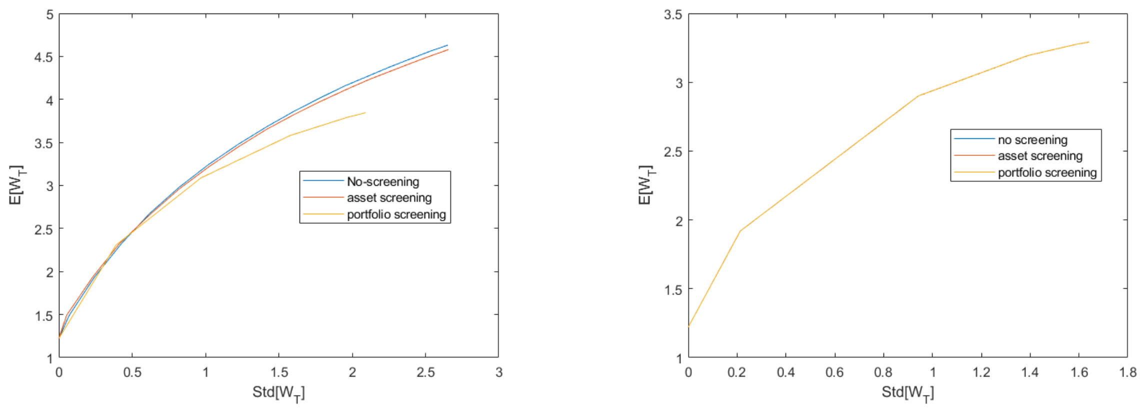


(a) Leverage ratio (b) Cash ratio

(c) Liquidity ratio

Figure 3.12: Beta-AR TLKM financial ratio

In this simulation, we have adopted the same assumption of permissible boundary as the one given by Equation (3.43). The initial values of the TLKM financial ratios $F_{j2,0}$ are taken to be the values at the end of 2018, which gives $F_{12,0} = 0.2296$, $F_{22,0} = 0.0677$, and $F_{32,0} = 0.1546$. We impose asset level screening constraint (3.10) and portfolio level screening constraint (3.13) to find the optimal solution for dividend purified Shariah portfolio with Case 1 and Case 2 constraint. The solutions of the Shariah portfolio problem for $\gamma \in [2, 18]$ are plotted in Figure 3.13. In Case 2, where leverage facility is not allowed, the portfolio level screening is better than the asset level screening in terms of its risk and return tradeoff. This result is consistent with the finding of Derigs and Marzban (2009) and our finding in Section 3.3.2, for one period Shariah portfolio consisting of risky assets only, and the finding in Section 2.4.2, for multiperiod Shariah portfolio with 1 risky asset and 1 risk-free asset. Table 3.9 provides the means and standard deviations of efficient frontiers in Figure 3.13 (b). The efficient frontier of the Shariah portfolio with the asset level screening constraint is slightly lower than the one with the portfolio level screening constraint. As we can see from Table 3.9, when the volatility level is equal to 1.5937, the mean of the Shariah portfolio with the asset level screening constraint is 3.2770, while the mean of the Shariah portfolio with the portfolio level screening constraint is 3.2773. We found interesting phenomenon in case 1 efficient frontiers: when leverage facility is allowed, the efficient frontier for Shariah portfolio with the asset level screening constraint outperform Shariah portfolio with the portfolio level screening constraint. The change of the efficient frontiers is caused by the redistribution of the asset allocations in order to fulfill the screening and leverage constraints which is depicted in Figure 3.14.



(a) Case 1 constraint

(b) Case 2 constraint

Figure 3.13: Shariah portfolio efficient frontiers with dividend purification with several Shariah screening constraints for no-shortselling case (a) and no shortselling and no-leverage case (b).

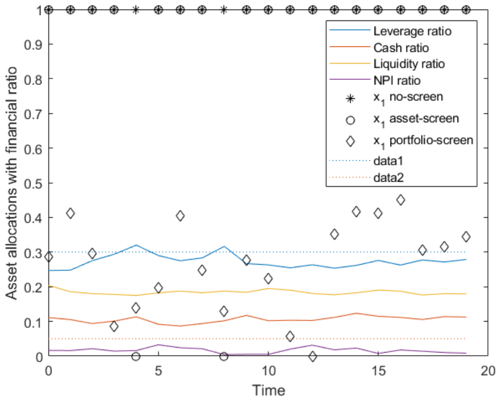
No screening		Asset-screening		Portfolio-screening	
$Std[W_T]$	$E[W_T]$	$Std[W_T]$	$E[W_T]$	$Std[W_T]$	$E[W_T]$
0	1.2202	0	1.2202	0	1.2202
0.2216	1.9165	0.2132	1.9193	0.2129	1.9194
0.9488	2.907	0.9422	2.8999	0.9417	2.9003
1.4016	3.2109	1.3908	3.1936	1.3903	3.1938
1.6018	3.2952	1.5937	3.277	1.5936	3.2773
1.6521	3.3106	1.6417	3.2917	1.6418	3.2921

Table 3.9: Optimal means and standard deviations of Shariah portfolios with several screening constraints and Case 2 constraint.

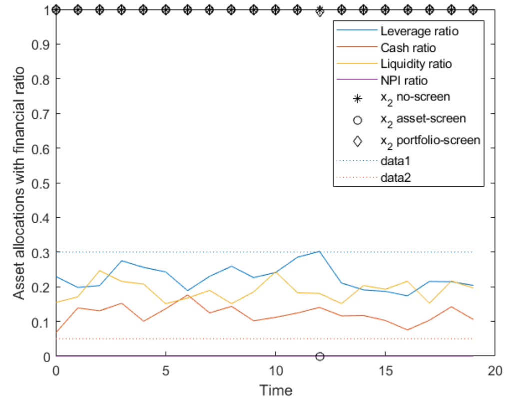
Figure 3.14 represents a simulated sample of financial ratios for ASII and TLKM, and the corresponding optimal asset allocations with screening constraints when $\gamma = 18$ (right-end point of the efficient frontiers in Figure 3.13). For Case 1 with asset level screening, once the leverage ratio of ASII exceeds the 30% permissible boundary at time $t = 4$ and $t = 8$, then $x_{1,t}$ is equal to zero. Similarly, when the liquidity ratio of TLKM reach 30.19% at time $t = 12$, then $x_{2,t}$ is equal to zero. The weights of these risky assets are re-distributed to the non-risky asset, which has lower return and lower risk. This redistribution leads to a lower level of frontier.

In Case 1, when the leverage facility is allowed, the optimal asset allocations without screening constraint are: buying ASII and TLKM as much as 100% of wealth for each assets, and borrowing 100% from the money market. When we add asset level screening constraint in Case 1 problem then we get similar optimal asset allocations as the ones without screening constraint except when the screening ratios exceed the boundary. However, when the portfolio level screening is applied, then the asset allocation for ASII are reduced to maintain the total of financial ratio from ASII and TLKM to be less than the permissible boundary. For example, at time $t = 0$, the optimal asset allocations are: buying 100% of TLKM (which has the highest return among other assets), buying 28.51% of ASII, and selling 28.51% of risk-free asset so that $x_{1,0}F_{11,0} + x_{2,0}F_{12,0} = 0.3$. This results explain the phenomenon that the efficient frontier of Shariah portfolio with the asset level screening constraint may outperform Shariah portfolio with the portfolio level screening constraint.

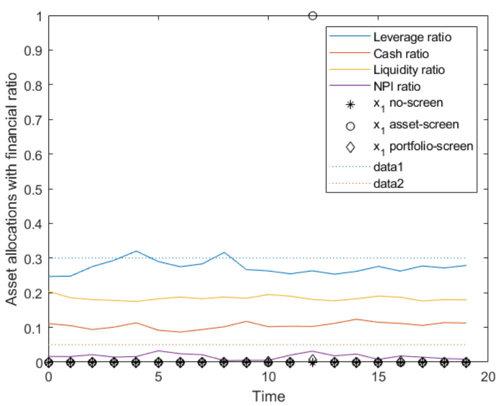
When the leverage facility is restricted (Case 2), then the optimal asset allocation without screening constraint is equivalent to buying 100% of wealth in TLKM. TLKM is chosen because it has the highest return and highest volatility among other assets, so that the highest return and highest volatility of portfolios (at $\gamma = 18$) is achieved. When the asset level screening constraint is processed, then TLKM is no longer acceptable at $t = 12$. In this case, the weight is re-distributed to ASII that has a higher return than the risk-free asset. If the portfolio level screening is applied to the Shariah portfolio, TLKM is still an admissible asset at $t = 10$, however the weight is reduced to 99%. The insignificant change of these optimal asset allocations causes the difference in efficient frontiers to be less conspicuous.



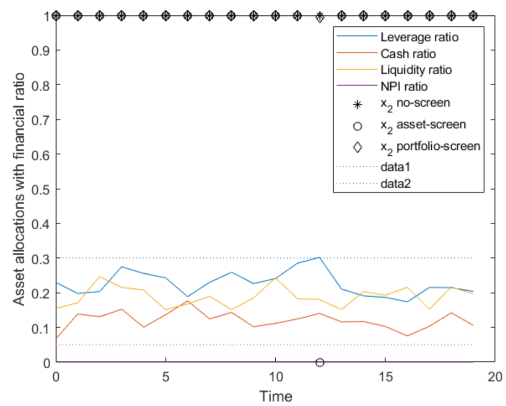
(a) ASII Case 1



(b) TLKM Case 1



(c) ASII Case 2



(d) TLKM Case 2

Figure 3.14: A simulated sample of asset allocations of ASII and TLKM with screening constraints for Case 1 and Case 2

3.4 Conclusion

In this chapter, we have extended the Shariah compliant portfolio model with purification and screening process discussed in Section 2 to a multi-period setting. We assume that the portfolio consist of 1 risk-free asset and N risky assets with their purified excess returns following geometric Brownian motions. We conduct 4 types of purification, namely dividend purification, investment purification, constant purification proposed by Derbali et al.

(2017) and Hazny et al. (2020), and the modified constant purification. We propose the modified version of Hazny et al. (2020) and Derbali et al. (2017) to address the negative values of purification in the existing model.

We solve the optimal multiperiod Shariah portfolio allocation problem by re-formulating the problem into a tractable LQ problem, solution of which is called the pre-commitment strategy. The advantage of this approach is that the Bellman dynamic programming principle is applicable, and hence a common backward recursive method can be applied to solve this optimization problem.

After constructing the portfolio problem, we introduce several Shariah constraints, namely no-shortselling, leverage restriction, and several screening constraints. For the screening process, we apply two assumptions to capture the temporal dynamics of the financial screening ratio: the independence assumption with a Beta distribution and the dependence assumption with a Beta-ARMA model. We have also incorporated the important non-compliance probabilistic constraint, which we have found to be efficient in maintaining the sustainability of an asset in the portfolio.

Chapter 4

Discrete Time Ruin Probability for Takaful (Islamic Insurance) with Investment and Qard-Hasan (Benevolent Loan) Activities

The contents of this chapter are adapted from a published paper titled "Discrete Time Ruin Probability for Takaful (Islamic Insurance) with Investment and Qard-Hasan (Benevolent Loan) Activities" by Puspita et al. (2020). This Chapter discusses the construction formula to calculate the finite-time ruin probability for Takaful, with the following contributions:

- We propose a risk model incorporating qard-hasan (benevolent loan) facility and mudarabah (profit-sharing) payment which are the unique and prominent features in Takaful business product.
- We construct a recursive formula to calculate a finite-time ruin probability for the proposed Takaful risk model. In constructing the formula, we follow the idea of Kim and Drekic (2016) by calculating the conditional survival probability of the first claim occurrence recursively. While the authors construct the formula for a conventional insurance product which considers interest loan and investment activity with a constant return, we construct the model with the unique Takaful features of qard-hasan and investment based mudarabah with stochastic return model.
- We also conduct numerical simulations to study the sensitivity of several prominent features in Takaful.

4.1 Hybrid (Mixed) Takaful Insurance Business Model with Qard-Hasan Facility

The Hybrid, or mixed, model is the most dominant model in the Takaful market, which can be explained by the fact that The Accounting and Auditing Organization for Islamic Financial Institutions (AAOIFI) recommends the practice of this model. In this model, a Wakalah (agent-based) contract is used for underwriting activities, while a Mudarabah (profit sharing) contract is adopted for investment activities. Based on the study conducted by Khan (2015) and Khan (2019), Hybrid Takaful model serves as the optimal structure for Takaful operation. In this study, we focus only on the Hybrid Takaful contract.

In regard to the underwriting activities, the Takaful operator acts as a Wakeel (agent) on behalf of participants to manage the Takaful fund. As shown in Figure (4.1), the operator manages the Takaful fund and pays all of the incurred expenses to the participants. In exchange for these tasks, the company charges each participant a predefined fee known as a Wakalah fee. This fee is deducted initially and goes to the shareholders' fund.

The Hybrid Takaful model applies the Mudarabah contract (profit-sharing basis) for the investment activities. The operator manages the assets and shares the income generated from the investment based on a predetermined profit share ratio. In this contract, the operator, as a fund manager or mudarib, will receive profit depending on the performance of the investment.

The income generated from an investment, after the deduction by mudarib's fee for the operator in conjunction with underwriting surplus, represents the surplus in Takaful (participants') fund. All of the surplus accounts are property of the Takaful participants. However, Takaful operator may receive an additional fee in case of a positive surplus in Takaful fund, which is called incentive or performance fee. This fee is determined as a percentage of the surplus generated in the Takaful fund. The remaining profit is distributed to the participants.

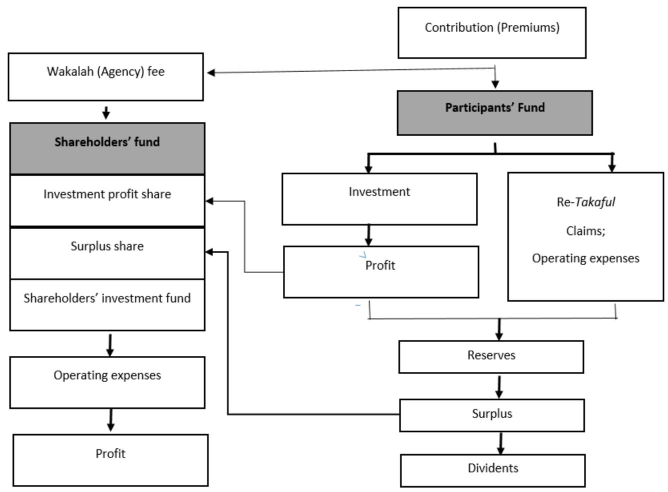


Figure 4.1: Hybrid Takaful Model. Source: Tolefat and Asutay Tolefat and Asutay (2013) pp. 39

In the case of a negative surplus in the Takaful fund, two options are usually considered. The first option is to charge the participants with additional contributions to cover the deficit value. However, this practice is not popular as it is not commercially feasible (Tolefat and Asutay, 2013). The second option, qard-hasan facility, is not only more popular in practice, but a mandatory requirement by some regulatory authorities and encouraged by the Islamic Financial Shariah Board (IFSB, 2010). Hence, in our model, we focus only on the qard-hasan option to overcome a deficit situation.

Qard-hasan, often called the benevolent loan, is an interest-free loan facility that is provided by shareholders as short term solvency resources to participants' funds in case of a deficit. Qard-hasan is a part of the shareholders' capital that is specially earmarked to maintain the solvency of participants' fund. According to the IFSB (2010) regulation, any drawn down qard-hasan need to be repaid out of the of future surpluses of participants' fund. In the event when the borrower is unable to settle the loan, the lender cannot force the borrower to make a repayment, and they must accept this transaction as a charitable act. Most Islamic scholars, including Onagun (2011); Yusof et al. (2011); Tolefat and Asutay (2013), and Rahim et al. (2017) agree with these IFSB (2010)'s rules in relation to the treatment of qard-hasan facility.

4.2 Surplus Process for Hybrid Takaful with Investment and Qard-Hassan facility

In this section, we propose a surplus model for Hybrid Takaful with investment activities and qard-hasan (non-interest loan) facility, which we use in the next section to evaluate the corresponding ruin probability. The model is inspired by Kim and Drekic (2016), who consider a discrete-time dependent Sparre Andersen risk model in the context of conventional insurance. The first difference between our approach and the one by Kim and Drekic (2016) is in the loan fund feature. In our model, there is no interest in undertaking loans, as in the conventional model. The second difference is in the loan repayment arrangement. While in the model proposed by Kim and Drekic (2016) the borrower is forced to pay the loan undertaking (including interest) when it exceeds a certain level (i.e. loan capacity), in our model the borrower will repay the loan only if they generate a positive surplus in the future. However, the lender has a right to get a part of each shared underwriting dividend to compensate for their effort to provide a benevolent loan. The third difference is in our assumption that the undrawn-down loan can be invested. We should note that Achlak (2016) has also developed a Takaful risk model based on Kim and Drekic (2016) paper with the assumption that the loan facility does not need to be repaid and can not be invested. In our study, we assume that the loan facility (i.e. qard-hasan facility) will be repaid from the future surplus, and the undrawn-down qard-hasan facility will be invested to enhance the facility. Finally, our model provides the option to invest in a risk-free or risky asset with a stochastic return, and takes into account Mudharabah or fund management fee for operator from each generating investment return.

In our Hybrid Takaful risk model, we incorporate the following four separate financial accounts:

U: surplus fund

F^I: investment fund

F^Q: qard-hasan fund

F^L: liability account

and three thresholds levels:

l^W: the minimal requirement of Takaful surplus level

l^I: trigger level for investment activities

l^D: trigger level for dividend payment

which satisfies $0 \leq l^W \leq l^I \leq l^D$.

We have explained in Chapter 4.1 that there are two separate financial accounts in Takaful, namely participants' funds and shareholders' funds. The participants' fund in our model are sub-divided into two separate financial accounts, namely, surplus fund U and investment fund F^I. The reason for the separation of both accounts is for better financial management, while the underwriting activities are represented in the surplus fund U, the financial activities are in the investment fund F^I. The qard-hasan fund is part of shareholders' fund that is specially allocated as a benevolent loan for participants in case of deficit which occurs due to underwriting activities. In our model, the drawn-down qard-hasan need to be repaid from future surplus of participants' fund. We introduce the liability account F^L to keep track of the total of qard-hasan borrowed and refunded. In our model we assume that all funds are in discrete monetary accounts. This assumption is built to facilitate the recursive calculation of finite-time ruin probability in Section 4.3.3.

By U_t, F_t^I, F_t^Q, and F_t^L we denote the value of surplus level, the investment fund, the qard-hasan fund, and the liability account, respectively, at the end of the time interval $(t-1, t]$, $t \in \mathbb{Z}^+$ (where $\mathbb{Z}^+ = \{1, 2, ...\}$). We assume that a constant contribution (premium) of $b \in \mathbb{Z}^+$ is received at $(t-1)+$, while claims are applied at $t-$. We also define $U_{t-}, F_{t-}^I, F_{t-}^Q$, and F_{t-}^L as the participants' surplus fund, investment fund, qard-hasan facility, and liability, respectively, immediately after a claim instance but before a withdrawal, borrowing, qard-hasan undertaking, and qard-hasan repayment instance.

A dividend trigger level l^D is a threshold that determines the dividend payment scenario. If $U_t \geq l^D$ a dividend amount of $\delta_t^i, i = \{1, 2\}$ from the underwriting surplus will be shared among participants and shareholders. In our model, we propose two options for the dividend value. In the first one for δ_t^1, we assume a constant dividend, while in the second for δ_t^2 we use a similar assumption to that adopted by Achlak (2016), namely, that the dividend is equal to $U_t - l^D$. We assume that the percentage of the dividends distributed to the participants and to the shareholders are given by x and $1-x$. respectively, where $x \in (0, 1)$.

We define P_t as the contribution received at time t and D_t^S as the total dividend distributed to shareholders at time t. Thus,

$$P_t = \begin{cases} b, & \text{if } U_t < l^D \\ b - x\delta_t^i, & \text{if } U_t \geq l^D \end{cases} \tag{4.1}$$

$$D_t^S = \begin{cases} 0, & \text{if } U_t < l^D \\ (1-x)\delta_t^i, & \text{if } U_t \geq l^D \end{cases} \tag{4.2}$$

where $\delta_t^1 = \delta \leq b,\ \delta \in \mathbb{Z}^+$ and $\delta_t^2 = U_t - l^D$.

A threshold l^I is a trigger point for investment activities. If $U_t \geq l^I$ a constant amount $d \in \mathbb{Z}^+$ is re-distributed to the investment fund at time $t+$. We assume that investment activities at each time interval are carried out after all of the outstanding debts and claims have been paid out. We denote the deposit amount corresponding to the time interval $(t, t+1]$ as D_t^I, thus

$$D_t^I = \begin{cases} 0, & \text{if } U_t < l^I \\ d, & \text{if } U_t \geq l^I. \end{cases} \tag{4.3}$$

Note that from equations (4.2) and (4.3), if $U_t > l^D$, then both D_t^S and D_t^I are paid.

We also assume that the operator, as a fund manager, may invest in the Shariah (permissible) non-risky or risky assets, like sukuk (Islamic bond) or a Shariah stock. In Takaful, the operator acts as a fund manager as well. Hence, they have the right to get "salary" from participants' funds due to this role. A Hybrid Takaful model applies Mudarabah (profit-share) for an investment activity in which it receives a dividend payment from investment generated profit. We assume that the fund manager receives $y \in (0, 1)$ part of an investment gain.

Threshold l^W represents the minimum level of the acceptable surplus of the participants' fund. If U_{t-} drops, at some time between $t-1$ and t, below l^W due to claims, we withdraw from F^I, or borrow from F^Q, to bring the surplus fund up to level l^W at time t. Withdrawal from the investment account F^I is utilized first. We consider undertaking a interest-free loan from qard-hasan facility if the investment fund F_{t-}^I is not sufficient to bring the surplus level back to l^W. The maximum qard-hasan that can be drawn down at time t is the maximum value of F_t^Q or the remaining money needed by U_t to reach l^W, whichever is smaller. The process will continue as long as the surplus-value is not negative. We assume that the un-drawn down qard-hasan fund will be invested in a risky or non-risky asset. The un-drawn down and investment gains will remain in the qard-hasan account to strengthen its facility. We denote the withdrawals and the qard-hasan undertaking amounts occurring during the time interval $(t-1, t]$ by W_t^I and W_t^Q, respectively.

$$W_t^I = \begin{cases} 0, & \text{if } U_{t-} \geq l^W \\ \min\{F_{t-}^1, (l^W - U_{t-})\}, & \text{if } U_{t-} < l^W \end{cases} \tag{4.4}$$

$$W_t^Q = \begin{cases} 0, & \text{if } U_{t-} \geq l^W \\ \min\{F_{t-}^Q, \max\{0, (l^W - U_{t-} - F_{t-}^I)\}\}, & \text{if } U_{t-} < l^W. \end{cases} \tag{4.5}$$

Participants need to repay their total qard-hasan undertaking to the qard-hasan fund F^Q in the future period when their surplus value is greater than l^W. We assume that the loan repayment will be paid instantly after the claim is paid out at $t-$. If the surplus-value after claim payout at $t-$ is greater than l^W, then the loan will be repaid at $t-$. The loan repayment amount should not make the surplus-value drop below l^W in any period. Unlike in the conventional counterpart, in the case when participants are not able to repay the qard-hasan, the undertaking qard-hasan will be counted as charity from shareholders to the participant. The participants are not to obligated to repay the loan in case of a deficit. So, in our model, the loan repayment will not be the reason for ruin, but it will affect the value of U_t and F_t^Q instead. We define D_t^Q as the qard-hasan repayment corresponding to the time interval $(t-1, t]$. Thus, we have

$$D_t^Q = \begin{cases} 0, & \text{if } U_{t-} \leq l^W \\ \min\{F_{t-}^L, (U_{t-} - l^W)\}, & \text{if } U_{t-} > l^W. \end{cases} \tag{4.6}$$

for calculation purpose, We identify the total liability of the participants' fund up to time t. Let us recall that F_t^L as the total liability at time t, and F_{t-}^L as the liability before qard-hasan undertaking and qard-hasan repayment. This account records only the qard-hasan undertaking and repayment without bearing any interest rate:

$$F_t^L = \sum_{i=1}^{t} W_i^Q - \sum_{i=1}^{t} D_i^W. \tag{4.7}$$

Finally, the surplus level at time t is the initial level of u plus the total cash inflows from: contributions, withdrawal from investment fund, and loan undertaking from qard-hasan facility, minus the total cash outflows to: deposit investment, qard-hasan repayment, and claim payments. Thus,

$$U_t = u + \sum_{i=0}^{t-1} P_i - \sum_{i=0}^{t-1} D_i^S - \sum_{i=0}^{t-1} D_i^I - \sum_{i=0}^{t-1} D_i^Q + \sum_{i=1}^{t} W_i^I + \sum_{i=1}^{t} W_i^Q - \sum_{i=1}^{N_t} X_i \tag{4.8}$$

where N_t represents the number of claims occurred by time t and X_i is the claim severity, represents the size of an individual claim. We assume that the claim distribution of N_t

and X_t has the same structure as in Sparre Andersen models Cheung et al. (2010). We also assume that the time between claims $(i-1)$ and i, $i \in \mathbb{Z}^+$, are defined by iid positive random variables $\{W_i, i \in \mathbb{Z}^+\}$ with a define pmf a_k and a corresponding survival function of A_k:

$$a_k = Pr\{W_i = k\},\ k = 1, 2, ..., n_a, \tag{4.9}$$

where $n_a \in \mathbb{Z}^+$ represents the upper bound for the interclaim times. Thus, for $k \leq n_a$, we have

$$A_k = Pr\{W_i > k\} = 1 - \sum_{j=1}^{k} a_j. \tag{4.10}$$

We denote by $\alpha_j(k)$ the conditional probability mass function (pmf) of X_i given $W_i = k$:

$$\alpha_j(k) = Pr\{X_i = j | W_i = k\},\ j \in \mathbb{Z}^+. \tag{4.11}$$

We assume that the pairs $\{(W_i, X_i), i \in \mathbb{Z}^+\}$ are iid, so that the joint pmf of (W_i, X_i) is of the form

$$Pr\{W_i = k, X_i = j\} = a_k \alpha_j(k). \tag{4.12}$$

In order to visualize the above descriptions of the money flow, below we present an illustrative example of the evolution of surplus fund, investment fund, qard-hasan facility fund, and the liability level given by Figures (4.2), (4.3), (4.4), and (4.5), respectively. The surplus fund starts from an initial level u, and the maximum capacity of the qard-hasan facility that is provided by shareholders at the initial point is f_Q. The investment fund and the liability level are zero at the initial period. In each period, at time $(t-1)+$ there is contribution income deducted by deposit from investment fund and dividend payment if the corresponding trigger points are reached at time t. If surplus drops below the level l^W due to claim payments at $t-$, we withdraw from the investment fund and/or qard-hasan facility. Every qard-hasan undertaking and repayment activities are recorded in the liability fund.

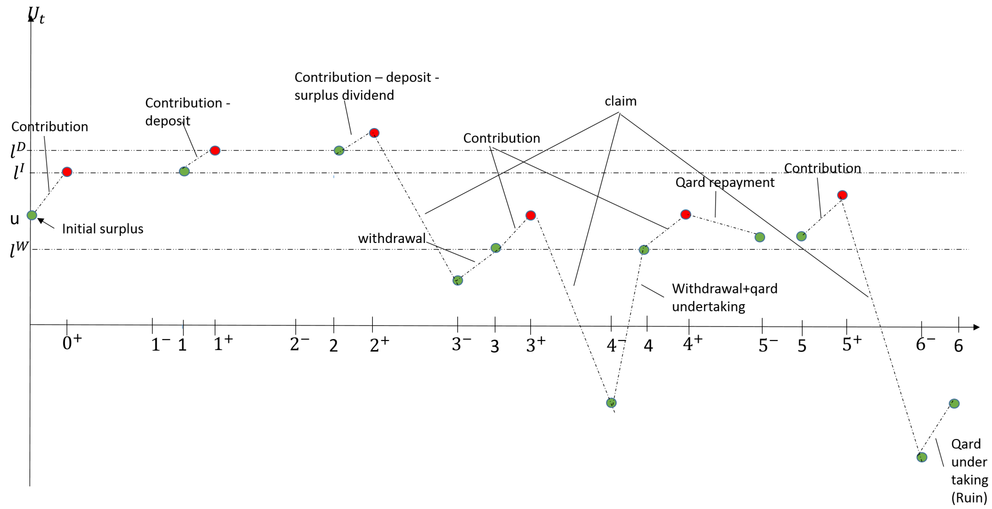


Figure 4.2: Example of a realization of the participants's fund process (U_t)

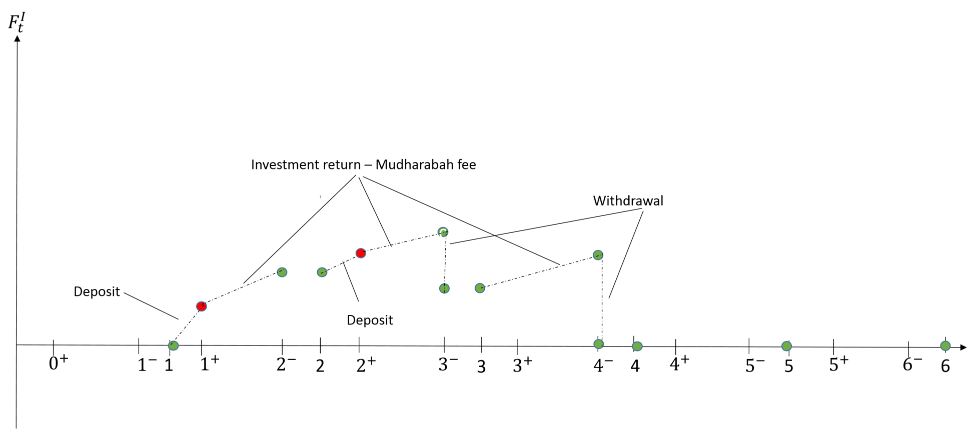


Figure 4.3: Example of a realization of the investment fund process (F_t^I)

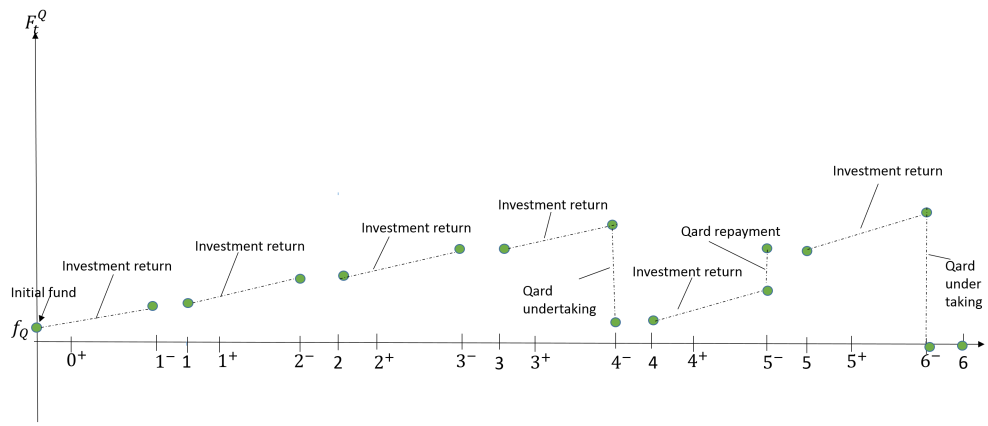


Figure 4.4: Example of a realization of the qard-hasan fund process (F_t^Q)

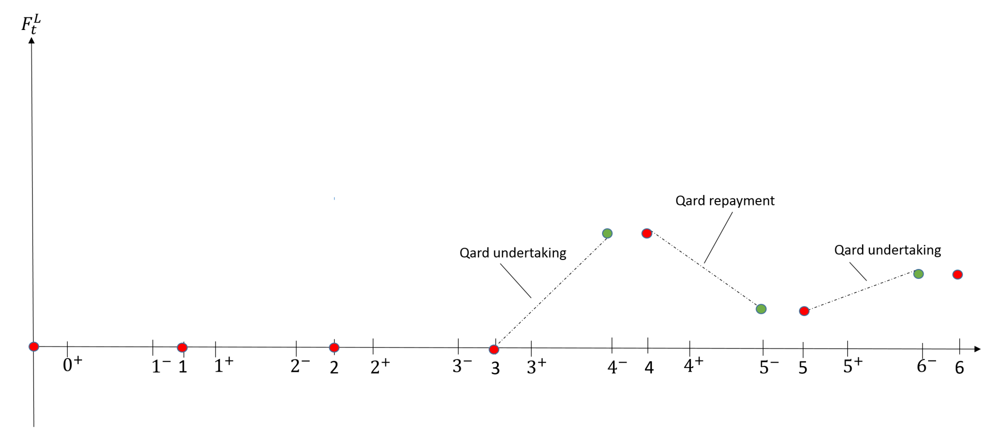


Figure 4.5: Example of a realization of the liability account process (F_t^L)

4.3 Finite-Time Ruin Probability for Hybrid-Takaful

This section describes a method of calculating a finite-time ruin probability associated with the Takaful risk model described by equation (4.8). In particular, our goal is to derive the probability of ruin occurring before time $\tau < \infty$, which we denote by $\Psi(v, g_I, g_Q, g_L, \tau)$:

$$\Psi(v, g_I, g_Q, g_L, \tau) = Pr\{T \leq \tau | U_0 = v, F_0^I = g_I, F_0^Q = g_Q, F_0^L = g_L\}, \ \tau \in \mathbb{Z}^*, \qquad (4.13)$$

where $\mathbb{Z}^* = \{0, 1, 2, ..\}$ and $T = \inf\{t | U_t < 0\}$ is a ruin time which defined as the first time when the surplus level drops below 0. In practice, τ represents the planning horizon of the insurance company. Typically, for non-life insurance, the managers set τ equal to four or five years (Burnecki et al., 2005).

To calculate the finite-time ruin probability, we will follow the idea of Cossette et al. (2006) and Kim and Drekic (2016) by calculating the conditional survival probability of the first claim occurrence recursively. We define $\sigma(u, f_I, f_Q, f_L, n, m)$ as the finite-time survival probability until time n given that the initial level of surplus, investment fund, qard-hasan facility, and liability level are u, f_I, f_Q and f_L respectively, and the elapsed time M_0 since the most recent claim occurrence is m:

$$\sigma(u, f_I, f_Q, f_L, n, m) = Pr\{T > n | U_0 = u, F_0^I = f_I, F_0^Q = f_Q, F_0^L = f_L, M_0 = m\}. \quad (4.14)$$

Then, the finite-time ruin probability (4.13) can be represented as:

$$\Psi(v, g_I, g_Q, g_L, \tau) = 1 - \sigma(v, g_I, g_Q, g_L, \tau, 0). \quad (4.15)$$

In Section 4.3.3 we develop a recursive formula for the finite-time survival probability. For this, we construct some auxiliary variables, namely the maximum value of the fund processes in Section 4.3.1 and a calling point in Section 4.3.2.

4.3.1 The Maximum Value of Funds Process

The surplus fund in Equation (4.8) is a stochastic function that the value might be decreasing or increasing depending on the claim payments. In this section we construct a formula to calculate the maximum value of the surplus fund $\hat{U}_{(t,u)}$, investment fund $\hat{F}^I_{(t,u,f_I)}$, qard-hasan fund $\hat{F}^Q_{(t,f_Q)}$, and liability level $\hat{F}^L_{(t,f_L)}$ representing levels of funds under the assumption of no claim, no withdrawal, no qard-hasan undertaking, and no qard-hasan repayment at time t given that the initial levels of $U_0 = u, F_0^I = f_I, F_0^Q = f_Q, F_0^L = f_L$. We should notice that $\hat{U}_{(t,u)}$ and $\hat{F}^L_{(t,f_L)}$ are non-decreasing functions of t, while $\hat{F}^I_{(t,u,f_I)}$ and $\hat{F}^Q_{(t,f_Q)}$ are non-decreasing functions of t if the investment returns are always positive. In our model, we use two different assumptions of investment return, the first one is a constant positive rate of return, and the second one is a stochastic return. The first assumption is similar with the one developed by Kim and Drekic (2016) and Achlak (2016).

To construct a formula to calculate the maximum value of surplus process $\hat{U}_{(t,u)}$ and the maximum value of the investment return $\hat{F^I}_{(t,u,f_I)}$, we need to identify the time points when the surplus level reaches the threshold levels l^W and l^I with the assumption of no claim, no withdrawal, and no qard-hasan undertaking and repayment. Under these assumptions, the surplus fund grows with a constant increase of b (i.e., the contribution payment) from time 0 until the surplus level reaches the threshold level l^I. From the time point when the surplus level reaches l^I until it reaches l^D, the surplus fund grows with a constant increase of $b-d$ due to deposit payment to the investment fund. Denote by $z^I_{(u)}$ and $z^D_{(u)}$ the time points when the surplus level with the initial value u reaches the trigger point l^I and l^D, respectively

$$z^I_{(u)} = \begin{cases} 0, & \text{if } u \geq l^I \\ \lceil \frac{l^I - u}{b} \rceil & \text{if } u < l^I \end{cases} \tag{4.16}$$

and

$$z^D_{(u)} = \begin{cases} 0, & \text{if } u \geq l^D \\ \lceil \frac{l^D - u - bz^I_{(u)}}{b-d} \rceil + z^I_{(u)} & \text{if } u < l^D. \end{cases} \tag{4.17}$$

where $\lceil x \rceil$ represents the least integer greater than or equal to x.

The Maximum Value of Surplus Fund

Under the assumption of no claim, no withdrawal, no borrowing, and no loan repayment, the surplus process on equation (4.8) becomes:

$$\hat{U}_{(t,u)} = u + \sum_{i=0}^{t-1} P_i - \sum_{i=0}^{t-1} D^S_i - \sum_{i=0}^{t-1} D^I_i. \tag{4.18}$$

By using the definition of premium P_t (Equation (4.1)), dividend payout to shareholders D^S_t (Equation (4.2)), deposit D^I_t (Equation (4.3)), and the time points when the surplus level reaches the threshold level l^W and l^I (i.e Equations (4.16) and (4.17)), it is easy to find the maximum value of surplus fund $\hat{U}_{(t,u)}$ as

$$\hat{U}_{(t,u)} = u + bt - d(t - z^I_{(u)})_+ - \delta(t - z^D_{(u)})_+, t \in \mathbb{Z}^+ \tag{4.19}$$

if we consider a constant dividend $\delta^1_t = \delta \in \mathbb{Z}^+$, or

$$\hat{U}_{(t,u)} = \begin{cases} u + bt - d(t - z^I_{(u)})_+ & \text{if } t \leq z^D_{(u)} \\ l^D + b - d & \text{if } t > z^D_{(u)} \end{cases} \tag{4.20}$$

if we consider dividend as $\delta_t^2 = U_t - l^D$ with $x_+ = \max\{x, 0\}$.

Equations (4.19) and (4.20) can be explained as the total cash inflow and outflow to the surplus fund. Before the time point $t = z^I_{(u)}$, there is only regular cash inflow, which is the constant contribution (tabaru) of b. Between $z^I_{(u)}$ and $z^D_{(u)}$, there is a regular outflow from the surplus process ($\hat{U}_{(t,u)}$) to the investment fund ($\hat{F}^I_{(t,u,f_I)}$), which is the deposit of d, in addition to the regular contribution payment. When the surplus reaches level l^D at $z^D_{(u)}$, the dividend of $\delta_t^i, i = \{1, 2\}$ will be distributed to the participants and shareholders. In addition to that, the surplus fund $\hat{U}_{(t,u)}$ will receive the contribution of b minus deposit d afterwards.

The Maximum Value of External Funds with Non-Risky Investment Return

In this section we assume that the operator, as a fund manager, invests the investment fund in the shariah (permissible) non-risky asset like sukuk (Islamic bond) with a constant rate of return $k_1 \geq 0$. We also assume that the un-drawn down qard-hasan fund will be invested at a constant investment gain of $k_2 \geq 0$. The un-drawn down qard-hasan and those investment gains will remain in the qard-hasan fund to strengthen the qard-hasan facility. It is noted that there is no interest in the undertaking loan from the qard-hasan account.

The initial value of the investment fund $\hat{F}^I_{(t,u,f_I)}$ will grow from its initial value f_I at the rate of k_1 due to investment activities. In addition to this, the investment fund will be increased by deposit d, regularly from $z^I_{(u)}$ until time t. In each period, y percentage of the investment gain will be shared to the operator as the Mudharabah fee. Therefore, the non-recursive form of the investment fund is given by

$$\hat{F}^1_{(t,u,f_I)} = \lfloor f_I(1 + k_1')^t + d^{k_1'}_{z^I_{(u)},t} \rfloor \tag{4.21}$$

where $\lfloor x \rfloor$ is a floor function of x represents the greatest integer less than or equal to x , and

$$k_1' = (1 - y)k_1 \tag{4.22}$$

denoting the investment gain after deducting the Mudharabah fee. The total future value of deposits made at times $z^{I+}_{(u)}$ up to time $t - 1$ with respect to the investment gain k_1' will be denoted by $d^{k_1'}_{z^I_{(u)},t}$:

$$d^{k_1'}_{z^I_{(u)},t} = d(1 + k_1') + d(1 + k_1')^2 + ... + d(1 + k_1')^{(t - z^I_{(u)})}.$$

The sum of the right-hand side can be calculated explicitly as

$$d^{k_1'}_{z^I_{(u)},t} = \begin{cases} 0, & \text{if } z^I_{(u)} > t \\ \frac{d(1+k_1')(1+k_1')^{t-z^I_{(u)}}-1}{k_1'}, & \text{if } z^I_{(u)} \leq t. \end{cases} \tag{4.23}$$

With the assumption of no qard-hasan undertaking and no qard-hasan repayment, there is no cash inflow or outflow except investment gains at the rate of k_2:

$$\hat{F}^Q_{(t,f_Q)} = \lfloor f_Q(1+k_2)^t \rfloor. \tag{4.24}$$

The investment fund and qard-hasan fund may take non-integer values due to interest accumulation. However, we apply the floor function in the Equation (4.21) and (4.24) to round down the value as we assume that all Funds are in the discrete monetary account. Taking the lower bound value of the investment and qard-hasan funds can be seen as a conservative approach.

The liability level under the assumption of no loan undertaking and repayment will remain the same as the initial liability level:

$$\hat{F}^L_{(t,f_L)} = f_L \tag{4.25}$$

for $t \in \mathbb{Z}^*$.

The Maximum Values of External Funds with Risky Investment Return

In this section we assume that the investment return is not constant. The Takaful operator invests the funds (investment and qard-hasan) in the same risky asset, for example, in the Shariah compliance stock or floating Sukuk (Islamic bond). In our study, we model the asset price in discrete time by a Markov chain that satisfies the recursive form $S_{n+1} = S_n Y_{n+1}$, where $\{Y_i\}$ are iid random variables. If the initial price is S_0, then expanding the recursion yields

$$S_n = S_0 \prod_{i=1}^{n} Y_i. \tag{4.26}$$

Therefore, the maximum value of qard-hasan facility ($\hat{F^Q}_{(t,f_Q)}$) with the initial value f_Q is:

$$\hat{F}^Q_{(f_Q,t)} = f_Q \prod_{i=1}^{t} Y_i. \tag{4.27}$$

To calculate the maximum value of investment fund $(\hat{F^I}_{(t,u,f_I)})$, we need to add the deposits and share the y part of Mudharabah fee. If the investment generates a positive return, then we need to share y part of the return with Takaful operator. Notice that the rate of return in the market model (4.26) is $\frac{S_i - S_{i-1}}{S_{i-1}} = Y_i - 1$. Then the real rate of return on the investment fund is: $(1-y)(Y_i - 1) = (Y_i - yY_i + y) - 1 = Y_i^* - 1$, where we assume that $Y_i > 1$. Thus, the rate of return for is $Y^* - 1$ with Y^* is defined as:

$$Y_i^* = \begin{cases} Y_i, & \text{if } Y_i \leq 1, \\ Y_i - yY_i + y, & \text{if } Y_i > 1. \end{cases} \tag{4.28}$$

Therefore, the maximum value of the investment fund $(\hat{F^I}_{(t,u,f_I)})$ with initial value f_I and deposit d is

$$\hat{F^I}_{(t,u,f_I)} = f_I \prod_{i=1}^{t} Y_i^* + \sum_{i=z^I_{(u)}}^{t-1} d \prod_{j=i+1}^{t} Y_i^*. \tag{4.29}$$

If we assume that $\{Y_i\}$ has a distribution $P(Y = j^u) = p$, $P(Y = j^d) = 1 - p$, then the asset price at $n+1$, given the value of S_n at time n, is

$$S_{n+1}|S_n = \begin{cases} j^u S_n, & \text{with probability} p, \\ j^d S_n, & \text{with probability} 1-p. \end{cases}$$

Under this model, in each time period, the asset price will go up by a constant factor of j^u with probability p, or go down by a constant factor of j^d with probability $1-p$, with $j^u \geq 1 \geq j^d \geq 0$. At the time n, there are 2^n possible state prices. This model is known in mathematical finance as a binomial market model. In insurance, binomial models have been applied to the problem of pricing of equity linked products (see, for example, Costabile et al. (2008); Costabile (2018)). One example of the most popular binomial market model is the one by Cox et al. (1979). In this model, the magnitude of the upward jump is $j^u = e^{\sqrt{\sigma^2}}$, while the magnitude of the down-ward jump is $j^d = 1/j^u$, and the probability of jump-up under risk-neutral probability measure is $p = \frac{1+r-j^d}{j^u - j^d}$, where σ^2 is the asset's variance and r is a fixed risk-free return.

Under the assumption of the binomial price model, we can calculate the maximum values of the external funds via the following algorithms. We define the maximum value of the investment fund at time t for state price l as

$$\hat{F}^I_{(t,u,f1,l)} = \lfloor F_I(t,u,f_I,l) \rfloor, \; l = 1, ..., 2^t, \tag{4.30}$$

where $F_I(t,u,f_I,l)$ can be calculated recursively by

$$F_I(t,u,f_I,l)=\begin{cases} f_I, & \text{if } t=0,\\ F_I(t-1,u,f_I,\lceil l/2\rceil)(j^u-yj^u+y), & \text{if } 0<t\le z^I_{(u)},\ l=\{1,3,5,..,N_l(t)-1\},\\ F_I(t-1,u,f_I,l/2)j^d, & \text{if } 0<t\le z^I_{(u)},\ l=\{2,4,6,..,N_l(t)\},\\ (F_I(t-1,u,f_I,\lceil l/2\rceil)+d)(j^u-yj^u+y), & \text{if } t>z^I_{(u)},\ l=\{1,3,5,..,N_l(t)-1\},\\ (F_I(t-1,u,f_I,l/2)+d)j^d, & \text{if } t>z^I_{(u)},\ l=\{2,4,6,..,N_l(t)\} \end{cases} \tag{4.31}$$

with $N_l(t)$ represents number of state price at time t, which equals to 2^t for Binomial model. Before the surplus reaches the dividend trigger level l^I at $z^I_{(u)}$, the investment fund grows at a rate of return of Y^*-1 (see Equation (4.28)). Starting from $z^I_{(u)}$, the investment fund receives a deposit d from the surplus fund, and it is also invested at the same rate of return Y^*. The state price $l\in\{1,3,5,..,N_l(t)-1\}$ represents the upward movements, while the state price $l\in\{2,4,6,..,N_l(t)\}$ represents the down-ward movements.

The maximum value of qard-hasan fund at time t for state price l as

$$F^Q_{(t,f_Q,l)}=\lfloor F_Q(t,f_Q,l)\rfloor,\ l=1,..,N_l(t) \tag{4.32}$$

where $f_Q(t,f_Q,l])$ can be calculated recursively by

$$F_Q(t,f_Q,l)=\begin{cases} f_Q, & \text{if } t=0,\\ F_Q(t-1,f_Q,\lceil l/2\rceil)j^u, & \text{if } t>0,\ l=\{1,3,5,..,N_l(t)-1\},\\ F_Q(t-1,f_Q,l/2)j^d, & \text{if } t>0,\ l=\{2,4,6,..,N_l(t)\}. \end{cases} \tag{4.33}$$

The investment fund grows at a rate of return $Y-1$ with a constant upward magnitude j^u and downward magnitude j^d.

If by $P(t,l)$ we denote the probability of the state price l at time t, then these probabilities can be calculated recursively by

$$P(t,l)=\begin{cases} 1, & \text{if } t=0,\\ P(t-1,\lceil l/2\rceil)p, & \text{if } t>0,\ l=\{1,3,5,..,N_l(t)-1\},\\ P(t-1,l/2)(1-p), & \text{if } t>0,\ l=\{2,4,6,..,N_l(t)\}. \end{cases} \tag{4.34}$$

In each period of time, the probability that the asset price will go up is p, while the probability that the asset will go down is $1-p$.

4.3.2 Calling Point

In this subsection we define a calling point, denoted by $c_{(t,m,u,f_L)}$, which represents the earliest time point before time t when the debt from qard-hasan facility needs to be repaid before the first claim occurs. If the elapsed waiting time at time 0 since the most recent claim occurrence is m, and the upper bound for the the interclaim times is n_a (see Equation (4.9)), then the next claim will occur before time $n_a - m$. Therefore the calling point $c_{(t,m,u,f_L)}$ is bounded by $\min\{n_a - m, t\}$. Qard-hasan repayment will be paid if participants have positive liability, and the surplus fund is greater than l^W. Note that the surplus and the liability at time t under the assumption of no claim occurrence are $\hat{U}_{(t,u)}$ and $\hat{F}^L_{(t,f_L)}$. Then

$$c_{t,m,u,f_L} = \begin{cases} \min\{n_a - m, t\}, \text{ if } (\hat{U}_{(i,u)} \leq l^W \text{ or} \hat{F}^L_{(i,f_L)} < 0) \ \forall i \in \{1, 2, ..., \min\{n_a - m, t\}\} \\ \min\{i \in \{1, 2, ..., \min\{n_a - m, t\}\} | \hat{U}_{(i,u)} > l^W\}, \text{ otherwise.} \end{cases} \tag{4.35}$$

In Equation (4.35), the earliest time point to make a loan repayment is the earliest time the liability level $\hat{F}^L_{(i,f_L)}$ becomes positive and the surplus value $\hat{U}_{(i,u)}$ is greater than l^W for $i = 1, 2..., \min\{n_a - m, t\}$.

4.3.3 Recursive Formula to Calculate the Finite-Time Survival Probability

The aim of this subsection is to develop the algorithm to calculate the finite-time survival probability in Equation (4.14). Once we find this value, our goal of finding the ruin probability can be achieved by using Equation (4.15). Under the condition of no external funds scenario, Cossette et al. (2006) propose a recursive algorithm to calculate the finite time survival probabilities as the sum of conditional finite time survival probabilities of the first claim occurring. Let us denote by $\sigma^C(u, n)$ and $\sigma^C(u, n, k)$, respectively, the finite-time survival probability and the finite time conditional survival probability given that the first claim occur at time k, with the absence of external funds (i.e $F^I_t = F^Q_t = F^L_t = 0$), then

$$\sigma^C(u, n) = \sum_{k=1}^{n_a} a_k \sigma^C(u, n, k) = \sum_{k=1}^{n} a_k \sigma^C(u, n, k) + \sum_{k=n+1}^{n_a} a_k, \tag{4.36}$$

$\sigma^C(u, n, k)$ in Equation (4.36) is equal to 1 for $k > n$, because the claim occurrence after time n implies that the process survives until n. Under the absence of investment activity ($d = 0$) and no dividend payment ($\delta^i_t = 0$), Cossette et al. (2006) define $\sigma^C(u, n, k), k \in$

$\{1, 2, .., n\}$ as the accumulation of weighted sum of $\sigma^C(u + bk - j, n - k)$, which is the probability of surviving the time interval $(k, n]$ with the level of surplus fund at time k after the claim payment is $u + bk - j$, for all possible values of claim severity j that does not cause ruin at time k. Then we have

$$\sigma^C(u, n) = \sum_{k=1}^{n} a_k \sum_{j=1}^{u+bk} \alpha_j(k)\sigma^C(u + bk - j, n - k) + A_k. \tag{4.37}$$

By following the idea of Kim and Drekic (2016) that expanding the recursive formula (4.37) with the existence of external funds, the finite time survival probability (4.14) can be found using the recursive algorithm in Theorem 4.3.1.

Theorem 4.3.1. *Let $\sigma(u, f_I, f_Q, f_L, n, m)$ is the finite time survival probability at time n, with the initial values of surplus fund, investment fund, qard-hasan fund, and liability level are u, f_I, f_Q and f_L, respectively, and the elapsed time at time 0 since the most recent claim occurrence is m as defined in Equation (4.14). Then $\sigma(u, f_I, f_Q, f_L, n, m)$ can be calculated recursively using Equation (4.38)*

$$\begin{aligned}
&\sigma(u, f_I, f_Q, f_L, n, m)\\
&= \sum_{k=1}^{c_{(n,m,u,f_L)}} \frac{a_{k+m}}{A_m} \sum_{l=1}^{N_l(k)} P(k,l) \sum_{j=1}^{\hat{U}_{(k,u)}+\hat{F}^I_{(k,u,f_I,l)}+\hat{F}^Q_{(k,f_Q,l)}} \alpha_j(k+m)\sigma(u^*(l), f_I^*(l), f_Q^*(l), f_L^*(l), n-k, 0)\\
&\quad + \frac{A_{c_{(n,m,u,f_L)}+m}}{A_m} \sum_{l=1}^{N_l(c_{(n,m,u,f_L)})} P(c_{(n,m,u,f_L)}, l)\sigma(u', f_I'(l), f_Q'(l), f_L', n - c_{(n,m,u,f_L)}, c_{(n,m,u,f_L)} + m),
\end{aligned} \tag{4.38}$$

with boundary condition

$$\sigma(u, f_I, f_Q, f_L, n, m) = \begin{cases} 0, & \text{if } u \in \mathbb{Z}^- \text{ or } m = n_a, \\ 1, & \text{if } u \in \mathbb{Z}^*, n = 0, \text{ and } m = 0, 1, ..., n_a - 1, \end{cases} \tag{4.39}$$

where $u^*(l), f_I^*(l), f_Q^*(l), f_L^*(l)$ can be found in Theorem 4.3.2, while $u', f_I'(l), f_Q'(l), f_L'$ can be found in Theorem 4.3.3. In the case of investment in a non-risky asset, then we set $N_l(k) = 1$, $P(k,l) = 1$, $\hat{F}^I_{(k,u,f_I,l)} = \hat{F}^I_{k,u,f_I}$ in Equation (4.21), and $\hat{F}^Q_{(k,f_Q,l)} = \hat{F}^Q_{k,f_Q}$ in Equation (4.24).

Theorem 4.3.2. *Denoted by $u^*(l), f_I^*(l), f_Q^*(l)$, and $f_L^*(l)$ be the initial levels for the next recursive in Equation (4.38), represents level of U, F^I, F^Q, and F^L, respectively, corresponding to state price $l \in \{1, 2, .., N_l(k)\}$ after the claim payment of $X_1 = j$ at time*

$k \in \{1, 2, .., c_{(n,m,u,f_L)}\}$. *Then* $u^*(l), f_I^*(l), f_Q^*(l)$, *and* $f_L^*(l)$ *can be calculated using the following equations:*

$$u^*(l) = \min\{\max\{\hat{U}_{(k,u)} - j - \hat{F}^L_{(k,f_L)}\ ,\ l^W\}\ ,\ \hat{U}_{(k,u)} - j + \hat{F}^I_{(k,u,f_I,l)} + \hat{F}^Q_{(k,f_Q,l)}\}, \tag{4.40}$$

$$f_I^*(l) = \min\{\max\{\hat{F}^I_{(k,u,f_I,l)} - l^W + \hat{U}_{(k,u)} - j\ ,\ 0\}\ ,\ \hat{F}^I_{(k,u,f_I,l)}\}, \tag{4.41}$$

$$\begin{aligned} f_Q^*(l) = \min\{\max\{\min\{\hat{F}^Q_{(k,f_Q,l)} + \hat{F}^L_{(k,f_L)}\ ,\ \hat{F}^Q_{(k,f_Q,l)} + \hat{U}_{(k,u)} - j - l^W\}\ ,\ \hat{F}^Q_{(k,f_Q,l)}\}\ , \\ \max\{\hat{F}^Q_{(k,f_Q,l)} + \hat{U}_{(k,u)} - j - l^W + \hat{F}^I_{(k,u,f_I,l)}\ ,\ 0\}\} \end{aligned} \tag{4.42}$$

$$\begin{aligned} f_L^* = \max\{\min\{\max\{0\ ,\ \hat{F}^L_{(k,f_L)} - \hat{U}_{(k,u)} + j + l^W\}\ ,\ \hat{F}^L_{(k,f_L)}\}\ , \\ \min\{\hat{F}^L_{(k,f_L)} + \hat{F}^Q_{(k,f_Q,l)}\ ,\ \hat{F}^L_{(k,f_L)} + l^W - \hat{U}_{(k,u)} + j - \hat{F}^I_{(k,u,f_I,l)}\}\}. \end{aligned} \tag{4.43}$$

Theorem 4.3.3. *Denoted by* $u', f_I'(l), f_Q'(l)$, *and* $f_L'(l)$ *be the initial levels for the next recursive in Equation (4.38), represents level of* U, F^I, F^Q, *and* F^L, *respectively, corresponding to state price* $l \in \{1, 2, .., N_l(k)\}$ *after qard-hasan repayment at time* $c_{(n,m,u,f_L)}$. *Then* $u', f_I'(l), f_Q'(l)$, *and* $f_L'(l)$ *can be calculated using the following equations:*

$$u' = \hat{U}_{(c_{(n,m,u,f_L)},u)} - \min\{(\hat{U}_{(c_{(n,m,u,f_L)},u)} - l^W)_+, \hat{F}^L_{(c_{(n,m,u,f_L)},f_L)}\}, \tag{4.44}$$

$$f_I'(l) = \hat{F}^I_{(c_{(n,m,u,f_L)},u,f_I,l)}, \tag{4.45}$$

$$f_Q'(l) = \hat{F}^Q_{(c_{(n,m,u,f_L)},f_Q,l)} + \min\{(\hat{U}_{(c_{(n,m,u,f_L)},u)} - l^W)_+, \hat{F}^L_{(c_{(n,m,u,f_L)},f_L)}\}, \tag{4.46}$$

$$f_L' = \hat{F}^L_{(c_{(n,m,u,f_L)},f_L)} - \min\{(\hat{U}_{(c_{(n,m,u,f_L)},u)} - l^W)_+, \hat{F}^L_{(c_{(n,m,u,f_L)},f_L)}\}. \tag{4.47}$$

4.4 Numerical Results

In this section, we implement the algorithm in section 4.3 to calculate the finite-time ruin probabilities based on the proposed Hybrid Takaful model (i.e. Equation (4.15)) through recursive formula (4.38). The objective is to study the effect some of the parameters may have on ruin probability and to investigate the performance of the Takaful proposed model in comparison with the conventional one. We use Wolfram-Mathematica version 9.0 to do all numerical calculations.

In our simulation study, we apply the set of input parameters as in Kim and Drekic (2016) and Achlak (2016), which will facilitate comparisons between the existing models and the proposed model. We assume that the interclaim time follows a truncated geometric distribution with $n_a = 25$:

$$a_k = \begin{cases} (2/11)(9/11)^(k-1) & \text{if } k = 1, 2, ..., 24, \\ (9/11)^{24} & \text{if } k = 25, \end{cases}$$

while the claim size distribution follows a discretized version of Pareto distribution with mean 10.5 and variance 120

$$\alpha_j(k) = \alpha_j = (1 + \frac{j-1}{30})^{-4} - (1 + \frac{j}{30})^{-4}, j \in \mathbb{Z}^+. \tag{4.48}$$

In all of the simulations, we set the contribution $b = 5$, the initial value for the participants' fund $v = 10$, the initial value of the investment fund $g_I = 0$, and the initial value of liability $g_L = 0$. All of these values are in the currency unit, for example, in million of dollars. In addition to that, we assume that the rate of return for the investment fund (k_1) is 1% per month.

Figure (4.6) shows ruin probabilities with time horizon $\tau = 25$, and trigger points $l^W = 0, l^I = 20$, and $l^D = 50$. We assume a constant dividend $\delta_t^1 = \delta = 3$, a deposit $d = 1$, and an investment rate of return for qard-hasan fund $k_2 = 0.02$. We calculate ruin time probabilities for several values of the maximum loan capacity g_Q. In our Takaful model, we need the percentage of Mudharabah fee (y), while in the conventional model, this fee is zero. We consider two different values of y, namely 0 and 50%.

For the surplus and investment fund, our proposed model has similar features as the conventional counterpart proposed by Kim and Drekic (2016). Hence, under the assumption of no loan facility and no Mudharabah fee, $g_Q = 0$ and $y = 0$, the ruin probabilities of our proposed Takaful model are the same as in the the conventional one in Kim and Drekic (2016). The difference between our model and Kim and Drekic (2016) model is in the features of the loan activities. In the conventional model, there is an interest rate charged in each loan. In addition to that, the borrower is forced to repay the loan if the loan undertaking (including the accumulative interest rate) exceeds the maximum loan capacity. This rule may cause the ruin to occur. The qard-hasan facility in our proposed Takaful model has two benefits; the loan capacity may positively grow as a result of investing the undrawn-down qard-hasan, and the loan will be repaid if there is enough money in the surplus fund. We may see from Figure (4.6) that the ruin probabilities of our proposed Takaful model with Mudharabah fee $y = 0$ or $y = 0.5$ are lower than the conventional counterpart. Moreover, the higher loan capacity produces a greater difference in ruin probabilities between the proposed Takaful model and the conventional one. The two different values of the Mudharabah fee that we have chosen in our implementation do not produce significantly different ruin probabilities.

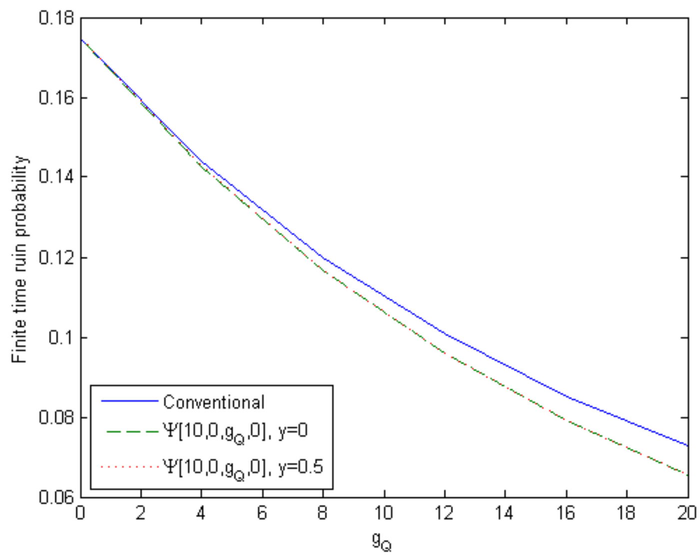


Figure 4.6: Finite time ruin probabilities of the conventional and the proposed Takaful models for several values of maximum loan capacity $0 \leq g_Q \leq 16$

Figure (4.7) compares the Takaful ruin time probabilities based on the proposed model with probabilities based on the model proposed by Achlak (2016). In the Achlak (2016) model, the qard-hasan fund is not invested, and there is no obligation for the participants to return the qard-hasan undertaking. We apply the same dividend value $\delta_t^2 = U_t - l^D$ with several values of the initial qard-hasan fund, g_Q. Other input parameters are the same as the input parameters that we use in the previous simulation.

We also set 3 different values of the rate of return for qard-hasan fund ($k_2 \in \{0, 0.01, 0.02\}$). In the case of no qard-hasan facility, we obtain the same ruin probabilities as that based on the Achlak (2016) model for all values of k_2. This is because the surplus and investment features in our proposed model are the same to the ones in Achlak (2016). For

$g_Q > 0$, when the qard-hasan fund is not invested ($k_2 = 0$), the qard-hasan repayment in our proposed model causes the ruin probability to be slightly higher than the one without loan repayment. This result is due to the loan repayment, causing a delay in depositing the investment fund. However, if we invest in the qard-hasan fund, the obligation of loan repayment leads to lower ruin probabilities than those without loan repayment. The difference between finite-time ruin probabilities with and without loan repayment becomes more visible when the initial qard-hasan fund (g_Q) increases.

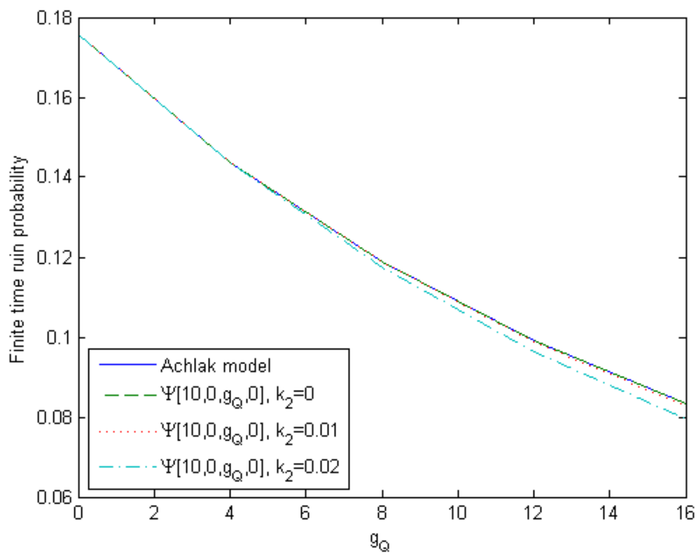


Figure 4.7: Finite time ruin probabilities with $l^W = 0$, $l^I = 20$, $l^D = 50$, $y = 0.5$, $\delta_t = U_t - l^D$ and $k_2 = 0$ for $\Psi[10, 0, g_Q, 0, 25]^1$, $k_2 = 0.01$ for $\Psi[10, 0, g_Q, 0, 25]^2$, and $k_2 = 0.02$ for $\Psi[10, 0, g_Q, 0, 25]^3$

Figure (4.8) represents finite-time ruin probabilities at $\tau = 12$ corresponding to time horizon $\tau = 12$ with trigger points $l^W = 0$, $0 \leq l^I \leq 50$, $l^D = 50$, initial qard-hasan fund $g_Q = 10$, Mudharabah fee $y = 5\%$, deposit $d = 1$, and dividend $\delta = 3$. We consider four different assumptions about the asset's return: the first one is a non risky asset with the rate of return of 1% per month. The other three are risky with the expected rate of return of 1% per month and variance 0.0001, 0.001, and 0.01. In this simulation we apply CRR model as explained in Chapter 4.3.1. The graph suggests that the finite-time ruin probabilities increase as the variance of the asset is increased. In addition to that we may see that the finite-time ruin probabilities for non risky assets ($\sigma^2 = 0$) and risky assets with low variance ($\sigma^2 = 0.0001$) increase as we increase the investment trigger level l^I. This phenomenon can be explained by the fact that when l^I increases then deposits in investment activities will be delayed. This, in turn, reduces the probability of ruin, since positive returns earned only from the investment activities in the external funds. However, the explanation does not apply to the case of investing in the risky asset with high volatility. It can be explained by the fact that investing in risky assets may produce negative returns on the investment fund that reduce the total reserve of Takaful fund. The higher volatility of the asset price will add the riskiness of the Takaful product. Hence, the higher volatility of the asset's return produces a higher ruin probability. For both $\sigma^2 = 0.001$ and 0.01, the optimal level to invest is $l^I = 45$, when using ruin probability as a criterion.

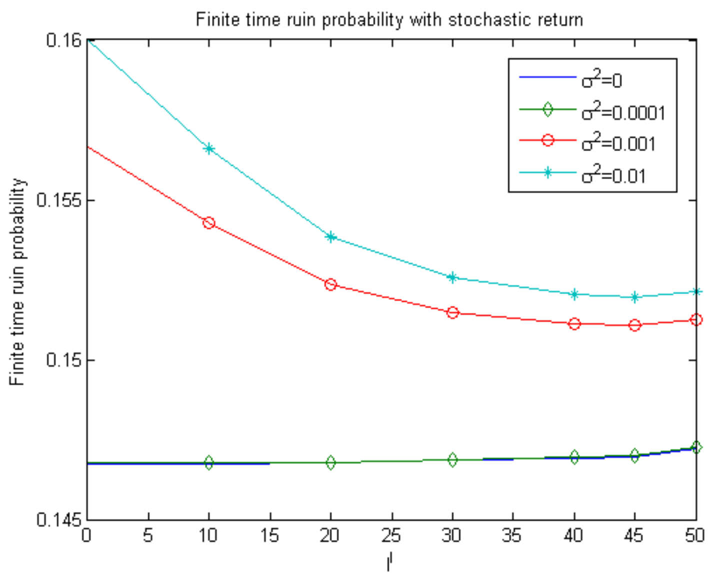


Figure 4.8: Finite time ruin probabilities corresponding to the interclaim time distribution (c) with $l^W = 0$, $0 \leq l^I \leq 45$, $l^D = 50$, $y = 0.05$, $\delta_t = 3$, $d = 1$, $k_1 = 0.01$, $k_2 = 0.02$, $v = 10, g_I = 0, g_Q = 10, g_L = 0$, and $\tau = 15$

4.5 Conclusion

In this chapter, we have proposed a framework of Hybrid-Takaful that incorporates investment activities and qard-hasan facility. Qard-hasan (benevolent loan) facility is the non-interest loan provided by shareholders to Takaful participants in the case of a deficit.

We assume that the qard-hasan undertaking will be returned if the participants' fund gains surplus in the future. We construct a surplus process of the participants' fund, and then derive a method of calculating finite-time ruin probabilities.

Based on our numerical simulations, we find that the qard-hasan facility improves the performance of the fund, as it decreases the finite-time ruin probabilities. This can be explained by the fact that this facility, unlike in the conventional insurance, provides loans at no cost and no mandatory repayment when there is a deficit. In addition, paying off the qard-hasan undertaking not only follows the Shariah rule but also has a positive effect on the business, if we invest the undrawn-down qard-hasan in non-risky assets. By paying off the loan undertaking to the qard-hasan fund, we can guarantee that the fund will grow at the corresponding rate of return. If the fund remains in the surplus account, then the investment return is delayed until the surplus account reaches the investment trigger level. Our study incorporates the option to invest the investment fund and qard-hasan fund in the same risky asset under the assumption of binomial CRR market model.

In theory, we can approximate the infinite-time ruin probability by calculating the finite-time ruin probability with time horizon $\tau \to \infty$. However, in practice this approach is expensive in terms of the numerical computations due to the recursive calculations. To overcome this problem, the simulation based method can be one possible approach to calculate the infinite-time ruin probability which is the future work of this research.

Chapter 5

Concluding Remarks and Future Works

This thesis presents a mathematical approach to quantify the risk associated with Islamic finance and insurance. In particular, we propose a Shariah-compliant portfolio model for single and multiperiod time settings, and a computational framework to calculate finite-time ruin probability in a Takaful product. This chapter provides some concluding remarks and possible future research topics for each component of this thesis.

5.1 A Single Period Markowitz-based Shariah Compliant Portfolio Model

In Chapter 2 we have proposed a Shariah-compliant portfolio optimization problem with a new purification model, which is more consistent with current practice than other existing models. We also have incorporated the importance of non-compliance probabilistic screening constraints, which are overlooked in the existing literature. We found that the proposed constraints are effective in maintaining the sustainability of the Shariah-compliance portfolio. However, our current model can not capture the cost incurred by the compliance change due to the screening process. Also, our study focuses on model construction and applies optimization software to solve the problem. Hence, several future works can be done to extend the research:

- Study the possible costs incurred due to changes in compliance status and include these variables into the Shariah portfolio model.

- Study the possibility to derive an analytical solution to solve the proposed Shariah portfolio problem. If the analytical solution can not be achieved, proposing a numerical approach to solve the Shariah portfolio problem is another interesting topic.
- In the current model, we use mean and variance to measure the risk of a portfolio. Considering other risk measure models can be an interesting topic in a Shariah compliance portfolio.

5.2 Multiperiod Mean-Variance Shariah Compliant Portfolio

In Chapter 3, we have extended the Shariah-compliant portfolio model with the purification and screening process in the multiperiod setting. The effectiveness of the proposed screening constraints is seen more profound in the case of the multiperiod Shariah portfolio problem. We found that the chosen assumption of the financial ratios model is sensitive to the Shariah portfolio efficient frontier. In the current numerical example, we only consider beta distribution and the Beta-AR(1) model to capture the temporal dynamics of financial screening ratios. We apply a pre-commitment strategy with a geometric Brownian motion model to solve the problem and overlook the time consistency issue. Several important future works can be done to improve the research:

- Conduct an indepth study to determine the best stochastic model to capture financial screening ratios.
- Impose time consistency constraint as proposed by Cong and Oosterlee (2017a) to the Shariah portfolio model to guarantee the time consistency condition.

5.3 Discrete Time Ruin Probability for Takaful (Islamic Insurance) with Investment and Qard-Hasan (Benevolent Loan) Activities

In Chapter 4, we have constructed a framework of Hybrid-Takaful that incorporates investment-based mudarabah (profit-sharing) activities and qard-hasan (benevolent loan) facility. In this study, we use ruin probability to quantify the risk associated with Takaful business.

We found that paying off the qard-hasan undertaking follows the Shariah rule and positively affects the business if we invest in the undrawn-down qard-hasan. In the current model, we consider the investment portfolio consists of 1 risk-free asset and 1 risky asset under the assumption of the binomial CRR model. There are several future interesting topics in Takaful model, including:

- Extend the model to invest in several risky assets for the investment fund and qard-hasan fund. For this case, we may follow the idea of Moon et al. (2008), who constructed the binomial state price for two dependent assets.
- Derive formula to compute the expected total discounted dividend payments from underwriting surplus, expected total mudarabah, and expected total wakalah fee for our proposed Hybrid Takaful model can be an alternative method to measure the performance of Takaful product. The recursive formula for dividend payments might have a faster convergence than the ruin probability calculation due to the discount factor as shown in Kim and Drekic (2016).
- Consider a simulation based approach to calculate a infinite time ruin probability.
- Conduct research on the optimal decision problem for Takaful products, including the study on the investment and dividend payment strategies.

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Appendix A

Descriptive Statistics of Data Used in Section 2.3

	TLKM	UNVR	PGAS	WIKA	KLBF	ASII
Mean	0.058605	0.059019	0.018663	0.066473	0.077166	0.096709
Variance	0.0211	0.016059	0.033481	0.066605	0.042362	0.052475
Std Dev	0.145259	0.126724	0.182977	0.25808	0.205821	0.229073
Skewness	0.647027	0.637435	0.752749	0.694834	0.417666	1.830642
Excess Kurtosis	1.109213	0.192871	1.514068	0.261214	0.313865	4.408252

Table A.1: Descriptive statistics of quarterly stock return data

	TLKM	UNVR	PGAS	WIKA	KLBF	ASII
Mean	0.069998	0.008359	0.009998	0.006350	0.006485	0.022681
Variance	0.006406	0.000014	0.000058	0.000016	0.000017	0.001222
Std Dev	0.080039	0.003795	0.007613	0.004058	0.004136	0.034957
Skewness	1.005547	0.022057	1.358758	1.38922	1.170942	0.617968
Excess Kurtosis	-0.642802	-1.830258	0.09621	0.240012	-0.489147	-1.102855

Table A.2: Descriptive statistics of quarterly dividend yield data

	TLKM	UNVR	PGAS	WIKA	KLBF	ASII
Mean	0.000158	0.000007	0.000691	0.000616	0.000188	0.007378
Variance	0.000000	0.000000	0.000002	0.000002	0.000000	0.000068
Std Dev	0.000246	0.000008	0.001479	0.001335	0.000347	0.008234
Skewness	1.235055	0.588716	2.434840	2.180649	1.452787	0.538026
Excess Kurtosis	0.343202	-1.104242	5.430587	3.349179	0.400690	-1.167754

Table A.3: Descriptive statistics of quarterly dividend purification rate data

	TLKM	UNVR	PGAS	WIKA	KLBF	ASII
Mean	0.002844	0.000021	0.002483	0.005245	0.000821	0.039644
Variance	0.000002	0.000000	0.000002	0.000024	0.000000	0.000450
Std Dev	0.001392	0.000018	0.001564	0.004853	0.000471	0.021224
Skewness	0.407361	1.247295	0.984013	1.220263	0.477745	0.398915
Excess Kurtosis	-0.846496	0.924108	0.627371	-0.069929	-1.072433	-0.919287

Table A.4: Descriptive statistics of quarterly investment purification rate data

Appendix B

Histograms and QQ-Plots of Financial Ratios Data

B.1 Histograms of Financial Ratios Data

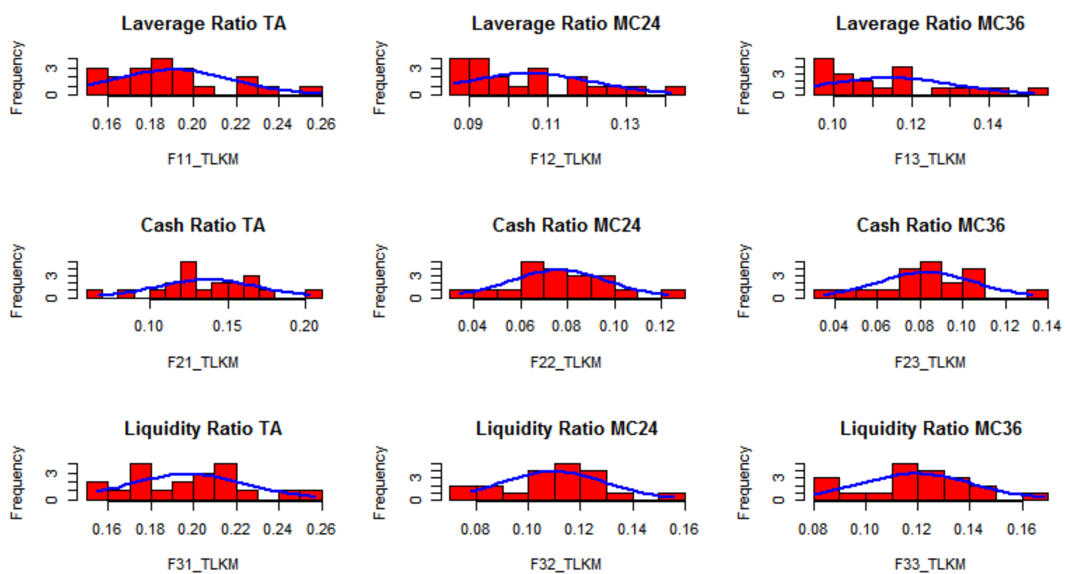


Figure B.1: Histograms TLKM Financial Ratios

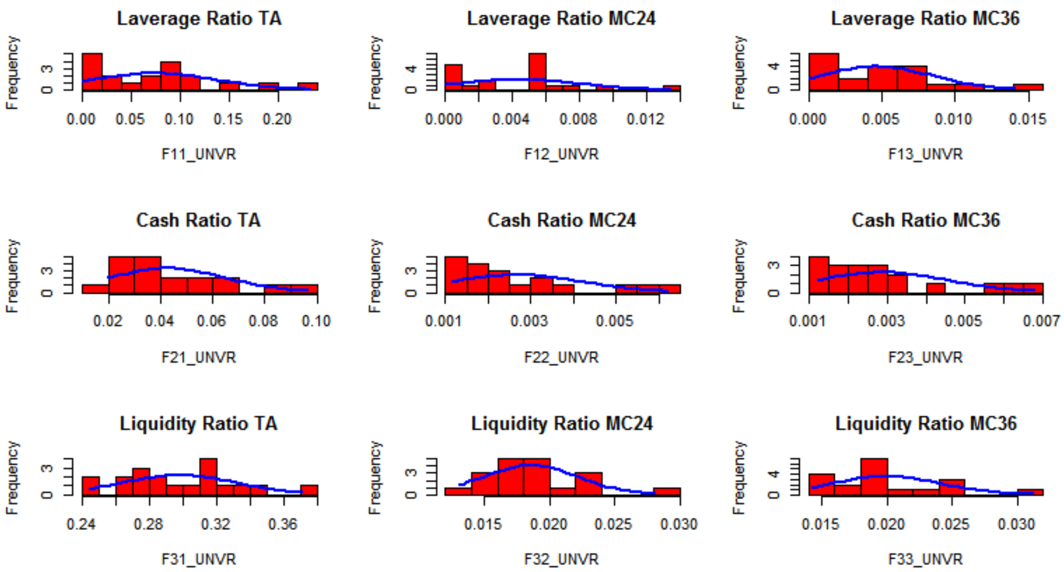


Figure B.2: Histograms UNVR Financial Ratios

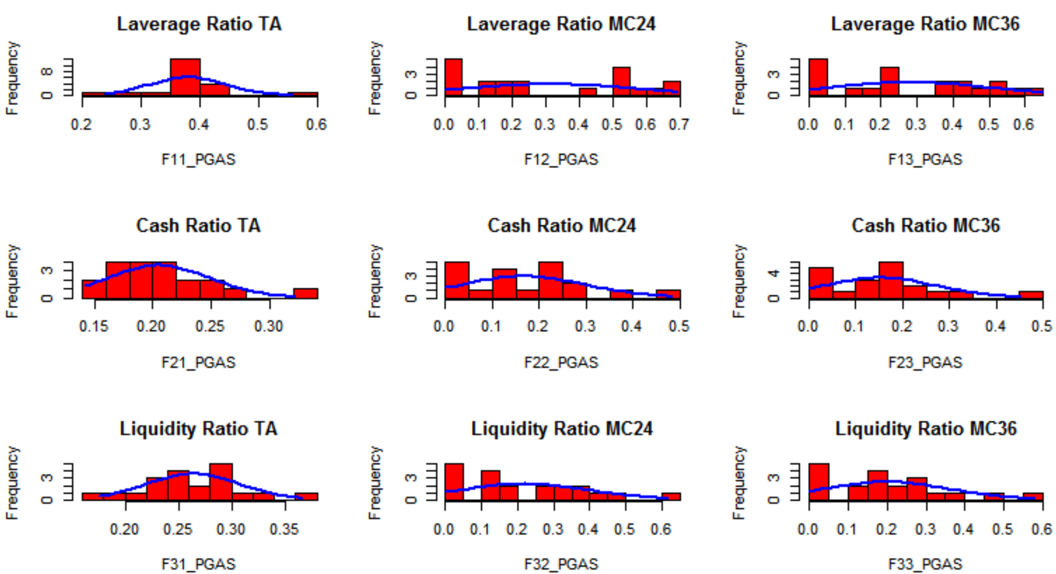


Figure B.3: Histograms PGAS Financial Ratios

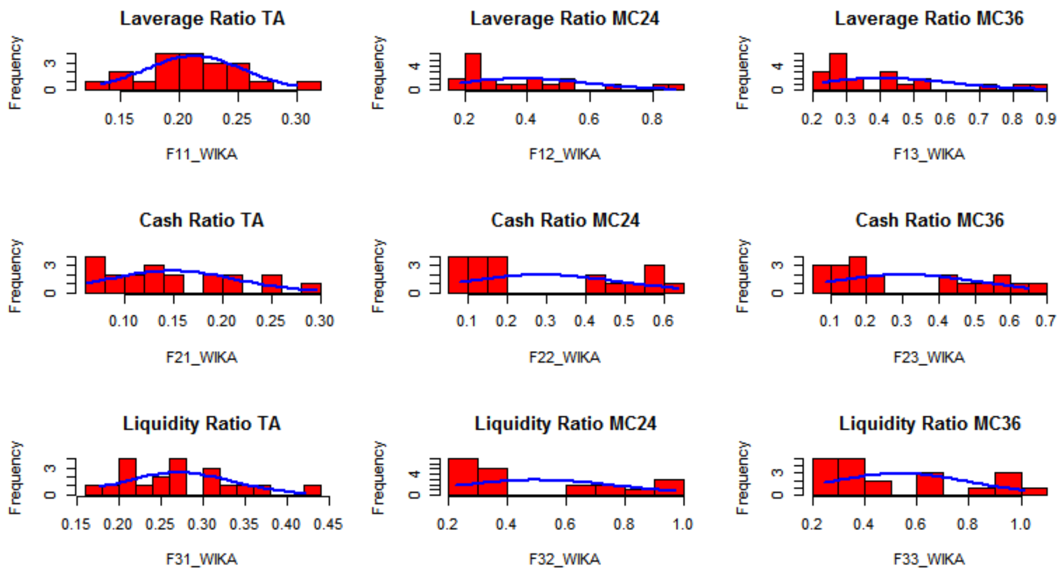


Figure B.4: Histograms WIKAFinancial Ratios

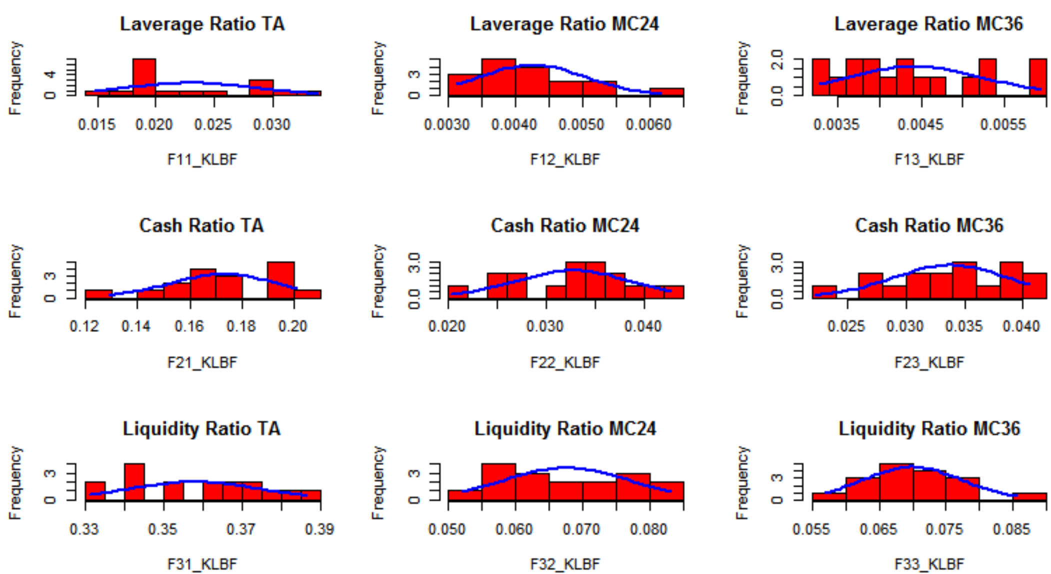


Figure B.5: Histograms KLBF Financial Ratios

Figure B.6: Histograms ASII Financial Ratio

B.2 QQ-Plots of Financial Ratios Data

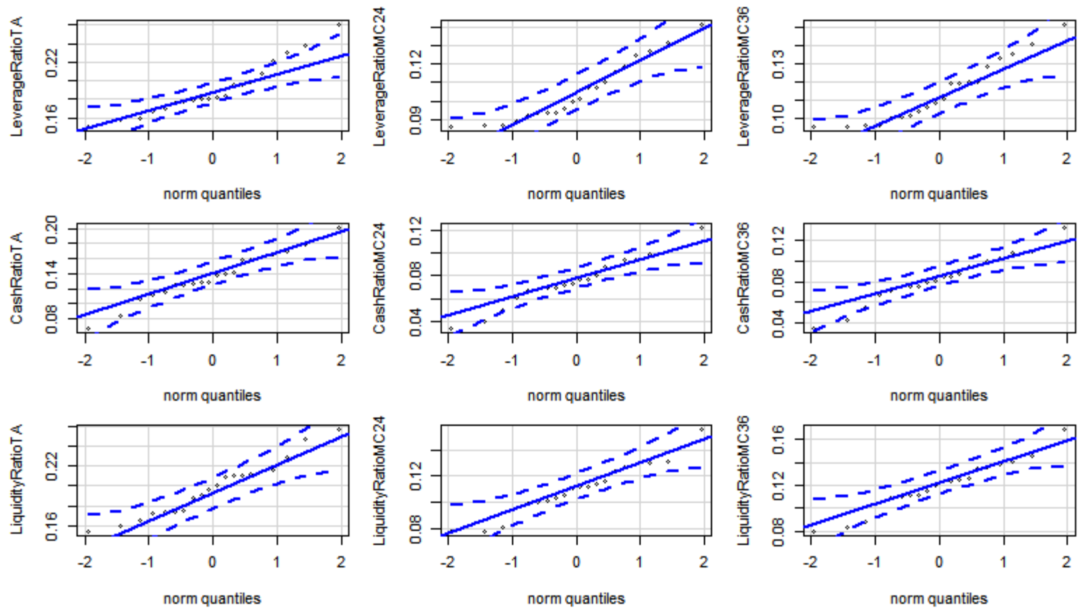


Figure B.7: qqPlots TLKM Financial Ratios

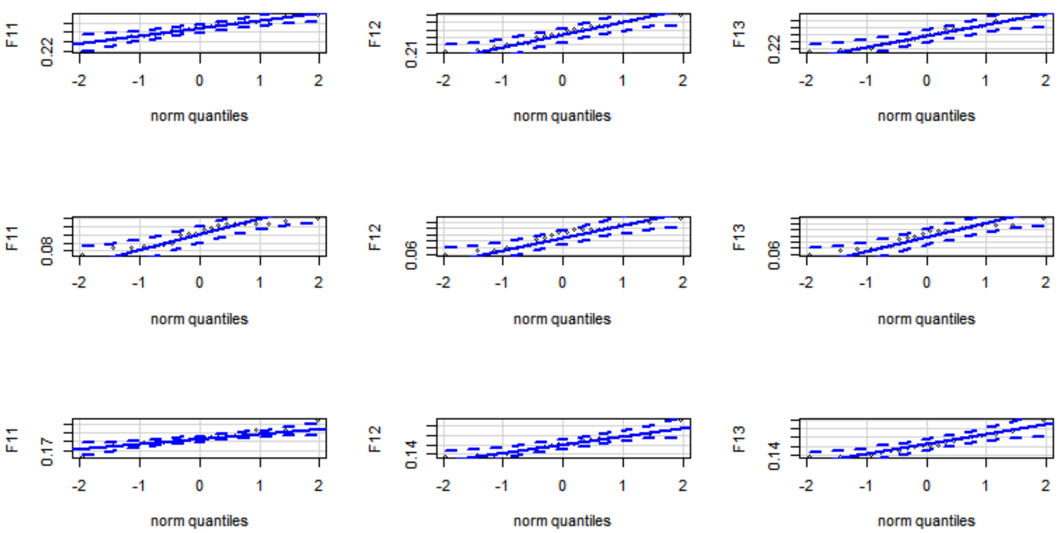


Figure B.8: qqPlots ASII Financial Ratiso

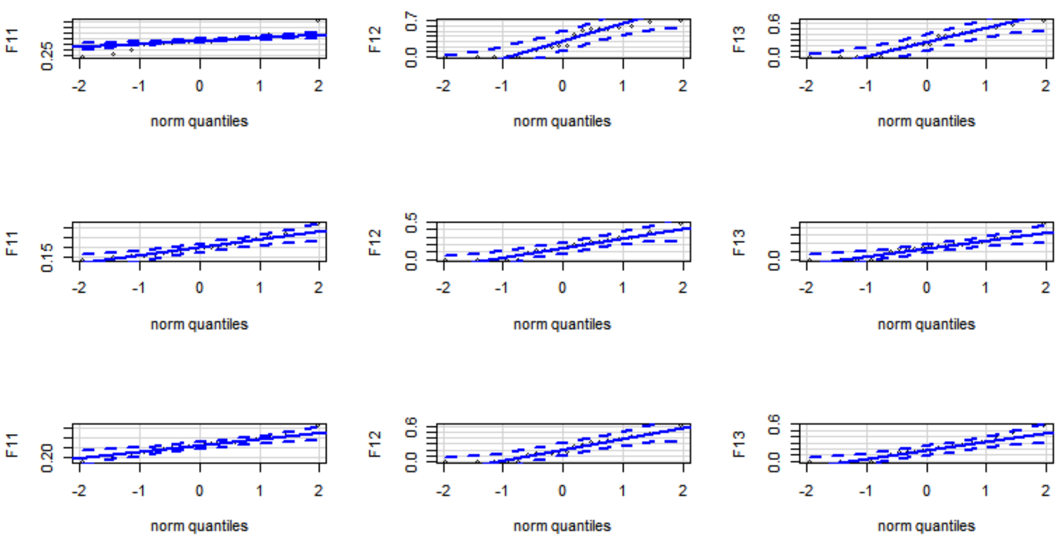


Figure B.9: qqPlots PGAS Financial Ratios

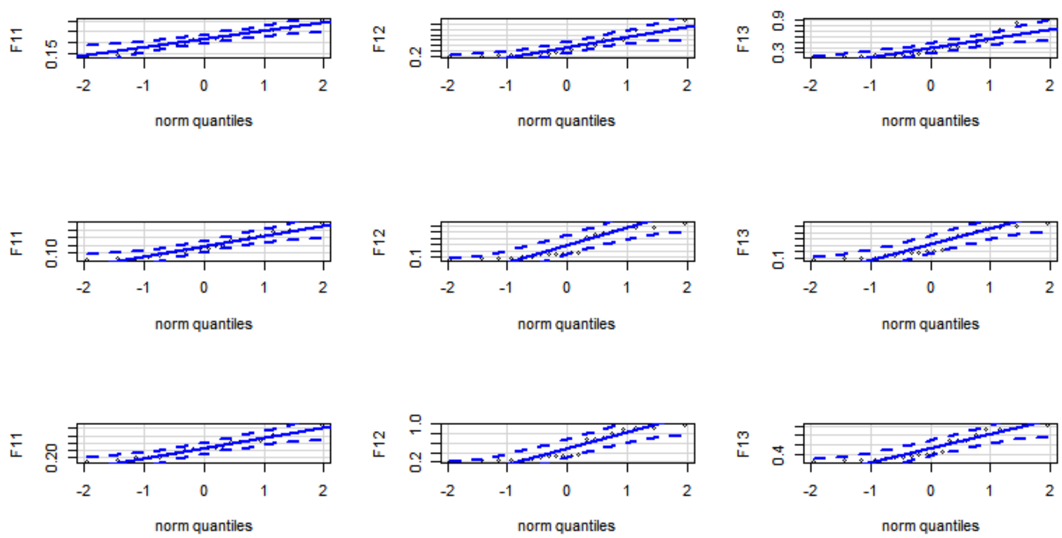


Figure B.10: qqPlots WIKA Financial Ratios

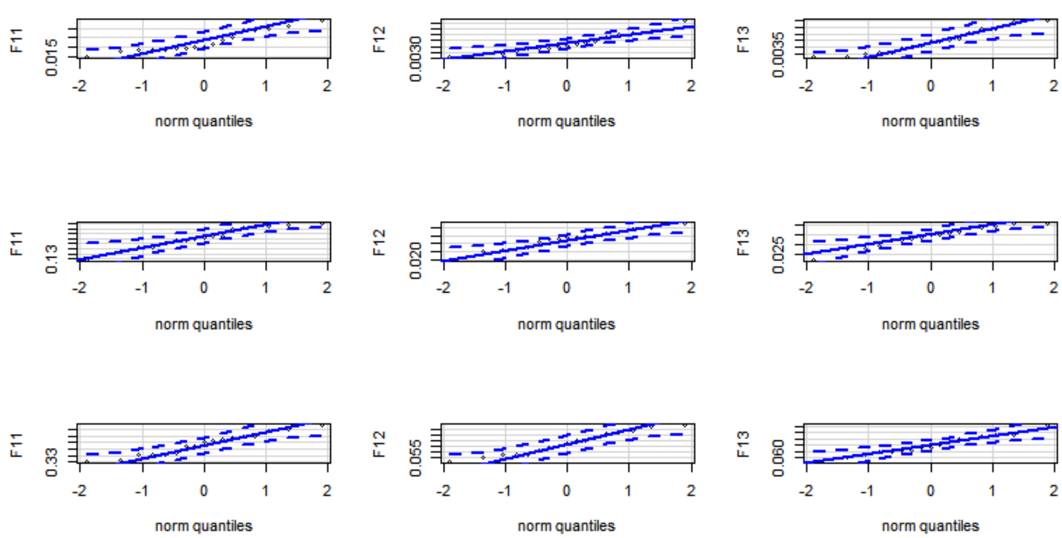


Figure B.11: qqPlots KLBF Financial Ratios

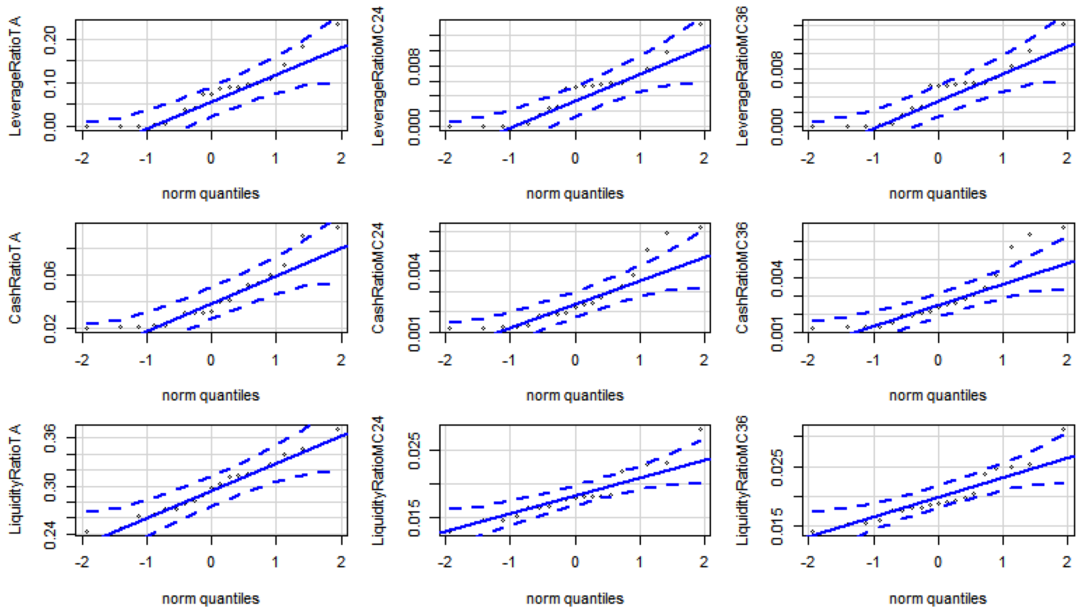


Figure B.12: qqPlots UNVR Financial Ratios

Appendix C

Data and Statistical Analysis Discussed in Section 3.3

C.1 Statistical Evidence of Returns Data

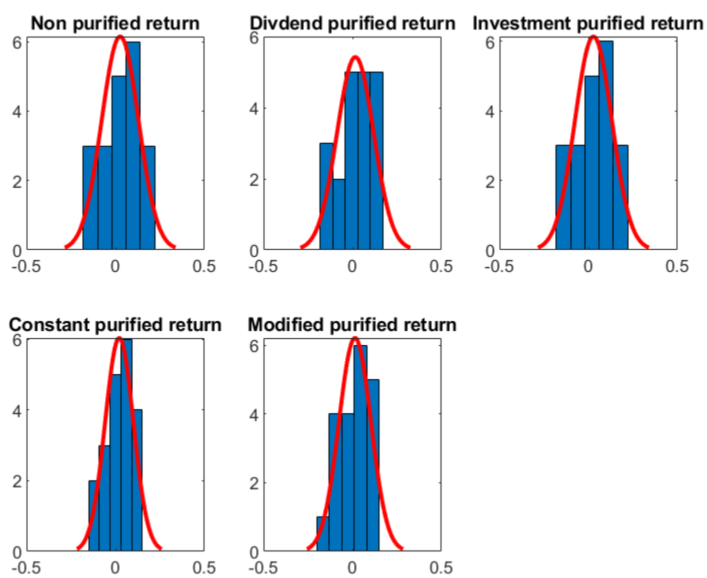


Figure C.1: Histograms of Return and Purified returns ASII

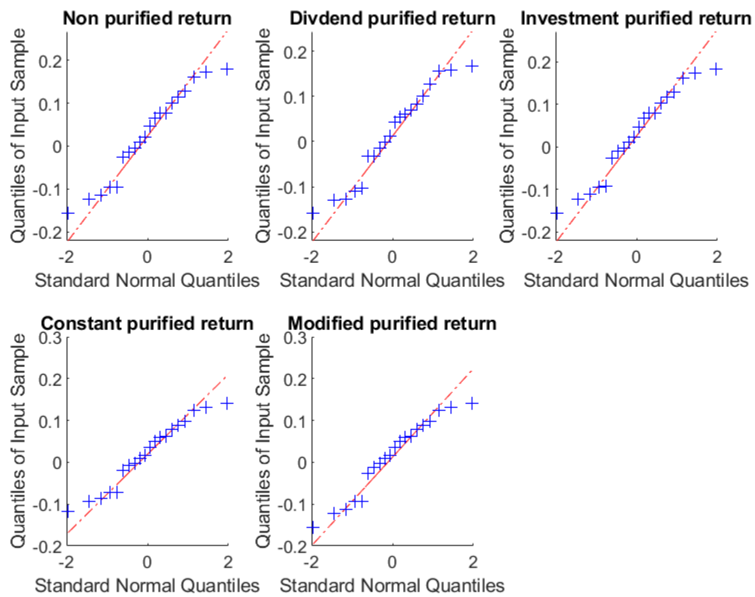


Figure C.2: QQ-Plots of Return and Purified returns ASII

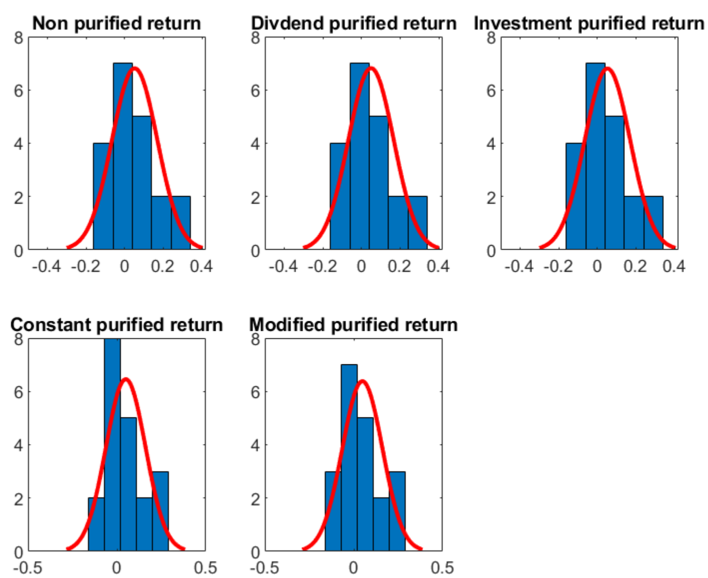


Figure C.3: Histograms of Return and Purified returns TLKM

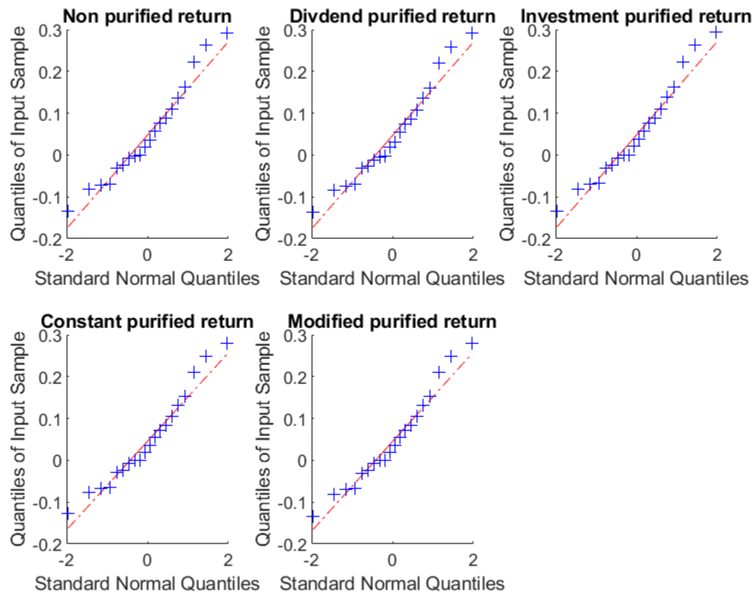


Figure C.4: QQ-Plots of Return and Purified returns TLKM

C.2 Algorithm to Estimate Beta-AR(1) Parameters

The conditional mean and variance of financial ratio $F_{ji,t}$, respectively, are:

$$E[F_{ji,t}|\mathcal{G}_t] = \pi_{ji,t}, \tag{C.1}$$

and

$$Var[F_{ji,t}|\mathcal{G}_t] = \frac{\pi_{ji,t}(1-\pi_{ji,t})}{\delta_{ji}} \tag{C.2}$$

where

$$\log(\frac{\pi_{ji,t}}{1-\pi_{ji,t}}) = \alpha^A_{ji} + \beta^{A1}_{ji}\log(\frac{F_{ji,t-\Delta t}}{1-F_{ji,t-\Delta t}}). \tag{C.3}$$

Then, we can estimate the Beta-AR(1) parameters by using the following algorithm:

Algorithm 1: The calculation of Beta-AR(1) parameters

Input: Historical time series data of financial ratio
$F_{ji,t}:\ y_{ji} = [y_{ji,1}, y_{ji,2}, \ldots, y_{ji,m}]$

Output: $\alpha^A_{ji}, \beta^A_{ji}, \delta_{ji}$

for $k \leftarrow 2$ ***to*** m **do**

- Calculate mean of sample data : $\pi_k = \frac{\sum_{s=1}^{k} y_{ji,s}}{k}$
- Calculate variance of sample data : $v_k = \frac{\sum_{s=1}^{k}(y_{ji,k}-\pi_k)^2}{k-1}$
- Calculate logit function of π : $g_{\pi_k} = \log(\frac{\pi_k}{1-\pi_k})$
- Calculate logit function of y : $g_{y_{ji,k}} = \log(\frac{y_{ji,k-1}}{1-y_{ji,k-1}})$

Find the linear regression coefficients : $g_\pi = \alpha^A_{ji} + \beta^{A1}_{ji} g_{y_{ji}}$

Find $\delta_{ji} := \arg\min(v - \frac{\pi(1-\pi)}{1+\delta_{ji}})^2$

return $\alpha^A_{ji}, \beta^{A1}_{ji}, \delta_{ji}$

C.3 Description of the Diagnostic Test

Under the Beta-AR(1) model, the predicted financial ratio follows:

$$F_{ji,t} \sim Beta(\pi_{ji,t}\delta_{ji}, \pi_{ji,t}\delta_{ji}) \tag{C.4}$$

where

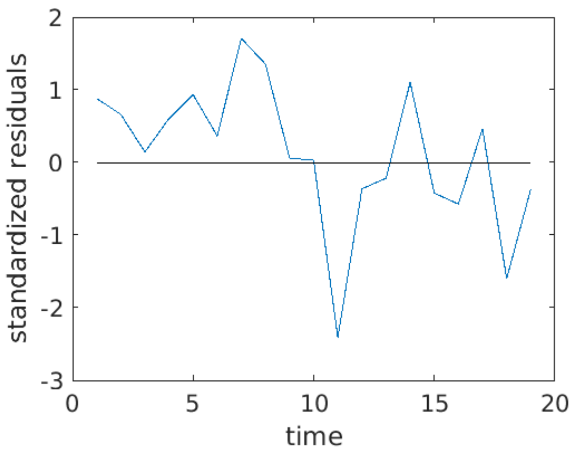
$$\log(\frac{\pi_{ji,t}}{1-\pi_{ji,t}}) = \alpha^A_{ji} + \beta^{A1}_{ji} \log(\frac{y_{ji,t-\Delta t}}{1-y_{ji,t-\Delta t}}) \tag{C.5}$$

with $y_{ji,t},\ t = 1, \ldots, m$ being the actual value.

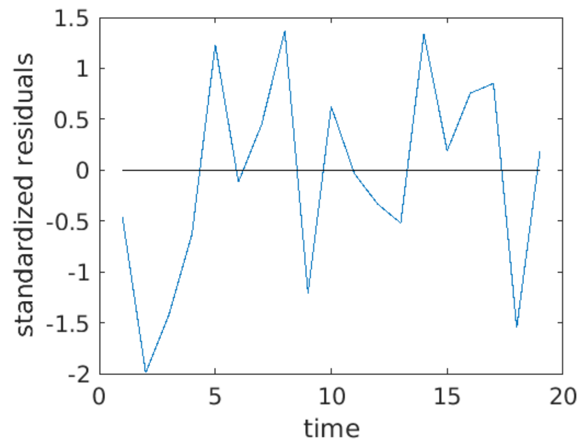
After obtaining the estimated coefficients $\alpha^A_{ji}, \beta^{A1}_{ji}$, and δ_{ji}, the standardized residuals are defined as

$$e_{ji,t} = \frac{y_{ji,t} - F_{ji,t}}{\sqrt{\frac{\sum_{t=1}^{m}(y_{ji,t}-F_{ji,t})^2}{m-1}}}. \tag{C.6}$$

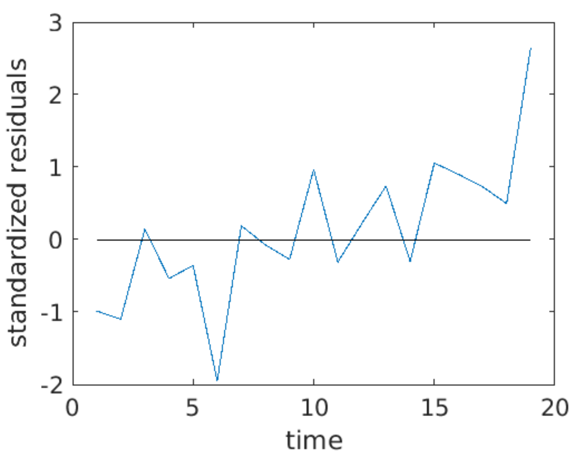
C.3.1 Plots of the Standardized Residuals



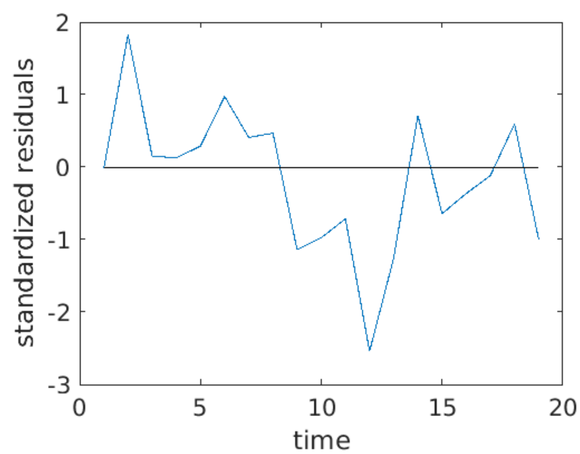
(a) Leverage ratio



(b) Cash ratio

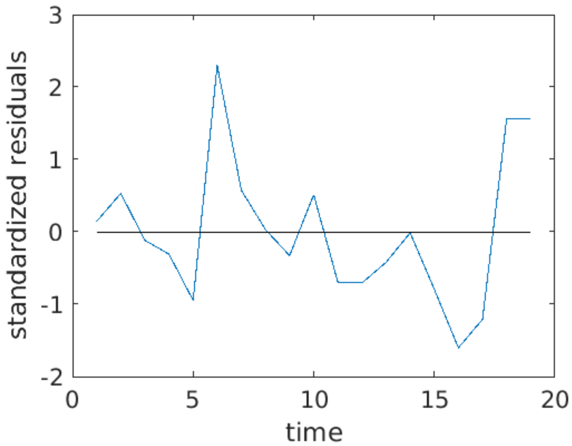


(c) Liquidity ratio

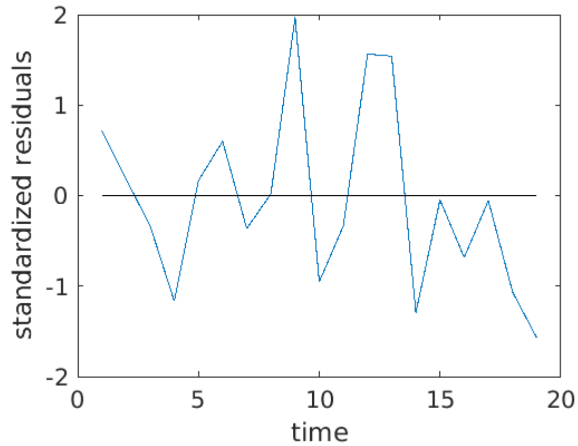


(d) NPI ratio

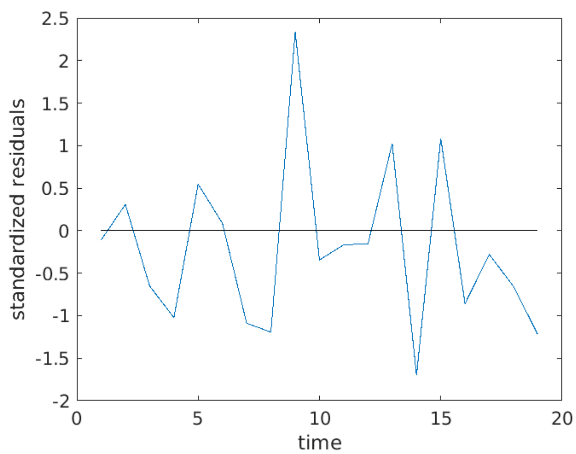
Figure C.5: Standardized residuals from Beta-AR(1) model for ASII financial ratios



(a) Leverage ratio



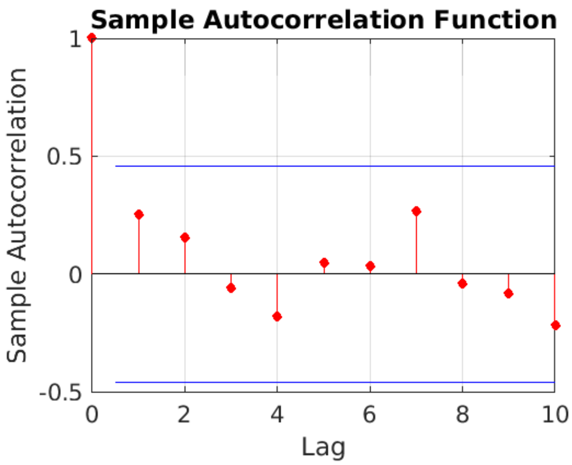
(b) Cash ratio



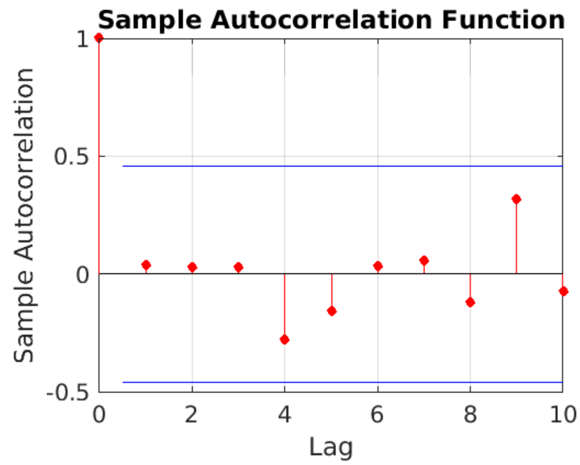
(c) Liquidity ratio

Figure C.6: Standardized residuals from Beta-AR(1) model for TLKM financial ratios

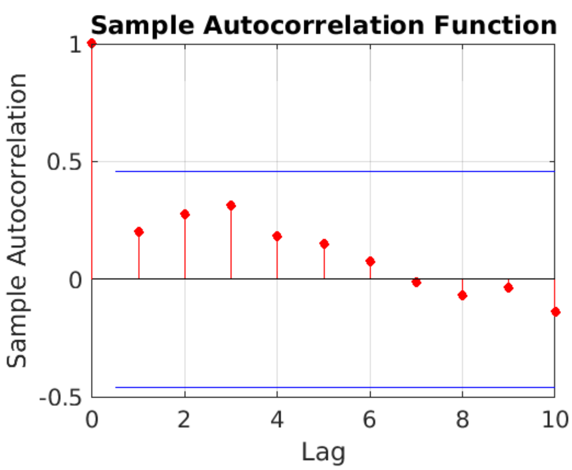
C.3.2 Autocorrelation of the Standardized Residuals



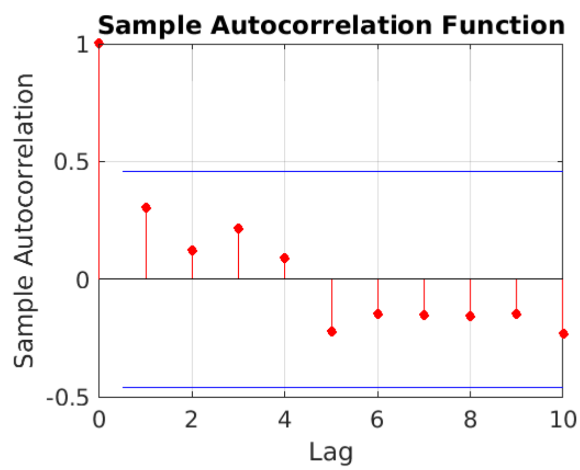
(a) Leverage ratio



(b) Cash ratio

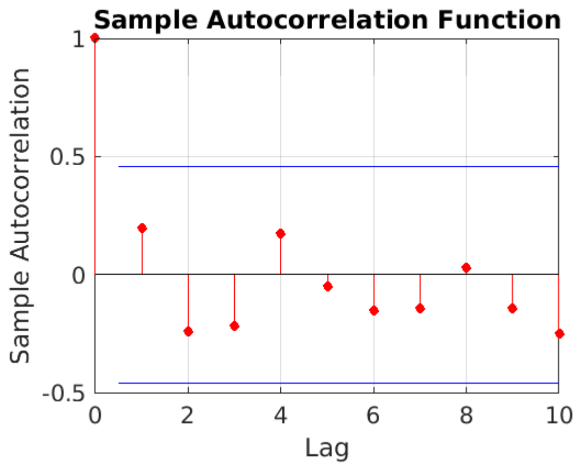


(c) Liquidity ratio

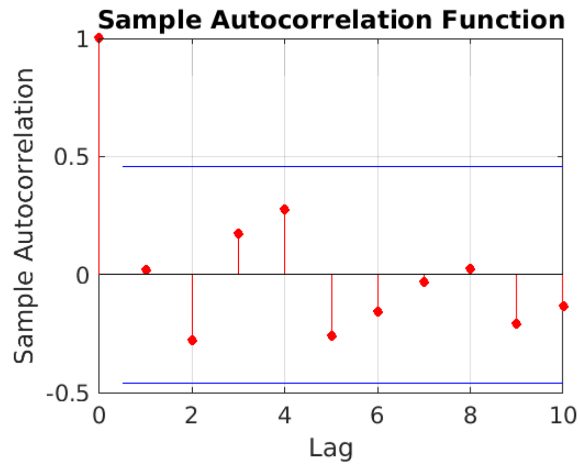


(d) NPI ratio

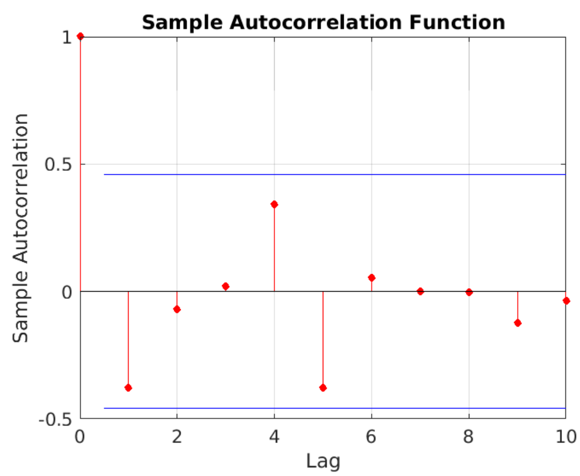
Figure C.7: Sample autocorrelation functions of the standardized residuals from Beta-AR(1) model for ASII financial ratios



(a) Leverage ratio



(b) Cash ratio



(c) Liquidity ratio

Figure C.8: Sample autocorrelation functions of the standardized residuals from Beta-AR(1) model for TLKM financial ratios

Appendix D

List of Symbols Used in Chapter 4

Symbol	Description
$a_k, k = \{1, .., n_a\}$	probability mass function (pmf) of W_i
$A_k, k \in \{1, .., n_a\}$	survival function of W_i
b	contribution
$c_{(t,m,u,f_L)}$	calling point for time horizon t with initial values m, u, f_L, and $M_0 = m$
d	deposit
$f_s,\ s \in \{I, Q, L\}$	initial value of F^s
$f_s^*(l),\ s \in \{I, Q, L\}$	updated initial value of $F^s,\ s \in \{I, Q, L\}$ after claim payment
$f_s'(l),\ s \in \{I, Q, L\}$	updated initial value of F^s after qard-hasan repayment
F^I	Investment fund
$\hat{F}^I_{(t,u,f_I,l)}$	Maximum value of F^I at time t with initial values u, f_I for asset price l
F^L	Libaility fund
$\hat{F}^L_{(t,f_L)}$	Maximum value of F^L at time t with initial values f_L
$\hat{F}^Q_{(t,f_Q,l)}$	Maximum value of F^Q at time t with initial values f_Q, and asset price l
$F_t^s,\ s \in \{I, Q, L\}$	Level of F^s at the end of period $(t-1, t]$
$F_{t-}^s,\ s \in \{I, Q, L\}$	Level of F^s after claim payment, before withdraw and borrowing corresponding to time interval (t-1,t]
j^d	magnitude of asset price down-ward movement
j^u	magnitude of asset price upward movement
k_1	investment gain of F^I under assumption risk-free asset
k_1'	real investment gain of F^I
k_1	investment gain of F^Q under assumption risk-free asset
l^D	trigger level for dividend payment

l^I	trigger level for investment activities
l^W	minimal requirement of surplus fund
M_0	elapsed waiting time since the most recent claim occurrence
n_a	upper bound of W_i
$N_l(t)$	number of asset state prices at time t
p	probability of asset price upward movement
$P(t,l)$	probability of the state price l at time t
T	ruin time
u	initial value of U
u^*	updated initial value of U after claim payment
u'	updated initial value of U after qard-hasan repayment
U	Surplus fund
U_t	Level of U at the end of period $(t-1,t]$
U_{t-}	Level of U after claim payment, before withdraw and borrowing corresponding to time interval $(t-1,t]$
$\hat{U}_{(t,u)}$	Maximum value of U at time t with initial value u
W_i	time between claims $(i-1)$ and i
X_i	i-th claim's severity
y	percentage of Mudharabah fee
$z^I(u)$	the earliest time U with initial value u reaches l^I
$z^D(u)$	the earliest time U with initial value u reaches l^D
$\alpha_j(k)$	conditional pmf of X_i given $W_i = k$
δ_t^1	a constant dividend
δ_t^2	excess surplus dividend
$\Psi(u, f_I, f_Q, f_L, \tau)$	finite-time ruin probability with values u, f_I, f_Q, f_L, τ
σ^2	variance of asset price rate of return
$\sigma(u, f_I, f_Q, f_L, n, m)$	finite-time survival probability with $\tau = n, M_0 = m$, and initial values u, f_I, f_Q, f_L
τ	time horizon of the finite-time ruin/survival probability
x_+	$\max\{x, 0\}$

Table D.1: List of Symbols

Appendix E

The proofs of Theorems in Chapter 4

E.1 Proof of Theorem 4.3.1

Proof. By incorporating debt fund scenario (i.e qard-hasan facility in Takaful), there is possibility to make a loan repayment before the first claim occurs. Hence, we need to reset the recursive calculation (4.37) at the time the qard-hasan repayment is made (i.e., at the calling point $c_{(n,m,u,f_L)}$). In practice we need to consider two cases when calculating $\sigma(u, f_I, f_Q, f_L, n, m)$, which correspond to the first claim occurring either before the calling point $c_{n,m,u,f}$ or after this point. While in the first case the calculations are similar to those in (4.37). But, the pmf of the interclaim time is now conditional on the value of m. While, in the second case the calculations need a different approach. From the explanation above, we get

$$\begin{aligned}\sigma(u, f_I, f_Q, f_L, n, m) = \sum_{k=1}^{c_{(n,m,u,f_L)}} \frac{a_{k+m}}{A_m} Pr\{T > n | U_0 = u, F_0^I = f_I, F_0^Q = f_Q,\\ F_0^L = f_L, M_0 = m, W_1(m) = k\} + \frac{A_{c_{(n,m,u,f_L)}}}{A_m} Pr\{T > n | U_0 = u, F_0^I = f_I,\\ F_0^Q = f_Q, F_0^L = f_L, M_0 = m, W_1(m) > c_{(n,m,u,f_L)}\}.\end{aligned} \tag{E.1}$$

The probabilities in the right hand side of Equation (E.1) represents the probabilities that the ruin does not occur until time point n, given that the first claim X_1 takes place at time $W_1(m) = k$, where $W_1(m)$ is the duration from the initial time point until the first claim occurring given that the elapsed waiting time since the most recent claim is m. Next, our objective is to prove that the right-hand side of Equation (E.1) is the same as the right-hand side of Equation (4.38). In particular, we will prove the following equations:

$$Pr\{T > n | U_0 = u, F_0^I = f_I, F_0^Q = f_Q, F_0^L = f_L, M_0 = m, W_1(m) = k\} = \sum_{l=1}^{N_l(k)} P(k,l) \sum_{j=1}^{\hat{U}_{(k,u)}+\hat{F}^I_{(k,u,f_I,l)}+\hat{F}^Q_{(k,f_Q,l)}} \alpha_j(k+m)\sigma(u^*(l), f_I^*(l), f_Q^*(l), f_L^*(l), n-k, 0),\ k \in \{1,2,..,c_{(n,m,u,f_L)}\}, \tag{E.2}$$

and

$$Pr\{T > n | U_0 = u, F_0^I = f_I, F_0^Q = f_Q, F_0^L = f_L, M_0 = m, W_1(m) > c_{(n,m,u,f_L)}\} = \sum_{l=1}^{N_l(c_{(n,m,u,f_L)})} P(c_{(n,m,u,f_L)}, l)\sigma(u', f_I'(l), f_Q'(l), f_L', n - c_{(n,m,u,f_L)}, c_{(n,m,u,f_L)} + m). \tag{E.3}$$

To prove Equation (E.2) we use similar approach as in Cossette et al. (2006). The conditional survival probability given the first claim occur at time $k \in \{1,2,..,c_{(n,m,u,f_L)}\}$ is the weighted sum of $\sigma(u^*, f_I^*, f_Q^*, f_L^*, n-k, 0)$, which is the probability of surviving the time interval $(k, n]$ with the level of funds' process at time k after the claim payment is u^*, f_I^*, f_Q^*, f_L^*, for all possible values of claim severity j that does not cause ruin. The maximum value of claim severity is bounded above by the amount of total available fund in the surplus fund, investment fund, and qard-hasan fund at time k before the first claim payment. This value is equal to $\hat{U}_{(k,u)} + \hat{F}^I_{(k,u,f_I,l)} + \hat{F}^Q_{(k,f_Q,l)}$, for all possible state price $l \in \{1,2,..,N_t(k)\}$. Where $\hat{U}_{k,u}$, $\hat{F}^I_{(k,u,f_I,l)}$, and $\hat{F}^Q_{(k,f_Q,l)}$ are the maximum values of fund processes that explained in Section 4.3.1. The initial surplus and external fund amounts for the next recursion, u^*, f_I^*, f_Q^*, f_L^*, are determined by the size of contributions and the incured claim j that can be found in Theorem 4.3.2. This explanation prove Equation (E.2).

Now, we follow the idea proposed by Kim and Drekic (2016) to prove Equation (E.3). In situations when $W_1(m) > c_{(n,m,u,f_L)}$, we do not consider on the claim payment, instead we consider on making the qard-hasan repayment at time $c_{(n,m,u,f_L)}$. Since, we are not considering the claim occurrence at time $c_{(n,m,u,f_L)}$, for the next recursion, the elapsed waiting time counter is increase by $c_{(n,m,u,f_L)}$, while n is reduced by $c_{(n,m,u,f_L)}$. The initial funds' process for the next recursion are $u', f_I'(l), f_Q'(l), f_L'$, which represents the funds' level after the qard-hasan repayment is made at time $c_{(n,m,u,f_L)}$. these values can be found in Theorem 4.3.3. □

E.2 Proof of Theorem 4.3.2

Proof. When explaining the formulas for $u^*(l), f_I^*(l), f_Q^*(l), f_L^*(l)$ it is convenient to split their values into 5 cases depending on the possible positions of the surplus level after a claim payment. We define array $(u^*(l), f_I^*(l), f_Q^*(l), f_L^*(l))$ in the following explanation for each cases.

- **case 1:** $\hat{U}_{(k,u)} - j - l^W \geq \hat{F}^L_{(k,f_L)}$. Since, in this case the surplus level after the claim payment exceeds l^W and is greater than the liability level, we make full qard-hasan repayment. Therefore we have $(\hat{U}_{(k,u)} - j - \hat{F}^L_{(k,f_L)}, \hat{F}^I_{(k,u,f_I,l)}, \hat{F}^Q_{(k,f_Q,l)} + \hat{F}^L_{(k,f_L)}, 0)$.

- **case 2:** $0 \leq \hat{U}_{(k,u)} - j - l^W < \hat{F}^L_{(k,f_L)}$. In this case the surplus level after the claim payment exceeds l^W but is not enough to cover all liability. Hence we make the qard-hasan repayment equal to the difference between the surplus and the level l^W . Therefore we have $(l^W, \hat{F}^I_{(k,u,f_I,l)}, \hat{F}^Q_{(k,f_Q,l)} + (\hat{U}_{(k,u)} - j - l^W), \hat{F}^L_{(k,f_L)} - (\hat{U}_{(k,u)} - j - l^W))$.

- **case 3:** $\hat{F}^I_{(k,u,f_I,l)} \geq l^W - (\hat{U}_{(k,u)} - j) > 0$. In this case the surplus level after the claim payment is less then l^W. Therefore, we need to withdraw from the investment fund to bring the surplus value back to the level l^W. Thus we have $(l^W, \hat{F}^I_{(k,u,f_I,l)}, \hat{F}^Q_{(k,f_Q,l)} + (\hat{U}_{(k,u)} - j - l^W), \hat{F}^L_{(k,f_L)} - (\hat{U}_{(k,u)} - j - l^W))$.

- **case 4:** $\hat{F}^Q_{(k,f_Q,l)} \geq l^W - \hat{U}_{(k,u)} + j - \hat{F}^I_{(k,u,f_I,l)} > 0$. In this case the surplus level after the claim payment is less then l^W, and the investment fund is not enough to cover the deficit of surplus fund, then we need to borrow from the qard-hasan fund to bring the surplus value back to the level l^W. Therefore we have $(l^W, 0, \hat{F}^Q_{(k,f_Q,l)} + \hat{U}_{(k,u)} - j - l^W + \hat{F}^I_{(k,u,f_I,l)}, \hat{F}^L_{(k,f_L)} - (\hat{U}_{(k,u)} - j - l^W + \hat{F}^I_{(k,u,f_I,l)}))$.

- **case 5:** $\hat{F}^Q_{(k,f_Q,l)} < l^W - \hat{U}_{(k,u)} + j - \hat{F}^I_{k,u,f_I,l}$. In this case the surplus level after the claim payment is less then l^W, however both the investment and qard-hasan funds are not enough to bring the surplus fund back to the level l^W. Thus, the maximum surplus level is equal to $\hat{U}_{(k,u)} - j + \hat{F}^I_{(k,u,f_I,l)} + \hat{F}^Q_{(k,f_Q,l)}$. As long as this value is non negative, the recursive calculation is still running. If this value is negative then the calculation is finished based on the boundary condition (4.39). Therefore we have $(\hat{U}_{(k,u)} - j + \hat{F}^I_{(k,u,f_I,l)} + \hat{F}^Q_{(k,f_Q,l)}, 0, 0, \hat{F}^L_{(k,f_L)} + \hat{F}^Q_{(k,f_Q,l)})$.

□

E.3 Proof of Theorem 4.3.3

Proof. At time $c_{(n,m,u,f_L)}$ we need to perform qard-hasan repayment by withdrawing from the surplus fund and add to qard-hasan fund. The amount of qard-hasan that borrowed by participants at time $c_{(n,m,u,f_L)}$ is $\hat{F}^L_{(c_{(n,m,u,f_L)},f_L)}$, hence this is the maximum value that need to be repaid. However, in our model, qard-hasan repayment should not bring the surplus level drop below l^W. By this assumption the maximum value that can be paid at time $c_{(n,m,u,f_L)}$ is $(\hat{U}_{(c_{(n,m,u,f_L)},u)} - l^W)_+$. We apply function $X_+ = \max\{0, X\}$ to make sure the criteria of qard-hasan repayment is $\hat{U}_{(c_{(n,m,u,f_L)},u)} \geq l^W$. Therefore total amount of qard-hasan repayment at time $c_{(n,m,u,f_L)}$ is $\min\{(\hat{U}_{(c_{(n,m,u,f_L)},u)} - l^W)_+, \hat{F}^L_{(c_{(n,m,u,f_L)},f_L)}\}$. After the qard-hasan fund is made, total liability at time $c_{(n,m,u,f_L)}$ is reduced. Qard-hasan repayment did not make any change to the level of investment fund. $\square$