# On First Passage Time Related Problems for Some Insurance Risk Processes

by

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## Author's Declaration

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

## Abstract

For many decades, the study of ruin theory has long been one of the central topics of interest in insurance risk management. Research in this area has largely focused on analyzing the insurer's solvency risk, which is essentially a standard first passage time problem. To model the manner in which the claim experience develops over time for a block of insurance business, various stochastic processes have been proposed and studied. Following the pioneer works of Lundberg [88] and Cramér [32], in which the classical compound Poisson model was proposed to model the insurer's surplus process, there has been a considerable amount of effort devoted to constructing more realistic risk models to better characterize some practical features of the insurer's surplus cash flows. This thesis aims to contribute to this line of research and enhance our general understanding of an insurer's solvency risk.

In most analyses of the main risk processes in risk theory, the income process is modelled by a deterministic process which accrues at a constant rate per time unit. As we know, this is a rather simplifying assumption which is far from being realistic in the insurance world, but one under which the solvency risk is typically assessed. To investigate the impact of income processes exhibiting a higher degree of variability on an insurer's solvency risk, the first part of the thesis focuses on analyzing risk models with random income processes. In Chapter 2, we consider a generalized Sparre Andersen risk model with a random income process which renews at claim instants. Under the setting of this particular generalization of the Sparre Andersen risk model, we investigate the impact of income processes on both infinite-time and finite-time ruin quantities. In Chapter 3, we further extend the results of the risk model proposed in Chapter 2 by analyzing a renewal insurance risk model with two-sided jumps and a random income process.

Another class of risk models that has drawn considerable interest in risk theory are the spectrally negative Lévy processes. Thanks to the development of the fluctuation theory of Lévy processes, first passage time analysis of Lévy insurance risk models has flourished in the last two decades, both in terms of models proposed and quantities analyzed. For example, risk models with dividends (or tax) payouts and exotic ruin have received considerable attention in the field of insurance mathematics. Leveraging the extensive literature on fluctuation identities for spectrally negative Lévy processes, the second part of the thesis considers some first passage problems in this context. In Chapter 4, we study a refracted Lévy risk model with delayed dividend pullbacks and obtain explicit expressions for two-sided exit identities for the proposed risk process. Chapter 5 introduces two types of random times with the goal of bridging the first and the last passage times' analyses. The Laplace transforms of these two random times are derived for the class of spectrally negative Lévy processes.

To ensure that the thesis flows smoothly, Chapter 1 introduces the background literature and main motivations of this thesis and provides the relevant mathematical preliminaries for the later chapters. Chapter 6 concludes the thesis with some remarks and potential directions for future research.

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**Dedication**

To my parents and my brother.

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# Chapter 1

# Introduction

First passage times measure the amount of time required for a stochastic process to enter or exit a specific state for the first time. It is a long-standing research topic in applied probability and has widespread applications in many fields, such as economics and finance. In an insurance context, the event of *"ruin"* is defined as having occurred if the insurer's surplus drops below 0, which is a first passage time problem of fundamental interest in risk theory.

In practice, insurers often need to determine the capital amount they shall hold against financial insolvency and meet their safety and competitive objectives subject to some regulatory constraints. Risk theory, with emphasis on the quantitative analysis of an insurer's surplus level, provides decision-makers with intuitive insights into measuring and managing the insurer's solvency risk. In the past twenty or so years, numerous models have been proposed to better characterize the cash flow dynamics of an insurance portfolio. The main objective of this thesis is to enhance our understanding of an insurer's solvency risk by studying first passage problems in insurance risk models that incorporate practical features in the industry.

The remainder of this chapter provides the background literature and main motivations of this thesis, introduces the insurance risk models of interest, and summarizes the relevant mathematical preliminaries used in the later chapters. The present chapter is concluded by a brief outline of the thesis.

## 1.1 Background and motivation

### 1.1.1 Ruin theory

In the early 1900s, Lundberg [88] and Cramér [32] proposed the use of the classical compound Poisson risk model (also known as the *Cramér-Lundberg model*), in which the claim arrival process is modelled by a compound Poisson process and the premium is collected at a constant rate over time. For over a century, there have been extensive analyses of the Cramér- Lundberg model, while early work primarily focused on analysing the probability of ruin in this framework. For a thorough investigation of the history and development of ruin theory, interested readers are referred to Asmussen and Albrecher [6].

In the setting of the Cramér-Lundberg model, Gerber and Shiu [46] introduced the expected discounted penalty function (it is often referred to as the *Gerber–Shiu discounted penalty function,* or *Gerber-Shiu function* for short) to analyze other functionals associated with the event of ruin (see Section 1.3.1 for more technical details). The Gerber-Shiu function has been widely studied and extended to different settings of surplus processes since then. For example, the Sparre Andersen (or renewal) risk model allows for a more general interclaim time distribution (see Section 1.2.1 for more details). Various researchers have contributed to the analysis of the Sparre Andersen risk model via the study of the

Gerber-Shiu function; interested readers are directed to many references in Section 1.2.1. Gerber-Shiu analysis has also been conducted in models with dividend strategies (see, e.g., Lin et al. [81], Yuen et al. [105] and Li and Garrido [77]) or with two-sided jumps (see, e.g., Cheung [24] and Labbé et al. [62]). Readers are referred to the special issue in volume 46, 2010 of *Insurance: Mathematics and Economics* for a selection of papers (and references therein) on the topic of Gerber-Shiu functions.

In the first passage time analysis of the main risk processes (such as the Cramér-Lundberg risk process and the Sparre Andersen risk model) in this research field, it is commonly assumed that the income process is highly predictable, exhibiting none to a very low degree of uncertainty in its future path dynamics. This assumption is mainly considered for purposes of mathematical tractability, but generally fails to accurately model an insurer's income dynamic. To better characterize the variability and uncertainty of an insurer's income process, there has been a number of generalizations proposed in the literature. A significant portion of this work relates to the addition of a random income process to some traditional surplus processes; see e.g., Boucherie et al. [21] and subsequently, Boikov [19], Temnov [95], Bao [10], Labbé and Sendova [63] and Zhang and Hu [108] in the Cramér-Lundberg risk process, and Zhang and Hu [107] and Cheung et al. [27] in the Sparre Andersen risk model. Characterizing variability in premium income is also considered in the context of Markov-additive risk processes, where the premium rate is assumed to vary depending on the state of an exogenous Markov process (see, e.g., Breuer and Badescu [23], Cheung and Landriault [26], Lu and Li [87], and Badescu et al. [9]). To a great extent, the main objective of the aforementioned papers is to carry out the Gerber-Shiu analysis.

However, to the best of the author's knowledge, far less attention has been paid to quantify the impact of more volatile income processes on the resulting ruin-related quantities. Therefore, in the first part of this thesis, we propose to fill this gap in the literature by quantitatively assessing the impact of the choice of income processes on solvency risk, in the context of a generalized Sparre Andersen risk model.

### 1.1.2 Fluctuation theory of spectrally negative Lévy processes

Another class of processes have been utilized by researchers to examine ruin-related quantities is the spectrally negative Lévy process. A spectrally negative Lévy process (also called a *Lévy insurance risk process*) is a Lévy process that does not experience positive jumps and does not have monotone paths (readers are referred to Section 1.2.2 for a formal definition of the process). The fluctuation theory of spectrally negative Lévy processes (SNLPs) has evolved parallelly with the Gerber-Shiu risk theory until the 2000s, when researchers observed important connections between the discounted penalty functions and the so-called *scale functions*. Zhou [109] noted some interesting connections between some known results for the classical risk model and those for SNLPs. Biffis and Kyprianou [17] provided an explicit characterization of a generalized version of the Gerber–Shiu function through the use of scale functions when the surplus is driven by a SNLP. As pointed out in Garrido and Morales [44], despite the fact that there are various intersections between these two theories, the methodologies and research focus are different.

Indeed, the advances in the fluctuation theory of SNLP have enriched the classical risk theory by enlarging the class of processes for which a ruin-related analysis can be carried out. For example, many researchers have conducted first passage analysis of Lévy insurance risk processes that are path-dependent due to tax payments (see, e.g., Albrecher et al. [5], Kyprianou and Zhou [61], and Avram et al. [8]) or dividend payments (see, e.g., Avram et al. [7], Loeffen and Renaud [85], and Czarna et al. [36]), which have broad risk management applications. The perturbed Lévy risk process has also attracted much interest; see, e.g., Huzak et al. [51], Zhang and Wu [106] and Kyprianou and Ott [59] for more details.

The fusion of ruin theory and the fluctuation theory of Lévy processes has pushed the classical theory even further into what one might call *exotic ruin theory*. In the last few years, exotic ruin times have attracted considerable attention in the context of Lévy insurance risk processes. One of the most well-known examples is the so-called *Parisian ruin time*, which was initially introduced to a ruin theory context by Dassios and Wu [38, 39]. In this context, the *Parisian ruin* occurs if an excursion below a pre-determined threshold is longer than a deterministic or stochastic time. Analyses of Parisian ruin times under a spectrally negative Lévy process have been performed by many researchers under different model settings; see, e.g., Landriault et al. [72], Baurdoux et al. [13], and Albrecher et al. [3] in the setting of a risk model with exponential implementation delays, and Czarna and Palmowski [33], Loeffen et al. [83], and Lkabous et al. [82] for the Parisian ruin with deterministic delays. Another example of exotic ruin events is the *cumulative Parisian ruin* proposed by Guérin and Renaud [50], which is based on the occupation time of an insurer's surplus process below level 0. We refer the reader to Landriault et al. [71, 65], Gerber et al. [48], Loeffen et al. [86] and references therein for additional references on the study of occupation times.

The refracted Lévy process naturally arises in the context of the so-called threshold dividend strategy (see, e.g., Lin and Pavlova [80], Yang et al. [104], and Albrecher and Hartinger [2]). For the refracted Lévy insurance risk process, various fluctuation identities related to classic and some exotic ruin times were obtained; see, e.g., Kyprianou and Loeffen [58] for the classic ruin time, Kyprianou et al. [60], Renaud [90] and Landriault et al. [65] for results on occupation times and Renaud [90] and Lkabous et al. [82] for

Parisian ruin times. The reader is also invited to consult Pérez and Yamazaki [89] for the study of a joint refracted and reflected Lévy process, and Czarna et al. [35] on the topic of multi-refracted Lévy processes. Indeed, under the threshold dividend strategy, dividends are paid when the insurer's surplus exceeds a pre-determined critical level, and dividends stop as soon as the insurer's surplus drops below the critical level. However, in practice, it seems rather unlikely that an insurer will immediately pull back the dividend payments as soon as its surplus level drops below the dividend threshold. Hence, in Chapter 4, we propose a refracted Lévy risk model with delayed dividend pullbacks and derive two-sided exit identities for the proposed risk model.

As alluded to in Section 1.1.1, research in ruin theory has largely focused on the analysis of the first passage time of a surplus process below a threshold level. More recently, there has been an accrued interest in the analysis of the last passage time below level 0, mainly in the framework of SNLPs. In risk theory, the last passage time is known to be useful in characterizing the ultimate recovery time. In the Cramér-Lundberg risk model, Gerber [45] used a martingale method to identify a closed-form representation for the Laplace transform of the last passage time. This work was later generalized by Chiu and Yin [29] to the class of spectrally negative Lévy processes. Baurdoux [12] further considered the analysis of the last passage time below 0 over an independent exponential time horizon. In other recent work, Li et al. [79] generalized the results in Baurdoux [12] by studying the joint Laplace transform of the last exit time, the value of the process at the last exit time, and the occupation time until the last exit time. Last passage times also play a key role in many optimal stopping problems. For example, Baurdoux and Pedraza [15, 14] obtained a stopping time that is close in $L^p$ sense to the last passage time for a spectrally negative Lévy process. In an effort to bridge the first and the last passage times' analyses, as well as provide a unified framework for theoretical studies, we propose and analyze two types of random times with the help of fluctuation identities of spectrally negative Lévy processes in Chapter 5.

## 1.2 Risk models

### 1.2.1 Sparre Andersen risk model

In the classical compound Poisson risk model, the interclaim times are assumed to be independent and identically distributed (iid) with an exponential distribution function. A substantial generalization of the compound Poisson model is the Sparre Andersen risk model (see Sparre Andersen [92]), which allows for a more general distribution for the time between claim events. Specifically, let $U_t$ denote the surplus level of the insurer at time $t$ which is defined as

$$U_t = u + ct - \sum_{i=1}^{N_t} X_i,$$

where $u \geq 0$ is the initial surplus, $c > 0$ is the constant premium rate per unit time, and $X_i$ is the $i$-th claim size. Let $\{N_t\}_{t\geq 0}$ be the claim number process with $N_t = \sup\{n \geq 0 : W_n \leq t\}$ where $W_0 = 0$ and $W_n = \sum_{i=1}^{n} T_i$ for $n \in \mathbb{N}_+$. The interclaim times $\{T_i\}_{i\in\mathbb{N}^+}$ is a sequence of iid non-negative random variables. In addition, we assume $\{X_i\}_{i\in\mathbb{N}^+}$ is a sequence of iid positive random variables (independent of the claim number process $N$). Besides, the *security loading condition* of $c\mathbb{E}[T_1] > \mathbb{E}[X_1]$ is imposed on the model so that ruin does not occur almost surely. We note that the Sparre Andersen model does not necessarily have the "independent and stationary increments" property; however, it is regenerative at claim times. The preserved random walk structure of $U_t$ at each claim instant enables us to apply conditioning arguments for the Gerber-Shiu analysis, such as conditioning on the first claim and on the first drop in surplus level.

In risk theory, an extensive literature exists on the analysis of the Sparre Andersen risk model. Existing results in the Sparre Andersen risk model are mainly based on assumptions about either the interclaim time distribution or the claim size distribution; see, e.g., Dickson and Hipp [40], Gerber and Shiu [47], and Landriault and Willmot [74]. We note that the ordinary Sparre Andersen risk model implicitly assumes that a claim occurs at time zero since the interclaim times $\{T_i\}_{i\in\mathbb{N}_+}$ are identically distributed. As an alternative, the *delayed* Sparred Andersen risk model is proposed to model the situation where a claim is not necessarily observed at time zero. In the setting of a delayed Sparre Andersen risk model, it is assumed that the distribution of $T_1$ is possibly different from that of $\{T_i\}_{i\geq 2}$; see, e.g., Willmot [99], Woo [103], and Kim and Willmot [52]. Furthermore, if $T_1$ follows an equilibrium distribution, the model is referred to as the *stationary* Sparre Andersen risk model; see, e.g., Willmot and Dickson [100] and Willmot et al. [101]. Another direction of generalizations is to incorporate a dependence structure into the aforementioned risk models. In a *dependent* Sparre Andersen risk model, we allow for a dependency between the distribution of the $i$-th claim $X_i$ and that of the time $T_i$ leading up to the claim. Readers are referred to Willmot and Woo [102] and references therein for more on this topic.

### 1.2.2 Spectrally negative Lévy process

A spectrally negative Lévy process is a natural generalization of the classical Cramér-Lundberg risk model. As illustrated in Section 2.7.1 of Kyprianou [57], the spectrally negative Lévy process (SNLP) is well suited to model the cash flow dynamics of an insurer. We first present the formal definition of the Lévy process (see Section 1 of Kyprianou [57] for more details).

**Definition 1.2.1 (Lévy process)** *A process $X = \{X_t\}_{t\geq 0}$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, is said to be a Lévy process if it has following properties:*

- $\mathbb{P}(X_0 = 0) = 1$;
- *The paths of $X$ are $\mathbb{P}$ almost surely right continuous with left limits;*
- *For $0 \leq s \leq t$, $X_t - X_s$ is equal in distribution to $X_{t-s}$;*
- *For $0 \leq s \leq t$, $X_t - X_s$ is independent of $\{X_u\}_{u\leq s}$.*

Specifically, if a Lévy process $X$ has no positive jumps and does not have monotone paths, then it is referred to a *spectrally negative Lévy process*. We now state the strong Markov property of Lévy processes, which will be heavily relied upon in the later analysis. Readers are referred to Section I.2 of Bertoin [16] and Section 3.1 of Kyprianou [57] for rigorous proofs and related discussions.

**Theorem 1.2.1 (Strong Markov property of Lévy processes)** *Let $X = \{X_t\}_{t\geq 0}$ be a Lévy process defined on $(\Omega, \mathcal{F}, \mathbb{P})$, and suppose $\tau$ is a stopping time with respect to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$. Associated with a given stopping time $\tau$ is the sigma-algebra $\mathcal{F}_\tau := \{A \in \mathcal{F} : A \cap \{\tau \leq t\} \in \mathcal{F}_t \text{ for all } t \geq 0\}$. Define on $\{\tau < \infty\}$ the process $\tilde{X} = \{\tilde{X}_t\}_{t\geq 0}$ where*

$$\tilde{X}_t = X_{\tau+t} - X_\tau, t \geq 0.$$

*Then, on the event $\{\tau < \infty\}$, the process $\tilde{X}$ is independent of $\mathcal{F}_\tau$, has the same law as $X$ and hence in particular is a Lévy process.*

For a Lévy process $X$, it is well known that, for any $t > 0$, $X_t$ is a random variable belonging to the class of infinitely divisible distributions. For any $\theta \in \mathbb{R}, t \geq 0$, define

$\Psi_t(\theta) = -\log \mathbb{E}\left[e^{i\theta X_t}\right]$ as the characteristic exponent of $X_t$, one can use the property of stationary independent increments (see page 4 of Kyprianou [57] for more details) to show that

$$\mathbb{E}\left[e^{i\theta X_t}\right] = e^{-t\Psi(\theta)}$$

for all $t \geq 0$, where $\Psi(\theta) := \Psi_1(\theta)$ is the characteristic exponent of $X_1$. The following theorem shows that a Lévy process can be characterized by a triplet.

**Theorem 1.2.2 (Lévy-Khintchine formula for Lévy processes)** *Suppose that $a \in \mathbb{R}, \sigma \in \mathbb{R}$ and $\Pi$ is a measure concentrated on $\mathbb{R}\backslash\{0\}$ such that $\int_{\mathbb{R}}(1 \wedge x^2)\Pi(\mathrm{d}x) < \infty$. From this triplet, define for each $\theta \in \mathbb{R}$,*

$$\Psi(\theta) = \mathrm{i}a\theta + \frac{1}{2}\sigma^2\theta^2 + \int_{\mathbb{R}} \left(1 - e^{\mathrm{i}\theta x} + \mathrm{i}\theta x \mathbb{1}_{\{|x|<1\}}\right) \Pi(\mathrm{d}x).$$

*Then there exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, on which a Lévy process is defined having the characteristic exponent $\Psi$.*

Generally speaking, a subordinator is a real-valued Lévy process whose sample paths are non-decreasing. The following lemma gives a necessary and sufficient condition for a Lévy process to be a subordinator:

**Lemma 1.2.1 (Subordinators)** *A Lévy process with Lévy-Khintchine exponent $\Psi(\theta)$ and triplet $(a, \sigma, \Pi)$ is a subordinator if and only if $\Pi(-\infty, 0) = 0$, $\int_{(0,\infty)}(1 \wedge x)\Pi(\mathrm{d}x) < \infty, \sigma = 0$ and the drift $d = -\left(a + \int_{(0,1)} x\Pi(\mathrm{d}x)\right) \geq 0$.*

For a subordinator $\{S_t\}_{t\geq 0}$, its Lévy-Khinchine exponent may be written as

$$\Psi(\theta) = -id\theta + \int_{\mathbb{R}_+} \left(1 - e^{i\theta x}\right) \Pi(\mathrm{d}x). \tag{1.1}$$

For a spectrally negative Lévy process $X$ (the measure $\Pi$ is necessarily concentrated on $(-\infty, 0)$ and $-X$ is not a subordinator), we often work with its *Laplace exponent*

$$\psi(\theta) = \frac{1}{t} \log \mathbb{E}\left(\mathrm{e}^{\theta X_t}\right) = -\Psi(-\mathrm{i}\theta), \tag{1.2}$$

which is finite for all $\theta \geq 0$. Moreover, the function $\psi : [0, \infty) \to \mathbb{R}$ is zero at zero, tends to infinity at infinity, is infinitely differentiable and strictly convex on $(0, \infty)$. Define the right inverse

$$\Phi(q) = \sup\{\theta \geq 0 : \psi(\theta) = q\}$$

for any $q \geq 0$, then $\Phi(0) = 0$ if $\psi'(0+) \geq 0$ and otherwise $\Phi(0) > 0$. Moreover, the process $X$ drifts to infinity if and only if $\psi'(0+) > 0$ (this is known as the security loading condition), oscillates if and only if $\psi'(0+) = 0$ and drifts to minus infinity if and only if $\psi'(0+) < 0$. The reader is referred to Theorem 7.2 of Kyprianou [57] for a rigorous proof for the asymptotic behaviour of $X$. Finally, we have that

$$\lim_{q \to 0} \frac{q}{\Phi(q)} = \psi'(0+) = \mathbb{E}[X_1], \tag{1.3}$$

if $\psi'(0+) \geq 0$. Otherwise, the aforementioned limit is zero.

## 1.3 Mathematical preliminaries

This section summarizes the mathematical preliminaries that will be used in the following chapters.

### 1.3.1 Ruin-related quantities and Gerber-Shiu functions

As alluded to in the earlier sections, the event of ruin is of practical interest in insurance risk management. Of central importance in this context is the probability that the insurer would become insolvent, which is referred to as the *ruin probability*. Other quantities associated with the ruin event, such as the severity of ruin, are also clearly important and natural extensions in the context of risk management.

For a surplus process $\{U_t\}_{t\geq 0}$, we define the time of ruin as

$$\tau = \inf\{t \geq 0 : U_t < 0\},$$

with the convention that $\inf \emptyset = \infty$. Then, the surplus prior to ruin and the deficit at ruin are $U_{\tau^-}$ and $|U_\tau|$, respectively. In the context of Lévy insurance risk processes, $U_{\tau^-}$ and $|U_\tau|$ are also called the *undershoot* and *overshoot* at first passage of level 0.

In 1998, Gerber and Shiu [46] proposed the expected discounted penalty function to incorporate other ruin-related quantities into the analysis. The Gerber-Shiu function is defined as

$$m_\delta(u) = \mathbb{E}[\mathrm{e}^{-\delta\tau} w(U_{\tau^-}, |U_\tau|) \mathbb{1}_{\{\tau<\infty\}} | U_0 = u], \tag{1.4}$$

for $\delta, u \geq 0$, where $w : (0,\infty)^2 \to [0,\infty)$ is a *penalty function* which satisfies mild integrability conditions. Information about any or all of the quantities $\tau, U_{\tau^-}$ and $|U_\tau|$ can be extracted from the Gerber-Shiu function $m_\delta(u)$ by imposing different assumptions on the penalty function $w$. For example, $m_\delta(u)$ reduces to the ruin probability when $\delta = 0$ and $w(x,y) \equiv 1$, and it can be interpreted as the joint and marginal defective distribution function of $(U_{\tau^-}, |U_\tau|)$ when $\delta = 0$ and $w(x,y) = \mathbb{1}_{\{x\leq x_1, y\leq y_1\}}$.

The parameter $\delta$ may be interpreted as a discount factor (i.e., force of interest) to incorporate the time value of money into the analysis. Moreover, $m_\delta(u)$ is the trivariate Laplace transform of $(\tau, U_{\tau^-}, |U_\tau|)$ when $w(x,y) = \mathrm{e}^{-sx-zy}$ and $\delta, s, z$ are complex numbers with non-negative real parts.

### 1.3.2 Rouché's theorem

In the Gerber-Shiu analysis, Rouché's theorem is usually used to verify that there are a certain number of roots to Lundberg's fundamental equation within a certain domain. Rouché's theorem and a modification to Rouché's theorem (see Titchmarsh [96] and Theorem 1 of Klimenok [54] ) are stated below.

**Theorem 1.3.1 (Rouché's theorem)** *If $f(z)$ and $g(z)$ are analytic inside and on a closed contour $D$ and $|g(z)| < |f(z)|$ on $D$, then $f(z)$ and $g(z) + f(z)$ have the same number of zeros inside $D$.*

**Theorem 1.3.2 (A modification to Rouché's theorem)** *Let the functions $f(z)$ and $\phi(z)$ be analytic in the open disk $|z| < 1$ and continuous on the boundary $|z| = 1$ and the following relations hold:*

$$|f(z)|_{|z|=1,z\neq 1} > |\phi(z)|_{|z|=1,z\neq 1}$$

*and*

$$f(1) = -\phi(1) \neq 0.$$

*Let also the functions $f(z)$ and $\phi(z)$ have the derivatives at the point $z = 1$ and the following inequality holds*

$$\frac{f'(1) + \phi'(1)}{f(1)} > 0.$$

*Then the numbers $N_{f+\phi}$ and $N_f$ of zeros of the functions $f(z) + \phi(z)$ and $f(z)$ in the domain $|z| < 1$ are related as follows*

$$N_{f+\phi} = N_f - 1.$$

### 1.3.3 Scale functions and exit problems

One appealing aspect of the class of spectrally negative Lévy processes is their analytic tractability, which is primarily attributed to the development of the two following families of functions.

**Definition 1.3.1 ($q$-scale functions)** *For a SNLP $X$ with Laplace exponent $\psi(\cdot)$ defined in (1.2), there exist a family of functions $W^{(q)}:\mathbb{R}\to[0,\infty)$ for any $q\geq 0$, where $W^{(q)}(x)=0$ for $x<0$ and $W^{(q)}$ is a strictly increasing and continuous function on $[0,\infty)$ whose Laplace transform satisfies*

$$\int_0^\infty \mathrm{e}^{-\beta x}W^{(q)}(x)\mathrm{d}x=\frac{1}{\psi(\beta)-q}\ \text{for}\ \beta>\Phi(q). \tag{1.5}$$

*The second function $Z^{(q)}$ is defined by*

$$Z^{(q)}(x):=1+q\int_0^x W^{(q)}(y)\mathrm{d}y,\quad x\in\mathbb{R}, \tag{1.6}$$

*for which a generalized form is given by*

$$Z^{(q)}(x,\theta)=\mathrm{e}^{\theta x}\left(1-(\psi(\theta)-q)\int_0^x \mathrm{e}^{-\theta y}W^{(q)}(y)\mathrm{d}y\right),\quad x\in\mathbb{R}, \tag{1.7}$$

*for $\theta\geq 0$. It is immediate that $Z^{(q)}(x,0)=Z^{(q)}(x)$ and $Z^{(q)}(x,\theta)=\mathrm{e}^{\theta x}$ for $x\leq 0$.*

For future reference, we refer to the functions $W^{(q)}$ and $Z^{(q)}$ as the *$q$-scale functions* (for the proof of the existence of $W^{(q)}$, one is referred to Chapter 8 in Kyprianou [57]). We state the following analytical properties of $q$-scale functions without proofs. The reader is referred to Kuznetsov et al. [56] and Kyprianou [57] for a thorough discussion of the properties of $q$-scale functions.

**Lemma 1.3.1 (Properties of $q$-scale functions)** *For a spectrally negative Lévy process $X$ with triplet $(a,\sigma,\Pi)$,*

1. *for all $q \geq 0$, the function $W^{(q)}$ has left and right derivatives on $(0,\infty)$, and $W^{(q)} \in C^1(0,\infty)$ if $X$ is a process of unbounded variation, or if the Lévy measure $\Pi$ is atomless when $X$ is of bounded variation;*

2. *for all $q \geq 0$,*

$$W^{(q)}(0+) = \begin{cases} 1/d, & \text{when } X \text{ has bounded variation,} \\ 0, & \text{otherwise,} \end{cases}$$

*and*

$$W^{(q)\prime}(0+) = \begin{cases} \frac{2}{\sigma^2}, & \text{when } \sigma \neq 0 \text{ or } \Pi(-\infty,0) = \infty, \\ \frac{\Pi(-\infty,0)+q}{d^2}, & \text{when } \sigma = 0 \text{ and } \Pi(-\infty,0) < \infty, \end{cases} \tag{1.8}$$

*where $d = -a - \int_{(-1,0)} x\Pi(\mathrm{d}x) > 0$ is the drift of $X$ and the first case of (1.8) is understood as $+\infty$ when $\sigma = 0$.*

We also recall the so-called *second-generation* scale functions (see Loeffen et al. [86] for more details), which are frequently used in the study of occupation times and are given by

$$\begin{aligned} \overline{W}_a^{(p,q)}(x) :&= W^{(p+q)}(x) - q\int_0^a W^{(p+q)}(x-y)W^{(p)}(y)\mathrm{d}y \\ &= W^{(p)}(x) + q\int_a^x W^{(p+q)}(x-y)W^{(p)}(y)\mathrm{d}y, \end{aligned} \tag{1.9}$$

$$\begin{aligned} \overline{Z}_a^{(p,q)}(x) :&= Z^{(p+q)}(x) - q\int_0^a W^{(p+q)}(x-y)Z^{(p)}(y)\mathrm{d}y \\ &= Z^{(p)}(x) + q\int_a^x W^{(p+q)}(x-y)Z^{(p)}(y)\mathrm{d}y, \end{aligned} \tag{1.10}$$

for $p, p+q \geq 0$ and $a, x \in \mathbb{R}$. Note that the expressions on the right-hand side of (1.9) and that of (1.10) can be shown to be equivalent, respectively, by using the following identities taken from Loeffen et al. [86]: for $p, q \geq 0$ and $x \in \mathbb{R}$,

$$(p-q)\int_0^x W^{(q)}(x-y)W^{(p)}(y)\mathrm{d}y = W^{(p)}(x) - W^{(q)}(x), \tag{1.11}$$

and

$$(p-q)\int_0^x W^{(q)}(x-y)Z^{(p)}(y)\mathrm{d}y = Z^{(p)}(x) - Z^{(q)}(x). \tag{1.12}$$

As we shall later see, the $q$-scale function plays a fundamental role in the quantitative analysis of SNLPs. In general, it is non-trivial to obtain explicit expressions of $q$-scale functions by inverting the Laplace transform. Nonetheless, explicit expressions of $q$-scale functions of some special cases of SNLPs are known, for example, the Brownian motion, the Cramér-Lundberg risk model with exponential claims, the tempered stable SNLP, etc. Also, in Landriault and Willmot [75], a non-standard analytic approach is proposed to derive explicit expression of $q$-scale functions for perturbed compound risk processes. We point out that one can resort to numerical approaches (see, e.g., Surya [93]) in other cases.

We now present some well known one- and two-sided exit identities for spectrally negative Lévy processes in terms of their scale functions. For any $a \in \mathbb{R}$, we define the first passage stopping times of a SNLP $X$ as

$$\tau_a^{+(-)} = \inf\{t \geq 0\colon X_t > (<) a\},$$

with the convention that $\inf \emptyset = \infty$.

**Theorem 1.3.3 (Exit identities for SNLPs)**

1. *For any $q \geq 0$ and $x \in \mathbb{R}$,*

$$\mathbb{E}_x\left[\mathrm{e}^{-q\tau_0^-}\mathbb{1}_{\{\tau_0^- < \infty\}}\right] = Z^{(q)}(x) - \frac{q}{\Phi(q)}W^{(q)}(x), \tag{1.13}$$

*where we understand $q/\Phi(q)$ in the limiting sense for $q = 0$, so that*

$$\mathbb{P}_x\left(\tau_0^- < \infty\right) = \begin{cases} 1 - \psi'(0+)W(x), & \text{if } \psi'(0+) \geq 0, \\ 1, & \text{if } \psi'(0+) < 0. \end{cases} \tag{1.14}$$

*2. For any* $q \geq 0$ *and* $x \leq a$,

$$\mathbb{E}_x \left[ \mathrm{e}^{-q\tau_a^+} \mathbb{1}_{\{\tau_a^+ < \tau_0^-\}} \right] = \frac{W^{(q)}(x)}{W^{(q)}(a)}, \tag{1.15}$$

*and*

$$\mathbb{E}_x \left[ \mathrm{e}^{-q\tau_0^-} \mathbb{1}_{\{\tau_0^- < \tau_a^+\}} \right] = Z^{(q)}(x) - \frac{W^{(q)}(x)}{W^{(q)}(a)} Z^{(q)}(a). \tag{1.16}$$

Note that without confusion, we adopt the convention that $\mathbb{P}_x$ and $\mathbb{E}_x$ are, respectively, the law and expectation when $X_0 = x \in \mathbb{R}$. For simplicity, we write $\mathbb{P}$ and $\mathbb{E}$ when $x = 0$.

## 1.4 Structure of the thesis

The thesis is organized as follows. In Chapter 2, we consider a generalized Sparre Andersen risk model with a random income process which renews at claim instants. For exponentially distributed claim sizes, we derive explicit expressions for some joint distributions involving the time to ruin and the number of claims until ruin. As special cases of the proposed insurance risk process, we consider income processes modelled by a subordinator or a particular varying premium rate model. The results in this chapter provide important risk management implications of a solvency nature for the insurer together with improving the existing literature on insurance risk models with random incomes.

In Chapter 3, we generalize the results of the risk model introduced in Chapter 2 by relaxing the restrictions on claim size distributions and on the independence structure between interclaim times and claim sizes. We consider a renewal insurance risk model with two-sided jumps and a random income process, where the income process and upward jumps represent random gains from different sources, while the downward jumps represent claim amounts. Moreover, we incorporate a fairly large class of dependence structures into the risk model where the claim sizes belong to the class of Coxian distributions. We analyze the Gerber-Shiu function when the penalty function depends only on the deficit at ruin and show that the Gerber-Shiu function can be characterized by a linear system of equations in this case.

Chapter 4 considers a refracted Lévy risk model with delayed dividend pullbacks triggered by a certain Poissonian observation scheme. With the help of fluctuation identities for spectrally negative Lévy processes, we obtain explicit expressions for two-sided exit identities of the proposed insurance risk process. Also, transaction costs are incorporated into the analysis of dividend payouts as a mechanism to penalize for the volatility of the dividend policy and account for an investor's typical preference for more stable cash flows. An explicit expression for the expected (discounted) dividend payouts net of transaction costs is derived. Finally, several numerical examples are considered to assess the impact of dividend delays on ruin-related quantities. We numerically show that dividend strategies with more steady dividend payouts can be preferred (over the well-known threshold dividend strategy) when transaction costs become too onerous.

In Chapter 5, we introduce two types of random times, namely $s_r$ and $l_r$, where the parameter $r$ can be interpreted as a measure of a decision maker's aversion to negative surplus. The two random times can not only recover the first and last passage times as limiting cases, but also capture more pathwise information of the underlying surplus process. For the class of spectrally negative Lévy processes, the Laplace transform of these two random times is explicitly derived in terms of scale functions. Concurrently, a few new results in fluctuation theory of SNLPs are obtained. Finally, we examine in more details some special cases of SNLPs, namely the Cramér-Lundberg risk model with exponential claims and the Brownian motion process with drift.

Chapter 6 concludes the thesis and discusses some directions for future research. Note that the results in Chapter 2 are published in Insurance: Mathematics and Economics (see Wang et al. [97]), and the results in Chapters 4 and 5 have been submitted for publication (see Landriault et al. [67] and Wang et al. [98]). As for Chapter 3, the work has been finalized recently and we expect to submit it for publication in the coming months. Finally, we remark that the chapters were written independently of one another. Despite efforts made to have consistent notations throughout the thesis, some inconsistencies may unfortunately remain. If this is the case, the reader should assume that the prevailing notation is the one stated in the corresponding chapter.

# Chapter 2

# An Insurance Risk Process with A Generalized Income Process

## 2.1 Introduction

In this chapter, we investigate the impact of income processes on both finite-time and infinite-time ruin quantities in the context of a generalized Sparre Andersen risk model.

The generalized Sparre Andersen risk model (see Section 2.2 for a detailed description of the risk model) is an extension of both the Cramér-Lundberg risk process and the Sparre Andersen risk model. In this model, we consider an income process which renews at claim instants that will preserve the regenerative property of the Sparre Andersen surplus process at claim instants. This general setup will also accommodate the more restrictive case where the income process renews at any time point to quantify the impact of more volatile income processes on an insurer's solvency risk. Also, it is worth pointing out that income processes that renew at claim instants may be more natural in the context of insurance business with low claim frequency (e.g., catastrophe insurance). That is, the premium rate is usually reset to a higher level shortly after a claim event and subsequently is expected to decrease as the no-claim period extends (until the next claim event which will again trigger a higher premium reset). This premium rate adjustment is somehow consistent with the well known theory of credibility in ratemaking (e.g., Klugman et al. [55]).

The main findings of our investigation in this chapter are summarized here:

- The dynamics of the income process play an important role in the assessment of an insurer's solvency risk.
- For a given positive security loading, insurance risk processes with more uncertain income processes may either display an accrued (e.g., a subordinator income process) or lowered (e.g., a preloaded premium strategy) solvency risk relative to the corresponding Sparre Andersen risk process.
- The distribution of the interclaim times has a direct impact on ruin related quantities. However, we observe that insurance risk processes with more volatile income processes are less sensitive to the change in the interclaim time distribution.

The rest of the chapter is organized as follows: in Section 2.2, the risk model of interest in this chapter is formally introduced together with some relevant ruin-related quantities. The main technical analysis is carried out in Section 2.3, where the joint distribution of the time to ruin, the number of claims until ruin and the surplus immediately prior to ruin is ultimately derived. The results in Section 2.3 are foundational to conduct the sensitivity analysis of income processes on an insurer's solvency risk in the later sections. In Section 2.4, we discuss in more details several special cases of income processes (including subordinators and a particular varying premium model) which can be accommodated in our general setup. The tractability of the results of Section 2.3 are further demonstrated for these special cases. In Section 2.5, various income processes are considered to assess the resulting impact on an insurer's solvency risk.

## 2.2 Risk model and notations

We assume that all processes are defined in a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$, where $\{\mathcal{F}_t\}_{t\geq 0}$ is the natural filtration generated by the surplus process $\{U_t\}_{t\geq 0}$ which is defined in Eq. (2.2) below. Let $\{N_t\}_{t\geq 0}$ be the claim number process with $N_t = \sup\{n \geq 0 : W_n \leq t\}$ where $W_0 = 0$ and $W_n = \sum_{i=1}^{n} T_i$ for $n \in \mathbb{N}_+$. The interclaim times $\{T_i\}_{i\in\mathbb{N}_+}$ are assumed to be a sequence of independent and identically distributed (iid) non-negative random variables (rv's) with cumulative distribution function (cdf) $K(t)$ and Laplace transform $\tilde{k}$. In addition, let $X_i$ be the size of the $i$-th claim, where $\{X_i\}_{i\in\mathbb{N}_+}$ is a sequence of iid rv's, independent of any other rv's in the model.

Let $\{R_t\}_{t\geq 0}$ be an independent non-decreasing process with monotone paths modelling the income process with $R_0 = 0$, and define $\{R_t^i\}_{t\geq 0}$ $(i = 1, 2, ...)$ to be independent copies of $\{R_t\}_{t\geq 0}$. We only consider the paths of $\{R_t^i\}_{t\geq 0}$ over the time interval $[0, T_i]$. The insurer's accumulated income process $\{R_t^*\}_{t\geq 0}$ is constructed by piecing together these paths of $\{R_t^i\}_{t\in[0,T_i]}$. More specifically, for $t \in [W_{i-1}, W_i)$, the income increment in $[W_{i-1}, t]$ is $R_{t-W_{i-1}}^i$, and the income increment between the $(i-1)$-th and $i$-th claims is $R_{W_i - W_{i-1}}^i = R_{T_i}^i$. Based on the above assumptions, we define $R_t^*$ as

$$R_t^* = \sum_{i=1}^{N_t} R_{T_i}^i + R_{t-W_{N_t}}^{N_t+1}, \quad t > 0, \tag{2.1}$$

with the convention that $\sum_{i=1}^{0}(\cdot) = 0$. We are now ready to introduce the risk model of interest in this chapter. Let $U_t$ be the insurer's surplus level at time $t \geq 0$ defined as

$$U_t = u + R_t^* - \sum_{i=1}^{N_t} X_i, \tag{2.2}$$

where $U_0 = u \geq 0$ is the initial surplus level. Given that ruin occurs at claim instants, we point out that the surplus level just before and after the $n$-th claim event ($n \in \mathbb{N}_+$) are

$$U_{W_n^-} = U_{W_{n-1}} + R_{T_n}^n, \quad U_{W_n} = U_{W_{n-1}} + R_{T_n}^n - X_n, \tag{2.3}$$

respectively. Furthermore, by the Strong Law of Large Numbers, the positive security loading condition is given by

$$\mathbb{E}[R_{T_1}^1] > \mathbb{E}[X_1], \tag{2.4}$$

which is assumed to hold so that ruin does not occur almost surely. We may write $\mathbb{E}[R_{T_1}^1] = (1+\theta)\mathbb{E}[X_1]$, where $\theta > 0$ is the so-called security loading factor.

Note that the surplus process (2.2) is a generalization of some known insurance risk processes in the literature. For instance,

- when $R_t = ct$, the surplus process (2.2) reduces to the well-known Sparre Andersen risk model;
- when $\{R_t\}_{t\geq 0}$ is an independent compound Poisson process with non-negative jumps, the surplus process (2.2) reduces to the ruin model studied by e.g., Boikov [19], Labbé et al. [63] and Temnov [95], which can be interpreted as modelling the evolution of an insurance company's surplus on a micro level;
- when $\{R_t\}_{t\geq 0}$ is a compound renewal process with a positive drift, the resulting surplus process (2.2) is a generalization of the risk model discussed in Cheung et al. [27], which was claimed to be suitable for insurers with business in both casualty insurance and life annuities.

Of risk management interest in connection to the surplus process $\{U_t\}_{t\geq 0}$ is the time of ruin $\tau = \inf\{t \geq 0 : U_t < 0\}$ (with the convention that $\inf \emptyset = \infty$). Let $N_\tau, U_{\tau^-}$ and $|U_\tau|$ be the number of claims until ruin, the surplus (immediately) prior to ruin and the deficit at ruin, respectively. Embedding the above four rv's related to the ruin event is the Gerber-Shiu function $\phi_{r,\delta,s,z}(u)$ defined as

$$\phi_{r,\delta,s,z}(u) = \mathbb{E}\left[r^{N_\tau} \mathrm{e}^{-\delta\tau} \mathrm{e}^{-sU_{\tau^-}} \mathrm{e}^{-z|U_\tau|} \mathbb{1}_{\{\tau<\infty\}} | U_0 = u\right],$$

for $0 < r \leq 1, \delta, s, z, u \geq 0$ [1]. By construction, the generalized Sparre Andersen risk model (2.2) preserves the random walk structure of the Sparre Andersen risk model at claim instants.

Given that the main objective of this chapter is to consider the impact of more volatile income processes on some finite-time and infinite-time ruin quantities, we henceforth assume that the claim sizes $\{X_i\}_{i\in\mathbb{N}_+}$ are exponentially distributed with mean $1/\alpha$. This is in consideration of mathematical tractability as explicit representation for some finite-time ruin quantities will be derived in this case. Let $h_j(t, x, y|u)$ be the joint defective density of the time to ruin of $t$, the surplus prior to ruin of $x$, and the deficit at ruin of $y$ for ruin occurring at time of the $j$th ($j = 1, 2, ...$) claim. For the surplus process (2.2) with exponential claim sizes, it is immediate that

$$\begin{aligned}
\phi_{r,\delta,s,z}(u) &= \sum_{j=1}^{\infty} \int_0^\infty \int_0^\infty \int_0^\infty r^j \mathrm{e}^{-\delta t} \mathrm{e}^{-sx} \mathrm{e}^{-zy} h_j(t, x, y|u) \mathrm{d}t\mathrm{d}x\mathrm{d}y \\
&= \sum_{j=1}^{\infty} \int_0^\infty \int_0^\infty \int_0^\infty r^j \mathrm{e}^{-\delta t} \mathrm{e}^{-sx} \mathrm{e}^{-zy} h_j(t, x|u) \alpha \mathrm{e}^{-\alpha y} \mathrm{d}t\mathrm{d}x\mathrm{d}y \\
&= \phi_{r,\delta,s}(u) \frac{\alpha}{\alpha + z},
\end{aligned}$$

[1]Note that $\delta, s$ and $z$ are respectively the Laplace transform arguments related to $\tau, U_{\tau^-}$ and $|U_\tau|$, while $r$ is the probability generating function argument related to $N_\tau$.

(i.e., the deficit at ruin is an independent exponential rv with mean $1/\alpha$) where $h_j(t,x|u) = \int_0^\infty h_j(t,x,y|u)\mathrm{d}y$ and

$$\phi_{r,\delta,s}(u) = \mathbb{E}\left[r^{N_\tau}\mathrm{e}^{-\delta\tau}\mathrm{e}^{-sU_{\tau^-}}\mathbb{1}_{\{\tau<\infty\}}|U_0 = u\right]. \tag{2.5}$$

Therefore, it suffices to consider $\phi_{r,\delta,s}(u)$ in the sequel. For notational convenience, we also denote its special case $\phi_{r,\delta,0}(u)$ by $\phi_{r,\delta}(u)$. Among other implications, it is not difficult to show that $\phi_{r,\delta,s,z}(u)$ satisfies a defective renewal equation.

## 2.3 General results

In this section, we extend the results of Landriault et al. [73] by deriving the joint distributions of $(\tau, N_\tau)$, and more general $(\tau, N_\tau, U_{\tau^-})$ for the generalized Sparre Andersen risk model (2.2) with exponential claim sizes. We do so by first providing a characterization of $\phi_{r,\delta,s}(u)$ in Proposition 2.3.1. These results will be quintessential to assess the impact of more volatile income processes on both finite-time and infinite-time ruin quantities in Sections 2.4 and 2.5.

Let $\nu$ be the Borel measure on space $(\mathbb{R}_+^2, \mathcal{B}(\mathbb{R}_+^2))$ induced by the pair $(R_{T_1}^1, T_1)$, i.e.,

$$\nu(\mathrm{d}z\times\mathrm{d}t) := \mathbb{P}\left(R_t^1 \in \mathrm{d}z, T_1 \in \mathrm{d}t\right) \quad \forall z\in\mathbb{R}_+, t\in\mathbb{R}_+,$$

with bivariate Laplace transform is defined as $\mathcal{L}\nu(s_1,s_2) = \int_{\mathbb{R}_+^2}\mathrm{e}^{-s_1 z}\mathrm{e}^{-s_2 t}\nu(\mathrm{d}z\times\mathrm{d}t)$ for $s_1, s_2 \geq 0$. Moreover, the $n$-fold convolution of $\nu$ is defined as

$$\nu^{*n}(A) = \int_{\mathbb{R}_+^2}\nu^{*(n-1)}\big(A-(z,t)\big)\nu(\mathrm{d}z\times\mathrm{d}t), \quad \forall A\in\mathcal{B}(\mathbb{R}_+^2).$$

**Proposition 2.3.1** *For $0 < r \leq 1, \delta, s, u \geq 0$, the Gerber-Shiu function (2.5) is given by*

$$\phi_{r,\delta,s}(u) = \frac{\alpha r\mathcal{L}\nu(\alpha+s,\delta)}{s+\alpha r\mathcal{L}\nu(\alpha+s,\delta)}\left(\frac{s}{\alpha}\mathrm{e}^{-(\alpha+s)u} + \phi_{r,\delta}(0)\mathrm{e}^{-\alpha u\left(1-\phi_{r,\delta}(0)\right)}\right), \tag{2.6}$$

*where $\phi_{r,\delta}(0)$ is the unique solution to*

$$z = r\mathcal{L}\nu\left(\alpha\left(1-z\right),\delta\right) \tag{2.7}$$

*in the interval* $(0,1)$.

**Proof.** Given that the descending ladder heights are exponentially distributed with mean $1/\alpha$ (independently of the path trajectories) and the risk process (2.2) renews at claim instants, it follows that

$$\phi_{r,\delta,s}\left(u\right) = \phi_{r,\delta}\left(0\right)\int_0^u \phi_{r,\delta,s}\left(u-x\right)\alpha \mathrm{e}^{-\alpha x}\mathrm{d}x + \phi_{r,\delta,s}\left(0\right)\mathrm{e}^{-su}\int_u^\infty \alpha \mathrm{e}^{-\alpha x}\mathrm{d}x.$$

By Laplace transform,

$$\widetilde{\phi}_{r,\delta,s}\left(z\right) = \phi_{r,\delta}\left(0\right)\frac{\alpha}{\alpha+z}\widetilde{\phi}_{r,\delta,s}\left(z\right) + \frac{\phi_{r,\delta,s}\left(0\right)}{s+\alpha+z},$$

or equivalently

$$\begin{aligned}\widetilde{\phi}_{r,\delta,s}\left(z\right) &= \frac{\phi_{r,\delta,s}\left(0\right)}{\left(s+\alpha+z\right)\left(1-\phi_{r,\delta}\left(0\right)\frac{\alpha}{\alpha+z}\right)} \\ &= \frac{\left(\alpha+z\right)\phi_{r,\delta,s}\left(0\right)}{\left(s+\alpha+z\right)\left(\alpha\left(1-\phi_{r,\delta}\left(0\right)\right)+z\right)},\end{aligned} \tag{2.8}$$

where $\widetilde{\phi}_{r,\delta,s}\left(z\right) = \int_0^\infty \mathrm{e}^{-zu}\phi_{r,\delta,s}\left(u\right)\mathrm{d}u$. By simple algebraic manipulations, (2.8) can be rewritten as

$$\widetilde{\phi}_{r,\delta,s}\left(z\right) = \phi_{r,\delta,s}\left(0\right)\left(\frac{s}{s+\alpha\phi_{r,\delta}\left(0\right)}\frac{1}{s+\alpha+z} + \frac{\alpha\phi_{r,\delta}\left(0\right)}{s+\alpha\phi_{r,\delta}\left(0\right)}\frac{1}{\alpha\left(1-\phi_{r,\delta}\left(0\right)\right)+z}\right),$$

whose Laplace transform inversion yields

$$\phi_{r,\delta,s}\left(u\right) = \phi_{r,\delta,s}\left(0\right)\left(\frac{s}{s+\alpha\phi_{r,\delta}\left(0\right)}\mathrm{e}^{-\left(\alpha+s\right)u} + \frac{\alpha\phi_{r,\delta}\left(0\right)}{s+\alpha\phi_{r,\delta}\left(0\right)}\mathrm{e}^{-\alpha\left(1-\phi_{r,\delta}\left(0\right)\right)u}\right), \tag{2.9}$$

for $u \geq 0$.

Now, by conditioning on $\left(R^1_{T_1}, T_1\right)$ and $X_1$, we also have

$$\phi_{r,\delta,s}(0) = r\int_{\mathbb{R}^2_+} \mathrm{e}^{-\delta t}\left(\int_0^z \phi_{r,\delta,s}(z-x)\,\alpha \mathrm{e}^{-\alpha x}\mathrm{d}x\right)\nu(\mathrm{d}z\times\mathrm{d}t) + r\int_{\mathbb{R}^2_+} \mathrm{e}^{-\delta t}\mathrm{e}^{-sz}\mathrm{e}^{-\alpha z}\nu(\mathrm{d}z\times\mathrm{d}t). \tag{2.10}$$

Substituting (2.9) into (2.10) leads to

$$\phi_{r,\delta,s}(0) = \phi_{r,\delta,s}(0)\frac{r\alpha}{s+\alpha\phi_{r,\delta}(0)}\left(\mathcal{L}\nu\left(\alpha\left(1-\phi_{r,\delta}(0)\right),\delta\right) - \mathcal{L}\nu(\alpha+s,\delta)\right) + r\mathcal{L}\nu(\alpha+s,\delta). \tag{2.11}$$

Firstly, when $s=0$ in Eq. (2.11), one deduces that $\phi_{r,\delta}(0) = \phi_{r,\delta,0}(0)$ must be a solution (in $z$) of

$$z = r\mathcal{L}\nu\left(\alpha(1-z),\delta\right). \tag{2.12}$$

Substituting (2.12) into (2.11) leads to

$$\phi_{r,\delta,s}(0) = \phi_{r,\delta,s}(0)\frac{\alpha}{s+\alpha\phi_{r,\delta}(0)}\left(\phi_{r,\delta}(0) - r\mathcal{L}\nu(\alpha+s,\delta)\right) + r\mathcal{L}\nu(\alpha+s,\delta),$$

or equivalently

$$\phi_{r,\delta,s}(0) = \left(s+\alpha\phi_{r,\delta}(0)\right)\frac{r\mathcal{L}\nu(\alpha+s,\delta)}{s+\alpha r\mathcal{L}\nu(\alpha+s,\delta)}. \tag{2.13}$$

Substituting (2.13) into (2.9) yields (2.6).

We are left to show that $\phi_{r,\delta}(0)$ is the unique solution of (2.7) in the interval $(0,1)$. For $\delta>0$ or $0<r<1$, we have

$$\begin{aligned}|r\mathcal{L}\nu(\alpha-\alpha z,\delta)| &\le r\int_{\mathbb{R}^2_+}\left|\mathrm{e}^{-(\alpha-\alpha z)y}\mathrm{e}^{-\delta t}\right|\nu(\mathrm{d}y\times\mathrm{d}t)\\ &< \int_{\mathbb{R}^2_+}\left|\mathrm{e}^{-(\alpha-\alpha z)y}\right|\nu(\mathrm{d}y\times\mathrm{d}t) \le 1 = |z|\end{aligned} \tag{2.14}$$

for any $z$ on the contour $D=\{z:|z|=1\}$. Then by Rouché's theorem, $z - r\mathcal{L}\nu(\alpha-\alpha z,\delta)$ has exactly one root inside the contour $D$. As for the case where $\delta=0$ and $r=1$,

$|\mathcal{L}\nu(\alpha - \alpha z, \delta)| < |z|$ on $D$ except at $z = 1$. We can use an extension of Rouché's theorem to prove the uniqueness of $\phi_{r,\delta}(0)$ in $D$. Specifically, note that

$$\frac{\mathrm{d}}{\mathrm{d}z}(z)\big|_{z=1} - \frac{\mathrm{d}}{\mathrm{d}z}(\mathcal{L}\nu(\alpha - \alpha z, \delta)\big|_{z=1} = 1 - \int_{\mathbb{R}_+^2} \alpha y \; \nu(\mathrm{d}y \times \mathrm{d}t) < 0$$

due to the positive security loading condition (2.4). Hence, by Theorem 1.3.2, $z - \mathcal{L}\nu(\alpha - \alpha z, 0)$ has exactly one root in $D$.

Finally, given that by definition, $\phi_{r,\delta}(0) > 0$, it follows that $\phi_{r,\delta}(0)$ is the unique solution of Eq. (2.7) in the interval $(0, 1)$. This completes the proof. ■

The following corollary is an immediate consequence of Proposition 2.3.1 when $\delta = s = 0$ and $r = 1$.

**Corollary 2.3.1** *(***Ruin probability***) For the insurance risk process $\{U_t\}_{t \geq 0}$ defined in (2.2), its infinite-time ruin probability for an initial surplus of $u \geq 0$ is given by*

$$\phi_{1,0}(u) = \phi_{1,0}(0)\mathrm{e}^{-\alpha u(1-\phi_{1,0}(0))}, \tag{2.15}$$

*where $\phi_{1,0}(0)$ is the unique solution to $z = \mathcal{L}\nu\left(\alpha(1 - z), 0\right)$ in the interval $(0, 1)$.*

**Remark 2.3.1** *From the proof of Proposition 2.3.1, we note that the positive security loading condition (2.4) is only required for the infinite-time ruin probability $\phi_{1,0}(u)$ (to ensure that $\phi_{1,0}(0)$ is the unique solution of Eq. (2.7) inside D). Otherwise, $\phi_{1,0}(u) = 1$ almost surely.*

*Under (2.4), from Eq. (2.7) and Takács lemma (see Cohen [30], pp. 653-656), we have*

$$\begin{aligned} \phi_{1,0}(0) &= \sum_{n=1}^{\infty} \frac{\alpha^{n-1}}{n!} \int_{\mathbb{R}_+^2} \mathrm{e}^{-\alpha z} z^{n-1} \mathbb{P}\left(R_t^* \in \mathrm{d}z, W_n \in \mathrm{d}t\right) \\ &= \sum_{n=1}^{\infty} \mathbb{E}\left[\mathrm{e}^{-\alpha R_{W_n}^*} \frac{\left(\alpha R_{W_n}^*\right)^{n-1}}{n!}\right]. \end{aligned} \tag{2.16}$$

*As a special example, when the interclaim times $\{T_i\}_{i\in\mathbb{N}_+}$ are exponentially distributed with mean $1/\lambda$, (2.16) becomes*

$$\begin{aligned}\phi_{1,0}(0) &= \sum_{n=1}^{\infty}\int_{\mathbb{R}_+}\int_{\mathbb{R}_+} \mathrm{e}^{-\alpha z}\frac{(\alpha z)^{n-1}}{n!}\frac{\lambda^n t^{n-1}\mathrm{e}^{-\lambda t}}{(n-1)!}\mathbb{P}\left(R_t^* \in \mathrm{d}z | W_n = t\right)\mathrm{d}t \\ &= \int_{\mathbb{R}_+}\int_{\mathbb{R}_+} \mathrm{e}^{-\alpha z-\lambda t}\sqrt{\frac{\lambda}{\alpha z t}}I_1(2\sqrt{\lambda\alpha z t})\mathbb{P}\left(R_t^* \in \mathrm{d}z | W_n = t\right)\mathrm{d}t, \end{aligned} \tag{2.17}$$

*where*

$$I_\nu(x) := \sum_{n=0}^{\infty}\frac{(x/2)^{\nu+2n}}{n!\Gamma(\nu+n+1)}$$

*is the modified Bessel function of the first kind. As a simple application of (2.17), one can consider the case where $R(t) = ct$, then*

$$\phi_{1,0}(0) = \sqrt{\frac{\lambda}{\alpha c}}\int_0^{\infty}\mathrm{e}^{-(\alpha c+\lambda)t}t^{-1}I_1(2\sqrt{\lambda\alpha c}\cdot t)\mathrm{d}t = \frac{\lambda}{\alpha c} = \frac{1}{1+\theta},$$

*as expected.*

In Theorems 2.3.1 and 2.3.2, explicit representations for the probability measures $\mathbb{P}_u(\tau \in \mathrm{d}t, N_\tau = n)$ and $\mathbb{P}_u(\tau \in \mathrm{d}t, N_\tau = n, U_{\tau^-} \in \mathrm{d}x)$ (with $\mathbb{P}_u(\cdot) := \mathbb{P}(\cdot|U_0 = u)$) are given. It should be noted that the joint distribution of $(\tau, N_\tau)$ in Theorem 2.3.1 can be obtained via a Lagrange expansion approach (e.g., Dickson and Willmot [41] and Landriault et al. [73]) similar to the proof of Theorem 2.3.2. However, we propose to prove this result by an interesting inductive argument instead.

**Theorem 2.3.1** *The joint distribution of the time to ruin and the number of claims until ruin is given by*

$$\mathbb{P}_u(\tau \in \mathrm{d}t, N_\tau = n) = \int_{\mathbb{R}_+} b_n(u,z)\mathrm{e}^{-\alpha(u+z)}\nu^{*n}(\mathrm{d}z \times \mathrm{d}t), \tag{2.18}$$

*for $t \in \mathbb{R}_+$ and $n \in \mathbb{N}_+$, where*

$$b_n(u,z) = \left(\frac{u+\frac{z}{n}}{u+z}\right)\frac{(\alpha u+\alpha z)^{n-1}}{(n-1)!}, \quad u \geq 0, z \in \mathbb{R}_+.$$

**Proof.** First, we observe that by conditioning $\mathbb{P}_u(\tau \in \mathrm{d}t, N_\tau = n)$ on $\left(\sum_{i=1}^n T_i, \sum_{i=1}^n R^i_{T_i}\right)$, it follows that

$$\mathbb{P}_u(\tau \in \mathrm{d}t, N_\tau = n) = \int_{\mathbb{R}_+} \mathbb{P}_u\left(N_\tau = n \,\middle|\, \sum_{i=1}^n T_i = t, \sum_{i=1}^n R^i_{T_i} = z\right)\nu^{*n}(\mathrm{d}z \times \mathrm{d}t).$$

Next, we prove by induction that

$$\mathbb{P}_u\left(N_\tau = n \,\middle|\, \sum_{i=1}^n T_i = t, \sum_{i=1}^n R^i_{T_i} = z\right) = b_n(u,z)\mathrm{e}^{-\alpha(u+z)}, \tag{2.19}$$

for $n = 1, 2, ..$ and $t, z \in \mathbb{R}_+$, which if true completes the proof of Theorem 2.3.1.

For $n = 1$, we have

$$\mathbb{P}_u\left(N_\tau = 1 \,\middle|\, T_1 = t, R^1_{T_1} = z\right) = \mathbb{P}_u(X_1 > u+z) = \mathrm{e}^{-\alpha(u+z)},$$

which corresponds to (2.19) with $n = 1$ (as $b_1(u,z) = 1$).

We now assume that (2.19) holds for $n = 1, 2, ..., m$, and show that (2.19) also holds for $n = m+1$. By conditioning on $\left(T_1, R^1_{T_1}\right)$, it follows that

$$\begin{aligned}
&\mathbb{P}_u\left(N_\tau = m+1 \,\middle|\, \sum_{i=1}^{m+1} T_i = t, \sum_{i=1}^{m+1} R^i_{T_i} = z\right) \\
= \;& \int_0^z \int_0^t \mathbb{P}_u\left(N_\tau = m+1 \,\middle|\, T_1 = s, R^1_{T_1} = w, \sum_{i=1}^{m+1} T_i = t, \sum_{i=1}^{m+1} R^i_{T_i} = z\right) \\
&\times \mathbb{P}\left(T_1 \in \mathrm{d}s, R^1_{T_1} \in \mathrm{d}w \,\middle|\, \sum_{i=1}^{m+1} T_i = t, \sum_{i=1}^{m+1} R^i_{T_i} = z\right),
\end{aligned} \tag{2.20}$$

where

$$\mathbb{P}_u\left(N_\tau = m+1 \,\middle|\, T_1 = s, R^1_{T_1} = w, \sum_{i=1}^{m+1} T_i = t, \sum_{i=1}^{m+1} R^i_{T_i} = z\right)$$

$$
\begin{aligned}
&= \int_0^{u+w} \mathbb{P}_u \left( N_\tau = m+1, U_{T_1} \in \mathrm{d}x \,\middle|\, T_1 = s, R^1_{T_1} = w, \sum_{i=1}^{m+1} T_i = t, \sum_{i=1}^{m+1} R^i_{T_i} = z \right) \\
&= \int_0^{u+w} \mathbb{P}_u \left( U_{T_1} \in \mathrm{d}x \,\middle|\, T_1 = s, R^1_{T_1} = w \right) \mathbb{P}_x \left( N_\tau = m \,\middle|\, \sum_{i=1}^{m} T_i = t-s, \sum_{i=1}^{m} R^i_{T_i} = z-w \right).
\end{aligned} \tag{2.21}
$$

Note the last equality in (2.21) follows from the regenerative property of the surplus process $\{U_t\}_{t\geq 0}$ at claim instants. Given that

$$
\mathbb{P}_u \left( U_{T_1} \in \mathrm{d}x \,\middle|\, T_1 = s, R^1_{T_1} = w \right) = \mathbb{P}\left( X_1 \in u+w-\mathrm{d}x \right) = \alpha \mathrm{e}^{-\alpha(u+w-x)} \mathrm{d}x,
$$

and from (2.19), (2.21) can be rewritten as

$$
\begin{aligned}
&\mathbb{P}_u \left( N_\tau = m+1 \,\middle|\, T_1 = s, R^1_{T_1} = w, \sum_{i=1}^{m+1} T_i = t, \sum_{i=1}^{m+1} R^i_{T_i} = z \right) \\
&= \int_0^{u+w} \alpha \mathrm{e}^{-\alpha(u+w-x)} b_m(x, z-w) \mathrm{e}^{-\alpha(x+z-w)} \mathrm{d}x \\
&= \alpha \mathrm{e}^{-\alpha(u+z)} \int_0^{u+w} b_m(x, z-w) \mathrm{d}x \\
&= \mathrm{e}^{-\alpha(u+z)} \frac{b_{m+1}(u,z)}{u + \frac{z}{m+1}} (u+w).
\end{aligned} \tag{2.22}
$$

Now, further substituting (2.22) into (2.20) yields

$$
\begin{aligned}
&\mathbb{P}_u \left( N_\tau = m+1 \,\middle|\, \sum_{i=1}^{m+1} T_i = t, \sum_{i=1}^{m+1} R^i_{T_i} = z \right) \\
&= \int_0^z \int_0^t \mathrm{e}^{-\alpha(u+z)} \frac{b_{m+1}(u,z)}{u + \frac{z}{m+1}} (u+w) \mathbb{P} \left( T_1 \in \mathrm{d}s, R^1_{T_1} \in \mathrm{d}w \,\middle|\, \sum_{i=1}^{m+1} T_i = t, \sum_{i=1}^{m+1} R^i_{T_i} = z \right) \\
&= \mathrm{e}^{-\alpha(u+z)} \frac{b_{m+1}(u,z)}{u + \frac{z}{m+1}} \int_0^z (u+w) \, \mathbb{P} \left( R^1_{T_1} \in \mathrm{d}w \,\middle|\, \sum_{i=1}^{m+1} T_i = t, \sum_{i=1}^{m+1} R^i_{T_i} = z \right) \\
&= \mathrm{e}^{-\alpha(u+z)} b_{m+1}(u,z),
\end{aligned} \tag{2.23}
$$

where the last equality in (2.23) follows from the fact that $\{R^i_{T_i}\}_{i\in\mathbb{N}_+}$ are iid. This completes the proof. ∎

**Remark 2.3.2** *Results for the Sparre Andersen risk model can be recovered by letting*

$$\nu^{*n}(\mathrm{d}z \times \mathrm{d}t) = \delta_0(\mathrm{d}z - ct)K^{*n}(\mathrm{d}t),$$

*where $\delta_0(\cdot)$ is the Dirac measure. In this case,*

$$\mathbb{P}_u\left(\tau \in \mathrm{d}t, N_\tau = n\right) = \int_{\mathbb{R}_+} b_n(u, ct)\mathrm{e}^{-\alpha(u+ct)}K^{*n}(\mathrm{d}t),$$

*which corresponds to Eq. (16) in Landriault et al. [73].*

**Remark 2.3.3** *An interesting observation arising from Eq. (2.19) is that the conditional probability of $N_\tau = n$ depends only on the accumulated income up to the $n$-th claim (namely, $\sum_{i=1}^n R^i_{T_i}$) and does not depend on the occurrence time of the $n$-th claim (namely, $\sum_{i=1}^n T_i$).*

**Remark 2.3.4** *We note that the marginal defective distribution of $N_\tau$ can be obtained by integrating Eq. (2.18) over $t$. Specifically, a binomial expansion yields*

$$\begin{aligned}
b_n(u,z) =& \frac{1}{n!}\alpha^{n-1}\left(nu+z\right)\left(u+z\right)^{n-2} \\
=& \frac{1}{n!}\left\{\alpha u\left(n-1\right)\left(\alpha u+\alpha z\right)^{n-2} + \left(\alpha u+\alpha z\right)^{n-1}\right\} \\
=& \frac{1}{n!}\left\{\alpha u\left(n-1\right)\sum_{j=0}^{n-2}\binom{n-2}{j}\left(\alpha u\right)^{n-2-j}\left(\alpha z\right)^j + \sum_{j=0}^{n-1}\binom{n-1}{j}\left(\alpha u\right)^{n-1-j}\left(\alpha z\right)^j\right\} \\
=& \frac{1}{n!}\left\{\left(\alpha z\right)^{n-1} + \sum_{j=0}^{n-2}\left\{\frac{(n-1)!\left(n-1-j\right)}{j!\left(n-1-j\right)!} + \frac{(n-1)!}{j!\left(n-1-j\right)!}\right\}\left(\alpha u\right)^{n-1-j}\left(\alpha z\right)^j\right\} \\
=& \frac{1}{n!}\left\{\left(\alpha z\right)^{n-1} + \sum_{j=0}^{n-2}\binom{n-1}{j}\left(n-j\right)\left(\alpha u\right)^{n-1-j}\left(\alpha z\right)^j\right\}.
\end{aligned}$$

*Thus,*

$$\begin{aligned}
&\mathbb{P}_u\left(N=n\right)\\
&=\frac{\mathrm{e}^{-\alpha u}}{n!}\int_{\mathbb{R}_+^2}\mathrm{e}^{-\alpha z}\left\{(\alpha z)^{n-1}+\sum_{j=0}^{n-2}\binom{n-1}{j}(n-j)\,(\alpha u)^{n-1-j}\,(\alpha z)^j\right\}\nu^{*n}(\mathrm{d}z\times\mathrm{d}t)\\
&=\frac{\mathrm{e}^{-\alpha u}}{n!}\left\{(-\alpha)^{n-1}\frac{\mathrm{d}^{n-1}}{\mathrm{d}\alpha^{n-1}}\left(\tilde{f}_R(\alpha)\right)^n+\sum_{j=0}^{n-2}\binom{n-1}{j}(n-j)\,(\alpha u)^{n-1-j}\,(-\alpha)^j\,\frac{\mathrm{d}^j}{\mathrm{d}\alpha^j}\left(\tilde{f}_R(\alpha)\right)^n\right\},
\end{aligned}\tag{2.24}$$

*for* $n\geq 1$*, where* $\tilde{f}_R(\alpha)=\mathcal{L}\nu\left(\alpha,0\right)$*. When* $u=0$*, it follows from Eq. (2.24) that*

$$\mathbb{P}_0\left(N=n\right)=\frac{(-\alpha)^{n-1}}{n!}\frac{\mathrm{d}^{n-1}}{\mathrm{d}\alpha^{n-1}}\left(\tilde{f}_R(\alpha)\right)^n.$$

*Also, Eq. (2.12) implies that*

$$\phi_{r,0}(0)=r\cdot M\left(\phi_{r,0}(0)\right),\tag{2.25}$$

*where* $M\left(z\right)=\tilde{f}_R\left(\alpha\left(1-z\right)\right)$ *is a mixed Poisson pgf. Generating function relationships of the form (2.25) are said to be Lagrangian in nature. Interested readers are referred to discussions in pp. 199-200 of Willmot and Woo [102] for more details.*

For the second probability measure $\mathbb{P}_u\left(\tau\in\mathrm{d}t,N_\tau=n,U_{\tau^-}\in\mathrm{d}x\right)$, the use of a similar inductive argument is more involved and we rather prove this result by directly inverting the Laplace transform $\phi_{r,\delta,s}(u)$.

**Theorem 2.3.2** *The joint distribution of the time of ruin, the number of claims until ruin and the surplus prior to ruin is given by*
*(i) For* $n=1,t,x>0$*,*

$$\mathbb{P}_u\left(\tau\in\mathrm{d}t,N_\tau=1,U_{\tau^-}\in\mathrm{d}x\right)=\mathbb{1}_{\{x>u\}}\mathrm{e}^{-\alpha x}\nu\left((\mathrm{d}x-u)\times\mathrm{d}t\right).\tag{2.26}$$

*(ii) For $n \geq 2, t, x > 0$,*

$$\begin{aligned}
&\mathbb{P}_u\left(\tau \in \mathrm{d}t, N_\tau = n, U_{\tau^-} \in \mathrm{d}x\right) \\
=&\sum_{m=1}^{n-1} \int_{(0,t)\times(0,x)} \frac{\alpha^m (z-x)^{m-1} \mathrm{e}^{-\alpha z}}{(m-1)!} \mathbb{P}_u\left(\tau \in \mathrm{d}t - y, N_\tau = n-m\right) \nu^{*m}(\mathrm{d}z \times \mathrm{d}y)\mathrm{d}x \\
&- \mathbb{1}_{\{x>u\}} \int_{(0,x-u)} \alpha^{n-1} \mathrm{e}^{-\alpha(u+z)} \frac{(z+u-x)^{n-2}}{(n-2)!} \nu^{*n}\left(\mathrm{d}z \times \mathrm{d}t\right)\mathrm{d}x. \qquad (2.27)
\end{aligned}$$

**Proof.** From Proposition 2.3.1 we know that

$$\phi_{r,\delta,s}(u) = \frac{\alpha r \mathcal{L}\nu(\alpha+s,\delta)}{s+\alpha r \mathcal{L}\nu(\alpha+s,\delta)} \left(\frac{s}{\alpha}\mathrm{e}^{-(\alpha+s)u} + \phi_{r,\delta}(u)\right) \qquad (2.28)$$

for $0 < r \leq 1, \delta, s, u \geq 0$. By series expansion, we have

$$\begin{aligned}
\frac{\alpha r \mathcal{L}\nu(\alpha+s,\delta)}{s+\alpha r \mathcal{L}\nu(\alpha+s,\delta)} &= \sum_{n=1}^{\infty} (-1)^{n-1} \left(\frac{\alpha r \mathcal{L}\nu(\alpha+s,\delta)}{s}\right)^n \\
&= \sum_{n=1}^{\infty} r^n \int_{\mathbb{R}_+^2} \int_z^\infty \mathrm{e}^{-\delta t} \frac{\alpha^n (z-x)^{n-1}}{(n-1)!} \mathrm{e}^{-\alpha z} \mathrm{e}^{-sx} \mathrm{d}x \; \nu^{*n}(\mathrm{d}z \times \mathrm{d}t), \qquad (2.29)
\end{aligned}$$

for $s > \alpha$, from which one easily deduces that

$$\begin{aligned}
&\frac{\alpha r \mathcal{L}\nu(\alpha+s,\delta)}{s+\alpha r \mathcal{L}\nu(\alpha+s,\delta)} \frac{s}{\alpha} \mathrm{e}^{-(\alpha+s)u} \\
=& \mathrm{e}^{-(\alpha+s)u} \sum_{n=1}^{\infty} r^n (-1)^{n-1} \frac{\alpha^{n-1}}{s^{n-1}} \int_{\mathbb{R}_+^2} \mathrm{e}^{-(\alpha+s)z} \mathrm{e}^{-\delta t} \nu^{*n}(\mathrm{d}z \times \mathrm{d}t) \\
=& r \int_{\mathbb{R}_+^2} \mathrm{e}^{-\delta t} \mathrm{e}^{-(\alpha+s)(u+z)} \nu(\mathrm{d}z \times \mathrm{d}t) \\
&- \sum_{n=2}^{\infty} r^n \int_{\mathbb{R}_+^2} \alpha^{n-1} \mathrm{e}^{-\delta t} \mathrm{e}^{-\alpha(u+z)} \int_z^\infty \frac{(z-x)^{n-2}}{(n-2)!} \mathrm{e}^{-s(x+u)} \mathrm{d}x \; \nu^{*n}(\mathrm{d}z \times \mathrm{d}t) \\
=& r \int_{\mathbb{R}_+^2} \mathrm{e}^{-\delta t} \mathrm{e}^{-(\alpha+s)(u+z)} \nu(\mathrm{d}z \times \mathrm{d}t) \\
&- \sum_{n=2}^{\infty} r^n \int_{\mathbb{R}_+^2} \alpha^{n-1} \mathrm{e}^{-\delta t} \mathrm{e}^{-\alpha(u+z)} \int_{z+u}^\infty \frac{(z+u-x)^{n-2}}{(n-2)!} \mathrm{e}^{-sx} \mathrm{d}x \; \nu^{*n}(\mathrm{d}z \times \mathrm{d}t). \qquad (2.30)
\end{aligned}$$

Then, by (2.29) and the convolution theorem, we have

$$
\begin{aligned}
&\frac{\alpha r \mathcal{L}\nu(\alpha+s,\delta)}{s+\alpha r \mathcal{L}\nu(\alpha+s,\delta)}\phi_{r,\delta}(u) \\
&= \left(\sum_{n=1}^{\infty} r^n \int_{\mathbb{R}_+^2}\int_z^{\infty} \mathrm{e}^{-\delta t}\frac{\alpha^n (z-x)^{n-1}}{(n-1)!}\mathrm{e}^{-\alpha z}\mathrm{e}^{-sx}\mathrm{d}x\ \nu^{*n}(\mathrm{d}z\times\mathrm{d}t)\right)\left(\sum_{n=1}^{\infty} r^n \int_{\mathbb{R}_+} \mathrm{e}^{-\delta t}\mathbb{P}_u\left(\tau\in\mathrm{d}t, N_\tau=n\right)\right) \\
&= \int_{\mathbb{R}_+} \mathrm{e}^{-\delta t}\sum_{n=2}^{\infty} r^n \sum_{m=1}^{n-1}\int_{(0,t)\times\mathbb{R}_+}\int_z^{\infty}\frac{\alpha^m (z-x)^{m-1}}{(m-1)!}\mathrm{e}^{-\alpha z}\mathrm{e}^{-sx}\mathrm{d}x\nu^{*m}\left(\mathrm{d}z\times\mathrm{d}y\right)\mathbb{P}_u\left(\tau\in\mathrm{d}t-y, N_\tau=n-m\right).
\end{aligned}
\tag{2.31}
$$

Substituting (2.30) and (2.31) into (2.28) results in

$$
\begin{aligned}
&\phi_{r,\delta,s}(u) \\
&= r\int_{\mathbb{R}_+^2} \mathrm{e}^{-\delta t}\mathrm{e}^{-(\alpha+s)(u+z)}\nu(\mathrm{d}z\times\mathrm{d}t) - \sum_{n=2}^{\infty} r^n \int_{\mathbb{R}_+^2} \alpha^{n-1}\mathrm{e}^{-\delta t}\mathrm{e}^{-\alpha(u+z)}\int_{z+u}^{\infty}\frac{(z+u-x)^{n-2}}{(n-2)!}\mathrm{e}^{-sx}\mathrm{d}x\ \nu^{*n}(\mathrm{d}z\times\mathrm{d}t) \\
&+ \int_{\mathbb{R}_+} \mathrm{e}^{-\delta t}\sum_{n=2}^{\infty} r^n \sum_{m=1}^{n-1}\int_{(0,t)\times\mathbb{R}_+}\int_z^{\infty}\frac{\alpha^m (z-x)^{m-1}}{(m-1)!}\mathrm{e}^{-\alpha z}\mathrm{e}^{-sx}\mathrm{d}x\nu^{*m}\left(\mathrm{d}z\times\mathrm{d}y\right)\mathbb{P}_u\left(\tau\in\mathrm{d}t-y, N_\tau=n-m\right).
\end{aligned}
$$

Interchanging the order of integration, one concludes that the joint distribution of $(\tau, N_\tau, U_{\tau^-})$ is as (2.26) and (2.27) given in Theorem 2.3.2. ■

## 2.4 Applications

To further examine the potential impact of different income processes on an insurer's solvency risk, we now consider several special cases of the insurance risk process (2.2). For each case, we focus on the application of Theorem 2.3.1, while expressions for the trivariate distribution in Theorem 2.3.2 are omitted but can be similarly obtained.

### 2.4.1 Subordinators

For a subordinator $\{S_t\}_{t\geq 0}$ with Lévy-Khinchine exponent $\Psi(\theta)$ given by (1.1), its Laplace exponent exists and is given by

$$\Phi(y) := -\frac{1}{t}\log \mathbb{E}\left(\mathrm{e}^{-yS_t}\right) = dy + \int_{\mathbb{R}_+}\left(1-\mathrm{e}^{-yx}\right)\Pi(\mathrm{d}x), \tag{2.32}$$

for $y \geq 0$.

We now study a special case of the generalized Sparre Andersen risk model (2.2) where $R_t = S_t$ is a subordinator with Laplace exponent (2.32) corresponding to the Lévy triplet $(a, 0, \Pi)$. Then, by the property of stationary and independent increments of $\{S_t\}_{t\geq 0}$, the accumulated income process $R_t^*$ is equal in distribution to $R_t$, $\forall t > 0$. Thus, the risk model (2.2) can be written as

$$U_t = u + S_t - \sum_{i=1}^{N_t} X_i. \tag{2.33}$$

See Figure 2.1 for a sample path illustration. For the risk model (2.33), it is immediate that the positive security loading condition (2.4) can be rewritten as

$$\mathbb{E}\left[S_{T_1}\right] = \Phi'(0+)\mathbb{E}[T_1] > \frac{1}{\alpha}, \tag{2.34}$$

while

$$\nu^{*n}(\mathrm{d}z \times \mathrm{d}t) = \mathbb{P}\left(S_t \in \mathrm{d}z, W_n \in \mathrm{d}t\right) = \mathbb{P}\left(S_t \in \mathrm{d}z\right) K^{*n}(\mathrm{d}t), \tag{2.35}$$

for $n \in \mathbb{N}_+$, with

$$\mathcal{L}\nu(s_1, s_2) = \int_{\mathbb{R}_+^2} \mathrm{e}^{-s_1 z}\mathrm{e}^{-s_2 t}\mathbb{P}\left(S_t \in \mathrm{d}z\right) K(\mathrm{d}t) = \tilde{k}\left(\Phi(s_1) + s_2\right). \tag{2.36}$$

By substituting (2.35) into Theorem 2.3.1, the joint distribution of $(\tau, N_\tau)$ is given by

$$\mathbb{P}_u\left(\tau \in \mathrm{d}t, N_\tau = n\right) = \mathbb{E}\left[b_n(u, S_t)\mathrm{e}^{-\alpha(u+S_t)}\right] K^{*n}(\mathrm{d}t), \quad t > 0, n \in \mathbb{N}_+. \tag{2.37}$$

Note that when $n = 1$, (2.37) reduces to

$$\mathbb{P}_u\left(\tau \in \mathrm{d}t, N_\tau = 1\right) = \mathrm{e}^{-\alpha u - \Phi(\alpha)t} K(\mathrm{d}t).$$

In the next Proposition, we state an interesting result for the infinite-time ruin probability $\phi_{1,0}(u)$ for the insurance risk process $\{U_t\}_{t\geq 0}$ defined in Eq. (2.33).

**Proposition 2.4.1** *For the insurance risk process $\{U_t\}_{t\geq 0}$ defined in (2.33) (satisfying the positive security loading condition (2.34)) where $\{S_t\}_{t\geq 0}$ is a subordinator with Laplace exponent $\Phi$, its infinite-time ruin probability satisfies*

$$\phi_{1,0}(u) \geq \phi^*_{1,0}(u),$$

*where $\phi^*_{1,0}(u)$ is the infinite-time ruin probability related to the insurance risk process (2.33) with $S_t = \Phi'(0+)t$.*

**Proof.** Given that

$$\Phi'(0+) = d + \int_{\mathbb{R}_+} x \Pi(\mathrm{d}x),$$

the Laplace exponent (2.32) can be rewritten as

$$\Phi(y) = \Phi'(0+)y + \int_{\mathbb{R}_+} \left(1 - \mathrm{e}^{-xy} - xy\right) \Pi(\mathrm{d}x),$$

for $y \geq 0$. It follows that

$$\Phi(y) \leq \Phi'(0+)y,$$

for $y \geq 0$ where $\Phi'(0+) > 0$ for Eq. (2.34) to hold. Hence,

$$0 \leq \Phi\left(\alpha(1-z)\right) \leq \Phi'(0+)\alpha(1-z),$$

for $z \in (0,1)$, which in turn implies that

$$f_1(z) = z - \tilde{k}\left(\Phi\left(\alpha(1-z)\right)\right) \le z - \tilde{k}\left(\Phi'(0+)\alpha(1-z)\right) = f_2(z),$$

for $\forall z \in (0,1)$. Both functions $f_1$ and $f_2$ are continuous in $z$, negative at $z = 0$ and equal to 0 at $z = 1$. From Corollary 2.3.1 (and Eq. (2.36)), we also know that $f_1$ and $f_2$ each have exactly one zero in the interval $(0,1)$. It follows that the zero of the left hand side is larger than the zero of the right hand side, i.e., $\phi_{1,0}(0) \ge \phi^*_{1,0}(0)$. The rest follows from the fact that $\phi_{1,0}(u)$ is an increasing function of $\phi_{1,0}(0)$ for $u \ge 0$ (again from Corollary 2.3.1). ■

Note that Proposition 2.4.1 has an important implication: among all insurance risk processes of the form (2.33) satisfying the positive loading condition (2.34) for a given $\Phi'(0+) > 0$, the insurance risk process with the smallest infinite-time ruin probability $\phi_{1,0}(u)$ for all $u \ge 0$ is the well known Sparre Andersen risk model (where the income process is deterministic and defined as $S_t = ct$ where $c = \Phi'(0+)$).

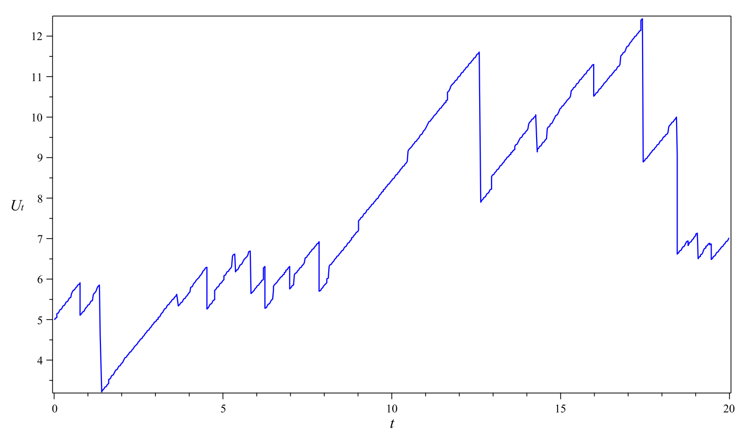



Figure 2.1: A sample path of the risk model (2.33) with a Gamma subordinator and compound Poisson aggregate claim process

In the next three subsections, we consider the following illustrative examples of subordinators, namely a compound Poisson process with drift, a gamma process, and an inverse Gaussian process.

### Compound Poisson Process

We first consider the case where $\{S_t\}_{t\geq 0}$ is a compound Poisson process with a drift of the form

$$S_t = ct + \sum_{i=1}^{M_t} Y_i, \quad t \geq 0, \tag{2.38}$$

where $c \geq 0$, $\{M_t\}_{t\geq 0}$ is an independent Poisson process with intensity $\mu > 0$ and the income sizes $\{Y_i\}_{i\in\mathbb{N}_+}$ are iid positive rv's with cdf $F_Y$ and Laplace transform $\tilde{f}_Y$ (independent of $\{M_t\}_{t\geq 0}$). The corresponding insurance risk process is a generalization of the ruin model studied by Boikov [19], Labbé et al. [63] and Temnov [95], in which the interclaim times are further assumed to be exponentially distributed.

In this setup,

$$\mathbb{P}\left(S_t \in \mathrm{d}z\right) = \sum_{m=0}^{\infty} F_Y^{*m}(\mathrm{d}z - ct)\frac{(\mu t)^m \mathrm{e}^{-\mu t}}{m!},$$

for $t > 0, z \geq ct, n \geq 1$, where $F_Y^{*m}$ is the cdf of the $m$-fold convolution of the cdf $F$ (with the convention that $F_Y^{*0}(\mathrm{d}z) = \delta_0(\mathrm{d}z)$). It follows from (2.37) that the joint distribution of $(\tau, N_\tau)$ is given by

$$\mathbb{P}_u\left(\tau \in \mathrm{d}t, N_\tau = n\right) = \mathrm{e}^{-\alpha(u+ct)-\mu t} \sum_{m=0}^{\infty} \frac{(\mu t)^m}{m!} \int_{\mathbb{R}_+} \mathrm{e}^{-\alpha y} b_n(u, ct+y) F_Y^{*m}(\mathrm{d}y) K^{*n}(\mathrm{d}t) \tag{2.39}$$

for $n \in \mathbb{N}_+$.

**Remark 2.4.1** (**Exponential upward jump sizes**) *More explicit expressions for (2.39) can be obtained under certain distributional assumptions for the upward jumps $\{Y_i\}_{i\in\mathbb{N}_+}$ (including but not limited to the class of mixed Erlang distributions). For ease of presentation, we provide an illustration when $\{Y_i\}_{i\in\mathbb{N}_+}$ form a sequence of iid exponential rv's with mean $1/\beta$, in which case (2.39) becomes*

$$\int_{\mathbb{R}_+} \mathrm{e}^{-\alpha y} b_1(u, ct+y) F_Y^{*m}(\mathrm{d}y) = \left(\frac{\beta}{\alpha+\beta}\right)^m$$

*for $n = 1$ and $m \geq 1$, and*

$$\begin{aligned}&\int_{\mathbb{R}_+} \mathrm{e}^{-\alpha y} b_n(u, ct+y) F_Y^{*m}(\mathrm{d}y)\\ =&\frac{\alpha^{n-1}}{n!}\sum_{k=0}^{n-2}\binom{n-2}{k}(u+ct)^{n-2-k}\frac{(k+m-1)!}{(m-1)!(\alpha+\beta)^k}\left(\frac{\beta}{\alpha+\beta}\right)^m\left(\frac{k+m}{\alpha+\beta}+nu+ct\right)\end{aligned}$$

*for $n \geq 2, m \geq 1$.*

For completeness, we conclude this section with the expression of the infinite-time ruin probability for the insurance risk process (2.33) with subordinator (3.30).

**Remark 2.4.2** (**Ruin probability**) *From Corollary 2.3.1, the infinite-time ruin probability is given by $\phi_{1,0}(u) = z_0 \mathrm{e}^{-\alpha u(1-z_0)}$, where $z_0$ is the unique solution (in $z$) to*

$$z = \tilde{k}\left(\mu + \alpha c - \alpha c z - \mu \tilde{f}_Y(\alpha - \alpha z)\right)$$

*in the interval $(0, 1)$.*

### Gamma Process

Let $S_t = ct + Z_t$, where $c \geq 0$ and $\{Z_t\}_{t\geq 0}$ is a gamma process with characteristic exponent $\Psi(\theta) = b\ln(1-\frac{i\theta}{a})$ and Lévy triplet $(-\int_{(0,1)} x\Pi(\mathrm{d}x), 0, bx^{-1}\mathrm{e}^{-ax}\mathrm{d}x)$. From its characteristic exponent, it is clear that $Z_t$ is a gamma rv with density $\gamma_{a,bt}(z) := a^{bt} z^{bt-1} \mathrm{e}^{-az}/\Gamma(bt)$ $(z > 0)$ for each fixed time $t > 0$. In this case, the $n$-fold convolution of $\nu$ is

$$\nu^{*n}(\mathrm{d}z \times \mathrm{d}t) = \gamma_{a,bt}(z - ct)\mathrm{d}z K^{*n}(\mathrm{d}t),$$

for $t > 0, z \geq ct, n \geq 1$.

It follows that the joint distribution of $(\tau, N_\tau)$ is of the form (2.37), where by simple algebraic manipulations, it can be seen that

$$\mathbb{P}_u (\tau \in \mathrm{d}t, N_\tau = 1) = \mathrm{e}^{-\alpha(u+ct)} \left( \frac{a}{a+\alpha} \right)^{bt} K(\mathrm{d}t)$$

for $n = 1$, and

$$\begin{aligned}
&\mathbb{P}_u (\tau \in \mathrm{d}t, N_\tau = n) \\
=&\frac{\alpha^{n-1}\mathrm{e}^{-\alpha(u+ct)}}{n!} \sum_{k=0}^{n-2} \binom{n-2}{k} (u+ct)^{n-2-k} \left( \mathbb{E}\left[ Z_t^{k+1} \mathrm{e}^{-\alpha Z_t} \right] + (ct+nu)\mathbb{E}\left[ Z_t^{k} \mathrm{e}^{-\alpha Z_t} \right] \right) K^{*n}(\mathrm{d}t) \\
=&\frac{\alpha^{n-1}\mathrm{e}^{-\alpha(u+ct)}}{n!} \sum_{k=0}^{n-2} \binom{n-2}{k} (u+ct)^{n-2-k} \frac{a^{bt}}{\Gamma(bt)} \frac{\Gamma(k+bt)}{(a+\alpha)^{k+bt}} \left( \frac{k+bt}{a+\alpha} + nu + ct \right) K^{*n}(\mathrm{d}t),
\end{aligned}$$

for $n \geq 2$.

**Remark 2.4.3** *(***Ruin probability***) From Corollary 2.3.1, the infinite-time ruin probability is given by* $\phi_{1,0}(u) = z_0 \mathrm{e}^{-\alpha u(1-z_0)}$*, where* $z_0$ *is the unique solution (in* $z$*) to*

$$z = \tilde{k} \left( c\alpha - c\alpha z - b \ln \frac{a}{a+\alpha-\alpha z} \right)$$

*in the interval* $(0, 1)$*.*

**Remark 2.4.4** *If* $u = c = 0$*, the joint distribution of* $(\tau, N_\tau)$ *reduces to*

$$\mathbb{P}_0 (\tau \in \mathrm{d}t, N_\tau = n) = \frac{\Gamma(n-1+bt)}{\Gamma(n+1)\Gamma(bt)} \left( \frac{\alpha}{a+\alpha} \right)^{n-1} \left( \frac{a}{a+\alpha} \right)^{bt} K^{*n}(\mathrm{d}t), \quad n \in \mathbb{N}_+.$$

**Inverse Gaussian Process**

Let $S_t = ct + G_t$, where $c \geq 0$, $\{G_t\}_{t\geq 0}$ is an inverse Gaussian process with characteristic exponent given by

$$\Psi(\theta) = \sqrt{\beta}\left(\sqrt{-2i\theta + \frac{\beta}{\mu^2}} - \frac{\sqrt{\beta}}{\mu}\right),$$

where $\beta, \mu > 0$, and its Lévy triplet is $(-2\mu\int_0^{\sqrt{\beta}/\mu}\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{y^2}{2}}\mathrm{d}y, 0, \frac{\sqrt{\beta}}{\sqrt{2\pi x^3}}\mathrm{e}^{-\frac{\beta x}{2\mu^2}}\mathrm{d}x)$. We refer readers to Chhikara and Folks [28] for a comprehensive study of the inverse Gaussian distribution. From the form of its characteristic exponent, it is well known that $G_t$ is inverse Gaussian distributed with density

$$f_{IG}(x;\mu t, \beta t^2) := \sqrt{\frac{\beta t^2}{2\pi x^3}}\exp\left(-\frac{\beta(x-\mu t)^2}{2\mu^2 x}\right), \quad x > 0,$$

for fixed $t > 0$. Moreover, the $n$-fold convolution of $\nu$ is

$$\nu^{*n}(\mathrm{d}z \times \mathrm{d}t) = \sqrt{\frac{\beta t^2}{2\pi(z-ct)^3}}\exp\left(-\frac{\beta(z-ct-\mu t)^2}{2\mu^2(z-ct)}\right)\mathrm{d}zK^{*n}(\mathrm{d}t),$$

for $t > 0, z \geq ct, n \geq 1$.

By simple algebraic manipulations one can show that

$$\exp\left(\frac{\beta t}{\mu}\left(1-(1+\frac{2\alpha\mu^2}{\beta})^{1/2}\right)\right)\cdot f_{IG}\left(x;\mu t(1+\frac{2\alpha\mu^2}{\beta})^{-1/2}, \beta t^2\right) = \exp(-\alpha x)\cdot f_{IG}(x;\mu t, \beta t^2).$$

Therefore, we have

$$\mathbb{E}\left[G_t^k \mathrm{e}^{-\alpha G_t}\right] = \exp\left(\frac{\beta t}{\mu}(1-\eta)\right)\left(\frac{\mu t}{\eta}\right)^k g_k(t),$$

where $\eta := \sqrt{1+2\alpha\mu^2/\beta}$ and

$$g_k(t) := \sum_{i=0}^{k-1}\frac{(k-1+i)!}{i!(k-1-i)!}\left(\frac{\mu}{2\beta\eta t}\right)^i,$$

for $k \geq 1$ and $g_0(t) = 1$. Using (2.37), the joint distribution of $(\tau, N_\tau)$ is given by

$$\mathbb{P}_u\left(\tau \in \mathrm{d}t, N_\tau = 1\right) = \mathrm{e}^{-\alpha(u+ct)} \exp\left(\frac{\beta t}{\mu}(1-\eta)\right) K(\mathrm{d}t)$$

for $n = 1$, and

$$\begin{aligned}
&\mathbb{P}_u\left(\tau \in \mathrm{d}t, N_\tau = n\right) \\
=&\frac{\alpha^{n-1}\mathrm{e}^{-\alpha(u+ct)}}{n!}\sum_{k=0}^{n-2}\binom{n-2}{k}(u+ct)^{n-2-k}\left(\mathbb{E}\left[G_t^{k+1}\mathrm{e}^{-\alpha G_t}\right] + (ct+nu)\mathbb{E}\left[G_t^{k}\mathrm{e}^{-\alpha G_t}\right]\right)K^{*n}(\mathrm{d}t) \\
=&\frac{\alpha^{n-1}\mathrm{e}^{-\alpha(u+ct)}}{n!}\exp\left(\frac{\beta t}{\mu}(1-\eta)\right)\sum_{k=0}^{n-2}\binom{n-2}{k}(u+ct)^{n-2-k}\left(\frac{\mu t}{\eta}\right)^k\left((ct+nu)g_k(t) + \frac{\mu t}{\eta}g_{k+1}(t)\right)K^{*n}(\mathrm{d}t),
\end{aligned}$$

for $n \geq 2$.

**Remark 2.4.5** *(***Ruin probability***) From Corollary 2.3.1, the infinite-time ruin probability is given by $\phi_{1,0}(u) = z_0\mathrm{e}^{-\alpha u(1-z_0)}$, where $z_0$ is the unique solution (in $z$) to*

$$z = \tilde{k}\left(c(\alpha - \alpha z) - \frac{\beta}{\mu}\left(1 - \left(1 + \frac{2\alpha\mu^2(1-z)}{\beta}\right)^{\frac{1}{2}}\right)\right)$$

*in the interval $(0, 1)$.*

### 2.4.2 Risk process with varying premium rates

Inspired by the No Claim Discount (NCD) system (e.g., Kliger and Levikson [53] and Constantinescu et al. [31]) and risk models with dependence between the claim arrival process and the premium rate (e.g., Cheung et al. [25]), we now consider an income process $\{R_t\}_{t\geq 0}$ defined as

$$R_t = \begin{cases} c_1 t, & t \leq \xi, \\ c_1\xi + c_2(t-\xi), & t > \xi, \end{cases} \tag{2.40}$$

where $\xi > 0$ and $c_1, c_2 > 0$ are two premium rates. The income process (2.40) can be further generalized to accommodate a finite number $m$ of premium rates $\{c_i\}_{i=1}^{m}$. For ease of presentation, we only consider the case $m = 2$.

For the surplus process $\{U_t\}_{t\geq 0}$ with income process (2.40), the premium rate effective at any given time $t \geq 0$ is $c_1$ $(c_2)$ if $t - W_{N_t} \leq (>)\, \xi$ (where $t - W_{N_t}$ corresponds to the time elapsed since the last claim event). When $c_1 > c_2$, the following interpretation can be given to the resulting income process: the insurer is assumed to collect the base premium rate of $c_1$ following a claim event. If the duration of the subsequent no-claim period exceeds $\xi$, the insurer is assumed to reduce the premium rate to $c_2$ once the duration $\xi$ is reached until the time of the next claim event.

The risk model with income process (2.40) may be appropriate for modelling the insurer's surplus level in relatively low frequency claim events (such as insurance portfolios covering natural catastrophe risks; e.g., Boudreault et al. [22]). It allows the insurer to partially reflect the recent claim experience by adjusting the premium rates over time.

Let $\rho := c_1 - c_2$, then the income process (2.40) can be rewritten as $R_t = c_1 t - \rho(t-\xi)_+$, which in turn implies that the $n$-th convolution of $\nu$ is given by

$$\nu^{*n}(\mathrm{d}z \times \mathrm{d}t) = \mathbb{P}\left(c_1 t - \rho V_n \in \mathrm{d}z, W_n \in \mathrm{d}t\right), \tag{2.41}$$

where $V_n := \sum_{i=1}^{n} (T_i - \xi)_+$. By substituting (2.41) into (2.18), the joint distribution of $(\tau, N_\tau)$ is given by

$$\mathbb{P}_u\left(\tau \in \mathrm{d}t, N_\tau = n\right) = \int_{[(t-n\xi)_+, t)} b_n(u, c_1 t - \rho v) \mathrm{e}^{-\alpha(u + c_1 t - \rho v)} \mathbb{P}\left(V_n \in \mathrm{d}v, W_n \in \mathrm{d}t\right), \tag{2.42}$$

for $t > 0, n \in \mathbb{N}_+$. As is clear from (2.42), finding the joint distribution of $(\tau, N_\tau)$ boils down to characterizing the joint distribution of $(V_n, W_n)$, which we briefly expand on below.

Indeed,

$$\mathbb{P}\left(V_n \in \mathrm{d}v, W_n \in \mathrm{d}w\right) = \sum_{k=0}^{n} \mathbb{P}\left(V_n \in \mathrm{d}v, W_n \in \mathrm{d}w, T_{(k)} \leq \xi < T_{(k+1)}\right),$$

where $T_{(k)}$ is the $k$-th order statistics of the random sample $\{T_k\}_{k=1}^n$ (with $T_{(0)} = 0$ and $T_{(n+1)} = \infty$). Given that $\{T_k\}_{k=1}^n$ is a sequence of iid rv's, it follows that

$$\begin{aligned}
&\mathbb{P}\left(V_n \in \mathrm{d}v, W_n \in \mathrm{d}w\right) \\
&= \sum_{k=0}^{n} \binom{n}{k} \mathbb{P}\left(V_n \in \mathrm{d}v, W_n \in \mathrm{d}w, \bigcap_{j=1}^{k} \{T_j < \xi\}, \bigcap_{j=k+1}^{n} \{T_j > \xi\}\right) \\
&= \sum_{k=0}^{n} \binom{n}{k} \mathbb{P}\left(\sum_{j=k+1}^{n} T_j \in \mathrm{d}v + (n-k)\xi, \sum_{j=1}^{k} T_j \in \mathrm{d}w - (n-k)\xi - v, \bigcap_{j=1}^{k} \{T_j < \xi\}, \bigcap_{j=k+1}^{n} \{T_j > \xi\}\right) \\
&= \sum_{k=0}^{n} \binom{n}{k} M_k\left(\xi, \mathrm{d}w - (n-k)\xi - v\right) S_{n-k}\left(\xi, \mathrm{d}v + (n-k)\xi\right), \qquad (2.43)
\end{aligned}$$

for $(w - n\xi)_+ \leq v \leq w$, where

$$S_n(x, \mathrm{d}y) := \mathbb{P}\left(\sum_{j=1}^{n} T_j \in \mathrm{d}y, \bigcap_{j=1}^{n} \{T_j > x\}\right),$$

and

$$M_n(x, \mathrm{d}y) := \mathbb{P}\left(\sum_{j=1}^{n} T_j \in \mathrm{d}y, \bigcap_{j=1}^{n} \{T_j < x\}\right),$$

for $n \in \mathbb{N}_+$, with the convention that $\sum_{i=1}^{0}(\cdot) = 0$ and $S_0(x, \mathrm{d}y) = M_0(x, \mathrm{d}y) = \delta_0(\mathrm{d}y)$.

When the probability measures $S_n(x, \mathrm{d}y)$ and $M_n(x, \mathrm{d}y)$ are fully characterized, so is the joint distribution of $(\tau, N_\tau)$ via Eqs. (2.42) and (2.43). We note that closed-form expressions for both $S_n(x, \mathrm{d}y)$ and $M_n(x, \mathrm{d}y)$ can be found for large classes of interclaim time distributions (including phase-type and mixed Erlang distributions). For simplicity, Proposition 2.4.2 states these results when the interclaim time distribution $K$ is exponential with mean $1/\lambda$.

**Proposition 2.4.2** *For $\{T_i\}_{i\in\mathbb{N}_+}$ a sequence of iid exponential rv's with mean $1/\lambda$ and $n \in \mathbb{N}_+$,*

$$S_n(x, \mathrm{d}y) = \lambda^n \mathrm{e}^{-\lambda y}\frac{(y-nx)^{n-1}}{(n-1)!}\mathrm{d}y,\ \ y > nx > 0,$$

*and*

$$M_n(x, \mathrm{d}y) = \frac{\lambda^n \mathrm{e}^{-\lambda y}}{(n-1)!}\sum_{k=0}^{n}\binom{n}{k}(-1)^k(y-kx)^{n-1}\mathbb{1}_{\{y\geq kx\}}\mathrm{d}y,\ \ 0 < y < nx.$$

**Proof.** By the memoryless property of the exponential distribution,

$$\begin{aligned} S_n(x, \mathrm{d}y) &= \mathbb{P}\left(\sum_{j=1}^{n} T_j \in \mathrm{d}y \middle| \bigcap_{j=1}^{n}\{T_j > x\}\right)\mathbb{P}\left(\bigcap_{j=1}^{n}\{T_j > x\}\right) \\ &= \mathbb{P}\left(\sum_{i=1}^{n}(T_i - x) \in \mathrm{d}y - nx \middle| \bigcap_{j=1}^{n}\{T_j > x\}\right)\mathbb{P}\left(\bigcap_{j=1}^{n}\{T_j > x\}\right) \\ &= \lambda^n \mathrm{e}^{-\lambda y}\frac{(y-nx)^{n-1}}{(n-1)!}\mathrm{d}y, \end{aligned}$$

for $n \geq 1$. As for $M_n(x, \mathrm{d}y)$, we first note that

$$\begin{aligned} \mathbb{E}\left[\mathrm{e}^{-s\sum_{j=1}^{n} T_j} \middle| \bigcap_{j=1}^{n}\{T_j < x\}\right] &= \left(\mathbb{E}\left[\mathrm{e}^{-sT_1}|T_1 < x\right]\right)^n \\ &= \left(\frac{\lambda}{1-\mathrm{e}^{-\lambda x}}\right)^n\left(\frac{1-\mathrm{e}^{-(\lambda+s)x}}{\lambda+s}\right)^n. \end{aligned} \tag{2.44}$$

Inverting (2.44) wrt $s$ leads to

$$\begin{aligned} &\mathcal{L}^{-1}\left\{\left(\frac{\lambda}{1-\mathrm{e}^{-\lambda x}}\right)^n\left(\frac{1-\mathrm{e}^{-(\lambda+s)x}}{\lambda+s}\right)^n\right\} \\ &= \left(\frac{\lambda}{1-\mathrm{e}^{-\lambda x}}\right)^n\frac{\mathrm{e}^{-\lambda y}}{(n-1)!}\sum_{k=0}^{n}\binom{n}{k}(-1)^k(y-kx)^{n-1}\mathbb{1}_{\{y\geq kx\}}\mathrm{d}y. \end{aligned} \tag{2.45}$$

Moreover, for $y \geq nx$,

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}(y-kx)^{n-1} = y^{n-1} + \sum_{k=1}^{n}\binom{n}{k}(-1)^k(y-kx)^{n-1}$$

$$= y^{n-1} + \sum_{k=1}^{n} \binom{n}{k} (-1)^k \sum_{i=0}^{n-1} \binom{n-1}{i} y^{n-1-i} (-x)^i k^i$$

$$= y^{n-1} + \sum_{i=0}^{n-1} \binom{n-1}{i} (-x)^i y^{n-1-i} \sum_{k=1}^{n} (-1)^k \binom{n}{k} k^i. \qquad (2.46)$$

Note that (2.46) is in fact equal to 0 because $\sum_{k=1}^{n}(-1)^k\binom{n}{k}k^i = 0$ for $i \geq 1$ and $\sum_{k=1}^{n}(-1)^k\binom{n}{k}k^i = -1$ for $i = 0$, which can be shown by taking derivative of $(1-z)^n = \sum_{k=0}^{n}(-1)^k\binom{n}{k}z^k$ wrt $z$ at $z = 1$. Combining (2.44), (2.45) and (2.46), we conclude that

$$\mathbb{P}\left(\sum_{j=1}^{n} T_j \in \mathrm{d}y \middle| \bigcap_{j=1}^{n} \{T_j < x\}\right)$$
$$= \left(\frac{\lambda}{1-\mathrm{e}^{-\lambda x}}\right)^n \frac{\mathrm{e}^{-\lambda y}}{(n-1)!} \sum_{k=0}^{n} \binom{n}{k} (-1)^k (y-kx)^{n-1} \mathbb{1}_{\{y \geq kx\}} \mathrm{d}y,$$

for $0 < y < nx$. Therefore, for $n \geq 1$,

$$M_n(x, \mathrm{d}y) = \mathbb{P}\left(\sum_{j=1}^{n} T_j \in \mathrm{d}y \middle| \bigcap_{j=1}^{n} \{T_j < x\}\right) \mathbb{P}\left(\bigcap_{j=1}^{n} \{T_j < x\}\right)$$
$$= \frac{\lambda^n \mathrm{e}^{-\lambda y}}{(n-1)!} \sum_{k=0}^{n} \binom{n}{k} (-1)^k (y-kx)^{n-1} \mathbb{1}_{\{y \geq kx\}} \mathrm{d}y.$$

This completes the proof. ■

For completeness, we conclude this sub-section by providing the expression for the infinite-time ruin probability $\phi_{1,0}(u)$ for the insurance risk process (2.2) with income process (2.40).

**Remark 2.4.6** *(***Ruin probability***) From Corollary 2.3.1, the infinite-time ruin probability for the varying premium rates risk model is* $\phi_{1,0}(u) = z_0 \mathrm{e}^{-\alpha u(1-z_0)}$, *where* $z_0$ *is the unique solution (in* $z$*) to*

$$z = \int_{(0,\xi)} \mathrm{e}^{-(\alpha - \alpha z)c_1 t} K(\mathrm{d}t) + \mathrm{e}^{-(\alpha-\alpha z)\rho\xi} \int_{(\xi,\infty)} \mathrm{e}^{-(\alpha-\alpha z)c_2 t} K(\mathrm{d}t)$$

*in the interval* $(0, 1)$.

## 2.5 Numerical examples

In this section, we concretely measure the impact of the choice of income processes on some infinite-time and finite-time ruin measures via some numerical examples. We limit the analysis to the following ruin quantities:

1. the probability of ruin occurring before or at the time of the $n$-th claim, namely $\psi_n(u) = \mathbb{P}_u\left(N_\tau \leq n\right)$;

2. the density of the time to ruin with no more than $n$ claims, i.e. $\mathbb{P}_u\left(\tau \in \mathrm{d}t, N_\tau \leq n\right)/\mathrm{d}t$.

The limiting case of (1) where $n \to \infty$ is also considered, namely the infinite-time ruin probability, which we shall denote by $\psi(u)$ (rather than $\phi_{1,0}(u)$) henceforth.

For each example, we use Corollary 2.3.1 to evaluate the infinite-time ruin probability $\psi(0)$ and then $\psi(u)$. With the help of *Maple*, the ruin probability $\psi_n(u)$ is evaluated by integrating (2.18) and truncating the infinite sum (where applicable) at a reasonable high integer value. The density of the time to ruin with no more than $n$ claims is obtained by summing (2.18) over $n$.

To isolate the effect of the income process on ruin measures, we consider a given aggregate claim process in Eq. (2.2) and consider different modelling options for $\{R_t\}_{t\geq 0}$. Unless otherwise stated, this aggregate claim process will be assumed to be a compound Poisson process where $\{N_t\}_{t\geq 0}$ is a Poisson process with arrival intensity $\lambda = 0.2$ and the claim sizes $\{X_i\}_{i\in\mathbb{N}_+}$ are exponentially distributed with mean $1/\alpha = 5$.

For illustrative purposes, we first consider a simple example of the subordinator considered in Section 2.4.1.

**Example 1.** We assume that the income process $\{R_t\}_{t\geq 0}$ is as defined in Section 2.4.1, i.e., the subordinator $\{S_t\}_{t\geq 0}$ is of the form (3.30) where $c = 1$, $\mu = 0.1$, and the income sizes $\{Y_i\}_{i\in\mathbb{N}_+}$ form a sequence of iid rv's with an Erlang-$k$ distribution with Laplace transform $\tilde{f}_Y(s) = (1/(1+s))^k$. Results for the ruin probabilities $\psi(u)$ and $\psi_n(u)$ ($n = 5, 10, 25, 50, 75, 100$) are provided in Table 2.1 for an initial surplus of $u = 10$ and $k = 1, 2$ and 5. The ratio $\psi_n(u)/\psi(u)$ is also provided in parenthesis.

Table 2.1: Values of $\psi(u)$ and $\psi_n(u)$ when $u = 10$ (Example 1)

| | $\psi(u)$ | $\psi_5(u)$ | $\psi_{10}(u)$ | $\psi_{25}(u)$ | $\psi_{50}(u)$ | $\psi_{75}(u)$ | $\psi_{100}(u)$ |
|---|---|---|---|---|---|---|---|
| $k=1$ | 0.7614 | 0.3127 | 0.4471 | 0.5866 | 0.6593 | 0.6905 | 0.7082 |
| | | (41.07%) | (58.72%) | (77.04%) | (86.59%) | (90.69%) | (93.01%) |
| $k=2$ | 0.6095 | 0.2886 | 0.4032 | 0.5145 | 0.5661 | 0.5852 | 0.5945 |
| | | (47.35%) | (66.15%) | (84.41%) | (92.88%) | (96.01%) | (97.54%) |
| $k=5$ | 0.3885 | 0.2408 | 0.3150 | 0.3699 | 0.3848 | 0.3875 | 0.3882 |
| | | (61.98%) | (81.08%) | (95.21%) | (99.05%) | (99.74%) | (99.92%) |

As expected, we observe that ruin is less likely to occur when the mean $k$ of the income sizes $\{Y_i\}_{i\in\mathbb{N}_+}$ increases. This is true for all ruin probabilities provided in Table 2.1 as we note that the positive security loading $\theta$ of the insurance risk process $\{U_t\}_{t\geq 0}$ is known to increase as the mean $k$ increases. We observe that, for a given $n$, the ratio $\psi_n(u)/\psi(u)$ increases as the mean $k$ increases. This is again not surprising from the above observation on the security loading $\theta$ of the corresponding insurance risk processes. The larger is the security loading factor $\theta$, the less likely ruin occurs late (if it does happen), which explains the relatively larger contribution to ruin from early claim events.

To investigate how the type of income process $\{R_t\}_{t\geq 0}$ may affect the ruin probability, we consider the following examples with different modelling assumptions for $\{R_t\}_{t\geq 0}$.

**Example 2.** We now consider the case where $R_t = ct$ (so the corresponding risk model is a Cramér-Lundberg risk process) and compare it with all three subordinators discussed in Section 2.4.1 under the following parameter settings:

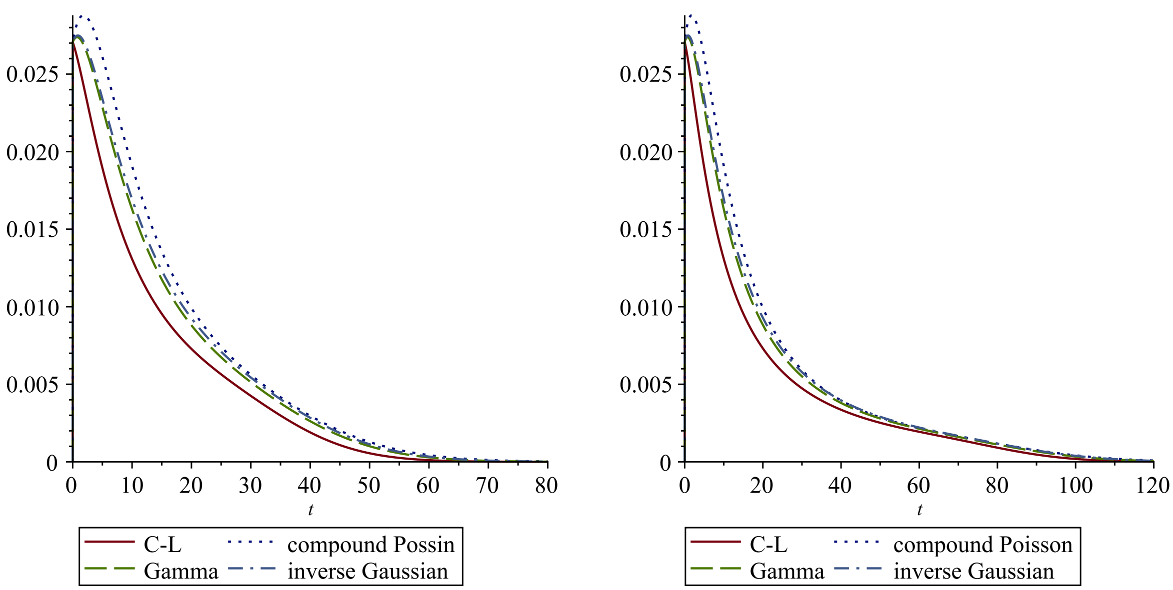
$$\begin{aligned}
\text{Cramér-Lundberg: } & c = 1.2, \\
\text{compound Poisson process: } & c = 0, \beta = 0.2, \mu = 0.24, \\
\text{Gamma process: } & c = 0, a = 0.2, b = 0.24, \\
\text{inverse Gaussian process: } & c = 0, \beta = 0.2, \mu = 1.2.
\end{aligned}$$

As a basis of comparison, all income processes are such that the corresponding surplus process $\{U_t\}_{t\geq 0}$ operates under a security loading factor $\theta$ of 20%. From Proposition 2.4.1, we already know that the Cramér-Lundberg setting will result in the lowest ruin probability $\psi(u)$ among the corresponding insurance risk processes.

Tables 2.2 and 2.3 display values of $\psi(u)$ and $\psi_n(u)$ ($n = 5, 25, 50$ and $100$) for $u = 5$ and $u = 10$, respectively. Figure 2.2 presents densities of the time to ruin with no more than 10 and 20 claims (left and right panel, respectively) for an initial surplus of $u = 10$.

Table 2.2: $\theta = 20\%$, $u = 5$ (Example 2)

| | Cramér-Lundberg | Compound Poisson | Gamma | Inverse Gaussian |
|---|---|---|---|---|
| $\psi(u)$ | 0.7054 | 0.8301 | 0.7833 | 0.8026 |
| $\psi_5(u)$ | 0.4493 | 0.5857 | 0.5273 | 0.5430 |
| | (63.69%) | (70.56%) | (67.31%) | (67.65%) |
| $\psi_{25}(u)$ | 0.6352 | 0.7543 | 0.7080 | 0.7256 |
| | (90.05%) | (90.87%) | (90.38%) | (90.40%) |
| $\psi_{50}(u)$ | 0.6740 | 0.7907 | 0.7463 | 0.7639 |
| | (95.55%) | (95.25%) | (95.27%) | (95.17%) |
| $\psi_{100}(u)$ | 0.6948 | 0.8105 | 0.7686 | 0.7866 |
| | (98.50%) | (97.64%) | (98.12%) | (98.00%) |




(a) $n = 10$ (b) $n = 20$

Figure 2.2: Density of the time of ruin with no more than $n$ claims for $u = 10$ (Example 2)

Table 2.3: $\theta = 20\%$, $u = 10$ (Example 2)

| | Cramér-Lundberg | Compound Poisson | Gamma | Inverse Gaussian |
|---|---|---|---|---|
| $\psi(u)$ | 0.5971 | 0.7580 | 0.6959 | 0.7213 |
| $\psi_5(u)$ | 0.2805 | 0.4154 | 0.3537 | 0.3675 |
| | (46.98%) | (54.81%) | (50.82%) | (50.96%) |
| $\psi_{25}(u)$ | 0.5038 | 0.6505 | 0.5914 | 0.6132 |
| | (84.37%) | (85.82%) | (84.98%) | (85.01%) |
| $\psi_{50}(u)$ | 0.5550 | 0.7020 | 0.6444 | 0.6668 |
| | (92.95%) | (92.62%) | (92.59%) | (92.44%) |
| $\psi_{100}(u)$ | 0.5828 | 0.7303 | 0.6754 | 0.6986 |
| | (97.60%) | (96.35%) | (97.04%) | (96.85%) |

From the results of Example 2, we observe that:

- For a given security loading factor $\theta$, insurance risk processes with different income processes display significantly different solvency risks. For instance, in Table 2.2, the ruin probability $\psi_{25}(5)$ for the risk processes with compound Poisson income process, gamma income process and inverse Gaussian income process is increased by 18.75%, 11.46% and 14.23% respectively, in comparison to the Cramér-Lundberg risk process.

- Moreover, as indicated in Proposition 2.4.1, in comparison to the Cramér-Lundberg risk process where the income process is deterministic, insurance risk processes with more uncertain income processes display an accrued risk of insolvency (i.e., the ruin probabilities are larger). This is also true for the finite-time ruin quantities considered here.

- Among the three subordinators discussed in Section 2.4.1, the insurance risk process with the gamma income process is the least risky, while the risk model with the compound Poisson income process is the riskiest across all ruin related quantities considered. A similar observation is also made by Garrido and Morales [44]. Hence, the dynamics of the income process play an important role in the assessment of an insurer's solvency risk.

- As pointed out by e.g., Landriault and Willmot [74] and Li et al. [78], there are some parallels that can be drawn between the variability of a surplus process' increments (as measured by its variance) and the corresponding ruin probabilities of this surplus process. More specifically, we often observe that for a given security loading factor, ruin probabilities tend to increase as the variance of the increments of a surplus process increases. This is also the case in the present example. Indeed, for the 4 surplus processes considered in Example 2, the variance of the increments of these surplus processes between claim instants follows the same ordering as the variance of their corresponding subordinators between claim instants which are 36, 96, 66 and 79.2, respectively. Hence, we note that here again the ruin probabilities displayed in Tables 2.2 and 2.3 follow the same ordering as the one given by the variance of their surplus process' increments between claim instants.

In the next example, we keep everything the same as in Example 2 but make a change to the claim arrival dynamics. Our main objective is to determine whether the Example 2 observations remain valid when the aggregate claim process is no longer a compound Poisson process.

**Example 3.** We now assume that the interclaim times $\{T_i\}_{i\in\mathbb{N}_+}$ follow an Erlang$(k,\lambda)$ distribution with cdf $K(t) = 1 - \sum_{n=0}^{k-1}(\lambda t)^n \mathrm{e}^{-\lambda t}/n!$. We consider the following two parameter settings:

1. $k = 2, \lambda = 0.4$;

2. $k = 3, \lambda = 0.6$.

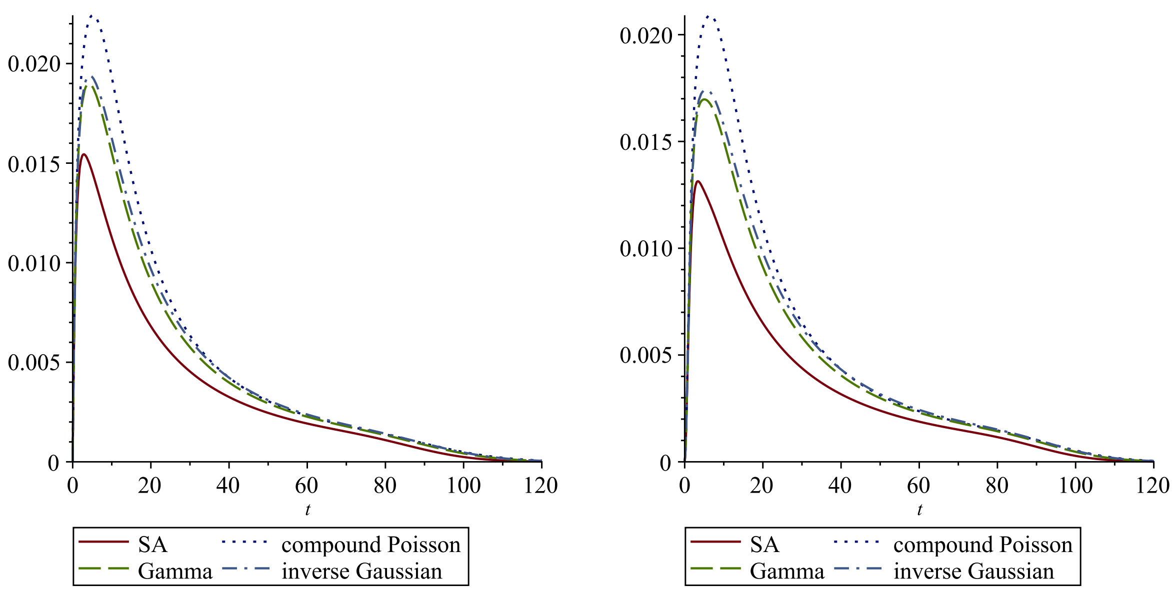
As all else remains the same as the setting of Example 2, the security loading factor $\theta$ is still 20% for both Erlang models. Tables 2.4 and 2.5 contain values of $\psi(u)$ and $\psi_n(u)$ ($n = 5, 25, 50$ and 100) for an initial surplus of $u = 10$ under the Erlang interclaim distribution (a) and (b), respectively. Figure 2.3 presents densities of the time to ruin with no more than 20 claims (left panel for the Erlang distribution (a) and right panel for the Erlang distribution (b)).

Table 2.4: $k = 2, \lambda = 0.4$ (Example 3)

| | Sparre Andersen | Compound Poisson | Gamma | Inverse Gaussian |
|---|---|---|---|---|
| $\psi(u)$ | 0.5060 | 0.7259 | 0.6442 | 0.6771 |
| $\psi_5(u)$ | 0.2293 | 0.3853 | 0.3142 | 0.3295 |
| | (45.32%) | (53.08%) | (48.78%) | (48.66%) |
| $\psi_{25}(u)$ | 0.4279 | 0.6202 | 0.5452 | 0.5722 |
| | (84.55%) | (85.44%) | (84.63%) | (84.50%) |
| $\psi_{50}(u)$ | 0.4738 | 0.6724 | 0.5975 | 0.6261 |
| | (93.64%) | (92.62%) | (92.75%) | (92.48%) |
| $\psi_{100}(u)$ | 0.4969 | 0.7036 | 0.6271 | 0.6572 |
| | (98.19%) | (96.92%) | (97.34%) | (97.07%) |

Table 2.5: $k = 3$, $\lambda = 0.6$ (Example 3)

| | Sparre Andersen | Compound Poisson | Gamma | Inverse Gaussian |
|---|---|---|---|---|
| $\psi(u)$ | 0.4664 | 0.7132 | 0.6227 | 0.6588 |
| $\psi_5(u)$ | 0.2093 | 0.3740 | 0.2991 | 0.3149 |
| | (44.89%) | (52.45%) | (48.03%) | (47.79%) |
| $\psi_{25}(u)$ | 0.3955 | 0.6084 | 0.5264 | 0.5555 |
| | (84.80%) | (85.31%) | (84.53%) | (84.32%) |
| $\psi_{50}(u)$ | 0.4385 | 0.6607 | 0.5782 | 0.6095 |
| | (94.02%) | (92.64%) | (92.84%) | (92.51%) |
| $\psi_{100}(u)$ | 0.4591 | 0.6917 | 0.6070 | 0.6402 |
| | (98.45%) | (96.99%) | (97.48%) | (97.17%) |




(a) $k = 2, \lambda = 0.4$ (b) $k = 3, \lambda = 0.6$

Figure 2.3: Densities of the time of ruin with no more than 20 claims (Example 3)

Reviewing the results of Example 3, the following observations are worthy of mention:

- The results of Example 3 are consistent with those of Example 2. In fact, the relative differences of ruin quantities between the risk process with deterministic income process (labelled SA) vs those with more uncertain income processes are noticeably more pronounced in Example 3 (than Example 2).

- The distribution of the interclaim times $\{T_i\}_{i\in\mathbb{N}_+}$ has a direct impact on all ruin quantities. However, we note that the ruin probabilities of the risk process with more volatile income processes seem less sensitive to the change in the distributional assumption for $\{T_i\}_{i\in\mathbb{N}_+}$.

In the last example, we consider the risk process with varying premium rates introduced in Section 2.4.2. Our goal is to investigate the effect of the premium rate adjustment mechanism on ruin quantities.

**Example 4.** We first consider an income process as defined in Section 2.4.2 with $c_1 = 2$. For an initial surplus of $u = 5$, values of $\psi(u)$ and $\psi_n(u)$ ($n = 5, 10, 15, 20, 25$ and $30$) for some specific values of $\rho$ are provided in Tables 2.6 and 2.7 when $\xi = 5$ and $\xi = 10$, respectively. We note that $\rho = c_1 - c_2$ can be interpreted as a premium discount for positive $\rho$.

As expected, all ruin probabilities increase as $\rho$ increases (or $\xi$ decreases). This is the case as the insurer is expected to receive less in premium incomes.

We now compare two varying premium insurance risk processes to an insurance risk process with no premium adjustment (i.e., with a constant premium rate). More precisely, we consider the following income process in the form of Section 2.4.2:
Case 1: single premium rate: $c = 1.2057$;

Table 2.6: Values of $\psi(u)$ and $\psi_n(u)$ when $\xi = 5$ (Example 4)

| | $\psi(u)$ | $\psi_5(u)$ | $\psi_{10}(u)$ | $\psi_{15}(u)$ | $\psi_{20}(u)$ | $\psi_{25}(u)$ | $\psi_{30}(u)$ |
|---|---|---|---|---|---|---|---|
| $\rho = 0$ | 0.3033 | 0.2642 | 0.2922 | 0.2993 | 0.3017 | 0.3026 | 0.3030 |
| | | (87.11%) | (96.34%) | (98.68%) | (99.47%) | (99.77%) | (99.90%) |
| $\rho = 0.2$ | 0.3099 | 0.2674 | 0.2973 | 0.3052 | 0.3080 | 0.3091 | 0.3095 |
| | | (86.29%) | (95.93%) | (98.48%) | (99.39%) | (99.74%) | (99.87%) |
| $\rho = 0.5$ | 0.3222 | 0.2732 | 0.3064 | 0.3159 | 0.3194 | 0.3209 | 0.3216 |
| | | (84.79%) | (95.10%) | (98.04%) | (99.13%) | (99.60%) | (99.81%) |
| $\rho = 1$ | 0.3525 | 0.2865 | 0.3277 | 0.3411 | 0.3467 | 0.3494 | 0.3508 |
| | | (81.28%) | (92.96%) | (96.77%) | (98.35%) | (99.12%) | (99.52%) |

Case 2: varying premium rates: $c_1 = 1.5, c_2 = 0.7, \xi = 5$;

Case 3: varying premium rates: $c_1 = 1, c_2 = 1.5591, \xi = 5$.

We point out that the security loading factor $\theta$ is the same ($\theta = 20.57\%$) under all three insurance risk processes. Table 2.8 presents the results of $\psi(u)$ and $\psi_n(u)$ ($n = 5, 10, 15, 20$ and 30) for $u = 5$.

From Table 2.8, we notice that a preloaded premium strategy (Case 2) may be desirable for the insurer in consideration of the ruin probability. This is the case as more premium is received after a claim event, improving the odds that the surplus process recovers from the claim event and avoids ruin. Note that this preloaded premium strategy mechanism provides some flexibility to the insurer to adjust the premium rate to reflect the recent claim experience. Not surprisingly, the opposite conclusion is observed in Case 3, where the ruin probabilities are higher than the constant premium rate risk model (Case 1). However, we point out that Case 3 may not be very realistic in insurance contexts.

Table 2.7: Values of $\psi(u)$ and $\psi_n(u)$ when $\xi = 10$ (Example 4)

| | $\psi(u)$ | $\psi_5(u)$ | $\psi_{10}(u)$ | $\psi_{15}(u)$ | $\psi_{20}(u)$ | $\psi_{25}(u)$ | $\psi_{30}(u)$ |
|---|---|---|---|---|---|---|---|
| $\rho = 0$ | 0.3033 | 0.2642 | 0.2922 | 0.2993 | 0.3017 | 0.3026 | 0.3030 |
| | | (87.11%) | (96.34%) | (98.68%) | (99.47%) | (99.77%) | (99.90%) |
| $\rho = 0.2$ | 0.3041 | 0.2645 | 0.2928 | 0.3001 | 0.3025 | 0.3034 | 0.3038 |
| | | (86.98%) | (96.28%) | (98.68%) | (99.47%) | (99.77%) | (99.90%) |
| $\rho = 0.5$ | 0.3057 | 0.2650 | 0.2938 | 0.3013 | 0.3039 | 0.3049 | 0.3053 |
| | | (86.69%) | (96.11%) | (98.56%) | (99.41%) | (99.74%) | (99.87%) |
| $\rho = 1$ | 0.3090 | 0.2662 | 0.2961 | 0.3041 | 0.3069 | 0.3081 | 0.3086 |
| | | (86.15%) | (95.83%) | (98.41%) | (99.32%) | (99.71%) | (99.87%) |

In Figure 2.4, we further emphasize the conclusions of Table 2.8 by plotting the infinite-time ruin probability $\psi(u)$ for an initial surplus of $u = 5$ in terms of the premium rate $c_1$ (for $\xi = 2.5, 5$ and 10). For a given $c_1$, we find the premium rate $c_2$ such that the corresponding insurance risk process has a security loading factor of 20.57%. It can be seen from Figure 2.4 that the infinite-time ruin probability $\psi(u)$ decreases as $c_1$ increases. Also, for $c_1 < 1.2057$ (which corresponds to a preloaded premium strategy), $\psi(u)$ is larger for larger $\xi$, while the opposite conclusion holds for $c_1 > 1.2057$.

Table 2.8: $u = 5$ (Example 4)

| | Case 1 | Case 2 | Case 3 |
|---|---|---|---|
| $\psi(u)$ | 0.6993 | 0.6433 | 0.7394 |
| $\psi_5(u)$ | 0.4473 | 0.4052 | 0.4818 |
| | (63.96%) | (62.98%) | (65.16%) |
| $\psi_{10}(u)$ | 0.5450 | 0.4983 | 0.5813 |
| | (77.93%) | (77.46%) | (78.62%) |
| $\psi_{15}(u)$ | 0.5889 | 0.5410 | 0.6253 |
| | (84.21%) | (84.09%) | (84.57%) |
| $\psi_{20}(u)$ | 0.6145 | 0.5659 | 0.6508 |
| | (87.80%) | (87.96%) | (88.02%) |
| $\psi_{30}(u)$ | 0.6435 | 0.5939 | 0.6798 |
| | (92.02%) | (92.32%) | (91.94%) |

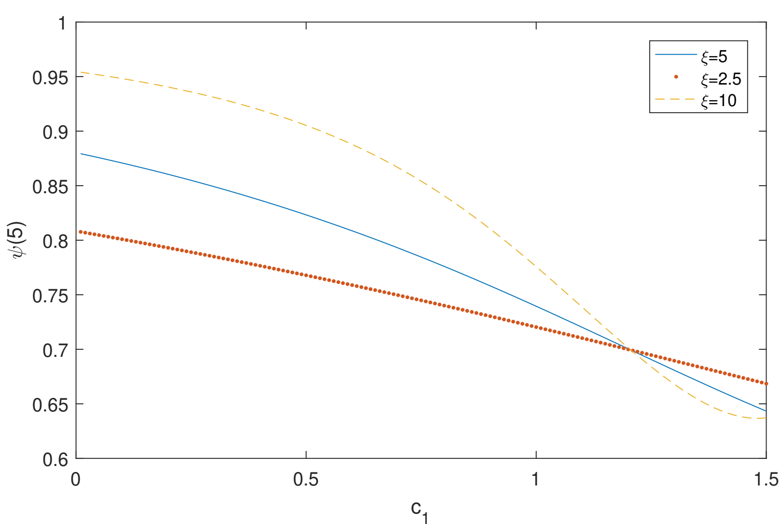



Figure 2.4: Ruin probability for varying premium rates model with $u = 5, \theta = 20.57\%$

# Chapter 3

# Deficit Analysis in an Insurance Risk Process with Dependence

## 3.1 Introduction

In this chapter, we extend the analysis of Chapter 2 to the renewal insurance risk process with a fairly general modelling assumption on the claim sizes. More specifically, the risk process of interest in this chapter is defined as

$$U_t = u + R_t^* + \sum_{i=1}^{N_t} X_i, \quad t \geq 0, \tag{3.1}$$

where $U_0 = u \geq 0$ is the initial surplus level and $\{R_t^*\}_{t\geq 0}$ is defined as in Section 2.2. Also, $\{N_t\}_{t\geq 0}$ is assume to be a renewal process defined as $N_t = \sup\{n \geq 0 : W_n \leq t\}$ where $W_0 = 0$ and $W_n = \sum_{i=1}^{n} T_i$ for $n \in \mathbb{N}_+$. The inter-arrival times $\{T_i\}_{i\in\mathbb{N}_+}$ are assumed to be a sequence of iid positive rv's with density $k(\cdot)$, while the jump sizes $\{X_i\}_{i\in\mathbb{N}_+}$ form a sequence of iid rv's taking values on $\mathbb{R}$. More precisely, $X_i$ follows a mixture distribution described as follows:

Let $\{B_i\}_{i\in\mathbb{N}_+}$ be a sequence of iid Bernoulli rv's with $\mathbb{P}(B_i = 1) = q_+ (0 \le q_+ < 1)$ and $\mathbb{P}(B_i = 0) = q_- = 1 - q_+$. Assume that downward jump (claim) sizes are given by the sequence of iid positive rv's $\{Y_i\}_{i\in\mathbb{N}_+}$ with density $f(\cdot)$, while the upward jump sizes are given by the sequence of iid positive rv's $\{P_i\}_{i\in\mathbb{N}_+}$ with density $p(\cdot)$. The two-sided jump size $X_i$ is then defined as

$$X_i = \mathbb{1}(B_i = 1)P_i - \mathbb{1}(B_i = 0)Y_i, \quad i \in \mathbb{N}_+. \tag{3.2}$$

We further assume that the random vectors $\{(T_i, B_i, Y_i, P_i)\}_{i\in\mathbb{N}_+}$ are iid with a common joint distribution of the form

$$\mathbb{P}(T_1 \in \mathrm{d}t, B_1 = j, Y_1 \in \mathrm{d}y, P_1 \in \mathrm{d}z) = \mathbb{P}(T_1 \in \mathrm{d}t, Y_1 \in \mathrm{d}y)\, \mathbb{P}(B_1 = j)\, p(z)\mathrm{d}z, \tag{3.3}$$

for $t, y, z > 0$ and $j = 0, 1$, where the joint distribution for $(T_1, Y_1)$ is assumed to be given by

$$\mathbb{P}(T_1 \in \mathrm{d}t, Y_1 \in \mathrm{d}y) := f(y|t)k(t) = \sum_{i=1}^{m}\sum_{j=1}^{n_i} g_{ij}(t)e_{\beta_i,j}(y), \quad t, y > 0, \tag{3.4}$$

where $e_{\beta,j}(y)$ stands for the Erlang density

$$e_{\beta,j}(y) = \frac{\beta(\beta y)^{j-1}\mathrm{e}^{-\beta y}}{(j-1)!}, \quad y > 0.$$

It is immediate that the marginal density of $T_i$ is

$$k(t) = \sum_{i=1}^{m}\sum_{j=1}^{n_i} g_{ij}(t), \quad t > 0.$$

We remark that the marginal density of $Y_i$ is a Coxian-$n$ pdf with $n = \sum_i^m n_i$. Also, by Eqs. (3.2) and (3.3), one deduces that

$$\mathbb{P}(X_i \in \mathrm{d}x, T_i \in \mathrm{d}t) = \begin{cases} q_+ p(x)k(t)\mathrm{d}x\mathrm{d}t, & x > 0, t > 0, \\ q_- f(-x|t)k(t)\mathrm{d}x\mathrm{d}t, & x < 0, t > 0. \end{cases}$$

We note the Erlang density $e_{\beta,j}(y)$ allows the factorization

$$e_{\beta_i,j}(x+y) = \frac{1}{\beta_i}\sum_{k=1}^{j} e_{\beta_i,j+1-k}(x)e_{\beta_i,k}(y), \tag{3.5}$$

which will be heavily relied upon in the later analysis. In this chapter, we denote the Laplace transform of an arbitrary function $a(\cdot)$ by $\tilde{a}(s) = \int_0^\infty \mathrm{e}^{-st}a(t)\mathrm{d}t$ (if it exists).

The rest of the chapter is organized as follows. In Section 3.2, we analyze the Gerber-Shiu function when the penalty function depends on the deficit at ruin and show that the Gerber-Shiu function can be characterized through a linear system of equations. Section 3.3 considers applications of the main result derived in Section 3.2 for particular choices of the income process, while Section 3.4 presents a few numerical examples.

## 3.2 General results

### 3.2.1 Density properties

Let $\tau := \inf\{t \geq 0 : U_t < 0\}$ (with the convention that $\inf\emptyset = \infty$) be the time to ruin for the surplus process $\{U_t\}_{t\geq 0}$ . Then $U_{\tau^-}$ and $|U_\tau|$ stand for the surplus (immediately) prior to ruin and the deficit at ruin, respectively. For a ruin event to occur on the first downward jump, the (defective) joint density of $(\tau, U_\tau-, |U_\tau|)$ at $(t,x,y)$ is given by

$$\begin{aligned} h_1(t,x,y|u) = &\sum_{n=0}^{\infty} q_+^n q_- \int_0^t \int_0^{x-u} k^{*n}(t-z)k(z)f(x+y|z)p^{*n}\left(x-u-v\right) \\ &\times \mathbb{P}\left(\sum_{i=1}^{n+1} R_{T_i}^i \in \mathrm{d}v \Big| \sum_{i=1}^{n} T_i = t-z, T_{n+1} = z\right)\mathrm{d}z, \end{aligned} \tag{3.6}$$

for $x > u, t, y > 0$, where $k^{*n}(\cdot)$ and $p^{*n}(\cdot)$ are the $n$-fold convolution densities of $k(\cdot)$ and $p(\cdot)$, respectively (with the convention that $k^{*0}(z) = p^{*0}(z) = \delta_0(z)$ and $\delta_0(z)$ is the well-known Dirac delta function). One can argue the validity of the probabilistic equation (3.6) as follows: conditioning on the event that there are $n(n=0,1,2,...)$ upward jumps before the first downward jump (with probability $q_+^n q_-$), to have a deficit of $y$, the amount of the first claim must be $x+y$, with conditional density $f(x+y|z)$, where $0<z<t$ is the inter-arrival time between the $n$ th upward jump and the first downward jump. Also, in order for $(\tau, U_{\tau^-})$ to be at $(t,x)$, the $n$ th jump arrival time should be $t-z$, and the sum of the total increments of the revenue process $R_t^*(=v)$ and $\sum_{i=1}^n P_i(=x-u-v)$ needs to be $x-u$.

Substituting (3.4) into (3.6), we have

$$h_1(t,x,y|u) = \sum_{i=1}^{m}\sum_{j=1}^{n_i} e_{\beta_i,j}(x+y)\eta_{ij}(t,x|u), \quad x>u, t, y>0, \tag{3.7}$$

where

$$\begin{aligned}\eta_{ij}(t,x|u) :=& \sum_{n=0}^{\infty} q_+^n q_- \int_0^t \int_0^{x-u} k^{*n}(t-z) g_{ij}(z) p^{*n}(x-u-v) \\ & \times \mathbb{P}\left(\sum_{i=1}^{n+1} R_i \in \mathrm{d}v \big| \sum_{i=1}^{n} T_i = t-z, T_{n+1}=z\right)\mathrm{d}z.\end{aligned}$$

For ruin occurring on a downward jump other than the first one, we denote by $h_2(t,r,x,y,v|u)$ the joint density of the time to ruin $(t)$, the time between the last two claims before ruin $(r)$, the surplus prior to ruin $(x)$, the deficit at ruin $(y)$ and the surplus at the second last claim before ruin $(v)$. Using probabilistic arguments and the regenerative property of $\{U_t\}_{t\geq 0}$ at jump times, we have

$$h_2(t,r,x,y,v|u) = \frac{h_1(r,x,y|v)}{\int_0^\infty h_1(r,x,y|v)\mathrm{d}y}\int_0^\infty h_2(t,r,x,y,v|u)\mathrm{d}y,$$

for $x>v>0, t>r>0, y>0$.

### 3.2.2 Conditioning on the first drop in surplus

We are interested in the Gerber-Shiu function

$$m_\delta(u) = \mathbb{E}_u\left[\mathrm{e}^{-\delta\tau}w(|U_\tau|)\mathbb{1}_{\{\tau<\infty\}}\right], \quad u \geq 0, \tag{3.8}$$

where $w(\cdot)$ is a penalty function involving the deficit at ruin. As discussed in Landriault et al. [64], a more general penalty function may be considered (see, e.g., Landriault and Willmot [74]) for the ordinary Sparre Andersen model. We opt for this slightly simpler penalty function $w(\cdot)$ which only involves the deficit at ruin because it allows us to carry out the subsequent analysis on the dependent risk model (3.1) with more simplicity.

By conditioning on the first drop in the surplus to a value below its initial level of $u$ (see Section 4.4 of Willmot and Woo [102] for more details), $m_\delta(u)$ satisfies the defective renewal equation

$$m_\delta(u) = \phi_\delta \int_0^u m_\delta(u-y)b_\delta(y)\mathrm{d}y + v_\delta(u), \tag{3.9}$$

where

$$\phi_\delta = \int_0^\infty \int_0^\infty \int_0^\infty \mathrm{e}^{-\delta t}\left(h_1(t,x,y|0) + h_2(t,x,y|0)\right)\mathrm{d}t\mathrm{d}x\mathrm{d}y,$$

$$b_\delta(y) = \frac{1}{\phi_\delta}\int_0^\infty \int_0^\infty \mathrm{e}^{-\delta t}\left(h_1(t,x,y|0) + h_2(t,x,y|0)\right)\mathrm{d}t\mathrm{d}x,$$

$$h_2(t,x,y|0) = \int_0^x \int_0^t h_2(t,r,x,y,v|0)\mathrm{d}r\mathrm{d}v,$$

and

$$v_\delta(u) = \phi_\delta \int_0^\infty w(y)b_\delta(y+u)\mathrm{d}y. \tag{3.10}$$

From (3.7) and (3.5), we have

$$
\begin{aligned}
&\int_0^\infty \int_0^\infty \mathrm{e}^{-\delta t} h_1(t,x,y|u)\mathrm{d}t\mathrm{d}x \\
&= \int_0^\infty \int_0^\infty \mathrm{e}^{-\delta t} \sum_{i=1}^{m}\sum_{j=1}^{n_i} e_{\beta_i,j}(x+y)\eta_{ij}(t,x|u)\mathrm{d}t\mathrm{d}x \\
&= \sum_{i=1}^{m}\sum_{k=1}^{n_i}\sum_{j=k}^{n_i} e_{\beta_i,k}(y)\frac{1}{\beta_i}\int_0^\infty \int_0^\infty \mathrm{e}^{-\delta t} e_{\beta_i,j+1-k}(x)\eta_{ij}(t,x|u)\mathrm{d}t\mathrm{d}x \\
&= \sum_{i=1}^{m}\sum_{k=1}^{n_i} e_{\beta_i,k}(y)\xi_{1,\delta,ik}(u),
\end{aligned}
$$

where

$$
\xi_{1,\delta,ik}(u) = \sum_{j=k}^{n_i}\frac{1}{\beta_i}\int_0^\infty \int_0^\infty \mathrm{e}^{-\delta t} e_{\beta_i,j+1-k}(x)\eta_{ij}(t,x|u)\mathrm{d}t\mathrm{d}x.
$$

Similarly,

$$
\begin{aligned}
&\int_0^\infty \int_0^\infty \mathrm{e}^{-\delta t} h_2(t,x,y|u)\mathrm{d}t\mathrm{d}x \\
&= \int_0^\infty \int_0^\infty \mathrm{e}^{-\delta t}\left\{\int_0^t \int_0^x \frac{h_1(r,x,y|v)}{\int_0^\infty h_1(r,x,y|v)\mathrm{d}y}\int_0^\infty h_2(t,r,x,y,v|u)\mathrm{d}y\mathrm{d}v\mathrm{d}r\right\}\mathrm{d}t\mathrm{d}x \\
&= \int_0^\infty \int_0^\infty \mathrm{e}^{-\delta t}\left\{\int_0^t \int_0^x \frac{\sum_{i=1}^{m}\sum_{j=1}^{n_i} e_{\beta_i,j}(x+y)\eta_{ij}(r,x|v)}{\int_0^\infty h_1(r,x,y|v)\mathrm{d}y}\int_0^\infty h_2(t,r,x,y,v|u)\mathrm{d}y\mathrm{d}v\mathrm{d}r\right\}\mathrm{d}t\mathrm{d}x.
\end{aligned}
\tag{3.11}
$$

By applying (3.5), Eq. (3.11) can be written as

$$
\int_0^\infty \int_0^\infty \mathrm{e}^{-\delta t} h_2(t,x,y|u)\mathrm{d}t\mathrm{d}x = \sum_{i=1}^{m}\sum_{k=1}^{n_i} e_{\beta_i,k}(y)\xi_{2,\delta,ik}(u),
$$

where

$$
\xi_{2,\delta,ik}(u) = \sum_{j=k}^{n_i}\frac{1}{\beta_i}\int_0^\infty \int_0^\infty \mathrm{e}^{-\delta t}\left\{\int_0^t \int_0^x \frac{e_{\beta_i,j+1-k}(x)\eta_{ij}(r,x|v)}{\int_0^\infty h_1(r,x,y|v)\mathrm{d}y}\int_0^\infty h_2(t,r,x,y,v|u)\mathrm{d}y\mathrm{d}v\mathrm{d}r\right\}\mathrm{d}t\mathrm{d}x.
$$

Therefore, we can rewrite $b_\delta(\cdot)$ as follows:

$$b_\delta(y) = \sum_{i=1}^{m}\sum_{k=1}^{n_i} e_{\beta_i,k}(y)\xi_{\delta,ik}, \tag{3.12}$$

where $\xi_{\delta,ik} = \frac{1}{\phi_\delta}\left(\xi_{1,\delta,ik}(0) + \xi_{2,\delta,ik}(0)\right)$. Substituting (3.12) into (3.10) and using the factorization formula (3.5) again, we have

$$v_\delta(u) = \sum_{i=1}^{m}\sum_{j=1}^{n_i} \xi^*_{\delta,ij} e_{\beta_i,j}(u), \tag{3.13}$$

where

$$\xi^*_{\delta,ij} = \phi_\delta \sum_{k=j}^{n_i} \frac{\xi_{\delta,ik}}{\beta_i}\mathbb{E}\left[w\left(E_{i,k+1-j}\right)\right]$$

and $E_{i,k+1-j}$ is a rv with Erlang density $e_{\beta_i,k+1-j}$.

Taking the Laplace transform of Eq. (3.9) and using Eqs. (3.12) and (3.13), one deduces that

$$\begin{aligned}\widetilde{m}_\delta(s) =& \frac{\tilde{v}_\delta(s)}{1-\phi_\delta\tilde{b}_\delta(s)} \\ =& \frac{\left\{\prod_{k=1}^{m}\left(\beta_k+s\right)^{n_k}\right\}\left\{\sum_{i=1}^{m}\sum_{j=1}^{n_i}\xi^*_{\delta,ij}\left(\frac{\beta_i}{\beta_i+s}\right)^j\right\}}{\left\{\prod_{k=1}^{m}\left(\beta_k+s\right)^{n_k}\right\}\left\{1-\phi_\delta\sum_{i=1}^{m}\sum_{j=1}^{n_i}\xi_{\delta,ij}\left(\frac{\beta_i}{\beta_i+s}\right)^j\right\}}.\end{aligned} \tag{3.14}$$

Noting that the denominator in Eq. (3.14) is a polynomial of degree $n$ in $s$, we can conclude that the denominator has $n$ roots in the complex plane, say $-R_{1,\delta}, -R_{2,\delta}, ..., -R_{n,\delta}$. If we further assume that these roots are all distinct, it follows from a partial fraction expansion that Eq. (3.14) may be re-expressed as

$$\widetilde{m}_\delta(s) = \sum_{k=1}^{n}\frac{C_{k,\delta}}{s+R_{k,\delta}}, \tag{3.15}$$

where $\{C_{k,\delta}\}_{k=1,...,n}$ are constants to be determined. The Laplace transform inversion of Eq. (3.15) leads to

$$m_\delta(u) = \sum_{k=1}^{n} C_{k,\delta} \mathrm{e}^{-R_{k,\delta} u}, \quad u \geq 0, \tag{3.16}$$

In what follows, we further assume that $m_\delta(u) \neq 0$ for $u \geq 0$, then $C_{k,\delta} \neq 0$ for all $k = 1, 2, ..., n$ (see Landriault et al. [64] for more technical details).

We now derive an integral equation for the Gerber-Shiu function $m_\delta(u)$ by conditioning on the time and the size of the first jump. If the first jump is an upward jump (i.e., $B_1 = 1$), then the process $\{U_t\}_{t\geq 0}$ restarts anew at level $u + R^1_{T_1} + P_1$. On the other hand, if the first jump is a downward jump (i.e., $B_1 = 0$), then there are two possible scenarios:

1. If the first jump causes ruin (i.e., $u + R^1_{T_1} - Y_1 < 0$), then we have $\tau = T_1$ and $|U_\tau| = Y_1 - u - R^1_{T_1}$;
2. If ruin does not occur at the first jump (i.e., $u + R^1_{T_1} - Y_1 > 0$), then the process $\{U_t\}_{t\geq 0}$ restarts anew at level $u + R^1_{T_1} - Y_1$.

Therefore, we have

$$\begin{aligned} m_\delta(u) =& q_+ \int_0^\infty \int_0^\infty \int_0^\infty \mathrm{e}^{-\delta t} m_\delta(u+v+y) k(t) p(y) \mathbb{P}\left(R^1_{T_1} \in \mathrm{d}v | T_1 = t\right) \mathrm{d}y\mathrm{d}t \\ &+ q_- \int_0^\infty \int_0^\infty \mathrm{e}^{-\delta t} \left\{ \int_{u+v}^\infty w(y-u-v) f(y|t) k(t) \mathrm{d}y + \int_0^{u+v} m_\delta(u+v-y) f(y|t) k(t) \mathrm{d}y \right\} \\ &\times \mathbb{P}\left(R^1_{T_1} \in \mathrm{d}v | T_1 = t\right) \mathrm{d}t. \end{aligned} \tag{3.17}$$

We now state the main theorem of this chapter.

**Theorem 3.2.1** *For the generalized dependent Sparre Andersen risk model defined in (3.1) with the joint density $f(y|t)k(t)$ given by (3.4), the Gerber-Shiu function $m_\delta(u)$ defined in*

*(3.8) is given by (3.16). Suppose that $\beta_1, \beta_2, ..., \beta_m$ and $R_{1,\delta}, R_{2,\delta}, ..., R_{n,\delta}$ are all distinct, then $R_{k,\delta}$ satisfies the Lundberg's equation:*

$$\mathbb{E}\left[\mathrm{e}^{-\delta T_1 - R_{k,\delta} R^1_{T_1}} \left(q_+ \tilde{p}(R_{k,\delta}) + q_- \mathrm{e}^{R_{k,\delta} Y_1}\right)\right] = 1, \tag{3.18}$$

*for $k = 1, 2, ..., n$. Furthermore, as long as*

$$\int_0^\infty \int_0^\infty \mathrm{e}^{-\delta t - \beta_i v} g_{i n_i}(t) \mathbb{P}\left(R^1_{T_1} \in \mathrm{d}v | T_1 = t\right) \mathrm{d}t \neq 0, \quad \textit{for } i = 1, 2, ..., m, \tag{3.19}$$

*it follows that $C_{1,\delta}, C_{2,\delta}, ..., C_{n,\delta}$ satisfy the following system of linear equations:*

$$\sum_{k=1}^n C_{k,\delta} \left(\frac{\beta_i}{\beta_i - R_{k,\delta}}\right)^j = \mathbb{E}\left[w\left(E_{i,j}\right)\right], \tag{3.20}$$

*for $i = 1, 2, ..., m$ and $j = 1, 2, ..., n_i$.*

**Proof.** Substituting (3.16) into the first term of (3.17), we have

$$\begin{aligned}
&\int_0^\infty \int_0^\infty \int_0^\infty \mathrm{e}^{-\delta t} m_\delta(u + v + y) k(t) p(y) \mathbb{P}\left(R^1_{T_1} \in \mathrm{d}v | T_1 = t\right) \mathrm{d}y \mathrm{d}t \\
= &\sum_{k=1}^n C_{k,\delta} \mathrm{e}^{-R_{k,\delta} u} \int_0^\infty \int_0^\infty \int_0^\infty \mathrm{e}^{-\delta t} \mathrm{e}^{-R_{k,\delta}(v+y)} k(t) p(y) \mathbb{P}\left(R^1_{T_1} \in \mathrm{d}v | T_1 = t\right) \mathrm{d}y \mathrm{d}t \\
= &\sum_{k=1}^n C_{k,\delta} \mathrm{e}^{-R_{k,\delta} u} \tilde{p}(R_{k,\delta}) \mathbb{E}\left[\mathrm{e}^{-\delta T_1 - R_{k,\delta} R^1_{T_1}}\right].
\end{aligned} \tag{3.21}$$

Substituting (3.4) and (3.16) into the second term of (3.17) and applying a similar procedure as in the proof of Theorem 1 in Landriault et al. [64], we obtain

$$\begin{aligned}
&\int_0^\infty \int_0^\infty \mathrm{e}^{-\delta t} \left\{ \int_{u+v}^\infty w(y - u - v) f(y|t) k(t) \mathrm{d}y + \int_0^{u+v} m_\delta(u + v - y) f(y|t) k(t) \mathrm{d}y \right\} \\
&\times \mathbb{P}\left(R^1_{T_1} \in \mathrm{d}v | T_1 = t\right) \mathrm{d}t \\
= &\sum_{i=1}^m \sum_{k=1}^n C_{k,\delta} \mathrm{e}^{-R_{k,\delta} u} \sum_{j=1}^{n_i} \left(\frac{\beta_i}{\beta_i - R_{k,\delta}}\right)^j \int_0^\infty \int_0^\infty \mathrm{e}^{-\delta t - R_{k,\delta} v} g_{ij}(t) \mathbb{P}\left(R^1_{T_1} \in \mathrm{d}v | T_1 = t\right) \mathrm{d}t
\end{aligned}$$

$$+\sum_{i=1}^{m}\sum_{j=1}^{n_i} e_{\beta_{i,j}}(u)\left\{\sum_{h=j}^{n_i}\sum_{l=j}^{h} M^*_{i,h,l-j+1}(\delta)\left(\mathbb{E}\left[w\left(E_{i,h-l+1}\right)\right]-\sum_{k=1}^{n} C_{k,\delta}\left(\frac{\beta_i}{\beta_i-R_{k,\delta}}\right)^{h-l+1}\right)\right\}, \tag{3.22}$$

where

$$M^*_{i,h,l}(\delta)=\frac{1}{\beta_i^2}\int_0^\infty\int_0^\infty \mathrm{e}^{-\delta t} g_{ih}(t) e_{\beta_i,l}(v)\mathbb{P}\left(R^1_{T_1}\in \mathrm{d}v|T_1=t\right)\mathrm{d}t.$$

Substituting (3.21) and (3.22) into (3.17) leads to

$$\begin{aligned}
&\sum_{k=1}^{n} C_{k,\delta}\mathrm{e}^{-R_{k,\delta}u}\Bigg(1-q_+\tilde{p}(R_{k,\delta})\mathbb{E}\left[\mathrm{e}^{-\delta T_1-R_{k,\delta}R^1_{T_1}}\right]\\
&-q_-\sum_{i=1}^{m}\sum_{j=1}^{n_i}\left(\frac{\beta_i}{\beta_i-R_{k,\delta}}\right)^j\int_0^\infty\int_0^\infty \mathrm{e}^{-\delta t-R_{k,\delta}v} g_{ij}(t)\mathbb{P}\left(R^1_{T_1}\in \mathrm{d}v|T_1=t\right)\mathrm{d}t\Bigg)\\
&=q_-\sum_{i=1}^{m}\sum_{j=1}^{n_i} e_{\beta_{i,j}}(u)\left\{\sum_{h=j}^{n_i}\sum_{l=j}^{h} M^*_{i,h,l-j+1}(\delta)\left(\mathbb{E}\left[w\left(E_{i,h-l+1}\right)\right]-\sum_{k=1}^{n} C_{k,\delta}\left(\frac{\beta_i}{\beta_i-R_{k,\delta}}\right)^{h-l+1}\right)\right\}.
\end{aligned} \tag{3.23}$$

Since Eq. (3.23) is true for all $u\geq 0$, the coefficients of $\mathrm{e}^{-R_{k,\delta}u}$ (for $k=1,2,...,n$) and $e_{\beta_{i,j}}(u)$ (for $j=1,2,...,n_i$ and $i=1,2,...,m$) must all be zero. Then, equating the coefficient of $\mathrm{e}^{-R_{k,\delta}u}$ to zero yields

$$\begin{aligned}
&q_+\tilde{p}(R_{k,\delta})\mathbb{E}\left[\mathrm{e}^{-\delta T_1-R_{k,\delta}R^1_{T_1}}\right]\\
&+q_-\sum_{i=1}^{m}\sum_{j=1}^{n_i}\left(\frac{\beta_i}{\beta_i-R_{k,\delta}}\right)^j\int_0^\infty\int_0^\infty \mathrm{e}^{-\delta t-R_{k,\delta}v} g_{ij}(t)\mathbb{P}\left(R^1_{T_1}\in \mathrm{d}v|T_1=t\right)\mathrm{d}t=1,
\end{aligned} \tag{3.24}$$

for $k=1,2,...,n$. Using (3.4), (3.24) can be written as (3.18). Also, equating the coefficient of $e_{\beta_{i,j}}(u)$ to zero yields

$$\sum_{h=j}^{n_i}\sum_{l=j}^{h} M^*_{i,h,l-j+1}(\delta)\left(\mathbb{E}\left[w\left(E_{i,h-l+1}\right)\right]-\sum_{k=1}^{n} C_{k,\delta}\left(\frac{\beta_i}{\beta_i-R_{k,\delta}}\right)^{h-l+1}\right)=0, \tag{3.25}$$

for $j=1,2,...,n_i$ and $i=1,2,...,m$. Following the same argument as in Landriault et al. [64] (Eqs. (29)-(35)), (3.25) can be simplified to (3.20) under the assumption (3.19). ■

We note that the arguments in Section 3 of Landriault et al. [64] still hold for the risk model (3.1). As a result, we obtain

$$\begin{aligned}\overline{G}_\delta(u) &:= \mathbb{E}\left[\mathrm{e}^{-\delta\tau}\mathbb{1}(\tau<\infty) \mid U_0=u\right]\\ &= \sum_{k=1}^{n} C^*_{k,\delta}\mathrm{e}^{-R_{k,\delta}u},\end{aligned}$$

where

$$C^*_{k,\delta} = \left\{\prod_{i=1}^{m}\left(\frac{\beta_i - R_{k,\delta}}{\beta_i}\right)^{n_i}\right\}\left\{\prod_{j=1,j\neq k}^{n}\left(\frac{R_{j,\delta}}{R_{j,\delta}-R_{k,\delta}}\right)\right\}. \tag{3.26}$$

The ruin probability $\psi(u)=\overline{G}_0(u)$ is the special case of $\overline{G}_\delta(u)$ with $\delta=0$. Also, the marginal discounted density of the deficit at ruin is given by

$$h_{\delta,2}(y|u) = \frac{\phi_\delta}{1-\phi_\delta}\left\{(1-\phi_\delta)\, b_\delta(u+y) + \int_0^u b_\delta(t+y)g_\delta(u-t)dt\right\}, \quad y>0,$$

where $g_\delta(y) = -\frac{\mathrm{d}}{\mathrm{d}y}\overline{G}_\delta(y) = \sum_{k=1}^{n} C^*_{k,\delta}R_{k,\delta}\mathrm{e}^{-R_{k,\delta}y}$.

## 3.3 Applications

To learn more about the impact of various dependence structures and income processes on an insurer's solvency risk, we now consider several special cases of the insurance risk process (3.1). For each case, we focus on certain applications of Theorem 3.2.1.

**Pure Drift**

We first consider the case where $R_t = ct$ $(c > 0)$ for all $t > 0$. Under this assumption, (3.24) reduces to

$$q_+\tilde{p}(R_{k,\delta})\tilde{k}\left(\delta + cR_{k,\delta}\right) + q_-\sum_{i=1}^{m}\sum_{j=1}^{n_i}\left(\frac{\beta_i}{\beta_i - R_{k,\delta}}\right)^j \tilde{g}_{ij}\left(\delta + cR_{k,\delta}\right) = 1, \tag{3.27}$$

which recovers Eq. (24) of Landriault et al. [64] by letting $q_+ = 0$. By further imposing different assumptions on the joint density (3.4), we have the following observations:

1. If we further assume that $n_i = 1$ for all $i$, then the claim size $Y_i$ is distributed as a combination of exponentials with dependence on the inter-arrival time $T_i$. In this case, the coefficients $C_{1,\delta}, C_{2,\delta}, ..., C_{n,\delta}$ satisfy the linear system of equations

$$\sum_{k=1}^{n}\frac{\beta_i C_{k,\delta}}{\beta_i - R_{k,\delta}} = \mathbb{E}\left[w\left(E_{i,1}\right)\right], \tag{3.28}$$

   for $i = 1, 2, ..., n$. We remark that Eq. (3.28) is of Cauchy type and $C_{k,\delta}$ can be solved explicitly (see Section 4 of Landriault et al. [64] for more details).

2. If we further assume $g_{i,j}(t) = g(t)$ $(t > 0)$ for all $i$ and $j$, then it is immediate that $k(t) = ng(t)$ and the claim size $Y_i$ is independent of the inter-arrival time $T_i$. Moreover, the claim size $Y_i$ has the common Coxian-$n$ density given by

$$f(y) = \frac{1}{n}\sum_{i=1}^{m}\sum_{j=1}^{n_i} e_{\beta_i,j}(y), y > 0.$$

   Under this assumption, (3.18) can be simplified to

$$\tilde{k}\left(\delta + cR_{k,\delta}\right)\left(q_+\tilde{p}(R_{k,\delta}) + q_-\tilde{f}(-R_{k,\delta})\right) = 1. \tag{3.29}$$

We note that Theorem 3.2.1 recovers a special case of Theorem 4 of Cheung et al. [27] (when $n = m = 0$) by letting $n_i = 1$ and $g_{i,1}(t) = g(t)$ for all $i$ .

### Compound Poisson process

We now consider the case where $\{R_t\}_{t\geq 0}$ is a compound Poisson process with a drift of the form

$$R_t = ct + \sum_{i=1}^{M_t} Z_i, \quad t \geq 0, \tag{3.30}$$

where $c \geq 0$, $\{M_t\}_{t\geq 0}$ is an independent Poisson process with intensity $\mu > 0$ and the jump sizes $\{Z_i\}_{i\in\mathbb{N}_+}$ are iid positive rv's with cdf $F_Z$ and Laplace transform $\tilde{f}_Z$ (independent of $\{M_t\}_{t\geq 0}$). In this setup,

$$\mathbb{P}\left(R_t \in \mathrm{d}z\right) = \sum_{m=0}^{\infty} F_Z^{*m}(\mathrm{d}z - ct)\frac{(\mu t)^m \mathrm{e}^{-\mu t}}{m!},$$

for $t > 0, z \geq ct, n \geq 1$, where $F_Z^{*m}$ is the cdf of the $m$-fold convolution of the cdf $F$ (with the convention that $F_Y^{*0}(\mathrm{d}z) = \delta_0(\mathrm{d}z)$). Then, (3.24) can be written as

$$\int_0^{\infty} \mathrm{e}^{-\left(\mu+\delta+cR_{k,\delta}\right)t} \left(\sum_{h=0}^{\infty} \frac{\left(\mu t \tilde{f}_Z(R_{k,\delta})\right)^h}{h!}\right) \times \left(q_+\tilde{p}(R_{k,\delta})k(t) + q_- \sum_{i=1}^{m}\sum_{j=1}^{n_i}\left(\frac{\beta_i}{\beta_i - R_{k,\delta}}\right)^j g_{ij}(t)\right)\mathrm{d}t = 1. \tag{3.31}$$

If we further assume that $g_{i,j}(t) = \frac{\lambda \mathrm{e}^{-\lambda t}}{n}$ for all $i$ and $j$, then it is immediate that $\sum_{i=1}^{N_t} X_i$ is a compound Poisson process with two sided jumps. The resulting risk model is a special case of the risk model considered in Labbé and Sendova [63] as one can split the Poisson process $\{N_t\}_{t\geq 0}$ into two independent Poisson processes. Under this assumption, (3.31) can be simplified to

$$\lambda\left\{q_+\tilde{p}(R_{k,\delta}) + \frac{q_-}{n}\sum_{i=1}^{m}\sum_{j=1}^{n_i}\left(\frac{\beta_i}{\beta_i - R_{k,\delta}}\right)^j\right\} = \lambda + \mu + \delta + cR_{k,\delta} - \mu\tilde{f}_Z(R_{k,\delta}). \tag{3.32}$$

**Gamma process**

Let $R_t = ct + Z_t$, where $c \geq 0$ and $\{Z_t\}_{t\geq 0}$ is a gamma process with characteristic exponent $\Psi(\theta) = b\ln(1 - \frac{i\theta}{a})$ and Lévy triplet $(-\int_{(0,1)} x\Pi(\mathrm{d}x), 0, bx^{-1}\mathrm{e}^{-ax}\mathrm{d}x)$. In this case,

$$\mathbb{P}\left(R_t \in \mathrm{d}z\right) = \frac{a^{bt}(z-ct)^{bt-1}\mathrm{e}^{-a(z-ct)}}{\Gamma(bt)}\mathrm{d}z, \quad z \geq ct, t > 0.$$

By simple algebraic manipulations, it can be seen from (3.24) that $R_{k,\delta}$ are positive roots to the equation

$$\begin{aligned}&q_+\tilde{p}(R_{k,\delta})\tilde{k}\left(\delta + cR_{k,\delta} - b\ln\frac{a}{a+R_{k,\delta}}\right)\\&+q_-\sum_{i=1}^{m}\sum_{j=1}^{n_i}\left(\frac{\beta_i}{\beta_i - R_{k,\delta}}\right)^j\tilde{g}_{ij}\left(\delta + cR_{k,\delta} - b\ln\frac{a}{a+R_{k,\delta}}\right) = 1.\end{aligned}$$

**Inverse Gaussian**

Let $R_t = ct + G_t$, where $c \geq 0$, $\{G_t\}_{t\geq 0}$ is an inverse Gaussian process with characteristic exponent given by

$$\Psi(\theta) = \sqrt{\beta}\left(\sqrt{-2i\theta + \frac{\beta}{\mu^2}} - \frac{\sqrt{\beta}}{\mu}\right),$$

where $\beta, \mu > 0$, and its Lévy triplet is $(-2\mu\int_0^{\sqrt{\beta}/\mu}\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{y^2}{2}}\mathrm{d}y, 0, \frac{\sqrt{\beta}}{\sqrt{2\pi x^3}}\mathrm{e}^{-\frac{\beta x}{2\mu^2}}\mathrm{d}x)$. Moreover, $G_t$ is inverse Gaussian distributed with density

$$f_{IG}(x;\mu t, \beta t^2) := \sqrt{\frac{\beta t^2}{2\pi x^3}}\exp\left(-\frac{\beta(x-\mu t)^2}{2\mu^2 x}\right), \quad x > 0,$$

for fixed $t > 0$. It is immediate that $\mathbb{P}\left(R_t \in \mathrm{d}z\right) = f_{IG}(z - ct; \mu t, \beta t^2)$ for $z \geq ct$. Then, using (3.24) one can show that $R_{k,\delta}$ are positive roots to the equation

$$q_+\tilde{p}(R_{k,\delta})\tilde{k}\left(h(R_{k,\delta})\right) + q_-\sum_{i=1}^{m}\sum_{j=1}^{n_i}\left(\frac{\beta_i}{\beta_i - R_{k,\delta}}\right)^j\tilde{g}_{ij}\left(h(R_{k,\delta})\right) = 1,$$

where $h(s) := \delta + cs + \frac{\beta}{\mu}\left(\left(1 + \frac{2\mu^2 s}{\beta}\right)^{1/2} - 1\right)$.

### Varying premiums rates

For the income process $\{R_t\}_{t\geq 0}$ defined by (2.40), we have

$$\mathbb{P}\left(R_t^1 \in \mathrm{d}s | T_1 = t\right) = \mathbb{1}(t \leq \xi)\delta(z - c_1 t) + \mathbb{1}(t > \xi)\delta\left(z - c_1\xi - c_2(t-\xi)\right).$$

It follows from (3.24) that $R_{k,\delta}$ are positive roots to the equation

$$\begin{aligned} 1 =& q_+\tilde{p}(R_{k,\delta})\left\{\int_0^{\xi} \mathrm{e}^{-\left(\delta + c_1 R_{k,\delta}\right)t} k(t)\mathrm{d}t + \mathrm{e}^{-(c_1-c_2)R_{k,\delta}\xi}\int_{\xi}^{\infty} \mathrm{e}^{-\left(\delta + c_2 R_{k,\delta}\right)t} k(t)\mathrm{d}t\right\} \\ &+ q_-\sum_{i=1}^{m}\sum_{j=1}^{n_i}\left(\frac{\beta_i}{\beta_i - R_{k,\delta}}\right)^j \left\{\int_0^{\xi} \mathrm{e}^{-\left(\delta + c_1 R_{k,\delta}\right)t} g_{ij}(t)\mathrm{d}t + \mathrm{e}^{-(c_1-c_2)R_{k,\delta}\xi}\int_{\xi}^{\infty} \mathrm{e}^{-\left(\delta + c_2 R_{k,\delta}\right)t} g_{ij}(t)\mathrm{d}t\right\}. \end{aligned}$$

It is clear that the above equation recovers (3.27) when $c_1 = c_2$.

## 3.4 Numerical analysis

In this section, we examine the impact of the choice of income processes and dependence structures on some ruin quantities via some numerical examples. Unless otherwise stated, we assume each jump $X_i$ follows a two-sided distribution with $q_+ = 0.2, q_- = 0.8$ and $p(y) = \mathrm{e}^{-y}$ for $y \geq 0$.

**Example 1.** We first consider the case where $R_t = ct$ and the corresponding risk model is a renewal risk model with two-sided jumps. We assume that the premium rate is $c = 2$ and inter-arrival times are exponentially distributed with density $k(t) = \mathrm{e}^{-t}$ $(t > 0)$. We compare four risk models with different dependence structures between the inter-arrival time $T_i$ and the downward jump size $Y_i$:

- Model IND: $f_{\mathrm{IND}}(y|t)k(t) = \frac{1}{2}\mathrm{e}^{-t}\mathrm{e}^{-y} + \frac{1}{2}\mathrm{e}^{-t}\frac{1}{4}\mathrm{e}^{-\frac{1}{4}y},\ y > 0.$

- Model A: $f_{\mathrm{A}}(y|t)k(t) = \mathrm{e}^{-2t}\mathrm{e}^{-y} + (\mathrm{e}^{-t} - \mathrm{e}^{-2t})\frac{1}{4}\mathrm{e}^{-\frac{1}{4}y},\ y > 0.$

- Model B: $f_{\mathrm{B}}(y|t)k(t) = \mathrm{e}^{-2t}\frac{1}{4}\mathrm{e}^{-\frac{1}{4}y} + (\mathrm{e}^{-t} - \mathrm{e}^{-2t})\mathrm{e}^{-y},\ y > 0.$

- Model C: $f_{\mathrm{C}}(y|t)k(t) = \frac{1}{2}\mathrm{e}^{-2t}\mathrm{e}^{-y} + \frac{1}{2}(\mathrm{e}^{-t} - \mathrm{e}^{-2t})y\mathrm{e}^{-y} + \frac{1}{2}\mathrm{e}^{-2t}\frac{3}{7}\mathrm{e}^{-\frac{3}{7}y} + \frac{1}{2}(\mathrm{e}^{-t} - \mathrm{e}^{-2t})\left(\frac{3}{7}\right)^2 y\mathrm{e}^{-\frac{3}{7}y},\ y > 0.$

We note that the mean of the downward jump size is the same between the four models ($\mathbb{E}[Y_i] = 2.5$), and thus the loading factor is also the same. Moreover, the marginal distributions of $Y$ for the first three models are identical and given by $f(y) = \frac{1}{2}\mathrm{e}^{-y} + \frac{1}{8}\mathrm{e}^{-\frac{1}{4}y}$ $(y > 0)$, namely a mixture of two exponentials.

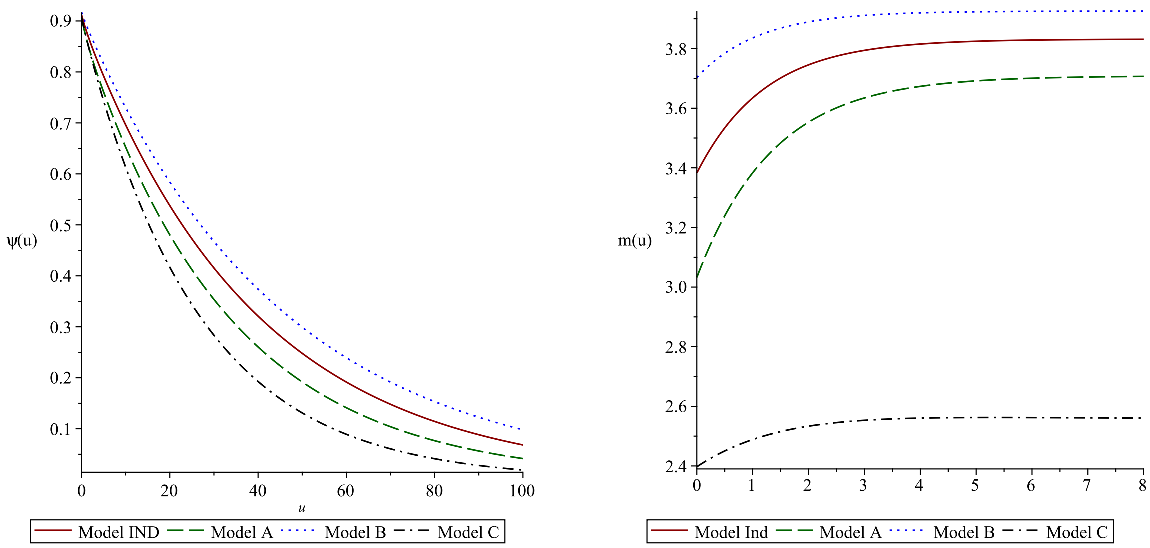
For notational convenience, we let $m(u) := \mathbb{E}_u\left[|U_\tau|\mathbb{1}_{\{\tau<\infty\}}\right]/\psi(u)$ $(u \geq 0)$, which is the conditional expectation of the deficit at ruin given that ruin occurs. By applying (3.28), (3.26) and (3.20), we obtain the conditional expected deficit at ruin for each model:

- $m_{\mathrm{IND}}(u) = (3.4512\mathrm{e}^{-0.0258u} - 0.3674\mathrm{e}^{0.8555u})\,/\,(0.9007\mathrm{e}^{-0.0258u} + 0.0109\mathrm{e}^{-0.8555u}).$

- $m_{\mathrm{A}}(u) = (3.2846\mathrm{e}^{-0.0306u} - 0.5395\mathrm{e}^{-0.7741u})\,/\,(0.8857\mathrm{e}^{-0.0306u} + 0.0195\mathrm{e}^{-0.7741u}).$

- $m_{\mathrm{B}}(u) = (3.5811\mathrm{e}^{-0.0223u} - 0.1856\mathrm{e}^{-0.9303u})\,/\,(0.9122\mathrm{e}^{-0.0223u} + 0.0047\mathrm{e}^{-0.9303u}).$

- $m_{\mathrm{C}}(u) = \frac{2.3054\mathrm{e}^{-0.0386u}+0.2030\mathrm{e}^{-0.5458u}-0.3918\mathrm{e}^{-0.7194u}+0.0577\mathrm{e}^{-1.1284u}}{0.9011\mathrm{e}^{-0.0386u}-0.0094\mathrm{e}^{-0.5458u}+0.0178\mathrm{e}^{-0.7194u}-0.0026\mathrm{e}^{-1.1284u}}.$




(a) (b)

Figure 3.1: (a) Ruin probability and (b) conditional expected deficit at ruin (Example 1).

Figure 3.1 plots the ruin probability $\psi(u)$ and the conditional expectation of $|U_\tau|$ against the initial surplus $u$ under different dependence structures between the inter-arrival times and the claim sizes. We observe that, for a given security loading factor, different dependence structures significantly impact an insurer's solvency risk. From Figure 3.1a, it is observed that $\psi_{\mathrm{A}}(u) < \psi_{\mathrm{IND}}(u) < \psi_{\mathrm{B}}(u)$ for relatively large $u$ (see Albrecher and Boxma [1] and Boudreault et al. [22] for similar observations in insurance risk models without upward jumps). Comparing Model A (B) to Model IND, we notice that ignoring the dependence structure may overestimate (underestimate) the solvency risk, depending on whether the claim sizes and inter-arrival times are positively associated or not. For Models A and C, a larger claim is likely to occur after a longer inter-arrival time due to the choice of parameters. However, we observe that $\psi_{\mathrm{C}}(u) < \psi_{\mathrm{A}}(u)$. This can be explained by the fact that the variance of the downward jump size $Y_i$ in Model A is larger (in comparison to Model

C) and $Y_i$ are more likely to take extreme values.

Figure 3.1b shows that the conditional expected deficit at ruin under different model settings follow the same ordering as the ruin probabilities. The conditional expected deficit at ruin are applied here in order to factor out the difference in the ruin probabilities from the unconditional expected deficit at ruin. Interestingly, it is observed that the conditional expected deficit under Model C is significantly lower than that under the other three models. We conclude that the dependence structures have a significant impact on the conditional expected deficit at ruin. Models with higher ruin probabilities are likely to have higher expected deficits at ruin.

In the next example, we keep everything the same as in Example 1 but consider different inter-arrival time assumptions.

**Example 2.** We consider two different inter-arrival time assumptions: an exponential distribution with $k_{\mathrm{A}}(t) = \mathrm{e}^{-t}$ $(t > 0)$ and an Erlang distribution with density $k_{\mathrm{B}}(t) = 4t\mathrm{e}^{-2t}$ $(t > 0)$. Both distributions have a common mean of 1 but the variance of the Erlang model is half of the one for the exponential model. Under each assumption on $\{T_i\}_{i\in\mathbb{N}_+}$, we further consider an independent risk model and a dependent risk model as follows:

- Model IND_A: $f_{\mathrm{IND}}(y|t)k_{\mathrm{A}}(t) = \frac{1}{2}\mathrm{e}^{-t}\mathrm{e}^{-y} + \frac{1}{2}\mathrm{e}^{-t}\frac{1}{4}\mathrm{e}^{-\frac{1}{4}y},\ y > 0.$

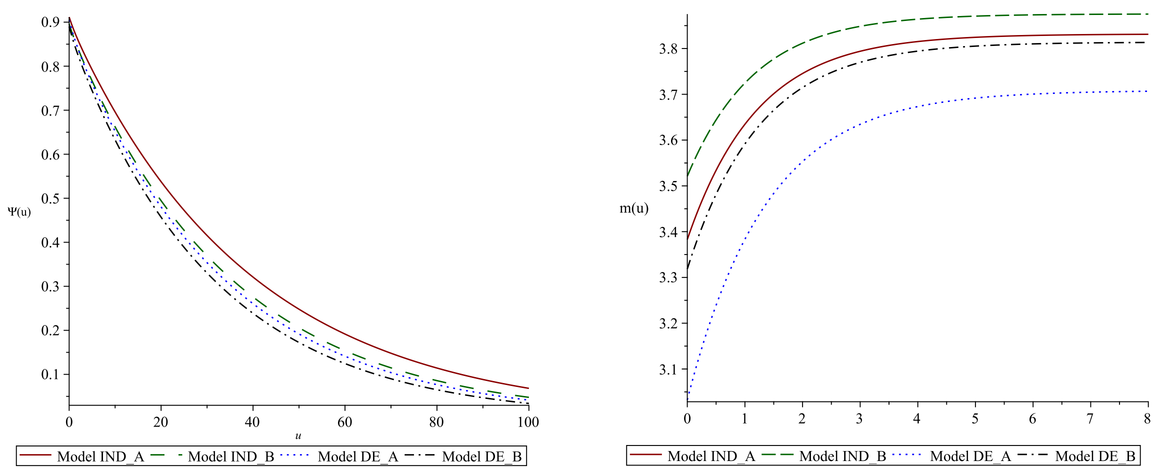
- Model IND_B: $f_{\mathrm{IND}}(y|t)k_{\mathrm{B}}(t) = 2t\mathrm{e}^{-2t}\mathrm{e}^{-y} + 2t\mathrm{e}^{-2t}\frac{1}{4}\mathrm{e}^{-\frac{1}{4}y},\ y > 0.$

- Model DE_A: $f_{\mathrm{DE}}(y|t)k_{\mathrm{A}}(t) = \mathrm{e}^{-2t}\mathrm{e}^{-y} + (\mathrm{e}^{-t} - \mathrm{e}^{-2t})\frac{1}{4}\mathrm{e}^{-\frac{1}{4}y},\ y > 0.$

- Model DE_B: $f_{\mathrm{DE}}(y|t)k_{\mathrm{B}}(t) = 4t\mathrm{e}^{-\sqrt{8}t}\mathrm{e}^{-y} + (4t\mathrm{e}^{-2t} - 4t\mathrm{e}^{-\sqrt{8}t})\frac{1}{4}\mathrm{e}^{-\frac{1}{4}y},\ y > 0.$

As a basis of comparison, the above four risk models have the identical marginal distribution for the rv $Y$ and the security loading factor is also the same. We note that Model IND_A and Model DE_A correspond to Model IND and Model A in Example 1. For Model IND_B and Model DE_B, we have the following results:

- $m_{\text{IND\_B}}(u) = (3.4361\text{e}^{-0.0292u} - 0.2805\text{e}^{-0.8895u}) \,/\, (0.8866\text{e}^{-0.0292u} + 0.0096\text{e}^{-0.8895u})$.
- $m_{\text{DE\_B}}(u) = (3.3384\text{e}^{-0.0325u} - 0.3843\text{e}^{-0.8432u}) \,/\, (0.8753\text{e}^{-0.0325u} + 0.0149\text{e}^{-0.8432u})$.




(a) (b)

Figure 3.2: (a) Ruin probability and (b) conditional expected deficit at ruin (Example 2).

Figure 3.2 presents the ruin probability and the conditional expected deficit at ruin for each risk model considered in this example. Model DE_A and Model DE_B have similar dependence structures but different inter-arrival times. It can be observed that $\psi_{\text{DE\_B}}(u) < \psi_{\text{IND\_B}}(u)$ and $m_{\text{DE\_B}}(u) < m_{\text{IND\_B}}(u)$ for $u > 0$, which is consistent with the results in Example 1.

Moreover, Figure 3.2a shows that $\psi_{\text{IND\_B}}(u) < \psi_{\text{IND\_A}}(u)$ (see Landriault and Willmot [74] for similar observations in an independent risk model without upward jumps) and

$\psi_{\text{DE\_B}}(u) < \psi_{\text{DE\_A}}(u)$ for $u > 0$. This can be explained by the fact that the exponentially distributed inter-arrival times are more likely to take on extreme values, in comparison to the Erlang distributed inter-arrival times with the same mean. However, it can be observed from Figure 3.2b that $m_{\text{IND\_B}}(u) > m_{\text{IND\_A}}(u)$ and $m_{\text{DE\_B}}(u) > m_{\text{DE\_A}}(u)$. The conditional expected deficit at ruin of risk processes with different inter-arrival times do not follow the same ordering as their ruin probabilities. Also, compared with risk models with Erlang inter-arrival times, risk models with exponential inter-arrival times are more sensitive to the change in the dependence assumption.

To investigate the impact of income processes and dependence structures on the ruin probability and the deficit at ruin, we consider the following example.

**Example 3.** We assume the inter-arrival times are Erlang distributed with density $k(t) = 4t\mathrm{e}^{-2t}$ $(t > 0)$ and the marginal density of $Y$ is $f_Y(y) = \frac{1}{2}\mathrm{e}^{-y} + \frac{1}{8}\mathrm{e}^{-\frac{1}{4}y}$ $(y > 0)$. We consider three subordinators discussed in Section 3.3 under the following parameter setting:

- Model CP_IND: $\{S_t\}_{t\geq 0}$ is a compound Poisson process where $\mu = 2$ and $f_Z(z) = \mathrm{e}^{-z}$ $(z > 0)$; $T_i$ is independent of $Y_i$.
- Model CP_DE: $\{S_t\}_{t\geq 0}$ is a compound Poisson process where $\mu = 2, f_Z(z) = \mathrm{e}^{-z}$ $(z > 0)$; $T_i$ depends on $Y_i$.
- Model Gam_DE: $\{S_t\}_{t\geq 0}$ is a Gamma process where $b = 2, a = 1$; $T_i$ depends on $Y_i$.
- Model IG_DE : $\{S_t\}_{t\geq 0}$ is an inverse Gaussian process where $\mu = 2, \beta = 2.5$; $T_i$ depends on $Y_i$.

For comparison purposes, $\mathbb{E}\left[S_{T_1}\right] = 2$ under each of the above four models and the joint density of $(T_i, Y_i)$ is $f(y|t)k(t) = 4t\mathrm{e}^{-\sqrt{8}t}\mathrm{e}^{-y} + (4t\mathrm{e}^{-2t} - 4t\mathrm{e}^{-\sqrt{8}t})\frac{1}{4}\mathrm{e}^{-\frac{1}{4}y}$ for the three dependent models. Using the results in Section 3.3, the conditional expected deficit at ruin of each model are given by:

- $m_{\text{CP\_IND}}(u) = (3.4433\mathrm{e}^{-0.0231u} - 0.4661\mathrm{e}^{-0.8154u}) \,/\, (0.9125\mathrm{e}^{-0.0231u} + 0.0122\mathrm{e}^{-0.8154u})$.
- $m_{\text{CP\_DE}}(u) = (3.2554\mathrm{e}^{-0.0252u} - 0.5603\mathrm{e}^{-0.7705u}) \,/\, (0.9063\mathrm{e}^{-0.0252u} + 0.0161\mathrm{e}^{-0.7705u})$.
- $m_{\text{Gam\_DE}}(u) = (3.3516\mathrm{e}^{-0.0284u} - 0.4838\mathrm{e}^{-0.8025u}) \,/\, (0.8928\mathrm{e}^{-0.0284u} + 0.0160\mathrm{e}^{-0.8025u})$.
- $m_{\text{IG\_DE}}(u) = (3.3701\mathrm{e}^{-0.0265u} - 0.5039\mathrm{e}^{-0.7953u}) \,/\, (0.9005\mathrm{e}^{-0.0265u} + 0.0154\mathrm{e}^{-0.7953u})$.

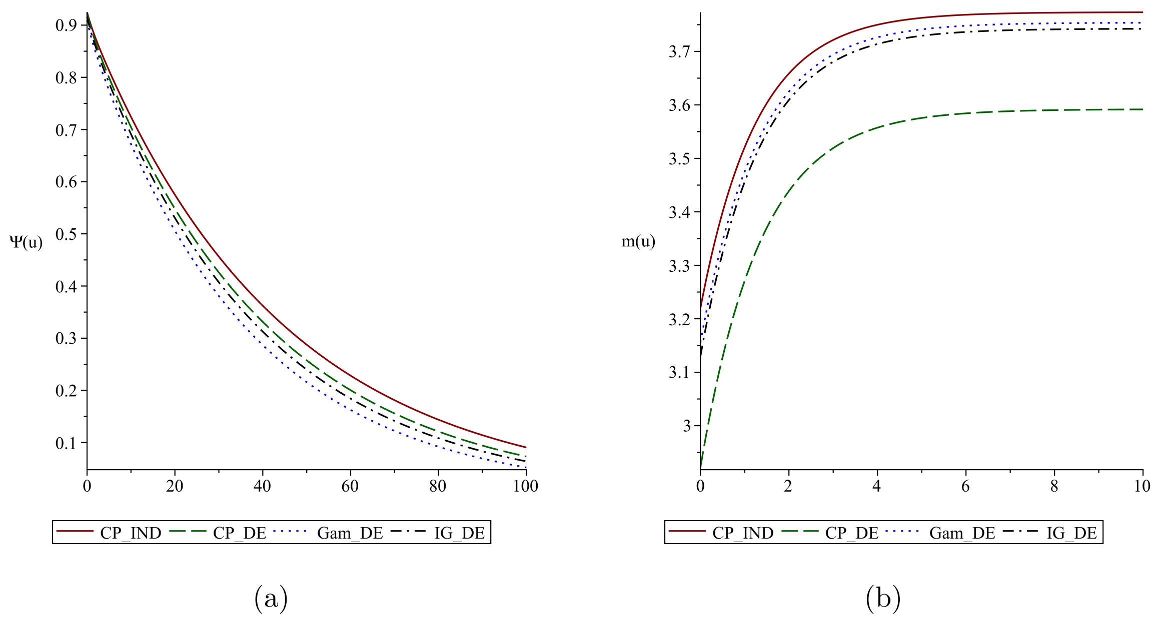



Figure 3.3: (a) Ruin probability and (b) conditional expected deficit at ruin (Example 3).

In Figure 3.3, we plot the ruin probabilities and the conditional expected deficit at ruin for different values of the initial surplus $u$. Comparing Models CP_IND and CP_DE, for risk processes with stochastic income processes, we have the same observation (as in the previous examples) that ignoring the dependence structure may overestimate the solvency risk. From Figure 3.3b, we observe that the conditional mean of the deficit at ruin of Model CP_DE is significantly lower than that of the other three models. Among these three dependent risk models with stochastic income processes, the model with the compound Poisson income process has the highest ruin probability but the lowest conditional expected deficit.

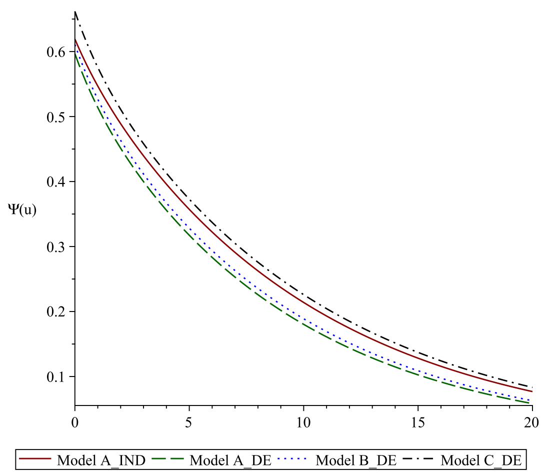
In the last example, we consider risk processes with varying premium rates studied in Section 3.3.

**Example 4.** We keep the same assumptions of the inter-arrival times, the downward jump sizes, and the joint density of $(T_i, Y_i)$ for dependent models as in Example 3. We further assume that $\xi = 1$. We consider the following risk models with different choices of premiums rates.

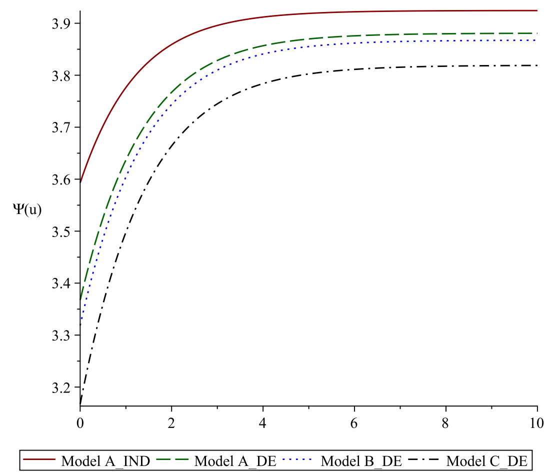
- Model A_IND: $c_1 = 3, c_2 = 2$, $T_i$ is independent on $Y_i$.
- Model A_DE: $c_1 = 3, c_2 = 2$, $T_i$ depends on $Y_i$.
- Model B_DE: $c_1 = c_2 = 2.7293$, $T_i$ depends on $Y_i$.
- Model C_DE: $c_1 = 2, c_2 = 4.6945$, $T_i$ depends on $Y_i$.

The conditional expected deficit at ruin of each model are given by

- $m_{\text{A\_IND}}(u) = (2.3374\mathrm{e}^{-0.1024u} - 0.1126\mathrm{e}^{-0.9298u}) \,/\, (0.5956\mathrm{e}^{-0.1024u} + 0.0236\mathrm{e}^{-0.9298u})\,.$
- $m_{\text{A\_DE}}(u) = (2.1622\mathrm{e}^{-0.1128u} - 0.1529\mathrm{e}^{-0.8936u}) \,/\, (0.5571\mathrm{e}^{-0.1128u} + 0.0396\mathrm{e}^{-0.8936u})\,.$
- $m_{\text{B\_DE}}(u) = (2.1990\mathrm{e}^{-0.1102u} - 0.1721\mathrm{e}^{-0.8831u}) \,/\, (0.5686\mathrm{e}^{-0.1102u} + 0.0422\mathrm{e}^{-0.8831u})\,.$
- $m_{\text{C\_DE}}(u) = (2.3412\mathrm{e}^{-0.0998u} - 0.2451\mathrm{e}^{-0.8469u}) \,/\, (0.6130\mathrm{e}^{-0.0998u} + 0.0488\mathrm{e}^{-0.8469u})\,.$




(a)




(b)

Figure 3.4: (a) Ruin probability and (b) conditional expected deficit at ruin (Example 4).

Comparing the three dependent risk models, we observe that the risk model with a preloaded premium strategy (Model A_DE) has the lowest ruin probability, which is consistent with the observations in Chapter 2. Comparing Model A_IND and Model C_DE, it can be observed from Figure 3.4 that the ruin probabilities are close but the conditional expected deficit under Model C_DE is much lower.

# Chapter 4

# A Refracted Lévy Process with Delayed Dividend Pullbacks

## 4.1 Introduction and description of the model

In this chapter, we propose and analyze a refracted Lévy risk model with delayed dividend pullbacks triggered by a certain Poissonian observation scheme. We recall that on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, the refracted Lévy process is a risk process $\tilde{U} = \{\tilde{U}_t\}_{t\geq 0}$ defined as

$$\tilde{U}_t = X_t - \delta \int_0^t \mathbb{1}_{\{\tilde{U}_s > b\}} \mathrm{d}s, \quad t \geq 0, \tag{4.1}$$

where $X = \{X_t\}_{t\geq 0}$ is a spectrally negative Lévy process (we exclude the case where $X$ has monotone paths), $\delta \geq 0$ is the refraction rate and $b > 0$ is the refraction level.

In the formulation (4.1), it is understood that a dividend rate of $\delta$ is paid whenever the insurer's surplus process $\tilde{U}$ exceeds level $b$, while no dividend is paid when the insurer's surplus lies below level $b$. However, in practice, it seems rather unlikely that an insurer will adopt a dividend strategy whose dividend pullbacks are as reactive as the one implied by the refracted Lévy process. Among other reasons, dividend pullbacks are generally regarded by the market as a signal of an entity's financial distress. More generally, investors are usually risk averse and tend to prefer stocks with steadier (i.e., less volatile) dividend payouts.

Hence, we propose a refracted Lévy process with delayed dividend pullbacks in this chapter. Heuristically speaking, this process will be such that dividend payouts will not immediately stop when the underlying risk process drops below the pre-determined dividend threshold $b > 0$. More specifically, dividend payouts will continue even if the insurer's surplus process drops below level $b$ as long as such excursions (below $b$) are considered "short" (relative to a given pre-specified *grace period*). If the surplus does not revisit the threshold level $b$ before the end of this *grace period*, dividend payouts stop and may only resume when the insurer's surplus creeps again above the threshold level $b$.

In light of recent contributions on Poissonian observations in the field of insurance mathematics (see, e.g., Landriault et al. [69] and Albrecher et al. [4]), we formally define the surplus process $U = \{U_t\}_{t\geq 0}$ of interest as

$$U_t = X_t - \delta \int_0^t \mathbb{1}_{\left\{U_s \geq b \text{ or } U_s < b, s - g_s < \mathrm{e}_\lambda^{g_s}, s \geq \kappa_b^+\right\}} \mathrm{d}s, \quad t \geq 0, \tag{4.2}$$

where

$$\kappa_a^{+(-)} = \inf\{t \geq 0 \colon U_t > (<) a\}, \quad a \in \mathbb{R},$$

and $g_t := \sup\{0 \leq s \leq t : U_s \geq b\}$, with the convention $\sup \emptyset = 0$. Also, let $G$ be the set of left-end points of excursions below $b$, and for each $g \in G$ we consider an iid copy of a generic independent (of $U$) exponential random variable $\mathrm{e}_\lambda$ with mean $1/\lambda > 0$. Note that $U_s < b$ implies that $g_s \in G$. Hence, $\mathrm{e}_\lambda^{g_t}$ is the length of the *grace period* associated to the excursion of $U$ below $b$ that started at time $g_t$. Figure 4.1 displays a sample path of $U$ to illustrate the dynamics of surplus process with delayed dividend pullbacks.

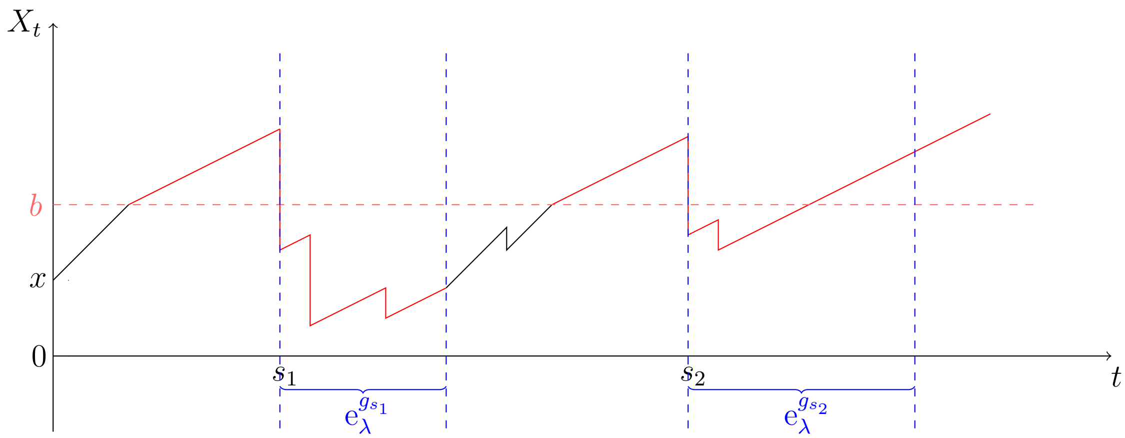



Figure 4.1: A sample path of the risk model $U$

For notational convenience, we define the binary dividend-paying process $Q = \{Q_t\}_{t\geq 0}$ as

$$Q_t = \begin{cases} 1, & \text{if } \{U_t \geq b\} \text{ or } \left\{U_t < b, t - g_t < \mathrm{e}_\lambda^{g_t}, \text{and } t \geq \kappa_b^+\right\}, \\ 0, & \text{otherwise,} \end{cases}$$

which allows to rewrite (4.2) as

$$U_t = X_t - \delta \int_0^t \mathbb{1}_{\{Q_s=1\}} \mathrm{d}s, \quad t \geq 0.$$

It can be shown that the two-dimensional process $\{(U_t, Q_t)\}_{t\geq 0}$ is a strong Markov process, which will be heavily relied upon in the analysis of the refracted Lévy process (with delays) $U$. In what follows, we shall denote by

$$\mathbb{P}_x(\cdot) := \begin{cases} \mathbb{P}\left(\cdot | U_0 = x, Q_0 = 0\right), & \text{if } x < b; \\ \mathbb{P}\left(\cdot | U_0 = x, Q_0 = 1\right), & \text{if } x \geq b. \end{cases}$$

Moreover, $\mathbb{E}_x$ will be the expectation operator associated to $\mathbb{P}_x$. For simplicity, we write $\mathbb{P}$ and $\mathbb{E}$ when $x = 0$. Also, note that there are obvious parallels to be drawn between the refracted Lévy process (with delays) $U$ and the work of Li et al. [76] on the non-refracted Lévy risk model with hybrid observation schemes.

Of particular interest for the refracted Lévy process (with delays) $U$ are the two-sided exit quantities

$$\mathbb{E}_x\left[\mathrm{e}^{-q\kappa_a^+}\mathbb{1}_{\left\{\kappa_a^+<\kappa_0^-\right\}}\right],$$

and

$$\mathbb{E}_x\left[\mathrm{e}^{-q\kappa_0^-}\mathbb{1}_{\left\{\kappa_0^-<\kappa_a^+\right\}}\right],$$

which will be derived in Theorem 4.3.1, and can be viewed as the counterparts of Theorem 4 in Kyprianou and Loeffen [58] for the refracted Lévy process $\tilde{U}$. Additionally, another quantity that has drawn much interest in the context of these refracted Lévy processes are the total discounted dividend payouts until ruin, whose expectation was derived by Kyprianou and Loeffen [58] for the refracted Lévy process $\tilde{U}$. Research on the optimality of dividend strategies is also very relevant in this context. We refer the reader to the work by e.g., Loeffen and Renaud [85], Czarna and Palmowski [34] and Renaud [91] for more details. It is generally assumed that maximizing the expected discounted dividend payouts until ruin is the optimality criterion. However, it is well known that investors have their own set of preferences, and as a result, their objectives may not necessarily be in line with maximizing the expected discounted dividend payouts until ruin. For instance, it is usually the case that investors are risk averse and reward steadier dividend payouts or alternatively, penalize for changes in dividend payouts. This could be the case as investors may have to rebalance their portfolios when a change in dividend payouts occurs (for instance, to maintain a given percentage of their portfolio in dividend-paying assets). Hence, in this chapter, we propose to incorporate transaction costs into the analysis of dividend payouts for the risk model $U$ by assuming that a lump-sum transaction fee of $\zeta$ is applied whenever the dividend payout rate changes. The expected (discounted) dividend payouts net of transaction costs is defined as

$$D(x) := \delta \mathbb{E}_x \left[ \int_0^{\kappa_0^-} \mathrm{e}^{-qt} \mathbb{1}_{\{Q_t=1\}} \mathrm{d}t \right] - \zeta \mathbb{E}_x \left[ \sum_{0 \le t < \kappa_0^-} \mathrm{e}^{-qt} \mathbb{1}_{\{\Delta Q_t \neq 0\}} \right], \tag{4.3}$$

where $\Delta Q_t := Q_t - Q_t^-$ and $q \geq 0$ can be interpreted as a force of interest. Note that $D(x)$ reduces to the expected discounted dividend payouts until ruin when $\zeta = 0$. We remark that the second term on the right-hand side of (4.3) is used to penalize for the volatility of the dividend policy. An explicit expression for $D(x)$ is given in Theorem 4.3.2. This is followed by a numerical study involving the quantity $D(x)$ in Section 4.4. More precisely, we identify a set of dividend strategies with identical ruin probability (for which the insurer is presumably indifferent), and pick the one maximizing $D(x)$ for an investor whose objective is consistent with this criterion. We observe that as the transaction cost $\zeta$ increases, a dividend strategy with less reactive dividend pullbacks (i.e., the dividend pullback rate $\lambda$ decreases) is preferred. We note that a similar exercise can be performed for other choices of $D(x)$ such as the one related to the traditional mean-variance criterion in the field of mathematical finance and insurance risk management, see e.g., [11, 18, 37]. For simplicity, we limit our analysis to $D(x)$ as defined in Eq. (4.3).

The rest of the chapter is organized as follows. In Section 4.2, we review some preliminary results for spectrally negative Lévy processes. Section 4.3 contains the main results of this chapter, while Section 4.4 presents a few numerical examples.

## 4.2 Preliminaries

Let $X = \{X_t\}_{t \geq 0}$ be a SNLP with the Laplace exponent $\psi : [0, \infty) \to \mathbb{R}$ given by

$$\psi(s) = \gamma s + \frac{1}{2}\sigma^2 s^2 + \int_0^\infty \left(\mathrm{e}^{-sz} - 1 + sz\mathbf{1}_{(0,1)}(z)\right) \Pi(\mathrm{d}z),$$

where $\gamma \in \mathbb{R}$, $\sigma \geq 0$, and Lévy measure $\Pi$ satisfies $\Pi(-\infty, 0) = 0$ and $\int_{(0,\infty)} (1 \wedge x^2)\, \Pi(\mathrm{d}x) < \infty$. Let $\Phi\colon [0,\infty) \to [0,\infty)$ be the right inverse of $\psi$. For any $q \geq 0$, the $q$-scale functions for $X$ are denoted as $W^{(q)}$ and $Z^{(q)}$, and the corresponding *second generation* scale functions defined in (1.9) and (1.10) are denoted by $\overline{W}_a^{(p,q)}(x)$ and $\overline{Z}_a^{(p,q)}(x)$, respectively. We write $W = W^{(0)}$ and $Z = Z^{(0)}$ when $q = 0$.

For convenience, we also introduce a second SNLP $Y = \{Y_t\}_{t\geq 0}$ (independent of $X$) with Laplace exponent $\psi(s) - \delta s$, whose right inverse is denoted by $\varphi(\cdot)$. It is well known that a SNLP must take the form of a strictly positive drift minus a pure jump subordinator if it has paths of bounded variation. To avoid the case where $Y$ has monotone path, we assume that $0 \leq \delta < \gamma + \int_0^1 z\Pi(\mathrm{d}z)$ if $X$ has paths of bounded variation throughout this chapter.

Let $\mathbb{W}^{(q)}$ (and $\mathbb{Z}^{(q)}$) be the counterpart of $W^{(q)}$ (and $Z^{(q)}$) for the SNLP $Y$. Similarly, we denote the counterparts of $\overline{W}_a^{(p,q)}(x)$ and $\overline{Z}_a^{(p,q)}(x)$ for the SNLP $Y$ by $\overline{\mathbb{W}}_a^{(p,q)}(x)$ and $\overline{\mathbb{Z}}_a^{(p,q)}(x)$, respectively. For completeness, expressions for $\overline{\mathbb{W}}_a^{(p,q)}(x)$ and $\overline{\mathbb{Z}}_a^{(p,q)}(x)$ are provided here. For $p, p+q \geq 0$ and $a, x \in \mathbb{R}$,

$$\overline{\mathbb{W}}_a^{(p,q)}(x) = \mathbb{W}^{(p+q)}(x) - q \int_0^a \mathbb{W}^{(p+q)}(x-y)\mathbb{W}^{(p)}(y)\mathrm{d}y, \tag{4.4}$$

and

$$\overline{\mathbb{Z}}_a^{(p,q)}(x) = \mathbb{Z}^{(p+q)}(x) - q \int_0^a \mathbb{W}^{(p+q)}(x-y)\mathbb{Z}^{(p)}(y)\mathrm{d}y. \tag{4.5}$$

For any $a \in \mathbb{R}$, we define the following stopping times

$$\tau_a^{+(-)} = \inf\{t \geq 0\colon X_t > (<) a\} \quad \text{and} \quad \nu_a^{+(-)} = \inf\{t \geq 0\colon Y_t > (<) a\},$$

with the convention that $\inf \emptyset = \infty$. The two-sided exit identities for $X$ are given in Lemma 1.3.3. For the SNLP $Y$, the same results also hold by substituting the scale functions of

$X$ by those of $Y$. For instance,

$$\mathbb{E}_x\left[\mathrm{e}^{-q\nu_a^+}\mathbb{1}_{\{\nu_a^+<\nu_0^-\}}\right]=\frac{\mathbb{W}^{(q)}(x)}{\mathbb{W}^{(q)}(a)}.$$

An extensive body of literature has recently emerged on "delayed" first passage times in which some grace period is given to the process before the first passage time is triggered/recognized. A common example is the so-called Parisian ruin time (below a critical level $b$) defined as

$$\nu_b^\lambda=\inf\{t\geq 0: Y_t<b, t-\tilde{g}_t>\mathrm{e}_\lambda^{\tilde{g}_t}\},$$

where $\tilde{g}_t=\sup\{s\leq t\colon Y_s\geq b\}$. In this context, each excursion below level $b$ of $Y$ is accompanied by an iid independent (of $Y$) exponentially distributed random variable $\mathrm{e}_\lambda^{\tilde{g}_t}$ with mean $1/\lambda>0$. We recall the following useful identities taken from Baurdoux et al. [13] and Landriault et al. [69]. For $q\geq 0, 0<b<a, x\in[0,a), y\in[0,b)$,

$$\mathbb{E}_x\left[\mathrm{e}^{-q\nu_b^\lambda}\mathbb{1}_{\left\{Y_{\nu_b^\lambda}\in\mathrm{d}y,\nu_b^\lambda<\nu_a^+\wedge\nu_0^-\right\}}\right]=\lambda\left(\frac{\overline{\mathbb{W}}_{x-b}^{(q,\lambda)}(x)}{\overline{\mathbb{W}}_{a-b}^{(q,\lambda)}(a)}\overline{\mathbb{W}}_{a-b}^{(q,\lambda)}(a-y)-\overline{\mathbb{W}}_{x-b}^{(q,\lambda)}(x-y)\right)\mathrm{d}y, \quad (4.6)$$

and

$$\mathbb{E}_x\left[\mathrm{e}^{-q\nu_a^+}\mathbb{1}_{\{\nu_a^+<\nu_b^\lambda\wedge\nu_0^-\}}\right]=\frac{\overline{\mathbb{W}}_{x-b}^{(q,\lambda)}(x)}{\overline{\mathbb{W}}_{a-b}^{(q,\lambda)}(a)}. \quad (4.7)$$

Moreover, using Theorem 1 in Loeffen et al. [86], one can show that for $0\leq b,x\leq a$,

$$\begin{aligned}
&\mathbb{E}_x[\mathrm{e}^{-q\nu_0^-}\mathbb{1}_{\{\nu_0^-<\nu_a^+\wedge\nu_b^\lambda\}}]\\
=&\mathbb{E}_x[\mathrm{e}^{-q\nu_0^-}\mathrm{e}^{-\lambda\int_0^{\nu_0^-}\mathbb{1}_{\{0<Y_t<b\}}\mathrm{d}t}\mathbb{1}_{\{\nu_0^-<\nu_a^+\}}]\\
=&\mathbb{Z}^{(q+\lambda)}(x)-\lambda\int_b^x\mathbb{W}^{(q)}(x-z)\mathbb{Z}^{(q+\lambda)}(z)\mathrm{d}z\\
-&\frac{\mathbb{Z}^{(q+\lambda)}(a)-\lambda\int_b^a\mathbb{W}^{(q)}(a-z)\mathbb{Z}^{(q+\lambda)}(z)\mathrm{d}z}{\mathbb{W}^{(q+\lambda)}(a)-\lambda\int_b^a\mathbb{W}^{(q)}(a-z)\mathbb{W}^{(q+\lambda)}(z)\mathrm{d}z}\left(\mathbb{W}^{(q+\lambda)}(x)-\lambda\int_b^x\mathbb{W}^{(q)}(x-z)\mathbb{W}^{(q+\lambda)}(z)\mathrm{d}z\right)\\
=&\overline{\mathbb{Z}}_b^{(q+\lambda,-\lambda)}(x)-\frac{\overline{\mathbb{W}}_{x-b}^{(q,\lambda)}(x)}{\overline{\mathbb{W}}_{a-b}^{(q,\lambda)}(a)}\overline{\mathbb{Z}}_b^{(q+\lambda,-\lambda)}(a). \quad (4.8)
\end{aligned}$$

### 4.2.1 Refracted Lévy processes

For the refracted Lévy process $\{\tilde{U}_t\}_{t\geq 0}$ defined in (4.1), we define its first passage times by

$$\tilde{\kappa}_a^{+(-)} = \inf\{t \geq 0\colon \tilde{U}_t > (<) a\}, a \in \mathbb{R}.$$

For $q \geq 0$ and $0 \leq x, b \leq a$,

$$\mathbb{E}_x\left[\mathrm{e}^{-q\tilde{\kappa}_a^+} \mathbb{1}_{\{\tilde{\kappa}_a^+ < \tilde{\kappa}_0^-\}}\right] = \frac{w_b^{(q)}(x)}{w_b^{(q)}(a)},$$

and

$$\mathbb{E}_x\left[\mathrm{e}^{-q\tilde{\kappa}_0^-} \mathbb{1}_{\{\tilde{\kappa}_0^- < \tilde{\kappa}_a^+\}}\right] = z_b^{(q)}(x) - \frac{w_b^{(q)}(x)}{w_b^{(q)}(a)} z_b^{(q)}(a),$$

where

$$w_b^{(q)}(x) = W^{(q)}(x) + \delta \mathbb{1}_{\{x \geq b\}} \int_b^x \mathbb{W}^{(q)}(x-y) W^{(q)\prime}(y)\mathrm{d}y, \tag{4.9}$$

and

$$z_b^{(q)}(x) = Z^{(q)}(x) + \delta q \mathbb{1}_{\{x \geq b\}} \int_b^x \mathbb{W}^{(q)}(x-y) W^{(q)}(y)\mathrm{d}y, \tag{4.10}$$

can be regarded as the scale functions of the refracted process $\tilde{U}$. For a refracted Lévy process, the probability of *classical* ruin is

$$\mathbb{P}_x\left(\tilde{\kappa}_0^- < \infty\right) = 1 - \left(\frac{\mathbb{E}[X_1] - \delta}{1 - \delta W(b)}\right) w_b^{(0)}(x),$$

if $0 < \delta < \mathbb{E}[X_1]$. A thorough derivation and discussion can be found in Kyprianou and Loeffen [58].

Also, for $p, q, x \geq 0$, we have the following useful identity taken from Renaud [90],

$$\delta \int_0^x \mathbb{W}^{(p)}(x-y) W^{(q)}(y)\mathrm{d}y + (p-q) \int_0^x \int_0^y \mathbb{W}^{(p)}(y-z) W^{(q)}(z)\mathrm{d}z\mathrm{d}y$$

$$= \int_0^x \mathbb{W}^{(p)}(y)\mathrm{d}y - \int_0^x W^{(q)}(y)\mathrm{d}y. \tag{4.11}$$

Differentiating (4.11) with respect to $x$ yields

$$\begin{aligned}&(q-p)\int_0^x \mathbb{W}^{(p)}(x-y)W^{(q)}(y)\mathrm{d}y\\ =&W^{(q)}(x) - \mathbb{W}^{(p)}(x) + \delta\int_{[0,x)} \mathbb{W}^{(p)}(x-y)W^{(q)}(\mathrm{d}y).\end{aligned} \tag{4.12}$$

Moreover, rearranging (4.11), one can show that

$$\int_0^x \mathbb{W}^{(p)}(x-y)\left(\delta W^{(q)}(y) - (q-p)\frac{Z^{(q)}(y)-1}{q}\right)\mathrm{d}y = \frac{\mathbb{Z}^{(p)}(x)-1}{p} - \frac{Z^{(q)}(x)-1}{q},$$

and thus

$$(q-p)\int_0^x \mathbb{W}^{(p)}(x-y)Z^{(q)}(y)\mathrm{d}y = \delta q\int_0^x \mathbb{W}^{(p)}(x-y)W^{(q)}(y)\mathrm{d}y - \mathbb{Z}^{(p)}(x) + Z^{(q)}(x). \tag{4.13}$$

## 4.3 Main results

In this section, we derive the main results of this chapter with regard to the two-sided exit problem for the refracted Lévy process (with delays) $U$. We later consider the expected (discounted) dividend payouts net of transaction costs defined in (4.3).

To better formulate the results, we define the following auxiliary functions: for $q, \lambda, x, b \geq 0$,

$$\xi_b^{(q,\lambda)}(x) = \overline{\mathbb{W}}_{x-b}^{(q,\lambda)}(x) - \left(W^{(q)}(x) + \lambda\int_0^b W^{(q)}(y)\overline{\mathbb{W}}_{x-b}^{(q,\lambda)}(x-y)\mathrm{d}y\right), \tag{4.14}$$

and

$$\alpha_b^{(q,\lambda)}(x) = \overline{\mathbb{Z}}_b^{(q+\lambda,-\lambda)}(x) - \left(Z^{(q)}(x) + \lambda\int_0^b Z^{(q)}(y)\overline{\mathbb{W}}_{x-b}^{(q,\lambda)}(x-y)\mathrm{d}y\right). \tag{4.15}$$

The next proposition provides alternative expressions for the above auxiliary functions.

**Proposition 4.3.1** *For $q, \lambda, x, b \geq 0$,*

$$\xi_b^{(q,\lambda)}(x) = \delta \int_{[0,b)} \left( \overline{\mathbb{W}}_{x-b}^{(q,\lambda)}(x-y) - \mathbb{W}^{(q)}(x-y) \right) W^{(q)}(\mathrm{d}y) + \mathbb{W}^{(q)}(x) - W^{(q)}(x), \quad (4.16)$$

*and*

$$\alpha_b^{(q,\lambda)}(x) = \delta q \int_0^x W^{(q)}(y) \overline{\mathbb{W}}_{x-b}^{(q,\lambda)}(x-y)\mathrm{d}y. \quad (4.17)$$

**Proof.** For $x < b$, (4.14) reduces to

$$\xi_b^{(q,\lambda)}(x) = \mathbb{W}^{(q+\lambda)}(x) - W^{(q)}(x) - \lambda \int_0^x W^{(q)}(y) \mathbb{W}^{(q+\lambda)}(x-y)\mathrm{d}y. \quad (4.18)$$

Using (4.12), it follows that (4.18) coincides with the right-hand side of (4.16).

For $x \geq b$, using (4.4), one deduces that

$$\begin{aligned}
\xi_b^{(q,\lambda)}(x) &= \overline{\mathbb{W}}_{x-b}^{(q,\lambda)}(x) - W^{(q)}(x) \\
&\quad - \lambda \int_0^b W^{(q)}(y) \left[ \mathbb{W}^{(q)}(x-y) + \lambda \int_{x-b}^{x-y} \mathbb{W}^{(q)}(z) \mathbb{W}^{(q+\lambda)}(x-y-z)\mathrm{d}z \right] \mathrm{d}y \\
&= \overline{\mathbb{W}}_{x-b}^{(q,\lambda)}(x) - W^{(q)}(x) - \lambda \int_{x-b}^x W^{(q)}(x-y) \mathbb{W}^{(q)}(y)\mathrm{d}y \\
&\quad - \lambda^2 \int_{x-b}^x \mathbb{W}^{(q)}(z) \left\{ \int_0^{x-z} W^{(q)}(y) \mathbb{W}^{(q+\lambda)}(x-y-z)\mathrm{d}y \right\} \mathrm{d}z.
\end{aligned} \quad (4.19)$$

Applying (4.12) again, the last term of (4.19) becomes

$$\begin{aligned}
&\lambda^2 \int_{x-b}^x \mathbb{W}^{(q)}(z) \left\{ \int_0^{x-z} W^{(q)}(y) \mathbb{W}^{(q+\lambda)}(x-y-z)\mathrm{d}y \right\} \mathrm{d}z \\
=& \lambda \int_{x-b}^x \mathbb{W}^{(q)}(z) \left[ - W^{(q)}(x-z) + \mathbb{W}^{(q+\lambda)}(x-z) - \delta \int_{[0,x-z)} \mathbb{W}^{(q+\lambda)}(x-z-y) W^{(q)}(\mathrm{d}y) \right] \mathrm{d}z.
\end{aligned} \quad (4.20)$$

Substituting (4.20) into (4.19) followed by simple algebraic manipulations completes the proof of (4.16).

We are left with the proof of Eq. (4.17). Using (4.13), one can show that (4.17) clearly holds for $x < b$. For $x \geq b$, by (4.4) and (4.13), it follows that

$$
\begin{aligned}
&\alpha_b^{(q,\lambda)}(x)\\
=&\overline{\mathbb{Z}}_b^{(q+\lambda,-\lambda)}(x) - Z^{(q)}(x) - \lambda \int_{x-b}^{x} Z^{(q)}(x-z)\mathbb{W}^{(q)}(z)\mathrm{d}z\\
&-\lambda^2 \int_{x-b}^{x}\int_0^{x-z} Z^{(q)}(y)\mathbb{W}^{(q)}(z)\mathbb{W}^{(q+\lambda)}(x-y-z)\mathrm{d}y\mathrm{d}z\\
=&\overline{\mathbb{Z}}_b^{(q+\lambda,-\lambda)}(x) - Z^{(q)}(x) - \lambda \int_{x-b}^{x} Z^{(q)}(x-z)\mathbb{W}^{(q)}(z)\mathrm{d}z\\
&-\lambda \int_{x-b}^{x} \mathbb{W}^{(q)}(z)\Big[ -\delta q \int_0^{x-z} \mathbb{W}^{(q+\lambda)}(x-y-z)W^{(q)}(y)\mathrm{d}y + \mathbb{Z}^{(q+\lambda)}(x-z) - Z^{(q)}(x-z)\Big]\mathrm{d}z\\
=&\mathbb{Z}^{(q)}(x) - Z^{(q)}(x) + \delta q\lambda \int_0^b \int_{x-b}^{x-y} \mathbb{W}^{(q)}(z)\mathbb{W}^{(q+\lambda)}(x-y-z)W^{(q)}(y)\mathrm{d}z\mathrm{d}y\\
=&\mathbb{Z}^{(q)}(x) - Z^{(q)}(x) + \delta q \int_0^b \left\{ \overline{\mathbb{W}}_{x-b}^{(q,\lambda)}(x-y) - \mathbb{W}^{(q)}(x-y)\right\} W^{(q)}(y)\mathrm{d}y. \qquad (4.21)
\end{aligned}
$$

Combining (4.21) and (4.13) completes the proof of (4.17). ■

In Theorem 4.3.1, we provide the two-sided exit results for the refracted Lévy process (with delays) $U$.

**Theorem 4.3.1** *For $q, \lambda \geq 0$ and $0 \leq x, b \leq a$,*

$$\mathbb{E}_x\left[\mathrm{e}^{-q\kappa_a^+}\mathbb{1}_{\{\kappa_a^+<\kappa_0^-\}}\right] = \frac{\mathcal{U}_b^{(q,\lambda)}(x)}{\mathcal{U}_b^{(q,\lambda)}(a)}, \qquad (4.22)$$

*and*

$$\mathbb{E}_x\left[\mathrm{e}^{-q\kappa_0^-}\mathbb{1}_{\{\kappa_0^-<\kappa_a^+\}}\right] = \mathcal{V}_b^{(q,\lambda)}(x) - \frac{\mathcal{U}_b^{(q,\lambda)}(x)}{\mathcal{U}_b^{(q,\lambda)}(a)}\mathcal{V}_b^{(q,\lambda)}(a), \qquad (4.23)$$

*where*

$$\mathcal{U}_b^{(q,\lambda)}(x) = W^{(q)}(x) + \mathbb{1}_{\{x\geq b\}}\left(\xi_b^{(q,\lambda)}(x) - \frac{\overline{\mathbb{W}}_{x-b}^{(q,\lambda)}(x)}{\mathbb{W}^{(q+\lambda)}(b)}\xi_b^{(q,\lambda)}(b)\right), \qquad (4.24)$$

*and*

$$\mathcal{V}_b^{(q,\lambda)}(x) = Z^{(q)}(x) + \mathbb{1}_{\{x\geq b\}}\left(\alpha_b^{(q,\lambda)}(x) - \frac{\overline{\mathbb{W}}_{x-b}^{(q,\lambda)}(x)}{\mathbb{W}^{(q+\lambda)}(b)}\alpha_b^{(q,\lambda)}(b)\right). \tag{4.25}$$

**Remark 4.3.1** *For* $\delta = 0$*, we have* $\mathbb{W}^{(q)}(x) = W^{(q)}(x)$ *and* $\overline{\mathbb{W}}_x^{(q,\lambda)}(y) = \overline{W}_x^{(q,\lambda)}(y)$*, which in turn implies from Eqs. (4.16) and (4.17) that* $\xi_b^{(q,\lambda)}(x) = \alpha_b^{(q,\lambda)}(x) = 0$*. Hence,* $\mathcal{U}_b^{(q,\lambda)}(x) = W^{(q)}(x)$ *and* $\mathcal{V}_b^{(q,\lambda)}(x) = Z^{(q)}(x)$ *for all* $x \geq 0$*, and as expected, (4.22) and (4.23) reduce to the classical two-sided exit results (1.15) and (1.16), respectively.*

**Remark 4.3.2** *We now compare the scale function* $w_b^{(q)}(x)$ *for the refracted Lévy process* $\{\tilde{U}_t\}_{t\geq 0}$ *to its counterpart* $\mathcal{U}_b^{(q,\lambda)}(x)$ *for the process* $U$ *by showing that* $\mathcal{U}_b^{(q,\lambda)}(x) \geq w_b^{(q)}(x)$ *for all* $x \geq 0$*. From Eq. (4.24) (together with Eqs. (4.16) and (4.12)) and the representation (4.9) for* $w_b^{(q)}(x)$*, it is not difficult to show that*

$$\mathcal{U}_b^{(q,\lambda)}(x) - w_b^{(q)}(x) = \delta\int_0^b W^{(q)\prime}(y)\left\{\overline{\mathbb{W}}_{x-b}^{(q,\lambda)}(x-y) - \frac{\overline{\mathbb{W}}_{x-b}^{(q,\lambda)}(x)}{\mathbb{W}^{(q+\lambda)}(b)}\mathbb{W}^{(q+\lambda)}(b-y)\right\}\mathrm{d}y, \tag{4.26}$$

*for* $x \geq b$*, and* $\mathcal{U}_b^{(q,\lambda)}(x) = w_b^{(q)}(x) = W^{(q)}(x)$ *for* $x \in [0, b)$*. Using (4.7), one observes that*

$$\frac{\mathbb{W}^{(q+\lambda)}(b-y)}{\overline{\mathbb{W}}_{x-b}^{(q,\lambda)}(x-y)} = \mathbb{E}\left[\mathrm{e}^{-q\nu_{x-b}^{+}}\mathbb{1}_{\left\{\nu_{x-b}^{+}<\nu_0^{\lambda}\wedge\nu_{-(b-y)}^{-}\right\}}\right],$$

*for all* $y \in [0, b)$*, which implies that* $\mathcal{U}_b^{(q,\lambda)}(x) \geq w_b^{(q)}(x)$ *for all* $x \geq 0$*. It immediately follows that*

$$\mathbb{E}_x\left[\mathrm{e}^{-q\kappa_a^{+}}\mathbb{1}_{\left\{\kappa_a^{+}<\kappa_0^{-}\right\}}\right] \leq \mathbb{E}_x\left[\mathrm{e}^{-q\tilde{\kappa}_a^{+}}\mathbb{1}_{\left\{\tilde{\kappa}_a^{+}<\tilde{\kappa}_0^{-}\right\}}\right],$$

*for* $x \leq b$*.*

**Proof.**

**1. Proof of (4.22)**

(i) For $0 \le x < b$, by the strong Markov property of $(U,Q)$ and the skip-free upward dynamic of $U$, it follows that

$$\begin{aligned}\mathbb{E}_x[\mathrm{e}^{-q\kappa_a^+}\mathbb{1}_{\{\kappa_a^+<\kappa_0^-\}}] =&\mathbb{E}_x\left[\mathrm{e}^{-q\kappa_b^+}\mathbb{1}_{\{\kappa_b^+<\kappa_0^-\}}\right]\mathbb{E}_b\left[\mathrm{e}^{-q\kappa_a^+}\mathbb{1}_{\{\kappa_a^+<\kappa_0^-\}}\right]\\ =&\mathbb{E}_x\left[\mathrm{e}^{-q\tau_b^+}\mathbb{1}_{\{\tau_b^+<\tau_0^-\}}\right]\mathbb{E}_b\left[\mathrm{e}^{-q\kappa_a^+}\mathbb{1}_{\{\kappa_a^+<\kappa_0^-\}}\right]\\ =&\frac{W^{(q)}(x)}{W^{(q)}(b)}\mathbb{E}_b\left[\mathrm{e}^{-q\kappa_a^+}\mathbb{1}_{\kappa_a^+<\kappa_0^-}\right],\end{aligned} \tag{4.27}$$

where the second equality follows from the fact that $\{X_t, t<\tau_b^+\}$ and $\{U_t, t<\kappa_b^+\}$ have the same distribution with respect to $\mathbb{P}_x$ when $x<b$.

(ii) Let

$$\kappa_b^\lambda = \inf\{t>0 : t-g_t > \mathrm{e}_\lambda^{g_t}\},\ b\in\mathbb{R}.$$

For $b \le x < a$, we first note that $\{Y_t, t<\nu_b^\lambda\}$ and $\{U_t, t<\kappa_b^\lambda\}$ have the same distribution with respect to $\mathbb{P}_x$. Once again, by the strong Markov property of $(U,Q)$ and the skip-free upward dynamic of $U$, we have

$$\begin{aligned}&\mathbb{E}_x\left[\mathrm{e}^{-q\kappa_a^+}\mathbb{1}_{\{\kappa_a^+<\kappa_0^-\}}\right]\\ =&\mathbb{E}_x\left[\mathrm{e}^{-q\kappa_a^+}\mathbb{1}_{\{\kappa_a^+<\kappa_0^-\wedge\kappa_b^\lambda\}}\right]+\mathbb{E}_x\left[\mathrm{e}^{-q\kappa_a^+}\mathbb{1}_{\{\kappa_b^\lambda<\kappa_a^+<\kappa_0^-\}}\right]\\ =&\mathbb{E}_x\left[\mathrm{e}^{-q\nu_a^+}\mathbb{1}_{\{\nu_a^+<\nu_0^-\wedge\nu_b^\lambda\}}\right]+\mathbb{E}_x\left[\mathrm{e}^{-q\nu_b^\lambda}\mathbb{E}_{Y_{\nu_b^\lambda}}\left[\mathrm{e}^{-q\kappa_a^+}\mathbb{1}_{\{\kappa_a^+<\kappa_0^-\}}\right]\mathbb{1}_{\{\nu_b^\lambda<\nu_0^-\wedge\nu_a^+\}}\right].\end{aligned} \tag{4.28}$$

Substituting (4.27) into (4.28) together with the help of Eqs. (4.6) and (4.7), it follows that

$$\mathbb{E}_x\left[\mathrm{e}^{-q\kappa_a^+}\mathbb{1}_{\{\kappa_a^+<\kappa_0^-\}}\right]$$

$$=\frac{\overline{\mathbb{W}}^{(q,\lambda)}_{x-b}(x)}{\overline{\mathbb{W}}^{(q,\lambda)}_{a-b}(a)}+\frac{\mathbb{E}_b\left[\mathrm{e}^{-q\kappa_a^+}\mathbb{1}_{\left\{\kappa_a^+<\kappa_0^-\right\}}\right]}{W^{(q)}(b)}\mathbb{E}_x\left[\mathrm{e}^{-q\nu_b^\lambda}W^{(q)}(Y_{\nu_b^\lambda})\mathbb{1}_{\left\{\nu_b^\lambda<\nu_0^-\wedge\nu_a^+\right\}}\right]$$

$$=\frac{\overline{\mathbb{W}}^{(q,\lambda)}_{x-b}(x)}{\overline{\mathbb{W}}^{(q,\lambda)}_{a-b}(a)}+\frac{\mathbb{E}_b\left[\mathrm{e}^{-q\kappa_a^+}\mathbb{1}_{\left\{\kappa_a^+<\kappa_0^-\right\}}\right]}{W^{(q)}(b)}\left(\xi_b^{(q,\lambda)}(x)+W^{(q)}(x)-\frac{\overline{\mathbb{W}}^{(q,\lambda)}_{x-b}(x)}{\overline{\mathbb{W}}^{(q,\lambda)}_{a-b}(a)}\left(\xi_b^{(q,\lambda)}(a)+W^{(q)}(a)\right)\right). \tag{4.29}$$

Given the expression of $\mathcal{U}_b^{(q,\lambda)}(x)$ as shown in Eq. (4.24), Eq. (4.29) can be rewritten as

$$\mathbb{E}_x\left[\mathrm{e}^{-q\kappa_a^+}\mathbb{1}_{\left\{\kappa_a^+<\kappa_0^-\right\}}\right]$$
$$=\frac{\overline{\mathbb{W}}^{(q,\lambda)}_{x-b}(x)}{\overline{\mathbb{W}}^{(q,\lambda)}_{a-b}(a)}+\frac{\mathbb{E}_b\left[\mathrm{e}^{-q\kappa_a^+}\mathbb{1}_{\left\{\kappa_a^+<\kappa_0^-\right\}}\right]}{W^{(q)}(b)\overline{\mathbb{W}}^{(q,\lambda)}_{a-b}(a)}\left(\overline{\mathbb{W}}^{(q,\lambda)}_{a-b}(a)\mathcal{U}_b^{(q,\lambda)}(x)-\overline{\mathbb{W}}^{(q,\lambda)}_{x-b}(x)\mathcal{U}_b^{(q,\lambda)}(a)\right), \tag{4.30}$$

for $b\le x<a$. In particular, by letting $x=b$ in Eq. (4.30), we obtain

$$\mathbb{E}_b\left[\mathrm{e}^{-q\kappa_a^+}\mathbb{1}_{\left\{\kappa_a^+<\kappa_0^-\right\}}\right]=\frac{W^{(q)}(b)}{\mathcal{U}_b^{(q,\lambda)}(a)}. \tag{4.31}$$

Substituting (4.31) into (4.27) and (4.30) completes the proof of (4.22).

**2. Proof of (4.23)**

We proceed similarly as for the proof of (4.22). Given the similarity, some details will be omitted here.

(i) For $0\le x<b\le a$, one can deduce that

$$\mathbb{E}_x\left[\mathrm{e}^{-q\kappa_0^-}\mathbb{1}_{\left\{\kappa_0^-<\kappa_a^+\right\}}\right]$$
$$=\mathbb{E}_x\left[\mathrm{e}^{-q\kappa_0^-}\mathbb{1}_{\left\{\kappa_0^-<\kappa_b^+<\kappa_a^+\right\}}\right]+\mathbb{E}_x\left[\mathrm{e}^{-q\kappa_0^-}\mathbb{1}_{\left\{\kappa_b^+<\kappa_0^-<\kappa_a^+\right\}}\right]$$
$$=\mathbb{E}_x\left[\mathrm{e}^{-q\tau_0^-}\mathbb{1}_{\left\{\tau_0^-<\tau_b^+\right\}}\right]+\mathbb{E}_x\left[\mathrm{e}^{-q\tau_b^+}\mathbb{1}_{\left\{\tau_b^+<\tau_0^-\right\}}\right]\mathbb{E}_b\left[\mathrm{e}^{-q\kappa_0^-}\mathbb{1}_{\left\{\kappa_0^-<\kappa_a^+\right\}}\right]$$
$$=Z^{(q)}(x)-\frac{Z^{(q)}(b)-\mathbb{E}_b\left[\mathrm{e}^{-q\kappa_0^-}\mathbb{1}_{\left\{\kappa_0^-<\kappa_a^+\right\}}\right]}{W^{(q)}(b)}W^{(q)}(x). \tag{4.32}$$

(ii) For $0 \le b \le x < a$, by conditioning on $\kappa_b^\lambda$ and making use of the fact that $\{Y_t, t < \nu_b^\lambda\}$ and $\{U_t, t < \kappa_b^\lambda\}$ have the same distribution with respect to $\mathbb{P}_x$, it follows that

$$\begin{aligned}
&\mathbb{E}_x\left[\mathrm{e}^{-q\kappa_0^-}\mathbb{1}_{\{\kappa_0^-<\kappa_a^+\}}\right] \\
=&\mathbb{E}_x\left[\mathrm{e}^{-q\kappa_0^-}\mathbb{1}_{\{\kappa_b^\lambda<\kappa_0^-<\kappa_a^+\}}\right]+\mathbb{E}_x\left[\mathrm{e}^{-q\kappa_0^-}\mathbb{1}_{\{\kappa_0^-<\kappa_a^+\wedge\kappa_b^\lambda\}}\right] \\
=&\mathbb{E}_x\left[\mathrm{e}^{-q\nu_b^\lambda}\mathbb{E}_{Y_{\kappa_b^\lambda}}\left[\mathrm{e}^{-q\kappa_0^-}\mathbb{1}_{\{\kappa_0^-<\kappa_a^+\}}\right]\mathbb{1}_{\{\nu_b^\lambda<\nu_0^-\wedge\nu_a^+\}}\right]+\mathbb{E}_x\left[\mathrm{e}^{-q\nu_0^-}\mathbb{1}_{\{\nu_0^-<\nu_a^+\wedge\nu_b^\lambda\}}\right].
\end{aligned} \tag{4.33}$$

Substituting (4.32), (4.6) and (4.8) into (4.33) yields

$$\begin{aligned}
&\mathbb{E}_x\left[\mathrm{e}^{-q\kappa_0^-}\mathbb{1}_{\{\kappa_0^-<\kappa_a^+\}}\right] \\
=&\frac{\overline{\mathbb{W}}_{x-b}^{(q,\lambda)}(x)}{\overline{\mathbb{W}}_{a-b}^{(q,\lambda)}(a)}\left(\lambda\int_0^b Z^{(q)}(y)\overline{\mathbb{W}}_{a-b}^{(q,\lambda)}(a-y)\mathrm{d}y\right)-\lambda\int_0^b Z^{(q)}(y)\overline{\mathbb{W}}_{x-b}^{(q,\lambda)}(x-y)\mathrm{d}y \\
&-\frac{Z^{(q)}(b)-\mathbb{E}_b\left[\mathrm{e}^{-q\kappa_0^-}\mathbb{1}_{\{\kappa_0^-<\kappa_a^+\}}\right]}{W^{(q)}(b)}\left(\mathcal{U}_b^{(q,\lambda)}(x)-\frac{\overline{\mathbb{W}}_{x-b}^{(q,\lambda)}(x)}{\overline{\mathbb{W}}_{a-b}^{(q,\lambda)}(a)}\mathcal{U}_b^{(q,\lambda)}(a)\right) \\
&+\overline{\mathbb{Z}}_b^{(q+\lambda,-\lambda)}(x)-\frac{\overline{\mathbb{W}}_{x-b}^{(q,\lambda)}(x)}{\overline{\mathbb{W}}_{a-b}^{(q,\lambda)}(a)}\overline{\mathbb{Z}}_b^{(q+\lambda,-\lambda)}(a).
\end{aligned} \tag{4.34}$$

In particular, for $x = b$, substituting (4.25) and (4.17) into (4.34) yields

$$\mathbb{E}_b\left[\mathrm{e}^{-q\kappa_0^-}\mathbb{1}_{\{\kappa_0^-<\kappa_a^+\}}\right]=Z^{(q)}(b)-\frac{W^{(q)}(b)}{\mathcal{U}_b^{(q,\lambda)}(a)}\mathcal{V}_b^{(q,\lambda)}(a). \tag{4.35}$$

Substituting (4.35) into (4.32) and (4.34) and together with (4.17) completes the proof (4.23). ■

**Corollary 4.3.1** *For $q, \lambda > 0$ and $x, b \ge 0$,*

$$\mathbb{E}_x\left[\mathrm{e}^{-q\kappa_0^-}\mathbb{1}_{\{\kappa_0^-<\infty\}}\right]=\mathcal{V}_b^{(q,\lambda)}(x)-\mathcal{L}_b^{(q,\lambda)}\mathcal{U}_b^{(q,\lambda)}(x), \tag{4.36}$$

*where*

$$\mathcal{L}_b^{(q,\lambda)} = \frac{\delta q \mathrm{e}^{\varphi(q)b}\int_b^\infty \mathrm{e}^{-\varphi(q)y} W^{(q)}(y)\mathrm{d}y + \frac{\delta q \mathbb{Z}^{(q+\lambda)}(b,\varphi(q))}{\lambda \mathbb{W}^{(q+\lambda)}(b)} W^{(q)}(b)}{\mathbb{Z}^{(q+\lambda)}(b,\varphi(q))\left(1-\frac{\xi_b^{(q,\lambda)}(b)}{\mathbb{W}^{(q+\lambda)}(b)}\right) - \lambda\int_0^b W^{(q)}(y)\mathbb{Z}^{(q+\lambda)}(b-y,\varphi(q))\mathrm{d}y} - \frac{\delta q}{\lambda}. \quad (4.37)$$

*In addition, under the condition* $0 \le \delta < \mathbb{E}(X_1)$*, letting* $q \to 0$ *one has the ruin probability*

$$\mathbb{P}_x\left(\kappa_0^- < \infty\right) = 1 - \mathcal{L}_b^{(0,\lambda)}\mathcal{U}_b^{(0,\lambda)}(x), \quad (4.38)$$

*where*

$$\mathcal{L}_b^{(0,\lambda)} = \frac{\psi'(0+)-\delta}{1+\delta\left(\mathbb{W}^{(\lambda)}(b) - W(b)\right) - \left(\delta + \frac{\mathbb{Z}^{(\lambda)}(b)}{\mathbb{W}^{(\lambda)}(b)}\right)\xi_b^{0,\lambda}(b)}. \quad (4.39)$$

**Remark 4.3.3** *It can be seen from (4.24) that* $\mathcal{U}_b^{(0,0)}(x) = \frac{W(b)\mathbb{W}(x)}{\mathbb{W}(b)}$ *for* $x \ge b$ *and* $\mathcal{U}_b^{(0,0)}(x) = W(x)$ *for* $x < b$*. As a result of Corollary 4.3.1, for* $\lambda = 0$*,*

$$\mathbb{P}_x\left(\kappa_0^- < \infty\right) = \begin{cases} 1 - \left(\psi'(0+) - \delta\right)\mathbb{W}(x), & x \ge b, \\ 1 - \frac{\psi'(0+)-\delta}{W(b)} W(x)\mathbb{W}(b), & x < b. \end{cases}$$

*Note that the ruin probability can be written as* $1 - \mathbb{P}_x(\tau_b^+ < \tau_0^-)\mathbb{P}_b(\nu_0^- = \infty)$ *for* $x < b$ *by using the classical exit identities (1.15) and (1.14).*

**Proof.** To prove Eq. (4.36), one shall identify

$$\lim_{a\to\infty}\frac{\mathcal{V}_b^{(q,\lambda)}(a)}{\mathcal{U}_b^{(q,\lambda)}(a)}, \quad (4.40)$$

which we will do by looking at the asymptotic behaviour of $\mathcal{U}_b^{(q,\lambda)}(a)$ and $\mathcal{V}_b^{(q,\lambda)}(a)$ in comparison to $\mathbb{W}^{(q)}(a)$.

By the dominated convergence theorem, we first point out that

$$\lim_{a\to\infty}\frac{\overline{\mathbb{W}}_{a-b}^{(q,\lambda)}(a)}{\mathbb{W}^{(q)}(a)} = 1 + \lambda\int_0^b \mathrm{e}^{-\varphi(q)y}\mathbb{W}^{(q+\lambda)}(y)\mathrm{d}y = \mathrm{e}^{-\varphi(q)b}\mathbb{Z}^{(q+\lambda)}\left(b,\varphi(q)\right). \quad (4.41)$$

It follows directly from (4.41) that

$$\lim_{a\to\infty}\frac{\alpha_b^{(q,\lambda)}(a)}{\mathbb{W}^{(q)}(a)}=\delta q\mathrm{e}^{-\varphi(q)b}\int_0^\infty W^{(q)}(y)\mathbb{Z}^{(q+\lambda)}\left(b-y,\varphi(q)\right)\mathrm{d}y,$$

and

$$\lim_{a\to\infty}\frac{\xi_b^{(q,\lambda)}(a)}{\mathbb{W}^{(q)}(a)}=\mathrm{e}^{-\varphi(q)b}\mathbb{Z}^{(q+\lambda)}\left(b,\varphi(q)\right)-\lambda\mathrm{e}^{-\varphi(q)b}\int_0^b W^{(q)}(y)\mathbb{Z}^{(q+\lambda)}\left(b-y,\varphi(q)\right)\mathrm{d}y.$$

Therefore,

$$\lim_{a\to\infty}\frac{\mathcal{V}_b^{(q,\lambda)}(a)}{\mathbb{W}^{(q)}(a)}=\delta q\mathrm{e}^{-\varphi(q)b}\int_0^\infty W^{(q)}(y)\mathbb{Z}^{(q+\lambda)}\left(b-y,\varphi(q)\right)\mathrm{d}y-\frac{\mathrm{e}^{-\varphi(q)b}\mathbb{Z}^{(q+\lambda)}\left(b,\varphi(q)\right)}{\mathbb{W}^{(q+\lambda)}(b)}\alpha_b^{(q,\lambda)}(b),\tag{4.42}$$

and

$$\begin{aligned}&\lim_{a\to\infty}\frac{\mathcal{U}_b^{(q,\lambda)}(a)}{\mathbb{W}^{(q)}(a)}\\ =&\mathrm{e}^{-\varphi(q)b}\mathbb{Z}^{(q+\lambda)}\left(b,\varphi(q)\right)\left(1-\frac{\xi_b^{(q,\lambda)}(b)}{\mathbb{W}^{(q+\lambda)}(b)}\right)-\lambda\mathrm{e}^{-\varphi(q)b}\int_0^b W^{(q)}(y)\mathbb{Z}^{(q+\lambda)}\left(b-y,\varphi(q)\right)\mathrm{d}y.\end{aligned}\tag{4.43}$$

By simple algebraic manipulations, one can show that the limit (4.40) is given by Eq. (4.37).

Moreover, taking the limit as $q\to 0$ in (4.37) and noting that $\varphi(0)=0$ and $\lim_{q\to 0}\frac{q}{\varphi(q)}=\psi'(0+)-\delta>0$ under the security loading condition, it follows that

$$\begin{aligned}\lim_{q\to 0}\lim_{a\to\infty}\frac{\mathcal{V}_b^{(q,\lambda)}(a)}{\mathcal{U}_b^{(q,\lambda)}(a)}&=\lim_{q\to 0}\frac{\delta q\mathrm{e}^{\varphi(q)b}\int_b^\infty \mathrm{e}^{-\varphi(q)y}W^{(q)}(y)\mathrm{d}y}{\mathbb{Z}^{(\lambda)}(b)\left(1-\frac{\xi_b^{(0,\lambda)}(b)}{\mathbb{W}^{(\lambda)}(b)}\right)-\lambda\int_0^b W(y)\mathbb{Z}^{(\lambda)}(b-y)\mathrm{d}y}\\ &=\frac{\psi'(0+)-\delta}{\mathbb{Z}^{(\lambda)}(b)\left(1-\frac{\xi_b^{(0,\lambda)}(b)}{\mathbb{W}^{(\lambda)}(b)}\right)-\lambda\int_0^b W(y)\mathbb{Z}^{(\lambda)}(b-y)\mathrm{d}y}.\end{aligned}\tag{4.44}$$

Note that

$$\mathbb{Z}^{(\lambda)}(b) = 1 + \lambda \int_0^b W(y) \left(\delta \mathbb{W}^{(\lambda)}(b-y) + \mathbb{Z}^{(\lambda)}(b-y)\right) \mathrm{d}y,$$

which can be proved by showing that the Laplace transforms on both sides are equal. Thus, we have

$$\lambda \int_0^b W(y) \mathbb{Z}^{(\lambda)}(b-y) \mathrm{d}y = \mathbb{Z}^{(\lambda)}(b) - 1 + \delta \left(W(b) - \mathbb{W}^{(\lambda)}(b)\right) + \delta \xi_b^{(0,\lambda)}(b). \tag{4.45}$$

Substituting (4.45) into (4.44) and noting that $\mathcal{V}_b^{(0,\lambda)}(x) = 1$ completes the proof of (4.38). ■

Now, we turn our attention to the expected discounted dividends net of transaction costs paid until ruin.

**Theorem 4.3.2** *For $q > 0$ and $\lambda, b, x \geq 0$,*

$$D(x) = \frac{\mathcal{B}_b^{(q,\lambda)}(b) - \zeta}{W^{(q)}(b) - \mathcal{A}_b^{(q,\lambda)}(b)} \mathcal{A}_b^{(q,\lambda)}(x) + \mathbb{1}_{\{x \geq b\}} \mathcal{B}_b^{(q,\lambda)}(x), \tag{4.46}$$

*where*

$$\mathcal{A}_b^{(q,\lambda)}(x) = \begin{cases} W^{(q)}(x), & x < b, \\ \lambda \int_0^b W^{(q)}(y) \left( \frac{\overline{\mathbb{W}}_{x-b}^{(q,\lambda)}(x) \mathbb{Z}^{(q+\lambda)}(b-y, \varphi(q))}{\mathbb{Z}^{(q+\lambda)}(b, \varphi(q))} - \overline{\mathbb{W}}_{x-b}^{(q,\lambda)}(x-y) \right) \mathrm{d}y, & x \geq b, \end{cases}$$

*and*

$$\begin{aligned} \mathcal{B}_b^{(q,\lambda)}(x) =& \frac{\delta}{q} + (\zeta\lambda - \delta) \left\{ \frac{\overline{\mathbb{Z}}_b^{(q+\lambda,-\lambda)}(x)}{q+\lambda} - \frac{\overline{\mathbb{W}}_{x-b}^{(q,\lambda)}(x)}{\varphi(q)} + \frac{\lambda \overline{\mathbb{W}}_{x-b}^{(q,\lambda)}(x) \mathbb{Z}^{(q+\lambda)}(b)}{\varphi(q) \mathbb{Z}^{(q+\lambda)}(b, \varphi(q))(q+\lambda)} \right\} \\ &+ \frac{\lambda \left(\zeta + \frac{\delta}{q}\right)}{q+\lambda} \left( \frac{q \overline{\mathbb{W}}_{x-b}^{(q,\lambda)}(x)}{\varphi(q) \mathbb{Z}^{(q+\lambda)}(b, \varphi(q))} - \mathbb{Z}^{(q)}(x-b) \right). \end{aligned}$$

**Proof.** (1) For $0 \le x < b$, since no dividends is payable until the process reaches level $b$, it follows that

$$\begin{aligned} D(x) =& \mathbb{E}_x \left[ \mathrm{e}^{-q\kappa_b^+} \mathbb{1}_{\left\{\kappa_b^+ < \kappa_0^-\right\}} \right] (D(b) - \zeta) \\ =& \frac{W^{(q)}(x)}{W^{(q)}(b)} (D(b) - \zeta) . \end{aligned} \tag{4.47}$$

(2) For $x \ge b$, dividends are continuously paid until $(\kappa_0^- \wedge \kappa_b^\lambda)$. If the stopping time $\kappa_b^\lambda$ occurs first, the dynamics of $U$ change from the one of process $Y$ to process $X$ and a transaction cost $\zeta$ is incurred. By the strong Markov property of $(U, Q)$ and Eq. (4.47), it follows that

$$\begin{aligned} D(x) =& \delta \mathbb{E}_x \left[ \int_0^{\kappa_0^- \wedge \kappa_b^\lambda} \mathrm{e}^{-qt} \mathrm{d}t \right] - \zeta \mathbb{E}_x \left[ \mathrm{e}^{-q\kappa_b^\lambda} \mathbb{1}_{\left\{\kappa_b^\lambda < \kappa_0^-\right\}} \right] + \mathbb{E}_x \left[ \mathrm{e}^{-q\kappa_b^\lambda} D(U_{\kappa_b^\lambda}) \mathbb{1}_{\left\{\kappa_b^\lambda < \kappa_0^-\right\}} \right] \\ =& \frac{\delta}{q} \left( 1 - \mathbb{E}_x \left[ \mathrm{e}^{-q\nu_0^-} \mathbb{1}_{\left\{\nu_0^- < \nu_b^\lambda\right\}} \right] \right) - \left( \zeta + \frac{\delta}{q} \right) \mathbb{E}_x \left[ \mathrm{e}^{-q\nu_b^\lambda} \mathbb{1}_{\left\{\nu_b^\lambda < \nu_0^-\right\}} \right] \\ &+ \mathbb{E}_x \left[ \mathrm{e}^{-q\nu_b^\lambda} W^{(q)} \left( Y_{\nu_b^\lambda} \right) \mathbb{1}_{\left\{\nu_b^\lambda < \nu_0^-\right\}} \right] \frac{D(b) - \zeta}{W^{(q)}(b)} . \end{aligned} \tag{4.48}$$

Taking limits as $a \to \infty$ in (4.8) and using Eq. (42) in Loeffen et al. [86], it follows that

$$\begin{aligned} \mathbb{E}_x \left[ \mathrm{e}^{-q\nu_0^-} \mathbb{1}_{\left\{\nu_0^- < \nu_b^\lambda\right\}} \right] &= \overline{\mathbb{Z}}_b^{(q+\lambda,-\lambda)}(x) - \overline{\mathbb{W}}_{x-b}^{(q,\lambda)}(x) \frac{\frac{q}{\varphi(q)} + \lambda \int_0^b \mathrm{e}^{-\varphi(q)y} \mathbb{Z}^{(q+\lambda)}(y) \mathrm{d}y}{1 + \lambda \int_0^b \mathrm{e}^{-\varphi(q)y} \mathbb{W}^{(q+\lambda)}(y) \mathrm{d}y} \\ &= \overline{\mathbb{Z}}_b^{(q+\lambda,-\lambda)}(x) - \overline{\mathbb{W}}_{x-b}^{(q,\lambda)}(x) \left( \frac{q+\lambda}{\varphi(q)} - \frac{\lambda \mathbb{Z}^{(q+\lambda)}(b)}{\varphi(q) \mathbb{Z}^{(q+\lambda)}(b, \varphi(q))} \right) . \end{aligned} \tag{4.49}$$

By Corollary 1.1 in Baurdoux et al. [13], we obtain

$$\begin{aligned} & \mathbb{E}_x \left[ \mathrm{e}^{-q\nu_b^\lambda} \mathbb{1}_{\left\{\nu_b^\lambda < \nu_0^-\right\}} \right] \\ =& \lambda \int_0^b \left\{ \frac{\overline{\mathbb{W}}_{x-b}^{(q,\lambda)}(x) \mathbb{Z}^{(q+\lambda)}(y, \varphi(q))}{\mathbb{Z}^{(q+\lambda)}(b, \varphi(q))} - \overline{\mathbb{W}}_{x-b}^{(q,\lambda)}(x - b + y) \right\} \mathrm{d}y . \end{aligned} \tag{4.50}$$

Using (1.7) and (4.4), one can show that

$$\int_0^b \mathbb{Z}^{(q+\lambda)}(y,\varphi(q))\mathrm{d}y = \frac{\mathbb{Z}^{(q+\lambda)}(b,\varphi(q)) - \frac{\lambda}{q+\lambda}\mathbb{Z}^{(q+\lambda)}(b) - \frac{q}{q+\lambda}}{\varphi(q)}, \tag{4.51}$$

and

$$\begin{aligned}\int_0^b \overline{\mathbb{W}}_{x-b}^{(q,\lambda)}(x-b+y)\mathrm{d}y =& \frac{\mathbb{Z}^{(q+\lambda)}(x) - \mathbb{Z}^{(q)}(x-b) - \lambda\int_0^{x-b}\mathbb{W}^{(q)}(z)\mathbb{Z}^{(q+\lambda)}(x-z)\mathrm{d}z}{q+\lambda} \\ =& \frac{\overline{\mathbb{Z}}_b^{(q+\lambda,-\lambda)}(x) - \mathbb{Z}^{(q)}(x-b)}{q+\lambda}.\end{aligned} \tag{4.52}$$

Substituting (4.51) and (4.52) into (4.50) yields

$$\begin{aligned}&\mathbb{E}_x\left[\mathrm{e}^{-q\nu_b^\lambda}\mathbb{1}_{\{\nu_b^\lambda<\nu_0^-\}}\right] \\ =&\lambda\left(\frac{\overline{\mathbb{W}}_{x-b}^{(q,\lambda)}(x)}{\varphi(q)}\left(1-\frac{\lambda\mathbb{Z}^{(q+\lambda)}(b)+q}{(q+\lambda)\mathbb{Z}^{(q+\lambda)}(b,\varphi(q))}\right) - \frac{\overline{\mathbb{Z}}_b^{(q+\lambda,-\lambda)}(x) - \mathbb{Z}^{(q)}(x-b)}{q+\lambda}\right).\end{aligned} \tag{4.53}$$

Also, applying (4.6) and then taking limits as $a\to\infty$ leads to

$$\begin{aligned}&\mathbb{E}_x\left[\mathrm{e}^{-q\nu_b^\lambda}W^{(q)}(Y_{\nu_b^\lambda})\mathbb{1}_{\{\nu_b^\lambda<\nu_0^-\}}\right] \\ =&\frac{\lambda\overline{\mathbb{W}}_{x-b}^{(q,\lambda)}(x)\int_0^b W^{(q)}(y)\mathbb{Z}^{(q+\lambda)}(b-y,\varphi(q))\mathrm{d}y}{\mathbb{Z}^{(q+\lambda)}(b,\varphi(q))} + \xi_b^{(q,\lambda)}(x) - \overline{\mathbb{W}}_{x-b}^{(q,\lambda)}(x) + W^{(q)}(x).\end{aligned} \tag{4.54}$$

Substituting (4.49), (4.53) and (4.54) into (4.47) and (4.48) completes the proof of (4.46).

■

In the next section, a numerical study of the impact of delayed dividend pullbacks on ruin probabilities and the expected (discounted) dividends net of transaction costs is performed. Among other things, it is shown that the strategy to delay the dividend pullbacks is preferred under certain model settings.

## 4.4 Numerical examples

### 4.4.1 Brownian risk processes

In this section, we consider the refracted Lévy process (with delays) $U$ where $X = \{X_t\}_{t\geq 0}$ is a drifted Brownian motion, i.e.,

$$X_t = X_0 + ct + B_t,$$

for $t \geq 0$, where $c > 0$ and $\{B_t\}_{t\geq 0}$ is a standard Brownian motion. It is well known that the Laplace exponent of $X$ is given by

$$\psi(s) = cs + \frac{1}{2}s^2, \; s \geq 0,$$

where the scale functions are

$$W^{(q)}(x) = \frac{1}{\Phi(q)+c}\left(\mathrm{e}^{\Phi(q)x} - \mathrm{e}^{-(\Phi(q)+2c)x}\right),$$

and

$$Z^{(q)}(x) = \frac{q}{\Phi(q)+c}\left(\frac{\mathrm{e}^{\Phi(q)x}}{\Phi(q)} + \frac{\mathrm{e}^{-(\Phi(q)+2c)x}}{\Phi(q)+2c}\right),$$

with $\Phi(q) = \sqrt{c^2+2q} - c$ for $q \geq 0$. The scale functions $\mathbb{W}^{(q)}$ and $\mathbb{Z}^{(q)}$ are the same as the ones for $W^{(q)}$ and $Z^{(q)}$, respectively, with $c$ replaced by $c-\delta$. We conduct a numerical study of the ruin probability $\mathbb{P}_x\left(\kappa_0^- < \infty\right)$ (as given in Eq. (4.38)) under the following parameter setting: $b = c = 1$ and $\delta = 0.5$. In Figure 4.2, we plot the ruin probability for the refracted Lévy process (with delays) $U$ with different delay rates $\lambda$ ($\lambda = 0.1, 10, 50, \infty$). As a basis of comparison, we also provide the values of the classical ruin probabilities for both processes $X$ and $Y$.

From Figure 4.2, the following observations are worthy of mention:

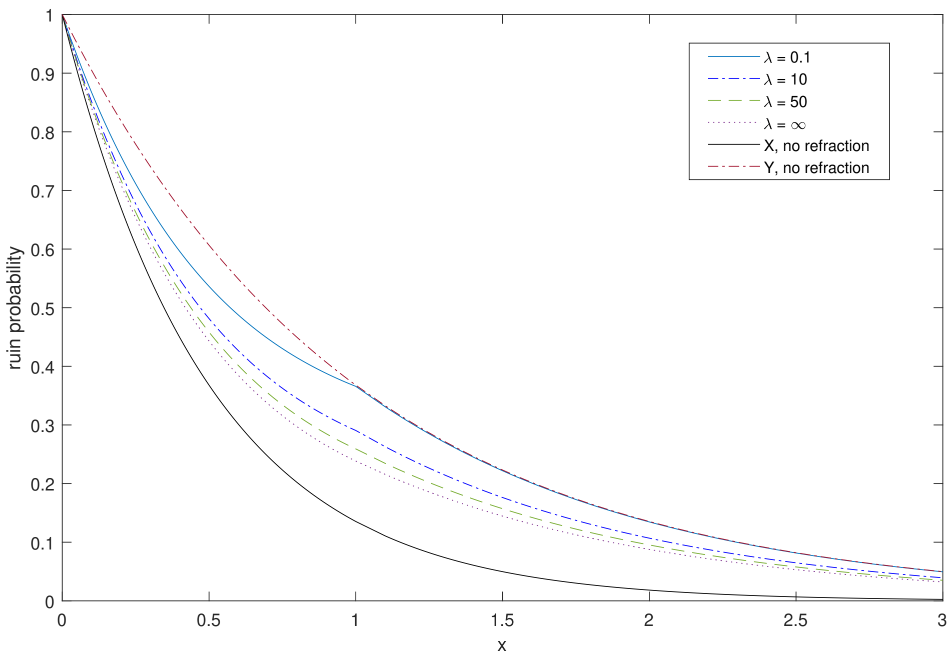



Figure 4.2: Ruin probabilities with different delay rates

- as expected, for a given initial surplus $x$, the ruin probability $\mathbb{P}_x\left(\kappa_0^- < \infty\right)$ decreases as the rate of dividend pullbacks $\lambda$ increases;

- the classical ruin probabilities for processes $X$ and $Y$ serve as lower and upper bounds, respectively, for the ruin probability $\mathbb{P}_x\left(\kappa_0^- < \infty\right)$;

- the rate of decrease in ruin probability is not smooth at the threshold level $b$ (given that a dividend payout is triggered precisely at $x = b$).

### 4.4.2 Crámer-Lundberg risk processes

We now consider the refracted Lévy process (with delays) $U$ where $X = \{X_t\}_{t\geq 0}$ is a Cramér-Lundberg risk process with exponentially distributed claims, namely

$$X_t = X_0 + ct - \sum_{i=1}^{N_t} C_i,$$

where $N = \{N_t\}_{t\geq 0}$ is a Poisson process with intensity $\eta > 0$ and $\{C_i\}_{i\in\mathbb{N}^+}$ is an iid sequence of exponential rv's with mean $1/\alpha$, independent of $N$. In what follows, we assume that $c > \delta + \frac{\eta}{\alpha}$ so that the ruin probability $\mathbb{P}_x\left(\kappa_0^- < \infty\right)$ is not trivially 1. In this case, it is well known that the Laplace exponent of $X$ is

$$\psi(s) = cs - \eta + \frac{\eta\alpha}{s+\alpha}, \quad s \geq 0,$$

Moreover, for $q > 0$ and $x \geq 0$, the scale functions $W^{(q)}$ and $Z^{(q)}$ are

$$W^{(q)}(x) = \frac{1}{c\left(\Phi(q) - \theta_q\right)}\left((\alpha + \Phi(q))\mathrm{e}^{\Phi(q)x} - \left(\alpha + \theta_q\right)\mathrm{e}^{\theta_q x}\right),$$

and

$$Z^{(q)}(x) = \frac{q}{\sqrt{\Delta_q}}\left(\frac{\alpha + \Phi(q)}{\Phi(q)}\mathrm{e}^{\Phi(q)x} - \frac{\alpha + \theta_q}{\theta_q}\mathrm{e}^{\theta_q x}\right),$$

where

$$\begin{aligned}\Phi(q) &= \frac{1}{2c}\left(q + \eta - c\alpha + \sqrt{\Delta_q}\right),\\ \theta_q &= \frac{1}{2c}\left(q + \eta - c\alpha - \sqrt{\Delta_q}\right),\end{aligned}$$

and

$$\Delta_q = (q + \eta - c\alpha)^2 + 4c\alpha q.$$

Here again, the scale functions $\mathbb{W}^{(q)}$ and $\mathbb{Z}^{(q)}$ are identically defined as $W^{(q)}$ and $Z^{(q)}$ respectively, but with $c$ replaced by $c - \delta$.

To investigate the impact of the dividend delay rate $\lambda$ on the expected (discounted) dividend payouts net of transaction costs, namely $D(x)$ defined in (4.3), we consider the refracted Lévy process (with delays) $U$ under the following parameter setting: $x = 2, b = 5, c = 4, \eta = 2, \alpha = 1$ and $q = 0.1$. Then, we identify different pairs $(\delta, \lambda)$ of dividend strategies whose corresponding ruin probability $\mathbb{P}_x\left(\kappa_0^- < \infty\right)$ (as given in Eq. (4.38)) are identical (where $\lambda < \infty$). We notice that for a given ruin probability, a dividend strategy with more reactive dividend pullbacks (i.e., a higher dividend pullback rate $\lambda$) must be accompanied by a higher dividend rate $\delta$. In other words, all else being equal, a decrease in the dividend pullback rate $\lambda$ comes at the expense of a decrease in the dividend rate $\delta$. We remark that dividend strategies with smaller $\lambda$ (or $\delta$) values are less volatile.

In Tables 4.1 and 4.2, we provide the values of $D(x)$ for the process $U$ with different choices of $\zeta$ under the constraint that the ruin probability is 0.22 and 0.32, respectively.

In what follows, we refer to the optimal dividend strategy as the strategy that maximizes $D(x)$ among the set of dividend strategies considered. From Tables 4.1 and 4.2, we observe that:

- In the absence of transaction costs (i.e., $\zeta = 0$), the optimal dividend strategy is the threshold dividend strategy $(\lambda \to \infty)$ in both Tables 4.1 and 4.2.
- As the transaction cost $\zeta$ increases, dividend strategies with smoother dividend payouts (smaller $\lambda/\delta$) are more optimal. For instance, in Table 4.2, the dividend strategy with $(\delta = 1.55, \lambda = 3.2724)$ leads the highest $D(x)$ when $\zeta = 0.5$, and the strategies with $(\delta = 1.45, \lambda = 1.0556)$ and $(\delta = 1.2952, \lambda = 0)$ are optimal when $\zeta = 0.7$ and $\zeta = 1$, respectively. The same observation holds for Table 4.1.

| $\delta$ | $\lambda$ | $D(x)$ | | | |
|---|---|---|---|---|---|
| | | $\zeta = 0$ | $\zeta = 1$ | $\zeta = 1.2$ | $\zeta = 1.5$ |
| 0.6890 | 0 | 4.9026 | 4.1422 | 3.9901 | **3.7620** |
| 0.80 | 1.0234 | 5.5926 | 4.2947 | 4.0351 | 3.6458 |
| 0.85 | 1.8362 | 5.8996 | 4.3550 | **4.0461** | 3.5827 |
| 0.90 | 3.1658 | 6.2039 | 4.3973 | 4.0360 | 3.4940 |
| 0.95 | 5.7322 | 6.5050 | **4.4097** | 3.9906 | 3.3621 |
| 1.00 | 12.7057 | 6.8023 | 4.3777 | 3.8928 | 3.1654 |
| 1.0595 | $\infty$ | **7.1493** | 4.2369 | 3.6544 | 2.7807 |

Table 4.1: Impact of $\lambda$ on $D(x)$ when the ruin probability is 0.22

In conclusion, we note that there is a trade-off between paying dividends at a higher rate (i.e., higher $\delta$) and being able to pay dividends more steadily (i.e., lower $\lambda$) under the consideration of transaction costs. As the transaction cost $\zeta$ increases, the above results confirm our intuition that dividend strategies with more steady dividend payouts would be preferred by investors.

| $\delta$ | $\lambda$ | $D(x)$ | | | |
|---|---|---|---|---|---|
| | | $\zeta = 0$ | $\zeta = 0.5$ | $\zeta = 0.7$ | $\zeta = 1$ |
| 1.2952 | 0 | 8.6144 | 8.2336 | 8.0812 | **7.8527** |
| 1.40 | 0.5837 | 9.1430 | 8.4549 | 8.1796 | 7.7667 |
| 1.45 | 1.0556 | 9.3927 | 8.5375 | **8.1954** | 7.6823 |
| 1.50 | 1.8172 | 9.6405 | 8.5953 | 8.1771 | 7.5500 |
| 1.55 | 3.2724 | 9.8860 | **8.6143** | 8.1056 | 7.3425 |
| 1.60 | 7.1910 | 10.1276 | 8.5694 | 7.9461 | 7.0112 |
| 1.6596 | $\infty$ | **10.4053** | 8.3471 | 7.5237 | 6.2888 |

Table 4.2: Impact of $\lambda$ on $D(x)$ when the ruin probability is 0.32

# Chapter 5

# Bridging the First and Last Passage Times for Lévy Models

## 5.1 Introduction

In previous chapters, we focused on studying first passage times related problems in different model settings. The last passage time has drawn considerable interest in recent years, mainly in the context of SNLPs. In this chapter, we introduce and study two types of random times to bridge the first and the last passage times' analyses.

As discussed in Gerber [45] and dos Reis [42], the ruin event is not always of great interest in certain risk management contexts. For instance, negative surplus may be inevitable for a start-up company or for a given subsidiary of a very large company. An entity may have funds available to support its negative surplus for some time with the hope that the business will recover in the future. For these cases, the analysis of the last recovery time from a negative surplus (also known as the last passage time below the solvency level) may be more relevant for risk management purposes. For a risk process $X$, the last passage time below 0 is defined as

$$g = \sup\{t \geq 0 : X_t \leq 0\}, \tag{5.1}$$

with the convention $\sup \emptyset = 0$.

From a risk management standpoint, it is certainly debatable whether an insurer should rely on risk measures involving the first or last passage time to formulate a comprehensive business plan on decisions related to its reinsurance, investment, capital injections, premium adjustments' strategy. In this chapter, as a possible remedy, we introduce two types of random times to bridge the first and last passage times. Both of these random times are triggered by negative excursions of an underlying risk process $X$. The first one is called *occupation-type first-last passage time* defined as

$$s_r = \sup\left\{t \geq \tau_0^- : \mathcal{O}_t > r \text{ and } X_t \leq 0\right\}, \quad r > 0, \tag{5.2}$$

where $\mathcal{O}_t = \int_0^t \mathbb{1}_{(-\infty,0)}(X_s)\,\mathrm{d}s$ is the *occupation time*[1] of the surplus process $X$ below level 0 up to time $t$. If the set in (5.2) is empty, we follow the convention that the supremum is reached at the smallest point, i.e., $\sup \emptyset = \tau_0^-$, where $\tau_0^- = \inf\{t > 0 \colon X_t < 0\}$ is the classical time of ruin. For a given sample path $\omega$, it is seen from (5.2) that $s_r(\omega) = \tau_0^-(\omega)$ if the total amount of time $X(\omega)$ stays below level 0 does not exceed $r$, and $s_r(\omega) = g(\omega)$ if the total amount of time $X(\omega)$ stays below level 0 exceeds $r$. As such, $s_r$ is like a "binary distribution" taking values in $\tau_0^-$ or $g$. Heuristically, we have $s_r \to \tau_0^-$ when $r \to \infty$, and $s_r \to g$ if $r \to 0$. This result will be formally proved in Propositions 5.5.1 and 5.5.2. Therefore, the parameter $r > 0$ can be interpreted as a decision maker's *aversion level to negative surplus*. A larger (resp. smaller) $r$ implies an insurer with a higher (resp. lower)

[1] There is an extensive literature on occupation times in applied probability and more specifically, in insurance mathematics. See, e.g. Loeffen et al. [86] and Landriault et al. [65].

aversion level to negative surplus and thus more weight is put on the first passage time $\tau_0^-$ (resp. last passage time $g$). In practice, it is expected that a start-up insurance company may experience financial distress at the onset and the associated aversion level to negative surplus should likely be lower. Besides, the aversion level $r$ is also related to a decision maker's risk preferences.

The second random time is the so-called *Parisian-type first-last passage time* defined as

$$l_r = \sup\left\{t \geq \tau_0^- :\ U_t > r\right\}, \quad r > 0, \tag{5.3}$$

where $U_t := t - g_t$ with $g_t := \sup\{0 \leq s \leq t : X_s \geq 0\}$. Note that $U_t$ corresponds to the length of the current excursion of the process $X$ below 0 at time $t$, a quantity which is known to play an important role in the definition of Parisian ruin times in Loeffen et al. [83] and Landriault et al. [71]. In this chapter, we first derive some distributional results for $U_t$, and make use of these results to study $l_r$. Intuitively, $l_r$ corresponds to the *ending time of negative excursions longer than* $r$. After $l_r$, the surplus process $X$ may still experience periods of negative surplus but none of these negative excursions will individually last longer than $r$ time units. It is seen from (5.3) that $\tau_0^- \leq l_r \leq g$ a.s. and $l_r$ may have mass point at $\tau_0^-$ and $g$. However, the main difference with $s_r$ is that, for a given sample path $\omega$, $l_r(\omega)$ may be such that $\tau_0^-(\omega) < l_r(\omega) < g(\omega)$. Hence, $l_r$ provides a smoother bridge (than $s_r$) between the first and last passage times. Similarly, the parameter $r$ in (5.3) can be interpreted as a decision maker's aversion to negative surplus. Furthermore, it will be shown that $l_r \to \tau_0^-$ if $r \to \infty$ and $l_r \to g$ if $r \to 0$ (see Propositions 5.5.1 and 5.5.2 for more details). See Figure 5.1 for a sample path illustration of $s_r$ and $l_r$.

The main objective of this chapter is to study the two hybrid random times, namely $s_r$ and $l_r$, through their Laplace transforms. To this end, we derive the joint distributions of $(\mathcal{O}_{e_q}, X_{e_q})$ and $(U_{e_q}, X_{e_q})$ for spectrally negative Lévy processes, which are new and of

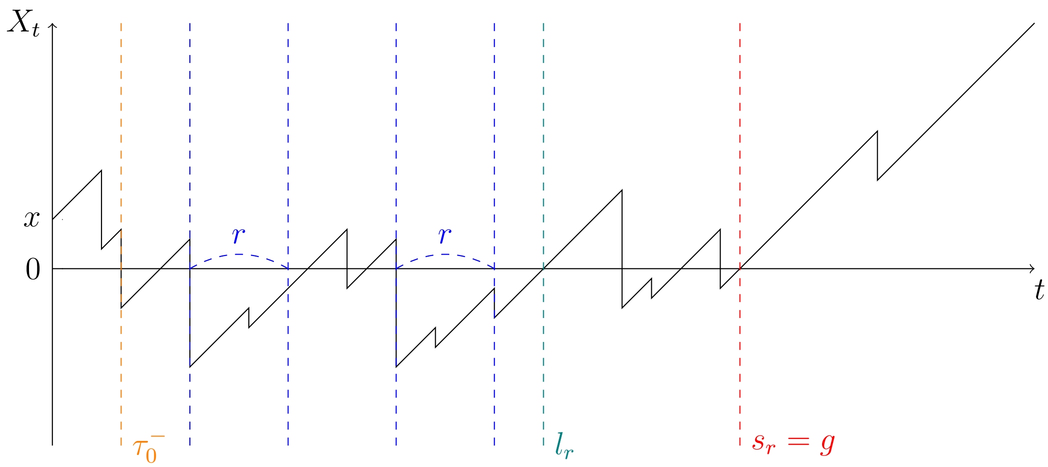



Figure 5.1: A sample path $\omega$ such that $\tau_0^-(\omega) < l_r(\omega) < g(\omega)$

interest in fluctuation theory. For the joint distribution of $(\mathcal{O}_{\mathrm{e}_q}, X_{\mathrm{e}_q})$, explicit expressions for a Brownian risk process with drift and a Cramér-Lundberg process with exponential jumps will also be derived.

The rest of the chapter is organized as follows. Section 5.2 presents necessary background materials. The main results of this chapter as they pertain to $s_r$ and $l_r$ are derived in Sections 5.3 and 5.4, respectively. Finally, in Section 5.5 we provide additional results on the convergence of the random times $s_r$ and $l_r$ when $r \to 0$ or $\infty$.

## 5.2 Preliminaries

For ease of notation, we will adopt the following conventions throughout the chapter:

- The Laplace exponent of the Lévy insurance risk process $X$ is denoted by $\psi(\cdot)$ and its right inverse is $\Phi_q = \sup\{\lambda \geq 0 \mid \psi(\lambda) = q\}$.

- We define the first-passage time of $X$ above a level $b \in \mathbb{R}$ as $\tau_b^+ = \inf\{t > 0\colon X_t > b\}$ with the convention $\inf \emptyset = \infty$. It is well known that

$$\mathbb{E}\left[\mathrm{e}^{-q\tau_b^+} \mathbb{1}_{\left\{\tau_b^+<\infty\right\}}\right] = \mathrm{e}^{-\Phi_q b}, \quad b > 0. \tag{5.4}$$

- For any $q \geq 0$, the $q$-scale functions for $X$ are denoted as $W^{(q)}$ and $Z^{(q)}$, and the corresponding *second generation* scale functions defined in (1.9) and (1.10) are denoted by $\overline{W}_a^{(p,q)}(x)$ and $\overline{Z}_a^{(p,q)}(x)$, respectively. We write $W = W^{(0)}$ and $Z = Z^{(0)}$ when $q = 0$.

- We denote the derivative of $W^{(q)}(x)$ with respect to $x$ by $W^{(q)\prime}(x)$ and the derivative of $Z^{(q)}(x, \theta)$ with respect to $x$ by

$$Z^{(q)\prime}(x, \theta) = \theta Z^{(q)}(x, \theta) - \psi_q(\theta) W^{(q)}(x), \tag{5.5}$$

where $\psi_q(\theta) = \psi(\theta) - q$. Also, the derivative of $\overline{W}_a^{(p,q)}(x)$ with respect to $x$ is given by

$$\overline{W}_a^{(p,q)\prime}(x) = W^{(p+q)\prime}(x) - q \int_0^a W^{(p+q)\prime}(x-y) W^{(p)}(y)\,\mathrm{d}y. \tag{5.6}$$

We also recall the following function introduced by Loeffen et al. [84] defined as

$$\Lambda^{(q)}(x, z) = \int_0^\infty W^{(q)}(x+u) \frac{u}{z} \mathbb{P}(X_z \in \mathrm{d}u),$$

and we write $\Lambda = \Lambda^{(0)}$ when $q = 0$. We denote the partial derivative of $\Lambda^{(q)}$ with respect to $x$ by

$$\Lambda^{(q)\prime}(x, z) = \frac{\partial \Lambda^{(q)}}{\partial x}(x, z) = \int_0^\infty W^{(q)\prime}(x+u) \frac{u}{z} \mathbb{P}(X_z \in \mathrm{d}u). \tag{5.7}$$

Besides the above functions, the following identity from Lemma 2.2 of Loeffen [86] is also recalled, which will be used in the later analysis. For $q \geq 0, x \in \mathbb{R}$,

$$\mathbb{E}_x\left[\mathrm{e}^{-q\tau_0^-} W\left(X_{\tau_0^-} + z\right) \mathbb{1}_{\left\{\tau_0^-<\infty\right\}}\right] = \overline{W}_x^{(q,-q)}(x+z) - W^{(q)}(x) Z(z, \Phi_q). \tag{5.8}$$

Also, by Theorem 3.1 of Chiu and Yin [29], it is known that

$$\mathbb{E}_x\left[\mathrm{e}^{-qg}\mathbb{1}_{\{g>0\}}\right] = \mathbb{E}\left[X_1\right]\ \left(\Phi_q'\mathrm{e}^{\Phi_q x} - W^{(q)}(x)\right), \quad q \geq 0, x \in \mathbb{R}, \tag{5.9}$$

if $\mathbb{E}\left[X_1\right] > 0$.

## 5.3 Occupation-type first-last passage time

### 5.3.1 Distribution of $s_r$

We begin our analysis with the occupation-type first-last passage time $s_r$ defined in (5.2). We recall that $s_r$ is a binary distribution taking values in $\tau_0^-$ and $g$. More specifically,

$$s_r = \begin{cases} \tau_0^-, & \text{if } \mathcal{O}_\infty \leq r, \\ g, & \text{if } \mathcal{O}_\infty > r, \end{cases}$$

where, from Corollary 5 of Landriault et al. [65],

$$\mathbb{P}_x\left(\mathcal{O}_\infty \leq r\right) = \mathbb{E}[X_1]\left(W(x) + \int_0^r \Lambda'\left(x,s\right)\mathrm{d}s\right). \tag{5.10}$$

We note that an expression for the Laplace transform and distribution function of $s_r$ are respectively given in Theorem 5.3.1 and Corollary 5.3.1. We first provide a preliminary result related to the joint distribution of $\left(\mathcal{O}_{\mathrm{e}_q}, X_{\mathrm{e}_q}\right)$, which will be used in the proof of Theorem 5.3.1.

**Lemma 5.3.1** *For $q > 0$, $x \in \mathbb{R}$ and $y, z \geq 0$,*

$$\begin{aligned}\mathbb{P}_x\left(\mathcal{O}_{\mathrm{e}_q} \in \mathrm{d}z, X_{\mathrm{e}_q} \in \mathrm{d}y\right) =& q\left(\mathrm{e}^{-\Phi_q y}W^{(q)}\left(x\right) - W^{(q)}\left(x-y\right)\right)\delta_0\left(\mathrm{d}z\right)\mathrm{d}y \\ &+ q\mathrm{e}^{-\Phi_q y}\mathrm{e}^{-qz}\left(\Lambda^{(q)\prime}\left(x,z\right) - \Phi_q\Lambda^{(q)}\left(x,z\right)\right)\mathrm{d}z\mathrm{d}y,\end{aligned} \tag{5.11}$$

*where $\delta_0(.)$ is the Dirac mass at 0.*

**Proof.** From Guérin and Renaud [49], we have the following potential measure discounted by its joint occupation time over the half line $(-\infty,0)$, that is : for $\lambda\geq 0$, $q>0$ and $x,y\in\mathbb{R}$,

$$\mathbb{E}_x\left[\mathrm{e}^{-\lambda\mathcal{O}_{\mathrm{e}_q}}, X_{\mathrm{e}_q}\in\mathrm{d}y\right]=q\left(\frac{\Phi_{q+\lambda}-\Phi_q}{\lambda}Z^{(q)}\left(x,\Phi_{\lambda+q}\right)Z^{(\lambda+q)}\left(-y,\Phi_q\right)-\overline{W}_x^{(q,\lambda)}\left(x-y\right)\right)\mathrm{d}y. \tag{5.12}$$

For $y\geq 0$, $\overline{W}_x^{(q,\lambda)}\left(x-y\right)=W^{(q)}(x-y)$ and $Z^{(\lambda+q)}\left(-y,\Phi_q\right)=\mathrm{e}^{-\Phi_q y}$, and Eq. (5.12) reduces to

$$\mathbb{E}_x\left[\mathrm{e}^{-\lambda\mathcal{O}_{\mathrm{e}_q}}, X_{\mathrm{e}_q}\in\mathrm{d}y\right]=q\mathrm{e}^{-\Phi_q y}\left(\frac{\Phi_{q+\lambda}Z^{(q)}\left(x,\Phi_{\lambda+q}\right)}{\lambda}-\Phi_q\frac{Z^{(q)}\left(x,\Phi_{\lambda+q}\right)}{\lambda}\right)-qW^{(q)}\left(x-y\right). \tag{5.13}$$

Finally, substituting the following identities (see, e.g., Landriault et al. [65]):

$$\frac{Z^{(q)}\left(x,\Phi_{\lambda+q}\right)}{\lambda}=\int_0^\infty \mathrm{e}^{-\lambda z}\left(\mathrm{e}^{-qz}\Lambda^{(q)}\left(x,z\right)\right)\mathrm{d}z, \tag{5.14}$$

and

$$\frac{\Phi_{\lambda+q}Z^{(q)}\left(x,\Phi_{\lambda+q}\right)-\lambda W^{(q)}\left(x\right)}{\lambda}=\int_0^\infty \mathrm{e}^{-\lambda z}\left(\mathrm{e}^{-qz}\Lambda^{(q)\prime}\left(x,z\right)\right)\mathrm{d}z, \tag{5.15}$$

into (5.13), one obtains Eq. (5.11) by Laplace transform inversion. ■

From Lemma 5.3.1, it is clear that

$$\mathbb{P}_x\left(\mathcal{O}_{\mathrm{e}_q}=0, X_{\mathrm{e}_q}\in\mathrm{d}y\right)=q\left(\mathrm{e}^{-\Phi_q y}W^{(q)}(x)-W^{(q)}(x-y)\right)\mathrm{d}y, \tag{5.16}$$

for $x,y\geq 0$. Also, it is worth noting that (5.16) corresponds to $\mathbb{P}_x\left(X_{\mathrm{e}_q}\in\mathrm{d}y,\tau_0^->\mathrm{e}_q\right)$, the $q$-potential measure of $X$ killed on exiting $[0,\infty)$.

We now derive an expression for the Laplace transform of $s_r$.

**Theorem 5.3.1** *For $q, r > 0$, $x \in \mathbb{R}$ and $\mathbb{E}[X_1] > 0$,*

$$\begin{aligned}\mathbb{E}_x\left[\mathrm{e}^{-q s_r}\right] =& \mathbb{E}[X_1] \int_r^\infty \mathrm{e}^{-qz}\left(\Lambda^{(q)\prime}(x,z) - \Phi_q \Lambda^{(q)}(x,z)\right)\mathrm{d}z \\ &+ \mathbb{E}[X_1] \int_0^r \int_0^\infty \left(\overline{W}_x^{(q,-q)\prime}(x+z) - W^{(q)}(x) Z'(z,\Phi_q)\right)\frac{z}{s}\mathbb{P}(X_s \in \mathrm{d}z)\,\mathrm{d}s. \quad (5.17)\end{aligned}$$

**Proof.** Using the fact that $\left\{s_r = \tau_0^-\right\} = \{\mathcal{O}_\infty \le r\}$ and $\left\{\tau_0^- < s_r < \mathrm{e}_q\right\} = \{\mathcal{O}_{\mathrm{e}_q} > r, X_s > 0 \text{ for all } s \ge \mathrm{e}_q\}$, it follows that

$$\begin{aligned}\mathbb{E}_x\left[\mathrm{e}^{-q s_r}\right] =& \mathbb{E}_x\left[\mathrm{e}^{-q s_r}\mathbb{1}_{\left\{s_r > \tau_0^-\right\}}\right] + \mathbb{E}_x\left[\mathrm{e}^{-q\tau_0^-}\mathbb{1}_{\left\{s_r = \tau_0^-\right\}}\right] \\ =& \mathbb{E}_x\left[\mathbb{P}_{X_{\mathrm{e}_q}}\left(\tau_0^- = \infty\right)\mathbb{1}_{\{\mathcal{O}_{\mathrm{e}_q} > r\}}\right] + \mathbb{E}_x\left[\mathrm{e}^{-q\tau_0^-}\mathbb{1}_{\{\mathcal{O}_\infty \le r\}}\right] \\ =& \mathbb{E}[X_1] \int_0^\infty \int_r^\infty W(y)\mathbb{P}_x\left(\mathcal{O}_{\mathrm{e}_q} \in \mathrm{d}z, X_{\mathrm{e}_q} \in \mathrm{d}y\right) + \mathbb{E}_x\left[\mathrm{e}^{-q\tau_0^-}\mathbb{1}_{\{\mathcal{O}_\infty \le r\}}\right], \quad (5.18)\end{aligned}$$

where the second equality follows from the Markov property (applied at time $\mathrm{e}_q$) and the third equality follows from the fact that $\mathbb{P}_x\left(\tau_0^- = \infty\right) = \mathbb{E}[X_1] W(x)$.

From (5.10) and Tonelli's theorem, we get

$$\begin{aligned}\mathbb{E}_x\left[\mathrm{e}^{-q\tau_0^-}\mathbb{1}_{\{\mathcal{O}_\infty \le r\}}\right] =& \mathbb{E}_x\left[\mathrm{e}^{-q\tau_0^-}\mathbb{P}_{X_{\tau_0^-}}\left(\mathcal{O}_\infty \le r\right)\right] \\ =& \mathbb{E}[X_1] \int_0^r \mathbb{E}_x\left[\mathrm{e}^{-q\tau_0^-}\Lambda'\left(X_{\tau_0^-}, s\right)\right]\mathrm{d}s \\ =& \mathbb{E}[X_1] \int_0^r \int_0^\infty \mathbb{E}_x\left[\mathrm{e}^{-q\tau_0^-} W'\left(X_{\tau_0^-} + z\right)\mathbb{1}_{\left\{\tau_0^- < \infty\right\}}\right]\frac{z}{s}\mathbb{P}(X_s \in \mathrm{d}z)\,\mathrm{d}s \\ =& \mathbb{E}[X_1] \int_0^r \int_0^\infty \left(\overline{W}_x^{(q,-q)\prime}(x+z) - W^{(q)}(x) Z'(z,\Phi_q)\right)\frac{z}{s}\mathbb{P}(X_s \in \mathrm{d}z)\,\mathrm{d}s,\end{aligned} \quad (5.19)$$

where (5.7) is applied in the third equality and the derivative of (5.8) is applied in the last equality. Substituting (5.11) and (5.19) into (5.18) completes the proof of Theorem 5.3.1.
■

In the following remark, we prove that $s_r$ converges in distribution to $\tau_0^-$ (as $r \to \infty$) and $g$ (as $r \to 0$) by showing that the Laplace transform of $s_r$ reduces to (1.13) and (5.9) in their respective limiting cases.

**Remark 5.3.1** *First, as $r \to \infty$,*

$$\begin{aligned}\lim_{r\to\infty} \mathbb{E}_x\left[\mathrm{e}^{-qs_r}\right] =& \mathbb{E}\left[X_1\right] \int_0^\infty \int_0^\infty \left(\overline{W}_x^{(q,-q)\prime}(x+z) - W^{(q)}(x) Z'(z,\Phi_q)\right) \frac{z}{s} \mathbb{P}\left(X_s \in \mathrm{d}z\right) \mathrm{d}s \\ =& \mathbb{E}[X_1] \lim_{\theta\to 0} \int_0^\infty \mathrm{e}^{-\theta s} \int_0^\infty \left(\overline{W}_x^{(q,-q)\prime}(x+z) - W^{(q)}(x) Z'(z,\Phi_q)\right) \frac{z}{s} \mathbb{P}\left(X_s \in \mathrm{d}z\right) \mathrm{d}s.\end{aligned} \tag{5.20}$$

*Applying Kendall's identity, and Eqs. (5.5) and Eq. (39) of Landriault et al. [66], we obtain*

$$\begin{aligned}&\int_0^\infty \mathrm{e}^{-\theta s} \int_0^\infty \left(\overline{W}_x^{(q,-q)\prime}(x+z) - W^{(q)}(x) Z'(z,\Phi_q)\right) \frac{z}{s} \mathbb{P}\left(X_s \in \mathrm{d}z\right) \mathrm{d}s \\ =& \int_0^\infty \mathrm{e}^{-\Phi_\theta z} \overline{W}_x^{(-q,q)\prime}(x+z)\,\mathrm{d}z - W^{(q)}(x) \int_0^\infty \mathrm{e}^{-\Phi_\theta z} Z'(z,\Phi_q) \mathrm{d}z \\ =& \Phi_\theta \int_0^\infty \mathrm{e}^{-\Phi_\theta z} \overline{W}_x^{(q,-q)}(x+z)\,\mathrm{d}z - W^{(q)}(x) - W^{(q)}(x) \left(\frac{\Phi_q(\theta - q)}{\theta(\Phi_\theta - \Phi_q)} - \frac{q}{\theta}\right) \\ =& \Phi_\theta \frac{Z^{(q)}(x,\Phi_\theta)}{\theta} - W^{(q)}(x) \frac{\Phi_\theta(\theta - q)}{\theta(\Phi_\theta - \Phi_q)}.\end{aligned} \tag{5.21}$$

*Substituting (5.21) into (5.20) and applying (1.3), one deduces that the Laplace transform of $s_r$ reduces to (1.13) as $r \to \infty$.*

*Now, we move on to the limiting case where $r \to 0$. With the help of (5.14) and (5.15), it follows that*

$$\begin{aligned}\lim_{r\to 0} \mathbb{E}_x\left[\mathrm{e}^{-qs_r}\right] =& \mathbb{E}\left[X_1\right] \int_0^\infty \mathrm{e}^{-qz} \left(\Lambda^{(q)\prime}(x,z) - \Phi_q \Lambda^{(q)}(x,z)\right) \mathrm{d}z \\ =& \mathbb{E}\left[X_1\right] \cdot \lim_{\lambda\to 0} \left(\frac{\left(\Phi_{\lambda+q} - \Phi_q\right) Z^{(q)}\left(x,\Phi_{\lambda+q}\right) - \lambda W^{(q)}(x)}{\lambda}\right)\end{aligned}$$

$$=\mathbb{E}\left[X_1\right]\left(\Phi_q'\mathrm{e}^{\Phi_q x}-W^{(q)}(x)\right), \tag{5.22}$$

*where* $Z^{(q)}(x,\Phi_q)=\mathrm{e}^{\Phi_q x}$ *and* $\lim_{\lambda\to 0}\frac{\Phi_{\lambda+q}-\Phi_q}{\lambda}=\Phi_q'$ *are applied in the last equation. Eq. (5.22) corresponds to the Laplace transform of* $g$ *given in (5.9).*

In fact, it can be shown that $s_r$ converges to $\tau_0^-$ and $g$ (as $r\to\infty$ and $r\to 0$, respectively) $\mathbb{P}_x$ almost surely. We refer the reader to Section 5.5 for the proof of this result.

The next result on the distribution of $s_r$ is an immediate consequence of Eq. (5.18).

**Corollary 5.3.1** *For* $t,r>0$, $x\in\mathbb{R}$ *and* $\mathbb{E}\left[X_1\right]>0$,

$$\mathbb{P}_x\left(s_r\le t\right)=\mathbb{E}\left[X_1\right]\int_0^\infty\int_r^\infty W(y)\mathbb{P}_x\left(\mathcal{O}_t\in\mathrm{d}s,X_t\in\mathrm{d}y\right)+\mathbb{P}_x\left(\mathcal{O}_\infty\le r,\tau_0^-\le t\right), \tag{5.23}$$

*where*

$$\mathbb{P}_x\left(\mathcal{O}_\infty\le r,\tau_0^-\le t\right)\;=\;\mathbb{E}\left[X_1\right]\int_0^r\int_0^\infty\mathbb{E}_x\left[W'\left(X_{\tau_0^-}+z\right)1_{\left\{\tau_0^-\le t\right\}}\right]\frac{z}{s}\mathbb{P}\left(X_s\in\mathrm{d}z\right)\mathrm{d}s.$$

For completeness, we also consider the random time $s^\theta$ defined as

$$s^\theta=\sup\left\{t\ge\tau_0^-:\mathcal{O}_t>\mathrm{e}_\theta\text{ and }X_t\le 0\right\}, \tag{5.24}$$

where the parameter $r$ in $s_r$ is replaced by an independent (of $X$) exponential rv $\mathrm{e}_\theta$. The following theorem gives an explicit expression (in terms of scale functions) of the Laplace transform of $s^\theta$.

**Theorem 5.3.2** *For* $q,\theta>0$, $x\in\mathbb{R}$ *and* $\mathbb{E}\left[X_1\right]>0$,

$$\mathbb{E}_x\left[\mathrm{e}^{-qs^\theta}\right]=\mathbb{E}[X_1]\left(\Phi_q'\mathrm{e}^{\Phi_q x}-\frac{\Phi_{q+\theta}-\Phi_q}{\theta}Z^{(q)}(x,\Phi_{q+\theta})+\frac{\Phi_\theta}{\theta}Z^{(q)}(x,\Phi_\theta)-\frac{\Phi_\theta}{\theta}\frac{\theta-q}{\Phi_\theta-\Phi_q}W^{(q)}(x)\right). \tag{5.25}$$

**Proof.** Similar to the proof of Theorem 5.3.1, from the strong Markov property of $X$, it follows that

$$\begin{aligned}\mathbb{E}_x\left[\mathrm{e}^{-qs^\theta}\right] =&\mathbb{E}_x\left[\mathbb{P}_{X_{\mathrm{e}_q}}\left(\tau_0^-=\infty\right)\mathbb{1}_{\{\mathcal{O}_{\mathrm{e}_q}>\mathrm{e}_\theta\}}\right]+\mathbb{E}_x\left[\mathrm{e}^{-q\tau_0^-}\mathbb{1}_{\{\mathcal{O}_\infty\le\mathrm{e}_\theta\}}\right]\\ =&\mathbb{E}[X_1]\mathbb{E}_x\left[W(X_{\mathrm{e}_q})\mathbb{1}_{\left\{\mathcal{O}_{\mathrm{e}_q}>\mathrm{e}_\theta\right\}}\right]+\mathbb{E}_x\left[\mathrm{e}^{-q\tau_0^-}\mathbb{E}_{X_{\tau_0^-}}\left[\mathrm{e}^{-\theta\mathcal{O}_\infty}\right]\right].\end{aligned} \tag{5.26}$$

Using the potential measure of $X$ (see Corollary 8.9 of Kyprianou [57]) and (5.13), one deduces that

$$\begin{aligned}&\mathbb{E}_x\left[W(X_{\mathrm{e}_q})\mathbb{1}_{\left\{\mathcal{O}_{\mathrm{e}_q}>\mathrm{e}_\theta\right\}}\right]\\ =&\mathbb{E}_x\left[W(X_{\mathrm{e}_q})\right]-\mathbb{E}_x\left[\mathrm{e}^{-\theta\mathcal{O}_{\mathrm{e}_q}}W(X_{\mathrm{e}_q})\right]\\ =&q\int_0^\infty W(y)\left(\Phi_q'\mathrm{e}^{-\Phi_q(y-x)}-W^{(q)}(x-y)\right)\mathrm{d}y\\ &-q\int_0^\infty W(y)\left(\frac{\left(\Phi_{q+\theta}-\Phi_q\right)Z^{(q)}\left(x,\Phi_{\theta+q}\right)\mathrm{e}^{-\Phi_q y}}{\theta}-W^{(q)}\left(x-y\right)\right)\mathrm{d}y\\ =&\Phi_q'\mathrm{e}^{\Phi_q x}-\frac{\Phi_{q+\theta}-\Phi_q}{\theta}Z^{(q)}(x,\Phi_{q+\theta}).\end{aligned} \tag{5.27}$$

Moreover, from Theorem 1 of Landriault et al. [71], we have

$$\begin{aligned}\mathbb{E}_x\left[\mathrm{e}^{-q\tau_0^-}\mathbb{E}_{X_{\tau_0^-}}\left[\mathrm{e}^{-\theta\mathcal{O}_\infty}\right]\right]=&\mathbb{E}_x\left[\mathrm{e}^{-q\tau_0^-}\mathbb{E}_{X_{\tau_0^-}}\left[\mathrm{e}^{-\theta\tau_0^+}\mathbb{E}\left[\mathrm{e}^{-\theta\mathcal{O}_\infty}\right]\right]\right]\\ =&\psi'(0+)\frac{\Phi_\theta}{\theta}\mathbb{E}_x\left[\mathrm{e}^{-q\tau_0^-+\Phi_\theta X_{\tau_0^-}}\right]\\ =&\psi'(0+)\frac{\Phi_\theta}{\theta}\left(Z^{(q)}(x,\Phi_\theta)-\frac{\theta-q}{\Phi_\theta-\Phi_q}W^{(q)}(x)\right).\end{aligned} \tag{5.28}$$

Substituting (5.27) and (5.28) into (5.26) completes the proof of Theorem 5.3.2. ■

Once again, it can be shown that $s^\theta$ converges in distribution to $\tau_0^-$ (and $g$) as $\theta\to 0(\infty)$.

**Remark 5.3.2** *By noting that $Z^{(q)}(x,\Phi_q)=\mathrm{e}^{\Phi_q x}$ and using (1.3), one observes from (5.25) that the Laplace transform of $s^\theta$ converges to the Laplace transform of $\tau_0^-$ as $\theta\to 0$.*

*Also, applying the initial value theorem, we obtain*

$$\lim_{\theta\to\infty}\frac{\Phi_{q+\theta}-\Phi_q}{\theta}Z^{(q)}(x,\Phi_{q+\theta})=W^{(q)}(x)$$

*and*

$$\lim_{\theta\to\infty}\frac{\Phi_\theta}{\theta}Z^{(q)}(x,\Phi_\theta)=W^{(q)}(x).$$

*Then, one concludes that (5.25) reduces to (5.9) as* $\theta\to\infty$.

As shown in Corollary 5.3.1, evaluating $\mathbb{P}_x\left(s_r\le t\right)$ boils down to deriving explicit expressions for $\mathbb{P}_x\left(\mathcal{O}_t\in\mathrm{d}s, X_t\in\mathrm{d}y\right)$ and $\mathbb{P}_x\left(X_{\tau_0^-}\in\mathrm{d}y,\tau_0^-\le t\right)$. In what follows, we provide their characterizations for two special cases of SNLPs, namely a Brownian motion with drift or a Cramér-Lundberg process with exponential claims.

### 5.3.2 Examples

#### Brownian risk model

Let $X_t=\mu t+B_t$, where $\mu>0$ and $\{B_t\}_{t\ge0}$ is a standard Brownian motion. Using Formula 2.0.2 of Borodin and Salminen [20], we obtain

$$\mathbb{P}_x\left(X_{\tau_0^-}\in\mathrm{d}y,\tau_0^-\le t\right)=x\int_0^t\frac{1}{\sqrt{2\pi}z^{3/2}}\exp\left(-\frac{(x+\mu z)^2}{2z}\right)\mathrm{d}z\delta_0(\mathrm{d}y)$$

for $x\ge0$. We recall that, for this risk process, the scale function $W^{(q)}(q\ge0)$ is given by

$$W^{(q)}(x)=\frac{1}{\Phi_q+\mu}\left(\mathrm{e}^{\Phi_q x}-\mathrm{e}^{-(\Phi_q+2\mu)x}\right),\quad x\ge0,\tag{5.29}$$

where

$$\Phi_q=\left(\sqrt{\mu^2+2q}-\mu\right).$$

Using Lemma 5.3.1, an expression for the joint distribution of $(\mathcal{O}_{\mathrm{e}_q}, X_{\mathrm{e}_q})$ is provided in the next corollary. This corresponds to Formulas 1.5.6 on page 258 of Borodin and Salminen [20].

**Corollary 5.3.2** *For $q > 0$ and $y, z \geq 0$,*

$$\begin{aligned}
&\mathbb{P}_x\left(\mathcal{O}_{\mathrm{e}_q} \in \mathrm{d}z, X_{\mathrm{e}_q} \in \mathrm{d}y\right) \\
=&q\mathrm{e}^{\mu(y-z)} A_{(\Phi_q+\mu)^2/2}(x,y,z)\,\mathrm{d}z\mathrm{d}y \\
+&\left\{\frac{q}{\Phi_q+\mu}\left(\mathrm{e}^{-(\Phi_q+2\mu)(x-y)}\mathbb{1}_{\{x>y\}} - \mathrm{e}^{-(\Phi_q+2\mu)x-\Phi_q y} + \mathrm{e}^{\Phi_q(x-y)}\mathbb{1}_{\{x\leq y\}}\right)\right\}\delta_0(\mathrm{d}z), \qquad (5.30)
\end{aligned}$$

*where*

$$A_\lambda(x,y,z) = \begin{cases} \frac{\sqrt{2}\mathrm{e}^{-y\sqrt{2\lambda}-\lambda z-x^2/(2z)}}{\sqrt{\pi z}} - \sqrt{2\lambda}\mathrm{e}^{-(y+x)\sqrt{2\lambda}} Erfc\left(\frac{\sqrt{2z}\lambda - x}{\sqrt{2\lambda}}\right), & \text{for } x \leq 0, \\ \mathrm{e}^{-(y+x)\sqrt{2\lambda}}\left(\frac{\sqrt{2}\mathrm{e}^{-\lambda z}}{\sqrt{\pi z}} - \sqrt{2\lambda} Erfc\left(\sqrt{\lambda z}\right)\right), & \text{for } x > 0. \end{cases}$$

**Proof.** First, we note that the first term on the right-hand side of Eq. (5.11) can be evaluated using (5.29). Now, we want to evaluate $\Lambda^{(q)\prime}(x,s) - \Phi_q \Lambda^{(q)}(x,s)$ for $x \in \mathbb{R}$. Given that $X_s$ has a normal distribution with mean $\mu s$ and variance $s$, we obtain

$$\Lambda^{(q)}(x,s) = \int_{(-x)\vee 0}^{\infty} W^{(q)}(x+z)\,\frac{z}{s\sqrt{2\pi s}}\mathrm{e}^{\frac{-(z-\mu s)^2}{2s}}\mathrm{d}z, \quad x \in \mathbb{R}.$$

For $x > 0$,

$$\begin{aligned}
\Lambda^{(q)}(x,s) &= \frac{W^{(q)}(x)}{2}\left(\frac{2\mathrm{e}^{-\frac{\mu^2 s}{2}}}{\sqrt{2\pi s}} + \mathrm{e}^{qs}\left(\Phi_q - 2(\Phi_q+\mu)\mathcal{N}\left(-\frac{s(\Phi_q+\mu)}{\sqrt{s}}\right)\right)\right) \\
&\quad + \frac{\mathrm{e}^{qs}}{2(\Phi_q+\mu)}\left((\Phi_q+2\mu)\mathrm{e}^{\Phi_q x} + \Phi_q \mathrm{e}^{-(\Phi_q+2\mu)x}\right),
\end{aligned}$$

and its derivative is given by

$$\Lambda^{(q)\prime}(x,s) = \frac{W^{(q)\prime}(x)}{2}\left(\frac{2\mathrm{e}^{-\frac{\mu^2 s}{2}}}{\sqrt{2\pi s}} + \mathrm{e}^{qs}\left(\Phi_q - 2(\Phi_q+\mu)\mathcal{N}\left(-\frac{s(\Phi_q+\mu)}{\sqrt{s}}\right)\right)\right)$$

$$+\frac{(\Phi_q+2\mu)\,\Phi_q \mathrm{e}^{qs}}{2\,(\Phi_q+\mu)}\left(\mathrm{e}^{\Phi_q x}-\mathrm{e}^{-(\Phi_q+2\mu)x}\right),$$

where $\mathcal{N}(\cdot)$ is the cumulative distribution function of the standard normal distribution. For $x \leq 0$, we obtain

$$\Lambda^{(q)}(x,s)=\mathrm{e}^{qs}\left(\mathrm{e}^{\Phi_q x}\mathcal{N}\left(\frac{x+s\,(\Phi_q+\mu)}{\sqrt{s}}\right)+\mathrm{e}^{-(\Phi_q+2\mu)x}\mathcal{N}\left(\frac{x-s\,(\Phi_q+\mu)}{\sqrt{s}}\right)\right),$$

and $\Lambda^{(q)\prime}(x,s)$ is given by

$$\begin{aligned}&\Lambda^{(q)\prime}(x,s)\\ =\;&\mathrm{e}^{qs}\left(\Phi_q\mathrm{e}^{\Phi_q x}\mathcal{N}\left(\frac{x+s\,(\Phi_q+\mu)}{\sqrt{s}}\right)-(\Phi_q+2\mu)\,\mathrm{e}^{-(\Phi_q+2\mu)x}\mathcal{N}\left(\frac{x-s\,(\Phi_q+\mu)}{\sqrt{s}}\right)\right)\\ &+\mathrm{e}^{qs}\left(\frac{\mathrm{e}^{\Phi_q x}\mathrm{e}^{-(x+s(\Phi_q+\mu))^2/(2s)}}{\sqrt{2\pi}}+\frac{\mathrm{e}^{-(\Phi_q+2\mu)x}\mathrm{e}^{-(x-s(\Phi_q+\mu))^2/(2s)}}{\sqrt{2\pi}}\right).\end{aligned}$$

Then, for $x \leq 0$,

$$\begin{aligned}\Lambda^{(q)\prime}(x,s)-\Phi_q\Lambda^{(q)}(x,s)=&\mathrm{e}^{qs}\left(\frac{\mathrm{e}^{\Phi_q x}\mathrm{e}^{-(x+s(\Phi_q+\mu))^2/(2s)}}{\sqrt{2\pi}}+\frac{\mathrm{e}^{-(\Phi_q+2\mu)x}\mathrm{e}^{-(x-s(\Phi_q+\mu))^2/(2s)}}{\sqrt{2\pi}}\right)\\ &-2\,(\Phi_q+\mu)\,\mathrm{e}^{qs}\mathrm{e}^{-(\Phi_q+2\mu)x}\mathcal{N}\left(\frac{x-s\,(\Phi_q+\mu)}{\sqrt{s}}\right),\end{aligned}\tag{5.31}$$

and for $x \geq 0$,

$$\begin{aligned}&\Lambda^{(q)\prime}(x,s)-\Phi_q\Lambda^{(q)}(x,s)\\ =&\mathrm{e}^{-(\Phi_q+2\mu)x}\left(\Phi_q\mathrm{e}^{qs}+\frac{2\mathrm{e}^{-\frac{\mu^2 s}{2}}}{\sqrt{2\pi s}}+\mathrm{e}^{qs}\left(\Phi_q-2\,(\Phi_q+\mu)\,\mathcal{N}\left(-\frac{s\,(\Phi_q+\mu)}{\sqrt{s}}\right)\right)\right).\end{aligned}\tag{5.32}$$

Rearranging and rewriting (5.31) and (5.32) using the complementary error function $Erfc(x) = \frac{2}{\sqrt{\pi}}\int_x^\infty \mathrm{e}^{-t^2}\mathrm{d}t = 2\mathcal{N}(-x\sqrt{2})$, we recover the result. ■

### Cramér-Lundberg risk model with exponential claims

Let $X = \{X_t\}_{t\geq 0}$ be a Cramér-Lundberg risk model with exponential claims, i.e.

$$X_t = X_0 + ct - \sum_{i=1}^{N_t} C_i,$$

where $N = \{N_t\}_{t\geq 0}$ is a Poisson process with rate $\eta > 0$ and $\{C_i\}_{i\in\mathbb{N}^+}$ is a sequence of iid exponential rv's with mean $1/\alpha$, independent of $N$. In this case, the law of $X$ is given by

$$\mathbb{P}\left(X_t \in \mathrm{d}z\right) = \mathrm{e}^{-\eta t}\left(\delta_{ct}(\mathrm{d}z) + \mathrm{e}^{-\alpha(ct-z)}\sum_{m=0}^{\infty}\frac{(\alpha\eta t)^{m+1}}{m!(m+1)!}(ct-z)^m \mathrm{d}z\right), \quad z \leq ct. \tag{5.33}$$

Also, from Eq. (20) of Landriault et al. [68], it is already known that

$$\begin{aligned}&\mathbb{P}_x\left(-X_{\tau_0^-} \in \mathrm{d}y, \tau_0^- \leq t\right)\\ &= \int_0^t \left\{\int_0^\infty \eta\alpha \mathrm{e}^{-\alpha(z+y)}\left(\mathbb{P}_x\left(X_w \in \mathrm{d}z\right) - \int_0^w \frac{z}{w-s}\mathbb{P}\left(X_{w-s} \in \mathrm{d}z\right) f(x+cs, s)\mathrm{d}s\right)\right\}\mathrm{d}w\mathrm{d}y,\end{aligned}$$

for $x \geq 0$, where

$$f(x,t) = \sum_{n=1}^{\infty}\frac{\mathrm{e}^{-\eta t}(\alpha\eta t)^n x^{n-1}\mathrm{e}^{-\alpha x}}{n!(n-1)!}.$$

For the process $X$, its scale function $W^{(q)} (q \geq 0)$ is given by

$$W^{(q)}(x) = \frac{1}{\sqrt{\Delta_q}}\left((\alpha+\Phi_q)\mathrm{e}^{\Phi_q x} - (\alpha+\theta_q)\mathrm{e}^{\theta_q x}\right), \quad x \geq 0, \tag{5.34}$$

where

$$\Phi_q = \frac{1}{2c}\left(q + \eta - c\alpha + \sqrt{\Delta_q}\right), \tag{5.35}$$

$$\theta_q = \frac{1}{2c}\left(q + \eta - c\alpha - \sqrt{\Delta_q}\right), \tag{5.36}$$

$$\Delta_q = (q+\eta-c\alpha)^2 + 4c\alpha q = (q+\eta+c\alpha)^2 - 4c\alpha\eta. \tag{5.37}$$

From Lemma 5.3.1, we now derive an expression for the joint distribution of $\left(\mathcal{O}_{\mathrm{e}_q}, X_{\mathrm{e}_q}\right)$.

**Corollary 5.3.3** *For $q > 0$, $x \geq -cs$ and $y \geq 0$,*

$$\begin{aligned}
&\mathbb{P}_x\left(\mathcal{O}_{\mathrm{e}_q} \in \mathrm{d}s, X_{\mathrm{e}_q} \in \mathrm{d}y\right) \\
=&\mathbb{P}_x\left(X_{\mathrm{e}_q} \in \mathrm{d}y, \tau_0^- > \mathrm{e}_q\right)\delta_0(\mathrm{d}s) + q\mathrm{e}^{\theta_q x - \Phi_q y + (c\theta_q - \eta - q)s}(\alpha + \theta_q)\mathrm{d}s\mathrm{d}y \\
&+ q\sum_{m=0}^{\infty} \frac{(\alpha\eta s)^{m+1}\mathrm{e}^{\theta_q x - \Phi_q y + (c\theta_q - \eta - q)s}}{(\alpha + \theta_q)^m m!(m+1)!}\left(f_{m+1}(s) - \frac{f_{m+2}(s)}{(\alpha + \theta_q)cs}\right)\mathrm{d}s\mathrm{d}y,
\end{aligned} \tag{5.38}$$

*where $f_m(s) := \gamma(m, (\alpha + \theta_q)(cs + 0 \wedge x))$ and $\gamma(m, x) = \int_0^x \mathrm{e}^{-z}z^{m-1}\mathrm{d}z$ is the incomplete gamma function. In addition,*

$$\begin{aligned}
&\mathbb{P}_x\left(X_{\mathrm{e}_q} \in \mathrm{d}y, \tau_0^- > \mathrm{e}_q\right) \\
=&\frac{q}{\sqrt{\Delta_q}}\left\{(\alpha + \theta_q)\mathrm{e}^{\theta_q x}\left(\mathbb{1}_{\{y \leq x\}}\mathrm{e}^{-\theta_q y} - \mathrm{e}^{-\Phi_q y}\right) + \mathbb{1}_{\{y > x\}}(\alpha + \Phi_q)\mathrm{e}^{\Phi_q(x-y)}\right\}\mathrm{d}y.
\end{aligned}$$

**Proof.** Using Eq. (5.34), it is straightforward to show that

$$\begin{aligned}
&\left(\mathrm{e}^{-\Phi_q y}W^{(q)}(x) - W^{(q)}(x-y)\right)\delta_0(\mathrm{d}s)\,\mathrm{d}y \\
=&\frac{1}{\sqrt{\Delta_q}}\left\{(\alpha + \theta_q)\mathrm{e}^{\theta_q x}\left(\mathbb{1}_{\{y \leq x\}}\mathrm{e}^{-\theta_q y} - \mathrm{e}^{-\Phi_q y}\right) + \mathbb{1}_{\{y > x\}}(\alpha + \Phi_q)\mathrm{e}^{\Phi_q(x-y)}\right\}\delta_0(\mathrm{d}s)\mathrm{d}y.
\end{aligned} \tag{5.39}$$

As for the second term of (5.11), using the relationship that $\Phi_q - \theta_q = \sqrt{\Delta_q}/c$, we obtain

$$\begin{aligned}
&\Lambda^{(q)'}(x, s) - \Phi_q\Lambda^{(q)}(x, s) \\
=&\int_{0\vee -x}^{\infty}\left(W^{(q)\prime}(x+z) - \Phi_q W^{(q)}(x+z)\right)\frac{z}{s}\mathbb{P}\left(X_s \in \mathrm{d}z\right) \\
=&\frac{(\alpha + \theta_q)\mathrm{e}^{\theta_q x}}{\sqrt{\Delta_q}}\int_{0\vee -x}^{\infty}\mathrm{e}^{\theta_q z}\left(\Phi_q - \theta_q\right)\frac{z}{s}\mathbb{P}\left(X_s \in \mathrm{d}z\right) \\
=&\frac{(\alpha + \theta_q)\mathrm{e}^{\theta_q x}}{cs}\int_{0\vee -x}^{\infty}\mathrm{e}^{\theta_q z}z\mathbb{P}\left(X_s \in \mathrm{d}z\right).
\end{aligned} \tag{5.40}$$

Applying Eq. (5.33), the integral on the right-hand side of (5.40) becomes

$$\int_{0\vee -x}^{\infty}\mathrm{e}^{\theta_q z}z\mathbb{P}\left(X_s \in \mathrm{d}z\right)$$

$$=\mathrm{e}^{-\eta s}\int_{0\vee -x}^{cs}\mathrm{e}^{\theta_q z}z\left(\delta_{cs}(\mathrm{d}z)+\mathrm{e}^{-\alpha(cs-z)}\sum_{m=0}^{\infty}\frac{(\alpha\eta s)^{m+1}}{m!(m+1)!}(cs-z)^m\mathrm{d}z\right)$$
$$=\mathrm{e}^{(c\theta_q-\eta)s}\left\{cs+\sum_{m=0}^{\infty}\frac{(\alpha\eta s)^{m+1}}{(\alpha+\theta_q)^{m+1}m!(m+1)!}\left(cs\cdot f_{m+1}(s)-\frac{f_{m+2}(s)}{\alpha+\theta_q}\right)\right\}, \tag{5.41}$$

Substituting (5.39), (5.40) and (5.41) into (5.11) completes the proof of Corollary 5.3.3.

■

For a fixed $t>0$, we can invert Eq. (5.11) wrt $q$ to obtain the joint distribution of $(\mathcal{O}_t, X_t)$ as follows.

**Theorem 5.3.3** *For $y, s\geq 0, t>0$ and $x\geq -cs$,*

$$\begin{aligned}
&\mathbb{P}_x\left(\mathcal{O}_t\in\mathrm{d}s, X_t\in\mathrm{d}y\right)\\
=&\left\{\mathbb{P}_x\left(X_t\in\mathrm{d}y\right)-y\mathrm{e}^{-\alpha x}\int_0^t\frac{\mathrm{e}^{-(\alpha c+\eta)u}}{t-u}\mathbb{P}\left(X_{t-u}\in\mathrm{d}y\right)\sum_{m=0}^{\infty}\frac{(\alpha\eta u)^{m+1}(x+cu)^m}{m!(m+1)!}\mathrm{d}u\right\}\delta_0(\mathrm{d}s)\\
&+\mathbb{1}_{\{t\geq s+\frac{y}{c}\}}\alpha\eta\cdot\exp\left(-\eta(t-s)-c\alpha v-\alpha x\right)\\
&\times\int_{0\vee -x}^{\infty}\frac{z\mathrm{e}^{-\alpha z}}{cs}\left\{I_0\left(w\right)-\frac{v\cdot I_2\left(w\right)}{\frac{x+z}{c}+t-s}\right\}\mathbb{P}\left(X_s\in\mathrm{d}z\right)\mathrm{d}s\mathrm{d}y,
\end{aligned} \tag{5.42}$$

*where $w=a\sqrt{v^2+\frac{v(x+y+z)}{c}}$, $v=t-s-\frac{y}{c}$ and $a=2\sqrt{c\alpha\eta}$. The function $I_\nu$ represents the modified Bessel function of the first kind of order $\nu$ and its integral representation is given by*

$$I_\nu(z)=\frac{\left(\frac{1}{2}z\right)^\nu}{\pi^{\frac{1}{2}}\Gamma\left(\nu+\frac{1}{2}\right)}\int_{-1}^{1}\left(1-t^2\right)^{\nu-\frac{1}{2}}\mathrm{e}^{-zt}\,\mathrm{d}t,\quad Re(z)>-\frac{1}{2}.$$

**Proof.** From Lemma 5.3.1, one observes that

$$\mathbb{P}_x\left(\mathcal{O}_t\in\mathrm{d}s, X_t\in\mathrm{d}y\right)$$

$$=\mathbb{P}_x\left(X_t\in \mathrm{d}y, t<\tau_0^-\right)\delta_0(\mathrm{d}s)+\mathcal{L}_q^{-1}\left\{\mathrm{e}^{-\Phi_q y}\mathrm{e}^{-qs}\left(\Lambda^{(q)'}(x,s)-\Phi_q\Lambda^{(q)}(x,s)\right)\right\}(t)\mathrm{d}s\mathrm{d}y, \tag{5.43}$$

where $\mathcal{L}_q^{-1}$ denotes the inverse Laplace transform wrt $q$.

In the Cramér-Lundberg risk process, it is known from Landriault et al. [68] that

$$\begin{aligned}&\mathbb{P}_x\left(X_t\in \mathrm{d}y, t<\tau_0^-\right)\\ =&\mathbb{P}_x\left(X_t\in \mathrm{d}y\right)-y\mathrm{e}^{-\alpha x}\int_0^t\frac{\mathrm{e}^{-(\alpha c+\eta)u}}{t-u}\mathbb{P}\left(X_{t-u}\in \mathrm{d}y\right)\sum_{m=0}^{\infty}\frac{(\alpha\eta u)^{m+1}(x+cu)^m}{m!(m+1)!}\mathrm{d}u, \quad t>0, y\geq 0.\end{aligned} \tag{5.44}$$

Hence, we are left with the evaluation of the second term on the right-hand side of Eq. (5.43).

First, using Eqs. (5.35) – (5.37), we have

$$\begin{aligned}&(\alpha+\theta_q)\mathrm{e}^{\theta_q(x+z)-\Phi_q y-qs}\\ =&\frac{2\alpha\eta\cdot\exp\left((\eta+c\alpha)s-\alpha(x-y+z)\right)}{a^2}\left(p-\sqrt{p^2-a^2}\right)\exp\left((b-s-\frac{y}{c})p-b\left(\sqrt{p^2-a^2}\right)\right),\end{aligned}$$

where $p:=q+\eta+c\alpha$ and $b:=\frac{x+y+z}{2c}$. From Eq. (D.17) in Drekic [43], we know that the inverse Laplace transform of $\left(p-\sqrt{p^2-a^2}\right)\cdot\exp\left(bp-b\left(\sqrt{p^2-a^2}\right)\right)$ wrt $p$ is

$$\begin{aligned}&\mathcal{L}_p^{-1}\left\{\left(p-\sqrt{p^2-a^2}\right)\exp\left(bp-b\left(\sqrt{p^2-a^2}\right)\right)\right\}(t)\\ =&\frac{a^2}{2}\left\{I_0\left(a\sqrt{t^2+2bt}\right)-\frac{t}{2b+t}I_2\left(a\sqrt{t^2+2bt}\right)\right\}.\end{aligned} \tag{5.45}$$

Then, it is immediate that

$$\begin{aligned}&\mathcal{L}_q^{-1}\left\{(\alpha+\theta_q)\mathrm{e}^{\theta_q(x+z)-\Phi_q y-qs}\right\}(t)\\ =&\mathbb{1}_{\{t\geq s+\frac{y}{c}\}}\alpha\eta\cdot\exp\left((\eta+c\alpha)(s-t)-\alpha(x-y+z)\right)\end{aligned}$$

$$\times\left\{I_0\left(a\sqrt{\left(t-s-\frac{y}{c}\right)^2+2b\left(t-s-\frac{y}{c}\right)}\right)-\frac{t-s-\frac{y}{c}}{2b+t-s-\frac{y}{c}}I_2\left(a\sqrt{\left(t-s-\frac{y}{c}\right)^2+2b\left(t-s-\frac{y}{c}\right)}\right)\right\}. \tag{5.46}$$

Therefore, letting $v=t-s-\frac{y}{c}$ and using (5.40) and (5.46), we obtain

$$\begin{aligned}
&\mathcal{L}_q^{-1}\left\{\mathrm{e}^{-\Phi_q y}\mathrm{e}^{-qs}\left(\Lambda^{(q)'}(x,s)-\Phi_q\Lambda^{(q)}(x,s)\right)\right\}(t)\\
=&\mathcal{L}_q^{-1}\left\{\int_{0\vee -x}^{\infty}(\alpha+\theta_q)\mathrm{e}^{\theta_q(x+z)-\Phi_q y-qs}\frac{z}{cs}\mathbb{P}\left(X_s\in\mathrm{d}z\right)\right\}(t)\\
=&\mathbb{1}_{\{t\geq s+\frac{y}{c}\}}\alpha\eta\cdot\exp\left(-\eta(t-s)-c\alpha v-\alpha x\right)\\
\times&\int_{0\vee -x}^{\infty}\frac{z\mathrm{e}^{-\alpha z}}{cs}\left\{I_0(w)-\frac{v\cdot I_2(w)}{\frac{x+z}{c}+t-s}\right\}\mathbb{P}\left(X_s\in\mathrm{d}z\right).
\end{aligned} \tag{5.47}$$

Substituting (5.44) and (5.47) into Eq. (5.43) completes the proof. ■

## 5.4 Parisian-type first-last passage times

In this section, we focus on the Parisian type first-last passage time $l_r$ defined in (5.3). We first note that the random time $l_r$ has ties with the Parisian ruin time with delay $r>0$ defined as

$$\kappa_r=\inf\{t>0\colon U_t>r\},$$

where $U_t=t-g_t$ with $g_t=\sup\{0\leq s\leq t: X_s\geq 0\}$. We recall that, for a spectrally negative Lévy insurance risk process $X$, Loeffen et al. [83] obtained an elegant expression for the probability of Parisian ruin, that is for $\mathbb{E}[X_1]>0$ and $x\in\mathbb{R}$,

$$\mathbb{P}_x(\kappa_r<\infty)=1-r\mathbb{E}[X_1]\frac{\Lambda(x,r)}{\int_0^\infty z\mathbb{P}(X_r\in\mathrm{d}z)}. \tag{5.48}$$

In the rest of this section, we denote by

$$\mathbb{P}_{u,x}(\cdot):=\mathbb{P}(\cdot|U_0=u,X_0=x),$$

for all $(u,x) \in \mathcal{S} := \{(0,\infty)\times(-\infty,0)\cup\{0\}\times[0,\infty)\}$ and by $\mathbb{E}_{u,x}$ its corresponding expectation operator.

**Theorem 5.4.1** *For $r,q>0$, $(u,x)\in\mathcal{S}$ and $\mathbb{E}[X_1]>0$,*

$$\mathbb{E}_{u,x}\left[\mathrm{e}^{-ql_r}\right] \;=\; r\mathbb{E}\left[X_1\right]\frac{\mathbb{E}_{u,x}\left[\Lambda\left(X_{\mathrm{e}_q}, r-U_{\mathrm{e}_q}\right)\mathbb{1}_{\left\{U_{\mathrm{e}_q}<r,\tau_0^-<\mathrm{e}_q\right\}}\right]}{\int_0^\infty z\mathbb{P}\left(X_r\in\mathrm{d}z\right)}. \tag{5.49}$$

**Proof.** We first note that

$$\mathbb{E}_{u,x}\left[\mathrm{e}^{-ql_r}\right] = \mathbb{P}_{u,x}\left(l_r<\mathrm{e}_q\right),$$

where $\mathrm{e}_q$ is an independent exponential rv with rate $q>0$. From the definition of $l_r$, we point out the following equivalence between the following two events:

$$\{l_r<\mathrm{e}_q\} = \left\{\tau_0^-<\mathrm{e}_q, U_t\le r \text{ for all } t\ge \mathrm{e}_q\right\}.$$

Hence,

$$\begin{aligned}\mathbb{E}_{u,x}\left[\mathrm{e}^{-ql_r}\right] &= \int_0^r\int_{-\infty}^\infty \mathbb{P}_{u,x}\left(\tau_0^-<\mathrm{e}_q, U_t\le r \text{ for all } t\ge\mathrm{e}_q, X_{\mathrm{e}_q}\in\mathrm{d}y, U_{\mathrm{e}_q}\in\mathrm{d}s\right)\\ &= \int_0^r\int_{-\infty}^\infty \mathbb{P}_{s,y}\left(\kappa_r=\infty\right)\mathbb{P}_{u,x}\left(\tau_0^-<\mathrm{e}_q, X_{\mathrm{e}_q}\in\mathrm{d}y, U_{\mathrm{e}_q}\in\mathrm{d}s\right),\end{aligned} \tag{5.50}$$

where the strong Markov property of $X$ at $\mathrm{e}_q$ is applied in (5.50). Given that that $U_{\mathrm{e}_q}=0$ when $X_{\mathrm{e}_q}\ge 0$, (5.50) can be written as

$$\begin{aligned}\mathbb{E}_{u,x}\left[\mathrm{e}^{-ql_r}\right] &= \int_0^\infty \mathbb{P}_y\left(\kappa_r=\infty\right)\mathbb{P}_x\left(\tau_0^-<\mathrm{e}_q, X_{\mathrm{e}_q}\in\mathrm{d}y\right)\\ &\quad+\int_0^r\int_{-\infty}^0 \mathbb{P}_{s,y}\left(\kappa_r=\infty\right)\mathbb{P}_{u,x}\left(X_{\mathrm{e}_q}\in\mathrm{d}y, U_{\mathrm{e}_q}\in\mathrm{d}s\right).\end{aligned} \tag{5.51}$$

Using (5.48) (note that $\Lambda(0,r)=1$) and the fact that $\mathbb{P}_y\left(\tau_0^+\le r\right)=\Lambda(y,r)$ for $y<0$ (see Loeffen et al. [83] for more details), we obtain

$$\mathbb{P}_{s,y}\left(\kappa_r=\infty\right) = \begin{cases}\mathbb{P}_y\left(\kappa_r=\infty\right) = r\mathbb{E}[X_1]\frac{\Lambda(y,r)}{\int_0^\infty z\mathbb{P}(X_r\in\mathrm{d}z)}, & \text{if } y\ge 0,\\ \mathbb{P}_y\left(\tau_0^+\le r-s\right)\mathbb{P}\left(\kappa_r=\infty\right) = r\mathbb{E}[X_1]\frac{\Lambda(y,r-s)}{\int_0^\infty z\mathbb{P}(X_r\in\mathrm{d}z)}, & \text{if } y<0.\end{cases} \tag{5.52}$$

Finally, substituting (5.52) into (5.51) yields the desired result. ■

From (5.49), we observe that the joint distribution of $(X_{e_q}, U_{e_q})$ plays a pivotal role in the characterization of the Laplace transform of $l_r$. Given that $U_{e_q} = 0$ when $X_{e_q} \geq 0$, we have

$$\mathbb{P}_{u,x}\left(X_{e_q} \in \mathrm{d}y, U_{e_q} = 0\right) = \mathbb{P}_x\left(X_{e_q} \in \mathrm{d}y\right), \tag{5.53}$$

for any $(u, x) \in \mathcal{S}$ and $y \geq 0$. Hence, we are left with the identification of the joint distribution of $(X_{e_q}, U_{e_q})$ only for $U_{e_q} > 0$ and $X_{e_q} < 0$.

**Lemma 5.4.1** *For $q, s > 0$, $y < 0$, and $(u, x) \in \mathcal{S}$,*

$$\begin{aligned}
&\mathbb{P}_{u,x}\left(X_{e_q} \in \mathrm{d}y, U_{e_q} > s\right) \\
=&\mathrm{e}^{-qs}\int_{-\infty}^{0} \mathbb{P}_x\left(X_{e_q} \in \mathrm{d}z, \tau_0^+ < \mathrm{e}_q\right)\mathbb{P}_z\left(X_s \in \mathrm{d}y, s < \tau_0^+\right) \\
&+\mathbb{1}(s \geq u)\int_{s-u}^{\infty} q\mathrm{e}^{-qt}\mathbb{P}_x\left(X_t \in \mathrm{d}y, t < \tau_0^+\right)\mathrm{d}t + \mathbb{1}(s < u)\mathbb{P}_x\left(X_{e_q} \in \mathrm{d}y, \mathrm{e}_q < \tau_0^+\right),
\end{aligned} \tag{5.54}$$

*where*

$$\mathbb{P}_{-z}\left(X_s \in \mathrm{d}y, s < \tau_0^+\right)\mathrm{d}z = \mathbb{P}_{-z}\left(X_s \in \mathrm{d}y\right)\mathrm{d}z - \int_0^s \frac{z}{t}\mathbb{P}\left(X_t \in \mathrm{d}z\right)\mathbb{P}\left(X_{s-t} \in \mathrm{d}y\right)\mathrm{d}t, \quad z \geq 0, \tag{5.55}$$

*and*

$$\mathbb{P}_x\left(\tau_0^+ < \mathrm{e}_q, -X_{e_q} \in \mathrm{d}z\right) = \begin{cases} q\left(\Phi_q'\mathrm{e}^{\Phi_q(x+z)} - W^{(q)}(x+z)\right)\mathrm{d}z, & x > 0, \\ q\mathrm{e}^{\Phi_q x}\left(\Phi_q'\mathrm{e}^{\Phi_q z} - W^{(q)}(z)\right)\mathrm{d}z, & x < 0. \end{cases} \tag{5.56}$$

**Proof.** To derive the joint distribution of $(X_{e_q}, U_{e_q})$, we consider separately the cases where $\left\{\tau_0^+ > \mathrm{e}_q\right\}$ (where $\tau_0^+$ is assumed to be 0 a.s. when $x \geq 0$) and $\left\{\tau_0^+ < \mathrm{e}_q\right\}$.

First, for the first case $\left\{\tau_0^+ > \mathrm{e}_q\right\}$, it follows that

$$
\begin{aligned}
&\mathbb{P}_{u,x}\left(X_{\mathrm{e}_q} \in \mathrm{d}y, \tau_0^+ > \mathrm{e}_q, U_{\mathrm{e}_q} > s\right) \\
=&\mathbb{P}_x\left(X_{\mathrm{e}_q} \in \mathrm{d}y, \tau_0^+ > \mathrm{e}_q, \mathrm{e}_q + u > s\right) \\
=&\mathbb{1}(s \geq u)\int_{s-u}^{\infty} q\mathrm{e}^{-qt}\mathbb{P}_x\left(X_t \in \mathrm{d}y, t < \tau_0^+\right)\mathrm{d}t + \mathbb{1}(s<u)\mathbb{P}_x\left(X_{\mathrm{e}_q} \in \mathrm{d}y, \mathrm{e}_q < \tau_0^+\right),
\end{aligned} \tag{5.57}
$$

for $s > 0$ and $y < 0$. We note that $\mathbb{P}_{u,x}\left(X_{\mathrm{e}_q} \in \mathrm{d}y, \tau_0^+ > \mathrm{e}_q, U_{\mathrm{e}_q} > s\right)$ is understood to be 0 when $x \geq 0$.

Now, for the second case $\left\{\tau_0^+ < \mathrm{e}_q\right\}$, from the definition of $U_t$,

$$
\begin{aligned}
\mathbb{P}_{u,x}\left(X_{\mathrm{e}_q} \in \mathrm{d}y, \tau_0^+ \leq \mathrm{e}_q, U_{\mathrm{e}_q} > s\right) &= \mathbb{P}_{u,x}\left(X_{\mathrm{e}_q} \in \mathrm{d}y, \tau_0^+ \leq \mathrm{e}_q - s, U_{\mathrm{e}_q} > s\right) \\
&= \mathrm{e}^{-qs}\mathbb{P}_{u,x}\left(X_{\mathrm{e}_q} \in \mathrm{d}y, \tau_0^+ \leq \mathrm{e}_q - s, U_{\mathrm{e}_q} > s \,|\mathrm{e}_q > s\right).
\end{aligned} \tag{5.58}
$$

Given that $\mathrm{e}_q - s\,|\mathrm{e}_q > s$ is also exponential with rate $q$, Eq. (5.58) can be rewritten as

$$
\begin{aligned}
\mathbb{P}_{u,x}\left(X_{\mathrm{e}_q} \in \mathrm{d}y, \tau_0^+ \leq \mathrm{e}_q, U_{\mathrm{e}_q} > s\right) &= \mathrm{e}^{-qs}\mathbb{P}_{u,x}\left(X_{\mathrm{e}_q+s} \in \mathrm{d}y, \tau_0^+ \leq \mathrm{e}_q, U_{\mathrm{e}_q+s} > s\right) \\
&= \mathrm{e}^{-qs}\int_{-\infty}^{0} \mathbb{P}_{u,x}\left(X_{\mathrm{e}_q+s} \in \mathrm{d}y, \tau_0^+ \leq \mathrm{e}_q, X_{\mathrm{e}_q} \in \mathrm{d}z, U_{\mathrm{e}_q+s} > s\right)
\end{aligned}
$$

Using the strong Markov property of $X$ at $\mathrm{e}_q$, it follows that

$$
\mathbb{P}_{u,x}\left(X_{\mathrm{e}_q} \in \mathrm{d}y, \tau_0^+ \leq \mathrm{e}_q, U_{\mathrm{e}_q} > s\right) = \mathrm{e}^{-qs}\int_{-\infty}^{0} \mathbb{P}_x\left(\tau_0^+ \leq \mathrm{e}_q, X_{\mathrm{e}_q} \in \mathrm{d}z\right)\mathbb{P}_z\left(X_s \in \mathrm{d}y, s < \tau_0^+\right). \tag{5.59}
$$

Combining (5.59) and (5.57) leads to (5.54).

It remains to prove Eq. (5.55) and (5.56). For Eq. (5.55), we have that

$$
\mathbb{P}_{-z}\left(X_s \in \mathrm{d}y, s < \tau_0^+\right) = \mathbb{P}_{-z}\left(X_s \in \mathrm{d}y\right) - \mathbb{P}_{-z}\left(X_s \in \mathrm{d}y, \tau_0^+ < s\right), \tag{5.60}
$$

where by conditioning on $\tau_0^+$ and then using Kendall's identity, the second term on the right-hand side of (5.60) becomes

$$\begin{aligned}\mathbb{P}_{-z}\left(X_s \in \mathrm{d}y, \tau_0^+ < s\right) &= \int_0^s \mathbb{P}\left(X_{s-t} \in \mathrm{d}y\right) \mathbb{P}_{-z}\left(\tau_0^+ \in \mathrm{d}t\right) \\ &= \int_0^s \frac{z}{t} \frac{\mathbb{P}\left(X_t \in \mathrm{d}z\right)}{\mathrm{d}z} \mathbb{P}\left(X_{s-t} \in \mathrm{d}y\right) \mathrm{d}t.\end{aligned}$$

Moreover, Eq. (5.56) is an immediate consequence of Theorem 8.7 and Corollary 8.9 of Kyprianou [57]. ■

Next, an expression for the joint Laplace transform of $U_{\mathrm{e}_q}$ and $X_{\mathrm{e}_q}$ is stated. This result follows from Lemma 5.4.1 and Eq. (5.53).

**Corollary 5.4.1** *For $\nu \geq 0, q > 0 \vee \psi(\nu)$, $p > \psi(\nu) - q$, and $(u, x) \in \mathcal{S}$,*

$$\mathbb{E}_{u,x}\left[\mathrm{e}^{-pU_{\mathrm{e}_q}+\nu X_{\mathrm{e}_q}}\right] = \mathbb{E}_{u,x}\left[\mathrm{e}^{-pU_{\mathrm{e}_q}+\nu X_{\mathrm{e}_q}} \mathbb{1}_{\{X_{\mathrm{e}_q} \geq 0\}}\right] + \mathbb{E}_{u,x}\left[\mathrm{e}^{-pU_{\mathrm{e}_q}+\nu X_{\mathrm{e}_q}} \mathbb{1}_{\{X_{\mathrm{e}_q} < 0\}}\right], \tag{5.61}$$

*where*

$$\begin{aligned}\mathbb{E}_{u,x}\left[\mathrm{e}^{-pU_{\mathrm{e}_q}+\nu X_{\mathrm{e}_q}} \mathbb{1}_{\{X_{\mathrm{e}_q} \geq 0\}}\right] =& \mathbb{E}_x\left[\mathrm{e}^{\nu X_{\mathrm{e}_q}} \mathbb{1}_{\{X_{\mathrm{e}_q} \geq 0\}}\right] \\ =& \frac{q\Phi_q' \mathrm{e}^{\Phi_q x}}{\Phi_q - \nu} + \frac{q\mathrm{e}^{\nu x} - qZ^{(q)}(x,\nu)}{q - \psi(\nu)}\end{aligned} \tag{5.62}$$

*and*

$$\begin{aligned}\mathbb{E}_{u,x}\left[\mathrm{e}^{-pU_{\mathrm{e}_q}+\nu X_{\mathrm{e}_q}} \mathbb{1}_{\{X_{\mathrm{e}_q} < 0\}}\right] &= \frac{pq}{p+q-\psi(\nu)} \frac{\Phi_q' \mathrm{e}^{\Phi_q x}(\Phi_{p+q} - \nu)}{(\Phi_q - \nu)(\Phi_{p+q} - \Phi_q)} \\ &- \frac{q\mathrm{e}^{-pu}\left(Z^{(q)}(x, \Phi_{p+q}) - Z^{(q)}(x,\nu)\right)}{p+q-\psi(\nu)} - \frac{q\Phi_q' \mathrm{e}^{\Phi_q x}}{\Phi_q - \nu}.\end{aligned} \tag{5.63}$$

Note that it is easy to verify that Eq. (5.63) holds more generally for any $q > 0$.

**Proof.** For $x \in \mathbb{R}, \nu \geq 0$ and $q > \psi(\nu)$, using the potential measure of $X$ without killing (see Corollary 8.8 of Kyprianou [57] for more details), one derives that

$$\begin{aligned}\mathbb{E}_x\left[\mathrm{e}^{\nu X_{\mathrm{e}_q}}\mathbb{1}_{\{X_{\mathrm{e}_q}\geq 0\}}\right] &= \int_{[0,\infty)} \mathrm{e}^{\nu y}\mathbb{P}_x\left(X_{\mathrm{e}_q}\in \mathrm{d}y\right)\\ &= \int_{[0,\infty)} \mathrm{e}^{\nu y} q\left(\Phi_q' \mathrm{e}^{-\Phi_q(y-x)} - W^{(q)}(x-y)\right)\mathrm{d}y\\ &= \frac{q\Phi_q' \mathrm{e}^{\Phi_q x}}{\Phi_q-\nu} + \frac{q\mathrm{e}^{\nu x} - qZ^{(q)}(x,\nu)}{q-\psi(\nu)},\end{aligned}$$

and

$$\begin{aligned}\mathbb{E}_x\left[\mathrm{e}^{\nu X_{\mathrm{e}_q}}\mathbb{1}_{\{X_{\mathrm{e}_q}< 0\}}\right] &= \mathbb{E}_x\left[\mathrm{e}^{\nu X_{\mathrm{e}_q}}\right] - \mathbb{E}_x\left[\mathrm{e}^{\nu X_{\mathrm{e}_q}}\mathbb{1}_{\{X_{\mathrm{e}_q}\geq 0\}}\right]\\ &= \frac{q\mathrm{e}^{\nu x}}{q-\psi(\nu)} - \frac{q\Phi_q' \mathrm{e}^{\Phi_q x}}{\Phi_q-\nu} - \frac{q\mathrm{e}^{\nu x} - qZ^{(q)}(x,\nu)}{q-\psi(\nu)}\\ &= \frac{qZ^{(q)}(x,\nu)}{q-\psi(\nu)} - \frac{q\Phi_q' \mathrm{e}^{\Phi_q x}}{\Phi_q-\nu}. \end{aligned} \tag{5.64}$$

One can show that Eq. (5.64) also holds for $q < \psi(\nu)$. Similarly, for $x \leq 0$ and $p+q > 0$, one can use the potential measure of $X$ killing at $\tau_0^+$ to verify that

$$\mathbb{E}_x\left[\mathrm{e}^{\nu X_{\mathrm{e}_{p+q}}}\mathbb{1}_{\{\mathrm{e}_{p+q}<\tau_0^+\}}\right] = \frac{p+q}{p+q-\psi(\nu)}\left(\mathrm{e}^{\nu x} - \mathrm{e}^{\Phi_{p+q}x}\right) \tag{5.65}$$

holds for any $\nu \geq 0$. Also, by the strong Markov property of $X$ and the memoryless property of the exponential distribution, it follows that

$$\begin{aligned}\mathbb{E}_x\left[\mathrm{e}^{\nu X_{\mathrm{e}_q}}\mathbb{1}_{\{X_{\mathrm{e}_q}<0,\tau_0^+<\mathrm{e}_q\}}\right] &= \mathbb{P}_x\left[\tau_0^+ < \mathrm{e}_q\right]\mathbb{E}\left[\mathrm{e}^{\nu X_{\mathrm{e}_q}}\mathbb{1}_{\{X_{\mathrm{e}_q}<0\}}\right]\\ &= q\mathrm{e}^{\Phi_q x}\left(\frac{1}{q-\psi(\nu)} - \frac{\Phi_q'}{\Phi_q-\nu}\right),\end{aligned} \tag{5.66}$$

for $\nu \geq 0$ and $x < 0$.

To prove Eq. (5.63), we consider separately the following cases:

*Case 1: $p > 0$ and $x \geq 0$.*

Using (5.64) and Lemma 5.4.1, we obtain

$$\begin{aligned}
&\mathbb{E}_{u,x}\left[\mathrm{e}^{-pU_{\mathrm{e}_q}+\nu X_{\mathrm{e}_q}}\mathbb{1}_{\left\{X_{\mathrm{e}_q}<0\right\}}\right]\\
=&\mathbb{E}_{u,x}\left[\mathrm{e}^{\nu X_{\mathrm{e}_q}}\mathbb{1}_{\left\{X_{\mathrm{e}_q}<0\right\}}\right]-\mathbb{E}_{u,x}\left[\mathrm{e}^{\nu X_{\mathrm{e}_q}}\mathbb{1}_{\left\{X_{\mathrm{e}_q}<0,U_{\mathrm{e}_q}>\mathrm{e}_p\right\}}\right]\\
=&\frac{qZ^{(q)}(x,\nu)}{q-\psi(\nu)}-\frac{q\Phi_q'\mathrm{e}^{\Phi_q x}}{\Phi_q-\nu}\\
&-\int_0^\infty\int_{-\infty}^0\mathrm{e}^{\nu y}p\mathrm{e}^{-ps}\mathrm{e}^{-qs}\int_{-\infty}^0\mathbb{P}_x\left(X_{\mathrm{e}_q}\in\mathrm{d}z,\tau_0^+<\mathrm{e}_q\right)\mathbb{P}_z\left(X_s\in\mathrm{d}y,s<\tau_0^+\right)\mathrm{d}s\\
=&\frac{qZ^{(q)}(x,\nu)}{q-\psi(\nu)}-\frac{q\Phi_q'\mathrm{e}^{\Phi_q x}}{\Phi_q-\nu}-\frac{p}{p+q}\int_{-\infty}^0\mathrm{e}^{\nu y}\int_{-\infty}^0\mathbb{P}_x\left(X_{\mathrm{e}_q}\in\mathrm{d}z\right)\mathbb{P}_z\left(X_{\mathrm{e}_{p+q}}\in\mathrm{d}y,\mathrm{e}_{p+q}<\tau_0^+\right)\\
=&\frac{qZ^{(q)}(x,\nu)}{q-\psi(\nu)}-\frac{q\Phi_q'\mathrm{e}^{\Phi_q x}}{\Phi_q-\nu}-\frac{p}{p+q-\psi(\nu)}\int_{-\infty}^0\mathbb{P}_x\left(X_{\mathrm{e}_q}\in\mathrm{d}z\right)\left(\mathrm{e}^{\nu z}-\mathrm{e}^{\Phi_{p+q}z}\right)\\
=&\frac{qZ^{(q)}(x,\nu)}{q-\psi(\nu)}-\frac{q\Phi_q'\mathrm{e}^{\Phi_q x}}{\Phi_q-\nu}-\frac{p}{p+q-\psi(\nu)}\left(\mathbb{E}_x\left[\mathrm{e}^{\nu X_{\mathrm{e}_q}}\mathbb{1}_{\left\{X_{\mathrm{e}_q}<0\right\}}\right]-\mathbb{E}_x\left[\mathrm{e}^{\Phi_{p+q}X_{\mathrm{e}_q}}\mathbb{1}_{\left\{X_{\mathrm{e}_q}<0\right\}}\right]\right)\\
=&\frac{qZ^{(q)}(x,\nu)}{q-\psi(\nu)}-\frac{q\Phi_q'\mathrm{e}^{\Phi_q x}}{\Phi_q-\nu}-\frac{pq}{p+q-\psi(\nu)}\left(\frac{Z^{(q)}(x,\nu)}{q-\psi(\nu)}-\frac{\Phi_q'\mathrm{e}^{\Phi_q x}}{\Phi_q-\nu}+\frac{Z^{(q)}(x,\Phi_{p+q})}{p}+\frac{\Phi_q'\mathrm{e}^{\Phi_q x}}{\Phi_q-\Phi_{p+q}}\right),
\end{aligned} \tag{5.67}$$

where the fourth equality follows from Eq. (5.65). Note that $u = 0$ in this case, (5.67) reduces to (5.63).

*Case 2: $p > 0$ and $x < 0$.*

By Lemma 5.4.1, we have

$$\begin{aligned}
&\mathbb{E}_{u,x}\left[\mathrm{e}^{-pU_{\mathrm{e}_q}+\nu X_{\mathrm{e}_q}}\mathbb{1}_{\left\{X_{\mathrm{e}_q}<0\right\}}\right]\\
=&\mathbb{E}_{u,x}\left[\mathrm{e}^{\nu X_{\mathrm{e}_q}}\mathbb{1}_{\left\{X_{\mathrm{e}_q}<0\right\}}\right]-\mathbb{E}_{u,x}\left[\mathrm{e}^{\nu X_{\mathrm{e}_q}}\mathbb{1}_{\left\{X_{\mathrm{e}_q}<0,U_{\mathrm{e}_q}>\mathrm{e}_p\right\}}\right]\\
=&\frac{qZ^{(q)}(x,\nu)}{q-\psi(\nu)}-\frac{q\Phi_q'\mathrm{e}^{\Phi_q x}}{\Phi_q-\nu}-\int_0^\infty\int_{-\infty}^0\mathrm{e}^{\nu y}p\mathrm{e}^{-ps}\mathrm{e}^{-qs}\int_{-\infty}^0\mathbb{P}_x\left(X_{\mathrm{e}_q}\in\mathrm{d}z,\tau_0^+<\mathrm{e}_q\right)\mathbb{P}_z\left(X_s\in\mathrm{d}y,s<\tau_0^+\right)\mathrm{d}s\\
&-\int_u^\infty\int_{-\infty}^0\mathrm{e}^{\nu y}p\mathrm{e}^{-ps}\int_{s-u}^\infty q\mathrm{e}^{-qt}\mathbb{P}_x\left(X_t\in\mathrm{d}y,t<\tau_0^+\right)\mathrm{d}t\mathrm{d}s
\end{aligned}$$

$$-\int_0^u \int_{-\infty}^0 \mathrm{e}^{\nu y} p \mathrm{e}^{-ps} \mathbb{P}_x \left( X_{\mathrm{e}_q} \in \mathrm{d}y, \mathrm{e}_q < \tau_0^+ \right) \mathrm{d}s. \tag{5.68}$$

Then, using Eqs. (5.65) and (5.66), we obtain

$$\begin{aligned}
&\int_0^\infty \int_{-\infty}^0 \mathrm{e}^{\nu y} p \mathrm{e}^{-ps} \mathrm{e}^{-qs} \int_{-\infty}^0 \mathbb{P}_x \left( X_{\mathrm{e}_q} \in \mathrm{d}z, \tau_0^+ < \mathrm{e}_q \right) \mathbb{P}_z \left( X_s \in \mathrm{d}y, s < \tau_0^+ \right) \mathrm{d}s \\
=& \frac{p}{p+q-\psi(\nu)} \int_{-\infty}^0 \mathbb{P}_x \left( X_{\mathrm{e}_q} \in \mathrm{d}z, \tau_0^+ < \mathrm{e}_q \right) \left( \mathrm{e}^{\nu z} - \mathrm{e}^{\Phi_{p+q} z} \right) \\
=& \frac{p}{p+q-\psi(\nu)} \left( \mathbb{E}_x \left[ \mathrm{e}^{\nu X_{\mathrm{e}_q}} \mathbb{1}_{\left\{ X_{\mathrm{e}_q} < 0, \tau_0^+ < \mathrm{e}_q \right\}} \right] - \mathbb{E}_x \left[ \mathrm{e}^{\Phi_{p+q} X_{\mathrm{e}_q}} \mathbb{1}_{\left\{ X_{\mathrm{e}_q} < 0, \tau_0^+ < \mathrm{e}_q \right\}} \right] \right) \\
=& \frac{p}{p+q-\psi(\nu)} \left( q \mathrm{e}^{\Phi_q x} \left( \frac{1}{q-\psi(\nu)} - \frac{\Phi_q'}{\Phi_q - \nu} \right) - q \mathrm{e}^{\Phi_q x} \left( \frac{1}{q-(p+q)} - \frac{\Phi_q'}{\Phi_q - \Phi_{p+q}} \right) \right) \\
=& \frac{pq \mathrm{e}^{\Phi_q x}}{p+q-\psi(\nu)} \left( \frac{1}{q-\psi(\nu)} - \frac{\Phi_q'}{\Phi_q - \nu} + \frac{1}{p} + \frac{\Phi_q'}{\Phi_q - \Phi_{p+q}} \right).
\end{aligned} \tag{5.69}$$

By Eq. (5.65), one derives that

$$\begin{aligned}
&\int_u^\infty \int_{-\infty}^0 \mathrm{e}^{\nu y} p \mathrm{e}^{-ps} \int_{s-u}^\infty q \mathrm{e}^{-qt} \mathbb{P}_x \left( X_t \in \mathrm{d}y, t < \tau_0^+ \right) \mathrm{d}t \mathrm{d}s + \int_0^u \int_{-\infty}^0 \mathrm{e}^{\nu y} p \mathrm{e}^{-ps} \mathbb{P}_x \left( X_{\mathrm{e}_q} \in \mathrm{d}y, \mathrm{e}_q < \tau_0^+ \right) \mathrm{d}s \\
=& p \mathrm{e}^{-pu} \int_0^\infty \int_{-\infty}^0 \mathrm{e}^{\nu y} \int_0^t \mathrm{e}^{-ps} q \mathrm{e}^{-qt} \mathbb{P}_x \left( X_t \in \mathrm{d}y, t < \tau_0^+ \right) \mathrm{d}s \mathrm{d}t + (1 - \mathrm{e}^{-pu}) \mathbb{E}_x \left[ \mathrm{e}^{\nu X_{\mathrm{e}_q}} \mathbb{1}_{\left\{ \mathrm{e}_q < \tau_0^+ \right\}} \right] \\
=& \mathrm{e}^{-pu} \left( \mathbb{E}_x \left[ \mathrm{e}^{\nu X_{\mathrm{e}_q}} \mathbb{1}_{\left\{ \mathrm{e}_q < \tau_0^+ \right\}} \right] - \frac{q}{p+q} \mathbb{E}_x \left[ \mathrm{e}^{\nu X_{\mathrm{e}_{p+q}}} \mathbb{1}_{\left\{ \mathrm{e}_{p+q} < \tau_0^+ \right\}} \right] \right) + (1 - \mathrm{e}^{-pu}) \mathbb{E}_x \left[ \mathrm{e}^{\nu X_{\mathrm{e}_q}} \mathbb{1}_{\left\{ \mathrm{e}_q < \tau_0^+ \right\}} \right] \\
=& \frac{q}{q-\psi(\nu)} \left( \mathrm{e}^{\nu x} - \mathrm{e}^{\Phi_q x} \right) - \mathrm{e}^{-pu} \frac{q}{p+q-\psi(\nu)} \left( \mathrm{e}^{\nu x} - \mathrm{e}^{\Phi_{p+q} x} \right).
\end{aligned} \tag{5.70}$$

Substituting Eqs. (5.70) and (5.69) into (5.68) completes the proof of Eq. (5.63) in this case.

*Case 3: $\psi(\nu) - q < p < 0$ and $x \geq 0$.*

By Theorem 2 of Baurdoux [12], one deduces that

$$\begin{aligned}
\mathbb{E}_{u,x} \left[ \mathrm{e}^{-pU_{\mathrm{e}_q}} \right] &= \mathbb{E}_x \left[ \mathrm{e}^{-p \left( \mathrm{e}_q - g_{\mathrm{e}_q} \right)} \right] \\
&= \int_0^\infty q \mathrm{e}^{-(p+q)t} \mathbb{E}_x \left[ \mathrm{e}^{p g_t} \right] \mathrm{d}t
\end{aligned}$$

$$
\begin{aligned}
&= \frac{q}{p+q}\mathbb{E}_x\left[\mathrm{e}^{pg_{\mathrm{e}_{p+q}}}\right] \\
&= \frac{q}{p+q}\left\{\frac{-p}{q}Z^{(q)}(x) - Z^{(q)}\left(x,\Phi_{p+q}\right) + \frac{p+q}{q} + \frac{p\Phi_{p+q}\Phi_q'\mathrm{e}^{\Phi_q x}}{\Phi_q\left(\Phi_{p+q}-\Phi_q\right)}\right\},
\end{aligned} \tag{5.71}
$$

for $-q < p < 0$. Since the process $\left\{\mathrm{e}^{cX_t-\psi(c)t}\right\}_{t\geq 0}$ is a martingale (with mean 1) for $c \geq 0$, we can introduce the change of measure

$$
\left.\frac{\mathrm{d}\mathbb{P}^c}{\mathrm{d}\mathbb{P}}\right|_{\mathcal{F}_t} = \mathrm{e}^{cX_t-\psi(c)t}.
$$

It is well known that if $(X,\mathbb{P})$ is a SNLP, then $(X,\mathbb{P}^c)$ is also a SNLP with the Laplace exponent given by $\psi_c(\lambda) = \psi(\lambda+c)-\psi(c)$. Readers are refereed to Chapter 3 of Kyprianou [57] for more details on this topic.

Applying the exponential change of measure and using (5.71), it follows that

$$
\begin{aligned}
&\mathbb{E}_{u,x}\left[\mathrm{e}^{-pU_{\mathrm{e}_q}+\nu X_{\mathrm{e}_q}}\right] \\
=&\mathrm{e}^{\nu x}\mathbb{E}_x\left[\mathrm{e}^{(-p+\psi(\nu))\mathrm{e}_q+pg_{\mathrm{e}_q}}\mathrm{e}^{\nu\left(X_{\mathrm{e}_q}-x\right)-\psi(\nu)\mathrm{e}_q}\right] \\
=&\mathrm{e}^{\nu x}\mathbb{E}_x^{\nu}\left[\mathrm{e}^{(-p+\psi(\nu))\mathrm{e}_q+pg_{\mathrm{e}_q}}\right] \\
=&\mathrm{e}^{\nu x}\mathbb{E}_x^{\nu}\left[\mathrm{e}^{-pU_{\mathrm{e}_q}}\mathrm{e}^{\psi(\nu)\mathrm{e}_q}\right] \\
=&\mathrm{e}^{\nu x}\int_0^\infty q\mathrm{e}^{-(q-\psi(\nu))t}\mathbb{E}_x^{\nu}\left[\mathrm{e}^{-pU_t}\right]\mathrm{d}t \\
=&\frac{q\mathrm{e}^{\nu x}}{q-\psi(\nu)}\mathbb{E}_x^{\nu}\left[\mathrm{e}^{-pU_{\mathrm{e}_{q-\psi(\nu)}}}\right] \\
=&\frac{q\mathrm{e}^{\nu x}}{q-\psi(\nu)}\left\{1-\mathrm{e}^{-\nu x}Z^{(q)}(x,\nu) + \frac{(q-\psi(\nu))p\mathrm{e}^{-\nu x}}{q-\psi(\nu)+p}\right. \\
&\left.\times\left\{\frac{\Phi_q'\mathrm{e}^{\Phi_q x}(\Phi_{q+p}-\nu)}{(\Phi_q-\nu)\left(\Phi_{q+p}-\Phi_q\right)} - \frac{\left(Z^{(q)}(x,\Phi_{q+p})-Z^{(q)}(x,\nu)\right)}{p}\right\}\right\} \\
=&\frac{q\mathrm{e}^{\nu x}-qZ^{(q)}(x,\nu)}{q-\psi(\nu)} + \frac{qp}{q-\psi(\nu)+p}\frac{\Phi_q'\mathrm{e}^{\Phi_q x}(\Phi_{q+p}-\nu)}{(\Phi_q-\nu)\left(\Phi_{q+p}-\Phi_q\right)} - \frac{q\left(Z^{(q)}(x,\Phi_{q+p})-Z^{(q)}(x,\nu)\right)}{q-\psi(\nu)+p},
\end{aligned}
$$

for $\psi(\nu)-q < p < 0$, which is the sum of Eqs. (5.62) and (5.63).

*Case 4: $\psi(\nu) - q < p < 0$ and $x < 0$.*

Applying the same techniques as in Eq. (5.71), we have

$$\begin{aligned}
\mathbb{E}_{u,x}\left[\mathrm{e}^{-pU_{\mathrm{e}_q}}\right] =& \mathbb{E}_x\left[\mathrm{e}^{-p\left(\mathrm{e}_q - g_{\mathrm{e}_q}\right)}\mathbb{1}_{\left\{\tau_0^+ \le \mathrm{e}_q\right\}}\right] + \mathrm{e}^{-pu}\mathbb{E}_x\left[\mathrm{e}^{-p\mathrm{e}_q}\mathbb{1}_{\left\{\tau_0^+ > \mathrm{e}_q\right\}}\right] \\
=& \frac{q}{p+q}\left\{\mathbb{E}_x\left[\mathrm{e}^{pg_{\mathrm{e}_{p+q}}}\mathbb{1}_{\tau_0^+ \le \mathrm{e}_{p+q}}\right] + \mathrm{e}^{-pu}\left(1 - \mathrm{e}^{\Phi_{p+q}x}\right)\right\} \\
=& \frac{q}{p+q}\left\{\mathbb{E}_x\left[\mathrm{e}^{p\tau_0^+}\mathbb{1}_{\tau_0^+ \le \mathrm{e}_{p+q}}\right]\mathbb{E}\left[\mathrm{e}^{pg_{\mathrm{e}_{p+q}}}\right] + \mathrm{e}^{-pu}\left(1 - \mathrm{e}^{\Phi_{p+q}x}\right)\right\} \\
=& \frac{q}{p+q}\left\{\mathrm{e}^{\Phi_q x}\left\{\frac{-p}{q} - 1 + \frac{p+q}{q} + \frac{p\Phi_{p+q}\Phi_q'}{\Phi_q(\Phi_{p+q} - \Phi_q)}\right\} + \mathrm{e}^{-pu}\left(1 - \mathrm{e}^{\Phi_{p+q}x}\right)\right\} \\
=& \frac{q}{p+q}\left\{\frac{p\Phi_{p+q}\Phi_q'\mathrm{e}^{\Phi_q x}}{\Phi_q(\Phi_{p+q} - \Phi_q)} + \mathrm{e}^{-pu}\left(1 - \mathrm{e}^{\Phi_{p+q}x}\right)\right\},
\end{aligned}$$

for $-q < p < 0$. Applying the exponential change of measure once again, it follows that

$$\begin{aligned}
&\mathbb{E}_{u,x}\left[\mathrm{e}^{-pU_{\mathrm{e}_q} + \nu X_{\mathrm{e}_q}}\right] \\
=& \frac{q\mathrm{e}^{\nu x}}{q - \psi(\nu)}\mathbb{E}_x^{\nu}\left[\mathrm{e}^{-pU_{\mathrm{e}_{q-\psi(\nu)}}}\right] \\
=& \frac{q\mathrm{e}^{\nu x}}{p + q - \psi(\nu)}\left\{\frac{p(\Phi_{p+q} - \nu)\Phi_q'\mathrm{e}^{(\Phi_q - \nu)x}}{(\Phi_q - \nu)((\Phi_{p+q} - \Phi_q)} + \mathrm{e}^{-pu}\left(1 - \mathrm{e}^{(\Phi_{p+q} - \nu)x}\right)\right\}
\end{aligned}$$

for $\psi(\nu) - q < p < 0$, which equals the sum of Eqs. (5.62) and (5.63). This completes the proof. ■

In the rest of this section, we consider a variation of the random time $l_r$ where the constant parameter $r$ is replaced by independent copies of a generic exponential rv $\mathrm{e}_\theta$ with rate $\theta > 0$. Specifically, we define $l^\theta$ as

$$l^\theta = \sup\left\{t \ge \tau_0^- \,:\, U_t > \mathrm{e}_\theta^{g_t}\right\}, \tag{5.72}$$

where $\mathrm{e}_\theta^{g_t}$ denotes an independent copy of $\mathrm{e}_\theta$ generated for the negative excursion that began at time $g_t$.

To study the distribution of $l^\theta$, we first recall the Parisian ruin with exponential delays defined as

$$\kappa^\theta = \inf\left\{t > 0 : U_t > \mathrm{e}_\theta^{g_t}\right\}. \tag{5.73}$$

An expression for the probability of Parisian ruin with exponential delays was first given in Landriault et al. [71], that is, for $\mathbb{E}[X_1] > 0$ and $x \in \mathbb{R}$,

$$\mathbb{P}_x\left(\kappa^\theta < \infty\right) = 1 - \mathbb{E}\left[X_1\right]\frac{\Phi_\theta}{\theta} Z\left(x, \Phi_\theta\right). \tag{5.74}$$

Applying Corollary 5.4.1, we have the following Laplace transform for $l^\theta$.

**Theorem 5.4.2** *Assume $\mathbb{E}[X_1] > 0$, for $q, \theta > 0$ and $(u, x) \in \mathcal{S}$,*

$$\mathbb{E}_{u,x}\left[\mathrm{e}^{-ql^\theta}\right] = \frac{\mathbb{E}\left[X_1\right]\Phi_\theta}{\theta}\left(\frac{(\theta - q)W^{(q)}(x)}{\Phi_q - \Phi_\theta} + \frac{\theta\Phi_q'\mathrm{e}^{\Phi_q x}}{\Phi_{q+\theta} - \Phi_q} - \mathrm{e}^{-\theta u}\left(Z^{(q)}(x, \Phi_{q+\theta}) - Z^{(q)}(x, \Phi_\theta)\right)\right). \tag{5.75}$$

**Proof.** First, one can observe from the definition of $l^\theta$ that the following two events are equivalent:

$$\left\{l^\theta < \mathrm{e}_q\right\} = \left\{\tau_0^- < \mathrm{e}_q, U_t \leq \mathrm{e}_\theta^{g_t} \text{ for all } t \geq \mathrm{e}_q\right\}.$$

Using a similar series of arguments as in the derivation of Eq. (5.51), we obtain

$$\begin{aligned}\mathbb{E}_{u,x}\left[\mathrm{e}^{-ql^\theta}\right] &= \int_0^\infty \mathbb{P}_y\left(\kappa^\theta = \infty\right)\mathbb{P}_x\left(\tau_0^- < \mathrm{e}_q, X_{\mathrm{e}_q} \in \mathrm{d}y\right) \\ &+ \int_0^\infty \int_{-\infty}^0 \mathbb{P}_{s,y}\left(\kappa^\theta = \infty\right)\mathbb{P}_{u,x}\left(X_{\mathrm{e}_q} \in \mathrm{d}y, U_{\mathrm{e}_q} \in \mathrm{d}s, s < \mathrm{e}_\theta\right).\end{aligned} \tag{5.76}$$

From Eqs. (5.74) and (5.4), one deduces that

$$\mathbb{P}_{s,y}\left(\kappa^\theta = \infty\right) = \begin{cases}\mathbb{P}_y\left(\kappa^\theta = \infty\right) = \frac{\mathbb{E}[X_1]\Phi_\theta}{\theta} Z\left(y, \Phi_\theta\right), & \text{if } y \geq 0, \\ \mathbb{P}_y\left(\tau_0^+ \leq \mathrm{e}_\theta - s\right)\mathbb{P}\left(\kappa^\theta = \infty\right) = \frac{\mathbb{E}[X_1]\Phi_\theta}{\theta}\mathrm{e}^{-\theta s}\mathrm{e}^{\Phi_\theta y}, & \text{if } y < 0,\end{cases} \tag{5.77}$$

which allows to rewrite (5.76) as

$$\mathbb{E}_{u,x}\left[\mathrm{e}^{-ql^{\theta}}\right]=\frac{\mathbb{E}\left[X_1\right]\Phi_\theta}{\theta}\left(\mathbb{E}_x\left[Z\left(X_{\mathrm{e}_q},\Phi_\theta\right)\mathbb{1}_{\left\{X_{\mathrm{e}_q}>0,\tau_0^-<\mathrm{e}_q\right\}}\right]+\mathbb{E}_{u,x}\left[\mathrm{e}^{-\theta U_{\mathrm{e}_q}+\Phi_\theta X_{\mathrm{e}_q}}\mathbb{1}_{\left\{X_{\mathrm{e}_q}\leq 0\right\}}\right]\right). \tag{5.78}$$

Using results on potential measures for the SNLP $X$ (see Chapter 8.4 of Kyprianou [57] for more details), one can show that

$$\mathbb{E}_x\left[Z\left(X_{\mathrm{e}_q},\Phi_\theta\right)\mathbb{1}_{\left\{X_{\mathrm{e}_q}>0,\tau_0^-<\mathrm{e}_q\right\}}\right]=\frac{(q-\theta)\left(\Phi_q'\mathrm{e}^{\Phi_q x}-W^{(q)}(x)\right)}{\Phi_q-\Phi_\theta}. \tag{5.79}$$

Also, from (5.63), it follows that

$$\begin{aligned}&\mathbb{E}_{u,x}\left[\mathrm{e}^{-\theta U_{\mathrm{e}_q}+\Phi_\theta X_{\mathrm{e}_q}}\mathbb{1}_{\left\{X_{\mathrm{e}_q}\leq 0\right\}}\right]\\=&\frac{\Phi_q'\mathrm{e}^{\Phi_q x}}{\Phi_q-\Phi_\theta}\left(\frac{\theta(\Phi_{q+\theta}-\Phi_\theta)}{\Phi_{q+\theta}-\Phi_q}-q\right)-\mathrm{e}^{-\theta u}\left(Z^{(q)}(x,\Phi_{q+\theta})-Z^{(q)}(x,\Phi_\theta)\right).\end{aligned} \tag{5.80}$$

Substituting (5.79) and (5.80) into (5.78) completes the proof of Theorem 5.4.2. ■

Using the same arguments as in Remark 5.3.2, it is straightforward to show that $l^\theta$ converges in distribution to $\tau_0^-$ (and $g$) as $\theta\to 0(\infty)$. We omit the details here.

## 5.5 Additional results on convergence of the first-last passage times

In this section, we study some limiting cases of $s_r$, $s^\theta$, $l_r$ and $l^\theta$ and show their consistency with known results in the literature. We note that the convergence (in distribution) results for $s^\theta$ and $l^\theta$ have already been discussed in Sections 5.3 and 5.4 through their Laplace transforms. In this section, we show that the convergence (in probability) results for $s^\theta$ and $l^\theta$ can be proved in a more direct way.

We begin by examining the limiting case of $s_r$, $s^\theta$, $l_r$ and $l^\theta$ when $r\to 0$ or $\theta\to\infty$.

**Proposition 5.5.1** *For any spectrally negative Lévy process $X$ with the condition that $\mathbb{E}[X_1] > 0$ and $x \in \mathbb{R}$,*

*(i) $s_r$ converges $\mathbb{P}_x$ almost surely to $g$ as $r \to 0$ when $\tau_0^- < \infty$.*

*(ii) $s^\theta$ converges in probability to $g$ as $\theta \to \infty$ when $\tau_0^- < \infty$.*

*(iii) $l_r$ converges $\mathbb{P}_x$ almost surely to $g$ as $r \to 0$ when $\tau_0^- < \infty$.*

*(iv) $l^\theta$ converges in probability to $g$ as $\theta \to \infty$ when $\tau_0^- < \infty$.*

**Proof.**

(i) First, we show that $s_r$ converges in probability to $g$. For any $\epsilon > 0$ and all $x \in \mathbb{R}$, we have

$$\begin{aligned}&\mathbb{P}_x\left(|s_r - g| > \epsilon, \tau_0^- < \infty\right)\\ =&\mathbb{P}_x\left(|s_r - g| > \epsilon, s_r = \tau_0^-, \tau_0^- < \infty\right) + \mathbb{P}_x\left(|s_r - g| > \epsilon, s_r > \tau_0^-, \tau_0^- < \infty\right),\end{aligned}$$

where the second term is zero as $s_r$ coincides with $g$ if $s_r > \tau_0^-$. For all $x \in \mathbb{R}$, we have

$$\begin{aligned}\lim_{r\to 0}\mathbb{P}_x\left(|s_r - g| > \epsilon, s_r = \tau_0^- < \infty\right) &= \lim_{r\to 0}\mathbb{P}_x\left(g - \tau_0^- > \epsilon, s_r = \tau_0^- < \infty\right)\\ &\leq \lim_{r\to 0}\mathbb{P}_x\left(\sigma_r = \infty, \tau_0^- < \infty\right)\\ &= \mathbb{P}_x\left(\tau_0^- = \infty, \tau_0^- < \infty\right) = 0. \qquad (5.81)\end{aligned}$$

where in the last equality we used the fact that $\sigma_r := \inf\{t > 0\colon \mathcal{O}_t > r\}$ converges $\mathbb{P}_x$ almost surely (a.s.) to the classical ruin time $\tau_0^-$ when $r \to 0$ (see Proposition 3.3 in Guérin and Renaud [50]). Therefore, for all $x \in \mathbb{R}$, we deduce that $s_r$ converges to $g$ in probability as $r \to 0$ when $\tau_0^- < \infty$. Also, note that $s_r$ is a non-decreasing function as

$r \to 0$, we conclude that $s_r$ converges to $g$ when $\tau_0^- < \infty$ $\mathbb{P}_x$ a.s. as $r \to 0$.

(ii) Similar to (i), for any $\epsilon > 0$ and all $x \in \mathbb{R}$,

$$\begin{aligned}
&\lim_{\theta\to\infty} \mathbb{P}_x \left(|s^\theta - g| > \epsilon, \tau_0^- < \infty\right) \\
&= \lim_{\theta\to\infty} \mathbb{P}_x \left(g - \tau_0^- > \epsilon, s^\theta = \tau_0^-, \tau_0^- < \infty\right) \\
&\le \lim_{\theta\to\infty} \mathbb{P}_x \left(\sigma_{\mathrm{e}_\theta} = \infty, \tau_0^- < \infty\right) = 0.
\end{aligned}$$

(iii) For any $\epsilon > 0$ and all $x \in \mathbb{R}$,

$$\begin{aligned}
&\lim_{r\to 0} \mathbb{P}_x \left(|l_r - g| > \epsilon, \tau_0^- < \infty\right) \\
&= \lim_{r\to 0} \mathbb{P}_x \left(g - l_r > \epsilon, \kappa_r < \infty, \tau_0^- < \infty\right) + \lim_{r\to 0} \mathbb{P}_x \left(g - l_r > \epsilon, \kappa_r = \infty, \tau_0^- < \infty\right) \\
&\le \lim_{r\to 0} \mathbb{P}_x \left(\mathbb{P}_{X_{l_r}} \left(\kappa_r = \infty, \tau_0^- < \infty\right)\right) + \lim_{r\to 0} \mathbb{P}_x \left(\kappa_r = \infty, \tau_0^- < \infty\right) \\
&= \lim_{r\to 0} \mathbb{P}_x \left(\mathbb{P} \left(\kappa_r = \infty, \tau_0^- < \infty\right)\right) + \lim_{r\to 0} \mathbb{P}_x \left(\kappa_r = \infty, \tau_0^- < \infty\right),
\end{aligned}$$

where the last step is by conditioning on $X_{l_r} = 0$. Since $\kappa_r$ has the same law as $\tau_0^-$ as $r \to 0$ (see Corollary 2.4 of Surya [94]) and $l_r$ is a non-decreasing function as $r \to 0$, we conclude that the above limits all approach zero and thus $l_r$ converges to $g$ a.s. when $\tau_0^- < \infty$ as $r \to 0$.

(iv) Similar to (iii), for any $\epsilon > 0$ and all $x \in \mathbb{R}$,

$$\begin{aligned}
&\lim_{\theta\to\infty} \mathbb{P}_x \left(|l^\theta - g| > \epsilon, \tau_0^- < \infty\right) \\
&= \lim_{\theta\to\infty} \mathbb{P}_x \left(g - l^\theta > \epsilon, \kappa^\theta < \infty, \tau_0^- < \infty\right) + \lim_{\theta\to\infty} \mathbb{P}_x \left(g - l^\theta > \epsilon, \kappa^\theta = \infty, \tau_0^- < \infty\right) \\
&\le \lim_{r\to 0} \mathbb{P}_x \left(\mathbb{P}_{X_{l^\theta}} \left(\kappa^\theta = \infty, \tau_0^- < \infty\right)\right) + \lim_{\theta\to\infty} \mathbb{P}_x \left(\kappa^\theta = \infty, \tau_0^- < \infty\right) \\
&= \lim_{\theta\to\infty} \mathbb{P}_x \left(\mathbb{P} \left(\kappa^\theta = \infty, \tau_0^- < \infty\right)\right) + \lim_{\theta\to\infty} \mathbb{P}_x \left(\kappa^\theta = \infty, \tau_0^- < \infty\right) = 0.
\end{aligned}$$

■

The following proposition is the counterpart to Proposition 5.5.1 when $r \to \infty$ or $\theta \to 0$.

**Proposition 5.5.2** *For any spectrally negative Lévy process $X$ with the condition that $\mathbb{E}[X_1] > 0$ and $x \in \mathbb{R}$,*

*(i) $s_r$ converges $\mathbb{P}_x$ almost surely to $\tau_0^-$ as $r \to \infty$.*

*(ii) $s^\theta$ converge $\mathbb{P}_x$ in probability to $\tau_0^-$ as $\theta \to 0$.*

*(iii) $l_r$ converge $\mathbb{P}_x$ almost surely to $\tau_0^-$ as $r \to \infty$.*

*(iv) $l^\theta$ converge $\mathbb{P}_x$ in probability to $\tau_0^-$ as $\theta \to 0$.*

**Proof.** As shown in the proof of Proposition 5.5.1, it is sufficient to prove the convergence in probability in cases (i) and (iii).

(i) For any $\epsilon > 0$ and all $x \in \mathbb{R}$,

$$\begin{aligned}
\lim_{r\to\infty} \mathbb{P}_x\left(|s_r - \tau_0^-| > \epsilon, \tau_0^- < \infty\right) \leq & \lim_{r\to\infty} \mathbb{P}_x\left(s_r > \tau_0^-, \tau_0^- < \infty\right) \\
= & \lim_{r\to\infty} \mathbb{P}_x\left(\sigma_r < \infty, \tau_0^- < \infty\right) \\
= & \lim_{r\to\infty} \mathbb{P}_x\left(\sigma_r < \infty\right) \\
= & 1 - \lim_{r\to\infty} \mathbb{P}_x\left(\int_0^\infty \mathbb{1}_{(-\infty,0)}(X_s)\mathrm{d}s \leq r\right) = 0,
\end{aligned}$$

the last equation holds because $X$ drifts to $+\infty$ when $\mathbb{E}[X_1] > 0$.

(ii) Similar to (i), for any $\epsilon > 0$ and all $x \in \mathbb{R}$,

$$\begin{aligned}
\lim_{\theta\to 0} \mathbb{P}_x\left(|s^\theta - \tau_0^-| > \epsilon, \tau_0^- < \infty\right) \leq & \lim_{\theta\to 0} \mathbb{P}_x\left(\sigma_{\mathrm{e}_\theta} < \infty\right) \\
= & 1 - \lim_{\theta\to 0} \mathbb{P}_x\left(\int_0^\infty \mathbb{1}_{(-\infty,0)}(X_s)\mathrm{d}s \leq \mathrm{e}_\theta\right) = 0.
\end{aligned}$$

(iii) For any $\epsilon > 0$ and all $x \in \mathbb{R}$,

$$\lim_{r\to\infty} \mathbb{P}_x\left(|l_r - \tau_0^-| > \epsilon, \tau_0^- < \infty\right) = \lim_{r\to\infty} \mathbb{P}_x\left(l_r - \tau_0^- > \epsilon, \kappa_r < \infty\right)$$

$$\leq \lim_{r\to\infty} \mathbb{P}_x\left(\kappa_r < \infty\right) \leq \lim_{r\to\infty} \mathbb{P}_x\left(\sigma_r < \infty\right) = 0.$$

(iv) For any $\epsilon > 0$ and all $x \in \mathbb{R}$,

$$\begin{aligned}
\lim_{\theta\to 0} \mathbb{P}_x\left(|l^\theta - \tau_0^-| > \epsilon, \tau_0^- < \infty\right) \leq & \lim_{\theta\to 0} \mathbb{P}_x\left(l^\theta > \tau_0^-, \tau_0^- < \infty\right) \\
= & \lim_{\theta\to 0} \mathbb{P}_x\left(\kappa^\theta < \infty, \tau_0^- < \infty\right) \\
= & \lim_{\theta\to 0} \mathbb{P}_x\left(\kappa^\theta < \infty\right) \\
= & 1 - \lim_{\theta\to 0} \mathbb{P}_x\left(\int_0^\infty \mathbb{1}_{(-\infty,0)}(X_s)\mathrm{d}s \leq \mathrm{e}_\theta\right) = 0.
\end{aligned}$$

■

# Chapter 6

# Concluding Remarks and Future Works

In this thesis, we have proposed insurance risk models that are more in line with reality and have drawn important risk management implications throughout. In Chapter 2, we build on the analysis of Landriault et al. [73] in the Sparre Andersen risk model with exponential claim sizes by considering a general non-decreasing income process. We derive explicit expressions for some joint distributions involving ruin-related quantities and assess the impact of income processes on an insurer's solvency risk. Our investigation shows that income processes play a determining role in the assessment of an insurer's solvency risk.

As an extension, in Chapter 3, a fairly large class of dependence structures between inter-arrival times and claim sizes is incorporated into the risk model studied in the previous chapter. We show that the Gerber-Shiu function can be characterized by a linear system of equations in this model setting and further examine the impact of risk models with different dependence structures.

Chapter 4 examines a refracted Lévy insurance risk model with delayed dividend pullbacks. Under the proposed model, we derive two-sided exit identities and show that there is a trade-off between paying dividends at a higher rate and being able to pay dividends more steadily, under the consideration of transaction costs. We note that exponential delays are assumed in the model setting. Extending the current results to a refracted risk model with deterministic delays may be of practical interest. The main technical difficulty is that the Gerber-Shiu measure at the Parisian ruin with a fixed implementation delay is unknown and non-trivial to derive. This quantity is also essential for studying the general expected penalty function at Parisian ruin with a fixed delay for Lévy risk processes. In the future, I intend to study this problem by applying advanced probability tools such as the excursion theory of Lévy processes.

Chapter 5 studies two hybrid random times with parameters measuring a decision maker's aversion level to negative surplus, which can recover the first and last passage time as limiting cases. It is worth pointing out that the techniques used in this chapter are different from those typically used in the first passage time analysis, as last passage times are not stopping times. It is of practical interest to study the proposed random times before a fixed time in the future. A potential starting point could be to study the random times before an independent and exponentially distributed random time.

Other than the aforementioned potential research directions, it is also interesting to study problems related to drawdown risks. Drawdown is a performance-based risk measure, measuring the drop in value (of a stock price or a surplus process) from the historical peak over a given period of time. There are various applications of drawdown related quantities in the context of finance and actuarial studies, such as their applications in option pricing, portfolio optimization, de Finetti's optimal dividend problem, etc.. To date, most of the existing research on drawdown risks have focused on the first drawdown time over a pre-specified size and severity of drawdowns. As pointed out in Landriault et al. [70], economic turmoil and volatile market fluctuations are better described by quantities containing more path-wise information. I am interested in investigating drawdown related quantities that are of broad risk management interest, such as the frequency of drawdowns in the context of general spectrally negative Lévy processes.

The last direction is to investigate optimization problems in the context of the aforementioned risk models and quantities. For example, under the risk model proposed in Chapter 4, it is of practical interest to examine how model parameters such as the dividend rate can be chosen to maximize (or minimize) some objective functions. Investigating optimal stopping (or prediction) problems in the context of last passage times is also interesting and has applications in financial modelling, such as predicting the time at which the process attains its ultimate maximum, which helps investors decide the optimal time to sell an asset.

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