Four-coloring P_6-free graphs

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Abstract

In this paper we present a polynomial time algorithm for the 4-COLORING PROBLEM and the 4-PRECOLORING EXTENSION problem restricted to the class of graphs with no induced six-vertex path, thus proving a conjecture of Huang. Combined with previously known results this completes the classification of the complexity of the 4-coloring problem for graphs with a connected forbidden induced subgraph.

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1 Introduction

All graphs in this paper are finite and simple. We use $[k]$ to denote the set $\{1, \dots, k\}$. Let G be a graph. A *k-coloring* of G is a function $f : V(G) \to [k]$. A k-coloring is *proper* if for every edge $uv \in E(G)$, $f(u) \neq f(v)$, and G is *k-colorable* if G has a proper k-coloring. The k-COLORING PROBLEM is the problem of deciding, given a graph G, if G is k-colorable. This problem is well-known to be NP-hard for all $k \geq 3$.

A function $L : V(G) \to 2^{[k]}$ that assigns a subset of $[k]$ to each vertex of a graph G is a *k-list assignment* for G. For a k-list assignment L, a function $f : V(G) \to [k]$ is an *L-coloring* if f is a k-coloring of G and $f(v) \in L(v)$ for all $v \in V(G)$. A graph G is *L-colorable* if G has a proper L-coloring. We denote by $X^0(L)$ the set of all vertices v of G with $|L(v)| = 1$. The k-LIST COLORING PROBLEM is the problem of deciding, given a graph G and a k-list assignment L, if G is L-colorable. Since this generalizes the k-coloring problem, it is also NP-hard for all $k \geq 3$.

A *k-precoloring* (G, X, f) of a graph G is a function $f : X \to [k]$ for a set $X \subseteq V(G)$ such that f is a proper k-coloring of $G|X$. Equivalently, a k-precoloring is a k-list assignment L in which $|L(v)| \in \{1, k\}$ for all $v \in V(G)$. A *k-precoloring extension* for (G, X, f) is a proper k-coloring g of G such that $g|_X = f|_X$, and the k-PRECOLORING EXTENSION PROBLEM is the problem of deciding, given a graph G and a k-precoloring (G, X, f), if (G, X, f) has a k-precoloring extension.

We denote by P_t the path with t vertices. Given a path P, its *interior* is the set of vertices that have degree two in P. We denote the interior of P by P^*. A *P_t in a graph G* is a sequence $v_1 - \dots - v_t$ of pairwise distinct vertices where for $i, j \in [t]$, v_i is adjacent to v_j if and only if $|i - j| = 1$. We denote by $V(P)$ the set $\{v_1, \dots, v_t\}$, and if $a, b \in V(P)$, say $a = v_i$ and $b = v_j$ and $i < j$, then $a - P - b$ is the path $v_i - v_{i+1} - \dots - v_j$. A graph is *$P_t$-free* if there is no P_t in G. Throughout the paper by "polynomial time" or "polynomial size" we mean running time, or size, that is polynomial in $|V(G)|$.

Since the k-COLORING PROBLEM and the k-PRECOLORING EXTENSION PROBLEM are NP-hard for $k \geq 3$, their restrictions to graphs with a forbidden induced subgraph have been extensively studied; see [2, 6] for a survey of known results. In particular, the following is known (given a graph H, we say that a graph G is *H-free* if no induced subgraph of G is isomorphic to H):

Theorem 1 ([6]). *Let H be a (fixed) graph, and let $k > 2$. If the k-*COLORING PROBLEM *can be solved in polynomial time when restricted to the class of H-free graphs, then every connected component of H is a path.*

Thus if we assume that H is connected, then the question of determining the complexity of k-coloring H-free graph is reduced to studying the complexity of coloring graphs with certain induced paths excluded, and a significant body of work has been produced on this topic. Below we list a few such results.

Theorem 2 ([1]). *The* 3-COLORING PROBLEM *can be solved in polynomial time for the class of P_7-free graphs.*

Theorem 3 ([4]). *The* k-COLORING PROBLEM *can be solved in polynomial time for the class of P_5-free graphs.*

Theorem 4 ([5]). *The* 4-COLORING PROBLEM *is NP-complete for the class of P_7-free graphs.*

Theorem 5 ([5]). *For all $k \geq 5$, the* k-COLORING PROBLEM *is NP-complete for the class of P_6-free graphs.*

The only cases for which the complexity of k-coloring P_t-free graphs is not known are $k = 4$, $t = 6$, and $k = 3$, $t \geq 8$.

The main result of this paper is the following:

Theorem 6. *The* 4-PRECOLORING EXTENSION PROBLEM *can be solved in polynomial time for the class of P_6-free graphs.*

In contrast, the 4-LIST COLORING PROBLEM restricted to P_6-free graphs is NP-hard as proved by Golovach, Paulusma, and Song [6]. As an immediate corollary of Theorem 6, we obtain that the 4-COLORING PROBLEM for P_6-free graphs is also solvable in polynomial time. This proves a conjecture of Huang [5], thus resolving the former open case above, and completes the classification of the complexity of the 4-COLORING PROBLEM for graphs with a connected forbidden induced subgraph.

1.1 Preliminary and Sketch of the Proof

We start with some notations. Let G be a graph. For $X \subseteq V(G)$ we denote by $G|X$ the subgraph induced by G on X, and by $G \setminus X$ the graph $G|(V(G) \setminus X)$. If $X = \{x\}$, we write $G \setminus x$ to mean $G \setminus \{x\}$. For disjoint subsets $A, B \subset V(G)$ we say that A is *complete* to B if every vertex of A is adjacent to every vertex of B, and that A is *anticomplete* to B if every vertex of A is non-adjacent to every vertex of B. If $A = \{a\}$ we write a is complete (or anticomplete) to B to mean $\{a\}$ that is complete (or anticomplete) to B. If $a \notin B$ is not complete and not anticomplete to B, we say that a is *mixed* on B. Finally, if H is an induced subgraph of G and $a \in V(G) \setminus V(H)$, we say that a is *complete to, anticomplete to*, or *mixed on* H if a is complete to, anticomplete to, or mixed on $V(H)$, respectively. For $v \in V(G)$ we write $N_G(v)$ (or $N(v)$ when there is no danger of confusion) to mean the set of vertices of G that are adjacent to v. Observe that since G is simple, $v \notin N(v)$. For $A \subseteq V(G)$, an *attachment* of A is a vertex of $V(G) \setminus A$ complete to A. For $B \subseteq V(G) \setminus A$ we denote by $B(A)$ the set of attachments of A in B. If $F = G|A$, we sometimes write $B(F)$ to mean $B(V(F))$.

Given a list assignment L for G, we say that the pair (G, L) is colorable if G is L-colorable. For $X \subseteq V(G)$, we write $(G|X, L)$ to mean the list coloring problem where we restrict the domain of the list assignment L to X. Let $X \subset V(G)$ be such that $|L(x)| = 1$ for every $x \in X$, and let $Y \subset V(G)$. We say that a list assignment M is *obtained from L by updating Y from X* if $M(v) = L(v)$ for every $v \notin Y$, and $M(v) = L(v) \setminus \bigcup_{x \in N(v) \cap X} \{L(x)\}$ for every $v \in Y$. If $Y = V(G)$, we say that M is *obtained from L by updating from X*. If M is obtained from L by updating from $X^0(L)$, we say that M is *obtained from L by updating.* Let $L = L_0$, and for $i \geq 1$ let L_i be obtained from L_{i-1} by updating. If $L_i = L_{i-1}$, we say that L_i is *obtained from L by updating exhaustively.* Since $0 \leq \sum_{v \in V(G)} |L_j(v)| < \sum_{v \in V(G)} |L_{j-1}(v)| \leq 4|V(G)|$ for all $j < i$, it follows that $i \leq 4|V(G)|$ and thus L_i can be computed from L in polynomial time.

An *excellent starred precoloring* of a graph G is a six-tuple $P = (G, S, X_0, X, Y^*, f)$ such that

(A) $f : S \cup X_0 \to \{1, 2, 3, 4\}$ is a proper coloring of $G|(S \cup X_0)$;

(B) $V(G) = S \cup X_0 \cup X \cup Y^*$;

(C) $G|S$ is connected and no vertex in $V(G) \setminus S$ is complete to S;

(D) every vertex in X has neighbors of at least two different colors (with respect to f) in S;

(E) no vertex in X is mixed on a component of $G|Y^*$; and

(F) for every component of $G|Y^*$, there is a vertex in $S \cup X_0 \cup X$ complete to it.

We call S the *seed* of P. We define two list assignments associated with P. First, define $L_P(v) = \{f(v)\}$ for every $v \in S \cup X_0$, and let $L_P(v) = \{1,2,3,4\} \setminus (f(N(v) \cap S))$ for $v \notin S \cup X_0$. Second, M_P is the list assignment obtained as follows. First, define M_1 to be the list assignment for $G|(X \cup X_0)$ obtained from $L_P|\{X \cup X_0\}$ by updating exhaustively; let $X_1 = \{x \in X \cup X_0 \ : \ |M_1(x_1)| = 1\}$. Now define $M_P(v) = L_P(v)$ if $v \notin X \cup X_0$, and $M_P(v) = M_1(v)$ if $v \in X \cup X_0$. Let $X^0(P) = X^0(M_P)$. Then $S \cup X_0 \subseteq X^0(P)$. A *precoloring extension of* P is a proper 4-coloring c of G such that $c(v) = f(v)$ for every $v \in S \cup X_0$; it follows that $M_P(v) = \{c(v)\}$ for every $v \in X^0(P)$. It will often be convenient to assume that $X_0 = X^0(P) \setminus S$, and this assumption can be made without loss of generality. Note that in this case, $M_P(v) = L_P(v)$ for all $v \in X$. A subset Q of X is *orthogonal* if there exist $a, b \in \{1,2,3,4\}$ such that for every $q \in Q$ either $M_P(q) = \{a,b\}$ or $M_P(q) = \{1,2,3,4\} \setminus \{a,b\}$. We say that P is *orthogonal* if $N(y) \cap X$ is orthogonal for every $y \in Y^*$.

For an excellent starred precoloring P and a collection excellent starred $\mathcal{L}$ of precolorings, we say that $\mathcal{L}$ is an *equivalent collection* for P (or that P is *equivalent* to $\mathcal{L}$) if P has a precoloring extension if and only if at least one of the precolorings in $\mathcal{L}$ has a precoloring extension, and a precoloring extension of P can be constructed from a precoloring extension of a member of $\mathcal{L}$ in polynomial time.

We break the proof of Theorem 6 into two independent parts. In one part, we reduce the 4-PRECOLORING EXTENSION PROBLEM for P_6-free graphs to determining if an excellent starred precolorings of a P_6-free graph has a precoloring extension, and finding one if it exists. In fact, we restrict the problem further, by ensuring that there is a universal bound (that works for all 4-precolorings of all P_6-free graphs) on the size of the seed of the excellent starred precolorings that we need to consider. More precisely, we prove:

Theorem 7. *There exists an integer $C > 0$ and a polynomial-time algorithm with the following specifications.*

Input: *A 4-precoloring (G, X_0, f) of a P_6-free graph G.*

Output: *A collection $\mathcal{L}$ of excellent starred precolorings of G such that*

1. *If for every $P' \in \mathcal{L}$ we can in polynomial time either find a precoloring extension of P', or determine that none exists, then we can construct a 4-precoloring extension of (G, X_0, f) in polynomial time, or determine that none exists:*
2. *$|\mathcal{L}| \leq |V(G)|^C$; and*
3. *for every $(G', S', X'_0, X', Y^*, f') \in \mathcal{L}$,*
   - *$|S'| \leq C$;*
   - *$X_0 \subseteq S' \cup X'_0$;*
   - *G' is an induced subgraph of G; and*
   - *$f'|X_0 = f|X_0$.*

The proof of Theorem 7 is hard and technical, so we outline the idea here and leave the detailed proof to the appendix. It consists of several steps. At each step we replace the problem that we are trying to solve by a polynomially sized collection of simpler problems, where by "simpler" we mean "closer to being an excellent starred precoloring". The strategy at every step is to "guess" (by exhaustively enumerating) a bounded number of vertices that have certain key properties, and their colors, add these vertices to the seed, and show that the resulting precoloring is better than the one we started with. The other part of the proof of Theorem 6 is an algorithm that tests in polynomial time if an excellent starred precoloring (where the size of the seed is fixed) has a precoloring extension. The goal of the present paper is to solve this problem. We prove:

Theorem 8. *For every positive integer C there exists a polynomial-time algorithm with the following specifications.*

Input: *An excellent starred precoloring $P = (G, S, X_0, X, Y^*, f)$ of a P_6-free graph G with $|S| \leq C$.*

Output: *A precoloring extension of P or a determination that none exists.*

Clearly, Theorem 7 and Theorem 8 together imply Theorem 6. The proof of Theorem 8 consists of several steps. At each step we replace the problem that we are trying to solve by a polynomially sized collection of simpler problems, and the problems created in the last step can be encoded via 2-SAT. Here is an outline of the proof. First we show that an excellent starred precoloring P of a P_6-free graph G can be replaced by a polynomially sized collection $\mathcal{L}$ of excellent starred precolorings of G that have an additional property (to which we refer as "being orthogonal") and P has a precoloring extension if and only if some member of $\mathcal{L}$ does. Thus in order to prove Theorem 8, it is enough to be able to test if an orthogonal excellent starred precoloring of a P_6-free graph has a precoloring extension. Our next step is an algorithm whose input is an orthogonal excellent starred precoloring P of a P_6-free graph G, and whose output is a "companion triple" for P. A companion triple consists of a graph H that may not be P_6-free, but certain parts of it are, a list assignment L for H, and a correspondence function h that establishes the connection between H and P. Moreover, in order to test if P has a precoloring extension, it is enough to test if (H, L) is colorable.

The next step of the algorithm is replacing (H, L) by a polynomially sized collection $\mathcal{M}$ of list assignments for H, such that (H, L) is colorable if and only if there exists $L' \in \mathcal{L}$ such that (H, L') is colorable, and in addition for every $L' \in \mathcal{L}$ the pair (H, L') is "insulated". Being insulated means that H is the union of four induced subgraphs $H_1, \ldots, H_4$, and in order to test if (H, L') is colorable, it is enough to test if (H_i, L') is colorable for each $i \in \{1, 2, 3, 4\}$. The final step of the algorithm is converting the problem of coloring each (H_i, L') into a 2-SAT problem, and solving it in polynomial time. Moreover, at each step of the proof, if a coloring exists, then we can find it, and convert in polynomial time into a precoloring extension of P.

This paper is organized as follows. In Section 2 we produce a collection $\mathcal{L}$ of orthogonal excellent starred precolorings. In Section 3 we construct a companion triple for an orthogonal precoloring. In Section 4 we start with a precoloring and its companion triple, and construct a collection $\mathcal{M}$ of lists L' such that every pair (H, L') is insulated. Finally, in Section 5 we describe the reduction to 2-SAT. Section 6 contains the proof of Theorem 8 and of Theorem 6. In the appendix, we give a detailed proof of Theorem 7.

2 From Excellent to Orthogonal

Let $P = (G, S, X_0, X, Y^*, f)$ be an excellent starred precoloring. For $v \in X \cup Y^*$, the *type* of v is the set $N(v) \cap S$. Thus the number of possible types for a given precoloring is at most $2^{|S|}$. In this section we will prove several lemmas that allow us to replace a given precoloring by an equivalent polynomially sized collection of "nicer" precolorings, with the additional property that the size of the seed of each of the new precolorings is bounded by a function of the size of the seed of the precoloring we started with. Keeping the size of the seed bounded allows us to maintain the property that the number of different types of vertices of $X \cup Y^*$ is bounded, and therefore, from the point of view of running time, we can always consider each type separately.

For $T \subseteq S$ we denote by $L_P(T)$ the set $\{1,2,3,4\} \setminus \bigcup_{v \in T}\{f(v)\}$. Thus if v is of type T, then $L_P(v) = L_P(T)$. For $T \subseteq S$ and $U \subseteq X \cup Y^*$ we denote by $U(T)$ the set of vertices of U of type T.

A subset Q of X is *orthogonal* if there exist $a, b \in \{1,2,3,4\}$ such that for every $q \in Q$ either $M_P(q) = \{a,b\}$ or $M_P(q) = \{1,2,3,4\} \setminus \{a,b\}$. We say that P is *orthogonal* if $N(y) \cap X$ is orthogonal for every $y \in Y^*$.

The goal of this section is to prove that for every excellent starred precoloring P of a P_6-free graph G, there is a an equivalent collection $\mathcal{L}(P)$ of orthogonal excellent starred precolorings of G. We start with a few technical lemmas.

Lemma 1. *Let $P = (G, S, X_0, X, Y^*, f)$ be an excellent starred precoloring of a P_6-free graph G. Let $i, j \in \{1,2,3,4\}$ and $k \in \{1,2,3,4\} \setminus \{i,j\}$. Let T_i, T_j be types such that $L_P(T_i) = \{i,k\}$ and $L_P(T_j) = \{j,k\}$, and let $x_i, x_i' \in X(T_i)$ and $x_j, x_j' \in X(T_j)$. Suppose that $y_i, y_j \in Y^*$ are such that $i, j \in M_P(y_i) \cap M_P(y_j)$, where possibly $y_i = y_j$. Suppose further that the only possible edge among x_i, x_i', x_j, x_j' is x_ix_j, and y_i is adjacent to x_i' and not to x_i, and y_j is adjacent to x_j' and not to x_j. Then there does not exist $y \in Y^*$ with $i, j \in M_P(y)$ and such that y is complete to $\{x_i, x_j\}$ and anticomplete to $\{x_i', x_j'\}$.*

Proof. Suppose such y exists. Since no vertex of X is mixed on a component of $G|Y^*$, it follows that y is anticomplete to $\{y_i, y_j\}$. Since $x_i, x_i' \in X$ and $i, k \in L_P(T_i)$, it follows that there exists $s_j \in T_i$ with $L_P(s_j) = \{j\}$. Similarly, there exists $s_i \in Tj$ with $L_P(s_i) = \{i\}$. Since $i \in L_P(T_i)$ and $j \in L_P(T_j)$, it follows that s_i is anticomplete to $\{x_i, x_i'\}$ and s_j is anticomplete to $\{x_j, x_j'\}$.

Since $i, j \in M_P(y_i) \cap M_P(y_j) \cap M_P(y)$ it follows that $\{s_i, s_j\}$ is anticomplete to $\{y_i, y_j, y\}$. Since $x_i' - s_j - x_i - y - x_j - s_i - x_j'$ (possibly shortcutting through x_ix_j) is not a P_6 in G, it follows that s_i is adjacent to s_j. If y_i is non-adjacent to x_j', and y_j is non-adjacent to x_i', then $y_i \neq y_j$, and since P is excellent, y_i is non-adjacent to y_j, and so $y_i - x_i' - s_j - s_i - x_j' - y_j$ is a P_6, a contradiction, so we may assume that y_i is adjacent to x_j'. But now $x_j' - y_i - x_i' - s_j - x_i - y$ is a P_6, a contradiction. This proves Lemma 1. □

Lemma 2. *Let $P = (G, S, X_0, X, Y^*, f)$ be an excellent starred precoloring of a P_6-free graph G. Let $\{i,j,k,l\} = \{1,2,3,4\}$. Let T_i, T_j be types such that $L_P(T_i) = \{i,k\}$ and $L_P(T_j) = \{j,k\}$, and let $x_i, x_i' \in X(T_i)$ and $x_j, x_j' \in X(T_j)$. Let $y_i^i, y_j^i \in Y^*$ with $i, l \in M_P(y_i^i) \cap M_P(y_j^i)$, and let $y_i^j, y_j^j \in Y^*$ with $j, l \in M_P(y_i^j) \cap M_P(y_j^j)$, where possibly $y_i^i = y_j^i$ and $y_i^j = y_j^j$. Assume that*

- *some component C_i of $G|Y^*$ contains both y_i^i, y_i^j;*
- *some component C_j of $G|Y^*$ contains both y_j^i, y_j^j;*
- *for every $t \in \{i,j\}$ there is a path M in C_t from y_t^i to y_t^j with $l \in M_P(u)$ for every $u \in V(M)$;*
- *the only possible edge among x_i, x_i', x_j, x_j' is x_ix_j;*
- *y_i^i, y_i^j are adjacent to x_i' and not to x_i;*
- *y_j^i, y_j^j are adjacent to x_j' and not to x_j.*

Then there do not exist $y^i, y^j \in Y^*$ with $i, l \in M_P(y^i)$, $j, l \in M_P(y^j)$ and such that

- *some component C of $G|Y^*$ contains both y^i and y^j, and*
- *$l \in M_P(u)$ for every $u \in V(C)$, and*
- *$\{y^i, y^j\}$ is complete to $\{x_i, x_j\}$ and anticomplete to $\{x'_i, x'_j\}$.*

Proof. Suppose such y^i, y^j exist. Since P is an excellent starred precoloring, no vertex of X is mixed on a component of $G|Y^*$, and therefore $V(C)$ is anticomplete to $V(C_i) \cup V(C_j)$. Since $x_i, x'_i \in X$ and $i, k \in L_P(T_i)$, it follows that there exists $s_j \in T_i$ with $L_P(s_j) = \{j\}$. Similarly, there exists $s_i \in Tj$ with $L_P(s_i) = \{i\}$. Since $i \in L_P(T_i)$ and $j \in L_P(T_j)$, it follows that s_i is anticomplete to $\{x_i, x'_i\}$ and s_j is anticomplete to $\{x_j, x'_j\}$. Since $i \in M_P(y^i) \cap M_P(y^i_i) \cap M_P(y^i_j)$, it follows that s_i is anticomplete to $\{y^i, y^i_i, y^i_j\}$, and similarly s_j is anticomplete to $\{y^j, y^j_i, y^j_j\}$.

First we prove that s_i is adjacent to s_j. Suppose not. Since $x'_i - s_j - x_i - x_j - s_i - x'_j$ is not a P_6 in G, it follows that x_i is non-adjacent to x_j. But now $x'_i - s_j - x_i - y^j - x_j - s_i$ or $x'_i - s_j - x_i - y^j - s_j - x'_j$ is a P_6 in G, a contradiction. This proves that s_i is adjacent to s_j.

If y^j_i is adjacent to x'_j, then $x'_j - y^j_i - x'_i - s_j - x_i - y^j$ is a P_6, a contradiction. Therefore x'_j is non-adjacent to y^j_i, and therefore x'_j is anticomplete to C_i. Similarly, x'_i is anticomplete to C_j. In particular it follows that $C_i \neq C_j$.

Since $L_P(T_j) = \{i, k\}$ there exists $s_l \in S$ with $L_P(s_l) = \{l\}$ such that s_l is complete to $X(T_j)$. Since $l \in M_P(y)$ for every $y \in \{y^i_i, y^j_i, y^i_j, y^j_j, y^i, y^j\}$, it follows that s_l is anticomplete to $\{y^i_i, y^j_i, y^i_j, y^j_j, y^i, y^j\}$. Recall that $x_i, x'_i \in X(T_i)$, and so no vertex of S is mixed on $\{x_i, x'_i\}$. Similarly no vertex of S is mixed on $\{x_j, x'_j\}$. If s_l is anticomplete to $\{x_i, x'_i\}$, then one of $y^j_i - x'_i - s_j - s_l - x'_j - y^j_j$, $x'_i - s_j - x_i - y^j - x_j - s_l$, $x'_i - s_j - x_i - x_j - s_l - x'_j$ is a P_6, so s_l is complete to $\{x_i, x'_i\}$.

Since $y^i_i - x'_i - s_j - s_i - x'_j - y^j_j$ is not a P_6, it follows that either s_j is adjacent to y^i_i, or s_i is adjacent to y^j_j. We may assume that s_j is adjacent to y^i_i.

Let M be a path in C_i from y^j_i to y^i_i with $l \in M_P(u)$ for every $u \in V(M)$. Since s_j is adjacent to y^i_i and not to y^j_i, there is exist adjacent $a, b \in V(M)$ such that s_j is adjacent to a and not to b. Since $l \in M_P(u)$ for every $u \in V(M)$, it follows that s_l is anticomplete to $\{a, b\}$. But now if s_l is non-adjacent to s_j, then $b - a - s_j - x_i - s_l - x'_j$ is a P_6, and if s_l is adjacent to s_j, then $b - a - s_j - s_l - x'_j - y^j_j$ is a P_6; in both cases a contradiction. This proves Lemma 2. □

Let $P = (G, S, X_0, X, Y^*, f)$ be an excellent starred precoloring of a P_6-free graph G. Let $S'' \subseteq X$, and let $X''_0 \subseteq X \cup Y^*$. Let $f' : S \cup X_0 \cup S'' \cup X''_0 \to \{1, 2, 3, 4\}$ be such that $f'|(S \cup X_0) = f|(S \cup X_0)$ and $(G, S \cup X_0 \cup S'' \cup X''_0, f')$ is a 4-precoloring of G. Let X'' be the set of vertices x of $X \setminus X''_0$ such that x as a neighbor $z \in S''$ with $f'(z) \in M_P(x)$. Let

$$S' = S \cup S''$$

$$X'_0 = X_0 \cup X'' \cup X''_0$$

$$X' = X \setminus (X'' \cup S'' \cup X''_0)$$

$$Y^{*\prime} = Y^* \setminus X''_0.$$

We say that $P' = (G, S', X'_0, X', Y^{*\prime}, f')$ is *obtained from P by moving S'' to the seed with colors $f'(S'')$, and moving X''_0 to X_0 with colors $f'(X''_0)$*. Sometimes we say that "we move S'' to S with colors $f'(S'')$, and X''_0 to X_0 with colors $f'(X''_0)$".

In the next lemma we show that this operation creates another excellent starred precoloring.

Lemma 3. *Let $P = (G, S, X_0, X, Y^*, f)$ be an excellent starred precoloring of a P_6-free graph G. Let $S'' \subseteq X$ and $X_0'' \subseteq X \cup Y^*$, and let $S', X_0', X', Y^{*\prime}, f'$ be as above. Then either $P' = (G, S', X_0', X', Y^{*\prime}, f')$ is an excellent starred precoloring.*

Proof. We need to check the following conditions:

1. $f' : S' \cup X_0' \to \{1, 2, 3, 4\}$ is a proper coloring of $G|(S' \cup X_0')$;
2. $V(G) = S' \cup X_0' \cup X' \cup Y^{*\prime}$;
3. $G|S'$ is connected and no vertex in $V(G) \setminus S'$ is complete to S';
4. every vertex in X' has neighbors of at least two different colors (with respect to f') in S';
5. no vertex in X' is mixed on a component of $G|Y^{*\prime}$; and
6. for every component of $G|Y^{*\prime}$, there is a vertex in $S' \cup X_0' \cup X'$ complete to it.

Next we check the conditions.

1. holds by the definition of P'.
2. holds since $S' \cup X_0' \cup X' \cup Y^{*\prime} = S \cup X_0 \cup X \cup Y^*$.
3. $G|S'$ is connected since $G|S$ is connected, and every $z \in S''$ has a neighbor in S. Moreover, since no vertex of $V(G) \setminus S$ is complete to S, it follows that no vertex of $V(G) \setminus S'$ is complete to S'.
4. follows from the fact that $X' \subseteq X$.
5. follows from the fact that $Y^{*\prime} \subseteq Y^*$ and $X' \subseteq X$.
6. follows from the fact that $Y^{*\prime} \subseteq Y^*$ and $S \cup X_0 \subseteq S' \cup X_0'$.

□

Let $P = (G, S, X_0, X, Y^*, f)$ be an excellent starred precoloring. Let $i, j \in \{1, 2, 3, 4\}$. Write $X_{ij} = \{x \in X$ such that $M_P(x) = \{i, j\}\}$. For $y \in Y^*$ let $C_P(y)$ (or $C(y)$ when there is no danger of confusion) denote the vertex set of the component of $G|Y^*$ that contains y.

Let $P = (G, S, X_0, X, Y^*, f)$ be an excellent starred precoloring, and let $\{i, j, k, l\} = \{1, 2, 3, 4\}$. We say that P is *kl-clean* if there does not exist $y \in Y^*$ with the following properties:

- $i, j \in M_P(y)$, and
- there is $u \in C(y)$ with $k \in M_P(u)$, and
- y has both a neighbor in X_{ik} and a neighbor in X_{jk}.

We say that P is *clean* if it is kl-clean for every $k, l \in \{1, 2, 3, 4\}$.

We say that P is *kl-tidy* if there do not exist vertices $y_i, y_j \in Y^*$ such that

- $i \in M_P(y_i)$, $j \in M_P(y_j)$, and
- $C(y_i) = C(y_j)$, and
- there is a path M from y_i to y_j in C such that $l \in M_P(u)$ for every $u \in V(M)$, and

- there is $u \in V(C)$ with $k \in M_P(u)$, and
- y_i has a neighbor in X_{ki} and a neighbor in X_{kj}

Observe that since no vertex of X is mixed on an a component of $G|Y^*$, it follows that $N(y_i) \cap X_{ki}$ is precisely the set of vertices of X_{ki} that are complete to $C(y_i)$, and an analogous statement holds for X_{kj}. We say that P is *tidy* if it is kl-tidy for every $k, l \in \{1, 2, 3, 4\}$.

We say that P is *kl-orderly* if for every y in Y^* with $\{i, j\} \subseteq M_P(y)$, $N(y) \cap X_{ik}$ is complete to $N(y) \cap X_{jk}$. We say that P is *orderly* if it is kl-orderly for every $k, l \in \{1, 2, 3, 4\}$

Finally, we say that P is *kl-spotless* if no vertex y in Y^* with $\{i, j\} \subseteq M_P(y)$ has both a neighbor in X_{ik} and a neighbor in X_{jk}. We say that P is *spotless* if it is kl-spotless for every $k, l \in \{1, 2, 3, 4\}$

Our goal is to replace an excellent starred precoloring by an equivalent collection of spotless precolorings. First we prove a lemma that allows us to replace an excellent starred precoloring with an equivalent collection of clean precolorings.

Lemma 4. *There is a function $q : \mathbb{N} \to \mathbb{N}$ such that the following holds. Let G be a P_6-free graph, and let $P = (G, S, X_0, X, Y^*, f)$ be an excellent starred precoloring of G. Then there is an algorithm with running time $O(|V(G)|^{q(|S|)})$ that outputs a collection $\mathcal{L}$ of excellent starred precolorings of G such that:*

- $|\mathcal{L}| \leq |V(G)|^{q(|S|)}$*;*
- $|S'| \leq q(|S|)$ *for every* $P' \in \mathcal{L}$*;*
- *every $P' \in \mathcal{L}$ is kl-clean for every (k, l) for which P is kl-clean;*
- *every $P' \in \mathcal{L}$ is 14-clean;*
- *$\mathcal{L}$ is an equivalent collection for P.*

Proof. Without loss of generality we may assume that $X_0 = X^0(P) \setminus S$. Thus $L_P(x) = M_P(x)$ for every $x \in X$. We may assume that P is not 14-clean for otherwise we may set $\mathcal{L} = \{P\}$. Let Y be the set of vertices of Y^* with $2, 3 \in M_P(y)$ and such that some $u \in C(y)$ has $1 \in M_P(u)$. Let $T_1, \ldots, T_p$ be the subsets of S with $L_P(T_s) = \{1, 2\}$ and $T_{p+1}, \ldots, T_m$ the subsets of S with $L_P(T_s) = \{1, 3\}$. Let $\mathcal{Q}$ be the collection of all m-tuples

$$((S_1, Q_1), (S_2, Q_2), \ldots, (S_m, Q_m))$$

where for every $r \in \{1, \ldots, m\}$

- $S_r \subseteq X(T_r)$ and $|S_r| \in \{0, 1\}$,
- if $S_r = \emptyset$, then $Q_r = \emptyset$
- $S_r = \{x_r\}$ then $Q_r = \{y\}$ where $y \in Y \cap N(x_r)$.

For $Q \in \mathcal{Q}$ construct a precoloring P_Q as follows. Let $r \in \{1, \ldots, m\}$. We may assume that $r \leq p$.

- Assume first that $S_r = \{x_r\}$. Then $Q_r = \{y_r\}$. Move $\{x_r\}$ to the seed with color 1, and for every $y \in Y$ such that $N(y) \cap X(T_r) \subset N(y_r) \cap X(T_r) \setminus \{x_r\}$, move $N(y) \cap X(T_r)$ to X_0 with the unique color of $L_P(T_r) \setminus \{1\}$.

- Next assume that $S_r = \emptyset$. Now for every $y \in Y$ move $N(y) \cap X(T_r)$ to X_0 with the unique color of $L_P(T_r) \setminus \{1\}$.

In the notation of Lemma 3, if the precoloring of $G|(X'_0 \cup S')$ thus obtained is not proper, remove Q form $\mathcal{Q}$. Therefore we may assume that the precoloring is proper. Repeatedly applying Lemma 3 we deduce that P_Q is an excellent starred precoloring. Observe that $Y^{*\prime} = Y^*$. Since $X' \subseteq X$ and $Y^{*\prime} = Y^*$, it follows that if P is kl-clean, then so is P_Q.

Now we show that P_Q is 14-clean. Let Y' be the set of vertices y of Y^* such that $2, 3 \in M_{P_Q}(y)$ and some vertex $u \in C(y)$ has $1 \in M_{P_Q}(u)$. Observe that $Y' \subseteq Y$. It is enough to check that no vertex of Y' has both a neighbor in X'_{12} and a neighbor in X'_{13}. Suppose this is false, and suppose that $y \in Y'$ has a neighbor $x_2 \in X'_{12}$ and a neighbor $x_3 \in X'_{13}$. Then $x_2 \in X_{12}$ and $x_3 \in X_{13}$. We may assume that $x_2 \in X(T_1)$ and $x_3 \in X(T_{p+1})$. Since $x_2, x_3 \notin X^0(P_Q)$, it follows that both $S_1 \neq \emptyset$ and $S_{p+1} \neq \emptyset$, and therefore $Q_1 \neq \emptyset$ and $Q_{p+1} \neq \emptyset$. Write $S_1 = \{x'_2\}$, $Q_1 = \{y_2\}$, $S_{p+1} = \{x'_3\}$ and $Q_{p+1} = \{y_3\}$. Since some $u \in C(y)$ has $1 \in M_{P_Q}(u)$, and since x'_2, x'_3 are not mixed on $C(y)$, it follows that y is anticomplete to $\{x'_2, x'_3\}$. Again since $x_2 \notin X^0(P_Q)$, it follows that $N(y) \cap X(T_1) \not\subseteq N(y_2) \cap X(T_1)$, and so we may assume that $x_2 \notin N(y_2)$. Similarly, we may assume that $x_3 \notin N(y_3)$. But now the vertices $x_2, x'_2, x_3, x'_3, y_2, y_3, y$ contradict Lemma 1. This proves that P_Q is 14-clean.

Since $S' = S \cup \bigcup_{i=1}^{m} S_i$, and since $m \leq 2^{|S|}$, it follows that $|S'| \leq |S| + m \leq |S| + 2^{|S|}$.

Let $\mathcal{L} = \{P_Q \,:\, Q \in \mathcal{Q}\}$. Then $|\mathcal{L}| \leq |V(G)|^{2m} \leq |V(G)|^{2^{|S|+1}}$. We show that $\mathcal{L}$ is an equivalent collection for P. Since every $P' \in \mathcal{L}$ is obtained from P by precoloring some vertices and updating, it is clear that if c is a precoloring extension of a member of $\mathcal{L}$, then c is a precoloring extension of P. To see the converse, let c be a precoloring extension of P. For every $i \in \{1, \dots, m\}$ define S_i and Q_i as follows. If no vertex of Y has a neighbor $x \in X(T_i)$ with $c(x) = 1$, set $S_i = Q_i = \emptyset$. If some vertex of Y has neighbor $x \in X(T_i)$ with $c(x) = 1$, let y be a vertex with this property and in addition with $N(y) \cap X(T_i)$ minimal; let $x \in X(T_i) \cap N(y)$ with $c(x) = 1$; and set $Q_i = \{y\}$ and $S_i = \{x\}$. Let $Q = ((S_1, Q_1), \dots, (S_m, Q_m))$. We claim that c is a precoloring extension of P_Q. Write $P_Q = (G, S', X'_0, X', Y', f')$. We need to show that $c(v) = f'(v)$ for every $v \in S' \cup X'_0$. Since c is a precoloring extension of P, it follows that $c(v) = f(v) = f'(v)$ for every $v \in S \cup X_0$. Since $S' \setminus S = \bigcup_{s=1}^{m} S_s$ and $c(v) = f'(v) = 1$ for every $v \in \bigcup_{s=1}^{m} S_s$, we deduce that $c(v) = f'(v)$ for every $v \in S'$. Finally let $v \in X'_0 \setminus X_0$. It follows that $v \in X$, $f'(v)$ is the unique color of $M_P(v) \setminus \{1\}$, and there are three possibilities.

1. $1 \in M_P(v)$ and v has a neighbor in $\bigcup_{s=1}^{m} S_s$, or
2. there is $i \in \{1, \dots, m\}$ with $S_i = \{x_i\}$ and $Q_i = \{y_i\}$, and there is $y \in Y^*$ such that $N(y) \cap X(T_i) \subseteq (N(y_i) \cap X(T_i)) \setminus \{x_i\}$, and $v \in N(y) \cap X(T_i)$, or
3. there is $i \in \{1, \dots, m\}$ with $S_i = Q_i = \emptyset$, and there is $y \in Y^*$ such that $v \in N(y) \cap X(T_i)$.

We show that in all these cases $c(v) = f'(v)$.

1. Let $x \in \bigcup_{s=1}^{m} S_s$. Then $c(x) = 1$, and so $c(v) \neq 1$, and thus $c(v) = f'(v)$.
2. By the choice of y_i and since $N(y) \cap X(T_i) \subseteq (N(y_i) \cap X(T_i)) \setminus \{x_i\}$, it follows that $c(u) \neq 1$ for every $u \in N(y) \cap X(T_i)$, and therefore $c(v) = f'(v)$.
3. Since $S_i = \emptyset$, it follows that for every $y' \in Y^*$ and for every $u \in N(y') \cap X(T_i)$ we have that $c(u) \neq 1$, and again $c(v) = f'(v)$.

This proves that c is a precoloring extension of P_Q, and completes the proof of Lemma 4. $\square$

Repeatedly applying Lemma 4 and using symmetry, we deduce the following:

Lemma 5. *There is a function $q : \mathbb{N} \to \mathbb{N}$ such that the following holds. Let G be a P_6-free graph. Let $P = (G, S, X_0, X, Y^*, f)$ be an excellent starred precoloring of G. Then there is an algorithm with running time $O(|V(G)|^{q(|S|)})$ that outputs a collection $\mathcal{L}$ of excellent starred precolorings of G such that:*

- $|\mathcal{L}| \leq |V(G)|^{q(|S|)}$*;*
- $|S'| \leq q(|S|)$ *for every* $P' \in \mathcal{L}$*;*
- *every* $P' \in \mathcal{L}$ *is clean;*
- $\mathcal{L}$ *is an equivalent collection for* P*.*

Next we show that a clean precoloring can be replaced with an equivalent collection of precolorings that are both clean and tidy.

Lemma 6. *There is a function $q : \mathbb{N} \to \mathbb{N}$ such that the following holds. Let G be a P_6-free graph. Let $P = (G, S, X_0, X, Y^*, f)$ be a clean excellent starred precoloring of G. Then there is an algorithm with running time $O(|V(G)|^{q(|S|)})$ that outputs a collection $\mathcal{L}$ of excellent starred precolorings of G such that:*

- $|\mathcal{L}| \leq |V(G)|^{q(|S|)}$*;*
- $|S'| \leq q(|S|)$ *for every* $P' \in \mathcal{L}$*;*
- *every* $P' \in \mathcal{L}$ *is clean;*
- *every* $P' \in \mathcal{L}$ *is* kl*-tidy for every* k, l *for which* P *is* kl*-tidy;*
- *every* $P' \in \mathcal{L}$ *is* 14*-tidy;*
- $\mathcal{L}$ *is an equivalent collection for* P*.*

Proof. Without loss of generality we may assume that $X_0 = X^0(P) \setminus S$, and thus $L_P(x) = M_P(x)$ for every $x \in X$. We may assume that P is not 14-tidy for otherwise we may set $\mathcal{L} = \{P\}$. Let Y be the set of all pairs (y_2, y_3) with $y_2, y_3 \in Y^*$ such that

- $2 \in M_P(y_2)$, $3 \in M_P(y_3)$,
- y_2, y_3 are in the same component C of $G|Y^*$,
- there is a path M from y_2 to y_3 in C such that $4 \in M_P(u)$ for every $u \in V(M)$, and
- for some $u \in V(C)$, $1 \in M_P(u)$,

Let $T_1, \ldots, T_p$ be the subsets of S with $L_P(T_s) = \{1, 2\}$ and let $T_{p+1}, \ldots, T_m$ be the subsets of S with $L_P(T_s) = \{1, 3\}$. Let $\mathcal{Q}$ be the collection of all m-tuples

$$((S_1, Q_1), (S_2, Q_2), \ldots, (S_m, Q_m))$$

where for $r \in \{1, \ldots, m\}$

- $S_r \subseteq X(T_r)$ and $|S_r| \in \{0, 1\}$,

- if $S_r = \emptyset$, then $Q_r = \emptyset$

- $S_r = \{x_r\}$ then $Q_r = \{(y_2^r, y_3^r)\}$ where $(y_2^r, y_3^r) \in Y$ and x_r is complete to $\{y_2^r, y_3^r\}$.

For $Q \in \mathcal{Q}$ construct a precoloring $P_Q = (G^Q, S^Q, X_0^Q, X^Q, Y^Q, f^Q)$ as follows. Let $r \in \{1, \dots, m\}$; for $r = 1, \dots, m$, we proceed as follows.

- Assume first that $S_r = \{x_r\}$. Then $Q_r = \{(y_2^r, y_3^r)\}$. Move x_r to the seed with color 1, and for every $(y_2, y_3) \in Y$ such that $N(y_2) \cap X(T_r) \subset N(y_2^r) \cap (X(T_r) \setminus \{x_r\})$, move $N(y_2) \cap X(T_r)$ to X_0 with the unique color of $L_P(T_r) \setminus \{1\}$.

- Next assume that $S_r = \emptyset$. Now for every $y \in Y$ move $N(y) \cap X(T_r)$ to X_0 with the unique color of $L_P(T_r) \setminus \{1\}$.

In the notation of Lemma 3, if the precoloring of $G|(X_0' \cup S')$ thus obtained is not proper, remove Q form $\mathcal{Q}$. Therefore we may assume that the precoloring is proper. Repeatedly applying Lemma 3 we deduce that P_Q is an excellent starred precoloring. Observe that $Y^Q = Y^*$, $M_{P_Q}(y) \subseteq M_P(y)$ for every $y \in Y^Q$, and $M_{P_Q}(x) = M_P(x)$ for every $x \in X^Q \setminus X^0(P_Q)$. It follows that P_Q is clean, and that if P is kl-tidy, then so is P_Q.

Now we show that P_Q is 14-tidy. Suppose that there exist $y_2, y_3 \in Y^Q$ that violate the definition of being 14-tidy. Let $x_2 \in X_{12}^Q$ and $x_3 \in X_{13}^Q$ be adjacent to y_2, say, and therefore complete to $\{y_2, y_3\}$. We may assume that $x_2 \in X(T_1)$ and $x_3 \in X(T_{p+1})$. Since $x_2, x_3 \notin X^0(P_Q)$, it follows that both $S_1 \neq \emptyset$ and $S_{p+1} \neq \emptyset$, and therefore $Q_1 \neq \emptyset$ and $Q_{p+1} \neq \emptyset$. Write $S_1 = \{x_2'\}$, $Q_1 = \{(y_2^2, y_3^2)\}$, $S_{p+1} = \{x_3'\}$ and $Q_{p+1} = \{y_2^3, y_3^3\}$.

Since there is a vertex u in the component of $G|Y^Q$ containing y_2, y_3 with $1 \in M_{P_Q}(u)$, and since no vertex of X is mixed on a component of Y^*, it follows that $\{y_2, y_3\}$ is anticomplete to $\{x_2', x_3'\}$. Since $x_2 \notin X^0(P_Q)$, it follows that $N(y_2) \cap X(T_1) \not\subseteq N(y_2^2) \cap (X(T_1) \setminus \{x_2'\})$, and so we may assume that $x_2 \notin N(y_2^2)$. Similarly, we may assume that $x_3 \notin N(y_2^3)$. But now, since no vertex of X is mixed on a component of Y^*, we deduce that the vertices $x_2, x_2', x_3, x_3', y_2^2, y_2^3, y_3^2, y_3^3, y_2, y_3$ contradict Lemma 2. This proves that P_Q is 14-tidy.

Since $S' = S \cup \bigcup_{i=1}^m S_i$, and since $m \leq 2^{|S|}$, it follows that $|S'| \leq |S| + m \leq |S| + 2^{|S|}$.

Let $\mathcal{L} = \{P_Q \,:\, Q \in \mathcal{Q}\}$. Then $|\mathcal{L}| \leq |V(G)|^{3m} \leq |V(G)|^{3 \times 2^{|S|}}$. We show that $\mathcal{L}$ is an equivalent collection for P. Since every $P' \in \mathcal{L}$ is obtained from P by precoloring some vertices and updating, it is clear that every precoloring extension of a member of $\mathcal{L}$ is a precoloring extension of P. To see the converse, suppose that P has a precoloring extension c. For every $i \in \{1, \dots, m\}$ define S_i and Q_i as follows. If there does not exist $(y_2^2, y_2^3) \in Y$ such that some $x \in X(T_i)$ with $c(x) = 1$ is complete to $\{y_2^2, y_2^3\}$, set $S_i = Q_i = \emptyset$. If such a pair exists, let (y_2^2, y_2^3) be a pair with this property and subject to that with the set $N(y_2^2) \cap X(T_i)$ minimal; let $x \in X(T_i)$ be complete to $\{y_2^2, y_2^3\}$ and with $c(x) = 1$; and set $Q_i = \{(y_2^2, y_2^3)\}$ and $S_i = \{x\}$. Let $Q = ((S_1, Q_1), \dots, (S_m, Q_m))$. We claim that c is a precoloring extension of P_Q. Write $P_Q = (G, S', X_0', X', Y', f')$. We need to show that $c(v) = f'(v)$ for every $v \in S' \cup X_0'$. Since c is a precoloring extension of P, it follows that $c(v) = f(v) = f'(v)$ for every $v \in S \cup X_0$. Since $S' \setminus S = \bigcup_{s=1}^m S_s$ and $c(v) = f'(v) = 1$ for every $v \in \bigcup_{s=1}^m S_s$, we deduce that $c(v) = f'(v)$ for every $v \in S'$. Finally let $v \in X_0' \setminus X_0$. Then $v \in X$, $f'(v)$ is the unique color of $M_P(v) \setminus \{1\}$, and there are three possibilities.

1. $1 \in M_P(v)$ and v has a neighbor in $\bigcup_{s=1}^m S_s$, or

2. there is $i \in \{1, \dots, m\}$ with $S_i = \{x_i\}$ and $Q_i = \{(y_i^2, y_i^3)\}$, and there exists $(y_2, y_3) \in Y$ such $N(y_2) \cap X(T_i) \subseteq X(T_i) \cap (N(y_i^2) \setminus \{x_i\})$, or

3. there is $i \in \{1, \dots, m\}$ with $S_i = Q_i = \emptyset$, and there exists $(y_2, y_3) \in Y$ such that $v \in X(T_i) \cap N(y_2)$.

We show that in all these cases $c(v) = f'(v)$.

1. Let $x \in \bigcup_{s=1}^{m} S_s$. Then $c(x) = 1$, and so $c(v) \neq 1$, and thus $c(v) = f'(v)$.
2. By the choice of y_i^2, y_i^3 and since $N(y_2) \cap X(T_i) \subseteq (N(y_i^2) \cap X(T_i)) \setminus \{x_i\})$, it follows that $c(u) \neq 1$ for every $u \in N(y_2) \cap X(T_i)$, and therefore $c(v) = f'(v)$.
3. Since $S_i = \emptyset$, it follows that for every $(y_2, y_3) \in Y$ and for every $u \in N(y_2) \cap X(T_i)$ we have $c(u) \neq 1$, and again $c(v) = f'(v)$.

This proves that c is an extension of P_Q, and completes the proof of Lemma 6. □

Repeatedly applying Lemma 6 and using symmetry, we deduce the following:

Lemma 7. *There is a function $q : \mathbb{N} \to \mathbb{N}$ such that the following holds. Let G be a P_6-free graph. Let $P = (G, S, X_0, X, Y^*, f)$ be a clean excellent starred precoloring of G. Then there is an algorithm with running time $O(|V(G)|^{q(|S|)})$ that outputs a collection $\mathcal{L}$ of excellent starred precolorings of G such that:*

- $|\mathcal{L}| \leq |V(G)|^{q(|S|)}$;
- $|S'| \leq q(|S|)$ *for every* $P' \in \mathcal{L}$;
- *every* $P' \in \mathcal{L}$ *is clean and tidy;*
- $\mathcal{L}$ *is an equivalent collection for* P.

Our next goal is to show that a clean and tidy precoloring can be replaced with an equivalent collection of orderly precolorings.

Lemma 8. *There is a function $q : \mathbb{N} \to \mathbb{N}$ such that the following holds. Let G be a P_6-free graph. Let $P = (G, S, X_0, X, Y^*, f)$ be a clean, tidy starred precoloring of G. Then there is an algorithm with running time $O(|V(G)|^{q(|S|)})$ that outputs a collection $\mathcal{L}$ of excellent starred precolorings of G such that:*

- $|\mathcal{L}| \leq |V(G)|^{q(|S|)}$;
- $|S'| \leq q(|S|)$ *for every* $P' \in \mathcal{L}$;
- *every* $P' \in \mathcal{L}$ *is clean and tidy;*
- *every* $P' \in \mathcal{L}$ *is* kl*-orderly for every* (k, l) *for which* P *is* kl*-orderly;*
- *every* $P' \in \mathcal{L}$ *is* 14*-orderly;*
- P *is equivalent to* $\mathcal{L}$.

Proof. Without loss of generality we may assume that $X_0 = X^0(P)$, and so $L_P(x) = M_P(x)$ for every $x \in X$. We may assume that P is not 14-orderly for otherwise we may set $\mathcal{L} = \{P\}$. Let $Y = \{y \in Y^*$ such that $\{2, 3\} \subseteq M_P(y)\}$. Let $T_1, \dots, T_p$ be the types with $L(T_s) = \{1, 2\}$ and $T_{p+1}, \dots, T_m$ the types with $L(T_s) = \{1, 3\}$. Let $\mathcal{Q}$ be the collection of all $p(m-p)$-tuples of pairs (S_i, Q_j) with $i \in \{1, \dots, p\}$ and $j \in \{p+1, \dots, m\}$, where

- $S_i, Q_j \subseteq Y$;
- $|S_i|, |Q_j| \in \{0, 1\}$;
- if $N(S_i) \cap X(T_i) = \emptyset$, then $S_i = \emptyset$;
- if $N(Q_j) \cap X(T_j) = \emptyset$, then $Q_j = \emptyset$.

For $Q \in \mathcal{Q}$ construct a precoloring P_Q as follows. Let $i \in \{1, \dots, p\}$ and $j \in \{p+1, \dots, m\}$.

- Assume first that $S_i = \{y_i\}$ $Q_j = \{y_j\}$. If there is an edge between $N(y_i) \cap X(T_i)$ and $N(y_j) \cap X(T_j)$, remove Q from $\mathcal{Q}$. Now suppose that $N(y_i) \cap X(T_i)$ is anticomplete to $N(y_j) \cap X(T_j)$. Move $T = (N(y_i) \cap X(T_i)) \cup (N(y_j) \cap X(T_j))$ into X_0 with color 1. For every $y \in Y$ complete to T and both with a neighbor in $X(T_i) \setminus T$ and a neighbor in $X(T_j) \setminus T$, proceed as follows: if $4 \in M_P(y)$, move y to X_0 with color 4; if $4 \notin M_P(y)$, remove Q from $\mathcal{Q}$.
- Next assume that exactly one of S_i, Q_j is non-empty. By symmetry we may assume that $S_i = \{y_i\}$ and $Q_j = \emptyset$. Move $T = N(y_i) \cap X(T_i)$ into X_0 with color 1. For every $y \in Y$ complete to T and both with a neighbor in $X(T_i) \setminus T$ and a neighbor in $X(T_j)$, proceed as follows: if $4 \in M_P(y)$, move y to X_0 with color 4; if $4 \notin M_P(y)$, remove Q from $\mathcal{Q}$.
- Finally assume that $S_i = Q_j = \emptyset$. For every $y \in Y$ with both a neighbor in $X(T_i)$ and a neighbor in $X(T_j)$, proceed as follows: if $4 \in M_P(y)$, move y to X_0 with color 4; if $4 \notin M_P(y)$, remove Q from $\mathcal{Q}$.

Let $Q \in \mathcal{Q}$, and let $P_Q = (G, S', X'_0, X', Y^{*\prime}, f')$. Since $X' \subseteq X$, $Y' \subseteq Y^*$ and $M_{P_Q}(v) \subseteq M_P(v)$ for every v, it follows that P_Q is excellent, clean, tidy, and that for $k, l \in \{1, 2, 3, 4\}$, if P is kl-orderly, then P_Q is kl-orderly.

Next we show that P_Q is 14-orderly. Suppose that some $y \in Y$ has a neighbor in $x_2 \in X'_{12}$ and a neighbor in $x_3 \in X'_{13}$ such that x_2 is non-adjacent to x_3. Then $x_2 \in X_{12}$ and $x_3 \in X_{13}$. We may assume that $x_2 \in X(T_1)$ and $x_3 \in X(T_{p+1})$. Since $x_2, x_3 \notin X^0(P_Q)$, it follows that both $S_1 \neq \emptyset$ and $Q_{p+1} \neq \emptyset$. Let $S_1 = \{y_2\}$ and $Q_{p+1} = \{y_3\}$. Since $x_2, x_3 \notin X^0(P_Q)$, it follows that y_2 is non-adjacent to x_2, and y_3 is non-adjacent to x_3. Since $y \notin X^0(P_Q)$, we may assume by symmetry that there is $x'_2 \in N(y_2) \cap X(T_1)$ such that y is non-adjacent to x'_2. Let $x'_3 \in N(y_3) \cap X(T_{p+1})$. Since $x_2, x_3, y \notin X^0(P_Q)$, it follows that $\{x'_2, x'_3\}$ is anticomplete to $\{x_2, x_3\}$. By the construction of Q, x'_2 is non-adjacent to x'_3. By Lemma 1, y is adjacent to x'_3. Since $L_P(T_1) = \{1, 2\}$, there is $s_3 \in S$ complete to $X(T_1)$. Since $3 \in M_{P_Q}(y) \cap L_P(y_2) \cap L_P(y_3) \cap L_P(x'_3) \cap L(x_3)$, it follows that s_3 is anticomplete to $\{y, y_2, y_3, x_3, x'_3\}$. Similarly, since $L_P(T_{p+1}) = \{1, 3\}$, there is $s_2 \in S$ complete to $X(T_{p+1})$. Since $2 \in M_{P_Q}(y) \cap L_P(y_2) \cap L_P(y_3) \cap L_P(x_2) \cap L_P(x'_2)$, it follows that s_2 is anticomplete to $\{y, y_2, y_3, x_2, x'_2\}$. Since $y_2 - x'_2 - s_3 - x_2 - y - t$ is not a P_6 for $t \in \{x_3, x'_3\}$, it follows that y_2 is complete to $\{x_3, x'_3\}$. Since $y_3 - x'_3 - y - x_2 - s_3 - x'_2$ is not a P_6, it follows that y_3 is adjacent to at least one of x_2, x'_2. Since the path $x_2 - y - x_3 - y_2 - x'_2$ cannot be extended to a P_6 via y_3, follows that y_3 is complete to $\{x_2, x'_2\}$. But now $s_2 - x_3 - y - x_2 - y_3 - x'_2$ is a P_6, a contradiction. This proves that P_Q is 14-orderly.

Observe that $S' = S$, and so $|S'| = |S|$. Observe also that also that $p(m-p) \leq (\frac{m}{2})^2$, and since $m \leq 2^{|S|}$, it follows that $p(m-p) \leq 2^{2|S|-2}$. Let $\mathcal{L} = \{P_Q \ : \ Q \in \mathcal{Q}\}$. Now $|\mathcal{L}| \leq |V(G)|^{2p(m-p)} \leq |V(G)|^{2^{2|S|-1}}$.

We show that $\mathcal{L}$ is an equivalent collection for P. Since every $P' \in \mathcal{L}$ is obtained from P by precoloring some vertices and updating, it is clear that if c is a precoloring extension of a member of $\mathcal{L}$, then c is a precoloring extension of P. To see the converse, suppose that P has a precoloring extension c. For every $i \in \{1, \dots, p\}$ and $j \in \{p+1, \dots, m\}$ define S_i and Q_j as follows. If every vertex of Y has a neighbor $x \in X(T_i)$ with $c(x) \neq 1$, set $S_i = \emptyset$, and if every vertex of Y has a neighbor $x \in X(T_j)$ with $c(x) \neq 1$, set $Q_j = \emptyset$. If some vertex of Y has no neighbor $x \in X(T_i)$ with $c(x) \neq 1$, let y_i be a vertex with this property and in addition with $N(y) \cap X(T_i)$ maximal; set $S_i = \{y_i\}$. If some vertex of Y has no neighbor $x \in X(T_j)$ with $c(x) \neq 1$, let y_j be a vertex with this property and in addition with $N(y) \cap X(T_j)$ maximal; set $Q_j = \{y_j\}$. We claim that c is a precoloring extension of P_Q. Write $P_Q = (G, S', X'_0, X', Y', f')$. We need to show that $c(v) = f'(v)$ for every $v \in S' \cup X'_0$. Since c is a precoloring extension of P, and since $S = S'$, it follows that $c(v) = f(v) = f'(v)$ for every $v \in S' \cup X_0$. Let $v \in X'_0 \setminus X_0$. It follows that either

1. $S_i = \{y_i\}$, $Q_j = \{y_j\}$, and $v \in X$ and $v \in (N(y_i) \cap X(T_i)) \cup (N(y_j) \cap X(T_j))$ and $f'(v) = 1$, or

2. $S_i = \{y_i\}$, $Q_j = \{y_j\}$, $v \in Y$, v is complete to $(N(y_i) \cap X(T_i)) \cup (N(y_j) \cap X(T_j))$, v has both a neighbor in $X(T_i) \setminus N(y_i)$ and a neighbor in $X(T_j) \setminus N(y_j)$, and $f'(v) = 4$, or

3. (possibly with the roles of i and j exchanged) $S_i = \{y_i\}$, $Q_j = \emptyset$, and $v \in X$ and $v \in N(y_i) \cap X(T_i)$, and $f'(v) = 1$, or

4. (possibly with the roles of i and j exchanged) $S_i = \{y_i\}$, $Q_j = \emptyset$, $v \in Y$, v is complete to $N(y_i) \cap X(T_i)$, v has both a neighbor in $X(T_i) \setminus N(y_i)$ and a neighbor in $X(T_j)$, and $f'(v) = 4$, or

5. $S_i = Q_j = \emptyset$, $v \in Y$, v has both a neighbor in $X(T_i)$ and a neighbor in $X(T_j)$, and $f'(v) = 4$.

We show that in all these cases $c(v) = f'(v)$.

1. By the choice of y_i, y_j, $c(u) = 1$ for every $u \in (N(y_i) \cap X(T_i)) \cup (N(y_j) \cap X(T_j))$, and so $c(v) = f'(v)$.

2. It follows from the maximality of y_i, y_j that v has both a neighbor $x_2 \in X(T_i)$ with $c(x_2) = 2$ and a neighbor $x_3 \in X(T_j)$ with $c(x_3) = 3$. Since P is clean, it follows that $1 \notin M_P(v)$, and therefore $c(v) = 4$.

3. By the choice of y_i, $c(u) = 1$ for every $u \in N(y_i) \cap X(T_i)$, and so $c(v) = f'(v)$.

4. It follows from the maximality of y_i that v has a neighbor $x_2 \in X(T_i)$ with $c(x_2) = 2$. Since $Q_j = \emptyset$, v has a neighbor $x_3 \in X(T_j)$ with $c(x_3) = 3$. Since P is clean, it follows that $1 \notin M_P(v)$, and so $c(v) = 4$.

5. Since $S_i = Q_j = \emptyset$, it follows that v has both a neighbor $x_2 \in X(T_i)$ with $c(x_2) = 2$, and a neighbor $x_3 \in X(T_j)$ with $c(x_3) = 3$. Since P is clean, it follows that $1 \notin M_P(v)$, and so $c(v) = 4$.

This proves that c is an extension of P_Q, and completes the proof of Lemma 8. □

Repeatedly applying Lemma 8 and using symmetry, we deduce the following:

Lemma 9. *There is a function $q : \mathbb{N} \to \mathbb{N}$ such that the following holds. Let G be a P_6-free graph. Let $P = (G, S, X_0, X, Y^*, f)$ be a clean and tidy excellent starred precoloring of G. Then there is an algorithm with running time $O(|V(G)|^{q(|S|)})$ that outputs a collection $\mathcal{L}$ of excellent starred precolorings of G such that:*

- $|\mathcal{L}| \leq |V(G)|^{q(|S|)}$;
- $|S'| \leq q(|S|)$ *for every* $P' \in \mathcal{L}$;
- *every* $P' \in \mathcal{L}$ *is clean, tidy and orderly;*
- P *is equivalent to* $\mathcal{L}$.

Next we show that a clear, tidy and orderly excellent starred precoloring can be replaced by an equivalent collection of spotless precolorings.

Lemma 10. *There is a function* $q : \mathbb{N} \to \mathbb{N}$ *such that the following holds. Let* G *be a* P_6*-free graph. Let* $P = (G, S, X_0, X, Y^*, f)$ *be a clean, tidy and orderly excellent starred precoloring of* G*. Then there is an algorithm with running time* $O(|V(G)|^{q(|S|)})$ *that outputs a collection* $\mathcal{L}$ *of excellent starred precolorings of* G *such that:*

- $|\mathcal{L}| \leq |V(G)|^{q(|S|)}$;
- $|S'| \leq q(|S|)$ *for every* $P' \in \mathcal{L}$;
- *every* $P' \in \mathcal{L}$ *is clean, tidy and orderly;*
- *every* $P' \in \mathcal{L}$ *is* kl*-spotless for every* (k, l) *for which* P *is* kl*-spotless;*
- *every* $P' \in \mathcal{L}$ *is* 14*-spotless;*
- P *is equivalent to* $\mathcal{L}$.

Proof. The proof follows closely the proof of Lemma 8, deviating from it only when we show that every $P' \in \mathcal{L}$ is 14-spotless. Without loss of generality we may assume that $X_0 = X^0(P)$, and so $L_P(x) = M_P(x)$ for every $x \in X$. We may assume that P is not 14-spotless for otherwise we may set $\mathcal{L} = \{P\}$. Let $Y = \{y \in Y^*$ such that $\{2, 3\} \subseteq M_P(y)\}$. Let $T_1, \ldots, T_p$ be the types with $L(T_s) = \{1, 2\}$ and $T_{p+1}, \ldots, T_m$ the types with $L(T_s) = \{1, 3\}$. Let $\mathcal{Q}$ be the collection of all $p(m-p)$-tuples (P_i, Q_j) with $i \in \{1, \ldots, p\}$ and $j \in \{p+1, \ldots, m\}$, where $S_i, Q_i \subseteq Y$ and $|P_i|, |Q_i| \in \{0, 1\}$.

For $Q \in \mathcal{Q}$ construct a precoloring P_Q as follows. Let $i \in \{1, \ldots, p\}$ and $j \in \{p+1, \ldots, m\}$.

- Assume first that $S_i = \{y_i\}$ $Q_j = \{y_j\}$. If there is an edge between $N(y_i) \cap X(T_i)$ and $N(y_j) \cap X(T_j)$, remove Q from $\mathcal{Q}$. Now suppose that $N(y_i) \cap X(T_i)$ is anticomplete to $N(y_j) \cap X(T_j)$. Move $T = (N(y_i) \cap X(T_i)) \cup (N(y_j) \cap X(T_j))$ into X_0 with color 1. For every $y \in Y$ complete to T and both with a neighbor in $X(T_i) \setminus T$ and a neighbor in $X(T_j) \setminus T$, proceed as follows: if $4 \in M_P(y)$, move y to X_0 with color 4; if $4 \notin M_P(y)$, remove Q from $\mathcal{Q}$.
- Next assume that exactly one of S_i, Q_j is non-empty. By symmetry we may assume that $S_i = \{y_i\}$ and $Q_j = \emptyset$. Move $T = N(y_i) \cap X(T_i)$ into X_0 with color 1. For every $y \in Y$ complete to T and both with a neighbor in $X(T_i) \setminus T$ and a neighbor in $X(T_j)$, proceed as follows. If $4 \in M_P(y)$, move y to X_0 with color 4; if $4 \notin M_P(y)$, remove Q from $\mathcal{Q}$.
- Finally assume that $S_i = S_j = \emptyset$. For every $y \in Y$ with both a neighbor in $X(T_i)$ and a neighbor in $X(T_j)$, proceed as follows: if $4 \in M_P(y)$, move y to X_0 with color 4; if $4 \notin M_P(y)$, remove Q from $\mathcal{Q}$.

Let $Q \in \mathcal{Q}$, and let $P_Q = (G, S', X_0', X_0, Y^{*\prime}, f')$. If f' is not a proper coloring of $G|(S' \cup X_0')$, remove Q from $\mathcal{Q}$. Since $X' \subseteq X$, $Y' \subseteq Y^*$ and $M_{P_Q}(v) \subseteq M_P(v)$ for every v, it follows that P_Q is excellent, clean, tidy and orderly, and that for $k, l \in \{1,2,3,4\}$, if P is kl-spotless, then P_Q is kl-spotless.

Next we show that P_Q is 14-spotless. Suppose that some $y \in Y$ has a neighbor in $x_2 \in X_{12}'$ and a neighbor in $x_3 \in X_{13}'$. Then $x_2 \in X_{12}$ and $x_3 \in X_{13}$. We may assume that $x_2 \in X(T_1)$ and $x_3 \in X(T_{p+1})$. Since $x_2, x_3 \notin X^0(P_Q)$, it follows that both $S_1 \neq \emptyset$ and $Q_{p+1} \neq \emptyset$. Let $S_1 = \{y_2\}$ and $Q_{p+1} = \{y_3\}$. Since $x_2, x_3 \notin X^0(P_Q)$, it follows that y_2 is non-adjacent to x_2, and y_3 is non-adjacent to x_3. Since $y \notin X^0(P_Q)$, we may assume by symmetry that there is $x_2' \in N(y_2) \cap X(T_1)$ such that y is non-adjacent to x_2'. Let $x_3' \in N(y_3) \cap X(T_{p+1})$. Since $x_2, x_3 \notin X^0(P_Q)$, it follows that $\{x_2', x_3'\}$ is anticomplete to $\{x_2, x_3\}$. By the construction of Q, x_2' is non-adjacent to x_3'. Now, since G is orderly, y is non-adjacent to x_3', contrary to Lemma 1. This proves that P_Q is 14-spotless.

Observe that $S = S'$, and so $|S| = |S'|$. Observe also that also that $p(m-p) \leq (\frac{m}{2})^2$, and since $m \leq 2^{|S|}$, it follows that $p(m-p) \leq 2^{2|S|-2}$. Let $\mathcal{L} = \{P_Q \,:\, Q \in \mathcal{Q}\}$. Now $|\mathcal{L}| \leq |V(G)|^{2p(m-p)} \leq |V(G)|^{2^{2|S|-1}}$.

The remainder of the proof follows word for word the proof of Lemma 8, and we omit it. This proves that P_Q has a precoloring extension, and completes the proof of Lemma 10. □

Observe that if an excellent starred precoloring is spotless, then it is clean and orderly. Repeatedly applying Lemma 10 and using symmetry, we deduce the following:

Lemma 11. *There is a function $q : \mathbb{N} \to \mathbb{N}$ such that the following holds. Let G be a P_6-free graph. Let $P = (G, S, X_0, X, Y^*, f)$ be a clean, tidy and orderly excellent starred precoloring of G. Then there is an algorithm with running time $O(|V(G)|^{q(|S|)})$ that outputs a collection $\mathcal{L}$ of excellent starred precolorings of G such that:*

- $|\mathcal{L}| \leq |V(G)|^{q(|S|)}$;
- $|S'| \leq q(|S|)$ *for every* $P' \in \mathcal{L}$;
- *every* $P' \in \mathcal{L}$ *is tidy and spotless;*
- P *is equivalent to* $\mathcal{L}$.

We now summarize what we have proved so far. Let $P = (G, S, X_0, X, Y^*, f)$ be an excellent starred precoloring of a P_6-free graph G. We say that $y \in Y^*$ is *wholesome* if $|M_P(y)| \geq 3$. A component of $G|Y^*$ if *wholesome* if it contains a wholesome vertex. We say that P is *near-orthogonal* if for every wholesome $y \in Y^*$ either

- y has orthogonal neighbors in X, or
- there exist $\{i, j, k, l\} = \{1, 2, 3, 4\}$ such that
  - $N(y) \cap X \subseteq X_{ki} \cup X_{kj}$, and
  - For every $u \in C(y)$, $|M_P(u) \cap \{i, j\}| \leq 1$, and
  - if there is $v_i \in C(y)$ with $i \in M_P(v_i)$ and $v_j \in C(y)$ with $j \in M_P(v_j)$, then for some $u \in C(y)$, $l \notin M_P(u)$.

Lemma 12. *There is a function $q : \mathbb{N} \to \mathbb{N}$ such that the following holds. Let $P = (G, S, X_0, X, Y^*, f)$ be an excellent starred precoloring of a P_6-free graph G. Then there is an algorithm with running time $O(|V(G)|^{q(|S|)})$ that outputs a collection $\mathcal{L}$ of excellent starred precolorings of G such that:*

- $|\mathcal{L}| \leq |V(G)|^{q(|S|)}$;
- $|S'| \leq q(|S|)$ *for every* $P' \in \mathcal{L}$;
- *every* $P' \in \mathcal{L}$ *is near-orthogonal;*
- P *is equivalent to* $\mathcal{L}$.

Proof. Let $\mathcal{L}_1$ be the collection of precolorings obtained by applying Lemma 5 to P. Let $\mathcal{L}_2$ be the union of the collections of precolorings obtained by applying Lemma 7 to each member of $\mathcal{L}_1$. Let $\mathcal{L}_3$ be the union of the collections of precolorings obtained by applying Lemma 9 to each member of $\mathcal{L}_2$. Let $\mathcal{L}$ be the union of the collections of precolorings obtained by applying Lemma 11 to each member of $\mathcal{L}_3$. Then $\mathcal{L}$ satisfies the first, second and fourth bullet in the statement of Lemma 12, and every $P' \in \mathcal{L}$ is tidy and spotless. Let $P' \in \mathcal{L}$, write $P' = (S', X'_0, X', Y', f')$. Suppose that P' is not near-orthogonal. Let $y \in Y'$, and assume that the neighbors of y are not orthogonal. We show that y satisfies the conditions in the definition of near-orthogonal. We may assume that y has a neighbor in X'_{12} and a neighbor in X'_{13}. Since P' is spotless, it follows that for every $u \in C(y)$, $|M_P(u) \cap \{2,3\}| \leq 1$. Since y is wholesome, we may assume that $M_P(y) = \{1,2,4\}$. Since P' is spotless, it follows that $N(y) \cap X' \subseteq X'_{12} \cup X'_{13}$. Since P' is tidy and $1 \in M_P(y)$, it follows that if there is $v_2 \in C(y)$ with $2 \in M_P(v_2)$ and $v_3 \in C(y)$ with $3 \in M_P(v_3)$, then for some $u \in C(y)$ $4 \notin M_P(u)$. This proves that y satisfies the conditions in the definition of near orthogonal, and completes the proof of Lemma 12. □

Let $P = (G, S, X_0, X, Y^*, f)$ be an excellent starred precoloring. Let $\{i,j,k,l\} = \{1,2,3,4\}$, let T^i be a type of X with $L_P(T^i) = \{i,k\}$ and let T^j be a type of X with $L_P(T^j) = \{j,k\}$. A *type A extension with respect to* (T^i, T^j) is a precoloring extension c of P such that there exists $y \in Y^*$ with $k, i \in M_P(y)$ and such that y has a neighbor $x_i \in X(T^i)$ and a neighbor $x_j \in X(T^j)$ with $c(x_i) = c(x_j) = k$.

Let $\mathcal{T}(P)$ be the set of all pairs (T^i, T^j) of types of X with $|L_P(T^j) \cap L_P(T^j)| = 1$. We say that P is *smooth* if P has a precoloring extension c such that for every $(T^i, T^j) \in \mathcal{T}(P)$, c is not of type A with respect to (T^i, T^j). A precoloring extension of P is *good* if it is not of type A for any $T \in \mathcal{T}(P)$.

We say that an excellent starred precoloring $P' = (G, S', X'_0, X', Y^{*\prime}, f')$ is a *refinement* of P if for every type T' of X', there is a type T of X such that $X'(T') \subseteq X(T)$.

Lemma 13. *There is a function* $q : \mathbb{N} \to \mathbb{N}$ *such that the following holds. Let* $P = (G, S, X_0, X, Y^*, f)$ *be a near-orthogonal excellent starred precoloring of a* P_6*-free graph* G*. There is an algorithm with running time* $O(|V(G)|^{q(|S|)}$ *that outputs a collection* $\mathcal{L}$ *of near-orthogonal excellent starred precolorings of* G *such that:*

- $|\mathcal{L}| \leq |V(G)|^{q(|S|)}$;
- $|S'| \leq q(|S|)$ *for every* $P' \in \mathcal{L}$;
- *a precoloring extension of a member of* $\mathcal{L}$ *is also a precoloring extension of* P;
- *if* P *has a precoloring extension, then some* $P' \in \mathcal{L}$ *is smooth.*

Proof. Let $\mathcal{T}(P) = \{(T_1, T'_1), \ldots, (T_t, T'_t)\}$. Let $\mathcal{Q}$ be the collection of t-tuples of triples $Q_{T_i,T'_i} = (Y_{T_i,T'_i}, A_{T_i,T'_i}, B_{T_i,T'_i})$ such that

- $|Y_{T_i,T'_i}| = |A_{T_i,T'_i}| = |B_{T_i,T'_i}| \leq 1$.

- $A_{T_i,T'_i} \subseteq X(T_i)$.
- $B_{T_i,T'_i} \subseteq X(T'_i)$.
- $Y_{T_i,T'_i} \subseteq Y^*$ and if $Y_{T_i,T'_i} = \{y\}$, then $L_P(T_i) \subseteq M_P(y)$.
- Y_{T_i,T'_i} is complete to $A_{T_i,T'_i} \cup B_{T_i,T'_i}$.
- A_{T_i,T'_i} is anticomplete to B_{T_i,T'_i}.

For $Q = (Q_{T_i,T'_i})_{(T_i,T'_i)\in\mathcal{T}(P)} \in \mathcal{Q}$, we construct a precoloring P_Q by moving $A_{T_i,T'_i} \cup B_{T_i,T'_i}$ to the seed with the unique color of $L_P(T_i) \cap L_P(T'_i)$ for all $(T_i,T'_i) \in \mathcal{T}(P)$. Let $P_Q = (G, S', X'_0, X', Y', f')$. Since $X' \subseteq X$ and $Y' \subseteq Y^*$, and $M_{P'}(v) \subseteq M_P(v)$ for every $v \in V(G)$, it follows that P_Q is excellent, near-orthogonal and for every type T' of X', there is a type T of X such that $X'(T') \subseteq X(T)$.

Let $\mathcal{L} = \{P\} \cup \{P_Q \ : \ Q \in \mathcal{Q}\}$. Observe that there are at most $2^{|S|}$ types, and therefore $t \leq 2^{2|S|}$. Now $|S'| \leq |S| + 2t \leq |S| + 2^{2|S|+1}$ and $|\mathcal{L}| \leq |V(G)|^{3t} \leq |V(G)|^{3\times 2^{2|S|}}$.

Since every member of $\mathcal{L}$ is obtained from P by precoloring some vertices and updating, it follows that every precoloring extension of a member of $\mathcal{L}$ is also a precoloring extension of P.

Now we prove the last assertion of Lemma 13. Suppose that P has a precoloring extension. We need to show that some $P' \in \mathcal{L}$ is smooth. Let c be a precoloring extension of P. For every $(T_i, T'_i) \in \mathcal{T}(P)$ such that c is of type A with respect to (T_i, T'_i), proceed as follows. We may assume that $L_P(T_i) = \{1, 2\}$ and $L_P(T'_i) = \{1, 3\}$. Let $y \in Y^*$ with $1, 2 \in M_P(y)$, $x_2 \in X(T_i)$ and $x_3 \in X(T'_i)$ such that y is adjacent to x_2, x_3 and $c(x_2) = c(x_3) = 1$, and subject to the existence of such x_2, x_3, choose y with the set $\{x \in N(y) \cap X(T'_i)$ such that $c(x) = 1\}$ minimal. Let $Q_{T_i,T'_i} = (\{y\}, \{x_2\}, \{x_3\})$. For every $(T_i, T'_i) \in \mathcal{T}(P)$ such that c is not of type A with respect to (T_i, T'_i), set $Q_{T_i,T'_i} = (\emptyset, \emptyset, \emptyset)$. Let $Q = (Q_{T_i,T'_i})_{(T_i,T'_i)\in\mathcal{P}}$; then $P_Q \in \mathcal{L}$.

We claim that c is a precoloring extension of P_Q that is not of type A for any $(T_i, T'_i) \in \mathcal{T}(P_Q)$. Write $P_Q = (G, S', X'_0, X', Y', f')$. Let $\{i, j, k, l\} = \{1, 2, 3, 4\}$. Suppose that T^i is a type of X' with $L_{P_Q}(T^i) = \{i, k\}$ and T^j is a type of X' with $L_{P_Q}(T^j) = \{j, k\}$, and such that $(T^i, T^j) \in \mathcal{T}(P_Q)$, and $y' \in Y'$ with $i, k \in M_{P_Q}(y')$ has neighbor $x'_i \in X'(T^i)$ and $x'_j \in X'(T^j)$ with $c(x'_i) = c(x'_j) = k$. Let $(\tilde{T^i}, \tilde{T^j}) \in \mathcal{T}(P)$ be such that $X'(T^i) \subseteq X(\tilde{T^i})$ and $X'(T^j) \subseteq X(\tilde{T^j})$. Since $i, k \in M_P(y)$, it follows that c is of type A for $(\tilde{T^i}, \tilde{T^j})$, and therefore $|Y_{\tilde{T^i},\tilde{T^j}}| = |A_{\tilde{T^i},\tilde{T^j}}| = |B_{\tilde{T^i},\tilde{T^j}}| = 1$. Let $Y_{\tilde{T^i},\tilde{T^j}} = \{y\}$ $A_{\tilde{T^i},\tilde{T^j}} = \{x_i\}$ and $B_{\tilde{T^i},\tilde{T^j}} = \{x_j\}$. Since $k \in M_{P_Q}(y')$ it follows that y' is anticomplete to $\{x_i, x_j\}$. By the choice of y, it follows that y' has a neighbor $x' \in X(\tilde{T^j}) \setminus N(y)$ with $c(x') = k$, and so we may assume that x'_j is non-adjacent to y. Since $L_P(T'_i) = \{j, k\}$ there exists $s_i \in S$ with $f(s_i) = i$ such that s_i is complete to $\{x_j, x'_j\}$. Since $i \in L_P(x_i) \cap L_P(y') \cap L_P(y)$, it follows that s_i is anticomplete to $\{x_i, y', y\}$. Since $c(x_i) = c(x'_i) = c(x_j) = c(x'_j)$, it follows that $\{x_i, x'_i, x_j, x'_j\}$ is a stable set. But now $x_i - y - x_j - s_i - x'_j - y'$ is a P_6 in G, a contradiction. This proves that c is a good precoloring extension of P_Q, and completes the proof of Lemma 13. □

We are finally ready to construct orthogonal precolorings.

Lemma 14. *There is a function $q : \mathbb{N} \to \mathbb{N}$ such that the following holds. Let $P = (G, S, X_0, X, Y^*, f)$ be a near-orthogonal excellent precoloring of a P_6-free graph G. There exist an induced subgraph G' of G and an orthogonal excellent starred precoloring $P' = (G', S', X'_0, X', Y', f')$ of G', such that*

- *$S = S'$,*
- *if P is smooth, then P' has a precoloring extension, and*

- *if c is a precoloring extension of P', then a precoloring extension of P can be constructed from c in polynomial time.*

Moreover, P' can be constructed in time $O(|V(G)|^{q(|S|)})$.

Proof. We may assume that P is not orthogonal. We say that a component C of $G|Y^*$ is *troublesome* if C is wholesome, and the set of attachments of C in X are not orthogonal. Let W be the union of the vertex sets of the component of $G|Y^*$ that are not wholesome.

We construct a set Z, starting with $Z = \emptyset$. For every troublesome component C, proceed as follows. We may assume that C has attachments in X_{12} and in X_{13}. Since P is near-orthogonal, and C is wholesome, we may assume that C contains a vertex z with $M_P(z) = \{1, 2, 4\}$.

- If there is $y \in V(C)$ with $M_P(y) = \{1, 3\}$, move $N(y) \cap X_{12}$ to X_0 with color 2.
- Suppose that there is no y as in the first bullet. If $|V(C)| > 2$, or $V(C) = \{z\}$ and z has a neighbor v in X_0 with $f(v) = \{4\}$, move $N(z) \cap X_{13}$ to X_0 with color 3.
- If none of the first two conditions hold, add $V(C)$ to Z. Observe that in this case $V(C) = \{y\}$, y has no neighbors in $Z \setminus \{y\}$. Moreover, since P is near-orthogonal, $V(C)$ is anticomplete to $X \setminus (X_{12} \cup X_{13})$, and so for every $u \in N(y)$, $4 \notin L_P(u)$. In this case we call 4 the *free color* of y.

Let $P'' = (G, S', X'_0, X'', Y'', f')$ be the precoloring we obtained after we applied the procedure above to all troublesome components. Let $G' = G \setminus Z$, and let $P' = (G', S', X'_0, X', Y', f')$ where $Y' = Y'' \setminus W \cup Z$ and $X' = X'' \cup W$. Since no vertex of W is wholesome, It follows from the definition of M_P that every vertex of W has neighbors of at least two different colors in S' (with respect to f'). Since W is anticomplete to Y', $X' \setminus W \subseteq X$, and $Y' \subseteq Y^*$, we deduce that P' is excellent and orthogonal. It follows from the construction of Z that every precoloring extension of P' can be extended to a precoloring extension of P by giving each member of Z its free color.

It remains to show that if P is smooth, then P' has a precoloring extension. Suppose that P is smooth, and let c be a good precoloring extension of P. We claim that $c|V(G')$ is a precoloring extension of P'. We need to show that $c(v) = f'(v)$ for every $v \in S' \cup X'_0$. Since $S = S'$, and $f(v) = f'(v)$ for every $v \in X_0$, it is enough to show that $c(v) = f'(v)$ for every $v \in X'_0 \setminus X_0$. Thus we may assume that there is a troublesome component C of $G|Y^*$ that has an attachment in X_{12} and an attachment in X_{13}, and $v \in X(C)$. Since P is near-orthogonal, we may assume that C contains a vertex y with $M_P(y) = \{1, 2, 4\}$, and $v \in X_{12} \cup X_{13}$. There are two possibilities.

1. There is $y \in V(C)$ with $M_P(y) = \{1, 3\}$, $v \in N(y) \cap X_{12}$ and $f'(v) = 2$, but $c(v) = 1$. We show that this is impossible. Since c is a good coloring, it follows that $c(u) = 3$ for every $u \in N(y) \cap X_{13}$, contrary to the fact that c is a coloring of G.
2. There is no y as in the first case, and either $|V(C)| > 2$, or $V(C) = \{z\}$ and z has a neighbor u in X_0 with $f(u) = \{4\}$, and $v \in X_{13} \cap N(z)$, $f'(v) = 3$ but $c(v) = 1$. We show that this too is impossible. It follows that there is a vertex $y' \in V(C)$ with $c(y) \neq 4$. Choose such y' with $4 \notin M_P(y')$ if possible. Since P is excellent, y' is adjacent to v. Since c is a good coloring, it follows that $c(u) = 2$ for every $u \in X_{12} \cap N(y')$. This implies that $c(y') = 3$. Since P is near-orthogonal and $3 \in M_P(y')$, it follows that $2 \notin M_P(y')$. Since $M_P(y') \neq \{1, 3\}$, it follows that $4 \in L(y')$. Since $1, 2 \in M_P(y)$ and $3 \in M_P(y')$, and since P is near-orthogonal, it follows that there is $z \in V(C)$ such that $4 \notin M_P(z)$. Since $c(v) = 1$ and $c(u) = 2$ for every attachment of $V(C)$ in X_{12}, it follows that $c(z) = 3$, contrary to the choice of y'.

Thus $c(v) = f'(v)$ for every $v \in S' \cup X'_0$, and so $c|V(G')$ is a precoloring extension of P'. This completes the proof of Lemma 14. □

We can now prove the main result of this section.

Theorem 9. *There is a function $q : \mathbb{N} \to \mathbb{N}$ such that the following holds. Let $P = (G, S, X_0, X, Y^*, f)$ be an excellent starred precoloring of a P_6-free graph G with $|S| \leq C$. Then there is an algorithm with running time $O(|V(G)|^{q(|S|)})$ that outputs a collection $\mathcal{L}$ of orthogonal excellent starred precolorings of induced subgraphs of G such that:*

- *$|\mathcal{L}| \leq |V(G)|^{q(|S|)}$;*
- *$|S'| \leq q(|S|)$ for every $P' \in \mathcal{L}$, and*
- *P has a precoloring extension, if and only if some $P' \in \mathcal{L}$ has a precoloring extension;*
- *given a precoloring extension of a member of $\mathcal{L}$, a precoloring extension of P can be constructed in polynomial time.*

Proof. By Lemma 12 there exist a function $q_1 : \mathbb{N} \to \mathbb{N}$ and a polynomial-time algorithm that outputs a collection $\mathcal{L}_1$ of excellent starred precolorings of G such that:

- $|\mathcal{L}_1| \leq |V(G)|^{q_1(|S|)}$;
- $|S'| \leq q_1(|S|)$ for every $P' \in \mathcal{L}_1$;
- every $P' \in \mathcal{L}_1$ is near-orthogonal; and
- P is equivalent to $\mathcal{L}_1$.

Let $P' \in \mathcal{L}_1$. Write $P' = (G, S(P'), X_0(P'), X(P'), Y^*(P'), f_{P'})$. By Lemma 13 there exist a function $q_2 : \mathbb{N} \to \mathbb{N}$ and a polynomial-time algorithm that outputs a collection $\mathcal{L}(P')$ of near-orthogonal excellent starred precolorings of G such that:

- $|\mathcal{L}(\mathcal{P}')| \leq |V(G)|^{q_2(|S(P')|)}$;
- $|S'| \leq q_2(|S(P')|)$ for every $P' \in \mathcal{L}$;
- if P' has a precoloring extension, then some $P'' \in \mathcal{L}(P')$ is smooth; and
- a precoloring extension of a member of $\mathcal{L}(P')$ is also a precoloring extension of P'.

Let $\mathcal{L}_2 = \bigcup_{P' \in \mathcal{L}} \mathcal{L}(P')$.

Clearly $\mathcal{L}_2$ has the following properties:

- $|\mathcal{L}_2| \leq |V(G)|^{q_1(q_2(|S|))}$;
- $|S'| \leq q_1(q_2(|S(P)|))$ for every $P' \in \mathcal{L}_2$;
- if P has a precoloring extension, then some $P'' \in \mathcal{L}(P')$ is smooth; and
- given a precoloring extension of a member of $\mathcal{L}_2$, one can construct in polynomial time a precoloring extension of P.

Let $P'' \in \mathcal{L}_2$. Write $P'' = (G, S(P''), X_0(P;'), X'(P'), Y^*(P''), f_{P''})$. By Lemma 14, there exists an induced subgraph G' of G and an orthogonal excellent starred precoloring $Orth(P'') = (G', S', X_0', X', Y', f')$ of G', such that

- $S(P'') = S'$;
- if P'' is smooth, then $Orth(P'')$ has a precoloring extension; and
- if c is a precoloring extension of $Orth(P'')$, then a precoloring extension of P'', and therefore of P, can be constructed from c in polynomial time.

Moreover, $Orth(P'')$ can be constructed in polynomial time.

Let $\mathcal{L} = \{Orth(P'') \ : \ P'' \in \mathcal{L}_2\}$. Now $\mathcal{L}$ has the following properties.

- $|\mathcal{L}| \leq |V(G)|^{q_1(q_2(|S|))}$;
- $|S'| \leq q_1(q_2(|S|))$ for every $P' \in \mathcal{L}$; and
- if c is a precoloring extension of $P' \in \mathcal{L}$, then a precoloring extension of P can be constructed from c in polynomial time.
- every $P' \in \mathcal{L}$ is orthogonal.

To complete the proof of the Theorem 9 we need to show that if P has a precoloring extension, then some $P' \in \mathcal{L}$ has a precoloring extension. So assume that P has a precoloring extension. Since $\mathcal{L}_1$ is equivalent to P, it follows that some $P_1 \in \mathcal{L}_1$ has a precoloring extension. This implies that some $P_2 \in \mathcal{L}(P_1) \subseteq \mathcal{L}_2$ is smooth. But now $Orth(P_2)$ has a precoloring extension, and $Orth(P_2) \in \mathcal{L}$. This completes the proof of Theorem 9. □

3 Companion triples

In view of Theorem 9 we now focus on testing for the existence of a precoloring extension for an orthogonal excellent starred precoloring.

Let G be a P_6-free graph, and let $P = (G, S, X_0, X, Y^*, f)$ be an orthogonal excellent starred precoloring of G. We may assume that $X_0 = X^0(P)$. Let $\mathcal{C}(P)$ be the set of components of $G|Y^*$, and let $C \in \mathcal{C}(P)$. It follows that $X \setminus X(C)$ is anticomplete to $V(C)$, and we may assume (using symmetry) that $X(C) \subseteq X_{12} \cup X_{34}$. We now define the precoloring obtained from P by *contracting the ij-neighbors of C*, or, in short, by *neighbor contraction*. We may assume that $\{i, j\} = \{1, 2\}$. Suppose that $X_{12} \cap X(C) \neq \emptyset$, and let $x_{12} \in X_{12} \cap X(C)$. Let $\tilde{G}$ be the graph define as follows:

$$V(\tilde{G}) = G \setminus (X_{12} \cap X(C)) \cup \{x_{12}\}$$

$$\tilde{G} \setminus \{x_{12}\} = G \setminus (X_{12} \cap X(C))$$

$$N_{\tilde{G}}(x_{12}) = \bigcup_{x \in X_{12} \cap X(C)} N_G(x) \cap V(\tilde{G}).$$

Moreover, let

$$\tilde{X} = X \setminus (X_{12} \cap X(C)) \cup \{x_{12}\}.$$

Then $\tilde{P} = (\tilde{G}, X_0, \tilde{X}, Y^*, f)$ is an orthogonal excellent starred precoloring of $\tilde{G}$. We say that $\tilde{P}$ is *obtained from P by contracting the 12-neighbors of C*, or, in short, *obtained from P by neighbor* *contraction.* We call x_{12} the image of $X_{12} \cap X(C)$, and define $x_{12}(C) = x_{12}$. Observe that $x_{12} \in X$ (this fact simplifies notation later), and that $M_P(v) = M_{\tilde{P}}(v)$ for every $v \in V(\tilde{G})$.

For $i, j \in \{1, 2, 3, 4\}$ and $t \in X_0 \cup S$ let $\tilde{G}_{ij}(t) = \tilde{G}|(\tilde{X}_{ij} \cup Y^* \cup \{t\})$. While graph $\tilde{G}$ may not be P_6-free, the following weaker statement holds:

Lemma 15. *Let P be an excellent orthogonal precoloring of a P_6-free graph G. Let $C \in \mathcal{C}(P)$ and assume that $X(C) \cap X_{12}$ is non-empty. Let $\tilde{P} = (\tilde{G}, X_0, \tilde{X}, Y^*, f)$ be obtained from P by contracting the 12-neighbors of C. Then $\tilde{G}_{ij}(t)$ is P_6-free for every $i, j \in \{1, 2, 3, 4\}$ and $t \in S \cup X_0$.*

Proof. If $\{i, j\} \neq \{1, 2\}$, then $\tilde{G}_{ij}(t)$ is an induced subgraph of G, and therefore it is P_6-free. So we may assume that $\{i, j\} = \{1, 2\}$. Suppose that $Q = q_1 - \ldots - q_6$ is a P_6 in $\tilde{G}_{ij}(t)$. Since $\tilde{G}_{ij}(t) \setminus x_{12}$ is an induced subgraph of G, it follows that $x_{12} \in V(Q)$. If the neighbors of x_{12} in Q have a common neighbor $n \in X(C) \cap X_{12}$, then $G|((V(Q) \setminus \{x_{12}\}) \cup \{n\})$ is a P_6 in G, a contradiction. It follows that x_{12} has two neighbors in Q, say a, b, each of a, b has a neighbor in $X_{12} \cap X(C)$, and no vertex of $X(C) \cap X_{12}$ is complete to $\{a, b\}$. Since $V(C)$ is complete to $X(C)$, it follows that $a, b \notin V(C)$, and so $a, b \in (X_{12} \setminus X(C)) \cup (Y^* \setminus V(C)) \cup \{t\}$. Let Q' be a shortest path from a to b with $Q'^* \subseteq X(C) \cup V(C)$. Since $V(Q) \setminus \{a, b, t\}$ is anticomplete to $V(C)$, and $V(Q) \setminus \{a, b\}$ is anticomplete to $X(C) \cap X_{12}$, it follows that $V(Q')$ is anticomplete to $V(Q) \setminus (\{x_{12}\} \cup \{a, b, t\})$. Moreover, if $t \neq a, b$, then t is anticomplete to $Q'^* \setminus V(C)$. If follows that if $t \notin V(Q) \setminus \{a, b, x_{12}\}$ or t is anticomplete to $V(Q') \cap V(C)$ then $q_1 - Q - a - Q' - b - Q - q_6$ is a path of length at least six in G, a contradiction; so $t \in V(Q) \setminus \{a, b, x_{12}\}$, and t has a neighbor in $V(Q') \cap V(C)$. Since $V(C)$ is complete to $X(C)$, it follows that $|V(C) \cap V(Q')| = 1$, and $|Q'^*| = 3$. Let $V(Q') \cap V(C) = \{q'\}$. We may assume that b has a neighbor $c \in V(Q) \setminus \{x_{12}\}$, and if $a = q_i$ and $b = q_j$, then $i < j$. Since $a - Q' - b - c$ is not a P_6 in G, it follows that $t = c$. But now $q_1 - a - Q' - q - t - Q - q_6$ is a P_6 in G, a contradiction. This proves Lemma 15. □

Let $P = (G, S, X_0, X, Y^*, f)$ be an orthogonal excellent starred precoloring. Let H be a graph, and let L be a 4-list assignment for H. Recall that $X^0(L)$ is the set of vertices of H with $|L(x_0)| = 1$. Let M be the list assignment obtained from M_P by updating Y^* from X_0. We say that (H, L, h) is a *near-companion triple for P with correspondence h* if there is an orthogonal excellent starred precoloring $\tilde{P} = (\tilde{G}, S, X_0, \tilde{X}, Y^*, f)$ obtained from P by a sequence of neighbor contractions, and the following hold:

- $V(H) = \tilde{X} \cup Z$;
- $h : Z \to \mathcal{C}(P)$;
- for every $z \in Z$, $N(z) = \tilde{X}(V(h(z)))$;
- $H|(Z \cup \tilde{X}_{ij})$ is P_6-free for all i, j;
- Z is a stable set;
- for every $x \in \tilde{X}$, $L(x) \subseteq M_P(x) = M(x)$;
- for every $z \in Z$ such that $L(z) \neq \emptyset$, if $q \in \{1, 2, 3, 4\}$ and $q \notin L(z)$, then some vertex $V(h(z))$ has a neighbor $u \in S \cup X_0 \cup X^0(L)$ with $f(u) = q$; and
- for every $z \in Z$ and every $q \in L(z)$, there is $v \in V(h(z))$ with $q \in M(v)$, and no vertex $u \in S \cup X_0$ with $f(u) = q$ is complete to $V(h(z))$.

For $z \in Z$, we call $h(z)$ the *image of z*.

If (H, L, h) is a near-companion triple for P, and in addition

- $\tilde{P}$ has a precoloring extension if and only if (H, L) is colorable, and a coloring of (H, L) can be converted to a precoloring extension of P in polynomial time.

we say that (H, L, h) is a *companion triple* for P.

For $i, j \in \{1, 2, 3, 4\}$ and $t \in S \cup X_0$ let $H_{ij}(t)$ be the graph obtained from $H|(\tilde{X}_{ij} \cup Z)$ by adding the vertex t and making t adjacent to the vertices of $N_{\tilde{G}}(t) \cap \tilde{X}_{ij}$. The following is a key property of near-companion triples.

Lemma 16. *Let G be a P_6-free graph, let $P = (G, S, X_0, X, Y^*, f)$ be an orthogonal excellent starred precoloring of G, and let (H, L, h) be a near-companion triple for P. Let M be the list assignment obtained from M_P by updating Y^* from X_0. Assume that $L(v) \neq \emptyset$ for every $v \in V(H)$. Let $i, j \in \{1, 2, 3, 4\}$ and $t \in X_0 \cup S$, and let Q be a P_6 in $H_{ij}(t)$. Then $t \in V(Q)$, and there exists $q \in V(Q) \setminus N(t)$ such that $f(t) \notin M(q)$.*

Proof. Since $H|(\tilde{X}_{ij} \cup Z)$ is P_6-free, it follows that $t \in V(Q)$. Suppose that for every $q \in V(Q) \setminus N(t)$, $f(t) \in L(q)$. Let $z \in V(Q) \cap Z$. Since t is anticomplete to Z, it follows that $f(t) \in L(z)$ By the definition of a near-companion triple, there is a vertex $q(z) \in V(h(z))$ such that $f(t) \in M(q(z))$. Since M is obtained from M_P by updating Y^* from X_0, it follows that t is non-adjacent to $q(z)$. Now replacing z with $q(z)$ for every $z \in V(Q) \cap Z$, we get a P_6 in $\tilde{G}_{ij}(t)$ that contradicts Lemma 15. This proves Lemma 16. □

The following is the main result of this section.

Theorem 10. *Let G be a P_6-free graph and let $P = (G, S, X_0, X, Y^*, f)$ be an orthogonal excellent starred precoloring of G. Then there is a polynomial time algorithm that outputs a companion triple for P.*

Proof. We may assume that $X_0 = X^0(P)$. Let M be the list assignment obtained from M_P by updating Y^* from X_0. Write $\mathcal{C} = \mathcal{C}(P)$. For $Q \subseteq \{1, 2, 3, 4\}$ and $C \in \mathcal{C}$, we say that a coloring c of (C, M) is a Q-coloring if $c(v) \in Q$ for every $v \in V(C)$. Given $Q \subseteq \{1, 2, 3, 4\}$, we say that Q is *good for* C if (C, M) admits a proper Q-coloring, and *bad for* C otherwise. By Theorem 2, for every Q with $|Q| \leq 3$, we can test in polynomial time if Q is good for C. Let $\mathcal{Q}(C)$ be the set of all inclusion-wise maximal bad subsets of $\{1, 2, 3, 4\}$. Observe that if Q is bad, then all its subsets are bad.

Here is another useful property of $\mathcal{Q}(C)$.

(1) *Let $Q \in \mathcal{Q}(C)$, and let $i \in Q$ be such that no $u \in S \cup X_0$ with $f(u) = i$ has a neighbor in $V(C)$. Then for every $j \in \{1, 2, 3, 4\} \setminus Q$, we have $(Q \setminus \{i\}) \cup \{j\} \in \mathcal{Q}(C)$.*

Suppose not. Let $Q' = Q \setminus \{i\} \cup \{j\}$. Let c be a proper Q'-coloring of (C, M). It follows from the definition of M that $i \in M(y)$ for every $y \in V(C)$. Recolor every vertex $u \in V(C)$ with $c(u) = j$ with color i. This gives a proper Q-coloring of (C, M), a contradiction. This proves (1).

First we describe a sequence of neighbor contractions to produce $\tilde{P}$ as in the definition of a companion triple. Let $C \in \mathcal{C}$ with $|V(C)| > 1$. Let $\{i, j, k, l\} = \{1, 2, 3, 4\}$ and let $X(C) \subseteq X_{ij} \cup X_{kl}$. We may assume (without loss of generality) that $X(C) \subseteq X_{12} \cup X_{34}$. If $X(C)$ meets both X_{12} and X_{34}, contract the $1,2$-neighbors of C, and the $3,4$-neighbors of C; observe that in this case $\tilde{X}(C) = \{x_{12}(C), x_{34}(C)\}$. If $X(C)$ meets exactly one of X_{12}, X_{34}, say $X(C) \subseteq X_{12}$, and $\{3, 4\}$ is bad for C, contract the 12-neighbors of C. Repeat this for every $Q \in \mathcal{Q}(C)$; let $\tilde{P} = (\tilde{G}, S, X_0, \tilde{X}, Y^*, f)$ be the resulting precoloring. Observe that $\tilde{X} \subseteq X$.

(2) *P has a precoloring extension if and only if $\tilde{P}$ has a precoloring extension, and a precoloring extension of $\tilde{P}$ can be converted into a precoloring extension of P in polynomial time.*

Since $|\mathcal{C}(P)| \leq |V(G)|$, it is enough to show that the property of having a precoloring extension, and the algorithmic property, do not change when we perform one step of the construction above.

Let us say that we start with $P_1 = (G_1, S, X_0, X_1, Y^*, f)$ and finish with $P_1 = (G_2, S, X_0, X_2, Y^*, f)$. We claim that in all cases, each of the sets that is being contracted (that is, replaced by its image) is monochromatic in every precoloring extension of P.

Let $C \in \mathcal{C}(P)$ with $|V(C)| > 1$, such that P_2 is obtained from P_1 by contracting neighbors of C. Let $\{i,j,k,l\} = \{1,2,3,4\}$ and let $X_1(C) \subseteq X_{ij} \cup X_{kl}$. If $\tilde{X}(C)$ meets both X_{ij} and X_{kl}, then since $|V(C)| > 1$, each of the sets $X_1(C) \cap X_{ij}$, $X_1(C) \cap X_{kl}$ is monochromatic in every precoloring extension of P_1, as required. So we may assume that $X_1(C) \subseteq X_{ij}$. Now $X_1(C)$ is monochromatic in every precoloring extension of P_1 because the set $\{k,l\}$ is bad for C. This proves the claim.

Now suppose that a set A was contracted to produce its image a. If P_1 has a precoloring extension, we can give a the unique color that appears in A, thus constructing an extension of P_2. On the other hand, if P_2 has a precoloring extension, then every vertex of A can be colored with the color of a. This proves (2).

Next we define $L : \tilde{X} \to 2^{[4]}$. Start with $L(x) = M_{\tilde{P}}(x)$ for every $x \in \tilde{X}$. Again let $C \in \mathcal{C}$ with $|V(C)| > 1$, let $\{i,j,k,l\} = \{1,2,3,4\}$, and let $\tilde{X}(C) \subseteq X_{ij} \cup X_{kl}$. For every $Q \in \mathcal{Q}(C)$ such that $Q = \{1,2,3,4\} \setminus \{i\}$, update L by removing i from $L(x)$ for every $x \in X_{ij} \cap \tilde{X}(C)$.

Next assume that $X(C)$ meets both X_{ij}, X_{kl} , the sets $\{i,k\}, \{i,l\}$ are good for C, and the sets $\{j,k\}, \{j,l\}$ are bad for C. Update L by removing i from $L(x_{ij}(C))$.

Finally, assume that $X(C)$ meets both X_{ij}, X_{kl} , the set $\{i,k\}$ is good for C, and the sets $\{i,l\}, \{j,k\}, \{j,l\}$ are bad for C. Update L by removing i from $L(x_{ij}(C))$ and by removing k from $L(x_{kl}(C))$.

Now the following holds.

(3) *Let $\{1,2,3,4\} = \{i,j,k,l\}$ and let $C \in \mathcal{C}$ such that $X(C) \subseteq X_{ij} \cup X_{kl}$.*

1. *If $\{1,2,3,4\} \setminus \{i\} \in \mathcal{Q}(C)$, then $i \notin \bigcup_{x \in \tilde{X}(C)} L(x)$.*
2. *If $\tilde{X}(C)$ meets both X_{ij} and X_{kl} and $\{i,k\}, \{i,l\}$ are both good for C, and $\{j,k\}, \{j,l\}$ are both bad for C, then $i \notin L(x_{ij}(C)) \cup L(x_{kl}(C))$.*
3. *If $\tilde{X}(C)$ meets both X_{ij} and X_{kl} and $\{i,k\}$ is good for C, and $\{i,l\}, \{j,k\}, \{j,l\}$ are bad for C, then $i,k \notin L(x_{ij}(C)) \cup L(x_{kl}(C))$.*

Next we show that:

(4) *If c is a precoloring extension of $\tilde{P}$, then $c(x) \in L(x)$ for every $x \in \tilde{X}$.*

This is clear for x such that $L(x) = M(x)$, so let $x \in \tilde{X}$ be such that $L(x) \neq M(x)$. Then there exists $C \in \mathcal{C}$ with $|V(C)| > 1$, and $\{i,j,k,l\} = \{1,2,3,4\}$ with $\tilde{X}(C) \subseteq X_{ij} \cup X_{kl}$, such that $x \in \tilde{X}(C)$. Suppose that $c(x) \in M(x) \setminus L(x)$. Observe that $c|V(C)$ is a coloring of (C, M). There are three possible situations in which $c(x)$ could have been removed from $M(x)$ to produce $L(x)$.

- $\{1,2,3,4\} \setminus \{i\}$ is bad for C, and $x \in X_{ij}$, and $c(x) = i$. In this case, since (C, M) is not $\{1,2,3,4\} \setminus \{i\}$-colorable, it follows that some $v \in V(C)$ has $c(v) = i$, but $V(C)$ is complete to $\tilde{X}(C)$, a contradiction.
- $\tilde{X}(C)$ meets both X_{ij} and X_{kl}, the sets $\{i,k\}, \{i,l\}$ are good for C, the sets $\{j,k\}, \{j,l\}$ are bad for C, $x = x_{ij}(C)$, and $c(x) = i$. Since $\tilde{X}(C) \cap X_{kl} \neq \emptyset$, it follows that $c(u) \in \{k,l\}$ for some $u \in \tilde{X}(C)$. Since the sets $\{j,k\}, \{j,l\}$ are bad for C and $|V(C)| > 1$, it follows that $c(v) = i$ for some $v \in V(C)$, but $x_{ij}(C)$ is complete to $V(C)$, a contradiction.

- $\tilde{X}(C)$ meets both X_{ij} and X_{kl}, the set $\{i,k\}$ is good for C, the sets $\{i,l\},\{j,k\},\{j,l\}$ are bad for C, and either $x = x_{ij}(C)$ and $c(x) = i$, or $x = x_{kl}(C)$ and $c(x) = l$. Since $\tilde{X}(C)$ meets both X_{ij} and X_{ik} and $|V(C)| > 1$, it follows that $|c(V(C)) \cap \{i,j\}| = 1$, and $|c(V(C)) \cap \{k,l\}| = 1$. Since $\{j,k\},\{j,l\}$ are bad for C, it follows that for some $v \in V(C)$ has $v(c) = i$, and so $c(x_{ij}(C)) \neq i$. Since $\{i,l\}$ is bad for C, it follows that $c(V(C)) = \{i,k\}$, and so $c(x) \neq k$, in both cases a contradiction.

This proves (4).

Finally, for every $C \in \mathcal{C}$, we construct the set $h^{-1}(C)$ and define $L(v)$ for every $v \in h^{-1}(C)$.

If $|V(C)| = 1$, say $C = \{y\}$, let $h^{-1}(C) = \{y\}$, and let $L(y) = M(y)$.

Now assume $|V(C)| > 1$. We may assume that $\tilde{X}(C) \subseteq X_{12} \cup X_{34}$.

If all subsets of $\{1,2,3,4\}$ of size three are bad, then set $h^{-1}C = \{z\}$ and $L(z) = \emptyset$. From now on we assume that there is a good subset for C of size at most three.

If $\tilde{X}(C) \subseteq X_{12}$ or $\tilde{X}(C) \subseteq X_{34}$, set $h^{-1}(C) = \emptyset$.

So we may assume that $\tilde{X}(C)$ meets both X_{12} and X_{34}. If all sets of size two, except $\{1,2\}$ and $\{3,4\}$ are bad for C, set $h^{-1}C = \{z\}$ and $L(z) = \emptyset$. Next let $Q \in \mathcal{Q}(C)$ with $|Q| = 2$; write $\{i,j,k,l\} = \{1,2,3,4\}$, and say $Q = \{i,j\}$. We say that Q is *friendly* if there exist $u_i, u_j \in S \cup X_0$, both with neighbors in C, and with $f(u_i) = i$ and $f(u_j) = j$. For every friendly set Q, let $v(C,Q)$ be a new vertex, and let $h^{-1}(C)$ consist of all such vertices $v(C,Q)$. Set $L(v(C,Q)) = \{1,2,3,4\} \setminus Q$.

Let $Z = \bigcup_{C \in \mathcal{C}} h^{-1}(C)$. Finally, define the correspondence function h by setting $h(z) = C$ for every $z \in h^{-1}(C)$ and $C \in \mathcal{C}$.

Now we define H. We set $V(H) = \tilde{X} \cup Z$, and $pq \in E(H)$ if and only if either

- $p, q \in \tilde{X}$ and $pq \in E(G)$, or
- there exists $C \in \mathcal{C}$ such that $p \in h^{-1}(C)$ and $q \in \tilde{X}(C)$.

The triple (H, L, h) that we have constructed satisfies the following.

- $\tilde{X} \subseteq V(H)$; write $Z = V(H) \setminus \tilde{X}$.
- $N(z) = \tilde{X}(V(h(x)))$ for every $z \in Z$.
- Z is a stable set.
- For every $x \in \tilde{X}$, $L(x) \subseteq M_P(x) = M(x)$.
- $h : Z \to \mathcal{C}(P)$.
- If $z \in Z$ with $L(z) \neq \emptyset$, and $q \in \{1,2,3,4\} \setminus L(z)$, then some vertex $V(h(z))$ has a neighbor $u \in S \cup X_0$ with $f(u) = q$. (This is in fact stronger than what is required in the definition of a companion triple; we will relax this condition later.)

To complete the proof of Theorem 10, it remains to show the following

1. For every $z \in Z$ and every $q \in L(z)$, there is $v \in V(h(z))$ with $q \in M(v)$, and no vertex $u \in S \cup X_0$ with $f(u) = q$ is complete to $V(h(z))$.
2. for every $i, j \in \{1,2,3,4\}$, $H|(\tilde{X_{ij}} \cup Z)$ is P_6-free.
3. P has a precoloring extension if and only if (H, L) is colorable, and a proper coloring of (H, L) can be converted to a precoloring extension of P in polynomial time.

We prove the first statement first. Let $z \in Z$ and $q \in L(z)$, and suppose that for every $v \in V(h(z))$ $q \notin M(v)$, or some vertex $u \in S \cup X_0$ with $f(u) = q$ is complete to $V(h(z))$. It follows that $|V(h(Z))| > 1$. Since $z \in Z$, it follows that there exists a set $\{i, j\} \in \mathcal{Q}(h(Z))$ and $L(z) = \{1, 2, 3, 4\} \setminus \{i, j\}$. But now it follows that $\{q, i, j\}$ is also bad for $h(Z)$, contrary to the maximality of $\{i, j\}$. This proves the first statement.

Next we prove the second statement. By Lemma 15, $\tilde{G}|(\tilde{X_{ij}} \cup Y^*)$ is P_6-free for every $i, j \in \{1, 2, 3, 4\}$. Suppose Q is a P_6 in H. Let $C \in \mathcal{C}(P)$. Since no vertex of $V(H) \setminus h^{-1}(C)$ is mixed on $h^{-1}(C)$, it follows that $|V(Q) \cap h^{-1}(C)| \leq 1$. Moreover, $\tilde{X}_{ij}(h^{-1}(C)) = \tilde{X}_{ij}(C)$. Let G' be obtained from $\tilde{G}$ by replacing each $C \in \mathcal{C}$ by a single vertex of C, choosing this vertex to be in $V(Q)$ if possible. Then G' is an induced subgraph of G, and Q is a P_6 in G', a contradiction. This proves the second statement.

Finally we prove the last statement. Let $\mathcal{C}_1 = \{C \in \mathcal{C} \,:\, |V(C)| = 1\}$, and let $Y = \bigcup_{C \in \mathcal{C}_1} V(C)$. Then $Y \subseteq Z$.

Suppose first that P has a precoloring extension. By (2), there exists a precoloring extension of $\tilde{P}$; denote it by c. By (4), $c|(\tilde{X} \cup Y)$ is a coloring of $(H|(X \cup Y), L)$. It remains to show that c can be extended to $Z \setminus Y$. Let $z \in Z$, and let $h(z) = C$. Then there is a friendly set $\{i, j\} \in \mathcal{Q}$ such that $z = v(C, Q)$. Since Z is a stable set, in order to show that c can be extended to $Z \setminus Y$, it is enough to show that

$$L(z) \not\subseteq c(\tilde{X}(C)).$$

Since $L(v(C, Q)) = \{1, 2, 3, 4\} \setminus Q$, it is enough to show that

$$\{1, 2, 3, 4\} \setminus c(\tilde{X}(C)) \not\subseteq Q.$$

But the latter statement is true because

$$c(V(C)) \subseteq \{1, 2, 3, 4\} \setminus c(\tilde{X}(C))$$

and $c(V(C))$ is a good set, and therefore $c(V(C)) \not\subseteq Q$. This proves that if $\tilde{P}$ has a precoloring extension, then (H, L) is colorable.

Now let c be a proper coloring of (H, L). By (2) it is enough to show that $\tilde{P}$ has a precoloring extension. We define a precoloring extension $\tilde{c}$ of $\tilde{P}$. Set $\tilde{c}(v) = f(v)$ for every $v \in S \cup X_0$, and $\tilde{c}(x) = c(x)$ for every $x \in \tilde{X} \cup Y$. It follows from the definition of L that $\tilde{c}$ is a precoloring extension of $(\tilde{G} \setminus (Y^* \setminus Y), S, X_0, \tilde{X}, Y)$.

Let $C \in \mathcal{C}$ with $|V(C)| > 2$. We extend $\tilde{c}$ to C. We will show that for every $Q \in \mathcal{Q}(C)$, $\{1, 2, 3, 4\} \setminus c(\tilde{X}(C)) \not\subseteq Q$. Consequently $T = \{1, 2, 3, 4\} \setminus c(\tilde{X}(C))$ is good for C. Since some vertex of $S \cup X_0 \cup \tilde{X}$ is complete to $V(C)$, it follows that $|T| \leq 3$. Therefore we can define $\tilde{c} : V(C) \to \{1, 2, 3, 4\}$ to be a proper T-coloring of (C, M), which can be done in polynomial time by Theorem 2.

So suppose that there is $Q \in \mathcal{Q}(C)$ such that $\{1, 2, 3, 4\} \setminus c(\tilde{X}(C)) \subseteq Q$. Then $\{1, 2, 3, 4\} \setminus Q \subseteq c(\tilde{X}(C))$. By (3.1), $|Q| < 3$.

We may assume that $\tilde{X}(C) \subseteq X_{12} \cup X_{34}$. Suppose first that $\tilde{X}(C)$ meets both X_{12} and X_{34}, and so $\tilde{X}(C) = \{x_{12}(C), x_{34}(C)\}$. Then $|c(\tilde{X}(C))| = 2$, and so $|Q| \neq 1$. Therefore may assume that $|Q| = 2$. If Q is friendly, then $c(v(C, Q)) \notin Q$, and so $\{1, 2, 3, 4\} \setminus Q \not\subseteq c(\tilde{X}(C))$, so we may assume that Q is not friendly. By symmetry, we may assume that $Q \in \{\{1, 2\}, \{1, 3\}\}$. If $Q = \{1, 2\}$, then since $L(x_{12}(C)) \subseteq \{1, 2\}$, it follows that $\{1, 2, 3, 4\} \setminus Q \not\subseteq c(\tilde{X}(C))$, so we may assume that $Q = \{1, 3\}$.

Suppose first that for every $i \in Q$, there is no vertex $u \in S \cup X_0$ with $c(u) = i$ and such that u has a neighbor in $V(C)$. Now (1) implies that every set of size two is bad for C. Therefore $h^{-1}(C) = \{z\}$ and $L(z) = \emptyset$, contrary to the fact that c is a proper coloring of (H, L).

We may assume from symmetry that

- there is a vertex $u \in S \cup X_0$ with $c(u) = 1$ and such that u has a neighbor in $V(C)$.
- there is no vertex $u \in S \cup X_0$ with $c(u) = 3$ and such that u has a neighbor in $V(C)$.

Now by (1) all the sets sets $\{1,2\}, \{1,3\}, \{1,4\}$ are bad. If the only good set is $\{3,4\}$, then $L(z) = \emptyset$, contrary to the fact that c is a coloring of (H, L). Therefore, at least one of $\{2,3\}, \{2,4\}$ is good, and (3.2) and (3.3) imply that $2 \notin L(u)$ for every $u \in \tilde{X}(C)$, contrary to the fact that $2 \in \{1,2,3,4\} \setminus Q \subseteq c(\tilde{X})$. This proves that not both $\tilde{X}(C) \cap X_{ij}$ and $\tilde{X}(C) \cap X_{kl}$ are non-empty.

We may assume that $\tilde{X}(C) \subseteq X_{12}$. Then $c(\tilde{X}(C)) \subseteq \{1,2\}$, and so $3, 4 \in Q$. Since $|Q| < 3$, we have $Q = \{3,4\}$. It follows from the construction of $\tilde{G}$ that $|\tilde{X}(C)| \leq 1$, contrary to the fact that $\{1,2,3,4\} \setminus Q \subseteq \bigcup_{u \in X(C)} \{c(u)\}$. This completes the proof of the second statement, and Theorem 10 follows. □

4 Insulating cutsets

Our next goal is to transform companion triples further, restricting them in such a way that we can test colorability.

Let H be a graph and let L be a 4-list assignment for H. We say that $D \subseteq V(H)$ is a *chromatic cutset* in H if $V(H) = A \cup B \cup D$, $A \neq \emptyset$, and $a \in A$ is adjacent to $b \in B$ only if $L(a) \cap L(b) = \emptyset$. For $i, j \in \{1,2,3,4\}$ let $D_{ij} = \{d \in D \ : \ L(d) \subseteq \{i,j\}\}$. The set A is called the *far side* of the chromatic cutset. We say that a chromatic cutset D is 12-*insulating* if $D = D_{12} \cup D_{34}$ and for every $\{p,q\} \in \{\{1,2\},\{3,4\}\}$ and every component $\tilde{D}$ of $H|D_{pq}$ the following conditions hold.

- $\tilde{D}$ is bipartite; let (D_1, D_2) be the bipartition.
- $|L(d)| = |L(d')|$ for every $d, d' \in D_1 \cup D_2$.
- There exists $a \in A$ with a neighbor in $\tilde{D}$ and with $L(a) \cap \{p,q\} \neq \emptyset$.
- Suppose that $|L(d)| = 2$ for every $d \in V(\tilde{D})$. Write $\{i,j\} = \{p,q\}$ and let $\{s,t\} = \{1,2\}$. If $a \in A$ has a neighbor in $d \in D_s$ and $i \in L(a)$, and $b \in B$ has a neighbor in $\tilde{D}$, then
  - if b has a neighbor in D_s, then $j \notin L(b)$, and
  - if b has a neighbor in D_t, then $i \notin L(b)$.

Insulating cutsets are useful for the following reason. We say that a component $\tilde{D}$ of $H|D_{pq}$ is *complex* if $|L(d)| = 2$ for every $d \in V(\tilde{D})$.

Theorem 11. *Let D be a 12-insulating chromatic cutset in (H, L), and let A, B be as in the definition of an insulating cutset. Let D' be the union of the vertex sets of complex components of $H|D_{12}$ and of $H|D_{34}$, and let $D'' = D \setminus D'$. If $(H|(B \cup D''), L)$ and $(H \setminus B, L)$ are both colorable, then (H, L) is colorable. Moreover, given proper colorings of $(H|(B \cup D''), L)$ and $(H \setminus B, L)$, a proper coloring of (H, L) can be found in polynomial time.*

Proof. Let c_1 be a proper coloring of $(H|(B \cup D''), L)$ and let c_2 be a proper coloring of $(H \setminus B, L)$.

A *conflict* in c_1, c_2 is a pair of adjacent vertices u, v such that $c_1(u) = c_2(v)$. Since c_1, c_2 are both proper colorings and D is a chromatic cutset, and $|L(d)| = 1$ for every $d \in D''$, we deduce that every conflict involves one vertex of D' and one vertex of B. Below we describe a polynomial-time procedure that modifies c_2 to reduce the number of conflicts (with c_1 fixed).

Let $u \in D'$ and $v \in B$ be a conflict. Then $uv \in E(H)$ and $c_1(u) = c_2(v)$. Let $\tilde{D}$ be the component of $G|D$ containing u. Then $V(\tilde{D}) \subseteq D'$ and $\tilde{D}$ is bipartite; let (D_1, D_2) be the bipartition of $\tilde{D}$. We may assume that $u \in D_1$. We may also assume that $L(d) = \{1, 2\}$ for every $d \in V(\tilde{D})$, and that $c_1(u) = c_2(v) = 2$. Since $L(d) = \{1, 2\}$ for every $d \in V(\tilde{D})$, it follows that for every $i \in \{1, 2\}$ and $d \in D_i$, we have $c_2(d) = i$. Let c_3 be obtained from c_2 by setting $c_3(d) = 1$ for every $d \in D_2$; $c_3(d) = 2$ for every $d \in D_1$; and $c_3(d) = c_2(d)$ for every $w \in (A \cup D) \setminus (D_1 \cup D_2)$. (This modification can be done in linear time).

First we show that c_3 is a proper coloring of $(H \setminus B, L)$. Since $L(d) = \{1, 2\}$ for every $d \in V(\tilde{D})$, $c_3(v) \in L(v)$ for every $v \in A \cup D$. Suppose there exist adjacent $xy \in D \cup A$ such that $c_3(x) = c_3(y)$. Since $\tilde{D}$ is a component of $H|D$, we may assume that $x \in D_1 \cup D_2$ and $y \in A$. Suppose first that $x \in D_1$. Then $c_3(y) = c_3(x) = 2$, and so $2 \in L(y)$ and y has a neighbor in D_1. But $v \in B$ has a neighbor in D_1 and $1 \in L(v)$, which is a contradiction. Thus we may assume that $x \in D_2$. Then $c_3(y) = c_3(x) = 1$, and so $1 \in L(y)$ and y has a neighbor in D_2. But $v \in B$ has a neighbor in D_1, and $1 \in L(b)$, again a contradiction. This proves that c_3 is a proper coloring of $(H \setminus B, L)$.

Clearly u, v is not a conflict in c_1, c_3. We show that no new conflict was created. Suppose that there is a new conflict, namely there exist adjacent $u' \in D'$ and $v' \in B$ such that $c_1(v') = c_3(u')$, but $c_1(v') \neq c_2(u')$. Then $u' \in V(\tilde{D})$. If $u' \in D_1$, then both v and v' have neighbors in D_1, and $1 \in L(v)$, and $2 \in L(v')$; if $u' \in D_2$, then v has a neighbor in D_1 and v' has a neighbor in D_2, and $1 \in L(v') \cap L(v)$; and in both cases we get a contradiction. Thus the number of conflicts in c_1, c_3 was reduced.

Now applying this procedure at most $|V(G)|^2$ times we obtained a proper coloring c'_1 of $(H|(B \cup D''), L)$ and a proper coloring c'_2 of $(H \setminus B, L)$ such that there is no conflict in c'_1, c'_2. Now define $c(v) = c'_1(v)$ if $v \in B \cup D''$ and $c(v) = c'_2(v)$ if $v \in V(H) \setminus B$; then c is a proper coloring of (H, L). This proves Theorem 11. □

Let G be a P_6-free graph, let $P = (G, S, X_0, X, Y^*, f)$ be an orthogonal excellent starred precoloring of G, and let (H, L, h) be a companion triple for P. Let $\{i, j, k, l\} = \{1, 2, 3, 4\}$. Let $Z^{ij} = \{z \in Z \ : \ N(z) \cap \tilde{X} \subset X_{ij} \cup X_{kl}\}$. It follows from the definition of a companion triple that $Z^{ij} = Z^{kl}$ and that $Z = \bigcup_{i,j \in \{1,2,3,4\}} Z^{ij}$. Next we prove a lemma that will allow us to replace a companion triple for P with a polynomially sized collection of near-companion triples for P, each of which has a useful insulating cutset. We will apply this lemma several times, and so we need to be able to apply it to near-companion triples for P, as well as to companion triples.

Lemma 17. *There is function $q : \mathbb{N} \to \mathbb{N}$ such that the following holds. Let G be a P_6-free graph, let $P = (G, S, X_0, X, Y^*, f)$ be an orthogonal excellent starred precoloring of G, and let (H, L, h) be a near-companion triple for P. Then there is an algorithm with running time $O(|V(G)|^{q(|S|)})$ that outputs a collection $\mathcal{L}$ of 4-list assignments for H such that*

- $|\mathcal{L}| \leq |V(G)|^{q(|S|)}$;
- *if $L' \in \mathcal{L}$ and c is a proper coloring of (H, L'), then c is a proper coloring of (H, L); and*
- *if (H, L) is colorable, then there exists $L' \in \mathcal{L}$ such that (H, L') is colorable.*

Moreover, for every $L' \in \mathcal{L}$,

- *$L'(v) \subseteq L(v)$ for every $v \in V(H)$;*
- *(H, L', h) is a near companion triple for P;*
- *if for some $i, j \in \{1, 2, 3, 4\}$ (H, L) has an ij-insulating cutset D' with far side Z^{ij}, then D' is an ij-insulating cutset with far side Z^{ij} in (H, L', h); and*

- *(H, L') has a 12-insulating cutset $D \subseteq \tilde{X}$ with far side Z^{12}.*

Proof. Let $\tilde{P} = (\tilde{G}, S, X_0, \tilde{X}, Y^*, f)$ be as in the definition of a near-companion triple. Assume that $Z^{12} \neq \emptyset$. If one of the graphs $\tilde{G}|\tilde{X}_{12}$ and $\tilde{G}|\tilde{X}_{34}$ is not bipartite, set $\mathcal{L} = \emptyset$. From now on we assume that $\tilde{G}|\tilde{X}_{12}$ and $\tilde{G}|\tilde{X}_{34}$ are bipartite. We may assume that $X_0 = X^0(\tilde{P})$. Let $T_1, \dots, T_p$ be types of $\tilde{X}$ with $|L_P(T_i)| = 2$ and such that $|L_P(T_i) \cap \{1, 2\}| = 1$. It follows that $|L_P(T_i) \cap \{3, 4\}| = 1$. Let $\mathcal{Q}$ be the set of all $2m$-tuples $Q = (Q_1, \dots, Q_m, P_1, \dots, P_m)$ such that

- $|Q_i| \leq 1$, $Q_i \subseteq \tilde{X}(T_i)$, and if $Q_i = \{q\}$, then $L(q) \cap \{1, 2\} \neq \emptyset$.
- $|P_i| \leq 1$, $P_i \subseteq \tilde{X}(T_i)$, and if $P_i = \{p\}$, then $L(p) \cap \{3, 4\} \neq \emptyset$.

For $x \in \tilde{X} \setminus (X_{12} \cup X_{34})$ and $z \in Z^{12}$ we say that z is a 12-*grandchild* of x if there is a component C of $\tilde{X}_{12}$ such that both x and z have neighbors in $V(C)$; a 34-*grandchild* is defined similarly. Let $G_{12}(x)$ be the set of 12-grandchildren of x; define $G_{34}(x)$ similarly.

We define a 4-list assignment L'_Q for H. Start with $L'_Q = L$. For every $i \in \{1, \dots, m\}$, proceed as follows. If $|Q_i| = 1$, say $Q_i = \{q_i\}$, set $L'_Q(q_i)$ to be the unique element of $L(q_i) \cap \{1, 2\}$. For every $x \in \tilde{X}(T_i)$ such that $G_{12}(q_i) \subset G(x)$ and $G_{12}(x) \setminus G_{12}(q_i) \neq \emptyset$, update $L'_Q(x)$ by removing from it the unique element of $L(x) \cap \{1, 2\}$. Next assume that $Q_i = \emptyset$. In this case, for every $x \in \tilde{X}(T_i) \setminus \{q_i, p_i\}$ such that x has a grandchild, update $L'_Q(x)$ by removing from it the unique element of $L(x) \cap \{1, 2\}$.

If $|P_i| = 1$, say $P_i = \{p_i\}$, set $L'_Q(p_i)$ to be the unique element of $L(p_i) \cap \{3, 4\}$. For every $x \in \tilde{X}(T_i)$ such that $G_{34}(p_i) \subset G(x)$ and $G_{12}(x) \setminus G_{12}(p_i) \neq \emptyset$, update $L'_Q(x)$ by removing from it the unique element of $L(x) \cap \{3, 4\}$. Next assume that $P_i = \emptyset$. In this case, for every $x \in X(T_i) \setminus \{p_i, q_i\}$ such that some component of $H|\tilde{X}_{34}$ contains both a neighbor of x and a neighbor of a vertex in Z^{12}, update $L'_Q(x)$ by removing from it the unique element of $L(x) \cap \{3, 4\}$.

If some vertex $z \in \tilde{X} \setminus \tilde{X}_{12}$ has neighbors on both sides of the bipartition of a component of $H|(\tilde{X}_{12})$, set $L'_Q(z) = L(z) \setminus \{1, 2\}$. If some vertex $z \in \tilde{X} \setminus \tilde{X}_{34}$ has neighbors on both sides of the bipartition of a component of $H|(\tilde{X}_{34})$, set $L'_Q(z) = L(z) \setminus \{3, 4\}$. Finally, set $L'_Q(v) = L(v)$ for every other $v \in V(H)$ not yet specified. Now let L_Q be obtained from L'_Q by updating exhaustively from $\bigcup_{i=1}^m (P_i \cup Q_i)$.

We need to check the following statements.

1. $L_Q(v) \subseteq L(v)$ for every $v \in V(H)$.
2. (H, L_Q, h) is a near-companion triple of P.
3. If for some $i, j \in \{1, 2, 3, 4\}$ (H, L) has an ij-insulating cutset D' with far side Z^{ij}, then D' is an ij-insulating cutset with far side Z^{ij} in (H, L_Q).
4. (H, L_Q) has a 12-insulating cutset with far side Z^{12}.

Clearly $L_Q(v) \subseteq L(v)$ for every $v \in V(H)$, and consequently it is routine to check that the third statement holds, and that in order to prove the second statement it is sufficient to prove the following:

(5) *Set $f(x) = L_Q(x)$ for every $x \in X^0(L_Q)$. Then for every $z \in Z$ with $L(z) \neq \emptyset$ and $q \in \{1, 2, 3, 4\}$ such that $q \notin L_Q(z)$, there is a vertex in $h(z)$ that has a neighbor $u \in S \cup X_0 \cup X^0(L_Q)$ with $f(u) = q$.*

We now prove this statement. Let $z \in Z$ and $q \in \{1, 2, 3, 4\}$ such that $q \notin L_Q(z)$. We need to show that there is a vertex in $h(z)$ that has a neighbor $u \in S \cup X_0 \cup X^0(L')$ with $f(u) = q$.

If $q \notin L(z)$, the claim follows from the fact that (H, L, h) is a near-companion triple for P, so we may assume that $q \in L(z)$, and therefore z has a neighbor u in $X^0(L_Q)$ with $f(u) = q$. Since Z is stable, it follows that $u \in \tilde{X}$, and therefore, by the definition of a companion triple, u is complete to $V(h(z))$. This proves (5).

Finally, we prove that (H, L_Q) has a 12-insulating cutset with far side Z^{12}. Let $D^1, \dots, D^t$ be the components of $H|\tilde{X}_{12}$ that contain a vertex x such that x has a neighbor z in Z^{12} with $L_Q(x) \cap L_Q(z) \neq \emptyset$. Let $F^1, \dots, F^s$ be defined similarly for $\tilde{X}_{34}$. Let $D = X^0(L_Q) \cup \bigcup_{i=1}^t V(D_i) \cup \bigcup_{j=1}^w V(F_j)$. We claim that D is the required cutset. Clearly D is a chromatic cutset, setting the far side to be Z^{12} and $B = V(H) \setminus (A \cup D)$, and the first two bullets of the definition of an insulating cutset are satisfied. Let $\tilde{D} \in \{D_1, \dots, D_t\}$ (the argument is symmetric for $F_1, \dots, F_s$). We need to check the following properties.

- *$\tilde{D}$ is bipartite.*
  This follows from the fact that $\tilde{G}|\tilde{X}_{ij} = H|\tilde{X}_{ij}$ is bipartite. Let (D_1, D_2) be the bipartition of $\tilde{D}$.

- *$|L(d)| = |L(d')|$ for every $d, d' \in D_1 \cup D_2$.*
  Since $L(d) \subseteq \{1, 2\}$ for every $d \in V(\tilde{D})$, and since we have updated exhaustively, it follows that if $V(\tilde{D})$ meets $X^0(L_Q)$, then $V(\tilde{D}) \subseteq X^0(L_Q)$.

- *There exists $a \in A$ with a neighbor in $\tilde{D}$ and with $L(a) \cap \{1, 2\} \neq \emptyset$.*
  This follows immediately from the definition of D.

- *Suppose that $|L(d)| = 2$ for every $d \in V(\tilde{D})$. We need to check that for $\{i, j\} = \{1, 2\}$, if $a \in A$ has a neighbor in $d \in D_1$ and $i \in L_Q(a)$, and $b \in B$ has a neighbor in $\tilde{D}$, then*
  - *if b has a neighbor in D_1, then $j \notin L_Q(b)$, and*
  - *if b has a neighbor in D_2, then $i \notin L_Q(b)$.*

We now check the condition of the last bullet. Let $a \in A$ have a neighbor $d \in D_1$ and $1 \in L_Q(a)$. Suppose $b \in B$ has a neighbor in $D_1 \cup D_2$, and violates the conditions above. It follows from the definition of Z^{12} and B that $b \in \tilde{X}$ and $|L_Q(b)| = 2$. We may assume that $b \in T_1(X)$. Since $|L_Q(b)| = 2$, we deduce that $L_Q(b) = L(b) = M_P(b) = L_P(T_1)$. Since b exists, $Q_1 \neq \emptyset$. Since $|L(d)| = 2$ for every $d \in V(\tilde{D})$, it follows that q_1 is anticomplete to $D_1 \cup D_2$. Since $b \notin X^0(L_Q)$, there is a component D_0 of $H|\tilde{X}_{12}$ such that q_1 has a neighbor $d_0 \in V(D_0)$ and b is anticomplete to $V(D_0)$. Let $\{i\} = L_Q(b) \cap \{1, 2\}$, and let $\{1, 2\} \setminus \{i\} = \{j\}$. Then $j \notin L_Q(b) = M_P(b)$, and so $j \not\subseteq L_P(T_1)$. Consequently, there is $s \in S$ with $f(s) = j$, such that s is complete to $\tilde{X}(T_1)$. Since $V(\tilde{D}) \cup V(D_0) \subseteq X_{12}$, it follows that s is anticomplete to $V(\tilde{D}) \cup V(D_0)$.

Suppose first that $V(\tilde{D}) \neq \{d\}$. Since b is not complete to $D_1 \cup D_2$ (because $L_Q(b) \cap \{1, 2\} \neq \emptyset$), there is an edge $d_1 d_2$ of $\tilde{D}$, such that b is adjacent to d_2 and not to d_1. Now $d_1 - d_2 - b - s - q_1 - d_0$ is a P_6 in $\tilde{G}_{12}(s)$, contrary to Lemma 15.

This proves that $V(\tilde{D}) = \{d\}$, and so b is adjacent to d, $i = 2$ and $j = 1$. Therefore $L_P(T_1) \cap \{1, 2\} = \{2\}$, and so $L_Q(q_1) = c(q_1) = 2$. Since $d_0 \in \tilde{X}_{12}$, it follows that $L_Q(d_0) = 1$. Since $1 \in L_Q(a)$ and L_Q is obtained by exhaustive updating, we deduce that a is non-adjacent to d_0. But now since $1 \in L_Q(a)$ and $f(s) = 1$, we deduce that $a - d - b - s - q_0 - d_0$ is a path in $H_{12}(s)$ contradicting Lemma 16. This proves that (H, L_Q) has a 12-insulating cutset with far side Z^{12}.

Let $\mathcal{L} = \{L_Q \ ; \ Q \in \mathcal{Q}\}$. Then $|\mathcal{L}| \leq |(V(G)|^{2m}$. Since $m \leq 2^{|S|}$, it follows that $|\mathcal{L}| \leq |V(G)|^{2^{|S|}}$. Since $L_Q(v) \subseteq L(v)$ for every $v \in V(H)$, it follows that every coloring of (H, L') is a coloring of (H, L).

Now suppose that (H, L) is colorable, and let c be a coloring. We show that some $L' \in \mathcal{L}$ is colorable. Let $i \in \{1, \ldots, m\}$. For a vertex $u \in \tilde{X}(T_i)$ define $val(u) = |G_{12}(u)|$. If some vertex u of $\tilde{X}(T_i)$ with a 12-grandchild has $c(u) \in L(u) \cap \{1, 2\}$, let q_i be such a vertex with $val(q_i)$ maximum and set $Q_i = \{q_i\}$. If no such u exists, let $Q_i = \emptyset$.

Define $P_1, \ldots, P_m$ similarly replacing $\tilde{X}_{12}$ with $\tilde{X}_{34}$. Let

$$Q = (Q_1, \ldots, Q_m, P_1, \ldots, P_m).$$

We show that $c(v) \in L_Q(v)$ for every $v \in V(H)$, and so (H, L_Q) is colorable. Since L_Q is obtained from L'_Q by updating, it is enough to prove that $c(v) \in L'_Q(v)$. Suppose not. There are two possibilities (possibly replacing 12 with 34).

1. $v \in \tilde{X}(T_i)$, $Q_i \neq \emptyset$, $G_{12}(q_i)$ is a proper subset of $G_{12}(v)$, and $c(v) \in \{1, 2\}$;

2. $v \in \tilde{X}(T_i)$, $Q_i = \emptyset$, $G_{12}(v) \neq \emptyset$, and $c(v) \in \{1, 2\}$.

We show that in both cases we get a contradiction.

1. In this case $val(v) > v(q_i)$, contrary to the choice of q_i.

2. The existence of v contradicts the fact that $Q_i = \emptyset$.

This proves that (H, L_Q) is colorable and completes the proof of Theorem 17. □

Let $P = (G, S, X_0, X, Y^*, f)$ be an orthogonal excellent starred precoloring of a P_6-free graph G. We say that a near-companion triple (H, L, h) is *insulated* if for every $i \in \{2, 3, 4\}$ such that Z^{1i} is non-empty, (H, L) has a $1i$-insulating cutset $D \subseteq \tilde{X}$ with far side Z^{1i}. We can now prove the main result of this section.

Theorem 12. *There is function $q : \mathbb{N} \to \mathbb{N}$ such that the following holds. Let G be a P_6-free graph, let $P = (G, S, X_0, X, Y^*, f)$ be an orthogonal excellent starred precoloring of G, and let (H, L, h) be a near-companion triple for P. There is an algorithm with running time $O(|V(G)|^{q(|S|)})$ that outputs a collection $\mathcal{L}$ of 4-list assignments for H such that*

- $|\mathcal{L}| \leq |V(G)|^{q(|S|)}$.
- *If $L' \in \mathcal{L}$ and c is a proper coloring of (H, L'), then c is a proper coloring of (H, L).*
- *If (H, L) is colorable, there exists $L' \in \mathcal{L}$ such that (H, L') is colorable.*

Moreover, for every $L' \in \mathcal{L}$.

- *$L'(v) \subseteq L(v)$ for every $v \in V(H)$.*
- *(H, L', h) is insulated.*

Proof. Let $\mathcal{L}_2$ be as in Lemma 17. By symmetry, we can apply Lemma 17 with 12 replaced by 13 to (H, L', h) for every $L' \in \mathcal{L}_2$; let $\mathcal{L}_3$ be the union of all the collections of lists thus obtained. Again by symmetry, we can apply Lemma 17 with 12 replaced by 14 to (H, L', h) for every $L' \in \mathcal{L}_3$; let $\mathcal{L}_4$ be the union of all the collections of lists thus obtained. Now $\mathcal{L}_4$ is the required collection of lists. □

5 Divide and Conquer

The main result of this section is the last piece of machinery that we need to solve the 4-precoloring-extension problem.

We need the following two facts.

Theorem 13. *[3] There is a polynomial time algorithm that tests, for graph H and a list assignment L with $|L(v)| \leq 2$ for every $v \in V(H)$, if (H, L) is colorable, and finds a proper coloring if one exists.*

Theorem 14. *[7] The 2-SAT problem can be solved in polynomial time.*

We prove:

Lemma 18. *Let G be a P_6-free graph and let $P = (G, S, X_0, X, Y^*, f)$ be an orthogonal excellent starred precoloring of G. Let (H, L', h) be a companion triple for P, where $V(H) = \tilde{X} \cup Z$, as in the definition of a companion triple. Assume that $D \subseteq \tilde{X}$ is a 12-insulating chromatic cutset in (H, L') with far side Z^{12}. There is a polynomial time algorithm that test if $(H|(Z^{12} \cup D), L')$ is colorable, and finds a proper coloring if one exists.*

Proof. We may assume that $X_0 = X^0(P)$. Let $\tilde{P} = (\tilde{G}, S, X_0, \tilde{X}, Y^*, f)$ be as in the definition of a companion triple, where $V(H) = \tilde{X} \cup Z$. By Theorem 13 we can test in polynomial time if $H|(D \cap \tilde{X}_{12}, L')$ and $H|(D \cap \tilde{X}_{34}, L')$ is colorable. If one of these pairs is not colorable, stop and output that $(H|(Z^{12} \cup D), L)$ is not colorable. So we may assume both the pairs are colorable, and in particular every component of $H|(D \cap \tilde{X}_{12})$ and $H|(D \cap \tilde{X}_{34})$ is bipartite.

We modify L' without changing the colorability property. First, let L'' be obtained from L' by updating exhaustively from $X^0(L')$. Next if $v \in V(H) \setminus \tilde{X}_{12}$ has a neighbor on both sides of the bipartition of a component of $H|\tilde{X}_{12}$, we remove both 1 and 2 from $L''(v)$, and the same for $\tilde{X}_{34}$; call the resulting list assignment L. (We have already done a similar modification while constructing list assignments L_Q in the proof of Lemma 17, but there we only modified lists of vertices in $\tilde{X}$, so this step is not redundant.) Set $f(u) = L(u)$ for every $u \in X^0(L)$. Clearly:

(6) *If $v \in V(H)$ is adjacent to $x \in X^0(L)$, then $L(v) \cap L(x) = \emptyset$.*

Next we prove:

(7) *Let $\{p, q\} \in \{\{1, 2\}, \{3, 4\}\}$ and let $z \in Z^{12}$ with $|L(z) \cap \{p, q\}| = 1$. Let $L(z) \cap \{p, q\} = \{i\}$ and $\{p, q\} \setminus L(z) = \{j\}$. Then there exists $y \in V(h(z))$ and $u \in S \cup X_0 \cup X^0(L)$ such that $f(u) = j$ and $uy \in E(\tilde{G})$.*

To prove (7) let $z \in Z$ with $L(z) \cap \{1, 2\} = \{1\}$ (the other cases are symmetric). Since $1 \in L(z)$, it follows that z does not have neighbors on both sides of the bipartition of a component of $H|\tilde{X}_{12}$, and therefore $L(z) = L''(z)$. If $2 \notin L'(z)$, then such u exists from the definition of a near-companion triple, so we may assume $2 \in L'(z)$. This implies that there is $u \in X^0(L)$ such that u is adjacent to z, and $f(u) = 2$. Since Z is stable, it follows that $u \in \tilde{X} \cup X_0 \cup S$, and so u is complete to $V(h(z))$, and (7) follows.

We define an instance I of the 2-SAT problem. The variables are the vertices of Z^{12}, and the clauses are as follows:

1. For every $z_1, z_2 \in Z^{12}$, if $L(z_i) \cap \{1, 2\} = \{i\}$ for $i = 1, 2$ and z_1, z_2 have neighbors on the same side of the bipartition of some component of $H|(D \cap \tilde{X}_{12})$, add the clause $(\neg z_1 \vee \neg z_2)$.

2. For every $z_1, z_2 \in Z^{12}$, if $L(z_1) \cap \{1,2\} = L(z_2) \cap \{1,2\} \in \{\{1\},\{2\}\}$ for $i = 1,2$ and z_1, z_2 have neighbors on opposite sides of the bipartition of some component of $H|(D \cap \tilde{X}_{12})$, add the clause $(\neg z_1 \vee \neg z_2)$.

3. For every $z_3, z_4 \in Z^{12}$, if $L(z_i) \cap \{3,4\} = \{i\}$ for $i = 3,4$ and z_3, z_4 have neighbors on the same side of the bipartition of some component of $H|(D \cap \tilde{X}_{34})$, add the clause $(z_3 \vee z_4)$.

4. For every $z_3, z_4 \in Z^{12}$, if $L(z_3) \cap \{3,4\} = L(z_4) \cap \{3,4\} \in \{\{3\},\{4\}\}$ for $i = 3,4$ and z_3, z_4 have neighbors on opposite sides of the bipartition of some component of $H|(D \cap \tilde{X}_{34})$, add the clause $(z_3 \vee z_4)$.

5. If $z \in Z^{12}$ and $L(z) \subseteq \{1,2\}$, add the clause $(z \vee z)$.

6. If $z \in Z$ and $L(z) \subseteq \{3,4\}$, add the clause $(\neg z \vee \neg z)$.

By Theorem 14 we can test in polynomial time if I is satisfiable.

We claim that I is satisfiable if and only if $(H|(Z^{12} \cup D), L)$ is colorable, and a proper coloring of $(H|(Z^{12} \cup D), L)$ can be constructed in polynomial time from a satisfying assignment for I.

Suppose first that $(H|(Z^{12} \cup D), L)$ is colorable, and let c be a proper coloring. For $z \in Z^{12}$, set $z = TRUE$ if $c(z) \in \{1,2\}$ and $z = FALSE$ if $c(z) \in \{3,4\}$. It is easy to check that every clause is satisfied.

Now suppose that I is satisfiable, and let g be a satisfying assignment. Let A' be the set of vertices $z \in Z^{12}$ with $g(z) = TRUE$, and let $B' = Z^{12} \setminus A'$. Let $A = A' \cup (D \cap \tilde{X}_{12})$ and $B = B' \cup (D \cap \tilde{X}_{34})$. For $v \in A$ let $L_A(v) = L'(v) \cap \{1,2\}$, and for $v \in B$ let $L_B(v) = L'(v) \cap \{3,4\}$. In order to show that $(H|(Z^{12} \cup D), L)$ is colorable and find a proper coloring, it is enough to prove that $(H|A, L_A)$ and $(H|B, L_B)$ are colorable, and find their proper colorings. We show that $(H|A, L_A)$ is colorable; the argument for $(H|B, L_B)$ is symmetric.

Since for every $z \in Z^{12}$ with $L(z) \subseteq \{3,4\}$ $(\neg z \vee \neg z)$ is a clause (of type 6) in I, it follows that $L(z) \cap \{1,2\} \neq \emptyset$ for every $z \in A$. Let $A_1 = \{v \in A : L_A(v) = \{1\}\}$, $A_2 = \{v \in A : L_A(v) = \{2\})$, and $A_3 = A \setminus (A_1 \cup A_2)$ Let F be a graph defined as follows. $V(F) = (A_3 \cup \{a_1, a_2\})$, where $F \setminus \{a_1, a_2\} = H|A_3$, $a_1 a_2 \in E(F)$, and for $i = 1,2$ $v \in A_3$ is adjacent to a_i if and only if v has a neighbor in A_i in H.

We claim that $(H|A, L_A)$ is colorable if and only if F is bipartite; and if F is bipartite, then a proper coloring of $(H|A, L_A)$ can be constructed in polynomial time. Suppose F is bipartite and let (F_1, F_2) be the bipartition. We may assume $a_i \in F_i$. Let $i \in \{1,2\}$. For every $v \in (F_i \cup A_i) \setminus \{a_i\}$, we have that $i \in L_A(v)$, and so we can set $c(v) = i$. This proves that $(H|A, L_A)$ is colorable, and constructs a proper coloring. Next assume that $(H|A, L_A)$ is colorable. For $i = 1,2$, let F_i' be the set of vertices of A colored i. Then $A_i \subseteq F_i'$, and setting $F_i = (F_i' \setminus A_i) \cup \{a_i\}$, we get that (F_1, F_2) is a bipartition of F. This proves the claim.

Finally we show that F is bipartite. Recall that the pair $(H|(D \cap \tilde{X}_{12}), L)$ is colorable, and therefore $H|(D \cap \tilde{X}_{12})$ is bipartite. Since $L_A(v) \subseteq L(v)$ for every $v \in A_3$, and $L_A(v) \cap \{1,2\} \neq \emptyset$ for every $v \in A$, it follows that no vertex of $A \cap Z^{12}$ has a neighbor on two opposite sides of a bipartition of a component of $H|(D \cap \tilde{X}_{12})$. Since Z^{12} is stable, this implies that the graph $H|A$ is bipartite.

Suppose that F is not bipartite. Then there is an odd cycle C in F, and so $V(C) \cap \{a_1, a_2\} \neq \emptyset$. In H this implies that there is a path $T = t_1 - \ldots - t_k$ with $\{t_2, \ldots, t_{k-1}\} \subseteq A_3$, such that either

- k is even, and for some $i \in \{1,2\}$ $t_1, t_k \in A_i$, or
- k is odd, $t_1 \in A_1$, and $t_k \in A_2$.

Since T is a path in $H|(Z \cup \tilde{X}_{12})$, it follows that $k \le 5$. If $t_1 \in \tilde{X}_{12} \cap D$, then $t_1 \in X^0(L)$, and so by (6), $t_2 \in A_1 \cup A_2$, a contradiction. This proves that $t_1 \in Z^{12}$, and similarly $t_k \in Z^{12}$.

Suppose first that k is even. Since Z^{12} is stable, it follows that $k \neq 2$, and so $k = 4$. Since $t_1, t_4 \in Z^{12}$ and since Z^{12} is stable, it follows that $t_2, t_3 \in \tilde{X}_{12}$. But now $(\neg t_1 \vee \neg t_4)$ is a clause (of type 2) in I, and yet $g(t_1) = g(t_4) = TRUE$, a contradiction.

This proves that k is odd. If $k = 3$ then, since Z^{12} is stable, $t_2 \in \tilde{X}_{12}$, and so $(\neg t_1 \vee \neg t_3)$ is a clause (of type 1) in I, and yet $g(t_1) = g(t_3) = TRUE$, a contradiction. This proves that $k = 5$. Since Z^{12} is stable, it follows that $t_2, t_4 \in \tilde{X}_{12}$. If $t_3 \in \tilde{X}_{12}$, then $(\neg t_1 \vee \neg t_5)$ is a clause (of type 1) in I, contrary to the fact that both $g(t_1) = g(t_5) = TRUE$, a contradiction. Therefore $t_3 \in Z^{12}$. We may assume that $t_1 \in A_1$. By (7) there exist $u \in S \cup X_0 \cup X^0(L)$ and $y_1 \in V(h(t_1))$ such that $f(u) = 2$ and $uy_1 \in E(\tilde{G})$. Since $t_2 \in \tilde{X}$, it follows that t_2 is complete to $V(h(t_1))$, and in particular t_2 is adjacent to y_1. Since $X_0 = X^0(P)$, it follows that u is anticomplete to $\{t_2, t_4\}$. Let $i \in \{3, 5\}$. By the definition of a companion triple, since $2 \in L(t_i)$, there exists $y_i \in V(h(t_i))$ such that u is non-adjacent to y_i in $\tilde{G}$. Now since no vertex of $\tilde{X}$ is mixed on a component to $\tilde{G}|Y^*$, it follows that $u - y_1 - t_2 - y_3 - t_4 - y_5$ is a P_6 in $\tilde{G}_{12}(u)$, contrary to Lemma 15. This proves Lemma 18. □

6 The complete algorithm

First we prove Theorem 8, which we restate.

Theorem 15. *For every integer C there exists a polynomial-time algorithm with the following specifications.*

Input: *An excellent starred precoloring $P = (G, S, X_0, X, Y^*, f)$ of a P_6-free graph G with $|S| \le C$.*

Output: *A precoloring extension of P or a determination that none exists.*

Proof. By Theorem 9 we can construct in polynomial time a collection $\mathcal{L}$ of orthogonal excellent starred precolorings of G, such that in order to determine if P has a precoloring extension (and find one if it exists), it is enough to check if each element of $\mathcal{L}$ has a precoloring extension, and find one if it exists. Thus let $P_1 \in \mathcal{L}$. By Theorem 10 we can construct in polynomial time a companion triple (H, L, h) for P_1, and it is enough to check if (H, L, h) is colorable.

Now proceed as follows. If $L(v) = \emptyset$ for some $v \in V(H)$, stop and output "no precoloring extension". So we may assume $L(v) \neq \emptyset$ for every $v \in V(H)$. Let $\mathcal{L}$ be a collection of lists as in Theorem 12. If $\mathcal{L} = \emptyset$, stop and output "no precoloring extension", so we may assume that $\mathcal{L} \neq \emptyset$. Let $L' \in \mathcal{L}$; then (H, L', h) is insulated. For every i let D^i be and insulating $1i$-cutset with far side Z^{1i}, and let $D^{i'} = \{d \in D_i \ : \ |L'(d)| = 2\}$. Let $H_i = H|(D^i \cup Z^{1i})$, and let $H_1 = H \setminus \bigcup_{i=2}^4 (D^{i'} \cup Z^{1i})$. Observe that $V(H_1) \subseteq \tilde{X}$. By Lemma 18, we can check if each of the pairs (H_i, L') with $i \in \{2, 3, 4\}$ is colorable, and by Theorem 13, we can check if (H_1, L') is colorable and find a proper coloring if one exists. If one of these pairs is not colorable, stop and output "no precoloring extension". So we may assume that (H_i, L') is colorable for every $i \in \{1, \ldots, 4\}$. Observe that D^2 is an insulating 12-cutset in $(H|(V(H_1) \cup V(H_2)), L')$ with far side Z^{12}, D^3 is an insulating 13-cutset in $(H|(V(H_1) \cup V(H_2) \cup V(H_3)), L')$ with far side Z^{13}, and D^4 is an insulating 14-cutset in (H, L') with far side Z^{14}. Now three applications of Theorem 11 show that (H, L) is colorable, and produce a proper coloring. This proves 15. □

We can now prove the main result of the series, the following.

Theorem 16. *There exists a polynomial-time algorithm with the following specifications.*

Input: *A 4-precoloring* (G, X_0, f) *of a* P_6*-free graph* G*.*

Output: *A precoloring extension of* (G, X_0, f) *or a determination that none exists.*

Proof. Let $\mathcal{L}$ be as in Theorem 7. Then $\mathcal{L}$ can be constructed in polynomial time, and it is enough to check if each element of $\mathcal{L}$ has a precoloring extension, and find one if it exists. Now apply the algorithm of Theorem 15 to every element of $\mathcal{L}$. □

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Appendix: Finding an Excellent Precoloring

A Establishing the Axioms on Y_0

Given a P_6-free graph G and a precoloring (G, A, f), our goal is to construct a polynomial number of seeded precolorings $P = (G, S, X_0, X, Y_0, Y, f)$ satisfying the following axioms, and such that if we can decide for each of them if it has a precoloring extension, then we can decide if (G, A, f) has a 4-precoloring extension, and construct one if it exists.

(i) $G \setminus X_0$ is connected.

(ii) S is connected and no vertex in $V(G) \setminus S$ is complete to S.

(iii) $Y_0 = V(G) \setminus (N(S) \cup X_0 \cup S)$.

(iv) No vertex $V(G) \setminus (Y_0 \cup X_0)$ is mixed on an edge of Y_0.

(v) If $|L_{S,f}(v)| = 1$ and $v \notin S$, then $v \in X_0$; if $|L_{S,f}(v)| = 2$, then $v \in X$; if $|L_{S,f}(v)| = 3$, then $v \in Y$; and if $|L_{S,f}(v)| = 4$, then $v \in Y_0$.

(vi) There is a color $c \in \{1, 2, 3, 4\}$ such for every vertex $y \in Y$ with a neighbor in Y_0, $f(N(y) \cap S) = \{c\}$. We let $L = \{1, 2, 3, 4\} \setminus \{c\}$.

(vii) With L as in (vi), we let Y_L^* be the subset of Y_L of vertices that are in connected components of $G|(Y_0 \cup Y_L)$ containing a vertex of Y_0. Then no vertex of $Y \setminus Y_L^*$ has a neighbor in $Y_0 \cup Y_L^*$, and no vertex in X is mixed on an edge of $Y_0 \cup Y_L^*$.

(viii) With Y_L^* as in (vii), for every component C of $G|(Y_0 \cup Y_L^*)$, there is a vertex v in X complete to C.

We start with a useful lemma.

Lemma 19. *Let G be a graph and let $X \subseteq V(G)$ be connected. If $v \in V(G) \setminus X$ is mixed on X, the there is an edge xy of X such that v is adjacent to x and not to y.*

Proof. Since v is mixed on X, both the sets $N(v) \cap X$ and $X \setminus N(v)$ are non-empty. Now since X is connected, there exist $x \in N(v) \cap X$ and $y \in X \setminus N(v)$ such that x is adjacent to y, as required. This proves Lemma 19. □

Now we establish the first axiom.

Lemma 20. *Given a 4-precoloring (G, X_0, f) of a P_6-free graph G, there is an algorithm with running time $O(|V(G)|^2)$ that outputs a collection $\mathcal{L}$ of seeded precolorings such that:*

- *$|\mathcal{L}| \leq |V(G)|$;*
- *every $P' \in \mathcal{L}$ is of the form $P' = (G|(V(C) \cup X_0), \emptyset, X_0, \emptyset, V(C), \emptyset, f)$ for a component C of $G \setminus X_0$;*
- *every $P' \in \mathcal{L}$ satisfies* (i)
- *(G, X_0, f) has a 4-precoloring extension if and only if each of the seeded precolorings $P' \in \mathcal{L}$ has a precoloring extension; and*

- *given a precoloring extension for each of the seeded precolorings $P' \in \mathcal{L}$, we can compute a 4-precoloring extension for (G, X_0, f) in polynomial time.*

Proof. For each connected component C of $G \setminus X_0$, the algorithm outputs the seeded precoloring $(G|(V(C) \cup X_0), \emptyset, X_0, \emptyset, V(C), \emptyset, f)$. Since the coloring is fixed on X_0, it follows that (G, X_0, f) has a 4-precoloring extension if and only if the 4-precoloring on X_0 can be extended to every connected component C of $G \setminus X_0$. This implies the statement of the lemma. □

The next lemma is used to arrange the following axioms, which we restate:

(ii) S is connected and no vertex in $V(G) \setminus S$ is complete to S.

(iii) $Y_0 = V(G) \setminus (N(S) \cup X_0 \cup S)$.

Lemma 21. *There is a constant C such that the following holds. Let $P = (G, \emptyset, X_0, \emptyset, Y_0, \emptyset, f)$ be a seeded precoloring of a P_6-free graph G with P satisfying* (i)*. Then there is an algorithm with running time $O(|V(G)|^C)$ that outputs an equivalent collection $\mathcal{L}$ for P such that*

- $|\mathcal{L}| \leq |V(G)|^C$*;*
- *every $P' \in \mathcal{L}$ is a normal subcase of G;*
- *every $P' = (G', S', X_0', X', Y_0', Y', f') \in \mathcal{L}$ with seed S' satisfies $|S'| \leq C$; and*
- *every $P' \in \mathcal{L}$ satisfies* (i), (ii) *and* (iii).

Moreover, for every $P' \in \mathcal{L}$, given a precoloring extension of P', we can compute a precoloring extension for P in polynomial time.

Proof. If $|V(G) \setminus X_0| \leq 5$, we enumerate all possible colorings. Now let $v \in V(G) \setminus X_0$, and let $S' = \{v\}$. While there is a vertex w in $V(G) \setminus S'$ complete to S', we add w to S'. Let S denote the set S' when this procedure terminates. If either $|S| \geq 5$ or $(G|(S \cup X_0), \emptyset, X_0, S, \emptyset, \emptyset, f)$ has no precoloring extension, then we output that P has no precoloring extension. Otherwise, we construct $\mathcal{L}$ as follows. For every proper coloring f' of $G|S$ such that $f \cup f'$ is a proper coloring of $G|(S \cup X_0)$, we add

$$P' = (G, S, X_0 \setminus S, N(S) \setminus X_0, V(G) \setminus (X_0 \cup S \cup N(S)), \emptyset, f \cup f')$$

to $\mathcal{L}$. Since $|S| \leq 4$, it follows that the first three bullets hold, and (iii) holds for P' by the definition of P'. Since X_0 is unchanged, it follows that (i) holds. Since S is a maximal clique, we have that (ii) holds for P'. This concludes the proof. □

The next four lemmas are technical tools that we use several times in the course of the proof. They are used to show that if we start with a seeded precoloring that has certain properties, and then move to its normal subcase, then these properties are preserved (or at least can be restored with a simple modification).

For a seeded precoloring $P = (G, S, X_0, X, Y_0, Y, f)$, a *type* is a subset of S. For $v \in V(G) \setminus (S \cup X_0)$, the *type of* v, denoted by $T_P(v) = T_S(v)$, is $N(v) \cap S$. For a type T and a set A, we let $A(T) = \{v \in A : T_P(v) = T\}$.

Lemma 22. *Let $P = (G, S, X_0, X, Y_0, Y, f)$ be a seeded precoloring of a P_6-free graph G satisfying* (ii) *and* (iii)*, and let Let $T, T' \subseteq S$ with $|f(T)| = |f(T')| = 1$ and such that $f(T) \neq f(T')$. Let $y, y' \in N(Y_0)$ such that $T(y) = T$ and $T(y') = T'$. Let $z, z' \in Y_0$ be such that yz and $y'z'$ are edges, and suppose that z is non-adjacent to z' and that y is non-adjacent to y'. Then either yz' or $y'z$ is an edge.*

Proof. Suppose both the pairs yz' and $y'z$ are non-adjacent. Since P satisfies (ii) and (iii), it follows that $G|S$ is connected and both y, y' have neighbors in S. Let Q be a shortest path from y to y' with interior in S. Since $|f(T)| = |f(T') = 1$ and $f(T) \neq f(T')$, it follows that $T \cap T' = \emptyset$, and so $|Q^*| > 1$. But now $z - y - Q - y' - z'$ is a path of length at least six in G, a contradiction. This proves Lemma 22. □

Lemma 23. *Let $P = (G, S, X_0, X, Y_0, Y, f)$ be a seeded precoloring of a P_6-free graph satisfying* (ii), (iii) *and* (iv), *and let $P' = (G', S', X_0', X', Y_0', Y', f')$ be a normal subcase of P satisfying* (iii). *Then no $v \in Y_0 \setminus (S' \cup Y_0')$ has both a neighbor in S' and a neighbor in Y_0'.*

Proof. Suppose such v exists. Let $y \in Y_0$ be a neighbor of v. Since P' is a normal subcase of P, P' satisfies (ii). Since v has both a neighbor in Y_0' and a neighbor in S', and since P' satisfies (iii), it follows that $v \in X' \cup Y' \cup X_0'$. Since $v \in Y_0$, it follows that v is anticomplete to S. Therefore v has a neighbor in $S' \setminus S \subseteq X \cup Y \cup Y_0$. Since P' satisfies (ii), there is a path Q from v to a vertex s of S with $Q^* \subseteq S'$. Then $V(Q) \setminus \{v\}$ is anticomplete to Y_0'. Let R be the maximal subpath of $v - Q - s$, with $v \in V(R)$, such that $V(R) \subseteq Y_0$. Then $s \notin V(R)$, and there is a unique vertex $t \in V(Q) \setminus V(R)$ with a neighbor in $V(R)$. Since $t \in N(Y_0)$, it follows that $t \notin S \cup Y_0$, and so $t \in X \cup Y$. But t is mixed on $V(R) \cup \{y\} \subseteq Y_0$, contrary to the fact that P satisfies (iv). This proves Lemma 23.

□

Lemma 24. *There is a constant C such that the following holds. Let $P = (G, S, X_0, X, Y_0, Y, f)$ be a seeded precoloring of a P_6-free graph G with P satisfying* (i), *and let $P' = (G', S', X_0', X', Y_0', Y', f')$ be a normal subcase of P satisfying* (iii) *and* (iv). *Then there is an algorithm with running time $O(|V(G)|^C)$ that outputs an equivalent collection $\mathcal{L}$ for P', such that $|\mathcal{L}| \leq 1$, and if $\mathcal{L} = \{P''\}$, then*

- *there is $Z \subseteq Y_0'$ such that $P'' = (G' \setminus Z, S', X_0', Y_0' \setminus Z, Y', f)$ and P'' is a normal subcase of P';*
- *P'' satisfies* (i)—(iv);
- *if P' satisfies* (v), *then P'' satisfies* (v).

Moreover, given a precoloring extension of P'', we can compute a precoloring extension for P in polynomial time.

Proof. Since P' is a normal subcase of P, it follows that P' satisfies (ii). We may assume that P' does not satisfy (i), for otherwise we can set $\mathcal{L} = \{P'\}$. Now let C be a connected component of $G' \setminus X_0'$ with $S' \cap V(C) = \emptyset$. It follows that $V(C) \subseteq Y_0'$ and C is a component of $G|Y_0'$.

Let $x \in N(V(C)) \cap (X_0' \setminus X_0)$. Since P satisfies (i), such a vertex x exists. By Lemma 23, $x \in X \cup Y$. Since P' satisfies (iv), it follows from Lemma 19 that x is complete to $V(C)$. Let $f'(x) = c$. Then in every precoloring extension d of P' we have $d(v) \neq c$ for every $v \in V(C)$.

Let $A = \{v \in X_0' : f'(v) \neq c\}$. By Theorem 2 and since G is P_6-free, we can decide in polynomial time if $(G'|(V(C) \cup A), A, f'|_A)$ has a precoloring extension with colors in $\{1, 2, 3, 4\} \setminus \{c\}$. If not, then P' has no precoloring extension, and we set $\mathcal{L} = \emptyset$. If $(G'|(V(C) \cup A), A, f'|_A)$ has a precoloring extension using only colors in $\{1, 2, 3, 4\} \setminus \{c\}$, then P' has a precoloring extension if and only if $(G' \setminus V(C), S', X_0', X', Y_0' \setminus V(C), Y', f')$ does.

We repeat this process a polynomial number of times until $G' \setminus X_0'$ is connected, and output the resulting seeded precoloring $P'' = (G'', S', X_0', X', Y_0'', Y', f')$ satisfying (i). Since $Y_0'' \subseteq Y_0'$, and the other sets of P'' remain the same as in P', it follows that the P'' satisfies (ii)–(iv), and if P' satisfies (v), then so does P''. This proves Lemma 24. □

Lemma 25. *Let $P = (G, S, X_0, X, Y_0, Y, f)$ be a seeded precoloring of a P_6-free graph satisfying* (ii), (iii) *and* (iv), *and let $P' = (G', S', X'_0, X', Y'_0, Y', f')$ be a normal subcase of P satisfying* (iii). *Then P' satisfies* (iv). *Moreover, if P satisfies* (vi), *then P' satisfies* (vi).

Proof. Since P' is a normal subcase of P, P' satisfies (ii). First we show that P' satisfies (iv). Suppose not, then there exists $v \in V(G) \setminus X'_0$ mixed on an edge xy of Y'_0, say v is adjacent to y and not to x. It follows that $v \in X' \cup Y'$, and since P satisfies (iv), $v \in Y_0$. Therefore v has a neighbor in S', contrary to Lemma 23. This proves that P' satisfies (iv).

Next assume that P satisfies (vi). We show that P' satisfies (vi). Let L as in (vi) applied to P. Suppose there exists $y \in N(Y'_0)$ with $L_{P'}(y) \neq L$ and $|L_{P'}(y)| = 3$. Since P satisfies (vi), it follows that $y \in Y_0 \setminus Y'_0$, and y has a neighbor $s \in S'$, contrary to Lemma 23. This proves that P' satisfies (vi).

This completes the proof of Lemma 25.

□

The next lemma is another technical tool, used to establish axioms (iv) and (vii).

Lemma 26. *There is a function $q : \mathbb{N} \to \mathbb{N}$ such that the following holds. Let $P = (G, S, X_0, X, Y_0, Y, f)$ be a seeded precoloring of a P_6-free graph G with P satisfying* (i), (ii) *and* (iii). *Let $L \subseteq [4]$ with $|L| = 3$, let c_4 be the unique element of $[4] \setminus L$. Let $R \subseteq Y_0 \cup Y_L$ such that $Y_0 \subseteq R$. Assume further that if $t \in (X \cup Y) \setminus R$ has a neighbor in R, then for every $z \in R$, $L_P(t) \neq L_P(z)$, and that there is no path $t - z_1 - z_2 - z_3$ with $t \in (X \cup Y) \setminus R$ and $z_1, z_2, z_3 \in R$. Then there is an algorithm with running time $O(|V(G)|^{q(|S|)})$ that outputs an equivalent collection $\mathcal{L}$ for P such that*

- *$|\mathcal{L}| \leq |V(G)|^{q(|S|)}$;*
- *every $P' \in \mathcal{L}$ is a normal subcase of P;*
- *every $P' = (G', S', X'_0, X', Y'_0, Y', f') \in \mathcal{L}$ with seed S' satisfies $|S'| \leq q(|S|)$;*
- *every $P' \in \mathcal{L}$ satisfies* (ii) *and* (iii).
- *no vertex of $(X' \cup Y') \setminus R$ is mixed on an edge of $(Y' \cup Y'_0) \cap R$.*

Moreover, for every $P' \in \mathcal{L}$, given a precoloring extension of P', we can compute a precoloring extension of P in polynomial time.

Proof. If G contains a K_5, then P has no precoloring extension; we output $\mathcal{L} = \emptyset$ and stop. Thus from now on we assume that G has no clique of size five. Let $Y_0^5 = R$ and let $Z^5 = (X \cup Y) \setminus R$. Let $\mathcal{T}^5$ be the set of types of vertices in Z^5, and set $j = 4$.

Let $\mathcal{Q}_j$ be the set of $|\mathcal{T}^j|$-tuples $(S^{j,T})_{T \in \mathcal{T}^{j+1}}$, where each $S^{j,T} \subseteq Z^{j+1}(T)$ and $S^{j,T}$ is constructed as follows (starting with $S^{j,T} = \emptyset$):

- If $R = Y_0$ or $c_4 \in f(T)$ proceed as follows. While there is a vertex $z \in Z^{j+1}(T)$ complete to $S^{j,T}$ and such that there is clique $\{a_1, \ldots, a_j\} \subseteq {Y_0}^{j+1}$ with $N(z) \cap \{a_1, \ldots, a_j\} = \{a_1\}$, choose such z with $N(z) \cap R$ maximal and add it to $S^{j,T}$.
- If $R \neq Y_0$ and $c_4 \notin f(T)$, while there is $z \in Z^{j+1}(T)$ complete to $S^{j,T}$ such that there is clique $\{a_1, \ldots, a_j\} \subseteq {Y_0}^{j+1}$ with $N(z) \cap \{a_1, \ldots, a_j\} = \{a_1\}$, add z to $S^{j,T}$. Let $X_0(z)$ be the set of all $z' \in Z^{j+1}(T)$ such that
  - z' is complete to $S^{j,T} \setminus \{z\}$

– there is a clique $\{b_1, \dots, b_j\} \subseteq {Y_0}^{j+1}$ such that $N(z') \cap \{b_1, \dots, b_j\} = \{b_1\}$, and
– $N(z') \cap R$ is a proper subset of $N(z) \cap R$.

When no such vertex z exists, let $X_0^{j,T} = \bigcup_{z \in S^{j,T}} X_0(z)$. Define $f'(z') = c_4$ for every $z' \in X_0^{j,T}$ (observe that since $c_4 \notin f(T)$, it follows that $c_4 \in L_P(z')$).

Since G has no clique of size five, it follows that $|S^{j,T}| \leq 4$ for all T. Let $Q \in \mathcal{Q}_j$; write $Q = (S^{j,T})_{T \in \mathcal{T}^{j+1}}$. Let $S^j = S^{j,Q} = \bigcup_{T \in \mathcal{T}^{j+1}} S^{j,T}$. Let $Y_0^j = Y_0^{j,Q} = Y_0^{j+1} \setminus N(S^j)$, $X_0^j = X_0^{j,Q} = \bigcup_{T \in \mathcal{T}^{j+1}} X_0^{j,T}$. $Z^j = Z^{j,Q} = (Z^{j+1} \setminus X_0^j) \cup (Y_0^{j+1} \setminus Y_0^j)$ and let $\mathcal{T}^j$ be the set of types of Z^j (in P). If $j > 2$, decrease j by 1 and repeat the construction above, to obtain a new set $\mathcal{Q}_{j-1}$; repeat this for each $Q \in \mathcal{Q}_j$.

Suppose $j = 2$. Then Q was constructed by fixing $Q_4 \in \mathcal{Q}_4$, constructing $\mathcal{Q}_3$ (with Q_4 fixed), fixing $Q_3 \in \mathcal{Q}_3$, constructing $\mathcal{Q}_2$ (with Q_3 fixed), and finally fixing $Q \in \mathcal{Q}_2$. Write $Q_2 = Q$. For consistency of notation we write $Q_5 = \emptyset$, $Z^5 = Z^{5,Q_5}$ and $Y_0^5 = Y_0^{5,Q_5}$. Let $S' = S \cup \bigcup_{j=2}^4 S^{j,Q_j}$. If $R \neq Y_0$, let $X'_0 = X_0 \cup \bigcup_{j=2}^4 X_0^{j,Q_j}$; if $R = Y_0$, let $X'_0 = X_0$.

For every function $f' : S' \setminus S \to \{1, 2, 3, 4\}$ such that $f \cup f'$ is a proper coloring of $G|(S' \cup X'_0)$, let

$$P_{f',Q} = (G, S', X'_0, Z^{2,Q} \cap X, Y_0^{2,Q}, Z^{2,Q} \cap Y, f \cup f').$$

Let $\mathcal{L}$ be the set of all $P_{Q,f'}$ as above. Observe that S' is obtained from S by adding a clique of size at most four for each type in $\mathcal{T}^j$ at each of the three steps ($j = 4, 3, 2$), and since $|\mathcal{T}^j| \leq 2^{|S|}$ for every j, it follows that $|S \cup S'| \leq |S| + 12 \times 2^{|S|}$. Since $|S' \setminus S| \leq 12 \times 2^{|S|}$, it follows that $|\mathcal{L}| \leq (4|V(G)|)^{12 \times 2^{|S|}}$.

In the remainder of the proof we show that every $P_{Q,f'} \in \mathcal{L}$ satisfies the required properties.

(8) *$S \cup \bigcup_{k=j}^4 S^k$ is connected for every $j \in \{2, \dots, 4\}$. In particular S' is connected.*

Since for every j, we have that $S^{j,Q_j} \subseteq Z^{j+1}$, it follows that every vertex of S^{j,Q_j} has a neighbor in $S \cup \bigcup_{k=j+1}^4 S^{k,Q_k}$, and (8) follows.

(9) *Let $j \in \{2, \dots, 5\}$. There is no path $z - a - b - c$ with $z \in Z^{j,Q_j}$ and $a, b, c \in Y_0^{j,Q_j}$.*

Suppose for a contradiction that there exist j and z violating (9); we may assume z is chosen with j maximum. By assumption $j \neq 5$ and $z \in Y_0^{j,Q_j} \setminus Y_0^{j+1,Q_{j+1}}$. It follows that z has a neighbor $z' \in S^{j,Q_j}$ and that z is anticomplete to $S \cup \bigcup_{k=j+1}^4 S^{k,Q_k}$. Since $z' \in S^{j,Q_j} \subseteq Z^{j+1,Q_j}$, it follows that z' has a neighbor $s \in S \cup \bigcup_{k=j+1}^4 S^{k,Q_k}$. But now $s - z' - z - a - b - c$ is a P_6 in G, a contradiction. This proves (9).

(10) *Let $j \in \{2, \dots, 4\}$. No vertex $z \in Z^{j,Q_j}$ has exactly one neighbor in a clique $\{a_1, \dots, a_j\} \subseteq Y_0^{j,Q_j}$.*

Suppose for a contradiction that there exist j and z violating (10); we may assume that z is chosen with j maximum. Write $Q_j = (S^{j,T})$. Let $\{a_1, \dots, a_j\} \subseteq Y_0^{j,Q_j}$ be a clique with $N(z) \cap \{a_1, \dots, a_j\} = \{a_1\}$.

Suppose first that $z \in R$. Let k be maximum such that $z \in Z^{k,Q_k}$. Then $z \notin Z^{k+1,Q_{k+1}}$, and thus $z \in Y_0^{k+1,Q_{k+1}}$, z has a neighbor $z' \in S^{k,Q_k}$, and z is anticomplete to $S \cup \bigcup_{l=k+1}^4 S^{l,Q_l}$. It follows that $z' \in Z^{k+1,Q_{k+1}}$. But now $z' - z - a_1 - a_j$ is a path with $z, a_1, a_j \in Y_0^{k+1,Q_{k+1}}$ contrary to (9). This proves that $z \notin R$.

It follows that $z \in Z^{j+1,Q_{j+1}} \cap (X \cup Y)$, and in particular z has a neighbor in S. Let $T = T_P(z)$. It follows that $S^{j,T} \neq \emptyset$; let $z' \in S^{j,T}$ be the first vertex that was added to $S^{j,T}$ that is non-adjacent to z (such a vertex exists by the definition of $S^{j,T}$). Then $L_P(z) = L_P(z')$. Since $z' \in S^{j,Q_j}$, it follows that z' is anticomplete to Y_0^{j,Q_j}. Since $a_1 \in Y_0^{j,Q_j} \subseteq Y_0^{j+1,Q_{j+1}}$, it follows that z has a neighbor in $Y_0^{j+1,Q_{j+1}}$ non-adjacent to z', and hence (by the choice of z' if $Y_0 = R$, and since $z \notin X_0(z')$ if $Y_0 \neq R$), it follows that z' has a neighbor $a' \in Y_0^{j+1}$ that is non-adjacent to z.

Suppose first that a' is complete to $\{a_1, \dots, a_j\}$. Since G contains no clique of size five, it follows that $j < 4$. But now $N(z') \cap \{a', a_1, \dots, a_j\} = \{a'\}$, contrary to the maximality of j.

Suppose next that a' is mixed on $\{a_1, \dots, a_j\}$. Let x be a neighbor and y be a non-neighbor of a' in $\{a_1, \dots, a_j\}$. Then $z' - a' - x - y$ is a path, which contradicts an assumption of the theorem.

It follows that a' is anticomplete to $\{a_1, \dots, a_j\}$. Since $z, z' \notin R$ and have neighbors in R, it follows that there is a vertex $t \in T$ that is anticomplete to R (this is immediate if $R = Y_0$, and follows from the fact that $L_P(z) \neq L$ if $R \neq Y_0$). Now $a' - z' - t - z - a_1 - a_j$ is a P_6 in G, a contradiction. This proves (10).

By (8) $P_{f',Q}$ satisfied (ii), and by construction (iii) holds. Now from (10) with $j = 2$ we deduce that no vertex of $(X' \cup Y') \setminus R$ is mixed on an edge of $(Y' \cup Y_0') \cap R$.

It remains to show that $\mathcal{L}$ is equivalent to P. Clearly for every $P' \in \mathcal{L}$, a precoloring extension of P' is also a precoloring extension of P.

Let d be a precoloring extension of P. We show that some $P' \in \mathcal{L}$ has a precoloring extension. Let $j \in \{2, 3, 4\}$; define $S^{j,T}$ and f' as follows (starting with $S^{j,T} = \emptyset$):

- If $R = Y_0$ or $c_4 \in f(T)$ proceed as follows. While there is a vertex $z \in Z^{j+1}(T)$ complete to $S^{j,T}$ and such that there is clique $\{a_1, \dots, a_j\} \subseteq {Y_0}^{j+1}$ with $N(z) \cap \{a_1, \dots, a_j\} = \{a_1\}$, choose such z is such that $N(z) \cap R$ maximal and add it to $S^{j,T}$; set $f'(z) = d(z)$.

- If $R \neq Y_0$ and $c_4 \notin f(T)$, while there is $z \in Z^{j+1}(T)$ complete to $S^{j,T}$ such that there is clique $\{a_1, \dots, a_j\} \subseteq {Y_0}^{j+1}$ with $N(z) \cap \{a_1, \dots, a_j\} = \{a_1\}$, choose such z with $d(z) \neq c_4$ and subject to that with $N(z) \cap R$ maximal; add z to $S^{j,T}$ and set $f'(z) = d(z)$. Let $X_0(z)$ be the set of all $z' \in Z^{j+1}(T)$ such that

  - z' is complete to $S^{j,T} \setminus \{z\}$,
  - there is a clique $\{b_1, \dots, b_j\} \subseteq {Y_0}^{j+1}$ such that $N(z') \cap \{b_1, \dots, b_j\} = \{b_1\}$, and
  - $N(z') \cap R$ is a proper subset of $N(z) \cap R$.

  It follows from the choice of z that $d(z') = c_4$ for every $z' \in X_0(z)$. When no such vertex z exists, let $X_0^{j,T} = \bigcup_{z \in S^{j,T}} X_0(z)$; thus $d(z') = c_4$ for every $z' \in X_0^{j,T}$. Define ${f'}_{j,T}(z') = c_4$ for every $z' \in X_0^{j,T}$, then ${f'}_{j,T}(z) = d(z)$ for every $z \in X_0^{j,T}$.

Let $Q_j = (S^{j,T})$ and let ${f'}_j = \bigcup_T {f'}_{j,T}$. It follows that $P_{f_2,Q_2} = (G, S', X_0', X', Y_0', Y', f \cup f')$ satisfies $d(v) = f_2(v)$ for every $v \in S' \cup X_0'$, and thus d is is a precoloring extension of P_{f_2,q_2}, as required. This proves Lemma 26. $\square$

The next lemma is used to arrange the following axiom, which we restate:

(iv) No vertex $V(G) \setminus (Y_0 \cup X_0)$ is mixed on an edge of Y_0.

Lemma 27. *There is a function $q : \mathbb{N} \to \mathbb{N}$ such that the following holds. Let $P = (G, S, X_0, X, Y_0, Y, f)$ be a seeded precoloring of a P_6-free graph G with P satisfying* (i), (ii) *and* (iii). *Then there is an algorithm with running time $O(|V(G)|^{q(|S|)})$ that outputs an equivalent collection $\mathcal{L}$ for P such that*

- $|\mathcal{L}| \leq |V(G)|^{q(|S|)}$;

- *every $P' \in \mathcal{L}$ is a normal subcase of P;*
- *every $P' = (G', S', X'_0, X', Y'_0, Y', f') \in \mathcal{L}$ with seed S' satisfies $|S'| \leq q(|S|)$; and*
- *every $P' \in \mathcal{L}$ satisfies* (i), (ii), (iii) *and* (iv).

Moreover, for every $P' \in \mathcal{L}$, given a precoloring extension of P', we can compute a precoloring extension for P in polynomial time.

Proof. Let $S^5 = \emptyset$. Let $Z = X \cup Y$. Since P satisfies (iii), it follows that every vertex of Z has a neighbor in S. While there is a vertex $z \in Z$ complete to S^5 and a path $z - a - b - c$ with $a, b, c \in Y_0$, we add z to S^5. If $|S^5| \geq 5$, then G contains a K_5 and thus it has no precoloring extension; set $\mathcal{L} = \emptyset$ and stop. Thus we may assume that $|S^5| \leq 4$. Let $Y_0^5 = Y_0 \setminus N(S^5)$ and let $Z^5 = Z \cup (Y_0 \setminus Y_0^5)$. Since S is connected, and since every vertex of S^5 has a neighbor in S, it follows that $S \cup S^5$ is connected.

(11) *There is no path $z - a - b - c$ with $z \in Z^5$ and $a, b, c \in Y_0^5$.*

Suppose for a contradiction that such a path exists, and suppose first that $z \in Z$. By the choice of S^5, it follows that there exists a vertex $z' \in Z \cap S^5$ non-adjacent to z. Since $S \cup S^5$ is connected, there exists a path Q connecting z and z' with interior in $S \cup S^5$. Since P satisfies (iii) and by the construction of S^5, it follows that Q^* is anticomplete to $\{a, b, c\}$. But now $z' - Q - z - a - b - c$ is a path of length at least six in G, a contradiction.

It follows that $z \in N(S^5) \setminus Z$, and thus $z \in Y_0 \setminus Y_0^5$. Let $s' \in S^5 \cap N(z)$. Then s' is anticomplete to $\{a, b, c\}$. Moreover, $s' \in Z$, and so s' has a neighbor $s \in S$. Since P satisfies (iii), s is anticomplete to Y_0, and so s is anticomplete to $\{z, a, b, c\}$. But now $s - s' - z - a - b - c$ is a P_6 in G, a contradiction. This proves (11).

For every $f' : S^5 \to [4]$ such that $f \cup f'$ is a proper coloring of $G|(S \cup S^5)$, let $P_{f'} = (G, S \cup S^5, X_0, Z^5, Y_0^5, \emptyset, f \cup f')$. Then $P_{f'}$ is a normal subcase of P that satisfies (i)-(iii).

Let $\mathcal{M}_{f'}$ be the collection of seeded precolorings obtained by applying Lemma 26 to P'_f with $R = Y_0^5$, and let $\mathcal{M}$ be the union of all such $\mathcal{M}_{f'}$. By (11) every $P'' \in \mathcal{M}$ satisfies (ii)–(iv).

Finally let $\mathcal{L}$ be obtained from $\mathcal{M}$ by applying Lemma 24 to every member of $\mathcal{M}$. Then every $P' \in \mathcal{L}$ satisfies (i)–(iv), as required. This proves Lemma 27. □

The purpose of Lemma 28 is to organize vertices according to their lists (which, in turn, arise from the colors of their neighbors in the seed) to satisfy the following axiom:

(v) If $|L_{S,f}(v)| = 1$ and $v \notin S$, then $v \in X_0$; if $|L_{S,f}(v)| = 2$, then $v \in X$; if $|L_{S,f}(v)| = 3$, then $v \in Y$; and if $|L_{S,f}(v)| = 4$, then $v \in Y_0$.

Moreover, we will construct new seeded precolorings in controlled ways from seeded precolorings satisfying (i), (ii), (iii), and (iv), to arrange that these axioms as well as (v) still hold for the new instances.

Lemma 28. *There is a constant C such that the following holds. Let $P = (G, S, X_0, X, Y_0, Y, f)$ be a seeded precoloring of a P_6-free graph G with P satisfying* (i), (ii), (iii) *and* (iv), *and let $P' = (G', S', X'_0, X', Y'_0, Y', f')$ be a normal subcase of P. Then there is an algorithm with running time $O(|V(G)|^C)$ that outputs an equivalent collection $\mathcal{L}$ for $\{P\}$ of seeded precoloring with $|\mathcal{L}| \leq 1$, such that if $\mathcal{L} = \{P''\}$ then*

- *P'' is a normal subcase of P', and*

- P'' *satisfies* (i), (ii), (iii), (iv) *and* (v).
- *If* P' (vi), *then* P'' *satisfies* (vi).
- *If* P' *satisfies* (vii), *then* P'' *satisfies* (vii).

Moreover, given a precoloring extension of P'', *we can compute a precoloring extension for* P *in polynomial time.*

Proof. Since P' is a normal subcase of P, it follows that P' satisfies (ii). By moving vertices between Y'_0 and $X' \cup Y'$, we may assume that P' satisfies (iii). By Lemma 25 P' satisfies (iv).

Let $Z_i = \{v \in V(G) \setminus (S' \cup X'_0) : |L_{P'}(v)| = i\}$. If $Z_0 \neq \emptyset$, then P' has no precoloring extension, and we output this and $\mathcal{L} = \emptyset$ and stop. Thus, we may assume that $Z_0 = \emptyset$. Let $f'' : Z_1 \to \{1, 2, 3, 4\}$ with $f''(v) = c$ if $L_{P'}(v) = \{c\}$. Since P' satisfies (iii), it follows that $Y'_0 = Z_4$, and so the seeded precoloring $\tilde{P} = (G', S', X'_0 \cup Z_1, Z_2, Z_4, Z_3, f' \cup f'')$ satisfies (iv). For the same reason, if P' satisfies (vi), then so does $\tilde{P}$, and if P' satisfies (vii), then so does $\tilde{P}$. Let P'' be obtained from the precoloring $\tilde{P}$ as in Lemma 24. It follows that P'' satisfies (i)–(v), and P'' is a normal subcase of P'. Clearly if $\tilde{P}$ satisfies (vi), then so does P'', and if $\tilde{P}$ satisfies (vii), then so does P''. This proves Lemma 28. □

In the next lemma we establish (vi), which we restate:

(vi) There is a color $c \in \{1, 2, 3, 4\}$ such for every vertex $y \in Y$ with a neighbor in Y_0, $f(N(y) \cap S) = \{c\}$. We let $L = \{1, 2, 3, 4\} \setminus \{c\}$.

Lemma 29. *There is a function* $q : \mathbb{N} \to \mathbb{N}$ *such that the following holds. Let* $P = (G, S, X_0, X, Y_0, Y, f)$ *be a seeded precoloring of a* P_6*-free graph* G *with* P *satisfying* (i), (ii), (iii), (iv) *and* (v). *Then there is an algorithm with running time* $O(|V(G)|^{q(|S|)})$ *that outputs an equivalent collection* $\mathcal{L}$ *for* P *such that*

- $|\mathcal{L}| \leq |V(G)|^{q(|S|)}$;
- *every* $P' \in \mathcal{L}$ *is a normal subcase of* P;
- *every* $P' = (G', S', X'_0, X', Y'_0, Y', f') \in \mathcal{L}$ *with seed* S' *satisfies* $|S'| \leq q(|S|)$; *and*
- *every* $P' \in \mathcal{L}$ *satisfies* (i), (ii), (iii), (iv), (v) *and* (vi).

Moreover, for every $P' \in \mathcal{L}$, *given a precoloring extension of* P', *we can compute a precoloring extension for* P *in polynomial time.*

Proof. A seeded precoloring $P = (G, S, X_0, X, Y_0, Y, f)$ is *acceptable* if for every precoloring extension c of P and for every non-adjacent $y, y' \in Y \cap N(Y_0)$ with $L_P(y) \neq L_P(y')$, we have $\{c(y), c(y')\} \not\subseteq L_P(y) \cap L_P(y')$.

First we construct a collection $\mathcal{M}$ of seeded precolorings that is an equivalent collection for P, and such that every member of $\mathcal{M}$ is acceptable. We proceed as follows. Let $\mathcal{T}$ be the set of all pairs (T, T') with $T, T' \subseteq S$ and $|f(T)| = |f(T')| = 1$ and $f(T) \neq f(T')$. Write $\mathcal{T} = \{(T_1, T'_1), \ldots, (T_t, T'_t)\}$. Let $\mathcal{Q}$ be the set of all t-tuples $Q = (Q_{T_1,T'_1}, \ldots, Q_{T_t,T'_t})$ such that $Q_{T_i,T'_i} = (P_{T_i,T'_i}, M_{T_i,T'_i}, N_{T_i,T'_i})$ where

- $|P_{T_i,T'_i}| = |M_{T_i,T'_i}| \leq |N_{T_i,T'_i}| \leq 1$.
- $P_{T_i,T'_i} \subseteq Y(T_i)$ and $N_{T_i,T'_i} \subseteq Y(T'_i)$.

- $M_{T_i,T'_i} \subseteq Y_0$.
- M_{T_i,T'_i} is complete to $P_{T_i,T'_i} \cup N_{T_i,T'_i}$.
- P_{T_i,T'_i} is anticomplete to N_{T_i,T'_i}.

Let $V(Q_{T_i,T'_i}) = P_{T_i,T'_i} \cup M_{T_i,T'_i} \cup N_{T_i,T'_i}$ and let $S(Q) = \bigcup_{i=1}^{t} V(Q_{T_i,T'_i})$. Let $(T_i, T'_i) \in \mathcal{T}$. Define $Z(T_i, T'_i)$ as follows.

- If $|P_{T_i,T'_i}| = |M_{T_i,T'_i}| = |N_{T_i,T'_i}| = 0$, then $Z(T_i, T'_i) = Y(T'{}_i) \cap N(Y_0)$.
- If $|P_{T_i,T'_i}| = |M_{T_i,T'_i}| = 0$ and $|N_{T_i,T'_i}| = 1$, then $Z(T_i, T'_i) = (Y(T'{}_i) \cap N(Y_0)) \setminus N(N_{T_i,T'_i})$.
- If $|P_{T_i,T'_i}| = |M_{T_i,T'_i}| = |N_{T_i,T'_i}| = 1$, then $Z(T_i, T'_i) = \emptyset$.

Let $Z(Q) = \bigcup_{(T_i,T'_i)\in\mathcal{T}} Z(T_i, T'_i)$. A function f' is said to be Q-*admissible* if $f' : S(Q) \cup Z(Q) \to \{1, \ldots, 4\}$ and for every $i \in \{1, \ldots, t\}$ it satisfies:

- $f'(P_{T_i,T'_i}), f'(N_{T_i,T'_i}) \in [4] \setminus (f(T_i) \cup f(T'_i))$.
- If $Z(T_i, T'_i) \subseteq Y(T'_i)$, then $f'(Z(T_i, T'_i)) = f(T_i)$.
- If $Z(T_i, T'_i) \subseteq Y(T_i)$, then $f'(Z(T_i, T'_i)) = f(T'_i)$.
- The coloring $f \cup f'$ of $G|(S \cup S(Q) \cup X_0 \cup Z(Q))$ is proper.

For every Q-admissible function f' with domain $S(Q) \cup Z(Q)$, let

$$P_{Q,f'} = (G, S\cup S(Q), X_0\cup Z(Q), X, Y_0\setminus(S(Q)\cup N(S(Q))), (Y\setminus(S(Q)\cup Z(Q)))\cup(N(S(Q))\cap Y_0), f\cup f').$$

Then $P_{Q,f'}$ is a normal subcase of P.

Since every vertex in $X\cup Y$ has a neighbor in S, it follows that $P_{Q,f'}$ satisfies (ii); by construction (iii) holds. By Lemma 25, $P_{Q,f'}$ satisfies (iv). Let $\mathcal{M}$ be the union of the collections obtained by applying Lemma 28, where the union is taken over all Q, f' as above. Then every member of $\mathcal{M}$ satisfies (i)–(v).

We show that there is a function $q_1 : \mathbb{N} \to \mathbb{N}$ such that $|S \cup S(Q)| \leq q_1(|S|)$ and $|\mathcal{M}| \leq |V(G)|^{q_1(|S|)}$. Since there are at most $2^{|S|}$ types, it follows that $t \leq 2^{2|S|}$. Now, since for every $(T_i, T'_i) \in \mathcal{T}$ we have that $|V(Q_{T_i,T'_i})| \leq 3$, it follows that for every $Q \in \mathcal{Q}$ we have $|S(Q)| \leq 3 \times 2^t$, and so $|S \cup S(Q)| \leq |S| + 3 \times 2^{2|S|}$ and $|\mathcal{Q}| \leq |V(G)|^{3\times 2^{2|S|}}$. Finally, for every Q, there are at most $4^{|S(Q)|} = 4^{3t}$ possible precoloring of $S(Q)$, since every precoloring of $S(Q)$ extends to an admissible function in a unique way, and we deduce that $|\mathcal{M}| \leq 4^{3t} \times |\mathcal{Q}| \leq 4^{3\times 2^{2|S|}} \times |V(G)|^{3\times 2^{2|S|}} \leq (4|V(G)|)^{3\times 2^{2|S|}}$ as required.

(12) *Let $P' \in \mathcal{M}$ with $P' = (G, S', X'_0, X', Y'_0, Y', f')$. If $y \in Y'$ has a neighbor $z \in Y'_0$, then $y \in Y$.*

Suppose that $y \notin Y$. Then $y \in Y_0 \cap Y'$ and there exist $s \in S' \setminus S$ such that y is adjacent to s, contrary to Lemma 23. This proves (12).

Next we show that every precoloring in $\mathcal{M}$ is acceptable. Let $P' = (G, S', X'_0, X', Y'_0, Y', f') \in \mathcal{M}$, and suppose there exist non-adjacent $y, y' \in N(Y'_0) \cap Y'$ with $L_{P'}(y) \neq L_{P'}(y')$ and such that there exists a precoloring extension c with $c(y), c(y') \in L_{P'}(y) \cap L_{P'}(y')$. Let $z \in N(y) \cap Y'_0$ and $z' \in N(y') \cap Y'_0$. Then $z, z' \in Y_0$, and so by Lemma 23, $y, y' \in Y$, $L_P(y) = L_{P'}(y)$ and $L_P(y') = L_{P'}(y')$. Let $T = T(y)$ and $T' = T(y')$ (in P). Then $T \cap T' = \emptyset$. By Lemma 22 we may assume that $z = z'$. Since $Y' \cap Y(T)$ and $Y' \cap Y(T')$ are both non-empty, it follows that $|V(Q_{T,T'})| > 1$. Let $P_{T,T'} = \{p\}$, $M_{T,T} = \{m\}$ and $N_{T,T'} = \{n\}$. Since $z \in Y'_0$, it follows that z is anticomplete to $V(Q_{T,T'})$. Since P' satisfies (v), $f'(p), f'(n) \in L_P(y) \cap L_P(y')$ and $|L_{P'}(y)| = |L_{P'}(y')| = 3$, it follows that $\{y, y', p, n\}$ is a stable set. By symmetry, we may assume that $f'(m) \in L_{P'}(y)$, and hence y is not adjacent to m. Let $s \in T \setminus T'$; then $z - y - s - p - m - n$ is a P_6 in G, a contradiction. This proves that every seeded precoloring in $\mathcal{M}$ is acceptable.

Next we show that $\mathcal{M}$ is equivalent to P. Clearly every precoloring extension of a member of $\mathcal{M}$ is a precoloring extension of P. For the converse, let c be a precoloring extension of P. For every pair of types $(T, T') \in \mathcal{T}$ for which there exist non-adjacent $y \in Y(T) \cap N(Y_0)$ and $y' \in Y(T') \cap N(Y_0)$, such that $c(y), c(y') \notin f(T) \cup f(T')$, choose such a pair y, y' and let z be a common neighbor of y, y' in Y_0 (such z exists by Lemma 22); set $P_{T,T'} = \{y\}$, $M_{T,T'} = \{z\}$ and $N_{T,T'} = \{y'\}$, and define $f'(y) = c(y)$, $f'(y') = c(y')$ and $f'(z) = c(z)$. Let $Z(T_i, T'_i) = \emptyset$.

Now let $(T, T') \in \mathcal{T}$ be such that no such y, y' exist. Suppose that there exists $y \in Y(T') \cap N(Y_0)$ with $c(y) \neq f(T)$, let $N_{T,T'} = \{y\}$, $P_{T,T'} = M_{T,T'} = \emptyset$, and let $f'(y) = c(y)$. Let $Z(T_i, T'_i) = (Y(T) \cap N(Y_0)) \setminus N(y)$, and set $f'(v) = f(T')$ for every $v \in Z(T_i, T'_i)$. Since (T, T') does not have the property described in the previous paragraph, it follows that $c((Y(T) \cap N(Y_0)) \setminus N(y)) = f(T')$, and so $c(v) = f'(v)$ for every $v \in Z(T_i, T'_i)$. Finally, suppose that $c(Y(T') \cap N(Y_0)) = f(T)$. Then set $P_{T,T'} = M_{T,T'} = N_{T,T'} = \emptyset$ and $Z(T_i, T'_i) = Y(T') \cap N(Y_0)$. Define $f'(v) = f(T)$ for every $v \in Z(T_i, T'_i)$. Let Q consist of all the triples $Q_{T,T'} = (P_{T,T'}, M_{T,T'}, N_{T,T'})$ as above. Let $S(Q) = \bigcup_{(T,T') \in \mathcal{T}} V(Q_{T,T'})$, and $Z(Q) = \bigcup_{(T,T') \in \mathcal{T}} Z(T_i, T'_i)$. Let

$$P_{Q,f'} = (G, S \cup S(Q), X_0 \cup Z(Q), X, Y_0 \setminus (S(Q) \cup N(S(Q))), (Y \setminus (S(Q) \cup Z(Q))) \cup (N(S(Q)) \cap Y_0), f \cup f').$$

Then c is a precoloring extension of $P_{Q,f'}$. Moreover, $P_{Q,f'}$ was one of the seeded precoloring we considered in the process of constructing $\mathcal{M}$, and so $\mathcal{M}$ contains the seeded precoloring obtained from $P_{Q,f'}$ by applying Lemma 28. It follows that $\mathcal{M}$ is an equivalent collection for P.

Let $P' = (G, S', X'_0, X', Y'_0, Y', f') \in \mathcal{M}$ be an acceptable seeded precoloring. For $c \in \{1, 2, 3, 4\}$ and a precoloring extension d of P', we say that is *c is active for L and d* if there exists a vertex $v \in Y' \cap N(Y'_0)$ with $L_{P'}(v) = L$ and $d(v) = c$.

Define $\mathcal{L}_1(P')$ as follows. For every function $g : Y' \cap N(Y'_0) \to [4]$ such that

- $g(v) \in L_{P'}(v)$ for every $v \in Y \cap N(Y'_0)$,
- $|g(Y'_L \cap N(Y'_0)| = 1$ for every $L \in \binom{[4]}{3}$, and
- $f' \cup g$ is a proper coloring of $G|(S' \cup X'_0 \cup (Y' \cap N(Y'_0)))$,

let

$$P''_g = (G, S', X'_0 \cup (Y' \cap N(Y'_0)), X', Y'_0, Y' \setminus N(Y'_0), f' \cup g).$$

It is easy to check that P''_g satisfies (ii)—(vi). Let P'_g be obtained from P''_g by applying Lemma 28. Then P'_g satisfies (i)—(vi). Let $\mathcal{L}_1(P')$ be the collections of all such P'_g.

Next we construct $\mathcal{L}_2(P')$. For every $L \in \binom{[4]}{3}$, for every $y_1, y_2 \in Y'_L \cap N(Y'_0)$, and for every $c_1, c_2 \in L$, define a function g as follows. Let $g(y_i) = c_i$. For every $L' \in \binom{[4]}{3} \setminus L$, let $Z(L')$ be the set of vertices $v \in Y'_{L'}$ such that v has a non-neighbor $n \in \{y_1, y_2\}$ with $g(n) \in L'$. For every $v \in Z(L')$, let $g(v)$ be the unique element of $L' \setminus L$. Finally, let $Z = \bigcup_{L' \in \binom{[4]}{3} \setminus L} Z(L')$.

If $f' \cup g$ is a proper coloring of $G|(S \cup X_0 \cup \{y_1, y_2\})$, let

$$P''_{L,y_1,y_2,c_1,c_2} = (G, S \cup \{y_1, y_2\}, X_0 \cup Z, X, Y_0 \setminus N(\{y_1, y_2\}), Y \setminus (Z \cup \{y_1, y_2\}), f' \cup g).$$

It is easy to check that P''_{L,y_1,y_2,c_1,c_2} satisfies (i)—(vi). Let P'_{L,y_1,y_2,c_1,c_2} be obtained from P''_{L,y_1,y_2,c_1,c_2} by applying Lemma 28. Let $\mathcal{L}_2(P')$ be the collection of all P'_{L,y_1,y_2,c_1,c_2} constructed this way; then every member of $\mathcal{L}_2$ satisfies (i)—(vi).

We claim that $\mathcal{L}(P') = \mathcal{L}_1(P') \cup \mathcal{L}_2(P')$ is an equivalent collection for $\{P'\}$. Clearly a precoloring extension of an element of $\mathcal{L}(P')$ is a precoloring extension of P. Now let c be a precoloring extension of P. If for every $L \in \binom{[4]}{3}$ there is at most one active color for L and c, then c is a precoloring extension of a member of $\mathcal{L}_1(P)$, so we may assume that there is $L_0 \in \binom{[4]}{3}$ such that at least two colors are active for L and c. We may assume that $L = \{1, 2, 3\}$ and the colors $1, 2$ are active. Let $y_i \in {Y'}_{L_0}$ with $c(y) = i$. We claim that c is a precoloring extension of $P''_{L_0,y_1,y_2,1,2}$. Let $L \in \binom{[4]}{3} \setminus L_0$. Since P' is acceptable, for every $v \in Y'_L$ that has a non-neighbor $n \in \{y_1, y_2\}$ with $c(n) \in L'$, we have that $c(v) \in L' \setminus L_0$. It follows that $c(v) = g(v)$, and the claim holds. This proves that $\mathcal{L}(P')$ is an equivalent collection for $\{P'\}$.

Finally, setting

$$\mathcal{L} = \bigcup_{P' \in \mathcal{M}} \mathcal{L}(P'),$$

Lemma 29 follows. This completes the proof. □

The next lemma is used to arrange the following axiom, which we restate:

(vii) With L as in (vi), we let Y^*_L be the subset of Y_L of vertices that are in connected components of $G|(Y_0 \cup Y_L)$ containing a vertex of Y_0. Then no vertex of $Y \setminus Y^*_L$ has a neighbor in $Y_0 \cup Y^*_L$, and no vertex of X is mixed on $Y_0 \cup Y^*_L$.

Lemma 30. *There is a function $q : \mathbb{N} \to \mathbb{N}$ such that the following holds. Let $P = (G, S, X_0, X, Y_0, Y, f)$ be a seeded precoloring of a P_6-free graph G with P satisfying* (i), (ii), (iii), (iv), (v) *and* (vi). *Then there is an algorithm with running time $O(|V(G)|^{q(|S|)})$ that outputs an equivalent collection $\mathcal{L}$ for P such that*

- $|\mathcal{L}| \leq |V(G)|^{q(|S|)}$;
- *every $P' \in \mathcal{L}$ is a normal subcase of P;*
- *for every $P' = (G', S', X'_0, X', Y'_0, Y', f') \in \mathcal{L}$, $|S'| \leq q(|S|)$;*
- *every $P' \in \mathcal{L}$ satisfies* (i), (ii), (iii), (iv), (v), (vi) *and* (vii);

Moreover, for every $P' \in \mathcal{L}$, given a precoloring extension of P', we can compute a precoloring extension for P in polynomial time.

Proof. We may assume that G contains no K_5, for otherwise, P does not have a precoloring extension and we output $\mathcal{L} = \emptyset$ and stop.

With L as in (vi) and Y^*_L as in (vii), let $Y^* = (X \cup (Y \setminus Y^*_L)) \cap N(Y_0 \cup Y^*_L)$. By the definition of Y^*_L, it follows that $L_P(y) \neq L$ for every $y \in Y^*$, and if $y \in Y^* \cap Y$, then y is anticomplete to Y_0. Let $\mathcal{T} = \{T_1, \ldots, T_t\}$ be the set of types of vertices in Y^*. Let $L = \{c_1, c_2, c_3\}$ and $\{c_4\} = \{1, 2, 3, 4\} \setminus L$. Let $\mathcal{Q}$ consist of all t-tuples $Q = ((S_{T_1}, R_{T_1}), \ldots, (S_{T_i}, R_{T_t}))$ such that

- $|R_{T_i}| \leq |S_{T_i}| \leq 1$.
- $S_{T_i} \cup R_{T_i} \subseteq Y^*(T_i)$.
- S_{T_i} is complete to R_{T_i}.

Let $V(Q) = \bigcup_{i=1}^{t}(S_{T_i} \cup R_{T_i})$. For every $Q \in \mathcal{Q}$ and for every $f' : V(Q) \to L$ with $f'(v) \in L_P(v) \setminus \{c_4\}$ for all $v \in V(Q)$, we proceed as follows. Let $\tilde{Y}^1_{Q,f'}$ be the set of all vertices v in Y^* such that $S_{T(v)} = \emptyset$. Let $\tilde{Y}^2_{Q,f'}$ be the set of all vertices v in Y^* such that $S_{T(v)} \neq \emptyset$, $R_{T(v)} = \emptyset$ and v is complete to $S_{T(v)}$. Let $\tilde{Y}_{Q,f'} = \tilde{Y}^1_{Q,f'} \cup \tilde{Y}^2_{Q,f'}$. Let $f'(v) = c_4$ for every $v \in \tilde{Y}_{Q,f'}$ Since $V(Q) \subseteq Y^*$, it follows that $G|(S \cup V(Q))$ is connected. Suppose that $f \cup f'$ is a proper coloring of $G|(S \cup X_0 \cup V(Q) \cup \tilde{Y}_{Q,f'})$. Let $\mathcal{L}'$ be obtained from the normal subcase

$$(G, S \cup V(Q), X_0 \cup \tilde{Y}_{Q,f'}, X \setminus (\tilde{Y}_{Q,f'} \cup V(Q)), Y_0, Y \setminus (\tilde{Y}_{Q,f'} \cup V(Q)), f \cup f' \cup g)$$

of P by applying Lemma 28. Suppose that $\mathcal{L}' = \{P_{Q,f'}\}$. Write $P_{Q,f'} = (G, S', X'_0, X', Y'_0, Y', f')$. Then $P_{Q,f'}$ satisfies (i)–(vi). Furthermore, $P_{Q,f'}$ has a precoloring extension if and only if P has a precoloring extension d such that $d(v) = f'(v)$ for every $v \in V(Q)$, and $d(v) = c_4$ for every $v \in Y^*$ such that either

- $S_{T(v)} = \emptyset$, or
- $S_{T(v)} \neq \emptyset$, $R_{T(v)} = \emptyset$, and v is complete to $S_{T(v)}$.

Moreover, $|V(Q)| \leq 2|\mathcal{T}| \leq 2^{|S|+1}$.

Let $\mathcal{L}_1$ be the set of all seeded precolorings $P_{Q,f'}$ as above (ranging over all $Q \in \mathcal{Q}$). Then $\mathcal{L}_1$ is an equivalent collection for P, and $|\mathcal{L}_1| \leq (3|V(G)|)^{2^{|S|+1}}$. Let $P' \in \mathcal{L}_1$ with $P' = (G, S', X'_0, X', Y'_0, Y', f')$. Since P' satisfies (vi), let L be as in (vi) and let ${Y'}^*_L$ be as in (vii).

(13) *There is no path $z - a - b - c$ with $z \in (X' \cup Y') \setminus {Y'}^*_L$ and $a, b, c \in {Y'}^*_L \cup Y'_0$.*

Suppose that such a path $z - a - b - c$ exists. First we show that $z \in X \cup Y$. Suppose not, then $z \in Y_0$ and z has a neighbor $s' \in S' \setminus S$. Since P satisfies (vi), it follows that $s' \in X$. Since $\{z, a, b, c\} \subseteq Y_0 \cup Y_L$, and since P satisfies (v), we deduce that there exists $s \in T(s')$ with $f(s) \in L$. Consequently, s is anticomplete to $\{z, a, b, c\}$. But now $s - s' - z - a - b - c$ is a P_6 in G, a contradiction. This proves that $z \in X \cup Y$.

Since $L_{S,f}(z) \neq L$, there exists $t \in T(z)$ with $f(t) \in L$. Since $z \notin X'_0$, it follows that $S_{T(v)} \neq \emptyset$, and either

- $R_{T(z)} \neq \emptyset$, or
- $R_{T(z)} = \emptyset$, and z is not complete to $S_{T(z)}$.

Let $S_{T(z)} = \{s\}$. Since $f'(s) \in L$, it follows that s is anticomplete to $\{a, b, c\}$. If z is non-adjacent to s, then $s - t - z - a - b - c$ is a P_6, a contradiction. It follows that z is adjacent to s, and therefore $R_{T(z)} \neq \emptyset$; say $R_{T(z)} = \{r\}$. Since s is adjacent to r, it follows that $f'(z) \neq f'(r)$. Since $z \notin X_0$, and since (v) holds, it follows that z is non-adjacent to r. Since $f'(r) \in L$, it follows that r is anticomplete to $\{a, b, c\}$. But now $r - s - z - a - b - c$ is a P_6 in G, a contradiction. This proves (13).

In view of (13), let $\mathcal{L}_2(P')$ be the collection of precolorings obtained from P' by applying Lemma 26 with $R = Y'_0 \cup {Y'}^*_L$. Let $P'' \in \mathcal{L}_2(P')$; write $P'' = (G, S'', X''_0, X'', Y''_0, Y'', f'')$. Then P'' satisfies (ii) and (iii) and no vertex of $(X'' \cup Y'') \setminus R$ is mixed on $(Y'' \cup Y''_0) \cap R$. By Lemma 25, P'' satisfies (iv) and (vi).

Let $\mathcal{L}_3(P'')$ be obtained by applying Lemma 28 to P'', and let $\tilde{P} \in \mathcal{L}_3(P'')$. Write $\tilde{P} = (\tilde{G}, \tilde{S}, \tilde{X}_0, \tilde{X}, \tilde{Y}_0, \tilde{Y}, \tilde{f})$. By Lemma 28, $\tilde{P}$ satisfies (i)–(vi). Since P'' satisfies (iii), $\tilde{S} = S''$ and $\tilde{Y}_0 = Y''_0$. Define $\tilde{Y}^*_L$ as in (vii), then $\tilde{Y}^*_L = R \cap \tilde{Y}$. Since no vertex of $(X'' \cup Y'') \setminus R$ is mixed on

$(Y'' \cup Y_0'') \cap R$, it follows that no vertex of $(\tilde{X} \cup \tilde{Y}) \setminus \tilde{Y}_L^*$ is mixed on $Y_0'' \cup \tilde{Y}_L^*$, and since $\tilde{P}$ satisfies (vi), we deduce that $\tilde{P}$ satisfies (vii). Now setting

$$\mathcal{L} = \bigcup_{P_1 \in \mathcal{L}_1} \bigcup_{P_2 \in \mathcal{L}_2(P_1)} \mathcal{L}_3(P_2)$$

Lemma 30 follows. □

We are now ready to prove the final lemma of this section, used to prove the following axiom, which we restate:

(viii) With Y_L^* as in (vii), for every component C of $G|(Y_0 \cup Y_L^*)$, there is a vertex v in X complete to C.

Lemma 31. *There is a constant c such that the following holds. Let $P = (G, S, X_0, X, Y_0, Y, f)$ be a seeded precoloring of a P_6-free graph G satisfying* (i), (ii), (iii), (iv), (v), (vi), *and* (vii). *Let L be as in* (vi) *and let Y_L^* as in* (vii). *There is an algorithm with running time $O(|V(G)|^c)$ that outputs an equivalent collection $\mathcal{L}$ of seeded precolorings, such that $|\mathcal{L}| \leq 1$, and if $\mathcal{L} = \{P'\}$, then*

- *there is $Z \subseteq Y_0 \cup Y_L^*$ such that $P' = (G \setminus Z, S, X_0, X, Y_0 \setminus Z, Y \setminus Z, f)$, and*
- *P' satisfies* (i)–(viii).

Proof. We may assume that P does not satisfy (viii) for otherwise we set $\mathcal{L} = \{P\}$. A component C of $G|(Y_0 \cup Y_L^*)$ is *deficient* if no vertex of X is complete to $V(C)$. Let C be a deficient component. It follows from (vii) that X is anticomplete to $V(C)$. Let $A = V(C) \cap Y_0$, $B = V(C) \setminus A$. For every vertex $v \in A \cup B$, let $L(v) = \{1,2,3,4\} \setminus (f(N(v) \cap (S \cup X_0)))$. It follows that $L(v) \subseteq L$ for $v \in B$. Moreover, by (i), it follows that $B \neq \emptyset$. Let $L = \{c_1, c_2, c_3\}$ and let $\{c_4\} = \{1,2,3,4\} \setminus L$.

For every component D of $G|A$, we proceed as follows.

Let $\mathcal{P}(D)$ be the set of lists $L^* \subseteq \{1,2,3,4\}$ with $|L^*| \leq 3$ such that D can be colored with list assignment $L'(x) = L(x) \cap L^*$ for $x \in V(D)$. Since G is P_6-free, it follows from Theorem 2 that $\mathcal{P}(D)$ can be computed in polynomial time. Since C is connected, it follows from (iv) that some vertex of B is complete to D. Consequently, in any precoloring extension of P, at most three colors appear in D, and at least one color of L does not appear in D. Therefore, if $\mathcal{P}(D) = \emptyset$, or if $L \subseteq L'$ for every $L' \in \mathcal{P}(D)$, then P has no precoloring extension we set $\mathcal{L} = \emptyset$ and stop. Let $\mathcal{P}^*(D)$ be the set of $L' \subseteq \{1,2,3,4\}$ such that $L' \notin \mathcal{P}(D)$, but for every proper superset $L'' \subseteq \{1,2,3,4\}$ of L' with $|L''| \leq 3$, we have that $L'' \in \mathcal{P}(D)$. Let $d \in V(D)$. We now replace D by a stable set $R(D) = \{d(L^*)\}_{L^*}$ of copies of d, one for each $L^* \in \mathcal{P}^*(D)$ with $c_4 \in L^*$, and set $L'(d(L^*)) = \{1,2,3,4\} \setminus L^*$. Then $L'(d(L^*)) \subseteq L$. Let C' denote the graph obtained by this process (repeated for every component of $C|Y_0$) from C. Let $L'(v) = L(v)$ for every $v \in V(C) \setminus Y_0$. Since C' is obtained from an induced subgraph of G by replacing vertices with stable sets, it follows that C' is P_6-free.

We claim that C has a proper L-coloring if and only if C' has a proper L'-coloring. Suppose that C has a proper L-coloring c. We need to show that $c|_{V(C) \setminus Y_0}$ can be extended to each $R(D)$. We can consider each D separately.

Let D be a component of $C|Y_0$. Let $L^* = c(D)$. Let $L^{**} = c(N(D))$. We claim that for every $r \in R(D)$, $L'(r) \setminus L^{**} \neq \emptyset$. Suppose $L'(r) \subseteq L^{**}$. Then $\{1,2,3,4\} \setminus L'(r) \in \mathcal{P}^*(D)$, but $L^* \subseteq \{1,2,3,4\} \setminus L^{**} \subseteq \{1,2,3,4\} \setminus L'(r)$, a contradiction. This proves that for every $r \in R(D)$, there exists $d(r) \in L'(r) \setminus L^{**}$, and setting $c(r) = d(r)$ we obtain a coloring of C'.

Next suppose that C' has a proper L'-coloring c. Let $L^* = \{1,2,3,4\} \setminus c(N(D))$. If $L^* \in \mathcal{P}(D)$, then we color D with an L-coloring using only those colors in L^*; this is possible by the definition of $\mathcal{P}(D)$. Thus we may assume that $L^* \notin \mathcal{P}(D)$. Since $L(x) \subseteq L$ for all $x \in N(D) \subseteq B$, it follows that $c_4 \in L^*$. From the definition of $\mathcal{P}^*(D)$, it follows that some superset L^{**} of L^* is in $\mathcal{P}^*(D)$. Then $L'(d'(L^{**}) = \{1,2,3,4\} \setminus L^{**} \subseteq \{1,2,3,4\} \setminus L^* = c(N(D))$. However, $c(d') \in L'(d')$, and thus $c(d) \in c(N(D)) = c(N(d))$, contrary to the fact that c is a proper coloring. This proves that C has a proper L-coloring if and only if C' has a proper L'-coloring.

We have so far proved the following:

- C' has a proper L'-coloring if and only if C has a proper L-coloring;
- C' is P_6-free; and
- for every $x \in V(C')$, we have that $L'(x) \subseteq L$.

By Theorem 2, we can decide in polynomial time if C' has a proper L'-coloring, and thus if C has a proper L-coloring. If not, then P has no precoloring extension; we set $\mathcal{L} = \emptyset$ and stop. If C has a proper L-coloring, then $(G \setminus V(C), S, X_0, X, Y_0 \setminus V(C), Y \setminus V(C), f)$ has a precoloring extension if and only if P does.

By repeatedly applying this algorithm to every deficient component C of $G|(Y_0 \cup Y_L^*)$, and setting $Z = \bigcup V(C)$ where the union is taken over all such components, we set $P' = (G \setminus Z, S, X_0, X, Y_0 \setminus Z, Y \setminus Z, f)$ and output $\mathcal{L} = \{P'\}$. Then P' satisfies (i)-(viii), and Lemma 31 follows. □

We call a seeded precoloring *good* if it satisfies (i), (ii), (iii), (iv), (v), (vi), (vii), and (viii).

By applying Lemmas 20, 21, 27, 28,29, 30 and 31, each to every seeded precoloring in the output of the previous one, we finally derive the main theorem of Section A.

Theorem 17. *There is a constant C such that the following holds. Let G be a P_6-free graph, and let (G, A, f) be a 4-precoloring of G. Then there exists a polynomial-time algorithm that computes a collection $\mathcal{L}$ of seeded precolorings such that*

- *$\mathcal{L}$ is equivalent for P.*
- *for every $(G', S', X'_0, X', Y'_0, Y', f') \in \mathcal{L}$, G' is an induced subgraph of G, $A \subseteq X'_0 \cup S'$ and $f'|_A = f|_A$.*
- *every $P \in \mathcal{L}$ is good*
- *every seeded precoloring in $\mathcal{L}$ has a seed of size at most C;*
- *$|\mathcal{L}| \leq |V(G)|^C$.*

By Theorem 17, to solve the 4-precoloring extension problem in polynomial time, it is sufficient to solve the precoloring extension problem for good seeded precolorings of P_6-free graphs (with seed size bounded by a constant) in polynomial time.

B Establishing the Axioms on Y

In the previous section, we arranged that components of $G|(Y_0 \cup Y)$ containing a vertex of Y_0 are well-behaved. In this section, we deal with components of $G|(Y_0 \cup Y)$ that do not contain a vertex of Y_0.

Let P be a starred precoloring. We say that a collection $\mathcal{L}$ of starred precolorings is an *equivalent collection* for P if P has a precoloring extension if and only if at least one of the starred precolorings in $\mathcal{L}$ does.

The following are the axioms we want to establish for starred precolorings.

(I) Every vertex y in Y satisfies $|L_P(y)| = 3$.

(II) Let $L_1, L_2 \subseteq \{1,2,3,4\}$ with $|L_1| = |L_2| = 3$ and $L_1 \neq L_2$. Then there is no path $a - b - c$ with $L_P(a) = L_1$, $L_P(b) = L_P(c) = L_2$ with $a, b, c \in Y$.

(III) Let $L_1, L_2, L_3 \subseteq \{1,2,3,4\}$ with $|L_1| = |L_2| = |L_3| = 3$ and $L_1 \neq L_2 \neq L_3 \neq L_1$. Then there is no path $a - b - c$ with $L_P(a) = L_1$, $L_P(b) = L_2$, $L_P(c) = L_3$ with $a, b, c \in Y$.

(IV) Let $L_1 \subseteq \{1,2,3,4\}$ with $|L_1| = 3$. Then there is no path $a - b - c$ with $L_P(b) = L_P(c) = L_1$ and $a \in X$, $b, c \in Y$.

(V) Let $L_1, L_2 \subseteq \{1,2,3,4\}$ with $|L_1| = |L_2| = 3$. Then there is no path $a - b - c$ with $L_P(b) = L_1, L_P(c) = L_2$ and $a \in X$ with $L_P(a) \neq L_1 \cap L_2$.

(VI) For every component C of $G|Y$, for which there is a vertex of X is mixed on C, there exist $L_1, L_2 \subseteq \{1,2,3,4\}$ with $|L_1| = |L_2| = 3$ such that C contains a vertex x_i with $L_P(x_i) = L_i$ for $i = 1, 2$, every vertex x in C satisfies $L_P(x) \in \{L_1, L_2\}$, and every $x \in X$ mixed on C satisfies $L_P(x) = L_1 \cap L_2$.

(VII) For every component C of $G|Y$ such that some vertex of X is mixed on C, and for L_1, L_2 as in (VI), $L_P(v) = L_1 \cap L_2$ for every vertex $v \in X$ with a neighbor in C.

(VIII) $Y = \emptyset$.

We begin by showing that starred precolorings exist, and we establish axiom (I).

Lemma 32. *Let P be a good seeded precoloring of a P_6-free graph G. Then*

$$P' = (G, S, X_0, X, Y \setminus Y_L^*, Y_L^* \cup Y_0, f)$$

(with Y_L^* as in (vii)*) is a starred precoloring satisfying* (I) *and P' has a precoloring extension if and only if P does, and every precoloring extension of P' is a precoloring extension of P.*

Proof. This is easily verified by checking the definition of a starred precoloring. ☐

Our next goal is to establish axiom (II), which we restate.

(II) Let $L_1, L_2 \subseteq \{1,2,3,4\}$ with $|L_1| = |L_2| = 3$ and $L_1 \neq L_2$. Then there is no path $a - b - c$ with $L_P(a) = L_1$, $L_P(b) = L_P(c) = L_2$ with $a, b, c \in Y$.

This lemma will also be useful for proving (IV).

Lemma 33. *There is a function $q : \mathbb{N} \to \mathbb{N}$ such that the following holds. Let $L_1 \subseteq \{1,2,3,4\}$ with $|L_1| = 3$, and let $P = (G, S, X_0, X, Y, Y^*, f)$ be a starred precoloring of a P_6-free graph G with P satisfying* (I)*. Then there is an algorithm with running time $O(|V(G)|^{q(|S|)})$ that outputs an equivalent collection $\mathcal{L}$ for P such that*

- $|\mathcal{L}| \leq |V(G)|^{q(|S|)}$*;*
- *every $P' \in \mathcal{L}$ is a starred precoloring of G;*
- *every $P' \in \mathcal{L}$ with seed S' satisfies $|S'| \leq q(|S|)$;*
- *every $P' = (G, S', X_0', X', Y', Y^*, f') \in \mathcal{L}$ satisfies* (I) *and $Y' \subseteq Y$;*

- *if there is no path $a-b-c$ with $L_P(a) \neq L'_1$, $L_P(b) = L_P(c) = L'_1$ with $a, b, c \in Y$ for some L'_1 with $|L'_1| = 3$, then there is no path $a-b-c$ with $L_{P'}(a) \neq L'_1$, $L_{P'}(b) = L_{P'}(c) = L'_1$ with $a, b, c \in Y'$;*

- *if P satisfies* (II)*, and if there is no path $a-b-c$ with $L_P(a) \neq L'_1$, $L_P(b) = L_P(c) = L'_1$ with $a, b, c \in X \cup Y$ for some L'_1 with $|L'_1| = 3$, then there is no path $a-b-c$ with $L_{P'}(a) \neq L'_1$, $L_{P'}(b) = L_{P'}(c) = L'_1$ with $a, b, c \in X' \cup Y'$; and*

- *there is no path $a-b-c$ with $L_{P'}(a) \neq L_1$, $L_{P'}(b) = L_{P'}(c) = L_1$ with $a, b, c \in X' \cup Y'$.*

Moreover, for every $P' \in \mathcal{L}$, given a precoloring extension of P', we can compute a precoloring extension for P in polynomial time, if one exists.

Proof. Let $L_1 \subseteq \{1, 2, 3, 4\}$ with $|L_1| = 3$, and let $P = (G, S, X_0, X, Y, Y^*, f)$ be a starred precoloring of a P_6-free graph G with P satisfying (I). We check in polynomial time if G contains a K_5. If so, then P does not have a precoloring extension and we output $\mathcal{L} = \emptyset$ as an equivalent collection. Therefore, for the remainder of the proof we may assume that G contains no K_5.

Let $\mathcal{L} = \emptyset$. Let $Y_1 = \{y \in Y : L_P(y) = L_1\}$. Let $\mathcal{T} = \{T_1, \dots, T_r\}$ be the set of types $T \subseteq S$ with $f(T) \neq \{1, 2, 3, 4\} \setminus L_1$ and $|f(T)| \leq 2$, and if P satisfies (II), $|f(T)| = 2$. Let $\mathcal{Q}$ be the set of all r-tuples of quadruples $((Q_1, R_1 c_1, d_1), \dots, (Q_r, R_r, c_r, d_r))$ such that for every $i \in \{1, \dots, r\}$,

- $c_i, d_i \in L_1$;
- $1 \geq |Q_i| \geq |R_i|$ and $Q_i \cup R_i$ is a clique; and
- $Q_i \cup R_i \subseteq (X \cup Y)(T_i)$.

For every $Q = ((Q_1, R_1, c_1, d_1), \dots, (Q_r, R_r, c_r, d_r)) \in \mathcal{Q}$, we proceed as follows. Let $S'^Q = Q_1 \cup R_1 \cup \dots \cup Q_r \cup R_r$, and let $f' : S' \to L_1$ be such that $f'(q_i) = c_i$ for all i for which $Q_i = \{q_i\}$, and $f'(r_i) = d_i$ for all i for which $R_i = \{r_i\}$. Let

$$\tilde{Y}^Q = \bigcup_{i: Q_i = \emptyset} (X \cup Y)(T_i),$$

and let $g^Q : \tilde{Y}^Q \to \{1, 2, 3, 4\} \setminus L_1$ be the constant function. Let

$$\tilde{Z}^Q = \bigcup_{i: R_i = \emptyset, Q_i \neq \emptyset} ((X \cup Y)(T_i) \cap N(Q_i)),$$

and let $g'^Q : \tilde{Z}^Q \to \{1, 2, 3, 4\} \setminus L_1$ be the constant function.

For $i \in \{1, \dots, r\}$, let $\tilde{X}_i$ and g''^Q_i be defined as follows. If $|f(T_i)| = 1$, we let $\tilde{X}_i = X(T_i) \cap N(Q_i) \cap N(R_i)$. If $|f(T_i)| = 2$, we let $\tilde{X}_i = X(T_i) \cap N(Q_i)$. We let $g''^Q(\tilde{X}_i) = \{1, 2, 3, 4\} \setminus (f'(T_i) \cup f'(Q_i) \cup f'(R_i))$. Let $\tilde{X}^Q = \tilde{X}_1 \cup \dots \cup \tilde{X}_r$.

Then, if $f \cup f' \cup g^Q \cup g'^Q \cup g''^Q$ is a proper coloring of $G|(S \cup S'^Q \cup X_0 \cup \tilde{Y}^Q \cup \tilde{Z}^Q \cup \tilde{X}^Q)$, we add the starred precoloring

$$\begin{aligned} P'^Q = (&G, S \cup S'^Q, \\ &X_0 \cup \tilde{X}^Q \cup \tilde{Y}^Q \cup \tilde{Z}^Q, \\ &X \setminus (\tilde{X}^Q \cup \tilde{Y}^Q \cup \tilde{Z}^Q \cup S'^Q), \\ &Y \setminus (\tilde{X}^Q \cup \tilde{Y}^Q \cup \tilde{Z}^Q \cup S'^Q), \\ &Y^*, f \cup f' \cup g^Q \cup g'^Q \cup g''^Q) \end{aligned}$$

to $\mathcal{L}$.

This starred precoloring satisfies (I). Every precoloring extension of P'^Q is a precoloring extension of P. Moreover, suppose that c is a precoloring extension of P. Let $Q = ((Q_1, R_1 c_1, d_1), \ldots, (Q_r, R_r, c_r, d_r))$ be defined as follows:

- For every type $T_i \in \mathcal{T}$ such that $c((X \cup Y)(T_i)) \subseteq \{1,2,3,4\} \setminus L_1$, we let $Q_i = R_i = \emptyset$ and $c_i, d_i \in L_1$ arbitrary.
- For every type $T_i \in \mathcal{T}$ such that there exist $x, y \in (X \cup Y)(T_i)$ with $c(x), c(y) \in L_1$ and $xy \in E(G)$, we let $Q_i = \{x\}$, $R_i = \{y\}$ and $c_i = c(x), d_i = c(y)$.
- For every type $T_i \in \mathcal{T}$ such that do not there exist x, y as above, but there is a vertex $v \in (X \cup Y)(T_i)$ with $c(v) \in L_i$, we let $Q_i = \{v\}, R_i = \emptyset, c_i = c(v), d_i = d(v)$.

Note that if $|Q_i \cup R_i| < 2$, then every vertex v in $(X \cup Y)(T_i)$ complete to $Q_i \cup R_i$ satisfies $c(v) \notin L_1$, and so g and g' agree with c on $\tilde{Y}$ and $\tilde{Z}$, respectively. It follows that $P'^Q \in \mathcal{L}$ and c is a precoloring extension of P'^Q. Consequently, that $\mathcal{L}$ is an equivalent collection for P.

We now prove that every $P'^Q \in \mathcal{L}$ satisfies the claims of the lemma. Let $Q = ((Q_1, R_1 c_1, d_1), \ldots, (Q_r, R_r, c_r, d_r))$ with $P'^Q \in \mathcal{L}$, and write $P' = P'^Q \in \mathcal{L}$ with $P' = (G, S', X'_0, X', Y', Y^*, f')$. Let $Y'_1 = \{y \in Y' : L_{P'}(y) = L_1\}$.

(14) *If there is no path $a - b - c$ with $L_P(a) \neq L'_1$, $L_P(b) = L_P(c) = L'_1$ with $a, b, c \in Y$ for some L'_1 with $|L'_1| = 3$, then there is no path $a - b - c$ with $L_{P'}(a) \neq L'_1$, $L_{P'}(b) = L_{P'}(c) = L'_1$ with $a, b, c \in Y'$; and if P satisfies* (II), *and if there is no path $a - b - c$ with $L_P(a) \neq L'_1$, $L_P(b) = L_P(c) = L'_1$ with $a, b, c \in X \cup Y$ for some L'_1 with $|L'_1| = 3$, then there is no path $a - b - c$ with $L_{P'}(a) \neq L'_1$, $L_{P'}(b) = L_{P'}(c) = L'_1$ with $a, b, c \in X' \cup Y'$.*

Suppose not; and let $a - b - c$ be such a path. Since $b, c \in Y' \subseteq Y$, it follows that $L_P(b) = L_P(c) = L'_1$. By the assumption of (14), it follows that $L_P(a) \neq L_{P'}(a)$, and so $a \in Y \cap X'$. This implies that $|L_{P'}(a)| = 2$. Since $a \notin Y'$, it follows that the first statement of (14) is proved.

Therefore, we may assume that (II) holds for P. Since P satisfies (II), it follows that $L_P(a) = L'_1$. Moreover, there is a vertex $s \in S' \setminus S$ with $f(s) \in L'_1$ and $as \in E(G)$. Since $b \in Y'$, it follows that $s - a - b$ is a path. But since P satisfies (II), it follows that $S' \setminus S \subseteq X$ by construction, and so $s \in X$. But then the path $s - a - b$ contradicts the assumption of (18). This implies (18).

(15) *There is no path $z - a - b - c$ with $z \in (X' \cup Y') \setminus Y'_1$ and $a, b, c \in Y'_1$.*

Suppose not; and let $z - a - b - c$ as in (15). It follows that $z \in X \cup Y$ and $a, b, c \in Y_1$. Let $T_i = N(z) \cap S \in \mathcal{T}$. Since $z \notin X'_0$, it follows that $z \notin \tilde{X}^Q \cup \tilde{Y}^Q \cup \tilde{Z}^Q$. Therefore, $Q_i \cup R_i$ contains a vertex y non-adjacent to z. Since $c_i, d_i \in L_1$, it follows that y is anticomplete to $\{z, a, b, c\}$. Let $s \in T_i$ with $f(s) \in L_1$; then s is a common neighbor of y and z. It follows that s is not adjacent to a, b, c. But then $y - s - z - a - b - c$ is a P_6 in G, a contradiction. This proves (15).

Let $\mathcal{L}_5 = \mathcal{L}$. We repeat the following procedure for $j = 4, 3, 2$. For every $P' = (G, S', X'_0, X', Y', Y^*, f') \in \mathcal{L}_{j+1}$, we proceed as follows. We let $\mathcal{L}_j(P') = \emptyset$. Let $Y'_1 = \{y \in Y' : L_{P'}(y) = L_1\}$. Let Y_1^* be the set of vertices y in $(X' \cup Y') \setminus Y'_1$ such that there is a clique $\{a_1, \ldots, a_j\} \subseteq Y'_1$ and $N(y) \cap \{a_1, \ldots, a_j\} = \{a_1\}$. Let $\mathcal{T}^j = \left\{T_1^j, \ldots, T_{r_j}^j\right\}$ be the set of all types $T \subseteq S'$ such that $f(T) \neq \{1,2,3,4\} \setminus L_1$ and $|f(T)| \leq 2$, and if P' satisfies (II), $|f(T)| = 2$. Let $\mathcal{Q}(P')$ be the set of all r_j-tuples of quadruples $((Q_1, R_1 c_1, d_1), \ldots, (Q_r, R_r, c_r, d_r))$ such that for every $i \in \{1, \ldots, r_j\}$,

- $c_i, d_i \in L_1$;
- $1 \geq |Q_i| \geq |R_i|$ and $Q_i \cup R_i$ is a clique; and

- $Q_i \cup R_i \subseteq (X \cup Y)(T_i)$.

For every $Q = ((Q_1, R_1 c_1, d_1), \dots, (Q_r, R_r, c_r, d_r)) \in \mathcal{Q}$, we proceed as follows. Let $S'^Q = Q_1 \cup R_1 \cup \cdots \cup Q_r \cup R_r$, and let $g^Q : S' \to L_1$ such that $g^Q(q_i) = c_i$ for all i such that $Q_i = \{q_i\}$, and $g^Q(r_i) = d_i$ for all i such that $R_i = \{r_i\}$.

For $i \in \{1, \dots, r_j\}$, we let Z_i be the set of vertices $z \in (X \cup Y)(T_i)$ such that one of the following holds:

- $Q_i = \emptyset$;
- $Q_i = \{q_i\}$, and $N(q_i) \cap Y_1' \subsetneq N(z) \cap Y_2'$;
- $Q_i = \{q_i\}$, $R_i = \{r_i\}$, z is adjacent to q_i and $N(r_i) \cap Y_2' \subsetneq N(z) \cap Y_1'$;

We let $\tilde{Z}^Q = Z_1 \cup \cdots \cup Z_{r_j}$ and $g'^Q : \tilde{Z}^Q \to \{1, 2, 3, 4\} \setminus L_1$. Let

$$\tilde{X}^Q = \bigcup_{i : R_i = \emptyset, Q_i \neq \emptyset} ((X \cup Y)(T_i) \cap N(S_i)),$$

and let $g''^Q : \tilde{X}^Q \to \{1, 2, 3, 4\} \setminus L_1$ be the constant function. Let

$$\begin{aligned} P'^Q = &(G, S' \cup S^Q, X_0' \cup \tilde{Z}^Q \cup \tilde{X}^Q, \\ &X' \setminus (S^Q \cup \tilde{Z}^Q \cup \tilde{X}^Q), \\ &Y' \setminus (S^Q \cup \tilde{Z}^Q \cup \tilde{X}^Q), Y^*, \\ &f' \cup g^Q \cup g'^Q \cup g''^Q). \end{aligned}$$

If $f' \cup g^Q \cup g'^Q \cup g''^Q$ is proper coloring of $G|(S' \cup S^Q \cup \tilde{Z}^Q \cup \tilde{X}^Q)$, then we add P'^Q to $\mathcal{L}_j(P')$.

It follows that for every $Q \in \mathcal{Q}(P')$, every precoloring extension of P'^Q is a precoloring extension of P'. Moreover, suppose that c is a precoloring extension of P'. We define $Q = ((Q_1, R_1 c_1, d_1), \dots, (Q_r, R_r, c_r, d_r))$ as follows:

- For every type $T_i \in \mathcal{T}$ such that $c((X \cup Y)(T_i)) \cap L_1 = \emptyset$, we let $Q_i = R_i = \emptyset$ and $c_i, d_i \in L_1$ arbitrary.
- For every type $T_i \in \mathcal{T}$ such that $c((X \cup Y)(T_i)) \cap L_1 \neq \emptyset$, we let v a vertex $v \in (X \cup Y)(T_i)$ with $c(v) \in L_1$ with $N(v) \cap Y_1$ maximal. We let $Q_i = \{v\}, c_i = c(v)$. If there is a vertex w in $N(v) \cap (X \cup Y)(T_i)$ with $c(w) \in L_1$, then we choose such a vertex with $N(w) \cap Y_1$ maximal and let $R_i = \{w\}, d_i = c(w)$; otherwise we let $R_i = \emptyset$ and $d_i \in L_1$ arbitrary.

The second bullet implies that $c(x) \notin L_1$ for every $x \in (X \cup Y)(T_i)$ such that $Q_i = \{q_i\}$ and $N(q_i) \cap Y_1' \subsetneq N(v) \cap Y_1'$. Similarly, $c(x) \notin L_1$ for every $x \in (X \cup Y)(T_i) \cap N(Q_i)$ such that $R_i = \{r_i\}$ and $N(r_i) \cap Y_1' \subsetneq N(v) \cap Y_1'$. It follows that $Q \in \mathcal{Q}(P')$, $P'^Q \in \mathcal{L}_j(P')$, and c is a precoloring extension of P'^Q. Thus $\mathcal{L}_j(P')$ is an equivalent collection for P'. By construction, P'^Q satisfies (I) for every $Q \in \mathcal{Q}(P')$.

Now let

$$\mathcal{L}_j = \bigcup_{P' \in \mathcal{L}_{j+1}} \mathcal{L}_j(P').$$

Since $\mathcal{L}_{j+1}$ is an equivalent collection for P and since $\mathcal{L}_j$ is the union of equivalent collections for every $P' \in \mathcal{L}_{j+1}$, it follows that $\mathcal{L}_j$ is an equivalent collection for P.

Let $P' \in \mathcal{L}_{j+1}$. Let $Q = ((Q_1, R_1 c_1, d_1), \dots, (Q_r, R_r, c_r, d_r)) \in \mathcal{Q}(P')$, and let $P'^Q = (G, S'', X_0'', X'', Y'', Y^*, f'') \in \mathcal{L}_j(P')$. Let $Y_1'' = \{y \in Y'' : L_{P'^Q}(y) = L_1\}$. From the previous step $(j+1)$ of our argument, we may assume that (16) and (15) hold for $j+1$ for P' and Y_1'. This is true when $j = 4$ as well, since G contains no K_5.

(16) *There is no vertex $z \in (X'' \cup Y'') \setminus Y_1''$ with $N(z) \cap \{a_1, \dots, a_j\} = \{a_1\}$ for a clique $\{a_1, \dots, a_j\} \subseteq Y_1''$.*

Suppose for a contradiction that z is such a vertex. Write $P' = (G, S', X_0', X', Y', Y^*, f')$. Let $Y_1' = \{y \in Y' : L_{P'}(y) = L_1\}$ for $i = 1, 2$. Suppose first that $z \in Y_1'$. Then z has a neighbor $s \in S'' \setminus S'$. It follows that $f''(s) \in L_1$ and $s \notin Y_1'$. Consequently, s is anticomplete to $\{a_1, \dots, a_j\}$. But then the path $s - z - a_1 - a_j$ contradicts the fact that (15) holds for P'.

It follows that $z \in (X' \cup Y') \setminus Y_1'$ and $\{a_1, \dots, a_j\} \subseteq Y_1'$. Let i such that $S' \cap N(z) = T_i$. Since $z \notin X_0''$, it follows $Q_i \neq \emptyset$; say $Q_i = \{q_i\}$. If z is non-adjacent to q_i, let $s = q_i$. Otherwise, it follows that $R_i = \{r_i\}$, say; let $s = r_i$. In both cases, it follows that s is non-adjacent to z.

Since $a_1, \dots, a_j \notin X''$, it follows that s is non-adjacent to $a_1, \dots, a_j$. The definition of Z_i implies that $N(s) \cap Y_2' \not\subseteq N(z) \cap Y_2'$. Since $a_1 \in (N(z) \setminus N(s)) \cap Y_1'$, we deduce that there exists a vertex $y \in (N(z) \setminus N(s)) \cap Y_1'$.

Let $s' \in T_i$ with $f(s') \in L_1$. Then, s' is non-adjacent to $a_1, \dots, a_j$. But $y - s - s' - z - a_1 - a_j$ is not a P_6 in G, and thus y has a neighbor in $\{a_1, \dots, a_j\}$. But y is not complete to $\{a_1, \dots, a_j\}$, since P' satisfies (16) for $j+1$. It follows that y is mixed on $\{a_1, \dots, a_j\}$, and thus by Lemma 19 there is a path $y - a - b$ with $a, b \in \{a_1, \dots, a_j\}$. But then $s - y - a - b$ is a path, contrary to the fact that P' satisfies (15). This concludes the proof of (16).

(17) *There is no path $z - a - b - c$ with $z \in (X'' \cup Y'') \setminus Y_1'$ and $a, b, c \in Y_1''$.*

Suppose not; and let $z - a - b - c$ be such a path. Since $Y_1'' \subseteq Y_1'$, the fact that P' satisfies (15) implies that $z \notin X' \cup Y'$, and thus $z \in Y_1'$. Thus z has a neighbor $s \in S'' \setminus S'$ with $f(s) \in L_1$. It follows that $s \in X' \cup Y'$, and thus $s - z - a - b$ is a path, contrary to the fact that (15) holds for P'. This proves (17).

(18) *If there is no path $a - b - c$ with $L_{P'}(a) \neq L_1'$, $L_{P'}(b) = L_{P'}(c) = L_1'$ with $a, b, c \in Y'$ for some L_1' with $|L_1'| = 3$, then there is no path $a - b - c$ with $L_{P''}(a) \neq L_1'$, $L_{P''}(b) = L_{P''}(c) = L_1'$ with $a, b, c \in Y''$; and if P' satisfies* (II)*, and if there is no path $a - b - c$ with $L_{P'}(a) \neq L_1'$, $L_{P'}(b) = L_{P'}(c) = L_1'$ with $a, b, c \in X \cup Y$ for some L_1' with $|L_1'| = 3$, then there is no path $a - b - c$ with $L_{P'}(a) \neq L_1'$, $L_{P''}(b) = L_{P''}(c) = L_1'$ with $a, b, c \in X'' \cup Y''$.*

Suppose not; and let $a - b - c$ be such a path. Since $b, c \in Y'' \subseteq Y'$, it follows that $L_{P'}(b) = L_{P'}(c) = L_1'$. By the assumption of (18), it follows that $L_{P'}(a) \neq L_{P''}(a)$, and so $a \in Y' \cap X''$. This implies that $|L_{P''}(a)| = 2$. Since $a \notin Y''$, it follows that the first statement of (18) is proved.

Therefore, we may assume that (II) holds for P'. Since P' satisfies (II), it follows that $L_{P'}(a) = L_1'$. Moreover, there is a vertex $s \in S'' \setminus S'$ with $f'(s) \in L_1'$ and $as \in E(G)$. Since $b \in Y''$, it follows that $s - a - b$ is a path. But since P' satisfies (II), it follows that $S'' \setminus S' \subseteq X'$ by construction, and so $s \in X'$. But then the path $s - a - b$ contradicts the assumption of (18). This implies (18).

It follows that (15) and (18) holds for P'^Q for every $P' \in \mathcal{L}_{j+1}$ and $Q \in \mathcal{Q}(P')$. Moreover, by construction, $\mathcal{L}_j$ is an equivalent collection for P. If $j > 2$, we repeat the procedure for $j - 1$; otherwise, we stop.

At termination, we have constructed an equivalent collection $\mathcal{L}_2$ for P and every $P' = (G, S', X_0', X', Y', Y_0', f') \in \mathcal{L}_2$ satisfies (I) and (16) for $j = 2$, and thus the last bullet of the lemma. The third-to-last and second-to-last bullets of the lemma follow from (14) and (18). Thus, $\mathcal{L}_2$ satisfies the properties of the lemma, and hence, the lemma is proved. □

Lemma 34. *There is a function $q : \mathbb{N} \to \mathbb{N}$ such that the following holds. Let $P = (G, S, X_0, X, Y, Y^*, f)$ be a starred precoloring of a P_6-free graph G with P satisfying* (I). *Then there is an algorithm with running time $O(|V(G)|^{q(|S|)})$ that outputs an equivalent collection $\mathcal{L}$ for P such that*

- $|\mathcal{L}| \leq |V(G)|^{q(|S|)}$*;*
- *every $P' \in \mathcal{L}$ is a starred precoloring of G;*
- *every $P' \in \mathcal{L}$ with seed S' satisfies $|S'| \leq q(|S|)$; and*
- *every $P' \in \mathcal{L}$ satisfies* (I) *and* (II).

Moreover, for every $P' \in \mathcal{L}$, given a precoloring extension of P', we can compute a precoloring extension for P in polynomial time, if one exists.

Proof. Let $\mathcal{L} = \{P\}$. We repeat the following for every pair L_1, L_2 of distinct lists of size three contained in $\{1, 2, 3, 4\}$. We apply Lemma 33 to every starred precoloring $P' \in \mathcal{L}$, and replace $\mathcal{L}$ by the union of the equivalent collections produced by Lemma 33. Then we move to the next pair of lists. □

The next lemma is a simple tool that we will use to establish further axioms.

Lemma 35. *Let G be a P_6-free graph with $u, v \in V(G)$ such that $V(G) = \{u, v\} \cup N(u) \cup N(v)$, $uv \notin E(G)$, $N(u) \cap N(v) = \emptyset$, and $N(u), N(v)$ stable. Then there is a partition $A_0, A_1, \ldots, A_k$ of $N(u)$ and a partition $B_0, B_1, \ldots, B_k$ of $N(v)$ with $k \geq 0$ such that*

- *A_0 is complete to $N(v)$;*
- *B_0 is complete to $N(u)$; and*
- *for $i = 1, \ldots, k$, $A_i, B_i \neq \emptyset$ and A_i is complete to $N(v) \setminus B_i$ and B_i is complete to $N(u) \setminus A_i$, and A_i is anticomplete to B_i.*

Proof. Let G, u, v as in the lemma. The result holds if $N(u) = \emptyset$ or $N(v) = \emptyset$; thus we may assume that both sets are non-empty. Let $a \in N(u), b \in N(v)$. If $ab \in E(G)$, we let $A_0 = \{a\}, B_0 = \{b\}$; otherwise, we let $A_1 = \{a\}, B_1 = \{b\}$. Now let $A_0, A_1, \ldots, A_k, B_0, B_1, \ldots, B_k$ be chosen such that their union is maximal subject to satisfying the conditions of the lemma. If their union is $V(G) \setminus \{u, v\}$, then there is nothing to show; thus we may assume that there is a vertex $x \notin \{u, v\}$ not contained in their union. Without loss of generality, we may assume that $x \in N(v)$.

If x is complete to $A = A_0 \cup A_1 \cup \cdots \cup A_k$, we can add x to B_0, contrary to the maximality of our choice of sets. Suppose first that x is complete to $A_1 \cup \cdots \cup A_k$. Let $A_{k+1} = A_0 \setminus N(x)$. Then A_{k+1} is non-empty, since x has a non-neighbor in A. But then $A_0 \setminus A_{k+1}, A_1, \ldots, A_k, A_{k+1}, B_0, B_1, \ldots, B_k, \{x\}$ satisfies the conditions of the lemma and has strictly larger union; a contradiction.

It follows that x has a non-neighbor in $A \setminus A_0$; without loss of generality we may assume that there is $y \in A_1$ non-adjacent to x. Let $w \in B_1$. Suppose that x has a neighbor $z \in A_1$. Then $w - v - x - z - u - y$ is a P_6 in G, a contradiction. It follows that x has no neighbor in A_1. If x is complete to $A \setminus A_1$, we can add x to B_1 and enlarge the structure, a contradiction; hence x has a non-neighbor z in $A \setminus A_1$. It follows that z is adjacent to w. But then $x - v - w - z - u - y$ is a P_6 in G, a contradiction. This concludes the proof of the lemma. □

The purpose of the following lemmas is to establish the following axiom, which we restate:

(III) Let $L_1, L_2, L_3 \subseteq \{1,2,3,4\}$ with $|L_1| = |L_2| = |L_3| = 3$ and $L_1 \neq L_2 \neq L_3 \neq L_1$. Then there is no path $a - b - c$ with $L_P(a) = L_1$, $L_P(b) = L_2$, $L_P(c) = L_3$ with $a, b, c \in Y$.

Lemma 36. *There is a function $q : \mathbb{N} \to \mathbb{N}$ such that the following holds. Let $L_1, L_2, L_3 \subseteq \{1,2,3,4\}$ with $|L_1| = |L_2| = |L_3| = 3$ and $L_1 \neq L_2 \neq L_3 \neq L_1$. Let $P = (G, S, X_0, X, Y, Y^*, f)$ be a starred precoloring of a P_6-free graph G with P satisfying* (I) *and* (II)*. Then there is an algorithm with running time $O(|V(G)|^{q(|S|)})$ that outputs an equivalent collection $\mathcal{L}$ for P such that*

- *$|\mathcal{L}| \leq |V(G)|^{q(|S|)}$;*
- *every $P' \in \mathcal{L}$ is a starred precoloring of G;*
- *every $P' \in \mathcal{L}$ with seed S' satisfies $|S'| \leq q(|S|)$;*
- *every $P' \in \mathcal{L}$ satisfies* (I) *and* (II)*;*
- *every $P' = (G, S', X'_0, X', Y', Y^*, f') \in \mathcal{L}$ satisfies that there is no path $a - b - c - d$ with $L_{P'}(a) = L_1$, $L_{P'}(b) = L_{P'}(d) = L_2$, $L_{P'}(c) = L_3$ with $a, b, c, d \in Y'$; and*
- *if P satisfies the previous bullet for L_1, L_2, L_3 and for L_3, L_2, L_1, then every $P' = (G, S', X'_0, X', Y', Y^*, f') \in \mathcal{L}$ satisfies that there is no path $a - b - c$ with $L_{P'}(a) = L_1$, $L_{P'}(b) = L_2$, $L_{P'}(c) = L_3$ with $a, b, c \in Y'$.*

Moreover, for every $P' \in \mathcal{L}$, given a precoloring extension of P', we can compute a precoloring extension for P in polynomial time.

Proof. We say that *the conditions of the last bullet hold for P* if P satisfies the second-to-last bullet for L_1, L_2, L_3 and L_3, L_2, L_1.

Let $Y_i = \{y \in Y : L_P(y) = L_i\}$ for $i = 1, 2, 3$. Let $\mathcal{T} = \{T_1, \ldots, T_r\}$ be the set of types $T \subseteq S$ with $f(T) = \{1,2,3,4\} \setminus L_1$. We let $\mathcal{Q}$ be the set of all r-tuples $(Q_1, \ldots, Q_r)$, where for each i, $Q_i = (S_i^1, S_i^2, R_i^1, R_i^2, c_i^1, c_i^2, c_i^3, c_i^4)$ such that the following hold:

1. $\{c_i^1, c_i^2\} \subseteq \{1,2,3,4\}$.
2. $1 \geq |S_i^1| \geq |S_i^2| \geq |R_i^1| \geq |R_i^2|$.
3. $S_i^1 \neq \emptyset$ if and only if one of the following holds:
    - there is a path $a - b - c - d$ with $a \in Y_1, b, d \in Y_2, c \in Y_3$ and $N(a) \cap S = T_i$; or
    - the conditions of the last bullet hold for P and there is a path $a - b - c$ with $a \in Y_1, b \in Y_2, c \in Y_3$ and $N(a) \cap S = T_i$.
4. $S_i^1 \cup S_i^2$ is a stable set, and $S_i^1 \cup S_i^2 \subseteq Y_1(T_i)$.
5. If $S_i^1 = \{s_i^1\}$, then s_i^1 has a neighbor in Y_2.
6. If $S_i^2 = \{s_i^2\}$, then s_i^2 has a neighbor in Y_2.
7. If $S_i^2 \neq \emptyset$, then $\{c_i^1, c_i^2\} = L_1 \setminus (L_2 \cap L_3)$ and $c_i^1 \in L_3, c_i^2 \in L_2$.
8. $R_i^1 \subseteq (N(S_i^1) \setminus N(S_i^2)) \cap Y_2$.
9. $R_i^2 \subseteq (N(S_i^2) \setminus N(S_i^1)) \cap Y_3$.
10. $R_i^1 \cup R_i^2$ is a stable set.

11. $\{c_i^3, c_i^4\} \subseteq L_2 \cap L_3$.

We let $S'^Q = \bigcup_{i=1}^r (S_i^1 \cup S_i^2)$ and $T'^Q = \bigcup_{i=1}^r (R_i^1 \cup R_i^2)$. Define $f'^Q : S'^Q \cup T'^Q \to \{1, 2, 3, 4\}$ by setting $f'^Q(v) = c_i^j$ if $S_i^j = \{v\}$ for $j = 1, 2$ and $f'^Q(v) = c_i^{j+2}$ if $R_i^j = \{v\}$ for $j = 1, 2$. Let S_1' be the set of $v \in (T'^Q \cup S'^Q)$ such that $f'^Q(v) \in L_2 \cap L_3$. Let S_2' be the set of $v \in (T'^Q \cup S'^Q)$ such that $f'^Q(v) \in L_2 \setminus L_3$, and let S_3' be the set of $v \in (T'^Q \cup S'^Q)$ such that $f'^Q(v) \in L_3 \cap L_2$. Let

$$\tilde{X}^Q = (N(S_1') \cap (Y_1 \cup Y_2 \cup Y_3)) \cup (N(S_2') \cap (Y_1 \cup Y_2)) \cup (N(S_3') \cap (Y_1 \cup Y_2)) \cup (N(T'^Q) \cap (Y_2 \cup Y_3)).$$

For $i \in \{1, \ldots, r\}$, we further define $\tilde{Z}_i = \emptyset$ if $|S_i^1 \cup S_i^2| < 2$ or $|R_i^1| > 0$, and $\tilde{Z}_i = (N(S_i^1) \setminus N(S_i^2)) \cap Y_2$ otherwise. We let $\tilde{Z}^Q = \bigcup_{i=1}^r \tilde{Z}_i$. Let $g^Q : \tilde{Z} \to L_2 \setminus L_3$ be the constant function. For $i \in \{1, \ldots, r\}$, we let $\tilde{Y}_i = \emptyset$ if $|S_i^1 \cup S_i^2| < 2$ or $|R_i^1 \cup R_i^2| \neq 1$, and $\tilde{Y}_i = (N(S_i^2) \setminus (N(S_i^1) \cup N(R_i^1))) \cap Y_3$ otherwise. We let $\tilde{Y}^Q = \bigcup_{i=1}^r \tilde{Y}_i$. Let $g'^Q : \tilde{Y} \to L_3 \setminus L_2$ be the constant function. For $i \in \{1, \ldots, r\}$, we let $\tilde{W}_i = \emptyset$ if $|S_i^1 \cup S_i^2| \neq 1$ or $c_i^1 \in L_1 \cap L_2 \cap L_3$, and $\tilde{W}_i = Y_1(T_i) \setminus S_i^1$ otherwise. We let $\tilde{W}^Q = \bigcup_{i=1}^r \tilde{W}_i$. We define $g''^Q : \tilde{W} \to L_1$ by setting $g''(\tilde{W}_i \setminus N(S_i^1)) = \{c_i^1\}$ and $g''(\tilde{W}_i \cap N(S_i^1)) = L_1 \setminus (\{c_i^1\} \cup (L_2 \cap L_3)$.

Let P'^Q be the starred precoloring

$$(G, S \cup S'^Q \cup T'^Q, X_0 \cup \tilde{W}^Q \cup \tilde{Y}^Q \cup \tilde{Z}^Q, X \cup \tilde{X}^Q, Y \setminus (S'^Q \cup T'^Q \cup \tilde{W}^Q \cup \tilde{X}^Q \cup \tilde{Y}^Q \cup \tilde{Z}^Q), Y^*, f \cup f'^Q \cup g^Q \cup g'^Q \cup g''^Q).$$

Since P satisfies (II), it follows that P' satisfies (II) as well. We let $\mathcal{L} = \{P'^Q : Q \in \mathcal{Q}, f \cup f'^Q \cup g^Q \cup g'^Q \cup g''^Q$ is a

(19) *$\mathcal{L}$ is an equivalent collection for P.*

Let $L_1 = \{c^1, c^2, c^3\}, L_2 = \{c^1, c^2, c^4\}$ and $L_3 = \{c^1, c^3, c^4\}$. Let Y_1^* denote the set of vertices in Y_1 with a neighbor in Y_2. Every precoloring extension for $P'^Q \in \mathcal{L}$ is a precoloring extension for P. Now suppose that P has a precoloring extension $c : V(G) \to \{1, 2, 3, 4\}$. We define an r-tuple $(Q_1, \ldots, Q_r)$, where for each i, $Q_i = (S_i^1, S_i^2, R_i^1, R_i^2, c_i^1, c_i^2, c_i^3, c_i^4)$. For $i \in \{1, \ldots, r\}$, we define $Q_i = (S_i^1, S_i^2, R_i^1, R_i^2, c_i^1, c_i^2, c_i^3, c_i^4)$ as follows:

- If neither bullet of 3 is satisfied, we let $Q_i = (\emptyset, \emptyset, \emptyset, \emptyset, c^1, c^1, c^1, c^1)$.
- If $Y_1^*(T_i)$ contains a vertex v with $c(v) = c^1$, we let $Q_i = (\{v\}, \emptyset, \emptyset, \emptyset, c^1, c^1, c^1, c^1)$.
- If $Y_1^*(T_i)$ contains a vertex v with $c(v) = c^2$ such that $c(Y_1^*(T_i) \setminus N(v)) \subseteq \{c^3\}$, we let $Q_i = (\{v\}, \emptyset, \emptyset, \emptyset, c^2, c^1, c^1, c^1)$.
- If $Y_1^*(T_i)$ contains a vertex v with $c(v) = c^3$ such that $c(Y_1^*(T_i) \setminus N(v)) \subseteq \{c^2\}$, we let $Q_i = (\{v\}, \emptyset, \emptyset, \emptyset, c^3, c^1, c^1, c^1)$.
- Let $u, v \in Y_1^*(T_i)$ such that $c(u) = c^2, c(v) = c^3$ and $uv \notin E(G)$. We let $A = (N(u) \setminus N(v)) \cap Y_2$ and $B = (N(v) \setminus N(u)) \cap Y_3$. We proceed as follows:
  - If $c(A) \subseteq L_2 \setminus L_3$, we let $Q_i = (\{u\}, \{v\}, \emptyset, \emptyset, c^2, c^3, c^1, c^1)$.
  - If there is a vertex $x \in A$ such that $c(x) \in L_2 \cap L_3$ and $c(B \setminus N(x)) \subseteq L_3 \setminus L_2$, we let $Q_i = (\{u\}, \{v\}, \{x\}, \emptyset, c^2, c^3, c(x), c^1)$.
  - If there is $x \in A$ and $y \in B$ such that $c(x), c(y) \in L_2 \cap L_3$ and $xy \notin E(G)$, we let $Q_i = (\{u\}, \{v\}, \{x\}, \{y\}, c^2, c^3, c(x), c(y))$.

It follows from the definitions of $\tilde{Y}^Q, \tilde{Z}^Q, \tilde{W}^Q$ that $c|_{(\tilde{Y}^Q \cup \tilde{Z}^Q \cup \tilde{W}^Q)} = g^Q|_{\tilde{Z}^Q} \cup g'^Q|_{\tilde{Y}^Q} \cup g''^Q|_{\tilde{W}^Q}$. It follows that $Q \in \mathcal{Q}$ and c is a precoloring extension of P'^Q. Thus $\mathcal{L}$ is an equivalent collection for P, which proves (19).

Let $Q \in \mathcal{Q}$ and let $P'^Q \in \mathcal{L}$ with $P'^Q = (G, S', X'_0, X', Y', Y^*, f')$, and let $Y'_i = \{y \in Y' : L_{P'}(y) = L_i\}$ for $i = 1, 2, 3$. We claim the following.

(20) *For every $i \in \{1, \ldots, r\}$ such that $S_i^1 = \{u\}, S_i^2 = \{v\}$, we have that $N(u) \cap (Y'_2 \cup Y'_3)$ is anticomplete to $N(v) \cap (Y'_2 \cup Y'_3)$.*

From the properties of Q, we know that $f'(u) \in L_1 \cap L_3$ and $f'(v) \in L_1 \cap L_2$. Since $u, v \in S'$, it follows that $N(u) \cap Y'_3 = \emptyset$, since $N(u) \cap Y_3 \subseteq \tilde{X}^Q$; similarly, $N(v) \cap Y'_2 = \emptyset$. We let $A = (N(u) \setminus N(v)) \cap Y_2$ and $B = (N(v) \setminus N(u)) \cap Y_3$. It follows that v is anticomplete to A and u is anticomplete to B. Let $a_1, \ldots, a_t$ be the components of $G|A$, and let $b_1, \ldots, b_s$ be the components of $G|B$. Since P satisfies (II), it follows that for every $i \in [t]$ and $j \in [s]$, $V(a_i)$ is either complete or anticomplete to $V(b_j)$.

Let H be the graph with vertex set $\{u, v\} \cup \{a_1, \ldots, a_t\} \cup \{b_1, \ldots b_s\}$; where $N_H(u) = \{a_1, \ldots, a_t\}$, $N_H(v) = \{b_1, \ldots, b_s\}$, the sets $\{a_1, \ldots, a_t\}$ and $\{b_1, \ldots, b_s\}$ are stable, and a_i is adjacent to b_j if and only if $V(a_i)$ is complete to $V(b_j)$ in G. Apply 35 to H, u and v to obtain a partition $A'_0, A'_1, \ldots, A'_k$ of $\{a_1, \ldots, a_t\}$ and a partition $B'_0, B'_1, \ldots, B'_k$ of $\{b_1, \ldots, b_t\}$. For $i \in [k]$, let $A_i = \bigcup_{a_j \in A_i} V(a_j)$ and $B_i = \bigcup_{b_j \in B_i} V(b_j)$.

It follows from the definition of H that in G,

- A_0 is complete to B;
- B_0 is complete to A; and
- for $j = 1, \ldots, k$, $A_j, B_j \neq \emptyset$ and A_j is complete to $B \setminus B_j$ and B_j is complete to $A \setminus A_j$, and A_j is anticomplete to B_j.

If $R_i^1 = \emptyset$, then $A \subseteq \tilde{Z}^Q$, and so $A \cap Y' = \emptyset$, and (20) follows. Thus $R_i^1 \neq \emptyset$. Suppose that $R_i^2 = \emptyset$. Then one of the following holds:

- $R_i^1 \subseteq A_0$, and so $B \subseteq \tilde{X}^Q$; or
- $R_i^1 \subseteq A_j$ for some $j > 0$, and so $B \setminus B_j \subseteq \tilde{X}^Q$ and $B_j \subseteq \tilde{Y}^Q$.

It follows that $N(v) \cap Y'_2 = \emptyset$, and (20) follows. Thus we may assume that $R_i^2 \neq \emptyset$, then there exists a $j > 0$ such that $R_i^1 \subseteq A_j$ and $R_i^2 \subseteq B_j$, and so $(A \setminus A_j) \cup (B \setminus B_j) \subseteq \tilde{X}^Q$, and again, (20) holds.

(21) *There is no path $z - a - b - c$ with $z \in Y'_1$, $a, c \in Y'_2$ and $b \in Y'_3$.*

Suppose that $z - a - b - c$ is such a path. Let $i \in \{1, \ldots, r\}$ such that $N(z) \cap S = T_i$. Since $z \notin X'_0$, it follows that $S_i^1 \neq \emptyset$. Write $S_i^1 = \{u\}$. Let $s \in T_i$; then $f'(s) \in L_2 \cup L_3$, and therefore s is anticomplete to $\{a, b, c\}$.

Suppose that $S_i^2 = \emptyset$. Then $f'(u) \in L_2 \cap L_3$, and thus u is non-adjacent to z, a, b, c. Now $u - s - z - a - b - c$ is a P_6 in G, a contradiction. Thus it follows that $S_i^2 = \{v\}$, and z is non-adjacent to u and v. By construction, it follows that $f'(u) \in L_2 \setminus L_3$, and $f'(v) \in L_3 \setminus L_2$. Since neither $u - s - z - a - b - c$ nor $v - s - z - a - b - c$ is a P_6 in G, it follows that u, v each have a neighbor in $\{a, b, c\}$. Since neighbors of u in Y_2 are in $\tilde{X}^Q$, it follows that u is non-adjacent to a and c, and hence u is adjacent to b. Since neighbors of v in Y_3 are in $\tilde{X}^Q$, it follows that v is non-adjacent to b, and v is adjacent to a or c. This contradicts (20), and thus (21) follows.

(22) *If the conditions of the last bullet hold for P, then there is no path $z - a - b$ with $z \in Y'_1$, $a \in Y'_2$ and $b \in Y'_3$.*

Suppose not, and let $z - a - b$ be such a path. Let $i \in \{1, \ldots, r\}$ such that $N(z) \cap S = T_i$. Let $s \in T_i$. Then $f'(s) \in L_2 \cap L_3$, since $f'(s) \notin L_1$, and hence s is anticomplete to a, b. Since $z \notin X'_0$, it follows that $S_i^1 \neq \emptyset$, say $S_i^1 = \{u\}$. Suppose first that $S_i^2 = \emptyset$. Since $z \notin X'_0$, it follows that

$f'(u) \in L_2 \cap L_3$, and thus u is non-adjacent to z, a, b. By construction, u has a neighbor y in Y_2, and since u is anticomplete to a, b, it follows that $y \neq a, b$. Since $y - u - s - z - a - b$ is not a P_6 in G, it follows that y has a neighbor in $\{z, a, b\}$. Since P satisfies (II), it follows that $u - y - a$ is not a path and so y is not adjacent to a. Since P satisfies the second-to-last bullet for L_1, L_2, L_3, it follows that $u - y - b - a$ is not a path, and so u is not adjacent to b. But then u is adjacent to z; and $b - a - z - u$ is a path contrary to the second-to-last bullet for L_3, L_2, L_1. This is a contradiction, and hence $S_i^2 \neq \emptyset$, say $S_i^2 = \{u\}$.

By construction, it follows that $f'(u) \in L_2 \setminus L_3$, and $f'(v) \in L_3 \setminus L_2$. If one of u, v has no neighbor in $\{a, b\}$, then we reach a contradiction as above. Since neighbors of u in Y_2 are in $\tilde{X}^Q$, it follows that u is adjacent to b, but not a. Since neighbors of v in Y_3 are in $\tilde{X}^Q$, it follows that v is adjacent to a, but not b. This contradicts (20), and proves (22).

We now replace every $P' \in \mathcal{L}$ by P'' satisfying (I) by moving vertices with lists of size less than three from Y' to X'. It follows that P'' still satisfies (II) and (21). This concludes the proof of the lemma. □

Lemma 37. *There is a function $q : \mathbb{N} \to \mathbb{N}$ such that the following holds. Let $P = (G, S, X_0, X, Y, Y^*, f)$ be a starred precoloring of a P_6-free graph G with P satisfying* (I) *and* (II)*. Then there is an algorithm with running time $O(|V(G)|^{q(|S|)})$ that outputs an equivalent collection $\mathcal{L}$ for P such that*

- $|\mathcal{L}| \leq |V(G)|^{q(|S|)}$*;*
- *every $P' \in \mathcal{L}$ is a starred precoloring of G;*
- *every $P' \in \mathcal{L}$ with seed S' satisfies $|S'| \leq q(|S|)$; and*
- *every $P' \in \mathcal{L}$ satisfies* (I)*,* (II) *and* (III)*.*

Moreover, for every $P' \in \mathcal{L}$, given a precoloring extension of P', we can compute a precoloring extension for P in polynomial time.

Proof. Let $\mathcal{L} = \{P\}$. For every triple (L_1, L_2, L_3) of distinct lists of size three included in $[4]$ we repeat the following. Apply Lemma 36 to every member of $\mathcal{L}$; replace $\mathcal{L}$ by the union of the collections thus obtained, and move to the next triple of lists. At the end of this process we have an equivalent collection $\mathcal{L}$ for P, in which every starred precoloring satisfies the second-to-last bullet of Lemma 36 for every (L_1, L_2, L_3).

Repeat the procedure described in the previous paragraph. Since the second-to-last bullet of the conclusion of Lemma 36 holds for each starred precoloring we input this time, it follows that the last bullet of Lemma 36 holds for the output for every (L_1, L_2, L_3). Thus (III) holds; this concludes the proof. □

Let $P = (G, S, X_0, X, Y, Y^*, f)$ be a starred precoloring. For $W \subseteq V(G)$ and $L \subseteq [4]$, we say that W *meets* L if $L_P(w) = L$ for some $w \in W$. We now have the following convenient property.

Lemma 38. *Let $P = (G, S, X_0, X, Y, Y^*, f)$ be a starred precoloring of a P_6-free graph G satisfying* (I)*,* (II) *and* (III)*. Let L_1, L_2, L_3, L_4 be the subsets of $[4]$ of size three. Let C be a component of $G|Y$ that meets at least three of the lists L_1, L_2, L_3, L_4. For $i \in [4]$, let $C_i = \{v \in V(C) : L_P(v) = L_i\}$. Then for every $i \neq j$, C_i is complete to C_j.*

Proof. Let $P = p_1 - \ldots - p_k$ be a path such that for some $i \neq j$ $p_1 \in C_i$, $p_k \in C_j$, p_1 is non-adjacent to p_k, and subject to that with k minimum. Since P satisfies (II), it follows that $p_2 \notin C_i$; say

$p_2 \in C_l$. Since P satisfies (II) and (III), it follows that $p_3 \in C_i$. Similarly, $p_4 \notin C_i$. By the minimality of k, we deduce that $k = 4$. By (III) applied to $p_2 - p_3 - p_4$, we deduce that $l = j$. Let C' be a component of $C|(C_i \cup C_j)$ with $p_1, \dots, p_4 \in V(C')$. Since C is connected, and since $V(C) \neq C_i \cup C_j$, there exists $c \in C_l$ with $l \neq i, j$ such that c has a neighbor in C'. Since P satisfies (II) and (III), it follows from Lemma 19 that c is complete to C'. But now $p_1 - c - p_4$ contradicts the fact that P satisfies (III). This proves Lemma 38. □

Our next goal is to establish axiom (IV), which we restate.

(IV) Let $L_1 \subseteq \{1, 2, 3, 4\}$ with $|L_1| = 3$. Then there is no path $a - b - c$ with $L_P(b) = L_P(c) = L_1$ and $a \in X$, $b, c \in Y$.

Lemma 39. *There is a function $q : \mathbb{N} \to \mathbb{N}$ such that the following holds. Let $P = (G, S, X_0, X, Y, Y^*, f)$ be a starred precoloring of a P_6-free graph G with P satisfying* (I). *Then there is an algorithm with running time $O(|V(G)|^{q(|S|)})$ that outputs an equivalent collection $\mathcal{L}$ for P such that*

- $|\mathcal{L}| \leq |V(G)|^{q(|S|)}$*;*
- *every $P' \in \mathcal{L}$ is a starred precoloring of G;*
- *every $P' \in \mathcal{L}$ with seed S' satisfies $|S'| \leq q(|S|)$; and*
- *every $P' \in \mathcal{L}$ satisfies* (I), (II), (III) *and* (IV).

Moreover, for every $P' \in \mathcal{L}$, given a precoloring extension of P', we can compute a precoloring extension for P in polynomial time.

Proof. Let $\mathcal{L} = \{P\}$. For every list $L \subseteq \{1, 2, 3, 4\}$ of size three, apply Lemma 33 to every member of $\mathcal{L}$, replace $\mathcal{L}$ by the union of the equivalent collections thus obtained, and move to the next list. At the end of the process we obtained the required equivalent collection for $\{P\}$. □

We now begin to establish the following axiom, which we restate below.

(V) Let $L_1, L_2 \subseteq \{1, 2, 3, 4\}$ with $|L_1| = |L_2| = 3$. Then there is no path $a - b - c$ with $L_P(b) = L_1, L_P(c) = L_2$ and $a \in X$ with $L_3 = L_P(a) \neq L_1 \cap L_2$.

We define the following auxiliary statement:

(23) *Let $L_1, L_2 \subseteq \{1, 2, 3, 4\}$ with $|L_1| = |L_2| = 3$. Then there is no path $a - b - c - d$ with $L_P(b) = L_P(d) = L_1, L_P(c) = L_2$ and $a \in X$ with $L_3 = L_P(a) \neq L_1 \cap L_2$.*

Lemma 40. *There is a function $q : \mathbb{N} \to \mathbb{N}$ such that the following holds. Let $L_1, L_2 \subseteq \{1, 2, 3, 4\}$ with $|L_1| = |L_2| = 3$ and $L_1 \neq L_2$, and let $L_3 \subseteq \{1, 2, 3, 4\}$ with $|L_3| = 2$ and $L_3 \neq L_1 \cap L_2$. Let $P = (G, S, X_0, X, Y, Y^*, f)$ be a starred precoloring of a P_6-free graph G with P satisfying* (I), (II), (III) *and* (IV). *Then there is an algorithm with running time $O(|V(G)|^{q(|S|)})$ that outputs an equivalent collection $\mathcal{L}$ for P such that*

- $|\mathcal{L}| \leq |V(G)|^{q(|S|)}$*;*
- *every $P' \in \mathcal{L}$ is a starred precoloring of G;*
- *every $P' \in \mathcal{L}$ with seed S' satisfies $|S'| \leq q(|S|)$;*
- *every $P' \in \mathcal{L}$ satisfies* (I), (II), (III) *and* (IV)*;*

- *every $P' \in \mathcal{L}$ satisfies* (23) *for every three lists L_1', L_2', L_3' such that P satisfies* (23) *for L_1', L_2', L_3';*
- *if P satisfies* (23) *for every three lists, then every $P' \in \mathcal{L}$ satisfies* (V) *for every three lists L_1', L_2', L_3' such that P satisfies* (V) *for L_1', L_2', L_3';*
- *every $P' \in \mathcal{L}$ satisfies* (23) *for L_1, L_2, L_3.*
- *if P satisfies* (23) *for every three lists L_1', L_2', L_3' such that $|L_1'| = |L_2'| = 3, L_1' \neq L_2', |L_3'| = 2, L_3' \neq L_1' \cap L_2'$, then every $P' = (G, S', X_0', X', Y', Y^*, f') \in \mathcal{L}$ satisfies that there is no path $a - b - c$ with $L_{P'}(a) = L_3$, $L_{P'}(b) = L_1$, $L_{P'}(c) = L_2$ with $a \in X$, $b, c \in Y'$.*

Moreover, for every $P' \in \mathcal{L}$, given a precoloring extension of P', we can compute a precoloring extension for P in polynomial time.

Proof. Let $\mathcal{L} = \emptyset$. Let $Y_i = \{y \in Y : L_P(y) = L_i\}$ for $i = 1, 2$, and let X_3 be the set of vertices v in X with list L_3 such that v starts a path $v - b - c - d$ ($v - b - c$ if the condition of the last bullet holds for P) with $v \in X, b, d \in Y_1, c \in Y_2$. Let L_4, L_5 be the two three-element lists in $\{1, 2, 3, 4\}$ that are not L_1, L_2, and let $Y_i = \{y \in Y : L_P(y) = L_i\}$ for $i = 4, 5$. We call a component C of $G|Y$ *bad* if $V(C) \cap Y_1 \neq \emptyset, V(C) \cap Y_2 \neq \emptyset$ and $V(C) \cap Y_i \neq \emptyset$ for some $i \in \{4, 5\}$.

Let $\mathcal{T} = \{T_1, \dots, T_r\}$ be the set of types $T \subseteq S$ with $f(T) = \{1, 2, 3, 4\} \setminus L_3$. We let $\mathcal{Q}$ be the set of all r-tuples $(Q_1, \dots, Q_r)$, where for each i,

$$Q_i = (S_i^1, S_i^2, R_i^1, R_i^2, R_i^3, R_i^4, C_i^1, C_i^2, X_i^{1,1}, X_i^{1,2}, X_i^{2,1}, X_i^{2,2}, f_i, case_i)$$

such that the following hold:

1. $f_i : S_i^1 \cup S_i^2 \cup R_i^1 \cup R_i^2 \cup R_i^3 \cup R_i^4 \cup X_i^{1,1} \cup X_i^{1,2} \cup X_i^{2,1} \cup X_i^{2,2} \to \{1, 2, 3, 4\}$.
2. $f_i(S_i^1 \cup S_i^2) \subseteq L_3$.
3. $1 \geq |S_i^1| \geq |S_i^2| \geq |R_i^1| \geq |R_i^2| \geq |R_i^3| \geq |R_i^4|$.
4. $S_i^1 \cup S_i^2$ is a stable set and $S_i^1 \cup S_i^2 \subseteq X_3(T_i)$.
5. If $S_i^1 = \emptyset$, then $X_3(T_i) = \emptyset$.
6. If $S_i^2 \neq \emptyset$, then $f_i(S_i^1 \cup S_i^2) = L_3$ and $L_3 \cap L_1 \cap L_2 = \emptyset$.
7. For $j = 1, 2$, if $S_i^j = \left\{s_i^j\right\}$ and s_i^j is mixed on a bad component, then C_i^j is the vertex set of a bad component on which s_i^j is mixed; otherwise, $C_i^j = \emptyset$.
8. For $j, k = 1, 2$, $|X_i^{j,k}| \leq 1$, and $|X_i^{j,k}| = 1$ if and only if $C_i^j \neq \emptyset$.
9. For $j = 1, 2$, if $C_i^j \neq \emptyset$, then there exist $p \neq q$ such that $X_i^{j,1} \cap C_i^j \cap Y_p \neq \emptyset$ and $X_i^{j,2} \cap C_i^j \cap Y_q \neq \emptyset$.
10. For $j = 1, 2, 3, 4$, $f_i(R_i^j) \subseteq L_1 \cap L_2$.
11. $case_i \in \{\emptyset, (a), (b), (c), (d), (e), (f)\}$.
12. $case_i \in \{\emptyset, (a), (b)\}$ if and only if $R_i^j = \emptyset$ for all $j \in \{1, 2, 3, 4\}$.
13. $case_i \in \{(c), (d), (e)\}$ if and only if $R_i^3, R_i^4 = \emptyset$ and $R_i^1, R_i^2 \neq \emptyset$.

14. $case_i = (f)$ if and only if $R_i^j \neq \emptyset$ for all $j \in \{1,2,3,4\}$.

15. If $S_i^2 = \emptyset$, then $case_i = \emptyset$.

16. If $case_i \neq \emptyset$, then let $\{u,v\} = S_i^1 \cup S_i^2$ such that $u \in S_i^1$ if and only if $f_i(u) \in L_1$; then $R_i^1, R_i^3 \subseteq N(u) \cap (Y_2 \setminus N(v))$ and $R_i^2, R_i^4 \subseteq N(v) \cap (Y_1 \setminus N(u))$.

17. If $case_i = (c)$, R_i^1 is anticomplete to R_i^2.

18. If $case_i \in \{(d),(e)\}$, R_i^1 is complete to R_i^2.

19. If $case_i = (f)$, then R_i^1 is complete to R_i^2 and anticomplete to R_i^4, and R_i^3 is anticomplete to R_i^2 and anticomplete to R_i^4.

We let

$$S'^Q = \bigcup_{i\in\{1,\dots,r\}} (S_i^1 \cup S_i^2 \cup R_i^3 \cup R_i^4 \cup X_i^{1,1} \cup X_i^{1,2} \cup X_i^{2,1} \cup X_i^{2,2}) \cup \bigcup_{i\in\{1,\dots,r\}, case_i \neq (c)} (R_i^1 \cup R_i^2),$$

and let $f'^Q = f_1 \cup \dots \cup f_r$.

For every $i \in \{1,\dots,r\}$, we let $\tilde{Y}_i = \bigcup_{j,k\in\{1,2\}} \bigcup_{p\in\{1,2,4,5\}, X_i^{j,k}\cap C_i^j \cap Y_p \neq \emptyset} (C_i^j \cap Y_p)$, and we let $h_i(C_i^j \cap Y_p \cap \tilde{Y}_i) \subseteq f_i(X_i^{j,k})$. Let $\tilde{Z}_i = (C_i^1 \cup C_i^2) \setminus \tilde{Y}_i$. Let $\tilde{Y}^Q = \bigcup_{i\in\{1,\dots,r\}} \tilde{Y}_i$ and $\tilde{Z}^Q = \bigcup_{i\in\{1,\dots,r\}} \tilde{Z}_i$ and $h^Q = h_1 \cup \dots \cup h_r$.

Let S_1' be the set of $v \in S'^Q$ such that $f'(v) \in L_1 \cap L_2$; let S_2' be the set of $v \in S'^Q$ such that $f'(v) \in L_1 \setminus L_2$, and let S_3' be the set of $v \in S'^Q$ such that $f'(v) \in L_2 \setminus L_1$. Let

$$\tilde{X}^Q = (N(S_1') \cap (Y_1 \cup Y_2)) \cup (N(S_2') \cap (Y_1)) \cup (N(S_3') \cap (Y_2)).$$

Let $\tilde{W}_i = X_3(T_i)$ if $S_i^1 = \{v\}, S_i^2 = \emptyset$ and $f'(v) \notin L_1 \cap L_2 \cap L_3$, and $\tilde{W}_i = \emptyset$ otherwise. If $\tilde{W}_i \neq \emptyset$, we let $g_i'' : \tilde{W}_i \to L_3$ such that $g''(y) = f'(v)$ is y if non-adjacent to v, and $g''(y)$ is the unique color in $L_3 \setminus (\{f'(v)\})$ otherwise. Let $\tilde{W}^Q = \bigcup_{i\in\{1,\dots,r\}} \tilde{W}_i$ and let $g''^Q = g_1''^Q \cup \dots \cup g_r''^Q$.

Let $\tilde{V}^Q$ be the set of vertices v in X with list L_3 such that S'^Q contains a neighbor s of v, and let $h'^Q : \tilde{V} \to L_3$ such that $h'^Q(v) \in L_3 \setminus (f'(s))$.

Let $\tilde{U}_i$ be the set of all vertices $x \in X_3(T_i)$ such that $S_i^1 = \{v\}$ and such that $f'(v) \in L_1 \cap L_2$ and $N(v) \cap Y_1 \subsetneq N(x) \cap Y_1$, and let $g_i : \tilde{U}_i \to L_3 \setminus (L_1 \cap L_2)$. Let $\tilde{U}^Q = \bigcup_{i\in\{1,\dots,r\}} \tilde{U}_i$ and $g^Q = g_1 \cup \dots \cup g_r$.

Let $\tilde{U'}_i$ be the set of all vertices $x \in X_3(T_i)$ such that $S_i^1 = \{u\}, S_i^2 = \{v\}$ such that $xu \notin E(G)$, and $N(v) \cap Y_1 \subsetneq N(x) \cap Y_1$, and let $g_i' : \tilde{U}_i \to \{f'(u)\}$. Let $\tilde{U'}^Q = \bigcup_{i\in\{1,\dots,r\}} \tilde{U'}_i$ and $g'^Q = g_1' \cup \dots \cup g_r'$.

Finally, we define $\tilde{T}_i$ as follows: If $case_i = \emptyset$, then $\tilde{T}_i = \emptyset$. Otherwise, let $\{u,v\} = S_i^1 \cup S_i^2$ such that $f'(u) \in L_1$, and let $A = N(u) \cap (Y_2 \setminus N(v))$ and $B = N(v) \cap (Y_1 \setminus N(u))$. If $case_i =$

(a) then $\tilde{T}_i = A$;

(b) then $\tilde{T}_i = B$;

(c) then $\tilde{T}_i = (A \cap N(R_i^2)) \cup (B \cap N(R_i^1))$;

(d) then $\tilde{T}_i = B \setminus N(R_i^1)$;

(e) then $\tilde{T}_i = A \setminus N(R_i^2)$;

(f) then $\tilde{T}_i = \emptyset$.

We let $\tilde{T}^Q = \bigcup_{i\in\{1,\ldots,r\}} \tilde{T}_i$ and let $h''^Q : \tilde{T}^Q \to (L_1 \setminus L_2) \cup (L_2 \setminus L_1)$ be the unique function such that $h''^Q(v) \in L_P(v)$ for all $v \in \tilde{T}^Q$.

The following statement could be proved using Lemma 35, but we give a shorter proof here:

(24) *Let i such that $\{u, v\} = S_i^1 \cup S_i^2$ and $f(u) \in L_1$. Let $R = R_i^1 \cup R_i^2 \cup R_i^3 \cup R_i^4$ if $case_i \neq (c)$ and $R = \emptyset$ otherwise. Then $(N(u) \cap Y_2) \setminus (N(v) \cup \tilde{T}_i \cup N(R)))$ is anticomplete to $(N(v) \cap Y_1) \setminus (N(u) \cup \tilde{T}_i \cup N(R)))$.*

Let $A' = A \setminus (\tilde{T}_i \cup N(R))$, $B' = B \setminus (\tilde{T}_i \cup N(R))$; then it suffices to prove that A' is anticomplete to B'. If $case_i = (a), (b), (d), (e)$, this follows since A' or B' is empty in each of these cases. In case (f), we have that $G|(\{u, v\} \cup R)$ is a six-cycle. Since the graph arising from a six-cycle by adding a vertex with exactly one neighbor in the cycle contains a P_6, it follows that $A', B' = \emptyset$. In case (c), we let $x'y'$ be an edge from A' to B', and we let $x \in A_i^1, y \in A_i^2$. Then $x - u - x' - y' - v - y$ is a P_6 in G, a contradiction. Again it follows that A' is anticomplete to B', and (24) follows.

Let P'^Q be the starred precoloring obtained from

$$
\begin{aligned}
(G, &S \cup S'^Q \\
&X_0 \cup \tilde{Y}^Q \cup \tilde{W}^Q \cup \tilde{V}^Q \cup \tilde{U}^Q \cup \tilde{U'}^Q \cup \tilde{T}^Q \\
&(X \setminus (\tilde{W}^Q \cup \tilde{V}^Q \cup \tilde{U}^Q \cup \tilde{U'}^Q)) \cup \tilde{Z}^Q \cup \tilde{X}^Q \\
&Y \setminus (\tilde{Y}^Q \cup \tilde{Z}^Q \cup \tilde{X}^Q \cup \tilde{T}^Q) \\
&Y^*, f \cup f'^Q \cup h^Q \cup h'^Q \cup h''^Q \cup g^Q \cup g_i'^Q \cup g''^Q)
\end{aligned}
$$

by moving every vertex with a list of size at most two X, and every vertex with a list of size at most one to X_0. Since P satisfies (II) and (III), it follows that P'^Q satisfies (II) and (III) as well. Moreover, P'^Q satisfies (I).

We let

$$\mathcal{L} = \left\{ P'^Q : Q \in \mathcal{Q}, f \cup f'^Q \cup h^Q \cup h'^Q \cup h''^Q \cup g^Q \cup g_i'^Q \cup g''^Q \text{ is a proper coloring} \right\}.$$

(25) *$\mathcal{L}$ is an equivalent collection for P.*

For every $P'^Q \in \mathcal{L}$, every precoloring extension of P'^Q is a precoloring extension of P. Conversely, let c be a precoloring extension of P, and define $Q = (Q_1, \ldots, Q_r)$, where for each i,

$$Q_i = (S_i^1, S_i^2, R_i^1, R_i^2, R_i^3, R_i^4, C_i^1, C_i^2, X_i^{1,1}, X_i^{1,2}, X_i^{2,1}, X_i^{2,2}, f_i', case_i)$$

is defined as follows:

- If $X_3(T_i) = \emptyset$, then $Q_i = (\emptyset, \emptyset, \emptyset, \emptyset, \emptyset, \emptyset, \emptyset, \emptyset, \emptyset, \emptyset, \emptyset, \emptyset, f_i, \emptyset)$, where f_i is the empty function.

- If $X_3(T_i)$ contains a vertex v with $c(v) \in L_1 \cap L_2$, we choose v with $N(v) \cap Y_1$ maximal and let $S_i^1 = \{v\}$, $case = \emptyset$. In this case, we let $S_i^2 = \emptyset$.

- If $X_3(T_i)$ contains no vertex v with $c(v) \in L_1 \cap L_2$, we let $u \in X_3(T_i)$ with $N(u) \cap Y_1$ maximal, and set $S_i^1 = \{u\}$. If there is a vertex $v \in X_3(T_i)$ with $c(v) \neq c(u)$ and $uv \notin E(G)$, we choose v with $N(v) \cap Y_1$ maximal and set $S_i^2 = \{v\}$; otherwise we let $S_i^2 = \emptyset$.

- If $S_i^2 = \emptyset$, we let $case_i = \emptyset$ and $R_i^j = \emptyset$ for $j = 1, 2, 3, 4$. Otherwise, we let $\{u, v\} = S_i^1 \cup S_i^2$ such that $c(u) \in L_1$. We let $A = N(u) \cap (Y_2 \setminus N(v))$ and $B = N(v) \cap (Y_1 \setminus N(u))$. Let $a_1, \ldots, a_t$ be the components of $G|A$, and let $b_1, \ldots, b_s$ be the components of $G|B$. Since P satisfies

(II), it follows that for every $i \in [t]$ and $j \in [s]$, $V(a_i)$ is either complete or anticomplete to $V(b_j)$.

Let H be the graph with vertex set $\{u,v\} \cup \{a_1,\dots,a_t\} \cup \{b_1,\dots b_s\}$; where $N_H(u) = \{a_1,\dots,a_t\}$, $N_H(v) = \{b_1,\dots,b_s\}$, the sets $\{a_1,\dots,a_t\}$ and $\{b_1,\dots,b_s\}$ are stable, and a_i is adjacent to b_j if and only if $V(a_i)$ is complete to $V(b_j)$ in G. Apply 35 to H, u and v to obtain a partition $A'_0, A'_1, \dots, A'_k$ of $\{a_1,\dots,a_t\}$ and a partition $B'_0, B'_1, \dots, B'_k$ of $\{b_1,\dots,b_t\}$. For $i \in [k]$, let $A_i = \bigcup_{a_j \in A_i} V(a_j)$ and $B_i = \bigcup_{b_j \in B_i} V(b_j)$.

It follows from the definition of H that in G,

- A_0 is complete to $N(v)$;
- B_0 is complete to $N(u)$; and
- for $j = 1,\dots,k$, $A_j, B_j \neq \emptyset$ and A_j is complete to $N(v) \setminus B_j$ and B_j is complete to $N(u) \setminus A_j$, and A_j is anticomplete to B_j.

If $A_0 = B_0 = \emptyset$ and $k = 1$, then A is anticomplete to B, and we let $case_i = \emptyset$. Otherwise, we consider the following cases, setting $case_i =$

(a) if $c(A) \subseteq L_2 \setminus L_1$;
(b) if $c(B) \subseteq L_1 \setminus L_2$;
(c) if there is an $i \in \{1,\dots,k\}$ such that $c(A \setminus A_i) \subseteq L_2 \setminus L_1$, and $c(B \setminus B_i) \subseteq L_1 \setminus L_2$;
(d) if there exist $x \in A$, $y \in B$ adjacent such that $c(x), c(y) \in L_2 \cap L_1$ and $c(B \setminus N(x)) \subseteq L_1 \setminus L_2$;
(e) if there exist $x \in A$, $y \in B$ adjacent such that $c(x), c(y) \in L_2 \cap L_1$ and $c(A \setminus N(y)) \subseteq L_2 \setminus L_1$;
(f) if there exist $x, x' \in A, y, y' \in B$, with x, y adjacent, x' non-adjacent to y, y' non-adjacent to x, and (consequently) x' adjacent to y', and $c(x), c(y), c(x'), c(y') \in L_2 \cap L_1$.

It is easy to verify that one of these cases occurs.

With the notation as above, if $case_i =$

(a) then we let $R_i^j = \emptyset$ for $j = 1,2,3,4$;
(b) then we let $R_i^j = \emptyset$ for $j = 1,2,3,4$;
(c) then we let $x \in A_i, y \in B_i$ and set $R_i^1 = \{x\}$, $R_i^2 = \{y\}$, $R_i^3 = R_i^4 = \emptyset$;
(d) then we let $R_i^1 = \{x\}$, $R_i^2 = \{y\}$, $R_i^3 = R_i^4 = \emptyset$;
(e) then we let $R_i^1 = \{x\}$, $R_i^2 = \{y\}$, $R_i^3 = R_i^4 = \emptyset$;
(f) then we let $R_i^1 = \{x\}$, $R_i^2 = \{y\}$, $R_i^3 = \{x'\}$, $R_i^4 = \{y'\}$.

- For $j = 1,2$, we proceed as follows. If $S_i^j = \emptyset$ or the vertex $v \in S_i^j$ is not mixed on a bad component, then we let $X_i^{j,1} = X_i^{j,2} = C_i^j = \emptyset$. Otherwise, let $v \in S_i^j$ and let C be a bad component of $G|Y$ on which v is mixed. We set $C_i^j = V(C)$. By Lemma 38 applied to C, it follows that for $p \neq q$, $V(C) \cap Y_p$ is complete to $V(C) \cap Y_q$. Since $Y_p \cap V(C) \neq \emptyset$ for at least three different $p \in \{1,2,4,5\}$, it follows that there exist $p, q \in \{1,2,4,5\}$ with $p \neq q$ such that $|c(V(C) \cap Y_p)| = 1$ and $|c(V(C) \cap Y_q)| = 1$. Let $X_i^{j,1} \subseteq V(C) \cap Y_p$, $X_i^{j,2} \subseteq V(C) \cap Y_q$, such that $|X_i^{j,k}| = 1$ for $k = 1,2$.

We let $f_i = c|_{S_i^1\cup S_i^2\cup R_i^1\cup R_i^2\cup R_i^3\cup R_i^4\cup X_i^{1,1}\cup X_i^{1,2}\cup X_i^{2,1}\cup X_i^{2,2}}$. It follows from the definition of Q that $Q \in \mathcal{Q}$. Moreover, c is a precoloring extension of P'^Q by the definition of Q and P'^Q. This proves (25).

Let $P' \in \mathcal{L}$ with $P' = (G, S', X_0', X', Y', Y^*, f')$ such that $P' = P'^Q$ for $Q = (Q_1, \ldots, Q_r)$, where for each i,

$$Q_i = (S_i^1, S_i^2, R_i^1, R_i^2, R_i^3, R_i^4, C_i^1, C_i^2, X_i^{1,1}, X_i^{1,2}, X_i^{2,1}, X_i^{2,2}, f_i, case_i).$$

Let $Y_i' = \{y \in Y' : L_{P'}(y) = L_i\}$ for $i = 1, 2$. We claim the following.

(26) *P' satisfies* (IV).

Suppose not; and let $x - a - b$ be a path with $x \in X'$ and $a, b \in Y'$ with $L_{P'}(a) = L_{P'}(b) = L$. Since P satisfies (II) and (IV), it follows that $x \notin X$, and so $x \in Y$ and $L_P(x) = L$. Moreover, since $x \in X' \setminus X$, it follows that x has a neighbor $s' \in S' \setminus S$ with $f'(s') \in L$. Since P satisfies (II) and (IV), and since s' is adjacent to x but not a, it follows that $s' \in Y$ and $L_P(s') = L$. Since s' has a neighbor $x \in Y$ with a neighbor $a \in Y'$, it follows that $x \notin \tilde{Y}^Q \cup \tilde{Z}^Q$. Since $s' \notin X$, it follows that $s' \notin S_i^1 \cup S_i^2$, and hence there exists $i \in \{1, \ldots, r\}$ such that $s' \in R_i^j$ for some $j \in \{1, 2, 3, 4\}$. Thus $L_P(s') \in \{L_1, L_2\}$. Let $\{u, v\} = S_i^1 \cup S_i^2$ such that $s' \in N(u) \setminus N(v)$. It follows that $case_i \in \{(d), (e), (f)\}$, and hence there is a vertex $t' \in R_i^1 \cup R_i^2 \cup R_i^3 \cup R_i^4$ such that t' is adjacent to s' and v, but t' is not adjacent to u, and $L_P(t') \in \{L_1, L_2\} \setminus \{L\}$, and $f'(t') \in L_1 \cap L_2$. But then $t' - s' - x$ or $t' - x - a$ is a path (since $a \in Y'$ it follows that a is not adjacent to t'); contrary to the fact that (II) holds for P. This is a contradiction, and (26) follows.

(27) *If P satisfies* (23) *for lists L_1', L_2', L_3', then P' satisfies* (23) *for L_1', L_2', L_3'.*

Suppose not; and let $x-a-b-c$ be a path such that $L_{P'}(x) = L_3'$ with $|L_3'| = 2$ and $L_3' \neq L_1' \cap L_2'$, $L_{P'}(a) = L_1' = L_{P'}(c)$, $L_{P'}(b) = L_2'$. Since P satisfies (II), (23) for L_1', L_2', L_3', and (III), it follows that $L_P(x) = L_2'$. Consequently, x has a neighbor s' in $S' \setminus S$ with $f'(s') \in L_2'$. Since $L_3' \neq L_1' \cap L_2'$, it follows that $f'(s') \in L_1'$. Thus $s'-x-a-b-c$ is a path. Suppose first that $s' \in Y$. It follows that $s' \notin S_i^1 \cup S_i^2$. Since s' has a neighbor $x \in Y$ with a neighbor $a \in Y'$, it follows that $x \notin \tilde{Y}^Q \cup \tilde{Z}^Q$. This implies that there exist $i \in \{1, \ldots, r\}$ and $j \in \{1, 2, 3, 4\}$ such that $s' \in R_i^j$. Since P satisfies (II) and (III), it follows that $L_P(s') = L_1'$. Let $\{u, v\} = S_i^1 \cup S_i^2$ such that u is adjacent to s' and v is not. It follows that $case_i \in \{(d), (e), (f)\}$, and hence there is a vertex $t' \in R_i^1 \cup R_i^2 \cup R_i^3 \cup R_i^4$ such that t' is adjacent to s' and v, but t' is not adjacent to u, and $f'(t') \in L_1' = L_P(s')$. Since $t' - s' - x - a - b - c$ is not a P_6 in G, it follows that $f'(t') \notin L_1' \cap L_2'$. Therefore, $L_P(t') \notin \{L_1', L_2'\}$. Since P satisfies (III), it follows that t' is adjacent to x (since $t' - s' - x$ is not a path). Since $f'(t') \in L_1'$, it follows that t' is not adjacent to a. Now $t' - x - a$ is a path in G, contrary to the fact that P satisfies (III). Thus, $s' \in X$.

Suppose that $L_P(s') \neq L_1' \cap L_2'$. Then s' has a neighbor s in S with $f'(s) \in L_1' \cap L_2'$. Now $s-s'-x-a-b-c$ is a P_6 in G, a contradiction. It follows that $s' \in X$ and $L_P(s') = L_1' \cap L_2'$. Since $s' \in S_i^1 \cup S_i^2$, it follows that there is a path $s' - y - z$ with $y \in L_1, z \in L_2$, and $L_P(s') \neq L_1 \cap L_2$. It follows that either $L_1 \notin \{L_1', L_2'\}$ or $L_2 \notin \{L_1', L_2'\}$. Since $z - y - s' - x - a - b - c$ is not a P_7 in G, it follows that $G|\{z, y, x, a, b, c\}$ is connected. Let $w \in \{y, z\}$ such that $L_P(w) \notin \{L_1', L_2'\}$. Since P satisfies (III), it follows that w is complete to x, a, b, c. But then $x - w - c$ is a path, contrary to the fact that (III) holds for P. This implies (27).

(28) *If P satisfies* (V) *for lists L_1', L_2', L_3', and P satisfies* (23) *for all lists, then P' satisfies* (V) *for L_1', L_2', L_3'.*

Suppose not; and let $x - a - b$ be a path such that $L_{P'}(x) = L_3'$ with $|L_3'| = 2$ and $L_3' \neq L_1' \cap L_2'$, $L_{P'}(a) = L_1'$, $L_{P'}(b) = L_2'$. Since P satisfies (II), (V) for L_1', L_2', L_3', and (III), it follows that

$L_P(x) = L'_2$. Consequently, x has a neighbor s' in $S' \setminus S$ with $f'(s') \in L'_2$. Since $L'_3 \neq L'_1 \cap L'_2$, it follows that $f'(s') \in L'_1$. Thus $s' - x - a - b$ is a path. Suppose first that $s' \in X$. Then there exist $i \in \{1, \ldots, r\}$ and $j \in \{1, 2\}$ such that $s' \in S_i^j$. It follows that $L_P(s') = L'_1 \cap L'_2$, since P satisfies (23) for all lists. By construction, it follows that there is a path $s' - y - z$ with $y \in L_1, z \in L_2$, and $L_P(s') \neq L_1 \cap L_2$. We choose such $y, z \in C_i^j$ if $C_i^j \neq \emptyset$. Since $z - y - s' - x - a - b$ is not a six-vertex path in G, it follows that $G|\{z, y, x, a, b\}$ is connected. Since $C_i^j \cap Y' = \emptyset$ by construction, it follows that $C_i^j = \emptyset$, and so s' is not mixed on a bad component. Since $L_1 \cap L_2 \neq L'_1 \cap L'_2$, it follows that either $L_1 \notin \{L'_1, L'_2\}$ or $L_2 \notin \{L'_1, L'_2\}$. Let $w \in \{y, z\}$ such that $L_P(w) \notin \{L'_1, L'_2\}$. Then $G|\{z, y, x, a, b\}$ is contained in a component of $G|Y$ containing vertices with lists L'_1, L'_2 and $L_P(w)$, hence a bad component. But since $s' - x - a - b$ is a path, s' is mixed on this bad component, a contradiction. It follows that $s' \in Y$.

Since P satisfies (II) and (III), it follows that $L_P(s') = L'_1$. Since s' has a neighbor $x \in Y$ with a neighbor $a \in Y'$, it follows that $s' \notin \tilde{Y}^Q \cup \tilde{Z}^Q$. Thus, there exist $i \in \{1, \ldots, r\}$ and $j \in \{1, 2, 3, 4\}$ such that $s' \in R_i^j$. By construction, it follows that $L_P(s') \in \{L_1, L_2\}$. Let $\{u, v\} = S_i^1 \cup S_i^2$ such that u is adjacent to s' and v is not. It follows that $case_i \in \{(d), (e), (f)\}$, and hence there is a vertex $t' \in R_i^1 \cup R_i^2 \cup R_i^3 \cup R_i^4$ such that t' is adjacent to s' and v, but t' is not adjacent to u, and $f'(t') \in L_1 \cap L_2$.

Suppose first that $\{L'_1, L'_2\} = \{L_1, L_2\}$. Then $f'(s'), f'(t') \in L'_1 \cap L'_2$. Let $s \in S$ be a common neighbor of u, v with $f'(s) \in L_1 \cap L_2$. Since $s - u - s' - x - a - b$ is not a P_6 in G, it follows that u is adjacent to a. Since $t' - v - s - u - a - b$ is not a P_6 in G, it follows that v has a neighbor in $\{u, a, b\}$. Since $f'(v) \in L_1$, it follows that v is non-adjacent to a. Thus v is adjacent to b. Since $a, b \notin \tilde{T}^Q$, it follows that $case_i = (f)$. By symmetry, we may assume that $s' \in R_i^1, t' \in R_i^2$. Let $x' \in R_i^3, y' \in R_i^4$. Then x', y' are non-adjacent to a, b. But then $x' - u - a - b - v - y$ is a P_6 in G, a contradiction. It follows that $\{L'_1, L'_2\} \neq \{L_1, L_2\}$.

Consequently, $L_P(t') \notin \{L'_1, L'_2\}$. Since P satisfies (III), it follows that $t' - s' - x$ is not a path, and so t' is adjacent to x. Since $f'(t') \in L'_1$, it follows that t' is not adjacent to a. Now $t' - x - a$ is a path, contrary to the fact that (III) holds for P. This proves (28).

(29) *P' satisfies* (23) *for L_1, L_2, L_3.*

Suppose not; and let $z - a - b - c$ be a path with $L_{P'}(z) = L_3$, $L_{P'}(a) = L_{P'}(c) = L_1$, $L_{P'}(b) = L_2$. Suppose first that $z \in X$. Let i such that $T_i = N(z) \cap S$. Then $S_i^1 \neq \emptyset$. Let $s' \in S_i^1 \cup S_i^2$, and let s be a common neighbor of s' and z in S with $f(s) \in L_1 \cap L_2$. Since $s' - s - z - a - b - c$ is not a path, it follows that z, a, b, c contains a neighbor of s' for every $s' \in S_i^1 \cup S_i^2$. But z is anticomplete to $S_i^1 \cup S_i^2$, for otherwise, $z \in \tilde{V}^Q$. If $S_i^2 = \emptyset$, then, since $z \notin X'_0$, it follows that $f(s') \in L_1 \cap L_2$ and so z is anticomplete to a, b, c, a contradiction. Therefore, $S_i^2 \neq \emptyset$. But then $S_i^1 \cup S_i^2 = \{u, v\}$ with $f'(u) \in L_2 \setminus L_1$, say. Since $a, b, c \in Y'$, it follows that u is adjacent to a or c, and v is adjacent to b; and no other edges between u, v and a, b, c exist. Now, Y' contains an edge between $N(u) \cap (Y_1 \setminus N(v))$ and $N(v) \cap (Y_2 \setminus N(u))$; but this contradicts (24).

Since P satisfies (II) and (III), it follows that $L_P(z) = L_2$. Then z has a neighbor $s' \in S' \setminus S$ with $f'(s') \in L_1 \cap L_2$ (for if $f'(s') \notin L_1$, then $L_{P'}(z) = L_1 \cap L_2 \neq L_3$), and $s' - z - a - b - c$ is a path. Suppose first that $s' \in Y$. Since P satisfies (II) and (III), it follows that $L_P(s') = L_1$. Moreover, by construction, s' has a neighbor $t' \in S'$ with $L_P(t') = L_2$ and $f'(t') \in L_1 \cap L_2$. But then $t' - s' - z - a - b - c$ is a P_6 in G, a contradiction. It follows that $s' \in X$.

Since $s' \in X$, it follows that $L(s') = L_3$, and so s' has a neighbor $s \in S$ with $f(s) \in L_1 \cap L_2$. But then $s - s' - z - a - b - c$ is a P_6 in G, a contradiction. This proves (29).

(30) *If P satisfies* (23) *for every three lists, then P' satisfies* (V) *for L_1, L_2, L_3.*

Suppose not; and let $z - a - b$ be a path with $L_{P'}(z) = L_3$, $L_{P'}(a) = L_1$, $L_{P'}(b) = L_2$.

Suppose first that $z \in X$. Let $i \in \{1, \ldots, r\}$ such that $T_i = N(z) \cap S$. By construction, it follows that $S_i^1 \neq \emptyset$. Let $s' \in S_i^1 \cup S_i^2$, and let s be a common neighbor of s' and z in S with $f(s) \in L_1 \cap L_2$. Let c be a neighbor of s' in Y_1; by construction, we may choose c to be non-adjacent to z. Then $c \neq a, b$ (since $b \notin Y_1$). Since $c - s' - s - z - a - b$ is not a path, it follows that either s' or c has a neighbor in $\{a, b\}$. Since P satisfies (IV), it follows that $s' - c - a$ is not a path. Since P satisfies (23) for all lists, it follows that $z - a - b - c$ is not a path. Consequently, s' has a neighbor in $\{a, b\}$. It follows that $f'(s') \notin L_1 \cap L_2$. Therefore, $S_i^1 \cup S_i^2 = \{u, v\}$ and both u, v have a neighbor in $\{a, b\}$. Since $a, binY'$, it follows that both a, b have a non-neighbor in $\{u, v\}$. This is a contradiction by (24).

Since $z \in Y$ and P satisfies (II) and (III), it follows that $L_P(z) = L_2$. Consequently, z has a neighbor s' in $S' \setminus S$ with $f'(s') \in L_2$. Since $L_3 \neq L_1 \cap L_2$, it follows that $f'(s') \in L_1$. Thus $s' - z - a - b$ is a path. Since s' has a neighbor $z \in Y$ with a neighbor $a \in Y'$, it follows that $s' \notin \tilde{Y}^Q \cup \tilde{Z}^Q$. Suppose first that $s' \in X$. Then there exist $i \in \{1, \ldots, r\}$ and $j \in \{1, 2\}$ such that $s' \in S_i^j$. It follows that $L_P(s') = L_1 \cap L_2$ since P satisfies (23) for all lists. But $S_i^j \subseteq X_3$ and so $L_P(s') \neq L_1 \cap L_2$, a contradiction. It follows that $s' \in Y$.

Since P satisfies (II) and (III), it follows that $L_P(s') = L_1$, and there exist $i \in \{1, \ldots, r\}$ and $j \in \{1, 2, 3, 4\}$ such that $s' \in R_i^j$. Moreover, $S_i^1 \cup S_i^2 = \{u, v\}$. By symmetry, we may assume that u is adjacent to s' and v is not. It follows that $case_i \in \{(d), (e), (f)\}$, and hence there is a vertex $t' \in R_i^1 \cup R_i^2 \cup R_i^3 \cup R_i^4$ such that t' is adjacent to s' and v, but t' is not adjacent to u. By construction, it follows that $f'(s'), f'(t') \in L_1 \cap L_2$. Let $s \in T_i$ with $f'(s) \in L_1 \cap L_2$. Since $s - u - s' - z - a - b$ is not a P_6 in G, it follows that u is adjacent to a or to z. Note that if $uz \in E(G)$, then z is adjacent to both s' and u, both of which are in S' and $f(s', u) \subseteq L_1$. This implies that $z \in X_0'$. It follows that u is adjacent to a. Since $t' - v - s - u - a - b$ is not a P_6 in G, it follows that v is adjacent to b. This contradicts (24) and concludes the proof of (30).

The statement of the lemma follows; we have proved every claim in (26), (27), (28), (29) and (30). □

Lemma 41. *There is a function $q : \mathbb{N} \to \mathbb{N}$ such that the following holds. Let $P = (G, S, X_0, X, Y, Y^*, f)$ be a starred precoloring of a P_6-free graph G with P satisfying* (I), (II), (III) *and* (IV). *Then there is an algorithm with running time $O(|V(G)|^{q(|S|)})$ that outputs an equivalent collection $\mathcal{L}$ for P such that*

- $|\mathcal{L}| \leq |V(G)|^{q(|S|)}$;
- *every $P' \in \mathcal{L}$ is a starred precoloring of G;*
- *every $P' \in \mathcal{L}$ with seed S' satisfies $|S'| \leq q(|S|)$; and*
- *every $P' \in \mathcal{L}$ satisfies* (I), (II), (III), (IV) *and* (V).

Moreover, for every $P' \in \mathcal{L}$, given a precoloring extension of P', we can compute a precoloring extension for P in polynomial time, if one exists.

Proof. Let $\mathcal{L} = \{P\}$. For every triple (L_1, L_2, L_3) of lists of size three, we repeat the following. Apply Lemma 40 to every member of $\mathcal{L}$, replace $\mathcal{L}$ with the union of the equivalent collections thus obtained, and move to the next triple. At the end of thus process (23) holds for every $P' \in \mathcal{L}$.

Now repeat the procedure of the previous paragraph. Since at this stage all inputs satisfy (23) for every triple of lists, it follows that (V) holds for every starred precoloring of the output. □

We now observe that the next axiom, which we restate, holds.

(VI) For every component C of $G|Y$, for which there is a vertex of X is mixed on C, there exist $L_1, L_2 \subseteq \{1,2,3,4\}$ with $|L_1| = |L_2| = 3$ such that C contains a vertex x_i with $L_P(x_i) = L_i$ for $i = 1, 2$, every vertex x in C satisfies $L_P(x) \in \{L_1, L_2\}$, and every $x \in X$ mixed on C satisfies $L_P(x) = L_1 \cap L_2$.

Lemma 42. *Let $P = (G, S, X_0, X, Y, Y^*, f)$ of a P_6-free graph G satisfying* (I)-(V)*, and let C be a component of $G|Y$ such that some vertex $x \in X$ is mixed on C. Then C meets exactly two lists L_1, L_2, and $L_P(x) = L_1 \cap L_2$.*

Proof. Since P satisfies (IV), Lemma 19 implies that C meets more than one list. By Lemma 19, there exist a, b in C such that $x - a - b$ is a path. By (IV) $L_P(a) \neq L_P(b)$, and by (V) $L_P(x) = L_P(a) \cap L_P(b)$. Let $c \in V(C)$ be such that $L_P(c) \neq L_P(a), L_P(b)$. By Lemma 38 c is complete to $\{a, b\}$. But then x is mixed on one of $\{a, c\}$, $\{b, c\}$, contrary to (V). This proves Lemma 42. □

The following lemma establishes that:

(VII) For every component C of $G|Y$ such that some vertex of X is mixed on C, and for L_1, L_2 as in (VI), $L_P(v) = L_1 \cap L_2$ for every vertex $v \in X$ with a neighbor in C.

Lemma 43. *There is a function $q : \mathbb{N} \to \mathbb{N}$ such that the following holds. Let $P = (G, S, X_0, X, Y, Y^*, f)$ be a starred precoloring of a P_6-free graph G with P satisfying* (I), (II), (III), (IV), (V) *and* (VI)*. Then there is an algorithm with running time $O(|V(G)|^{q(|S|)})$ that outputs an equivalent collection $\mathcal{L}$ for P such that*

- *$|\mathcal{L}| \leq |V(G)|^{q(|S|)}$;*
- *every $P' \in \mathcal{L}$ is a starred precoloring of G;*
- *every $P' \in \mathcal{L}$ with seed S' satisfies $|S'| \leq q(|S|)$; and*
- *every $P' \in \mathcal{L}$ satisfies* (I), (II), (III), (IV), (V), (VI) *and* (VII).

Moreover, for every $P' \in \mathcal{L}$, given a precoloring extension of P', we can compute a precoloring extension for P in polynomial time, if one exists.

Proof. Let $\mathcal{R} = \{T_1, \ldots, T_r\}$ be the set of all $T \subseteq S$ with $|f(T)| = 2$, let $S = \{s_1, \ldots, s_s\}$, and let $\mathcal{T} = \{(L_1^1, L_2^1), \ldots, (L_1^t, L_2^t)\}$ be the set of all pairs (L_1, L_2) with $|L_1| = |L_2| = 3$ and $L_1 \neq L_2$. We let $\mathcal{Q}$ be the set of all $(rst + 1)$-tuples $Q = (Q_{1,1,1}, \ldots, Q_{r,s,t}, f')$, where $i \in [r], j \in [s]$ and $k \in [t]$, and for each i, j, k the following statements hold:

- $Q_{i,j,k} \subseteq X(T_i)$ and $|Q_{i,j,k}| \leq 1$;
- $Q_{i,j,k} = \emptyset$ if $[4] \setminus f(T_i) = L_1^k \cap L_2^k$ or $f(s_j) \in f(T_i)$;
- if $Q_{i,j,k} = \{x\}$, then there is a component C of $G|Y$ such that
  - s_j has a neighbor in $V(C)$;
  - some vertex of X is mixed on C, and C meets L_1^k, L_2^k as in (VI);
  - x has neighbors in $V(C)$

  and x has the maximum number of such components C among all vertices in $X(T_i)$;
- if $Q_{i,j,k} = \emptyset$, then no vertex $x \in X(T_i)$ and component C as above exist,

- Let $\tilde{Q} = \bigcup_{i \in \{1,\dots,r\}, j \in \{1,\dots,s\}, k \in \{1,\dots,t\}} Q_{i,j,k}$, then $f' : \tilde{Q} \to \{1,2,3,4\}$ satisfies that $f' \cup f$ is a proper coloring of $G|(S \cup X_0 \cup \tilde{Q})$.

For $Q \in \mathcal{Q}$, we construct a starred precoloring P^Q from P as follows. We let $\tilde{Z}^Q$ be the set of vertices z in $X \setminus \tilde{Q}$ such that $\tilde{Q}$ contains a neighbor x of z with $f'(x) \in L_P(z)$, and let $g^Q : \tilde{Z}^Q \to \{1,2,3,4\}$ be the unique function such that $g^Q(z) \in L_P(z) \setminus f'(N(z) \cap \tilde{Q})$. We let $\tilde{X}^Q$ be the set of vertices z in Y such that $\tilde{Q}$ contains a neighbor x of z with $f'(x) \in L_P(z)$.

We let

$$P^Q = (G, S \cup \tilde{Q}, X_0 \cup \tilde{Z}^Q, (X \setminus (\tilde{Z}^Q \cup \tilde{Q})) \cup \tilde{X}^Q, Y \setminus \tilde{X}^Q, Y^*, f \cup f' \cup g^Q),$$

and let $\mathcal{L} = \{P^Q : Q \in \mathcal{Q}, f \cup f' \cup g^Q \text{ is a proper coloring}\}$. It is easy to check that $\mathcal{L}$ is an equivalent collection for P.

Let $Q \in \mathcal{Q}$, and let $P^Q = (G', S', X'_0, X', Y', Y^*, f')$. By construction, P^Q satisfies (I). Since P satisfies (II), (III), so does P^Q. Since P satisfies (II), it follows that P^Q satisfies (IV).

(31) P^Q *satisfies* (V).

Suppose not; and let $a - b - c$ be a path with $a \in X'$, $b, c \in Y'$ such that $L_{P^Q}(a) = L_3, L_{P^Q}(b) = L_1, L_{P^Q}(c) = L_2$ and $L_1 \neq L_2$, $L_3 \neq L_1 \cap L_2$. Since P satisfies (V), it follows that $a \in Y$. Since P satisfies (II) and (III), it follows that $L_P(a) = L_2$, and there is a vertex $x \in \tilde{Q}$, say $x \in Q_{i,j,k}$ such that x is adjacent to a and $f'(x) \in L_P(a)$. Since $c \in Y'$, it follows that x is not adjacent to c. Since x is mixed on a component of $G|Y$ meeting L_1 and L_2, and since P satisfies (VI), it follows that $L_P(x) = L_1 \cap L_2$. Thus $x - a - b - c$ is a path, and there is a component C of $G|Y$ such that $V(C)$ meets L_1^k, L_2^k and x has a neighbor in C and $L_1^k \cap L_2^k \neq L_P(x) = L_1 \cap L_2$. It follows that $a, b, c \notin V(C)$, and so $V(C)$ is anticomplete to a, b, c. By symmetry, we may assume that $L_1^k \notin \{L_1, L_2\}$. Let $d \in V(C)$ with $L_P(d) = L_1^k$. Since P satisfies (V) and (IV), and since x has a neighbor in C, it follows that x is complete to C and thus adjacent to d. Since $L_P(d) \notin \{L_1, L_2\}$, it follows that there is a vertex $s \in S$ with $f(s) \in L_1 \cap L_2$ and s adjacent to d. But then $c - b - a - x - d - s$ is a P_6 in G, a contradiction. This proves (31).

Now by Lemma 42, P^Q satisfies (VI).

(32) P^Q *satisfies* (VII).

Suppose not. Let C be a component of $G'|Y'$ such that some vertex of X' is mixed on C, and with L_1, L_2 as in (VI), and let $v \in X'$ with $N(v) \cap C \neq \emptyset$ such that $L_{P^Q}(v) \neq L_1 \cap L_2$.

Since $L_{P^Q}(v) \neq L_1 \cap L_2$, we may assume that $[4] \setminus L_1 \subseteq L_P(v)$. Let $s \in S$ with $f(s) = [4] \setminus L_1$, such that s has a neighbor in C. Since P^Q satisfies (VI), it follows that v is complete to C.

We claim that every $x \in X' \cap Y$ is complete to C. Suppose that $x \in Y \cap X'$ is mixed on C. Since P^Q satisfies (VI), it follows that $L_{P^Q}(x) = L_1 \cap L_2$. By symmetry, we may assume that $L_P(x) = L_1$, and therefore, x has a neighbor s in $\tilde{Q} \cap X$ and $f(s) = L_1 \setminus L_2$. But then s is mixed on the component $\tilde{C}$ of $G|Y$ containing $V(C) \cup \{x\}$, $\tilde{C}$ meets L_1 and L_2, and $L_P(s) \neq L_1 \cap L_2$, contrary to the fact that P satisfies (VI). This proves the claim. Now since some vertex of X' is mixed on C, it follows that some vertex of X is mixed on C.

Next we claim that $v \in X$. Suppose $v \in Y$. Then there is a component $\tilde{C}$ of $G|Y$ such that $V(C) \cup \{v\} \subseteq V(\tilde{C})$. Since some $x \in X$ is mixed on C, and since P satisfies (VI), we deduce that $L_P(v) \in \{L_1, L_2\}$. Consequently, v has a neighbor s in $\tilde{Q}$. Therefore $q \in X$. Since v is complete to C, it follows that v has a neighbor n in C with $L_P(n) = L_P(v)$. But then x is mixed on the edge vn, contrary to the fact that P satisfies (IV). This proves that $v \in X$.

By construction, Q contains an entry $Q_{i,j,k}$ with $T_i = T(v)$, $s_j = s$ and $(L_1^k, L_2^k) = (L_1, L_2)$, and in view of the claims of the previous two paragraphs, $Q_{i,j,k} \neq \emptyset$. Write $Q_{i,j,k} = \{z\}$. Let C' be a component of $G|Y$ meeting both L_1 and L_2, such that some vertex of X is mixed on C', and both s and z have a neighbor in C'. Since $f'(z) \in L_1 \cup L_2$, it follows that z is not complete to C. Since $L_P(z) \neq L_1 \cap L_2$, it follows from the fact that P satisfies (VI) that z is not mixed on either of C, C'. Consequently, z is complete to C', and z is anticomplete to C. Now by the maximality of z we may assume that v is anticomplete to C'. Since $[4] \setminus L_1 \subseteq L_P(z) = L_P(v)$, it follows that s is anticomplete to $\{z, v\}$.

Let $a \in V(C) \cap N(s)$ and $a' \in V(C') \cap N(s)$. Since each of C, C' meets L_2, we can also choose $b \in V(C) \setminus N(s)$ and $b' \in V(C') \setminus N(s)$. $L_P(z) \neq L_1 \cap L_2$, there exists $t \in T_i$ with $f(t) \in L_1 \cap L_2$. Then t is anticomplete to $V(C) \cup V(C')$. It t is non-adjacent to s, then $s - a - v - t - z - a'$ is a P_6 in G, so t is adjacent to s. If a is non-adjacent to b, then $b - v - a - s - a' - z$ is a P_6, so a is adjacent to b. But now $b - a - s - t - z - a'$ is a P_6, a contradiction. Thus, (32) follows.

This concludes the proof of the Lemma 43. □

We are now ready to prove the final axiom.

(VIII) $Y = \emptyset$.

Lemma 44. *There is a function $q : \mathbb{N} \to \mathbb{N}$ such that the following holds. Let $P = (G, S, X_0, X, Y, Y^*, f)$ be a starred precoloring of a P_6-free graph G with P satisfying* (I), (II), (III), *(IV), (V), (VI), (VII). Then there is an algorithm with running time $O(|V(G)|^{q(|S|)})$ that outputs collection $\mathcal{L}$ of starred precolorings such that*

- *if we know for every $P' \in \mathcal{L}$ whether P' has a precoloring extension or not, then we can decide if P has a precoloring extension in polynomial time;*
- $|\mathcal{L}| \leq |V(G)|^{q(|S|)}$;
- *every $P' \in \mathcal{L}$ is a starred precoloring of G;*
- *every $P' \in \mathcal{L}$ with seed S' satisfies $|S'| \leq q(|S|)$; and*
- *every $P' \in \mathcal{L}$ satisfies* (VIII).

Moreover, for every $P' \in \mathcal{L}$, given a precoloring extension of P', we can compute a precoloring extension for P in polynomial time, if one exists.

Proof. Let $P = (G, S, X_0, X, Y, Y^*, f)$. For every component C of $G \setminus (S \cup X_0)$, Let P_C be the starred precoloring

$$(G|(V(C) \cup S \cup X_0), S, X_0, X \cap V(C), Y \cap V(C), Y^* \cap V(C), f).$$

Then P_C satisfies (I)–(VII). Let $\mathcal{L}_0$ be the collection of all such starred precolorings P_C. Clearly P has a precoloring extension if and only if every member of $\mathcal{L}_0$ does, so from now on we focus on constructing an equivalent collection for each P_C separately. To simplify notation, from now we will simply assume that $G \setminus (X_0 \cup S)$ is connected.

In the remainder of the proof we either find that P has no precoloring extension, output $\mathcal{L} = \emptyset$ and stop, or construct two disjoint subsets U and W of Y, and a subset $\tilde{X}_0$ of X such that

- $U \cup W = Y$,

- No vertex of X is mixed on a component of $G|W$,
- For every component C of $G|W$, some vertex of $X \cup X_0 \cup S$ is complete to C.
- There is a set F with $|F| \leq 2^6$ of colorings of $G|\tilde{X}_0$ that contains every coloring of $G|\tilde{X}_0$ that extends to a precoloring extension of P, and F can be computed in polynomial time.
- P has a precoloring extension if and only if for some $f' \in F$

  $$(G \setminus U, S, X_0 \cup \tilde{X}_0, X \setminus \tilde{X}_0, W, Y^*, f \cup f')$$

  has a precoloring extension.

Having constructed such $U, W, \tilde{X}_0$ and F, for each $f' \in F$ we set

$$P_{f'} = (G \setminus U, S, X_0 \cup \tilde{X}_0, X \setminus \tilde{X}_0, \emptyset, Y^* \cup W, f \cup f')$$

and output the collection $\mathcal{L} = \{P_{f'}\}_{f' \in F}$, which has the desired properties.

Start with $U = W = \tilde{X}_0 = \emptyset$. For $v \in Y$, let $M(v) = L_P(v) \setminus f(N(v) \cap (S \cup X_0))$. For $L \subsetneq [4]$, we denote by M_L the list assignment $M_L(v) = M(v) \cap L$. To construct U, W and $\tilde{X}_0$, we first examine each component of $G|Y$ separately. Every time we enlarge U, we will "restart" the algorithm with $(G, S, X_0, X, Y, Y^*, f)$ replaced by $(G \setminus U, S, X_0, X, Y \setminus U, Y^*, f)$. Since we only do this when U is enlarged, there will be at most $|V(G)|$ such iterations, and so it is enough to ensure that each iteration can be done in polynomial time.

Let C be a component of $G|Y$. If no vertex of X is mixed on C, and some vertex of $S \cup X_0 \cup X$ is complete to C, we add $V(C)$ to W. So we may assume that either some vertex of X is mixed on C, or no vertex of X is complete to C. Let $C_i = \{v \in V(C) : L_P(v) = [4] \setminus \{i\}\}$. Since P satisfies (I), it follows that $V(C) = \bigcup_{i=1}^{4} C_i$,

Suppose first that C meets exactly one list L. Since P satisfies (VI), it follows that no vertex of X is mixed on C, and so $N(V(C)) \subseteq S \cup X_0$. By Theorem 2, we can test in polynomial time if (C, M) is colorable. If not, then P has no precoloring extension, we set $\mathcal{L} = \emptyset$ and stop. If (C, M) is colorable, then deleting $V(C)$ does not change the existence of a precoloring extension for P, and we add $V(C)$ to U.

Now suppose that C meets at least three lists. By Lemma 38 C_i is complete to C_j for every $i \neq j$. Since P satisfies (VI), it follows that no vertex of X is mixed on C, and so $N(V(C)) \subseteq S \cup X_0$. Since C_i is non-empty for at least three values of i, it follows that in every proper coloring of C, at most two colors appear in C_i, and for $i \neq j$ the sets of colors that appear in C_i and C_j are disjoint. By Theorem 13, for every $L \subset [4]$ with $|L| \leq 2$ and for every i, we can test in polynomial time if $(C|C_i, M_L)$ is colorable. If there exist disjoint lists $L_1, \ldots, L_4$ such that (G_i, M_{L_i}) is colorable for all i, then deleting $V(C)$ does not change the existence of a precoloring extension for P, and we add $V(C)$ to U. If no such $L_1, \ldots, L_i$ exist, then P has no precoloring extension, we set $\mathcal{L} = \emptyset$ and stop.

Thus we may assume that C meets exactly two lists, say $V(C) = C_3 \cup C_4$. Let $A_1, \ldots, A_k$ be the components of $C|C_3$ and $A_{k+1}, \ldots, A_t$ be the components of $C|C_4$. Since P satisfies (II), for every $i \in [k]$ and $j \in \{k+1, \ldots, t\}$, A_i is either complete or anticomplete to A_j, and since P satisfies (IV), for every $i \in [t]$ no vertex of X is mixed on A_i. Since P satisfies (VII), if $x \in X$ has a neighbor in C, then $L_P(x) = \{1, 2\}$. By Theorem 2, for every A_i and for every $L \subseteq [4]$ with $|L \cap \{1, 2\}| \leq 1$, we can test in polynomial time if (A_i, M_L) is colorable. If (A_i, M_L) is colorable, we say that the set $M_L \cap \{1, 2\}$ *works* for A_i. Suppose that $\emptyset$ works for i. We may assume $i = 1$. It follows that (A_1, M) can be colored with color 3. Since $N(V(A_1)) \subseteq S \cup X_0 \cup X_{\{1,2\}} \cup C_4$, it follows that deleting A_i does not change the existence of a precoloring extension for P, and so we add $V(A_i)$ to U. Thus we may assume that $\emptyset$ does not work for any i.

Since C is connected and both C_3, C_4 are non-empty, it follows that for every i there is j such that A_i is complete to A_j, and so in every proper coloring of C, at most one of the colors $1, 2$ appears in each $V(A_i)$. Since $\emptyset$ does not work for any i, it follows that in every precoloring extension of P, exactly one of the colors $1, 2$ appears in each $V(A_i)$, and both 1 and 2 appear in $V(C)$. If some $x \in X$ is complete to C, then $x \in X_{\{1,2\}}$, and so G has no precoloring extension; we set $\mathcal{L} = \emptyset$, and stop. Thus we may assume that no vertex of X is complete to $V(C)$.

Let X_C be the set of vertices of X that are mixed on $V(C)$. Then $X_C \subseteq X_{\{1,2\}}$, and $N(V(C)) \subseteq S \cup X_0 \cup X_C$. Let $A_C = \{a_1, \ldots, a_t\}$. Let H_C be the graph with vertex set $X_C \cup A_C$, where

- $a_i a_j \in E(H_C)$ if and only if A_i is complete to A_j,
- for $x \in X_C$, $xa_i \in E(H_C)$ if and only if x is complete to A_i, and
- $H_C|(X_C) = G|(X_C)$.

Let $T_C(a_i)$ be the the union of all the sets that work for i. Suppose first that $X_C = \emptyset$. Then $N(V(C)) \subseteq S \cup X_0$. By Theorem 13 we can test in polynomial time if (H_C, T_C) is colorable. If (H_C, T_C) is not colorable, then P has no precoloring extension; we output $\mathcal{L} = \emptyset$ and stop. Thus we may assume that (H_C, T_C) is colorable. Since $N(V(C)) \subseteq S \cup X_0$, deleting $V(C)$ does not change the existence of a precoloring extension, and we add $V(C)$ to U. Thus we may assume that $X_C \neq \emptyset$.

Now let $C^1, \ldots, C^l$ be all the components of $G|Y$ for which $V(C^i) = C_3^i \cup C_4^i$ and $X_C \neq \emptyset$. Let H be the graph with vertex set $\bigcup_{i=1}^{l} V(H_{C^i})$ and such that $uv \in E(H)$ if and only if either

- $uv \in E(H_{C^i})$ for some i, or
- $u, v \in X$ and $uv \in E(G)$.

Let $T(v) = T_C(v)$ if $v \in V(H) \setminus X$, and let $T(v) = M(v)$ if $v \in V(H) \cap X$. By Theorem 13, we can test in polynomial time if (H, T) is colorable. If (H, T) is not colorable, then P has no precoloring extension; we output $\mathcal{L} = \emptyset$ and stop. Thus we may assume that (H, T) is colorable. Note that $T(v) \subseteq \{1, 2\}$ for every $v \in V(H)$.

Next we will show H is connected, and therefore (H, T) has at most two proper colorings, and we can compute the set of all proper colorings of (H, T) in polynomial time. Suppose that H is not connected. Since each C^i is connected, it follows that $H|A_{C^i}$ is connected for all i, and since for every i, every vertex of X_{C^i} has a neighbor in A_{C^i}, it follows that $H|V(H_{C^i})$ is connected for every i. Let D_1, D_2 be distinct components of H. Since $G \setminus (S \cup X_0)$ is connected, there is exist $p, q \in [l]$ such that $V(H_{C^p}) \subseteq D_1$, $V(H_{C^q}) \subseteq D_2$, and there is a path $P = p_1 - \ldots - p_m$ in $G \setminus (S \cup X_0)$ with $p_1 \in V(C^p) \cup X_{C^p}$, $p_m \in V(C^q) \cup X_{C^q}$, and P^* is disjoint from $\bigcup_{i=1}^{l}(V(C^i) \cup X_{C^i})$. Since for every i, $N(V(C^i)) \subseteq S \cup X_0 \cup X_{C^i}$, it follows that $p_1 \in X_{C^p}$ and $p_m \in X_{C^q}$, and P^* is anticomplete to $V(C^p) \cup V(C^q)$. By Lemma 19, there exist $a_p, b_p \in V(C^p)$ such that $p_m - a_p - b_p$ is a path, and there exist $a_q, b_q \in V(C^q)$ such that $p_m - a_q - b_q$ is a path. But now $b_p - a_p - p_1 - P - p_m - a_q - b_q$ is a path of length at least six in G, a contradiction. This proves that H is connected.

Let $\tilde{X}_0^{3,4} = V(H) \cap X$, and let $F^{3,4}$ be the set of all proper colorings of $(G|\tilde{X}_0^{3,4}, M)$ that extend to a coloring of (H, T). Then $|F^{3,4}| \leq 2$, and we can compute $F^{3,4}$ in polynomial time. Let $U^{3,4} = \bigcup_{i=1}^{l} V(C^i)$. Since for each i, $N(C^i) \subseteq \tilde{X}_0^{3,4} \cup S \cup X_0$, it follows that

P has a precoloring extension if and only if for some $f' \in F^{3,4}$

(33) $$(G \setminus U^{3,4}, S, X_0 \cup \tilde{X}_0^{3,4}, X \setminus \tilde{X}_0^{3,4}, Y \setminus U^{3,4}, Y^*, f \cup f')$$

has a precoloring extension.

For every $i, j \in [4]$ with $i \neq j$ define $U^{i,j}$, $F^{i,j}$ and $\tilde{X}_0^{i,j}$ similarly. Let $\tilde{X}_0 = \bigcup \tilde{X}_0^{i,j}$. Let F be the set of all functions $f' : \tilde{X}_0 \to [4]$ such that $f'|_{\tilde{X}_0{}^{i,j}} \in F^{i,j}$. Then $|F| \leq 2^6$. Let $U' = \bigcup U^{i,j}$.

If follows from (33) that P has a precoloring extension if and only if

$$(G \setminus U', S, X_0 \cup \tilde{X}, X \setminus \tilde{X}_0, Y \setminus U', Y^*, f \cup f')$$

has a precoloring extension for some $f' \in F$. Now we add U' to U, and Lemma 44 follows. □

We are now ready to prove our the main result, which we restate:

Theorem 18. *There exists an integer $C > 0$ and a polynomial-time algorithm with the following specifications.*

Input: *A 4-precoloring (G, X_0, f) of a P_6-free graph G.*

Output: *A collection $\mathcal{L}$ of excellent starred precolorings of G such that*

1. *$|\mathcal{L}| \leq |V(G)|^C$,*
2. *for every $(G', S', X_0', X', \emptyset, Y^*, f') \in \mathcal{L}$*
   - *$|S'| \leq C$,*
   - *$X_0 \subseteq S' \cup X_0'$,*
   - *G' is an induced subgraph of G, and*
   - *$f'|_{X_0} = f|_{X_0}$.*
3. *if we know for every $P \in \mathcal{L}$ whether P has a precoloring extension, then we can decide in polynomial time if (G, X_0, f) has a 4-precoloring extension; and*
4. *given a precoloring extension for every $P \in \mathcal{L}$ such that P has a precoloring extension, we can compute a 4-precoloring extension for (G, X_0, f) in polynomial time, if one exists.*

Proof. Let (G, X_0, f) be a 4-precoloring of a P_6-free graph G. We apply Theorem 17 to (G, X_0, f) to obtain a collection $\mathcal{L}_0$ of good seeded precolorings with the desired properties. Then we apply Lemma 32 to each seeded precoloring in $\mathcal{L}_0$ to obtain a starred precoloring satisfying (I); let $\mathcal{L}_1$ be the collection thus obtained. Next, starting with $\mathcal{L}_1$, apply Lemma 34, Lemma 37, Lemma 39, Lemma 41, Lemma 42, Lemma 43 and Lemma 44 to each element in the output of the previous one, to finally obtain a collection $\mathcal{L}$. Then $\mathcal{L}$ is an equivalent collection for P, and every element of $\mathcal{L}$ satisfies (II), (III), (IV), (V), (VI), (VII) and (VIII). Finally, (VIII) implies that each starred precoloring in $\mathcal{L}$ is excellent, as claimed. □