On Amenability Properties and Coideals of Quantum Groups

by

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Author's Declaration

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

We study amenability type properties of locally compact quantum groups and subobjects of quantum groups realized as submodules of their von Neumann algebras. An important class of such subobjects are the coideals, which offer a way of defining a "quasi-subgroup" for locally compact quantum groups. Chapters 3, 4, and 5 are based on [3], [2], and [4] respectively.

In Chapter 3, we establish the notion of a non-commutative hull of a left ideal of $L^1(\widehat{\mathbb{G}})$ for a discrete quantum group $\mathbb{G}$. Non-commutative spectral synthesis is defined too, and is related to a certain Ditkin's property at infinity, allowing for a description of the closed left ideals of $L^1(\widehat{\mathbb{G}})$ for many known compact quantum groups $\widehat{\mathbb{G}}$ from the literature. We apply this work to study weak* closed in ideals in the quantum measure algebra of coamenable compact quantum groups and certain closed ideals in $L^1(\widehat{\mathbb{G}})$ which admit bounded right approximate identities in relation to coamenability of $\widehat{\mathbb{G}}$ (Theorem 3.3.14).

In Chapter 4, we study relative amenability and amenability of coideals of a discrete quantum group, and coamenability of coideals of a compact quantum group. Making progress towards answering a coideal version of a question of [65], we prove a duality result that generalizes Tomatsu's theorem [122] (lemmas 4.4.14 and 4.1.9). Consequently, we characterize the reduced central idempotent states of a compact quantum group (Corollary 4.1.2).

In Chapter 5, we study tracial and $\mathbb{G}$-invariant states of discrete quantum groups. A key result here is that tracial idempotent states are equivalently $\mathbb{G}$-invariant idempotent states (Proposition 5.3.12). A consequence is the resolution of an open problem in [96, 22] in the discrete case, namely that amenability of $\mathbb{G}$ is equivalent to nuclearity of and the existence of a tracial state on $C_r(\widehat{\mathbb{G}})$ (Corollary 5.3.14). We also obtain that simplicity of $C_r(\widehat{\mathbb{G}})$ implies no $\mathbb{G}$-invariant states exist (Corollary 5.3.15). Finally, we prove existence and uniqueness results of traces in terms of the cokernel, $\mathbb{H}_F$, of the Furstenberg boundary and the canonical Kac quotient of $\widehat{\mathbb{G}}$.

In Chapter 6, we develop a notion of operator amenability and operator biflatness of the action of a completely contractive Banach algebra on another completely contractive Banach algebra. We study these concept on various actions defined for locally compact quantum groups and their quantum subgroups, and relate them to usual operator amenability and other related properties, including amenability, coamenability, and compactness.

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Chapter 1

Introduction

The von Neumann algebraic formulation of locally compact quantum groups (due to Kustermans and Vaes [80], and pioneered by Woronowicz [133]) offers a generalization of the category of locally compact groups, where the function algebras typically considered by harmonic analysts and operator algebraists are quantized, i.e., are non-commutative. An important aspect of this formulation is that the deformed versions of $SU(2)$, $SU_q(2)$, are realized in this framework as compact quantum groups, something that the Kac algebras, a precedent to locally compact quantum groups [36], failed to achieve. More generally, many natural examples of quantum groups can be found in terms of non-commutative symmetries or "liberated" versions of groups (quantum permutations, free unitary and free orthogonal quantum groups, etc) and extensions of groups (tensor products, crossed products, free products, etc).

Perhaps the most important feature of locally compact quantum groups is Pontryagin duality, a feature of abelian locally compact groups that is lost in the more general setting of locally compact groups, but is regained at the level of locally compact quantum groups. In particular, various operator algebras (e.g., group von Neumann algebras and essentially bounded function algebras) are unified under the umbrella of locally compact quantum groups, allowing one to study properties that broadly apply to these algebras and their dual spaces. Moreover, beyond unification of classical locally compact groups and their duals, quantum groups offer a broader range of operator algebras for operator algebraists and harmonic analysts to study from their viewpoints.

Our work primarily studies the operator algebras and their dual spaces associated with locally compact quantum groups. Chapter 2 is expository, where we cover the basic features of locally compact quantum groups that will appear in this work. We discuss the aspects of Pontryagin duality and representation theory that will be relevant to us, with particular attention on the discrete quantum groups and their duals, compact quantum groups. Here, we will present a comprehensive account of the theory of closed quantum subgroups and their quotients. This will lead into a discussion of coideals and their relation with idempotent states and group-like projections. At the end of the chapter, we will outline discrete and compact quantum groups arising from discrete crossed product constructions, a set of examples we will use several times in this thesis to illustrate our work.

Amenability is among the most important and deeply studied properties of locally compact groups. For example, for operator algebraists, amenability relates to nuclearity of the reduced C^*-algebras of locally compact groups [95, 83]. More specifically, a locally compact group G is amenable if and only if its reduced C^*-algebra $C_r(\widehat{G})$ is nuclear and admits a tracial state. If G is discrete then $C_r(\widehat{G})$ always has a tracial state, called the canonical trace by operator algebraists and the Haar state by quantum group theorists.

When G is discrete, the question of whether or not the canonical trace is the unique tracial state on $C_r(\widehat{G})$ (the unique trace property) has interesting connections to amenability. It is the normal subgroups of a discrete group that detect unique trace property [15, 66]. More precisely, the amenable radical is trivial if and only if a discrete group has the unique trace property.

For harmonic analysts, amenability of a locally compact group G relates to cohomological properties of its group algebra $L^1(G)$ and Fourier algebra $A(G)$. Most notably, operator amenability of $L^1(G)$ is equivalent to operator amenability of $A(G)$, which is equivalent to amenability of G [101, 61].

While of many of the same theorems for amenability of locally compact groups generalize to locally compact quantum groups (and especially those of Kac type), there often lies phenomena that only exists in the quantum setting, that can both reveal new distinctions invisible for locally compact groups and exhibit obstructions that require new techniques.

In this thesis, we focus mainly on amenability of discrete quantum groups, and its dual property for compact quantum groups, coamenability. We also study related properties that generalize various aspects of (co)amenability. In Chapter 3, we study the left ideals of quantum convolution algebras (quantum measure algebras and quantum group algebras). We identify a non-commutative notion of a hull for the left ideal of the quantum group algebra of a compact quantum group. This notion unifies the usual notion of a hull of an ideal in the Fourier algebra of a discrete group and a known description of the closed left ideals of the group algebra of a compact group. On the other hand, this introduces a notion of a quantum subset of a discrete quantum group, in analogy with the fact that the hulls of the ideals of Fourier algebras of a discrete group G are the subsets of G (see [69, 51]).

A fundamental property of closed ideals of a Fourier algebra is spectral synthesis, which occurs when a hull is the zero set of exactly one closed ideal. By definition, when every closed subset of a group G has synthesis, we have a complete characterization of the closed ideals in $A(G)$. In this work, spectral synthesis of non-commutative hulls is defined as well, and we achieve a result that shows every quantum subset of a discrete quantum group $\mathbb{G}$ has spectral synthesis if and only if $\mathbb{G}$ has a certain Ditkin's property at infinity, generalizing the work of [71]. In these cases, we have a characterization of the compact quantum groups in which the closed left ideals of their quantum group algebras are fully classified, covering many known examples from the literature with the approximation property (including $SU_q(2)$, free unitary groups, etc). We are also able to fully classify the weak* closed left ideals of quantum measure algebras of coamenable compact quantum groups, generalizing the work of [132].

For a compact group G, a notoriously difficult problem is the classification of the idempotents in its measure algebra. Equivalently, one is tasked with identifying the closed left ideals of the group algebra which admit bounded right approximate identities (see [87]). Dually, the idempotents in the Fourier-Stieltjes algebra of a locally compact group are fully classified [53, 19]. An interesting problem here is identifying the closed ideals of the Fourier algebra which admit bounded approximate identities. When a discrete group is amenable, like with the measure algebra of a compact group, we achieve a correspondence of such closed ideals with the idempotents in the Fourier-Stieltjes algebra [87]. For a non-amenable discrete group G, however, we find that the closed ideals whose hull is a subgroup never admit a bounded approximate identity, and when G is amenable, every such ideal has a bounded approximate identity.

The problems in the previous paragraph can be stated for compact quantum groups as well. For instance, for a coamenable compact quantum group, the closed left ideals which admit a bounded right approximate identity are in one-to-one correspondence with idempotents in the quantum measure algebra [87]. The quantum subsets of a discrete quantum group $\mathbb{G}$ can be used to describe the structure of the compact quasi-subgroups (denoted N) of the compact quantum group $\widehat{\mathbb{G}}$, in the same sense that closed subgroups of a discrete group are also subsets (hence hulls of closed ideals of the Fourier algebra). In particular, each compact quasi-subgroup N has a corresponding closed left ideal $J^1(N)$ in the quantum group algebra. In Chapter 3, we make progress towards proving $J^1(N)$ has a bounded approximate identity if and only if $\widehat{\mathbb{G}}$ is coamenable. We achieve this equivalence in certain instances where N admits a "quantum coset," meaning, there exists a certain translation of N that generalizes a coset of a subgroup, and whenever N is a "coamenable coideal".

As alluded to above, characterizing traces and simplicity of C^*-algebras is a problem of interest to operator algebraists (e.g. for classification of C^*-algebras). The C^*-algebras that admit a unique trace have an especially simple description of their traces, as there is only one. An important class of examples of C^*-algebras where important progress has been made on understanding simplicity and their traces are the reduced group C^*-algebras. It was shown with [15] that the unique trace property of a discrete group G is equivalent to the faithfulness of its action of G on its Furstenberg boundary, which is equivalent to triviality of the amenable radical. Then, using the work of [66], we find that simplicity of reduced C^*-algebras implies the action of G on its Furstenberg boundary is faithful, and thus the unique trace property. This characterization asserts that the unique trace property is determined by the class of traces on $C_r(\widehat{G})$ of the form 1_N where N is a normal and amenable subgroup of G: if $1_{R_a(G)} = 1_{\{e\}}$, where $R_a(G)$ is the amenable radical, then $1_{\{e\}}$ is the only trace.

In [65], the Furstenberg boundary $\partial_F(\mathbb{G})$ for a discrete quantum group $\mathbb{G}$ was developed and it was shown that faithfulness of $\mathbb{G} \curvearrowright \partial_F(\mathbb{G})$ implies the unique trace property of $\mathbb{G}$ whenever $\mathbb{G}$ is unimodular. Towards their construction of $\partial_F(\mathbb{G})$, the authors of [65] define relative amenability of coideals and coamenability of quotients. On one hand, a closed quantum subgroup $\widehat{\mathbb{H}_F} \leq \widehat{\mathbb{G}}$ is identified as the cokernel of $\mathbb{G} \curvearrowright \partial_F(\mathbb{G})$ so that $\ell^\infty(\mathbb{H}_F)$ is a minimal, relatively amenable two-sided coideal, and provides a "quantum way" of accessing the kernel of the Furstenberg boundary.

For locally compact groups in general, Caprace and Monod [17] developed a notion of relative amenability of a closed subgroup H of G. On one hand, relative amenability of H was shown to be equivalent to the existence of a G-equivariant unital completely positive map $L^\infty(G) \to L^\infty(G/H)$ and amenability of H was shown to be equivalent to the existence of a G-equivariant unital completely positive projection $L^\infty(G) \to L^\infty(G/H)$. In general, for locally compact quantum groups, we define relative amenability and amenability of a coideal N as the existence of a $\mathbb{G}$-equivariant ucp map and $\mathbb{G}$-equivariant projection $L^\infty(\mathbb{G}) \to N$ respectively.

For discrete groups, relative amenability and amenability are known to coincide, however, for locally compact groups in general, their equivalence remains open. For discrete quantum groups, coincidence of relative amenability and amenability for coideals remains unknown too. For discrete G, the key to the equivalence between relative amenability and amenability is the exploitation of H-invariant means on $\ell^\infty(G)$. In Chapter 4, we develop a notion of a P-invariant mean for a coideal $N \subseteq \ell^\infty(\mathbb{G})$, where $P \in \ell^\infty(\mathbb{G})$ is a so-called "group-like projection" associated with N, which generalizes the notion of an H-

invariant mean from the classical cases. With this, we achieve characterizations of relative amenability and amenability of coideals in terms of P-invariant means.

For a discrete group G and subgroup H, it is known that H is amenable if and only if the indicator function 1_H is reduced, i.e., is an element of $C_r(\widehat{G})$. In [65], coamenable quotients of compact quantum groups were defined. In Chapter 4 we extend this definition to compact quasi-subgroups of compact quantum groups. We make progress towards generalizing the equivalence stated at the start of this paragraph. As one of the main theorems of Chapter 4, we prove that the central idempotent states on the reduced C^*-algebra of a compact quantum group $\widehat{\mathbb{G}}$ are in one-to-one correspondence with the amenable quantum subgroups of $\mathbb{G}$.

The Haar state of a compact quantum group is not necessarily tracial. Here, the problem of characterizing the unique trace property is now a problem of the both the existence and uniqueness of traces on the reduced C^*-algebra. A result that explains how the trace $1_{R_a(G)}$ determines the unique trace property was obtained in [66]. It states that every trace on $C_r(\widehat{G})$ "concentrates" on $R_a(G)$. The underlying technique for the proof involves extending G-invariant states on $C_r(\widehat{G})$, to G-invariant states on the crossed product of G with its Furstenberg boundary. Here, G-invariance is invariance with respect to the conjugation action $G \curvearrowright C_r(\widehat{G})$. A quantum conjugation action exists $\mathbb{G} \curvearrowright C_r(\widehat{\mathbb{G}})$ exists for discrete quantum groups as well, and it is known that for unimodular $\mathbb{G}$ (and in particular, discrete groups) the $\mathbb{G}$-invariant states and traces coincide (for example, see [65]). An obstruction for arbitrary discrete quantum groups is that their coincidence is unknown in general (see [96]).

It was shown in [65] that the Haar state of a compact quantum group $\widehat{\mathbb{G}}$ is $\mathbb{G}$-invariant if and only if it is tracial. In Chapter 5, we prove that a Haar idempotent is $\mathbb{G}$-invariant if and only if it is tracial. It also turns out that every tracial idempotent is a Haar idempotent, so, this shows that the tracial states and $\mathbb{G}$-invariant states coincide for the idempotent states. We deduce that $\mathbb{G}$-invariant states exist if and only if tracial states exist. In [96, 22], it was shown that $\mathbb{G}$ is amenable if and only if $C_r(\widehat{\mathbb{G}})$ is nuclear and has a $\mathbb{G}$-invariant state. So, we have achieved that $\mathbb{G}$ is amenable if and only if $C_r(\widehat{\mathbb{G}})$ is nuclear and has a tracial state, a proper generalization of the classical case.

Then we prove that the $\mathbb{G}$-invariant states on $C_r(\widehat{\mathbb{G}})$ "concentrate" on $\widehat{\mathbb{G}}/\widehat{\mathbb{H}}_F$, generalizing the fact that tracial states on a discrete group concentrate on the amenable radical. In particular, this shows that whenever $\widehat{\mathbb{G}}/\widehat{\mathbb{H}}_F$ is coamenable, it is the minimal Kac quantum subgroup of $\widehat{\mathbb{G}}$ where its quotient is coamenable. As a consequence, we prove that the existence of tracial states on $C_r(\widehat{\mathbb{G}})$ is equivalent to having that $\mathbb{H}_F$ is unimodular and $\widehat{\mathbb{G}}/\widehat{\mathbb{H}}_F$ is coamenable. We also obtain a uniqueness condition of tracial states on $\mathbb{G}$ in terms of

$\mathbb{H}_F$. As a consequence, for non-unimodular $\mathbb{G}$, we obtain that if $C_r(\widehat{\mathbb{G}})$ is simple, then it has no traces.

In Chapter 6, we develop a notion of operator amenability of a completely contracte Banach algebraic action, which is a virtual diagonal type condition on the action of a completely contractive Banach algebra on another completely contractive Banach algebra, where usual operator amenability can be realized as operator amenability of the multiplication action of a completely contractive Banach algebra on itself. Analogously, we define relative operator biflatness of an action by the existence of certain completely bounded left inverses of the action. Similarly, this generalizes the notion of operator biflatness.

We study relative operator and operator biflatness of an action with an eye towards usual operator amenability and operator biflatness. We characterize operator amenability of the action of a locally compact quantum group $\mathbb{G}$ on its dual with co-amenability of $\mathbb{G}$ and amenability of $\widehat{\mathbb{G}}$. For discrete and compact quantum groups, we show operator biflatness entails the Kac property and coamenability. On one hand, this provides interesting connections of operator biflatness and amenability of an action with the notoriously difficult problems of characterizing operator amenability of the quantum group algebras of discrete quantum groups and establishing duality between amenability and coamenability for locally compact quantum groups. On the other hand, it reveals a distinction between operator biflatness and operator amenability of an action.

Later in Chapter 6, we focus on actions given by completely bounded multipliers. In this framework, we achieve a generalization of the relationship between operator amenability and operator biflatness. This framework covers the examples given by the action of the various quantum convolution algebras of a quantum subgroup $\mathbb{H}$ on the quantum convolution algebras of its parent quantum group $\mathbb{G}$. We study discrete quantum groups, locally compact groups and their duals, where we relate operator amenbility of an action to operator amenability. In particular, for the Fourier-Stieltjes algebra $B(G)$ of a locally compact group G, we find connections between operator amenability of certain translation invariant subalgebras with operator amenability of $B(G)$ and compactness of G.

Chapter 2

Locally Compact Quantum Groups

2.1 Basics of Operator Theory

Here, we will review the basics of operator theory and establish the notation we will use throughout this thesis. Good references are [98, 33].

A (concrete) **operator space** is a subspace $X \subseteq \mathcal{B}(\mathcal{H})$ where $\mathcal{B}(\mathcal{H})$ is the algebra of linear operators on a Hilbert space $\mathcal{H}$. Tensoring with a matrix algebra M_n gives us another operator space

$$M_n \otimes X \cong M_n(X) \subseteq M_n(\mathcal{B}(\mathcal{H})) \cong \mathcal{B}(\oplus_{i=1}^n \mathcal{H}).$$

Equivalently, X is an operator space if it is a **matricially normed** vector space, which is when there exists a sequence of norms $\{||\cdot||_{M_n(X)}\}_{n\in\mathbb{N}}$ where each $||\cdot||_{M_n(X)}$ is a norm on $M_n(X)$ satisfying

$$||axb||_{M_n(X)} \leq ||a||_{M_n}||x||_{M_n(X)}||b||_{M_n},\ a, b \in M_n, x \in M_n(X)$$

and

$$\left|\left|\begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix}\right|\right|_{M_{n+m}(X)} = \max\{||x||_{M_n(X)}, ||y||_{M_m(Y)}\},\ x \in M_n(X), y \in M_m(X).$$

With this framework, we obtain a sequence of matrix norms $||\cdot||_{M_n(X)}$. Then, for any linear map $\varphi : X \to Y$ between operator spaces X and Y, we can define the linear map $\varphi^n : M_n(X) \to M_n(Y)$ where $\varphi^n([x_{i,j}]) = [\varphi(x_{i,j})]$. The natural morphisms in the category of operator spaces are **completely bounded (cb)** linear maps, where a linear map $\varphi : X \to Y$ is cb if

$$||\varphi||_{cb} := \sup_{n\in\mathbb{N}} ||\varphi^n|| < \infty.$$

We say φ is **completely contractive (cc)** if $||\varphi||_{cb} \leq 1$ and is **completely isometric (ci)** if $||\varphi||_{cb} = 1$. The symbol $\mathcal{CB}(X, Y)$ denotes the cb maps from X to Y. Then $M_n(\mathcal{CB}(X, Y)) = \mathcal{CB}(X, M_n(Y))$, which entails that $\mathcal{CB}(X, Y)$ is an operator space. In particular, the duals and preduals of operator spaces are operator spaces. A bounded linear functional $\varphi : X \to \mathbb{C}$ is automatically completely bounded, and satisfies $||\varphi||_{cb} = ||\varphi||$.

An **operator algebra** is a subalgebra of some $\mathcal{B}(\mathcal{H})$. A (concrete) C^***-algebra** is an (operator norm) closed $*$-subalgebra $A \subseteq \mathcal{B}(\mathcal{H})$, where the involution is given by the adjoint of operators. A C^*-algebra is in particular an operator space. Equivalently, A is a Banach $*$-algebra that has a C^*-norm: a norm $||\cdot||$ such that $||a^*a|| = ||a||^2$. The morphisms we give the category of C^*-algebras are the non-degenerate $*$-homomorphisms $A \to M(B)$, where $M(B)$ is the multiplier algebra of B. The morphisms in the category of unital C^*-algebras (as a subcategory of C^*-algebras) are unital $*$-homomorphisms.

With the involutive structure of a C^*-algebra, a notion of positivity arises, where $a^*a \geq 0$. A linear map $\varphi : A \to B$ is **positive** if $\mu(a) \geq 0$ for every positive $a \in A$. A positive functional φ is automatically bounded and when A is unital, it satisfies $||\varphi|| = \varphi(1)$. A **state** is a unital positive functional.

We can formulate positivity matricially as well. We say a linear map $\varphi : A \to B$ is **completely positive (cp)** if φ^n is positive for every n. A cp map is automatically cb and when A is unital, it satisfies $||\varphi||_{cb} = \varphi(1)$, so, a **unital cp (ucp)** map is automatically ci. A positive linear functional is automatically cp and a state is automatically ucp.

A **von Neumann algebra** M is a (unital) C^*-algebra that is additionally weak* closed, where the trace class operators $\mathcal{T}(\mathcal{H}) = \mathcal{B}(\mathcal{H})_*$ are unique (up to isomorphism) as a predual of $\mathcal{B}(\mathcal{H})$. The morphisms in the category of von Neumann algebras are the normal unital $*$-homomorphisms, where normalilty of $\sigma : M \to N$ is the property where $\sup \sigma(x_i) = \sigma(\sup x_i)$ for every increasing net (x_i) in M. Normality is equivalent to weak*-weak* continuity. Every von Neumann algebra $M \subseteq \mathcal{B}(\mathcal{H})$ has a predual M_* that is unique up to isomorphism (in the category of Banach spaces). Von Neumann's double commutant theorem states that given a C^*-algebra A, $A'' = \overline{A}^{wk*}$ is a von Neumann algebra.

Let X and Y be operator spaces. A matricial norm on $X \otimes Y$ is subcross if $||x \otimes y||_{M_{n_1 n_2}(X\otimes Y)} \leq ||x||_{M_{n_1}(X)}||y||_{M_{n_2}(Y)}$ for every $x \in M_{n_1}(X)$ and $y \in M_{n_2}(Y)$. There exists a largest sub-cross matrical norm, say $(||\cdot||^{\gamma}_{M_n(X\otimes Y)})$. The **operator projective tensor**

product is the completion

$$M_n(X\widehat{\otimes}Y) = \overline{X \otimes Y}^{(||\cdot||^{\gamma}_{M_n(X\otimes Y)})}.$$

A **completely contractive (cc)** Banach algebra is a Banach algebra A that has an operator space structure and multiplication that extends to a cc map $A\widehat{\otimes}A \to A$.

For C^*-algebras A and B, we identify two canonical C^*-algebraic cross norms on $A \otimes B$. There is a maximal C^*-cross norm on $A \otimes B$ given by

$$||\cdot||_{max} := \sup\{||\pi(\cdot)|| : \pi : A \otimes B \to \mathcal{B}(\mathcal{H}_\pi) \text{ is a } *\text{-representation}\}.$$

We denote

$$A \otimes_{max} B = \overline{A \otimes B}^{||\cdot||_{max}}.$$

Using the embeddings $A \subseteq \mathcal{B}(\mathcal{H})$ and $B \subseteq \mathcal{B}(\mathcal{K})$, we obtain an injective inclusion $A \otimes B \subseteq \mathcal{B}(\mathcal{H} \otimes_2 \mathcal{K})$, where $\mathcal{H} \otimes_2 \mathcal{K}$ is the Hilbert space tensor product. The norm induced by this inclusion is independent of the faithful representations and is denoted $||\cdot||_{min}$. We write

$$A \otimes_{min} B = \overline{A \otimes B}^{||\cdot||_{min}}.$$

For every C^*-cross norm $||\cdot||$, we have $||\cdot||_{min} \leq ||\cdot|| \leq ||\cdot||_{max}$.

Let $M \subseteq \mathcal{B}(\mathcal{H})$ and $N \subseteq \mathcal{B}(\mathcal{K})$ be von Neumann algberas. The **spatial tensor product** of M and N is

$$M\overline{\otimes}N = \{X \in \mathcal{B}(\mathcal{H} \otimes_2 \mathcal{K}) : (\varphi \otimes \mathrm{id})(X) \in N, (\mathrm{id} \otimes \psi)(Y) \in M \ \forall \ \varphi \in \mathcal{T}(\mathcal{H}), \psi \in \mathcal{T}(\mathcal{K})\}.$$

Then $M\overline{\otimes}N$ is a von Neumann algebra, and we have $(M\overline{\otimes}N)_* = M_*\widehat{\otimes}N_*$. It turns out that $M\overline{\otimes}N$ is the the wot closure of $M \otimes N$ in $\mathcal{B}(\mathcal{H} \otimes_2 \mathcal{K})$, which makes it the von Neumann algebra generated by $M \otimes N$ (see [33]).

2.2 Weights On von Neumann Algebras

In this section, we review the essential theory for von Neumann algebras. We recommend [118, 119] as references. For a von Neumann algebra M, we denote $M_{\geq 0} = \{x \in M : x \geq 0\}$.

Definition 2.2.1. A **weight** on M is a function $\varphi : M_{\geq 0} \to [0, \infty]$ satisfying

- $\varphi(x + y) = \varphi(x) + \varphi(y)$ for all $x, y \in M_{\geq 0}$;

- $\varphi(rx) = r\varphi(x)$ for all $r \in \mathbb{R}_{\geq 0}$ and $x \in M_{\geq 0}$.

We use the following standard notation:

- $\mathcal{M}_\varphi^+ := \{x \in M_{\geq 0} : \varphi(x) < \infty\}$;
- $\mathcal{N}_\varphi := \{x \in M : \varphi(x^*x) < \infty\}$;
- $\mathcal{M}_\varphi := \text{Span}\{x^*y : x, y \in \mathcal{N}_\varphi\}$.

It turns out that $\mathcal{N}_\varphi$ is a left ideal in M and $\mathcal{M}_\varphi$ is a $*$-subalgebra of M, so, we may extend φ linearly to $\mathcal{M}_\varphi$.

Definition 2.2.2. Let M be a von Neumann algebra and φ a weight. We have that φ is:

- **semifinite** if $\mathcal{M}_\varphi$ is weak* dense in M;
- **faithful** if $\varphi(x) = 0$ implies $x = 0$ for every $x \in M_{\geq 0}$;
- **normal** if $\varphi(\sup x_i) = \sup \varphi(x_i)$ for every increasing net (x_i) in $M_{\geq 0}$.

Given a von Neumann algebra M and normal semifinite faithful weight φ, we can define an inner product on $\mathcal{N}_\varphi$ by setting

$$\langle x, y \rangle_\varphi = \varphi(y^*x).$$

We let H_φ be the Hilbert space completion of $\mathcal{N}_\varphi$ with respect to $\langle \cdot, \cdot \rangle_\varphi$. We obtain a faithful $*$-representation $\pi_\varphi : M \to \mathcal{B}(H_\varphi)$ by setting $\pi_\varphi(x)\Lambda_\varphi(y) = \Lambda_\varphi(xy)$ for every $x \in M$ and $y \in \mathcal{N}_\varphi$, where $\Lambda_\varphi : \mathcal{N}_\varphi \to H_\varphi$ is the canonical inclusion. Then $(H_\varphi, \Lambda_\varphi, \psi_\varphi)$ is the **GNS construction** for φ. It turns out that through the identification $M \cong \pi_\varphi(M)$, M is standardly represented on H_φ (see [49] for more about the standard representation).

2.3 Definition and Basic Theory

To motivate the notion of a quantum group used in this thesis, we highlight important Banach algebraic aspects of locally compact groups. A locally compact group (lcg) is a group G equipped with a locally compact topology τ_G that makes the group product and inverse continuous. Every lcg has a left Haar measure m, which is a left translation invariant Radon measure [52]. We stress that the Haar measure is fundamental to the underlying topology of G. For example, G is discrete if and only if m is the counting measure.

The L^∞-space $L^\infty(G) := L^\infty(G,m)$ is a (commutative) von Neumann algebra via its representation as multiplication operators on $L^2(G) := L^2(G,m)$, giving us $L^\infty(G) \subseteq \mathcal{B}(L^2(G))$. The **group algebra** is the (cc) Banach $*$-algebra $L^1(G) := L^1(G,m)$ whose product is defined by

$$f * g(\cdot) = \int_G f(s)g(s^{-1}\cdot)\,ds \text{ (convolution)}$$

and involution is defined by

$$f^\# = \Delta(\cdot^{-1})\overline{f}(\cdot^{-1})$$

where Δ is the modular function on G. It turns out that $L^1(G)$ is a complete invariant for G: we have $L^1(G) \cong L^1(H)$ as a Banach algebras if and only if $G \cong H$ as locally compact groups [131]. On the other hand, the von Neumann algebraic structure of $L^\infty(G)$ is not enough to encode G. For this, consider the **coproduct:** the normal unital $*$-homomorphism

$$\Delta_G : L^\infty(G) \to L^\infty(G\times G)$$
$$\Delta_G(f)(s,t) = f(st) \text{ a.e.}$$

It turns out that convolution satisfies the formula $f * g = (f \otimes g)\Delta_G$. Then, there exists a normal unital $*$-isomorphism $\sigma : L^\infty(G) \to L^\infty(H)$ such that $(\sigma\otimes\sigma)\Delta_G = \Delta_H \circ \sigma$ if and only if $G \cong H$ as locally compact groups. So, the von Neumann algebraic structure of $L^\infty(G)$ plus the coproduct is enough to determine G.

Our definition of a quantum group is the von Neumann algebraic one due to Kustermans and Vaes [82], originally pioneered by Woronowicz [133].

Definition 2.3.1. A **locally a compact quantum group (LCQG)** is a quadruple $\mathbb{G} = (L^\infty(\mathbb{G}), \Delta_\mathbb{G}, h_L^\mathbb{G}, h_R^\mathbb{G})$ where

- $L^\infty(\mathbb{G})$ is a von Neumann algebra;
- $\Delta_\mathbb{G} : L^\infty(\mathbb{G}) \to L^\infty(\mathbb{G})\overline{\otimes}L^\infty(\mathbb{G})$ is a normal unital $*$-homomorphism satisfying

$$(\Delta_\mathbb{G}\otimes \mathrm{id})\Delta_\mathbb{G} = (\mathrm{id}\otimes\Delta_\mathbb{G})\Delta_\mathbb{G} \text{ (coassocaitivity).}$$

- h_L is a normal semifinite weight satisfying

$$(\mathrm{id} \otimes h_L^{\mathbb{G}})\Delta_{\mathbb{G}}(x) = h_L^{\mathbb{G}}(x) \text{ for all } x \in \mathcal{M}_{h_L^{\mathbb{G}}} \text{ (left invariance);}$$

- h_R is a normal semifinite weight satisfying

$$(h_R^{\mathbb{G}} \otimes \mathrm{id})\Delta_{\mathbb{G}}(x) = h_R^{\mathbb{G}}(x) \text{ for all } x \in \mathcal{M}_{h_R^{\mathbb{G}}} \text{ (right invariance);}$$

We call $\Delta_{\mathbb{G}}$ the **coproduct**, h_L the **left Haar weight**, and h_R the **right Haar weight** of $\mathbb{G}$ respectively.

Remark 2.3.2.

- The Grothendieck formula $L^\infty(G \times G) \cong L^\infty(G)\overline{\otimes}L^\infty(G)$ (which implies $L^1(G \times G) \cong L^1(G)\widehat{\otimes}L^1(G)$) ensures that the quantized definition of a coproduct is a bonafide generalization of the coproduct for locally compact groups. Furthermore, every left / right Haar measure on a locally compact group defines a left / right Haar weight on $L^\infty(G)$ by integration.

- The LCQGs where $L^\infty(\mathbb{G})$ is commutative are exactly the lcgs [117].

We will denote $L^1(\mathbb{G}) := L^\infty(\mathbb{G})_*$ and the accompanying norm by $||\cdot||_1$. We have that $L^1(\mathbb{G})$ is a cc Banach algebra with respect to **convolution:**

$$f * g := (f \otimes g)\Delta_{\mathbb{G}}.$$

We refer to $L^1(\mathbb{G})$ as the L^1**–algebra** or **quantum group algebra** of $\mathbb{G}$.

An important feature of a group algebra is that it is a Banach $*$–algebra. From this, one might think the correct notion of a quantum group would have a $*$–algebraic L^1–algebra. This does not generally happen, however, the L^1–algebra of a LCQG does at least contain a dense $*$–algebra. To get to such an object, we must discuss the antipode. In service of not making too much of a digression, we will not construct the antipode but will rather highlight its important properties.

Let $(L^2(\mathbb{G}, h_L^{\mathbb{G}}), \Lambda_{h_L^{\mathbb{G}}}, \pi_{h_L^{\mathbb{G}}})$ denote the GNS construction for the left Haar weight of $\mathbb{G}$. It turns out that if we let $L^2(\mathbb{G}, h_R^{\mathbb{G}})$ denote the GNS Hilbert space associated with $\psi_R^{\mathbb{G}}$, then $L^2(\mathbb{G}, h_R^{\mathbb{G}}) \cong L^2(\mathbb{G}, h_L^{\mathbb{G}})$ as Hilbert spaces. So, we simply consider $L^2(\mathbb{G}) = L^2(\mathbb{G}, h_L^{\mathbb{G}})$ to be the canonical Hilbert space representing $L^\infty(\mathbb{G})$.

The **antipode** of a LCQG always exists [80]. It is a weak* closed (unbounded) antihomomorphism $S_{\mathbb{G}} : D(S_{\mathbb{G}}) \subseteq L^\infty(\mathbb{G}) \to L^\infty(\mathbb{G})$, where $D(S_{\mathbb{G}})$ is the weak* dense domain of $S_{\mathbb{G}}$. It satisfies $S_{\mathbb{G}} \circ * \circ S_{\mathbb{G}} \circ * = \mathrm{id}$. Then we define

$$L^1_\#(\mathbb{G}) := \{f \in L^1(\mathbb{G}) : f \circ S_{\mathbb{G}} \in L^1(\mathbb{G})\}$$

which is a dense subalgebra of $L^1(\mathbb{G})$. Moreover, it is a $*$–algebra with respect to the involution $\#$ defined by $f^{\#}(x) = \overline{f(S_{\mathbb{G}}(x)^*)}$ and is a Banach $*$–algebra with respect to $\#$ and the norm $f \mapsto \max\{||f||_1, ||f^{\#}||_1\}$ (see [80]).

Definition 2.3.3. We say $\mathbb{G}$ is of **Kac** type whenever $S_{\mathbb{G}}$ is norm bounded. In this case $S_{\mathbb{G}}$ is an isometric $*$-antiautomorphism of $L^\infty(\mathbb{G})$.

A clear consequence of $\mathbb{G}$ being of Kac type is that $L^1(\mathbb{G}) = L^1_{\#}(\mathbb{G})$, so in other words, $L^1(\mathbb{G})$ is an involutive Banach algebra when $\mathbb{G}$ is of Kac type.

Remark 2.3.4. We in particular have that every locally compact group is of Kac type.

Important related objects are the unitary antipode and the scaling group. For details on the following discussion, see [80]. There is a one parameter group $(\tau_t^{\mathbb{G}})_{t\in\mathbb{R}}$ of automorphisms on $L^\infty(\mathbb{G})$ such that $S_{\mathbb{G}} = R_{\mathbb{G}} \circ \tau^{\mathbb{G}}_{-1/2}$ is the polar decomposition of $S_{\mathbb{G}}$ (see [80]). We call $(\tau_t^{\mathbb{G}})$ the **scaling group** of $\mathbb{G}$ and $R_{\mathbb{G}}$ the **unitary antipode**.

We let $(\sigma_t^{h_L^{\mathbb{G}}})_{t\in\mathbb{R}}$ be the modular group associated with $h_L^{\mathbb{G}}$, so in particular satisfies $h_L^{\mathbb{G}}(xy) = h_L^{\mathbb{G}}(y\sigma_{-1}(x))$ for $y \in \mathcal{N}_{h_L^{\mathbb{G}}} \cap \mathcal{N}_{h_L^{\mathbb{G}}}$ and $x \in \mathcal{N}_{h_L^{\mathbb{G}}} \cap D(\sigma_{-1})$. The objects described above satisfy the following:

$$\begin{aligned} h_L^{\mathbb{G}} \circ R_{\mathbb{G}} &= h_R \\ \Sigma(R_{\mathbb{G}} \otimes R_{\mathbb{G}})\Delta_{\mathbb{G}} &= \Delta_{\mathbb{G}} \circ R_{\mathbb{G}} \\ \Delta_{\mathbb{G}} \circ \tau_t^{\mathbb{G}} &= (\tau_t^{\mathbb{G}} \otimes \tau_t^{\mathbb{G}})\Delta_{\mathbb{G}}. \end{aligned}$$

We call $(\sigma_t^{h_L^{\mathbb{G}}})_{t\in\mathbb{R}}$ the **modular group** of $\mathbb{G}$. We say $\mathbb{G}$ is **unimodular** if $h_L^{\mathbb{G}} = h_R^{\mathbb{G}}$.

2.4 Quantum Group Duality

There are unitaries $W_{\mathbb{G}} \in L^\infty(\mathbb{G})\overline{\otimes}\mathcal{B}(L^2(\mathbb{G}))$ and $V_{\mathbb{G}} \in \mathcal{B}(L^2(\mathbb{G}))\overline{\otimes}L^\infty(\mathbb{G})$ satisfying the pentagonal relation

$$(W_{\mathbb{G}})_{12}(W_{\mathbb{G}})_{13}(W_{\mathbb{G}})_{23} = (W_{\mathbb{G}})_{23}(W_{\mathbb{G}})_{12},$$

where we are using the following standard leg numbering notation

$$(W_{\mathbb{G}})_{12} = W_{\mathbb{G}} \otimes \mathrm{id}, \Sigma_{23}(W_{\mathbb{G}})_{12}\Sigma_{23}, \text{ and } (W_{\mathbb{G}})_{23} = \mathrm{id} \otimes W_{\mathbb{G}},$$

where

$$\Sigma : L^2(\mathbb{G}) \otimes_2 L^2(\mathbb{G}) \to L^2(\mathbb{G}) \otimes_2 L^2(\mathbb{G}), a \otimes b \mapsto b \otimes a,$$

and implements the coproduct:

$$\Delta_{\mathbb{G}}(x) = W_{\mathbb{G}}^*(1 \otimes x)W_{\mathbb{G}} = V_{\mathbb{G}}(x \otimes 1)V_{\mathbb{G}}^*, x \in L^\infty(\mathbb{G}).$$

The unitaries $W_{\mathbb{G}}$ and $V_{\mathbb{G}}$ are known as the **left and right fundemantal unitary operators** of $\mathbb{G}$ respectively. The antipode satisfies $(S \otimes \mathrm{id})(W_{\mathbb{G}}) = W_{\mathbb{G}}^*$.

The maps

$$\lambda_{\mathbb{G}} : L^1(\mathbb{G}) \to \mathcal{B}(L^2(\mathbb{G})),\ f \mapsto (f \otimes \mathrm{id})W_{\mathbb{G}}$$

and

$$\rho_{\mathbb{G}} : L^1(\mathbb{G}) \to \mathcal{B}(L^2(\mathbb{G})),\ f \mapsto (\mathrm{id} \otimes f)V_{\mathbb{G}}$$

are representations of $L^1(\mathbb{G})$, known as the **left regular** and **right regular representation** respectively. Set

$$L^\infty(\widehat{\mathbb{G}}) = \lambda_{\mathbb{G}}(L^1(\mathbb{G}))'' \subseteq \mathcal{B}(L^2(\mathbb{G})).$$

Then, there exists a coproduct $\Delta_{\widehat{\mathbb{G}}}$, and left and right Haar weights $\widehat{h_L^{\mathbb{G}}}$ and $\widehat{h_R^{\mathbb{G}}}$ such that

$$\widehat{\mathbb{G}} := (L^\infty(\widehat{\mathbb{G}}), \Delta_{\widehat{\mathbb{G}}}, \widehat{h_L^{\mathbb{G}}}, \widehat{h_R^{\mathbb{G}}})$$

is a LCQG known as the **dual** of $\mathbb{G}$.

The following holds:

$$\hat{\hat{\mathbb{G}}} = \mathbb{G} \text{ (Pontryagin Duality)}.$$

In particular, the $L^2(\mathbb{G}) \cong L^2(\widehat{\mathbb{G}})$ as Hilbert spaces, and we have $W_{\widehat{\mathbb{G}}} = \Sigma W_{\mathbb{G}}^* \Sigma \in L^\infty(\widehat{\mathbb{G}}) \overline{\otimes} L^\infty(\mathbb{G})$. So, we have the description

$$L^\infty(\mathbb{G}) = \lambda_{\widehat{\mathbb{G}}}(L^1(\widehat{\mathbb{G}}))'' = \overline{\{(\mathrm{id} \otimes u)W_{\mathbb{G}} : u \in L^1(\widehat{\mathbb{G}})\}}^{wk*}.$$

Note that we have $V_{\mathbb{G}} \in L^\infty(\mathbb{G}) \overline{\otimes} L^\infty(\widehat{\mathbb{G}})'$.

Remark 2.4.1. Given a lcg G, the left regular representation is a τ_G-*wot* continuous unitary representation $\lambda_G : G \to \mathcal{U}(L^2(G))$ (where, here, τ_G is the topology on G), defined by setting $\lambda_G(s)\xi(\cdot) = \xi(s^{-1}\cdot)$ m-a.e. for $\xi \in L^2(G)$. This induces the left regular representation $\lambda_G : L^1(G) \to \mathcal{B}(L^2(G))$,

$$\lambda_G(f)\xi = \int_G f(s)\lambda(s)\xi(\cdot)\, ds,\ \xi \in L^2(G).$$

Then

$$L^\infty(\widehat{G}) = VN(G) = \overline{\mathrm{Span}(\lambda_G(G))}^{wk*}.$$

The coproduct satisfies

$$\Delta_{\widehat{G}}(\lambda(s)) = \lambda(s) \otimes \lambda(s).$$

In particular, $\Sigma \circ \Delta_{\widehat{G}} = \Delta_{\widehat{G}}$. A LCQG $\mathbb{G}$ satisfying $\Sigma \circ \Delta_{\mathbb{G}} = \Delta_{\mathbb{G}}$ is called **cocommutative**. Equivalently, $L^1(\mathbb{G})$ is commutative, and a LCQG $\mathbb{G}$ is cocommutative if and only if it is the dual of a lcg.

2.5 Representations and C^*-algebras

The τ_G-*wot* continuous unitary representations of a lcg G are in 1-1 correspondence with the non-degenerate $*$-representations of $L^1(G)$. Informed by this correspondence, by a representation of a LCQG $\mathbb{G}$, we will mean a representation of $L^1(\mathbb{G})$. A **corepresentation operator** is an operator $U \in L^\infty(\mathbb{G})\overline{\otimes}\mathcal{B}(\mathcal{H}_U)$, where $\mathcal{H}_U$ is a Hilbert space, such that

$$(\Delta_{\mathbb{G}} \otimes \mathrm{id})(U) = U_{12}U_{23}.$$

Then we have a correspondence between non-degenerate representations π of $L^1(\mathbb{G})$ and corepresentation operators U via the relation

$$\pi(f) = (f \otimes \mathrm{id})(U).$$

The unitary corepresentation operators correspond to non-degenerate representations on $L^1(\mathbb{G})$ that restrict to $*$–representations on $L^1_{\#}(\mathbb{G})$. We simply refer to such representations as $*$–representations. We will use the notation π, U^π, and $\mathcal{H}_\pi$ to denote a representation, its corepresentation operator, and Hilbert space respectively.

Representations π and σ are **unitarily equivalent** if there exists a unitary $U : \mathcal{H}_\pi \to \mathcal{H}_\sigma$ such that

$$(1 \otimes U)U^\pi(1 \otimes U^*) = U^\sigma.$$

We will use $[\pi]$ to denote the equivalence classes of π with respect to unitary equivalence.

Remark 2.5.1. The regular representations are $*$-representations and the corresponding fundamental unitaries that implement them are their corresponding corepresentation operators.

What were are about to describe is part of the C^*-algebraic formulation of a LCQG (cf. [81]). The **reduced C^*-algebra** of $\mathbb{G}$ is the C^*-algebra

$$C_r(\widehat{\mathbb{G}}) = \overline{\lambda_{\mathbb{G}}(L^1(\mathbb{G}))}^{\|\cdot\|_r} \subseteq \mathcal{B}(L^2(\mathbb{G}))$$

where $||\cdot||_r$ denotes the norm on $\mathcal{B}(L^2(\mathbb{G}))$. Clearly, $C_r(\widehat{\mathbb{G}})$ is weak* dense in $L^\infty(\widehat{\mathbb{G}})$.

The coassociative, non-degenerate $*$-homomorphism

$$\Delta^r_{\mathbb{G}} := \Delta_{\mathbb{G}}|_{C_r(\mathbb{G})} : C_r(\mathbb{G}) \to M(C_r(\mathbb{G}) \otimes_{min} C_r(\mathbb{G}))$$

is the **copoduct** of $C_r(\mathbb{G})$.

The space $M^r(\mathbb{G}) = C_r(\mathbb{G})^*$ is a cc Banach algebra with respect to the product

$$\mu * \nu := (\mu \otimes \nu)\Delta^r_{\mathbb{G}},\ \ \mu, \nu \in M^r(\mathbb{G}).$$

We call $M^r(\mathbb{G})$ the **reduced measure algebra** of $\mathbb{G}$.

In what follows we recount the universal setting of a LCQG (cf. [79]). Consider the C^*-norm $||\cdot||_u$ defined by setting

$$||f||_u = \sup\{||f||_{C^*} : ||\cdot||_{C^*} \text{ is a } C^*\text{-seminorm on } L^1_{\#}(\mathbb{G})\},\ \ f \in L^1_{\#}(\mathbb{G}).$$

Then we define $C_u(\widehat{\mathbb{G}}) := \overline{L^1_{\#}(\mathbb{G})}^{||\cdot||_u}$. Through this construction, we obtain the universal representation $\varpi_{\mathbb{G}} : L^1(\mathbb{G}) \to \mathcal{B}(\mathcal{H}_u)$. We denote

$$C_u(\widehat{\mathbb{G}}) := \overline{\varpi_{\mathbb{G}}(L^1(\mathbb{G}))}^{||\cdot||_u}.$$

We call $C_u(\widehat{\mathbb{G}})$ the **universal C^*-algebra** of $\mathbb{G}$.

From the universal property of $C_u(\widehat{\mathbb{G}})$, one can obtain a non-degenerate $*$-homomorphism

$$\Delta^u_{\mathbb{G}} : C_u(\mathbb{G}) \to M(C_u(\mathbb{G}) \otimes_{min} C_u(\mathbb{G})),$$

which we call the **coproduct** for $C_u(\mathbb{G})$.

The space $M^u(\mathbb{G}) = C_u(\mathbb{G})^*$ is a c.c. Banach algbera with respect to the product $\mu * \nu = (\mu \otimes \nu)\Delta^u_{\mathbb{G}}$ where $\mu, \nu \in M^u(\mathbb{G})$.

The universal property gives us the surjective, non-degenerate $*$-homomorphism

$$\Gamma_{\mathbb{G}} : C_u(\mathbb{G}) \to C_r(\mathbb{G}),\ \ \varpi_{\widehat{\mathbb{G}}}(f) \mapsto \lambda_{\widehat{\mathbb{G}}}(f).$$

We call $\Gamma_{\mathbb{G}}$ the **reducing morphism** of $\mathbb{G}$. The adjoint induces a completely isometric homomorphism

$$(\Gamma_{\mathbb{G}})^* : M^r(\mathbb{G}) \to M^u(\mathbb{G})$$

with which $M^r(\mathbb{G})$ embeds as a weak* closed ideal in $M^u(\mathbb{G})$. Moreover, the inclusion $C_r(\mathbb{G}) \subseteq L^\infty(\mathbb{G})$ induces a completely isometric weak* dense inclusion $L^1(\mathbb{G}) \subseteq M^r(\mathbb{G})$ and with which $L^1(\mathbb{G})$ is a closed ideal in $M^u(\mathbb{G})$.

It turns out $M^u(\mathbb{G})$ is unital, whose identity element we denote by $\epsilon^u_{\mathbb{G}}$. It turns out that $\epsilon^u_{\mathbb{G}} : C_u(\mathbb{G}) \to \mathbb{C}$ is a non-degenerate $*$-homomorphism and satisfies the counit property:

$$(\epsilon^u_{\mathbb{G}} \otimes \mathrm{id})\Delta^u_{\mathbb{G}} = \mathrm{id} = (\mathrm{id} \otimes \epsilon^u_{\mathbb{G}})\Delta^u_{\mathbb{G}}.$$

The functional $\epsilon^u_{\mathbb{G}}$ is often called the **counit** of $\mathbb{G}$.

2.6 Compact and Discrete Quantum Groups

There are several equivalent formulations of compact quantum groups (CQGs) (cf. [133] and Runde [105]) generalizing the notion of compactness from groups. We will say a LCQG $\mathbb{G}$ is **compact** if we have $h^{\mathbb{G}}_L = h^{\mathbb{G}}_R = h_{\mathbb{G}} \in L^1(\mathbb{G}) := L^\infty(\mathbb{G})_*$ and call $\mathbb{G}$ a **compact quantum group (CQG)**. The state $h_{\mathbb{G}}$ (after normalization) is known as the **Haar state** of $\mathbb{G}$. Equivalently, $C_u(\mathbb{G})$ and $C_r(\mathbb{G})$ are unital.

We let $Irr(\mathbb{G})$ denote the set of equivalence classes of irreducible $*$-representations of the CQG $\mathbb{G}$. We take care to point out that whenever we choose an irreducible representation $\pi \in Irr(\mathbb{G})$, we are choosing a representative π.

Given $\pi \in Irr(\mathbb{G})$ and an orthonormal basis (ONB) $\{e^\pi_j\}$ of $\mathcal{H}_\pi$, we write

$$[u^\pi_{i,j}] = U^\pi \in L^\infty(\mathbb{G})\overline{\otimes}M_{n_\pi}$$

so that

$$\pi(f) = (f \otimes \mathrm{id})U^\pi = [f(u^\pi_{i,j})], \ f \in L^1(\mathbb{G}).$$

We let $\overline{\pi}$ denote the representation $\overline{U^\pi} = [(u^\pi_{i,j})^*]_{i,j}$. A celebrated feature of CQGs is the extension of Peter-Weyl theory from the compact groups.

Theorem 2.6.1. *Let $\mathbb{G}$ be a CQG. The following hold.*

1. *Every irreducible representation of $\mathbb{G}$ is finite dimensional.*

2. *There exists a maximal family of irreducible representations such that every $*$–representation decomposes into a direct sum of elements of the maximal family.*

3. *The left regular representation decomposes into a direct sum of the irreducibles, each with multiplicity* $\dim(\mathcal{H}_\pi) = n_\pi$.

In particular, we have

$$L^2(\mathbb{G}) \cong \ell^2 - \bigoplus_{\pi \in Irr(\widehat{\mathbb{G}})} \mathcal{H}_\pi \otimes \mathcal{H}_{\overline{\pi}}$$

and we may write

$$W_{\mathbb{G}} = \bigoplus_{\pi \in Irr(\mathbb{G})} U^\pi \otimes I_{n_{\overline{\pi}}}.$$

Then

$$\mathrm{Pol}(\mathbb{G}) := \mathrm{span}\{u^\pi_{i,j} : \pi \in Irr(\mathbb{G}), 1 \leq i, j \leq n_\pi\}$$

is a $*$-algebra that is weak* dense in $L^\infty(\mathbb{G})$ and norm dense in $C_r(\mathbb{G})$. By placing a universal norm $|| \cdot ||_u$ on $\mathrm{Pol}(\mathbb{G})$ such as we did for $L^1(\mathbb{G})$ in the previous section, we get that $C_u(\mathbb{G}) \cong \overline{\mathrm{Pol}(\mathbb{G})}^{||\cdot||_u}$ as C^*-algebras. Given $\pi \in Irr(\mathbb{G})$ and fixed i_0 and j_0, the set

$$\{u^\pi_{i,j} : 1 \leq i, j \leq n_\pi\}$$

is linearly independent.

The coproduct, antipode, and counit satisfy the following formulas:

$$\begin{aligned}
\Delta_{\mathbb{G}}(u^\pi_{i,j}) &:= \Delta_{\mathbb{G}}|_{Pol(\mathbb{G})}(u^\pi_{i,j}) = \sum_{t=1}^{n_\pi} u^\pi_{i,t} \otimes u^\pi_{t,j} \\
S_{\mathbb{G}}(u^\pi_{i,j}) &= (u^\pi_{i,j})^* \\
\epsilon_{\mathbb{G}}(u^\pi_{i,j}) &:= \epsilon^u_{\mathbb{G}}|_{\mathrm{Pol}(\mathbb{G})}(u^\pi_{i,j}) = \delta_{i,j}.
\end{aligned}$$

For each $\pi \in Irr(\widehat{\mathbb{G}})$, there exists a unique positive, invertible matrix F_π, satisfying $tr(F_\pi) = tr(F_\pi^{-1}) > 0$ such that

$$((U^\pi)^t)^{-1} = (1 \otimes F_\pi)\overline{U^\pi}^{-1}(1 \otimes F_\pi^{-1}). \tag{2.1}$$

We will say F_π is the F**-matrix** associated with π. It was shown in [27] that a representative π and ONB may be chosen so that F_π is diagonal. We will not fix such a choice here, however, because we will sometimes be choosing representatives and ONBs for other reasons. Note that when F_π is chosen to be diagonal, the elements $u^\pi_{i,j}$ are linearly independent.

A critical use of the F-matrices of a CQG is in the Schur's orthogonality relations, extended from the theory of compact groups.

Theorem 2.6.2. *Let $\mathbb{G}$ be a compact quantum group. Then*

$$h_{\mathbb{G}}((u^{\pi}_{i,j})^* u^{\sigma}_{k,l}) = \delta_{\pi,\sigma}\delta_{j,l}\frac{(F_\pi^{-1})_{i,k}}{tr(F_\pi)} \text{ and } h_{\mathbb{G}}(u^{\pi}_{i,j}(u^{\sigma}_{k,l})^*) = \delta_{\pi,\sigma}\delta_{i,k}\frac{(F_\pi)_{j,l}}{tr(F_\pi)}$$

A **discrete quantum group** is a dual of a CQG. Equivalently the quantum group algebra $L^1(\mathbb{G})$ is unital, which is equivalent to having $L^1(\mathbb{G}) = M^u(\mathbb{G})$. We have

$$L^\infty(\widehat{\mathbb{G}}) = \ell^\infty(\mathbb{G}) := \ell^\infty - \bigoplus_{\pi \in Irr(\mathbb{G})} M_{n_\pi}$$

and

$$\widehat{h}_L(\cdot) = \bigoplus_{\pi \in Irr(\mathbb{G})} tr(F_\pi)tr(F_\pi^{-1}\cdot) \text{ and } \widehat{h}_R(\cdot) = \bigoplus_{\pi \in Irr(\mathbb{G})} tr(F_\pi)tr(F_\pi\cdot).$$

Moreover,

$$C_u(\mathbb{G}) = C_r(\mathbb{G}) = c_0(\mathbb{G}) := c_0 - \bigoplus_{\pi \in Irr(\widehat{\mathbb{G}})} M_{n_\pi}$$

and hence

$$L^1(\mathbb{G}) = M^u(\mathbb{G}) = M^r(\mathbb{G}) = \ell^1(\mathbb{G}) := \ell^1 - \bigoplus_{\pi \in Irr(\widehat{\mathbb{G}})} (M_{n_\pi})_*.$$

Notation: bearing in mind the duality between discreteness and compactness, we will normally use $\mathbb{G}$ to denote a DQG and $\widehat{\mathbb{G}}$ to denote a CQG.

Remark 2.6.3.

- Given a DQG $\mathbb{G}$, we have the following Fourier series decomposition: if $\pi \in Irr(\widehat{\mathbb{G}})$, then

  $$E^{\pi}_{i,j} = \sum_{t=1}^{n_\pi} \frac{1}{\mathrm{tr}(F_\pi)} (F_\pi)^{-1}_{i,k} \lambda_{\widehat{\mathbb{G}}}(\widehat{(u^{\pi}_{k,j})^*})$$

  where $x \mapsto \hat{x}$ is the linear contraction $L^\infty(\widehat{\mathbb{G}}) \to L^1(\widehat{\mathbb{G}})$ given by $\widehat{x}(y) = h_{\widehat{\mathbb{G}}}(xy)$.

  Conversely, for every basis functional $\delta^{\pi}_{i,j} \in (M_\pi)_* \subseteq \ell^1(\mathbb{G})$ defined by the formula $\delta^{\pi}_{i,j}(E^{\sigma}_{k,l}) = \delta_{\pi,\sigma}\delta_{i,k}\delta_{j,l}$, we have $(u^{\pi}_{j,i})^* = \lambda_{\mathbb{G}}(\delta^{\pi}_{i,j})$.

- The LCQGs $G = (L^\infty(G), \Delta, m)$, where G is a compact group, comprise the CQGs where $L^\infty(\mathbb{G})$ is commutative (cf. [117]). The LCQGs $\widehat{G} = (VN(G), \Delta, 1_{\{e\}})$, where G is discrete, $\Delta(\lambda(s)) = \lambda(s) \otimes \lambda(s)$, and $1_{\{e\}} \in A(G)$ is the indicator function at $\{e\}$, comprise the CQGs where $L^1(\mathbb{G})$ is commutative.

2.7 Amenability and Coamenability

Amenability is among the most well studied group properties, admitting a host of characterizations and generalizations to approximation properties. It has been a driving force behind the development of abstract Harmonic analysis, where various Banach algebraic characterizations of amenability have been sought, including nuclearity of the reduced group C^*-algebras, injectivity of the group von Neumann algebras, amenability of the group algebra, operator amenability of the Fourier algebra, and the list goes on []. Using these various characterizations, amenability is easily extended to the quantum setting and a large research programme exists to obtain analogous theorems at this level. We will review the important pieces here of the theory here, starting with the most basic definition of amenability.

Remark 2.7.1. Before proceeding, we make a technical remark. We obtain an action of $L^1(\mathbb{G})$ on $L^\infty(\mathbb{G})^*$ by taking the adjoint of the action of $L^1(\mathbb{G})$ on $L^\infty(\mathbb{G})$: we set

$$\omega * f(x) := \omega(f * x) = \omega(\mathrm{id} \otimes f)\Delta_{\mathbb{G}}(x)$$

for $f \in L^1(\mathbb{G})$, $\omega \in L^\infty(\mathbb{G})^*$, and $x \in L^\infty(\mathbb{G})$. Given von Neumann algebras N and M, it is clear that the slice maps $\varphi \otimes \mathrm{id} : N\overline{\otimes}M \to M$ are defined for normal functionals $\varphi \in N_*$. While less clear, it is the case that slice maps are still defined if we drop normality and additionally satisfy $\varphi(\mathrm{id} \otimes \Phi) = \Phi(\varphi \otimes \mathrm{id})$ for any normal $*$–homomorphism $\Phi : M \to K$ to another von Neumann algebra K (consult [30] or [92]). Thus we are justified in writing

$$\omega * f(x) = (\omega \otimes f)\Delta_{\mathbb{G}}(x) = f(x * \omega)$$

and similarly for actions on the left.

Definition 2.7.2. A functional $\omega \in L^\infty(\mathbb{G})^*$ is said to be **left invariant** if

$$f * \omega = \omega(f \otimes \mathrm{id})\Delta_{\mathbb{G}} = f(1)\omega$$

for all $f \in L^1(\mathbb{G})$. We define **right invariance similarly**. We say ω is **invariant** if it is both left and right invariant.We say $\mathbb{G}$ is **amenable** if there exists a left invariant state $m \in L^\infty(\mathbb{G})^*$.

By starting with a non-zero left invariant functional $\omega \in L^\infty(\mathbb{G})^*$, it is possible to show $|\omega|$ is a left invariant state (cf. [102]).

Proposition 2.7.3. *If there exists a non-zero left invariant functional on $L^\infty(\mathbb{G})$, then $\mathbb{G}$ is amenable.*

The following set of characterizations follow from standard tricks with the unitary antipode and convexity arguments.

Proposition 2.7.4. *[30] The following are equivalent:*

1. *$\mathbb{G}$ is amenable;*

2. *there is a right invariant state on $L^\infty(\mathbb{G})$;*

3. *and there is an invariant state on $L^\infty(\mathbb{G})$;*

4. *there is a net of states $(\omega_i) \subseteq L^1(\mathbb{G})$ such that*

$$||f * \omega_i - f(1)\omega_i||_1 \to 0$$

for all $f \in L^1(\mathbb{G})$;

In the locally compact group case, the structure of the left regular representation is intimately linked to amenability. For example, a locally compact group G is amenable if and only if the reducing morphism is injective, so $C^*(G) \cong C^*_{\lambda_G}(G)$ as C^*-algebras [56, 57].

Definition 2.7.5. We say $\mathbb{G}$ is **coamenable** if the reducing morphism $\Gamma_{\mathbb{G}} : C_u(\mathbb{G}) \to C_r(\mathbb{G})$ is injective.

One of the keys to understanding the equivalence of coamenability with amenability for locally compact groups is a certain kind of "closeness of the trivial representation to λ_G." To elaborate, first notice that we have $C^*(G) \cong C^*_{\lambda_G}(G)$ if and only if $C^*_{\lambda_G}(G)^* = B_r(G) \cong B(G)$ weak* homeomorphically and isometrically as Banach algebras. It is a consequence of Leptin's theorem [85] that this happens if and only if $1_G \in B(G)$ can be weakly approximated from $B_r(G)$. More precisely, Leptin's theorem says G is amenable if and only if $A(G)$ admits a bai (and recall $A(G)$ is weak* dense in $B_r(G)$), which can be shown to be equivalent to having $1_G \in B_r(G)$. This characterization of coamenability carries over to LCQGs.

Theorem 2.7.6. *[102] The following are equivalent:*

1. *$\mathbb{G}$ is coamenable;*

2. *$L^1(\mathbb{G})$ admits a bai;*

3. *and $\epsilon^u_{\mathbb{G}} \in M^r(\mathbb{G})$.*

It is well-known that coamenability of $\mathbb{G}$ implies amenability of $\widehat{\mathbb{G}}$, but the converse currently remains open. The converse as been achieved for a wide range of classes of LCQGs, however, including those with the approximation property [22]. In particular, this includes the locally compact groups and their duals, and the compact and discrete quantum groups, the latter of which was originally proved by Tomatsu.

Theorem 2.7.7. *[122] Let $\mathbb{G}$ be a DQG. Then $\mathbb{G}$ is amenable if and only if $\widehat{\mathbb{G}}$ is coamenable.*

2.8 Quantum Subgroups and Related Objects

2.8.1 Closed Quantum Subgroups

Let G and H be lcgs. Gelfand duality tells us that we have a continuous group homomorphism $\phi : H \to G$ if and only there is is a non-degenerate $*$-homomorphism $\pi_H : C_0(G) \to M(C_0(H))$ satisfying $(\pi_H \otimes \pi_H)\Delta_G = \pi_H \circ \Delta_H$. The map π_H is obtain from ϕ by setting $\pi_H(f) = f \circ \phi$. The map ϕ is closed and injective if and only if $\phi_H(C_0(G)) = C_0(H)$, which identifies H with a closed subgroup of G. This informs Woronowicz's notion of a closed quantum subgroup.

Definition 2.8.1. [28] Let $\mathbb{G}$ and $\mathbb{H}$ be LCQGs. We say $\mathbb{H}$ is a **(Woronowicz) closed quantum subgroup** of $\mathbb{G}$ if there exists a surjective non-degenerate $*$-homomorphism $\pi^u_{\mathbb{H}} : C_u(\mathbb{G}) \to C_u(\mathbb{H})$ satisfying

$$(\pi^u_{\mathbb{H}} \otimes \pi^u_{\mathbb{H}})\Delta^u_{\mathbb{G}} = \Delta^u_{\mathbb{H}} \circ \pi^u_{\mathbb{H}}.$$

Remark 2.8.2. Let G be a locally compact group. The (Woronowicz) closed quantum subgroups of $\widehat{G}$ are of the form $\widehat{G/N}$ where N is a closed normal subgroup of G.

The above framework is not the only way to express $\mathbb{H}$ as a subgroup of $\mathbb{G}$. For locally compact groups G and H, H is a closed subgroup of G if and only if there is an embedding $VN(H) \to VN(G)$ such that $\Delta_{\widehat{G}}|_{VN(H)} = \Delta_{\widehat{H}}$ [28]. This informs the following alternative notion of a subgroup.

Definition 2.8.3. [28] Let $\mathbb{G}$ and $\mathbb{H}$ be LCQGs. Then $\mathbb{H}$ is a **(Vaes) closed quantum subgroup** of $\mathbb{G}$ if and only if there exists an injective normal unital $*$-homomorphism $\gamma_{\mathbb{H}} : L^\infty(\widehat{\mathbb{H}}) \to L^\infty(\widehat{\mathbb{G}})$ satisfying

$$(\gamma_{\mathbb{H}} \otimes \gamma_{\mathbb{H}})\Delta_{\widehat{\mathbb{G}}} = \Delta_{\mathbb{H}} \circ \gamma_{\mathbb{H}}.$$

We will usually ignore the embedding $\gamma_{\mathbb{H}}$ and simply consider $L^\infty(\widehat{\mathbb{H}}) \subseteq L^\infty(\widehat{\mathbb{G}})$ whenever $\mathbb{H}$ is a (Vaes) closed quantum subgroup of $\mathbb{G}$.

Remark 2.8.4. 1. A **Baaj-Vaes** algebra of a LCQG $\mathbb{G}$ is a von Neumann subalgebra $N \subseteq L^\infty(\mathbb{G})$ such that $\Delta_{\mathbb{G}}(N) \subseteq N \overline{\otimes} N$, $R_{\mathbb{G}}(N) = N$, and $\tau_t^{\mathbb{G}}(N) = N$ for all $t \in \mathbb{R}$. In particular, if $\widehat{\mathbb{H}}$ is a (Vaes) closed quantum subgroup of $\widehat{\mathbb{G}}$, then $\gamma_{\widehat{\mathbb{H}}}(L^\infty(\mathbb{H}))$ is a Baaj-Vaes algebra of $\mathbb{G}$ [6]. It turns out that Baaj-Vaes algebras arise from (Vaes) closed quantum subgroups, giving us a one-to-one correspondence between (Vaes) closed quantum subgroups of $\widehat{\mathbb{G}}$ and Baaj-Vaes algebras of $\mathbb{G}$ (see [6, Proposition 10.5]).

2. When $\mathbb{G}$ is discrete or compact, then the Baaj-Vaes subalgebras of $\mathbb{G}$ are simply the subalgebras satisfying $\Delta_{\mathbb{G}}(N) \subseteq N \overline{\otimes} N$ (see [6]).

It turns out that a (Vaes) closed quantum subgroup is always a (Woronowicz) closed quantum subgroup as well [28, 64], but the converse is unknown in general. We do, however, have that a (Woronwicz) closed quantum subgroup is (Vaes) closed in the case of CQGs, DQGs, and locally compact groups and co-groups. With that in mind, in such cases, we will simply refer to (Woronowicz) and (Vaes) closed quantum subgroups as closed quantum subgroups. In the case of CQGs, with the equivalent category of Hopf $*$-algebras and their morphisms, we obtain the following alternative description.

Theorem 2.8.5. *[133] Let $\mathbb{G}$ and $\mathbb{H}$ be CQGs. Then $\mathbb{H}$ is a closed quantum subgroup of $\mathbb{G}$ if and only if there exists a surjective unital $*$-homomorphism $\pi_{\mathbb{H}} : \mathrm{Pol}(\mathbb{G}) \to \mathrm{Pol}(\mathbb{H})$ such that*

$$(\pi_{\mathbb{H}} \otimes \pi_{\mathbb{H}})\Delta_{\mathbb{G}} = \Delta_{\mathbb{H}} \circ \pi_{\mathbb{G}}.$$

It is an elementary topological fact that a closed subgroup of a compact subgroup is compact, and that a closed subgroup of a discrete subgroup is again discrete. Although less trivial, this fact remains for LCQGs.

Theorem 2.8.6. *[28, 64] Let $\mathbb{G}$ be a LCQG and $\mathbb{H}$ a (Woronowicz) closed quantum subgroup of $\mathbb{G}$. Then*

- *if $\mathbb{G}$ is compact then $\mathbb{H}$ is compact;*
- *and if $\mathbb{G}$ is discrete then $\mathbb{H}$ is discrete.*

2.8.2 Open Quantum Subgroups

For locally compact group G and H, openness of an inclusion $H \to G$ is equivalent to the existence of a surjective normal $*$-homomorphism $L^\infty(G) \to L^\infty(H)$. This informs the following definition.

Definition 2.8.7. [64] Let $\mathbb{G}$ an $\mathbb{H}$ be LCQGs. Then $\mathbb{H}$ is an **open quantum subgroup** of $\mathbb{G}$ if there exists a surjective normal unital $*$-homomorphism $\sigma_{\mathbb{H}} : L^\infty(\mathbb{G}) \to L^\infty(\mathbb{H})$ such that

$$(\sigma_{\mathbb{H}} \otimes \sigma_{\mathbb{H}})\Delta_{\mathbb{G}} = \Delta_{\mathbb{H}} \circ \sigma_{\mathbb{H}}.$$

Remark 2.8.8. Let G be a locally compact group. Then the open quantum subgroups of $\widehat{G}$ are of the form $\widehat{G/K}$ where K is a compact normal subgroup.

It is an elementary fact of topological group theory that an open subgroup of a topological group is also closed. This fact remains at the level of LCQGs.

Theorem 2.8.9. *[64] Let $\mathbb{G}$ and $\mathbb{H}$ be LCQGs where $\mathbb{H}$ is an open quantum subgroup of $\mathbb{G}$. Then $\mathbb{H}$ is (Woronowicz) closed.*

Openness allows us to embed $L^\infty(\mathbb{H})$ into $L^\infty(\mathbb{G})$, and this is done as follows: let $1_{\mathbb{H}}$ be the central support of $\sigma_{\mathbb{H}}$ and define the $*$-homomorphism

$$\iota_{\mathbb{H}} : L^\infty(\mathbb{H}) \to L^\infty(\mathbb{G}),\ \iota_{\mathbb{H}}(\sigma_{\mathbb{H}}(x)) = 1_{\mathbb{H}}x$$

which we note is an isomorphism onto its range. We use can these projections $1_{\mathbb{H}}$ to characterize open quantum subgroups by realizing them as a subset of the group-like projections.

Definition 2.8.10. Let $\mathbb{G}$ be a LCQG. A **(left or right) group-like projection** is an orthogonal projection $P \in L^\infty(\mathbb{G})$ satisfying

$$(P \otimes 1)\Delta_{\mathbb{G}}(P) = P \otimes P \text{ or } (1 \otimes P)\Delta_{\mathbb{G}}(P) = P \otimes P$$

respectively. We say P is group-like if it is both left and right.

In particular, given an open quantum subgroup $\mathbb{H}$, the projections $1_{\mathbb{H}}$ as defined above are group-like [64].

Theorem 2.8.11. *[64] Let $\mathbb{G}$ be a LCQG. Then the open quantum subgroups are in one-to-one correspondence with the central group-like projections via $\mathbb{H} \iff 1_{\mathbb{H}}$.*

Remark 2.8.12. Let G be a lcg.

1. The group-like projections of G are the indicator functions 1_H where H is an open subgroup of G.

2. The group-like projections of $\widehat{G}$ are the orthogonal projections $P_K = \int_K \lambda_G(s)\, dm_K(s)$ where K is a compact subgroup of G and $m_K \in M(G)$ is its Haar state [75]. We have that P_K is central if and only if K is normal.

Discreteness of a group automatically entails openess of any subset, and in particular, any subgroup of it. For a DQG $\mathbb{G}$ and closed quantum subgroup $\mathbb{H}$, the second adjoint $(\pi^u_{\mathbb{H}})^{**} : \ell^\infty(\mathbb{G}) \to \ell^\infty(\mathbb{H})$ realizes $\mathbb{H}$ as an open quantum subgroup of $\mathbb{G}$.

2.8.3 Quotients

Now will we discuss the formulation of a quotient space associated with a LCQG. In order to do so, we must discuss equivalent ways of approach (Woronowicz) closed quantum subgroups. These equivalent formulations are fundamentally linked to the action of "left multiplication of $\mathbb{H}$ on $\mathbb{G}$."

Definition 2.8.13. Let $\mathbb{G}$ be a LCQG and N a von Neumann algebra. A **left coaction** is a normal injective $*$-homomorphism $\alpha : N \to L^\infty(\mathbb{G})\overline{\otimes}N$ satisfying

$$(\Delta_{\mathbb{G}} \otimes \mathrm{id})\alpha = (\mathrm{id} \otimes \alpha)\alpha.$$

Given such a coaction, we say $\mathbb{G}$ acts on N. Right coactions are defined analogously, with maps $\beta : N \to N\overline{\otimes}L^\infty(\mathbb{G})$.

An action of $\mathbb{G}$ on a von Neumann algebra N bestows us with a right $L^1(\mathbb{G})$-module structure by setting

$$n *_\alpha f := (f \otimes \mathrm{id})\alpha(n),\ n \in N, f \in L^1(\mathbb{G}).$$

Then we obtain a left $L^1(\mathbb{G})$-module structure on N_* with

$$f *_\alpha \omega := (f \otimes \omega)\alpha,\ \omega \in N_*, f \in L^1(\mathbb{G}).$$

Our most basic example of a coaction is $\mathbb{G}$ acting on $L^\infty(\mathbb{G})$ with the coproduct. Then, $*_{\Delta_{\mathbb{G}}}$ is just convolution. The following theorem is what allows us to build the left action of $\mathbb{H}$ on $\mathbb{G}$.

Theorem 2.8.14. *[28] Let $\mathbb{G}$ and $\mathbb{H}$ be LCQGs. There is a one-to-one correspondence between non-degenerate $*$-homomorphisms $C_u(\mathbb{G}) \to C_u(\mathbb{H})$ and unitaries $V \in M(C_r(\mathbb{H}) \otimes_{min} C_r(\widehat{\mathbb{G}}))$ satisfying*

$$(id \otimes \Delta^r_{\widehat{\mathbb{G}}})(V) = V_{23}V_{13} \text{ and } (\Delta^r_{\mathbb{H}} \otimes id)(V) = V_{12}V_{13}.$$

The unitaries in the above theorem are called **bicharacters**, and given a non-degenerate $*$-homomorphism $\pi : C_u(\mathbb{G}) \to C_u(\mathbb{H})$, we build the corresponding bicharacter by setting $V = (\Gamma_{\mathbb{H}} \circ \pi_{\mathbb{H}} \otimes \Gamma_{\widehat{\mathbb{G}}})(W_{\mathbb{G}})$. We can build a right version V_r with the right fundamental unitary.

Now, let $\mathbb{H}$ be a (Woronowicz) closed quantum subgroup. Then the map

$$r_{\mathbb{H}} : L^\infty(\mathbb{G}) \to L^\infty(\mathbb{G}) \overline{\otimes} L^\infty(\mathbb{H}),\ x \mapsto V_r(x \otimes 1)V_r^*$$

is a coaction of $\mathbb{H}$ on $L^\infty(\mathbb{G})$ (cf. [28]). Then we define the right quotient space via the following fixed point von Neumann algebra:

$$L^\infty(\mathbb{G}/\mathbb{H}) = \{x \in L^\infty(\mathbb{G}) : r_{\mathbb{H}}(x) = x \otimes 1\} =: \mathrm{Fix}(r_{\mathbb{H}}).$$

With the bicharacter constructed from $\pi_{\mathbb{H}}$, we obtain a left coaction $l_{\mathbb{H}}$, and the quotient space $\mathbb{H}\backslash\mathbb{G}$ is the defined to be the fixed point algebra of $l_{\mathbb{H}}$: $L^\infty(\mathbb{H}\backslash\mathbb{G}) := \mathrm{Fix}(l_{\mathbb{H}})$.

In the special case where $\mathbb{H}$ is open, the canonical left action of $\mathbb{H}$ on $\mathbb{G}$ can be realized by setting

$$l_{\mathbb{H}} = (\mathrm{id} \otimes \sigma_{\mathbb{H}})\Delta_{\mathbb{G}} \text{ [64]}$$

and similarly for the right coaction. We also have the following description of $\mathbb{G}/\mathbb{H}$:

$$L^\infty(\mathbb{G}/\mathbb{H}) = \{x \in L^\infty(\mathbb{G}) : (1 \otimes 1_{\mathbb{H}})\Delta_{\mathbb{G}}(x) = x \otimes 1_{\mathbb{H}}\}.$$

Remark 2.8.15. The quotient spaces $L^\infty(\mathbb{H}\backslash\mathbb{G})$ and $L^\infty(\mathbb{G}/\mathbb{H})$ are $\mathbb{G}$-spaces via the left and right coactions $\Delta_{\mathbb{G}}|_{L^\infty(\mathbb{H}\backslash\mathbb{G})}$ and $\Delta_{\mathbb{G}}|_{L^\infty(\mathbb{G}/\mathbb{H})}$ respectively.

For a (Woronowicz) closed quantum subgroup $\mathbb{H} \leq \mathbb{G}$, we will denote the corresponding $L^1(\mathbb{H})$-bimodule action on $L^1(\mathbb{G})$ by setting

$$g *_{\mathbb{H}} f = (g \otimes f)l_{\mathbb{H}} \text{ and } f *_{\mathbb{H}} g = (f \otimes g)r_{\mathbb{H}},\ f \in L^1(\mathbb{G}), g \in L^1(\mathbb{H}).$$

Then, using the equations

$$\rho^l_{\mathbb{H}}|_{C_r(\mathbb{G})} \circ \Gamma_{\mathbb{G}} = ((\Gamma_{\mathbb{H}} \circ \pi^u_{\mathbb{H}}) \otimes \Gamma_{\mathbb{G}})\Delta^u_{\mathbb{G}}$$
$$\rho^r_{\mathbb{H}}|_{C_r(\mathbb{G})} \circ \Gamma_{\mathbb{G}} = (\Gamma_{\mathbb{G}} \otimes (\Gamma_{\mathbb{H}} \circ \pi^u_{\mathbb{H}}))\Delta^u_{\mathbb{G}} \text{ [14, Section 1]}$$

we get, for $f \in L^1(\mathbb{G})$ and $g \in L^1(\mathbb{H})$,

$$\begin{aligned}
\overbrace{(g \circ \Gamma_{\mathbb{H}} \circ \pi^u_{\mathbb{H}})}^{\in M^u(\mathbb{G})} * \overbrace{(f \circ \Gamma_{\mathbb{G}})}^{\in M^u(\mathbb{G})} &= ((g \circ \Gamma_{\mathbb{H}} \circ \pi^u_{\mathbb{H}}) \otimes (f \circ \Gamma_{\mathbb{G}}))\Delta^u_{\mathbb{G}} \\
&= (g *_{\mathbb{H}} f)\Gamma_{\mathbb{G}} \\
(f \circ \Gamma_{\mathbb{G}}) * (g \circ \Gamma_{\mathbb{H}} \circ \pi^u_{\mathbb{H}}) &= (f *_{\mathbb{H}} g)\Gamma_{\mathbb{G}}
\end{aligned}$$

which says the action of $L^1(\mathbb{H})$ on $L^1(\mathbb{G})$ is really just convolution in $M^u(\mathbb{G})$. Note that whenever $\mathbb{H}$ is open, we have for $f \in L^1(\mathbb{G})$ and $g \in L^1(\mathbb{H})$,

$$g *_{\mathbb{H}} f = (g \circ \sigma_{\mathbb{H}}) * f \text{ and } f *_{\mathbb{H}} g = f * (g \circ \sigma_{\mathbb{H}}).$$

We denote $J^1(\mathbb{G}, \mathbb{H}) := L^\infty(\mathbb{G}/\mathbb{H})_\perp$, which satisfies

$$L^1(\mathbb{G})/J^1(\mathbb{G}, \mathbb{H}) \cong L^1(\mathbb{G}/\mathbb{H}) := L^\infty(\mathbb{G}/\mathbb{H})_*.$$

In particular, $J^1(\mathbb{G}, \mathbb{H}) = \ker(T_{\mathbb{H}})$ where we let $T_{\mathbb{H}} : L^1(\mathbb{G}) \to L^1(\mathbb{G}/\mathbb{H})$ be the above quotient map. Since $L^\infty(\mathbb{G}/\mathbb{H})$ is a $\mathbb{G}$-space via the coaction coming from restriction of the coproduct, we have $L^\infty(\mathbb{G}/\mathbb{H}) * L^1(\mathbb{G}) \subseteq L^\infty(\mathbb{G}/\mathbb{H})$. The former condition is easily seen to be equivalent to the fact $J^1(\mathbb{G}, \mathbb{H})$ is a left ideal. Two-sidedness of these properties is describing normality of $\mathbb{H}$.

Theorem 2.8.16. *[6] Let $\mathbb{G}$ be a LCQG and $\mathbb{H}$ a (Woronowicz) closed quantum subgroup. TFAE:*

1. $L^\infty(\mathbb{H}\backslash\mathbb{G}) = L^\infty(\mathbb{G}/\mathbb{H})$;
2. $\mathbb{G}/\mathbb{H}$ *is a LCQG with coproduct* $\Delta_{\mathbb{G}/\mathbb{H}} := \Delta_{\mathbb{G}}|_{L^\infty(\mathbb{G}/\mathbb{H})}$;
3. $\Delta_{\mathbb{G}}(L^\infty(\mathbb{G}/\mathbb{H})) \subseteq L^\infty(\mathbb{G}/\mathbb{H})\overline{\otimes}L^\infty(\mathbb{G}/\mathbb{H})$;
4. $J^1(\mathbb{G}, \mathbb{H})$ *is two-sided.*

Definition 2.8.17. Let $\mathbb{G}$ be a LCQG and $\mathbb{H}$ a (Woronowicz) closed quantum subgroup. We say $\mathbb{H}$ is **normal** if it satisfies one of the equivalent conditions of Theorem 2.8.16. We will write $\mathbb{H} \trianglelefteq \mathbb{G}$.

For an abelian locally compact group G, it is an elementary result that we have duality between its normal subgroups and quotient groups of its Pontryagin dual via $G \trianglerighteq N \iff \widehat{G/N} \trianglelefteq \widehat{G}$. This correspondence persists into the setting of LCQGs:

$$\mathbb{G} \trianglerighteq \mathbb{H} \iff \widehat{\mathbb{G}/\mathbb{H}} \trianglelefteq \widehat{\mathbb{G}} \text{ [28]}.$$

In the case of CQGs, we have additional structure to consider coming from the underlying Hopf $*$-algebras. Indeed, given a CQG $\mathbb{G}$ and closed quantum subgroup $\mathbb{H}$, we will set

$$\mathrm{Pol}(\mathbb{G}/\mathbb{H}) := \{a \in \mathrm{Pol}(\mathbb{G}) : (\mathrm{id} \otimes \pi_{\mathbb{H}})\Delta_{\mathbb{G}} = a \otimes 1\}.$$

Then $L^\infty(\mathbb{G}/\mathbb{H}) = \overline{\mathrm{Pol}(\mathbb{G}/\mathbb{H})}^{wk*}$. The Haar state of $\mathbb{H}$ gives us a projection

$$R_{h_{\mathbb{H}} \circ \pi_{\mathbb{H}}} := (\mathrm{id} \otimes h_{\mathbb{H}} \circ \pi_{\mathbb{H}})\Delta_{\mathbb{G}} : \mathrm{Pol}(\mathbb{G}) \to \mathrm{Pol}(\mathbb{G}/\mathbb{H})$$

This allows us to see the following.

Theorem 2.8.18. *[129] Let $\mathbb{G}$ be a CQG and $\mathbb{H}$ a closed quantum subgroup. For all $\pi \in Irr(\mathbb{G})$, $\pi(\omega_{\mathbb{H}}) = I_{d_\pi}$ where $1 \leq d_\pi \leq n_\pi$. Furthermore, $\mathbb{H}$ is normal if and only if $\pi(\omega_{\mathbb{H}}) = I_{n_\pi}$ or 0. Whenever $\mathbb{H}$ is normal, we have*

$$\mathrm{Pol}(\mathbb{G}/\mathbb{H}) = \mathrm{Span}\{u^\pi_{i,j} : \pi(\omega_{\mathbb{H}}) = I_{n_\pi}, \pi \in Irr(\mathbb{G})\}.$$

2.9 Ideals and Coideals

2.9.1 Ideals and Invariant Subspaces

Before proceeding, we will setup the notion of invariance. Let X and Y be weak* closed subspaces of $\mathcal{B}(\mathcal{H})$ and $\mathcal{B}(\mathcal{K})$ respectively, where $\mathcal{H}$ and $\mathcal{K}$ are Hilbert spaces. Then the **Fubini tensor** product is the tensor product space

$$X \overline{\otimes}_{\mathcal{F}} Y = \{x \in \mathcal{B}(\mathcal{H} \otimes_2 \mathcal{K}) : (\mathrm{id} \otimes \omega)(x) \in X, (\mu \otimes \mathrm{id})(x) \in Y, \omega \in \mathcal{B}(\mathcal{H})_*, \mu \in \mathcal{B}(\mathcal{K})_*\}.$$

Note that clearly we always have $X \overline{\otimes} Y \subseteq X \overline{\otimes}_{\mathcal{F}} Y$ and for von Neumann algebras N and M we have $N \overline{\otimes} M = N \overline{\otimes}_{\mathcal{F}} M$ [77]. Also, if we pick preduals X_* and Y_* of X and Y respectively, then $(X_* \widehat{\otimes} Y_*)^* = X \overline{\otimes}_{\mathcal{F}} Y$ (see [33]).

Definition 2.9.1. For a LCQG $\mathbb{G}$, we say a subset $E \subseteq L^\infty(\mathbb{G})$, $C_r(\mathbb{G})$, or $C_u(\mathbb{G})$ is **right $\mathbb{G}$-invariant** or **right invariant** if $E * L^1(\mathbb{G}) \subseteq E$, $E * M^r(\mathbb{G}) \subseteq E$, or $E * M^u(\mathbb{G}) \subseteq E$ respectively. If $\mathbb{G}$ is compact, we say $E \subseteq \mathrm{Pol}(\mathbb{G})$ is right invariant if $E * \mathrm{Pol}(\mathbb{G})^* \subseteq E$. We will write

$$X \trianglelefteq_r L^\infty(\mathbb{G}),\ X \trianglelefteq_r C_r(\mathbb{G}), C_u(\mathbb{G}),\ \text{or } X \trianglelefteq_r \mathrm{Pol}(\mathbb{G})$$

whenever X is a right invariant weak* closed subspace, norm closed subspace, or subspace respectively. We analogously define **left invariance**. Whenever a subspace such as above is both left and right invariant, we will simply use the notation $\trianglelefteq$ instead.

Remark 2.9.2. It is easy to see that a weak* closed subspace $X \subseteq L^\infty(\mathbb{G})$ is right invariant if and only if $\Delta_{\mathbb{G}}(X) \subseteq L^\infty(\mathbb{G})\overline{\otimes}_{\mathcal{F}}X$. If $\mathbb{G}$ is compact, then a subspace $X \subseteq \mathrm{Pol}(\mathbb{G})$ is right invariant if and only if $\Delta_{\mathbb{G}}(X) \subseteq \mathrm{Pol}(\mathbb{G}) \otimes X$.

In this thesis, we will have particular interest in the closed left ideals of $L^1(\mathbb{G})$. It is straightforward to show, for example, that if $X \subseteq L^\infty(\mathbb{G})$ is a right invariant subspace, then its preannihilator is a closed left ideal $X_\perp \trianglelefteq_l L^1(\mathbb{G})$, and whenever I is a left ideal of $L^1(\mathbb{G})$, its annihilator is a right invariant weak* closed subspace $I^\perp \trianglelefteq_r L^\infty(\mathbb{G})$. Using the bipolar theorem, we then have $(X_\perp)^\perp = \overline{X}^{wk*}$ and $(I^\perp)_\perp = \overline{I}^{\|\cdot\|_1}$. Thus we achieve the following linear duality between right invariant subspaces and left ideals.

Proposition 2.9.3. *Let $\mathbb{G}$ be a LCQG. We have the following one-to-one correspondences:*

1. *weak* closed right invariant subspaces of $L^\infty(\mathbb{G})$ and closed left ideals of $L^1(\mathbb{G})$, via*
$$X \trianglelefteq_r L^\infty(\mathbb{G}) \iff X_\perp \trianglelefteq_l L^1(\mathbb{G});$$
2. *closed right invariant subspaces of $C_r(\mathbb{G})$ and weak* closed left ideals of $M^r(\mathbb{G})$, via*
$$X \trianglelefteq_r C_r(\mathbb{G}) \iff X^\perp \trianglelefteq_l M^r(\mathbb{G});$$
3. *closed right invariant subspaces of $C_u(\mathbb{G})$ and weak* closed left ideals of $M^u(\mathbb{G})$, via*
$$X \trianglelefteq_r C_u(\mathbb{G}) \iff X^\perp \trianglelefteq_l M^u(\mathbb{G});$$
4. *and whenever $\mathbb{G}$ is compact, right invariant subspaces of $\mathrm{Pol}(\mathbb{G})$ and left ideals of $\mathrm{Pol}(\mathbb{G})$, via*
$$X \trianglelefteq_r \mathrm{Pol}(\mathbb{G}) \iff X^\perp \trianglelefteq_l \mathrm{Pol}(\mathbb{G})^*.$$
Note that $\mathrm{Pol}(\mathbb{G})^*$ is the vector space dual of $\mathrm{Pol}(\mathbb{G})$.

2.9.2 Coideals

Definition 2.9.4. Let $\mathbb{G}$ be a LCQG. A **right coideal** of $L^\infty(\mathbb{G})$, or $C_r(\mathbb{G}), C_u(\mathbb{G})$ is a right invariant von Neumann subalgebra or C^*-subalgebra respectively. If $\mathbb{G}$ is compact, then a **coideal** of $\mathrm{Pol}(\mathbb{G})$ is a right invariant subalgebra. We define **left coideals** similarly, and call the subalgebras that are both left and right coideals **two-sided coideals**. We will call a right coideal of the form $L^\infty(\mathbb{G}/\mathbb{H})$, for a (Woronowicz) closed quantum subgroup $\mathbb{H}$, a right coideal of **quotient type**.

Remark 2.9.5. 1. By definition, the Baaj-Vaes subalgebras of $L^\infty(\mathbb{G})$ are two-sided coideals. In particular, every (Vaes) closed quantum subgroup $\mathbb{H}$ of $\mathbb{G}$ admits a canonical two-sided coideal of $\widehat{\mathbb{G}}$, $L^\infty(\widehat{\mathbb{H}})$, and a right coideal of $\mathbb{G}$, $L^\infty(\mathbb{H}/\mathbb{G})$.

2. If G is a LCG, then every right coideal of $L^\infty(G)$ is of quotient type and every coideal of $VN(G)$ is two-sided (see [69, 28]). Then there is a correspondence

$$\begin{aligned} H \leq G &\iff \text{right coideals of } G \\ &\iff \text{two-sided coideals of } \widehat{G}. \end{aligned}$$

Now, we will set up notation generalizing what we had for (Woronowicz) closed quantum subgroups in the previous section. Fix a LCQG $\mathbb{G}$ and right coideal N of $L^\infty(\mathbb{G})$. We will denote the preannihilator $J^1(N) = N_\perp$, and again, we note $N_* \cong L^1(\mathbb{G})/J^1(N)$, so $J^1(N) = (T_N)_\perp$ where we let $T_N : L^1(\mathbb{G}) \to N_*$ be the quotient map. The following is straightfoward.

Proposition 2.9.6. *Let $\mathbb{G}$ be a LCQG and N a right coideal of $L^\infty(\mathbb{G})$. TFAE:*

1. *N is a coideal;*
2. *$J^1(N)$ is two-sided;*
3. *and T_N is an algebraic homomorphism.*

Remark 2.9.7. 1. Clearly, whenever $\mathbb{H}$ is a (Woronowicz) closed quantum subgroup of $\mathbb{G}$, $J^1(L^\infty(\mathbb{G}/\mathbb{H})) = J^1(\mathbb{G}, \mathbb{H})$, and $T_{L^\infty(\mathbb{G}/\mathbb{H})} = T_{\mathbb{H}}$. In particular, for a locally compact group G and closed subgroup H, we already pointed out in the previous section that $J^1(L^\infty(G/H)) = J^1(G, H)$, which we remark covers every case of a right coideal. On the Pontryagin dual side, we have $J^1(VN(H)) = I(H)$.

2. For a LCQG $\mathbb{G}$, every right coideal of $L^\infty(\mathbb{G})$ offers a natural action of $\mathbb{G}$ on $L^\infty(\mathbb{G})$, generalizing the natural action of a locally compact group on a quotient space. Indeed, given a right coideal N, the map $\Delta_{\mathbb{G}}|_N : N \to L^\infty(\mathbb{G})\overline{\otimes}N$ is a left coaction of $\mathbb{G}$ on N.

We have a general way of transporting coideals between between $L^\infty(\mathbb{G})$ and $L^\infty(\widehat{\mathbb{G}})$.

Definition 2.9.8. Let $\mathbb{G}$ be a LCQG and N a right coideal of $L^\infty(\mathbb{G})$. The **codual** of N is the left coideal $\widetilde{N} := N' \cap L^\infty(\widehat{\mathbb{G}})$.

Remark 2.9.9. 1. It is the case that $\tilde{\tilde{N}} = N$ [72, 123].

2. If $\mathbb{G}$ is a LCQG and $\mathbb{H}$ a (Vaes) closed quantum subgroup, then $L^\infty(\widehat{\mathbb{H}}) = \widetilde{L^\infty(\mathbb{G}/\mathbb{H})}$ [72, 123].

3. If $\mathbb{G}$ is discrete or compact, then the Baaj-Vaes algebras are simply the two-sided coideals (see [6]).

2.9.3 Compact Quasi-Subgroups

At the level of LCQGs, there are nested classes of right coideals, which are equal in the classical setting. We have already seen, for example, that not every right coideal of $VN(G)$, for a locally compact group G, is the quotient space of a closed quantum subgroup. We also discussed how open quantum subgroups arise from central group-like projections, thus leading one to wonder about the relationship between idempotent states and group-like projections given the Pontryagin duality between openness and compactness. This has recently been an active line research from which a compelling set of solutions has been found [46, 111, 112, 74, 73]. The short answer is, we do achieve a correspondence between certain group-like projection of a LCQG and idempotents states in its Pontryagin dual.

Before proceeding, we set-up the basics for multipliers and completely bounded multipliers. We recommend [13, 54] as references for the following discussions.

Definition 2.9.10. A **left multiplier** of a Banach algebra A is a bounded linear right A–module map $m : A \to A$. We denote the left multipliers on A by $M^l(A)$. We denote the **completely bounded left multipliers** by $M^l_{cb}(A) := M^l(A) \cap \mathcal{CB}(A)$.

Remark 2.9.11. Note that the elements $m \in M^l_{cb}(A)$ are exactly those such that m^* is completely bounded.

We note $M^l(A)$ is a Banach algebra, viewed as a subalgebra of $\mathcal{B}(A)$, which has A embedded contractively as an ideal via the map $a \mapsto m_a$ where $m_a(b) = ab$ for $b \in A$, and we will denote the adjoint $M_a := m_a^*$. Similarly, $M^l_{cb}(A)$ is a c.c. Banach algebra, viewed as an operator subspace of $\mathcal{CB}(A)$, which has A embedded into $M^l_{cb}(A)$ completely contractively. For a LCQG $\mathbb{G}$, because $M^u(\mathbb{G})$ contains $L^1(\mathbb{G})$ as an ideal, we get that $M^u(\mathbb{G})$ embeds completely contractively into $M^l_{cb}(L^1(\mathbb{G}))$ via the map $\mu \mapsto m_\mu$ where $m_\mu(f) = \mu * f$ for $f \in L^1(\mathbb{G})$ and again we denote the adjoint by M_μ. Note also that $M_{\epsilon^u_{\mathbb{G}}} = \text{id}$.

The **double centralizers** of a Banach algebra A are pairs (L, R) of left and right multipliers $L \in M^l(A)$ and $R \in M^r(A)$ satisfying $aL(b) = R(a)b$. We denote the double centralizers by $M(A)$, which also turns out to be a Banach algebra, and has A contractively embedded as an ideal via the map $a \mapsto (l_a, r_a)$ where $l_a(b) = ab$ and $r_a(b) = ba$. There is also a contractive embedding $M(A) \subseteq M^l(A)$. Similarly, we define **completely bounded double centralizers** of a c.c. Banach algebra A, which are double centralizers whose associated bounded linear maps and completely bounded. We denote the completely bounded double centralizers by $M_{cb}(A)$. Similarly, we have completely contractive embeddings $A \subseteq M_{cb}(A) \subseteq M^l_{cb}(A)$. For LCQGs, we have $M^u(\mathbb{G}) \subseteq M^l_{cb}(L^1(\mathbb{G}))$.

The adjoint $M = m^* : L^\infty(\mathbb{G}) \to L^\infty(\mathbb{G})$ of a left multiplier m is a normal linear right $L^1(\mathbb{G})$-module map. Note that we have a contractive inclusion $M^u(\mathbb{G}) \to M^l_{cb}(L^1(\mathbb{G}))$ via $\mu \mapsto m^l_\mu$ where $m^l_\mu(f) = f * \mu$ for all $f \in L^1(\mathbb{G})$. For $\mu \in M^u(\mathbb{G})$, we will denote $R_\mu = (m^l_\mu)^*$. The following outlines how the various properties of functionals in $M^u(\mathbb{G})$ correspond to properties of the left multipliers they generate.

Proposition 2.9.12. *[112] Let $\mathbb{G}$ be LCQG and $\mu \in M^u(\mathbb{G})$. Then, for $\mu, \omega \in M^u(\mathbb{G})$,*

1. *$R_\mu \circ R_\omega = R_{\omega * \mu}$*
2. *R_μ is positive if and only if μ is positive;*
3. *and R_μ is unital if and only if μ is unital.*

We we will denote the adjoints of thhe maps $\mu \mapsto \mu * f$ by L_μ.

In particular, condition 1 of the above proposition says M^l_μ is a projection if and only if μ is idempotent. It turns out we can characterize the states in terms of completely positive left multipliers.

Theorem 2.9.13. *[112] Let $\mathbb{G}$ be a LCQG and $L : L^\infty(\mathbb{G}) \to L^\infty(\mathbb{G})$ be a normal unital completely positive map. TFAE:*

1. *L is right $L^1(\mathbb{G})$-module map;*
2. *L is the adjoint of a left multiplier;*
3. *$\Delta_{\mathbb{G}} \circ L = (id \otimes L)\Delta_{\mathbb{G}}$;*
4. *there exists a state $\mu \in M^u(\mathbb{G})$ such that $L = R_\mu$.*

From the above we can glean that the idempotent states $\omega \in M^u(\mathbb{G})$ correspond with normal unital completely positive right $L^1(\mathbb{G})$-module projections. In the case where $\mathbb{G}$ is compact,

$$R_\omega|_{\mathrm{Pol}(\mathbb{G})} = (\mathrm{id} \otimes \omega)\Delta_{\mathbb{G}} : \mathrm{Pol}(\mathbb{G}) \to \mathrm{Pol}(\mathbb{G})$$

is a projection. Most importantly for us is that the objects $N_\omega := R_\omega(L^\infty(\mathbb{G}))$, where $\omega \in M^u(\mathbb{G})$ is an idempotent state are coideals.

Definition 2.9.14. A **compact quasi-subgroup** of $\mathbb{G}$ is a right coideal of the form N_ω for some idempotent state $\omega \in M^u(\mathbb{G})$. We will say N_ω is **open** if $\omega \in L^1(\mathbb{G})$.

Whenever $\omega \in M^u(\mathbb{G})$ is an idempotent state, the orthogonal projection $P_\omega = \lambda_{\mathbb{G}}(\omega) \in L^\infty(\widehat{\mathbb{G}})$ is (left and right) group-like (see Section 2.8.2 for definition).

We will say a group-like projection $P \in L^\infty(\mathbb{G})$ is **invariant under the scaling group** of $\mathbb{G}$ if $\tau_t^{\mathbb{G}}(P) = P$ for all $t \in \mathbb{R}$. By an **integrable** coideal, we mean a coideal $N \subseteq L^\infty(\mathbb{G})$ such that $h_L^{\mathbb{G}}|_N$ is semifinite.

Theorem 2.9.15. *[38] Let $\mathbb{G}$ be a LCQG. Then there is a one-to-one correspondence between the following:*

- *idempotent states $\omega \in M^u(\mathbb{G})$;*
- *group-like projections $P_\omega \in L^\infty(\widehat{\mathbb{G}})$ that are invariant under the scaling group;*
- *and integrable right coideals $N_\omega \subseteq L^\infty(\mathbb{G})$ that are invariant under the scaling group.*

Given an idempotent state $\omega \in M^u(\mathbb{G})$, the corresponding group-like projections are given by

$$P_\omega = \lambda_{\mathbb{G}}(\omega).$$

The codual of N_ω has the following description:

$$\widetilde{N_\omega} = \{x \in L^\infty(\widehat{\mathbb{G}}) : (P_\omega \otimes \mathrm{id})\Delta_{\widehat{\mathbb{G}}}(x) = P_\omega \otimes x\} =: \widetilde{N^L_{P_\omega}}$$

and is the minimal left coideal containing P_ω [38]. In the above case, we say P_ω **generates** N_{P_ω}. From Proposition 2.9.12, we can see that for idempotent states $\omega, \mu \in M^u(\mathbb{G})$, $\omega * \mu = \omega$ if and only if $N_\omega \subseteq N_\mu$, which is, of course, equivalent to having $\widetilde{N^L_{P_\omega}} \supseteq \widetilde{N^L_{P_\mu}}$ (cf. [74]).

More generally, we will consider the right coideals of the form

$$\widetilde{N_P} := \{x \in L^\infty(\mathbb{G}) : (1 \otimes P)\Delta_{\mathbb{G}}(x) = x \otimes P\},$$

where $P \in L^\infty(\mathbb{G})$ is a (right) group-like projection. Thus, we propose the following definition.

Definition 2.9.16. Let $\mathbb{G}$ be a LCQG. An **open quasi-subgroup** of $\mathbb{G}$ is a right coideal of $L^\infty(\mathbb{G})$ of the form N_P for a group-like projection P.

Remark 2.9.17.
1. If $\mathbb{G}$ is discrete, then every right coideal of $\mathbb{G}$ is open [72, Proposition 1.5 and Corollary 1.6].
2. An open compact quasi-subgroup is an open coideal in our sense (see [74]).
3. If $\mathbb{G}$ is compact, the open compact quasi-subgroups are the finite dimensional coideals (see [74]).

Much like the statement that the open compact subgroups are dual to open compact subgroups of abelian locally compact groups, we get the following.

Theorem 2.9.18. *[74] Let $\mathbb{G}$ be a LCQG and N an open compact quasi-subgroup. Then $\widetilde{N}$ is also an open compact quasi-subgroup.*

2.9.4 Haar Idempotents

As we alluded to for CQGs, the quotients of CQGs by quantum subgroups are examples of compact quasi-subgroups. More generally, a compact (Woronowicz) closed quantum subgroup $\mathbb{H}$ of a LCQG $\mathbb{G}$ has a quotient space that is a compact quasi-subgroup. The Haar state of $\mathbb{H}$ induces the idempotent state $h_{\mathbb{H}} \circ \pi_{\mathbb{H}} \in M^u(\mathbb{G})$. Then $L^\infty(\mathbb{G}/\mathbb{H}) = N_{h_{\mathbb{H}} \circ \pi_{\mathbb{H}}}$.

Definition 2.9.19. An idempotent state of the form $h_{\mathbb{H}} \circ \pi_{\mathbb{H}}$ for a compact (Woronowicz) closed quantum subgroup of $\mathbb{G}$ is called a **Haar idempotent**.

Remark 2.9.20. It is a theorem of Kawada and Itô [75] that the closed subgroups of a compact group G are in 1–1 correspondence with the idempotent states in the measure algebra via $H \leq G \iff m_H \in M(G)$. Likewise, for a discrete group Γ, we have a 1–1 correspondence between subgroups and idempotent states in $B(\Gamma)$ via $\Lambda \leq \Gamma \iff 1_\Lambda \in B(\Gamma)$, and we have $VN(\Gamma) \cdot 1_\Lambda = VN(\Lambda)$ (see [58]). The idempotent states in $B(\Gamma)$ that are Haar idempotents are those where Λ is normal.

The Haar idempotents are distinguished in the class of idempotent states with the following.

Theorem 2.9.21. *[46, 112] Let $\mathbb{G}$ be a LCQG. An idempotent state $\omega \in M^u(\mathbb{G})$ is Haar if and only if*

$$I_\omega = \{a \in C^u(\mathbb{G}) : \omega(a^*a) = 0\}$$

is a two-sided ideal.

Remark 2.9.22. An application of the Cauchy-Schwarz inequality easily tells that N_ω is always a left ideal, so what must be determined is whether or not it is right invariant. Equivalently, I_ω is self-adjoint. An inspection of the proof reveals that if $\omega = h_{\mathbb{H}} \circ \pi_{\mathbb{H}}$ then $C_u(\mathbb{G})/I_\omega \cong C_r(\mathbb{H})$ as ω corresponds to a faithful state on $C_u(\mathbb{G})/I_\omega$. As noted in [112], the quotient map $C_u(\mathbb{G}) \to C_r(\mathbb{H})$ extends to a morphism $C_u(\mathbb{G}) \to C_u(\mathbb{H})$ which realizes $\mathbb{H}$ as a quantum subgroup of $\mathbb{G}$. If $\mathbb{G}$ is compact, then $\mathrm{Pol}(\mathbb{G})/\mathrm{Pol}(\mathbb{G}) \cap I_\omega \cong \mathrm{Pol}(\mathbb{H})$.

2.10 Examples of C/DQGs

In this section we will explain the main examples we will be seeing in this thesis. Since we are primarily interested in C/DQGs, we will restrict our attention to such cases.

2.10.1 Tensor Products

Compact Products

Let $\mathbb{G}$ and $\mathbb{H}$ be CQGs. Taking inspiration from the completely isometric isomorphisms $VN(G)\overline{\otimes}VN(H) \cong VN(G \times H)$ and $L^\infty(G)\overline{\otimes}L^\infty(H) \cong L^\infty(G \times H)$, the product of $\mathbb{G}$ and $\mathbb{H}$ is given by the von Neumann algebra

$$L^\infty(\mathbb{G} \times \mathbb{H}) := L^\infty(\mathbb{G})\overline{\otimes}L^\infty(\mathbb{H}),$$

with coproduct $\Delta_{\mathbb{G}\times\mathbb{H}} := (\mathrm{id} \otimes \Sigma \otimes \mathrm{id})(\Delta_{\mathbb{G}} \otimes \Delta_{\mathbb{G}})$, and Haar state $h_{\mathbb{G}\times\mathbb{H}} := h_{\mathbb{G}} \otimes h_{\mathbb{H}}$ [127]. We further realize the structure of $\mathbb{G} \times \mathbb{H}$ as follows.

Proposition 2.10.1. *[127] The following hold:*

1. $Irr(\mathbb{G} \times \mathbb{H}) = \{(u^\pi_{i,j} \otimes u^\sigma_{k,l})^{n_\pi n_\sigma}_{i,j,k,l=1},\ \pi \in Irr(\mathbb{G}), \sigma \in Irr(\mathbb{H})\}$;

2. $\mathrm{Pol}(\mathbb{G} \times \mathbb{H}) = \mathrm{Pol}(\mathbb{G}) \otimes \mathrm{Pol}(\mathbb{H})$*;*

3. and $L^1(\mathbb{G} \times \mathbb{H}) = L^1(\mathbb{G}) \widehat{\otimes} L^1(\mathbb{H})$*.*

From the definition, it is easy to see that convolution in $L^1(\mathbb{G}\times\mathbb{H})$ satisfies the equation

$$(f_1 \otimes g_1) * (f_2 \otimes g_2) = (f_1 * f_2) \otimes (g_1 * g_2), \;\; f_1, f_2 \in L^1(\mathbb{G}), g_1, g_2 \in L^1(\mathbb{H}).$$

It is also straightforward to see that any right coideal N of $L^\infty(\mathbb{G})$ is also right coideal of $L^\infty(\mathbb{G} \times \mathbb{H})$ and similarly for right coideals in $L^\infty(\mathbb{H})$. Furthermore, both $\mathbb{G}$ and $\mathbb{H}$ are normal closed quantum subgroups of $\mathbb{G} \times \mathbb{H}$, which are realized via the C^*-quantum homomorphisms $\pi^u_{\mathbb{G}} = (\mathrm{id} \otimes \epsilon^u_{\mathbb{H}})$ and $\pi^u_{\mathbb{H}} = (\epsilon^u_{\mathbb{G}} \otimes \mathrm{id})$ respectively [127].

Example 2.10.2. Let K be a compact group and G a discrete group. Then $K \times \widehat{G}$ is a compact quantum group. It is a 'genuine' example of a quantum group when neither G nor K is abelian. Their duals give us examples of discrete quantum groups.

2.10.2 Discrete Crossed Products

CQGs from Crossed Products

With what follows, we use [26] as a reference.

Definition 2.10.3. A **discrete C^*–dynamical system** is a triple (A, Γ, α) where A is a unital C^*–algebra, Γ is a discrete group, and $\alpha : \Gamma \to Aut(A)$ is a continuous homomorphism.

Given a discrete C^*–dynamical system, we denote the finitely supported A–valued functions on Γ

$$A[\Gamma] = \mathrm{span}\{as : a \in A, s \in \Gamma\}.$$

We view the symbols $a \in A$ and $s \in \Gamma$ as the elements $a = ae$ and $1s = s$ in $A[\Gamma]$, which we assert to satisfy $sas^{-1} = \alpha(s)(a)$ for all $s \in \Gamma$ and $a \in A$, and has the following $*$–algebraic structure

$$(as)(bt) = a\alpha(s)(b)s^{-1}t \text{ and } (as)^* = s^{-1}a^*$$

for $t \in \Gamma$ and $b \in A$ (note that we only needed A to be a unital $*$–algebra). In other words, $A[\Gamma]$ is a $*$–algebra that contains a copy of A and a copy of Γ as unitaries such that α is inner.

A **covariant representation** of (A, Γ, α) is a pair (π_A, π_Γ) such that $\pi_A : A \to \mathcal{B}(\mathcal{H})$ is a $*$-representation and $\pi : \Gamma \to \mathcal{B}(\mathcal{H})$ a unitary representation satisfying the covariance equation

$$\pi_A(\alpha(s)(a)) = \pi_\Gamma(s)\pi_A(a)\pi_\Gamma(s)^*.$$

A covariant representation gives rise to a representation $\pi_A \rtimes_\alpha \pi_\Gamma : A[\Gamma] \to \mathcal{B}(\mathcal{H})$ by setting $\pi_A \rtimes_\alpha \pi_\Gamma(as) = \pi_A(a)\pi(s)$. If we let $\theta : A \to \mathcal{B}(\mathcal{H}_\theta)$ be a faithful $*$-representation, then we can define a canonical covariant representation $(\pi^\theta, \lambda^\theta)$ of (A, Γ, α) by defining

$$\pi^\theta : A \to \mathcal{B}(L^2(\Gamma, \mathcal{H}_\theta)),\ \pi^\theta(a)\xi(s) = \theta(\alpha(s^{-1})(a))\xi(s)$$

and

$$\lambda^\theta : \Gamma \to \mathcal{B}(L^2(\Gamma, \mathcal{H}_A)),\ \lambda^\theta(t)\xi(s) = \xi(t^{-1}s)$$

for $a \in A$, $s, t \in \Gamma$, and $\xi \in L^2(\Gamma, \mathcal{H}_\theta)$. Then the **reduced crossed product** is the C^*-algebra

$$A \rtimes_{r\alpha} \Gamma := \overline{\pi^\theta \rtimes_\alpha \lambda^\theta(A[\Gamma])}^{||\cdot||_r},$$

which we note is independent of the choice of faithful representation on A [26]. We also obtain the **universal crossed product** by setting

$$A \rtimes_{u\alpha} \Gamma := \overline{A[\Gamma]}^{||\cdot||_u}$$

where

$$||sa||_u = \sup\{||\pi_A \rtimes_\alpha \pi_\Gamma(sa)|| : (\pi_A, \pi_\Gamma) \text{ is a covariant representation of } (A, \Gamma, \alpha)\}.$$

Note that $A \rtimes_{r\alpha} \Gamma$ and $A \rtimes_{u\alpha} \Gamma$ contain copies isometric of A, and $C^*_\lambda(\Gamma)$ and $C^*(\Gamma)$ respectively, and in each case, we will abuse notation and denote the copy of each element $a \in A$ and $s \in \Gamma$ by a and s respectively.

We also have a von Neumann algebraic version of the crossed product. Our main reference will be [119, Chapter X].

Definition 2.10.4. A **discrete W^*-dynamical system** is a triple (M, Γ, β) where M is a von Neumann algbera, Γ is a discrete group, and $\alpha : \Gamma \to Aut(M)$ is a weak* continuous homomorphism.

In this case, if $\theta : M \to \mathcal{B}(\mathcal{H}_\theta)$ is a normal $*$-representation, using similar definitions as in the C^*-algebra case, we can build a canonical pair of representations $(\pi^\theta, \lambda^\theta)$. Then

$$M \overline{\rtimes}_\beta \Gamma := (\pi^\theta(M)\lambda^\theta(\Gamma))''$$

is the **discrete von Neumann crossed product** of (M, Γ, β). We note that $M\overline{\rtimes}_\beta\Gamma$ contains an isometric copy of M and $VN(\Gamma)$ and, as before, we will abuse notation and denote the copy of each $x \in M$ and $s \in \Gamma$ by x and s respectively.

If $\mathbb{G}$ is a CQG and α intertwines $\Delta^u_{\mathbb{G}}$, then we call $(C_u(\mathbb{G}), \Gamma, \alpha)$ a **Woronowicz C^*-dynamical system**. A discrete crossed product of a Woronowicz C^*-dynamical system has an underlying CQG whose structure is described in the following.

Theorem 2.10.5. *[129, 127] Let $(C_u(\mathbb{G}), \Gamma, \alpha)$ be a Woronowicz C^*-dynamical system. Then α induces an action of Γ on $C_r(\mathbb{G})$ and $L^\infty(\mathbb{G})$ (which intertwines their respect coproducts and we again denote by α), and there exists a CQG (denoted $\mathbb{G} \rtimes_\alpha \widehat{\Gamma}$) such that:*

1. $Irr(\mathbb{G} \rtimes_\alpha \widehat{\Gamma}) = \{su^\pi_{i,j} : \pi \in Irr(\mathbb{G}),\ s \in \Gamma\}$;
2. $\mathrm{Pol}(\mathbb{G})[\Gamma] = \mathrm{Pol}(\mathbb{G} \rtimes_\alpha \widehat{\Gamma})$;
3. $C_u(\mathbb{G}) \rtimes_{u\alpha} \Gamma = C^u(\mathbb{G} \rtimes_\alpha \widehat{\Gamma})$;
4. $C_r(\mathbb{G}) \rtimes_{r\alpha} \Gamma = C^r(\mathbb{G} \rtimes_\alpha \widehat{\Gamma})$;
5. $L^\infty(\mathbb{G})\overline{\rtimes}_\alpha\Gamma = L^\infty(\mathbb{G} \rtimes_\alpha \widehat{\Gamma})$;
6. $h_{\mathbb{G}\rtimes_\alpha\widehat{\Gamma}} = h_{\mathbb{G}} \rtimes_\alpha 1_{\{e\}}$;
7. $\Delta_{\mathbb{G}\rtimes_\alpha\widehat{\Gamma}}|_{L^\infty(\mathbb{G})} = \Delta_{\mathbb{G}}$ *and* $\Delta_{\mathbb{G}\rtimes_\alpha\widehat{\Gamma}}|_{VN(\Gamma)} = \Delta_{\widehat{\Gamma}}$;
8. *and $\widehat{\Gamma}$ is a normal closed quantum subgroup of $\mathbb{G} \rtimes_\alpha \widehat{\Gamma}$ and $\mathbb{G} = (\mathbb{G} \rtimes_\alpha \widehat{\Gamma})/\widehat{\Gamma}$ via the Hopf $*$-homomorphism*

$$\mathrm{Pol}(\mathbb{G} \rtimes_\alpha \widehat{\Gamma}) \to \mathbb{C}[\Gamma],\ \pi_{\widehat{\Gamma}}(sa) = \epsilon_{\mathbb{G}}(a)s$$

for $s \in \Gamma$ and $a \in Pol(\mathbb{G})$.

Remark 2.10.6.

1. Note for $u \in B(\Gamma) = C^*(\Gamma)^*$ and $\varphi \in M^u(\mathbb{G})$, $u \rtimes_\alpha \varphi \in M^u(\mathbb{G} \rtimes_\alpha \widehat{\Gamma})$ denotes the functional such that

$$u \rtimes_\alpha \varphi(ta) = u(t)\varphi(a),\ t \in \Gamma, a \in C_u(\mathbb{G}).$$

2. Following Theorem 2.10.5 8., clearly every closed quantum subgroup of $\widehat{\Gamma}$ is a closed quantum subgroup of $\mathbb{G} \rtimes_\alpha \widehat{\Gamma}$. More generally, we see from 7. that every right invariant subspace of $L^\infty(\mathbb{G})$ and $VN(\Gamma)$ is also a right invariant subspace of $L^\infty(\mathbb{G} \rtimes_\alpha \widehat{\Gamma})$.

3. We obtain many examples from the pair of any CQG $\mathbb{G}$ and discrete group Γ via the trivial action $\text{id} : \Gamma \to Aut(L^\infty(\mathbb{G}))$, which is defined by $\text{id}(s)(x) = x$ for all $s \in \Gamma$ and $x \in L^\infty(\mathbb{G})$. In this case, we get $L^\infty(\mathbb{G} \rtimes_{id} \widehat{\Gamma}) \cong L^\infty(\mathbb{G})\overline{\otimes} VN(\Gamma)$ as von Neumann algebras.

Maintaining the same notation as Theorem 2.10.5, we will call $\mathbb{G} \rtimes_\alpha \widehat{\Gamma}$ the **crossed product** of $\mathbb{G}$ and $\widehat{\Gamma}$ by α.

We can use the ideas from [48] to describe the universal and reduced measure algebras of a crossed product. Indeed, we identify any $\mu \in M^u(\mathbb{G} \rtimes_\alpha \widehat{\Gamma})$ with an element of $C_b(\Gamma, M^u(\mathbb{G}))$ by setting $\mu(s)(a) = \mu(sa)$ for $s \in \Gamma$ and $a \in C_u(\mathbb{G})$. Then, by definition,

$$M^u(\mathbb{G} \rtimes_\alpha \widehat{\Gamma}) = \left\{ \mu \in C_b(\Gamma, M^u(\mathbb{G})) : \sup_{\sum_{s\in\Gamma} sa_s \in B_1(\text{Pol}(\mathbb{G})[\Gamma])} |\sum_{s\in\Gamma} \mu(sa_s)| < \infty \right\}.$$

Convolution can be realized as follows, which is a result we believe is well-known.

Proposition 2.10.7. *Let $\mathbb{G} \rtimes_\alpha \widehat{\Gamma}$ be a crossed product. For $\mu, \nu \in M^u(\mathbb{G} \rtimes_\alpha \widehat{\Gamma})$, when viewed as elements of $C_b(\Gamma, M^u(\mathbb{G}))$, we have $\mu * \nu(s) = \mu(s) * \nu(s)$ for all $s \in \Gamma$. In particular, for $u, v \in B(\Gamma)$ and $\varphi, \psi \in M^u(\mathbb{G})$, $(u \rtimes_\alpha \varphi) * (v \rtimes_\alpha \psi) = (uv) \rtimes_\alpha (\varphi * \psi)$.*

Proof. For $s \in \Gamma$ and $a \in C_u(\mathbb{G})$, we compute:

$$\mu * \nu(sa) = (\mu \otimes \nu)\Delta^u_{\mathbb{G}\rtimes_\alpha\widehat{\Gamma}}(sa) = (\mu \otimes \nu)(s \otimes s)\Delta^u_{\mathbb{G}}(a) = (\mu(s) * \nu(s))(a).$$

□

We will, in particular, care about the L^1-algebras of CQGs arising from the crossed product constructions. Their form can be gleaned from [48] where the structure of "Fourier spaces" of crossed products has been described. For more on the general theory, we recommend the reference [88].

Theorem 2.10.8. *[88, Definition 3.1] Let $\mathbb{G} \rtimes_\alpha \widehat{\Gamma}$ be a crossed product. The elements of $L^1(\mathbb{G} \rtimes_\alpha \widehat{\Gamma})$ identify with continuous functions $f : \Gamma \to L^1(\mathbb{G})$ such that $f(t)(x) = \tilde{f}(tx)$ for $t \in \Gamma$ and $x \in L^\infty(\mathbb{G})$, where $\tilde{f} \in L^\infty(\mathbb{G} \rtimes_\alpha \widehat{\Gamma})^*$ is of the form*

$$\tilde{f}(T) = \sum_{n\geq 1} \langle T\xi_n, \eta_n \rangle$$

for some $(\xi_n), (\eta_n) \subseteq L^2(\Gamma, L^2(\mathbb{G}))$ with $\sum_{n\geq 1} ||\xi_n||, \sum_{n\geq 1} ||\eta_n|| < \infty$ and $T \in L^\infty(\mathbb{G} \rtimes_\alpha \widehat{\Gamma})$.

DQGs from Crossed Products

Now let $\mathbb{G}$ be a DQG. We will briefly discuss the structure of the duals of CQGs coming from discrete crossed products. Before proceeding, we point out the following.

Proposition 2.10.9. *[39, Corollary 6.4] We have that $\widehat{\mathbb{G}} \ltimes_\alpha \widehat{\Gamma}$ is co–amenable if and only if Γ is amenable and $\widehat{\mathbb{G}}$ is co–amenable.*

Given a DQG $\mathbb{G}$, from Theorem 2.10.5 2. we can see that the DQG $\widehat{\widehat{\mathbb{G}} \ltimes_\alpha \widehat{\Gamma}}$ is realized such that $\ell^\infty(\widehat{\mathbb{G} \ltimes_\alpha \widehat{\Gamma}}) = \ell^\infty(\widehat{\mathbb{G}})\overline{\otimes}\ell^\infty(\Gamma)$, and so using the duality formula of Effros and Ruan [32], $\ell^1(\widehat{\widehat{\mathbb{G}} \ltimes_\alpha \widehat{\Gamma}}) = \ell^1(\widehat{\mathbb{G}})\widehat{\otimes}\ell^1(\Gamma)$ isometrically. Also note that $c_0(\widehat{\widehat{\mathbb{G}} \ltimes_\alpha \widehat{\Gamma}}) = c_0(\widehat{\mathbb{G}}) \otimes_{min} c_0(\Gamma)$ isometrically. In the above, by $\widehat{\otimes}$ we mean the operator projective tensor product, however, since $\ell^1(\Gamma)$ is a factor, it coincides with the usual projective tensor product.

As usual, we are particularly interested in the L^1–algebra $\ell^1(\widehat{\mathbb{G} \ltimes_\alpha \widehat{\Gamma}})$. Its convolution product may be explicitly described as in the following proposition (which we suspect is well-known, however, we have been unable to find a reference).

Proposition 2.10.10. *Let $u, v \in \ell^1(\mathbb{G})$ and $s, t \in \Gamma$. Then we have the product formula in $\ell^1(\widehat{\widehat{\mathbb{G}} \ltimes_\alpha \widehat{\Gamma}})$:*

$$(u \otimes \delta_s) * (v \otimes \delta_t) = u * (\lambda_{\widehat{\mathbb{G}}}^{-1} \circ \alpha_s \circ \lambda_{\widehat{\mathbb{G}}})(v) \otimes \delta_{s^{-1}t}. \tag{2.2}$$

Proof. The left regular representation of $\widehat{\mathbb{G}} \ltimes \widehat{\Gamma}$ is $\lambda_{\mathbb{G}} \ltimes \lambda_\Gamma$. So, to prove (2.2), just apply $\lambda^{-1}_{\widehat{\mathbb{G} \ltimes \widehat{\Gamma}}}$ to the equation

$$\lambda_{\widehat{\mathbb{G}}}(u)s\lambda_{\widehat{\mathbb{G}}}(v)t = \lambda_{\widehat{\mathbb{G}}}(u)\alpha_s(\lambda_{\widehat{\mathbb{G}}}(v))s^{-1}t,\ u, v \in L^1(\widehat{\mathbb{G}}), s, t \in \Gamma.$$

□

Remark 2.10.11. If G is a compact group, then the left regular representation on $\ell^1(\widehat{G}) = A(G)$ is just the inclusion $A(G) \subseteq C(G)$. So, in this case $(u \otimes \delta_s) * (v \otimes \delta_t) = u\alpha_s(v) \otimes \delta_{s^{-1}t}$.

Chapter 3

Ideals of L^1-algebras of Compact Quantum Groups

3.1 Introduction

Describing the ideals of a Banach algebra is a fundamental problem. As is done with Hilbert's Nullstellensatz, the closed ideals of a semi-simple Tauberian commutative Banach algebra A of certain types (without going into details) can be distinguished by their zero sets in the Gelfand spectrum $\sigma(A)$. A semi-simple commutative Banach algebra A is Tauberian if $\{a : \hat{a} \text{ has compact support}\}$, where $\hat{a}$ is the Gelfand transform of a, is dense in A. More precisely, every closed ideal I of such A has that $I = I(E) = \{a \in A : \hat{a}|_E = 0\}$ for some closed subset $E \subseteq \sigma(A)$, and E is called the **hull** of I.

This correspondence lends itself nicely to commutative Banach algebras studied in abstract harmonic analysis. The Fourier algebra $A(G) \subseteq C_0(G)$ of a locally compact group G is the commutative Banach algebra of coefficient functions of the left regular representation of G, and naturally identifies with the predual of the group von Neumann algebra $VN(G)$ (for the basics of Fourier algebras, see [37, 69]). Alternatively, we have that $A(G) = L^1(\widehat{G})$ where $\widehat{G}$ is the quantum group dual of G (see Section 2) where we note that if G is abelian then $\widehat{G}$ is the Pontryagin dual of G and $L^1(\widehat{G}) = A(G)$ has a bounded approximate identity. In general, $A(G)$ has a bounded approximate identity if and only if G is amenable [85], $\sigma(A(G)) = G$, and $A(G)$ is Tauberian. It turns out that if $G = \Gamma$ is discrete and amenable, then every closed ideal $I \subseteq A(\Gamma)$ has $I = I(E)$ where $E = \text{hull}(I) \subseteq \Gamma$.

More generally, for a closed subset $E \subseteq G$, we will write

$$I(E) = \{u \in A(G) : u|_E = 0\}$$

and

$$j(E) = \{u \in A(G) : u \text{ has compact support disjoint from } E\}.$$

The ideal $I(E)$ is always closed. Since $A(G)$ is Tauberian, for any ideal $I \subseteq A(G)$ we have $j(E) \subseteq I \subseteq I(E)$ where $E = \mathrm{hull}(I)$ [51, Chap. X Section 1]. The closed subset E is said to be a **set of synthesis** if $\overline{j(E)} = I(E)$ and so with this language, the closed ideal structure of $A(G)$ is completely characterized when every closed subset of G is a set of synthesis. The locally compact groups where such a thing holds have been characterized.

Theorem 3.1.1. *[71] Let G be a locally compact group. Then every closed subset of G is a set of synthesis if and only if G is discrete and $u \in \overline{uA(G)}$ for all $u \in A(G)$.*

Whenever $u \in \overline{uA(G)}$ for all $u \in A(G)$, we say G has **Ditkin's property at infinity** or **property** D_∞. This property covers a broad range of groups, which clearly includes all of those which admit an approximate identity, and is poorly understood. Indeed, there are no known examples of locally compact groups without property D_∞ (see Section 3.3 for more on property D_∞ (Definition 3.2.12) and its quantization). On the other hand, we understand the closed ideals of $A(G)$ for many examples of discrete groups in the literature (which includes all discrete groups with the approximation property (see Section 2.4).

Because of Schur's lemma, the Gelfand spectrum of a commutative Banach algebra A is the set of irreducible representations A. So, it seems natural to try to use irreducible representations to try to build a "quantum hull" for a left ideal of a non-commutative Banach algebra to glean information on its structure. Such a thing was achieved for group algebras of compact groups. Let us fix a compact group G. Recall that for a unitary representation $\pi : G \to \mathcal{B}(\mathcal{H}_\pi)$, the corresponding L^1-representation is given by $\pi(f) = \int_G f(t)\pi(t)\,dt$ for $f \in L^1(G)$. The closed left ideals have a representation theoretic description as follows.

Theorem 3.1.2. *[51] The closed left ideals of $L^1(G)$ are of the form*

$$I(E) = \{f \in L^1(G) : \pi(f)(E_\pi) = 0,\ \pi \in Irr(G)\}$$

where $E = (E_\pi)_{\pi \in Irr(G)}$ for subspaces $E_\pi \subseteq \mathcal{H}_\pi$.

The symbol $Irr(G)$ denotes the irreducible $*$-representations on $L^1(G)$. So, we might consider the sequence $E = (E_\pi)_{\pi \in Irr(G)}$ to be the hull of $I(E)$ and we might say that every "closed quantum subset" of $\widehat{G}$ is a "set synthesis", where we might regard the (discrete) quantum group $\widehat{G}$ as the "quantum spectrum" of $L^1(G)$. Note that the sequence (E_π) depends on the choice of representatives we are choosing for each $\pi \in Irr(G)$.

In the general scheme of locally compact quantum groups (LCQGs), the compact and discrete quantum groups are dual to one another (cf. [133] and [105]). So, it is reasonable to attempt to unify Theorems 3.1.2 and 3.1.1 at the level of compact quantum groups (CQGs). Using the analogies found between the representation theory of compact quantum groups (CQGs) in general and compact groups (cf. [133]), we formulate notions of hull and synthesis. In particular, we have that the sequences $E = (E_\pi)_{\pi\in Irr(\mathbb{G})}$, where each E_π is a subspace of the Hilbert space $\mathcal{H}_\pi$ where $L^1(\mathbb{G})$ acts by π and $Irr(\mathbb{G})$ are the irreducible $*$-representations on $L^1(\mathbb{G})$, are hulls of left ideals of $L^1(\mathbb{G})$ and we say E is a set of synthesis if $\overline{j(E)} = I(E)$ where $j(E)$ and $I(E)$ are defined in Section 3.1. More precisely, let $\mathbb{G}$ be a CQG and $I \subseteq L^1(\mathbb{G})$ a left ideal. We prove that there exists a sequence $E = (E_\pi)_{\pi\in Irr(\mathbb{G})}$ such that:

- $j(E) \subseteq I \subseteq I(E)$ (Proposition 3.2.4);
- every $E = (E_\pi)_{\pi\in Irr(\mathbb{G})}$ is a set of synthesis if and only if $f \in \overline{L^1(\mathbb{G}) * f}$ for every $f \in L^1(\mathbb{G})$ (Theorem 3.2.10).

In particular, for every CQG $\mathbb{G}$ where $\widehat{\mathbb{G}}$ satisfies property (left) D_∞, which is the property where $f \in \overline{L^1(\mathbb{G}) * f}$ for every $f \in L^1(\mathbb{G})$, we establish a complete description of the closed left ideals of $L^1(\mathbb{G})$. This includes many examples of CQGs from the literature, including $SU_q(2)$, free unitary quantum groups, free orthogonal quantum groups, etc (see Section 3.3).

A coamenable CQG is a CQG $\mathbb{G}$ where $L^1(\mathbb{G})$ admits a bounded approximate identity, and are CQGs such that their duals have property left D_∞. In this case, our above stuctural result for closed left ideals of $L^1(\mathbb{G})$ lend themselves towards weak* closed left ideals of their measure algebras. The measure algebra $M(G)$ of a locally compact group, is a unital Banach algebra that identifies naturally with the dual space of the continuous functions vanishing at infinity $C_0(G)$ via integration. Then, here, we have that $\overline{L^1(G)}^{wk*} = M(G)$ with respect to the $\sigma(M(G), C_0(G))$ topology. Switching perspectives to the duals of locally compact amenable G, the Fourier-Stieltjes algebra $B(G)$, which is the subalgebra of the bounded continuous functions $C_b(G)$ consisting of coefficient functions of continuous unitary representations, is a unital commutative Banach algebra that identifies with the dual space of the (unique) group C^*-algebra $C^*(G)$. Here, we have the analogous fact that $\overline{A(G)}^{wk*} = B(G)$ with respect to the $\sigma(B(G), C^*(G))$ topology. For coamenable CQGs, the quantum measure algebra $M(\mathbb{G})$ is the dual space of the C^*-algebra $C(\mathbb{G})$ of $\mathbb{G}$, and likewise satisfies $\overline{L^1(\mathbb{G})}^{wk*} = M(\mathbb{G})$. It is a generalization of both the measure algebra of a compact group and the Fourier-Stieltjes algebra of an amenable discrete group.

The quantum measure algebra is a formidable Banach algebra to study, and obtaining information about its structure is often very difficult. For instance, the Gelfand spectrum $\sigma(B(G))$ is often much larger than G, as exhibited by the Wiener-Pitt phenomenon for non-compact abelian groups (see [125, 124]). It is possible to make traction, however, if the natural weak* topology is taken into account. Recently, White [132] achieved a description of the weak* closed left ideals of the measure algebra $M(G)$ of a compact group G by exploiting weak* approximation by elements in $L^1(G)$. With Theorem 3.2.16, we show White's techniques extend directly to coamenable CQGs in order to achieve a similar description. Indeed, we prove the following.

- Let $\mathbb{G}$ be a coamenable CQG. Every weak* closed left ideal $I \subseteq M(\mathbb{G})$ is of the form

  $$I = I^u(E) = \{\mu \in M(\mathbb{G}) : \pi(\mu)(E_\pi) = 0 \text{ for all } \pi \in Irr(\mathbb{G})\}$$

  where $E = (E_\pi)_{\pi \in Irr(\mathbb{G})}$ is a closed quantum subset of $\widehat{\mathbb{G}}$ and $\pi : M(\mathbb{G}) \to \mathcal{B}(\mathcal{H}_\pi)$ is the natural extension of $\pi \in Irr(\mathbb{G})$ to $M(\mathbb{G})$.

Another fundamental problem in Banach algebras is describing their idempotents. The idempotents of $B(G)$ for locally compact G were completely characterized as the characteristic functions on sets in the sigma ring generated by clopen subgroups of G by the Cohen-Host idempotent theorem (see [69]). Other than in the abelian case, the classification of idempotents in $M(G)$ remains an open problem, even for $SO(3)$. On the other hand, the idempotent states in $B(G)$ are exactly the characteristic functions on clopen subgroups of G (see [58]) and the idempotent states in $M(G)$ are exactly the Haar measures coming from compact subgroups of G (see [76]). The idempotent states in $M(\mathbb{G})$, where $\mathbb{G} = U_q(2), SU_q(2)$, and $SO_q(3)$ were completely classified in [46].

For a CQG $\mathbb{G}$, an intimately related problem is the determination of the closed left ideals in $L^1(\mathbb{G})$ that admit bounded right approximate identities. When $\mathbb{G}$ is coamenable, there is a one-to-one correspondence between idempotents in $M(\mathbb{G})$ and closed left ideals of $L^1(\mathbb{G})$ that have bounded right approximate identities (see [87]). For a (possibly non-amenable) discrete group Γ, the ideals in $A(\Gamma)$ with bounded approximate identities were completely characterized by Forrest in [41]. A more specific result in [41] is that an ideal of the form $I(\Lambda)$ for some subgroup Λ has a bounded approximate identity if and only if Γ is amenable. In other words, if Γ is amenable, then every such $I(\Lambda)$ has a bounded approximate identity and when Γ is non-amenable, no such $I(\Lambda)$ has a bounded approximate identity. We point out also that for any $s \in \Gamma$, $I(s\Lambda)$ has a bounded approximate identity if and only if $I(\Lambda)$ does, and thus this characterization easily applies to cosets of subgroups as well.

A compact quasi-subgroup is a von Neumann subalgebra of $L^\infty(\mathbb{G})$ that corresponds to an idempotent state in the universal measure algebra $M^u(\mathbb{G})$ in a sense that generalizes the identifications in the above paragraph between subgroups of discrete / compact groups (denoted Γ / G) and idempotent states in $B(\Gamma)$ / $M(G)$ respectively (see Section 4.1). For example, if Γ is a discrete group, the compact quasi-subgroups are the subalgebras $VN(\Lambda) \subseteq VN(\Gamma)$ where Λ is a subgroup of Γ. Then

$$I(\Lambda) = VN(\Lambda)_\perp = \{u \in A(\Gamma) : u(x) = 0 \text{ for all } x \in VN(\Lambda)\}.$$

We make progress towards generalizing Forrest's result by proving the following.

- Let $\mathbb{G}$ be a CQG and $N \subseteq L^\infty(\mathbb{G})$ be a compact quasi-subgroup with associated idempotent state $\omega \in M^r(\mathbb{G})$, where $M^r(\mathbb{G})$ is the reduced measure algebra. Then $J^1(N) := N_\perp$ has a bounded right approximate identity if and only if $\mathbb{G}$ is coamenable.

We point out that H is amenable if and only if $1_H \in B_r(G)$, where $B_r(G)$ is the reduced Fourier-Stieltjes algebra. So, our result is a generalization of Forrest's result applied to amenable subgroups of G.

Section 2 will comprise the preliminaries for locally compact quantum groups where we will in particular recall the theory behind closed quantum subgroups and more generally, invariant subspaces.

In Section 3, we will develop the notion of a hull E of a closed left ideal $I \subseteq L^1(\mathbb{G})$ and then will classify the compact quantum groups such that $\overline{j(E)} = I = I(E)$, for each hull E, in terms of Ditkin's property at infinity (or property left D_∞, a property which has recently achieved a new characterization [5]), (see Theorem 3.2.10). In particular, we can describe the closed left ideals of compact quantum groups whose dual has the approximation property. Then we will show White's techniques [132] for classifying the weak* closed left ideals of the measure algebra of a compact group extend to the setting of coamenable compact quantum groups (see Theorem 3.2.16). We will conclude the section with a brief discussion of property left D_∞ and provide examples of CQGs which are weakly amenable and consequently have property left D_∞.

Finally, in Section 4 we study the closed left ideals of $L^1(\mathbb{G})$ which admit a brai, with special emphasis on the preannihilator space $J^1(N)$ of a compact quasi–subgroup N (the natural quantum analogue of a closed subgroup of a compact group). We also study the associated weak* closed left ideal $J^u(N)$ in $M^u(\mathbb{G})$ and in a certain case, show $J^u(N) = \overline{J^1(N)}^{wk*}$ if and only if $\mathbb{G}$ is coamenable if and only if $J^1(N)$ admits a bounded right approximate identity (see Theorems 3.3.12 and 3.3.13, and Corollary 3.3.14). We conclude the section by showing whenever $N \neq X = Nx$ for some $x \in Gr(\mathbb{G})$, that $X_\perp$ possesses a bounded approximate identity if and only if $\mathbb{G}$ is coamenable (see Theorem 3.3.23). In this context, we think of X as being a "quantum coset" of the compact quasi–subgroup N. We end by illustrating these last results on discrete crossed products equipped with the structure of a compact quantum group.

3.1.1 Approximation Property and Weak Amenability

We will be interested in weakened versions of amenability. In lieu of the duality between coamenability and amenability (for CQG/DQGs), the natural choice is to weaken boundedness of a left or right bai (blai or brai) in $L^1(\widehat{G})$. Before proceeding, we make the following observation regarding multipliers of quantum groups.

Remark 3.1.3. Whenever $\mathbb{G}$ is coamenable, it is the case that $M(L^1(\mathbb{G})) = M^l(L^1(\mathbb{G})) = M(\mathbb{G})$. For locally compact co–groups, this property characterizes amenability. That $M(A(G)) = B(G)$ implies amenability for discrete G is due to [90] (and generally is due to Losert [86]), and Losert extended this to the case of $M_{cb}A(G)$ in an unpublished manuscript. See also [84]. For discrete G, however, see [12]. For a LCQG $\mathbb{G}$ in general, we also have that the completely isometric equalities $M^l_{cb}(L^1(\mathbb{G})) = M^r(\mathbb{G}) = M^u(\mathbb{G})$ characterizes coamenability (cf. [54]).

A first weakening, then, would be to loosen the boundedness criterion of the bai.

Definition 3.1.4. We say a LCQG $\mathbb{G}$ is **weakly amenable** if there exists a net $(f_i) \subseteq L^1(\mathbb{G})$ such that $f_i * f \to f$ for all $f \in L^1(\mathbb{G})$ and $\sup_i ||f_i||_{M^l_{cb}} < \infty$.

There is another relevant, even weaker version of amenability.

Definition 3.1.5. We say a LCQG $\mathbb{G}$ has the **approximation property (AP)** if there exists a net $(f_i) \subseteq L^1(\widehat{\mathbb{G}})$ such that $L_{f_i} \to \mathrm{id}_{L^\infty(\widehat{\mathbb{G}})}$ in the stable point weak* topology of $\mathcal{CB}(L^\infty(\widehat{\mathbb{G}}))$, by which we mean, for a separable Hilbert space $\mathcal{H}$, we have

$$\varphi(L_{f_i} \otimes \mathrm{id})(a) \to \varphi(a)$$

for all $a \in L^\infty(\widehat{\mathbb{G}})\overline{\otimes}\mathcal{B}(\mathcal{H})$ and $\varphi \in L^1(\widehat{\mathbb{G}})\widehat{\otimes}\mathcal{B}(\mathcal{H})_*$.

We have the following description of the AP, which we point out implies $L^1(\mathbb{G})$ has a left approximate identity when we let $H = \{e\}$, where H is as denoted in the following proposition. Note that when H is compact, 1_H is the identity element in $A(H)$.

Proposition 3.1.6. *For a LCQG $\mathbb{G}$, TFAE:*

1. *$\widehat{\mathbb{G}}$ has the AP;*
2. *for every compact group H, there is a net $(f_i) \subseteq L^1(\mathbb{G})$ such that*

$$||(f_i \otimes 1_H) * g - g||_1 \to 0$$

for all $g \in L^1(\mathbb{G})\widehat{\otimes}A(H)$;

3. *and there is a net $(f_i) \subseteq L^1(\mathbb{G})$ such that*

$$||(f_i \otimes 1_{SU(2)}) * g - g||_1 \to 0$$

for all $g \in L^1(\mathbb{G})\widehat{\otimes}A(SU(2))$.

Proof. The proof follows verbatim of the proof of [50, Theorem 1.11]: the techniques are entirely functional analytic and pass directly to LCQGs. □

Note that the map $M_{cb}(L^1(\mathbb{G})) \ni m \mapsto m^* \in \mathcal{CB}^{\sigma}_{L^1(\mathbb{G})}(L^\infty(\mathbb{G}))$ is a completely isometric isomorphism, where $\mathcal{CB}^{\sigma}_{L^1(\mathbb{G})}(L^\infty(\mathbb{G}))$ denotes the normal completely bounded right $L^1(\mathbb{G})$-module maps on $L^\infty(\mathbb{G})$. Crann pointed out in [20, Proposition 3.2] that the work of Kraus and Ruan [78, Theorem 2.2] extends directly from Kac algebras to LCQGs so that we obtain a predual

$$M^l_{cb}(L^1(\mathbb{G}))_* = Q^l_{cb}(L^1(\mathbb{G})) = \{\omega_{A,\varphi} : A \in C^r_0(\mathbb{G}) \otimes_{min} \mathcal{K}(\mathcal{H}), \varphi \in L^1(\mathbb{G})\widehat{\otimes}\mathcal{T}(\mathcal{H})\}$$

where $\omega_{A,\varphi}(M) = \varphi(M \otimes \mathrm{id})(A)$ for $M \in M^l_{cb}(L^1(\mathbb{G}))$ and $\mathcal{H}$ is a separable Hilbert space.

For $T \in L^\infty(\mathbb{G})\overline{\otimes}\mathcal{B}(\mathcal{H})$ and $\varphi \in L^1(\mathbb{G})\widehat{\otimes}\mathcal{T}(\mathcal{H})$, we similarly use $\omega_{T,\varphi}$ to denote the functional satisfying $\omega_{T,\varphi}(M) = \varphi(M \otimes \mathrm{id})(T)$. Then, the argument used by Kraus and Ruan in [78, Proposition 5.2] extends directly to CQGs to give us the following.

Proposition 3.1.7. *Let $\mathbb{G}$ be a CQG. Then $\omega_{T,\varphi} \in Q^l_{cb}(L^1(\mathbb{G}))$ for all $T \in L^\infty(\mathbb{G})\overline{\otimes}\mathcal{B}(\mathcal{H})$ and $\varphi \in L^1(\mathbb{G})\widehat{\otimes}\mathcal{T}(\mathcal{H})$.*

With Proposition 3.1.7 in hand, a proof verbatim to the proof in [78, Theorem 5.4] shows the following, which allows us to view the AP of the duals of CQGs within our framework of weakening coamenability. For convenience, we will supply a proof.

Proposition 3.1.8. *We have that a DQG $\mathbb{G}$ has the AP if and only if there exists a net $(e_i) \subseteq L^1(\widehat{\mathbb{G}})$ such that $e_i \to \epsilon^u_{\widehat{\mathbb{G}}}$ in the $\sigma(M^l_{cb}(L^1(\widehat{\mathbb{G}})), Q_{cb}(L^1(\widehat{\mathbb{G}})))$ topology on $M^l_{cb}(L^1(\widehat{\mathbb{G}}))$.*

Proof. Suppose $\mathbb{G}$ has the AP. Let $(e_i) \subseteq L^1(\widehat{\mathbb{G}})$ be a net such that L_{e_i} converges in the stable point weak* topology to id. In particular, for $A \in C_r(\widehat{\mathbb{G}}) \otimes_{min} \mathcal{K}(\mathcal{H})$ and $\varphi \in L^1(\widehat{\mathbb{G}})\widehat{\otimes}\mathcal{T}(\mathcal{H})$

$$\omega_{A,\varphi}(L_{e_i}) = \varphi(L_{e_i} \otimes \mathrm{id})(A) \to \varphi(A) = \omega_{A,\varphi}(L_{\epsilon_{\widehat{\mathbb{G}}^u}})$$

which says exactly that $e_i \to \epsilon^u_{\widehat{\mathbb{G}}}$ in the weak* topology on $M^l_{cb}(L^1(\widehat{\mathbb{G}}))$.

Conversely, suppose $e_i \to \epsilon^u_{\widehat{\mathbb{G}}}$ in the weak* topology on $M^l_{cb}(L^1(\widehat{\mathbb{G}}))$. Then, for $T \in L^\infty(\widehat{\mathbb{G}})\overline{\otimes}\mathcal{B}(\mathcal{H})$ and $\varphi \in L^1(\widehat{\mathbb{G}})\widehat{\otimes}\mathcal{T}(\mathcal{H})$, using Proposition 3.1.7, we have

$$\varphi(L_{e_i} \otimes \mathrm{id})(T) = \omega_{T,\varphi}(L_{e_i}) \to \omega_{T,\varphi}(L_{\epsilon^u_{\widehat{\mathbb{G}}}}) = \varphi(T)$$

which says exactly that $L_{e_i} \to \mathrm{id}$ in the stable point weak* topology. □

Remark 3.1.9. We are unaware of a version of Proposition 3.1.8 for general LCQGs. To prove the result, we would require a general version of Proposition 3.1.7, however, their proof makes essential use of the underlying Hopf algebras of CQGs and does not clearly extend to general LCQGs.

An immediate observation from Proposition 3.1.8 is the following.

Corollary 3.1.10. *A weakly amenable DQG has the AP.*

3.2 Structure of Left Ideals

3.2.1 Left Ideals of L^1–algebras

For the rest of this chapter, $\mathbb{G}$ will always denote a CQG unless otherwise specified. This section contains the most fundamental results of this Chapter. A structure theorem for the left ideals of $L^1(\mathbb{G})$ is obtained, which, as we will make explicit in Section 3.3, allows us to describe the closed left ideals of $L^1(\mathbb{G})$ for many CQGs $\mathbb{G}$, including those that have the approximation property. In lieu of the previous section, this includes the weakly amenable CQGs.

Recall that for $x \in \mathrm{Pol}(\mathbb{G})$, $\hat{x} = h \cdot x$. So, fix a CQG $\mathbb{G}$ and let

$$E = (E_\pi)_{\pi \in Irr(\mathbb{G})}$$

where each $E_\pi \subseteq \mathcal{H}_\pi$ is a subspace (possibly trivial or all of $\mathcal{H}_\pi$). We will write

$$I(E) = \{f \in L^1(\mathbb{G}) : \pi(f)(E_\pi) = 0,\ E_\pi \in E\}$$

and

$$j(E) = I(E) \cap \widehat{\mathrm{Pol}(\mathbb{G})} = I(E) \cap \lambda_{\mathbb{G}}^{-1}(c_{00}(\widehat{\mathbb{G}})),$$

It is easy to check $j(E)$ is a left ideal and $I(E)$ a closed left ideal in $L^1(\mathbb{G})$. Then we will refer to such a set E as the **hull** of any ideal I containing $j(E)$ and contained in $I(E)$. We also refer to the collection of subspaces $E = (E_\pi)_{\pi \in Irr(\mathbb{G})}$ as a **closed quantum subset** of $\widehat{\mathbb{G}}$.

Remark 3.2.1. It should be addressed that $I(E)$, and hence $j(E)$, depends on the choice of representatives in $Irr(\mathbb{G})$ (where the subspaces E_π are chosen up to isomorphism). Indeed, suppose π is unitarily equivalent to ρ, and write $(1 \otimes U^*)U^\pi(1 \otimes U) = U^\rho$. Note that the unitary U is a Hilbert space isomorphism $U : \mathcal{H}_\rho \to \mathcal{H}_\pi$. Then $\pi(f)\xi = U\rho(f)U^*\xi$ shows $\pi(f)\xi = 0$ if and only if $\rho(f)U^*\xi = 0$. In particular, we have $UE_\rho = E_\pi$.

Definition 3.2.2. We say E is a **set of synthesis** if $I(E) = \overline{j(E)}$.

Before proceeding, we recall the following well-known fact.

Lemma 3.2.3. *Let I be a left ideal in some matrix algebra M_n. Then $I = \{A \in M_n : A(E) = 0\}$ for some subspace $E \subseteq \mathbb{C}^n$.*

Proof. The proof can be found, for example, in [51, Lemma 38.11]. The idea of the proof is as follows: there exists $A_0 \in I \subseteq M_n$ such that that $E = \ker(A_0) = \{\xi \in \mathbb{C}^n : A_0\xi = 0\}$. Then one obtains $I = \{A \in M_n : \ker(A) \supseteq \ker(A_0)\} = \{A \in M_n : A(E) = 0\}$. □

Proposition 3.2.4. *Let $\mathbb{G}$ be a CQG and $I \trianglelefteq L^1(\mathbb{G})$ a left ideal. Then there is a hull E such that $j(E) \subseteq I \subseteq I(E)$.*

Proof. We follow the methods used for compact groups in [51]. Let 1_π be the projection onto M_{n_π}. Then $\pi(f) = 1_\pi \lambda_{\mathbb{G}}(f)$, and so from density, combined with the fact M_{n_π} is finite dimensional, $\pi(L^1(\mathbb{G})) = M_{n_\pi}$. Consequently, $\pi(I)$ is a left ideal in M_{n_π}. Then, using Lemma 3.2.3, we can write $\pi(I) = \{\pi(f) \in M_{n_\pi} : f \in L^1(\mathbb{G}), \pi(f)(E_\pi) = 0\}$ for some subspace $E_\pi \subseteq \mathcal{H}_\pi$. Let $E = (E_\pi)_{\pi \in Irr(\mathbb{G})}$, where each $E_\pi \subseteq \mathcal{H}_\pi$ is the aforementioned subspace for each π. From here, it is easy to see that $I \subseteq I(E)$.

Now take $f \in j(E)$, so $\lambda_{\mathbb{G}}(f) = \oplus_{i=1}^n \pi_i(f)$ for some $\pi_1, \ldots, \pi_n \in Irr(\mathbb{G})$. Since $\oplus_{i=1}^n \pi_i(j(E)) = \oplus_{i=1}^n \pi_i(I)$, we can find $g \in I$ so that $\oplus_{i=1}^n \pi_i(g) = \lambda_{\mathbb{G}}(f)$. Set $P = \oplus_{i=1}^n I_{n_{\pi_i}}$ and let $e \in L^1(\mathbb{G})$ be such that $\lambda_{\mathbb{G}}(e) = P$. Then

$$\lambda_{\mathbb{G}}(I) \ni \lambda_{\mathbb{G}}(e * g) = P\lambda_{\mathbb{G}}(g) = \bigoplus_{i=1}^n \pi_i(g) = \lambda_{\mathbb{G}}(f).$$

□

The two-sided case is as follows.

Corollary 3.2.5. *Let $\mathbb{G}$ be a CQG and $I \trianglelefteq L^1(\mathbb{G})$ an ideal. Then there exists a hull of $\mathbb{G}$, say E, such that $j(E) \subseteq I \subseteq I(E)$ where each $E_\pi \in E$ is either $\mathcal{H}_\pi$ or $\{0\}$.*

Proof. Following Proposition 3.2.4, what is left is noticing that each $E_\pi \in E$ must satisfy either $E_\pi = \mathcal{H}_\pi$ or $E = \{0\}$. Inspecting the proof of Proposition 3.2.4, the result follows because $\pi(I)$ is a two sided ideal M_{n_π}: we either have $\pi(I) = M_{n_\pi}$ or $\pi(I) = \{0\}$. □

Given the duality between right invariant subspaces of $L^1(\mathbb{G})$ and left ideals of $L^1(\mathbb{G})$ observed in Section 2.3, we immediately have that for any weak* closed right invariant subspace X of $L^\infty(\mathbb{G})$ that there exists a hull E such that

$$I(E)^\perp \subseteq X \subseteq j(E)^\perp.$$

Then, a natural question is, can we describe $I(E)^\perp$ and $j(E)^\perp$ explicitly in terms of E? It turns out we can, and the answer to this question will be important for us when we characterize the CQGs such that every hull is a set of synthesis.

Given a hull E, we will denote

$$\mathrm{Pol}(\widehat{E}) = \{u^\pi_{\xi,\eta} : \eta \in E_\pi, \xi \in \mathcal{H}_\pi \ \pi \in Irr(\mathbb{G})\},$$

where $u^\pi_{\xi,\eta} = (\mathrm{id} \otimes w_{\eta,\xi})(U^\pi)$ and $w_{\eta,\xi}(T) = \langle T\eta, \xi\rangle$ for $T \in M_{n_\pi}$. Then, we will denote

$$C_r(\widehat{E}) = \overline{\mathrm{Pol}(\widehat{E})}^{||\cdot||_r}, \ C_u(\widehat{E}) = \overline{\mathrm{Pol}(\widehat{E})}^{||\cdot||_u}, \text{ and } L^\infty(\widehat{E}) = \overline{C_r(\widehat{E})}^{wk*}.$$

Proposition 3.2.6. *Let $\mathbb{G}$ be a CQG and E a hull. Then*

1. *Where we recall that $\hat{x} = h \cdot x = h(x\cdot)$,*

$$j(E)^{\perp} = \overline{\{x \in L^{\infty}(\mathbb{G}) : \mathrm{im}(\pi(f \circ S_{\mathbb{G}})) \subseteq \ker(\pi(\widehat{x})),\ f \in I(E),\ \pi \in Irr(\mathbb{G})\}}^{wk*}$$
$$= \bigcap_{f \in I(E)} \ker((\mathrm{id} \otimes f)\Delta_{\mathbb{G}});$$

2. *and $I(E)^{\perp} = L^{\infty}(\widehat{E}) = \overline{\{u^{\pi}_{\xi,\eta} : \pi(f)\xi \neq 0,\ f \in I(E), \pi \in Irr(\mathbb{G}), \xi \in E_{\pi}\}}^{wk*}$.*

Remark 3.2.7. Because the symbol $f \circ S_{\mathbb{G}}$ is defined only for $f \in L^1_{\#}(\mathbb{G})$, an explanation of the notation in Proposition 3.2.6 is in order. We set

$$\pi(f \circ S_{\mathbb{G}}) := (f \otimes \mathrm{id})(S_{\mathbb{G}} \otimes \mathrm{id})U^{\pi}$$

which is defined because $S_{\mathbb{G}}|_{\mathrm{Pol}(\mathbb{G})} : \mathrm{Pol}(\mathbb{G}) \to \mathrm{Pol}(\mathbb{G})$ is a bijection.

Proof. 1. We will first notice

$$\overline{\{x \in L^{\infty}(\mathbb{G}) : \mathrm{im}(\pi(f \circ S_{\mathbb{G}})) \subseteq \ker(\pi(\widehat{x})),\ f \in I(E), \pi \in Irr(\mathbb{G})\}}^{wk*}$$
$$= \overline{\{x \in L^{\infty}(\mathbb{G}) : \mathrm{im}(\pi(f \circ S_{\mathbb{G}})) \subseteq \ker(\pi(\widehat{x})),\ f \in j(E), \pi \in Irr(\mathbb{G})\}}^{wk*}$$

and then will show

$$j(E)^{\perp} = \overline{\{x \in L^{\infty}(\mathbb{G}) : \mathrm{im}(\pi(f \circ S_{\mathbb{G}})) \subseteq \ker(\pi(\widehat{x})),\ f \in j(E), \pi \in Irr(\mathbb{G})\}}^{wk*}.$$

Accordingly, suppose $x \in L^{\infty}(\mathbb{G})$ satisfies $\mathrm{im}(\pi(h \circ S_{\mathbb{G}}^{-1})) \subseteq \ker(\pi(\widehat{x}))$ for all $h \in j(E)$. For $f \in I(E)$, find $g \in j(E)$ such that $\pi(f \circ S_{\mathbb{G}}) = \pi(g \circ S_{\mathbb{G}})$. Then

$$\pi(\widehat{x})\pi(f \circ S_{\mathbb{G}}) = \pi(\widehat{x})\pi(g \circ S_{\mathbb{G}}) = 0,$$

as desired. The reverse containment is obvious.

Moving on to the main proof, let $x \in j(E)^{\perp}$. Then, for $f \in j(E)$,

$$f * x = (\mathrm{id} \otimes f)\Delta_{\mathbb{G}}(x) = 0$$

thanks to right invariance of $j(E)^{\perp}$. Therefore,

$$0 = \pi(\widehat{f * x}) = \pi(\widehat{x})\pi(f \circ S_{\mathbb{G}}^{-1}), \tag{3.1}$$

in other words, $\mathrm{im}(\pi(f \circ S_{\mathbb{G}})) \subseteq \ker(\pi(\widehat{x}))$.

Conversely, take $x \in L^{\infty}(\mathbb{G})$ such that $\mathrm{im}(\pi(f \circ S_{\mathbb{G}})) \subseteq \ker(\pi(\widehat{x}))$ for all $f \in j(E)$. Following equation 3.1 in reverse tells us $f * x = 0$. Since $f \in \lambda_{\mathbb{G}}^{-1}(c_{00}(\widehat{\mathbb{G}}))$, we can find $\pi_1, \ldots, \pi_n \in Irr(\mathbb{G})$ so that $\pi(f) = 0$ if $\pi_i \neq \pi \in Irr(\mathbb{G})$ for all i. Let $\epsilon_{\oplus_{i=1}^n \pi_i} = \lambda_{\mathbb{G}}^{-1}(1_{\oplus_{i=1}^n M_{n_{\pi_i}}})$ via the identification $\oplus_{i=1}^n M_{n_{\pi_i}} \subseteq c_{00}(\widehat{\mathbb{G}})$. Then

$$\lambda_{\mathbb{G}}(\epsilon_{\oplus_{i=1}^n \pi_i} * f) = (\oplus_{i=1}^n \pi_i)(\epsilon_{\oplus_{i=1}^n \pi_i})(\oplus_{i=1}^n \pi_i)(f) = \lambda_{\mathbb{G}}(f),$$

so $\epsilon_{\oplus_{i=1}^n \pi_i} * f = f$. Similarly, $f * \epsilon_{\oplus_{i=1}^n \pi_i} = f$. Therefore,

$$0 = \epsilon_{\oplus_{i=1}^n \pi_i}(f * x) = f(x),$$

as desired.

Now we justify

$$j(E)^{\perp} = \bigcap_{f \in I(E)} \ker((\mathrm{id} \otimes f)\Delta_{\mathbb{G}}).$$

All of the work for this part of the claim has already been done. For $x \in j(E)^{\perp}$ and $f \in I(E)$, notice that we can repeat the steps of the converse above to get $f * x = 0$. Conversely, notice the forward implication above actually depended on having $f * x = 0$.

2. For $\pi \in Irr(\mathbb{G})$, pick an ONB $\{e_j^{\pi}\}$ by choosing an ONB for E_{π} and then extending it to $\mathcal{H}_{\pi}$. Then for $f \in I(E)$ we have

$$0 = \pi(f)(E_{\pi}) = [f(u_{i,j}^{E_{\pi}})](E_{\pi})$$

if and only if $f(u_{i,j}^{\pi}) = 0$ for every $e_j^{\pi} \in E_{\pi}$. The rest is clear. ☐

Using the duality observed between right invariant subspaces of $L^{\infty}(\mathbb{G})$ and left ideals in $L^1(\mathbb{G})$, Proposition 3.2.4 and the above proposition gives us an equivalent statement in $L^{\infty}(\mathbb{G})$. For the notation $\trianglelefteq_L$, we refer the reader back to Chapter 2.9.1.

Corollary 3.2.8. *Let $\mathbb{G}$ be a CQG. If $X \trianglelefteq_l L^{\infty}(\mathbb{G})$ is a right invariant subspace, then there exists a hull E such that*

$$L^{\infty}(\widehat{E}) \subseteq X \subseteq \bigcap_{f \in I(E)} \ker((\mathrm{id} \otimes f)\Delta_{\mathbb{G}}).$$

Proof. This follows immediately from Proposition 3.2.6 and Proposition 3.2.4. ☐

Before getting to the main theorem, we still need to think about the singly generated ideals.

Lemma 3.2.9. *Let $\mathbb{G}$ be a CQG. Fix $f \in L^1(\mathbb{G})$ and let E be the hull of $\mathbb{G}$ associated with the closed principal left ideal $\overline{L^1(\mathbb{G}) * f}$. The following hold:*

1. *we have $E_\pi = \ker(\pi(f))$ for each $E_\pi \in E$;*
2. *$f \in I(E)$;*
3. *if E is a set of synthesis, then $f \in \overline{L^1(\mathbb{G}) * f}$;*
4. *$\overline{L^1(\mathbb{G}) * f}^{\perp} = \ker((\mathrm{id} \otimes f)\Delta_{\mathbb{G}})$;*

Proof. 1. This follows easily from the fact $\pi(g * f)(E_\pi) = \pi(g)\pi(f)(E_\pi) = 0$ for each $g \in L^1(\mathbb{G})$.

2. This follows immediately by definition of $I(E)$ and from 1.

3. If E is a set of synthesis, then from 1., 2., and Proposition 3.2.4,

$$\overline{j(E)} = \overline{L^1(\mathbb{G}) * f} = I(E) \ni f.$$

4. If $f * x = 0$, then $g * f(x) = g(f * x) = 0$ for each $f \in L^1(\mathbb{G})$, that is, $x \in (\overline{L^1(\mathbb{G}) * f})_\perp$. Conversely, if $x \in (\overline{L^1(\mathbb{G}) * f})_\perp$, then $0 = g * f(x) = g(f * x)$ for all $g \in L^1(\mathbb{G})$, which implies $f * x = 0$. □

Theorem 3.2.10. *Let $\mathbb{G}$ be a CQG. Then every hull is a set of synthesis if and only if $f \in \overline{L^1(\mathbb{G}) * f}$ for all $f \in L^1(\mathbb{G})$.*

Proof. If we assume every hull is a set of synthesis, then in particular, from Lemma 3.2.9 we have $f \in \overline{L^1(\mathbb{G}) * f}$ for every $f \in L^1(\mathbb{G})$. Conversely, because of Proposition 3.2.6, all we need to show is $I(E) \subseteq [\bigcap_{f \in I(E)} \ker((\mathrm{id} \otimes f)\Delta_{\mathbb{G}})]_\perp$. So, take $f \in I(E)$ and let $x \in L^\infty(\mathbb{G})$ satisfy $f * x = 0$. Find a net $(g_i) \subseteq L^1(\mathbb{G})$ such that $g_i * f \to f$. Then

$$0 = g_i(f * x) = g_i * f(x) \to f(x).$$

□

From Proposition 3.2.4, the hull of a closed left ideal I in $L^1(\mathbb{G})$ is a set of synthesis if and only if I is the unique closed left ideal corresponding to E. So, when every E is a set of synthesis, we have a description of every closed left ideal of $L^1(\mathbb{G})$ in terms of the hull E. To be explicit, from Theorem 3.2.10 we immediately conclude the following.

Corollary 3.2.11. *Let $\mathbb{G}$ be a CQG such that $f \in \overline{L^1(\mathbb{G}) * f}$ for all $f \in L^1(\mathbb{G})$. The closed left ideals of $L^1(\mathbb{G})$ are of the form $I(E)$ for some hull E.*

In light of Theorem 3.2.10, we make the following definition.

Definition 3.2.12. We say a LCQG $\mathbb{G}$ has **Ditkin's left property at infinity (or property left D_∞)**, if $f \in \overline{L^1(\widehat{\mathbb{G}}) * f}$ for every $f \in L^1(\widehat{\mathbb{G}})$.

Note that property left D_∞ is a property of $\widehat{\mathbb{G}}$ rather than $\mathbb{G}$ because in the classical case of a discrete group Γ, property D_∞ is defined to be a property of the group Γ, which is the dual of the CQG $\widehat{\Gamma}$. Hence, in the general case we regard property left D_∞ as a property of the discrete dual of $\mathbb{G}$.

3.2.2 Weak* Closed Left Ideals of Measure Algebras

The main result of this subsection is that we achieve a characterization of the weak* closed ideals of the measure algebra of a coamenable CQG. Essentially, we will show White's techniques [132] generalize to the setting of CQGs. Before getting to this, however, we will begin by discussing some more general things about ideals of measure algebras of (not necessarily coamenable) CQGs.

First note, from the hulls of a CQG $\mathbb{G}$, we identify the closed left ideals $I(E)$ of $L^1(\mathbb{G})$. Then, using that $L^1(\mathbb{G}) \subseteq M^r(\mathbb{G}), M^u(\mathbb{G})$ isometrically as an ideal, we identify the weak* closed left ideals $\overline{I(E)}^{wk*} \subseteq M^r(\mathbb{G})$ and $\overline{I(E)}^{wk*} \subseteq M^u(\mathbb{G})$. As we will see shortly, for coamenable $\mathbb{G}$, this process identifies all weak* closed left ideals of $M(\mathbb{G})$. We are interested in another process to find weak* closed left ideals in $M^u(\mathbb{G})$ and $M^r(\mathbb{G})$.

Because we have the embedding $\mathrm{Pol}(\mathbb{G}) \subseteq C_u(\mathbb{G})$, we can immediately extend $\pi \in Irr(\mathbb{G})$ to a representation $\pi : M^u(\mathbb{G}) \to M_{n_\pi}$ by setting

$$\pi(\mu) = (\mu \otimes \mathrm{id})(U^\pi) = [\mu(u^\pi_{i,j})].$$

With this in hand, given a hull E, we will define

$$I^u(E) = \{\mu \in M^u(\mathbb{G}) : \pi(\mu)(E_\pi) = 0,\ E_\pi \in E\}$$

and

$$I^r(E) = \{\mu \in M^r(\mathbb{G}) : \pi(\mu)(E_\pi) = 0,\ E_\pi \in E\},$$

which are both easily checked to be weak* left closed ideals in $M^u(\mathbb{G})$ and $M^r(\mathbb{G})$ respectively. We also have the following.

Proposition 3.2.13. *Let $\mathbb{G}$ be a CQG. Then $I^u(E) = C_u(\widehat{E})^\perp$ and $I^r(E) = C_r(\widehat{E})^\perp$.*

Proof. The proof follows similarly to the analogous result in Proposition 3.2.6. □

Now we will work towards the main result of this section. The techniques involve exploiting the following sort of objects.

Definition 3.2.14. A Banach algebra A is **compliant** if there exists a Banach space X such that $M(A) = X^*$ and the maps $M(A) \to A$, $\mu \mapsto \mu a$ and $\mu \mapsto a\mu$, for $a \in A$, are weak*-weakly continuous (where weak$^* = \sigma(M(A), X)$).

Recall, for coamenable $\mathbb{G}$ we have $M(\mathbb{G}) = M^l(L^1(\mathbb{G}))$ (cf. [54]).

Proposition 3.2.15. *Let $\mathbb{G}$ be a coamenable LCQG. Then $L^1(\mathbb{G})$ is compliant if and only if $\mathbb{G}$ is compact.*

Proof. According to [132, Proposition 5.8 (i)], compliance of $L^1(\mathbb{G})$ implies it is an ideal in $L^1(\mathbb{G})^{**}$. Then [106, Theorem 3.8] implies $\mathbb{G}$ is compact. Conversely, if $\mathbb{G}$ is compact then, thanks to [105, Theorem 2.3],

$$C(\mathbb{G}) = L^1(\mathbb{G}) * L^\infty(\mathbb{G}) = L^\infty(\mathbb{G}) * L^1(\mathbb{G})$$

where we have used Cohen's factorization theorem. From here, [132, Lemma 5.7] says $L^1(\mathbb{G})$ is compliant. □

As we will see in the proof of the following theorem, as observed with [132, Theorem 5.10], the distinct advantage of compliance of $L^1(\mathbb{G})$ is that the weak* closed left ideals in $M(\mathbb{G})$ are exactly of the form $\overline{I}^{wk*}$ where I is a left ideal in $L^1(\mathbb{G})$. This allows us to use the structure of closed ideals in $L^1(\mathbb{G})$ obtained in the previous section.

Theorem 3.2.16. *Let $\mathbb{G}$ be a coamenable CQG. The weak* closed left ideals of $M(\mathbb{G})$ are of the form $I^u(E)$ for some hull E of $\mathbb{G}$.*

Proof. Since $L^1(\mathbb{G})$ is compliant, because of [132, Theorem 5.10], the weak* closed left ideals of $M(\mathbb{G})$ are of the form $\overline{I}^{wk*}$ for a closed left ideal I of $L^1(\mathbb{G})$. Now apply Corollary 3.2.11 to get that $I = I(E)$ for some hull E. By definition $L^1(\mathbb{G}) \cap I^u(E) = I(E)$, and so using [132, Theorem 5.10], we get $I^u(E) = \overline{I}^{wk*}$ as desired. □

Corollary 3.2.17. *Let $\mathbb{G}$ be a coamenable CQG. The closed right invariant subspaces of $C(\mathbb{G})$ are of the form $C(\mathbb{G}, E)$ for some hull E of $\mathbb{G}$.*

Proof. This follows from Theorem 3.2.16 and Proposition 3.2.6. □

3.2.3 Ditkin's Property at Infinity and Examples

Even for locally compact groups, property left D_∞ is a rather opaque condition and, to our knowledge, there are no known examples of locally compact groups without property left D_∞ [69, Section 6.7]. Recently a characterization of property left D_∞ for locally compact co–groups has been obtained by Andreou [5]. Using the techniques developed there, Andreou obtained a new proof that AP implies property left D_∞ using techniques based around Fubini tensor products (a result which may also be read from [50, Theorem 1.11]). For this section, we will write down some basic equivalent formulations of property left D_∞ (which we were recorded by Andreou for locally compact groups). Then we will provide examples of CQGs with property left D_∞.

We will say $x \in L^\infty(\mathbb{G})$ satisfies **condition** (H) if $x \in \overline{L^1(\mathbb{G}) * x}^{wk*} \subseteq L^\infty(\mathbb{G})$.

Proposition 3.2.18. *Let $\mathbb{G}$ be a CQG. TFAE:*

1. *$\widehat{\mathbb{G}}$ has property left D_∞;*
2. *every $x \in L^\infty(\mathbb{G})$ satisfies condition (H);*
3. *for $x \in L^\infty(\mathbb{G})$ and $f \in L^1(\mathbb{G})$, $f * x = 0$ implies $f(x) = 0$;*
4. *and for all $X \trianglelefteq_r L^\infty(\mathbb{G})$ we have $x * f \in X$ for all $f \in L^1(\mathbb{G})$ if and only if $x \in X$.*

Proof. First, $(4 \implies 2)$ follows verbatim to the corresponding statement in [5, Proposition 6.7]. Now we note that commutativity of the Fourier Algebra appears in the proof of the corresponding statement in [5, Proposition 6.7] of $(1 \implies 3)$, so we must supply our own proof in the CQG setting here to obtain $(1 \iff 2 \iff 3)$. With that said, $(2 \implies 1)$ does follow verbatim from [5, Proposition 6.7] and the converse follows from a similar Hahn–Banach argument. Then $(1 \iff 3)$ follows from the observation

$$f \in \overline{L^1(\mathbb{G}) * f} \iff \ker(f \otimes \mathrm{id})\Delta_{\mathbb{G}} = \overline{L^1(\mathbb{G}) * f}^{\perp} \subseteq \{f\}^{\perp}.$$

To reiterate, we have $(4 \implies 3 \iff 2 \iff 1)$.

For $(3 \implies 4)$, take $f \in L^1(\mathbb{G})$, so $x * f \in X$, which means for $g \in X_\perp$ that

$$0 = g(x * f) = f(g * x).$$

Since $f \in L^1(\mathbb{G})$ was arbitrary, we deduce that $g * x = 0$, which means $g(x) = 0$, that is $x \in X$ as desired. ∎

The following is an immediate consequence of Proposition 3.1.6.

Proposition 3.2.19. *If a DQG $\widehat{\mathbb{G}}$ has the AP, then it has property left D_∞.*

Now we point out weakly amenable examples in the literature.

Example 3.2.20. The duals of the free unitary and orthogonal compact quantum groups U_F^+ and O_F^+, quantum permutation groups S_n^+, and quantum reflection groups $H_n^{+(s)}$ are all weakly amenable [13].

3.3 Coamenability and Ideals

3.3.1 Compact Quasi–Subgroups

In this section, we make progress towards understanding the closed left ideals of $L^1(\mathbb{G})$ that admit **bounded right approximate identities (brais)**. Recall that if Γ is discrete, then the ideals $I(\Lambda) \subseteq A(\Gamma)$, where Λ is a subgroup of Γ, admit a **bounded approximate identity (bai)** if and only if Γ is amenable [41]. Moreover, it is not too difficult to prove that $I(s\Lambda)$ has a bai if and only if $I(\Lambda)$ does, where $s \in \Gamma$. So, if Γ is amenable, we have identified many ideals that admit bais and otherwise, many ideals that do not. Note that this was generalized to amenable locally compact groups in [42].

Recall that the right coideals of $VN(\Gamma)$ are of the form $VN(\Lambda)$, where Λ is a subgroup of Γ and $J^1(VN(\Lambda)) := VN(\Lambda)_\perp = I(\Lambda)$. So, for CQGs in general, we replace $VN(\Lambda)$ with a right coideal $N \subseteq L^\infty(\mathbb{G})$, and $I(\Lambda)$ with $J^1(N) := N_\perp$. As we will elaborate on shortly, we are forced to restrict our attention to a certain subclass of right coideals. This starts with the following result of [87].

Theorem 3.3.1. *[87, Theorem 3.1] Let $\mathbb{G}$ be a LCQG and $I \trianglelefteq L^1(\mathbb{G})$ be a closed left ideal. Then I has a brai only if there exists a right $L^1(\mathbb{G})$–module projection $L^\infty(\mathbb{G}) \to I^\perp$.*

Proof. See the corresponding reference for a proof. We will point out, however, that the projection onto $I^\perp$ is of the form $x \mapsto e * x$, where $e \in L^\infty(\mathbb{G})^*$ is a weak* cluster point of the given brai, and $e * x$ denotes the natural action of $L^\infty(\mathbb{G})^*$ on $L^\infty(\mathbb{G})$. □

Remark 3.3.2. According to [55, Theorem 3.1], (recall, $\mathbb{G}$ is compact) the bounded right $L^1(\mathbb{G})$–module maps $\mathcal{B}^R_{L^1(\mathbb{G})}(L^\infty(\mathbb{G}))$ are normal, i.e., for every $M \in \mathcal{B}^R_{L^1(\mathbb{G})}(L^\infty(\mathbb{G}))$ there exists a left $L^1(\mathbb{G})$-module map $m : L^1(\mathbb{G}) \to L^1(\mathbb{G})$ such that $M = m^* : L^\infty(\mathbb{G}) \to L^\infty(\mathbb{G})$.

Now, if I is a closed left that has a brai with weak* cluster point $e \in L^\infty(\mathbb{G})^*$ (afforded by Banach–Alaoglu), then we get $I = L^1(\mathbb{G}) * e$, where $f * e$ denotes the natural action of f on $L^\infty(\mathbb{G})^*$ (see also the proof of [87, Theorem 2.2]).

In light of Theorem 3.3.1 and Remark 3.3.2, we will be focusing on coideals that are projections of right $L^1(\mathbb{G})$-module maps. In fact, we will restrict our attention to the coideals that are compact quasi-subgroups.

Recall, for a two-sided coideal N, N_* has a Banach algebra structure inherited directly from the quotient $L^1(\mathbb{G})/J^1(N) \cong N_*$. If we assume N_* has a bai (so in the case $N = L^\infty(\mathbb{G}/\mathbb{H})$ for some normal closed quantum subgroup $\mathbb{H}$, $\mathbb{G}/\mathbb{H}$ is coamenable), then we can easily transfer bais between $L^1(\mathbb{G})$ and $J^1(N)$ from the results found in [31] and [40].

Proposition 3.3.3. *Let $I \trianglelefteq L^1(\mathbb{G})$ be a closed two–sided ideal and suppose $L^1(\mathbb{G})/I$ has a bai. Then $\mathbb{G}$ is coamenable if and only if I has a bai.*

Proof. If $L^1(\mathbb{G})/I$ and I both have bais, then we can build a bai for $L^1(\mathbb{G})$ [31, Pg. 43]. The converse is covered by the more general fact that given a Banach algebra A which has a bai and closed left ideal J, J has a brai if and only if $J^\perp$ is right invariantly complemented in A^*, i.e., there is a right A–module projection $P : A^* \to J^\perp$ (cf. [40, 4.1.4 Pg. 42]). $\square$

When Γ is a discrete group and Λ is a subgroup, the compact quasi-subgroup $VN(\Lambda) \subseteq VN(\Gamma)$ is explicitly written in terms of its hull Λ via

$$VN(\Lambda) \cong \overline{\operatorname{Span}\{\lambda_\Gamma(s) : s \in \Lambda\}}^{wk*} = L^\infty(\widehat{\Gamma}, \Lambda),$$

where $L^\infty(\widehat{\Gamma}, \Lambda)$ is our notation from Section 3.3.1. Our techniques for the main results of this section will use a similar description for the compact quasi-subgroups of CQGs in general.

Given a compact quasi–subgroup N, we will denote the corresponding idempotent state by ω_N. Then, for compact $\mathbb{G}$, we have a projection

$$R_{\omega_N} = (\operatorname{id} \otimes \omega_N)\Delta_{\mathbb{G}} : \operatorname{Pol}(\mathbb{G}) \to R_{\omega_N}(\operatorname{Pol}(\mathbb{G}))$$

onto a right invariant subalgebra of $\operatorname{Pol}(\mathbb{G})$ satisfying

$$\overline{R_{\omega_N}(\operatorname{Pol}(\mathbb{G}))}^{wk*} = N.$$

See also [45, Section 2] for a discussion in the case of CQGs.

Remark 3.3.4. Wang [128] showed that normality is equivalent to having $[\omega_{L^\infty(\mathbb{G}/\mathbb{H})}(u^\pi_{i,j})] = I_{n_\pi}$ or 0 for all $\pi \in Irr(\mathbb{G})$, from which it was also shown for normal $\mathbb{H}$,

$$\mathrm{Pol}(\mathbb{G}/\mathbb{H}) = \mathrm{Pol}(E_\mathbb{H})$$

where $E_\mathbb{H} = (E_\pi)_{\pi\in Irr(\mathbb{G})}$ is the hull such that $E_\pi = \mathcal{H}_\pi$ if $[\omega_{L^\infty(\mathbb{G}/\mathbb{H})}(u^\pi_{i,j})] = I_{n_\pi}$ and $E_\pi = \{0\}$ otherwise. In particular, $L^\infty(\widehat{E_\mathbb{H}}) = L^\infty(\mathbb{G}/\mathbb{H})$.

The above remark generalizes to the following for compact quasi–subgroups (and uses the same techniques as Wang).

Lemma 3.3.5. *Let N be a compact quasi–subgroup. Then, there exists an orthonormal basis $\{e^\pi_i\}$ of $\mathcal{H}_\pi$ so that $u^\pi_{i,j} \in N$ if and only if $\omega_N(u^\pi_{j,j}) = 1$, and*

$$R_{\omega_N}(\mathrm{Pol}(\mathbb{G})) = \mathrm{span}\{u^\pi_{i,j} : 1 \le i \le n_\pi, e^\pi_j \in E_\pi\}.$$

Proof. Fix $\pi \in Irr(\mathbb{G})$. Since ω_N is an idempotent state, $\pi(\omega_N)$ is an orthogonal projection. Choose an ONB $\{e^\pi_i\}$ so that $\pi(\omega_N)$ is diagonal, so, $\omega_N(u^\pi_{i,j}) = \delta_{i,j}$ or 0. If $u^\pi_{i,j} \in N$, then

$$u^\pi_{i,j} = R_{\omega_N}(u^\pi_{i,j}) = (\mathrm{id} \otimes \omega_N)\Delta_\mathbb{G}(u^\pi_{i,j}) = \omega_N(u^\pi_{j,j})u^\pi_{i,j}$$

implies $\omega_N(u^\pi_{j,j}) = 1$ and otherwise, $u^\pi_{i,j} \neq \omega_N(u^\pi_{j,j})u^\pi_{i,j}$, which means $\omega_N(u^\pi_{j,j}) = 0$.

Notice that we have shown $R_{\omega_N}(u^\pi_{i,j}) = u^\pi_{i,j}$ or 0. The second claim follows. ☐

Corollary 3.3.6. *Let $\mathbb{G}$ be a CQG and N a compact quasi–subgroup with hull E_N. Then* $\mathrm{Pol}(\widehat{E_N}) = R_{\omega_N}(\mathrm{Pol}(\mathbb{G}))$, and furthermore, $L^\infty(\widehat{E_N}) = N$.

This establishes an explicit description of compact quasi-subgroups in terms of their underlying hull that is reminiscent of the embedding $VN(\Lambda) \subseteq VN(\Gamma)$ when Λ is a subgroup of Γ.

Now fix a compact quasi–subgroup N. We will build from it canonical "continuous function spaces" and "measure spaces". Through these spaces we identify a certain weak* closed left ideal, $J^u(N)$, in $M^u(\mathbb{G})$ corresponding to N. The importance of this left ideal will reveal itself in the main results of this section. Inspired by the techniques of White [132] that we exploited in Section 3.3.2, we relate the problem of determining the existence of brais in $J^1(N)$ with the problem of determining when $\overline{J^1(N)}^{wk*} = J^u(N)$.

Accordingly, we will define

$$C_u(\widehat{E_N}) := \overline{\mathrm{Pol}(\widehat{E_N})}^{||\cdot||_u} \text{ and } C_r(\widehat{E_N}) := \Gamma_\mathbb{G}(C_u(\widehat{E_N})),$$

where we recall that $\Gamma_{\mathbb{G}} : C_u(\mathbb{G}) \to C_r(\mathbb{G})$ is the reducing morphism, and so, by definition, we have a surjective $*$–homomorphism $\Gamma_{\mathbb{G}}|_{C_u(\widehat{E_N})} : C_u(\widehat{E_N}) \to C_r(\widehat{E_N})$. Note that since $\Gamma_{\mathbb{G}}(\mathrm{Pol}(\widehat{E_N})) \subseteq N$, we have $C_r(\widehat{E_N}) \subseteq N$ and by weak density of $\mathrm{Pol}(\widehat{E_N})$ in N, we have $\overline{C_r(\widehat{E_N})}^{wk*} = N$. We also have the right $M^u(\mathbb{G})$–module conditional expectation

$$R^u_{\omega_N} := (\mathrm{id} \otimes \omega_N)\Delta^u_{\mathbb{G}} : C_u(\mathbb{G}) \to C_u(\widehat{E_N}),$$

which satisfies $\Gamma_{\mathbb{G}} \circ R^u_{\omega_N} = R_{\omega_N} \circ \Gamma_{\mathbb{G}}$, where we note that $R_{\omega_N} = R_{\omega_N}|_{C_r(\mathbb{G})} : C_r(\mathbb{G}) \to C_r(\widehat{E_N})$ is a projection as well. Then we will set

$$M^u(\widehat{E_N}) := C_u(\widehat{E_N})^* \text{ and } M^r(\widehat{E_N})^* := C_r(\widehat{E_N}).$$

Then, by definition, the adjoint is a completely isometric embedding:

$$(\Gamma_{\mathbb{G}}|_{C_u(\widehat{E_N})})^* : M^r(\widehat{E_N}) \to M^u(\widehat{E_N}).$$

Now, by taking the adjoint of the inclusion $C_u(\widehat{E_N}) \subseteq C_u(\mathbb{G})$, we obtain a surjective weak*–weak* continuous linear map

$$T^u_N : M^u(\mathbb{G}) \to M^u(\widehat{E_N})$$

whose kernel we denote by $J^u(N)$, which of course satisfies $J^u(N) = C_u(\widehat{E_N})^{\perp}$.

Remark 3.3.7. Note that in the case of a quotient $\mathbb{G}/\mathbb{H}$, where $E_{\mathbb{G}/\mathbb{H}}$ is the associated hull, it is not hard to show that

$$C_u(\mathbb{G}/\mathbb{H}) := \{a \in C_u(\mathbb{G}) : (\mathrm{id} \otimes \pi^u_{\mathbb{H}})\Delta^u_{\mathbb{G}}(a) = a \otimes 1\} = C^u(L^{\infty}(\mathbb{G}/\mathbb{H})).$$

Furthermore, we will denote $C_r(\mathbb{G}/\mathbb{H}) = C_r(\widehat{E_{\mathbb{G}/\mathbb{H}}})$ etc.

For the moment we will consider quotients of closed quantum subgroups. The following notion was formulated in [65].

Definition 3.3.8. For a CQG $\mathbb{G}$, we say a quotient $\mathbb{G}/\mathbb{H}$ is **coamenable** if $\pi^u_{\mathbb{H}} : C_u(\mathbb{G}) \to C^u(\mathbb{H})$ admits a reduced version, that is, there exists $\pi^r_{\mathbb{H}} : C_r(\mathbb{G}) \to C^r(\mathbb{H})$ such that $\Gamma_{\mathbb{H}} \circ \pi^u_{\mathbb{H}} = \pi^r_{\mathbb{H}} \circ \Gamma_{\mathbb{G}}$ (where $\Gamma_{\mathbb{G}} : C_u(\mathbb{G}) \to C_r(\mathbb{G})$ is the reducing morphism).

When Γ is discrete, it is well-known that a subgroup Λ is amenable if and only if $1_{\Lambda} \in B_r(\Gamma)$. So, in general, coamenability of $\mathbb{G}/\mathbb{H}$ is a bonafide generalization of amenability for "quantum quotients."

Now we state a useful necessary condition for coamenability of a quotient motivating the condition $\omega_N \in M^r(\mathbb{G}) = \overline{L^1(\mathbb{G})}^{wk*}$, which we will be using for the main results of this subsection.

Proposition 3.3.9. *Let $\mathbb{G}$ be a CQG and $\mathbb{H}$ a closed quantum subgroup. If $\mathbb{G}/\mathbb{H}$ is coamenable, then $\omega_{L^\infty(\mathbb{G}/\mathbb{H})} \in M^r(\mathbb{G}) \subseteq M^u(\mathbb{G})$.*

Proof. Recall that $\omega_{L^\infty(\mathbb{G}/\mathbb{H})} = h^u_{\mathbb{H}} \circ \pi^u_{\mathbb{H}}$ and the completely isometric embedding $M^r(\mathbb{G}) \subseteq M^u(\mathbb{G})$ is given by the adjoint of $\Gamma_{\mathbb{G}}$. Recall also that we can factorize $h^u_{\mathbb{H}} = h^r_{\mathbb{H}} \circ \Gamma_{\mathbb{H}}$. Then

$$M^r(\mathbb{G}) \ni h^r_{\mathbb{H}} \circ \pi^r_{\mathbb{H}} \circ \Gamma_{\mathbb{G}} = h^r_{\mathbb{H}} \circ \Gamma_{\mathbb{H}} \circ \pi^u_{\mathbb{H}} = h^u_{\mathbb{H}} \circ \pi^u.$$

□

So, by assuming $\omega_N \in M^r(\mathbb{G})$, we know that this condition holds at least for coamenable quotients (compare Corollary 3.3.14 with Proposition 3.3.3). Next we take a look at the associated left ideals in $M^u(\mathbb{G})$.

Proposition 3.3.10. *Let $\mathbb{G}$ be a CQG and N a compact quasi–subgroup. Then $J^u(N)$ has a right unit.*

Proof. Let $\mathbb{G}$ be a CQG and N a compact quasi–subgroup. First notice that for $\mu \in J^u(N)$ and $a \in C_u(\mathbb{G})$,

$$0 = \mu(E^R_{C_u(\widehat{E_N})}(a)) = \mu(\mathrm{id} \otimes \omega_N)\Delta^u_{\mathbb{G}}(a) = \mu * \omega_N(a)$$

Therefore, $\mu * (\epsilon^u_{\mathbb{G}} - \omega_N) = \mu$ for all $\mu \in J^u(N)$. Finally, by choosing an ONB as in Lemma 3.3.5,

$$(\epsilon^u_{\mathbb{G}} - \omega_N)(u^\pi_{i,j}) = \delta_{i,j} - \delta_{i,j} = 0$$

for all $u^\pi_{i,j} \in \mathrm{Pol}(\widehat{E_N})$. Then, from density of $\mathrm{Pol}(\widehat{E_N})$ in $C_u(\widehat{E_N})$, we have $\epsilon^u_{\mathbb{G}} - \omega_N|_{C_u(\widehat{E_N})} = 0$, that is, $\epsilon^u_{\mathbb{G}} - \omega_N \in J^u(N)$. □

Corollary 3.3.11. *Let $\mathbb{G}$ be a CQG and N an invariant subalgebra. Then $J^u(N)$ has an identity element.*

Proof. A similarly proof to Proposition 3.3.10 shows $\epsilon^u_{\mathbb{G}} - \omega_N$ is also a left identity. □

Accordingly, we will denote the right (or two–sided when appropriate) identity of $J^u(N)$ by e^u. Notice then that

$$J^u(N) = J^u(N) * e^u \subseteq M^u(\mathbb{G}) * e^u \subseteq J^u(N),$$

so

$$J^u(N) = M^u(\mathbb{G}) * e^u.$$

A natural question to ask is, when can we approximate e^u from $J^1(N)$? In particular, how does this relate to the existence of a brai in $J^1(N)$? The answer is as follows.

Theorem 3.3.12. *Let N be a compact quasi–subgroup. If $J^1(N)$ has a brai then $\overline{J^1(N)}^{wk*} = J^u(N)$, where the weak* topology is the one induced by $C_u(\mathbb{G})$.*

Proof. Assume $J^1(N)$ has a brai (e_j) and pass to a weak* convergent subnet with limit point $e \in \overline{J^1(N)}^{wk*} \subseteq M^r(\mathbb{G})$. Before proceeding with the proof, we point out some intermediate facts. We will first show $\overline{J^1(N)}^{wk*} = M^r(\mathbb{G}) * e$. Since $L^1(\mathbb{G})$ is an ideal in $M^r(\mathbb{G})$, for $\mu \in M^r(\mathbb{G})$ and $f \in J^1(N)$ we have $\mu * f * e_j \in J^1(N)$ for all $j \in J^1(N)$, from which we conclude that $\mu * f \in J^1(N)$. In particular, we have $\mu * e_j \in J^1(N)$ for all j and so by taking limits, $\mu * e \in \overline{J^1(N)}^{wk*}$.

Next we will show

$$J^u(N) * e = \overline{J^1(N)}^{wk*}. \tag{3.2}$$

Clearly $J^1(N) \subseteq J^u(N)$, from which we immediately deduce $e * e^u = e$. Then,

$$J^1(N) \subseteq J^u(N) * e = J^u(N) * e * e \subseteq M^r(\mathbb{G}) * e = \overline{J^1(N)}^{wk*},$$

using that $M^r(\mathbb{G})$ is an ideal in $M^u(\mathbb{G})$, and so $J^u(N) * e = \overline{J^1(N)}^{wk*}$ as desired.

Set $\omega_N^r = \epsilon_{\mathbb{G}}^u - e$. For $f \in L^1(\mathbb{G})$, we have

$$f \circ R_{\omega_N^r} = 0 \iff f * \omega_N^r = 0 \iff f * e = f.$$

So, $f \circ R_{\omega_N^r} = 0$ for all $f \in J^1(N)$, which implies $R_{\omega_N^r}(L^\infty(\mathbb{G})) \subseteq N$. Then, since $R_{\omega_N}|_N = \mathrm{id}_N$,

$$R_{\omega_N^r * \omega_N} = R_{\omega_N} \circ R_{\omega_N^r} = R_{\omega_N^r}.$$

Recall from the proof of Proposition 3.3.10 that $\omega_N = \epsilon_{\mathbb{G}}^u - e^u$. Then, by injectivity of $\mu \mapsto R_\mu$ we get

$$(\epsilon_{\mathbb{G}}^u - e) * (\epsilon_{\mathbb{G}}^u - e^u) = \omega_N * \omega_N^r = \omega_N^r = \epsilon_{\mathbb{G}}^u - e,$$

from which we have $e * e^u = e^u$. Therefore, using (3.2),

$$\overline{J^1(N)}^{wk*} = \overline{J^1(N)}^{wk*} * e^u = J^u(N) * e * e^u = J^u(N) * e^u = J^u(N).$$

$\square$

Our question of weakly approximating elements of $J^u(N)$ by elements of $J^1(N)$ turns out to relate to coamenability of $\mathbb{G}$.

Theorem 3.3.13. *Let $\mathbb{G}$ be a CQG and N a compact quasi–subgroup. If $\mathbb{G}$ is coamenable then $\overline{J^1(N)}^{wk*} = J^u(N)$. Conversely, if $\omega_N \in M^r(\mathbb{G})$ and $\overline{J^1(N)}^{wk*} = J^u(N)$, then $\mathbb{G}$ is coamenable.*

Proof. Assume $\mathbb{G}$ is coamenable. We first note that $C_u(\widehat{E_N}) = C_r(\widehat{E_N})$, so we will simply write $C(N)$. Because of [132, Theorem 5.10] (cf. Theorem 3.2.16), it suffices to show $J^u(N) \cap L^1(\mathbb{G}) = J^1(N)$. First, clearly $J^1(N) \subseteq J^u(N)$. For the reverse containment, take $a \in C(\mathbb{G})$. Then for $a \in C(N)$ and $f \in J^u(N) \cap L^1(\mathbb{G})$,

$$0 = T_N^u(f)(a) = f(a) = T_N(f)(a)$$

which implies $f(N) = 0$ by weak* density of $C(N)$ in N and normality of f.

Conversely,

$$M^r(\mathbb{G}) = \overline{L^1(\mathbb{G})}^{wk*} \ni \omega_N + e^u = \epsilon^u_{\mathbb{G}},$$

where the equality was noted in the proof of Proposition 3.3.10. This implies coamenability of $\mathbb{G}$. □

A coamenability result we are looking for presents itself as follows.

Corollary 3.3.14. *Let $\mathbb{G}$ be a CQG and N an compact quasi–subgroup such that $\omega_N \in M^r(\mathbb{G})$. Then $J^1(N)$ has a brai if and only if $\mathbb{G}$ is coamenable.*

Proof. If $J^1(N)$ has a brai, then apply Theorems 3.3.12 and 3.3.13 to get coamenability of $\mathbb{G}$. The converse is a special case of the following more general fact: if A is a Banach algebra with a bai, then a closed left ideal J has a brai if and only if there is a right A–module projection $A^* \to J^\perp$ (cf. [40, 4.1.4 Pg. 42]). □

From Proposition 3.3.9 and Corollary 3.3.14, we also deduce the following.

Corollary 3.3.15. *Let $\mathbb{G}$ be a CQG and $\mathbb{H}$ a closed quantum subgroup such that $\mathbb{G}/\mathbb{H}$ is a coamenable quotient. Then $\mathbb{G}$ is coamenable if and only if $J^1(\mathbb{G}, \mathbb{H})$ has a brai.*

A compact quasi–subgroup N is **open** if $\omega_N \in L^1(\mathbb{G})$. It was shown in [74] that the open quasi–subgroups of a CQG are the finite dimensional right coideals. Using Corollary 3.3.14 we obtain the following.

Corollary 3.3.16. *Let $\mathbb{G}$ be a CQG and N an open compact quasi–subgroup. Then $\mathbb{G}$ is coamenable if and only if $J^1(N)$ has a brai.*

3.3.2 Quantum Cosets of Compact Quasi–Subgroups

Our main result of the previous subsection is a generalization of the result of Forrest [41] that for a discrete group Γ, the ideal $I(\Lambda)$ has a bai if and only if Γ is amenable, restricted to the case where Λ is an *amenable* subgroup of Γ (so that $1_\Lambda \in B_r(\Gamma)$). The techniques of Forrest exploited the fact that $I(\Lambda)$ has a bai if and only if $I(s\Lambda)$ has a bai and the use of the Hahn-Banach theorem, for $s \in \Gamma \setminus \Lambda$.

To get past the condition where $\omega_N \in M^r(\mathbb{G})$, we will consider (invariant) compact quasi-subgroups that "admit a quantum coset". Along the way, we achieve a generalization of the fact $I(\Lambda)$ has a bai if and only if $I(s\Lambda)$ has a bai, and so we are able to characterize coamenability of $\mathbb{G}$ in terms of the closed left ideals coming from the "quantum cosets" of compact quasi-subgroups. A quantum coset will turn out to be a translation of a compact quasi-subgroup by an element of the intrinsic group of $\mathbb{G}$.

Definition 3.3.17. The group

$$Gr(\mathbb{G}) = \{x \in L^\infty(\mathbb{G})^{-1} : \Delta_{\mathbb{G}}(x) = x \otimes x\},$$

where $L^\infty(\mathbb{G})^{-1}$ is the set of invertibles in $L^\infty(\mathbb{G})$, is called the **intrinsic group** of $\widehat{\mathbb{G}}$.

Remark 3.3.18. Our reference for the following discussion is [68]. We actually have that each element of $Gr(\mathbb{G})$ is unitary and is a locally compact group when equipped with the weak* topology. It is straightforward seeing that $Gr(\mathbb{G}) = sp(L^1(\mathbb{G}))$. Alternatively, one can identify $Gr(\mathbb{G}) \subseteq Irr(\mathbb{G})$ as the 1–dimensional unitary representations, so in particular, we have $Gr(\mathbb{G}) \subseteq \mathrm{Pol}(\mathbb{G})$. Note that whenever $\mathbb{G}$ is compact, the von Neumann algebra generated by $Gr(\mathbb{G})$ is of the form $VN(\Gamma)$ for some discrete group Γ. We will abuse notation and simply write $Gr(\mathbb{G}) = \Gamma$.

When Γ is discrete and Λ is a subgroup, we have $\lambda(s)VN(\Lambda) = VN(s\Lambda)$. Then $I(s\Lambda) = (\lambda(s)VN(\Lambda))_\perp$. So, more generally, for a CQG $\mathbb{G}$ and compact quasi-subgroup $N \subseteq L^\infty(\mathbb{G})$, we consider sN, where $s \in Gr(\mathbb{G})$, to be a **quantum coset** of N.

We will proceed by writing down a series of lemmas that allow us to use a generalization of Forrest's argument in [41], as alluded to at the start of the subsection. To begin, the following essentially says that the quantum cosets of a compact quasi-subgroup are disjoint from the compact quasi-subgroup. This is reminiscent of the fact that proper cosets of a subgroup are disjoint with the subgroup.

Lemma 3.3.19. *Let N be a compact quasi–subgroup. Then for $x \in Gr(\mathbb{G})$,*

$$xN \cap N = \begin{cases} N & \text{if } x \in N \\ \{0\} & \text{otherwise} \end{cases}.$$

Proof. From Lemma 3.3.5 we know that $\omega_N(x) = 1$ if $x \in N$ and $\omega_N(x) = 0$ if $x \notin N$. Then the equation

$$(\mathrm{id} \otimes \omega_N)\Delta_{\mathbb{G}}(x) = \omega_N(x)x$$

tell us $R_{\omega_N}(x) = x$ if $x \in N$and $R_{\omega_N}(x) = 0$ otherwise. Then for $y \in N$, using that R_{ω_N} is a conditional expectation, we have

$$xy = R_{\omega_N}(xy) = R_{\omega_N}(x)y$$

if and only if $R_{\omega_N}(x) = x$. □

Given $x \in L^\infty(\mathbb{G})$, we denote $x \cdot f \in L^1(\mathbb{G})$ as the action such that $(f \cdot x)(y) = f(xy)$ for all $y \in L^\infty(\mathbb{G})$. If $x \in Gr(\mathbb{G})$, then, since x is a unitary, $\cdot x : L^1(\mathbb{G}) \to L^1(\mathbb{G})$ is an isometric algebra automorphism.

Lemma 3.3.20. *Let $X \trianglelefteq_r L^\infty(\mathbb{G})$ be a right invariant weak* closed subspace. For $x \in Gr(\mathbb{G})$,*

$$X_\perp \cdot x = (xX)_\perp$$

is a closed left ideal. If $X_\perp$ is two–sided, then $X_\perp \cdot x$ is two–sided.

Proof. Since $(X_\perp)^\perp = X$, it is clear that $X_\perp \cdot x \subseteq (xX)_\perp$. For $f \in (xX)_\perp$, it can be shown using a Hahn–Banach argument that $f \cdot x^{-1} \in X_\perp$. Then $f = (f \cdot x^{-1}) \cdot x \in X_\perp \cdot x$. For the remaining claim, it is easy to see that $X_\perp \cdot x$ is closed. Then for $f \in L^1(\mathbb{G})$ and $y \in X$,

$$(yx) * f = (f \otimes \mathrm{id})(x \otimes x)\Delta_{\mathbb{G}}(y) \in xX$$

because $((f \cdot x) \otimes \mathrm{id})\Delta_{\mathbb{G}}(y) \in X$. So xX is right invariant, meaning $(xX)_\perp = X_\perp \cdot x$ is a left ideal. If X is also left invariant, then left invariance of xX follows similarly. □

Remark 3.3.21. Let N be a compact quasi–subgroup. Note that for $x \in Gr(\mathbb{G}) \setminus N$, xN does not contain 1 and so cannot be a von Neumann algebra and is not a compact quasi–subgroup. We did see, however, in the above lemma that xN is a weak* closed right invariant subspace of $L^\infty(\mathbb{G})$. Next we will note $R_{\omega \cdot x^{-1}} : L^\infty(\mathbb{G}) \to L^\infty(\mathbb{G})$ is a projection onto xN. To see this, first notice $x\,\mathrm{Pol}(\widehat{E_N})$ is weak* dense in xN because $L^\infty(\mathbb{G}) \ni y \mapsto xy \in L^\infty(\mathbb{G})$ is a weak*–weak* homeomorphic linear bijection. Therefore it suffices to check $(\mathrm{id} \otimes \omega_N \cdot x^{-1})\Delta_{\mathbb{G}}$ is a projection onto $x\,\mathrm{Pol}(\widehat{E_N})$. For this, if we take $y \in \mathrm{Pol}(\widehat{E_N})$,

$$(\mathrm{id} \otimes \omega_N \cdot x^{-1})\Delta_{\mathbb{G}}(xy) = x(\mathrm{id} \otimes \omega_N)\Delta_{\mathbb{G}}(y) = xy$$

and if we take $y \in \mathrm{Pol}(\mathbb{G})$,

$$(\mathrm{id} \otimes \omega_N \cdot x^{-1})\Delta_{\mathbb{G}}(y) = x(\mathrm{id} \otimes \omega_N)\Delta_{\mathbb{G}}(x^{-1}y) \in x\,\mathrm{Pol}(\widehat{E_N}).$$

We point out that the idempotent functional $\omega_N \cdot x^{-1}$ is easily seen to be a contractive idempotent. Contractive idempotents and their associated weak* closed right invariant subspaces were studied in [93, 72] (at the level LCQGs). While given a contractive idempotent $\omega \in M^u(\mathbb{G})$, $R_\omega(L^\infty(\mathbb{G}))$ is not an algebra, it is a **ternary ring of operators (TRO)**, i.e., whenever $x, y, z \in R_\omega(L^\infty(\mathbb{G}))$, $xy^*z \in R_\omega(L^\infty(\mathbb{G}))$.

Lemma 3.3.22. *Let N be a compact quasi–subgroup. For $x \in Gr(\mathbb{G}) \cap (L^\infty(\mathbb{G}) \setminus N)$, $T_N(J^1(N) \cdot x) = N_*$.*

Proof. For each $y \in N$, using $xN \cap N = \{0\}$ from Lemma 3.3.19, find $f \in J^1(N) \cdot x$ so that $f(y) \neq 0$. Then $T_N(f)(y) = f(y) \neq 0$, from which, using a straightforward Hahn–Banach argument and that T_N is open (open mapping theorem) and hence closed, we see that $T_N(J^1(N) \cdot x) = N_*$ as desired. □

The following theorem is the statement that $\mathbb{G}$ is coamenable if and only if the preannihilator of an invariant quantum coset has a bai.

Theorem 3.3.23. *Let $\mathbb{G}$ be a CQG and X a weak* closed invariant subspace of $L^\infty(\mathbb{G})$. Suppose $\{sX : s \in Gr(\mathbb{G})\}$ has a compact quasi–subgroup and at least two elements. Then $\mathbb{G}$ is coamenable if and only if $X_\perp$ has a bai.*

Proof. Let $N \in \{sX : s \in Gr(\mathbb{G})\}$ denote the compact quasi–subgroup. As discussed in Remark 3.3.21, we know $N = x_0X$ for some $x_0 \in Gr(\mathbb{G})$ and from Lemma 3.3.20, $J^1(N) = X_\perp \cdot x_0$ is a two–sided ideal (and so N is actually an invariant compact quasi–subgroup).

The proof is a generalization of the argument employed by Forrest [41]. Suppose $X_\perp$ has a bai. Now, for $f \in X_\perp$, $y \in L^\infty(\mathbb{G})$, and $x \in Gr(\mathbb{G})$

$$\begin{aligned}
||(e_j \cdot x) * (f \cdot x) - f \cdot x||_1 &= \sup_{y \in B_1(L^\infty(\mathbb{G}))} |(e_j \otimes f)\Delta_{\mathbb{G}}(xy) - f(xy)| \\
&= \sup_{y \in x^{-1}B_1(L^\infty(\mathbb{G}))} |(e_j \otimes f)\Delta_{\mathbb{G}}(y) - f(y)| \\
&= \sup_{y \in B_1(L^\infty(\mathbb{G}))} |e_j * f(y) - f(y)| \\
&= ||e_j * f - f||_1 \to 0
\end{aligned}$$

where in the second last equality, we used the fact x is a unitary. A similar proof shows $f * (e_j \cdot x) \to f$, so $e_j \cdot x$ is a bai on $X_\perp \cdot x$. Now, we know $X \cdot Gr(\mathbb{G})$ has two elements, one of which is N. Without loss of generalization, we will suppose the other element is X. So, we have that $J^1(N)$ and $X_\perp = J^1(N) \cdot x_0^{-1}$ both have bais. Then from invariance of N, we know T_N is an algebraic homomorphism and coupling this fact with Lemma 3.3.22 finds us a bai on $N_* = T_N(J^1(N) \cdot x_0^{-1})$. Then we apply Proposition 3.3.3.

For the converse, from the discussion in Remark 3.3.21, we have a right $L^1(\mathbb{G})$–module projection $L^\infty(\mathbb{G}) \to X$ induced by the idempotent functional $\omega_N \cdot x_0^{-1}$. The rest is identical to the proof of the converse of Proposition 3.3.3. □

With our last corollary, we drop the invariance condition of N, but we are forced to put back the condition that $\omega_N \in M^r(\mathbb{G})$. What is distinct from before is that we incorporate quantum cosets of N.

Corollary 3.3.24. *Let $\mathbb{G}$ be a CQG and X a weak* closed right invariant subspace of $L^\infty(\mathbb{G})$. Suppose $\{sX : s \in Gr(\mathbb{G})\}$ has a compact quasi-subgroup N such that $\omega_N \in M^r(\mathbb{G})$ and at least two elements. Then $\mathbb{G}$ is coamenable if and only if $X_\perp$ has a brai.*

Proof. Let $N \in \{sX : s \in Gr(\mathbb{G})\}$ be the given compact quasi–subgroup. In the proof of Theorem 3.3.23, we showed $J^1(N)$ also has a brai. Then from Corollary 3.3.14 we know $\mathbb{G}$ is coamenable. The proof of the converse is identical to the proof of the converse in Theorem 3.3.23. □

Remark 3.3.25. In particular, if we have a compact quasi–subgroup $N \subseteq L^\infty(\mathbb{G})$ such that $Gr(\mathbb{G}) \setminus N$ is non–trivial, then we are in a situation satisfying hypothesis of Theorem 3.3.23.

3.3.3 Examples: Discrete Crossed Products

The main results of Section 3.4.2 are applicable only to CQGs where the intrinsic group is non-trivial, and avoids the compact quasi-subgroups in question. Unfortunately, it can be the case where the intrinsic group is trivial, which happens exactly when the trivial representation is the only 1-dimensional representation (eg $SU_q(2)$). In this subsection we present a class of examples which do have a non-trivial intrinsic group, and hence are where we can apply our results from Section 3.4.2. These examples come from the CQGs in the form of crossed products of discrete groups.

Proposition 3.3.26. *Let $(\mathbb{G}, \Gamma, \alpha)$ be a Woronowicz dynamical system. Then the following hold:*

1. *if we assume $Gr(\mathbb{G}) \neq \{1\}$, $\mathbb{G} \rtimes \Gamma$ is coamenable if and only if $J^1(VN(\Gamma)) \cdot x$ has a bai, where $x \in Gr(\mathbb{G})$;*

2. *$\mathbb{G} \rtimes \Gamma$ is coamenable if and only if any $J^1(N) \cdot s$ has a bai, where $s \in \Gamma$ and $Ns = VN(\Lambda s)$ for some proper subgroup Λ of Γ or Ns is an (invariant) quantum coset of $L^\infty(\mathbb{G})$.*

Proof. First note that $\Gamma, Gr(\mathbb{G}) \subseteq Gr(\mathbb{G} \rtimes \Gamma)$ and $Gr(\mathbb{G}) \cap \Gamma = \{1\}$. For 1, because $Gr(\mathbb{G}) \neq \{1\}$, we can find $x \in Gr(\mathbb{G}) \subseteq Gr(\mathbb{G} \rtimes \Gamma) \setminus \Gamma$ and we apply Theorem 3.3.23.

Likewise, for 2, we can find $x \in \Gamma \subseteq Gr(\mathbb{G} \rtimes \Gamma) \setminus Gr(\mathbb{G})$ or non–trivial $x \in \Lambda \setminus \Gamma \subseteq Gr(\mathbb{G} \rtimes \Gamma)$, and then we apply Theorem 3.3.23. □

3.4 Open Problems

We will present problems left over from our investigations.

We have characterized the CQGs where every hull is a set of synthesis (Theorem 3.2.10) as the CQGs with property left D_∞. This means the closed left ideals (and consequently the weak* closed right invariant subspaces of $L^\infty(\mathbb{G})$) are classified for the CQGs satisfying property left D_∞. This leaves us with the following very open ended question.

Question 3.4.1. *Which hulls of a CQG are always sets of synthesis?*

For example, the closed subgroups of a locally compact groups are always sets of synthesis (cf. [69]). So, we ask the following more specific question.

Question 3.4.2. *Are the hulls of right coideals sets of synthesis?*

We have made partial progress towards identifying when the left ideals $J^1(N)$ associated with a compact quasi–subgroup admit a brai. While we have a complete characterization in terms of the condition $J^u(N) = \overline{J^1(N)}^{wk*}$ (Theorem 3.3.12), our characterization in terms of coamenability of $\mathbb{G}$ (Corollary 3.3.14) requires what is essentially a coamenability type condition on N. This leaves us with the following question.

Question 3.4.3. *Given a CQG and compact quasi–subgroup N, if $J^1(N)$ has brai, then do we have $\omega_N \in M^r(\mathbb{G})$?*

Successfully answering the above question means we can say $\mathbb{G}$ is coamenable if and only if $J^1(N)$ admits a brai.

We have also characterized coamenability of $\mathbb{G}$ in terms the existence of brais on the associated left ideals of a very small class of TROs associated with a contractive idempotent (Theorem 3.3.23 and Corollary 3.3.24). Namely, if we set $X = R_\omega(L^\infty(\mathbb{G}))$ where $\omega \in M^u(\mathbb{G})$ is a contractive idempotent, we require $Gr(\mathbb{G}) \cap (L^\infty(\mathbb{G}) \setminus X) \neq \varnothing$ and one of two things: either X is invariant or $\omega \in M^r(\mathbb{G})$. Therefore we ask the following general question.

Question 3.4.4. *Let $\mathbb{G}$ be a CQG and $\omega \in M^u(\mathbb{G})$ a contractive idempotent. Do we have that $R_\omega(L^\infty(\mathbb{G}))_\perp$ admits a brai if and only if $\mathbb{G}$ is coamenable?*

Chapter 4

Coamenable and Amenable Coideals

4.1 Introduction

Definition 4.1.1. Let A be a C^*-algebra. A trace is a state $\mu : A \to \mathbb{C}$ satisfying $\mu(ab) = \mu(ba)$ for all $a, b \in A$.

Understanding the tracial states of C^*-algebras and simplicity of C^*-algebras are problems of interest to operator algebraists (eg. in classification theory). For a discrete group G, whenever $C_r(\widehat{G})$ has the unique trace property (which would be the Haar state), then the traces are well-understood: they are comprised of the Haar state alone. When studying these properties of the reduced C^*-algebras of groups, an important class of traces to consider are the indicator functions (which are idempotent states) $1_N \in C_r(\widehat{G})^* \subseteq \ell^\infty(G)$, where N is a normal and amenable subgroup of G. Besides the Haar state, a distinguished example is $1_{R_a(G)}$ where $R_a(G)$ is the **amenable radical** of G, the largest amenable normal subgroup. More precisely, it was achieved in [15, 66] that $C_r(\widehat{G})$ has a unique trace if and only if $R_a(G) = \{e\}$ and if $C_r(\widehat{G})$ is simple, then it has a unique trace.

More generally, the idempotent states on the universal C^*-algebra $C_u(\widehat{G})$ are exactly those of the form 1_H where H is a subgroup of G. The traces are those where $H \leq G$ is normal. The universal idempotent states have received a lot of attention in the literature lately (see [111, 112, 74, 73, 46, 72]), specifically on their connection to group-like projections and certain coideals. As far as we can tell, aside from the results in [74] concerning normal idempotent states, the reduced idempotent states have been left untouched, leaving a gap in the understanding of the unique trace property as mentioned at the start of this introduction. Towards this, we characterize the reduced central idempotent states.

Corollary 4.1.2. *Let $\mathbb{G}$ be a discrete quantum group. There is a one-to-one correspondence between the amenable quantum subgroups of $\mathbb{G}$ and the central idempotent states on $C_r(\widehat{\mathbb{G}})$.*

Generalizing the classical case, it then follows that the normal amenable quantum subgroups of $\mathbb{G}$ with unimodular quotient are in one-to-one correspondence with the reduced central idempotent tracial states of $\widehat{\mathbb{G}}$.

For this, we will be studying amenability and coamenability of right coideals. As such, will make progress towards establishing duality between relative amenability and coamenability across Pontryagin duality, which we are about to define.

The following was coined in [65].

Definition 4.1.3. Let $\mathbb{G}$ be a locally compact quantum group. A right coideal $N \subseteq L^\infty(\mathbb{G})$ is **relatively amenable** if there exists a unital completely positive (ucp) right $L^1(\mathbb{G})$-module map $L^\infty(\mathbb{G}) \to N$.

Inspired by the above, we make the following definition.

Definition 4.1.4. Let $\mathbb{G}$ be a locally compact quantum group. A right coideal $N \subseteq L^\infty(\mathbb{G})$ is **amenable** if there exists a ucp right $L^1(\mathbb{G})$-module projection $L^\infty(\mathbb{G}) \to N$.

We will also make use of the terms amenability and relative amenability in reference to weak$*$ closed right invariant subspaces of $\ell^\infty(\mathbb{G})$ as well.

The terms relative amenability and amenability are motivated by their equivalence with relative amenability and amenability of a closed subgroup of a locally compact group [17]. In the case of a discrete group G, amenability and relative amenability are equivalent, and have the following characterization.

Theorem 4.1.5. *[29, 1, 100] Let G be a discrete group and H a subgroup. The following are equivalent:*

1. *H is amenable;*
2. *$\ell^\infty(G/H)$ is amenable;*
3. *$\ell^\infty(G/H)$ is relatively amenable;*

4. $J^1(G,H) = \ell^\infty(G/H)_\perp$ *has a bounded right approximate identity (brai);*

5. $J^1(G,H)$ *has a brai in* $\ell^1_0(G) = \{f \in \ell^1(G) : \int_G f = 0\}$*;*

6. $J^1(G,H)$ *has a brai in* $\ell^1_0(H)$*.*

At the level of locally compact groups, conditions 1., 2., 4., and 6. are equivalent, and 2. and 5. are equivalent (cf. [17]). It is unknown if relative amenability and amenability coincide for locally compact groups in general. We note, however, that the techniques are trivialized in the discrete setting. In the discrete quantum group setting, where we replace subgroups with right coideals, we have to work significantly harder to reach similar results. Both the fact that quantum groups exist only as virtual objects, and that coideals have no underlying closed quantum subgroup (cf. [28]) each introduce barriers of their own. For the coideals of that are of quotient type, we achieve a complete generalization in Section 4.4.6.

Essential to Caprace and Monod's work on (relative) amenability is the notion of an H-invariant state on $\ell^\infty(G)$. While an $\mathbb{H}$-invariant state on $\ell^\infty(\mathbb{G})$ is a coherent notion (see Section 4.4.6), coideals in general are not necessarily quotients by quantum subgroups. Every coideal of a discrete quantum group, however, can be assigned a group-like projection (see Remark 2.9.17). Then, to get around this obstruction, we develop a notion of a P-invariant state (see Definition 4.3.14) and prove that a coideal $\widetilde{N_P}$ is relatively amenable if and only if a P-invariant state exists (see Theorem 4.3.13).

Given a group-like projection P, we prove that certain weak* closed right invariant subspaces $M_P \supseteq \widetilde{N_P}$ are essential to amenability (see Section 4.2.2). The link between amenability and relative amenability can be revealed through amenability of these right invariant subspaces: M_P is amenable if and only if a P-invariant state m exists that satisfies $m(P) \neq 0$ (see Theorem 4.3.19). We also achieve an analogue of Theorem 4.1.5 for the subspaces M_P and their left ideals $J^1(M_P)$ (see Theorem 4.3.20).

Recall that the compact quasi-subgroups are in one-to-one correspondence with the idempotent states in the universal measure algebras. Quantum group duality gives us a one-to-one correspondence between right coideals of $L^\infty(\mathbb{G})$ and right coideals of $L^\infty(\widehat{\mathbb{G}})$ via their codual coideals (see Section 4.2). In particular, given an idempotent state ω and its compact quasi-subgroup $N_\omega \subseteq L^\infty(\mathbb{G})$, we identify its codual coideal $\widetilde{N_{P_\omega}} \subseteq L^\infty(\widehat{\mathbb{G}})$. For a locally compact group G, this duality reduces to the identification of the coideals $L^\infty(G/H)$ and $VN(H) = L^\infty(\widehat{H})$.

In general it is not that hard to show coamenability of a LCQG $\mathbb{G}$ implies amenability of its dual $\widehat{\mathbb{G}}$. It is a highly non-trivial result of Tomatsu [122] that a compact quantum group $\widehat{\mathbb{G}}$ is coamenable only if the discrete quantum group $\mathbb{G}$ is amenable, generalizing the case of compact and discrete Kac algebras due to Ruan [103], which generalizes Leptin's theorem from the classical discrete setting. Kalantar et al. [65] coined the notion of a coamenable right coideal of quotient type for compact quantum groups. This notion generalizes to compact quasi-subgroups (cf. Section 4.2), which we prove has the following characterization.

Corollary 4.1.6. *Let $\mathbb{G}$ be a discrete quantum group and N_ω a compact quasi-subgroup of $\widehat{\mathbb{G}}$. We have that N_ω is* **coamenable** *if and only if $\omega \in M^r(\widehat{\mathbb{G}})$.*

See sections 4.1 and 4.2 for more.

Remark 4.1.7. One advantage of the characterization of coamenability in Corollary 4.1.6 is that it immediately generalizes to locally compact quantum groups.

Kalantar et al. posed the following question.

Question 4.1.8. *[65] Let $\mathbb{G}$ be a discrete quantum group. Let $\widehat{\mathbb{H}}$ be a closed quantum subgroup of $\widehat{\mathbb{G}}$. Is it true that $L^\infty(\widehat{\mathbb{G}}/\widehat{\mathbb{H}})$ is coamenable if and only if $\ell^\infty(\mathbb{H})$ is relatively amenable?*

This questions extends to compact quasi-subgroups of compact quantum groups in the following manner: is it true that $N_\omega \subseteq L^\infty(\widehat{\mathbb{G}})$ is coamenable if and only if its codual coideal $\widetilde{N_{P_\omega}}$ is relatively amenable?

One of our main results makes progress towards the compact quasi-subgroup version of Question 4.1.8. We achieve the converse when we use amenability of the subspace M_P.

Theorem 4.1.9. *Let $\mathbb{G}$ be a discrete quantum group and $N_\omega \subseteq L^\infty(\widehat{\mathbb{G}})$ a compact quasi-subgroup with $P = \lambda_{\widehat{\mathbb{G}}}(\omega)$. If N_ω is coamenable then M_P is amenable.*

Our progress towards the forwards direction is with Lemma 4.4.14, which then we use to prove our characterization of reduced central idempotent states in Corollary 4.1.2.

The remainder of the Chapter is organized as follows: in Section 3, for a group-like projection P we develop the notion of P-invariant states and relate it to relative amenability of $\widetilde{N}_P$ (Theorem 4.3.17). We achieve similar characterizations of both relative amenability and amenability of the subspaces of the form M_P. With these characterizations in hand, we are able to establish a version of Theorem 4.1.5 (2. $\iff$ 4. $\iff$ 6.) (Theorem 4.3.20).

In Section 4 we shift gears towards compact quantum groups and their right coideals, with special attention to their compact quasi-subgroups. We prove that a compact quasi-subgroup N_ω is coamenable if and only if the associated idempotent state ω factors through the reduced C^*-algebra (Corollary 4.1.6). We then classify the central idempotent states on $C_r(\widehat{\mathbb{G}})$ (Theorem 4.1.2).

Finally, in Section 5 we cover examples in the form of discrete crossed products.

Remark 4.1.10. Kalantar et al. independently achieved Theorem 4.1.5 (1. $\iff$ 2. $\iff$ 3.) for discrete quantum groups [65, Theorem 4.7].

4.2 Discrete Quantum Group Dynamics

Let $\mathbb{G}$ be a DQG and A be a unital C^*-algebra.

Definition 4.2.1. *A is a* $\mathbb{G}$**-C^*-algebra** if there exists a unital injective $*$-homomorphism $\alpha : A \to M(c_0(\mathbb{G}) \otimes_{min} A)$ satisfying

- $(\mathrm{id} \otimes \alpha)\alpha = (\Delta_{\mathbb{G}} \otimes \mathrm{id})\alpha$;
- the closed linear span of $(c_0(\mathbb{G}) \otimes 1)\alpha(A)$ is norm dense in $c_0(\mathbb{G}) \otimes_{min} A$.

We call α a (left) **coaction** of $c_0(\mathbb{G})$ on A or an **action** of $\mathbb{G}$ on A.

Remark 4.2.2. Given a $\mathbb{G}$-C^*-algebra A, we may write $A = C(X)$, where X is the underlying compact quantum space, and say that $\mathbb{G}$ acts on X.

For a $\mathbb{G}$-C^*-algebra A, will use the notation $a * f = (f \otimes \mathrm{id})\alpha(a)$ for $a \in A$ and $f \in \ell^1(\mathbb{G})$. Given $\mathbb{G}$-C^*-algebras A and B, we will say a ucp map $\phi : A \to B$ is $\mathbb{G}$**-equivariant** if for all $a \in A$ and $f \in \ell^1(\mathbb{G})$, we have $\phi(a) * f = \phi(a * f)$. For any $\mathbb{G}$-equivariant ucp map $\phi : \ell^\infty(\mathbb{G}) \to \ell^\infty(\mathbb{G})$, the space $\phi(\ell^\infty(\mathbb{G}))$ is a $\mathbb{G}$-C^*-algebra when considered as a C^*-algebra with the Choi-Effros product.

Example 4.2.3. The unital C^*-algebra $C_r(\widehat{\mathbb{G}})$ is a $\mathbb{G}$-C^*-algebra with coaction $\Delta^l(\hat{a}) =$

$W_{\mathbb{G}}^*(1 \otimes \hat{a})W_{\mathbb{G}}$. Using the decomposition $W_{\mathbb{G}} = \bigoplus_{\pi \in Irr(\widehat{\mathbb{G}}} \sum_{i,j}^{n_\pi} E_{i,j}^\pi \otimes (u_{j,i}^\pi)^*$, we obtain

$$\begin{aligned}\Delta_{\mathbb{G}}^l(\hat{a}) &= \sum_{\pi \in Irr(\widehat{\mathbb{G}})} \sum_{i,j,k,l} E_{i,j}^\pi E_{k,l}^\pi \otimes u_{i,j}^\pi \hat{a}(u_{l,k}^\pi)^* \\ &= \sum_{\pi \in Irr(\widehat{\mathbb{G}})} \sum_{i,l=1}^{n_\pi} E_{i,l}^\pi \otimes \sum_{k=1}^{n_\pi} u_{i,k}^\pi \hat{a}(u_{l,k}^\pi)^* \\ &= \sum_{\pi \in Irr(\widehat{\mathbb{G}})} \sum_{i,j=1}^{n_\pi} E_{i,j}^\pi \otimes L_{i,j}^\pi(\hat{a})\end{aligned}$$

where $L_{i,j}^\pi(\hat{a}) = \sum_{t=1}^{n_\pi} u_{i,t}^\pi \hat{a}(u_{j,t}^\pi)^*$.

The **reduced crossed product** of a $\mathbb{G}$-C^*-algebra A and $\mathbb{G}$ is the closed linear span of $(C_r(\widehat{\mathbb{G}}) \otimes 1)\alpha(A)$ in $M(\mathcal{K}(\ell^2(\mathbb{G})) \otimes_{min} A)$. We denote the reduced crossed product of A with $\mathbb{G}$ by $A \rtimes_r \mathbb{G}$. It turns out that $A \rtimes_r \mathbb{G}$ is a $\mathbb{G}$-C^*-algebra (see [65, Lemma 2.11] and the preceding sections), with coaction $\beta : A \rtimes_r \mathbb{G} \to M(c_0(\mathbb{G}) \otimes_{min} A \rtimes_r \mathbb{G})$ defined by setting $\beta(A) = W_{12}^* A_{23} W_{12}$. The coaction β satisfies

- $\beta|_{C_r(\widehat{\mathbb{G}}) \otimes 1} = \Delta_{\mathbb{G}}^l \otimes \mathrm{id}$;
- $\beta|_{\alpha(A)} = \mathrm{id} \otimes \alpha$.

This makes the canonical embeddings of A and $C_r(\widehat{\mathbb{G}})$ into $A \rtimes_r \mathbb{G}$ $\mathbb{G}$-equivariant.

Given a $\mathbb{G}$-C^*-algebra A and $\mu \in A^*$, the **Poisson transform** of μ is the ucp $\mathbb{G}$-equivariant map $\mathcal{P}_\mu : A \to \ell^\infty(\mathbb{G})$ defined by $\mathcal{P}_\mu(a) = (\mathrm{id} \otimes \mu)\alpha(a)$ for $a \in A$.

Definition 4.2.4. A $\mathbb{G}$-C^*-algebra A is a $\mathbb{G}$**-boundary** if the Poisson transform $\mathcal{P}_\mu$ of every state $\mu \in A^*$ is completely isometric.

We define a partial ordering on the $\mathbb{G}$-equivariant ucp maps $\ell^\infty(\mathbb{G}) \to \ell^\infty(\mathbb{G})$ by setting

$$\phi \leq \psi \text{ if } ||\phi(x)|| \leq ||\psi(x)|| \text{ for all } x \in \ell^\infty(\mathbb{G})$$

for such $\phi, \psi : \ell^\infty(\mathbb{G}) \to \ell^\infty(\mathbb{G})$. There exists minimal elements with respect to this poset. The $\mathbb{G}$-C^*-algebra $\phi(\ell^\infty(\mathbb{G}))$, where ϕ is minimal with respect to the above poset, is a $\mathbb{G}$-boundary that does not depend on the choice of minimal ϕ.

Definition 4.2.5. The **Furstenberg boundary** is the $\mathbb{G}$-boundary $C(\partial_F(\mathbb{G}))$ that is isomorphic to any $\phi(\ell^\infty(\mathbb{G}))$ where ϕ is a minimal $\mathbb{G}$-equivariant ucp map.

In other words, the Furstenberg boundary is constructed out of minimal relatively amenable spaces. It turns out to be the **universal boundary** in the sense that for any $\mathbb{G}$-boundary A, there is a completely isometric ucp $\mathbb{G}$-equivariant embedding $A \to C(\partial_F(\mathbb{G}))$ (see [65, Theorem 4.16]). It also satisfies the following three properties (see [65, Proposition 4.10] and [65, Proposition 4.13]).

Definition 4.2.6. Let A be a $\mathbb{G}$-boundary. We say A is **$\mathbb{G}$-essential** if any ucp $\mathbb{G}$-equivariant ucp $A \to B$ where B is a $\mathbb{G}$-C^*-algebra, is completely isometric. We say A is **$\mathbb{G}$-rigid** if identity map is the unique $\mathbb{G}$-equivariant ucp map $A \to A$. We say A is **$\mathbb{G}$-injective** if for any $\mathbb{G}$-C^*-algebras A and B with ucp $\mathbb{G}$-equivariant ucp maps $\psi : A \to C(\partial_F(\mathbb{G}))$ and $\iota : A \to B$, where ι is completely isometric, there exists a ucp $\mathbb{G}$-equivariant map $\phi : B \to C(\partial_F(\mathbb{G}))$ such that $\psi = \phi \circ \iota$.

Of critical importance to us is the cokernel of the Furstenberg bounday.

Definition 4.2.7. The **cokernel** of $\partial_F(\mathbb{G})$ is the two-sided coideal [65, Proposition 2.9]

$$N_F := \{\mathcal{P}_\mu(a) : a \in C(\partial_F(\mathbb{G})), \mu \in C(\partial_F(\mathbb{G}))^*\}'' \subseteq \ell^\infty(\mathbb{G}).$$

Then there exists a closed quantum subgroup $\mathbb{H}_F$ of $\widehat{\mathbb{G}}$ such that $\ell^\infty(\widehat{\mathbb{H}}_F) = N_F$. We also call $\widehat{\mathbb{H}}_F$ the cokernel of $\partial_F(\mathbb{G})$. We say the action of $\mathbb{G}$ on $\partial_F(\mathbb{G})$ is **faithful** when $\widehat{\mathbb{H}}_F = \mathbb{G}$. We let $P_F = \lambda_{\widehat{\mathbb{G}}}(\omega_F)$ denote the associated group-like projection and idempotent state.

Remark 4.2.8. For a discrete group G, the kernel of the action of G on $\partial_F(G)$ is $R_a(G)$, the amenable radical of G. Then the cokernel is $G/R_a(G)$.

For DQGs, the cokernel of the Furstenberg boundary has a similar structure. The cokernel of $\partial_F(\mathbb{G})$ turns out to be the unique smallest relatively amenable two-sided coideal of $\mathbb{G}$ [65, Theorem 5.1]. Then we have $\ell^\infty(\widehat{\mathbb{H}}_F) \subseteq \ell^\infty(\mathbb{G}/R_a(\mathbb{G}))$, where we recall that $R_a(\mathbb{G})$ is the amenable radical of $\mathbb{G}$. Whether the reverse containment holds or not remains open (see [65, Question 8.3]).

4.3 Amenability and Relative Amenability of Coideals

4.3.1 Annihilator Ideals

We introduce certain ideals and subspaces of the L^1-algebra of a DQG which turn out to be fundamental to amenability and relative amenability of the right coideals (and left and two-sided coideals).

Remark 4.3.1. Before proceeding, we make a technical remark. We obtain an action of $L^1(\mathbb{G})$ on $L^\infty(\mathbb{G})^*$ by taking the adjoint of the action of $L^1(\mathbb{G})$ on $L^\infty(\mathbb{G})$: we set

$$\omega * f(x) := \omega(f * x) = \omega(\mathrm{id} \otimes f)\Delta_{\mathbb{G}}(x), \;\; f \in L^1(\mathbb{G}), \omega \in L^\infty(\mathbb{G})^*, x \in L^\infty(\mathbb{G}).$$

Given von Neumann algebras N and M, it is clear that the slice maps $\varphi \otimes \mathrm{id} : N \overline{\otimes} M \to M$ are defined for normal functionals $\varphi \in N_*$. While less clear, it is the case that slice maps are still defined if we drop normality and additionally satisfy $(\varphi \otimes \mathrm{id})(\mathrm{id} \otimes \Phi) = \Phi(\varphi \otimes \mathrm{id})$ for any normal $*$–homomorphism $\Phi : M \to K$ to another von Neumann algebra K (consult [30] or [92]). Thus we are justified in writing

$$\omega * f(x) = (\omega \otimes f)\Delta_{\mathbb{G}}(x) = f(x * \omega)$$

and similarly for actions on the left.

From now on, we will assume $\mathbb{G}$ is discrete.

For a functional $m \in \ell^\infty(\mathbb{G})^*$ and $x \in \ell^\infty(\mathbb{G})$, we will use the notation

$$Inv_L(m, x) = \{f \in \ell^1(\mathbb{G}) : f * m = f(x)m\}$$

and likewise for $Inv_R(m, x)$ but for normal functionals acting on the right of m. Then we set $Inv(m, x) = Inv_L(m, x) \cap Inv_R(m, x)$. We will denote

$$Ann_L(m) := Inv_L(m, 0)$$

and by $Ann_R(m)$ and $Ann(m)$ we mean the analogous thing. In the following special case, we will simply write $Inv_L(m, 1) = Inv_L(m)$ etc.

Remark 4.3.2. We point out that $Inv_L(m, x)$ is always a closed subspace and is moreover a closed subalgebra whenever x is a **character** ($x \in \ell^\infty(\mathbb{G})$ such that $x^* = x^{-1}$ and $\Delta_{\mathbb{G}}(x) = x \otimes x$). It is also easy to see that $Ann_L(m)$ is a left ideal.

Using our above notation, amenability is this: there exists a state $m \in \ell^\infty(\mathbb{G})^*$ such that $\ell^1(\mathbb{G}) = Inv_L(m)$. It is easily seen that amenability is equivalent to the existence of a state $m \in \ell^\infty(\mathbb{G})^*$ such that

$$\ell^1_0(\mathbb{G}) := \{f \in \ell^1(\mathbb{G}) : f(1) = 0\} = Ann_L(m).$$

We have that $\ell^1_0(\mathbb{G})$ is an ideal of codimension one in $\ell^1(\mathbb{G})$, which means, if there is an invariant state $m \in \ell^\infty(\mathbb{G})^*$, then

$$Inv_L(m) = \ell^1(\mathbb{G}) = \ell^1_0(\mathbb{G}) + \mathbb{C}\epsilon_{\mathbb{G}}.$$

The generalization of this relationship is as follows.

Proposition 4.3.3. *Assume $\mathbb{G}$ is discrete. Let $m \in \ell^\infty(\mathbb{G})^*$ and fix a character $x \in \ell^\infty(\mathbb{G})$ with $m(x) \neq 0$. Then*

1. *$Ann_L(m) + \mathbb{C}\epsilon_{\mathbb{G}} = Inv_L(m, x)$.*
2. *If $m' \in \ell^\infty(\mathbb{G})$ satisfies $m'(x) \neq 0$, then $Inv_L(m, x) = Inv_L(m', x)$ if and only if $Ann_L(m) = Ann_L(m')$.*

Proof. 1. First note $\ker(f \mapsto f(x)) \cap Ann_L(m) = Ann_L(m)$. To see this, notice $f * m = 0$ implies

$$0 = f * m(x) = f(x)m(x)$$

so $f(x) = 0$ because $m(x) \neq 0$. So, if $f \in Ann_L(m)$ and $c \in \mathbb{C}$, then $(f + c\epsilon_{\mathbb{G}})(x) = c$. Then

$$(f + c\epsilon_{\mathbb{G}}) * m = cm = [(f + c\epsilon_{\mathbb{G}})(x)]m.$$

On the other hand, given $f \in Inv_L(m, x)$, if $f(x) = 0$, then $f \in Ann_L(m)$ is automatic, and otherwise, $(f - f(x)\epsilon_{\mathbb{G}}) * m = 0$. So $f = (f - f(x)\epsilon_{\mathbb{G}}) + f(x)\epsilon_{\mathbb{G}}$ is the desired decomposition.

2. For the second claim, assume $Inv_L(m, x) = Inv_L(m', x)$. Then for $f \in Ann_L(m)$ and $c \in \mathbb{C}$, $g = f + c\epsilon_{\mathbb{G}} \in Inv_L(m) = Inv_L(m')$ and in particular, $g(x) = c$. So $f = g - g(x)\epsilon_{\mathbb{G}} \in Ann_L(m')$. Symmetry means $Ann_L(m') = Ann_L(m)$. The converse is clear from the first claim. □

Remark 4.3.4. An immediate consequence of Proposition 4.3.3 is that if $x, y \in \ell^\infty(\mathbb{G})$ are characters satisfying $m(x) \neq 0 \neq m(y)$, then $Inv_L(m, x) = Inv_L(m, y)$. In particular, for $f \in Inv_L(m, x)$ we obtain $f(x)m = f(y)m$ which implies $f(x) = f(y)$.

There is well-known a correspondence between bounded linear right $\ell^1(\mathbb{G})$–module maps $E_\omega : \ell^\infty(\mathbb{G}) \to \ell^\infty(\mathbb{G})$ and functionals $\omega \in \ell^\infty(\mathbb{G})^*$ via the assignment $\omega = \epsilon_{\mathbb{G}} \circ E_\omega$ where

$$E_\omega(x) := \omega * x.$$

We will call ω **right idempotent** if $\omega(\omega * x) = \omega(x)$ for all $x \in \ell^\infty(\mathbb{G})$.

Remark 4.3.5.

1. It is the case that ω is right idempotent exactly when E_ω is idempotent, and likewise for (complete) positivity and unitality. In particular, E_ω is a unital completely positive (ucp) projection exactly when ω is a right idempotent state (see [63, 55] for an account of right $L^1(\mathbb{G})$–module maps in the setting of LCQGs).

2. We also point out that the easy general fact

$$(B + \mathbb{C}\epsilon_{\mathbb{G}})^\perp = \ker(\epsilon_{\mathbb{G}}) \cap B^\perp,$$

where $B \subseteq \ell^1(\mathbb{G})$ is a subset, combined with Proposition 4.3.3 tells us

$$Inv_L(\omega, x)^\perp = \ker(\epsilon_{\mathbb{G}}) \cap Ann_L(\omega)^\perp$$

whenever $x \in \ell^\infty(\mathbb{G})$ is a character such that $\omega(x) \neq 0$.

We will be begin by studying lattice properties of right idempotent states in $\ell^\infty(\mathbb{G})^*$ for a DQG $\mathbb{G}$. The following propositions run parallel to the ideas in [74] but we note the idempotent states considered there lived in $\ell^1(\mathbb{G})$ (and note that in our case, the compact quasi-subgroups of a DQG are finite dimensional (cf. [74])). Nevertheless, our main concern is of the spaces $Ann_L(m)$ and $Inv_L(m)$ which were not considered in [74]. The following is simple enough yet we point it out for convenience.

Proposition 4.3.6. *Assume $\mathbb{G}$ is discrete and let $m, \omega \in \ell^\infty(\mathbb{G})^*$ be right idempotent states. TFAE:*

1. $m(\omega * x) = \omega(x)$ *for all* $x \in \ell^\infty(\mathbb{G})$;
2. $E_m \circ E_\omega = E_\omega$;
3. $\omega * \ell^\infty(\mathbb{G}) \subseteq m * \ell^\infty(\mathbb{G})$;
4. $Ann_L(m) \subseteq Ann_L(\omega)$;
5. $Inv_L(m) \subseteq Inv_L(\omega)$.

Proof. As noted before, the equivalences ($1 \iff 2 \iff 3$) is nothing new (the techniques are identical to those seen in [74]). That we have ($4 \iff 5$) follows similarly to the proof of Proposition 4.3.3. Then ($3 \iff 4$) follows from the more general fact that for any Banach spaces X and Y we have $X \subseteq Y$ if and only if $Y_\perp \subseteq X_\perp$. □

Remark 4.3.7. We note that the equivalence of the conditions $3. \iff 4. \iff 5.$ did not require the idempotence condition. The Furstenberg boundary $\partial_F(\mathbb{G})$ offers an idempotent state m_F that is maximal among the orderings $Ann_L(\omega) \subseteq Ann_L(m_F)$ and $Inv_L(\omega) \subseteq Inv_L(m_F)$.

Of course, we can use approximation arguments.

Lemma 4.3.8. *Let $m \in L^\infty(\mathbb{G})^*$ be a state. There exists a net of states $(\omega_i) \subseteq L^1(\mathbb{G})$ such that*

$$\omega_i * f - f(1)\omega_i \to 0$$

for all $f \in Inv_L(m) \cup Ann_L(m)$.

Proof. The argument follows from the proof of the corresponding statement for left invariant means with minor adjustments (see [35]). To elaborate, the first claim follows from weak density of $\overline{B_1(L^1(\mathbb{G}))_+}$ in $\overline{B_1(L^\infty(\mathbb{G})^*)_+}$, and the second follows from a convexity argument on the space $\prod_{f \in Inv_R(m) \cup Ann_R(m)} L^1(\mathbb{G})$. □

With the above observations in hand, we quickly deduce the following.

Proposition 4.3.9. *Let $\mathbb{G}$ be discrete N a right coideal. TFAE:*

1. *there exists a state $m \in \ell^\infty(\mathbb{G})^*$ such that $J^1(N) \subseteq Ann_L(m)$;*

2. *there exists a state $m \in \ell^\infty(\mathbb{G})^*$ such that $J^1(N) + \mathbb{C}\epsilon_{\mathbb{G}} \subseteq Inv_L(m)$;*

3. *N is relatively amenable.*

Proof. The equivalence ($1 \iff 2$) is due to Proposition 4.3.3, and then that

$$J^1(N) \subseteq Ann_L(m) \iff m * \ell^\infty(\mathbb{G}) = Ann_L(m)^\perp \subseteq N$$

gives us ($1 \iff 3$). □

We achieve the corresponding result for amenability just as easily.

Corollary 4.3.10. *Suppose $\mathbb{G}$ is discrete and N a right coideal. TFAE:*

1. *there exists a right idempotent state $m \in \ell^\infty(\mathbb{G})^*$ such that $J^1(N) = Ann_L(m)$;*
2. *there exists a right idempotent state $m \in \ell^\infty(\mathbb{G})^*$ such that $J^1(N) + \mathbb{C}\epsilon_{\mathbb{G}} = Inv_L(m)$;*
3. *N is amenable in $\mathbb{G}$.*

Proof. ($1 \iff 2$) is accomplished by Proposition 4.3.3 and ($1 \iff 3$) follows because $J^1(N) = Ann_L(m)$ if and only if $m * \ell^\infty(\mathbb{G}) = N$ (cf. Remark 4.3.5). □

4.3.2 Amenable and Relatively Amenable Coideals

We continue to assume $\mathbb{G}$ is a DQG. Recall that the right coideals of $\ell^\infty(\mathbb{G})$ are the open quasi-subgroups of $\mathbb{G}$. Let $\widetilde{N}_P$ be the open quasi-subgroup of $\mathbb{G}$ generated by the group-like projection P. We will establish the role P plays in amenability and relative amenability of $\widetilde{N}_P$ as a right coideal.

The following useful lemma is probably well known to experts, but we provide a proof for convenience.

Lemma 4.3.11. *Assume $\mathbb{G}$ is discrete and P is a group-like projection. Let m be a functional such that $m * \ell^\infty(\mathbb{G}) \subseteq \widetilde{N}_P$. Then $P(m * x) = m(x)P$ for all $x \in \ell^\infty(\mathbb{G})$.*

Proof. First, notice for $x \in \widetilde{N}_P$ that

$$P(\epsilon_{\mathbb{G}}(x)) = \epsilon_{\mathbb{G}}(x) \otimes P = (\epsilon_{\mathbb{G}} \otimes \mathrm{id})(1 \otimes P)\Delta_{\mathbb{G}}(x) = 1 \otimes Px = Px.$$

We point out that above fact appears explicitly in the literature (see the proof of [38, Theorem 3.1]). Now, for $x \in \ell^\infty(\mathbb{G})$, by assumption $m * x \in \widetilde{N}_P$, so,

$$P(m * x) = P\epsilon_{\mathbb{G}}(m * x) = m(x)P.$$

□

As mentioned in the introduction, in classical setting of a discrete group G, amenability of a subgroup $H \leq G$ is equivalent to relative amenability of $\ell^\infty(G/H)$, which, in turn, is equivalent to the existence of an H-invariant state on $\ell^\infty(G)$. The following theorem establishes an analogue of an H-invariant state for coideals of DQGs.

Remark 4.3.12. We denote the canonical predual action of $\ell^\infty(\mathbb{G})$ on $\ell^1(\mathbb{G})$ as follows:

$$xf(y) = f(yx) \text{ and } fx(y) = f(xy),\ x, y \in \ell^\infty(\mathbb{G}),\ f \in \ell^1(\mathbb{G}).$$

Theorem 4.3.13. *Let P be a group-like projection and $\omega \in \ell^\infty(\mathbb{G})^*$. Then $\omega * \ell^\infty(\mathbb{G}) \subseteq \widetilde{N}_P$ if and only if $\ell^1(\mathbb{G})P \subseteq Inv_L(\omega)$.*

Proof. Notice that, given $x \in \ell^\infty(\mathbb{G})$ and $f \in \ell^1(\mathbb{G})$,

$$(fP) * \omega(x) = f(P(\omega * x))$$

so $(fP) * \omega = (fP)\omega = f(P)\omega$ for all $f \in \ell^1(\mathbb{G})$ if and only if

$$(\mathrm{id} \otimes \omega)(P \otimes 1)\Delta_{\mathbb{G}}(x) = P(\omega * x) = P\omega(x). \tag{4.1}$$

So, if we assume $\ell^1(\mathbb{G})P \subseteq Inv_L(\omega)$ then

$$\begin{aligned}(1 \otimes P)\Delta_{\mathbb{G}}(\omega * x) &= (\mathrm{id} \otimes \mathrm{id} \otimes \omega)(1 \otimes P \otimes 1)(\Delta_{\mathbb{G}} \otimes \mathrm{id})\Delta_{\mathbb{G}}(x) \\ &= (\mathrm{id} \otimes \mathrm{id} \otimes \omega)(1 \otimes P \otimes 1)(\mathrm{id} \otimes \Delta_{\mathbb{G}})\Delta_{\mathbb{G}}(x) \\ &= (1 \otimes P)(\mathrm{id} \otimes \omega)\Delta_{\mathbb{G}}(x) \text{ (using (4.1))} \\ &= (x * \omega) \otimes P.\end{aligned}$$

Conversely,

$$(\omega * \ell^\infty(\mathbb{G})) \cap \{x \in \ell^\infty(\mathbb{G}) : Px = 0\} = (\omega * \ell^\infty(\mathbb{G})) \cap \ker \epsilon_{\mathbb{G}} = Inv_L(\omega)^\perp,$$

where the first equality clearly follows from the more general fact $Px = \epsilon_{\mathbb{G}}(x)P$, for $x \in \widetilde{N}_P$, as pointed out in Lemma 4.3.11, and the second was pointed out in Remark 4.3.5. We have that $x \in \ell^\infty(\mathbb{G})$ satisfies $0 = (fP)(x) = f(Px)$ for all $f \in \ell^1(\mathbb{G})$ if and only if $Px = 0$. So

$$(\ell^1(\mathbb{G})P)^\perp = \{x \in \ell^\infty(\mathbb{G}) : Px = 0\}.$$

Hence $Inv_L(\omega)^\perp \subseteq (\ell^1(\mathbb{G})P)^\perp$, which implies $\ell^1(\mathbb{G})P \subseteq Inv_L(\omega)$. □

Notice that the above claims follow through if we replace $\ell^1(\mathbb{G})P$ with $P\ell^1(\mathbb{G})$. With this result in hand, we formulate the following notion of left invariance.

Definition 4.3.14. Let P be a group-like projection. We say $\omega \in \ell^\infty(\mathbb{G})^*$ is **P-left invariant** if either $fP * \omega = f(P)\omega$ or $Pf * \omega = f(P)\omega$ holds for every $f \in \ell^1(\mathbb{G})$.

In particular, $\widetilde{N}_P$ is relatively amenable if and only if there exists a P-left invariant state.

Remark 4.3.15. 1. Take $f, g \in \ell^1(\mathbb{G})$. An easy calculation shows $(fP) * (gP) = ((fP) * g)P$, which means $\ell^1(\mathbb{G})P$ is a closed subalgebra of $\ell^1(\mathbb{G})$.

2. The algebra $\ell^1(\mathbb{G})P$ was studied in [44] for the dual of a compact group $\widehat{G}$. In that setting, $\ell^1(\widehat{G})P = A(G/K)$, which is the Fourier algebra of the coset space G/K for a closed subgroup K.

Given a group–like projection P, we will denote the weak$*$ closed right invariant subspaces

$$M_P := \{x \in \ell^\infty(\mathbb{G}) : (1 \otimes P)\Delta_{\mathbb{G}}(x)(1 \otimes P) = x \otimes P\} \supseteq \widetilde{N}_P.$$

We will also use the notation

$$J^1(M_P) := (M_P)_\perp.$$

These subspaces allow us to establish a relationship between amenability and P-invariant states on $\ell^\infty(\mathbb{G})$. The key property is that $x \mapsto PxP$ is a positive map so that states that are conjugated by P remain positive. This will be indispensible when we relate amenability of M_P with brais on $J^1(M_P)$.

Remark 4.3.16. We have been unable to determine whether not we generally have $M_P = \widetilde{N}_P$. If $\mathbb{H}$ is a closed quantum subgroup of $\mathbb{G}$, since $1_{\mathbb{H}}$ is central, we have $\ell^\infty(\mathbb{G}/\mathbb{H}) = \widetilde{N}_{1_{\mathbb{H}}} = M_{1_{\mathbb{H}}}$.

Theorem 4.3.17. *Assume $\mathbb{G}$ is discrete and $0 \neq P$ is a group–like projection. Then M_P is amenable in $\mathbb{G}$ if and only if M_P is relatively amenable via a state $m \in \ell^\infty(\mathbb{G})^*$ such that $m(P) \neq 0$ (and $m * \ell^\infty(\mathbb{G}) \subseteq M_P$).*

Proof. First assume M_P is amenable with right idempotent state $m \in \ell^\infty(\mathbb{G})^*$ such that $\widetilde{N}_P = m * \ell^\infty(\mathbb{G})$. Assume for a contradiction that $m(P) = 0$. Since P is group-like and generates M_P, $P \in M_P$, and so $m * P = P$ by assumption. But then

$$P = P(m * P) = m(P)P = 0 \text{ (Lemma 4.3.11)}.$$

which contradicts our assumption.

Now we will prove the converse. We will first see that PmP is a right idempotent state. Since $x \mapsto PxP$ is positive, PmP is a positive functional and since $\frac{1}{m(P)}(PmP)$ is unital, it is a state. For right idempotency, take $x \in \ell^\infty(\mathbb{G})$. Then,

$$\begin{aligned}(\frac{1}{m(P)}(PmP))\left((\frac{1}{m(P)}(PmP)) * x\right) &= \frac{1}{m(P)}m[(\mathrm{id}\otimes\frac{1}{m(P)}m)(P\otimes P)\Delta_{\mathbb{G}}(x)(P\otimes P)] \\ &= \frac{1}{m(P)^2}m[(1\otimes P)(\mathrm{id}\otimes m)\Delta_{\mathbb{G}}(PxP)(1\otimes P)] \\ &= \frac{1}{m(P)^2}\overbrace{m[P(m*(PxP))P]}^{=m(Pm(PxP)P)\ \text{(Lemma 4.3.11)}} \\ &= \frac{1}{m(P)}m(PxP).\end{aligned}$$

For the remainder of the proof we will show PmP satisfies $(PmP) * \ell^\infty(\mathbb{G}) = M_P$, where we replace m with $\frac{1}{m(P)}m$, (so $PmP(1) = 1$). Note that Lemma 4.3.11 still applies to m after scaling. First, take $x \in M_P$. Then

$$PmP * x = (\mathrm{id}\otimes m)(1\otimes P)\Delta_{\mathbb{G}}(x)(1\otimes P) = x,$$

shows $M_P \subseteq (PmP) * \ell^\infty(\mathbb{G})$. On the other hand, for $x \in \ell^\infty(\mathbb{G})$,

$$\begin{aligned}&(1\otimes P)\Delta_{\mathbb{G}}((PmP)*x)(1\otimes P)\\ &= (\mathrm{id}\otimes\mathrm{id}\otimes PmP)(1\otimes P\otimes 1)(\Delta_{\mathbb{G}}\otimes\mathrm{id})\Delta(x)(1\otimes P\otimes 1)\\ &= (\mathrm{id}\otimes\mathrm{id}\otimes m)(1\otimes P\otimes P)[(\mathrm{id}\otimes\Delta_{\mathbb{G}})\Delta(x)](1\otimes P\otimes P)\\ &= (\mathrm{id}\otimes\mathrm{id}\otimes m)(1\otimes P\otimes 1)[(\mathrm{id}\otimes\Delta_{\mathbb{G}})(1\otimes P)\Delta(x)(1\otimes P)](1\otimes P\otimes 1)\ \text{(group-likeness)}\\ &= (1\otimes P\otimes 1)(\mathrm{id}\otimes\mathrm{id}\otimes m)(\mathrm{id}\otimes\Delta_{\mathbb{G}})[(1\otimes P)\Delta(x)(1\otimes P)](1\otimes P\otimes 1)\\ &= (\mathrm{id}\otimes m\otimes\mathrm{id})(1\otimes P\otimes 1)(\Delta_{\mathbb{G}}(x)\otimes P)\ \text{(Lemma 4.3.11)}\\ &= (PmP)*x\otimes P.\end{aligned}$$

We conclude that $(PmP) * x \in M_P$. □

Proposition 4.3.18. *Let P be a group–like projection and $\omega \in \ell^\infty(\mathbb{G})^*$. Then $\omega * \ell^\infty(\mathbb{G}) \subseteq M_P$ if and only if $P\ell^1(\mathbb{G})P \subseteq Inv_L(\omega)$.*

Proof. The proof is completely analogous to the proof of Theorem 4.3.13. Indeed, we have $P\ell^1(\mathbb{G})P \subseteq Inv_L(\omega)$ if and only if $P(\omega * x)P = P\omega(x)$. Then, if $P\ell^1(\mathbb{G})P \subseteq Inv_L(\omega)$, it is readily checked that $(1\otimes P)\Delta_{\mathbb{G}}(\omega * x)(1\otimes P) = \omega * x \otimes P$ and that $Inv_L(\omega)^\perp = \{(\omega * \ell^\infty(\mathbb{G})) \cap \{x \in \ell^\infty(\mathbb{G}) : PxP = 0\}$ as in the proof of Theorem 4.3.13 (indeed, it can be checked that $Px = P\epsilon_{\mathbb{G}}(x)$ for every $x \in M_P$). Conversely, $f(PxP) = 0$ for every $f \in \ell^1(\mathbb{G})$ if and only if $PxP = 0$, so $(P\ell^1(\mathbb{G})P)^\perp = \{x \in \ell^\infty(\mathbb{G}) : PxP = 0\}$. Hence, $Inv_L(\omega)^\perp \subseteq \{x \in \ell^\infty(\mathbb{G}) : PxP\}$. □

As a consequence, we obtain the following characterization of amenability of M_P.

Corollary 4.3.19. *Let P be a group-like projection. We have that M_P is amenable if and only if there exists a state $m : \ell^\infty(\mathbb{G}) \to \mathbb{C}$ such that $m(P) \neq 0$ and $P\ell^1(\mathbb{G}) \subseteq Inv_L(\omega)$.*

Proof. This is a straightforward application of Proposition 4.3.18 and Theorem 4.3.17 □

Now we characterize amenability of M_P in terms of the existence of brais for $J^1(M_P)$.

Theorem 4.3.20. *Assume $\mathbb{G}$ is discrete and P is a group–like projection. TFAE:*

1. *M_P is amenable;*
2. *$J^1(M_P)$ admits a brai;*
3. *$J^1(M_P)$ admits a brai in $\{P\}_\perp$.*

By 3., we mean that there exists a bounded net $(e_i) \subseteq \{P\}_\perp$ such that $f * e_j \to f$ for every $f \in J^1(M_P)$.

Proof. ($1 \implies 2$) It is a standard argument for a Banach algebra A that admits a bai (in particular, if it is unital), that a closed left ideal I admits a brai if and only if there exists a right A-module projection $A \to I^\perp$ (see [97] and [17, Theorem 7]). We apply this argument to the left ideal $J^1(M_P) \subseteq \ell^1(\mathbb{G})$.

($2 \implies 3$) follows because $J^1(M_P) \subseteq \{P\}_\perp$. What remains is showing ($3 \implies 1$). To this end, let $(e_i) \subseteq \{P\}_\perp$ be a brai for $J^1(M_P)$. Let e be a weak$*$ cluster point and set $\omega = \epsilon_\mathbb{G} - e$. Notice that for all $f \in J^1(M_P)$, $f * \omega = f * \epsilon_\mathbb{G} - f * e = f - f = 0$, i.e., $Ann_L(\omega) \supseteq J^1(M_P)$, which implies $M_P \supseteq \omega * \ell^\infty(\mathbb{G})$. Notice also that $\omega(P) = 1 \neq 0$.

Using Theorem 4.3.13, we have

$$P\ell^1(\mathbb{G})P \subseteq Inv_L(\omega).$$

From here, we point out that $P\ell^1(\mathbb{G})P$ is spanned by states since the map $f \mapsto PfP$ preserves positive elements. So, take a state $f \in P\ell^1(\mathbb{G})P$. We can assume ω is Hermitian since the decomposition $\omega = \Re(\omega) + \imath\Im(\omega)$ is unique and we must have $\Re(\omega)(P) \neq 0$ or $\Im(\omega)(P) \neq 0$. Now let $\omega = \omega_+ - \omega_-$ be the Jordan decomposition, uniquely determined so that $||\omega_+|| + ||\omega_-|| = ||\omega||$ (cf. [118, Theorem 4.2]). Then, since $f * \omega_+$ is positive,

$$||f * \omega_+|| = (f \otimes \omega_+)\Delta_\mathbb{G}(1) = \omega_+(1) = ||\omega_+||$$

and similarly $||f * \omega_-|| = ||\omega_-||$. So, by uniqueness, we must have $f * \omega_+ = \omega_+$ and $f * \omega_- = \omega_-$. Without loss of generality, suppose $\omega_+(P) \neq 0$, so we denote $m = \omega_+$, and what we have shown is $P\ell^1(\mathbb{G})P \subseteq Inv_L(m)$. From here, Proposition 4.3.18 and Theorem 4.3.19 give us amenability of M_P. □

We can use Theorem 4.3.17 to generalize the statement that a discrete group is amenable if and only if a subgroup and its quotient are amenable. First, we formulate the following definition found in [47] for quotients of DQGs and later extended to quotients of LCQGs in [21], for left coideals.

Definition 4.3.21. We say a right coideal N **acts amenably on** $\mathbb{G}$ if there exists a state $m \in N^*$ such that $(\mathrm{id} \otimes m)\Delta_{\mathbb{G}}|_N(x) = m(x)$ for all $x \in N$. We will call such m a **left invariant state** on N.

The following was originally shown by Crann [21] in the context of LCQGs with the right coideal being of quotient type. An analogous statement was also shown for DQGs acting on von Neumann algebras (with amenability of the right coideal replaced with amenability of the action (cf. [89])). The idea of the proof is similar here too, but simplified in the present setting. We provide the proof for convenience.

Corollary 4.3.22. *Let $\mathbb{G}$ be a DQG and P a group–like projection. Then $\mathbb{G}$ is amenable if and only if $\widetilde{N}_P$ is amenable and acts amenably on $\mathbb{G}$.*

Proof. If we assume $\mathbb{G}$ is amenable, then Theorem 4.3.17 tells us $\widetilde{N}_P$ is amenable. Furthermore, if we let $m \in \ell^\infty(\mathbb{G})$ be a left invariant state, then $m|_{\widetilde{N}_P}$ is a left invariant state on $\widetilde{N}_P$.

Conversely, let m be a right invariant state on $\widetilde{N}_P$ and let $E : \ell^\infty(\mathbb{G}) \to \widetilde{N}_P$ be the associated ucp projection. Recall that right $\ell^1(\mathbb{G})$-modularity of a ucp map $F : \ell^\infty(\mathbb{G}) \to \ell^\infty(\mathbb{G})$ is equivalent to $\mathbb{G}$-equivariance (cf. [112]): $(\mathrm{id} \otimes F)\Delta_{\mathbb{G}} = \Delta_{\mathbb{G}} \circ F$. Then for $x \in \ell^\infty(\mathbb{G})$,

$$\begin{aligned}(\mathrm{id} \otimes m \circ E)\Delta_{\mathbb{G}}(x) &= (\mathrm{id} \otimes m)\Delta_{\mathbb{G}}|_{\widetilde{N}_P}(E(x)) \text{ (equivariance)}\\ &= m(E(x))\end{aligned}$$

by definition. □

4.4 Amenability and Coamenability of Coideals

4.4.1 $L^1(\widehat{\mathbb{G}})$-submodules in Compact Quantum Groups

For a LCQG $\mathbb{G}$, recall that the **unitary antipode** is the $*$-antiautomorphism $R_{\mathbb{G}} : L^\infty(\mathbb{G}) \to L^\infty(\mathbb{G})$ defined by setting $R_{\mathbb{G}} = Jx^*J$, where $J : L^2(\mathbb{G}) \to L^2(\mathbb{G})$ is the modular conjugation for h_L. For a CQG $\widehat{\mathbb{G}}$ and $\pi \in Irr(\widehat{\mathbb{G}})$, the unitary antipode satisfies

$$(R_{\widehat{\mathbb{G}}} \otimes \mathrm{id})U^\pi = (1 \otimes F_\pi^{1/2})(U^\pi)^*(1 \otimes F_\pi^{-1/2}). \tag{4.2}$$

In general, for locally compact $\mathbb{G}$, the unitary antipode satisfies

$$(R_{\mathbb{G}} \otimes R_{\mathbb{G}})\Delta_{\mathbb{G}} = \Sigma\Delta_{\mathbb{G}} \circ R_{\mathbb{G}}$$

and so it is straightforward to see that if N is a right coideal, then $R_{\mathbb{G}}(N)$ is a left coideal.

Let $\mathbb{G}$ be a DQG. For each $\pi \in Irr(\widehat{\mathbb{G}})$, let $P_\pi \in M_{n_\pi}$ be the orthogonal projection onto $E_\pi \subseteq \mathcal{H}_\pi$. We will denote $P_E = \oplus_{\pi\in Irr(\widehat{\mathbb{G}})} P_\pi \in \ell^\infty(\mathbb{G})$. Now, $\ell^\infty(\mathbb{G})P_E$ is a weak$*$ closed right ideal in $\ell^\infty(\mathbb{G})$. Conversely, for any weak$*$ closed right ideal I in $\ell^\infty(\mathbb{G})$, there is an orthogonal projection $P = \oplus_{\pi\in Irr(\widehat{\mathbb{G}})} P_\pi \in \ell^\infty(\mathbb{G})$ such that $I = \ell^\infty(\mathbb{G})P$. Then

$$E = (P\mathcal{H}_\pi)_{\pi\in Irr(\widehat{\mathbb{G}})} = (P_\pi\mathcal{H}_\pi)_{\pi\in Irr(\widehat{\mathbb{G}})}$$

is a closed quantum subset of $\widehat{\mathbb{G}}$. So, we have a one-to-one correspondence between closed quantum subsets of $\widehat{\mathbb{G}}$, orthogonal projections in $\ell^\infty(\mathbb{G})$, and weak$*$ closed left ideals in $\ell^\infty(\mathbb{G})$. They may also be detected as follows (see the analogous result for coideals of LCQGs [72, Proposition 1.5]).

Finally, we note that if E is a closed quantum subset, then

$$R_{\widehat{\mathbb{G}}}(\mathrm{Pol}(\widehat{E})) = \mathrm{span}\{u^\pi_{\xi,\eta} : \xi \in P_E\mathcal{H}_\pi, \eta \in \mathcal{H}_\pi, \pi \in Irr(\widehat{\mathbb{G}})\}$$

is left $L^1(\widehat{\mathbb{G}})$-invariant. Given a left or right coideal $N \subseteq L^\infty(\widehat{\mathbb{G}})$, we will let $L^2(N)$ be the closure of the elements of N inside $L^2(\widehat{\mathbb{G}})$. Note, also, that $P_E \in \ell^\infty(\mathbb{G}) \subseteq \mathcal{B}(\ell^2(\mathbb{G})) \cong \mathcal{B}(L^2(\widehat{\mathbb{G}}))$ since $\ell^2(\mathbb{G}) \cong L^2(\widehat{\mathbb{G}})$.

Proposition 4.4.1. *The following hold:*

1. *P_E is the orthogonal projection onto the left coideal $L^2(R_{\widehat{\mathbb{G}}}(L^\infty(\widehat{E})))$;*

2. $\ell^\infty(\mathbb{G})(1-P_E) = \overline{\lambda_{\widehat{\mathbb{G}}}(j(E))}^{wk*}$*;*

3. $\overline{\lambda_{\mathbb{G}}(\ell^1(\mathbb{G})P_E)}^{wk*} = X^*(E)$ *where* $X^*(E) = \{x^* : x \in L^\infty(\widehat{E})\}$*;*

4. $\overline{\lambda_{\mathbb{G}}(P_E\ell^1(\mathbb{G}))}^{wk*} = R_{\widehat{\mathbb{G}}}(L^\infty(\widehat{E}))$.

Proof. 1. We refer the reader to [130, Section 2.1]. For each $\pi \in Irr(\widehat{\mathbb{G}})$ fix an ONB $\{e_i^\pi\}$ so that F_π is diagonal. It was established with [130, Proposition 2.1.2] that the Fourier transform

$$\mathcal{F} : L^2(\widehat{\mathbb{G}}) \to \ell^2(\mathbb{G}),\ \eta_{\widehat{\mathbb{G}}}(x) \mapsto \eta_{\mathbb{G}}(\lambda_{\widehat{\mathbb{G}}}(\hat{x})),$$

where $\widehat{x}(y) = h(x^*y)$, is a unitary operator, and furthermore, the elements $E_{i,j}^\pi$ are identified with the elements $tr(F_\pi)(F_\pi)_{i,i}^{-1}u_{i,j}^\pi$. With [130, Proposition 2.1.2], one obtains the decomposition $\ell^2(\mathbb{G}) \cong \ell^2 - \bigoplus_{\pi\in Irr(\widehat{\mathbb{G}})} S_2(\mathcal{H}_\pi)$.

Recall that

$$M_{n_\pi} \cong C_{n_\pi} \otimes_{min} R_{n_\pi},\ E_{i,j}^\pi \mapsto e_i^\pi \otimes e_j^\pi$$

where C_{n_π} and R_{n_π} are the column and row Hilbert spaces on $\mathcal{H}_\pi$, and each R_{n_π} is invariant with respect to the left regular representation of $\widehat{\mathbb{G}}$. The latter implies $P_E\ell^2(\mathbb{G}) = \oplus_{\pi\in Irr(\widehat{\mathbb{G}})} P_E\mathcal{H}_\pi$. So, if we let $x_\pi = \sum_{i,j}^{n_\pi} c_{i,j}^\pi E_{i,j}^\pi \in M_{n_\pi}$, then $\eta_{\mathbb{G}}(x_\pi) \in P_E\ell^2(\mathbb{G})$ if and only if $\eta = \sum_{i=1}^{n_\pi} c_{i,j}^\pi e_i^\pi \in P_E\mathcal{H}_\pi$, which equivalently says $\eta \in P_E\ell^2(\mathbb{G})$ if and only if $u_{\xi,\eta}^\pi \in R_{\mathbb{G}}(\mathrm{Pol}(\widehat{E}))$ for arbitrary $\eta \in \mathcal{H}_\pi$. We deduce that $P_E\ell^2(\mathbb{G}) = L^2(R_{\mathbb{G}}(L^\infty(\widehat{E})))$ using the above identification between $E_{i,j}^\pi$ and $tr(F_\pi)(F_\pi)_{i,i}^{-1}u_{i,j}^\pi$.

2. We established in Chapter 3 that for any $\pi \in Irr(\widehat{\mathbb{G}})$

$$\pi(j(E)) = \{A \in M_{n_\pi} : A(P_E\mathcal{H}_\pi) = 0\} = M_{n_\pi}(1-P_E).$$

Then

$$\lambda_{\widehat{\mathbb{G}}}(j(E)) = c_{00}(\mathbb{G})(1-P_E),$$

and the rest follows from weak$*$ density of $c_{00}(\mathbb{G})$ in $\ell^\infty(\mathbb{G})$.

3. For each $\pi \in Irr(\widehat{\mathbb{G}})$ choose an ONB so that P_E is diagonal. So,

$$\delta_{i,j}^\pi P_E = \begin{cases} \delta_{i,j}^\pi & \text{if } E_{i,j} \in P_E\ell^\infty(\mathbb{G}) \\ 0 & \text{otherwise} \end{cases} = \begin{cases} \delta_{i,j}^\pi & \text{if } e_i^\pi \in P_E\mathcal{H}_\pi \\ 0 & \text{otherwise} \end{cases}$$

and $\lambda_{\mathbb{G}}(\delta_{i,j}^\pi) = (u_{j,i}^\pi)^*$ So, $\lambda_{\mathbb{G}}(\ell_F^1(\mathbb{G})P_E) = (\mathrm{Pol}(\widehat{E}))^*$ and the rest is clear.

4. This follows from a similar argument to 3. and by using (4.2) (cf. Section 4.1). $\square$

Remark 4.4.2. First note that if N is a right coideal, then $N = L^\infty(\widehat{E})$. Indeed, it follows from the work in [72] that the orthogonal projection P onto $L^2(R_{\widehat{\mathbb{G}}}(N))$ is the associated group-like projection for $R_{\widehat{\mathbb{G}}}(N)$. Since $P \in \ell^\infty(\mathbb{G})$, it must be the case that $P = P_E$, and then from 4. of Proposition 4.4.1 and [72, Proposition 1.5] we deduce that $R_{\widehat{\mathbb{G}}}(N) = R_{\widehat{\mathbb{G}}}(L^\infty(\widehat{E}))$ (note that in [72] the right regular representation is used but we are using the left regular representation, and hence the corresponding results in [72] are on right coideals whereas ours are on left coideals).

If $L^\infty(\widehat{E})$ is a right coideal and $\widetilde{N}_P \subseteq \ell^\infty(\mathbb{G})$ is the codual, Proposition 4.4.1 1. tells us $P_E = R_{\mathbb{G}}(P)$. Indeed, the orthogonal projection onto $L^2(R_{\widehat{\mathbb{G}}}(L^\infty(\widehat{E}))$ is a group-like projection that generates the *left* coideal $R_{\mathbb{G}}(\widetilde{N}_P)$ (see [72]).

This means we should be able to glean information from $L^\infty(\widehat{E})$ using the projection P_E. For instance, the right coideals are in 1-1 one correspondence with the group-like projections.

Proposition 4.4.3. *[72, Proposition 1.5] We have that $L^\infty(\widehat{E})$ is a right coideal if and only if P_E is group-like.*

Our next result is concerned about two-sidedness of invariant subspaces. It is something that is well-known for coideals.

Proposition 4.4.4. *Let $\mathbb{G}$ be a DQG and E a closed quantum subset. We have that $L^\infty(\widehat{E})$ is invariant (two-sided) if and only if P_E is central.*

Proof. If P_E is central, then we have $P_E\mathcal{H}_\pi = \mathcal{H}_\pi$ or $\{0\}$. Consequently, it follows by definition of $\mathrm{Pol}(\widehat{E})$ that if $u^\pi_{i,j} \in \mathrm{Pol}(\widehat{E})$ for any i, j, then $u^\pi_{i,j} \in \mathrm{Pol}(\widehat{E})$ for every i, j. It is then clear that $\Delta_{\widehat{\mathbb{G}}}(u^\pi_{i,j}) \in \mathrm{Pol}(\widehat{E}) \otimes \mathrm{Pol}(\widehat{E})$. So $\Delta_{\widehat{\mathbb{G}}}(a) \in \mathrm{Pol}(\widehat{E}) \otimes \mathrm{Pol}(\widehat{E})$ for every $a \in \mathrm{Pol}(\widehat{E})$. By weak$*$ density, we conclude that $\Delta_{\widehat{\mathbb{G}}}(L^\infty(\widehat{E})) \subseteq L^\infty(\widehat{E})\overline{\otimes}L^\infty(\widehat{E})$.

Conversely, if $L^\infty(\widehat{E})$ is two-sided, using linear independence of the sets $\{u^\pi_{i,j_0} : 1 \leq i \leq n_\pi\}$ and $\{u^\pi_{i_0,j} : 1 \leq j \leq n_\pi\}$ for fixed i_0 and j_0, and the fact

$$\Delta_{\widehat{\mathbb{G}}}(u^\pi_{i,j}) = \sum_{t=1}^{n_\pi} u^\pi_{i,t} \otimes u^\pi_{t,j} \in \mathrm{Pol}(\widehat{E}) \otimes \mathrm{Pol}(\widehat{E})$$

it follows that if $u^\pi_{i,j} \in \mathrm{Pol}(\widehat{E})$, then $u^\pi_{i,j} \in \mathrm{Pol}(\widehat{E})$ for every i, j. Consider $P = \oplus_{\pi \in Irr(\widehat{\mathbb{G}})} P_\pi$ where $P_\pi = I_{n_\pi}$ if $u^\pi_{i,j} \in \mathrm{Pol}(\widehat{E})$ and 0 otherwise. Then

$$\mathrm{Pol}(\widehat{E}) = \{u^\pi_{\xi,\eta} : \xi, \eta \in P\mathcal{H}_\pi, \pi \in Irr(\widehat{\mathbb{G}})\}.$$

□

The correspondence of quantum subgroups of DQGs with central group-like projections combined with Proposition 4.4.4 achieves the following well-known result.

Corollary 4.4.5. *Let $\mathbb{G}$ be a DQG. A right coideal $L^\infty(\widehat{E}) \subseteq L^\infty(\widehat{\mathbb{G}})$ is invariant if and only if $L^\infty(\widehat{E}) = L^\infty(\widehat{\mathbb{H}})$ for a closed quantum subgroup $\mathbb{H}$ of $\mathbb{G}$.*

4.4.2 Kac Property for Compact Quantum Groups

It was shown by Daws [27] that a set of representatives in $Irr(\widehat{\mathbb{G}})$ may be chosen so that the F-matrices are diagonal, which is something we sometimes do, but not always. In this case, given $F_\pi = diag(\lambda_1, \ldots, \lambda_{n_\pi})$, Schur's orthogonality is realized as the formulas:

$$h_{\widehat{\mathbb{G}}}((u^\pi_{i,j})^* u^\sigma_{k,l}) = \delta_{\pi,\sigma}\delta_{i,k}\delta_{j,l}\frac{\lambda_i^{-1}}{tr(F_\pi)} \text{ and } h_{\widehat{\mathbb{G}}}((u^\pi_{i,j})^* u^\sigma_{k,l}) = \delta_{\pi,\sigma}\delta_{i,k}\delta_{j,l}\frac{\lambda_j}{tr(F_\pi)}.$$

Definition 4.4.6. A DQG $\mathbb{G}$ is **unimodular** if $h_L = h_R$. We say $\widehat{\mathbb{G}}$ is **Kac** if $h_{\widehat{\mathbb{G}}}$ is a tracial state.

Unimodularity of $\mathbb{G}$ is well-known to be equivalent to Kacness of $\widehat{\mathbb{G}}$. There is the further well-known characterization (see, for example [27]).

Theorem 4.4.7. *Let $\mathbb{G}$ be a DQG. The following are equivalent:*

1. *$\mathbb{G}$ is unimodular;*
2. *$\widehat{\mathbb{G}}$ is Kac;*
3. *every $\pi \in Irr(\widehat{\mathbb{G}})$ has $F_\pi = I_{n_\pi}$;*
4. *$((U^\pi)^t)^{-1} = \overline{U^\pi}$.*

4.4.3 Coamenable Compact Quasi-Subgroups

Recall that the projection $P = \lambda_{\widehat{\mathbb{G}}}(\omega)$ is group-like, where $\omega : C_u(\widehat{\mathbb{G}}) \to \mathbb{C}$ is an idempotent state. For this chapter we say the right coideals N_ω and $\widetilde{N}_P$ are **codual** coideals. We sometimes use the notation $\widetilde{N_\omega} = \widetilde{N}_P$ and $\widetilde{\widetilde{N}_P} = N_\omega$. In the particular case where $\widehat{\mathbb{H}}$ is a closed quantum subgroup of $\widehat{\mathbb{G}}$, $\ell^\infty(\widehat{\mathbb{H}}) = L^\infty(\mathbb{G}/\mathbb{H})$.

Remark 4.4.8. Our terminology is not faithful to the literature nor our definitions in the preliminaries of this thesis. The codual of a right coideal N of a LCQG $\mathbb{G}$ is typically defined to be the *left* coideal $N' \cap L^\infty(\widehat{\mathbb{G}})$. For discrete $\mathbb{G}$, it turns out that

$$\widetilde{N}_P = R_{\mathbb{G}}(N_\omega' \cap \ell^\infty(\mathbb{G})) \text{ [123, Lemma 2.6].}$$

Using the formulas for ω on the matrix coefficients (Lemma 3.3.5) and the decomposition of the left regular representation into irreducibles, it is straightforward checking that we have

$$(R_\omega \otimes \mathrm{id})W_{\widehat{\mathbb{G}}} = W_{\widehat{\mathbb{G}}}(1 \otimes P).$$

See [74] for an account of compact quasi-subgroups at the level of LCQGs.

Given an idempotent state $\omega \in M^u(\widehat{\mathbb{G}})$, we let

$$R^u_\omega = (\mathrm{id} \otimes \omega)\Delta^u_{\widehat{\mathbb{G}}} : C_u(\widehat{\mathbb{G}}) \to C_u(\widehat{\mathbb{G}})$$

denote the universal version, and

$$R^r_\omega = R_\omega|_{C_r(\widehat{\mathbb{G}})} : C_r(\widehat{\mathbb{G}}) \to C_r(\widehat{\mathbb{G}})$$

denote the reduced version. Likewise with L^u_ω and L^r_ω. It turns out that $\Gamma_{\widehat{\mathbb{G}}} \circ R^u_\omega = R^r_\omega \circ \Gamma_{\widehat{\mathbb{G}}}$.

We first recount what was established in Chapter 3 (see also [45, Section 2]). Let N_ω be a compact quasi-subgroup of $L^\infty(\widehat{\mathbb{G}})$, so $N_\omega = R_\omega(L^\infty(\widehat{\mathbb{G}})) = X(E_\omega)$ for some idempotent state $\omega \in M^u(\widehat{\mathbb{G}})$ and hull E_ω. We have that $P_{E_\omega} = \lambda_{\widehat{\mathbb{G}}}(\omega)$. Then

$$\mathrm{Pol}(\widehat{E}_\omega) = R_\omega(\mathrm{Pol}(\widehat{\mathbb{G}})).$$

Now, let

$$C_u(\widehat{E}_\omega) = \overline{\mathrm{Pol}(\widehat{E}_\omega)}^{||\cdot||_u} \subseteq C_u(\widehat{\mathbb{G}}),$$

and

$$C_r(\widehat{E}_\omega) = \Gamma_{\widehat{\mathbb{G}}}(C_u(\widehat{E}_\omega)).$$

Note, then, that it follows that

$$C_r(\widehat{E}_\omega) = R^r_\omega(C_r(\widehat{\mathbb{G}})) \text{ and } C_u(\widehat{E}_\omega) = R^u_\omega(C_u(\widehat{\mathbb{G}})).$$

We will also set $M^r(\widehat{E}_\omega) = C_r(\widehat{E}_\omega)^*$ and $M^u(\widehat{E}_\omega) = C_u(\widehat{E}_\omega)^*$.

Definition 4.4.9. Let $\mathbb{G}$ be a discrete quantum group and N_ω a compact quasi-subgroup of $\widehat{\mathbb{G}}$. We say N_ω is **coamenable** if there exists a state $\epsilon_N \in M^r(\widehat{E}_\omega)$ such that

$$\epsilon_N \circ \Gamma_{\widehat{\mathbb{G}}}|_{C_u(\widehat{E}_\omega)} = \epsilon^u_{\widehat{\mathbb{G}}}|_{C_u(\widehat{E}_\omega)}.$$

Suppose N_ω is coamenable, with associated state ϵ_N. A consequence of coameability of N_ω is that

$$(u \otimes \epsilon_N)\Delta_{\widehat{\mathbb{G}}}(x) = u(x)$$

for all $x \in N_\omega$ and $u \in L^1(\widehat{\mathbb{G}})$, or,

$$u * (\epsilon_N \circ R_\omega) = (u * \epsilon_N) \circ R_\omega = u \circ R_\omega$$

for all $u \in L^1(\widehat{\mathbb{G}})$.

Proposition 4.4.10. *N_ω is coamenable if and only if there exists a state $\epsilon^r_N \in M^r(\widehat{\mathbb{G}})$ such that $\epsilon^r_N \circ \Gamma_{\widehat{\mathbb{G}}}|_{C_u(\widehat{E}_\omega)} = \epsilon^u_{\widehat{\mathbb{G}}}|_{C_u(\widehat{E}_\omega)}$.*

Proof. Suppose N_ω is coamenable with state $\epsilon_N \in M^r(\widehat{E}_\omega)$ as in the definition. Then $\epsilon_N \circ R^r_\omega \in M^r(\widehat{\mathbb{G}})$ is a state, and we have

$$\epsilon_N \circ R^r_\omega \circ \Gamma_{\widehat{\mathbb{G}}}|_{C_u(\widehat{E}_\omega)} = \epsilon_N \circ \Gamma_{\widehat{\mathbb{G}}} \circ R^u_\omega|_{C_u(\widehat{E}_\omega)} = \epsilon^u_{\widehat{\mathbb{G}}}|_{C_u(\widehat{E}_\omega)}.$$

Conversely, if $\epsilon^r_N \in M^r(\widehat{\mathbb{G}})$ is a state such as in the hypothesis, then it is straightforward to show that $\epsilon^r_N|_{C_r(\widehat{E}_\omega)} \in M^r(\widehat{E}_\omega)$ is a state that makes N_ω coamenable. □

As we are about to see, the counit associated with N_ω is actually ω. Thus N_ω is coamenable if and only if $\omega \in M^r(\widehat{\mathbb{G}})$.

Proof of Corollary 4.1.6. Suppose $\omega \in M^r(\widehat{\mathbb{G}})$. Using Lemma 3.3.5, for each $\pi \in Irr(\widehat{\mathbb{G}})$, choose an ONB $\{e^\pi_j\}$ for $\mathcal{H}_\pi$ that diagonalizes ω. Then for any $u^\pi_{i,j} \in \mathrm{Pol}(\widehat{E}_\omega)$ we have $\omega(u^\pi_{i,j}) = \delta_{i,j} = \epsilon^u_{\mathbb{G}}(u^\pi_{i,j})$. By density $\omega \circ \Gamma_{\widehat{\mathbb{G}}}|_{C_u(\widehat{E}_\omega)} = \epsilon^u_{\widehat{\mathbb{G}}}|_{C_u(\widehat{E}_\omega)}$.

Conversely, from Proposition 4.4.10, there exists a state $\epsilon^r_N \in M^r(\widehat{\mathbb{G}})$ such that $\epsilon^r_N \circ \Gamma_{\widehat{\mathbb{G}}}|_{C_u(\widehat{E}_\omega)} = \epsilon^u_{\widehat{\mathbb{G}}}|_{C_u(\widehat{E}_\omega)}$. In the proof of Proposition 4.4.10 we see that it can be arranged that there exists $\epsilon_N \in M^r(\widehat{E}_\omega)$ such that $\epsilon_N \circ R^r_\omega = \epsilon^r_N$. In particular, we may arrange the property $\epsilon^r_N \circ R^r_\omega = \epsilon^r_N$. Thence,

$$R^r_\omega = (\mathrm{id} \otimes \epsilon^r_N)\Delta^r_{\widehat{\mathbb{G}}} \circ R^r_\omega = (\mathrm{id} \otimes (\epsilon^r_N \circ R^r_\omega))\Delta^r_{\widehat{\mathbb{G}}} = R^r_{\epsilon^r_N}.$$

By injectivity of the map $\mu \mapsto R^r_\mu$ we deduce that $\omega = \epsilon^r_N \in M^r(\widehat{\mathbb{G}})$. □

Recall that $P = \lambda_{\widehat{\mathbb{G}}}(\omega) = (\omega \otimes \mathrm{id})W_{\widehat{\mathbb{G}}}$ is the group-like projection generating the codual of N_ω. Coamenability of N_ω means that we may weak$*$ approximate $\omega \in M^r(\widehat{\mathbb{G}})$ with states $(e_j) \subseteq L^1(\widehat{\mathbb{G}})$. These states satisfy the property

$$u(e_j \otimes \mathrm{id})W_{\widehat{\mathbb{G}}} \to u(\omega \otimes \mathrm{id})W_{\widehat{\mathbb{G}}} = u(P).$$

With these observations, we can establish coamenability of a compact quasi-subgroup in terms of almost invariant vectors in $\ell^2(\mathbb{G})$.

Corollary 4.4.11. *If N_ω is coamenable then there exists a net of unit vectors $(\xi_j) \subseteq P\ell^2(\mathbb{G})$ such that for $\eta \in \ell^2(\mathbb{G})$,*

$$||W_{\widehat{\mathbb{G}}}(\xi_j \otimes P\eta) - \xi_j \otimes P\eta||_2 \to 0.$$

Proof. From Corollary 4.1.6, we have that $\omega \in M^r(\widehat{\mathbb{G}})$. Let $(w_j) \subseteq L^1(\widehat{\mathbb{G}})$ be a net of states weak$*$ approximating ω. By idempotency of ω, $(w_j \circ R_\omega) \subseteq L^1(\widehat{\mathbb{G}})$ is still a net of states that weak$*$ approximates ω. Since $R_\omega \circ R_\omega = R_\omega$, we may assume $w_j \circ R_\omega = w_j$.

The restriction $w_j|_{N_\omega} \in (N_\omega)_*$ is a state, so, we can find a unit vector $\xi_j \in L^2(N_\omega)$ such that $w_j|_{N_\omega} = w_{\xi_j}|_{N_\omega}$. We want to show $w_j = w_{\xi_j}|_{L^\infty(\widehat{\mathbb{G}})}$. For $x \in L^\infty(\widehat{\mathbb{G}})$,

$$w_j(x) = w_{\xi_j}(R_\omega(x)) = \langle R_\omega(x)\xi_j, \xi_j\rangle.$$

Using the equation,

$$P\eta_{\widehat{\mathbb{G}}}(x) = \eta_{\widehat{\mathbb{G}}}(R_\omega(x)) \text{ (cf. [74])},$$

for $y \in N_\omega$ and $\zeta \in L^2(N_\omega)$ we get

$$\begin{aligned} w_{\eta_{\widehat{\mathbb{G}}}(y),\zeta}(R_\omega(x)) &= \langle R_\omega(x)\eta_{\widehat{\mathbb{G}}}(y), \zeta\rangle = \langle \eta_{\widehat{\mathbb{G}}}(R_\omega(x)y), \zeta\rangle \\ &= \langle \eta_{\widehat{\mathbb{G}}}(R_\omega(xy)), \zeta\rangle = \langle Px\eta_{\widehat{\mathbb{G}}}(y), \zeta\rangle = \langle x\eta_{\widehat{\mathbb{G}}}(y), \zeta\rangle \end{aligned}$$

where we used the fact R_ω is a N_ω-bimodule map and that $L^2(N_\omega) = P\ell^2(\mathbb{G})$. Using density of $\eta_{\widehat{\mathbb{G}}}(N_\omega)$ in $L^2(N_\omega)$ we get

$$w_j(x) = w_{\xi_j}(x).$$

The rest of the proof is an adaptation of the case where $\omega = \epsilon^u_{\widehat{\mathbb{G}}}$ (cf. [13, Theorem 3.12]). For $\eta \in \ell^2(\mathbb{G})$,

$$\begin{aligned} ||W_{\widehat{\mathbb{G}}}(\xi_j \otimes P\eta) - \xi_j \otimes P\eta||_2 &= 2||P\eta||^2 - 2Re\langle W_{\widehat{\mathbb{G}}}(\xi_j \otimes P\eta), \xi_j \otimes P\eta\rangle \\ &= 2||P\eta||^2 - 2Re(w_{\xi_j} \otimes w_{P\eta})(W_{\widehat{\mathbb{G}}}) \\ &\to 2||P\eta||^2 - 2w_{P\eta}(P) = 0. \end{aligned}$$

□

4.4.4 Central Idempotents and Amenable Quantum Subgroups of $\mathbb{G}$

We will fix an idempotent state $\omega \in M^u(\widehat{\mathbb{G}})$ (and hence a compact quasi-subgroup $N_\omega \subseteq L^\infty(\widehat{\mathbb{G}})$ and its hull E_ω). We set $P = \lambda_{\widehat{\mathbb{G}}}(\omega)$. We require a certain lemma before proceeding.

Set $B_\omega = R_\omega \circ L_\omega$, which, from coassociativity, is a ucp projection (but possibly without $L^1(\widehat{\mathbb{G}})$-module properties). We will denote the subspace

$$\mathrm{Pol}(\widehat{E}_\omega) \cap R_{\widehat{\mathbb{G}}}(\mathrm{Pol}(\widehat{E}_\omega)) = \mathrm{Pol}_B(\widehat{E}_\omega) = \{u^\pi_{\xi,\eta} : \xi, \eta \in P\mathcal{H}_\pi,\ \pi \in Irr(\widehat{\mathbb{G}})\} = B_\omega(\mathrm{Pol}(\widehat{\mathbb{G}})).$$

So,

$$\overline{\mathrm{Pol}_B(\widehat{E}_\omega)}^{wk*} = N_\omega \cap R_{\widehat{\mathbb{G}}}(N_\omega) = B_\omega(L^\infty(\widehat{\mathbb{G}})).$$

Set $C^u_B(\widehat{E}_\omega) = \overline{\mathrm{Pol}_B(\widehat{E}_\omega)}^{||\cdot||_u}$, $C^r_B(\widehat{E}_\omega) = \Gamma_{\widehat{\mathbb{G}}}(C^u_B(\widehat{E}_\omega))$, and $B^r_B(E_\omega) = C^r_B(\widehat{E}_\omega)^*$. A similar proof to Proposition 4.4.1 will show $\overline{\lambda_{\mathbb{G}}(P\ell^1(\mathbb{G})P)}^{wk*} = B_\omega(L^\infty(\widehat{\mathbb{G}}))$.

Lemma 4.4.12. *If there exists a net of unit vectors $(\xi_j) \subseteq \ell^2(\mathbb{G})$ such that*

$$||\lambda_{\mathbb{G}}(PfP)\xi_j - f(P)\xi_j||_2 \to 0,\ f \in \ell^1(\mathbb{G}) \tag{4.3}$$

then N_ω is coamenable.

Proof. The proof follows from a similar statement in the proof that amenability of $\mathbb{G}$ implies coamenability of $\widehat{\mathbb{G}}$ (cf. [13, Theorem 3.15]). Consider

$$\epsilon_P : P\ell^1(\mathbb{G})P \to \mathbb{C},\ f \mapsto f(1).$$

Then (4.3) tells us $||\lambda_{\mathbb{G}}(PfP)|| \leq |f(P)|$ for every $f \in \ell^1(\mathbb{G})$, so $\epsilon_{\widetilde{N}_P}$ extends to a functional $\widetilde{\epsilon_P} \in M^r_B(\widehat{E}_\omega)$. For each $\pi \in Irr(\widehat{\mathbb{G}})$, using Lemma 3.3.5, we can choose an ONB $\{e^\pi_j\}$ of $\mathcal{H}_\pi$ that diagonalizes ω. Then $\omega(u^\pi_{i,j}) = \delta_{i,j}$ if $u^\pi_{i,j} \in \mathrm{Pol}(\widehat{E}_\omega)$ and zero otherwise. Then, since $\omega \circ R_{\widehat{\mathbb{G}}} = \omega$ [112, Proposition 4], we have $\omega \circ R_{\widehat{\mathbb{G}}}(u^\pi_{i,j}) = \delta_{i,j}$ if $u^\pi_{i,j} \in R_{\widehat{\mathbb{G}}}(\mathrm{Pol}(\widehat{E}_\omega))$ and zero otherwise. Note that this entails P is diagonal. So, for $u^\pi_{i,j} \in \mathrm{Pol}(\widehat{\mathbb{G}})$,

$$\begin{aligned}
\widetilde{\epsilon_P} \circ B_\omega(u^\pi_{i,j}) &= (\omega \otimes \widetilde{\epsilon_P} \otimes \omega)(\sum_{t,s=1}^{n_\pi} u^\pi_{i,t} \otimes u^\pi_{t,s} \otimes u^\pi_{s,j}) \\
&= \omega(u^\pi_{i,i})\omega(u^\pi_{j,j})\widetilde{\epsilon_P}(u^\pi_{i,j}) \\
&= \omega(u^\pi_{i,i})\omega(u^\pi_{j,j})\delta^\pi_{i,j}(P) \\
&= \omega(u^\pi_{i,j}).
\end{aligned}$$

By density of $\mathrm{Pol}(\widehat{\mathbb{G}})$ in $C_r(\widehat{\mathbb{G}})$, we deduce that $\omega = \widetilde{\epsilon_P} \circ B_\omega \in M^r(\widehat{\mathbb{G}})$ and we apply Corollary 4.1.6. □

We will need to use the standard representations of M_{n_π} and $\ell^\infty(\mathbb{G})$, and of their tensor product.

Lemma 4.4.13. *Let $P_\pi \in M_{n_\pi} \subseteq \ell^\infty(\mathbb{G})$ be an orthogonal projection, $\xi \in L^2(\widehat{\mathbb{G}})^+$, and $\pi \in Irr(\widehat{\mathbb{G}})$. Then $P_\pi F_\pi^{1/2} P_\pi \otimes \xi_j$ and $(P_\pi F_\pi^{1/2} \otimes 1)[\lambda(\delta^\pi_{k,l})\xi]_{l,k}(P_\pi \otimes 1)$ lie in the positive cone $(S_2(\mathcal{H}_\pi) \otimes L^2(\mathbb{G}))^+$.*

Proof. It was shown in [11] (see also [13]) that $F_\pi^{1/2} \otimes \xi_j$ and $(F_\pi^{1/2} \otimes 1)[\lambda(\delta^\pi_{k,l})\xi]_{l,k}$ lie in $(L^2(M_{n_\pi}) \otimes L^2(\mathbb{G}))^+$. We will review the proof here. It is then evident that $P_\pi F_\pi^{1/2} P_\pi \otimes \xi_j$ and $(P_\pi F_\pi^{1/2} \otimes 1)[\lambda(\delta^\pi_{k,l})\xi]_{l,k}(P_\pi \otimes 1)$ lie in $(L^2(M_{n_\pi}) \otimes L^2(\mathbb{G}))^+$ as well.

We will begin with some preliminaries first. Recall that $\delta = (F_\pi^{-2})_{\pi \in Irr(\widehat{\mathbb{G}})}$, where F_π is the F-matrix for π, is the modular element for $\ell^\infty(\mathbb{G})$. Moreover, $\tau_t(x) = \sigma_t(x) = \delta^{-it/2} x \delta^{it/2}$. So, for $x \in M_{n_\pi}$,

$$\tau_{-i/2}(x) = F_\pi^{1/2} x F_\pi^{-1/2}.$$

Recall, also, that $S_{\mathbb{G}} = R_{\mathbb{G}} \circ \tau_{-i/2}$. Finally, we remark that since we are considering M_{n_π} as a tracial von Neumann algebra, the positive cone for $L^2(M_{n_\pi})$ is just the positive matrices and the positive cone for $\ell^2(\mathbb{G})$ is the closure of elements of the form $\eta_{\mathbb{G}}(x\delta^{1/4})$ where $x \in c_{00}(\mathbb{G})$ and is positive.

It is clear that $F_\pi^{1/2} \otimes \xi$ lies in the positive cone. Since the positive cone in $L^2(M_{n_\pi})$ is the closure of the positive matrices, it suffices to prove that $(\mu \otimes \mathrm{id})[(F_\pi^{1/2} \otimes 1)[\lambda(\delta^\pi_{k,l})\xi]_{l,k}] \in \ell^2(\mathbb{G})^+$ for every positive $\mu \in \ell^2(\mathbb{G})$. So, fix a positive functional $\mu \in (M_{n_\pi})_*$ and observe that

$$(\mu \otimes \mathrm{id})(F_\pi^{1/2} \otimes 1)[\lambda(\delta^\pi_{k,l})\xi]_{l,k} = \sum_{k,l} \mu(F_\pi^{1/2} E^\pi_{k,l})(u^\pi_{l,k})^* \xi = ((\mu F_\pi^{1/2}) \otimes \mathrm{id}) \left((\Sigma(U^\pi)^*\Sigma)^t\right) \xi$$

Given $x \in M_{n_\pi}$, $F_\pi x = \delta^{-1/2} x$ and $x F_\pi = x\delta^{-1/2}$, and

$$(\mu F_\pi^{1/2})(x^t) = \mu((x F_\pi^{1/2})^t) = \mu(R_{\mathbb{G}}(S_{\mathbb{G}}(x) F_\pi^{1/2})^t) = (F_\pi^{1/2}\mu) R^t_{\mathbb{G}} \circ S_{\mathbb{G}}(x)$$

where $R^t_{\mathbb{G}}(x) = (R_{\mathbb{G}}(x))^t$. Note that since $\mu \in (M_{n_\pi})_*$,

$$(\delta^{-1/4}\mu \otimes \mathrm{id})(W_{\mathbb{G}})^t = (F_\pi^{1/2}\mu \otimes \mathrm{id})\Sigma((U^\pi)^*)^t\Sigma.$$

Since $\xi \in \ell^2(\mathbb{G})^+$, there exists elements of the form $\eta_{\mathbb{G}}(x\delta^{1/4})$ that approximate ξ, where $x \in c_{00}(\mathbb{G})$ is positive. Then, replacing ξ with $\eta_{\mathbb{G}}(x\delta^{1/4})$, we obtain,

$$\begin{aligned}((\mu F_\pi^{1/2}) \otimes \mathrm{id})\left(\Sigma((U^\pi)^*)^t\Sigma\right)\eta_{\mathbb{G}}(x\delta^{1/4}) &= ((\delta^{-1/4}\mu) \circ R^t_{\mathbb{G}} \otimes \mathrm{id})\left((S_{\mathbb{G}} \otimes \mathrm{id})W_{\mathbb{G}}\right)\eta_{\mathbb{G}}(x\delta^{1/4})\\ &= ((\delta^{-1/4}\mu \circ R^t_{\mathbb{G}}) \otimes \mathrm{id})\left(W^*_{\mathbb{G}}\right)\eta_{\mathbb{G}}(x\delta^{1/4})\\ &= \eta_{\mathbb{G}}(((\delta^{-1/4}\mu) \circ R^t_{\mathbb{G}} \otimes \mathrm{id})(\Delta_{\mathbb{G}}(x\delta^{1/4})))\\ &= \eta_{\mathbb{G}}((\mu \circ R^t_{\mathbb{G}} \otimes \mathrm{id})\Delta_{\mathbb{G}}(x)\delta^{1/4})\end{aligned}$$

which is positive because μ is positive. The third equality follows because $W^*_{\mathbb{G}}(\eta_{\mathbb{G}}(x) \otimes \eta_{\mathbb{G}}(y)) = (\eta_{\mathbb{G}} \otimes \eta_{\mathbb{G}})(\Delta_{\mathbb{G}}(y)(x \otimes 1))$ and $\Delta_{\mathbb{G}}(\delta) = \delta \otimes \delta$. □

The following lemmas illustrate how can we relative amenability and coamenability of coideals via Pontryagin duality. In both proofs we use an adaptation of the proof that $\mathbb{G}$ is amenable if and only if $\widehat{\mathbb{G}}$ is coamenable (due to [122] but we follow [13]).

Lemma 4.1.9 *Let $\mathbb{G}$ be a discrete quantum group and $N_\omega \subseteq L^\infty(\widehat{\mathbb{G}})$ a compact quasi-subgroup with $P = \lambda_{\widehat{\mathbb{G}}}(\omega)$. If N_ω is coamenable then M_P is amenable.*

Proof. Assume N_ω is coamenable. Using Lemma 4.4.11, obtain a net of unit vectors $(\xi_j) \subseteq P\ell^2(\mathbb{G})$ such that

$$||W_{\widehat{\mathbb{G}}}(\xi_j \otimes P\eta) - \xi_j \otimes P\eta||_2 \to 0.$$

Since $W_{\widehat{\mathbb{G}}} = \Sigma(W_{\mathbb{G}})^*$, we have

$$||W_{\mathbb{G}}(P\eta \otimes \xi_j) - (P\eta \otimes \xi_j)||_2 \to 0.$$

So, for $w_{\eta,\zeta} = f \in \ell^1(\mathbb{G})$ and $x \in \ell^\infty(\mathbb{G})$,

$$\begin{aligned}&|(Pw_{\eta,\zeta}P * w_{\xi_j}(x) - w_{\eta,\zeta}(P)w_{\xi_j}(x)|\\ &\le |\langle(1 \otimes x)[W_{\mathbb{G}}(P\eta \otimes \xi_j) - (P\eta \otimes \xi_j)], W_{\mathbb{G}}(P\zeta \otimes \xi_j)\rangle| + |\langle(1 \otimes x)P\eta \otimes \xi_j, W_{\mathbb{G}}(P\zeta \otimes \xi_j) - P\zeta \otimes \xi_j\rangle|\\ &= |\langle(1 \otimes x)[W_{\mathbb{G}}(P\eta \otimes \xi_j) - (P\eta \otimes \xi_j)], W_{\mathbb{G}}(P\zeta \otimes \xi_j)\rangle| + |\langle(1 \otimes x)P\eta \otimes \xi_j, W_{\mathbb{G}}(P\zeta \otimes \xi_j) - P\zeta \otimes \xi_j\rangle|\\ &\le ||x||\,||P\zeta||\,||W_{\mathbb{G}}(P\eta \otimes \xi_j) - P\eta \otimes \xi_j||_2 + ||x||\,||P\eta||\,||W_{\mathbb{G}}(P\zeta \otimes \xi_j) - P\zeta \otimes \xi_j||_2\\ &\to 0.\end{aligned}$$

If we let m be a weak$*$ cluster point of the net $(w_{\xi_j}|_{\ell^\infty(\mathbb{G})})$, then it is straightforward to show m is a state satisfying $(PfP)*m = f(P)m$ for all $f \in \ell^1(\mathbb{G})$. Finally, since $(\xi_j) \subseteq P\ell^2(\mathbb{G})$, $w_{\xi_j}(P) = 1$ for all j, so $m(P) = 1$. Using Proposition 4.3.18 and Proposition 4.3.19, we deduce that M_P is amenable. □

It follows from the work of [64] that the central group-like projections in $\ell^\infty(\mathbb{G})$ are in one-to-one correspondence with the quantum subgroups of $\mathbb{G}$, which are in one-to-one correspondence with the central idempotent states in $C_u(\widehat{\mathbb{G}})^*$. Here, we then have that $\widetilde{N_P} = \ell^\infty(\mathbb{G}/\mathbb{H})$ where $\mathbb{H}$ is a quantum subgroup of $\mathbb{G}$ and $N_\omega = L^\infty(\widehat{\mathbb{H}})$ (see the proof of Corollary 4.1.2 for a full justification).

Lemma 4.4.14. *Let $\mathbb{G}$ be a discrete quantum group and $N_\omega \subseteq L^\infty(\widehat{\mathbb{G}})$ a compact quasi-subgroup such that ω is central. Denote $P = \lambda_{\widehat{\mathbb{G}}}(\omega)$. If $\widetilde{N}_P$ is relatively amenable then N_ω is coamenable.*

Proof. The proof follows with very few changes to the proof in [11]. We give it in detail for the benefit of the reader nonetheless. Let $m \in \ell^\infty(\mathbb{G})^*$ be a P-left invariant state. Using Lemma 4.3.8 and that $\ell^\infty(\mathbb{G})$ is in standard form, we can find a net of unit vectors $(\xi_\alpha) \subseteq \ell^2(\mathbb{G})$ such that $(w_{\xi_\alpha}|_{\ell^\infty(\mathbb{G})})$ weak$*$ approximates m, so that we have

$$||Pf * w_{\xi_\alpha} - f(P)w_{\xi_\alpha}||_1 \to 0, \;\; f \in \ell^1(\mathbb{G}).$$

Note that since P is central, we either have $P_\pi = I_{n_\pi}$ or $P_\pi = \{0\}$. Define the functionals $\mu_\alpha, \eta_\alpha \in (M_{n_\pi}(\ell^\infty(\mathbb{G})))_*$ by setting

$$\eta_\alpha(x) = (\mathrm{tr} \otimes w_{\xi_\alpha})\left((P_\pi \otimes 1)(x)\right) = (\mathrm{tr} \otimes w_{\xi_\alpha})\left((P_\pi \otimes 1)(x)(P_\pi \otimes 1)\right), \;\; x = [x_{m,n}] \in M_{n_\pi}(\ell^\infty(\mathbb{G}))$$

and

$$\mu_\alpha(x) = \sum_{n,m}^{n_\pi} (P_\pi \delta^\pi_{m,n} P_\pi) * w_{\xi_\alpha}(x_{m,n}), \;\; x = [x_{m,n}] \in M_{n_\pi}(\ell^\infty(\mathbb{G})).$$

It is clear that η_α is positive since $tr \otimes \omega_{\xi_\alpha}$ and $x \mapsto (P_\pi \otimes 1)x(P_\pi \otimes 1)$ are positive. Then, from

$$\mu_\alpha(x) = \sum_{t,n,m} \langle x_{m,n} \lambda_{\mathbb{G}}(P_\pi \delta^\pi_{t,n})\xi_\alpha, \lambda_{\mathbb{G}}(P_\pi \delta^\pi_{t,m})\xi_\alpha \rangle = (\mathrm{tr} \otimes w_{\xi_\alpha})\left([\lambda_{\mathbb{G}}(P_\pi \delta^\pi_{k,l})]^*_{l,k} x [\lambda_{\mathbb{G}}(P_\pi \delta^\pi_{k,l})]_{l,k}\right) \tag{4.4}$$

we are able to deduce that μ_α is positive. To justify (4.4), recall that $\Delta_{\mathbb{G}}(y) = W^*_{\mathbb{G}}(1 \otimes y) W_{\mathbb{G}}$

and $W_{\mathbb{G}} = \Sigma W^*_{\widehat{\mathbb{G}}}\Sigma = \bigoplus_{\pi\in Irr(\widehat{\mathbb{G}})} \sum_{i,j=1}^{n_\pi} E_{i,j}\otimes (u^\pi_{j,i})^*$. Then,

$$\begin{aligned}
\mu_\alpha(x) &= \sum_{n,m}^{n_\pi} (P_\pi \delta^\pi_{m,n} P_\pi) * w_{\xi_\alpha}(x_{m,n}) \\
&= \sum_{m,n} \sum_{\sigma\in Irr(\widehat{\mathbb{G}})} \sum_{i,j,k,l} (P_\pi \delta^\pi_{m,n} P_\pi)(E^\sigma_{i,j} E^\sigma_{k,l}) \omega_{\xi_\alpha}(u^\sigma_{i,j} x_{m,n} (u^\sigma_{l,k})^*) \\
&= \sum_{t,n,m} \delta^\pi_{m,n} (P_\pi E^\pi_{m,n} P_\pi) \langle x_{m,n} \lambda_{\mathbb{G}}(\delta^\pi_{t,n})\xi_\alpha, \lambda_{\mathbb{G}}(\delta^\pi_{t,m})\xi_\alpha\rangle \text{ (since } P_\pi \text{ is diagonal and } P^*_\pi = P_\pi = P^2_\pi) \\
&= (\mathrm{tr}\otimes w_{\xi_\alpha}) \left([\lambda_{\mathbb{G}}(\delta^\pi_{k,l})]^*_{l,k} (P_\pi\otimes 1) x (P_\pi \otimes 1) [\lambda_{\mathbb{G}}(\delta^\pi_{k,l})]_{l,k} \right).
\end{aligned}$$

Then,

$$\begin{aligned}
(\eta_\alpha - \mu_\alpha)([x_{n,m}]) &= \sum_{t=1}^{n_\pi} (P_\pi)_t w_{\xi_\alpha}(x_{t,t}) - \sum_{n,m} P\delta^\pi_{m,n} P * w_{\xi_\alpha}(x_{m,n}) \\
&= \sum_{t=1}^{n_\pi} (\delta^\pi_{t,t} P_\pi)(P_\pi) w_{\xi_\alpha}(x_{t,t}) - \sum_{n,m} P\delta^\pi_{m,n} P * w_{\xi_\alpha}(x_{m,n}) \\
&= \sum_{n,m}^{n_\pi} (\delta^\pi_{m,n} P_\pi)(P_\pi) w_{\xi_\alpha}(x_{m,n}) - P_\pi \delta^\pi_{m,n} P_\pi * w_{\xi_\alpha}(x_{m,n})
\end{aligned}$$

and so we have that

$$||\eta_\alpha - \mu_\alpha||_{(M_{n_\pi}(\ell^\infty(\mathbb{G})))_*} \to 0$$

since $f(P)\omega_{\xi_\alpha} - Pf * \omega_{\xi_\alpha} \to 0$ for all $f \in \ell^1(\mathbb{G})$. Then,

$$||(F^{1/2}_\pi P_\pi \otimes 1)\mu_\alpha(P_\pi F^{1/2}_\pi \otimes 1) - (F^{1/2}_\pi P_\pi \otimes 1)\eta_\alpha(P_\pi F^{1/2}_\pi \otimes 1)||_{(M_{n_\pi}(\ell^\infty(\mathbb{G})))_*} \to 0$$

where F_π is the F-matrix associated with π. Consider

$$P_\pi F^{1/2}_\pi P_\pi \otimes \xi_\alpha \in L^2(M_{n_\pi}) \otimes \ell^2(\mathbb{G})$$

and

$$(P_\pi F^{1/2}_\pi \otimes 1)[\lambda_{\mathbb{G}}(\delta^\pi_{k,l})\xi_\alpha]_{l,k}(P_\pi \otimes 1) \in L^2(M_{n_\pi}) \otimes \ell^2(\mathbb{G}).$$

We now claim that we have

$$w_{PF^{1/2}_\pi P_\pi \otimes \xi_\alpha} = (P_\pi F^{1/2}_\pi \otimes 1)\eta_\alpha (F^{1/2}_\pi P_\pi \otimes 1) \tag{4.5}$$

and

$$w_{(P_\pi F_\pi^{1/2}\otimes 1)[\lambda_{\mathbb{G}}(\delta^\pi_{k,l})\xi_\alpha]_{l,k}(P_\pi\otimes 1)} = (P_\pi F_\pi^{1/2}\otimes 1)\mu_\alpha(F_\pi^{1/2}P_\pi\otimes 1). \tag{4.6}$$

Indeed, given $x\in M_{n_\pi}(\ell^\infty(\mathbb{G}))$, for (4.5),

$$\begin{aligned}\omega_{P_\pi F_\pi^{1/2}P_\pi\otimes\xi_\alpha}(x) &= \omega_{P_\pi F_\pi^{1/2}P_\pi}\otimes\omega_{\xi_\alpha}(x)\\ &= (\mathrm{tr}\otimes w_{\xi_\alpha})\left((P_\pi F_\pi^{1/2}P\otimes 1)x(PF_\pi^{1/2}P_\pi\otimes 1)\right)\\ &= \eta_\alpha\left((F_\pi^{1/2}P_\pi\otimes 1)x(P_\pi F_\pi^{1/2}\otimes 1)\right)\\ &= \left((P_\pi F_\pi^{1/2}\otimes 1)\eta_\alpha(F_\pi^{1/2}P_\pi\otimes 1)\right)(x)\end{aligned}$$

and for (4.6),

$$\begin{aligned}&w_{(P_\pi F_\pi^{1/2}\otimes 1)[\lambda(P_\pi\delta^\pi_{k,l})\xi_\alpha]_{l,k}}(x)\\ &= (\mathrm{tr}\otimes w_{\xi_\alpha})\left((P_\pi\otimes 1)[\lambda(\delta^\pi_{k,l})\xi_\alpha]^*_{l,k}\left((F_\pi^{1/2}P_\pi\otimes 1)x(P_\pi F_\pi^{1/2}\otimes 1)\right)[\lambda(\delta^\pi_{k,l})\xi_\alpha]_{l,k}(P_\pi\otimes 1)\right)\\ &= \left((PF_\pi^{1/2}\otimes 1)\mu_\alpha(F_\pi^{1/2}P\otimes 1)\right)(x)\end{aligned}$$

where we used (4.4) in the last equality.

From Lemma 4.4.13 we know that $PF_\pi^{1/2}P\otimes\xi_\alpha$ and $(PF_\pi^{1/2}\otimes 1)[\lambda_{\mathbb{G}}(\delta^\pi_{l,k})\xi_\alpha]_{k,l}$ lie in the positive cone of $L^2(M_{n_\pi})\otimes\ell^2(\mathbb{G})$. Using the Powers-Størmer inequality (cf. [49]), we have

$$\begin{aligned}&||(P_\pi F_\pi^{1/2}\otimes 1)[\lambda(\delta^\pi_{k,l})\xi_\alpha]_{l,k} - P\otimes\xi_\alpha||_{L^2(M_{n_\pi})\otimes\ell^2(\mathbb{G})}\\ &\le ||w_{(P_\pi F_\pi^{1/2}\otimes 1)[\lambda(\delta^\pi_{k,l})\xi_\alpha]_{l,k}} - \omega_{P_\pi F_\pi^{1/2}P_\pi\otimes\xi_\alpha}||_{(M_{n_\pi}(\ell^\infty(\mathbb{G})))_*}\to 0\end{aligned}$$

Thus we deduce the following limit

$$\begin{aligned}&||(F_\pi^{1/2}P_\pi\otimes 1)[\lambda(\delta^\pi_{k,l})\xi_\alpha]_{l,k} - (F_\pi^{1/2}P_\pi\otimes\xi_\alpha)||_{L^2(M_{n_\pi})\otimes\ell^2(\mathbb{G})}\to 0\\ \iff\ &||(P_\pi\otimes 1)[\lambda(\delta^\pi_{k,l})\xi_\alpha]_{l,k} - P_\pi\otimes\xi_\alpha||_{L^2(M_{n_\pi})\otimes\ell^2(\mathbb{G})}\to 0.\end{aligned}$$

Then, for $\sum\alpha_{i,j}\delta^\pi_{i,j} = f_\pi\in(M_{n_\pi})_*\subseteq\ell^1(\mathbb{G})$

$$||\lambda_{\mathbb{G}}(Pf_\pi P)\xi_\alpha - f_\pi(P)\xi_\alpha||_2 = ||(Pf_\pi^t\otimes\mathrm{id})((P_\pi\otimes 1)[\lambda_{\mathbb{G}}(\delta^\pi_{k,l})\xi_\alpha]_{l,k} - P_\pi\otimes\xi_\alpha)||_2\to 0.$$

where $f_\pi^t = \sum\alpha_{l,k}\delta^\pi_{k,l}$. Density and Lemma 4.4.12 tell us N_ω is coamenable. □

We showed in the proof of Lemma 4.4.14 that if $\widetilde{N}_P = M_P$, then coamenability of N_ω implies amenability of $\widetilde{N}_P$. This occurs in the particular case where P is central, so that $\ell^\infty(\mathbb{G}/\mathbb{H}) = \widetilde{N}_P$ for some quantum subgroup $\mathbb{H}\le\mathbb{G}$. Therefore, lemmas 4.4.14 and 4.1.9 give us the following.

Corollary 4.4.15. *Let $\mathbb{G}$ be a DQG and $\mathbb{H} \leq \mathbb{G}$ a quantum subgroup. Then $\ell^\infty(\mathbb{G}/\mathbb{H})$ is relatively amenable if and only if it is amenable.*

Remark 4.4.16. Let us maintain the same notation as in Lemma 4.4.14 and the paragraph above it. Since P is central, $M_P = \widetilde{N_P} = \ell^\infty(\mathbb{G}/\mathbb{H})$. Using the definition of coamenability of $N_\omega = L^\infty(\widehat{\mathbb{H}})$, it is not too difficult to prove that coamenability of $L^\infty(\widehat{\mathbb{H}})$ as a coideal is equivalent to coamenability of $\widehat{\mathbb{H}}$ as a CQG. Thus we have established a proof that $\ell^\infty(\mathbb{G}/\mathbb{H})$ is relatively amenable if and only if $\widehat{\mathbb{H}}$ is coamenable using the techniques of Vaes and Blanchard [11] for Tomatsu's theorem [122] and our work on coamenable compact quasi-subgroups in Section 4.4.3. Another application of Tomatsu's theorem gives us that $\ell^\infty(\mathbb{G}/\mathbb{H})$ is relatively amenable if and only if $\mathbb{H}$ is amenable. Thus, we have found a different approach to obtain [65, Theorem 3.7]. In their work, they use the natural action of $\mathbb{H}$ on $\ell^\infty(\mathbb{G})$ and achieve their result by working with amenability of $\mathbb{H}$. In our work, we work on the 'dual side' of amenability, and work with coamenability of $\widehat{\mathbb{H}}$ instead. In Section 4.4.6, we expand on this equivalence of relative amenability of $\ell^\infty(\mathbb{G}/\mathbb{H})$ with amenability of $\mathbb{H}$ (see also Remark 4.4.31).

Another consequence of our above lemmas is the following.

Corollary 4.1.2 *Let $\mathbb{G}$ be a discrete quantum group. There is a one-to-one correspondence between the amenable quantum subgroups of $\mathbb{G}$ and the central idempotent states on $C_r(\widehat{\mathbb{G}})$.*

Proof. It was shown with [65, Theorem 4.3] that there is a one-to-one correspondence between central group-like projections in $\ell^\infty(\mathbb{G})$ and amenable quantum subgroups of $\mathbb{G}$. Let $\mathbb{H}$ be a quantum subgroup of $\mathbb{G}$ and $1_\mathbb{H}$ the central group-like projection that generates $\ell^\infty(\mathbb{G}/\mathbb{H})$. It follows from [65, Lemma 4.2] and [38, Theorem 4.3] that $1_\mathbb{H} = \lambda_{\widehat{\mathbb{G}}}(\omega)$ for some central idempotent state $\omega : C_u(\widehat{\mathbb{G}}) \to \mathbb{C}$. Since $1_\mathbb{H}$ is central, $M_{1_\mathbb{H}} = \ell^\infty(\mathbb{G}/\mathbb{H}) = \widetilde{N}_{1_\mathbb{H}}$, which gives a correspondence between central idempotent states on $C_u(\widehat{\mathbb{G}})$ and central group-like projections in $\ell^\infty(\mathbb{G})$. It was shown with [65, Theorem 3.7] that $\mathbb{H}$ is amenable if and only if $\ell^\infty(\mathbb{G}/\mathbb{H})$ is relatively amenable. With Corollary 4.4.15 we then know that amenability of $\mathbb{H}$ is equivalent to amenability of $\ell^\infty(\mathbb{G}/\mathbb{H})$. Then, the combination of lemmas 4.4.14 and 4.1.9 gives us the result. ☐

Remark 4.4.17. The tracial central idempotent states on $C_r(\widehat{\mathbb{G}})$ are in one-to-one correspondence with the amenable normal quantum subgroups of $\mathbb{G}$ for which their quotients are unimodular. This follows from the duality between normal quantum subgroups of $\mathbb{G}$

and normal quantum subgroups of $\widehat{\mathbb{G}}$ (see Chapter 2.8). Indeed, $\mathbb{H}$ is normal if and only if $\widehat{\mathbb{G}/\mathbb{H}}$ is a closed quantum subgroup of $\widehat{\mathbb{G}}$.

Suppose $1_{\mathbb{H}} = \lambda_{\widehat{\mathbb{G}}}(\omega_{\widehat{\mathbb{G}/\mathbb{H}}})$ where $\omega_{\widehat{\mathbb{G}/\mathbb{H}}} = h_{\widehat{\mathbb{G}/\mathbb{H}}} \circ \pi_{\widehat{\mathbb{G}/\mathbb{H}}}$ is the Haar state on $\widehat{\mathbb{G}/\mathbb{H}}$. It is easy to see that $\omega_{\widehat{\mathbb{G}/\mathbb{H}}}$ is tracial whenever $h_{\widehat{\mathbb{G}/\mathbb{H}}}$ is tracial.

Conversely, if $1_{\mathbb{H}} = \lambda_{\widehat{\mathbb{G}}}(\omega)$ and ω is tracial, then ω must be of Haar type because then

$$\{a \in C_u(\widehat{\mathbb{G}}) : \omega(a^*a) = 0\}$$

is a two-sided ideal (see [112, Theorem 5]). It follows that $\widehat{\mathbb{H}}$ is a quotient of $\widehat{\mathbb{G}}$, and hence $\omega = \omega_{\widehat{\mathbb{G}/\mathbb{H}}}$ is the tracial Haar state on $\widehat{\mathbb{G}/\mathbb{H}}$ (see [28] for more).

4.4.5 A Remark on Simplicity and Traces

Definition 4.4.18. A DQG $\mathbb{G}$ is said to be **unimodular** if $h_L = h_R$.

Remark 4.4.19. It is well-known that $\mathbb{G}$ is unimodular if and only if $\widehat{\mathbb{G}}$ is Kac.

Recall the Furstenberg boundary discussed in Section 4.2. The cokernel $N_F = N_{P_F} = \ell^\infty(\mathbb{H}_F)$ of the Furstenberg boundary was shown to be the codual of a compact quasi-subgroup with [65, Proposition 2.9].

As observed with [65, Proposition 4.18], we find that $\mathbb{G}$ is amenable if and only if $N_F = \mathbb{C}1$, which follows essentially because N_F is relatively amenable. Therefore, in a certain sense, the larger N_F is the "less amenable" $\mathbb{G}$ is. We can refine this intuition. For a discrete group G, the cokernel of $\partial_F(G)$ is the quotient space $\ell^\infty(G/R_a(G))$, where $R_a(G)$ is the amenable radical of G, which is known to be the kernel of the action of G on $\partial_F(G)$. In particular, the action of G on $\partial_F(G)$ is faithful if and only if $R_a(G)$ is trivial, i.e., $N_F = \ell^\infty(G)$ (see [65]).

Definition 4.4.20. We say the action of $\mathbb{G}$ on $\partial_F(\mathbb{G})$ is **faithful** if $N_F = \ell^\infty(\mathbb{G})$.

Let $h_{\widehat{\mathbb{H}}_F} \circ \pi_{\mathbb{H}_F} = \omega_F \in M^u(\widehat{\mathbb{G}})$ be the idempotent state such that $\lambda_{\widehat{\mathbb{G}}}(\omega_F) = P_F$. It remains open to determine whether or not $\widehat{\mathbb{G}}/\mathbb{H}_F$ is coamenable. A positive answer to Question 4.1.8 would establish that we always have $\omega_F \in C_r(\widehat{\mathbb{G}})^*$.

Corollary 4.4.21. *Suppose $\mathbb{G}$ is a unimodular DQG and $\widehat{\mathbb{G}}/\mathbb{H}_F$ is coamenable. If $\mathbb{G}$ has the unique trace property then the action of $\mathbb{G}$ on $\partial_F(\mathbb{G})$ is faithful.*

Proof. Since $\mathbb{G}$ is unimodular, it follows that $\mathbb{H}_F$ is unimodular because $\widehat{\mathbb{H}}_F$ is a closed quantum subgroup of $\widehat{\mathbb{G}}$. Suppose $N_F \neq \ell^\infty(\mathbb{G})$. Hence $\mathbb{H}_F \neq \mathbb{G}$, and we have that $h_{\widehat{\mathbb{H}}_F} \circ \pi_{\mathbb{H}_F} = \omega_F \neq h_{\widehat{\mathbb{G}}}$ is a tracial state in $M^r(\widehat{\mathbb{G}})$. □

We are grateful to Mehrdad Kalantar for pointing out to us the following.

Corollary 4.4.22. *Suppose $\mathbb{G}$ is a unimodular DQG and $\widehat{\mathbb{G}}/\mathbb{H}_F$. If $\mathbb{G}$ is C^*-simple then $\mathbb{G}$ has the unique trace property.*

Proof. If $C_r(\widehat{\mathbb{G}})$ is simple then ω_F must be faithful, so $\omega_F = h_{\widehat{\mathbb{G}}}$. In particular, $N_F = \ell^\infty(\mathbb{G})$. Then $\mathbb{G}$ has the unique trace property because of Corollary 4.4.21. □

4.4.6 Amenability of Quantum Subgroups

Given a DQG $\mathbb{G}$ and closed quantum subgroup $\mathbb{H}$, we will show that amenability and relative amenability of $\ell^\infty(\mathbb{G}/\mathbb{H})$ characterizes amenability of $\mathbb{H}$. Since the group-like projection $1_\mathbb{H} \in \ell^\infty(\mathbb{G})$ associated with $\ell^\infty(\mathbb{G}/\mathbb{H})$ is central (cf. [64]), we point out that $\ell^\infty(\mathbb{G}/\mathbb{H}) = N_{1_\mathbb{H}} = M_{1_\mathbb{H}}$.

We denote the natural bimodule action of $\ell^1(\mathbb{H})$ on $\ell^1(\mathbb{G})$ as follows:

$$\varphi *_\mathbb{H} f = (\varphi \otimes f)l_\mathbb{H} = (\varphi \circ \sigma_\mathbb{H}) * f \text{ and } f *_\mathbb{H} \varphi = f * (\varphi \circ \sigma_\mathbb{H}),\ \varphi \in \ell^1(\mathbb{H}), f \in \ell^1(\mathbb{G}),$$

where $\sigma_\mathbb{H}$ and $l_\mathbb{H}$ are defined in Section 2.

Definition 4.4.23. We will say $m \in \ell^\infty(\mathbb{G})^*$ is **$\mathbb{H}$-invariant** if

$$\varphi(\sigma_\mathbb{H} \otimes m)\Delta_\mathbb{G} = \varphi *_\mathbb{H} m = \varphi(1)m,\ \varphi \in \ell^1(\mathbb{H}).$$

In correspondence with a module map on $\ell^\infty(\mathbb{G})$, we establish the case where m is $\mathbb{H}$–invariant.

Lemma 4.4.24. *Suppose $m \in \ell^\infty(\mathbb{G})^*$ is an $\mathbb{H}$-invariant state. Then E_m is a ucp $\ell^1(\mathbb{G})$–module map $\ell^\infty(\mathbb{G}) \to \ell^\infty(\mathbb{G}/\mathbb{H})$.*

Proof. For $x \in \ell^\infty(\mathbb{G})$, we compute,

$$\begin{aligned}(\mathrm{id} \otimes \sigma_\mathbb{H})\Delta_\mathbb{G}(E_m(x)) &= (\mathrm{id} \otimes \sigma_\mathbb{H} \otimes m)(\mathrm{id} \otimes \Delta_\mathbb{G})\Delta_\mathbb{G}(x) \\ &= (\mathrm{id} \otimes m)\Delta_\mathbb{G}(x) = E_m(x) \otimes 1.\end{aligned}$$

□

Remark 4.4.25. Consider the forward direction of [21, Theorem 3.2]. There, an invariant state $m \in \ell^\infty(\mathbb{H})$ was taken, and the projection was defined by $P = (m \otimes \mathrm{id})l_{\mathbb{H}}$. Note the difference with our projection E_m in the above lemma, where $m \in \ell^\infty(\mathbb{G})^*$.

We can immediately characterize the $\mathbb{H}$-invariant functionals as those that the annihilate the left ideals $J^1(\mathbb{G}, \mathbb{H})$ using our preceding work.

Lemma 4.4.26. *A non–zero functional $\mu \in \ell^\infty(\mathbb{G})^*$ is $\mathbb{H}$-invariant if and only if $f * \mu = 0$ for all $f \in J^1(\mathbb{G}, \mathbb{H})$.*

Proof. We first claim

$$\sigma_{\mathbb{H}}(\ell^\infty(\mathbb{G}/\mathbb{H})) = \mathbb{C}\ . \tag{4.7}$$

Indeed, if $x \in \ell^\infty(\mathbb{G}/\mathbb{H})$ then

$$\Delta_{\mathbb{H}}(\sigma_{\mathbb{H}}(x)) = (\sigma_{\mathbb{H}} \otimes \sigma_{\mathbb{H}})\Delta_{\mathbb{G}}(x) = \sigma_{\mathbb{H}}(x) \otimes 1,$$

which means $\sigma_{\mathbb{H}}(x) \in \ell^\infty(\mathbb{H}/\mathbb{H}) = \mathbb{C}$.

Now, to proceed with the proof, take $f \in J^1(\mathbb{G}, \mathbb{H})$. Then

$$f * \mu(x) = f \otimes \mu(\Delta_{\mathbb{G}}(x)) = f(E_\mu(x)) = 0$$

since $E_\mu(x) \in \ell^\infty(\mathbb{G}/\mathbb{H}) = J^1(\mathbb{G}, \mathbb{H})^\perp$ from Lemma 4.4.24.

Conversely, because

$$f(E_\mu(x)) = f * \mu(x) = 0$$

for all $f \in J^1(\mathbb{G}, \mathbb{H})$, it follows that $E_\mu(x) \in \ell^\infty(\mathbb{G}/\mathbb{H}) = J^1(\mathbb{G}, \mathbb{H})^\perp$. So, if we take $\varphi \in \ell^1(\mathbb{H})$ and $x \in \ell^\infty(\mathbb{G})$, then

$$\begin{aligned} \varphi *_{\mathbb{H}} \mu(x) &= (\varphi \circ \sigma_{\mathbb{H}})(E_\mu(x)) = \varphi(1) \overbrace{\sigma_{\mathbb{H}}(E_\mu(x))}^{\in \mathbb{C}} \text{ (using (4.7))} \\ &= \varphi(1)\sigma_{\mathbb{H}}(E_\mu(x)) = \varphi(1)\epsilon_{\mathbb{G}}(E_\mu(x)) = \varphi(1)\mu(\sigma_{\mathbb{H}}(x)). \end{aligned}$$

□

Recall that $1_{\mathbb{H}}$ is the group-like projection that generates $\ell^\infty(\mathbb{G}/\mathbb{H})$. A consequence of Theorem 4.3.13 is the following.

Corollary 4.4.27. *A functional $m \in \ell^1(\mathbb{G})^*$ is $\mathbb{H}$-invariant if and only if it is left $1_{\mathbb{H}}$-invariant.*

Before proceeding, we require the following result shown in the proof of [102, Theorem 2.1] at the level of Hopf von Neumann algebras.

Proposition 4.4.28. *[103] Let m be an invariant non–zero linear functional on $\ell^\infty(\mathbb{G})$. Then there exists an invariant mean on $\ell^\infty(\mathbb{G})$.*

As one would hope, $\mathbb{H}$-invariance is due to amenability of $\mathbb{H}$.

Proposition 4.4.29. *There exists a non–zero $\mathbb{H}$–invariant functional on $\ell^\infty(\mathbb{G})$ that is non-vanishing on $1_{\mathbb{H}}$ if and only if $\mathbb{H}$ is amenable.*

Proof. Let m be an $\mathbb{H}$–invariant functional. Let $\iota_{\mathbb{H}} : \ell^\infty(\mathbb{H}) \to \ell^\infty(\mathbb{G})$ be the injective $*$-homomorphism defined by setting

$$\iota_{\mathbb{H}}(\sigma_{\mathbb{H}}(x)) = 1_{\mathbb{H}}x, x \in \ell^\infty(\mathbb{G}).$$

We will show $\iota_{\mathbb{H}}^*(m)$ is a non–zero invariant functional on $\ell^\infty(\mathbb{H})$. First notice that

$$\iota_{\mathbb{H}}(\sigma_{\mathbb{H}}(x)) = 1_{\mathbb{H}}x = \iota_{\mathbb{H}}(\sigma_{\mathbb{H}}(1_{\mathbb{H}}x))$$

so $\sigma_{\mathbb{H}}(1_{\mathbb{H}}x) = \sigma_{\mathbb{H}}(x)$. Then,

$$\begin{aligned}
(\mathrm{id} \otimes \iota_{\mathbb{H}}^*(m))\Delta_{\mathbb{H}}(\sigma_{\mathbb{H}}(x)) &= (\mathrm{id} \otimes \iota_{\mathbb{H}}^*(m))\Delta_{\mathbb{H}}(\sigma_{\mathbb{H}}(1_{\mathbb{H}}x)) \\
&= (\mathrm{id} \otimes \iota_{\mathbb{H}}^*(m))(\sigma_{\mathbb{H}} \otimes \sigma_{\mathbb{H}})\Delta_{\mathbb{G}}(1_{\mathbb{H}}x) \\
&= (\sigma_{\mathbb{H}} \otimes m)(1 \otimes 1_{\mathbb{H}})\Delta_{\mathbb{G}}(1_{\mathbb{H}}x) \\
&= (\sigma_{\mathbb{H}} \otimes m)(1_{\mathbb{H}} \otimes 1)\Delta_{\mathbb{G}}(1_{\mathbb{H}}x) \\
&= (\sigma_{\mathbb{H}} \otimes m)(1 \otimes 1)\Delta_{\mathbb{G}}(1_{\mathbb{H}}x) \\
&= m(1_{\mathbb{H}}x) \\
&= \iota_{\mathbb{H}}^*(m)(\sigma_{\mathbb{H}}(x)).
\end{aligned}$$

We have that $\iota_{\mathbb{H}}^*(m)$ is non-zero because $m(1_{\mathbb{H}}) \neq 0$. Using Proposition 4.4.28 we get an invariant mean.

Conversely, just take $m \circ \sigma_{\mathbb{H}}$. □

We have thus acquired enough to obtain the following.

Corollary 4.4.30. *Let $\mathbb{G}$ be a DQG and $\mathbb{H}$ a closed quantum subgroup. The following are equivalent:*

1. $\mathbb{H}$ *is amenable;*
2. $\ell^\infty(\mathbb{G}/\mathbb{H})$ *is amenable;*
3. $\ell^\infty(\mathbb{G}/\mathbb{H})$ *is relatively amenable;*
4. $J^1(\mathbb{G},\mathbb{H})$ *has a brai;*
5. $J^1(\mathbb{G},\mathbb{H})$ *has a brai in* $\ell^1_0(\mathbb{G})$*;*
6. $J^1(\mathbb{G},\mathbb{H})$ *has a brai in* $\ell^1_0(\mathbb{H})$*.*

Proof. (1. $\iff$ 2.) This follows from lemmas 4.4.14 and 4.1.9 and Tomatsu's theorem [122] (see Remark 4.4.16).

(2. $\iff$ 3.) This follows from Corollary 4.4.15.

(1. $\implies$ 4.) This follows from Theorem 4.3.20 after noting that $\ell^\infty(\mathbb{G}/\mathbb{H}) = M_{1_\mathbb{H}}$ and $J^1(\mathbb{G},\mathbb{H}) = J^1(M_{1_\mathbb{H}})$ because $1_\mathbb{H}$ is central.

(4. $\implies$ 6.) is clear.

(6. $\implies$ 5.) Since $\sigma_\mathbb{H}$ is unital, we deduce that $\ell^1_0(\mathbb{H})\circ\sigma_\mathbb{H} \subseteq \ell^1_0(\mathbb{G})$. Then, if $(e_j) \subseteq \ell^1_0(\mathbb{H})$ is a brai for $J^1(\mathbb{G},\mathbb{H})$, it is clear that $(e_j \circ \sigma_\mathbb{H}) \subseteq \ell^1_0(\mathbb{G})$ is a brai for $J^1(\mathbb{G},\mathbb{H})$.

(5. $\implies$ 1.) Let $(e_j) \subseteq \ell^1_0(\mathbb{G})$ be a brai for $J^1(\mathbb{G},\mathbb{H})$, with weak$*$ cluster point $\mu \in \ell^\infty(\mathbb{G})^*$. Then, $f * (\epsilon_\mathbb{G} - \mu) = 0$ for every $f \in J^1(\mathbb{G},\mathbb{H})$, and so an application of Proposition 4.4.26 and Proposition 4.4.29 tells us $\mathbb{H}$ is amenable. □

Remark 4.4.31. We must point out that Kalantar et al. [65] independently achieved Corollary 4.4.30 1 $\iff$ 2 $\iff$ 3. To obtain their result, they build an injective right $\ell^1(\mathbb{G})$-module map $\ell^\infty(\mathbb{H}) \to \ell^\infty(\mathbb{G})$, generalizing how one builds such a map $\ell^\infty(H) \to \ell^\infty(G)$ for a discrete group G and subgroup H, using a set of representatives for the coset space G/H.

As discussed in Remark 4.4.16, we prove this same result on the 'dual side.' We prove that relative amenability of $\ell^\infty(\mathbb{G}/\mathbb{H})$ is equivalent to coamenability of the compact quasi-subgroup $L^\infty(\widehat{\mathbb{H}})$ using a minor adjustment of Blanchard and Vaes' proof of Tomatsu's theorem ([11] and [122]) and our work on coamenable compact quasi-subgroups in Section 4.4.3.

4.5 Examples: Discrete Crossed Products

Here, we assume $\mathbb{G}$ is a DQG and $\widehat{\mathbb{G}} \rtimes_\alpha \widehat{\Gamma}$ denotes the crossed product of a discrete group Γ with the CQG $\widehat{\mathbb{G}}$.

Proposition 4.5.1. *Let $\mathbb{G}$ be a DQG and $(\widehat{\mathbb{G}}, \Gamma, \alpha)$ a Woronowicz C^*–dynamical system. The following are equivalent.*

1. *$\mathbb{G}$ is amenable;*
2. *$\ell^1_0(\mathbb{G})\widehat{\otimes}\ell^1(\Gamma)$ has a bai;*
3. *$\ell^1_0(\mathbb{G})\widehat{\otimes}\ell^1(\Gamma)$ has a bai in $\ker(F \mapsto F(1 \otimes 1))$;*
4. *$\ell^1_0(\mathbb{G})\widehat{\otimes}\ell^1(\Gamma)$ has a bai in $\ell^1_0(\mathbb{G})$.*
5. *there is a right $\ell^1(\mathbb{G})\widehat{\otimes}\ell^1(\Gamma)$–module conditional expectation $\ell^\infty(\mathbb{G})\overline{\otimes}\ell^\infty(\Gamma) \to \ell^\infty(\Gamma)$.*

Proof. We will show $J^1(\widehat{\widehat{\mathbb{G}} \rtimes_\alpha \widehat{\Gamma}}, \mathbb{G}) = \ell^1_0(\mathbb{G})\widehat{\otimes}\ell^1(\Gamma)$. The rest follows from Corollary 4.4.30 and Proposition 2.10.5. The inclusion $\ell^\infty(\Gamma) \subseteq \ell^\infty(\mathbb{G})\overline{\otimes}\ell^\infty(\Gamma)$ is given by $x \mapsto 1 \otimes x$. The preadjoint of the inclusion is $(1 \otimes \mathrm{id}) = T_{\mathbb{G}} : \ell^1(\mathbb{G})\widehat{\otimes}\ell^1(\Gamma) \to \mathbb{C} \otimes \ell^1(\Gamma) = \ell^1(\Gamma)$, where we view $1 : \ell^1(\Gamma) \ni f \mapsto f(1) \in \mathbb{C}$ as a complete quotient map. Notice that $J^1(\widehat{\widehat{\mathbb{G}} \rtimes_\alpha \widehat{\Gamma}}, \mathbb{G}) = \ker(T_{\mathbb{G}})$ Then from [33, Proposition 7.1.7]

$$\ker(T_{\mathbb{G}}) = \ker(1)\widehat{\otimes}\ell^1(\Gamma) = \ell^1_0(\mathbb{G})\widehat{\otimes}\ell^1(\Gamma).$$

□

There are canonical ways of building a crossed product out of a CQG $\widehat{\mathbb{G}}$ (see [39, 127, 126]). Let

$$\chi(\widehat{\mathbb{G}}) = Sp(C_u(\widehat{\mathbb{G}})) := \{s \in M^u(\widehat{\mathbb{G}}) : s \text{ is a homomorphism}\}.$$

Under convolution, $\chi(\widehat{\mathbb{G}})$ is a compact group (see [126, 2.14]). Now, the map $\chi(\widehat{\mathbb{G}}) \to Aut(C_u(\widehat{\mathbb{G}}))$ defined by $s \mapsto \alpha_s = (s^{-1} \otimes \mathrm{id} \otimes s)(\mathrm{id} \otimes \Delta^u_{\widehat{\mathbb{G}}})\Delta^u_{\widehat{\mathbb{G}}}$ is a continuous group homomorphism such that $(\alpha_s \otimes \alpha_s)\Delta^u_{\widehat{\mathbb{G}}} = \Delta^u_{\widehat{\mathbb{G}}} \circ \alpha_s$. From any subgroup $\Gamma \leq \chi(\widehat{\mathbb{G}})$ equipped with the discrete topology, we get a Woronowicz discrete C^*–dynamical system $(C_u(\widehat{\mathbb{G}}), \Gamma, \alpha)$.

Alternatively we can consider the intrinsic group (see [68] for a reference). Define

$$Int(\widehat{\mathbb{G}}) = \{u \in \mathcal{U}(C_u(\widehat{\mathbb{G}})) : \Delta^u_{\widehat{\mathbb{G}}}(u) = u \otimes u\}.$$

It is true that $Int(\widehat{\mathbb{G}})$ is a co–commutative closed quantum subgroup of $\widehat{\mathbb{G}}$. By defining the map $Int(\widehat{\mathbb{G}}) \to Aut(C_u(\widehat{\mathbb{G}}))$, $s \mapsto \alpha_s$, such that $\alpha_s(a) = s^*as$, we obtain a homomorphism, which we also mention satisfies $\Delta^u_{\widehat{\mathbb{G}}} \circ \alpha_s = (\alpha_s \otimes \alpha_s)\Delta^u_{\widehat{\mathbb{G}}}$. Then, for any subgroup $\Gamma \leq Int(\widehat{\mathbb{G}})$, we get a Woronowicz discrete C^*–dynamical system $(C_u(\widehat{\mathbb{G}}), \Gamma, \alpha)$.

Whenever $\mathbb{G}$ is co–amenable and Γ is non–amenable, we obtain examples of non–amenable DQGs containing an amenable closed quantum subgroup outside of the realm of discrete groups.

Example 4.5.2. 1. Let G be a compact group and Γ a discrete group acting on G by continuous automorphisms $\alpha_* : \Gamma \to Aut(G)$. Then α_* canonically induces an action, denoted α, on $C(G)$. It is straightforward to see α intertwines Δ_G so that the crossed product of $G \rtimes_\alpha \widehat{\Gamma}$ gives rise to a generally non–commutative and non–cocommutative CQG [127]. We can easily build examples using $\Gamma = \mathbb{Z}$ and by having $\mathbb{Z}$ act on G via inner automorphisms. For example, we can let $\alpha_* = \alpha^h_z(g) = h^z g h^{-z}$ for some fixed $h \in G$. In this case, $G \rtimes_\alpha \widehat{\mathbb{Z}}$ is a co–amenable CQG.

Here is a non–co–amenable example. Let $G = SU(2)$. We have that $\mathbb{F}_2$ is a discrete subgroup of $SU(2)$, and so we obtain continuous automorphisms $\alpha_* : \mathbb{F}_2 \to Aut(SU(2))$ by setting $(\alpha_*)_s(u) = sus^{-1}$. Therefore, since $\mathbb{F}_2$ is non–amenable, $SU(2) \rtimes_\alpha \widehat{\mathbb{F}_2}$ is non–co–amenable. Then $\widehat{SU(2) \rtimes_\alpha \widehat{\mathbb{F}_2}}$ contains $\widehat{SU(2)}$ as an amenable quantum subgroup.

We thank Nico Spronk for pointing out the following set of examples. We consider the wreath product construction. Let G and Γ be arbitrary compact and discrete groups respectively. Set $G^\Gamma = \prod_{s\in\Gamma} G$, which, because of Tychonoff's theorem, is itself a compact group. Then we can define a action, say α_*, of Γ on G^Γ by setting $(\alpha_*)_s((k_t)_{t\in\Gamma}) = (k_{s^{-1}t})_{t\in\Gamma}$. Then $\widehat{G^\Gamma \rtimes_\alpha \Gamma}$ is a DQG containing $\widehat{G^\Gamma}$ as an amenable quantum subgroup.

2. If we consider the trivial action $id : \Gamma \to Aut(C_u(\widehat{\mathbb{G}}))$, then the discrete crossed product reduces to the tensor product:

$$L^\infty(\widehat{\mathbb{G}} \rtimes_{id} \widehat{\Gamma}) = L^\infty(\widehat{\mathbb{G}}) \overline{\otimes} VN(\Gamma) =: L^\infty(\widehat{\mathbb{G}} \times \widehat{\Gamma}).$$

Here we can build CQGs using any pair of a CQG and discrete group. As an explicit example, we can let $\widehat{\mathbb{G}} = SU_q(2)$ with $q \in [-1, 1] \setminus \{0\}$ (cf. [135, 134]) and Γ be a non-amenable discrete group. Recall that $SU_q(2)$ is a non-Kac, coamenable CQG [8, Corollary 6.2]. Then $SU_q(2) \times \widehat{\Gamma}$ is non-co-amenable (and non-Kac), so $\widehat{SU_q(2) \times \widehat{\Gamma}}$ contains $\widehat{SU_q(2)}$ as an amenable quantum subgroup.

Chapter 5

Tracial and $\mathbb{G}$-equivariant States

5.1 Introduction

In [15] and [73], group dynamical characterizations of the unique trace property and simplicity of reduced group C^*-algebras were achieved, where they showed that the reduced C^*-algebra $C_r(\widehat{G})$ of a discrete group G has a unique tracial state if and only if the action of G on its Furstenburg boundary $\partial_F(G)$ is faithful and in turn that simplicity of $C_r(\widehat{G})$ implies it has a unique trace (the Haar state). Given that $R_a(G) = \ker(G \curvearrowright \partial_F(G))$, where $R_a(G)$ is the amenable radical, this shows that the unique trace property is equivalent to having $R_a(G) = \{e\}$. The key point of this result is that in determining the tracial structure of $C_r(\widehat{G})$, the tracial states of the form $1_N \in C_r(\widehat{G})^*$ where N is an amenable normal subgroup of G are fundamental to the study of traces on reduced group C^*-algebras.

Serving as a stepping stone towards establishing quantum group dynamic machinery for quantizing the unique trace property and simplicity of $C_r(\widehat{\mathbb{G}})$ where $\mathbb{G}$ is a discrete quantum group, Kalantar et al. [65] constructed the Furstenberg boundary $\partial_F(\mathbb{G})$ (see Section 4.1). Moreover, the cokernel of $\ker(\mathbb{G} \curvearrowright \partial_F(\mathbb{G}))$ was shown in [65] to be equal to $\ell^\infty(\widehat{\mathbb{H}_F})$ where $\mathbb{H}_F$ is a quantum subgroup of $\widehat{\mathbb{G}}$ that is minimal as an object where $\ell^\infty(\mathbb{H}_F)$ is relatively amenable (see Section 4.1). In particular, we might call $\widehat{\mathbb{H}_F}$ the 'relatively amenable coradical' of $\mathbb{G}$. There is a catch, though. At the quantum level, the canonical trace is replaced with the Haar state $h_{\widehat{\mathbb{G}}} \in C_r(\widehat{\mathbb{G}})^*$, however, $h_{\widehat{\mathbb{G}}}$ may not be tracial and there are known examples where $C_r(\widehat{\mathbb{G}})$ has no trace (e.g., see [7]). It turns out the Haar state is tracial if and only if $\mathbb{G}$ is unimodular. Therefore, in [65], the unique trace property was considered for unimodular discrete quantum groups.

Definition 5.1.1. We say a discrete quantum group $\mathbb{G}$ is **C^*-simple** if the reduced C^*-algebra $C_r(\widehat{\mathbb{G}})$ is simple. We say a unimodular discrete quantum group $\mathbb{G}$ has the **unique trace property** if the Haar state $h_{\widehat{\mathbb{G}}}$ of $\widehat{\mathbb{G}}$ is the unique tracial state.

Theorem 5.1.2. *[65] Let $\mathbb{G}$ be a unimodular discrete quantum group. If the action of $\mathbb{G}$ on $\partial_F(\mathbb{G})$ is faithful then $\mathbb{G}$ has the unique trace property.*

It currently remains open to determine the converse and if simplicity of $C_r(\widehat{\mathbb{G}})$ implies the unique trace property.

In this Chapter, we make a study of tracial states on reduced C^*-algebras of arbitrary discrete quantum groups. The position of $\mathbb{H}_F$ as a closed quantum subgroup of $\widehat{\mathbb{G}}$ in relation to the canonical Kac quotient $\mathbb{H}_{Kac}$ (see Section 4.2) turns out to be a fundamental property that governs the existence and uniqueness of traces.

Our result, which informs the position of $\mathbb{H}_F$ as a closed quantum subgroup of $\widehat{\mathbb{G}}$ is the following, which generalizes the fact that every G-invariant state (invariant with respect to the conjugation action) in $C_r(\widehat{G})$ concentrates on the amenable radical.

Theorem 5.3.19. *Let $\mathbb{G}$ be a DQG. Every $\mathbb{G}$-invariant state $\tau \in C_r(\widehat{\mathbb{G}})^*$ concentrates on $\widehat{\mathbb{G}}/\mathbb{H}_F$.*

We must note that most of the above theorem was proven in the proof of [65, Theorem 5.3].

For classical groups, it turns out the G-invariant states are the tracial states on $C_r(\widehat{G})$ and one immediately sees from the above result that faithfulness of the Furstenburg boundary gives the unique trace property. This fact remains true for unimodular discrete quantum groups but fails in the arbitrary case, where it was shown in [65] that the $\mathbb{G}$-invariant tracial states are the KMS states of the scaling automorphism group of $\mathbb{G}$ (see [65]). They also prove that faithfulness of $\mathbb{G} \curvearrowright \partial_F(\mathbb{G})$ implies $C_r(\widehat{\mathbb{G}})$ has no $\mathbb{G}$-invariant states.

Despite the apparent disparity between traces and $\mathbb{G}$-invariant states in general, it turns out that Haar idempotents (see Section 2.4) are $\mathbb{G}$-invariant if and only if they are tracial (see Proposition 5.3.12), generalizing the well-known fact that the Haar state is $\mathbb{G}$-invariant if and only if it is tracial (see Section 4.3). In particular, using the work of Crann [22], we can prove the following.

Corollary 5.3.14. *Let $\mathbb{G}$ be a DQG. We have that $C_r(\widehat{\mathbb{G}})$ is nuclear and has a tracial state if and only if $\mathbb{G}$ is amenable..*

It is straightforward to prove that if $C_r(\widehat{\mathbb{G}})$ is simple then it has no tracial states whenever $\mathbb{G}$ is non-unimodular. We are able to obtain the following.

Corollary 5.3.15. *Let $\mathbb{G}$ be a non-unimodular discrete quantum group. If $C_r(\widehat{\mathbb{G}})$ is simple it has no $\mathbb{G}$-invariant states.*

After analyzing how the existence and uniqueness of traces relates to the canonical Kac quotient $\mathbb{H}_{Kac}$ (see Proposition 5.3.3), we obtain the following as a Corollary of Theorem 5.3.19.

Corollary 5.3.21. *Let $\mathbb{G}$ be a DQG. Then $\mathbb{H}_F$ is a closed quantum subgroup of every Kac closed quantum subgroup $\mathbb{H}$ of $\widehat{\mathbb{G}}$, where $\widehat{\mathbb{G}}/\mathbb{H}$ is coamenable. In particular, the following hold whenever $\widehat{\mathbb{G}}/\mathbb{H}_F$ is coamenable:*

1. *$\mathbb{H}_F$ is Kac if and only if $C_r(\widehat{\mathbb{G}})$ has a tracial state;*
2. *$\mathbb{H}_F = \mathbb{H}_{Kac}$ and $C_{rKac}(\mathbb{H}_F) = C_r(\mathbb{H}_F)$ if and only if $C_r(\widehat{\mathbb{G}})$ has a unique tracial state;*
3. *$\mathbb{H}_F = \mathbb{H}_{Kac}$ if and only if $C_r(\widehat{\mathbb{G}})$ has a unique idempotent tracial state.*

See Section 3.2 for coamenability of quotients and coideals.

So, it is the Haar state coming from the cokernel of $\mathbb{G} \curvearrowright \partial_F(\mathbb{G})$ that governs the existence and uniqueness of traces of discrete quantum groups (at least when it is itself reduced).

Let $\mathbb{H}$ be a closed quantum subgroup of $\widehat{\mathbb{G}}$ with idempotent state $\omega \in C_u(\widehat{\mathbb{G}})^*$. Set $P = \lambda_{\widehat{\mathbb{G}}}(\omega)$. From Lemma 4.1.9 it follows that if $\widehat{\mathbb{G}}/\mathbb{H}$ is coamenable, then $M_P \supseteq \ell^\infty(\widehat{\mathbb{H}}) = \widetilde{N_P}$ is amenable in $\ell^\infty(\mathbb{G})$ (see Section 4.1 for the definition of relative amenability and amenability). This marked progress towards [65, Question 8.1], where rather than relative amenability of $\ell^\infty(\widehat{\mathbb{H}})$, it is amenability of M_P that is necessary for coamenability of $L^\infty(\widehat{\mathbb{G}}/\mathbb{H})$. We establish the forwards direction of [65, Question 8.1] when $\mathbb{H}$ is of Kac type.

Corollary 5.3.23. *Let $\mathbb{G}$ be DQG. If $\mathbb{H}$ is a Kac closed quantum subgroup of $\widehat{\mathbb{G}}$ and $\widehat{\mathbb{G}}/\mathbb{H}$ is a coamenable quotient then $\ell^\infty(\widehat{\mathbb{H}})$ is relatively amenable in $\ell^\infty(\mathbb{G})$.*

We discuss now the organization of this chapter. Section 2 is reserved for preliminary concepts. We discuss locally compact quantum groups, $\mathbb{G}$-boundaries, closed quantum subgroups, and coideals and idempotent states. In Section 3 we discuss coamenability of coideals. We introduce a C^*-algebraic framework for coideals, and prove basic facts that generalize known characterizations of coamenability of compact quantum groups for coideals. We also touch on the lattice structure of reduced idempotent states. We reserve Section 4 for our main theorems. We recall the construction of the Furstenberg boundary, the Kac and unimodularity properties, the construction of the canonical Kac quotient, and the basics of $\mathbb{G}$-invariant states. We prove an analogue of the fact the Haar state is $\mathbb{G}$-invariant if and only if $\widehat{\mathbb{G}}$ is Kac [65, Lemma 5.2] for discrete quantum groups: we prove that the Haar measure is unimodular if and only if it is $\widehat{\mathbb{G}}$-invariant (Theorem 5.3.16). We spend the remainder of the Chapter proving our main theorems highlighted above.

5.2 Coamenable Coideals

5.2.1 Amenable and Coamenable Quantum Groups

The coproduct on $\ell^\infty(\mathbb{G})$ extends to a coproduct on $\mathcal{B}(\ell^2(\mathbb{G}))$ by defining

$$\Delta^l_{\mathbb{G}}(T) = W^*_{\mathbb{G}}(1 \otimes T)W_{\mathbb{G}}.$$

Then $\mathcal{B}(\ell^2(\mathbb{G}))$ is a $\mathcal{T}(\ell^2(\mathbb{G}))$-bimodule with respect to the actions

$$\mu \triangleleft T = (\mathrm{id} \otimes \mu)\Delta^l_{\mathbb{G}}(T) \text{ and } T \triangleleft \mu = (\mu \otimes \mathrm{id})\Delta^l_{\mathbb{G}}(T),\ \mu \in \mathcal{T}(\ell^2(\mathbb{G}), T \in \mathcal{B}(\ell^2(\mathbb{G})).$$

One last characterization of amenability of a DQG we will mention is the following (which holds for LCQGs in general).

Theorem 5.2.1. *[23] Let $\mathbb{G}$ be a DQG. The following are equivalent:*

1. *$\mathbb{G}$ is amenable;*
2. *there exists a right $\mathcal{T}(\ell^2(\mathbb{G}))$-module conditional expectation $E : \mathcal{B}(\ell^2(\mathbb{G})) \to L^\infty(\widehat{\mathbb{G}})$ such that $E(\ell^\infty(\mathbb{G})) = \mathbb{C}1$;*
3. *there exists a left $\mathcal{T}(\ell^2(\mathbb{G}))$-module conditional expectation $E : \mathcal{B}(\ell^2(\mathbb{G})) \to L^\infty(\widehat{\mathbb{G}})'$ such that $E(\ell^\infty(\mathbb{G})) = \mathbb{C}1$.*

Remark 5.2.2. A similar result was independently achieved in [121].

5.2.2 C^*-algebraic Coideals

Let $E = (E_\pi)$ be the hull of a right coideal $N \subseteq L^\infty(\mathbb{G})$, where $\mathbb{G}$ is a CQG. We will denote

$$C_u(\widehat{E}) = \overline{\mathrm{Pol}(\widehat{E})}^{||\cdot||_u} \subseteq C_u(\widehat{\mathbb{G}})$$

and

$$C_r(\widehat{E}) = \overline{\mathrm{Pol}(\widehat{E})}^{||\cdot||_r} \subseteq C_r(\widehat{\mathbb{G}}).$$

As mentioned in Remark 4.4.2, $L^\infty(\widehat{E}) = N$. After choosing an ONB $\{e_j^\pi\}$ that diagonalizes P_E, where $P_E\mathcal{H}_\pi = E_\pi$ for all $\pi \in Irr(\widehat{\mathbb{G}})$, so that $\mathrm{Pol}(\widehat{E}) = \mathrm{span}\{u_{i,j}^\pi : e_j^\pi \in E_\pi, 1 \leq i \leq n_\pi, \pi \in Irr(\widehat{\mathbb{G}})\}$, using a Hahn Banach argument, we can deduce that $L^\infty(\widehat{E}) \cap \mathrm{Pol}(\widehat{\mathbb{G}}) = \mathrm{Pol}(\widehat{E})$. In particular, $C_u(\widehat{E})$ and $C_r(\widehat{E})$ are C^*-algebras.

Notice that $\Gamma_E := \Gamma_{\mathbb{G}}|_{C_u(\widehat{E})} : C_u(\widehat{E}) \to C_r(\widehat{E})$ is a surjective unital $*$-homomorphism satisfying

$$(\Gamma_{\mathbb{G}} \otimes \Gamma_E)\Delta^u = \Delta^r \circ \Gamma_E.$$

Let $C_{env}(\widehat{E})$ be the closure of $\mathrm{Pol}(\widehat{E})$ with respect to the universal norm:

$$||a||_E^{env} = \sup\{||a|| : ||\cdot|| \text{ is a } C^*\text{-seminorm}\}.$$

The standard argument will show that there is an identification $S(\mathrm{Pol}(\widehat{E})) \cong S(C_{env}(\widehat{E}))$: for each $\mu \in S(\mathrm{Pol}(\widehat{E}))$ there exists a unique $\tilde{\mu} \in S(C_{env}(\widehat{E}))$ such that $\tilde{\mu}|_{\mathrm{Pol}(\widehat{E})} = \mu$. Then for $a \in \mathrm{Pol}(\widehat{E})$,

$$||aa^*||_u = \sup_{\mu \in S(Pol(\widehat{E}))}\{|\mu(aa^*)|\} = \sup_{\mu \in S(C_{env}(\widehat{E}))}\{|\mu(aa^*)|\}.$$

In the compact case, it turns out $C_u(\widehat{E_\omega})$ is $C_{env}(\widehat{E_\omega})$.

Proposition 5.2.3. *Let $E = E_\omega$ be compact. Then $C_u(\widehat{E_\omega}) = C_{env}(\widehat{E_\omega})$. In other words, $C_{env}(\widehat{E_\omega})$ isometrically injects into $C_u(\widehat{\mathbb{G}})$.*

Proof. The proof is essentially the same as the proof that $C^*(H)$ isometrically embeds into $C^*(G)$ whenever G is a discrete group and H is a subgroup. The projection R_ω :

$\mathrm{Pol}(\widehat{\mathbb{G}}) \to \mathrm{Pol}(\widehat{E_\omega})$ induces a linear inclusion $S(\mathrm{Pol}(\widehat{E_\omega})) \to S(\mathrm{Pol}(\widehat{\mathbb{G}}))$, $\mu \mapsto \mu \circ R_\omega$. Consider $a \in \mathrm{Pol}(\widehat{E_\omega})$. Then

$$||a||_u^2 = ||aa^*||_u = \sup_{\mu \in S(\mathrm{Pol}(\widehat{\mathbb{G}}))} \{|\mu(aa^*)|\} = \sup_{\mu \in S(\mathrm{Pol}(\widehat{E_\omega}))} \{|\mu \circ R_\omega(aa^*)|\}$$
$$= \sup_{\mu \in S(\mathrm{Pol}(\widehat{E_\omega}))} \{|\mu(aa^*)|\} = (||a||_{E_\omega}^{env})^2.$$

□

With the above proposition in hand, it is now straightforward to deduce that for two-sided coideals, their reduced C^*-algebras are equal to the reduced C^*-algebras that comprise their underlying quantum group.

Proposition 5.2.4. *We have that* $C_r(\widehat{E_{\widehat{\mathbb{H}}}}) = C_r(\widehat{\mathbb{H}})$.

Proof. Note that Proposition 5.2.3 says that $C_u(\widehat{E_{\widehat{\mathbb{H}}}}) = C_u(\widehat{\mathbb{H}})$. Denote $I_{\mathbb{G}}^h = \{a \in C_u(\widehat{\mathbb{G}}) : h_{\mathbb{G}}(a^*a) = 0\}$. Then $I_{\mathbb{H}}^h = I_{\mathbb{G}}^h \cap C_u(\widehat{\mathbb{H}})$ and so

$$C_r(\widehat{E_{\widehat{\mathbb{H}}}}) = C_u(\widehat{E_{\widehat{\mathbb{H}}}})/I_{\mathbb{G}}^h = C_u(\widehat{\mathbb{H}})/I_{\mathbb{H}}^h = C_r(\widehat{\mathbb{H}}).$$

□

The functional

$$\epsilon_E^u := \epsilon_{\widehat{\mathbb{G}}}^u|_{C_u(\widehat{E})} : C_u(\widehat{E}) \to \mathbb{C}$$

is a state that satisfies $\epsilon_E^u(u_{i,j}^\pi) = \delta_{i,j}$ for all $u_{i,j}^\pi \in \mathrm{Pol}(\widehat{E})$. As was shown in Lemma 3.3.5, if $E = E_\omega$ is compact, then $\omega|_{C_u(\widehat{E})} = \epsilon_E^u$.

Definition 5.2.5. Let E be a hull for a coideal. We say E is **coamenable** if there exists $\epsilon_E^r \in C_r(\widehat{E})^*$ such that $\epsilon_E^r \circ \Gamma_E = \epsilon_E^u$.

This a direct extension of the notion of a coamenable quotient from [65] and a coamenable compact quasi-subgroup from Chapter 4. Recall from the latter the following.

Proposition 5.2.6. *(Corollary 4.1.6) Let* $\mathbb{G}$ *be a DQG and* E_ω *be a hull of a compact quasi-subroup. Then* E_ω *is coamenable if and only if* $\omega \in M^r(\widehat{\mathbb{G}})$.

Recall that $\mathbb{G}$ is coamenable if and only if $C_u(\widehat{\mathbb{G}}) \cong C_r(\widehat{\mathbb{G}})$. An argument verbatim to the argument used for [9, Theorem 2.2] will prove the following.

Proposition 5.2.7. *We have that E is coamenable if and only if $\Gamma_E : C_u(\widehat{E}) \to C_r(\widehat{E})$ is injective.*

For quotients, we have the following characterization.

Theorem 5.2.8. *[65, Theorem 3.11] Let $\mathbb{G}$ be a DQG. We have that $\widehat{\mathbb{G}}/\mathbb{H}$ is a coamenable quotient if and only if there exists a $*$-homomorphism $\pi^r_{\mathbb{H}} : C_r(\widehat{\mathbb{G}}) \to C_r(\mathbb{H})$ such that $\Gamma_{\mathbb{G}} \circ \pi^u_{\mathbb{H}} = \Gamma_{\widehat{\mathbb{H}}} \circ \pi^r_{\mathbb{H}}$.*

Given that $\mathbb{H} \leq \widehat{\mathbb{G}}$, because of the identity $\Gamma_{\mathbb{G}} \circ \pi^u_{\mathbb{H}} = \Gamma_{\widehat{\mathbb{H}}} \circ \pi^r_{\mathbb{H}}$, we will normally just write $\pi_{\mathbb{H}}$ for $\pi^u_{\mathbb{H}}$ and $\pi^r_{\mathbb{H}}$ unless there is risk of confusion.

Let $\mathbb{G}$ be discrete. The following characterization of relative amenability and amenability of coideals was achieved.

Theorem 5.2.9. *(Theorem 4.3.13 and Theorem 4.3.17) Let $\mathbb{G}$ be a DQG and $\widetilde{N}_P$ be a right coideal where $P = \lambda_{\widehat{\mathbb{G}}}(\omega)$ for an idempotent state $\omega \in M^u(\widehat{\mathbb{G}})$. The following is true:*

1. *$\widetilde{N}_P$ is relatively amenable if and only if there exists a state $m \in \ell^\infty(\mathbb{G})$ such that $P(m \otimes id)\Delta_{\mathbb{G}}(x) = m(x)P$ for all $x \in \ell^\infty(\mathbb{G})$;*

2. *M_P is amenable if and only if there exists a state $m \in \ell^\infty(\mathbb{G})$ such that $m(P) = 1$ and $P(m \otimes id)\Delta_{\mathbb{G}}(x) = m(x)P$ for all $x \in \ell^\infty(\mathbb{G})$.*

Proof. In Theorem 5.2.9, what was shown is that $\widetilde{N}_P$ is relatively amenable if and only if there exists a state $m \in \ell^\infty(\mathbb{G})^*$ satisfying $(\mathrm{id} \otimes m)\Delta_{\mathbb{G}}(x)(P \otimes 1) = m(x)P$. Since P is preserved by the unitary antipode (which we will not define here) [72, Lemma 1.3], the standard trick of turning right invariant states into left invariant states with the unitary antipode will work here to show $P(m \otimes \mathrm{id})\Delta_{\mathbb{G}}(x) = m(x)P$. □

We call a state m satisfying 1. of Theorem 5.2.9 a ***P*-invariant** state.

Before getting to our main result, we require more elaboration on some finer details of the interplay between codual coideals in $L^\infty(\widehat{\mathbb{G}})$ and in $\ell^\infty(\mathbb{G})$. Given that $N = L^\infty(\widehat{E})$ is a right coideal in $L^\infty(\widehat{\mathbb{G}})$, we denote

$$N^L = \overline{\{u^\pi_{\xi,\eta} : \xi \in \mathcal{H}_\pi, \eta \in E_\pi, \pi \in Irr(\widehat{\mathbb{G}})\}}^{wk*},$$

which is a left coideal in $L^\infty(\widehat{\mathbb{G}})$ that satisfies $N^L = L^\infty(\widehat{\mathbb{G}}) \cap (\widetilde{N}_{P_E})'$ and $\widetilde{N}_{P_E} = \ell^\infty(\mathbb{G}) \cap (N^L)'$. Recall that we have an action of $\ell^\infty(\mathbb{G})$ on $\ell^1(\mathbb{G})$ denoted by setting

$$fx(y) = f(xy) \text{ and } xf(y) = f(yx),\ \ f \in \ell^1(\mathbb{G}), x, y \in \ell^\infty(\mathbb{G}).$$

We will have use for the fact

$$\overline{\lambda_{\mathbb{G}}(P\ell^1(\mathbb{G}))}^{wk*} = \overline{\{(f \otimes \mathrm{id})(W_{\mathbb{G}}(P \otimes 1)) : f \in \ell^1(\mathbb{G})\}}^{wk*} = N^L,$$

(see [123, 72] and Chapter 4). Finally, we denote the space

$$M_P = \{x \in \ell^\infty(\mathbb{G}) : (1 \otimes P)\Delta_{\mathbb{G}}(x)(1 \otimes P) = x \otimes P\} \supseteq \widetilde{N_P}$$

as defined in Section 4.3.2. A significant fact about M_P is that coamenability of N_ω implies amenability of M_P.

As with Theorem 5.2.9, it can be proved that M_P is amenable if and only if there exists a state $m \in \ell^\infty(\mathbb{G})^*$ such that $P(m \otimes \mathrm{id})\Delta_{\mathbb{G}}(x)P = m(x)P$ for all $x \in \ell^\infty(\mathbb{G})$.

Remark 5.2.10. Contrary to our use of the word in this work, N^L is usually referred to as the codual of $\widetilde{N}_P$.

Theorem 5.2.11. *Let $\mathbb{G}$ be a DQG. Let $\widetilde{N}_P$ be a right coideal in $\ell^\infty(\mathbb{G})$ such that $P = \lambda_{\widehat{\mathbb{G}}}(\omega)$ for an idempotent state $\omega \in M^u(\widehat{\mathbb{G}})$. We have that M_P is amenable if and only if there exists a left $P\mathcal{T}(\ell^2(\mathbb{G}))P$-module ucp projection $E : \mathcal{B}(\ell^2(\mathbb{G})) \to P(N^L)'P$ such that $E(\ell^\infty(\mathbb{G})) = \mathbb{C}P$.*

Proof. The assertion in 1. follows by replacing each conjugation PXP in the proof with PX. Assume that $\widetilde{N}_P$ is amenable with P-invariant state m. We use an adaptation of the proof for [10, Theorem 3.3] (see also [121, Theorem 2.1]). Define $E : \mathcal{B}(\ell^2(\mathbb{G})) \to \mathcal{B}(\ell^2(\mathbb{G}))$ by setting

$$E(T) = (m \otimes \mathrm{id})W_{\mathbb{G}}^*(1 \otimes PTP)W_{\mathbb{G}} = (m \otimes \mathrm{id})\Delta^l_{\mathbb{G}}(PTP), T \in \mathcal{B}(\ell^2(\mathbb{G})).$$

This map is clearly ucp. From the proof of Theorem 4.3.17, we find that the property

$$m(x) = m(Px) = m(xP) = m(PxP),\ \ x \in \ell^\infty(\mathbb{G})$$

may be arranged. Consequently, we have

$$E(T) = (m \otimes \mathrm{id})\Delta^l_{\mathbb{G}}(PTP) = (1 \otimes P)(m \otimes \mathrm{id})\Delta^l_{\mathbb{G}}(T)(1 \otimes P) = PE(T)P \tag{5.1}$$

where we used group-likeness of P. Given this, notice that

$$\begin{aligned}(1\otimes P)\Delta^l_{\mathbb{G}}\circ E(T)(1\otimes P) &= (1\otimes 1\otimes P)(m\otimes \mathrm{id})(\mathrm{id}\otimes\Delta^l_{\mathbb{G}})\Delta^l_{\mathbb{G}}(PTP)(1\otimes 1\otimes P)\\ &= (m\otimes \mathrm{id}\otimes \mathrm{id})(\Delta_{\mathbb{G}}\otimes \mathrm{id})(1\otimes P)\Delta^l_{\mathbb{G}}(PTP)(1\otimes P)\\ &= (1\otimes P\otimes P)(m\otimes \mathrm{id}\otimes \mathrm{id})(\Delta_{\mathbb{G}}\otimes \mathrm{id})\Delta^l_{\mathbb{G}}(T)(1\otimes P\otimes P)\\ &\quad ((5.1)\text{ and group-likeness})\\ &= (P\otimes 1\otimes P)(1\otimes m\otimes \mathrm{id})(1\otimes \Delta^l_{\mathbb{G}}(T))(1\otimes 1\otimes P)\\ &\quad (P\text{-invariance})\\ &= P\otimes E(T)\ (5.1).\end{aligned}$$

Then, since $m(P)=1$ and $E(T)=PE(T)P$,

$$E\circ E(T) = (m\otimes \mathrm{id})\Delta^l_{\mathbb{G}}\circ E(T) = E(T).$$

We showed $\Delta^l_{\mathbb{G}}(E(T)) = P\otimes E(T)$. On the other hand, if $\Delta^l_{\mathbb{G}}(T)=P\otimes T$, then

$$E(T) = P(m\otimes \mathrm{id})\Delta^l_{\mathbb{G}}(T)P = PTP$$

so

$$E(\mathcal{B}(\ell^2(\mathbb{G}))) = \{T\in P\mathcal{B}(\ell^2(\mathbb{G}))P : \Delta^l_{\mathbb{G}}(T) = P\otimes T\}.$$

Now, notice that

$$\begin{aligned}W^*_{\mathbb{G}}(1\otimes T)W_{\mathbb{G}} = P\otimes T &\iff (1\otimes T)W_{\mathbb{G}} = W_{\mathbb{G}}(P\otimes T)\\ &\implies (1\otimes T)W_{\mathbb{G}}(P\otimes 1) = W_{\mathbb{G}}(P\otimes T)\\ &\iff T\hat{x} = \hat{x}T \text{ for all } \hat{x}\in N^L,\end{aligned}$$

which implies $E(\mathcal{B}(\ell^2(\mathbb{G}))\subseteq P(N^L)'P$, and for $\hat{x}\in (N^L)'$,

$$\begin{aligned}E(P\hat{x}P) &= (m\otimes \mathrm{id})(P\otimes P)W^*_{\mathbb{G}}(1\otimes P\hat{x}P)W_{\mathbb{G}}(P\otimes 1)\\ &= (1\otimes P)(m\otimes \mathrm{id})(W_{\mathbb{G}}(P\otimes 1))^*(1\otimes P\hat{x}P)W_{\mathbb{G}}(P\otimes 1)\\ &= (1\otimes P\hat{x}P)(m\otimes \mathrm{id})(W_{\mathbb{G}}(P\otimes 1))^*W_{\mathbb{G}}(P\otimes 1)\ (\text{since } P\hat{x}P\in (N^L)')\\ &= (1\otimes P\hat{x}P)(m\otimes \mathrm{id})(P\otimes 1)W^*_{\mathbb{G}}W_{\mathbb{G}}(P\otimes 1)\\ &= P\hat{x}P.\end{aligned}$$

Therefore, $E(\mathcal{B}(\ell^2(\mathbb{G}))) = P(N^L)'P$.

For the claim that $E|_{\ell^\infty(\mathbb{G})} = \mathbb{C}P$, let $x\in\ell^\infty(\mathbb{G})$. Then

$$E(x) = (m\otimes \mathrm{id})\Delta_{\mathbb{G}}(PxP) = P(m\otimes \mathrm{id})\Delta_{\mathbb{G}}(x)P = m(x)P.$$

Finally, we show E is left $P\mathcal{T}(\ell^2(\mathbb{G}))P$-module. For this, let $\tau \in \mathcal{T}(\ell^2(\mathbb{G}))$ and $T \in \mathcal{B}(\ell^2(\mathbb{G}))$. Then,

$$\begin{aligned}E(P\tau P \triangleleft T) &= (m \otimes \mathrm{id} \otimes P\tau P)(\Delta^l_{\mathbb{G}} \otimes \mathrm{id})(P \otimes P)\Delta^l_{\mathbb{G}}(T)(P \otimes P)\\ &= (m \otimes \mathrm{id} \otimes P\tau P)(\mathrm{id} \otimes \Delta^l_{\mathbb{G}})\Delta^l_{\mathbb{G}}(PTP) \text{ (group-likeness)}\\ &= P\tau P \triangleleft E(T).\end{aligned}$$

Conversely, set $m = \epsilon_{\mathbb{G}} \circ E|_{\ell^\infty(\mathbb{G})}$. Since E is upc, m is a state. If we let $E(x) = Pc_x$, where $c_x \in \ell^\infty(\mathbb{G})$, then

$$m(x) = \epsilon_{\mathbb{G}}(Pc_x) = c_x,$$

so $E(x) = Pm(x)$. Since $\triangleleft$ is a lifting of convolution from $\ell^1(\mathbb{G})$ (see [67]), E is left $P\ell^1(\mathbb{G})P$-module. Then, for $f \in \ell^1(\mathbb{G})$,

$$\begin{aligned}m * (PfP)(x) &= (\epsilon_{\mathbb{G}} \otimes \mathrm{id})(E \otimes PfP)\Delta_{\mathbb{G}}(x)\\ &= (\epsilon_{\mathbb{G}} \otimes \mathrm{id})(\mathrm{id} \otimes PfP)\Delta_{\mathbb{G}}(E(x)) = f(PE(x)P) = f(P)m(x).\end{aligned}$$

This shows $P(m \otimes \mathrm{id})\Delta_{\mathbb{G}}(x) = m(x)P$ as desired. □

Remark 5.2.12.

1. We have that Theorem 5.2.11 is a bona fide generalization of Theorem 5.2.11, $i \iff iii$. Indeed, we have that $\mathbb{G}$ is amenable if and only if $\mathbb{C}1 = \widetilde{N}_1$ is an amenable coideal and in this case, $N^L_{\epsilon^u_{\widehat{\mathbb{G}}}} = (\widetilde{N}_1)' \cap L^\infty(\widehat{\mathbb{G}}) = L^\infty(\widehat{\mathbb{G}})$.

2. Suppose $\mathbb{H} \leq \mathbb{G}$ is a quantum subgroup and $\widetilde{N_P} = \ell^\infty(\mathbb{G}/\mathbb{H})$. Then $P = 1_{\mathbb{H}}$ is a central group-like projection achieved from the central support of the morphism $\sigma_{\mathbb{H}} : \ell^\infty(\mathbb{G}) \to \ell^\infty(\mathbb{H})$ implementing the quantum subgroup (see [64]). It follows from the characterizations of compact quasi-subgroups (see [74]) and the work in [64] that there exists an idempotent state ω such that $\lambda_{\widehat{\mathbb{G}}}(\omega) = 1_{\mathbb{H}}$. Also, $\ell^\infty(\mathbb{G}/\mathbb{H})' \cap L^\infty(\widehat{\mathbb{G}}) = L^\infty(\widehat{\mathbb{H}})$ (see [123]). Since $1_{\mathbb{H}}$ is central, $M_{1_{\mathbb{H}}} = \ell^\infty(\mathbb{G}/\mathbb{H})$ and from the work in [65] or Chapter 4, we find that $\mathbb{H}$ is amenable if and only if $M_{1_{\mathbb{H}}}$ is amenable. Then, Theorem 5.2.11 shows that $\mathbb{H}$ is amenable if and only if there exists a left $\mathcal{T}(\ell^2(\mathbb{G}))1_{\mathbb{H}}$-module ucp projection $E : \mathcal{B}(\ell^2(\mathbb{G})) \to 1_{\mathbb{H}}\ell^\infty(\mathbb{G}/\mathbb{H})'$ such that $E(\ell^\infty(\mathbb{G})) = \mathbb{C}1_{\mathbb{H}}$.

5.2.3 The Semi-Lattice of Reduced Idempotent States

We will use the notation $\mathrm{Idem}(\widehat{\mathbb{G}}) \subseteq M^u(\widehat{\mathbb{G}})$ to denote the idempotent states and $\mathrm{Idem}_r(\widehat{\mathbb{G}}) = \mathrm{Idem}(\widehat{\mathbb{G}}) \cap M^r(\widehat{\mathbb{G}})$ to denote the reduced idempotent states. Similarly, we let $\mathrm{Idem}^H(\widehat{\mathbb{G}}) \subseteq \mathrm{Idem}(\widehat{\mathbb{G}})$ denote the Haar idempotents and $\mathrm{Idem}^H_r(\widehat{\mathbb{G}}) = \mathrm{Idem}^H(\widehat{\mathbb{G}}) \cap M^r(\widehat{\mathbb{G}})$. We let

$\text{ZIdem}(\widehat{\mathbb{G}})$ etc denote the central idempotents in $M^u(\widehat{\mathbb{G}})$. We equip $\text{Idem}(\widehat{\mathbb{G}})$ with the following poset from [74]:

$$\mu \leq \nu \text{ if } \mu * \nu = \nu.$$

We have the following equivalent ways of realizing this poset structure.

Lemma 5.2.13. *[74, Lemma 2.1] Let $\mathbb{G}$ be a LCQG. The following are equivalent for $\mu, \nu \in Idem(\mathbb{G})$:*

1. $\mu \leq \nu$;
2. $E_\mu \circ E_\nu = E_\nu$;
3. $N_\nu \subseteq N_\mu$.

Remark 5.2.14. If $\mathbb{G}$ is discrete and $L^\infty(\widehat{E}_i) \subseteq L^\infty(\widehat{\mathbb{G}})$ are right coideals for $i = 1, 2$ with coduals $\widetilde{N}_{P_i} \subseteq \ell^\infty(\mathbb{G})$, then we obtain the additional relation:

$$L^\infty(\widehat{E}_1) \subseteq L^\infty(\widehat{E}_2) \iff \widetilde{N}_{P_2} \subseteq \widetilde{N}_{P_1}.$$

In [74], a meet and join operation was also defined on the idempotent states of a LCQG:

$$\mu \vee \nu = \inf\{\omega \in Irr(\mathbb{G}) : \omega \geq \mu, \omega \geq \nu\}$$

and

$$\mu \wedge \nu = \sup\{\omega \in Irr(\mathbb{G}) : \omega \leq \mu, \omega \leq \nu\}.$$

With respect to this ordering for a DQG $\mathbb{G}$, in both $\text{Idem}(\widehat{\mathbb{G}})$ and $\text{Idem}^H(\widehat{\mathbb{G}})$, we automatically see that $\epsilon^u_{\widehat{\mathbb{G}}}$ is the unique smallest element and $h_{\widehat{\mathbb{G}}}$ is the unique largest element in the sense that

$$\epsilon^u \leq \omega \text{ and } \omega \leq h_{\widehat{\mathbb{G}}} \text{ for all } \omega \in \text{Idem}(\widehat{\mathbb{G}}).$$

As a consequence, we always have that $\mu \vee \nu, \mu \wedge \nu \in \text{Idem}(\widehat{\mathbb{G}})$.

We also have that $h_{\widehat{\mathbb{G}}}$ is the unique largest element in $\text{ZIdem}^H_r(\widehat{\mathbb{G}})$, and so here, for any $\mu, \nu \in \text{ZIdem}^H_r(\widehat{\mathbb{G}})$, we have that $\mu \vee \nu \in \text{ZIdem}^H_r(\widehat{\mathbb{G}})$. What is interesting is finding a unique smallest element of the various reduced idempotent state spaces. For instance, for unimodular $\mathbb{G}$, the smallest element of $\text{Idem}_r(\widehat{\mathbb{G}})$ is $h_{\widehat{\mathbb{G}}}$ if and only if $\widehat{\mathbb{G}}$ has the unique trace property. It turns out that there always exists minimal elements.

Proposition 5.2.15. *Let $\mathbb{G}$ be a DQG. The following hold:*

1. *$\mathrm{Idem}_r(\widehat{\mathbb{G}})$ has minimal elements;*

2. *$\mathrm{Idem}_r^H(\widehat{\mathbb{G}})$ has minimal elements;*

3. *$\mathrm{ZIdem}_r(\widehat{\mathbb{G}})$ has minimal elements.*

Proof. 1. We will first show that there exists minimal elements in $\mathrm{Idem}_r(\widehat{\mathbb{G}})$ using Zorn's lemma. Consider a chain $\{\omega_i : i \in I\}$ in $\mathrm{Idem}_r(\widehat{\mathbb{G}})$, where $\omega_i \leq \omega_j$ if $i \geq j$. Let ω be a weak$*$ cluster point of the bounded net $(\omega_i)_i$. It is straightforward showing ω is an idempotent state, and since $M^r(\widehat{\mathbb{G}})$ is weak$*$ closed, $\omega \in \mathrm{Idem}_r(\widehat{\mathbb{G}})$. What remains is showing $\omega \leq \omega_i$ for all i. Indeed, take $i_0 \in I$, and for $a \in C_r(\widehat{\mathbb{G}})$, suppose that $E_{\omega_{i_0}}(a) = a$. Let $\epsilon > 0$. For $u \in M^u(\widehat{\mathbb{G}})$, find $i \geq i_0$ such that $|u \circ E_\omega(a) - u \circ E_{\omega_i}(a)| < \epsilon$. Then,

$$|u \circ E_\omega(a) - u(a)| = |u \circ E_\omega(a) - u \circ E_{\omega_i}(a)| < \epsilon.$$

We have shown $N_{\omega_{i_0}} \subseteq N_\omega$, which is the desired outcome.

2. If, as above, we instead have $(\omega_i)_i \subseteq \mathrm{Idem}_r^H(\widehat{\mathbb{G}})$, then $\omega \in \mathrm{Idem}_r^H(\widehat{\mathbb{G}})$. Indeed, we can prove that I_ω is self-adjoint and apply Theorem 2.9.21. For this, if $a \in I_\omega$ then,

$$\omega(aa^*) = \lim_i \omega_i(aa^*) = 0,$$

because $\omega \leq \omega_i$ and each I_{ω_i} is self-adjoint for all i.

3. This is a straightforward adjustment of the above Zorn's lemma argument. □

While $\mathrm{Idem}_r(\widehat{\mathbb{G}})$ may have minimal elements, it may not be the case that there exists a minimal element ω where $\omega \leq \mu$ for all $\mu \in \mathrm{Idem}_r(\widehat{\mathbb{G}})$ as the following example illustrates.

Example 5.2.16. Consider the free group on 2 generators, $\mathbb{F}_2 = \langle s_1, s_2 \rangle$. The amenable subgroups $\mathbb{Z} \cong \langle s_1 \rangle$ are maximal amenable subgroups, so that the coideals $VN(\langle s_i \rangle) \subseteq VN(\mathbb{F}_2)$ are coamenable. This means that $1_{\langle s_i \rangle} \in M^r(\widehat{\mathbb{F}_2})$ are minimal (non-Haar, central) idempotent states that are distinct from the Haar state, but are incomparable to one another. In particular, we have that $1_{\langle s_1 \rangle} \wedge 1_{\langle s_2 \rangle} = 1_{\mathbb{F}_2} \notin \mathrm{Idem}_r(\widehat{\mathbb{F}_2})$. So, this gives us an example of a discrete quantum group where the meet operation is well-defined in neither $\mathrm{ZIdem}_r(\widehat{\mathbb{G}})$ nor $\mathrm{Idem}_r(\widehat{\mathbb{G}})$.

It is well-known that $\mathbb{F}_2$ is C^*-simple, thus the Haar state is the minimal (and only) element of $\mathrm{Idem}_r^H(\widehat{\mathbb{F}_2}) = \mathrm{ZIdem}_r^H(\widehat{\mathbb{F}_2})$. We do not know if $\mathrm{Idem}_r^H(\widehat{\mathbb{G}})$ has a smallest element in general.

Given a closed quantum subgroup $\mathbb{H}$ of a DQG $\mathbb{G}$, it is clear from the definition that coamenability of $L^\infty(\widehat{\mathbb{H}})$ as a coideal is just coamenability of $\widehat{\mathbb{H}}$. So, we see that $L^\infty(\widehat{\mathbb{H}})$ is a coamenable coideal if and only if $\ell^\infty(\mathbb{G}/\mathbb{H})$ is relatively amenable.

Recall that $\mathbb{H}$ is normal if and only if $\omega_{\widehat{\mathbb{H}}}$ is Haar. The amenable radical $R_a(\mathbb{G})$ is the largest normal amenable closed quantum subgroup of $\mathbb{G}$ (see [65, Proposition 3.15] and the preceding section). This makes $\ell^\infty(\mathbb{G}/R_a(\mathbb{G}))$ the smallest relatively amenable quotient by a normal closed quantum subgroup. Since $\ell^\infty(\mathbb{G}/\mathbb{H}_1) \subseteq \ell^\infty(\mathbb{G}/\mathbb{H}_2)$ if and only if $L^\infty(\widehat{\mathbb{H}_2}) \subseteq L^\infty(\widehat{\mathbb{H}_1})$ if and only if $\omega_{\widehat{\mathbb{G}/\mathbb{H}_1}} \leq \omega_{\widehat{\mathbb{G}/\mathbb{H}_2}}$, we are able to deduce the following.

Proposition 5.2.17. *Let $\mathbb{G}$ be a DQG. We have that $\omega_{\widehat{\mathbb{G}/R_a(\mathbb{G})}} = \min ZIdem_r^H(\widehat{\mathbb{G}})$.*

5.3 Traces on Quantum Groups

5.3.1 The Kac Property and Canonical Kac Quotient

Let us recall the canonical Kac quotient constructed by Sołtan [120]. Define the ideal

$$I_{Kac} = \{a \in C_u(\widehat{\mathbb{G}}) : \tau(aa^*) = 0 \text{ for every tracial state } \tau \in M^u(\widehat{\mathbb{G}})\}.$$

Then, $\mathbb{H}_{Kac}$ is a Kac quantum subgroup of $\widehat{\mathbb{G}}$ such that $C_u(\widehat{\mathbb{G}})/I_{Kac} \cong C_u(\mathbb{H}_{Kac})$. We call $\mathbb{H}_{Kac}$ the **canonical Kac quotient** of $\widehat{\mathbb{G}}$. Sołtan showed that $\mathbb{H}_{Kac}$ is a closed quantum subgroup of $\widehat{\mathbb{G}}$, and it follows more or less from the definitions that every Kac closed quantum subgroup of $\widehat{\mathbb{G}}$ is a closed quantum subgroup of $\mathbb{H}_{Kac}$. We denote the corresponding Haar idempotent by ω_{Kac}.

Remark 5.3.1. If we let

$$I^r_{Kac} = \{a \in C_r(\widehat{\mathbb{G}}) : \tau(aa^*) = 0 \text{ for every tracial state } \tau \in M^r(\widehat{\mathbb{G}})\}$$

when tracial states exist, then Sołtan's construction yields a CQG C^*-algebra $C_{rKac}(\mathbb{H}_{Kac}) \cong C_r(\widehat{\mathbb{G}})/I^r_{Kac}$. Because of the quotient $C_r(\widehat{\mathbb{G}}) \to C_{rKac}(\mathbb{H}_{Kac})$, if $\widehat{\mathbb{G}}$ is not coamenable, then $C_{rKac}(\mathbb{H}_{Kac}) \neq C_u(\mathbb{H}_{rKac})$. On the other hand, we have been unable to determine whether or not $C_{rKac}(\mathbb{H}_{Kac}) \neq C_r(\mathbb{H}_{Kac})$, i.e., whether or not $C_{rKac}(\mathbb{H}_{Kac})$ is exotic, without the assumption that ω is the unique tracial state on $C_r(\widehat{\mathbb{G}})$.

Recall that because of Theorem 2.9.21, tracial idempotent states are automatically Haar. In fact, we see that $\mathbb{H}$ is a Kac closed quantum subgroup of $\widehat{\mathbb{G}}$ if and only if $\omega_{\widehat{\mathbb{G}}/\mathbb{H}} = h_{\mathbb{H}} \circ \pi^u_{\mathbb{H}}$ is tracial. Then if $\mathbb{G}$ is unimodular, we find that an idempotent state is Haar if and only if it is tracial.

Remark 5.3.2. If we have a tracial state $\tau \in M^u(\widehat{\mathbb{G}})$, then the idempotent state achieved by taking a weak$*$ cluster point of the Cesaro sums $\frac{1}{n}\sum_{k=1}^n \tau^{*k}$ is a tracial idempotent state (see [94]). In particular, if a tracial state exists, then a tracial Haar idempotent exists.

Proposition 5.3.3. *The following hold:*

1. *$C_r(\widehat{\mathbb{G}})^*$ has a tracial state if and only if there exists a Kac closed quantum subgroup $\mathbb{H} \leq \widehat{\mathbb{G}}$ such that $\widehat{\mathbb{G}}/\mathbb{H}$ is co-amenable;*

2. *$C_r(\widehat{\mathbb{G}})^*$ has a unique idempotent tracial state if and only if $\mathbb{H}_{Kac}$ is the only Kac closed quantum subgroup such that $\widehat{\mathbb{G}}/\mathbb{H}_{Kac}$ is coamenable;*

3. *$C_r(\widehat{\mathbb{G}})^*$ has a unique tracial state if and only if $\mathbb{H}_{Kac}$ is the only Kac closed quantum subgroup such that $\widehat{\mathbb{G}}/\mathbb{H}_{Kac}$ is coamenable and $C_{rKac}(\mathbb{H}_{Kac}) = C_r(\mathbb{H}_{Kac})$.*

Proof. 1. As discussed in Remark 5.3.2, a tracial Haar idempotent exists, and hence a Kac closed quantum subgroup with coamenable quotient. Conversely, $\omega_{\widehat{\mathbb{G}}/\mathbb{H}}$ is a tracial state.

2. Suppose $C_r(\widehat{\mathbb{G}})^*$ has a unique idempotent tracial state. Then, for any Kac closed quantum subgroup $\mathbb{H} \leq \widehat{\mathbb{G}}$ where $\widehat{\mathbb{G}}/\mathbb{H}$ is coamenable, we find that $\omega_{\widehat{\mathbb{G}}/\mathbb{H}} = \omega_{Kac}$ by uniqueness. Conversely, if ω is a tracial idempotent state, it is a tracial Haar idempotent and it has an associated Kac closed quantum subgroup $\mathbb{H} \leq \widehat{\mathbb{G}}$ such that $\widehat{\mathbb{G}}/\mathbb{H}$ is coamenable. Then $\mathbb{H} = \mathbb{H}_{Kac}$, which implies $\omega_{\widehat{\mathbb{G}}/\mathbb{H}} = \omega_{Kac}$ by uniqueness.

3. The forward direction is similar to the forward direction of 2.. What we must check is that $C_{rKac}(\mathbb{H}_{Kac}) = C_r(\mathbb{H}_{Kac})$. Since $\omega_{Kac} \in C_r(\widehat{\mathbb{G}})^*$ is the unique trace,

$$I^r_{Kac} = \{a \in C_r(\widehat{\mathbb{G}}) : \omega_{Kac}(a^*a) = 0\} =: I^r_{\omega_{Kac}}$$

and $C_r(\widehat{\mathbb{G}})/I^r_{\omega_{Kac}} = C_r(\mathbb{H}_F)$.

Conversely, let $\tau \in C_r(\widehat{\mathbb{G}})^*$ be a tracial state. The closed quantum subgroups of $\mathbb{H}_{Kac}$ are the Kac closed quantum subgroups of $\widehat{\mathbb{G}}$. Moreover, if $\mathbb{H}_{Kac}/\mathbb{H}$ were to be coamenable, then so would be $\widehat{\mathbb{G}}/\mathbb{H}$ because then we obtain the quotient map

$$\pi = \pi^r_{\mathbb{H}} \circ \pi^r_{\mathbb{H}_{Kac}} : C_r(\widehat{\mathbb{G}}) \to C_r(\mathbb{H}_{Kac}) \to C_r(\mathbb{H}),$$

where $\pi^r_{\mathbb{H}} : C_r(\mathbb{H}_{Kac}) \to C_r(\mathbb{H})$ and $\pi^r_{\mathbb{H}_{Kac}} : C_r(\widehat{\mathbb{G}}) \to C_r(\mathbb{H}_{Kac})$ are as in Theorem 5.2.8. It is straightforward to check that $\pi \circ \Gamma_{\mathbb{G}} = \Gamma_{\widehat{\mathbb{H}}} \circ \pi$ so we deduce that $\widehat{\mathbb{G}}/\mathbb{H}$ is coamenable.

In particular, $(\mathbb{H}_{Kac})_F = \mathbb{H}_{Kac}$, i.e., the action of $\widehat{\mathbb{H}_{Kac}}$ on its Furstenburg boundary is faithful, so $C_r(\mathbb{H}_{Kac})$ has a unique trace [65, Theorem 5.3]. Since τ is a tracial state, and $I^r_{Kac} \subseteq \ker(\tau)$ using a standard Cauchy-Schwarz argument, there exists a state $\tilde{\tau} \in C_{rKac}(\mathbb{H}_{Kac})^*$ such that $\tilde{\tau} \circ q = \tau$ where $q : C_r(\widehat{\mathbb{G}}) \to C_{rKac}(\mathbb{H}_{Kac}) = C_{rKac}(\mathbb{H}_{Kac})$ is the quotient map discussed in the above remark. Clearly $\tilde{\tau}$ is tracial and since $C_{rKac}(\mathbb{H}_{Kac}) = C_r(\mathbb{H}_{Kac})$, $\tilde{\tau} = h_{\mathbb{H}_{Kac}}$ by uniqueness. Therefore, $\tau = \omega_{Kac}$. □

Remark 5.3.4. Notice that we showed that $\widehat{\mathbb{G}}/\mathbb{H}_{Kac}$ is coamenable if there exists a tracial state in $C_r(\widehat{\mathbb{G}})$.

5.3.2 $\mathbb{G}$-Invariant States

For the moment, we consider a general LCQG $\mathbb{G}$.

Definition 5.3.5. A state $\mu \in M^r(\widehat{\mathbb{G}})$ or $\mu \in L^\infty(\widehat{\mathbb{G}})^*$ is **$\mathbb{G}$-invariant** if $(\mathrm{id} \otimes \mu)\Delta^l_{\mathbb{G}} = \mu$.

Remark 5.3.6. Instead of $\mathbb{G}$-invariance, Crann [22] used the terminology **inner invariance**, and said that $\mathbb{G}$ is **topologically inner amenable** if there exists an inner invariant state in $M^r(\widehat{\mathbb{G}})$ and is **inner amenable** if there exists a state in $L^\infty(\widehat{\mathbb{G}})^*$.

Remark 5.3.7. Let $\mathbb{G} = G$ be a discrete group. Then for $\hat{a} \in C^*_r(G)$,

$$\Delta^l_G(\hat{a}) = \sum_{s \in G} \delta_s \otimes \lambda(s)\hat{a}\lambda(s)^*.$$

We recover the conjugation action of G on $C^*_r(G)$ as follows: for $s_0 \in G$,

$$(\delta_{s_0} \otimes \mathrm{id})\Delta^l_G(\hat{a}) = \lambda(s_0)\hat{a}\lambda(s_0)^* = s_0 \cdot \hat{a}.$$

We find a noncommutative analogue of the conjugation action by the irreducibles of $\widehat{\mathbb{G}}$. We define the convolution product:

$$f \triangleleft \mu = (f \otimes \mu)\Delta^l_{\mathbb{G}},\ f \in \ell^1(\mathbb{G}),\ \mu \in M^r(\widehat{\mathbb{G}}).$$

If we let $\delta^\pi_{i,j}$ be the dual basis element with respect to the matrix units $E^\pi_{i,j}$, then $\delta^\pi_{i,j} \triangleleft \mu = \mu \circ L^\pi_{i,j}$, where we recall that $L^\pi_{i,j}(\hat{a}) = \sum_{t=1}^{n_\pi} u^\pi_{i,t}\hat{a}(u^\pi_{j,t})^*$. In particular, we immediately see the following.

Proposition 5.3.8. *Let $\mathbb{G}$ be a DQG. We have that $\mu \in M^r(\mathbb{G})$ is $\mathbb{G}$-invariant if and only if $\mu \circ L^\pi_{i,j} = \delta^\pi_{i,j}\mu$ for every $\pi \in Irr(\widehat{\mathbb{G}})$ and i, j.*

Remark 5.3.9. The above characterization of $\mathbb{G}$-invariance allows us to easily see that that $h_{\widehat{\mathbb{G}}}$ is $\mathbb{G}$-invariant whenever it is tracial. Indeed, recall that the condition $(U^\pi)^t\overline{U^\pi} = I_{n_\pi} = \overline{U^\pi}(U^\pi)^t$ for every irreducible π is equivalent to $\mathbb{G}$ being Kac. Then for $a \in \mathrm{Pol}(\widehat{\mathbb{G}})$,

$$h_{\widehat{\mathbb{G}}}(\sum_{t=1}^{n_\pi} u^\pi_{i,t} a (u^\pi_{j,t})^*) = h_{\widehat{\mathbb{G}}}(a \sum_{t=1}^{n_\pi} (u^\pi_{j,t})^* u^\pi_{i,t}) = \delta_{i,j} h_{\widehat{\mathbb{G}}}(a).$$

We will remind the reader of the scaling group of a DQG, along with an explicit formula.

Definition 5.3.10. Let $\mathbb{G}$ be a DQG. The scaling group of $\widehat{\mathbb{G}}$ is the one-parameter group of automorphisms $(\tau_t)_{t\in\mathbb{R}}$ of $L^\infty(\widehat{\mathbb{G}})$, with analytic extension to $\mathbb{C}$, that satisfies

$$\tau_z(u^\pi_{i,j}) = \sum_{k,l=1}^{n_\pi} (F^\pi_{i,k})^{iz} (F^\pi_{l,j})^{-iz} u^\pi_{k,l}.$$

For a discrete group G, the G-invariant functionals on $C^*_r(G)$ are exactly the tracial states on $C^*_r(G)$. For unimodular discrete $\mathbb{G}$, a state is tracial if and only if it is $\mathbb{G}$-invariant [65, Lemma 5.2]. This changes for non-unimodular $\mathbb{G}$, however. We have that the $\mathbb{G}$-invariant states are the KMS states of the scaling group (with inverse temperature 1) on $C_r(\widehat{\mathbb{G}})$ [64, Lemma 5.2]. More precisely, a state $\tau \in C_r(\widehat{\mathbb{G}})^*$ is $\mathbb{G}$-invariant if and only if $\tau(ab) = \tau(\tau_i(b)a)$ for every $a, b \in C_r(\widehat{\mathbb{G}})$.

Remark 5.3.11. Let $\tau \in C_r(\widehat{\mathbb{G}})^*$ be a $\mathbb{G}$-invariant state. Since it is KMS, as with tracial states (see Remark 5.3.2) we find that a weak$*$ limit ω of the Cesaro sums of τ is KMS, and hence ω is a $\mathbb{G}$-invariant idempotent state. Indeed, it it can be shown that $(\tau_z \otimes \tau_z)\Delta_{\widehat{\mathbb{G}}} = \Delta_{\widehat{\mathbb{G}}} \circ \tau_z$ for every $z \in \mathbb{C}$. Then, for $a, b \in C_r(\widehat{\mathbb{G}})$,

$$\begin{aligned}
\tau * \tau(ab) &= (\tau \otimes \tau)\left(\Delta_{\widehat{\mathbb{G}}}(a)\Delta_{\widehat{\mathbb{G}}}(b)\right) \\
&= (\tau \otimes \tau)\left([(\tau_i \otimes \tau_i)\Delta_{\widehat{\mathbb{G}}}(b)]\Delta_{\widehat{\mathbb{G}}}(a)\right) \\
&= (\tau \otimes \tau)\Delta_{\widehat{\mathbb{G}}}(\tau_i(b))\Delta_{\widehat{\mathbb{G}}}(a) \\
&= \tau * \tau(\tau_i(b)a).
\end{aligned}$$

So, the convolution powers of a KMS state is still KMS, and hence so would be ω.

A straightforward application of the Cauchy-Schwarz inequality informs us that ω is a Haar idempotent: indeed, if $a \in I_\omega \cap \mathrm{Pol}(\widehat{\mathbb{G}})$, then

$$|\omega(aa^*)|^2 = |\omega(\tau_i(a^*)a)|^2 \leq \omega(\tau_i(a^*)^*\tau_i(a^*))\omega(a^*a) = 0,$$

so $a^* \in I_\omega \cap \mathrm{Pol}(\widehat{\mathbb{G}})$.

The Haar state is $\mathbb{G}$-invariant if and only if it is tracial. This was proven with [65, Lemma 5.2] and is also something that is relatively straightforward to deduce from [73, Corollary 3.20]. It turns out this can be witnessed by Haar states realized as Haar idempotents on larger CQGs.

Proposition 5.3.12. *Let $\mathbb{G}$ be a DQG. A Haar idempotent state on $C_r(\widehat{\mathbb{G}})$ is $\mathbb{G}$-invariant if and only if it is tracial.*

Before completing the proof, we remind the reader of important features of closed quantum subgroups of CQGs (see [123] for the compact case and [28] for the locally compact case). If $\mathbb{H} \leq \widehat{\mathbb{G}}$ and $\widehat{\mathbb{G}}/\mathbb{H}$ is coamenable (so $\pi_{\mathbb{H}} : C_r(\widehat{\mathbb{G}}) \to C_r(\mathbb{H})$), then there is an injective, normal unital $*$-homomorphism $\gamma_{\mathbb{H}} : \ell^\infty(\widehat{\mathbb{H}}) \to \ell^\infty(\mathbb{G})$ satisfying $\Delta_{\mathbb{G}} \circ \gamma_{\mathbb{H}} = (\gamma_{\mathbb{H}} \otimes \gamma_{\mathbb{H}})\Delta_{\widehat{\mathbb{H}}}$ so that $\gamma_{\mathbb{H}}(\ell^\infty(\widehat{\mathbb{H}}))$ is the corresponding two-sided coideal mentioned in Section 2.4. The pre-adjoint is a surjective algebra homomorphism $(\gamma_{\mathbb{H}})_* : \ell^1(\mathbb{G}) \to \ell^1(\widehat{\mathbb{H}})$. It turns out that $(\gamma_{\mathbb{H}} \otimes \mathrm{id})W_{\widehat{\mathbb{H}}} = (\mathrm{id} \otimes \pi_{\mathbb{H}})W_{\mathbb{G}}$.

Proof. Let $\omega = h_{\mathbb{H}} \circ \pi_{\mathbb{H}}$ be a tracial Haar idempotent. In particular, $\mathbb{H}$ is Kac. We have that $U^\pi_{\mathbb{H}} = (\pi_{\mathbb{H}} \otimes \mathrm{id})U^\pi = [\pi_{\mathbb{H}}(u^\pi_{i,j})]$ is a unitary corepresentation matrix of $\mathbb{H}$. Then, since $\mathbb{H}$ is Kac and $*$-representations decompose into irreducibles, we have $\overline{U^\pi_{\mathbb{H}}}(U^\pi_{\mathbb{H}})^t = I_{n_\pi}$. Therefore, for $a \in C_r(\widehat{\mathbb{G}})$,

$$\begin{aligned} h_{\mathbb{H}} \circ \pi_{\mathbb{H}}(L^\pi_{i,j}(a)) &= h_{\mathbb{H}}(\sum_{t=1}^{n_\pi} \pi_{\mathbb{H}}(u^\pi_{i,t})\pi_{\mathbb{H}}(a)\pi_{\mathbb{H}}((u^\pi_{j,t})^*)) \\ &= h_{\mathbb{H}}(\pi_{\mathbb{H}}(a)\sum_{t=1}^{n_\pi} \pi_{\mathbb{H}}((u^\pi_{j,t})^*)\pi_{\mathbb{H}}(u^\pi_{i,t})) = \delta_{i,j} h_{\mathbb{H}} \circ \pi_{\mathbb{H}}(a). \end{aligned}$$

Conversely, assume that $h_{\mathbb{H}} \circ \pi_{\mathbb{H}}$ is $\mathbb{G}$-invariant. Given a state $f \in \ell^1(\widehat{\mathbb{H}})$, find a state $\varphi \in \ell^1(\mathbb{G})$ such that $\varphi \circ \gamma_{\mathbb{H}} = f$. Then, for $a \in C_r(\widehat{\mathbb{G}})$

$$\begin{aligned} (f \otimes h_{\mathbb{H}})\Delta^l_{\widehat{\mathbb{H}}}(\pi_{\mathbb{H}}(a)) &= (\varphi \circ \gamma_{\mathbb{H}} \otimes h_{\mathbb{H}})W^*_{\widehat{\mathbb{H}}}(1 \otimes \pi_{\mathbb{H}}(a))W_{\widehat{\mathbb{H}}} \\ &= (\varphi \otimes h_{\mathbb{H}})(\gamma_{\mathbb{H}} \otimes \mathrm{id})W^*_{\widehat{\mathbb{H}}}(1 \otimes \pi_{\mathbb{H}}(a))(\gamma_{\mathbb{H}} \otimes \mathrm{id})W_{\widehat{\mathbb{H}}} \\ &= (\varphi \otimes h_{\mathbb{H}})(\mathrm{id} \otimes \pi_{\mathbb{H}})(W^*_{\mathbb{G}})(1 \otimes \pi_{\mathbb{H}}(a))(\mathrm{id} \otimes \pi_{\mathbb{H}})(W_{\mathbb{G}}) \\ &= (\varphi \otimes h_{\mathbb{H}} \circ \pi_{\mathbb{H}})W^*_{\mathbb{G}}(1 \otimes a)W_{\mathbb{G}} \\ &= h_{\mathbb{H}}(\pi_{\mathbb{H}}(a)). \end{aligned}$$

So, $h_{\mathbb{H}}$ is $\widehat{\mathbb{H}}$-invariant which implies it is tracial, and we deduce $h_{\mathbb{H}} \circ \pi_{\mathbb{H}}$ is tracial. □

By taking Cesaro sums of convolution powers of traces or KMS states, we immediately deduce the following with Proposition 5.3.12 in hand, which, despite the apparent disparity between $\mathbb{G}$-invariant states and tracial states, a relationship remains none-the-less.

Corollary 5.3.13. *Let $\mathbb{G}$ be a DQG. Then $C_r(\widehat{\mathbb{G}})$ has $\mathbb{G}$-invariant state if and only if it has a tracial state.*

This partially resolves an open problem from [96, 22] at the level of CQGs, which generalizes the equivalence between nuclearity of $C_r(\widehat{G})$ with amenability of a discrete group G.

Corollary 5.3.14. *Let $\mathbb{G}$ be a DQG. We have that $C_r(\widehat{\mathbb{G}})$ is nuclear and has a tracial state if and only if $\mathbb{G}$ is amenable.*

Proof. It is was proven in [22] (combined with [122] and building off the work in [96]) that $C_r(\widehat{\mathbb{G}})$ is nuclear and has a $\mathbb{G}$-invariant sate if and only if $\mathbb{G}$ is amenable. The proof is complete with Corollary 5.3.13. □

This also leads to a statement regarding simplicity of the reduced C^*-algebra of $\mathbb{G}$.

Corollary 5.3.15. *Let $\mathbb{G}$ be a non-unimodular discrete quantum group. If $C_r(\widehat{\mathbb{G}})$ is simple it has no $\mathbb{G}$-invariant states.*

Proof. Suppose a $\mathbb{G}$-invariant state τ exists. As shown in Remark 5.3.11, a $\mathbb{G}$-invariant Haar idempotent state ω exists. By simplicity, ω is faithful, and hence $\omega = h_{\widehat{\mathbb{G}}}$, which is impossible because $h_{\widehat{\mathbb{G}}}$ is not $\mathbb{G}$-invariant. □

For the remainder of this subsection, we will prove the right Haar weight of a DQG $\mathbb{G}$ is $\widehat{\mathbb{G}}$-invariant if and only if $\mathbb{G}$ is unimodular. By $\widehat{\mathbb{G}}$-invariance, we mean that $(\mathrm{id} \otimes h_R)\Delta^l_{\widehat{\mathbb{G}}}(X) = h_R(X)$ for every $X \in c_{00}(\mathbb{G})$. We should note that this result is also obtained by easy adjustments of the argument used for [65, Lemma 5.2]. Our argument uses only the definition of the right Haar weight h_R and a well-known characterization of the Kac property whereas in [65, Lemma 5.2] a general technique involving the modular and scaling groups is used (see Remark 5.3.17).

Before getting to our result, we wish to write out the action of $\widehat{\mathbb{G}}$ on $\ell^\infty(\mathbb{G})$, $\Delta^l_{\widehat{\mathbb{G}}}$, more explicitly. Given $X = [x_{i,j}] \in M_{n_\pi}$,

$$\begin{aligned}\Delta^l_{\widehat{\mathbb{G}}}(X) = W^*_{\widehat{\mathbb{G}}}(1 \otimes X)W_{\widehat{\mathbb{G}}} &= n_\pi \sum_{i,j,k,l=1}^{n_\pi} (u^\pi_{i,j})^* u^\pi_{k,l} \otimes E^\pi_{j,i} X E^\pi_{k,l} \\ &= \sum_{i,j,k,l=1}^{n_\pi} (u^\pi_{i,j})^* u^\pi_{k,l} \otimes x_{i,k} E^\pi_{j,l}.\end{aligned}$$

Recall that $\mathbb{G}$ is Kac if and only if for all $\pi \in Irr(\widehat{\mathbb{G}})$ we have $((U^\pi)^t)^{-1} = \overline{U^\pi}$. Recall then that if $\widehat{\mathbb{G}}$ is unimodular:

$$h(X) = tr(I_{n_\pi})tr(X) = n_\pi X, \ X \in M_{n_\pi}.$$

Theorem 5.3.16. *Let $\mathbb{G}$ be a DQG. We have that $\mathbb{G}$ is unimodular if and only if h_R is $\widehat{\mathbb{G}}$-invariant.*

Proof. Fix $\pi \in Irr(\widehat{\mathbb{G}})$ and $X \in M_{n_\pi} \subseteq L^\infty(\widehat{\mathbb{G}})$. Suppose that $\widehat{\mathbb{G}}$ is unimodular, which implies $\mathbb{G}$ is Kac, so the F-matrix for π is trivial. Then

$$\begin{aligned}(\mathrm{id} \otimes h)\Delta^l_{\widehat{\mathbb{G}}}(X) &= n_\pi \sum_{i,j,k,l=1}^{n_\pi} (u^\pi_{i,j})^* u^\pi_{k,l} x_{i,k} \,\mathrm{tr}(E^\pi_{j,l}) = n_\pi \sum_{i,k=1}^{n_\pi} x_{i,k} \overbrace{\left(\sum_{j=1}^{n_\pi} (u^\pi_{i,j})^* u^\pi_{k,j}\right)}^{=\delta_{i,k}} \\ &= n_\pi \sum_{i=1}^{n_\pi} x_{i,i} = h(X).\end{aligned}$$

We note that the third equation above follows from the Kac property (in particular, from $((U^\pi)^t)^{-1} = \overline{U^\pi}$). Weak$*$ density of $\oplus_{\pi \in Irr(\mathbb{G})} M_{n_\pi}$ in $\ell^\infty(\mathbb{G})$ gives us the result.

Conversely, choose a representative π so that $F_\pi = diag(\lambda_1, \ldots, \lambda_{n_\pi})$. Given $X \in M_{n_\pi}$, a simple calculation shows $h_R(X) = \mathrm{tr}(F_\pi) \sum_{i=1}^{n_\pi} \lambda_i^{-1} x_{i,i}$. By assumption,

$$\begin{aligned}h_R(X) &= \sum_{i,j,k,l=1}^{n_\pi} (u^\pi_{i,j})^* u^\pi_{k,l} x_{i,k} \,\mathrm{tr}(F_\pi)\,\mathrm{tr}(F_\pi E^\pi_{j,l}) \\ &= \sum_{i,j,k=1}^{n_\pi} (u^\pi_{i,j})^* u^\pi_{k,j} x_{i,k} \,\mathrm{tr}(F_\pi)\lambda_j.\end{aligned}$$

Then, since the Haar state $h_{\widehat{\mathbb{G}}}$ is a state,

$$h_R(X) = (h_{\widehat{\mathbb{G}}} \otimes h_R)\Delta^l_{\widehat{\mathbb{G}}}(X) = \sum_{i,j,k=1}^{n_\pi} h_{\widehat{\mathbb{G}}}((u^\pi_{i,j})^* u^\pi_{k,j}) x_{i,k} \operatorname{tr}(F_\pi)\lambda_j$$
$$= \sum_{i,j=1}^{n_\pi} \frac{\lambda_i^{-1}}{\operatorname{tr}(F_\pi)} x_{i,i} \operatorname{tr}(F_\pi)\lambda_j = \sum_{i=1}^{n_\pi} \operatorname{tr}(F_\pi)\lambda_i^{-1} x_{i,i} = h_L(X).$$

By weak$*$ density of $\oplus_{\pi \in Irr(\mathbb{G})} M_{n_\pi}$ in $\ell^\infty(\mathbb{G})$, we have the result. □

Remark 5.3.17. In the proof of [65, Lemma 5.2], it was shown that $\mathbb{G}$-invariance of $h_{\widehat{\mathbb{G}}}$ is equivalent to having $\tau_i^{\widehat{\mathbb{G}}} = \sigma_i$ where σ is the modular group of $h_{\widehat{\mathbb{G}}}$. If we replace $\mathbb{G}$ with $\widehat{\mathbb{G}}$ and h with h_R, then mild adjustments of the proof will obtain that h_R is $\widehat{\mathbb{G}}$-invariant if and only if $\tau_i^{\mathbb{G}} = \sigma_i^R$. It turns out that $\tau_i^{\mathbb{G}} = \sigma_i^R$ is equivalent to unimodularity of $\mathbb{G}$.

5.3.3 Existence and Uniqueness of Traces

Definition 5.3.18. Let $\mathbb{G}$ be a DQG and $\mathbb{H}$ a closed quantum subgroup of $\widehat{\mathbb{G}}$. We say a state $\omega \in M^u(\widehat{\mathbb{G}})$ **concentrates** on $\widehat{\mathbb{G}}/\mathbb{H}$ if $\tau \circ R_{\epsilon_{\widehat{\mathbb{G}}} - \omega_{\widehat{\mathbb{G}}/\mathbb{H}}}|_{\mathrm{Pol}(\widehat{\mathbb{G}})} = 0 = \tau \circ L_{\epsilon_{\widehat{\mathbb{G}}} - \omega_{\widehat{\mathbb{G}}/\mathbb{H}}}|_{\mathrm{Pol}(\widehat{\mathbb{G}})}$.

The following was essentially shown in the proof of [65, Theorem 5.3]. We complete the proof here.

Theorem 5.3.19. *Let $\mathbb{G}$ be a DQG. Every $\mathbb{G}$-invariant state $\tau \in C_r(\widehat{\mathbb{G}})^*$ concentrates on $\widehat{\mathbb{G}}/\mathbb{H}_F$.*

Proof. Let τ be a $\mathbb{G}$-invariant state, and using $\mathbb{G}$-injectivity of $C(\partial_F(\mathbb{G}))$, we obtain a ucp $\mathbb{G}$-equivariant extension $M_\tau : C(\partial_F(\mathbb{G})) \rtimes_r \mathbb{G} \to C(\partial_F(\mathbb{G}))$ of τ. It was shown in the proof of [65, Theorem 5.3] that for all $y \in \widetilde{N}_F$, $\lambda_{\widehat{\mathbb{G}}}(\tau)y = \epsilon_{\mathbb{G}}(y)x$. We remind the reader how this is done here.

By $\mathbb{G}$-rigidity, the restriction of M_τ to $\alpha(C(\partial_F(\mathbb{G}))$ is equal to α^{-1}, so we conclude that $\alpha(C(\partial_F(\mathbb{G}))$ lies in the multiplicative domain of M_τ. In particular,

$$M_\tau(\alpha(x)(\hat{a} \otimes 1)) = \tau(\hat{a})x = M_\tau((\hat{a} \otimes 1)\alpha(x))$$

for all $x \in C(\partial_F(\mathbb{G}))$ and $\hat{a} \in C_r(\widehat{\mathbb{G}})$. Let β be the coaction of $\mathbb{G}$ on $C(\partial_F(\mathbb{G})) \rtimes_r \mathbb{G}$ (see Section 2.2). The equation $(\mathrm{id} \otimes \beta)\beta = (\Delta_{\mathbb{G}} \otimes \mathrm{id})\beta$ implies that

$$(W^*_{\mathbb{G}} \otimes 1)(1 \otimes \alpha(x)) = \beta(\alpha(x))(W^*_{\mathbb{G}} \otimes 1), \ x \in C(\partial_F(\mathbb{G})).$$

By applying $\mathrm{id}\otimes M_\tau$ to both sides of the above equation and using $\beta|_{\alpha(C(\partial_F(\mathbb{G}))} = \mathrm{id}\otimes\alpha$, we obtain,

$$\alpha(x)((\mathrm{id}\otimes\tau)(W_{\mathbb{G}}^*)\otimes 1) = (\mathrm{id}\otimes\tau)(W_{\mathbb{G}}^*)\otimes x = ((\mathrm{id}\otimes\tau)(W_{\mathbb{G}}^*)\otimes 1)\alpha(x).$$

Therefore $(\mathrm{id}\otimes\tau)(W_{\mathbb{G}}^*)\mathcal{P}_\mu(x) = \mu(x)(\mathrm{id}\otimes\tau)(W_{\mathbb{G}}^*) = \mathcal{P}_\mu(x)(\mathrm{id}\otimes\tau)(W_{\mathbb{G}}^*)$ for every $\mu\in C(\partial_F(\mathbb{G}))^*$ and $x\in C(\partial_F(\mathbb{G}))$. Then, since $\mu(x) = \epsilon_{\mathbb{G}}\circ\mathcal{P}_\mu(x)$ and $W_{\widehat{\mathbb{G}}} = \Sigma(W_{\mathbb{G}}^*)$, we deduce that

$$\epsilon_{\mathbb{G}}(y)\lambda_{\widehat{\mathbb{G}}}(\tau) = \lambda_{\widehat{\mathbb{G}}}(\tau)y = y\lambda_{\widehat{\mathbb{G}}}(\tau), \text{ for all } y\in N_F.$$

In particular, for $P_F = \lambda_{\widehat{\mathbb{G}}}(\omega_F)\in\widetilde{N}_F$,

$$\lambda_{\widehat{\mathbb{G}}}(\tau*\omega_F) = \lambda_{\widehat{\mathbb{G}}}(\tau)P_F = \lambda_{\widehat{\mathbb{G}}}(\tau) = \lambda_{\widehat{\mathbb{G}}}(\omega_F*\tau).$$

Then, $\tau\circ R_{\epsilon_{\widehat{\mathbb{G}}}-\omega_F}|_{\mathrm{Pol}(\widehat{\mathbb{G}})} = \tau*(\epsilon_{\widehat{\mathbb{G}}}-\omega_F)|_{\mathrm{Pol}(\widehat{\mathbb{G}})} = 0 = \tau\circ L_{\epsilon_{\widehat{\mathbb{G}}}-\omega_F}|_{\mathrm{Pol}(\widehat{\mathbb{G}})}$. □

Remark 5.3.20. 1. Arrange a set of representatives of the irreducibles of $\widehat{\mathbb{G}}$ so that $\lambda_{\widehat{\mathbb{G}}}(\omega_F)$ is diagonal (see Lemma 3.3.5). Then Theorem 5.3.19 says that $\tau(u_{i,j}^\pi) = 0$ whenever $u_{i,j}^\pi\notin\mathrm{Pol}(\widehat{\mathbb{G}}/\mathbb{H}_F)$ or $u_{i,j}^\pi\notin\mathrm{Pol}(\mathbb{H}_F\backslash\widehat{\mathbb{G}})$.

More specifically, let $\mathbb{H}$ be a Kac closed quantum subgroup of $\widehat{\mathbb{G}}$ where $\widehat{\mathbb{G}}/\mathbb{H}$ is coamenable. From Theorem 5.3.19 and Proposition 5.3.12, we have that the associated (tracial) Haar idempotent $\omega_{\widehat{\mathbb{G}}/\mathbb{H}}\in C_r(\widehat{\mathbb{G}})$ satisfies $\omega_{\widehat{\mathbb{G}}/\mathbb{H}}*\omega_F = \omega_{\widehat{\mathbb{G}}/\mathbb{H}}$. This means that $L^\infty(\widehat{\mathbb{G}}/\mathbb{H})\subseteq L^\infty(\widehat{\mathbb{G}}/\mathbb{H}_F)$ using Lemma 5.2.13, and so $\mathbb{H}_F$ is a closed quantum subgroup of $\mathbb{H}$.

2. The above observation makes Theorem 5.3.19 a noncommutative version of [15, Theorem 4.1], which states that every tracial state of $C_r(G)$, where G is a discrete group, concentrates on the amenable radical of G, i.e., $\tau(\lambda(s)) = 0$ for every $s\in G\setminus R_a(G)$.

From here, we can completely settle the existence and uniqueness of traces for arbitrary discrete quantum groups.

Corollary 5.3.21. *Let $\mathbb{G}$ be a DQG. Then $\mathbb{H}_F$ is a closed quantum subgroup of every Kac closed quantum subgroup $\mathbb{H}$ of $\widehat{\mathbb{G}}$, where $\widehat{\mathbb{G}}/\mathbb{H}$ is coamenable. In particular, the following hold whenever $\widehat{\mathbb{G}}/\mathbb{H}_F$ is coamenable:*

1. *$\mathbb{H}_F$ is Kac if and only if $C_r(\widehat{\mathbb{G}})$ has a tracial state;*
2. *$\mathbb{H}_F = \mathbb{H}_{Kac}$ and $C_{rKac}(\mathbb{H}_F) = C_r(\mathbb{H}_F)$ if and only if $C_r(\widehat{\mathbb{G}})$ has a unique tracial state.*

3. $\mathbb{H}_F = \mathbb{H}_{Kac}$ *if and only if* $C_r(\widehat{\mathbb{G}})$ *has a unique idempotent tracial state.*

Proof. The statements regarding $\mathbb{G}$-invariant states are due to Proposition 5.3.12.

1. If $C_r(\widehat{\mathbb{G}})$ has a tracial state, then Theorem 5.3.3 implies that a closed quantum subgroup $\mathbb{H}$ such that $\widehat{\mathbb{G}}/\mathbb{H}$ is coamenable exists, from which we have that $\mathbb{H}_F$ is Kac because it is a closed quantum subgroup of $\mathbb{H}$. Conversely, if $C_r(\widehat{\mathbb{G}})$ has no tracial states, then $\mathbb{H}_F$ could not be Kac because then $\omega_F \in C_r(\widehat{\mathbb{G}})^*$ would be tracial.

2. Recall from the proof of Theorem 5.3.3 that the extistence of a tracial state implies $\widehat{\mathbb{G}}/\mathbb{H}_{Kac}$ is coamenable. Then, as discussed in Remark 5.3.20, $\mathbb{H}_F$ is a closed quantum subgroup of $\mathbb{H}_{Kac}$. If $C_r(\widehat{\mathbb{G}})$ has a unique trace, then $\omega_F = \omega_{Kac}$, so $\mathbb{H}_F = \mathbb{H}_{Kac}$. Moreover, since $\omega_F \in C_r(\widehat{\mathbb{G}})^*$ is the unique trace,

$$I^r_{Kac} = \{a \in C_r(\widehat{\mathbb{G}}) : \omega(a^*a) = 0\} =: I^r_{\omega_F}$$

and $C_r(\widehat{\mathbb{G}})/I^r_{\omega_F} = C_r(\mathbb{H}_F)$.

Conversely, for every Kac closed quantum subgroup $\mathbb{H}$ of $\widehat{\mathbb{G}}$ where $\widehat{\mathbb{G}}/\mathbb{H}$ is coamenable, we have $L^\infty(\widehat{\mathbb{G}}/\mathbb{H}_{Kac}) \subseteq L^\infty(\widehat{\mathbb{G}}/\mathbb{H}) \subseteq L^\infty(\widehat{\mathbb{G}}/\mathbb{H}_F)$. So, $\mathbb{H} = \mathbb{H}_F = \mathbb{H}_{Kac}$, which then Theorem 5.3.3 implies $C_r(\widehat{\mathbb{G}})$ has a unique trace.

3. Similar to proof of 2.. □

5.3.4 The Coamenable Coradical and Duality of Relative Amenability and Coamenability

In terms of the lattice of idempotent states of a CQG, given the existence of tracial state on $C_r(\widehat{\mathbb{G}})$ and the $\widehat{\mathbb{G}}/\mathbb{H}_F$ is coamenable, Theorem 5.3.19 says that $\widehat{\mathbb{G}}/\mathbb{H}_F$ is the largest coamenable quotient where $\mathbb{H}_F$ is Kac. So, whenever $\mathbb{G}$ is unimodular, $\widehat{\mathbb{G}}/\mathbb{H}_F$ is the largest coamenable quotient. For unimodular $\mathbb{G}$, this partially answers [65, Question 8.2].

Definition 5.3.22. Let $\mathbb{G}$ be a DQG such that $C_r(\widehat{\mathbb{G}})$ has a tracial state and $\widehat{\mathbb{G}}/\mathbb{H}_F$ is coamenable. We call $\mathbb{H}_F$ the **Kac coamenable coradical** of $\widehat{\mathbb{G}}$. When $\mathbb{G}$ is unimodular, we just call $\mathbb{H}_F$ the coamenable coradical.

It turns out that the Kac closed quantum subgroups of $\widehat{\mathbb{G}}$ have that coameability implies relative amenability across Pontryagin duality.

Corollary 5.3.23. *Let $\mathbb{G}$ be DQG. If $\mathbb{H}$ is a Kac closed quantum subgroup of $\widehat{\mathbb{G}}$ and $\widehat{\mathbb{G}}/\mathbb{H}$ is a coamenable quotient then $\ell^\infty(\widehat{\mathbb{H}})$ is relatively amenable in $\ell^\infty(\mathbb{G})$.*

Proof. Suppose $\widehat{\mathbb{G}}/\mathbb{H}$ is a coamenable quotient. From Theorem 5.3.19 we have $L^\infty(\widehat{\mathbb{G}}/\mathbb{H}) \subseteq L^\infty(\widehat{\mathbb{G}}/\mathbb{H}_F)$. Then $\widetilde{N}_F \subseteq \ell^\infty(\widehat{\mathbb{H}})$, which implies $\ell^\infty(\widehat{\mathbb{H}})$ is relatively amenable because $\widetilde{N}_F$ is. □

Remark 5.3.24. Suppose $\mathbb{G}$ is a unimodular DQG and $\widehat{\mathbb{G}}/\mathbb{H}_F$ is coamenable. Then we have that $\omega_F = \min \operatorname{Idem}_r^H(\widehat{\mathbb{G}})$.

5.4 Open Problems

The result [15, Theorem 4.1] says that G-invariant states (and equivalently, tracial states) concentrate on the amenable radical. We have a generalization of this result. With Theorem 5.3.19. A more general statement is [16, Theorem 5.2] for which a consequence is as follows: given a G-C^*-algebra A, every G-invariant state (and equivalently, every tracial state) on $A \rtimes_r G$ concentrates on $A \rtimes_r R_a(G)$. A key step in the proof is lifting a G-invariant state up to $(A \otimes C(\partial_F(G)) \rtimes_r G$, where we are using the diagonal action of G on $A \otimes C(\partial_F(G))$. A major obstruction for proving this for DQGs is the absence of a "diagonal coaction."

Choose a set of representatives of $Irr(\widehat{\mathbb{G}})$ so that $\lambda_{\widehat{\mathbb{G}}}(\omega_F)$ is diagonal. We will say a $\mathbb{G}$-invariant state τ on $A \rtimes_r \mathbb{G}$ **concentrates on**

$$A \rtimes_r \mathbb{G}/\widehat{\mathbb{H}_F} := \overline{\operatorname{span}(C_r(\widehat{\mathbb{G}}/\mathbb{H}_F) \otimes 1)\alpha(A)} \subseteq A \rtimes_r \mathbb{G}$$

if τ satisfies $\tau((u^\pi_{i,j} \otimes 1)\alpha(a)) = 0$ if $u^\pi_{i,j} \notin \operatorname{Pol}(\widehat{\mathbb{G}}/\mathbb{H}_F)$.

Question 5.4.1. *Let $\mathbb{G}$ be a DQG and A be a $\mathbb{G}$-C^*-algebra. Is it true that every $\mathbb{G}$-invariant state on $A \rtimes_r \mathbb{G}$ concentrates on $A \rtimes_r \mathbb{G}/\widehat{\mathbb{H}_F}$?*

If $\mathbb{G} \curvearrowright \partial_F(\mathbb{G})$ is faithful, then a state τ on $A \rtimes_r \mathbb{G}$ concentrates on A if $\tau((u^\pi_{i,j} \otimes 1)\alpha(a)) = 0$ for every $u^\pi_{i,j} \neq 1$. A more specific question is the following.

Question 5.4.2. *Let $\mathbb{G}$ be a DQG such such that $\mathbb{G} \curvearrowright \partial_F(\mathbb{G})$ is faithful and A be a $\mathbb{G}$-C^*-algebra. Does every $\mathbb{G}$-invariant state on $A \rtimes_r \mathbb{G}$ concentrate on A?*

Whenever $\mathbb{G} \curvearrowright \partial_F(\mathbb{G})$ is faithful (for instance, when $C_r(\widehat{\mathbb{G}})$ is simple), this would tell us that the $\mathbb{G}$-invariant states on $A \rtimes_r \mathbb{G}$ are in one-to-one correspondence with the $\mathbb{G}$-invariant states on A.

Chapter 6

Relative Operator Amenability

6.1 Introduction

Let A be a completely contractive (cc) Banach algebra. We denote the operator projective tensor product by $A\widehat{\otimes}A$, which comes equipped with a canonical operator A-bimodule structure (cf. Section 3.1). A **virtual diagonal** of A is an element $D \in (A\widehat{\otimes}A)^{**}$ such that $a \cdot D = D \cdot a$ and $m(D)a = a$ for all $a \in A$.

Definition 6.1.1. We say A is **operator amenable** if A has a virtual diagonal.

As noted in [101], and as shown in the Banach module setting in [61, 60], an equivalent formulation of operator amenability of A is that every completely bounded derivation from A into a dual operator A-bimodule is inner, i.e., given any operator A-bimodule X and completely bounded derivation $d : A \to X^*$, there exists $x_0 \in X^*$ such that $d = ad_{x_0}$, where

$$ad_{x_0}(a) = a \cdot x_0 - x_0 \cdot a,\ a \in A.$$

For a locally compact group G, operator amenability of $L^1(G)$ reduces to amenability as defined by Johnson in his memoir [61] for Banach space modules, where it was shown that G is amenable if and only if $L^1(G)$ is amenable.

The Fourier algebra, $A(G)$, of a locally compact group G, was introduced in Eymard's seminal paper [37], and is the Pontryagin dual object of the group algebra, namely, $L^1(\widehat{G}) = A(G)$ where $\widehat{G}$ is the quantum group dual of G (cf. [82]). So, it was expected that the Fourier algebra reflects many of the same properties of the group algebra. However, in [62] it was shown $A(SO(3))$ is non-amenable, and then in [43] it was shown $A(G)$ is amenable if and only if G is virtually abelian. It was then the seminal theorem of Ruan [101] that G is amenable if and only if $A(G)$ is operator amenable. With that, we see it is operator amenability of $A(G)$ and $L^1(G)$, and not amenability, that governs amenability of G. For a more comprehensive survey on the development of operator amenability with an eye towards Banach algebras coming from abstract harmonic analysis, we recommend [113].

Speaking at the level of locally compact quantum groups (LCQGs), we get that G is amenable if and only if $L^1(G)$ is (operator) amenable and $\widehat{G}$ is coamenable if and only if $A(G) = L^1(\widehat{G})$ is operator amenable. So, it is natural to compare operator amenability of the L^1-algebra of a quantum group with coamenability and amenability. Indeed, it is a straightforward exercise to show operator amenability of $L^1(\mathbb{G})$ of a LCQG $\mathbb{G}$ implies $\mathbb{G}$ is amenable and coamenable. Then, for Kac compact quantum groups (CQGs), Ruan [103] showed that a Kac CQG $\mathbb{G}$ is coamenable if and only if $L^1(\mathbb{G})$ is operator amenable if and only if $L^1(\widehat{\mathbb{G}})$ is operator amenable.

Due to the success in the case of Kac compact quantum groups, it was conjectured that operator amenability of $L^1(\mathbb{G})$ would coincide with amenability and coamenability of $\mathbb{G}$ for general LCQGs. But, in the compact case, it turned out operator amenability also governs the Kac property. Building off the work of Daws [27], Caspers et al. [18] showed that if $\mathbb{G}$ is a CQG then operator amenability if $L^1(\mathbb{G})$ implies $\mathbb{G}$ is Kac.

In this article we introduce operator amenability of the left action of a cc Banach algebra A on an operator A-bimodule B which is itself a cc Banach algebra. Our definition is a generalization of operator amenability, where we have operator amenability of right multiplication of A on itself reduces to operator amenability of A. While the term "relative amenability" has appeared in the literature before, (for example, in [99], which we address in Remark 6.3.9), we believe our definition is new.

In some instances, operator amenability of an action captures operator amenability of the constituent cc Banach algebras, and in other instances it captures the amenability properties. To begin, we prove a "true analogue" of Johnson's theorem for locally compact quantum groups. We show the left action of $L^1(\widehat{\mathbb{G}})$ on $L^1(\mathbb{G})$, β_*, induced by the convolution product on $\mathcal{B}(L^2(\mathbb{G}))_* := \mathcal{T}(L^2(\mathbb{G}))$ that is defined by the left fundamental unitary of $\widehat{\mathbb{G}}$, is operator amenable if and only if $\mathbb{G}$ is coamenable and $\widehat{\mathbb{G}}$ is amenable (see Proposition 6.3.13). This curiously relates to the major open problem of determining whether or not coamenability of a LCQG $\mathbb{G}$ is equivalent to amenability of $\widehat{\mathbb{G}}$.

We introduce a notion of operator biflatness of an action, which we note relates to a notion of relative operator biflatness that already exists in the literature (see Remark 6.3.9).

As a direct extension of classical operator amenability and classical opertator biflatness, we are able to show that operator biflatness of an action and the existence of central bounded approximate identity implies the action is operator amenable. Unlike the classical case, however, we show that if $\mathbb{G}$ is compact, then β_* is operator biflat only if $\mathbb{G}$ is Kac (see Proposition 6.3.21), hence the converse could not be achieved.

The case where A acts on B as completely bounded multipliers gives us a bit more traction. Here we say A is operator amenable relative to B if the right action of A on B is operator amenable and likewise with relative operator biflatness. For instance, we prove A is operator amenable relative to B if and only if A is operator biflat relative to B and there exists a bounded approximate identity in B that is central with respect to the A-bimodule action (Proposition 6.4.4). A rich set of examples is given by the natural action of $L^1(\mathbb{H})$ on $M^u(\mathbb{G})$, where $\mathbb{H}$ is a closed (Woronowicz) quantum subgroup of the locally compact quantum group $\mathbb{G}$. In the case where $\mathbb{G}$ is compact and $\widehat{\mathbb{H}}$ is a closed quantum subgroup of $\widehat{\mathbb{G}}$, we characterize relative operator amenability of $L^1(\widehat{\mathbb{H}})$ in $L^1(\widehat{\mathbb{G}})$: in particular, we show the action of $L^1(\widehat{\mathbb{H}})$ on $L^1(\widehat{\mathbb{G}})$ is operator amenable if and only if $L^1(\widehat{\mathbb{H}})$ is operator amenable (see Theorem 6.4.9). We produce analogous results for locally compact groups, thus obtaining an extension of Johnson's theorem to relative operator amenability. We achieve a different result for their duals. If G is a locally compact group and N is a closed subgroup, then $A(G/N)$ is operator amenable relative to $A(G)$ if and only if G is amenable (see Proposition 6.4.15). We also analyze relative operator amenability of certain translation invariant subalgebras of the Fourier-Stieltjes algebra of G. In particular, we relate relative operator amenability to operator amenability and compactness of G.

This chapter is presented as follows. In Section 2 we review the basics of compact quantum groups and then we discuss convolution products on $\mathcal{T}(L^2(\mathbb{G}))$ induced by the left and right fundamental unitaries of locally compact quantum groups and their duals.

In Section 3, we define operator amenability and operator biflatness of an action. Then we briefly discuss how the aforementioned convolution products on $\mathcal{T}(L^2(\mathbb{G}))$ allow us to turn $L^1(\mathbb{G})$ into an operator $L^1(\widehat{\mathbb{G}})$-bimodule. We prove that operator amenability of β_* is equivalent to coamenability of $\mathbb{G}$ and amenability of $\widehat{\mathbb{G}}$. Then we show operator biflatness of β_* implies $\mathbb{G}$ is Kac, a result which may be of independent interest.

In Section 4 we study operator amenability of an action by completely bounded multipliers. We make basic observations, before proving characterizing relative operator amenability of $L^1(\widehat{\mathbb{H}})$ in $L^1(\widehat{\mathbb{G}})$. We end by discussing various instances of cc Banach algebras acting as completely bounded multipliers on other cc Banach algebras in the realm of locally compact groups and their duals.

6.2 Convolution on $\mathcal{T}(L^2(\mathbb{G}))$

In this section, we will review convolution products at the level $\mathcal{T}(L^2(\mathbb{G}))$, originally introduced at the level of locally compact groups by Neufang [91]. Our main reference is [67]. We note that some of the concepts here have been teased previously in this thesis.

We can use the left fundamental unitary to induce a right coaction of $\mathbb{G}$ on $\mathcal{B}(L^2(\mathbb{G}))$ via

$$\begin{aligned}\Delta^l : \mathcal{B}(L^2(\mathbb{G})) &\to L^\infty(\mathbb{G})\overline{\otimes}\mathcal{B}(L^2(\mathbb{G})) \subseteq \mathcal{B}(L^2(\mathbb{G}))\overline{\otimes}\mathcal{B}(L^2(\mathbb{G}))\\ x &\mapsto W^*(1\otimes x)W\end{aligned}$$

and the right fundamental unitary to induce a right coaction of $\mathbb{G}$ on $\mathcal{B}(L^2(\mathbb{G}))$ by setting

$$\begin{aligned}\Delta^r : \mathcal{B}(L^2(\mathbb{G})) &\to \mathcal{B}(L^2(\mathbb{G}))\overline{\otimes}L^\infty(\mathbb{G}) \subseteq \mathcal{B}(L^2(\mathbb{G}))\overline{\otimes}\mathcal{B}(L^2(\mathbb{G})),\\ x &\mapsto V(x\otimes 1)V^*.\end{aligned}$$

Clearly $\Delta^l|_{L^\infty(\mathbb{G})} = \Delta^l|_{L^\infty(\mathbb{G})} = \Delta$, however, these actions differ at the level of $\mathcal{B}(L^2(\mathbb{G}))$. For example, for every $\hat{x} \in L^\infty(\widehat{\mathbb{G}})$ we have $\Delta^r(\hat{x}) = \hat{x}\otimes 1$ and for every $\hat{x}' \in L^\infty(\widehat{\mathbb{G}})'$ we have $\Delta^l(\hat{x}') = 1\otimes\hat{x}'$ but not vice versa (see [67] for more). In fact, we obtain a canonical right action of $\mathbb{G}$ on $L^\infty(\widehat{\mathbb{G}})$ by restriction:

$$\Delta^l|_{L^\infty(\widehat{\mathbb{G}})} : L^\infty(\widehat{\mathbb{G}}) \to L^\infty(\mathbb{G})\overline{\otimes}L^\infty(\widehat{\mathbb{G}}).$$

Perhaps most importantly about Δ^l and Δ^r is that they define normal coassociative coproducts on $\mathcal{B}(L^2(\mathbb{G}))$, which allows us to define the following completely contractive products on $\mathcal{T}(L^2(\mathbb{G}))$:

$$\varphi \triangleleft \omega := (\varphi\otimes\omega)\Delta^l \text{ and } \varphi \triangleright \omega := (\varphi\otimes\omega)\Delta^r,\ \varphi,\omega \in \mathcal{T}(L^2(\mathbb{G})).$$

We call the above products left and right convolution products respectively, and with them, we induce an operator $\mathcal{T}(L^2(\mathbb{G}))$-bimodule structure on $\mathcal{B}(L^2(\mathbb{G}))\overline{\otimes}\mathcal{B}(L^2(\mathbb{G}))$ by setting

$$\varphi \triangleright T = (\mathrm{id}\otimes\varphi)\Delta^r(T) \text{ and } T \triangleleft \varphi = (\varphi\otimes\mathrm{id})\Delta^l(T),\ \varphi \in \mathcal{T}(L^2(\mathbb{G})), T \in \mathcal{B}(L^2(\mathbb{G})).$$

On the dual side, we denote the analogous right and left coactions $\widehat{\Delta}^l$ and $\widehat{\Delta}^r$ of $\widehat{\mathbb{G}}$ on $\mathcal{B}(L^2(\mathbb{G}))$, and the corresponding convolution products and modules actions by $\widehat{\triangleleft}$ and $\widehat{\triangleright}$.

We have that the space $L^\infty(\mathbb{G})_\perp \subseteq \mathcal{T}(L^2(\mathbb{G}))$ is a closed ideal in both $(\mathcal{T}(L^2(\mathbb{G})), \triangleleft)$ and $(\mathcal{T}(L^2(\mathbb{G}), \triangleleft)$, and also the completely quotient map $\mathcal{T}(L^2(\mathbb{G})) \to L^1(\mathbb{G})$ is also an algebra homomorphism with respect to both $\triangleleft$ and $\triangleright$ [67, Proposition 3.1]. So, the convolutions on $\mathcal{T}(L^2(\mathbb{G}))$ are in fact liftings of convolution on $L^1(\mathbb{G})$ (and $L^1(\widehat{\mathbb{G}})$).

Example 6.2.1. Consider the case of a locally compact group G and the coaction

$$\Delta^l : \mathcal{B}(L^2(G)) \to L^\infty(G)\overline{\otimes}\mathcal{B}(L^2(G)).$$

In this case, the fundamental unitary is the unitary $W \in L^\infty(G)\overline{\otimes}VN(G)$ defined by setting $W\xi(s,t) = \xi(s, s^{-1}t)$ for a.e. $s, t \in G$ and $\xi \in L^2(G \times G)$. It is not hard to show that for $u \in A(G)$, $f \in L^1(G)$, and $\hat{x} \in VN(G)$ we have

$$f \triangleleft u(\hat{x}) = (f \otimes u)\Delta^l|_{VN(G)}(\hat{x}) = \int_G u(\lambda(s)\hat{x}\lambda(s)^*)f(s)\, ds.$$

So $f \triangleleft u \in A(G)$ is identified with the function

$$t \mapsto \int_G u(sts^{-1})f(s)\, ds.$$

Now consider the coaction

$$\widehat{\Delta}^l : \mathcal{B}(L^2(G)) \to VN(G)\overline{\otimes}\mathcal{B}(L^2(G)).$$

Since $L^\infty(G)$ is commutative and $\widehat{W} \in VN(G)\overline{\otimes}L^\infty(G)$, we have for $x \in L^\infty(G)$,

$$\widehat{\Delta}^l|_{L^\infty(G)}(x) = \widehat{W}^*(1 \otimes x)\widehat{W} = 1 \otimes x.$$

So, $u \mathbin{\widehat{\triangleleft}} f = u(e)f$, where we recall from earlier in this section that $\widehat{\triangleleft}$ is the convolution produced induced by $\widehat{\Delta}^l$.

6.3 Operator Amenability

6.3.1 Operator Amenability of Actions

For references on operator spaces, we recommend the texts of Effros and Ruan [33] and Paulsen [98]. A cc Banach algebra is a Banach algebra A equipped with the structure of an operator space (as introduced in Section 2.1 of this thesis) and whose multiplication m extends to a complete contraction $m : A\widehat{\otimes}A \to A$. An operator space X is a left operator A-module if there exists a left action of A on X that extends to a complete contraction $A\widehat{\otimes}X$ and which is algebraically a left A-module. We define right operator A-modules similarly. The dual of a left operator A-module X is canonically a right operator A-module via the dual action. We say X is an operator A-bimodule if X is both a left and right operator A-module for which X is algebraically an A-bimodule. The dual of an operator A-bimodule is again an operator A-bimodule. Given a left and a right operator A-module X and Y respectively, $X\widehat{\otimes}Y$ is naturally an operator A-bimodule by setting

$$a \cdot (x \otimes y) = x \otimes (a \cdot y) \text{ and } (x \otimes y) \cdot a = (x \cdot a) \otimes y,\ x \otimes y \in X\widehat{\otimes}Y, a \in A.$$

Note that we could alternatively choose the bimodule structure given by setting $a\cdot(x\otimes y) = (a\cdot x)\otimes y$ and $(x\otimes y)\cdot a = x\otimes(y\cdot a)$. Then the dual $(X\widehat{\otimes}Y)^*$ is canonically an A-bimodule, whose adjoint actions satisfy

$$a \cdot (\varphi \otimes \psi) = \varphi \otimes (a \cdot \psi) \text{ and } (\varphi \otimes \psi) \cdot a = (\varphi \cdot a) \otimes \psi,\ \varphi \otimes \psi \in (X\widehat{\otimes}Y)^*, a \in A.$$

Operator amenability relates to splitting properties of the operator projective tensor product. We first define the following.

Definition 6.3.1. We say A is **operator biflat** if there exists a completely bounded A-bimodule map $\theta : (A\widehat{\otimes}A)^* \to A^*$ such that $\theta \circ m^* = \mathrm{id}$. We call θ a **splitting morphism**.

The relation alluded to above is the following.

Theorem 6.3.2. *[97, Theorem 1.3] A is operator amenable if and only if A has a bai and is operator biflat.*

Remark 6.3.3. Note, again, that only the Banach space version of amenability and biflatness was considered in [97], however, we note that the operator theoretic versions follows from the same proof mutatis mutandis.

Now suppose B is another cc Banach algebra that is an operator A-bimodule, and denote the left action by $\alpha : A\widehat{\otimes}B \to B$. Now, we formulate the following.

Definition 6.3.4. We say α is **(left) operator biflat** if there exists an A-bimodule and left B-module completely bounded map $\theta : (A\widehat{\otimes}B)^* \to B^*$ such that $\theta \circ \alpha^* = \mathrm{id}$. We call θ a **splitting morphism** for α.

Remark 6.3.5.

1. Clearly, operator biflatness of m is exactly operator biflatness of A, so, what we have defined in Definition 6.3.21 generalizations operator biflatness.
2. Crann and Tanko defined relative operator biflatness in [24], which applies more generally to cc Banach algebras acting on operator spaces. They said A is operator biflat relative to a right operator A-module E if there exists a completely bounded left A-module map $\theta : (A\widehat{\otimes}E)^* \to E^*$ such that $\theta \circ \alpha^* = \mathrm{id}$ where, as usual, α denotes the completely contractive extension of the left action to $A\widehat{\otimes}E$. It is clear that Definition 6.3.4 is a specialized version of Crann and Tanko's definition of relative operator biflatness.

Using an analogy with Theorem 6.3.2, we formulate a generalization of operator amenability.

Definition 6.3.6. We say α is **operator amenable** if there exists $D \in (A\widehat{\otimes}B)^{**}$ such that $a \cdot D = D \cdot a$ for all $a \in A$ and $b\alpha^{**}(D) = \alpha^{**}(D)b = b$. We call D a **virtual diagonal** for α.

An **approximate diagonal** is a bounded net $(d_i) \subseteq A\widehat{\otimes}A$ such that $a \cdot d_i - d_i \cdot a \to 0$ and $m(d_i)a \to a$ for all $a \in A$. Standard techniques allow one to pass between virtual and approximate diagonals.

Theorem 6.3.7. *[60] We have that A is operator amenable if and only if A has an approximate diagonal.*

An **approximate diagonal** for α is a bounded net $(d_i) \subseteq A\widehat{\otimes}B$ such that $a\cdot d_i - d_i \cdot a \to 0$ for all $a \in A$, and $\alpha(d_i)b \to b$ and $b\alpha(d_i) \to b$ for all $b \in B$. Then, the argument used by Johnson [60] follows mutatis mutandis for operator amenability of an action.

Proposition 6.3.8. *We have that α is operator amenable if and only if α has an approximate diagonal.*

Remark 6.3.9.
1. Clearly, operator amenability of m is exactly operator amenability of A, so, what we have defined in Definition 6.3.6 generalizations operator amenability.
2. As we pointed out in Remark 6.3.5, operator biflatness of an action is nothing new. To the author's knowledge, however, operator amenability of an action is new. Relative amenability of a Banach algebra was defined by Read [99] for unital Banach algebras $A \subseteq B$ with $1_A = 1_B = 1$. Formulating Read's definition for operator spaces, Read says A is operator amenable relative to B if there exists $D \in (B\widehat{\otimes}B)^{**}$ such that $a \cdot D = D \cdot a$ and $m^{**}(D) = 1$ for all $a \in A$. We call this notion Read's relative operator amenability. It is clear operator amenability of the left action of A on B in the above situation implies Read's relative operator amenability. We do not know whether converse holds, however.

   We can extend Read's relative operator amenability to our more general scheme where B is just an operator A-bimodule by saying A is operator amenable relative to B if there exists $D \in (B\widehat{\otimes}B)^{**}$ such that $a \cdot D = D \cdot a$ for all $a \in A$ and $b \cdot m^{**}(D) = m^{**}(D) \cdot b = b$ for all $b \in B$. At this level, we can see Read's relative amenability does not imply operator amenability of α. In fact, if B is operator amenable and A is a subalgebra, then A always has Read's relative operator amenabililty. So, in this case, Read's relative operator amenability of A is independent of operator amenability of A. For example, if $A = I$ is a closed ideal, then A is operator amenable if and only if I admits a bai (cf. [107]).

On the other hand, let α denote the left multiplication of I on B. Then, if we suppose α is operator amenable with approximate diagonal $(d_i) \subseteq I\widehat{\otimes}B$, then $(\alpha(d_i)) \subseteq I$ is a bai for B, so $I = B$. In other words, such an α is never operator amenable if I is a proper ideal of B.

Note also, as we mentioned above, if B is operator amenable and I has a bai, then I is operator amenable. So, if I is proper we obtain an example where operator amenability does not imply operator amenability of an action. We can see such an example by taking $B = L^1(G)$ and $I = L^1_0(G) = \{f \in L^1(G) : \int f = 0\}$ for amenable G.

While at this time we do not have an analogous result to Theorem 6.3.2, we can say the following.

Proposition 6.3.10. *If α is operator biflat and there exists $E \in B^{**}$ such that $bE = b = Eb$ and $a \cdot E = E \cdot a$ for all $a \in A$ and $b \in B$, then α is operator amenable.*

Proof. Let θ be a splitting morphism of α^* and set $D = \theta^*(E)$. For $a \in A$, that we have $a \cdot D = D \cdot a$ follows directly from the fact $a \cdot E = E \cdot a$ and A-bimodularity of θ^*. Then,

$$\alpha^{**}(D)b = \alpha^{**}(\theta^*(E))b = Eb = b$$

and $b\alpha^{**}(D) = b$ is similar. □

6.3.2 Operator Amenability of β_*

We set

$$\beta = \widehat{\Delta}^l|_{L^\infty(\mathbb{G})} : L^\infty(\mathbb{G}) \to L^\infty(\widehat{\mathbb{G}})\overline{\otimes}L^\infty(\mathbb{G}),\ x \mapsto \widehat{W}^*(1 \otimes x)\widehat{W}.$$

Let us establish the operator $L^1(\widehat{\mathbb{G}})$-bimodule structures we will be working with. For $u, v \in L^1(\widehat{\mathbb{G}})$

$$u \mathbin{\widehat{\lhd}} v = u * v,\ v \in L^1(\widehat{\mathbb{G}})$$

and for $f \in L^1(\mathbb{G})$

$$f \mathbin{\widehat{\rhd}} u = u(1)f$$

because $\widehat{\Delta}^l|_{L^\infty(\widehat{\mathbb{G}})} = \widehat{\Delta}$ and $\widehat{\Delta}^r(x) = x \otimes 1$ for all $x \in L^\infty(\mathbb{G})$ respectively. From the coaction property, we see that $\widehat{\triangleleft}$ makes $L^1(\mathbb{G})$ into a left operator $L^1(\widehat{\mathbb{G}})$-module and from the "triviality" of the right action of $L^1(\widehat{\mathbb{G}})$ on $L^1(\mathbb{G})$ via $\widehat{\triangleright}$, we see that $\widehat{\triangleright}$ makes $L^1(\mathbb{G})$ a right operator $L^1(\widehat{\mathbb{G}})$-module. Then, we use $\widehat{\triangleleft}$ and $\widehat{\triangleright}$ to impart an $L^1(\widehat{\mathbb{G}})$-bimodule structure on $L^1(\mathbb{G})$:

$$(u \mathbin{\widehat{\triangleleft}} f) \mathbin{\widehat{\triangleright}} v = v(1) u \mathbin{\widehat{\triangleleft}} f = u \mathbin{\widehat{\triangleleft}} (f \mathbin{\widehat{\triangleright}} v).$$

The natural $L^1(\widehat{\mathbb{G}})$-bimodule structure on $L^\infty(\widehat{\mathbb{G}}) \overline{\otimes} L^\infty(\mathbb{G})$ satisfies

$$u \cdot (\hat{x} \otimes x) = \hat{x} \otimes (u \mathbin{\widehat{\triangleright}} x) = \hat{x} \otimes u(1) x$$

and

$$(\hat{x} \otimes x) \cdot u = (\hat{x} \triangleleft u \otimes x) = (\hat{x} * u \otimes x),$$

while the natural left operator $L^1(\mathbb{G})$-module structure satisfies

$$f \cdot (\hat{x} \otimes x) = \hat{x} \otimes f * x$$

for $u, v \in L^1(\widehat{\mathbb{G}}), f \in L^1(\mathbb{G}), \hat{x} \in L^\infty(\widehat{\mathbb{G}})$, and $x \in L^\infty(\mathbb{G})$. For the rest of this section, these will be the operator module structures we will be considering.

Remark 6.3.11. If β_* is operator amenable with virtual diagonal D, then by triviality of the right action of $L^1(\widehat{\mathbb{G}})$ on $L^1(\mathbb{G})$ we have

$$u \cdot D = D \cdot u = u(1) D, u \in L^1(\widehat{\mathbb{G}}).$$

What mediates the connection between coamaneability of $\mathbb{G}$ and operator amenability of β_* is inner amenability of $\widehat{\mathbb{G}}$, introduced in [22], which is a definition we alluded to in the previous chapter.

Definition 6.3.12. We say $\widehat{\mathbb{G}}$ is **inner amenable** if there exists a $\widehat{\mathbb{G}}$-invariant state $m \in L^\infty(\mathbb{G})^*$, i.e., a state satisfying,

$$m(x \mathbin{\widehat{\triangleleft}} u) = m(x) u(1), \ x \in L^\infty(\mathbb{G}), u \in L^1(\widehat{\mathbb{G}}).$$

We say $\widehat{\mathbb{G}}$ is **strongly inner amenable** if there exists a net of unit vectors $(\xi_j) \subseteq L^2(\mathbb{G})$ satisfying

$$||\widehat{W} \Sigma(\widehat{V})(\eta \otimes \xi_j) - \eta \otimes \xi_j||_2 \to 0.$$

We call such a net (ξ_j) a net of **almost $\widehat{\mathbb{G}}$-invariant vectors**. It turns out that strong inner amenability implies inner amenability [22, Proposition 3.4] and that coamenability implies strong inner amenability [22, Proposition 3.8].

Proposition 6.3.13. *Let $\mathbb{G}$ be a LCQG. Then $\mathbb{G}$ is coamenable and $\widehat{\mathbb{G}}$ is amenable if and only if β_* is operator amenable.*

Proof. Suppose β_* is operator amenable. It is immediate from the definition that $\mathbb{G}$ is coamenable. Define the functional $m = D(x \otimes 1)$. It is readily shown that m is left invariant because $u \cdot D = u(1)D$. Then $\widehat{\mathbb{G}}$ has a left invariant state and so is amenable.

Conversely, let $(e_i) \subseteq L^1(\mathbb{G})$ be a net. Using the standard form of $\mathcal{B}(L^2(\mathbb{G}))$, find ξ_i so that $e_i = w_{\xi_i}|_{L^\infty(\mathbb{G})}$. Let $e \in L^\infty(\mathbb{G})^*$ be a weak* cluster point of (e_i). In the proof of [22, Proposition 3.8] it was shown that (ξ_i) is a net of almost $\widehat{\mathbb{G}}$-invariant vectors. Then in the proof of [22, Proposition 3.4] it was shown that e is a $\widehat{\mathbb{G}}$-invariant state.

Now let $(w_j) \subseteq L^1(\widehat{\mathbb{G}})$ be a net of asymptotic invariant states. Set $d_j = w_j \otimes e$ and let $D \in (L^\infty(\widehat{\mathbb{G}})\overline{\otimes}L^\infty(\mathbb{G}))^*$ be a weak* cluster point. Since e is inner invariant, $D \circ \beta = e$ and for $u \in L^1(\widehat{\mathbb{G}})$, since w_j is asymptotically left invariant, $u \cdot D = u(1)D = D \cdot u$. □

Recall that [22, Corllary 7.4] states that for a LCQG $\mathbb{G}$ such that $\widehat{\mathbb{G}}$ has the approximation property, $\mathbb{G}$ is coamenable if and only $\widehat{\mathbb{G}}$ is amenable.

Corollary 6.3.14. *Let $\mathbb{G}$ be a LCQG and suppose that $\widehat{\mathbb{G}}$ has the approximation property. Then β_* is operator amenable if and only if $\mathbb{G}$ is coamenable.*

We let $\widehat{\beta} = \Delta^l|_{L^\infty(\widehat{\mathbb{G}})}$. Following immediately from the previous corollary is the following.

Corollary 6.3.15. *Let $\mathbb{G}$ be a LCQG. Then $\mathbb{G}$ is amenable and coamenable if and only if β_* and $\widehat{\beta}_*$ are operator amenable.*

6.3.3 Operator Biflatness of β_*

We maintain the same operator $L^1(\widehat{\mathbb{G}})$-bimodule structure on $L^1(\mathbb{G})$ from the previous section.

Remark 6.3.16. By triviality of the right action of $L^1(\widehat{\mathbb{G}})$ on $L^1(\mathbb{G})$, any completely bounded $\theta : L^\infty(\widehat{\mathbb{G}})\overline{\otimes}L^\infty(\mathbb{G}) \to L^\infty(\mathbb{G})$ is automatically a left $L^1(\widehat{\mathbb{G}})$-module map. This means that left $L^1(\widehat{\mathbb{G}})$ and $L^1(\mathbb{G})$-modularity of such a map are the interesting module properties.

For classical operator amenability, we see from Proposition 6.3.2 that operator biflatness is the apriori weaker property. We will see a similar relationship in the subsequent section when the actions are equipped with additional structure. In the present more general setting, however, we will see that this is not the case.

Proposition 6.3.17. *Let $\mathbb{G}$ be a LCQG. If β_* is operator biflat then $\widehat{\mathbb{G}}$ is amenable.*

Proof. Let θ be a splitting homomorphism for β_*. Fix $\hat{x} \in L^\infty(\widehat{\mathbb{G}})$. Then $f * \theta(\hat{x} \otimes 1) = \theta(\hat{x} \otimes f * 1) = f(1)\theta(\hat{x} \otimes 1)$ for all $f \in L^1(\mathbb{G})$, so $\theta(\hat{x} \otimes 1) \in \mathbb{C}$. It is then straight forward to show the functional $L^\infty(\widehat{\mathbb{G}}) \ni \hat{x} \mapsto \theta(\hat{x} \otimes 1)$ is left invariant. □

In particular, if $\widehat{\mathbb{G}}$ has the approximation property, then coamenability of $\mathbb{G}$ follows from operator biflatness of β_*. This is in stark contrast to classical operator biflatness.

Locally Compact Groups

As far as we are aware, it is an open problem to determine whether or not $A(G)$ is always operator biflat, with progress having been made in [24] and [116]. On the other hand, since G is coamenable, $L^1(G)$ is operator biflat if and only if $L^1(G)$ is operator amenable. On the other hand, it is not difficult to determine that β_* is always operator biflat for groups.

Proposition 6.3.18. *Let $\mathbb{G} = G$ be a locally compact group. We have that β_* is always operator biflat.*

Proof. Recall from Remark 6.2.1 that for $x \in L^\infty(G)$ we have $\beta(x) = 1 \otimes x$. Let m be an invariant state on $VN(G)$ and set $\theta = m \otimes \mathrm{id}_{L^\infty(G)}$. It is clear that $\theta : VN(G)\overline{\otimes}L^\infty(G) \to L^\infty(G)$ is a completely bounded linear map satisfying $\theta \circ \beta = \mathrm{id}$. It is also clear that θ is a left $L^1(G)$-module map. To see right $A(G)$-modularity, take $u \in A(G)$ and observe that

$$\theta(Z \cdot u) = (m \otimes \mathrm{id})((u \otimes \mathrm{id})\widehat{\Delta} \otimes \mathrm{id})(Z)) = u(e)(m \otimes \mathrm{id})(Z) = \theta(Z) \mathbin{\widehat{\triangleleft}} u$$

for $Z \in VN(G)\overline{\otimes}L^\infty(G)$. □

Remark 6.3.19. On the other hand, we have been unable to determine when $\widehat{\beta}_*$ is operator biflat in the case where $\mathbb{G} = G$ is a locally compact group. We know from Proposition 6.3.17 that G must at least be amenable.

Compact Quantum Groups

In the compact case, it turns out that $\widehat{\mathbb{G}}$ is Kac if and only if h is $\mathbb{G}$-invariant (see [65]). Thus, from Theorem 6.3.13 we can express operator amenability of $L^1(\mathbb{G})$ for compact $\mathbb{G}$ entirely in terms of amenability properties of β_*.

Corollary 6.3.20. *Let $\mathbb{G}$ be a CQG. We have that β_* is operator amenable and h is $\widehat{\mathbb{G}}$-invariant if and only if $L^1(\mathbb{G})$ is operator amenable.*

Thus we achieve the following.

Theorem 6.3.21. *Let $\mathbb{G}$ be a CQG. If β_* is operator biflat then $\mathbb{G}$ is coamenable and Kac.*

Proof. Let θ be the splitting morphism for β. For $x \in L^\infty(\mathbb{G})$,

$$\begin{aligned} h(x) &= h \circ \theta \circ \beta(x) \\ &= (\mathrm{id} \otimes h)\Delta(\theta \circ \beta(x)) \\ &= (\theta \otimes h)(\mathrm{id} \otimes \Delta)\beta(x) \text{ (left } L^1(\mathbb{G})\text{-modularity)} \\ &= \theta((\mathrm{id} \otimes h)\beta(x)) \text{ (left invariance)} \\ &= \theta((\mathrm{id} \otimes h)\beta(x) \otimes 1). \end{aligned}$$

Then, for $f \in L^1(\widehat{\mathbb{G}})$

$$\begin{aligned} (f \otimes h)\beta(x) &= h(x \triangleleft f) \\ &= \theta((\mathrm{id} \otimes h)\beta(x \triangleleft f) \otimes 1) \\ &= \theta([(\mathrm{id} \otimes h)\beta(x)] * f \otimes 1) \text{ (coaction property)} \\ &= \theta((\mathrm{id} \otimes h)\beta(x) \otimes 1) \triangleleft f \\ &= h(x)f(1). \end{aligned}$$

So, we have $(\mathrm{id} \otimes h)\beta(x) = h(x)$ and then [65, Lemma 5.2] says $\mathbb{G}$ is Kac and coamenability of $\mathbb{G}$ follows from Proposition 6.3.17. □

Combined with Ruan's result, we have the following immediate consequence.

Corollary 6.3.22. *Let $\mathbb{G}$ be a CQG. If β_* is operator biflat then $L^1(\mathbb{G})$ is operator amenable.*

6.4 Relative Operator Amenability of Quantum Subgroups

6.4.1 Operator Modules as Multipliers

We will be focused on the following situation for the rest of the paper.

Definition 6.4.1. Let A be a c.c Banach algebra and B an operator A-bimodule that is itself a cc Banach algebra. We say A **acts on** B **as completely bounded (cb) multipliers** if

$$(a \cdot b)b' = a \cdot (bb') \text{ for all } a \in A, b, b' \in B, \tag{6.1}$$

and

$$b(a \cdot b') = (b \cdot a)b' \text{ for all } a \in A, b, b' \in B, \tag{6.2}$$

We thank Nico Spronk for pointing out the contents of the following remark.

Remark 6.4.2. If A acts on B as cb multipliers, it is a relatively simple exercise to show A embeds contractively into the completely bounded double centralizers of B.

6.4.2 Operator Amenability of an Action by Multipliers

We will often use the following terminology.

Definition 6.4.3. Let A be a cc Banach algebra acting as cb multipliers on the cc Banach algebra B. Then, if the given right action is operator amenable, we will say A is **operator amenable relative to** B. In this situation, we will often call the given virtual diagonals and approximate diagonals of the action **relative virtual diagonals** and **relative approximate diagonals**. If the given right action is operator biflat, we will say A is **operator biflat relative to** B.

The first advantage of working with an action by multipliers is that we obtain the converse of Proposition 6.3.10.

Proposition 6.4.4. *Let A and B be cc Banach algebras such that A acts by cb multipliers on B with right action α. Then A is operator amenable relative to B if and only if A is operator biflat relative to B and B has a bai (e_i) such that $a \cdot e_i - e_i \cdot a \to 0$ for all $a \in A$.*

Proof. The proof is similar to the proof of [97, Theorem 1.3]. For convenience of the reader, we provide the proof here. First, if $(D_i) \subseteq B\widehat{\otimes}A$ is a relative approximate diagonal, then by definition, $(\alpha(D_i)) \subseteq B$ is a bai. That any weak* cluster point E satisfies $a \cdot E = E \cdot a$ for all $a \in A$ easily follows from the fact that any weak* cluster point of (D_i) is a relative virtual diagonal. Now let D be a relative virtual diagonal, and for $F \in (B\widehat{\otimes}A)^*$ and $b \in B$, we set $\theta(F)(b) = D(b \cdot F)$. Then for $f \in B^*$,

$$\langle \theta \circ \alpha^*(f), b\rangle = D(b \cdot \alpha^*(f)) \overbrace{=}^{(6.2)} D(\alpha^*(b \cdot f)) = \alpha^{**}(D)(b \cdot f) = f(b).$$

So $\theta \circ \alpha^* = \mathrm{id}$ as desired. What remain are the module properties of θ. For A-bimodularity, take $a \in A$. Then

$$\begin{aligned} \theta(F \cdot a)(b) &= D(b \cdot F \cdot a) = (a \cdot D)(b \cdot F) \\ &= (D \cdot a)(b \cdot F) = D(a \cdot (b \cdot F)) \\ &\overbrace{=}^{(6.2)} D((a \cdot b) \cdot F) = (a \cdot \theta(F))(b) \end{aligned}$$

and

$$\theta(a \cdot F)(b) = D(b \cdot (a \cdot F)) \overbrace{=}^{(6.1)} D((b \cdot a) \cdot F) = \theta(F)(b \cdot a) = (a \cdot \theta(F))(b).$$

Finally, to see left B-modularity, for $b' \in B$,

$$\theta(b' \cdot F)(b) = D(b \cdot (b' \cdot F)) = D((bb') \cdot F) = (b' \cdot \theta(F))(b).$$

The converse is covered by Proposition 6.3.10. □

We will primarily be interested in comparing operator amenability with operator amenability. Of course, a cc Banach algebra is always operator amenable relative to itself if and only if it is operator amenable.

Now, if we assume the constituent cc Banach algebras have coinciding identity elements, then operator amenability implies relative operator amenability.

Proposition 6.4.5. *Let $A \subseteq B$ be unital cc Banach algebras such that $1_A = 1_B = 1$. Let $\alpha : B\widehat{\otimes}A \to B$ denote the right action. If A is operator amenable, it is relatively operator amenable in B.*

Proof. This follows using a similar idea as the one employed in [104, Theorem 2.2.4, (i) $\implies$ (iii)]. We have that $\ker(\alpha)$ is an operator A–bimodule and similarly to the argument used originally used by Johnson [61] (see also [104, Theorem 2.2.4, (i) $\implies$ (iii)]), one can show $\ker[(\alpha)^{**}] \cong [\ker(\alpha)]^{**}$ is a dual operator A–bimodule. In light of this, set $E = 1 \otimes 1$. For $a \in A$, $\alpha(a \cdot (1 \otimes 1) - (1 \otimes 1) \cdot a) = 0$, so

$$Ad_E(A) \subseteq \ker(\alpha) \subseteq [\ker(\alpha)]^{**}.$$

What we have just shown is $Ad_E : A \to [\ker(\alpha)]^{**}$ is a derivation and operator amenability of A means we can find $N \in [\ker(\alpha)]^{**}$ such that $Ad_E = Ad_N$. From here it can be easily shown $D = E - N \in B\widehat{\otimes}E$ is a relative virtual diagonal. □

In general, however, it is not the case that operator amenability implies relative operator amenability and vice versa. Consider a cc Banach algebra A and a closed ideal I, whose left multiplication on A we denote by α. If we suppose I is operator amenable relative to A with bounded approximate diagonal $(D_i) \subseteq A\widehat{\otimes}I$, then $(\alpha(D_i)) \subseteq I$ is a bai for A, so $I = A$. This gives us the following.

Proposition 6.4.6. *Let A be a cc Banach algebra and I a closed ideal. We have that I is operator amenable relative to A if and only if $I = A$ and is operator amenable.*

Recall that a LCQG $\mathbb{G}$ is discrete if and only if $L^1(\mathbb{G}) = M^u(\mathbb{G})$ and is coamenable if and only if $M^r(\mathbb{G}) = M^u(\mathbb{G})$. Then Proposition 6.4.6 gives us the following.

Corollary 6.4.7. *Let $\mathbb{G}$ be a LCQG. The following hold:*

1. *$L^1(\mathbb{G})$ is operator amenable relative to $M^u(\mathbb{G})$ if and only if $\mathbb{G}$ is discrete and $L^1(\mathbb{G})$ is operator amenable;*

2. *$M^r(\mathbb{G})$ is operator amenable relative to $M^u(\mathbb{G})$ if and only if $\mathbb{G}$ is coamenable and $M^r(\mathbb{G})$ is operator amenable.*

Example 6.4.8. By definition, $\mathbb{C}$ acts on any cc Banach algebra as cb multipliers. Then, since $\mathbb{C}\widehat{\otimes}A \cong A$ completely isometrically, we can see that $\mathbb{C}$ is operator amenable relative to A if and only if A has a bai. For a LCQG $\mathbb{G}$, this means that $\mathbb{G}$ is coamenable if and only if $\mathbb{C}$ is operator amenable relative to $L^1(\mathbb{G})$.

6.4.3 Discrete Quantum Groups

Let X be an operator $\ell^1(\mathbb{G})$–bimodule. For $x \in X$ and $f \in \ell^1(\mathbb{H})$,

$$f \circ \sigma_{\mathbb{H}} \cdot x = (f \circ \sigma_{\mathbb{H}} \otimes \epsilon_{\widehat{\mathbb{G}}})\Delta_{\widehat{\mathbb{G}}} \cdot x = (f *_{\mathbb{H}} \epsilon_{\widehat{\mathbb{G}}}) \cdot x$$

so X is an operator $\ell^1(\mathbb{H})$–bimodule by setting $f \cdot x = f \circ \sigma_{\mathbb{H}} \cdot x$ and $x \cdot f = x \cdot f \circ \sigma_{\mathbb{H}}$. Let $d : \ell^1(\widehat{\mathbb{G}}) \to X^*$ be a completely bounded derivation. Then $d|_{(\sigma_{\mathbb{H}})_*(\ell^1(\mathbb{H}))} : \ell^1(\mathbb{H}) \to X^*$ is also a completely bounded derivation and furthermore, given $f \in \ell^1(\mathbb{H})$,

$$d|_{(\sigma_{\mathbb{H}})_*(\ell^1(\mathbb{H}))}(f \circ \sigma_{\mathbb{H}}) = d \circ (\sigma_{\mathbb{H}})_*(f)$$

so we will write

$$d|_{\ell^1(\mathbb{H})} := d \circ (\sigma_{\mathbb{H}})_* : \ell^1(\mathbb{H}) \to X^*$$

as the "restriction" of our completely bounded derivation to $\ell^1(\mathbb{H})$.

Theorem 6.4.9. *Assume $\mathbb{G}$ is discrete and let $\mathbb{H}$ be an quantum subgroup. TFAE:*

1. *$\ell^1(\mathbb{H})$ is operator amenable;*
2. *for every operator $\ell^1(\mathbb{G})$–bimodule E, every operator $\ell^1(\mathbb{H})$–module F such that F^* is a submodule of E^* satisfies the property that whenever $d : \ell^1(\mathbb{G}) \to E^*$ is a derivation with $d|_{\ell^1(\mathbb{H})}(\ell^1(\mathbb{H})) \subseteq F^*$, it follows that $d|_{\ell^1(\mathbb{H})} : \ell^1(\mathbb{H}) \to F^*$ is inner;*
3. *$\ell^1(\mathbb{H})$ is operator amenable relative to $\ell^1(\mathbb{G})$;*
4. *and $\ell^1(\mathbb{G})$ is operator biflat relative to $\ell^1(\mathbb{H})$.*

Before we begin, recall that

$$r_{\mathbb{H}} = (\sigma_{\mathbb{H}} \otimes \mathrm{id})\Delta_{\mathbb{G}} \text{ and } l_{\mathbb{H}} = (\mathrm{id} \otimes \sigma_{\mathbb{H}})\Delta_{\mathbb{G}}$$

so that

$$f *_{\mathbb{H}} g = (f \otimes g)r_{\mathbb{H}} \text{ and } g *_{\mathbb{H}} f = (g \otimes f)l_{\mathbb{H}},\ f \in \ell^1(\mathbb{H}), g \in \ell^1(\mathbb{G}).$$

Proof. ($1 \implies 2$) is obvious. We will show ($2 \implies 3$) using a similar idea to Johnson's original proof [61] (see also [104, Theorem 2.2.4, $(i) \implies (iii)$]). Set $E = \epsilon_{\mathbb{G}} \otimes \epsilon_{\mathbb{H}}$ and notice that for $f \in \ell^1(\mathbb{H})$,

$$\begin{aligned}(l_{\mathbb{H}})^*(f \cdot E - E \cdot f) &= (l_{\mathbb{H}})^*((f \circ \sigma_{\mathbb{H}}) \otimes \epsilon_{\mathbb{H}} - \epsilon_{\mathbb{G}} \otimes f) \\ &= f \circ \sigma_{\mathbb{H}} - f \circ \sigma_{\mathbb{H}} = 0.\end{aligned}$$

We have shown $ad_E(\ell^1(\mathbb{H})) \subseteq \ker((l_{\mathbb{H}})_*) \subseteq (\ker(l_{\mathbb{H}})_*)^{**}$. After minor adjustments, in [104, Theorem 2.2.4, $(i) \implies (iii)$] it was shown that $\ker(\Delta_{\mathbb{G}})^* \cong (\ker(\Delta_{\mathbb{G}})_*)^{**}$ is an operator $\ell^1(\mathbb{G})$–bimodule and the same reasoning implies $\ker(l_{\mathbb{H}}^*) \cong (\ker(l_{\mathbb{H}})_*)^{**}$ mutatis mutandis. We see that $(\ker((l_{\mathbb{H}})_*))^{**}$ is an $\ell^1(\mathbb{H})$–bimodule since $f *_{\mathbb{H}} g = (f \circ \sigma_{\mathbb{H}}) * g$ and $g *_{\mathbb{H}} f = g * (f \circ \sigma_{\mathbb{H}})$ for $f \in \ell^1(\mathbb{H})$ and $g \in \ell^1(\mathbb{G})$, and using associativity in $\ell^1(\mathbb{G})$. Now $F \in \ker(l_{\mathbb{H}}^*)$ by definition satisfies

$$0 = l_{\mathbb{H}}^*(F) = F(\mathrm{id} \otimes \sigma_{\mathbb{H}})\Delta_{\mathbb{G}} = \Delta_{\mathbb{G}}^* \circ (\mathrm{id} \otimes \sigma_{\mathbb{H}}^*)F,$$

so $F(\mathrm{id} \otimes \sigma_{\mathbb{H}}) \in \ker \Delta_{\mathbb{G}}^*$, and for $f \in \ell^1(\mathbb{H})$ we have

$$\begin{aligned} f \cdot [F(\mathrm{id} \otimes \sigma_{\mathbb{H}})] &= ((f \circ \sigma_{\mathbb{H}}) \otimes F)(\Delta_{\mathbb{G}} \otimes \sigma_{\mathbb{H}}) \\ &= ((f \circ \sigma_{\mathbb{H}}) \cdot F)(\mathrm{id} \otimes \sigma_{\mathbb{H}}) \end{aligned}$$

and

$$\begin{aligned} [F(\mathrm{id} \otimes \sigma_{\mathbb{H}})] \cdot f &= F \otimes (f \circ \sigma_{\mathbb{H}})(\mathrm{id} \otimes \sigma_{\mathbb{H}} \otimes \mathrm{id})\Delta_{\mathbb{G}} \\ &= (F \otimes f)(\mathrm{id} \otimes \Delta_{\mathbb{H}})(\mathrm{id} \otimes \sigma_{\mathbb{H}}) \\ &= (F \cdot f)(\mathrm{id} \otimes \sigma_{\mathbb{H}}). \end{aligned}$$

What we have shown is

$$(\mathrm{id} \otimes \sigma_{\mathbb{H}}^*) : \ker(l_{\mathbb{H}}^*) \to \ker \Delta_{\mathbb{G}}^*$$

is a completely isometric $\ell^1(\mathbb{H})$–bimodule homomorphism. In particular, $\ker(l_{\mathbb{H}}^*)$ is an $\ell^1(\mathbb{H})$–submodule of $\ker \Delta_{\mathbb{G}}^*$. Set $E' = \epsilon_{\mathbb{G}} \otimes \epsilon_{\mathbb{G}}$. Then, for $f \in \ell^1(\mathbb{H})$,

$$\begin{aligned} ad_{E'}|_{\ell^1(\mathbb{H})}(f) &= ad_{E'}(f \circ \sigma_{\mathbb{H}}) \\ &= (f \circ \sigma_{\mathbb{H}}) \otimes \epsilon_{\mathbb{G}} - \epsilon_{\mathbb{G}} \otimes (f \circ \sigma_{\mathbb{H}}) \\ &= [(f \circ \sigma_{\mathbb{H}}) \otimes \epsilon_{\mathbb{H}} - \epsilon_{\mathbb{G}} \otimes f](\mathrm{id} \otimes \sigma_{\mathbb{H}}) \\ &= ad_E(f)(\mathrm{id} \otimes \sigma_{\mathbb{H}}), \end{aligned}$$

thus $ad_E = ad_N$ for some $N \in \ker(l_{\mathbb{H}})^*$ by assumption. Then it is straightforward to see that $D = E - N \in (\ell^\infty(\mathbb{G})\overline{\otimes}\ell^\infty(\mathbb{H}))^*$ is a relative virtual diagonal.

$(3 \implies 1)$ Set $D' = (\iota_{\mathbb{H}}^* \otimes \mathrm{id})D \in (\ell^\infty(\mathbb{H})\overline{\otimes}\ell^\infty(\mathbb{H}))^*$. Then for $f \in \ell^1(\mathbb{H})$ and $z \in$

$\ell^\infty(\mathbb{G})\overline{\otimes}\ell^\infty(\mathbb{H})$,

$$\begin{aligned}
f\cdot D'(\sigma_{\mathbb{H}}\otimes\mathrm{id})(z) &= (f\otimes D)(\mathrm{id}\otimes\iota_{\mathbb{H}}\otimes\mathrm{id})(\Delta_{\mathbb{H}}\circ\sigma_{\mathbb{H}}\otimes\mathrm{id})(1_{\mathbb{H}}\otimes 1)(z)\\
&= (f\otimes D)(1\otimes 1_{\mathbb{H}}\otimes 1)(\sigma_{\mathbb{H}}\otimes\mathrm{id}\otimes\mathrm{id})(\Delta_{\mathbb{G}}\otimes\mathrm{id})(1_{\mathbb{H}}\otimes 1)(z)\\
&= ((f\circ\sigma_{\mathbb{H}})\otimes D)(1\otimes 1_{\mathbb{H}}\otimes 1)(\Delta_{\mathbb{G}}\otimes\mathrm{id})(1_{\mathbb{H}}\otimes 1)(z)\\
&= ((f\circ\sigma_{\mathbb{H}})\otimes D)(1_{\mathbb{H}}\otimes 1\otimes 1)(\Delta_{\mathbb{G}}\otimes\mathrm{id})(1_{\mathbb{H}}\otimes 1)(z)\\
&= ((f\circ\sigma_{\mathbb{H}})\otimes D)(\Delta_{\mathbb{G}}\otimes\mathrm{id})(1_{\mathbb{H}}\otimes 1)(z)\\
&= (D\otimes f)(\mathrm{id}\otimes\Delta_{\mathbb{H}})(1_{\mathbb{H}}\otimes 1)(z)\\
&= (D\otimes f)(\iota_{\mathbb{H}}\otimes\mathrm{id}\otimes\mathrm{id})(\sigma_{\mathbb{H}}\otimes\Delta_{\mathbb{H}})(z)\\
&= (D'\cdot f)(\sigma_{\mathbb{H}}\otimes\mathrm{id})(z).
\end{aligned}$$

For the remaining claim, for $x\in\ell^\infty(\mathbb{G})$,

$$\begin{aligned}
(\Delta_{\mathbb{H}})^*(D')(\sigma_{\mathbb{H}}(x)) &= D(\iota_{\mathbb{H}}\otimes\mathrm{id})\Delta_{\mathbb{H}}(\sigma_{\mathbb{H}}(1_{\mathbb{H}}x))\\
&= D(\mathrm{id}\otimes\sigma_{\mathbb{H}})(1_{\mathbb{H}}\otimes 1)\Delta_{\mathbb{G}}(1_{\mathbb{H}}x)\\
&= D(\mathrm{id}\otimes\sigma_{\mathbb{H}})\Delta_{\mathbb{G}}(1_{\mathbb{H}}x)\\
&= \epsilon_{\mathbb{G}}(1_{\mathbb{H}}x)\\
&= \epsilon_{\mathbb{H}}(\sigma_{\mathbb{H}}(1_{\mathbb{H}}x))\\
&= \epsilon_{\mathbb{G}}(x).
\end{aligned}$$

($3 \iff 4$) is Proposition 6.4.4. □

Corollary 6.4.10. *Assume $\mathbb{G}$ is discrete and let $\mathbb{H}$ be a quantum subgroup. Then $\ell^1(\mathbb{H})$ is operator amenable if and only if there exists $D\in(\ell^\infty(\mathbb{G})\overline{\otimes}\ell^\infty(\mathbb{G}))^*$ such that $f\cdot D = D\cdot f$ for all $f\in\ell^1(\mathbb{H})$ and $D(1\otimes 1_{\mathbb{H}})\Delta_{\mathbb{G}} = \epsilon_{\mathbb{G}}$.*

Proof. Let $D'\in(\ell^\infty(\mathbb{H})\overline{\otimes}\ell^\infty(\mathbb{H}))^*$ be a virtual diagonal. Set $D = D'(\sigma_{\mathbb{H}}\otimes\sigma_{\mathbb{H}})$. Then for $f\in\ell^1(\mathbb{H})$,

$$\begin{aligned}
f\cdot D &= ((f\circ\sigma_{\mathbb{H}})\otimes D)(\Delta_{\mathbb{G}}\otimes\mathrm{id})\\
&= (f\otimes D')(\Delta_{\mathbb{H}}\otimes\mathrm{id})(\sigma_{\mathbb{H}}\otimes\sigma_{\mathbb{H}}\otimes\sigma_{\mathbb{H}})\\
&= (D'\otimes f)(\mathrm{id}\otimes\Delta_{\mathbb{H}})(\sigma_{\mathbb{H}}\otimes\sigma_{\mathbb{H}}\otimes\sigma_{\mathbb{H}}) = D\cdot f
\end{aligned}$$

and

$$D(1\otimes 1_{\mathbb{H}})\circ\Delta_{\mathbb{G}} = D'(\sigma_{\mathbb{H}}\otimes\sigma_{\mathbb{H}})(1\otimes 1_{\mathbb{H}})\Delta_{\mathbb{G}} = D'(\sigma_{\mathbb{H}}\otimes\sigma_{\mathbb{H}})\Delta_{\mathbb{G}} = \epsilon_{\mathbb{G}}.$$

Conversely, set $D' = D(\mathrm{id} \otimes \iota_{\mathbb{H}})$. Then, for $f \in \ell^1(\mathbb{H})$ and $z = (\mathrm{id} \otimes \sigma_{\mathbb{H}})(y) \in \ell^\infty(\mathbb{G})\overline{\otimes}\ell^\infty(\mathbb{H})$,

$$\begin{aligned}
(f \cdot D')(z) &= ((f \circ \sigma_{\mathbb{H}}) \otimes D)(\Delta_{\mathbb{G}} \otimes \iota_{\mathbb{H}})(\mathrm{id} \otimes \sigma_{\mathbb{H}})(y) \\
&= (D \otimes (f \circ \sigma_{\mathbb{H}}))(\mathrm{id} \otimes \Delta_{\mathbb{G}})(1 \otimes 1_{\mathbb{H}})(y) \\
&= (D \otimes f)(\mathrm{id} \otimes \mathrm{id} \otimes \sigma_{\mathbb{H}})(1 \otimes 1 \otimes 1_{\mathbb{H}})(\mathrm{id} \otimes \Delta_{\mathbb{G}})(1 \otimes 1_{\mathbb{H}})(y) \\
&= (D \otimes f)(\mathrm{id} \otimes \mathrm{id} \otimes \sigma_{\mathbb{H}})(1 \otimes 1_{\mathbb{H}} \otimes 1)(\mathrm{id} \otimes \Delta_{\mathbb{G}})(1 \otimes 1_{\mathbb{H}})(y) \\
&= (D \otimes f)(\mathrm{id} \otimes \iota_{\mathbb{H}} \otimes \mathrm{id})(\mathrm{id} \otimes \sigma_{\mathbb{H}} \otimes \sigma_{\mathbb{H}})(\mathrm{id} \otimes \Delta_{\mathbb{G}})(1 \otimes 1_{\mathbb{H}})(y) \\
&= (D' \otimes f)(\mathrm{id} \otimes \Delta_{\mathbb{H}})(z) = (D' \cdot f)
\end{aligned}$$

and

$$(l_{\mathbb{H}})^*(D') = D(1 \otimes \iota_{\mathbb{H}})(\mathrm{id} \otimes \sigma_{\mathbb{H}})\Delta_{\mathbb{G}} = D(1 \otimes 1_{\mathbb{H}})\Delta_{\mathbb{G}} = \epsilon_{\mathbb{G}}.$$

Thanks to Theorem 6.4.9, we know $\ell^1(\mathbb{H})$ is operator amenable. □

6.4.4 Locally Compact Groups

Let G be a locally compact groups and H a closed subgroup. The canonical left and right actions of H on G are given by left and right translation respectively. Define the maps

$$l_H(x)(s,t) = x(st) \text{ and } r_H(y)(s,t) = y(st)$$

for $x \in L^\infty(G \times H)$ and $y \in L^\infty(H \times G)$. The $L^1(H)$-bimodule action on $L^1(G)$ is explicitly given by

$$f *_H g(s) = \int_H f(t)g(t^{-1}s)\, dm_H(t) \text{ and } g *_H f(s) = \int_H g(st)f(t^{-1})\, dm_H(t)$$

for $f \in L^1(H)$ and $g \in L^1(G)$, and where m_H is the Haar measure on H.

Remark 6.4.11. Recall that we can manufacture a bai for $L^1(G)$ by setting $e_j^G = \frac{1}{m(K_j)} 1_{K_j}$ where (K_j) is a neighbourhood base of compact sets for the identity. A similar calculation which shows (e_j^G) is a bai for $L^1(G)$ will show $f *_H e_j^H \to f$ and $e_j^H *_H f \to f$ for every $f \in L^1(G)$, i.e., $(e_j^H) \subseteq L^1(H)$ is a bai for $L^1(G)$.

Proposition 6.4.12. *Let G be a locally compact groups with closed subgroup H. Then H is amenable if and only if $L^1(H)$ is operator amenable relative to $L^1(G)$.*

Proof. Suppose H is amenable. By Johnson's theorem, $L^1(H)$ is (operator) amenable. Now, let $(e_i^G) \subseteq L^1(G)$ be a bai and $(e_j^H) \subseteq L^1(H)$ a bai for $L^1(G)$. Let $E \in (L^1(G)\widehat{\otimes}L^1(H))^{**}$ be a weak* cluster point of the net $(e_i^G \otimes e_j^H)_{(i,j)} \subseteq L^1(G)\widehat{\otimes}L^1(H)$ given by the product of directed sets. After passing to a subnet if necessary, $(e_i^G * e_i^H)_{i,j}$ is a bai for $L^1(G)$, and so $l_H^*(E) * f = f = f * l_H^*(E)$ for all $f \in L^1(G)$. From here, we can use a proof similar to Johnson's original proof [61] (see also [104, Theorem 2.2.4, $(i) \implies (iii)$]) to obtain a relative virtual diagonal.

Conversely, take a relative virtual diagonal D. For $x \in L^\infty(H)$, define $m(x) = D(1 \otimes x)$. It can easily be seen that m is right invariant. □

Corollary 6.4.13. *We have that $L^1(G)$ is operator amenable relative to $M(G)$ if and only if $M(G)$ is operator amenable.*

Proof. From Corollary 6.4.7, we know $L^1(G)$ is operator amenable relative to $M(G)$ if and only if G is discrete and $L^1(G)$ is operator amenable, which, using Johnson's theorem, is equivalent to G being discrete and amenable. This last statement is equivalent to operator amenability of $M(G)$ [25]. □

6.4.5 Duals of Locally Compact Groups

Recall that the closed quantum subgroup of $\widehat{G}$ are of the form $\widehat{G/N}$ where N is a closed normal subgroup of G (cf. [28]). Define the maps

$$l_{\widehat{G/N}}(\lambda_G(s)) = \lambda_G(s) \otimes \lambda_{G/N}(sN) \text{ and } r_{\widehat{G/N}}(\lambda_G(s)) = \lambda_{G/N}(sN) \otimes \lambda_G(s).$$

Then

$$u *_{\widehat{G/N}} v(s) = u(s)v(sN) \text{ and } v *_{\widehat{G/N}} u(s) = v(sN)u(s)$$

for $s \in G$, $u \in A(G)$, and $v \in A(G/N)$.

For the following discussion, we review bicharacters on locally compact groups from [28]. Let

$$U_{\widehat{G/N}} = (\mathrm{id} \otimes \gamma_{\widehat{G/N}})(W_{\widehat{G/N}})$$

be the bicharacter for $\widehat{G/N}$ as a quantum subgroup of G, where $\gamma_{\widehat{G/N}} : L^\infty(G/N) \to L^\infty(G)$ is the adjoint of the map

$$T_N : L^1(G) \to L^1(G/N),\ f(\cdot) \mapsto \int_N f(s\cdot)dm_N(s) \text{ a.e..}$$

Note that $\gamma_{\widehat{G/N}}$ is an isometric $*$-homomorphism that realizes $\widehat{G/N}$ as a (Vaes) closed quantum subgroup of $\widehat{G}$. The unitary $U_{\widehat{G/N}}$ is in particular a unitary in $M(C_r^*(G/N) \otimes_{min} C_0(G)) \subseteq VN(G/N)\overline{\otimes}L^\infty(G)$ that satisfies

$$r_{\widehat{G/N}}(x) = U^*_{\widehat{G/N}}(1 \otimes x)U_{\widehat{G/N}}. \tag{6.3}$$

Recall that a locally compact quantum group $\mathbb{G}$ is coamenable if and only if the counit admits a reduced version, $\epsilon^r_{\mathbb{G}} \in M^r(\mathbb{G})$, which is such that $(\epsilon^r_{\mathbb{G}} \otimes \mathrm{id})(W_{\mathbb{G}}) = 1$. Equivalently, there exists a net of unit vectors $(\xi_j) \subseteq L^2(\mathbb{G})$ such that

$$||W_{\mathbb{G}}(\xi_j \otimes \xi) - (\xi_j \otimes \xi)||_2 \to 0$$

for every $\xi \in L^2(\mathbb{G})$. We call such (ξ_j) a net of **almost invariant vectors**. It is well-known that the vector functional $\omega_{\xi_j,\xi_j}|_{L^1(\mathbb{G})}$ is a bounded approximate identity for $L^1(\mathbb{G})$.

The following discussion is probably well-known to experts, however, we have failed to find a reference in the literature, so we establish the consequences therein here. Our techniques follow as it does with usual coamenability. Suppose $\widehat{G/N}$ is coameanable and $(\xi_j) \subseteq L^2(G/N)$ is the allotted net of almost invariant vectors. Then, for $\xi \in L^2(G)$,

$$(\omega_{\xi_j,\xi_j} \otimes \omega_{\xi,\xi})(U_{\widehat{G/N}}) = \omega_{\xi_j,\xi_j}(\mathrm{id} \otimes T_N(\omega_{\xi,\xi}))(W_{\widehat{G/N}}) \to T_N(\omega_{\xi,\xi})(1).$$

Then, for $\eta \in L^2(G)$, using a similar argument as in the usual coamenable case that

$$||U_{\widehat{G/N}}(\xi_j \otimes \xi) - \xi_j \otimes \xi||_2 \to 0.$$

Let $(\lambda_G)_{\xi,\eta} \in A(G)$ denote the corresponding vector functional so, $(\lambda_G)_{\xi,\eta}(s) = \langle \lambda_G(s)\xi, \eta\rangle$. Then, for $\xi \in L^2(G)$ and $x \in VN(G)$,

$$\begin{aligned}
&|(\lambda_{G/N})_{\xi_j,\xi_j} *_{\widehat{G/N}} (\lambda_G)_{\xi,\xi} - (\lambda_G)_{\xi,\xi}|(x) = |\langle U^*_{\widehat{G/N}}(1 \otimes x)U_{\widehat{G/N}}(\xi_j \otimes \xi), \xi_j \otimes \xi\rangle - \langle x\xi, \xi\rangle| \\
&= |\langle (1 \otimes x)[U_{\widehat{G/N}}(\xi_j \otimes \xi) - (\xi_j \otimes \xi)], U_{\widehat{G/N}}(\xi_j \otimes \xi)\rangle + \langle (1 \otimes x)(\xi_j \otimes \xi), U_{\widehat{G/N}}(\xi_j \otimes \xi) - (\xi_j \otimes \xi)\rangle| \\
&\leq ||x||\,||\xi||\,||U_{\widehat{G/N}}(\xi_j \otimes \xi) - (\xi_j \otimes \xi)||_2 + ||x||\,||\xi||\,||U_{\widehat{G/N}}(\xi_j \otimes \xi) - (\xi_j \otimes \xi)||_2 \to 0.
\end{aligned}$$

Therefore, by setting $\epsilon_j = (\lambda_G)_{\xi_j,\xi_j}$, for every $u \in A(G)$, $\epsilon_j *_{\widehat{G/N}} u \to u$ and using a symmetric argument,

$$u *_{\widehat{G/N}} \epsilon_j \to u.$$

In other words $(\epsilon_j) \subseteq A(G/N)$ is a bai for $A(G)$.

Remark 6.4.14. Our methods in the above paragraph used nothing specific to cocommutative quantum groups. The same methods will pass to LCQGs with only minor adjustmenents to give us the following: if a (Vaes) closed quantum subgroup $\mathbb{H}$ of a LCQG $\mathbb{G}$ is coamenable with almost invariant vectors $(\xi_j) \subseteq L^2(\mathbb{H})$, then $\omega_{\xi_j,\xi_j}|_{L^1(\mathbb{H})} \subseteq L^1(\mathbb{H})$ is a bai for $L^1(\mathbb{G})$.

The following is exactly what one would expect to happen.

Proposition 6.4.15. *Let G be a locally compact group and N a closed normal subgroup. Then $A(G/N)$ is operator amenable relative to $A(G)$ if and only if G is amenable.*

Proof. Relative operator amenability implies $A(G)$ has a bai. Then, Leptin's theorem implies G is amenable. Conversely, if G is amenable, then G/N is amenable as well. So, from above paragraph, we can find a net of states $(e_j^{G/N}) \subseteq A(G/N)$ such that $u *_{\widehat{G/N}} e_j \to u$ for every $u \in A(G)$. Then, given a bai $(e_j^G) \subseteq A(G)$, $e_i^G *_{\widehat{G/N}} e_j^{G/N}$ is still a bai for $A(G)$ (where the net is taken with respect to the product of directed sets). Now, let $E \in (VN(G)\overline{\otimes}VN(G/N))^*$ be a weak* cluster point of the net $(e_i^G \otimes e_j^{G/N})_{i,j} \subseteq A(G)\widehat{\otimes}A(G/N)$ respectively. Then $l^*_{\widehat{G/N}}(E)u = u = ul^*_{\widehat{G/N}}(E)$ for every $u \in A(G)$, and proceed as in Johnson's original argument [61] (see also [104, Theorem 2.2.4, $(i) \implies (iii)$]) to obtain a relative virtual diagonal. □

Remark 6.4.16. The above proposition provides us with more examples of cc Banach algebras that are operator amenable but not relatively operator amenable. If G is a non-amenable locally compact group and N a closed normal subgroup such that G/N is amenable, then $A(G/N)$ is operator amenable, but, because of Proposition 6.4.15, $A(G/N)$ is not operator amenable relative to $A(G)$.

This means our notion of relative operator amenability of $A(G/N)$ with respect to $A(G)$ is capturing amenability of G. Compare that with the analogous fact that G is amenable if and only if the ideal $I(H)$ has a bai for any closed subgroup H (cf. [107]).

We will spend the rest of this section analyzing relative operator amenability of certain subalgebras of the Fourier-Stieltjes algebra $B(G) := M^u(\widehat{G})$ and reduced Fourier-Stieltjes algebra $B_r(G) := M^r(\widehat{G})$. Recall that the counit is $1_G \in B(G)$.

Recall that given a von Neumann algebra, a subspace $X \subseteq M_*$ is said to be **translation invariant** if $M \cdot X, X \cdot M \subseteq X$, where $m \cdot x$ for a $m \in M$ and $x \in X$ denotes the canonical predual action of M on M_*. Recall that a closed translation invariant subspace A of M_* has a unique central projection $P_A \in M$ such that $A = M_* \cdot P_A$.

For a LCQG, recall that the co-product on $C_0^u(\mathbb{G})$ lifts to a co-product, denoted $\Delta_{\mathbb{G}}^u$, on $C_0^u(\mathbb{G})^{**}$ (cf. [79]). Similarly to the case for $L^\infty(\mathbb{G})$, we will say a projection $P_A \in W^*(G) := B(G)^* = C^*(G)^{**}$ is **group-like** if

$$(P_A \otimes 1)\overline{\Delta}_{\widehat{G}}^u(P_A) = P_A \otimes P_A = (1 \otimes P_A)\overline{\Delta}_{\widehat{G}}^u(P_A).$$

Notice that if P_A is group-like, then

$$(u \cdot P_A)(v \cdot P_A) = (u \otimes v)(P_A \otimes P_A)\Delta_{\widehat{G}}^u = ((u \cdot P_A)v) \cdot P_A,$$

which says $A = B(G) \cdot P_A$ is an algebra.

In the proof of [108, Theorem 2.3], the following was noted.

Proposition 6.4.17. *[108] Let $A \subseteq B(G)$ be a closed translation invariant subalgebra. TFAE:*

1. *$B(G) \ni u \mapsto u \cdot P_A \in A$ is an algebra homomorphism ($P_A \in \sigma(B(G))$);*
2. *$\ker(B(G) \ni u \mapsto u \cdot P_A \in A)$ is a closed ideal;*
3. *and $\overline{\Delta}_{\widehat{G}}^u(P_A) = P_A \otimes P_A$.*

Note in particular that if $\overline{\Delta}_{\widehat{G}}^u(P_A) = P_A \otimes P_A$, then P_A is group-like. So, we have a distinguished class of group-like projections, which are those such that $B(G) \cdot (1 - P_A)$ is a closed ideal of $B(G)$. We thank Nico Spronk for pointing out the following proposition.

Proposition 6.4.18. *Let $A \subseteq B(G)$ be a translation invariant subalgebra such that $P_A \in B(G)$ is group-like. Then A is unital.*

Proof. Since $1_G \in B(G)$ is the counit,

$$P_A 1_G(P_A) = (\mathrm{id} \otimes 1_G)(P_A \otimes 1)\Delta_{\widehat{G}}^u(P_A) = P_A,$$

which implies $1_G(P_A) = 1$. Since $1_G : W^*(G) \to \mathbb{C}$ is a $*$-homomorphism,

$$1_G \cdot P_A(x) = 1_G(P_A x) = 1_G(x),\ x \in W^*(G),$$

which shows $1_G \cdot P_A = 1_G$. □

This allows us to characterize relative operator amenability of the translation invariant subalgebras of $B(G)$ that are "implemented" by group-like projections.

Proposition 6.4.19. *Let G be a locally compact group and $A \subseteq B(G)$ a closed translation invariant subalgebra such that P_A is group-like. Then A is operator amenable relative to $B(G)$ if and only if A is operator amenable.*

Proof. Denote the right action of A on $B(G)$ by α. Let $(D_i) \subseteq B(G)\widehat{\otimes}A$ be a bounded approximate relative diagonal. Take $u \in A = B(G) \cdot P_A$ and $v \in B(G)$. Then, since $u(P_A \cdot v) = P_A \cdot (uv)$, we have

$$u \cdot ((P_A \otimes 1) \cdot D_i) - ((P_A \otimes 1) \cdot D_i) \cdot u = (P_A \otimes 1) \cdot (u \cdot D_i - D_i \cdot u) \to 0$$

and

$$\alpha((P_A \otimes 1) \cdot D_i)u = (P_A \cdot \alpha(D_i))u = P_A \cdot (\alpha(D_i)u) \to P_A \cdot u = u.$$

Then, $((P_A \otimes 1) \cdot D_i) \subseteq A\widehat{\otimes}A$ is a bounded approximate diagonal for A.

The converse is covered by Proposition 6.4.5. □

A general scheme for obtaining translation invariant subspaces of $B(G)$ was obtained in [114] by looking at the "unitarizable topologies" on G. The following proposition, its proof, and the following remark is due to Nico Spronk.

Proposition 6.4.20. *Let G be a locally compact group and N a closed normal subgroup. The group-like projection $P \in W^*(G)$ such that $P \cdot B(G) = B(G/N)$ is group-like.*

Proof. The spectrum of $B(G)$,

$$\sigma(B(G)) = \{x \in W^*(G) : \Delta^u_{\widehat{G}}(x) = x \otimes x\}$$

is a weak* closed and conjugate closed subsemigroup of the unit ball of $W^*(G)$ (cf. [124, 125]). We have $\varpi(G) \subseteq \sigma(B(G))$, however, it may be the case that $\overline{\varpi(G)}^{wk*}$ is a proper subset of $\sigma(B(G))$ as is exhibited by the Wiener-Pitt phenonemon of non-compact abelian groups.

We let $\sigma(B(G))_P$ denote the idempotents in $\sigma(B(G))$ and set

$$G_P = \{u \in \sigma(B(G)) : uu^* = P = u^*u\}.$$

Since $\sigma(B(G))$ is a semitopological semigroup, G_P is complete with respect to two-sided uniformity (cf. [110, II.4.4 & II.4.6]).

Suppose K is a compact normal subgroup of G_P for some $P \in \sigma(B(G))_P$. Then the projection P_K,

$$u(P_K) = \int_K u(s)\, dm_K(s),\ u \in B(G),$$

where m_K is the Haar probability measure on K, satisfies $B(G) \cdot P_K = B(G_P/K)$ and is clearly group-like.

Let

$$ZP = \{P \in \overline{\varpi(G)}^{wk*} \cap \sigma(B(G))_P : P\varpi(s) = \varpi(s)P \text{ for all } \ s \in G\}.$$

As shown in [114, 5.1] any group of unitaries that is complete with respect to two-sided uniformity and admits a continuous dense image of G is of the form G_P/K for some $P \in ZP$ and compact normal subgroup K of G_P. A sample of the scope of such is discussed in [114]. For a non-compact abelian group G, ZP is infinite, however, there may only be finitely many locally compact G_P (see, for example, [59, 34]). In particular, G/N is of the form G_P/K for some G_P and K above, and we have $B(G) \cdot P_K = B(G/N)$. □

In particular, we have the following.

Corollary 6.4.21. *Let G be a locally compact group and N a closed normal subgroup. Then $B(G/N)$ is operator amenable relative to $B(G)$ if and only if $B(G/N)$ is operator amenable.*

Remark 6.4.22. Maintaining the same notation as in the proof of Proposition 6.4.20, for $P \in PZ$, we have $\varpi(G)P \subseteq G_P$ and so for $s \in G$,

$$\varpi(s)P_K\varpi(s)^* = \int_K \varpi(s)t\varpi(s)^*\, dm_K(t) = P_K$$

since $t \mapsto \varpi(s)t\varpi(s)^*$ is an automorphism of K, and is invariant for the Haar measure on K.

It was shown in [125] that there is a minimal element $z \in \sigma(B(G))_P$ such that $G_z = \overline{z\varpi(G)}^{wk*}$ is the almost periodic compactification of G. We note that P_{G_z} is central and minimal among the group-like projections P_K. Indeed, from the first remark of this paragraph we have $\varpi(s)P_{G_z} = P_{G_z}$. Hence if

$$x = wk^* \lim_i \sum_{j=1}^{n_i} x_{i,j}\varpi(s_j) \in W^*(G),$$

then

$$xP_{G_z} = wk^* \lim_i \sum_{j=1}^{n_i} x_{i,j} P_{G_z} = wk^* \lim_i \sum_{j=1}^{n_i} x_{i,j} 1_G(\varpi(s_j)) P_{G_z} = 1_G(x) P_{G_z}.$$

Consequently,

$$P_K P_{G_z} = \int_K s P_{G_z} \, dm_K(s) = \int_K 1_G(s) P_{G_z} \, dm_K(s) = P_{G_z}$$

since $1_G(s) = 1$, as $s \in \sigma(B(G))$.

Let us focus on the Fourier / reduced Fourier-Stieltjes algebras of the quotients G/N, which we note are generally non-unital subalgebras of $B(G)$. From Proposition 6.4.18 we know the central projections "implementing" them are necessarily not group-like.

Using that $A(G) = B(G)$ if and only if G is compact, and $B(G) = B_r(G)$ if and only if G is amenable, we immediately deduce the following from Corollary 6.4.7.

Corollary 6.4.23. *For a locally compact group G, the following hold:*

1. *G is compact if and only if $A(G)$ is operator amenable relative to $B(G)$;*
2. *$B(G)$ is operator amenable if and only if $B_r(G)$ is operator amenable relative to $B(G)$.*

Spronk [115] showed connected groups are compact if and only if $B(G)$ is operator amenable, So, for a connected group G, Corollary 6.4.23 says $A(G)$ is operator amenable relative to $B(G)$ if and only if $B(G)$ is operator amenable. In general, however, there are non-compact groups with operator amenable Fourier-Stieltjes algebras (cf. [109]). With the above, and Corollary 6.4.21, we can produce examples that show relative operator amenability of the subalgebras $B(G/N) = A(G/N)$ of $B(G)$ is distinct from both compactness and operator amenability of $B(G)$.

Example 6.4.24. 1. The non-compact groups with operator amenable Fourier-Stieltjes algebras considered in [109] were of the form $A \rtimes K$ for an abelian normal subgroup $A \subseteq G$ and compact group K. They admitted the decomposition

$$B(G) \cong A(K) \oplus_1 A(G) \cong A(G/A) \oplus_1 A(G).$$

Note that such a decomposition is impossible if G is compact and A is proper since then we would have $A(G) = B(G)$. Given such a decomposition, it follows that

$A(G/N)$ is the closed linear span of all matrix coefficients of representations disjoint from the left regular representation of G (cf. [69, Section 2.8]). In particular, $1_G \in A(G/N)$, which shows G/N is compact and so $A(G/N) = B(G/N)$ is operator amenable relative to $B(G)$.

2. We thank Nico Spronk for pointing out the following examples. The **Rajchman algebra** of a locally compact group G is the closed subalgebra $B_0(G) = B(G) \cap C_0(G)$. The euclidean motion groups are of the form $G_d := \mathbb{R}^d \rtimes SO(d)$ for $d \geq 2$, where $SO(d)$ acts on $\mathbb{R}^d$ by rotation. Given an euclidean motion group G_d, it was shown in [70] that we have the decomposition

$$B(G_d) = B_0(G_d) \oplus_1 A(SO(d)).$$

Now, $SO(d)$ is compact, so $A(SO(d))$ is operator amenable, hence operator amenable relative to $B(G_d)$. On the other hand, as noted in [70], $B_0(G_d)$ does not admit a (bounded) approximate identity, so it could not be operator amenable. Therefore, $B(G_d)$ is not operator amenable.

We conclude the paper with a broad classes of examples of translation invariant subalgebras of $B(G)$ with which relative operator amenability is characterized with compactness of G.

Corollary 6.4.25. *Let G be a locally compact group and N a normal subgroup.*

1. *Suppose $N = K$ is compact. We have that $A(G/K)$ is operator amenable relative to $B(G)$ if and only if G is compact.*

2. *Suppose N is amenable. We have that $B_r(G/N)$ is operator amenable relative to $B(G)$ if and only if G is amenable and $B_r(G/N)$ is operator amenable.*

Proof. 1. If G is compact then G/K is compact. Then $A(G/K) = B(G/K)$, and we apply Corollary 6.4.21. Conversely, let $(D_i) \subseteq B(G)\widehat{\otimes}A(G/K)$ be a bounded approximate relative diagonal. Since $A(G/K) \subseteq A(G)$, $(\alpha(D_i)) \subseteq A(G)$ is a bai for $B(G)$, which implies $B(G) = A(G)$.

2. If G is amenable then G/N is amenable. Then $B_r(G/N) = B(G/N)$, and we apply Corollary 6.4.21. Conversely, since $B_r(G)$ is an ideal in $B(G)$, we apply a similar proof as in 1. to deduce that $B_r(G) = B(G)$ with which it follows that G is amenable. Then G/N is amenable, so $B_r(G/N) = B(G/N)$, and we apply Corollary 6.4.21 to get that $B_r(G/N)$ is operator amenable relative to $B(G)$.

□

Remark 6.4.26. By choosing an amenable non-compact group G and compact normal subgroup K, we see that $A(G/K)$ is operator amenable and yet from Corollary 6.4.25 we see that $A(G/K)$ is not operator amenable relative to $B(G)$.

6.5 Open Problems

In this next section we present problems left over from our investigations.

A few questions naturally arise given what we already know about operator amenability in certain contexts. We start with the compact quantum version of Proposition 6.4.15.

Question 6.5.1. *Is it true for a CQG $\mathbb{G}$ and closed quantum subgroup $\mathbb{H}$ that $L^1(\mathbb{H})$ is operator amenable relative to $L^1(\mathbb{G})$ if and only if $L^1(\mathbb{G})$ is operator amenable?*

We should note that the classification of operator biflatness of Fourier algebras remains an open problem. Progress, for instance, has been made in [24] and [116].

Question 6.5.2. *Let G be a locally compact group and N a closed normal subgroup. Is $A(G/N)$ always operator biflat relative to $A(G)$?*

For compact G/N, we see that $A(G/N) = B(G/N)$ is always operator amenable relative to $B(G)$, and so with Corollary 6.4.21 we see that relative operator amenability of $A(G/N)$ is independent of operator amenability of $B(G)$ and compactness of G, as pointed out with specific examples in Example 6.4.24. On the other hand, if $N = K$ is compact, with Proposition 6.4.25 we showed $A(G/K)$ is operator amenable relative to $B(G)$ if and only if G is compact, which is equivalent to compactness of G/K.

Question 6.5.3. *Let G be a locally compact group and N a closed normal subgroup. Is $A(G/N)$ operator amenable relative to $B(G)$ if and only if G/N is compact?*

We have the corresponding question for reduced Fourier-Stieltjes algebras.

Question 6.5.4. *Is $B_r(G/N)$ operator amenable relative to $B(G)$ if and only if $B_r(G/N)$ is operator amenable?*

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