Analysis of a pressure-robust hybridized discontinuous Galerkin method for the stationary Navier–Stokes equations

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Abstract

We present well-posedness and an a priori error analysis of the hybridized discontinuous Galerkin method for the stationary form of the Navier–Stokes problem proposed in (J Sci Comput, 76(3):1484–1501, 2018). This scheme was shown to result in an approximate velocity field that is pointwise divergence-free and divergence-conforming. As a consequence we show that the velocity error estimate is independent of the pressure. Furthermore, we show that estimates for both the velocity and pressure are optimal. Numerical examples demonstrate pressure-robustness and optimality of the scheme.

Keywords: Navier–Stokes, finite element method, hybridized, discontinuous Galerkin, pressure-robust

1. Introduction

In recent years pressure-robust numerical methods for incompressible flows have gained much popularity. These methods have the appealing property that the error in the approximate velocity is independent of the pressure. This is in contrast to non-pressure-robust methods for incompressible flows, such as the Taylor–Hood finite element [16], Crouzeix–Raviart [11], MINI elements [1], and certain discontinuous Galerkin methods [6, 7, 12], where the velocity error estimates depend on the best approximation error of the pressure scaled by the inverse of the viscosity. It was shown in [24] that the underlying mechanism behind pressure-robustness is a fundamental invariance property of incompressible flows; namely, the fact that perturbing the external force by a gradient field affects only the pressure, and not the velocity. Failure by a numerical scheme to reproduce this invariance property at the discrete level gives rise to a lack of pressure-robustness. As discussed in [19], a numerical scheme resulting in point-wise solenoidal, divergence-conforming approximate velocity field satisfies this invariance property at the discrete level.

For H^1-conforming elements, balancing the velocity and pressure spaces such that the approximate velocity field is point-wise solenoidal while satisfying the discrete inf-sup condition is a formidable challenge [19]. A possible solution is the use of discontinuous Galerkin methods in the construction of pressure-robust schemes. The relaxation of H^1-conformity allows the use of

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[1]SR gratefully acknowledges support from the Natural Sciences and Engineering Research Council of Canada through the Discovery Grant program (RGPIN-05606-2015) and the Discovery Accelerator Supplement (RGPAS-478018-2015).

divergence-conforming velocity spaces for which the approximate velocity field is trivially point-wise divergence-free. This was demonstrated in [8, 37] where pressure-robust discontinuous Galerkin methods for Stokes and Navier–Stokes flows were introduced.

Unfortunately, discontinuous Galerkin methods are known to be expensive compared to standard finite element methods. This led to the creation of hybridized discontinuous Galerkin (HDG) methods in [9]. HDG methods are constructed such that globally coupled unknowns are defined on cell boundaries only as opposed to the interior cell based unknowns of the discontinuous Galerkin method. As a result, HDG methods have significantly smaller systems of algebraic equations. In combination with preconditioning techniques [10, 34] the HDG method is computationally competitive with traditional continuous Galerkin approaches [20, 39]. Many HDG methods have been proposed for incompressible flow problems. Examples include [2, 5, 13, 27, 31, 33] for Stokes and Oseen flows, [3, 17, 21, 22, 26, 28, 30] for Navier–Stokes flows, and more recently [29] for incompressible magneto-hydrodynamics.

The work of Lehrenfeld and Schöberl [23] presents the first pressure-robust HDG method of the Navier–Stokes equation without post-processing. Like [8] they suggest the use of a divergence-conforming velocity space. Conversely, the divergence-conformity of the HDG method for incompressible flows by Rhebergen and Wells [31, 32] is a consequence of the construction of the HDG discretization. This HDG method uses 'standard' cell and facet discontinuous Galerkin spaces and does not involve a divergence-conforming finite element space for the velocity. Numerical results in [32] suggest the method is pressure-robust for the Navier–Stokes problem, but without supporting analysis. The analysis of the method by Rhebergen and Wells for the Stokes problem was presented in [31]. The purpose of this paper is to provide stability and error analysis of the method in [32] for the Navier–Stokes problem.

The outline of this paper is as follows. We present the Navier–Stokes problem in section 2 and the HDG method is introduced in section 3. Notation and properties of the multilinear forms involved are discussed in section 4. Existence and uniqueness of the discrete solution are shown in section 5. We derive optimal pressure-robust error estimates for the velocity in a mesh dependent energy norm, the pressure in the L^2-norm and optimal L^2-error estimates for the velocity in section 6. Numerical examples in section 7 demonstrate pressure-robustness of the scheme and we draw conclusions in section 8.

2. The Navier–Stokes equations

Let $\Omega \subset \mathbb{R}^d$ be a polygonal ($d = 2$) or polyhedral ($d = 3$) domain with boundary Γ. We consider the Navier–Stokes equations: given a body force $f : \Omega \to \mathbb{R}^d$ and kinematic viscosity $\nu \in \mathbb{R}^+$, find the velocity $u : \Omega \to \mathbb{R}^d$ and pressure $p : \Omega \to \mathbb{R}$ such that

$$-\nu\nabla^2 u + \nabla \cdot (u \otimes u) + \nabla p = f \qquad \text{in } \Omega, \tag{1a}$$

$$\nabla \cdot u = 0 \qquad \text{in } \Omega, \tag{1b}$$

$$u = 0 \qquad \text{on } \Gamma. \tag{1c}$$

It is well known, e.g., [36], that given a body force $f \in \left[L^2(\Omega)\right]^d$, the variational formulation of the Navier–Stokes problem eq. (1): find $(u, p) \in \left[H_0^1(\Omega)\right]^d \times L_0^2(\Omega)$ such that

$$\int_\Omega \nu \nabla u : \nabla v \,\mathrm{d}x + \int_\Omega (u \cdot \nabla u) \cdot v \,\mathrm{d}x - \int_\Omega p \nabla \cdot v \,\mathrm{d}x = \int_\Omega f \cdot v \,\mathrm{d}x \qquad \forall v \in \left[H_0^1(\Omega)\right]^d \tag{2a}$$

$$\int_\Omega q \nabla \cdot u \,\mathrm{d}x = 0 \qquad \forall q \in L_0^2(\Omega), \tag{2b}$$

admits a unique solution provided

$$\|f\| \leq \nu^2 (C_o C_p)^{-1}, \tag{3}$$

where, $\|\cdot\|$ is the $L^2(\Omega)$-norm, C_p is the Poincaré constant and C_o is a constant depending only on Ω and d. In addition, the velocity satisfies the stability estimate

$$\|u\|_1 \leq C_p \nu^{-1} \|f\|, \tag{4}$$

where $\|\cdot\|_1$ is the $H^1(\Omega)$-norm.

3. The hybridizable discontinuous Galerkin method

Let $\mathcal{T} = \{K\}$ denote the triangulation of the domain Ω into simplices K. Furthermore let $\mathcal{F}$ denote the set of all facets and let Γ^0 denote the mesh skeleton, i.e., the union of all facets. We denote the characteristic length of a cell K by h_K and we denote the outward unit normal vector on the boundary of a cell, ∂K, by n.

We introduce discontinuous finite element approximation spaces for the velocity and pressure:

$$V_h := \left\{ v_h \in \left[L^2(\Omega)\right]^d,\ v_h \in \left[P_k(K)\right]^d\ \forall K \in \mathcal{T} \right\}, \tag{5a}$$

$$Q_h := \left\{ q_h \in L^2(\Omega),\ q_h \in P_{k-1}(K)\ \forall K \in \mathcal{T} \right\}. \tag{5b}$$

In addition, we introduce also discontinuous finite element approximation spaces for the approximate traces of the velocity and pressure:

$$\bar{V}_h := \left\{ \bar{v}_h \in \left[L^2(\mathcal{F})\right]^d,\ \bar{v}_h \in \left[P_k(F)\right]^d\ \forall F \in \mathcal{F},\ \bar{v}_h = 0 \text{ on } \Gamma \right\}, \tag{6a}$$

$$\bar{Q}_h := \left\{ \bar{q}_h \in L^2(\mathcal{F}),\ \bar{q}_h \in P_k(F)\ \forall F \in \mathcal{F} \right\}, \tag{6b}$$

where $P_l(D)$ denotes the space of polynomials of degree $l > 0$ on a domain D. For notational purposes, we introduce the spaces $V_h^\star = V_h \times \bar{V}_h$, $Q_h^\star = Q_h \times \bar{Q}_h$ and $X_h^\star = V_h^\star \times Q_h^\star$. For notational convenience, we denote function pairs in $V_h^\star$ and $Q_h^\star$ by boldface, e.g., $\boldsymbol{v}_h = (v_h, \bar{v}_h) \in V_h^\star$ and $\boldsymbol{q}_h = (q_h, \bar{q}_h) \in Q_h^\star$.

The HDG formulation for the Navier–Stokes problem eq. (1) is given by [32]: given $f \in \left[L^2(\Omega)\right]^d$, find $(\boldsymbol{u}_h, \boldsymbol{p}_h) \in X_h^\star$ such that

$$a_h(\boldsymbol{u}_h, \boldsymbol{v}_h) + o_h(u_h; \boldsymbol{u}_h, \boldsymbol{v}_h) + b_h(\boldsymbol{p}_h, v_h) = \sum_{K \in \mathcal{T}} \int_K f \cdot v_h \,\mathrm{d}x \qquad \forall \boldsymbol{v}_h \in V_h^\star, \tag{7a}$$

$$b_h(\boldsymbol{q}_h, u_h) = 0 \qquad \forall \boldsymbol{q}_h \in Q_h^\star, \tag{7b}$$

where

$$a_h(\boldsymbol{u},\boldsymbol{v}) := \sum_{K\in\mathcal{T}}\int_K \nu\nabla u : \nabla v \,\mathrm{d}x + \sum_{K\in\mathcal{T}}\int_{\partial K}\frac{\alpha\nu}{h_K}(u-\bar{u})\cdot(v-\bar{v})\,\mathrm{d}s, \tag{8a}$$

$$- \sum_{K\in\mathcal{T}}\int_{\partial K}\left[\nu(u-\bar{u})\cdot\partial_n v + \nu\partial_n u\cdot(v-\bar{v})\right]\mathrm{d}s,$$

$$o_h(w;\boldsymbol{u},\boldsymbol{v}) := -\sum_{K\in\mathcal{T}}\int_K u\otimes w : \nabla v\,\mathrm{d}x + \sum_{K\in\mathcal{T}}\int_{\partial K}\tfrac{1}{2}w\cdot n(u+\bar{u})\cdot(v-\bar{v})\,\mathrm{d}s \tag{8b}$$

$$+ \sum_{K\in\mathcal{T}}\int_{\partial K}\tfrac{1}{2}|w\cdot n|\,(u-\bar{u})\cdot(v-\bar{v})\,\mathrm{d}s,$$

$$b_h(\boldsymbol{p},v) := -\sum_{K\in\mathcal{T}}\int_K p\nabla\cdot v\,\mathrm{d}x + \sum_{K\in\mathcal{T}}\int_{\partial K} v\cdot n\bar{p}\,\mathrm{d}s. \tag{8c}$$

To ensure stability, the penalty parameter $\alpha > 0$ needs to be chosen sufficiently large [31, 38].

It was shown in [31] for the Stokes problem and [32] for the Navier–Stokes problem that the approximate velocity $u_h \in V_h$ obtained from the hybridized discontinuous Galerkin discretization eq. (7) possesses two appealing properties, namely, $\nabla\cdot u_h = 0$ pointwise and $u_h \in H(\mathrm{div};\Omega)$. These properties are key to proving a pressure-robust error estimate for the velocity field in section 6.

4. Preliminaries

In this section we present some stability and boundedness results of the hybridizable discontinuous Galerkin method eq. (7) and some other preliminaries. To set notation, let

$$V(h) := V_h + \left[H_0^1(\Omega)\right]^d \cap \left[H^2(\Omega)\right]^d, \qquad Q(h) := Q_h + L_0^2(\Omega)\cap H^1(\Omega), \tag{9a}$$

$$\bar{V}(h) := \bar{V}_h + \left[H_0^{3/2}(\Gamma^0)\right]^d, \qquad \bar{Q}(h) := \bar{Q}_h + H_0^{1/2}(\Gamma^0), \tag{9b}$$

and $V^\star(h) := V(h)\times\bar{V}(h)$, $Q^\star(h) := Q(h)\times\bar{Q}(h)$ and $X^\star(h) := V^\star(h)\times Q^\star(h)$. Frequent use will also be made of functions in the following space:

$$V_h^{\mathrm{div}} := \left\{v_h \in V_h :\ b_h(\boldsymbol{q}_h, v_h) = 0\ \forall \boldsymbol{q}_h \in Q_h^\star\right\}. \tag{10}$$

We denote the trace operator by $\gamma : H^k(\Omega)\to H^{k-1/2}(\Gamma^0)$ to restrict functions in $H^s(\Omega)$ to Γ^0. The trace operator is applied component-wise for functions in $\left[H^s(\Omega)\right]^d$.

Given D an open subset of $\mathbb{R}^d$ we denote for scalar-valued functions $p, q \in L^2(D)$ the standard inner-product by $(p,q)_D := \int_D pq\,\mathrm{d}x$ and its corresponding norm $\|p\|_D := \sqrt{(p,p)_D}$. Furthermore, we define $(p,q)_{\mathcal{T}} := \sum_{K\in\mathcal{T}}(p,q)_K$ and denote the usual L^2-norm on Ω by $\|p\| := \sqrt{(p,p)_{\mathcal{T}}}$. For a scalar-valued function $p\in H^k(\Omega)$, we denote by $\|p\|_k$ the usual $H^k(\Omega)$-norm.

For scalar-valued functions $p, q \in L^2(F)$, where $F\subset\mathbb{R}^{d-1}$, we define the inner-product $\langle p,q\rangle_F := \int_F pq\,\mathrm{d}s$ with norm $\|p\|_F = \sqrt{\langle p,p\rangle_F}$. Similar definitions hold for vector-valued functions.

We introduce the following mesh-dependent inner-product and norms:

$$(\boldsymbol{u},\boldsymbol{v})_v := (\nabla u, \nabla v)_{\mathcal{T}} + \sum_{K\in\mathcal{T}} \alpha h_K^{-1}\langle \bar{u}-u, \bar{v}-v\rangle_{\partial K} \qquad \boldsymbol{u},\boldsymbol{v}\in V^\star(h), \tag{11a}$$

$$|||\boldsymbol{v}|||_v^2 := \sum_{K\in\mathcal{T}} \|\nabla v\|_K^2 + \sum_{K\in\mathcal{T}} \alpha h_K^{-1}\|\bar{v}-v\|_{\partial K}^2 \qquad \boldsymbol{v}\in V^\star(h), \tag{11b}$$

$$|||\boldsymbol{v}|||_{v'}^2 := |||\boldsymbol{v}|||_v^2 + \sum_{K\in\mathcal{T}} \frac{h_K}{\alpha}\left\|\frac{\partial v}{\partial n}\right\|_{\partial K}^2 \qquad \boldsymbol{v}\in V^\star(h), \tag{11c}$$

$$|||\boldsymbol{q}|||_p^2 := \|q\|^2 + \sum_{K\in\mathcal{T}} h_K\|\bar{q}\|_{\partial K}^2 \qquad \boldsymbol{q}\in Q^\star(h), \tag{11d}$$

where we note that $|||\cdot|||_v$ and $|||\cdot|||_{v'}$ are equivalent on $V_h^\star$, see [31]. We define also

$$|||(\boldsymbol{v}_h,\boldsymbol{q}_h)|||_{v,p}^2 := \nu|||\boldsymbol{v}_h|||_v^2 + \nu^{-1}|||\boldsymbol{q}_h|||_p^2 \qquad (\boldsymbol{v}_h,\boldsymbol{q}_h)\in X_h^\star, \tag{12a}$$

$$\begin{aligned}|||(\boldsymbol{v},\boldsymbol{q})|||_{v',p'}^2 &:= |||(\boldsymbol{v},\boldsymbol{q})|||_{v,p}^2 + \sum_{K\in\mathcal{T}} \frac{\nu h_K}{\alpha}\left\|\frac{\partial v}{\partial n}\right\|_{\partial K}^2 \\ &= \nu|||\boldsymbol{v}|||_{v'}^2 + \nu^{-1}|||\boldsymbol{q}|||_p^2 \qquad (\boldsymbol{v},\boldsymbol{q})\in X^\star(h).\end{aligned} \tag{12b}$$

The standard discrete H^1-norm for $v\in V(h)$ is defined as $\|v\|_{1,h} := |||(v_h, \{\!\{v_h\}\!\})|||_v$, where $\{\!\{v\}\!\} := \frac{1}{2}(v^+ + v^-)$ is the average operator and $v^\pm$ denote the trace of v from the interior of $K^\pm$. Furthermore, use will be made of the following discrete Poincaré inequality:

$$\|v_h\| \le c_p\|v_h\|_{1,h} \le c_p|||\boldsymbol{v}_h|||_v \quad \forall \boldsymbol{v}_h\in V_h^\star, \tag{13}$$

where c_p is a constant independent of h_K [12, Theorem 5.3].

Previously it was shown [31, Lemmas 4.2 and 4.3] that for sufficiently large α, the bilinear form $a_h(\cdot,\cdot)$ is coercive and bounded, i.e., there exist constants $c_a^s > 0$ and $c_a^b > 0$, independent of h, such that for all $\boldsymbol{v}_h\in V_h^\star$ and $\boldsymbol{u},\boldsymbol{v}\in V^\star(h)$

$$a_h(\boldsymbol{v}_h,\boldsymbol{v}_h) \ge \nu c_a^s|||\boldsymbol{v}_h|||_v^2 \quad \text{and} \quad |a_h(\boldsymbol{u},\boldsymbol{v})| \le \nu c_a^b|||\boldsymbol{u}|||_{v'}|||\boldsymbol{v}|||_{v'}. \tag{14}$$

The boundedness of $b_h(\cdot,\cdot)$ was shown in the proof of [31, Lemma 4.8], i.e., there exists a constant $c_b^b > 0$, independent of h, such that for all $\boldsymbol{v}\in V^\star(h)$ and $\boldsymbol{q}\in Q^\star(h)$,

$$|b_h(\boldsymbol{q},v)| \le c_b^b|||\boldsymbol{v}|||_v|||\boldsymbol{q}|||_p, \tag{15}$$

while the stability of $b_h(\cdot,\cdot)$ was proven in [34, Lemma 1]: there exists a constant $\beta_p > 0$, independent of h, such that for all $\boldsymbol{q}_h\in Q_h^\star$

$$\beta_p|||\boldsymbol{q}_h|||_p \le \sup_{\boldsymbol{v}_h\in V_h^\star} \frac{b_h(\boldsymbol{q}_h,v_h)}{|||\boldsymbol{v}_h|||_v}. \tag{16}$$

Discrete inf-sup stability follows from coercivity of $a_h(\cdot,\cdot)$ eq. (14) and the stability of $b_h(\cdot,\cdot)$ eq. (16), e.g. [12, Lemma 6.13]: there exists a constant $\sigma > 0$, independent of h and ν, such that for all $(\boldsymbol{v}_h,\boldsymbol{q}_h)\in X_h^\star$

$$\sigma|||(\boldsymbol{v}_h,\boldsymbol{q}_h)|||_{v,p} \le \sup_{(\boldsymbol{w}_h,\boldsymbol{r}_h)\in X_h^\star} \frac{a_h(\boldsymbol{v}_h,\boldsymbol{w}_h) + b_h(\boldsymbol{q}_h,w_h) - b_h(\boldsymbol{r}_h,v_h)}{|||(\boldsymbol{w}_h,\boldsymbol{r}_h)|||_{v,p}}. \tag{17}$$

For the trilinear form $o_h(\cdot;\cdot,\cdot)$ it was shown [3, Proposition 3.6] that for $w_h \in V_h^{\text{div}}$

$$o_h(w_h; \boldsymbol{v}_h, \boldsymbol{v}_h) = \frac{1}{2}\sum_{K\in\mathcal{T}}\int_{\partial K} |w_h \cdot n||v_h - \bar{v}_h|^2 \,\mathrm{d}s \geq 0 \qquad \forall \boldsymbol{v}_h \in V_h^\star. \tag{18}$$

It was also shown [3, Proposition 3.4] that for $w_1, w_2 \in V(h)$, $\boldsymbol{u} \in V^\star(h)$ and $\boldsymbol{v} \in V^\star(h)$ that

$$\left|o_h(w_1; \boldsymbol{u}, \boldsymbol{v}) - o_h(w_2; \boldsymbol{u}, \boldsymbol{v})\right| \leq c_o \|w_1 - w_2\|_{1,h} \,|||\boldsymbol{u}|||_v |||\boldsymbol{v}|||_v. \tag{19}$$

Finally, we note that if $(u,p) \in \left(\left[H_0^1(\Omega)\right]^d \cap \left[H^2(\Omega)\right]^d\right) \times \left(L_0^2(\Omega) \cap H^1(\Omega)\right)$, letting $\boldsymbol{u} = (u, \gamma(u))$ and $\boldsymbol{p} = (p, \gamma(p))$, then

$$a_h(\boldsymbol{u}, \boldsymbol{v}_h) + o_h(u; \boldsymbol{u}, \boldsymbol{v}_h) + b_h(\boldsymbol{p}, v_h) + b_h(\boldsymbol{q}_h, u) = \int_\Omega f \cdot v_h \,\mathrm{d}x \quad \forall (\boldsymbol{v}_h, \boldsymbol{q}_h) \in X_h^\star. \tag{20}$$

This consistency result follows immediately from [31, Lemma 4.1] and noting that, after integration by parts, using that u and $\bar{v}_h$ are single-valued on cell boundaries, and that $\bar{v}_h = 0$ on Γ,

$$o_h(u; \boldsymbol{u}, \boldsymbol{v}_h) = \sum_{K\in\mathcal{T}}\int_K \nabla\cdot(u \otimes u)\cdot v_h \,\mathrm{d}x. \tag{21}$$

5. Existence and uniqueness

The hybridizable discontinuous Galerkin method for the Navier–Stokes problem eq. (7) results in a system of nonlinear algebraic equations. To show existence and uniqueness of this nonlinear system, we use the classic Brouwer's fixed point theorem, e.g., [14].

Lemma 5.1 (Existence and uniqueness)**.** *Assuming*

$$\|f\| < \frac{(\nu c_a^s)^2}{c_o c_p}, \tag{22}$$

where c_p is the constant from eq. (13), c_a^s is the constant from eq. (14), and c_o is the constant from eq. (19), there exists a unique solution $(\boldsymbol{u}_h, \boldsymbol{p}_h) \in X_h^\star$ to the hybridizable discontinuous Galerkin method for the Navier–Stokes problem eq. (7). Furthermore,

$$|||\boldsymbol{u}_h|||_v \leq c_p(c_a^s\nu)^{-1}\|f\| \quad \text{and} \quad \sigma|||(\boldsymbol{u}_h, \boldsymbol{p}_h)|||_{v,p} \leq c_p\|f\| + \frac{c_o c_p^2}{(c_a^s\nu)^2}\|f\|^2, \tag{23}$$

where σ is the discrete inf-sup constant eq. (17).

Proof. We prove first existence of a solution $\boldsymbol{u}_h \in V_h^{\text{div}} \times \bar{V}_h$ to eq. (7). We start by defining a mapping $\Psi : V_h^{\text{div}} \times \bar{V}_h \to V_h^{\text{div}} \times \bar{V}_h$ by

$$\forall \boldsymbol{w}_h, \boldsymbol{v}_h \in V_h^{\text{div}} \times \bar{V}_h, \quad \left(\Psi(\boldsymbol{w}_h), \boldsymbol{v}_h\right)_v = a_h(\boldsymbol{w}_h, \boldsymbol{v}_h) + o_h(w_h; \boldsymbol{w}_h, \boldsymbol{v}_h) - (f, v_h)_{\mathcal{T}}. \tag{24}$$

Taking $\boldsymbol{v}_h = \boldsymbol{w}_h$ in eq. (24) we find by coercivity of $a_h(\cdot,\cdot)$ eq. (14), positivity of $o_h(\cdot;\cdot,\cdot)$ eq. (18), Cauchy–Schwarz and eq. (13),

$$\left(\Psi(\boldsymbol{w}_h), \boldsymbol{w}_h\right)_v \geq \left(\nu c_a^s |||\boldsymbol{w}_h|||_v - c_p\|f\|\right)|||\boldsymbol{w}_h|||_v. \tag{25}$$

For all $\boldsymbol{w}_h \in V_h^{\text{div}} \times \bar{V}_h$ that satisfy $\lVert\!\lvert \boldsymbol{w}_h \rVert\!\rvert_v = c_p (c_a^s \nu)^{-1} \lVert f \rVert$ we therefore find that $(\Psi(\boldsymbol{w}_h), \boldsymbol{w}_h)_v \geq 0$. Brouwer's fixed point theorem implies the existence of $\boldsymbol{u}_h \in \mathcal{B}_h := \{\boldsymbol{v}_h \in V_h^{\text{div}} \times \bar{V}_h \;:\; \lVert\!\lvert \boldsymbol{v}_h \rVert\!\rvert_v \leq c_p (c_a^s \nu)^{-1} \lVert f \rVert\}$ such that $\Psi(\boldsymbol{u}_h) = 0$. Equivalently, there exists $\boldsymbol{u}_h \in V_h^{\text{div}} \times \bar{V}_h$ satisfying the first estimate in eq. (23) and

$$a_h(\boldsymbol{u}_h, \boldsymbol{v}_h) + o_h(u_h; \boldsymbol{u}_h, \boldsymbol{v}_h) = (f, v_h)_{\mathcal{T}} \quad \forall \boldsymbol{v}_h \in V_h^{\text{div}} \times \bar{V}_h, \tag{26}$$

proving the existence of a solution $\boldsymbol{u}_h$ to eq. (7) restricted to $V_h^{\text{div}} \times \bar{V}_h$.

Next we prove uniqueness of $\boldsymbol{u}_h \in V_h^{\text{div}} \times \bar{V}_h$ to eq. (7). For this, assume two solutions $\boldsymbol{u}_{h,1} \in V_h^{\text{div}} \times \bar{V}_h$ and $\boldsymbol{u}_{h,2} \in V_h^{\text{div}} \times \bar{V}_h$ that both solve eq. (7). We will show that $\boldsymbol{u}_{h,1} = \boldsymbol{u}_{h,2}$ under the smallness assumption eq. (22). We first note that coercivity of $a_h(\cdot,\cdot)$ eq. (14) implies

$$\nu c_a^s \lVert\!\lvert \boldsymbol{u}_{h,1} - \boldsymbol{u}_{h,2} \rVert\!\rvert_v^2 \leq a_h(\boldsymbol{u}_{h,1} - \boldsymbol{u}_{h,2}, \boldsymbol{u}_{h,1} - \boldsymbol{u}_{h,2}). \tag{27}$$

Furthermore, note that for any $\boldsymbol{v}_h \in V_h^{\text{div}} \times \bar{V}_h$,

$$a_h(\boldsymbol{u}_{h,1} - \boldsymbol{u}_{h,2}, \boldsymbol{v}_h) + o_h(u_{h,1}; \boldsymbol{u}_{h,1}, \boldsymbol{v}_h) - o_h(u_{h,2}; \boldsymbol{u}_{h,2}, \boldsymbol{v}_h) = 0. \tag{28}$$

Combining eq. (27) and eq. (28),

$$\begin{aligned} \nu c_a^s \lVert\!\lvert \boldsymbol{u}_{h,1} - \boldsymbol{u}_{h,2} \rVert\!\rvert_v^2 \leq & o_h(u_{h,2}; \boldsymbol{u}_{h,1}, \boldsymbol{u}_{h,1} - \boldsymbol{u}_{h,2}) - o_h(u_{h,1}; \boldsymbol{u}_{h,1}, \boldsymbol{u}_{h,1} - \boldsymbol{u}_{h,2}) \\ & - o_h(u_{h,2}; \boldsymbol{u}_{h,1} - \boldsymbol{u}_{h,2}, \boldsymbol{u}_{h,1} - \boldsymbol{u}_{h,2}) \\ \leq & o_h(u_{h,2} - u_{h,1}; \boldsymbol{u}_{h,1}, \boldsymbol{u}_{h,1} - \boldsymbol{u}_{h,2}), \end{aligned} \tag{29}$$

since $o_h(u_{h,2}; \boldsymbol{u}_{h,1} - \boldsymbol{u}_{h,2}, \boldsymbol{u}_{h,1} - \boldsymbol{u}_{h,2}) \geq 0$ by eq. (18). Next, by eq. (19) and eq. (23)

$$\begin{aligned} \nu c_a^s \lVert\!\lvert \boldsymbol{u}_{h,1} - \boldsymbol{u}_{h,2} \rVert\!\rvert_v^2 &\leq c_o \lVert u_{h,2} - u_{h,1} \rVert_{1,h} \lVert\!\lvert \boldsymbol{u}_{h,1} \rVert\!\rvert_v \lVert\!\lvert \boldsymbol{u}_{h,1} - \boldsymbol{u}_{h,2} \rVert\!\rvert_v \\ &\leq c_o \lVert\!\lvert \boldsymbol{u}_{h,1} \rVert\!\rvert_v \lVert\!\lvert \boldsymbol{u}_{h,1} - \boldsymbol{u}_{h,2} \rVert\!\rvert_v^2 \\ &\leq c_o c_p (c_a^s \nu)^{-1} \lVert f \rVert \, \lVert\!\lvert \boldsymbol{u}_{h,1} - \boldsymbol{u}_{h,2} \rVert\!\rvert_v^2, \end{aligned} \tag{30}$$

implying

$$\left((\nu c_a^s)^2 - c_o c_p \lVert f \rVert \right) \lVert\!\lvert \boldsymbol{u}_{h,1} - \boldsymbol{u}_{h,2} \rVert\!\rvert_v^2 \leq 0. \tag{31}$$

By eq. (22) it follows that $\boldsymbol{u}_{h,1} = \boldsymbol{u}_{h,2}$, proving uniqueness of $\boldsymbol{u}_h \in V_h^{\text{div}} \times \bar{V}_h$.

We next prove the existence and uniqueness of $\boldsymbol{p}_h$. Given the solution $\boldsymbol{u}_h \in V_h^{\text{div}} \times \bar{V}_h$, the pressure $\boldsymbol{p}_h \in Q_h^{\star}$ is the solution to

$$b_h(\boldsymbol{p}_h, v_h) = (f, v_h)_{\mathcal{T}} - a_h(\boldsymbol{u}_h, \boldsymbol{v}_h) - o_h(u_h; \boldsymbol{u}_h, \boldsymbol{v}_h) \quad \forall \boldsymbol{v}_h \in V_h^{\star}. \tag{32}$$

Since $a_h(\boldsymbol{u}_h, \cdot)$ and $o_h(u_h; \boldsymbol{u}_h, \cdot)$ are bounded linear functionals on $V_h^{\star}$ by, respectively eq. (14) and eq. (19), the right hand side itself is a bounded linear functional on $V_h^{\star}$. Existence of a unique solution $\boldsymbol{p}_h \in Q_h^{\star}$ to eq. (32) is now guaranteed by the inf-sup condition eq. (16).

Lastly, we prove the second estimate of eq. (23). By eq. (7) $a_h(\boldsymbol{u}_h, \boldsymbol{w}_h) + b_h(\boldsymbol{p}_h, w_h) - b_h(\boldsymbol{r}_h, u_h) = (f, w_h)_{\mathcal{T}} - o_h(u_h; \boldsymbol{u}_h, \boldsymbol{w}_h)$. Discrete inf-sup stability eq. (17), and boundedness of $o_h(\cdot;\cdot,\cdot)$ eq. (19) therefore result in

$$\sigma \lVert\!\lvert (\boldsymbol{u}_h, \boldsymbol{p}_h) \rVert\!\rvert_{v,p} \leq \sup_{(\boldsymbol{w}_h, \boldsymbol{r}_h) \in X_h^{\star}} \frac{(f, w_h)_{\mathcal{T}} - o_h(u_h; \boldsymbol{u}_h, \boldsymbol{w}_h)}{\lVert\!\lvert (\boldsymbol{w}_h, \boldsymbol{r}_h) \rVert\!\rvert_{v,p}} \leq c_p \lVert f \rVert + c_o \lVert\!\lvert \boldsymbol{u}_h \rVert\!\rvert_v^2. \tag{33}$$

The result follows from the first estimate in eq. (23). □

6. Error analysis

In this section we prove that the HDG method eq. (7) for the Navier–Stokes problem is pressure-robust, i.e., the velocity error is pressure-independent. Let $\Pi_{\mathrm{BDM}} : \left[H^1(\Omega)\right]^d \to V_h$ be the usual Brezzi–Douglas–Marini (BDM) interpolation operator as given in the following lemma [15, Lemma 7].

Lemma 6.1. *If the mesh consists of triangles in two dimensions or tetrahedra in three dimensions there is an interpolation operator $\Pi_{\mathrm{BDM}} : [H^1(\Omega)]^d \to V_h$ with the following properties for all $u \in [H^{k+1}(K)]^d$:*

(i) $[\![n \cdot \Pi_{\mathrm{BDM}} u]\!] = 0$, *where* $[\![a]\!] = a^+ + a^-$ *and* $[\![a]\!] = a$ *on, respectively, interior and boundary faces is the usual jump operator.*

(ii) $\|u - \Pi_{\mathrm{BDM}} u\|_{m,K} \le c h_K^{l-m} \|u\|_{l,K}$ *with* $m = 0, 1, 2$ *and* $m \le l \le k+1$.

(iii) $\left\|\nabla \cdot (u - \Pi_{\mathrm{BDM}} u)\right\|_{m,K} \le c h_K^{l-m} \|\nabla \cdot u\|_{l,K}$ *with* $m = 0, 1$ *and* $m \le l \le k$.

(iv) $\int_K q(\nabla \cdot u - \nabla \cdot \Pi_{\mathrm{BDM}} u) \,\mathrm{d}x = 0$ *for all* $q \in P_{k-1}(K)$.

(v) $\int_F \bar{q}(n \cdot u - n \cdot \Pi_{\mathrm{BDM}} u) \,\mathrm{d}s = 0$ *for all* $\bar{q} \in P_k(F)$, *where* F *is a face on* ∂K.

Furthermore, let $\bar{\Pi}_V$, Π_Q and $\bar{\Pi}_Q$ be the standard L^2-projection operators onto $\bar{V}_h$, Q_h and $\bar{Q}_h$, respectively. We then introduce the approximation and interpolation errors

$$\begin{aligned}
&\xi_u = u - \Pi_{\mathrm{BDM}} u, && \zeta_u = u_h - \Pi_{\mathrm{BDM}} u, && \bar{\xi}_u = \gamma(u) - \bar{\Pi}_V u, && \bar{\zeta}_u = \bar{u}_h - \bar{\Pi}_V u,\\
&\xi_p = p - \Pi_Q p, && \zeta_p = p_h - \Pi_Q p, && \bar{\xi}_p = \gamma(p) - \bar{\Pi}_Q p, && \bar{\zeta}_p = \bar{p}_h - \bar{\Pi}_Q p,
\end{aligned}$$

and, for notational convenience, $\boldsymbol{\xi}_u = (\xi_u, \bar{\xi}_u)$, $\boldsymbol{\zeta}_u = (\zeta_u, \bar{\zeta}_u)$, $\boldsymbol{\xi}_p = (\xi_p, \bar{\xi}_p)$ and $\boldsymbol{\zeta}_p = (\zeta_p, \bar{\zeta}_p)$. Subtracting now the HDG method eq. (7) from the consistency result eq. (20) and splitting the errors, we obtain the following error equation:

$$\begin{aligned}
a_h(\boldsymbol{\zeta}_u, \boldsymbol{v}_h) + b_h(\boldsymbol{\zeta}_p, v_h) + b_h(\boldsymbol{q}_h, \zeta_u) =& a_h(\boldsymbol{\xi}_u, \boldsymbol{v}_h) + b_h(\boldsymbol{\xi}_p, v_h) + b_h(\boldsymbol{q}_h, \xi_u)\\
&- o_h(u; \boldsymbol{\zeta}_u, \boldsymbol{v}_h) - o_h(\zeta_u; \boldsymbol{u}_h, \boldsymbol{v}_h)\\
&+ o_h(u; \boldsymbol{\xi}_u, \boldsymbol{v}_h) + o_h(\xi_u; \boldsymbol{u}_h, \boldsymbol{v}_h).
\end{aligned} \tag{34}$$

In the following lemma we will find an energy estimate for the velocity error.

Theorem 6.1 (Pressure robust velocity error estimate)**.** *Let C_p and C_o be the constants in eq. (3). Furthermore let c_p be the discrete Poincaré constant of eq. (13), c_o the constant in eq. (19) and c_a^s the constant in eq. (14). Let $u \in \left[H^{k+1}(\Omega)\right]^d$ be the velocity solution to the Navier–Stokes problem eq. (1), $\boldsymbol{u} = (u, \gamma(u))$, and $\boldsymbol{u}_h \in \boldsymbol{V}_h^\star$ the velocity solution of the HDG discretization eq. (7). Then assuming the smallness condition*

$$c_o' c_p' \|f\| \le \tfrac{1}{2} \nu^2 (c_a')^2, \tag{35}$$

where $c_p' = \max\{C_p, c_p\}$, $c_o' = \max\{C_o, c_o\}$ and $c_a' = \min\{1, c_a^s\}$ we obtain the pressure-robust velocity error estimate

$$\|\!|\boldsymbol{u} - \boldsymbol{u}_h|\!\|_v \le c h^k \|u\|_{k+1}, \tag{36}$$

where $c > 0$ a constant independent of h and ν.

Proof. In the error equation eq. (34) take $(\boldsymbol{v}_h, \boldsymbol{q}_h) = (\boldsymbol{\zeta}_u, -\boldsymbol{\zeta}_p)$. Then, by coercivity of $a_h(\cdot,\cdot)$ eq. (14)

$$\begin{aligned}\nu c_a^s \|\!|\!|\boldsymbol{\zeta}_u|\!|\!|_v^2 \le a_h(\boldsymbol{\zeta}_u, \boldsymbol{\zeta}_u) =& a_h(\boldsymbol{\xi}_u, \boldsymbol{\zeta}_u) + b_h(\boldsymbol{\xi}_p, \zeta_u) - b_h(\boldsymbol{\zeta}_p, \xi_u) \\ & - o_h(u; \boldsymbol{\zeta}_u, \boldsymbol{\zeta}_u) - o_h(\zeta_u; \boldsymbol{u}_h, \boldsymbol{\zeta}_u) \\ & + o_h(u; \boldsymbol{\xi}_u, \boldsymbol{\zeta}_u) + o_h(\xi_u; \boldsymbol{u}_h, \boldsymbol{\zeta}_u).\end{aligned} \tag{37}$$

By properties of the BDM interpolation operator and using that u_h is pointwise divergence-free and divergence-conforming, we note that $b_h(\boldsymbol{\xi}_p, \zeta_u) = 0$ and $b_h(\boldsymbol{\zeta}_p, \xi_u) = 0$. Furthermore, $o_h(u; \boldsymbol{\zeta}_u, \boldsymbol{\zeta}_u) \ge 0$ so that

$$\nu c_a^s \|\!|\!|\boldsymbol{\zeta}_u|\!|\!|_v^2 \le a_h(\boldsymbol{\xi}_u, \boldsymbol{\zeta}_u) - o_h(\zeta_u; \boldsymbol{u}_h, \boldsymbol{\zeta}_u) + o_h(u; \boldsymbol{\xi}_u, \boldsymbol{\zeta}_u) + o_h(\xi_u; \boldsymbol{u}_h, \boldsymbol{\zeta}_u). \tag{38}$$

We next bound each term on the right hand side separately.

By boundedness of $a_h(\cdot,\cdot)$ eq. (14),

$$a_h(\boldsymbol{\xi}_u, \boldsymbol{\zeta}_u) \le \nu c_a^b \|\!|\!|\boldsymbol{\xi}_u|\!|\!|_{v'} \|\!|\!|\boldsymbol{\zeta}_u|\!|\!|_{v'} \le \nu c c_a^b \|\!|\!|\boldsymbol{\xi}_u|\!|\!|_{v'} \|\!|\!|\boldsymbol{\zeta}_u|\!|\!|_v, \tag{39}$$

where the second inequality is by equivalence of $\|\!|\!|\cdot|\!|\!|_{v'}$ and $\|\!|\!|\cdot|\!|\!|_v$ on $V_h^\star$.

From eq. (4) and eq. (35) it follows that $\|u\|_1 \le \frac{1}{2} c_a' c_o^{-1} \nu$. Furthermore, from eq. (23) and eq. (35) it follows that $\|\!|\!|\boldsymbol{u}_h|\!|\!|_v \le \frac{1}{2} c_a' c_o^{-1} \nu$. Then, by eq. (19),

$$o_h(u; \boldsymbol{\xi}_u, \boldsymbol{\zeta}_u) \le c_o \|u\|_1 \|\!|\!|\boldsymbol{\xi}_u|\!|\!|_v \|\!|\!|\boldsymbol{\zeta}_u|\!|\!|_v \le \tfrac{1}{2} c_a' \nu \|\!|\!|\boldsymbol{\xi}_u|\!|\!|_v \|\!|\!|\boldsymbol{\zeta}_u|\!|\!|_v, \tag{40a}$$

$$o_h(\xi_u; \boldsymbol{u}_h, \boldsymbol{\zeta}_u) \le c_o \|\xi_u\|_{1,h} \|\!|\!|\boldsymbol{u}_h|\!|\!|_v \|\!|\!|\boldsymbol{\zeta}_u|\!|\!|_v \le \tfrac{1}{2} c_a' \nu \|\!|\!|\boldsymbol{\xi}_u|\!|\!|_v \|\!|\!|\boldsymbol{\zeta}_u|\!|\!|_v, \tag{40b}$$

$$o_h(\zeta_u; \boldsymbol{u}_h, \boldsymbol{\zeta}_u) \le c_o \|\zeta_u\|_{1,h} \|\!|\!|\boldsymbol{u}_h|\!|\!|_v \|\!|\!|\boldsymbol{\zeta}_u|\!|\!|_v \le \tfrac{1}{2} c_a' \nu \|\!|\!|\boldsymbol{\zeta}_u|\!|\!|_v^2. \tag{40c}$$

Combining eq. (38)–eq. (40) and dividing by $\|\!|\!|\boldsymbol{\zeta}_u|\!|\!|_v$,

$$\tfrac{1}{2} c_a' \nu \|\!|\!|\boldsymbol{\zeta}_u|\!|\!|_v \le (\nu c_a^s - \tfrac{1}{2} c_a' \nu) \|\!|\!|\boldsymbol{\zeta}_u|\!|\!|_v \le \nu (c_a' + c c_a^b) \|\!|\!|\boldsymbol{\xi}_u|\!|\!|_{v'}. \tag{41}$$

The result follows by a triangle inequality and the interpolation estimates of the BDM interpolation operator defined in lemma 6.1 and the L^2-projection operator. □

Given the velocity error estimate of the previous theorem we can now state an error estimate for the pressure in the L^2-norm.

Lemma 6.2 (Pressure error estimate in the L^2-norm)**.** *Let* $(u,p) \in \left[H^{k+1}(\Omega)\right]^d \times H^k(\Omega)$ *be the solution to the Navier–Stokes problem eq. (1) and* $\boldsymbol{u} = (u, \gamma(u))$ *and* $\boldsymbol{p} = (p, \gamma(p))$*. Let* $(\boldsymbol{u}_h, \boldsymbol{p}_h) \in X_h^\star$ *solve eq. (7), then*

$$\|p - p_h\| \le c \left(h^k \|p\|_k + h^k \|u\|_{k+1} \right), \tag{42}$$

with $c > 0$ *a constant independent of* h *and* ν*.*

Proof. By the triangle inequality and the inf-sup condition eq. (16),

$$\begin{aligned}\|p - p_h\| &\le \|\!|\!|\boldsymbol{p} - \boldsymbol{q}_h|\!|\!|_p + \|\!|\!|\boldsymbol{p}_h - \boldsymbol{q}_h|\!|\!|_p \\ &\le \|\!|\!|\boldsymbol{p} - \boldsymbol{q}_h|\!|\!|_p + \beta_p^{-1} \sup_{\boldsymbol{v} \in V_h^\star} \frac{b_h(\boldsymbol{p} - \boldsymbol{p}_h, v_h)}{\|\!|\!|\boldsymbol{v}_h|\!|\!|_v} + \beta_p^{-1} \sup_{\boldsymbol{v} \in V_h^\star} \frac{b_h(\boldsymbol{p} - \boldsymbol{q}_h, v_h)}{\|\!|\!|\boldsymbol{v}_h|\!|\!|_v}.\end{aligned} \tag{43}$$

Bounding the third term on the right hand side using the boundedness of $b_h(\cdot,\cdot)$ eq. (15),

$$\|p - p_h\| \le \left(1 + \beta_p^{-1} c_b^b\right) |||\boldsymbol{p} - \boldsymbol{q}_h|||_p + \beta_p^{-1} \sup_{\boldsymbol{v}\in V_h^\star} \frac{b_h(\boldsymbol{p} - \boldsymbol{p}_h, v_h)}{|||\boldsymbol{v}_h|||_v}. \tag{44}$$

Proceeding as in the velocity error estimate,

$$\begin{aligned} b_h(\boldsymbol{p} - \boldsymbol{p}_h, v_h) &= a_h(\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{v}_h) + o_h(\boldsymbol{u} - \boldsymbol{u}_h; \boldsymbol{u}, \boldsymbol{v}_h) - o_h(\boldsymbol{u}_h; \boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{v}_h) \\ &\le \left(c_a^b + c_a'\right) \nu |||\boldsymbol{u} - \boldsymbol{u}_h|||_{v'} |||\boldsymbol{v}_h|||_v. \end{aligned} \tag{45}$$

Combining eq. (44) and eq. (45), and since $\boldsymbol{q}_h \in Q_h^\star$ is arbitrary,

$$\|p - p_h\| \le \left(1 + \beta_p^{-1} c_b^b\right) \inf_{\boldsymbol{q}_h\in Q_h^\star} |||\boldsymbol{p} - \boldsymbol{q}_h|||_p + \beta_p^{-1} \left(c_a^b + c_\alpha\right) \nu |||\boldsymbol{u} - \boldsymbol{u}_h|||_{v'}. \tag{46}$$

Standard interpolation estimates for the L^2-projection can be used to show that

$$\inf_{\boldsymbol{q}_h\in Q_h^\star} |||\boldsymbol{p} - \boldsymbol{q}_h|||_p \le ch^k \|p\|_k\,, \tag{47}$$

where c is a constant independent of h. To bound the second term on the right hand side of eq. (46), note that

$$|||\boldsymbol{u} - \boldsymbol{u}_h|||_{v'} \le |||\boldsymbol{\xi}_u|||_{v'} + |||\boldsymbol{\zeta}_u|||_{v'} \le |||\boldsymbol{\xi}_u|||_{v'} + c|||\boldsymbol{\zeta}_u|||_v \le c|||\boldsymbol{\xi}_u|||_{v'}, \tag{48}$$

where the last inequality is by eq. (41). The result follows from eq. (46), eq. (47), eq. (48) and the interpolation estimates of the BDM interpolation operator defined in lemma 6.1 and the L^2-projection operator. □

We end this section by showing the velocity error estimate in the L^2-norm. For this we require the solution (ϕ, ψ) to the following dual problem [18, Chapter 6]:

$$-\nu\nabla^2\phi - \nabla\cdot(u\otimes\phi) - \nabla\psi - (\nabla\phi)^T u = g \qquad \text{in } \Omega, \tag{49a}$$

$$\nabla\cdot\phi = 0 \qquad \text{in } \Omega, \tag{49b}$$

$$\phi = 0 \qquad \text{on } \Gamma. \tag{49c}$$

We assume the following regularity estimate:

$$\|\phi\|_2 + \|\psi\|_1 \le c_r\|g\|\,, \tag{50}$$

with $c_r > 0$ a constant independent of h. This regularity estimate holds for convex polyhedron Ω assuming $u \in \left[L^\infty(\Omega)\right]^d$ [3]. It will be convenient to introduce the interpolation errors

$$\xi_\phi = \phi - \Pi_{\mathrm{BDM}}\phi, \qquad \bar{\xi}_\phi = \gamma(\phi) - \bar{\Pi}_V\phi,$$

$$\xi_\psi = \psi - \Pi_Q\psi, \qquad \bar{\xi}_\psi = \gamma(\psi) - \bar{\Pi}_Q\psi.$$

and $\boldsymbol{\xi}_\phi = (\xi_\phi, \bar{\xi}_\phi)$, $\boldsymbol{\xi}_\psi = (\xi_\psi, \bar{\xi}_\psi)$.

Lemma 6.3 (Velocity error estimate in the L^2-norm)**.** *Let $u \in \left[H^{k+1}(\Omega)\right]^d \cap \left[L^\infty(\Omega)\right]^d$ be the velocity solution to the Navier–Stokes problem eq. (1), $\boldsymbol{u} = (u, \gamma(u))$, and $\boldsymbol{u}_h \in V_h^\star$ the velocity solution of the HDG discretization eq. (7). Subject to the regularity condition eq. (50), there exists a constant $c > 0$, independent of h, such that*

$$\|u - u_h\| \le ch^{k+1}\|u\|_{k+1}\,. \tag{51}$$

Proof. By definition of $a_h(\cdot,\cdot)$ eq. (8a), integration by parts, using the single-valuedness of u, $\partial_n\phi$ and $\bar{u}_h$ across cell boundaries, and that $u = \bar{u}_h = 0$ on Γ, we note that

$$a_h(\boldsymbol{u} - \boldsymbol{u}_h, (\phi, \gamma(\phi))) = -\sum_{K\in\mathcal{T}_h} \int_K \nu(u - u_h) \cdot \nabla^2\phi \,\mathrm{d}x. \tag{52}$$

Furthermore, by definition of eq. (8b), using that $\phi = \gamma(\phi)$ on cell boundaries and the identity $(a \otimes b) : C = b \cdot C^T a$ for vectors $a, b \in \mathbb{R}^m$ and tensor $C \in \mathbb{R}^{n\times n}$

$$o_h(u - u_h; (u, \gamma(u)), (\phi, \gamma(\phi))) = -\sum_{K\in\mathcal{T}_h} \int_K (u - u_h) \cdot (\nabla\phi)^T u \,\mathrm{d}x. \tag{53}$$

Similarly, using again the identity $(a \otimes b) : C = b \cdot C^T a$,

$$o_h(u; \boldsymbol{u} - \boldsymbol{u}_h, (\phi, \gamma(\phi))) = -\sum_{K\in\mathcal{T}_h} \int_K (u - u_h) \cdot \nabla \cdot (u \otimes \phi) \,\mathrm{d}x, \tag{54}$$

where we used also that $(\nabla\phi)u = (u \cdot \nabla)\phi$ and, for divergence-free u, $\nabla \cdot (u \otimes \phi) = (u \cdot \nabla)\phi$.

Next, by definition of b_h eq. (8c), integration by parts, using that u, $u_h{\cdot}n$ and ψ are single-valued across cell boundaries, and that $u = u_h = 0$ on Γ,

$$-b_h((\psi, \gamma(\psi)), u - u_h) = -\sum_{K\in\mathcal{T}} \int_K \nabla\psi \cdot (u - u_h) \,\mathrm{d}x. \tag{55}$$

Once again from the definition of b_h eq. (8c),

$$b_h(\boldsymbol{p} - \boldsymbol{p}_h, \phi) = -\int_\Omega (p - p_h)\nabla \cdot \phi \,\mathrm{d}x = 0. \tag{56}$$

since $\nabla \cdot \phi = 0$.

Adding eq. (52)–eq. (56),

$$\begin{aligned} a_h(\boldsymbol{u} - \boldsymbol{u}_h, (\phi, \gamma(\phi))) &+ o_h(u - u_h; (u, \gamma(u)), (\phi, \gamma(\phi))) \\ &+ o_h(u; \boldsymbol{u} - \boldsymbol{u}_h, (\phi, \gamma(\phi))) + b_h(\boldsymbol{p} - \boldsymbol{p}_h, \phi) - b_h((\psi, \gamma(\psi)), u - u_h) \\ &= \sum_{K\in\mathcal{T}} \int_K (u - u_h) \cdot \left(-\nu\nabla^2\phi - \nabla \cdot (u \otimes \phi) - \nabla\psi - (\nabla\phi)^T u\right) \mathrm{d}x. \end{aligned} \tag{57}$$

Taking $g = u - u_h$ in eq. (49) we therefore find that

$$\begin{aligned} \|u - u_h\|^2 = a_h(\boldsymbol{u} - \boldsymbol{u}_h, (\phi, \gamma(\phi))) &+ b_h(\boldsymbol{p} - \boldsymbol{p}_h, \phi) + o_h(u - u_h; (u, \gamma(u)), (\phi, \gamma(\phi))) \\ &+ o_h(u; \boldsymbol{u} - \boldsymbol{u}_h, (\phi, \gamma(\phi))) - b_h((\psi, \gamma(\psi)), u - u_h). \end{aligned} \tag{58}$$

Next, from the consistency of the scheme eq. (20),

$$a_h(\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{v}_h) + b_h(\boldsymbol{p} - \boldsymbol{p}_h, v_h) - o_h(u_h; \boldsymbol{u}_h, \boldsymbol{v}_h) + o_h(u; \boldsymbol{u}, \boldsymbol{v}_h) - b_h(\boldsymbol{q}_h, u - u_h) = 0. \tag{59}$$

Subtract now eq. (59) from eq. (58) and choose $\boldsymbol{v}_h = (\Pi_{\text{BDM}}\phi, \bar{\Pi}_V\phi)$ and $\boldsymbol{q}_h = (\Pi_Q\psi, \bar{\Pi}_Q\psi)$. Algebraic manipulation then results in

$$\begin{aligned} \|u - u_h\|^2 =& a_h(\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{\xi}_\phi) + b_h(\boldsymbol{p} - \boldsymbol{p}_h, \xi_\phi) + o_h(u - u_h; (u, \gamma(u)), \boldsymbol{\xi}_\phi) \\ & - o_h(u - u_h; \boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{\xi}_\phi) + o_h(u; \boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{\xi}_\phi) \\ & + o_h(u - u_h; \boldsymbol{u} - \boldsymbol{u}_h, (\phi, \gamma(\phi))) - b_h(\boldsymbol{\xi}_\psi, u - u_h) \\ =& T_1 + T_2 + T_3 + T_4 + T_5 + T_6 + T_7. \end{aligned} \tag{60}$$

Note first that

$$\begin{aligned} T_2 = b_h(\boldsymbol{p} - \boldsymbol{p}_h, \xi_\phi) = - \sum_{K \in \mathcal{T}_h} \int_K (p - p_h) \nabla \cdot (\phi - \Pi_{\text{BDM}}\phi) \,\mathrm{d}x \\ + \sum_{K \in \mathcal{T}_h} \int_{\partial K} (\phi - \Pi_{\text{BDM}}\phi) \cdot n \bar{p}_h \,\mathrm{d}s = 0, \end{aligned} \tag{61}$$

by properties of the BDM interpolation operator and the L^2-projection operator Π_Q, see [4].

We next bound the remaining terms in eq. (60). By boundedness of $a_h(\cdot,\cdot)$ eq. (14),

$$T_1 \le c_a^b \nu \|\!|\boldsymbol{u} - \boldsymbol{u}_h|\!\|_{v'} \left\|\!\left|\boldsymbol{\xi}_\phi\right|\!\right\|_{v'}. \tag{62}$$

Next, the interpolation property (ii) of the BDM projection in lemma 6.1 and the interpolation properties of the L^2-projection $\bar{\Pi}_V$ followed by assumption eq. (50), yield

$$\left\|\!\left|\boldsymbol{\xi}_\phi\right|\!\right\|_{v'} \le h\|\phi\|_2 \le h\|u - u_h\|, \tag{63}$$

so that

$$T_1 \le c\nu h \|\!|\boldsymbol{u} - \boldsymbol{u}_h|\!\|_{v'} \|u - u_h\|. \tag{64}$$

From boundedness of the trilinear form $o_h(\cdot;\cdot,\cdot)$ eq. (19), the smallness assumption eq. (35), and eq. (63)

$$T_3 \le \tfrac{1}{2} c_\alpha \nu \|\!|\boldsymbol{u} - \boldsymbol{u}_h|\!\|_{v'} \left\|\!\left|\boldsymbol{\xi}_\phi\right|\!\right\|_{v'} \le c\nu h \|\!|\boldsymbol{u} - \boldsymbol{u}_h|\!\|_{v'} \|u - u_h\|, \tag{65}$$

and, similarly,

$$T_4 + T_5 \le c\nu h \|\!|\boldsymbol{u} - \boldsymbol{u}_h|\!\|_{v'} \|u - u_h\|. \tag{66}$$

For T_6, using the boundedness of the trilinear form $o_h(\cdot;\cdot,\cdot)$ eq. (19), the fact that $\|\phi\|_1 \le \|\phi\|_2$, and eq. (36),

$$\begin{aligned} T_6 \le C_O \|\!|\boldsymbol{u} - \boldsymbol{u}_h|\!\|_{v'}^2 \|\phi\|_1 \le ch \|\!|\boldsymbol{u} - \boldsymbol{u}_h|\!\|_{v'} \|u\|_2 \|\phi\|_1 \\ \le ch \|\!|\boldsymbol{u} - \boldsymbol{u}_h|\!\|_{v'} \|u\|_2 \|u - u_h\|. \end{aligned} \tag{67}$$

To bound T_7, we use the boundedness of $b_h(\cdot,\cdot)$ eq. (15), standard interpolation estimates for the L^2-projections Π_Q and $\bar{\Pi}_Q$, and the regularity assumption eq. (50) to find

$$T_7 \le c_b^b \left\|\!\left|\boldsymbol{\xi}_\psi\right|\!\right\|_p \|\!|\boldsymbol{u} - \boldsymbol{u}_h|\!\|_{v'} \le ch\|\psi\|_1 \|\!|\boldsymbol{u} - \boldsymbol{u}_h|\!\|_{v'} \le ch \|\!|\boldsymbol{u} - \boldsymbol{u}_h|\!\|_{v'} \|u - u_h\|. \tag{68}$$

The result follows after collecting eq. (64)–eq. (68), dividing both sides by $\|u - u_h\|$ and applying the interpolation estimates of the BDM interpolation operator defined in lemma 6.1 and the L^2-projection operator. □

7. Numerical examples

In this section we present numerical examples that demonstrate optimality and pressure-robustness of the scheme. All numerical examples have been implemented with the penalty parameter $\alpha = 10k^2$ using the high order finite element library NGSolve [35].

In all test cases below, we compare the HDG method analyzed in this paper to the HDG method proposed in [21]. The method proposed in [21] considers a smaller pressure trace function space in that $\bar{Q}_h$ eq. (6b) is replaced by

$$\tilde{Q}_h := \left\{ \bar{q}_h \in L^2(\mathcal{F}),\ \bar{q}_h \in P_{k-1}(F)\ \forall F \in \mathcal{F} \right\}.$$

The velocity and pressure estimates of this scheme are optimal, see [31] for the analysis of the Stokes problem. Despite the velocity field obtained by the discretization in [21] being pointwise divergence free, the method is not pressure robust. This can be attributed to the fact that the approximate velocity field is not divergence-conforming.

7.1. No flow problem

In this first example we consider the no flow problem from [19, Example 1.1] adapted to the stationary Navier–Stokes problem. For this we take $\Omega = (0,1)^2$, set $\nu = 1$, and apply homogeneous Dirichlet boundary conditions. The source term is taken to be $f = (0, r(1 - y + 3y^2))$, where $r > 0$ is a parameter. The exact solution to this problem is $u = 0$ and $p = r(y^3 - \frac{1}{2}y^2 + y - \frac{7}{12})$. Changing the parameter r should affect only the pressure solution. This example tests whether the numerical scheme mimics this property.

In fig. 1 we plot the velocity and pressure errors using a polynomial approximation with $k = 2$ for $r = 1$ and $r = 10^6$. We observe in fig. 1a that the velocity error using the HDG method that is not divergence-conforming is, as expected, not pressure-robust. Although the velocity converges optimally, increasing the parameter r increases the error in the velocity. Conversely, the error in the velocity of the divergence-conforming method is of machine-precision, no matter the grid size. Although the error in the velocity increases as r increases, this can be attributed to an increase in the condition number of the matrix that needs to be inverted at each Picard iteration.

The pressure-errors are identical for both HDG methods, see fig. 1b. The errors in the pressure converge optimally and increase as r increases.

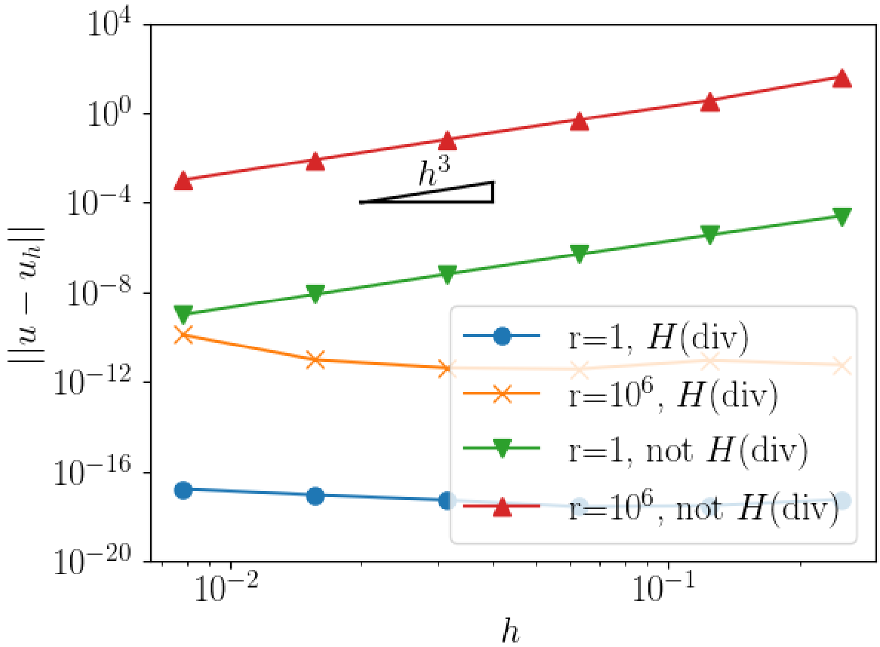
7.2. Potential flow problem

We next consider the potential flow problem from [25, Example 4]. Setting $f = 0$, this test case was constructed such that pressure is balanced by the nonlinear convection terms, and serves to show that nonlinear convection terms can also induce a lack of pressure-robustness [19].

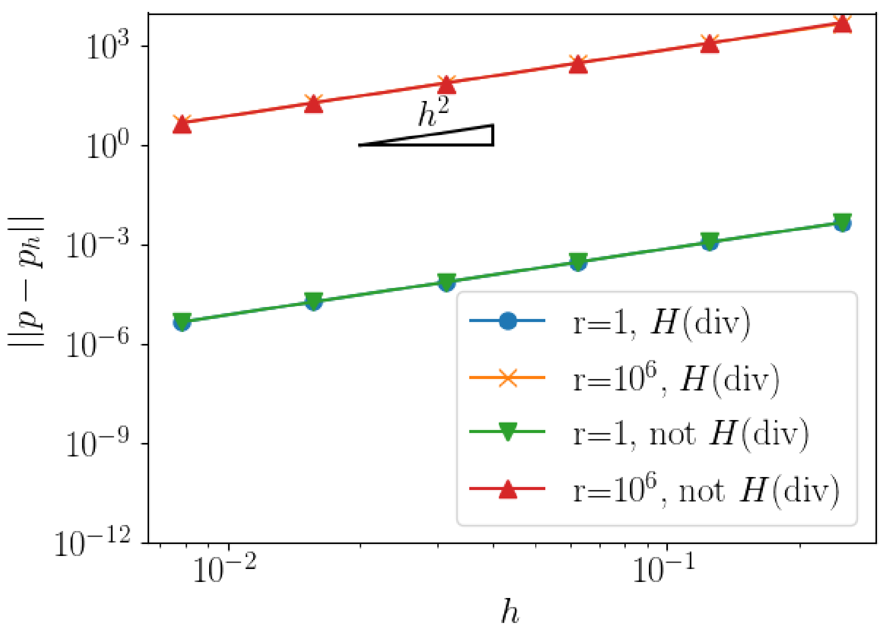
On the domain $\Omega = (-\frac{1}{2}, \frac{1}{2})^2$, we consider the problem where the exact solution is given by $u = \nabla\phi$ and $p = -\frac{1}{2}|u|^2$, with the harmonic function $\phi = y^5 + 5x^4y - 10x^2y^3$.

In fig. 2 we plot the velocity and pressure errors using a polynomial approximation with $k = 2$ for $\nu = 10^5$ and $\nu = 10^{-5}$. We observe optimal rates of convergence for both methods for velocity and pressure.

For the HDG method that is not divergence-conforming, however, the errors in the velocity and pressure increase significantly as the viscosity is decreased. Furthermore, there is no convergence of the non-linear solvers for large h for the case that $\nu = 10^{-5}$. This was observed also in [19, 25] for schemes that are not pressure-robust.



(a) Velocity error.



(b) Pressure error.

Figure 1: Results for the no flow problem section 7.1 using polynomial degree $k = 2$.

For the divergence-conforming method, the errors in velocity and pressure are unaffected by the decrease in viscosity and there are no problems associated with the convergence of the non-linear solver.

8. Conclusions

In this paper we have proven optimal error estimates of a hybridizable discontinuous Galerkin method for the stationary Navier–Stokes problem. Furthermore, due to the scheme resulting in an approximate velocity field that is divergence-conforming and pointwise divergence free, the error estimates obtained for the velocity field do not depend on the pressure. The pressure-robust error estimates were confirmed by numerical examples. In addition, the numerical examples demonstrated that if a solenoidal velocity field is not divergence-conforming, the HDG method will fail to be pressure-robust.

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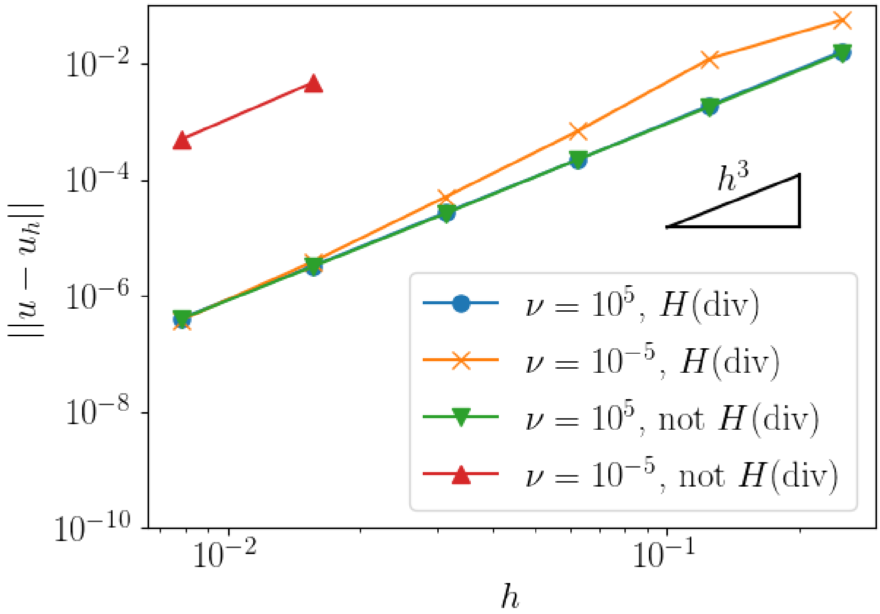
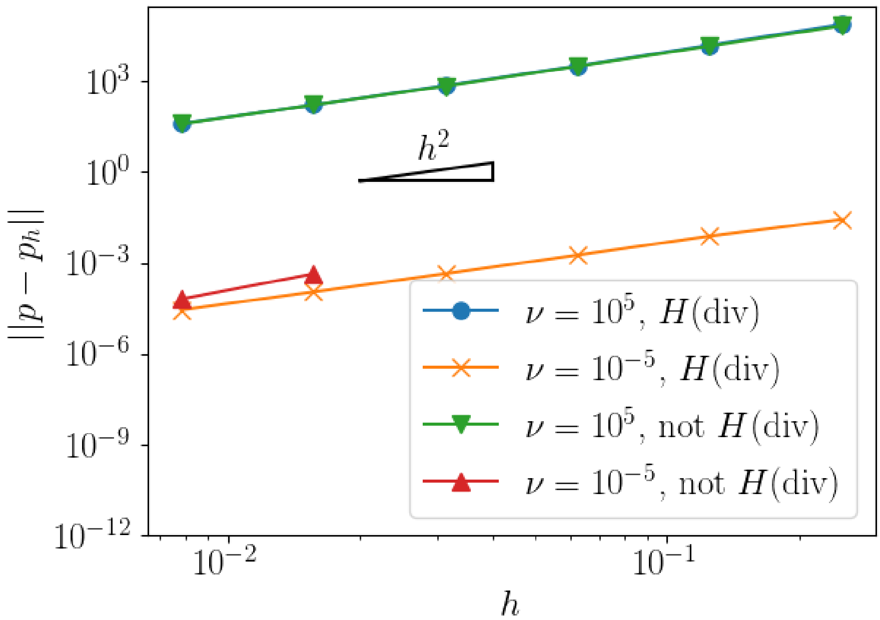


(a) Example 2, Velocity error.



(b) Example 2, Pressure error.

Figure 2: Results for the potential flow problem section 7.2 using polynomial degree $k = 2$.

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