Tracial and ideal structure of crossed products and related constructions

by

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Author's Declaration

This thesis consists of material all of which I authored or co-authored: see Statement of Contributions included in the thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Statement of Contributions

Chapter 3 was solely authored by Dan Ursu, and was fully published as [Urs21]. We repeat the citation here:

Dan Ursu, *Characterizing traces on crossed products of noncommutative C*-algebras*, Advances in Mathematics **391** (2021), 107955. DOI : 10.1016/j.aim.2021.107955

Chapter 4 was solely authored by Dan Ursu, and was not written for publication.

Chapter 5, was solely authored by Dan Ursu, and was fully published as [Urs22]. We repeat the citation here:

Dan Ursu, *Relative C*-simplicity and characterizations for normal subgroups*, Journal of Operator Theory **87** (2022), no. 2, 471–486.

Chapter 6 was coauthored by Tattwamasi Amrutam and Dan Ursu, and is a paper fully published as [AU22]. Both authors contributed roughly equally to the development of the manuscript. We repeat the citation here:

Tattwamasi Amrutam and Dan Ursu, *A generalized Powers averaging property for commutative crossed products*, Transactions of the American Mathematical Society **375** (2022), no. 3, 2237–2254. DOI : 10.1090/tran/8567

Chapter 7 was coauthored by Matthew Kennedy, Se-Jin Kim, Xin Li, Sven Raum, and Dan Ursu, and is a preprint currently submitted for publication. The preprint can be found at [KKL+21]. All five authors contributed roughly equally to the development of the manuscript.

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Appendix A is a short compilation of some of the existing work in the literature. The work itself was not authored by Dan Ursu, but the presentation is original, and some new proofs of old results are also sometimes given.

Abstract

In this thesis, we concern ourselves with asking questions about the basic structure of group C*-algebras $C^*_\lambda(G)$, crossed products $A \rtimes G$ and $A \rtimes_\lambda G$, and groupoid C*-algebras $C^*_r(\mathcal{G})$ and $C^*_{\text{ess}}(\mathcal{G})$. Specifically, we are concerned with two main topics. One is the simplicity of these algebras, and we either extend work that was already done in the case of group C*-algebras and crossed products, or characterize simplicity altogether in the case of groupoid C*-algebras. The other is the structure of traces on these algebras, in particular in the case of crossed products.

In the third chapter, we give complete descriptions of the tracial states on both the universal and reduced crossed products $A \rtimes G$ and $A \rtimes_\lambda G$ of a C*-dynamical system consisting of a unital C*-algebra A and a discrete group G. In particular, we also answer the question of when the tracial states on the crossed products are in canonical bijection with the G-invariant tracial states on A. This generalizes the unique trace property for discrete groups. The analysis simplifies greatly in various cases, for example when the conjugacy classes of the original group G are all finite, and in other cases gives previously known results, for example when the original C*-algebra A is commutative. We also obtain results and examples in the case of abelian groups that contradict existing results in the literature of Bédos and Thomsen. Specifically, we give a finite-dimensional counterexample, and provide a correction to the result of Thomsen.

The fourth chapter is a short note on results in the von Neumann crossed product case that were never submitted for publication, and the author suspects might potentially be folklore, but cannot actually find anywhere. We extend the results on C*-crossed products from the third chapter to the case of von Neumann crossed products $M \overline{\rtimes} G$. In particular, we obtain results that characterize when a G-invariant normal tracial state on M has a unique normal tracial extension to the crossed product. As a consequence, we also characterize when such crossed products are finite factors (i.e. either of type II_1, or isomorphic to M_n).

In the fifth chapter, we consider the notion of a plump subgroup that was recently introduced by Amrutam. This is a relativized version of Powers' averaging property, and it is known that Powers' averaging property for G is equivalent to C*-simplicity, i.e. simplicity of $C^*_\lambda(G)$. With this in mind, we introduce a relativized notion of C*-simplicity, and show that for normal subgroups $N \lhd G$ it is equivalent to plumpness, along with several other characterizations.

For the sixth chapter, we prove a generalized version of Powers' averaging property that characterizes simplicity of reduced crossed products $C(X) \rtimes_\lambda G$, where G is a countable

discrete group, and X is a compact Hausdorff space which G acts on minimally by homeomorphisms. As a consequence, we generalize results of Hartman and Kalantar on unique stationarity to the state space of $C(X) \rtimes_\lambda G$ and to Kawabe's generalized space of amenable subgroups $\mathrm{Sub}_a(X, G)$. This further lets us generalize a result of the first named author and Kalantar on simplicity of intermediate C*-algebras. We prove that if $C(Y) \subseteq C(X)$ is an inclusion of unital commutative G-C*-algebras with X minimal and $C(Y) \rtimes_\lambda G$ simple, then any intermediate C*-algebra A satisfying $C(Y) \rtimes_\lambda G \subseteq A \subseteq C(X) \rtimes_\lambda G$ is simple.

For the seventh chapter, we characterise, in several complementary ways, étale groupoids with locally compact Hausdorff space of units whose essential groupoid C*-algebra has the ideal intersection property, assuming that the groupoid is either Hausdorff or σ-compact. This leads directly to a characterisation of the simplicity of this C*-algebra which, for Hausdorff groupoids, agrees with the reduced groupoid C*-algebra. Specifically, we prove that the ideal intersection property is equivalent to the absence of essentially confined amenable sections of isotropy groups. For groupoids with compact space of units we moreover show that this is equivalent to the uniqueness of equivariant pseudo-expectations. A key technical idea underlying our results is a new notion of groupoid action on C*-algebras including the essential groupoid C*-algebra itself. For minimal groupoids, we further obtain a relative version of Powers averaging property. Examples arise from suitable group representations into simple groupoid C*-algebras. This is illustrated by the example of the quasi-regular representation of Thompson's group T with respect to Thompson's group F, which satisfies the relative Powers averaging property in the Cuntz algebra $\mathcal{O}_2$.

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For Chapter 3, the author would like to thank his supervisor, Matthew Kennedy, for giving detailed comments and suggestions throughout the original paper [Urs21] being written. In addition, the author would also like to thank Erik Bédos, Mehrdad Kalantar, Sven Raum, and Klaus Thomsen for taking the time to look through a near-finished draft of the paper and giving helpful feedback. The author would also like to thank Se-Jin (Sam) Kim and Nicholas Manor for discussions on the validity of Proposition 3.2.5 and Proposition 3.2.10, respectively.

For Chapter 4, this was a short note that was never submitted for publication. The author would like to thank Erik Bédos for some interesting discussion on whether the results are already known or worthwhile, and some discussion on the twisted case, relating this back to Kleppner's results for twisted group von Neumann algebras.

For Chapter 5, the author would like to thank his supervisor, Matthew Kennedy, for giving detailed comments and suggestions throughout the development of the original paper [Urs22]. In addition, the author would also like to thank Tattwamasi Amrutam, Mehrdad Kalantar, and Sven Raum for looking through a draft of the paper and giving helpful feedback.

For Chapter 6, this was originally a paper [AU22] coauthored with Tattwamasi Amrutam. The authors owe a huge debt of gratitude to Mehrdad Kalantar and Matthew Kennedy, who are their advisors respectively, for many useful discussions surrounding the problem. The authors also thank Sven Raum, Eusebio Gardella, Shirley Geffen and Yongle Jiang for taking the time to carefully read a near complete draft of the paper and giving helpful feedback. The authors would also like to thank the anonymous referee for the detailed reading of the paper and for their comments and suggestions which enhanced the exposition of the paper.

For Chapter 7, this was originally a paper [KKL+21] coauthored with Matthew Kennedy, Se-Jin Kim, Xin Li, and Sven Raum. The authors are grateful to Jean Renault for pointing out the references [FS82] and [Ren91].

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Other coauthors may have their own specific acknowledgements or funding information. These can be found in the original papers.

Dedication

This is dedicated to the people who helped me out in life when I needed them the most. You know who you are, and I am forever grateful.

Table of Contents

Chapter 1

Introduction

This thesis is largely a collection of papers that the author wrote over his PhD program, either singly-authored or coauthored. Each chapter, with the exception of Chapter 4, corresponds to an individual paper that is either already published, or posted on arXiv. Information on these papers and any coauthors can be found in both the Statement of Contributions section and the Acknowledgements section. While each chapter comes with its own introduction and motivation, we provide a brief, somewhat more unified overview here as well.

We will concern ourselves with C*-algebras and von Neumann algebras, in particular those arising out of some kind of construction involving groups or groupoids, such as reduced group C*-algebras $C^*_\lambda(G)$, crossed products $A \rtimes_\lambda G$, and groupoid C*-algebras $C^*_r(\mathcal{G})$ and $C^*_{\text{ess}}(\mathcal{G})$. Most basic facts about these constructions, and C*-algebras in general, can either be found in Chapter 2, or in one of the chapter-specific preliminary sections.

The first couple of chapters are concerned with the question of describing the tracial states on many of these algebras. This is often of interest, as for example, tracial data shows up in the Elliot invariant in classification theory. In the case of commutative crossed products and groupoid C*-algebras, the problem is essentially solved in the case of universal groupoid C*-algebras $C^*(\mathcal{G})$ by Neshveyev in [Nes13] (and consequently in universal commutative crossed products $C(X) \rtimes G$), and results can at least be obtained on reduced commutative crossed products $C(X) \rtimes_\lambda G$ by a result of Bryder and Kennedy [BK16, Theorem 5.2] reducing things to the amenable radical of G.

In Chapter 3, we focus on noncommutative crossed products $A \rtimes G$ and $A \rtimes_\lambda G$, obtaining complete descriptions of tracial extensions of G-invariant traces on A. In particular, we also attempt to answer the question of when a G-invariant trace on A has a *unique* tracial extension to each of the crossed products, with respect to various assumptions on G. It was already known that some form of *proper outerness* of the action is usually sufficient, but we provide weaker conditions that turn out to be true *if and only if* conditions in many cases. We also give counterexamples to results of Bédos [Béd96, Proposition 11] and Thomsen [Tho95, Theorem 4.3] in the case of G being abelian, providing corrections to Thomsen's result (with an analogous correction likely holding for Bédos' result, but not being explicitly worked out). We also work out the cases when G is an FC-group, and when $G = \mathbb{Z}$. Interestingly enough, this last case thankfully turns out to agree with the original result of Thomsen, and it is often a case that people are interested in.

Chapter 4 is a short unpublished note that takes inspiration from the results of Chapter 3, but this time works in the case of von Neumann crossed products $M\overline{\rtimes}G$. We answer the question of when G-invariant normal tracial states on M have unique normal tracial extension to the entire crossed product, and as a result, also deduce exactly when the crossed product is a finite factor. I suspect that some of the results in this chapter might already be folklore among the von Neumann algebra community, but I cannot actually find them anywhere.

Moving away from traces, another basic question about the various C*-algebraic constructions mentioned earlier that gathered much interest over time is that of figuring out when these are simple. While this is an older question, with one of the first major results being the result of Powers in [Pow75] that $C^*_\lambda(\mathbb{F}_2)$ is simple, the first true *if and only if* results only started arising over the last few years, due to Breuillard, Kalantar, Kennedy, and Ozawa over several papers. A brief history of their results, along with a short proof, can be found in Appendix A.

One of the results that arose out of this, and we will draw our attention to, was Kennedy and Haagerup independently demonstrating in [Ken20] and [Haa16], respectively, that C*-simplicity of G is equivalent to an averaging property originally considered by Powers to show that $\mathbb{F}_2$ is C*-simple. In short, Powers' averaging property can be formulated as follows: given any $a \in C^*_\lambda(G)$, we have that

$$\tau_\lambda(a) \in \overline{\operatorname{conv}} \{g \cdot a \mid g \in G\},$$

where τ_λ is the canonical trace on $C^*_\lambda(G)$.

Subsequent results of a similar flavour that arose afterwards was some work done by Amrutam in [Amr21]. In particular, he was interested in characterizing a property of subgroups $H \leq G$ which he called *plump*, which is a relativized version of Powers' averaging property, as it allowed him to easily describe in certain cases C*-subalgebras lying between

$C_r^*(G)$ and $A \rtimes_\lambda G$, as long as the kernel of the action of G on A contains a plump subgroup. We say that $H \leq G$ is *plump* if for any $a \in C_\lambda^*(G)$, we have

$$\tau_\lambda(a) \in \overline{\text{conv}}\,\{h \cdot a \mid h \in H\}\,.$$

In Chapter 5, we give complete characterizations for plumpness for normal subgroups $N \triangleleft G$, as the interest originally arose out of studying kernels anyways. In particular, we also show that it is equivalent to a relativized notion of C*-simplicity, along with other relativized characterizations inspired from the existing C*-simplicity results.

Chapter 6 also takes inspiration from Powers' averaging property, but in a different direction. The simplicity of commutative crossed products $C(X) \rtimes_\lambda G$ was already characterized by Kawabe in [Kaw17], but an equivalent version of Powers' averaging property was lacking. This is what is done in this chapter. Namely, we show that $C(X) \rtimes_\lambda G$ is simple if and only if the action of G on X is minimal and

$$E(a) \in \overline{C(X) - \text{conv}}\,\{g \cdot a \mid g \in G\}\,,$$

where we now make use of $C(X)$-convex combinations. From here, we proceed to generalize results of Hartman and Kalantar [HK17] on unique stationarity of states on either the reduced group C*-algebra $C_\lambda^*(G)$ and the space of subgroups $C(\text{Sub}_a(G))$, except this time to the crossed product $C(X) \rtimes_\lambda G$, and to Kawabe's space of amenable stabilizer subgroups $C(\text{Sub}_a(X, G))$. We also generalize results of Amrutam and Kalantar [AK20] on simplicity of intermediate C*-algebras $C_\lambda^*(G) \subseteq A \subseteq C(X) \rtimes_\lambda G$ in the case of C*-simple groups G, but this time in the case of intermediate C*-algebras between arbitrary simple commutative crossed products.

Going back to the question of characterizing when such algebras are simple, the next logical step after commutative crossed products is the case of étale groupoids. It is worth noting that groupoid C*-algebras provide quite a large class of C*-algebras, as it is the combined result of several authors that every classifiable C*-algebra is the reduced twisted groupoid C*-algebra of a Hausdorff étale groupoid. While the case of commutative crossed products was solved without much difficulty in [AS94] for universal crossed products $C(X) \rtimes G$, and generalized from reduced group C*-algebras $C_\lambda^*(G)$ to reduced crossed products $C(X) \rtimes_\lambda G$ in [Kaw17] without too much hassle either, the case of groupoid C*-algebras proved to be much harder. A generalization of Archbold and Spielberg's result for $C(X) \rtimes G$ would only arrive for the universal groupoid C*-algebra $C^*(\mathcal{G})$ twenty years later in [BCFS14], and partial results in the reduced case very recently by Borys in [Bor19] and [Bor20]. Our main contribution in Chapter 7 was making Borys' results a true *if and only if*, but also obtaining far more general results that don't assume the groupoid is minimal, has compact unit space, or is even Hausdorff. Generalizations of the generalized Powers' averaging property mentioned earlier for crossed products $C(X) \rtimes_\lambda G$ are also obtained.

Chapter 2

General preliminaries

Each individual chapter has chapter-specific preliminaries. However, there are some common facts that the reader should be aware of, and will often be used without reference. A good reference for completely positive maps and related topics is [Pau03], and a good reference for group C*-algebras, amenability, and many other things is [BO08] and [Dav96]. It is assumed that the reader is at least familiar with basic facts about C*-algebras. The C*-algebras that we work with will always be unital unless otherwise specified, and likewise the groups will always be discrete.

2.1 Completely positive maps

First, we recall the basics of unital and completely positive maps. Given a C*-algebra A, consider the matrix algebra $M_n(A)$. By faithfully representing $A \subseteq B(H)$, we can view $M_n(A) \subseteq B(H^n)$. This gives us a norm structure on $M_n(A)$ that makes it a C*-algebra as well. The norm structure is unique and hence does not depend on the choice of representation $A \subseteq B(H)$.

A linear map $\phi : A \to B$ between C*-algebras is called *positive* if whenever $a \in A$ with $a \geq 0$, we have $\phi(a) \geq 0$. Such maps will automatically satisfy $\phi(a^*) = \phi(a)^*$ for all $a \in A$. We may also consider the linear map $\phi^{(n)} : M_n(A) \to M_n(B)$ given by

$$\phi^{(n)}([a_{ij}]) = [\phi(a_{ij})]$$

The map ϕ is called *completely positive* if $\phi^{(n)}$ is positive for each $n \in \mathbb{N}$. It is known that if a map $\phi : A \to B$ is unital and completely positive, then it is also *completely contractive* as well, in the sense that each $\phi^{(n)}$ is contractive. A map $\phi : A \to B$ is called a *complete order embedding* if given any $x \in M_n(A)$, we have $x \geq 0$ if and only if $\phi^{(n)}(x) \geq 0$. Every unital injective *-homomorphism is automatically a complete order embedding.

In the category of unital C*-algebras, together with unital and completely positive maps as morphisms, we automatically have that isomorphisms in this category exactly coincide with *-isomorphisms. That is, if $\phi : A \to B$ and $\psi : B \to A$ are unital and completely positive maps with the property that $\psi \circ \phi = \mathrm{id}_A$ and $\phi \circ \psi = \mathrm{id}_B$, then they are automatically *-isomorphisms. In particular, if $\phi : A \to B$ is a unital, surjective, complete order embedding, then it is automatically a *-isomorphism.

Given any unital and completely positive map $\phi : A \to B$, we canonically have a left ideal and a right ideal associated to it by:

$$L_\phi = \{a \in A \mid \phi(a^*a) = 0\}$$

and

$$R_\phi = \{a \in A \mid \phi(aa^*) = 0\},$$

respectively.

Every unital and completely positive map also has a *multiplicative domain* given by

$$\mathrm{mult}(\phi) = \{a \in A \mid \phi(a^*a) = \phi(a)^*\phi(a) \text{ and } \phi(aa^*) = \phi(a)\phi(a)^*\}.$$

As it turns out, this is in fact a unital C*-subalgebra of A, and moreover, given any $a \in A$ and $b \in \mathrm{mult}(\phi)$, we have that

$$\phi(ab) = \phi(a)\phi(b) \text{ and } \phi(ba) = \phi(b)\phi(a)$$

2.2 Amenability

The standard definition of amenability is as follows. A discrete group G is called *amenable* if there is a left-invariant state on $\ell^\infty(G)$, where we are considering the left-translation action given by $(s \cdot f)(t) = f(s^{-1}t)$.

There is a more geometric interpretation of amenability. Consider a compact convex subset K of a locally convex Hausdorff vector space V. An example that will show up very often is the state space of a unital C*-algebra. We say that a map $\phi : K \to K$ is *affine* if it respects convex combinations, i.e.

$$\phi(\alpha k + (1-\alpha)l) = \alpha\phi(k) + (1-\alpha)\phi(l).$$

A discrete group G is amenable if and only if any action on a compact convex set K by affine homeomorphisms has a fixed point, i.e. a point $k \in K$ such that $s \cdot k = k$ for all $s \in G$.

Amenability is preserved under taking subgroups $H \leq G$. Moreover, given a group G and a normal subgroup N, we have that G is amenable if and only if both N and G/N are. It is also preserved under taking direct limits.

The two most basic examples of amenable groups are all finite groups and all abelian groups. The most basic example of a nonamenable group is $\mathbb{F}_n$, the free group on n generators, for $n \geq 2$.

2.3 Group C*-algebras and crossed products

Before we begin, recall that, given an inclusion $A \subseteq B$ of unital C*-algebra, a *conditional expectation* $E : B \to A$ is a unital and completely positive map such that $E|_A = \mathrm{id}_A$.

Given a discrete group G, we may consider the left-regular representation on $\ell^2(G)$, the Hilbert space with orthonormal basis $\{\delta_s \mid s \in G\}$. The group G acts on this space by unitaries λ_s, which permute the basis vectors around by left translation as follows: $\lambda_s \delta_t = \delta_{st}$. It is easy to check that $\lambda_s^* = \lambda_{s^{-1}}$ and $\lambda_s \lambda_t = \lambda_{st}$. This representation $\lambda : G \to U(\ell^2(G))$ extends to a *-representation of the group ring $\lambda : \mathbb{C}[G] \to B(\ell^2(G))$, which is in fact injective on the group ring. From here, we may define the *reduced group C*-algebra* as follows:

$$C^*_\lambda(G) := \overline{\mathrm{span}}^{\|\cdot\|} \{\lambda_s \mid s \in G\} = \overline{\lambda(\mathbb{C}[G])}^{\|\cdot\|}.$$

It is easy to see that every element $a \in C^*_\lambda(G)$ has a unique associated Fourier series $a \sim \sum_{t \in G} \alpha_t \lambda_t$ (convergence is irrelevant), which corresponds to a matrix in $B(\ell^2(G))$ whose (r, s)-entry is $\alpha_{rs^{-1}}$. Thus, we get a faithful trace $\tau_\lambda : C^*_\lambda(G) \to \mathbb{C}$ that sends an element $a \sim \sum_{t \in G} \alpha_t \lambda_t$ to α_e. In other words, $\tau_\lambda(a) = \langle\, a\delta_e \,|\, \delta_e \,\rangle$.

The reduced group C*-algebras play quite nicely with respect to subgroups. Given a subgroup $H \leq G$, we canonically have an inclusion $C^*_\lambda(H) \subseteq C^*_\lambda(G)$, and a conditional expectation $E_H : C^*_\lambda(G) \to C^*_\lambda(H)$ sending λ_s to itself if $s \in H$, and zero otherwise.

There is also a *universal group C*-algebra* $C^*(G)$, which is the norm-completion of $\mathbb{C}[G]$ under the norm given by

$$\|a\| = \sup \{\|\pi(a)\| \mid \pi : G \to U(H) \text{ representation}\}$$

for every $a \in \mathbb{C}[G]$ (we are using the fact that every representation $\pi : G \to U(H)$ extends to a *-representation $\pi : \mathbb{C}[G] \to B(\ell^2(G))$). This C*-algebra has the property that given any representation $\pi : G \to U(H)$, there is an extension $\widetilde{\pi} : C^*(G) \to B(H)$ with the property that $\widetilde{\pi}(u_s) = \pi(s)$ for all $s \in G$. Observe that, as a consequence of applying this to the left-regular representation on $\ell^2(G)$, we have that $\mathbb{C}[G]$ canonically embeds into $C^*(G)$ as well.

Amenability of groups is characterized by these two group C*-algebras. Namely, a discrete group G is amenable if and only if the canonical *-homomorphism $\pi : C^*(G) \to C^*_\lambda(G)$ is given by $\pi(u_s) = \lambda_s$ is in fact a *-isomorphism. This is also equivalent to the trivial representation of G on $\mathbb{C}$ (sending each group element to 1) extending to a *-homomorphism $1_G : C^*_\lambda(G) \to \mathbb{C}$.

Crossed products are a similar story. Consider a unital C*-algebra A, and a discrete group G acting on A by *-automorphisms. We say that a unital C*-algebra B is a *C*-crossed product* of A and G if:

- $A \subseteq B$ as a unital C*-subalgebra.
- $G \subseteq B$ as unitaries u_s.
- The action of G on A is inner inside of B, in the sense that $u_s a u_s^* = s \cdot a$.
- B is generated as a C*-algebra by A and G.

Observe that we canonically have that if B is a crossed product of A and G, then

$$B = \overline{\left\{ \sum_{\text{finite}} a_t u_t \;\middle|\; t \in G, a_t \in A \right\}}^{\|\cdot\|}.$$

Multiplying two elements au_s and bu_t occurs as follows:

$$au_s bu_t = au_s bu_s^* u_s \lambda_t = a(s \cdot b)u_{st}.$$

It will sometimes be useful to consider the *algebraic crossed product*

$$A[G] = \left\{ \sum_{\text{finite}} a_t u_t \;\middle|\; t \in G, a_t \in A \right\},$$

where the above is formal finite sums. Given any C*-crossed product B, there is always canonically a *-homomorphism $\pi : A[G] \to B$ such that π is the identity on A and G, and has dense range.

There are again two canonical C*-crossed products of A and G. The first is the reduced crossed product $A \rtimes_\lambda G$. To construct this, we first faithfully represent $A \subseteq B(H)$, and then consider the Hilbert space $H \otimes \ell^2(G)$, and a *-homomorphism $\pi : A \to B(H \otimes \ell^2(G))$ and representation $\lambda : G \to U(H \otimes \ell^2(G))$ given by:

$$\pi(a)(h \otimes \delta_t) = ((t^{-1} \cdot a)h) \otimes \delta_t$$

and

$$\lambda_s(h \otimes \delta_t) = h \otimes \delta_{st}.$$

The *reduced crossed product* $A \rtimes_\lambda G$ is the C*-algebra generated by $\pi(A)$ and $\lambda(G)$, or equivalently, the norm closure of the representation of $A[G]$ induced by π and λ. This norm completion of $A[G]$ is independent of the choice of faithful representation $A \subseteq B(H)$ from earlier. It is also the case that we canonically have an embedding $A[G] \subseteq A \rtimes_\lambda G$.

We also again get matrix representations and Fourier series. Every element $x \in A \rtimes_\lambda G$ has a unique associated Fourier series $x \sim \sum_{t \in G} a_t \lambda_t$ (convergence is irrelevant), which in turn corresponds to the matrix in $B(H \otimes \ell^2(G))$ whose (r, s)-entry is $r^{-1} a_{rs^{-1}}$.

Observe that, using the matrix representations from earlier, the projection to the (e, e)-corner gives us a faithful conditional expectation $E : A \rtimes_\lambda G \to A$, which maps $x \sim \sum_{t \in G} a_t \lambda_t$ to a_e. Moreover, equipping the C*-algebra $A \rtimes_\lambda G$ with the conjugation action $s \cdot x = \lambda_s x \lambda_s^*$, the expectation is G-equivariant. Finally, the expectation can actually uniquely determine the reduced crossed product, as we will see later.

There is a *universal crossed product* $A \rtimes G$ constructed as follows. Given any representation $\pi : A \to B(H)$, and any representation $\sigma : G \to U(H)$, with the property that $\pi(s \cdot a) = \sigma(s)\pi(a)\sigma(s)^*$, we say that (π, σ) are a covariant pair. These correspond to *-representations $\rho : A[G] \to B(H)$ satisfying $\rho|_A = \pi$ and $\rho|_G = \sigma$. The universal crossed product $A \rtimes G$ is given by the norm completion of $A[G]$ under the norm

$$\|x\| = \sup \{\rho_{\pi,\sigma}(x) \mid \pi : A \to B(H), \sigma : G \to U(H), \text{with } (\pi, \sigma) \text{ a covariant pair}\}.$$

It has the property that whenever (π, σ) are a covariant pair for A and G on the Hilbert space H, there is a *-homomorphism $\rho : A \rtimes G \to B(H)$ with the property that $\rho|_A = \pi$ and $\rho|_G = \sigma$. Again, $A[G]$ canonically embeds into the universal crossed product.

It is worth noting that by letting $A = \mathbb{C}$ in the above constructions, we have that $A \rtimes_\lambda G = C^*_\lambda(G)$ and $A \rtimes G = C^*(G)$. Furthermore, given a subgroup $H \leq G$ and an H-C*-subalgebra $B \subseteq A$, we canonically have $B \rtimes_\lambda H \subseteq A \rtimes_\lambda G$. For the universal crossed products, we still obtain a *-homomorphism, but we might no longer obtain an embedding.

Going back to amenability of groups, we have that if G is amenable, then the reduced and universal crossed products coincide. More specifically, the canonical *-homomorphism $\pi : A \rtimes G \to A \rtimes_\lambda G$ satisfying $\pi|_A = \mathrm{id}_A$ and $\pi(u_s) = \lambda_s$ is in fact a *-isomorphism. For a fixed A, the converse is not true, but it is now clearly true if A is allowed to vary over all C*-algebras (just let $A = \mathbb{C}$).

A convenient fact that I have not seen anywhere, but is probably well-known, is the following: the reduced crossed product $A \rtimes_\lambda G$ is characterized as the unique (semi)norm completion of $A[G]$ such that the map $E : A[G] \to A$ given by $E(\sum a_t \lambda_t) = a_e$ extends to a faithful conditional expectation on the completion. In other words, the norm is big enough so that the map is continuous, but small enough so that the map is faithful. Observe that by letting $A = \mathbb{C}$, a similar statement can be made about the reduced group C*-algebra.

To see this, assume that $A \rtimes_w G$ is some arbitrary norm completion of $A[G]$ with $E : A \rtimes_w G \to A$ being the canonical conditional expectation on $A[G]$. Consider the canonical *-homomorphism $\pi : A \rtimes G \to A \rtimes_w G$, and consider the composition $\widetilde{E} = E \circ \pi : A \rtimes G \to A$. Observe that this map is G-equivariant with respect to the conjugation action on $A \rtimes G$. We claim that the left ideal

$$I = \left\{ x \in A \rtimes G \;\middle|\; \widetilde{E}(x^* x) = 0 \right\}$$

in fact is equal to $\ker \pi$. To see this, assume $x \in I$. Then

$$0 = \widetilde{E}(x^* x) = E(\pi(x^* x)) = E(\pi(x)^* \pi(x)),$$

and by faithfulness, we have $\pi(x)^* \pi(x) = 0$, i.e. $\pi(x) = 0$. Conversely, if we start with $x \in \ker \pi$, then

$$\widetilde{E}(x^* x) = E(\pi(x^* x)) = E(\pi(x)^* \pi(x)) = 0,$$

i.e. $x \in I$. Thus, $I = \ker \pi$. It follows that there is an isomorphism

$$(A \rtimes G)/I \cong A \rtimes_w G$$

that canonically maps the image of $A[G]$ in one to the image of $A[G]$ in the other. However, by density of $A[G]$, the map $\widetilde{E} : A \rtimes G \to A$ does not depend on the choice of completion $A \rtimes_w G$, and therefore the ideal I does not either. Thus, any two completions of the aforementioned form are also isomorphic in the same way, or in other words, there is only one possible completion.

The von Neumann algebra constructions are similar. For groups, $L(G)$ is obtained by taking a completion of $\mathbb{C}[G]$ in $B(\ell^2(G))$ under your favourite topology that is not the norm topology (weak operator, weak*, strong operator, etc... will all give the same result), and the trace τ_λ is now a faithful normal trace. For $M\overline{\rtimes}G$, it is again a closure of $M[G]$ under an appropriate topology in $B(H \otimes \ell^2(G))$ for a faithful normal representation $M \subseteq B(H)$, and we obtain a faithful normal conditional expectation $E : M\overline{\rtimes}G \to M$. The universal equivalents are not usually studied, as they are usually far too massive. For example, the universal group von Neumann algebra would be $C^*(G)^{**}$.

Chapter 3

Characterizing traces on crossed products of noncommutative C*-algebras

3.1 Introduction and statement of main results

Understanding the tracial states on a C*-algebra is often of interest, for example in classification theory. In this paper, we concern ourselves with both the reduced and universal crossed products arising from C*-dynamical systems consisting of a unital C*-algebra A and a discrete group G. Given any G-invariant tracial state on A, we give complete descriptions of the tracial extensions to the crossed products. We also translate our characterization into an equivalent condition for when the tracial extension is unique. Finally, in various special cases, we simplify this condition on uniqueness of tracial extension.

To establish notation, A will always denote a unital C*-algebra, and G a discrete group acting on A by *-automorphisms. In addition, the term "automorphism" will always mean *-automorphism. The reduced crossed product of this action will be denoted by $A \rtimes_\lambda G$, and the universal crossed product by $A \rtimes G$. The unitary corresponding to $t \in G$ in $A \rtimes_\lambda G$ will be denoted by λ_t, and in $A \rtimes G$ will simply be denoted by t. Furthermore, $T(A)$ will denote the set of all tracial states on A, and $T_G(A)$ the set of tracial states that are invariant under the action of G. Finally, "trace" will only be used to refer to tracial states.

A key idea in our paper takes inspiration from one of the techniques used by Kennedy and Schafhauser in [KS19]. In their paper, they study the intersection property of reduced crossed products, i.e. the property that every nonzero ideal of $A \rtimes_\lambda G$ has nonzero intersection with A. A key point in their paper is that what they call *pseudoexpectations*, introduced in [KS19, Section 6], can be used to characterize the intersection property. These are G-equivariant, unital, completely positive maps $\phi : A \rtimes_\lambda G \to I_G(A)$ with $\phi|_A$ being the identity map, where $I_G(A)$ is the G-injective envelope of A. It is worth noting that this notion of pseudoexpectation is based on the original, different notion of pseudoexpectation introduced by Pitts in [Pit17], and studied by both Pitts and Zarikian in subsequent papers.

We adapt the notion of pseudoexpectations used by Kennedy and Schafhauser to one that can instead be used to characterize tracial extensions of $\tau \in T_G(A)$ to both the universal and reduced crossed products. It has been previously recognized (for example in [Béd93, Section 2]) that the dynamics of G on $\pi(A)''$, where $\pi : A \to B(H_\tau)$ is the GNS representation of τ, plays an important role in determining the tracial extensions to the crossed products. See the review we give in Section 3.2.1 for why we have an action of G on this von Neumann algebra, along with other basic properties. This seems to suggest that $\pi(A)''$ is the appropriate object to consider in place of the G-injective envelope $I_G(A)$.

As we will make use of the amenable radical of G when dealing with the reduced crossed product, the notion of pseudoexpectation that we introduce works relative to any normal subgroup of G. Note that, given a normal subgroup $N \triangleleft G$, we canonically have an action of G on $A \rtimes N$ satisfying $s \cdot (at) = (s \cdot a)(sts^{-1})$ for $s \in G$ and $t \in N$, by the universal property of $A \rtimes N$.

We also note ahead of time that we will not use the term "pseudoexpectation", which as mentioned above refers to maps whose codomain is some kind of injective envelope. Instead, for our purposes, we will adopt the term "weak expectation", which instead typically refers to maps whose codomain is an enveloping von Neumann algebra. For example, the usual notion of a weak expectation of a unital inclusion of C*-algebras $A \subseteq B$ is a unital and completely positive map $F : B \to A^{**}$ extending the canonical inclusion $\iota : A \hookrightarrow A^{**}$. In our case, we replace the universal enveloping von Neumann algebra A^{**} with the von Neumann algebra $\pi(A)''$ generated under the GNS representation $\pi : A \to B(H_\tau)$ of our fixed trace $\tau \in T_G(A)$. Note that the canonical map $\pi : A \to \pi(A)''$ is not necessarily faithful, contrary to the case of A^{**}, $I(A)$, or $I_G(A)$.

Definition 3.1.1. Let $\tau \in T_G(A)$, let $\pi : A \to B(H_\tau)$ be the GNS representation of (A, τ), let $M = \pi(A)''$, and let $N \triangleleft G$ be a normal subgroup. A map $F : A \rtimes N \to M$ is called a *weak expectation* for (A, τ, G, N) if it is unital, completely positive, G-equivariant, and satisfies $F|_A = \pi$. If $N = G$, then we call such a map a weak expectation for (A, τ, G).

Theorem 3.1.2. *Let $\tau \in T_G(A)$, let $\pi : A \to B(H_\tau)$ be the GNS representation of (A, τ), let $M = \pi(A)''$, and let τ_M denote the corresponding faithful normal trace on M. Then the following sets are in natural bijection with each other:*

1. *The set of all weak expectations $F : A \rtimes G \to M$ for (A, τ, G).*
2. *The set of all $\{x_t\}_{t \in G} \subseteq M$ satisfying:*
   (a) *$x_e = 1$.*
   (b) *$x_t y = (t \cdot y)x_t$ for all $y \in M$ and $t \in G$.*
   (c) *$s \cdot x_t = x_{sts^{-1}}$ for all $s, t \in G$.*
   (d) *The matrix $[x_{st^{-1}}]_{s,t \in \mathcal{F}}$ is positive for all finite $\mathcal{F} \subseteq G$.*
3. *$\{\sigma \in T(A \rtimes G) \mid \sigma|_A = \tau\}$.*

The natural map from (1) to (2) is given by letting $x_t = F(t)$, and the natural map from (2) to (3) is given by defining a trace $\sigma \in T(A \rtimes G)$ by $\sigma(at) = \tau_M(\pi(a)x_t)$.

For the case of the reduced crossed product, we replace almost all instances of G with the amenable radical $R_a(G)$, which is the largest amenable normal subgroup of G. This was originally introduced by Day in [Day57, Section 4, Lemma 1]. The main idea making the case of the reduced crossed product tractable is that, by a result of Bryder and Kennedy [BK16, Theorem 5.2], any trace on $A \rtimes_\lambda G$ concentrates on $A \rtimes_\lambda R_a(G)$, in the sense that it vanishes on $a\lambda_t$ whenever $t \notin R_a(G)$. However, $A \rtimes_\lambda R_a(G) = A \rtimes R_a(G)$ by amenability of $R_a(G)$, and so we may apply the results we obtained in the case of universal crossed products.

Theorem 3.1.3. *Let $\tau \in T_G(A)$, let $\pi : A \to B(H_\tau)$ be the GNS representation of (A, τ), let $M = \pi(A)''$, and let τ_M denote the corresponding faithful normal trace on M. Then the following sets are in natural bijection with each other:*

1. *The set of all weak expectations $F : A \rtimes R_a(G) \to M$ for $(A, \tau, G, R_a(G))$.*
2. *The set of all $\{x_t\}_{t \in R_a(G)} \subseteq M$ satisfying:*
   (a) *$x_e = 1$.*
   (b) *$x_t y = (t \cdot y)x_t$ for all $y \in M$ and $t \in R_a(G)$.*
   (c) *$s \cdot x_t = x_{sts^{-1}}$ for all $s \in G$ and $t \in R_a(G)$.*

(d) The matrix $[x_{st^{-1}}]_{s,t\in\mathcal{F}}$ is positive for all finite $\mathcal{F} \subseteq R_a(G)$.

3. $\{\sigma \in T(A \rtimes_\lambda G) \mid \sigma|_A = \tau\}$.

The natural map from (1) to (2) is given by letting $x_t = F(\lambda_t)$, and the natural map from (2) to (3) is given by defining a trace $\sigma \in T(A \rtimes_\lambda G)$ by $\sigma(a\lambda_t) = \tau_M(\pi(a)x_t)$ for $t \in R_a(G)$, and $\sigma(a\lambda_t) = 0$ for $t \notin R_a(G)$.

Traces on $A \rtimes G$ and $A \rtimes_\lambda G$ are easiest to understand when they correspond exactly to G-invariant traces on A. Let $E : A \rtimes_\lambda G \to A$ denote the canonical expectation. There is also a canonical expectation from $A \rtimes G$ to A given by composing E with the canonical *-homomorphism from $A \rtimes G$ to $A \rtimes_\lambda G$.

Remark 3.1.4. Any trace $\sigma \in T(A \rtimes_\lambda G)$ satisfies $\sigma|_A \in T_G(A)$. Conversely, any $\tau \in T_G(A)$ gives rise to a trace on $A \rtimes_\lambda G$ by composing with the canonical expectation. That is, $\tau \circ E \in T(A \rtimes_\lambda G)$. Analogous results hold for the universal crossed product.

Keeping the above in mind, we generalize the notion of the unique trace property for discrete groups. Recall that G is said to have the unique trace property if the only trace on the reduced group C*-algebra $C^*_\lambda(G) \subseteq B(\ell^2(G))$ is the canonical one, given by $\tau_\lambda(a) = \langle a\delta_e \,|\, \delta_e \rangle$. This was shown in [BKKO17, Corollary 4.3] to be equivalent to G having trivial amenable radical.

Definition 3.1.5. Given any $\sigma \in T(A \rtimes_\lambda G)$, we will say that σ is *canonical* if it is of the form $\sigma = \tau \circ E$ for some $\tau \in T_G(A)$, or equivalently, if $\sigma = \sigma \circ E$. Canonical traces on $A \rtimes G$ are defined analogously. Given $\tau \in T_G(A)$, we will say that it has *unique tracial extension to $A \rtimes_\lambda G$* (or $A \rtimes G$) if the only $\sigma \in T(A \rtimes_\lambda G)$ (respectively, $T(A \rtimes G)$) satisfying $\sigma|_A = \tau$ is the canonical one.

Before proceeding further, we note that setting $A = \mathbb{C}$ in both of the above theorems indeed gives back previously known results.

Remark 3.1.6. Setting $A = \mathbb{C}$ in Theorem 3.1.2 gives back the well-known result that traces on the universal group C*-algebra $C^*(G)$ correspond to positive definite functions $f : G \to \mathbb{C}$ that are constant on conjugacy classes and satisfy $f(e) = 1$. In addition, setting $A = \mathbb{C}$ in Theorem 3.1.3 gives back the fact that the unique trace property for the reduced group C*-algebra $C^*_\lambda(G)$ is equivalent to having $R_a(G) = \{e\}$, as in set (2), we may always let $x_t = 1$ for $t \in R_a(G)$.

In the above theorems in set (2), condition (b) in particular highlights a link with proper outerness of the action (see the review in Section 3.2.2 for definitions). Thus, we obtain an immediate corollary which is perhaps a slight generalization of some previously known results (see [Béd96, Proposition 9], for example).

Corollary 3.1.7. *Let $\tau \in T_G(A)$, let $\pi : A \to B(H_\tau)$ be the GNS representation of (A, τ), and let $M = \pi(A)''$. If the action of $R_a(G)$ on M is properly outer, then τ has unique tracial extension to $A \rtimes_\lambda G$. If the action of G on M is properly outer, then τ has unique tracial extension to $A \rtimes G$.*

Another key idea in our paper is developed in Section 3.3, where we convert the conditions in set (2) in Theorems 3.1.2 and 3.1.3 into conditions on what we call a *partial almost inner action*, with an optional property which we call *positively compatible*. This is similar to the notion of a partial group representation—see, for example, the book of Exel [Exe17]—together with its applications in the work done by Kennedy and Schafhauser in [KS19]. We adapt the notion of a properly outer action (for single automorphisms), giving us what we call a *jointly almost properly outer action*. This condition is again sufficient, and in some special cases necessary, for $\tau \in T_G(A)$ to have unique tracial extension. All of the definitions required for this theorem and its corollaries can be found in Definitions 3.3.1 and 3.3.3.

Theorem 3.1.8. *Let $\tau \in T_G(A)$, let $\pi : A \to B(H_\tau)$ be the GNS representation of (A, τ), and let $M = \pi(A)''$.*

1. *The trace τ has unique tracial extension to the reduced crossed product $A \rtimes_\lambda G$ if and only if the action of G on M is not partially almost inner relative to the normal subgroup $R_a(G)$ with respect to some nontrivial positively compatible $\{(p_t, u_t)\}_{t \in R_a(G)}$. In particular, it is sufficient for the action to be jointly almost properly outer relative to $R_a(G)$.*

2. *The trace τ has unique tracial extension to the universal crossed product $A \rtimes G$ if and only if the action of G on M is not partially almost inner with respect to some nontrivial positively compatible $\{(p_t, u_t)\}_{t \in G}$. In particular, it is sufficient for the action to be jointly almost properly outer.*

The rest of our results are simplifications of the above theorem in certain special cases. Recall that the FC center of a group G is the set of all elements of G with finite conjugacy classes. An FC group is a group in which every conjugacy class is finite, i.e. one that is equal to its FC center. It is known that FC groups are amenable, and so in particular this next result applies to such groups.

Corollary 3.1.9. *Assume G is a group with the property that the amenable radical $R_a(G)$ and the FC center coincide. Let $\tau \in T_G(A)$, let $\pi : A \to B(H_\tau)$ denote the GNS representation, and let $M = \pi(A)''$. Then τ has unique tracial extension to $A \rtimes_\lambda G$ if and only if the action of G on M is jointly almost properly outer relative to the normal subgroup $R_a(G)$.*

The conditions of the above theorem simplify even further in the case of groups whose amenable radical is equal to the center. In particular, the following corollary applies to abelian groups.

Corollary 3.1.10. *Assume G is a group with the property that $R_a(G) = Z(G)$. Let $\tau \in T_G(A)$, let $\pi : A \to B(H_\tau)$ denote the GNS representation, and let $M = \pi(A)''$. Then τ has unique tracial extension to $A \rtimes_\lambda G$ if and only if for any $t \in R_a(G) \setminus \{e\}$, there does not exist a central projection $p \neq 0$ in M and $u \in U(Mp)$ with the properties that:*

1. *$s \cdot p = p$ and $s \cdot u = u$ for all $s \in G$.*
2. *t acts by $\operatorname{Ad} u$ on Mp.*

In the case of abelian groups, various results of Bédos and Thomsen on finite factoriality of von Neumann crossed products [Béd96, Proposition 11], and unique tracial extension for C*-crossed products [Tho95, Theorem 4.3], respectively, already exist in the literature, but they are incorrect. This is investigated in Section 3.5.1, where a finite-dimensional counterexample is given. Corollary 3.1.10 serves as a correction to the result of Thomsen. Interestingly enough, even though Section 3.5.2 gives a counterexample in the case of finite cyclic groups, Thomsen's result still holds in the case of integer actions.

Theorem 3.1.11. *Assume $\alpha \in \operatorname{Aut}(A)$, and consider the corresponding action of $\mathbb{Z}$ on A. Let $\tau \in T_{\mathbb{Z}}(A)$, let $\pi : A \to B(H_\tau)$ denote the GNS representation, and let $M = \pi(A)''$. Then τ has unique tracial extension to $A \rtimes_\lambda \mathbb{Z}$ if and only if the action of $\mathbb{Z}$ on M is properly outer.*

Another case in which the characterization simplifies is in the case of crossed products of commutative C*-algebras. The case of the universal crossed product is already known—see [KTT90, Theorem 2.7]. Essential freeness and its relation to proper outerness are reviewed in Section 3.2.2.

Corollary 3.1.12. *Assume G acts on a compact Hausdorff space X by homeomorphisms, and μ is a G-invariant Radon probability measure on X. Then μ has unique tracial extension to $C(X) \rtimes_\lambda G$ if and only if the action of $R_a(G)$ on (X, μ) is essentially free. For the universal crossed product $C(X) \rtimes G$, μ has unique tracial extension if and only if the action of G on (X, μ) is essentially free.*

3.2 Preliminaries

3.2.1 Tracial GNS representations

Throughout this paper, we will make heavy use of passing from a tracial C*-algebra to the von Neumann algebra it generates under the GNS representation. Here, we establish the basic facts that we will use. This first proposition is well-known—see, for example, [Tak02, Chapter V, Proposition 3.19].

Proposition 3.2.1. *Assume A is a unital C*-algebra, and $\tau \in T(A)$. Let $\pi : A \to B(H_\tau)$ denote the GNS representation and let $M = \pi(A)''$. Then there is a faithful normal trace τ_M on M satisfying $\tau_M \circ \pi = \tau$.*

Observe that the above proposition makes no assumptions on $\tau \in T(A)$ being faithful—it is always the case that $\tau_M \in T(M)$ is faithful. In addition, τ_M is uniquely determined by normality.

It is also a basic fact of von Neumann algebras that we do not need to worry about normality when dealing with *-isomorphisms. The following can be found, for example, in [Tak02, Chapter III, Corollary 3.10].

Proposition 3.2.2. *Assume $\pi : M \to N$ is a *-isomorphism of von Neumann algebras. Then π is automatically normal and has normal inverse.*

It is well-known that any trace-preserving group action on a C*-algebra will extend to the GNS von Neumann algebra. First, we note the following result, the proof of which is straightforward and left as an exercise to the reader.

Lemma 3.2.3. *Assume A and B are unital C*-algebras, $\tau \in T(A)$ and $\sigma \in T(B)$, and $\rho : A \to B$ is a *-isomorphism satisfying $\sigma \circ \rho = \tau$. Let $\pi_\tau : A \to B(H_\tau)$ and $\pi_\sigma : B \to B(H_\sigma)$ denote the GNS representations, let $M = \pi_\tau(A)''$ and $N = \pi_\sigma(B)''$, and let $\tau_M \in T(M)$ and $\tau_N \in T(N)$ the corresponding faithful normal traces. There is a unique *-isomorphism $\widetilde{\rho} : M \to N$ that satisfies $\widetilde{\rho} \circ \pi_\tau = \pi_\sigma \circ \rho$. In addition, $\sigma_N \circ \widetilde{\rho} = \tau_M$.*

Proposition 3.2.4. *Assume A is a unital C*-algebra and $\tau \in T_G(A)$. Let $\pi : A \to B(H_\tau)$ denote the GNS representation, let $M = \pi(A)''$, and let τ_M denote the corresponding faithful normal trace on M. Letting $\alpha_t : A \to A$ denote the action of $t \in G$ on A, there are *-automorphisms $\widetilde{\alpha}_t : M \to M$ satisfying $\widetilde{\alpha}_t \circ \pi = \pi \circ \alpha_t$, and each $\widetilde{\alpha}_t$ is uniquely determined. In addition, $t \mapsto \widetilde{\alpha}_t$ defines a valid group action, and with respect to this action, we have that $\tau_M \in T_G(M)$, and $\pi : A \to M$ is G-equivariant.*

Proof. Existence and uniqueness of $\widetilde{\alpha}_t : M \to M$ satisfying $\widetilde{\alpha}_t \circ \pi = \pi \circ \alpha_t$ immediately follows from Lemma 3.2.3. It remains to check that $t \mapsto \widetilde{\alpha}_t$ indeed gives a group homomorphism:

$$\widetilde{\alpha}_s(\widetilde{\alpha}_t(\pi(a))) = \widetilde{\alpha}_s\pi(\alpha_t(a)) = \pi(\alpha_s(\alpha_t(a))) = \pi(\alpha_{st}(a)).$$

Our earlier remark on the uniqueness of these *-automorphisms tells us that $\widetilde{\alpha}_s \circ \widetilde{\alpha}_t = \widetilde{\alpha}_{st}$. Finally, the fact that $\tau_M \circ \widetilde{\alpha}_t = \tau_M$ tells us $\tau_M \in T_G(M)$, and the fact that $\widetilde{\alpha}_t \circ \pi = \pi \circ \alpha_t$ tells us $\pi : A \to M$ is G-equivariant. ∎

GNS representations also behave nicely with respect to trace-preserving inclusions of C*-algebras. The following lemma is likely already known—we offer a proof here for convenience.

Proposition 3.2.5. *Assume $A \subseteq B$ is a unital embedding of C*-algebras, and $\tau \in T(B)$. Let $\pi : A \to B(L^2(A,\tau))$ and $\sigma : B \to B(L^2(B,\tau))$ be the GNS representations of $(A, \tau|_A)$ and (B,τ), respectively, let $M = \pi(A)''$ and $N = \sigma(B)''$, and let $\tau_M \in T(M)$ and $\tau_N \in T(N)$ be the corresponding faithful normal traces. Then we have an embedding $\iota : M \to N$ with the properties that $\iota(M)$ is a von Neumann subalgebra of N, $\iota : M \to \iota(M)$ is a normal *-isomorphism with normal inverse, $\iota \circ \pi = \sigma|_A$, and $\tau_N \circ \iota = \tau_M$.*

Proof. Observe that we canonically have $L^2(A,\tau) \subseteq L^2(B,\tau)$, and let $F : B(L^2(B,\tau)) \to B(L^2(A,\tau))$ denote the compression map. Given that $L^2(A,\tau)$ is $\sigma(A)$-invariant, we have that $F(\sigma(a)) = \pi(a)$ for all $a \in A$. By normality, we have $F(\sigma(A)'') \subseteq \pi(A)''$.

We claim that $F|_{\sigma(A)''} : \sigma(A)'' \to \pi(A)''$ is injective. Observe that $\tau_M \circ F$ and τ_N agree on $\sigma(A)$, and so by normality, on $\sigma(A)''$. But τ_N is faithful, and this forces F to be faithful on $\sigma(A)''$.

Surjectivity of $F|_{\sigma(A)''} : \sigma(A)'' \to \pi(A)''$ is also easy enough to deduce—the unit ball of $\sigma(A)''$ is weak*-compact, and hence by normality it maps to a weak*-closed subset of $\pi(A)''$. In addition, the unit ball of $\sigma(A)$ maps to a norm-dense subset of the unit ball of $\pi(A)$ (this is true for any quotient map of C*-algebras). These two facts, combined with Kaplansky density, tell us that the image of the unit ball of $\sigma(A)''$ is the entire unit ball of $\pi(A)''$. Linearity takes care of the rest.

In summary, we have shown that $F|_{\sigma(A)''} : \sigma(A)'' \to \pi(A)''$ is a *-isomorphism. We claim that $\iota \coloneqq (F|_{\sigma(A)''})^{-1} : \pi(A)'' \to \sigma(A)''$ is the embedding we are looking for. By construction, we have $\iota(\pi(a))$ to $\sigma(a)$. From here, we see that

$$\tau_N(\iota(\pi(a))) = \tau_N(\sigma(a)) = \tau(a) = \tau_M(\pi(a)),$$

and so by normality, $\tau_N \circ \iota$ and τ_M agree on all of M. ∎

3.2.2 Properly outer automorphisms

It has long been recognized that proper outerness of an action of G on a C*-algebra or a von Neumann algebra leads to nice structure theory for the corresponding crossed product, particularly in the von Neumann algebra case—see, for example, [Kal69, Theorem 3.3]. This is a generalization of essential freeness for measure spaces. To establish notation, given a set X and a map $\alpha : X \to X$, we denote the set of fixed points $\{x \in X \mid \alpha(x) = x\}$ by $\mathrm{Fix}(\alpha)$.

Definition 3.2.6. Assume X is a compact Hausdorff space, $\alpha : X \to X$ a homeomorphism, and μ an α-invariant Radon probability measure on X. We say that α is *essentially free* on (X, μ) if $\mu(\mathrm{Fix}(\alpha)) = 0$. If G is a group acting on X by μ-invariant homeomorphisms α_t, we say that the action is essentially free on (X, μ) if each α_t is essentially free for $t \in G \setminus \{e\}$.

Kallman introduced in [Kal69, Definition 1.3] a notion of freely acting automorphisms for general von Neumann algebras, as opposed to just $L^\infty(X, \mu)$:

Definition 3.2.7. Let M be a von Neumann algebra and $\alpha \in \mathrm{Aut}(M)$. We say that α is *freely acting* if whenever $xy = \alpha(y)x$ for all $y \in M$, we have $x = 0$.

General automorphisms on von Neumann algebras enjoy a very nice decomposition theory into an inner part and a freely acting part—see [Kal69, Theorem 1.11], along with its proof.

Theorem 3.2.8. *Let M be a von Neumann algebra and $\alpha \in \mathrm{Aut}(M)$. There is a largest α-invariant central projection $p \in M$ with the property that $\alpha|_{Mp}$ is inner. In addition, $\alpha|_{M(1-p)}$ is freely acting. Finally, the decomposition $\alpha = \alpha_1 \oplus \alpha_2$ and $M = M_1 \oplus M_2$, with $\alpha_i \in \mathrm{Aut}(M_i)$ and the property that α_1 is inner and α_2 is freely acting, is unique.*

Proper outerness is an equivalent formulation of freeness (nowadays, the terms are often used interchangeably), and is usually defined as follows:

Definition 3.2.9. Let M be a von Neumann algebra and $\alpha \in \mathrm{Aut}(M)$. We say that α is *properly outer* if there is no nonzero α-invariant central projection $p \in M$ with the property that $\alpha|_{Mp}$ is inner. If G is a group acting on M by *-automorphisms α_t, we say that the action is properly outer if each α_t is properly outer for $t \in G \setminus \{e\}$.

We will not make use of this following definition, but it is worth noting that we call a group action $\alpha : G \to \mathrm{Aut}(M)$ *inner* if there is a group homomorphism $\beta : G \to U(M)$

with the property that $\alpha(t) = \operatorname{Ad} \beta(t)$. It is important to keep in mind that this is not equivalent to having each $\alpha(t)$ be inner—indeed, it is not hard to check that the example in Section 3.5.1 is in fact a finite-dimensional example of this phenomenon. In addition, we call the action *outer* if each $\alpha(t)$, $t \in G \setminus \{e\}$, is outer. Observe that if M is a factor, then outer and properly outer are equivalent.

This next result is well-known, and highlights the fact that proper outerness truly does generalize the notion of essential freeness. The proof is slightly nontrivial and hard to find in the literature, and so we include it here.

Proposition 3.2.10. *Assume X is a compact Hausdorff space, $\alpha : X \to X$ a homeomorphism, and μ an α-invariant Radon probability measure. Then α is essentially free on (X, μ) if and only if the corresponding automorphism on $L^\infty(X, \mu)$ is properly outer.*

Proof. If α were not essentially free on (X, μ), then $p = 1_{\mathrm{Fix}(\alpha)}$ is a nonzero α-invariant central projection in $L^\infty(X, \mu)$, and the action on $(L^\infty(X, \mu))p$ is trivial.

Conversely, assume such a projection $p \in L^\infty(X, \mu)$ exists, and let $E = \operatorname{supp} p$. Replacing E by $\cup_{n\in\mathbb{Z}} \alpha^n(E)$, we may assume without loss of generality that E is α-invariant. We claim that $Y \coloneqq E \setminus \mathrm{Fix}(\alpha)$ is a null set. Given $y \in Y$, we may choose an open neighborhood U_y with the property that $\alpha(U_y) \cap U_y = \emptyset$. Observe that $U_y \cap Y$ is a null set by our assumption that α acts trivially on $L^\infty(E, \mu)$. Now, μ is inner regular on all sets (it is outer regular, and we may take complements), and so given any $\varepsilon > 0$, we may choose a compact set $K \subseteq Y$ with $\mu(Y \setminus K) < \varepsilon$. By compactness, K admits a finite subcover from $\{U_y\}_{y\in Y}$, and using the fact that every $U_y \cap K$ is a null set, we deduce that K is a null set. Consequently, so is Y. Thus, without loss of generality, we have $E \subseteq \mathrm{Fix}(\alpha)$, and so the action of α on (X, μ) is not essentially free. ■

Although we will not make use of this fact, it is worth keeping in mind that "central" is often omitted from Theorem 3.2.8 and Definition 3.2.9. It is a result of Borchers, [Bor74, Lemma 5.7], that if e is any projection in M, not necessarily central, and $\alpha \in \mathrm{Aut}(M)$ satisfies $\alpha(e) = e$ and is inner on eMe, then it is inner on Mp, where p is the central cover of e.

3.3 Almost inner actions

This section builds on what was reviewed in Section 3.2.2. As previously mentioned, we aim to convert the conditions in Theorems 3.1.2 and 3.1.3, set (2), into conditions on inner actions on corners of the von Neumann algebra M, which are often much easier to check in practice. This will be done by taking the polar decomposition of the elements x_t, and this is the motivation behind the definitions that follow. Observe that condition (b) in these theorems is precisely the identity used in the definition of freely acting automorphisms. Condition (d), however, has no obvious nice resulting condition on the unitaries that we obtain, and so we do not include any analogous condition in our definition of *partially almost inner* below. Instead, we include it as a separate property which we call *positively compatible*.

Definition 3.3.1. Assume M is a von Neumann algebra, $N \triangleleft G$ is normal, and G acts on M by *-automorphisms. We say that the action is *partially almost inner relative to N* with respect to $\{(p_t, u_t)\}_{t\in N}$ if:

1. Given any $t \in N$, p_t is a central projection in M satisfying $t \cdot p_t = p_t$, u_t is a unitary in Mp_t, and moreover, t acts on Mp_t by $\operatorname{Ad} u_t$.

2. $p_e = 1$ and $u_e = 1$.

3. $p_t = p_{t^{-1}}$ and $u_t^* = u_{t^{-1}}$ for all $t \in N$.

4. $s \cdot p_t = p_{sts^{-1}}$ and $s \cdot u_t = u_{sts^{-1}}$ for all $s \in G$ and $t \in N$.

If $p_t = 0$ for all $t \in N \setminus \{e\}$, we call $\{(p_t, u_t)\}_{t\in N}$ *trivial*, and *nontrivial* otherwise. If there exists a choice of $\{(p_t, u_t)\}_{t\in N}$ with $p_t = 1$ for all $t \in N$, then we say that the action is *almost inner relative to N*. If, in addition, $N = G$, then we simply call the action *almost inner*. We say that the action is *jointly almost properly outer relative to N* if the only $\{(p_t, u_t)\}_{t\in N}$ with respect to which it is partially almost inner is the trivial one. We will simply call the action *jointly almost properly outer* if it is jointly properly outer relative to G.

Remark 3.3.2. Consider the above definition in the case of $N = G$. It is worth noting that if $p_t = 1$ for all $t \in G$, then the map $t \mapsto u_t$ is not necessarily an inner action of G on M, i.e. we are not guaranteed that $u_{st} = u_s u_t$. This is the motivation behind the term *almost*. In addition, the usual definition of a properly outer (or just outer) action $\alpha : G \to \operatorname{Aut}(M)$ is that each individual $\alpha(t)$, $t \neq e$, is properly outer (respectively, outer). The term *jointly* highlights the fact that we require compatibility conditions between each of the individual $\alpha(t)$.

Definition 3.3.3. Let $\{(p_t, u_t)\}_{t\in N}$ be as in Definition 3.3.1. We say that $\{(p_t, u_t)\}_{t\in N}$ are *positively compatible* if there exist elements $\{y_t\}_{t\in N} \subseteq M$ such that:

1. $y_t \in Z(M)$ and $y_t \geq 0$ for all $t \in N$.

2. The projection onto $\overline{\text{ran}} y_t$ is p_t for all $t \in N$.

3. $y_e = 1$.

4. $y_t = y_{t^{-1}}$ for all $t \in N$.

5. $s \cdot y_t = y_{sts^{-1}}$ for all $s \in G$ and $t \in N$.

6. Given any finite $\mathcal{F} \subseteq N$, the matrix $[u_{st^{-1}} y_{st^{-1}}]_{s,t \in \mathcal{F}}$ is positive.

It is worth noting that positive compatibility is not a redundant condition, as the following example shows:

Example 3.3.4. It is known that it is possible to construct an infinite group G with only two conjugacy classes, using HNN extensions. A proof can be found in the original paper by Higman, Neumann, and Neumann—see [HNN49, Theorem III]. Let $A = \mathbb{C}$, let $p_t = 1$ for all $t \in G$, and let $u_t = -1$ for $t \neq e$. It is clear that the trivial action is almost inner with respect to $\{(p_t, u_t)\}_{t \in G}$. However, we claim that $\{(p_t, u_t)\}_{t \in G}$ is not positively compatible. To this end, assume otherwise and let $\{y_t\}_{t \in G}$ be as in Definition 3.3.3, and observe that y_t are all some positive constant $\gamma > 0$ for $t \neq e$. Now letting $\mathcal{F} \subseteq G$ be any finite subset with $|\mathcal{F}| = n$, we have that the matrix

$$\begin{bmatrix} 1 & -\gamma & \dots & -\gamma \\ -\gamma & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & -\gamma \\ -\gamma & \dots & -\gamma & 1 \end{bmatrix}$$

is positive. Letting $[-\gamma]$ denote the $n \times n$ matrix with all entries being $-\gamma$, the above matrix is equal to $(1+\gamma)I + [-\gamma]$, and basic linear algebra tells us that the eigenvalues of this matrix are $(1+\gamma) - n\gamma$ and $1+\gamma$. In particular, $(1+\gamma) - n\gamma < 0$ if n is sufficiently large, contradicting the positivity of the above matrix.

Proposition 3.3.5. *Assume M is a von Neumann algebra, $N \triangleleft G$ is normal, G acts on M by *-automorphisms, and $\{x_t\}_{t \in N} \subseteq M$ is such that:*

1. *$x_t y = (t \cdot y) x_t$ for all $y \in M$ and $t \in N$.*

2. *$x_e = 1$.*

3. $x_t^* = x_{t^{-1}}$ *and* $t \in N$.

4. $s \cdot x_t = x_{sts^{-1}}$ *for all* $s \in G$ *and* $t \in N$.

Given $t \in N$*, consider the polar decomposition of* x_t*, i.e. let* u_t *be the unique partial isometry such that both* $x_t = u_t |x_t|$ *and* $u_t^* u_t$ *is the projection onto* $\overline{\text{ran}}\, |x_t|$*, and furthermore denote this projection by* p_t*. Then the action is partially almost inner relative to* N *with respect to* $\{(p_t, u_t)\}_{t \in N}$*. Moreover, if for every finite* $\mathcal{F} \subseteq N$*, we have that the matrix* $[x_{st^{-1}}]_{s,t \in \mathcal{F}}$ *is positive, then* $\{(p_t, u_t)\}_{t \in N}$ *is positively compatible with respect to* $\{|x_t|\}_{t \in N}$*.*

Conversely, if the action is partially almost inner relative to N *with respect to* $\{(p_t, u_t)\}_{t \in N}$*, then* $x_t = u_t$ *satisfy the above conditions. If, in addition,* $\{(p_t, u_t)\}_{t \in N}$ *is positively compatible with respect to* $\{y_t\}_{t \in N}$*, then* $x_t = u_t y_t$ *satisfy the above conditions, and also satisfy the property that for any finite* $\mathcal{F} \subseteq N$*, the matrix* $[x_{st^{-1}}]_{s,t \in \mathcal{F}}$ *is positive.*

Proof. This proof is somewhat similar to the proof of [Kal69, Theorem 1.1]. For convenience, we recreate the necessary parts here. Assume $\{x_t\}_{t \in N}$ is such a collection. Observe that for $w \in U(M)$, we have

$$w^* x_t^* x_t w = x_t^* (t \cdot w)^* (t \cdot w) x_t = x_t^* x_t,$$

which shows $x_t^* x_t \in Z(M)$. Thus, we have that $|x_t|$ and $p_t \in W^*(|x_t|)$ also lie in the center. Given x_t is fixed by t, so is p_t. In addition,

$$x_t x_t^* = (t \cdot x_t^*) x_t = x_t^* x_t,$$

i.e. x_t is normal. Now, the equality

$$\overline{\text{ran}} x_t = (\ker x_t^*)^\perp = (\ker |x_t^*|)^\perp = \overline{\text{ran}}\, |x_t^*| = \overline{\text{ran}}\, |x_t|$$

tells us that $u_t u_t^* = u_t^* u_t$, i.e. u_t is a unitary in $M p_t$. Furthermore, we note that

$$u_t y\, |x_t| = u_t\, |x_t|\, y = x_t y = (t \cdot y) x_t = (t \cdot y) u_t\, |x_t|\,,$$

which shows that t acts by $\operatorname{Ad} u_t$ on $M p_t$. This gives us property (1) of partial almost inner actions. Furthermore, it is clear that we have both $u_e = 1$ and $p_e = 1$ (property (2)).

Given that $|x_{t^{-1}}| = |x_t^*| = |x_t|$, we have $p_{t^{-1}} = p_t$. Now observe that

$$u_t^*\, |x_t| = |x_t|^*\, u_t^* = x_t^* = x_{t^{-1}} = u_{t^{-1}}\, |x_{t^{-1}}| = u_{t^{-1}}\, |x_t|\,.$$

Given that u_t^* and $u_{t^{-1}}$ share the same initial projection p_t, it follows from uniqueness of polar decomposition that $u_t^* = u_{t^{-1}}$. This is property (3).

Finally, given $s \in G$ and $t \in N$, we see that

$$u_{sts^{-1}} |x_{sts^{-1}}| = x_{sts^{-1}} = s \cdot x_t = (s \cdot u_t)(s \cdot |x_t|) = (s \cdot u_t) |x_{sts^{-1}}| .$$

We wish to conclude that $u_{sts^{-1}}$ and $s \cdot u_t$ have the same initial projection. By the crossed product construction, we may assume without loss of generality that $M \subseteq B(H)$, where G acts on H by unitaries λ_t, and $t \cdot y = \lambda_t y \lambda_t^*$ for all $y \in M$. This gives us that $(s \cdot u_t)^*(s \cdot u_t) = \lambda_s(u_t^* u_t)\lambda_s^*$ is the projection onto

$$\lambda_s(\overline{\operatorname{ran}} |x_t|) = \overline{\operatorname{ran}}(\lambda_s |x_t|) = \overline{\operatorname{ran}}(\lambda_s |x_t| \lambda_s^*) = \overline{\operatorname{ran}} |x_{sts^{-1}}| .$$

Thus, $u_{sts^{-1}}$ and $s \cdot u_t$ share the same initial projection $p_{sts^{-1}}$. Uniqueness of polar decomposition tells us that they are therefore equal, which also gives us that $s \cdot p_t = p_{sts^{-1}}$. This is property (4).

If $[x_{st^{-1}}]_{s,t \in \mathcal{F}}$ is positive for all finite $\mathcal{F} \subseteq N$, then it follows immediately from the definition and from the work that was done above that $\{(p_t, u_t)\}_{t \in N}$ are positively compatible with respect to $\{|x_t|\}_{t \in N}$.

The converse given for converting $\{(p_t, u_t)\}_{t \in N}$ back into elements x_t satisfying the given properties follows from the definitions and is straightforward to verify. ■

The intersection property for noncommutative reduced crossed products is studied in [KS19]. Their results show that if the action $G \curvearrowright I_G(A)$, where $I_G(A)$ denotes the G-injective envelope of A, is properly outer, then $A \rtimes_\lambda G$ has the intersection property. Moreover, if the action $G \curvearrowright I(A)$, where $I(A)$ denotes the usual injective envelope of A, has a property they call *vanishing obstruction*, then the converse to this result holds. Here, we show that a very mild adaptation of the intersection property is enough to guarantee that a partial almost inner action is positively compatible.

Proposition 3.3.6. *Assume M is a von Neumann algebra, $N \triangleleft G$ is normal, G acts on M by *-automorphisms, and the action is partially almost inner relative to N with respect to $\{(p_t, u_t)\}_{t \in N}$. If, in addition, we have that $p_s p_t \leq p_{st}$ and $u_s u_t = u_{st} p_s p_t$ for all $s, t \in N$, then $\{(p_t, u_t)\}_{t \in N}$ is positively compatible with respect to $\{p_t\}_{t \in N}$.*

Proof. Let $\{s_1, \ldots, s_n\}$ be a finite subset of N. We wish to show that the matrix $[u_{s_i s_j^{-1}}]$ is positive. Let $Z(M) = C(X)$, and consider the sets $\operatorname{supp} p_{s_i s_j^{-1}} \subseteq X$. We may choose finitely many disjoint sets $E_k \subseteq X$ such that $\sqcup_k E_k = X$, and for any i, j, and k, we have $E_k \subseteq \operatorname{supp} p_{s_i s_j^{-1}}$ or $E_k \cap \operatorname{supp} p_{s_i s_j^{-1}} = \emptyset$. These sets can be chosen to be finite intersections of sets of the form $\operatorname{supp} p_{s_i s_j^{-1}}$ and their complements, making each E_k clopen. We will prove that $[u_{s_i s_j^{-1}} 1_{E_k}] \geq 0$ for every k.

To this end, fix k, and define a relation on $\{1, \ldots, n\}$ by $i \sim j$ if and only if $E_k \subseteq \operatorname{supp} p_{s_i s_j^{-1}}$. This is in fact an equivalence relation—it is clear that this is reflexive and symmetric. Transitivity follows from the fact that $p_{s_{i_1} s_{i_2}^{-1}} p_{s_{i_2} s_{i_3}^{-1}} \leq p_{s_{i_1} s_{i_3}^{-1}}$. Thus, if we assume without loss of generality that $\{s_1, \ldots, s_n\}$ are ordered such that the equivalence classes are of the form $\{m, m+1, \ldots, m+l\}$, then $[u_{s_i s_j^{-1}} 1_{E_k}]$ becomes a block diagonal matrix, where each block of the diagonal is of the form $[u_{s_i s_j^{-1}} 1_{E_k}]_{i,j=m,\ldots,m+l}$, and $E_k \subseteq \operatorname{supp} p_{s_i s_j^{-1}}$ for every element in this submatrix. Hence, to prove our original matrix is positive, we may assume without loss of generality that $E_k \subseteq \operatorname{supp} p_{s_i s_j^{-1}}$ for all i and j. This matrix is positive, as

$$\begin{bmatrix} u_{s_1 s_1^{-1}} 1_{E_k} \\ \vdots \\ u_{s_n s_1^{-1}} 1_{E_k} \end{bmatrix} \begin{bmatrix} u_{s_1 s_1^{-1}} 1_{E_k} \\ \vdots \\ u_{s_n s_1^{-1}} 1_{E_k} \end{bmatrix}^* = [u_{s_i s_j^{-1}} 1_{E_k}].$$

■

3.4 Proof of main results

As before, A denotes a unital C*-algebra and G a discrete group acting on A by *-automorphisms. Throughout this section, we will fix an invariant trace $\tau \in T_G(A)$, denote by $\pi : A \to B(H_\tau)$ the GNS representation, let $M = \pi(A)''$, and let τ_M be the corresponding faithful normal trace on M.

This first lemma is likely already known, and we give a quick proof for convenience. We will denote $A[G] \coloneqq \{\sum_{\text{finite}} a_t w_t\}$, i.e. the set of finitely-supported functions from G to A, together with the usual *-algebraic operations obtained by viewing this as a subset of $A \rtimes G$. A function $\phi : A[G] \to \mathbb{C}$ is said to be *positive definite* if for any $f \in A[G]$, we have $\phi(f^* f) \geq 0$.

Lemma 3.4.1. *Assume $\phi : A[G] \to \mathbb{C}$ is a positive definite function satisfying $\phi(1) = 1$. Then ϕ extends to a state on $A \rtimes G$.*

Proof. This proof is essentially a modified GNS construction. Define a sesquilinear form on $A[G]$ by $\langle f_1 \mid f_2 \rangle \coloneqq \phi(f_2^* f_1)$, and observe that this is positive, as ϕ is positive definite. Letting $N = \{f \in A[G] \mid \langle f \mid f \rangle = 0\}$, we have that the completion of $A[G]/N$ with respect to the corresponding quotient inner product becomes a Hilbert space, which we will denote by H.

It is clear that we have a unitary representation $u : G \to U(H)$ given by $u(s)f = w_s f$ for $f \in A[G]$. We also have a *-representation $\rho : A \to B(H)$ given by $\rho(a)f = af$, as

$$\langle af \mid af \rangle = \phi(f^* a^* a f) \leq \|a\|^2 \phi(f^* f),$$

where this last equality holds due to the fact that

$$\phi(f^*(\|a\|^2 - a^* a)f) = \phi(f^*(\|a\|^2 - a^* a)^{1/2}(\|a\|^2 - a^* a)^{1/2} f) \geq 0.$$

(It is a subtle but important point that $\|a\|^2 - a^* a$ still admits a positive square root in $A[G]$. This is not necessarily true anymore if we replace a with an arbitrary element of $A[G]$). Moreover, ρ and u form a covariant pair. By the universal property of $A \rtimes G$, we obtain a *-homomorphism $\widetilde{\rho} : A \rtimes G \to B(H)$ given by $\widetilde{\rho}(as) = \rho(a)u(s)$. Consequently, we obtain a positive functional $\sigma \in (A \rtimes G)^*$ given by $\sigma(at) = \langle \widetilde{\rho}(at) w_e \mid w_e \rangle = \phi(a w_t)$. It is clear that, in addition, $\sigma(1) = 1$. ∎

Proposition 3.4.2. *Assume $\{x_t\}_{t \in G} \subseteq M$ satisfy the assumptions of Theorem 3.1.2, set (2). Then there is a trace $\sigma \in T(A \rtimes G)$ satisfying $\sigma(at) = \tau_M(\pi(a)x_t)$.*

Proof. In light of Lemma 3.4.1, to show that we at least obtain a state $\sigma \in S(A \rtimes G)$ with the above property, it suffices to show that the function $\sigma : A[G] \to \mathbb{C}$ given by $\sigma(a w_t) = \tau_M(\pi(a)x_t)$ is positive definite. To this end, assume $f = \sum_{i=1}^n a_{s_i} w_{s_i} \in A[G]$. We have that

$$\begin{aligned}
\sigma(f^* f) &= \sum_{i=1}^n \sum_{j=1}^n \sigma(w_{s_i}^* a_{s_i}^* a_{s_j} w_{s_j}) \\
&= \sum_{i=1}^n \sum_{j=1}^n \sigma((s_i^{-1} \cdot (a_{s_i}^* a_{s_j})) w_{s_i^{-1} s_j}) \\
&= \sum_{i=1}^n \sum_{j=1}^n \tau_M((s_i^{-1} \cdot \pi(a_{s_i})^*)(s_i^{-1} \cdot \pi(a_{s_j})) x_{s_i^{-1} s_j}) \\
&= \tau_M\left(\sum_{i=1}^n \sum_{j=1}^n (s_i^{-1} \cdot \pi(a_{s_i})^*) x_{s_i^{-1} s_j} (s_j^{-1} \cdot \pi(a_{s_j})) \right).
\end{aligned}$$

Observe, however, that

$$\begin{bmatrix} s_1^{-1} \cdot \pi(a_{s_1}) \\ \vdots \\ s_n^{-1} \cdot \pi(a_{s_n}) \end{bmatrix}^* \begin{bmatrix} x_{s_1^{-1}s_1} & \dots & x_{s_1^{-1}s_n} \\ \vdots & & \vdots \\ x_{s_n^{-1}s_1} & \dots & x_{s_n^{-1}s_n} \end{bmatrix} \begin{bmatrix} s_1^{-1} \cdot \pi(a_{s_1}) \\ \vdots \\ s_n^{-1} \cdot \pi(a_{s_n}) \end{bmatrix}$$
$$= \sum_{i=1}^{n} \sum_{j=1}^{n} (s_i^{-1} \cdot \pi(a_{s_i})^*) x_{s_i^{-1}s_j} (s_j^{-1} \cdot \pi(a_{s_j})),$$

guaranteeing that σ is positive definite. It remains to show that the extension to $A \rtimes G$ is still a trace:

$$\begin{aligned} \sigma((as)(bt)) &= \sigma((a(s \cdot b))st) \\ &= \tau_M(\pi(a)(s \cdot \pi(b))x_{st}) \\ &= \tau_M(s^{-1} \cdot (\pi(a)(s \cdot \pi(b))x_{st})) \\ &= \tau_M((s^{-1} \cdot \pi(a))\pi(b)x_{ts}) \\ &= \tau_M(\pi(b)x_{ts}(s^{-1} \cdot \pi(a))) \\ &= \tau_M(\pi(b)(t \cdot \pi(a))x_{ts}) \\ &= \sigma((b(t \cdot a))ts) \\ &= \sigma((bt)(as)) \end{aligned}$$

■

Remark 3.4.3. The first half of the proof of Proposition 3.4.2 does not use the fact that τ (and hence τ_M) is a trace. Thus, if we assume that τ is only a G-invariant state, we still obtain a state $\sigma \in S(A \rtimes G)$ given by $\sigma(at) = \tau_M(\pi(a)x_t)$, except σ is of course not necessarily a trace anymore.

Lemma 3.4.4. *Assume $\sigma_1, \sigma_2 \in T(A \rtimes G)$ are two states satisfying $\sigma_1(at) = \tau_M(\pi(a)x_t)$ and $\sigma_2(at) = \tau_M(\pi(a)y_t)$ for some $\{x_t\}_{t\in G}, \{y_t\}_{t\in G} \subseteq M$. If $\sigma_1 = \sigma_2$, then $x_t = y_t$ for all $t \in G$.*

Proof. Assume otherwise, and fix some $t \in G$ with $x_t \neq y_t$. Letting $(a_\lambda) \subseteq A$ be a net with the property that $(\pi(a_\lambda))$ is weak*-convergent to $(x_t - y_t)^*$, we see that

$$(\sigma_1 - \sigma_2)(a_\lambda t) = \tau_M(\pi(a_\lambda)(x_t - y_t)) \to \tau_M((x_t - y_t)^*(x_t - y_t)).$$

This limit value is nonzero, as τ_M is faithful. Thus, there is some λ such that $\sigma_1(a_\lambda t)$ and $\sigma_2(a_\lambda t)$ differ. ■

Proof of Theorem 3.1.2. Starting with any weak expectation $F : A \rtimes G \to M$ for (A, τ, G) and letting $x_t = F(t)$, we note that A lies in the multiplicative domain of F—see, for example, [BO08, Proposition 1.5.7], for a review of multiplicative domain. Thus, $F(at) = \pi(a)x_t$, and so the map between sets (1) and (2) is necessarily injective. It remains to show that x_t indeed satisfy all of the aforementioned properties. We have that $x_e = 1$ follows from F being unital, and $s \cdot x_t = x_{sts^{-1}}$ follows from F being G-equivariant. Now, given any $a \in A$ and $t \in G$, observe that

$$x_t\pi(a) = F(ta) = F((t \cdot a)t) = (t \cdot \pi(a))x_t.$$

Given that $\pi(A)$ is weak*-dense in M, taking limits allows us to conclude that $x_t y = (t \cdot y)x_t$ holds for all $y \in M$. Finally, given $s_1, \dots, s_n \subseteq G$, we note that

$$F^{(n)}\left(\begin{bmatrix} s_1 \\ \vdots \\ s_n \end{bmatrix}\begin{bmatrix} s_1 \\ \vdots \\ s_n \end{bmatrix}^*\right) = \begin{bmatrix} x_{s_1 s_1^{-1}} & \dots & x_{s_1 s_n^{-1}} \\ \vdots & & \vdots \\ x_{s_n s_1^{-1}} & \dots & x_{s_n s_n^{-1}} \end{bmatrix}.$$

Complete positivity of F says that $[x_{s_i s_j^{-1}}]$ is therefore positive.

Now, starting with any $\{x_t\}_{t \in G} \subseteq M$ as in (2), Proposition 3.4.2 tells us that $\sigma(at) = \tau_M(\pi(a)x_t)$ indeed defines a valid trace. Moreover, this map from (2) to (3) is injective by Lemma 3.4.4.

Finally, we can show the maps from sets (1) to (2) and (2) to (3) are bijective by showing that their composition is surjective. That is, we need to show that for any $\sigma \in T(A \rtimes G)$ satisfying $\sigma|_A = \tau$, there exist some weak expectation $F : A \rtimes G \to M$ for (A, τ, G) such that $\sigma = \tau_M \circ F$.

To this end, fix such a σ, let $\rho : A \rtimes G \to B(H_\rho)$ be the GNS representation of $(A \rtimes G, \sigma)$, let $N = \rho(A \rtimes G)''$, and let σ_N denote the corresponding faithful normal trace on N. Given that $(A, \tau) \subseteq (A \rtimes G, \sigma)$ is a trace-preserving embedding, this canonically gives a trace-preserving embedding $(M, \tau_M) \subseteq (N, \sigma_N)$ sending $\pi(a)$ to $\sigma(a)$ by Proposition 3.2.5. There is a unique normal conditional expectation $F' : N \to M$ satisfying $\sigma_N = \tau_M \circ F'$—see, for example, [BO08, Lemma 1.5.11]. We let $F = F' \circ \rho$, and show that this is the map we are looking for. Observe that

$$\tau_M(F(at)) = \tau_M(F'(\rho(at))) = \sigma_N(\rho(at)) = \sigma(at),$$

i.e. $\tau_M \circ F = \sigma$. The only non-trivial fact remaining is to show that F is G-equivariant.

Given that τ_M and σ are G-invariant, we have

$$\begin{aligned}\tau_M(\pi(a)(s^{-1}\cdot F(sts^{-1}))) &= \tau_M(\pi(s\cdot a)F(sts^{-1}))\\ &= \sigma((s\cdot a)sts^{-1})\\ &= \sigma(at)\\ &= \tau_M(\pi(a)F(t)),\end{aligned}$$

and so we may apply Lemma 3.4.4 to conclude that $s^{-1}\cdot F(sts^{-1}) = F(t)$, i.e. $s\cdot F(t) = F(sts^{-1})$. This is enough to guarantee G-equivariance on the entire domain, as

$$F(s\cdot(at)) = F((s\cdot a)sts^{-1}) = \pi(s\cdot a)F(sts^{-1}) = s\cdot(\pi(a)F(t)) = s\cdot F(at).$$

■

Proof of Theorem 3.1.3. The proof that the given map from set (1) to set (2) is well-defined and injective is analogous to what was done in the proof of Theorem 3.1.2.

To go from (2) to (3), we first note that $A\rtimes_\lambda R_a(G) = A\rtimes R_a(G)$ by amenability of $R_a(G)$, and so there is a trace $\sigma'\in T(A\rtimes_\lambda R_a(G))$ satisfying $\sigma'(a\lambda_t) = \tau_M(\pi(a)x_t)$ by Theorem 3.1.2. Composing with the canonical conditional expectation $E_{R_a(G)} : A\rtimes_\lambda G\to A\rtimes_\lambda R_a(G)$, which maps $a\lambda_t$ to itself if $t\in R_a(G)$ and zero otherwise, gives us a state $\sigma := \sigma'\circ E_{R_a(G)}\in S(A\rtimes_\lambda G)$. It remains to check that this is indeed still a trace on $A\rtimes_\lambda G$. Note that for $s,t\in G$, we have $st\in R_a(G)$ if and only if $ts\in R_a(G)$ by normality of $R_a(G)$. Hence, if $st\notin R_a(G)$, then

$$\sigma(a\lambda_s b\lambda_t) = \sigma(a(s\cdot b)\lambda_{st}) = 0 = \sigma(b(t\cdot a)\lambda_{ts}) = \sigma(b\lambda_t a\lambda_s).$$

The case of $st\in R_a(G)$ is identical to what was done in the proof of Proposition 3.4.2.

Finally, we again wish to show that the composition of the maps from (1) to (2) and (2) to (3) is surjective, i.e. given $\sigma\in T(A\rtimes_\lambda G)$ with $\sigma|_A = \tau$, there exists some weak expectation $F : A\rtimes R_a(G)\to M$ for $(A,\tau,G,R_a(G))$ satisfying $\sigma = \tau_M\circ F\circ E_{R_a(G)}$. (This last composition makes sense, as $A\rtimes R_a(G) = A\rtimes_\lambda R_a(G)$ by amenability). Letting $\rho : A\rtimes G\to A\rtimes_\lambda G$ be the canonical *-homomorphism, we note that $\sigma\circ\rho\in T(A\rtimes G)$, and so there is some weak expectation $F' : A\rtimes G\to M$ for (A,τ,G) satisfying $\sigma\circ\rho = \tau_M\circ F'$ by Theorem 3.1.2. Observe that we canonically have $A\rtimes R_a(G)\subseteq A\rtimes G$—this is because the following composition of canonical maps yields the identity map:

$$A\rtimes R_a(G)\to A\rtimes G\to A\rtimes_\lambda G\to A\rtimes_\lambda R_a(G) = A\rtimes R_a(G)$$

We claim that $F := F'|_{A\rtimes R_a(G)}$ is the map we are looking for. This follows from [BK16, Theorem 5.2], which says that $\sigma = \sigma\circ E_{R_a(G)}$. ■

Proof of Corollary 3.1.7. This follows immediately from Theorems 3.1.2 and 3.1.3, and the fact that properly outer and freely acting are equivalent (see the review in Section 3.2.2). ■

Proof of Theorem 3.1.8. This follows immediately from Theorems 3.1.2 and 3.1.3, together with the correspondence given in Proposition 3.3.5. ■

Proof of Corollary 3.1.9. If the action is jointly almost properly outer relative to $R_a(G)$, then Theorem 3.1.8 tells us that τ has unique tracial extension. Conversely, assume the action is partially almost inner relative to $R_a(G)$ with respect to some nontrivial $\{(p_t, u_t)\}_{t\in R_a(G)}$. Pick $t_0 \neq e$ such that $p_{t_0} \neq 0$, and let C denote the conjugacy class of t_0 in G. Now define

$$v_t = \begin{cases} 1 & \text{if } t = e \\ u_t & \text{if } t \in C \cup C^{-1} \\ 0 & \text{otherwise} \end{cases}, q_t = \begin{cases} 1 & \text{if } t = e \\ p_t & \text{if } t \in C \cup C^{-1} \\ 0 & \text{otherwise} \end{cases}, y_t = \begin{cases} 1 & \text{if } t = e \\ \frac{1}{2|C|}p_t & \text{if } t \in C \cup C^{-1} \\ 0 & \text{otherwise} \end{cases}$$

Observe that $[v_{st^{-1}}y_{st^{-1}}]_{s,t\in G} = 1 + \sum_{t\in C\cup C^{-1}} \frac{1}{2|C|}u_t \otimes \lambda_t$ is in fact a positive element in $M \otimes_{\min} C^*_\lambda(G)$, as $\sum_{t\in C\cup C^{-1}} \frac{1}{2|C|}u_t \otimes \lambda_t$ is a self-adjoint element of norm at most 1, and so the action is partially almost inner relative to $R_a(G)$ with respect to the nontrivial and positively compatible set $\{(q_t, v_t)\}_{t\in R_a(G)}$. By Theorem 3.1.8, we are done. ■

Proof of Corollary 3.1.10. If τ does not have unique tracial extension, then Corollary 3.1.9 says that the action is partially almost inner relative to $R_a(G)$ with respect to some nontrivial $\{(p_t, u_t)\}_{t\in R_a(G)}$. Choosing $t \in R_a(G) \setminus \{e\}$ with $p_t \neq 0$ gives us what we want.

Conversely, assume that we do have such a $t \in R_a(G)\setminus\{e\}$, $p \neq 0$ in M, and $u \in U(Mp)$. If $t \neq t^{-1}$, then letting $p_t = p_{t^{-1}} = p$, $u_t = u$, $u_{t^{-1}} = u^*$, and $p_s, u_s = 0$ for $s \neq e, t, t^{-1}$ gives us a nontrivial $\{(p_t, u_t)\}_{t\in R_a(G)}$, and so τ cannot have unique tracial extension by Corollary 3.1.9.

The case of $t = t^{-1}$ requires just a bit more work. Letting $w = u^2$, observe that $\operatorname{Ad} w = \mathrm{id}$, and so $w \in (Z(Mp))^G$, which is a commutative von Neumann algebra. Even in the non-separable setting, every such algebra is isomorphic to $L^\infty(Y, \nu)$ for some locally compact Y and positive Radon measure ν on Y—see, for example, [Tak02, Chapter III, Theorem 1.18]. (This is the space of all measurable, locally essentially bounded functions from Y to $\mathbb{C}$, modulo agreeing locally almost everywhere). Thus, we may choose a unitary $v \in (Z(Mp))^G$ with the property that $v^2 = w^*$. Now letting $p_t = p$, $u_t = uv$, and $p_s, u_s = 0$ for $s \neq e, t$, we obtain a nontrivial $\{(p_t, u_t)\}_{t\in R_a(G)}$ as before, and again τ cannot have unique tracial extension by Corollary 3.1.9. ■

The following results describe how the spectrum of a commutative von Neumann algebra breaks up with respect to a periodic automorphism, and are needed for the proof of Theorem 3.1.11. Recall that an extremally disconnected topological space is one where the closure of any open set is open, and that the spectrum of a commutative von Neumann algebra is always extremally disconnected—see, for example, [Tak02, Chapter III, Theorem 1.18]. In terms of notation, $d|n$ will denote "d divides n".

Lemma 3.4.5. *Assume X is an extremally disconnected compact Hausdorff space, and $\alpha : X \to X$ is a homeomorphism satisfying $\alpha^n = \mathrm{id}$ for some $n \in \mathbb{N}$. Then X breaks up as $X = \sqcup_{d|n} X_d$, where each X_d is clopen, α-invariant, and has the property that every $x \in X_d$ has orbit of size d.*

Proof. Letting $Y_d = \mathrm{Fix}(\alpha^d)$ for $d|n$, we know that Y_d is clopen by Frolík's theorem—see [Fro71, Theorem 3.1]. This is the set of all points whose orbit size divides d. From here, we can let $X_d = Y_d \setminus (\cup_{\substack{m|n \\ m<d}} Y_m)$. ■

Lemma 3.4.6. *Assume X is an extremally disconnected compact Hausdorff space, and $\alpha : X \to X$ is a homeomorphism with the property that every orbit is finite and of the same size $n \in \mathbb{N}$. Then there is a clopen transversal of the orbits, i.e. there is some clopen $E \subseteq X$ with the property that $X = \sqcup_{k=0}^{n-1} \alpha^k(E)$.*

Proof. We claim that there is at least one nonempty open subset $U \subseteq X$ with the property that all of $U, \alpha(U), \dots, \alpha^{n-1}(U)$ are pairwise disjoint. To see this, choose any $x \in X$, and let U_k, $k = 0, \dots, n-1$, be pairwise disjoint open sets satisfying $p^k(x) \in U_k$. Now letting

$$U = U_0 \cap p^{-1}(U_1) \cap \cdots \cap p^{-(n-1)}(U_{n-1}),$$

we have that $U, p(U), \dots, p^{n-1}(U)$ are all pairwise disjoint.

Given an ascending chain (U_λ) of such open sets, the union $\cup U_\lambda$ is still such a set, and so by Zorn's lemma, there is a maximal open set U with this property. We claim that it is in fact clopen. This follows from the following fact: if $V, W \subseteq X$ are open and $V \cap W = \emptyset$, then $V \cap \overline{W} = \emptyset$, and as $\overline{W}$ is open, we have $\overline{V} \cap \overline{W} = \emptyset$.

Finally, we claim that our maximal set U is in fact the set we are looking for, i.e. $X = \sqcup_{k=0}^{n-1} \alpha^k(U)$. Assume otherwise, and consider the smaller space $X \setminus \sqcup_{k=0}^{n-1} \alpha^k(U)$ (a clopen, α-invariant subset of X). Obtaining as before a nonempty open subset $V \subseteq X \setminus \sqcup_{k=0}^{n-1} \alpha^k(U)$ with the property that $V, \alpha(V), \dots, \alpha^{n-1}(V)$ are all pairwise disjoint, the set $U \cup V$ again satisfies this property, contradicting maximality of U. ■

Proof of Theorem 3.1.11. First, if the action of $\mathbb{Z}$ on M is properly outer, then τ has unique tracial extension by Corollary 3.1.7. Conversely, assume the action of $\mathbb{Z}$ on M is not properly outer, and let $n \geq 1$ be such that α^n is not properly outer on M. Let p be the largest α^n-invariant central projection such that $\alpha^n|_{Mp}$ is inner, and fix a unitary $u \in U(Mp)$ implementing this action. Observe that for any $x \in M\alpha(p)$, we have

$$\alpha(u)x\alpha(u)^* = \alpha(u\alpha^{-1}(x)u^*) = \alpha\alpha^n\alpha^{-1}x = \alpha^n x.$$

In other words, α^n is inner on $M\alpha(p)$. By assumption, $\alpha(p) \leq p$. But then

$$p = \alpha^n(p) \leq \alpha^{n-1}(p) \leq \cdots \leq \alpha(p) \leq p.$$

This shows p is in fact α-invariant. In general, even though the choice of unitary $u \in U(Mp)$ satisfying $\alpha^n = \operatorname{Ad} u$ is not unique, we still cannot guarantee that there is some choice that also satisfies $\alpha(u) = u$—see [Con77, Proposition 1.6] for an example of this phenomenon on the separable hyperfinite II_1 factor. However, we will show that it is always possible to choose an α-invariant unitary implementing the action of α^{n^2}. From here, Corollary 3.1.10 will apply, giving us the fact that τ cannot have unique tracial extension.

To simplify notation, we can assume without loss of generality that $p = 1$. Observe that our previous computations above show that $\operatorname{Ad}\alpha(u) = \operatorname{Ad} u$, and so $\alpha(u) = uv$ for some $v \in Z(M)$. Moreover, the fact that $\alpha^n = \operatorname{Ad} u$ tells us that

$$u = \alpha^n(u) = \alpha^{n-1}(uv) = \cdots = uv\alpha(v)\ldots\alpha^{n-1}(v),$$

or in other words,

$$v\alpha(v)\ldots\alpha^{n-1}(v) = 1.$$

Now let $Z(M) = C(X)$. We know that α induces a homeomorphism on X, which we will denote by α_X. Given that α^n is inner, we know that α_X^n is the identity map. By Lemma 3.4.5, we have $X = \sqcup_{d|n} X_d$, where X_d is the set of all $x \in X$ with the size of the α_X-orbit being exactly d, and furthermore each X_d is clopen and α_X-invariant.

We will show for every $d|n$ that there is some central unitary $w_d \in M1_{X_d}$ with the property that $\alpha(u^n w_d) = u^n w_d$. Again to simplify notation, we may assume without loss of generality that $X = X_d$ for a single $d|n$. Applying Lemma 3.4.6, there exists a clopen transversal $E \subseteq X$ of the orbits of α_X. Let $q = 1_E$, and observe that $q, \ldots, \alpha^{d-1}(q)$ are pairwise orthogonal projections that sum to 1. Keeping this in mind, we may decompose v as follows: let $v_k = \alpha^{-k}(v)q \in Mq$ for $k = 0, \ldots, d-1$, so that

$$v = v_0 + \cdots + \alpha^{d-1}(v_{d-1}).$$

Now define w as follows:

$$w = q + \alpha(v_1^n) + \alpha^2(v_1^n v_2^n) + \cdots + \alpha^{d-1}(v_1^n \ldots v_{d-1}^n),$$

and note that

$$\alpha(w)^* = (v_1^n \ldots v_{d-1}^n)^* + \alpha(q) + \alpha^2(v_1^n)^* + \cdots + \alpha^{d-1}(v_1^n + \ldots v_{d-2}^n)^*,$$

so that

$$w\alpha(w)^* = (v_1^n \ldots v_{d-1}^n)^* + \alpha(v_1^n) + \alpha^2(v_2^n) + \cdots + \alpha^{d-1}(v_{d-1}^n).$$

We claim that we in fact have $w\alpha(w)^* = v^n$. Our earlier equality $v \ldots \alpha^{n-1}(v) = 1$ gives us $(v_0 \ldots v_{d-1})^{n/d} = 1$, and so we obtain the equality $(v_1^n \ldots v_{d-1}^n)^* = v_0^n$.

In summary, we have obtained a central unitary w with the property that $\alpha(w) = (v^n)^* w$. Keeping in mind that $\alpha^{n^2} = \operatorname{Ad} u^n = \operatorname{Ad}(u^n w)$, and also that

$$\alpha(u^n w) = \alpha(u)^n \alpha(w) = u^n v^n (v^n)^* w = u^n w,$$

we may apply Corollary 3.1.10 to conclude that τ cannot have unique tracial extension to $A \rtimes_\lambda \mathbb{Z}$. ■

Proof of Corollary 3.1.12. The proofs for the universal crossed product and reduced crossed product are almost identical. Hence, we only prove the reduced case.

First, assume that the action of $R_a(G)$ on (X, μ) is essentially free. This is equivalent to the action of $R_a(G)$ on $L^\infty(X, \mu)$ being properly outer. By Corollary 3.1.7, μ has unique tracial extension.

Now assume that the action of $R_a(G)$ on (X, μ) is not essentially free, and let $p_t = u_t = 1_{\mathrm{Fix}(t)}$. It is straightforward to check that the action of G on $L^\infty(X, \mu)$ is partially almost inner relative to $R_a(G)$ with respect to the (nontrivial by assumption) $\{(p_t, u_t)\}_{t \in R_a(G)}$. Furthermore, it is clear that the additional assumptions of Proposition 3.3.6 are satisfied, and so $\{(p_t, u_t)\}_{t \in R_a(G)}$ is also positively compatible. Again by Theorem 3.1.8, μ cannot have unique tracial extension. ■

3.5 Examples

3.5.1 A finite-dimensional (counter)example

Here, we give an action of $\mathbb{Z}_2 \times \mathbb{Z}_2$ on M_2 such that the action of each $t \in \mathbb{Z}_2 \times \mathbb{Z}_2$ is inner, but the crossed product is isomorphic to M_4. Observe that, letting τ be the unique (hence, automatically invariant) trace on M_2, and $\pi : M_2 \to B(H_\tau)$ be the GNS representation, we canonically have $\pi(M_2)'' \cong M_2$. In particular, the action of $\mathbb{Z}_2 \times \mathbb{Z}_2$ on $\pi(M_2)''$ is not properly outer, but the only invariant trace on M_2 extends to a unique trace on the crossed product. Of course, one could also view the crossed product as a von Neumann crossed product, and from this perspective it is a finite factor of type I.

This contradicts [Tho95, Theorem 4.3], which claims that if G is countable and abelian and A is unital and separable, then four various conditions are equivalent. In particular, condition (1), which states that $T(A \rtimes_\lambda G)$ and $T_G(A)$ are in canonical bijection, is equivalent to condition (4), which states that for any $\tau \in \partial_e(T_G(A))$, letting $\pi : A \to B(H_\tau)$ be the GNS representation, the action of G on $\pi(A)''$ is properly outer. The example in this section contradicts (1) $\implies$ (4). The converse, along with the equivalence between (1), (2), and (3), still appear to be correct.

Similarly, this example also contradicts the precursor result [Béd96, Proposition 11], which again gives an equivalence between three conditions. It, in particular, claims that if G is abelian and acts on a finite factor N, then condition (a) stating the von Neumann crossed product $N \overline{\rtimes} G$ is a finite factor is equivalent to condition (c) stating the action is properly outer. Again, (a) $\implies$ (c) is false, but the converse, along with (a) $\iff$ (b), still appear to be correct.

We first present a proof of the following example using purely elementary techniques, and afterwards show how our results apply.

Example 3.5.1. Consider $G = \mathbb{Z}_2 \times \mathbb{Z}_2 = \langle u \rangle \times \langle v \rangle$, acting on $A = M_2$, where the action $\alpha : G \to \mathrm{Aut}(M_2)$ is given by $\alpha_u = \mathrm{Ad}\, U$ and $\alpha_v = \mathrm{Ad}\, V$, where

$$U = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad V = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Then this is a well-defined action, and the crossed product isomorphic to M_4, and is hence simple and has unique trace.

Proof using elementary techniques. It is easy to check that α_s and α_t are commuting automorphisms, both of order 2, and so we obtain a group homomorphism $\alpha : G \to \mathrm{Aut}(M_2)$. Given that the crossed product is 16-dimensional, it suffices to show that it has trivial center in order to prove it is isomorphic to M_4.

To this end, assume that $\sum_s a_s \lambda_s \in Z(A \rtimes_\lambda G)$. Then given any $t \in G$, we have

$$\lambda_t \left(\sum_s a_s \lambda_s \right) = \sum_s (t \cdot a_s) \lambda_{ts} = \sum_s (t \cdot a_s) \lambda_{st},$$

while

$$\left(\sum_s a_s\lambda_s\right)\lambda_t = \sum_s a_s\lambda_{st}.$$

This shows each a_s is invariant under the action of each $t \in G$. In other words, a_s commutes with each of the matrices I, U, V, and UV. But these matrices are easily seen to span M_2, and so $a_s \in Z(M_2) = \mathbb{C}$.

Now letting $b \in M_2$ be arbitrary, we have

$$b\left(\sum_s a_s\lambda_s\right) = \sum_s (ba_s)\lambda_s,$$

while

$$\left(\sum_s a_s\lambda_s\right)b = \sum_s (a_s(s \cdot b))\lambda_s.$$

If $s \in G$ is such that $a_s \neq 0$, then $b = s \cdot b$ for all b. Writing $\alpha_s = \operatorname{Ad} W$, this tells us $W \in Z(M_2) = \mathbb{C}$, so $s = e$. ■

Proof using Corollary 3.1.10. As the crossed product is 16-dimensional, it suffices to prove that it has unique trace. Note that M_2 has a unique trace, and the double commutant under its GNS representation is again M_2. Assume there is a nontrivial element $t \in \mathbb{Z}_2 \times \mathbb{Z}_2 \setminus \{e\}$, a nontrivial central projection $p \in M_2$, and a unitary $w \in (M_2)p$ with the properties that

1. $s \cdot p = p$ and $s \cdot w = w$ for all $s \in \mathbb{Z}_2 \times \mathbb{Z}_2$.
2. t acts by $\operatorname{Ad} w$ on $(M_2)p$.

Clearly, we must have $p = 1$. Also, as $Z(M_2) = \mathbb{C}$, if there is one unitary $w \in M_2$ implementing the action of t and satisfying $s \cdot w = w$ for all $s \in \mathbb{Z}_2 \times \mathbb{Z}_2$, then all unitaries implementing this inner action necessarily satisfy this invariance property. But $UV = -VU$, and so $v \cdot U = -U$, $u \cdot V = -V$, and $v \cdot (UV) = -UV$. Thus, the above situation cannot occur, and so by Corollary 3.1.10, we are done. ■

3.5.2 A finite cyclic group (counter)example

This section aims to give another counterexample to results cited in Section 3.5.1, but in the case of G being a finite cyclic group instead, and also show how our results apply.

Before proceeding further, we first recall the notion of *separably inheritable* in the sense of Blackadar [Bla06, Definition II.8.5.1]. We say that a property (P) is *separably inheritable* if the following two hold:

1. Whenever A is a C*-algebra satisfying (P), and $B \subseteq A$ is a separable C*-subalgebra, then there is an intermediate C*-algebra C with $B \subseteq C \subseteq A$ with C separable and satisfying (P).

2. Whenever $A_1 \hookrightarrow A_2 \hookrightarrow \dots$ is an inductive system of separable C*-algebras, each satisfying (P), with injective connecting maps, then the direct limit $\varinjlim A_n$ also satisfies (P).

It is remarked that, in the unital category, the property of having a unique trace is separably inheritable [Bla06, II.8.5.5]. The following lemma shows that this works in the unital *equivariant* category as well, and will allow us to construct a counterexample in the separable setting.

Lemma 3.5.2. *Let G be a countable discrete group, and consider the category of unital G-C*-algebras.*

1. *If A is a C*-algebra in this category with a unique trace, and B is any separable C*-subalgebra (not necessarily unital or G-invariant), then there is a unital G-invariant C*-subalgebra $C \subseteq A$ with C separable and having a unique trace, and also containing B.*

2. *Whenever $A_1 \hookrightarrow A_2 \hookrightarrow \dots$ is an inductive system of separable C*-algebras in this category, each with a unique trace, with injective connecting maps, then the direct limit $\varinjlim A_n$ also has a unique trace.*

Proof. The fact that the inductive limit property works is clear. Hence, we prove the intermediate C*-algebra property. Let A and B be as above. We claim that there is always a unital G-invariant intermediate C*-subalgebra $\widetilde{B}$ satisfying $B \subseteq \widetilde{B} \subseteq A$. To see this, let $(b_n) \subseteq B$ be a norm-dense sequence, and let

$$\widetilde{B} \coloneqq C^*(1, \{gb_n \mid n \in \mathbb{N}, g \in G\}).$$

From here, we may construct a sequence of subalgebras of A satisfying

$$A_1 \subseteq \widetilde{A}_1 \subseteq A_2 \subseteq \widetilde{A}_2 \subseteq \dots,$$

where $A_1 = B$, $\widetilde{A}_n$ is defined as before in relation to A_n, and A_{n+1} has a unique trace. Then it is clear that

$$C := \overline{\bigcup_{n\in\mathbb{N}} A_n} = \overline{\bigcup_{n\in\mathbb{N}} \widetilde{A}_n}$$

will satisfy the properties we want. ∎

Example 3.5.3. There is a separable C*-algebra A and an automorphism $\alpha \in \mathrm{Aut}(A)$ of order 4 such that the following is true: A has a unique trace τ, and if we let $\pi : A \to B(H_\tau)$ be the GNS representation of (A, τ) and $M = \pi(A)''$, then:

1. M is the separable hyperfinite II_1 factor.
2. The corresponding action of $\mathbb{Z}_4$ on M is not (properly) outer. In fact, the action of $2 \in \mathbb{Z}_4$ on A is inner (and hence also inner on M).
3. The C*-crossed product $A \rtimes_\lambda \mathbb{Z}_4$ has a unique trace.
4. The von Neumann crossed product $M \overline{\rtimes} \mathbb{Z}_4$ is a II_1 factor.

Proof. Let R be the separable hyperfinite II_1 factor. It was shown in [Con77, Proposition 1.6] that for any $p \in \mathbb{N}$ and any p-th root of unity γ, there is an automorphism $s_p^\gamma \in \mathrm{Aut}(R)$ with the properties that p is the smallest positive integer with $(s_p^\gamma)^p$ being inner, $(s_p^\gamma)^p = \mathrm{Ad}\, U_\gamma$, where writing $R = \overline{\otimes}_{n=1}^\infty M_p$, we have

$$U_\gamma = \begin{bmatrix} \gamma & & & \\ & \gamma^2 & & \\ & & \ddots & \\ & & & \gamma^p \end{bmatrix} \otimes \left(\otimes_{n=2}^\infty I\right),$$

and moreover, $s_p^\gamma(U_\gamma) = \gamma U_\gamma$. Observe that, as $Z(R) = \mathbb{C}$, then $s_p^\gamma(W) = \gamma W$ for any unitary $W \in R$ satisfying $(s_p^\gamma)^p = \mathrm{Ad}\, W$. For our purposes, we will let $p = 2$ and $\gamma = -1$, and fix an outer automorphism $\alpha \in \mathrm{Aut}(R)$ with $\alpha^2 = \mathrm{Ad}\, u$ and $\alpha(u) = -u$. Observe that $u = U_2$ as defined above guarantees $u^2 = 1$, and so $\alpha^4 = \mathrm{id}$.

Consider the $\mathbb{Z}_4$ action on R induced by α, and observe that by weak*-separability of R, we have that $u \in R$ is contained in a norm-separable, weak*-dense C*-subalgebra. By Lemma 3.5.2, there is a $\mathbb{Z}_4$-invariant, norm-separable, weak*-dense unital C*-subalgebra $A \subseteq R$ containing u and having a unique trace (denote the unique trace on both R and A by τ). We claim that this is the C*-algebra we are looking for.

First, we verify that if $\pi : A \to B(H_\tau)$ is the GNS representation, then we get $\pi(A)'' = R$. Denote the GNS Hilbert spaces of (A, τ) and (R, τ) by $L^2(A, \tau)$ and $L^2(R, \tau)$, respectively, and note that $L^2(A, \tau) \subseteq L^2(R, \tau)$. As A is SOT-dense in $R \subseteq B(L^2(R, \tau))$, then if $x \in R$ with $(a_\lambda) \subseteq A$ SOT-convergent to x, it is easy to see that (a_λ) is also $\|\cdot\|_2$-convergent to x. It follows that $L^2(A, \tau) = L^2(R, \tau)$, and so $\pi(A)'' = R$.

By normality, the unique extension of $\alpha|_A \in \mathrm{Aut}(A)$ to $\pi(A)'' = R$ is again α. By construction, the corresponding action of $\mathbb{Z}_4$ on R is not (properly) outer, as α^2 is inner (and is in fact inner on A by construction).

We wish to apply Corollary 3.1.10 to conclude that the crossed product $A \rtimes_\lambda \mathbb{Z}_4$ has a unique trace. This follows from our previous computations—the only nontrivial $n \in \mathbb{Z}_4$ that admits a nontrivial central projection $p \in R$ with the properties that $\alpha(p) = p$ and α^n is inner on Rp is $n = 2$ (together with $p = 1$). However, as we saw earlier, it is impossible to choose a unitary w satisfying both $\alpha^2 = \mathrm{Ad}\, w$ and $\alpha(w) = w$. This gives us that τ has unique tracial extension to the crossed product $A \rtimes_\lambda \mathbb{Z}_4$.

From here, we can conclude that the von Neumann crossed product $R \overline{\rtimes} \mathbb{Z}_4$ is still a II_1 factor. We know it admits at least one faithful normal trace, namely $\tau \circ E$, where $E : R \overline{\rtimes} \mathbb{Z}_4 \to R$ is the canonical expectation. Given any other normal trace, it necessarily agrees with $\tau \circ E$ on $A \rtimes_\lambda \mathbb{Z}_4$, and by normality and weak*-density therefore agrees with $\tau \circ E$ on all of $R \overline{\rtimes} \mathbb{Z}_4$. ■

3.5.3 Various crossed products with reduced group C*-algebras

Let $C^*_\lambda(G)$ denote the reduced group C*-algebra, $L(G)$ the group von Neumann algebra, and $\tau_\lambda \in T(L(G))$ the canonical trace. We will, furthermore, denote by $\mathrm{Char}(G)$ the set of all group homomorphisms from G to the circle group $\mathbb{T}$. We say that $\mathrm{Char}(G)$ separates the points of G if for any $s \neq t$ in G, there is some $\chi \in \mathrm{Char}(G)$ such that $\chi(s) \neq \chi(t)$. Equivalently, for any $t \neq e$, there is some character $\chi \in \mathrm{Char}(G)$ with $\chi(t) \neq 1$. This definition generalizes to any $H \leq G$ and $K \leq \mathrm{Char}(G)$.

The following facts are likely well-known. In particular, this first proposition is proven in greater generality in [Beh69, Theorem 5.2]. We provide quick proofs of them for convenience.

Proposition 3.5.4. *Every $\chi \in \mathrm{Char}(G)$ induces an automorphism α_χ on $C^*_\lambda(G)$ and $L(G)$ given by mapping λ_t to $\chi(t)\lambda(t)$. If G is ICC, then α_χ is (properly) outer on $L(G)$ for every $\chi \neq 1$.*

Proof. Viewing $C^*_\lambda(G) \subseteq L(G) \subseteq B(\ell^2(G))$, we may define a unitary $U_\chi \in U(\ell^2(G))$ mapping δ_t to $\chi(t)\delta_t$. From here, we see that

$$U_\chi \lambda_s U_\chi^* \delta_t = U_\chi \lambda_s (\overline{\chi(t)}\delta_t) = \overline{\chi(t)} U_\chi \delta_{st} = \overline{\chi(t)}\chi(st)\delta_{st} = \chi(s)\lambda_s\delta_t,$$

i.e. $U_\chi \lambda_s U_\chi^* = \chi(s)\lambda_s$. It follows that $\alpha_\chi := \operatorname{Ad} U_\chi$ induces the automorphism we want on $C^*_\lambda(G)$ and $L(G)$.

Now assume that G is ICC and that $\chi \in \operatorname{Char}(G)$ satisfies $\alpha_\chi = \operatorname{Ad} u$ for some $u \in L(G)$. Write $u \sim \sum_t \alpha_t \lambda_t$, and observe that

$$\begin{aligned} & u\lambda_s u^* = \chi(s)\lambda_s \\ \iff & u\lambda_s = \chi(s)\lambda_s u \\ \iff & \sum_t \alpha_t u_{ts} = \sum_t \chi(s)\alpha_t u_{st} \\ \iff & \sum_t \alpha_{ts^{-1}} u_t = \sum_t \chi(s)\alpha_{s^{-1}t} u_t \\ \iff & \alpha_{ts^{-1}} = \chi(s)\alpha_{s^{-1}t} \\ \iff & \alpha_{sts^{-1}} = \chi(s)\alpha_t \end{aligned}$$

It follows from square-summability of $(\alpha_t)_{t\in G}$ that $\alpha_t = 0$ for $t \neq e$. Hence, $\alpha_\chi = \operatorname{id}$, so $\chi = 1$. ■

Lemma 3.5.5. *For any groups G and H, we have $R_a(G \times H) = R_a(G) \times R_a(H)$.*

Proof. Let $\pi_G : G\times H \to G$ and $\pi_H : G\times H \to H$ denote the canonical projections. Observe that $\pi_G(R_a(G \times H))$ is an amenable normal subgroup of G, and hence $\pi_G(R_a(G \times H)) \subseteq R_a(G)$. Similarly, $\pi_H(R_a(G \times H)) \subseteq R_a(H)$, and so $R_a(G \times H) \subseteq R_a(G) \times R_a(H)$. But $R_a(G) \times R_a(H)$ is an amenable normal subgroup of $G \times H$, and so we get equality. ■

It is easy to check that for any $t \in G$ and $\chi \in \operatorname{Char}(G)$, we have that $\operatorname{Ad}\lambda_t$ and α_χ commute. Thus, for any $H \leq G$ and $K \leq \operatorname{Char}(G)$, we have an action of $H \times K$ on $C^*_\lambda(G)$. This action of course cannot be properly outer on $L(G)$ if $H \neq \{e\}$. However, as this next example shows, τ_λ can still have unique tracial extension to the corresponding reduced crossed product.

Example 3.5.6. Assume G is ICC, and let $H \leq G$ and $K \leq \operatorname{Char}(G)$. Then τ_λ has unique tracial extension to $C^*_\lambda(G) \rtimes_\lambda (H \times K)$ if and only if K separates the points of $R_a(H)$.

Proof. We know that the GNS representation of $(C^*_\lambda(G), \tau_\lambda)$ is the canonical representation $\pi : C^*_\lambda(G) \to B(\ell^2(G))$, and so $\pi(C^*_\lambda(G))'' = L(G)$.

First, assume that K separates the points of $R_a(H)$, and assume the action is partially almost inner relative to $R_a(H) \times K$ with respect to $\{(p_{t,\chi}, u_{t,\chi})\}_{(t,\chi)\in R_a(H)\times K}$ (note that $R_a(H \times K) = R_a(H) \times K$ by Lemma 3.5.5). Observe that by Proposition 3.5.4, in order for $p_{t,\chi} \neq 0$, it must be the case that $\chi = 1$, as the action of $t \in H$ is always inner. Assume $p_{t,e} = 1$ for some nontrivial $t \in H \setminus \{e\}$. Then $u_{t,e} = \gamma\lambda_{t,e}$ for some $\gamma \in \mathbb{T}$. By assumption, there is some $\chi \in K$ with the property that $\chi(t) \neq 1$, and so

$$\chi \cdot (\gamma\lambda_t) = \gamma\chi(t)\lambda_t \neq \gamma\lambda_t.$$

This contradicts the definition of being partially almost inner, and therefore the set $\{(p_{t,\chi}, u_{t,\chi})\}_{(t,\chi)\in R_a(H)\times K}$ is trivial. By Theorem 3.1.8, τ_λ must have unique tracial extension.

Now assume K does not separate the points of $R_a(H)$, and let

$$N \coloneqq \{h \in R_a(H) \mid \chi(h) = 1 \text{ for all } \chi \in K\} \neq \{e\}.$$

Observe that this is still an amenable normal subgroup of H. Now, we will define a partial almost inner action as follows. Given $(t, \chi) \in R_a(H) \times K$, let

$$p_{t,\chi} = \begin{cases} 1 & \text{if } t \in N \text{ and } \chi = 1 \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad u_{t,\chi} = \begin{cases} \lambda_t & \text{if } t \in N \text{ and } \chi = 1 \\ 0 & \text{otherwise} \end{cases}.$$

It is straightforward to check that the action of $H \times K$ on $C^*_\lambda(G)$ is indeed partially almost inner relative to $R_a(H) \times K$ with respect to $\{(p_{t,\chi}, u_{t,\chi})\}_{(t,\chi)\in R_a(H)\times K}$. Moreover, this is positively compatible by Proposition 3.3.6. Thus, by Theorem 3.1.8, τ_λ cannot have unique tracial extension. ■

It is worth noting that the above example is not vacuous. For example, we could let $G = \mathbb{F}_2$ with canonical generators a and b, $H = \langle a \rangle$, and K any subgroup that contains a character mapping a to $e^{2\pi i\theta}$, where θ is an irrational number.

Chapter 4

A note on traces on von Neumann crossed products

4.1 Introduction and statement of main results

This is a short note, meant to take the results in Chapter 3, which dealt with C*-crossed products, and generalize them to von Neumann crossed products. I suspect that many of the results in this short note might be folklore, but I cannot find them anywhere, and hence still find it worthwhile to write up a note on them.

Throughout this note, M will denote a von Neumann algebra, and G a discrete group acting on M by *-automorphisms. The von Neumann crossed product will be denoted by $M\overline{\rtimes}G$. In Chapter 3, which is essentially just [Urs21], we gave complete descriptions of the tracial states on both the universal and reduced crossed products of a unital C*-algebra A and a discrete group G acting on A by *-automorphisms. We denote these crossed products by $A \rtimes G$ and $A \rtimes_\lambda G$, respectively. The proof in both cases relies heavily on universality of the crossed product one way or another (in the reduced case, universality of $A \rtimes_\lambda R_a(G)$ is used, where $R_a(G)$ denotes the amenable radical of G). While it appears unlikely that an analogous description of normal tracial states on a von Neumann crossed product $M\overline{\rtimes}G$ holds, due to the lack of any universal property, it is still possible to use the results in the C*-algebraic case to characterize when a G-invariant normal tracial state on M has unique normal tracial extension to the crossed product. Recall that there always exists at least one such extension, given by composing with the canonical conditional expectation $E : M\overline{\rtimes}G \to M$.

This result makes use of the FC-center of G, denoted $FC(G)$, which is the set of all elements of G with a finite conjugacy class. It is known that $FC(G)$ is always amenable, which we prove for convenience in Section 4.2. Moreover, we denote the centralizer of an element $t \in G$ by $C_G(t)$. Finally, we also make use of an action being *jointly almost properly outer*, which was a property weaker than proper outerness introduced in Section 3.3. We briefly recall the necessary concepts in Section 4.2.

Theorem 4.1.1. *Assume τ is a G-invariant normal tracial state on M, and let p_τ denote the support projection of τ. The following are equivalent:*

1. *τ has unique normal tracial extension to $M \overline{\rtimes} G$.*
2. *The action of G on Mp_τ is jointly almost properly outer relative to the normal subgroup $FC(G)$, in the sense of Definition 4.2.2.*
3. *There is no $t \in FC(G) \setminus \{e\}$, nonzero t-invariant central projection $q \in M$ with $q \leq p_\tau$, and unitary $u \in Mq$ such that:*
   - *(a) t acts by $\operatorname{Ad} u$ on Mq.*
   - *(b) $s \cdot q = q$ and $s \cdot u = u$ for all $s \in C_G(t)$. Optionally, we may additionally require that $s \cdot p = p$ and $s \cdot u = u^*$ for any $s \in G$ with the property that $sts^{-1} = t^{-1}$.*

Given that it is well known that a tracial von Neumann algebra (that is, one admitting a faithful normal tracial state) is a factor if and only if it has a unique normal tracial state, it is possible to convert the above equivalence into one that characterizes exactly when $M \overline{\rtimes} G$ is a finite factor. It is already well-known that if M admits a G-invariant normal tracial state, and the action of G on M is *properly outer* in the sense of Definition 4.2.1, then $M \overline{\rtimes} G$ is a factor if and only if the action of G on $Z(M)$ is ergodic, i.e. $Z(M)^G = \mathbb{C}$.

Corollary 4.1.2. *The von Neumann crossed product $M \overline{\rtimes} G$ is a finite factor if and only if the following conditions hold:*

1. *M admits a G-invariant normal tracial state.*
2. *$Z(M)^G = \mathbb{C}$. That is, G acts ergodically on $Z(M)$.*
3. *There is no $t \in FC(G) \setminus \{e\}$, nonzero t-invariant central projection $q \in M$, and unitary $u \in Mq$ satisfying:*
   - *(a) t acts by $\operatorname{Ad} u$ on Mq.*
   - *(b) $s \cdot q = q$ and $s \cdot u = u$ for all $s \in C_G(t)$. Optionally, we may additionally require that $s \cdot p = p$ and $s \cdot u = u^*$ for any $s \in G$ with the property that $sts^{-1} = t^{-1}$.*

4.2 Preliminaries

It is well-known that *freeness* or *proper outerness* of the action of G on a von Neumann algebra M leads to nice structure theory of the crossed product $M\bar{\rtimes}G$. We briefly recall the notion here:

Definition 4.2.1. An automorphism α on a von Neumann algebra M is said to be *properly outer* if there is no nonzero α-invariant central projection $p \in M$ with the property that $\alpha|_{Mp}$ is inner. A group action by *-automorphisms $G \curvearrowright M$ is called properly outer if the action of every $t \in G \setminus \{e\}$ is properly outer.

A weaker notion which adds compatibility conditions between the individual group elements was introduced in Section 3.3, and used for studying tracial states on crossed products of C*-algebras. See the discussion presented there for more details, along with an explanation of the naming convention.

Definition 4.2.2. Assume G acts on a von Neumann algebra M by *-automorphisms, and $N \triangleleft G$ is a normal subgroup. We say that the action is *partially almost inner* relative to N and with respect to $\{(p_t, u_t)\}_{t\in N}$ if:

1. p_t is a t-invariant central projection in M, u_t is a unitary in Mp_t, and t acts by $\mathrm{Ad}\, u_t$ on Mp_t.
2. $p_e = 1$ and $u_e = 1$.
3. $p_t = p_{t^{-1}}$ and $u_t^* = u_{t^{-1}}$.
4. $s \cdot p_t = p_{sts^{-1}}$ and $s \cdot u_t = u_{sts^{-1}}$ for all $t \in N$ and $s \in G$.

The set $\{(p_t, u_t)\}_{t\in N}$ above is called *trivial* if $p_t = 0$ for all $t \in N \setminus \{e\}$, and *nontrivial* otherwise. The action is called *jointly almost properly outer* relative to N if the only such $\{(p_t, u_t)\}_{t\in N}$ is the trivial one.

Finally, given the difficulty I had in finding a paper written in English showing that the FC-center $FC(G)$ of a group G is always amenable, a short proof is included here.

Proposition 4.2.3. *The FC-center $FC(G)$ of a group G is always amenable.*

Proof. Clearly, this is equivalent to showing that if every element of G has finite conjugacy class, then G is amenable. Given that amenability is preserved under taking direct limits, and G is the direct limit of its finitely-generated subgroups, we may assume without loss of generality that G is finitely-generated as well.

To this end, let $G = \langle h_1, \ldots, h_n \rangle$. Consider the action of G on itself by conjugation. The orbit-stabilizer theorem tells us that because the orbit of an element h_i is finite, the stabilizer $C_G(h_i)$ is finite-index. But because $h_1, \ldots, h_n$ generate G, we have that

$$Z(G) = C_G(h_1) \cap \cdots \cap C_G(h_n),$$

and so the center $Z(G)$ is finite-index as well. Given that $Z(G)$ is abelian, hence amenable, this forces G to be amenable. ■

4.3 Proof of main results

As before, M will denote a von Neumann algebra, G a discrete group acting on M by *-automorphisms, and the von Neumann crossed product by $M \overline{\rtimes} G$.

Lemma 4.3.1. *Every normal G-invariant state $\sigma \in S(M \overline{\rtimes} G)$ concentrates on $M \overline{\rtimes} FC(G)$, in the sense that $\sigma(y\lambda_t) = 0$ whenever $t \notin FC(G)$. In particular, this applies to normal tracial states.*

Proof. Consider an element $y\lambda_t \in M \overline{\rtimes} G$, where $t \notin FC(G)$, and assume that $\sigma(y\lambda_t) = \alpha \neq 0$. Consider a sequence of subsets $F_n \subseteq G$ with the property that $|F_n| = n$, and sts^{-1} are distinct for $s \in F_n$. Letting

$$a_n = \frac{1}{|F_n|} \sum_{s \in F_n} s \cdot (y\lambda_t) = \sum_{s \in F_n} \frac{1}{n} (s \cdot y) \lambda_{sts^{-1}},$$

any weak*-cluster point of (a_n) necessarily converges to zero, as the Fourier series coefficients converge (in norm) to zero. This, however, is a problem, as $\sigma(a_n) = \alpha \neq 0$, and so the normality of σ is violated. ■

Remark 4.3.2. The above result is, in a sense, analogous to [BK16, Theorem 5.2], which says that tracial states on a reduced C*-crossed product $A \rtimes_\lambda G$ concentrate on $A \rtimes_\lambda R_a(G)$, where $R_a(G)$ denotes the amenable radical of G.

Before we begin the proof of Theorem 4.1.1, we again note that we will make heavy use of the results in Chapter 3 that deal with the case of C*-crossed products. To apply these results, we first make two observations. The first is that norm-separability of the C*-algebra is not a requirement for any of the main results in Chapter 3 to hold. (It is worth noting that, even if it were, it would still be possible in the case of separable M to modify the following proof to work with a norm-separable, weak*-dense C*-subalgebra). The second observation is that, given a normal tracial state τ on M with support projection p_τ, letting $\pi : M \to B(H)$ be the GNS representation, we have that $\pi(M)''$ is canonically isomorphic to Mp_τ.

Proof of Theorem 4.1.1. To show (2) $\implies$ (1), assume that τ has a nontrivial normal tracial extension σ to $M\overline{\rtimes}G$. Letting p_τ be the support projection of τ, we have by Theorem 3.1.3 that the restriction to the C*-crossed product $\sigma|_{M\rtimes_\lambda G}$ corresponds to coefficients $\{x_t\}_{t\in R_a(G)} \in Mp_\tau$ (satisfying a list of properties which will not be repeated here), and σ is given by

$$\sigma(y\lambda_t) = \tau((yp_\tau)x_t)$$

for $t \in R_a(G)$, and zero otherwise. Observe that by Lemma 4.3.1, these coefficients x_t are in fact zero if $t \notin FC(G)$, as we have

$$0 = \sigma(x_t^*\lambda_t) = \tau(x_t^*x_t),$$

and τ is faithful on Mp_τ. Moreover, we have that these values determine σ on the von Neumann crossed product by normality, and so at least one of the coefficients x_t is nonzero for some $t \in FC(G) \setminus \{e\}$. By Proposition 3.3.5, taking polar decomposition shows that the action on Mp_τ cannot be jointly almost properly outer relative to $FC(G)$.

Now we show (3) $\implies$ (2). Assume the action were not jointly almost properly outer relative to $FC(G)$, and let $\{(p_t, u_t)\}_{t\in FC(G)}$ be as in Definition 4.2.2 with $p_{t_0} \neq 0$ for some fixed $t_0 \in FC(G) \setminus \{e\}$. Then it is clear that $t = t_0$, $q = p_{t_0}$, and $u = u_{t_0}$ violate the requirements of (3), including the optional one.

Finally, to show (1) $\implies$ (3), assume t, q, and u are as in (3), not assuming the additional optional assumption, and let C be the conjugacy class of t. Define $x_{sts^{-1}} := s \cdot u$, and observe that this is well-defined on the conjugacy class of t. It is also not hard to check that $s \cdot x_r = x_{srs^{-1}}$ for all $s \in G$ and $r \in C$, and that moreover, $x_r y = (r \cdot y)x_r$ for all $y \in Mp_\tau$. From here, we let

$$z := \sum_{r\in C} x_t^*\lambda_t,$$

which is a non-scalar central element of $M\overline{\rtimes}G$. Thus, either $\frac{1}{2}(z+z^*)$ or $\frac{1}{2i}(z-z^*)$ is also non-scalar, and now self-adjoint. Adding a sufficiently large positive scalar, we obtain an element $w \in Z(M\overline{\rtimes}G)$ with the property that w is non-scalar, positive, supported on $\{e\} \cup C \cup C^{-1}$, and all Fourier coefficients in $w = \sum_r w_r \lambda_r$ except w_e lie in Mp_τ. This can be rescaled to guarantee that $w_e = 1$. From here, we define a non-trivial normal tracial extension as follows. Let $E : M\overline{\rtimes}G \to M$ be the canonical conditional expectation, and observe that $\tau \circ E$ is a tracial state, and define

$$\sigma(a) = (\tau \circ E)(wa).$$

It is not hard to check that σ is a normal tracial state on $M\overline{\rtimes}G$, and it is nontrivial, as letting $w = \sum_r w_r \lambda_r$ and choosing some $r \neq e$ with $w_r \neq 0$, we have

$$\sigma(\lambda_r^* w_r^*) = \tau(w_r w_r^*) > 0,$$

by faithfulness of τ on Mp_τ. ■

Proof of Corollary 4.1.2. First, assume that the crossed product $M\overline{\rtimes}G$ is a finite factor. Let $\sigma \in T(M\overline{\rtimes}G)$ be the unique normal tracial state, and let $\tau = \sigma|_M$. It is clear that τ is G-invariant. Furthermore, as $Z(M)^G \subseteq Z(M\overline{\rtimes}G)$, we also have that $Z(M)^G = \mathbb{C}$. Finally, by faithfulness of τ, its support projection is 1, and so by Theorem 4.1.1, we get the last condition that we require.

Conversely, assume that all of the stated conditions hold. First, we claim that from $Z(M)^G = \mathbb{C}$, we can only have at most one G-invariant normal tracial state on M. To see this, assume τ_1 and τ_2 are two distinct such states. Recall that the Dixmier averaging property for von Neumann algebras says that for any $x \in M$, we have

$$\overline{\operatorname{conv}}\,\{uxu^* \mid u \in U(M)\} \cap Z(M) \neq \emptyset.$$

Thus, if τ_1 and τ_2 are different on some $x \in M$, it follows that they are necessarily different on $Z(M)$ as well. Fix any weak*-dense C*-subalgebra $C(X) \subseteq Z(M)$ (or just let them be equal). It is a consequence of Proposition 3.2.5 that $Z(M) = L^\infty(X, \tau_1)$. Given that τ_2 is also a normal trace on this algebra, it necessarily corresponds to some function in $f \in L^1(X, \tau_1)$ satisfying

$$\int F\, d\tau_2 = \int F f\, d\tau_1$$

for all $F \in L^\infty(X, \tau_1)$. Given that both τ_1 and τ_2 are G-invariant, it follows that f is also G-invariant. But then if we choose a concrete representative from the almost-everywhere equivalence class of f, and consider the set

$$\{x \in X \mid f(x) \leq n\}$$

for any $n \in \mathbb{N}$, this set is necessarily almost-everywhere nonempty for a large enough n, and corresponds to some G-invariant projection $p \in L^\infty(X, \tau_1)$. Thus, $p = 1$, and $f \in L^\infty(X, \tau_1)$ in fact, thus being constant by G-invariance. It follows that $\tau_1 = \tau_2$.

Observe that the unique G-invariant normal tracial state $\tau \in T(M)$ has support projection p_τ lying in $Z(M)^G = \mathbb{C}$, and hence $p_\tau = 1$. It follows from Theorem 4.1.1 that the only normal tracial extension to $M \overline{\rtimes} G$ is $\tau \circ E$. Given that this is a composition of faithful maps, this just says that $M \overline{\rtimes} G$ has a unique normal faithful tracial state, which is equivalent to being a finite factor. ■

4.4 A subnote on the non-tracial and twisted cases

The following example is meant to highlight the fact that if we do not require our von Neumann algebra M to admit a normal G-invariant tracial state, even if we have the remaining conditions of Corollary 4.1.2, we still cannot expect the crossed product $M \overline{\rtimes} G$ to be a factor. Recall that it is known that it is possible to construct infinite groups with exactly two conjugacy classes. See [HNN49, Theorem III].

Example 4.4.1. Let G be an infinite group with only two conjugacy classes. It is clear that G is ICC, i.e. $FC(G) = \{e\}$. Let $M = \ell^\infty(G \setminus \{e\})$, equipped with the action of $(s \cdot f)(t) = f(s^{-1}ts)$. It is clear that $Z(M)^G = \mathbb{C}$. We claim, however, that $M \overline{\rtimes} G$ is not a factor.

Observe that, in general, $\sum_t x_t \lambda_t \in M \overline{\rtimes} G$ lies in the center if and only if $s \cdot x_t = x_{sts^{-1}}$ (commutes with λ_s) and $x_t y = (t^{-1} \cdot y) x_t$ (commutes with $y \in M$). We claim that there is an element with Fourier series given by $x_t = \delta_t$ if $t \neq e$, and $x_e = 0$. Observe that these Fourier coefficients, if they define a valid element of the crossed product, indeed satisfy the required invariance conditions.

To show that they define a valid element, we will work in a concrete representation of the crossed product. Consider the Hilbert spaces $H = \ell^2(G \setminus \{e\})$. Any Fourier series $\sum_t x_t \lambda_t$ corresponds to an operator on $\ell^2(G, H)$ whose entries are elements of $\ell^\infty(G \setminus \{e\})$, and whose (r, s)-entry given by $r^{-1} \cdot x_{rs^{-1}} = x_{s^{-1}r}$. For our particular choice of x_t from earlier, note that if $r_1 \neq r_2$, then the (r_1, s) and (r_2, s) entries, assuming they are nonzero, are $\delta_{s^{-1}r_1} \neq \delta_{s^{-1}r_2}$. A similar result holds for $s_1 \neq s_2$ for the (r, s_1) and (r, s_2) entries. Thus, viewing $\ell^2(G, H) = \ell^2(G \times (G \setminus \{e\}))$, every row and every column of the corresponding matrix has at most a single nonzero entry, namely 1. From here, it is not hard to convince yourself that such a matrix is indeed a bounded linear operator on $\ell^2(G \times (G \setminus \{e\}))$ of norm at most 1. The same goes for the matrix corresponding to any sum $\sum_{t \in F} x_t \lambda_t$ for

$F \subseteq G$ finite. Thus, the net consisting of such finite sums, and indexed by finite sums ordered under inclusion, has the element $\sum_t x_t \lambda_t$ as a weak*-cluster point, and this finishes the example.

Going back to the tracial case for a second, one can suddenly ask about the case of twisted crossed products. After some scrap work, and some helpful feedback from Erik Bédos, I strongly suspect it is possible to show that if $M \overline{\rtimes}_\sigma G$ is a twisted von Neumann crossed product corresponding to an action of G on M and a 2-cocycle $\sigma : G \times G \to U(M)$, then the exact same results should hold with some small modification to the invariance conditions presented. Namely, the following should hold true:

Conjecture 4.4.2. *Assume $M \overline{\rtimes}_\sigma G$ is a twisted von Neumann crossed product with 2-cocycle $\sigma : G \times G \to U(M)$. Then Theorem 4.1.1 (1) $\iff$ (3), and Corollary 4.1.2 still hold if one replaces the condition "$s \cdot u = u$ for all $s \in C_G(t)$" with "$s \cdot u = \sigma(s,t)\sigma(t,s)^* u$ for all $s \in C_G(t)$" (ignoring the optional condition presented afterwards).*

This can be seen as generalizing the work done by Kleppner in [Kle62], where he shows in the case of $M = \mathbb{C}$ that the twisted group von Neumann algebra $W^*(G, \sigma)$ is a factor if and only if whenever we have an element $t \in G$ satisfying $\sigma(s,t) = \sigma(t,s)$ for all $s \in C_G(t)$ (in this case, t is said to be σ-regular), we necessarily have that t has infinite conjugacy class. Specifically, this follows from [Kle62, Lemma 3] and [Kle62, Lemma 4], which say that the σ-regular elements are indeed closed under conjugation, and the dimension of the center of the algebra corresponds to the number of σ-regular finite conjugacy classes.

Rigorously proving the above conjecture would likely involve doing similar work to what was done in the non-twisted case, except with lots of $\sigma(\cdot,\cdot)$ floating around.

Chapter 5

Relative C*-simplicity and characterizations for normal subgroups

5.1 Introduction and statement of main results

Throughout this paper, unless specified otherwise, G denotes a discrete group, H a subgroup of G, N a normal subgroup of G, and A a C*-algebra equipped with an action of G by *-automorphisms. The reduced group C*-algebra of G is denoted by $C^*_\lambda(G)$, the canonical trace on $C^*_\lambda(G)$ by τ_λ, and the reduced crossed product of A and G by $A \rtimes_\lambda G$. All topological G-spaces will be assumed to be compact and Hausdorff.

A recent result of Amrutam [Amr21, Theorem 1.1] gives a sufficient condition for all intermediate C*-subalgebras B satisfying $C^*_\lambda(G) \subseteq B \subseteq A \rtimes_\lambda G$ to be of the form $A_1 \rtimes_\lambda G$ for some G-C*-subalgebra $A_1 \subseteq A$. Namely, he introduces the notion of a *plump subgroup*, and proves that the above intermediate subalgebra property holds if G has the approximation property (AP), and the kernel of the action $G \curvearrowright A$ contains a plump subgroup of G. For convenience, we recall the definition here:

Definition 5.1.1. A subgroup $H \leq G$ is *plump* if for any $\varepsilon > 0$ and any finite $F \subseteq G \setminus \{e\}$, there are $s_1, \ldots, s_m \in H$ such that

$$\left\| \frac{1}{m} \sum_{j=1}^{m} \lambda_{s_j} \lambda_t \lambda_{s_j}^* \right\| < \varepsilon \quad \forall t \in F.$$

However, the following remark shows that, for [Amr21, Theorem 1.1], it suffices to consider only normal subgroups:

Remark 5.1.2. Assume $H \leq K \leq G$, and H is plump in G. Then it is clear that K is also plump in G. In particular, the kernel of the action $G \curvearrowright A$ contains a plump subgroup of G if and only if the kernel itself is plump.

Sufficient characterizations of plumpness are given in [Amr21, Section 3]. Recall that, if N is a normal subgroup of G, then the action of N on its Furstenberg boundary $\partial_F N$ extends uniquely to an action of G - see the review given in Section 5.2.1. It is shown that if N is C*-simple and has trivial centralizer in G, then G acts freely on $\partial_F N$, which in turn implies N is plump in G [Amr21, Corollary 3.2]. One of the results we will show is that the converses to these statements also hold:

Theorem 5.1.3. *Assume $N \triangleleft G$ is normal. The following are equivalent:*

1. *N is plump in G.*
2. *The action $G \curvearrowright \partial_F N$ is free.*
3. *There exists some G-minimal, N-strongly proximal, G-topologically free space.*
4. *N is C*-simple and $C_G(N) = \{e\}$.*
5. *G is C*-simple and $C_G(N) = \{e\}$.*

Setting $N = G$ in the above theorem gives back various equivalences between C*-simplicity and other characterizations. For a review of these characterizations, together with necessary definitions, see the review in Section 5.2.2.

Remark 5.1.4. Plumpness is a relativized version of Powers' averaging property, and so setting $N = G$ in Theorem 5.1.3, we get back that G is C*-simple if and only if it satisfies Powers' averaging property. In fact, if G contains any (not necessarily normal) plump subgroup H, then we see that both H and G satisfy Powers' averaging property, and so both are C*-simple. Similarly, one also obtains the various dynamical characterizations of C*-simplicity by setting $N = G$.

From here, it is natural to ask if plumpness is equivalent to some generalized notion of C*-simplicity. To answer this question, we introduce the notion of relative simplicity of C*-algebras, and using this, relative C*-simplicity for groups.

Definition 5.1.5. Assume A is a unital C*-algebra, and $B \subseteq A$ is a unital sub-C*-algebra. We say that B is *relatively simple* in A if any unital completely positive map $\phi : A \to B(\mathcal{H})$ which is a *-homomorphism on B is faithful on A. Given $H \leq G$, we say that H is *relatively C*-simple* in G if $C^*_\lambda(H)$ is relatively simple in $C^*_\lambda(G)$.

Theorem 5.1.6. *Assume $N \triangleleft G$ is normal. The following are equivalent:*

1. *N is plump in G.*
2. *N is relatively C*-simple in G.*
3. *$C^*_\lambda(N)$ is relatively simple in $C(\partial_F N) \rtimes_\lambda G$.*
4. *$C(\partial_F N) \rtimes_\lambda N$ is relatively simple in $C(\partial_F N) \rtimes_\lambda G$.*

Remark 5.1.7. For consistency, we will use the term *relatively C*-simple* in place of *plump* throughout the rest of this paper when it comes to normal subgroups.

We may also ask what other characterizations of C*-simplicity generalize to an equivalent characterization of relative C*-simplicity. Kennedy's intrinsic characterization is one such result. For a review of this, along with a review of the Chabauty topology on the space of subgroups $\mathrm{Sub}(G)$, again see the review in Section 5.2.2. Note that instead of using the former term *residually normal*, all instances were updated to the current terminology of *confined*.

Definition 5.1.8. Assume $H \leq G$. An *H-uniformly recurrent subgroup* of G is a (nonempty) closed H-minimal subset of $\mathrm{Sub}(G)$. It is called amenable if one (equivalently all) of its elements are amenable. It is called nontrivial if it is not $\{\{e\}\}$. A subgroup $K \leq G$ is called *H-confined* if the closed H-orbit of K in $\mathrm{Sub}(G)$ does not contain the trivial subgroup $\{e\}$. Algebraically, $K \leq G$ is H-confined if and only if there exists a finite $F \subseteq G \setminus \{e\}$ such that $F \cap sKs^{-1} \neq \emptyset$ for any $s \in H$.

Theorem 5.1.9. *Assume $N \triangleleft G$ is normal. The following are equivalent:*

1. *N is relatively C*-simple in G.*
2. *There is no amenable N-confined subgroup of G.*
3. *There is no nontrivial amenable N-uniformly recurrent subgroup of G.*

5.2 Preliminaries

5.2.1 Boundary theory

Boundary theory was originally developed by Furstenberg in [Fur73], and played an important role in [KK17] and [BKKO17], which study C*-simplicity of discrete groups. For convenience, we recall all of the basic facts that we will use here. To establish notation, G will always denote a discrete group.

Definition 5.2.1. Let X be a compact Hausdorff space, and assume that G acts by homeomorphisms on X. The action is *minimal* if X has no nontrivial closed G-invariant subsets. The action is *strongly proximal* if for any Borel Radon probability measure $\mu \in P(X)$, the weak*-closed G-orbit of μ contains a Dirac mass δ_x. A *boundary* is a minimal and strongly proximal compact Hausdorff space.

The appropriate notion of morphism between boundaries is a G-equivariant, continuous map.

Proposition 5.2.2. *Morphisms between boundaries are unique, assuming they exist.*

Proof. This follows almost immediately from [Fur73, Proposition 4.2]. ∎

Proposition 5.2.3. *There is a universal boundary $\partial_F G$, in the sense that every other boundary is the image of $\partial_F G$ under some morphism. This universal boundary is also unique up to isomorphism.*

The universal boundary $\partial_F G$ given above is nowadays called the *Furstenberg boundary*, and a proof of its existence can be found in [Fur73, Proposition 4.6].

Recall that an *extremally disconnected* space is one where the closure of any open set is open. The following is a well-known theorem of Frolík, and can be found in [Fro71, Theorem 3.1].

Theorem 5.2.4. *The fixed point set of any homeomorphism of an extremally disconnected space is clopen.*

Corollary 5.2.5. *The fixed point set of any homeomorphism of $\partial_F G$ is clopen.*

Proof. It is known that the Furstenberg boundary of a discrete group is always extremally disconnected - see [KK17, Remark 3.16] or [BKKO17, Proposition 2.4]. ∎

A well-known relativization fact that will come in extremely useful is the following:

Proposition 5.2.6. *Assume $N \triangleleft G$ is normal. The action of N on $\partial_F N$ extends uniquely to an action of G.*

A proof of this fact can be found in [Oza14, Lemma 20]. Note that, by uniqueness, there is no ambiguity when referring to *the* action of G on $\partial_F N$.

5.2.2 C*-simplicity

Again, G will always denote a discrete group. The group G is called C*-simple if its reduced group C*-algebra $C^*_\lambda(G)$ is simple. Here, we collect the various characterizations of C*-simplicity that we will make use of throughout this paper.

The Furstenberg boundary $\partial_F G$ (see the review in Section 5.2.1) played a central role in the original characterizations of C*-simplicity. Recall that an action $G \curvearrowright X$ is said to be *free* if the fixed point sets $\mathrm{Fix}(t)$ are empty for $t \neq e$. If X is a topological space, a weaker notion is *topologically free*, where the fixed point sets $\mathrm{Fix}(t)$ have empty interior for $t \neq e$. The following theorem is collectively proven in [BKKO17, Theorem 3.1] (Theorem 5.2.7, $(i) \iff (iii) \iff (iv)$) and [KK17, Theorem 6.2] (Theorem 5.2.7, $(i) \iff (ii)$, along with other equivalences).

Theorem 5.2.7. *The following are equivalent:*

1. *G is C*-simple.*
2. *$C(\partial_F G) \rtimes_\lambda G$ is simple.*
3. *The action of G on $\partial_F G$ is free.*
4. *The action of G on some boundary is topologically free.*

It is now known that C*-simplicity is equivalent to an averaging property originally considered by Powers. The definition we present here is easily seen to be equivalent to the definition presented in [Ken20, Definition 6.2].

Definition 5.2.8. A discrete group G is said to satisfy *Powers' averaging property* if for any $\varepsilon > 0$ and any finite $F \subseteq G \setminus \{e\}$, there are $s_1, \ldots, s_m \in G$ such that

$$\left\| \frac{1}{m} \sum_{j=1}^{m} \lambda_{s_j} \lambda_t \lambda_{s_j}^* \right\| < \varepsilon \quad \forall t \in F.$$

The equivalence between C*-simplicity and Powers' averaging property was independently proven in [Ken20, Theorem 6.3] and [Haa16, Theorem 4.5].

Theorem 5.2.9. *The following are equivalent:*

1. *G is C*-simple.*
2. *G has Powers' averaging property.*

Finally, there is an intrinsic characterization of C*-simplicity by Kennedy. Recall that we may view the space of subgroups of G, i.e. $\mathrm{Sub}(G)$, as a closed (hence compact) subset of 2^G. The corresponding topology on $\mathrm{Sub}(G)$ is known as the *Chabauty topology*, and the space of amenable subgroups $\mathrm{Sub}_a(G)$ is again a closed (hence compact) subset of this space. More information can be found in [Ken20, Section 4]. The following definitions can be found at the start of [Ken20, Section 4] and in [Ken20, Definition 5.1].

Definition 5.2.10. A *uniformly recurrent subgroup* of G is a nonempty closed minimal subset of $\mathrm{Sub}(G)$. It is called *amenable* if one (equivalently all) of its elements are amenable, and *nontrivial* if it is not $\{\{e\}\}$. A subgroup $K \leq G$ is called *confined* (formerly called *residually normal*, but we have updated all occurrences of the term) if the closed orbit in $\mathrm{Sub}(G)$ does not contain the trivial subgroup $\{e\}$. Algebraically, $K \leq G$ is confined if and only if there exists a finite $F \subseteq G \setminus \{e\}$ such that $F \cap sKs^{-1} \neq \emptyset$ for any $s \in G$.

The following equivalence is proven in [Ken20, Theorem 4.1] and [Ken20, Theorem 5.3]:

Theorem 5.2.11. *The following are equivalent:*

1. *G is C*-simple.*
2. *G has no nontrivial amenable uniformly recurrent subgroup.*
3. *G has no amenable confined subgroup.*

Remark 5.2.12. It is worth noting that countability of G is not a requirement for any of the characterizations of C*-simplicity listed here. We will use some of these equivalences to prove our results, which in turn will be used for some examples, some of which involve uncountable groups.

5.3 Proof of main results

We first prove Theorem 5.1.3, most of which is already proven in [Amr21]. First, we dualize the definition of plumpness to pass from the C*-algebra $C^*_\lambda(G)$ to its state space:

Lemma 5.3.1. *Assume $H \leq G$. Then H is plump in G if and only if for any $\phi \in S(C^*_\lambda(G))$, the closed convex hull $\overline{\mathrm{conv}} H\phi$ contains the canonical trace τ_λ.*

Proof. The proof is analogous to the proof given in [Haa16, Theorem 4.5]. ∎

Lemma 5.3.2. *Assume $N \lhd G$ is normal. Then any G-minimal, N-strongly proximal space X is also N-minimal.*

Proof. Given that N-minimal components are always disjoint, it follows from strong proximality that there can only be exactly one N-minimal component in X - call it M. Further, we note that any translate tM (where $t \in G$) is still N-invariant. Indeed, $NtM = tNM = tM$, and so $M \subseteq tM$ by uniqueness. Using this, we also obtain $tM \subseteq t(t^{-1}M) = M$, and so M is G-invariant. But X is assumed to be G-minimal, and so $M = X$, i.e. X is N-minimal. ∎

Lemma 5.3.3. *Assume that N and X are as in Lemma 5.3.2, and $\phi : \partial_F N \to X$ is a G-equivariant continuous map. If, in addition, X is G-topologically free, then the action of G on $\partial_F N$ is free.*

Proof. Assume otherwise, so that there is some $t \in G$ with $U \coloneqq \mathrm{Fix}_{\partial_F N}(t)$ nonempty. It is known that U is necessarily clopen (see Corollary 5.2.5). We know that $\partial_F N = s_1 U \cup \cdots \cup s_n U$ for some $s_i \in N$, by minimality and compactness. Thus, $X = s_1\phi(U) \cup \cdots \cup s_n\phi(U)$, and so $\phi(U)$ being closed implies it has non-empty interior. But $\phi(U) \subseteq \mathrm{Fix}_X(t)$, contradicting topological freeness. This shows that the action of G on $\partial_F N$ is free. ∎

Remark 5.3.4. This argument is analogous to the proof of [BKKO17, Lemma 3.2], which claims that maps $\pi : Y \to X$ between minimal G-spaces map closed sets of nonempty interior to closed sets of nonempty interior. However, without the additional assumption that U is clopen (hence $\phi(U)$ is closed), one cannot conclude that $\phi(U)$ has interior just from having $X = s_1\phi(U) \cup \cdots \cup s_n\phi(U)$. Consider, for example, $[0,1] = ([0,1] \cap \mathbb{Q}) \cup ([0,1] \cap \mathbb{Q}^\complement)$. Hence, the above lemma also serves as a slight correction to the proof of [BKKO17, Theorem 3.1, (2) $\iff$ (3)], which uses [BKKO17, Lemma 3.2] as a prerequisite.

Remark 5.3.5. Before beginning the proof of Theorem 5.1.3, the author would like to thank Sven Raum for giving a much cleaner proof of $(iii) \implies (ii)$, which is the argument presented here. The original proof can be found in Section 5.4.

Proof of Theorem 5.1.3. The implications $(iv) \implies (ii) \implies (i)$ are given in [Amr21, Theorem 3.1, Corollary 3.2]. Further, $(iv) \iff (v)$ follows easily from [BKKO17, Theorem 1.4]. It is clear that $(ii) \implies (iii)$ holds, as $\partial_F N$ is such a space.

To show that $(iii) \implies (ii)$ holds, assume X is such a space. By Lemma 5.3.2, we have that X is in fact N-minimal, and therefore an N-boundary. Hence, we obtain an N-equivariant continuous map $\phi : \partial_F N \to X$. We claim it is G-equivariant. Letting $s \in N$ and $t \in G$, and slightly abusing notation by directly viewing these as automorphisms on $\partial_F N$ and X, we have that

$$(t \circ \phi \circ t^{-1})(sy) = (t \circ \phi)(t^{-1}stt^{-1}y) = tt^{-1}st(\phi(t^{-1}y)) = s((t \circ \phi \circ t^{-1})(y)).$$

As morphisms between boundaries (in this case, N-equivariant continuous maps) are necessarily unique, we have that $t \circ \phi \circ t^{-1} = \phi$, or in other words, ϕ is G-equivariant. By Lemma 5.3.3, we have that the action of G on $\partial_F N$ is free.

It remains to show $(i) \implies (iv)$. To this end, we note that N is C*-simple by Remark 5.1.4. Assume it is not the case that $C_G(N) = \{e\}$, and choose a nontrivial amenable subgroup $K \leq C_G(N)$. We know that the canonical character $1_K : K \to \mathbb{C}$ extends to a *-homomorphism $1_K : C^*_\lambda(K) \to \mathbb{C}$, and that there is also a canonical conditional expectation $E_K : C^*_\lambda(G) \to C^*_\lambda(K)$ mapping λ_t to itself if $t \in K$, and zero otherwise. It is easy to check that the composition $1_K \circ E_K : C^*_\lambda(G) \to \mathbb{C}$ is an N-fixed state, which contradicts Lemma 5.3.1. ∎

We now aim to prove Theorem 5.1.6. Some easy observations about relative simplicity as defined in Definition 5.1.5 are in place.

Proposition 5.3.6. *Let A, B, and C denote unital C*-algebras.*

1. *A is relatively simple in itself if and only if it is simple.*
2. *If $A \subseteq B \subseteq C$ with A relatively simple in C, then B is simple.*
3. *If $A \subseteq B \subseteq C$ with A relatively simple in C, then A is relatively simple in B and B is relatively simple in C.*

Proof. First, to prove (iii), let $\phi : B \to B(\mathcal{H})$ be a unital completely positive map which is a *-homomorphism on A. This extends to a unital completely positive map $\widetilde{\phi} : C \to B(\mathcal{H})$, which is faithful by assumption, and so ϕ is faithful, showing A is relatively simple in B. It is clear that B is relatively simple in C almost by definition. Claim (i) follows from the fact that for *-homomorphisms, faithfulness and injectivity are equivalent. Finally, (ii) follows by applying (iii) to get that A is relatively simple in B, then applying (iii) again to the containment $A \subseteq B \subseteq B$ to conclude that B is relatively simple in itself, and finally applying (i). ■

We also make use of the following lemma:

Lemma 5.3.7. *Assume $H \leq G$. Then H is plump in G if and only if the only H-equivariant unital completely positive map $\phi : C^*_\lambda(G) \to C(\partial_F H)$ is the canonical trace τ_λ.*

Proof. Observe that Lemma 5.3.1 tells us that, because τ_λ is H-invariant, H is plump if and only if there are no nontrivial H-irreducible closed convex subsets of $S(C^*_\lambda(G))$. As the closure of the extreme points of any such subset is an H-boundary [Gla76, Theorem III.2.3], this is equivalent to there being no nontrivial H-boundaries in $S(C^*_\lambda(G))$. From here, the proof is analogous to that of [Ken20, Proposition 3.1]. ■

Proof of Theorem 5.1.6. $(i) \implies (iii)$ This argument is adapted from part of the proof of $(2) \implies (1)$ in [BKKO17, Theorem 3.1]. Assume $\phi : C(\partial_F N) \rtimes_\lambda G \to B(\mathcal{H})$ is unital and completely positive, and also a *-homomorphism on $C^*_\lambda(N)$. We may equip $B(\mathcal{H})$ with an N-action given by $s \cdot T = \phi(\lambda_s)T\phi(\lambda_s)^*$ for $s \in N$ and $T \in B(\mathcal{H})$. Further, ϕ is N-equivariant with respect to this action on $B(\mathcal{H})$, as $C^*_\lambda(N)$ lies in the multiplicative domain of ϕ. We also have that, by injectivity, there is an N-equivariant unital completely positive map $\psi : B(\mathcal{H}) \to C(\partial_F N)$. We note that $\psi \circ \phi : C(\partial_F N) \rtimes_\lambda G \to C(\partial_F N)$ restricts to the canonical trace on $C^*_\lambda(G)$ by Lemma 5.3.7. Furthermore, $\psi \circ \phi$ is the identity on $C(\partial_F N)$ by rigidity, and so $C(\partial_F N)$ lies in the multiplicative domain of this map. Combining these two observations yields that $\psi \circ \phi$ is the canonical expectation, which is faithful, and so ϕ is faithful.

$(iii) \implies (ii)$ This follows from Proposition 5.3.6.

$(ii) \implies (i)$ Assume (i) does not hold. We know that N must be C*-simple by Proposition 5.3.6, and so looking back at Theorem 5.1.3, it must be the case that $C_G(N) \neq \{e\}$. Choose any nontrivial amenable subgroup K of $C_G(N)$, and note that $N \cap K = \{e\}$, as N has trivial center (C*-simplicity implies trivial amenable radical). Thus, $NK \cong N \times K$, and so $C^*_\lambda(NK) \cong C^*_\lambda(N) \otimes C^*_\lambda(K)$. Letting $\lambda_N : C^*_\lambda(N) \to B(\ell^2(N))$ denote the extension of the left-regular representation of N to $C^*_\lambda(N)$, and $1_K : C^*_\lambda(K) \to \mathbb{C}$ the extension of the trivial character, we have that $\lambda_N \otimes 1_K : C^*_\lambda(N) \otimes C^*_\lambda(K) \to B(\ell^2(N))$ is a non-faithful *-homomorphism. Thus, any unital completely positive extension $\phi : C^*_\lambda(G) \to B(\ell^2(N))$ will be non-faithful, yet be a *-homomorphism on $C^*_\lambda(N)$.

$(iii) \implies (iv)$ This follows from Proposition 5.3.6.

$(iv) \implies (i)$ This implication will be quite similar to $(ii) \implies (i)$. Assume that (i) doesn't hold, and observe that by Proposition 5.3.6, $C(\partial_F N) \rtimes_\lambda N$ is simple, which is known to imply that N is C*-simple (see Theorem 5.2.7). Again, $C_G(N) \neq \{e\}$, and choosing any nontrivial amenable subgroup $K \leq C_G(N)$, we have $NK \cong N \times K$. Further, K acts trivially on $\partial_F N$ by [BKKO17, Lemma 5.3], and so $C(\partial_F N) \rtimes_\lambda (NK) \cong (C(\partial_F N) \rtimes_\lambda N) \otimes C^*_\lambda(K)$. Letting $\pi : C(\partial_F N) \rtimes_\lambda N \to B(\mathcal{H})$ be any (necessarily faithful) representation, and $1_K : C^*_\lambda(K) \to \mathbb{C}$ be the extension of the trivial character, we have that $\pi \otimes 1_K : (C(\partial_F N) \rtimes_\lambda N) \otimes C^*_\lambda(K) \to B(\mathcal{H})$ is a non-faithful *-homomorphism. Any unital completely positive extension to $C^*_\lambda(G)$ will contradict the assumption of (iv). ∎

Proof of Theorem 5.1.9. $(ii) \implies (i)$ Assume (i) does not hold. Applying Theorem 5.1.3, we either have that N is not C*-simple, or $C_G(N) \neq \{e\}$. If N is not C*-simple, then by Kennedy's intrinsic characterization of C*-simplicity (see Theorem 5.2.11), N has a nontrivial amenable N-confined subgroup. If, on the other hand, $C_G(N) \neq \{e\}$, then any nontrivial amenable subgroup of $C_G(N)$ is N-confined.

$(i) \implies (ii)$ This is analogous to [Ken20, Remark 4.2]. For convenience, we give the modified argument here. Assume (ii) doesn't hold, and K is a nontrivial amenable N-confined subgroup of G. Amenability tells us that there is some K-invariant measure $\mu \in P(\partial_F N)$. Strong proximality tells us that there is a net $(s_\lambda) \subseteq N$ with $s_\lambda \mu \to \delta_x$ for some $x \in \partial_F N$. Dropping to a subnet, we may assume that $(s_\lambda K s_\lambda^{-1})$ is also convergent to some L, and $L \neq \{e\}$ by assumption. Chopping off the start of our net, we may in addition assume that there is some $l \in L \setminus \{e\}$ with $l \in s_\lambda K s_\lambda^{-1}$ for all λ, i.e. $l = s_\lambda k_\lambda s_\lambda^{-1}$ for some $k_\lambda \in K$. From here, we note that

$$l s_\lambda \mu \to l\delta_x = \delta_{lx},$$

while we also have

$$l s_\lambda \mu = s_\lambda k_\lambda \mu = s_\lambda \mu \to \delta_x.$$

This shows $lx = x$, and so G cannot act freely on $\partial_F N$.

$(ii) \iff (iii)$ If K is any nontrivial amenable N-confined subgroup of G, then any N-minimal subset of the closed N-orbit of K is an N-uniformly recurrent subgroup.

Conversely, any element of an N-uniformly recurrent subgroup is N-confined by definition. ∎

We conclude this section with some remarks.

Remark 5.3.8. Countability of N or G is not a requirement for any of the above proofs, nor is it required for any of the C*-simplicity analogues of the above characterizations obtained by setting $N = G$ (see Remark 5.2.12), some of which were applied here.

Remark 5.3.9. Some of the characterizations we have given are closed under taking supergroups. Namely, if $H \leq G$ is any (not necessarily normal) subgroup that satisfies any of Theorem 5.1.3 (*iii*) or (*v*), Theorem 5.1.6 (*ii*), or Theorem 5.1.9 (*ii*) or (*iii*), then so does any subgroup of G containing H. In particular, any normal subgroup of G containing H is relatively C*-simple.

5.4 The universal G-minimal, H-strongly proximal space

This section was originally dedicated to proving (*iii*) $\implies$ (*ii*) in Theorem 5.1.3, until a much cleaner proof was suggested by Sven Raum - see Remark 5.3.5. The existence of a type of relative Furstenberg boundary with respect to arbitrary (not necessarily normal) subgroups might still be interesting, and for this reason this section is still kept.

Proposition 5.4.1. *Assume H is a (not necessarily normal) subgroup of G. There is a universal G-minimal, H-strongly proximal G-space $B(G, H)$, in the sense that any other such space X is a G-equivariant continuous image of $B(G, H)$. Further, this space is unique up to G-equivariant homeomorphism.*

Proof. The proof is quite similar to the topological proof of the existence of the Furstenberg boundary, a very brief sketch of which is given in [Fur73, Proposition 4.6]. We fill in the details and modify the argument appropriately here.

Let $\{X_\alpha\}_{\alpha \in A}$ denote the set of all up-to-isomorphism G-minimal, H-strongly proximal spaces, where isomorphism refers to G-equivariant homeomorphism. Note that these can indeed be put into a set, as all of these spaces are necessarily the continuous image of βG by minimality, so there is a limit on the cardinality of these spaces.

We claim that the space $X := \prod_\alpha X_\alpha$ is still H-strongly proximal. To see this, given any measure $\mu \in P(X)$, we note that for any $\alpha \in A$, there is a net $(h_\lambda) \subseteq H$ with $(h_\lambda \mu)$ converging to a Dirac mass when restricted to $C(X_\alpha)$. From here, it is easy to see that this can be done on finitely many $\alpha_1, \dots, \alpha_n \in A$. Now for each finite $F \subseteq A$, letting $\mu_F \in \overline{H\mu}$ be a Dirac mass when restricted to each $C(X_\alpha)$ for $\alpha \in F$, we note that any cluster point of the net (μ_F) (indexed over finite subsets of A, ordered under inclusion) is necessarily a Dirac mass on the entire space X.

Let $B(G, H)$ be a G-minimal subset of X. It is clear that this space is still H-strongly proximal. We will also show that it is universal. Given any X_α, consider the coordinate projection map $\pi_\alpha : X \to X_\alpha$. We see that $\pi_\alpha|_{B(G,H)} : B(G, H) \to X_\alpha$ is still surjective, as the image of this map is closed and G-invariant, and X_α is G-minimal.

Finally, this space is unique up to isomorphism. Indeed, if B' is another universal space, then there are G-equivariant continuous maps $\phi_1 : B(G, H) \to B'$ and $\phi_2 : B' \to B(G, H)$. But their compositions $\phi_2 \circ \phi_1 : B(G, H) \to B(G, H)$ and $\phi_1 \circ \phi_2 : B' \to B'$ are necessarily the respective identity maps between these spaces, as these spaces are both G-boundaries, and, assuming they exist, morphisms between G-boundaries are unique. ∎

Remark 5.4.2. A different notion of relative Furstenberg boundary is presented in [Mon21], and so we avoid using the term *Furstenberg boundary* and notation $\partial(G, H)$ to describe the universal object from Proposition 5.4.1. Our notation $B(G, H)$ is derived from Furstenberg's notation $B(G)$ for the Furstenberg boundary of G, given in [Fur73, Proposition 4.6].

Corollary 5.4.3. *If $N \triangleleft G$ is normal, the universal G-minimal, N-strongly proximal space is $\partial_F N$.*

Proof. Letting $B(G, N)$ denote the universal such space, there is a G-equivariant continuous surjection $\phi_1 : B(G, N) \to \partial_F N$. However, Lemma 5.3.2 tells us that $B(G, N)$ is in fact an N-boundary, and so there is an N-equivariant continuous surjection $\phi_2 : \partial_F N \to B(G, N)$. The composition $\phi_1 \circ \phi_2 : \partial_F N \to \partial_F N$ is N-equivariant, and thus necessarily the identity map. This shows ϕ_2 is injective, hence bijective. Thus, ϕ_1 is also bijective, and therefore the isomorphism we are looking for. ∎

It is worth emphasizing a subtle point - Lemma 5.3.2 tells us that any G-minimal, N-strongly proximal space is the N-equivariant image of $\partial_F N$. However, Corollary 5.4.3 gives us a G-equivariant map.

Proof of Theorem 5.1.3, (iii) $\implies$ (ii). Assume X is G-minimal, N-strongly proximal, and G-topologically free. By Corollary 5.4.3, there is a G-equivariant continuous map $\phi : \partial_F N \to X$. By Lemma 5.3.3, the action of G on $\partial_F N$ is free. ∎

5.5 Examples

It is worth noting that the characterization of being C*-simple and having trivial centralizer, originally given as a sufficient condition in [Amr21, Corollary 3.2], is perhaps the "nicest" characterization of relative C*-simplicity. As such, some of the examples below will still be proven with this result, as opposed to our new results.

5.5.1 Free products

Given that the canonical example of a C*-simple group is $\mathbb{F}_2$, the free group on two generators, it is worthwhile to use this as an easy example. We will re-prove the following special case of [Amr21, Example 3.8] using one of our new results.

Example 5.5.1. Let $\mathbb{F}_2 = \langle a, b\rangle$ denote the free group on two generators. Then the normal closure of a is relatively C*-simple in $\mathbb{F}_2$.

Proof. The Nielsen-Schreier theorem tells us that any subgroup of a free group is free. Thus, the only nontrivial amenable subgroups of $\mathbb{F}_2$ are the cyclic subgroups. Given any such subgroup $\langle x\rangle$, assume first that the reduced word of x starts with b or b^{-1}. Then the reduced word length of $a^n x a^{-n}$ is eventually strictly increasing, showing that $\langle a^n x a^{-n}\rangle \to \{e\}$ in the Chabauty topology. This is also true if the reduced word of x ends with b or b^{-1}. Finally, if both the start and end of x lie in $\{a, a^{-1}\}$, then the reduced word length of $(bab^{-1})^n x (bab^{-1})^{-n}$ is strictly increasing, and so $\langle (bab^{-1})^n x (bab^{-1})^{-n}\rangle$ is Chabauty-convergent to $\{e\}$. By Theorem 5.1.9, we are done. ■

We will also generalize [Amr21, Example 3.8] to free products as follows:

Theorem 5.5.2. *Assume $G = H * K$, where H and K are nontrivial, and they are also not both $\mathbb{Z}_2$. Then any nontrivial normal subgroup of G is relatively C*-simple.*

Proof. It is known that any such group is C*-simple [PS79]. Hence, any normal subgroup $N \lhd G$ is C*-simple as well by [BKKO17, Theorem 1.4]. It remains to show that any nontrivial normal subgroup has trivial centralizer. Assume otherwise, so that there exists some normal subgroup $N \neq \{e\}$ with $C_G(N) \neq \{e\}$, and pick nontrivial elements $x \in N$ and $y \in C_G(N)$. C*-simplicity of N tells us that N has trivial center, i.e. $N \cap C_G(N) = \{e\}$, and so $\langle x, y\rangle \cong \langle x\rangle \times \langle y\rangle$. But the Kurosh subgroup theorem tells us that

$$\langle x\rangle \times \langle y\rangle = \mathbb{F}_X * \prod_{i\in I}^{*} s_i H_i s_i^{-1} * \prod_{j\in J}^{*} t_j K_j t_j^{-1}$$

for some subset $X \subseteq G$ and subgroups $H_i \leq H$, $K_j \leq K$. There are two cases when such a subgroup is abelian. The first case is if X is a singleton, and I and J are empty. This is impossible, as $\langle x \rangle \times \langle y \rangle$ is not isomorphic to $\mathbb{Z}$. The second case is if, without loss of generality, $\langle x \rangle \times \langle y \rangle$ is some conjugate $s_i H_i s_i^{-1}$, where $H_i \leq H$. Equivalently, $\langle s_i^{-1} x s_i \rangle \times \langle s_i^{-1} y s_i \rangle$ is a subgroup of H. But both N and $C_G(N)$ are normal subgroups of G, and so this says that there are some nontrivial $s \in N$ and $t \in C_G(N)$ that both lie in H. However, if we choose any nontrivial $r \in K$, then t cannot commute with $rsr^{-1} \in N$, a contradiction. ■

5.5.2 Direct products

Taking direct sums and direct products of existing examples can provide some easy new examples:

Lemma 5.5.3. *Let (G_i) be a family of C*-simple groups. Then both $\oplus G_i$ and $\prod G_i$ are C*-simple.*

Proof. We know that each G_i acts freely on its Furstenberg boundary $\partial_F G_i$ (see Theorem 5.2.7). From here, it is not hard to check that $\prod \partial_F G_i$ is a free boundary action for both $\oplus G_i$ and $\prod G_i$, and so both of these groups are C*-simple by the same theorem. ■

Theorem 5.5.4. *Let (G_i) be a family of groups with relatively C*-simple normal subgroups $N_i \triangleleft G_i$. The direct sum $\oplus N_i$ is relatively C*-simple in the direct product $\prod G_i$.*

Proof. Observe that $\oplus N_i$ is normal in $\prod G_i$, and the commutator of this subgroup is $\prod C_{G_i}(N_i)$. By Theorem 5.1.3, this commutator is trivial, and so applying this theorem again together with Lemma 5.5.3, $\oplus N_i$ is relatively C*-simple in $\prod G_i$. ■

Remark 5.5.5. This shows that there exists an uncountable group with a countable relatively C*-simple normal subgroup, for example $\oplus_{n \in \mathbb{N}} \mathbb{F}_2 \triangleleft \prod_{n \in \mathbb{N}} \mathbb{F}_2$. From the C*-algebras perspective, there is a non-separable C*-algebra with a separable relatively simple sub-C*-algebra.

5.5.3 Wreath products

Recall that the (unrestricted) wreath product $G \wr H$ is $(\prod_H G) \rtimes H$, where H acts by left-translation on $\prod_H G$.

Theorem 5.5.6. *Assume N is a relatively C*-simple subgroup of some group $G \neq \{e\}$, and let H be any arbitrary group. Then the direct sum $\oplus_H N$ is relatively C*-simple in $G \wr H$.*

Proof. Note that $\oplus_H N$ is normal in $G \wr H$. It is easy to check that the canonical action of $\prod_H G$ on $\prod_H \partial_F N$, together with the action of H on $\prod_H \partial_F N$ by left-translation, extend to an action of all of $G \wr H$. It is also not hard to see that $\oplus_H N$ acts strongly proximally and $G \wr H$ acts minimally on this space.

It remains to show that the action of $G \wr H$ is still topologically free. To this end, first consider any nontrivial element of the form $((g_h), e) \in G \wr H$. Its fixed point set is given by $\prod_H \mathrm{Fix}(g_h)$, which is empty by Theorem 5.1.3 and the assumption that at least one $g_h \neq e$. Now given any element $((g_h), h_0) \in G \wr H$ with $h_0 \neq e$, we note that

$$((g_h), h_0)(x_h) = (g_h x_{h_0^{-1}h}),$$

and so (x_h) lies in the fixed point set of this element if and only if $g_h x_{h_0^{-1}h} = x_h$ for all h. In particular, setting $h = h_0$ gives us $g_{h_0} x_e = x_{h_0}$. If $\mathrm{Fix}((g_h), h_0)$ were to have interior, then it would contain a basic open subset of the form $\prod_H U_h$, where $U_h \subseteq X$ is open, and all but finitely many $U_h = X$. Given that N is C*-simple and $N \neq \{e\}$ (as N is relatively C*-simple in $G \neq \{e\}$ by assumption) we know that $\partial_F N$ has no isolated points [KK17, Proposition 3.15], and so no U_h is a singleton. But given that x_{h_0} is entirely determined by the value x_e takes, this cannot be the case. We conclude that $\mathrm{Fix}((g_h), h_0)$ has empty interior, and so by Theorem 5.1.3, we are done. ∎

Remark 5.5.7. The sufficient condition for plumpness given in [Amr21, Lemma 3.5] assumes the group is countable and has countable fixed point sets. The proof of Theorem 5.5.6, however, gives a natural class of topologically free boundary actions admitting uncountably many fixed points. Here, we see that $H \leq G \wr H$ fixes the diagonal of $\prod_H \partial_F N$, and $\partial_F N$ is uncountable as it is a nontrivial compact Hausdorff space with no isolated points. One could also replace $\partial_F N$ by any G-minimal, N-strongly proximal, G-topologically free space X, and so any element $((g_h), e) \in G \wr H$ admits the fixed point set $\prod_H \mathrm{Fix}(g_h)$, which is uncountable if, for example, $\mathrm{Fix}(g_h)$ are nonempty for all h, and at least one $g_h = e$. Finally, while wreath products $G \wr H$ are often uncountable (for example, if $G \neq \{e\}$ and H is infinite), the same observations hold for the restricted wreath product $(\oplus_H G) \rtimes H$ as well, which is countable if G and H are countable.

5.5.4 Groups with trivial center and only cyclic amenable subgroups

It is sometimes the case that the only amenable subgroups of a given group are the cyclic subgroups. For example, this property is true of the free groups by the Nielsen-Schreier theorem. Our aim is to show the following:

Theorem 5.5.8. *Assume G is such that any amenable subgroup is cyclic, $Z(G) = \{e\}$, and in addition, no two elements have finite coprime order. Then any nontrivial normal subgroup of G is relatively C*-simple.*

Remark 5.5.9. It is worth noting that this last requirement that G should have no two elements of finite coprime order is true for a large class of groups, including torsion-free groups and p-groups.

Whether or not G has trivial center is surprisingly sufficient in determining whether G is C*-simple or not. In the case of countable groups, [BKKO17, Theorem 6.12] tells us it suffices to prove that $R_a(G) \subseteq Z(G)$. The argument we present here avoids countability, but requires a bit of a detour into theory on the Furstenberg boundary.

Lemma 5.5.10. *Let G denote any discrete group, and let $x \in \partial_F G$ be arbitrary. Letting G_x denote the point-stabilizer of x, if $s \in G_x$ is nontrivial, and $y_1, \dots, y_n \in \partial_F G$, then $G_{y_1} \cap \cdots \cap G_{y_n}$ always contains some conjugate of s.*

Proof. This is a special case of [BKKO17, Lemma 3.7] obtained by setting $U = \mathrm{Fix}(s)$ (necessarily clopen - see Corollary 5.2.5), $\varepsilon = \frac{1}{n}$, and $\mu = \frac{1}{n}(\delta_{y_1} + \cdots + \delta_{y_n})$. Our end result is that there is some $r \in G$ with $ry_i \in \mathrm{Fix}(s)$ for all i, or in other words, $r^{-1}sr \in G_{y_i}$ for all i. ∎

Proposition 5.5.11. *Assume G has the property that any amenable subgroup is cyclic. Then G is C*-simple if and only if $Z(G) = \{e\}$.*

Proof. As the center is always an amenable normal subgroup, any C*-simple group G must have trivial center. Conversely, assume G has trivial center. We will first show that G has trivial amenable radical. Given any $t \in G$, we have that $(\langle t \rangle R_a(G))/R_a(G) \cong \langle t \rangle / (\langle t \rangle \cap R_a(G))$, which is amenable, and so by extension, $\langle t \rangle R_a(G)$ is amenable, thus cyclic. This shows t commutes with every element of $R_a(G)$. Since t was arbitrary, $R_a(G) \subseteq Z(G) = \{e\}$.

Now we wish to show that none of the point-stabilizers G_x for $x \in \partial_F G$ can be nontrivial. Assume otherwise. Recall that G_x is always amenable - see, for example [BKKO17, Proposition 2.7]. This tells us that $G_x = \langle s \rangle$ for some $x \in \partial_F G$ and $s \neq e$. If G_x were finite, it follow from $\bigcap_{y \in \partial_F G} G_y = R_a(G) = \{e\}$ that there are $y_1, \ldots, y_n \in \partial_F G$ with $G_x \cap G_{y_1} \cap \cdots \cap G_{y_n} = \{e\}$. This contradicts Lemma 5.5.10. If G_x is infinite cyclic, this tells us that there is some $y \in \partial_F G$ with $G_x \neq G_y$. Without loss of generality, $G_x \not\subseteq G_y$, and so $G_x \cap G_y = \langle s^n \rangle$ for some $|n| \geq 2$. Again, Lemma 5.5.10 gives us that there is some $r \in G$ with $rsr^{-1} = s^m$ for some $|m| \geq 2$. It is easy to show that, inductively, $r^k s r^{-k} = s^{m^k}$, and so $r^k \langle s \rangle r^{-k}$ converges to $\{e\}$ in the Chabauty topology. This can never happen if $G_x \neq \{e\}$, as $\{G_x\}_{x \in \partial_F G}$ is an amenable uniformly recurrent subgroup - see [Ken20, Remark 4.3]. ■

Proof of Theorem 5.5.8. By Proposition 5.5.11, any such group is C*-simple. Assume N is a nontrivial normal subgroup, and $C_G(N) \neq \{e\}$. We know that G being C*-simple implies N is C*-simple by [BKKO17, Theorem 1.4], and so $Z(N) = N \cap C_G(N)$ is trivial. Thus, if we choose nontrivial $x \in N$ and $y \in C_G(N)$, then $\langle x, y \rangle \cong \langle x \rangle \times \langle y \rangle$. Such a group is both amenable and non-cyclic, contradicting our assumption. Hence, any nontrivial normal subgroup has trivial centralizer, and so by Theorem 5.1.3, we are done. ■

Recall that the free Burnside group $B(m, n)$ is the universal group on m generators satisfying $x^n = e$ for all elements x in the group. The Burnside problem, which was one of the largest open problems in group theory, asked whether such groups are always finite. The answer, as it turns out, is no, and in addition, some of these groups are C*-simple - see [BKKO17, Corollary 6.14]. In particular, they remark that for any $m \geq 2$ and n odd and sufficiently large, any non-cyclic subgroup contains a copy of the non-amenable group $B(2, n)$. Hence, we obtain the following:

Example 5.5.12. Assume $m \geq 2$ and n is prime and sufficiently large. Then any nontrivial normal subgroup of $B(m, n)$ is relatively C*-simple.

5.5.5 A remark on Thompson's group F

Thompson's group F was the original candidate counterexample for the now-disproven Day-von Neumann conjecture, which stated that a group is non-amenable if and only if it contains a copy of $\mathbb{F}_2$, the free group on two generators. A good introduction to this, and related groups, can be found in [CFP96]. It is known that F does not contain a copy of $\mathbb{F}_2$, but whether or not it is amenable is still a large open question in group theory. However, it is known that F is non-amenable if and only if it is C*-simple - see [LM18, Theorem 1.6]. Hence, with a bit of extra work, we obtain the following equivalence:

Theorem 5.5.13. *Thompson's group F is non-amenable if and only if its derived subgroup $[F, F]$ is relatively C*-simple in F.*

Proof. Relative simplicity of $[F, F]$ in F would imply that both $[F, F]$ and F are C*-simple, in particular non-amenable. It remains to prove the converse.

Assume F is non-amenable, hence C*-simple. It is known that every proper quotient of F is abelian [CFP96, Theorem 4.3], or equivalently, $[F, F] \subseteq N$ for any normal subgroup $N \triangleleft F$ with $N \neq \{e\}$. In particular, we must have that $C_F([F, F]) = \{e\}$, otherwise $[F, F]$ would be abelian (and thus F would be amenable). By Theorem 5.1.3, we are done. ■

This shows, for example, that to prove amenability of F, it would suffice to construct a non-faithful unital completely positive map $\phi : C^*_\lambda(F) \to B(\mathcal{H})$ that is a *-homomorphism on $C^*_\lambda([F, F])$.

Chapter 6

A generalized Powers averaging property for commutative crossed products

6.1 Introduction

The notion of Powers' averaging property for discrete groups has played an important role in recent years in questions about simplicity related to reduced group C*-algebras and reduced crossed products. In this paper, we introduce a generalized version of Powers' averaging property for reduced crossed products of the form $C(X) \rtimes_\lambda G$, and prove that it is equivalent to simplicity of the crossed product. We then derive various consequences.

First, we recall the notion of Powers' averaging property, along with a brief history of recent applications. Let G be a countable discrete group, and let $P(G)$ denote the set of probability measures on G. For convenience, we will denote the finitely supported probability measures on G by $P_f(G)$. Recall that we canonically have an action of $P(G)$ on any G-C*-algebra A as follows: given $\mu \in P(G)$ and $a \in A$, we let

$$\mu a = \sum_{g \in G} \mu(g)(g \cdot a).$$

The group G is said to be C*-simple if its reduced group C*-algebra $C^*_\lambda(G)$ is simple. It was shown independently in [Ken20, Theorem 6.3] and [Haa16, Theorem 4.5] that C*-simplicity is equivalent to an averaging property originally considered by Powers, which can most conveniently be stated as follows: G is said to have *Powers' averaging property* if for any $a \in C^*_\lambda(G)$, we have

$$\tau_\lambda(a) \in \overline{\{\mu a \mid \mu \in P_f(G)\}}.$$

Here, τ_λ denotes the canonical trace on $C^*_\lambda(G)$, and we are canonically viewing $\mathbb{C} \subseteq C^*_\lambda(G)$. The set $P_f(G)$ above can be replaced by $P(G)$ instead. It is also clear that it suffices to check only the $a \in C^*_\lambda(G)$ satisfying $\tau_\lambda(a) = 0$, as it is always possible to "normalize" an arbitrary $a \in C^*_\lambda(G)$ by considering $a - \tau_\lambda(a)$.

It was later shown by Hartman and Kalantar in the proof of [HK17, Theorem 5.1] that averaging with respect to all of $P_f(G)$ is not necessary, and that if G is countable, then Powers' averaging property for $C^*_\lambda(G)$ is equivalent to the existence of a single measure $\mu \in P(G)$ (not guaranteed to have finite support) satisfying $\mu^n a \to \tau_\lambda(a)$ for all $a \in C^*_\lambda(G)$.

A generalization of Powers' averaging property to reduced (twisted) crossed products of unital C*-algebras and C*-simple groups was given by [BK16, Section 3]. They showed that the same averaging property holds for elements $a \in A \rtimes_\lambda G$ satisfying $\mathbb{E}(a) = 0$, where $\mathbb{E} : A \rtimes_\lambda G \to A$ denotes the canonical conditional expectation, i.e.

$$0 \in \overline{\{\mu a \mid \mu \in P_f(G)\}}.$$

The ideas mentioned above were used by the first named author and Kalantar in [AK20] to show that if G is C*-simple and the action of G on a compact Hausdorff space X is minimal, then not only is the reduced crossed product $C(X) \rtimes_\lambda G$ simple, but so is any intermediate C*-algebra lying between $C^*_\lambda(G)$ and $C(X) \rtimes_\lambda G$.

Of course, if G is not C*-simple, then $A \rtimes_\lambda G$ can never have Powers' averaging property, but it is still possible for the crossed product to be simple. An easy example is $C(\mathbb{T}) \rtimes_\lambda \mathbb{Z}$, where $\mathbb{Z}$ acts on the circle $\mathbb{T}$ by an irrational rotation. For this reason, we introduce a generalized version of Powers' averaging which does turn out to be equivalent to simplicity in the end. Let X be a compact Hausdorff space equipped with an action of G by homeomorphisms.

We define in Section 6.2, and in particular in Definition 6.2.3, the spaces $P(G, C(X))$ and $P_f(G, C(X))$ of what we call *generalized* $(G, C(X))$*-probability measures*. Given an inclusion of unital G-C*-algebras $C(X) \subseteq A$, the space $P(G, C(X))$ canonically admits a left action on A, and a right action on the state space $S(A)$. With this, we are able to conveniently generalize Powers' averaging property to crossed products as follows:

Definition 6.1.1. Let G be a countable discrete group acting on a compact Hausdorff space X by homeomorphisms, and let $\mathbb{E} : C(X) \rtimes_\lambda G \to C(X)$ denote the canonical conditional expectation. We say that $C(X) \rtimes_\lambda G$ has the *generalized Powers' averaging property* if for every $a \in C(X) \rtimes_\lambda G$ with $\mathbb{E}(a) = 0$, we have

$$0 \in \overline{\{\mu a \mid \mu \in P_f(G, C(X))\}}.$$

One might define other generalized analogues of Powers' averaging property, for example requiring that $\mathbb{E}(a)$ lie in the above set given any $a \in C(X) \rtimes_\lambda G$ not necessarily satisfying $\mathbb{E}(a) = 0$. It is not immediately obvious, however, that this is equivalent with the version in Definition 6.1.1, as considering $a - \mathbb{E}(a)$ for an arbitrary $a \in C(X) \rtimes_\lambda G$ just tells us that for any $\varepsilon > 0$, there is a $\mu \in P_f(G, C(X))$ with the property that $\|\mu a - \mu\mathbb{E}(a)\| < \varepsilon$. However, unlike in the case where $C(X) = \mathbb{C}$, we do not have that $\mu\mathbb{E}(a) = \mathbb{E}(a)$ in general. Hence, our first main result is perhaps a bit surprising:

Theorem 6.1.2. *Let G be a countable discrete group acting on a compact Hausdorff space X by homeomorphisms, and assume that the action is minimal. Let $\mathbb{E} : C(X) \rtimes_\lambda G \to C(X)$ denote the canonical conditional expectation. The following are equivalent:*

1. *$C(X) \rtimes_\lambda G$ is simple.*
2. *$C(X) \rtimes_\lambda G$ has the generalized Powers' averaging property.*
3. *$\mathbb{E}(a) \in \overline{\{\mu a \mid \mu \in P_f(G, C(X))\}}$ for all $a \in C(X) \rtimes_\lambda G$.*
4. *$\nu(\mathbb{E}(a)) \in \overline{\{\mu a \mid \mu \in P_f(G, C(X))\}}$ for all $a \in C(X) \rtimes_\lambda G$ and $\nu \in P(X)$.*

Next we generalize Hartman and Kalantar's results. It is worth noting that they operate under a slightly different action of $P(G)$ on any G-C*-algebra, with a convolution product given by

$$\mu * a = \sum_{g \in G} \mu(g)(g^{-1} \cdot a)$$

for any $\mu \in P(G)$, and a left action of $P(G)$ on $S(A)$ given by

$$\mu * \phi = \sum_{g \in G} \mu(g)(g \cdot \phi).$$

However, this is only a minor technicality to keep in mind, and it is easy to rephrase their results (which we do) in terms of the actions we use in our paper.

As previously mentioned, they show that Powers' averaging property for $C^*_\lambda(G)$ is equivalent to the existence of a measure $\mu \in P(G)$ with the property that $\mu^n a \to \tau_\lambda(a)$ for any

$a \in C^*_\lambda(G)$ [HK17, Theorem 5.1]. As a consequence, the only state $\phi \in S(C^*_\lambda(G))$ that is *μ-stationary* (that is, satisfying $\phi\mu = \phi$) is the canonical trace τ_λ, and this is in fact a characterization of C*-simplicity of G [HK17, Theorem 5.2]. Similar result holds in the crossed product setting:

Theorem 6.1.3. *Let G be a countable discrete group acting on a compact Hausdorff space X by homeomorphisms, and assume that the action is minimal. Let $\mathbb{E} : C(X) \rtimes_\lambda G \to C(X)$ denote the canonical conditional expectation. If $C(X) \rtimes_\lambda G$ is simple, then there is a generalized measure $\mu \in P(G, C(X))$ with the property that $\mu^n a \to 0$ whenever $\mathbb{E}(a) = 0$. Optionally, we may also require that μ have full support.*

Corollary 6.1.4. *Let G be a countable discrete group acting on a compact Hausdorff space X by homeomorphisms, and assume that the action is minimal. Let $\mathbb{E} : C(X) \rtimes_\lambda G \to C(X)$ denote the canonical conditional expectation. Then the crossed product $C(X) \rtimes_\lambda G$ is simple if and only if there is some $\mu \in P(G, C(X))$ with full support and with the property that any μ-stationary state on $C(X) \rtimes_\lambda G$ is of the form $\nu \circ \mathbb{E}$ for some μ-stationary $\nu \in P(X)$.*

It is worth noting in Corollary 6.1.4 that, given a generalized measure $\mu \in P(G, C(X))$, there is no guarantee that there be a *unique* μ-stationary measure $\nu \in P(X)$. If one could strengthen the averaging in Theorem 6.1.2 (4) to work with a single measure, for example if there were some $\nu \in P(X)$ and $\mu \in P(G, C(X))$ such that $\mu^n a \to \nu(\mathbb{E}(a))$ for any $a \in C(X) \rtimes_\lambda G$, then it would be possible to obtain uniqueness of ν as well. However, we were unable to prove such a result.

Our first application of Powers' averaging property is a natural generalization of the main result in [AK20].

Theorem 6.1.5. *Let G be a countable discrete group, and assume that $C(Y) \subseteq C(X)$ is an equivariant inclusion of commutative unital G-C*-algebras. Assume moreover that the action of G on X is minimal. If $C(Y) \rtimes_\lambda G$ is simple, then every intermediate C*-algebra A satisfying $C(Y) \rtimes_\lambda G \subseteq A \subseteq C(X) \rtimes_\lambda G$ is simple.*

One other result of Hartman and Kalantar that we generalize is the following. Denote the space of amenable subgroups of G by $\mathrm{Sub}_a(G)$. This is naturally a compact Hausdorff space if we equip it with the Chabauty topology, which is the topology induced by viewing this canonically as a subset of 2^G (the power set of G), and it also carries a G-action by homeomorphisms given by conjugation. It has been known for a few years that the dynamics on this space characterizes C*-simplicity, with [Ken20, Theorem 4.1] essentially stating that G is C*-simple if and only if $\{\{e\}\}$ is the unique minimal component in

$\mathrm{Sub}_a(G)$, and [HK17, Corollary 5.7] stating that C*-simplicity is equivalent to unique stationarity of $\delta_{\{e\}}$ with respect to some $\mu \in P(G)$.

The dynamical analogue for crossed products $C(X) \rtimes_\lambda G$ (where X is minimal) is a result of Kawabe [Kaw17, Theorem 6.1]. Consider the space

$$\mathrm{Sub}_a(X, G) := \{(x, H) \mid x \in X, H \leq G_x, \text{ and } H \text{ amenable}\},$$

where G_x denotes the stabilizer subgroup of x. This is again a compact Hausdorff space with G-action given by $s \cdot (x, H) = (sx, sHs^{-1})$, and Kawabe's result amounts to saying that $C(X) \rtimes_\lambda G$ is simple if and only if the only minimal component in $\mathrm{Sub}_a(X, G)$ is $X \times \{\{e\}\}$. This hints that there should also be a "unique stationarity result" involving measures supported on this minimal component.

Corollary 6.1.6. *Let G be a countable discrete group acting on a compact Hausdorff space X by homeomorphisms, and assume that the action is minimal. Let $\mathrm{Sub}_a(X, G)$ denote Kawabe's generalized space of amenable subgroups, and view $C(X) \subseteq C(\mathrm{Sub}_a(X, G))$ as dual to the canonical projection $\mathrm{Sub}_a(X, G) \twoheadrightarrow X$ mapping (x, H) to x. The crossed product $C(X) \rtimes_\lambda G$ is simple if and only if there is some $\mu \in P(G, C(X))$ with the property that any μ-stationary measure in $P(\mathrm{Sub}_a(X, G))$ is supported on $X \times \{\{e\}\}$.*

6.2 The space of generalized probability measures

To establish notation, G will denote a countable discrete group, and X will denote a compact Hausdorff space which G acts on by homeomorphisms. All C*-algebras and morphisms are assumed to be unital.

We define the notion of a generalized probability measure. As motivation, consider the case of a G-C*-algebra A. Given a fixed $a \in A$, any probability measure $\mu \in P(G)$ provides a convenient way of representing a convex combination of the elements $\{g \cdot a \mid g \in G\}$. Namely, we may define $\mu a := \sum_{g \in G} \mu(g) g \cdot a$.

With this in mind, we want a space of generalized probability measures which represents *C(X)-convex combinations*. For convenience, we first review this notion here:

Definition 6.2.1. Assume $C(X) \subseteq A$ is an inclusion of unital C*-algebras, and let $K \subseteq A$. We say that K is $C(X)$-*convex* if, given finitely many $f_1, \ldots, f_n \in C(X)$ with $\sum_{i=1}^n f_i^2 = 1$, and any $a_1, \ldots, a_n \in K$, we have $\sum_{i=1}^n f_i a_i f_i \in K$. Such a sum is called a $C(X)$-*convex combination* of $a_1, \ldots, a_n$.

Remark 6.2.2. The usual notion of $C(X)$-convex combinations is slightly more general, and deals with sums of the form $\sum_{i=1}^{n} f_i^* a_i f_i$, where $\sum_{i=1}^{n} f_i^* f_i = 1$ and f_i is no longer assumed to be positive. For our purposes, we will stick with the definition given in Definition 6.2.1, as working with positive f_i is in particular necessary for Lemma 6.4.1 later on.

If $C(X) \subseteq A$ is an inclusion of unital G-C*-algebras, and $a \in A$, we want our notion of generalized probability measures to represent $C(X)$-convex combinations of $\{g \cdot a \mid g \in G\}$.

Definition 6.2.3. Consider a formal sum

$$\mu = \sum_{s \in G} \sum_{i \in I_s} f_i s f_i$$

(where all I_s are disjoint), with the properties $f_i \geq 0$, $f_i \neq 0$, and $\sum_{s \in G} \sum_{i \in I_s} f_i^2 = 1$. Equivalenty, we may also combine the above double-sum into a single sum

$$\mu = \sum_{i \in I} f_i s_i f_i$$

if we allow repetition among the group elements s_i. We say that μ is a *generalized* $(G, C(X))$*-probability measure*, and denote the set of all such generalized measures by $P(G, C(X))$. The set of all finite-sum generalized measures is denoted by $P_f(G, C(X))$. Given a unital G-C*-algebra A containing an equivariant copy of $C(X)$, and any $\mu = \sum_{i \in I} f_i s_i f_i \in P(G, C(X))$, we have a unital and completely positive map on A given by

$$\mu a = \sum_{i \in I} f_i (s_i \cdot a) f_i.$$

Moreover, this induces a right action on the state space $S(A)$, given by $(\phi\mu)(a) = \phi(\mu a)$.

Remark 6.2.4. We note a couple of things. First, observe that for a fixed $s \in G$, it is in general not possible to simplify an expression of the form $f_1 s f_1 + f_2 s f_2$ as a single hsh for some $h \in C(X)$. This is a consequence of noncommutativity. If $C(X) \subseteq A$ and we were to consider the action of this element on some $a \in A$, this becomes

$$f_1(s \cdot a) f_1 + f_2(s \cdot a) f_2,$$

which in general is not equal to any $h(s \cdot a)h$ for any $h \in C(X)$. Because of this, repetition of group elements is allowed, and this is also reasoning behind the choice of terminology and notation, namely "finite-sum" and $P_f(G, C(X))$, as opposed to "compactly supported" and $P_c(G, C(X))$. Given a generalized probability measure $\mu = \sum_{i\in I} f_i s_i f_i$, it is possible to have infinitely many elements $i \in I$ with s_i all being equal. In other words, it is possible to have an infinite sum (which doesn't necessarily simplify to a finite sum) that is still "compactly supported" on G.

The rest of this section is dedicated to proving various technicalities and basic properties of the space $P(G, C(X))$. First, the following is an easy exercise in functional analysis:

Lemma 6.2.5. *Assume X is a Banach space, and $\sum_{i\in I} x_i$ is an infinite unordered sum. Then this sum converges if and only if for all $\varepsilon > 0$, there exists a finite set $F \subseteq I$ such that for all finite sets $J \subseteq I \setminus F$, we have $\left\|\sum_{j\in J} x_j\right\| < \varepsilon$.*

From this, we obtain two important results:

Corollary 6.2.6. *Any $\mu = \sum_{i\in I} f_i s_i f_i$ in $P(G, C(X))$ has the property that I is countable. In particular, $P(G, C(X))$ is indeed a set.*

Proof. Consider the sum $\sum_{i\in I} f_i^2 = 1$ and $\varepsilon = \frac{1}{n}$ ($n \in \mathbb{N}$) in Lemma 6.2.5. Then necessarily, only finitely many f_i^2 can have norm at least $\frac{1}{n}$. Hence, at most countably many f_i can be nonzero. ■

Corollary 6.2.7. *Assume $C(X) \subseteq A$ is an inclusion of G-C^*-algebras, $\mu = \sum_{i\in I} f_i s_i f_i \in P(G, C(X))$, and $a \in A$. Then the sum given by $\sum_{i\in I} f_i(s_i \cdot a) f_i$ is convergent, or in other words, the value μa is well-defined. Moreover, the map $a \mapsto \mu a$ is unital and completely positive.*

Proof. We first prove this for positive a. Let $\varepsilon > 0$, and let $F \subseteq I$ be such that for all finite $J \subseteq I \setminus F$, we have $\left\|\sum_{j\in J} f_j^2\right\| < \varepsilon$. Then

$$\left\|\sum_{j\in J} f_j(s_j \cdot a) f_j\right\| \leq \|a\| \left\|\sum_{j\in J} f_j^2\right\| < \varepsilon \|a\|.$$

To see that general values of a work, let $\mu_F = \sum_{i\in F} f_i s_i f_i$ for finite $F \subseteq I$. Writing a as a finite linear combination of four positive elements $\sum_{k=1}^4 c_k a_k$, we have that each of the nets $(\mu_F a_k)_F$ converges for each k. In particular, the net

$$\mu_F a = \sum_{k=1}^{4} c_k \mu_F a_k$$

must therefore also be convergent. The fact that $a \mapsto \mu a$ is completely positive follows from the fact that $a \mapsto \mu_F a$ is completely positive for each finite $F \subseteq I$. ■

Similar to how $P(G)$ is a convex semigroup, we have that the spaces $P_f(G, C(X))$ and $P(G, C(X))$ also form semigroups, and moreover satisfy an appropriate notion of $C(X)$-convexity.

Proposition 6.2.8. *The space $P_f(G, C(X))$ is $C(X)$-convex, in the sense that given finitely many $\{g_j\}_{j\in J} \subseteq C(X)$ with $\sum_{j\in J} g_j^2 = 1$ and any $\{\mu_j\}_{j\in J} \subseteq P_f(G, C(X))$ with $\mu_j = \sum_{i\in I_j} f_i s_i f_i$ (and all index sets I_j disjoint for distinct values of j), we have that*

$$\sum_{j\in J} g_j \mu_j g_j \coloneqq \sum_{j\in J} \sum_{i\in I_j} g_j f_i s_i f_i g_j$$

also lies in $P_f(G, C(X))$. The same is true for $P(G, C(X))$, except that J can be infinite.

Proof. We prove the case of $P(G, C(X))$, as the case of $P_f(G, C(X))$ is almost the same except without needing to worry about convergence. Observe that, given any finite subsets $F \subseteq J$ and $F_j \subseteq I_j$ for $j \in F$, we have

$$\sum_{j\in F} \sum_{i\in F_j} g_j^2 f_i^2 = \sum_{j\in F} g_j^2 \sum_{i\in F_j} f_i^2 \le \sum_{j\in F} g_j^2 \cdot 1 \le 1.$$

Moreover, given $\varepsilon > 0$, if we choose $F \subseteq J$ finite with $\sum_{j\in J} g_j^2 \ge 1 - \varepsilon$ and finite $F_j \subseteq I_j$ for $j \in F$ with $\sum_{i\in F_j} f_i^2 \ge 1 - \varepsilon$, then

$$\sum_{j\in F} \sum_{i\in F_j} g_j^2 f_i^2 = \sum_{j\in F} g_j^2 \sum_{i\in F_j} f_i^2 \ge \sum_{j\in F} g_j^2 \cdot (1 - \varepsilon) \ge (1 - \varepsilon)^2.$$

This proves that $\sum_{j\in J} \sum_{i\in I_j} g_j^2 f_i^2 = 1$. ■

It is perhaps worthwhile to do an example of a $C(X)$-convex combination.

Example 6.2.9. Consider $X = [0, 1]$, let G be some group acting on X, let s and t be distinct group elements, and consider the following set of generalized probability measures and coefficients:

$$\begin{aligned}
\mu_1 &= \sqrt{x} s \sqrt{x} + \sqrt{1 - x} t \sqrt{1 - x} \\
\mu_2 &= \sqrt{1 - x} s \sqrt{1 - x} + \sqrt{x} t \sqrt{x} \\
g_1 &= \frac{1}{\sqrt{2}} \\
g_2 &= \frac{1}{\sqrt{2}}
\end{aligned}$$

We have that

$$g_1\mu_1g_1 + g_2\mu_2g_2 = (\frac{1}{\sqrt{2}}\sqrt{x})s(\frac{1}{\sqrt{2}}\sqrt{x}) + (\frac{1}{\sqrt{2}}\sqrt{1-x})s(\frac{1}{\sqrt{2}}\sqrt{1-x})$$
$$+ (\frac{1}{\sqrt{2}}\sqrt{x})t(\frac{1}{\sqrt{2}}\sqrt{x}) + (\frac{1}{\sqrt{2}}\sqrt{1-x})t(\frac{1}{\sqrt{2}}\sqrt{1-x})$$

which cannot be simplified further (see Remark 6.2.4), and is left as-is, with group elements being repeated in the expression.

Remark 6.2.10. Given an inclusion of G-C*-algebras $C(X) \subseteq A$, $a \in A$, finitely many $\{g_j\}_{j\in J} \subseteq C(X)$ with $\sum_{j\in J} g_j^2 = 1$, and $\{\mu_j\}_{j\in J} \subseteq P_f(G, C(X))$, we have that

$$\left(\sum_{j\in J} g_j\mu_j g_j\right)(a) = \sum_{j\in J} g_j\mu_j(a)g_j.$$

Consequently,

$$\{\mu a \mid \mu \in P_f(G, C(X))\}$$

is $C(X)$-convex as well. In fact, it is the smallest G-invariant, $C(X)$-convex subset of A containing a.

Now we define a semigroup structure on $P_f(G, C(X))$ and $P(G, C(X))$.

Proposition 6.2.11. *The space $P(G, C(X))$ is a semigroup under the following multiplication: given $\mu = \sum_{i\in I} f_i s_i f_i$ and $\nu = \sum_{j\in J} g_j t_j g_j$, let*

$$\mu\nu := \sum_{i\in I}\sum_{j\in J}(f_i(s_ig_j))(s_it_j)((s_ig_j)f_i).$$

Moreover, $P_f(G, C(X))$ is a subsemigroup of $P(G, C(X))$.

Proof. Observe that, given any finite subsets $F_I \subseteq I$ and $F_J \subseteq J$, we have

$$\sum_{i\in F_I}\sum_{j\in F_J}(f_i^2(s_ig_j))^2 = \sum_{i\in F_I} f_i^2 s_i \left(\sum_{j\in F_J} g_j^2\right) \leq \sum_{i\in F_I} f_i^2 s_i 1 \leq 1.$$

Moreover, any finite subset of $I \times J$ is contained in a set of the form $F_I \times F_J$. Finally, given $\varepsilon > 0$, if one chooses F_I and F_J to be such that $\sum_{i\in F_I} f_i^2 \geq 1-\varepsilon$ and $\sum_{j\in F_J} g_j^2 \geq 1-\varepsilon$, then we have

$$\sum_{i\in F_I}\sum_{j\in F_J}(f_i(s_ig_j))^2 = \sum_{i\in F_I} f_i s_i \left(\sum_{j\in F_J} g_j^2\right) \geq \sum_{i\in F_I} f_i s_i(1-\varepsilon) \geq (1-\varepsilon)^2.$$

This proves that $\sum_{i\in I}\sum_{j\in J}(f_i(s_ig_j))^2 = 1$, and so this multiplication on $P(G, C(X))$ is well-defined. Associativity is tedious but not hard to check. The fact that $P_f(G, C(X))$ is a subsemigroup is clear. ∎

Remark 6.2.12. The multiplication on $P(G, C(X))$ is defined in such a way so that if $C(X) \subseteq A$ is an inclusion of unital G-C*-algebras, $\mu_1, \mu_2 \in P(G, C(X))$, and $a \in A$, then

$$(\mu_1\mu_2)(a) = \mu_1(\mu_2 a).$$

In other words, we canonically have a left semigroup action of $P(G, C(X))$ on A, and consequently a right semigroup action on $S(A)$.

Let $C(X) \subseteq A$ be an inclusion of unital G-C*-algebras, and let $\mu \in P(G, C(X))$ be a generalized measure. We say that a state $\phi \in S(A)$ is *μ-stationary* if $\phi\mu = \phi$, and denote the set of all μ-stationary states on A by $S_\mu(A)$. Observe that this definition makes sense even for C*-subalgebras that don't necessarily contain $C(X)$, but are at least μ-invariant. It is not hard to see that μ-stationary states always exist and can be extended to a larger C*-algebra. The proof is a mere modification of [HK17, Proposition 4.2]. We include it for the sake of completeness.

Proposition 6.2.13. *Suppose that $C(X) \subseteq A$ is an inclusion of unital G-C*-algebras, $\mu \in P(G, C(X))$, and $B \subseteq A$ is a μ-invariant unital C*-subalgebra. Then every μ-stationary state $\tau \in S_\mu(B)$ can be extended to a μ-stationary state $\eta \in S_\mu(A)$. In particular, $S_\mu(A)$ is always nonempty.*

Proof. Let $K = \{\zeta \in S(A) \mid \zeta|_B = \tau\}$, a compact convex set. For any $\mu \in P(G, C(X))$, the map $\Phi_\mu : K \to K$ defined by $\Phi_\mu(\zeta) = \zeta\mu$ is an affine continuous map. It is well-known that Φ_μ has a fixed point, say η. Then, $\eta \in S_\mu(A)$ and $\eta|_B = \tau$. To see that $S_\mu(A)$ is nonempty, let $B = \mathbb{C}$. ∎

We wish to define an appropriate notion of full support for measures in $P(G, C(X))$. For this, the following observation will come in useful.

Lemma 6.2.14. *Assume X is a Banach space, and $\sum_{i\in I} x_i$ is an infinite unordered sum that converges in norm. Then given any $J \subseteq I$, the sum $\sum_{j\in J} x_j$ also converges.*

Proof. We know by Lemma 6.2.5 that given any $\varepsilon > 0$, there is a finite subset $F \subseteq I$ such that for any finite subset $E \subseteq I \setminus F$, we have $\left\|\sum_{i\in E} x_i\right\| < \varepsilon$. But then, letting $F' = F \cap J$, it is clear that for any finite set $E' \subseteq J \setminus F'$, we also have $\left\|\sum_{j\in E'} x_j\right\| < \varepsilon$. ∎

Definition 6.2.15. We say that a generalized measure $\mu \in P(G, C(X))$ has full support if, writing

$$\mu = \sum_{s \in G} \sum_{i \in I_s} f_i s f_i,$$

we have that for each $s \in G$, there is some $\delta > 0$ such that $\sum_{i \in I_s} f_i^2 \geq \delta$. Equivalently (by compactness of X), given any $s \in G$ and $x \in X$, we can find $i \in I_s$ such that $f_i(x) > 0$.

6.3 Proof of generalized Powers averaging

In this section, we prove Theorem 6.1.2. To give a brief overview, we first recall how this is proven in the case of the usual reduced group C*-algebra.

Both in [Ken20] and [Haa16], which independently prove C*-simplicity is equivalent to Powers' averaging property, the key tool used was the dynamical characterization of C*-simplicity of G in terms of its action on the Furstenberg boundary $\partial_F G$ (see [KK17, Theorem 1.5] or [BKKO17, Theorem 1.1]). The Furstenberg boundary of a group was originally developed by Furstenberg [Fur73] (see also [Fur63]) as a topological object, but it can also be realized as the G-injective envelope of the complex numbers $\mathbb{C}$ (see [KK17, Definition 3.1, Theorem 3.11]). Briefly, we recall the topological characterization below:

Definition 6.3.1. Let X be a G-space. A measure $\nu \in P(X)$ is called *contractible* if $\{\delta_x \mid x \in X\} \subseteq \overline{G\nu}^{\mathrm{w}^*}$. A *$G$-boundary* is a G-space X with the additional property that every measure $\nu \in P(X)$ is contractible.

It is worth noting that, from the perspective of convexity, being a G-boundary is the same as saying that $P(X)$ is irreducible as a compact convex G-space.

Theorem 6.3.2. *There is a universal G-boundary $\partial_F G$ (known as the Furstenberg boundary of G) with the property that every G-boundary X is the image of $\partial_F G$ under a surjective, G-equivariant map.*

It is shown in [Ken20] that C*-simplicity of G is equivalent to $\{\tau_\lambda\}$ being the unique G-boundary in $S(C^*_\lambda(G))$. A similar result can consequently be achieved in the entire dual space of $C^*_\lambda(G)$, and a Hahn-Banach separation argument yields that this is equivalent to Powers' averaging property.

There is a generalized notion of boundaries introduced in [KS19, Section 7], which is used for dealing with noncommutative crossed products $A \rtimes_\lambda G$. However, this notion is more technical, as it involves working with matrix convex sets and matrix state spaces. It is possible to develop a similar notion, but using only usual convex sets instead of matrix convex ones, and use this in the case of commutative crossed products $C(X) \rtimes_\lambda G$. However, [Kaw17], which deals with proving equivalences of simplicity of such crossed products, does not develop such a theory of generalized boundaries. Instead, the generalized notion of boundary necessary here is developed in [Nag20, Section 3], albeit from the perspective of compact sets and their measures rather than from convex sets.

From Naghavi's results, we use (part of) [Nag20, Theorem 3.2] and the discussion following it, which we quickly paraphrase as follows:

Theorem 6.3.3. *For a countable discrete group G, let X be a minimal G-space, and let $C(X) \subseteq C(Y)$ be an inclusion of unital G-C*-algebras. The following are equivalent:*

1. *$C(Y)$ is a G-essential extension of $C(X)$.*
2. *Given any measure $\nu \in P(Y)$ with the property that the restriction $\nu|_{C(X)} \in P(X)$ is contractible, we have that ν is also contractible.*

In particular, the above is true for the G-injective envelope $C(Y) = I_G(C(X))$, which is a maximal G-essential extension of $C(X)$.

Definition 6.3.4. Assume G is a countable discrete group, X is a minimal G-space, and $C(X) \subseteq C(Y)$ is an inclusion of unital G-C*-algebras. We say that Y is a (G, X)-*boundary* if $C(Y)$ satisfies any of the equivalent conditions above. Furthermore, we let $\partial_F(G, X)$ denote the Gelfand spectrum of the G-injective envelope of $C(X)$, i.e. $I_G(C(X)) = C(\partial_F(G, X))$. It can be shown that $\partial_F(G, X)$ is universal among all (G, X)-boundaries.

We also recall the result [Kaw17, Theorem 3.4] of Kawabe characterizing the ideal intersection property for $C(X) \rtimes_\lambda G$ in the special case of minimal dynamical systems (in which case, simplicity and the intersection property coincide).

Theorem 6.3.5. *For a countable discrete group G, let X be a minimal G-space, and let $\partial_F(G, X)$ be the Gelfand spectrum of the G-injective envelope of $C(X)$. Then the following are equivalent:*

1. *The crossed product $C(\partial_F(G, X)) \rtimes_\lambda G$ is simple.*
2. *The crossed product $C(X) \rtimes_\lambda G$ is simple.*
3. *The action of G on $\partial_F(G, X)$ is free.*

With the above, we are ready to tackle proving our generalized version of Powers' averaging property. The following lemma shows that, although measures on minimal spaces need not be contractible in general, they have the weaker property that arbitrary measures can still be pushed to Dirac masses using $P(G, C(X))$.

Lemma 6.3.6. *Assume X is a minimal G-space, and fix any $x \in X$. There is a net $(\mu_\lambda) \subseteq P_f(G, C(X))$ with the property that for any $\nu \in P(X)$, we have $\nu\mu_\lambda \xrightarrow{\text{w}^*} \delta_x$.*

Proof. Fix an open neighbourhood V of x. Observe that, given any $y \in X$, we have that Gy is dense in X by minimality. In particular, there exists some $s \in G$ with the property that $sy \in V$, or equivalently, $y \in s^{-1}V$. It follows that the sets sV form an open cover of X, and so there is some finite subcover $s_1V, \dots, s_nV$. Now let $F_1, \dots, F_n \in C(X)$ be a partition of unity subordinate to this open cover, let $f_i = F_i^{1/2}$, and let

$$\mu_V = \sum_{i=1}^{n} f_i s_i f_i.$$

It is not hard to see that, given any $\nu \in P(X)$, $\nu\mu_V$ is a measure with support contained in $\overline{V}$. It follows that the net $(\nu\mu_V)$, indexed by open neighbourhoods of x ordered under reverse inclusion, converges weak* to δ_x. ∎

This allows us to push arbitrary measures towards the trivial boundary in $S(C(X) \rtimes_\lambda G)$ in the case of simple crossed products.

Proposition 6.3.7. *Let X be a minimal G-space, and assume that the crossed product $C(X) \rtimes_\lambda G$ is simple. Then given any state $\phi \in S(C(X) \rtimes_\lambda G)$, we have that*

$$\{\nu \circ \mathbb{E} \mid \nu \in P(X)\} \subseteq \overline{\{\phi\mu \mid \mu \in P_f(G, C(X))\}}^{\text{w}^*}.$$

Proof. For convenience, denote the latter set above by K. By G-invariance, convexity, and weak*-closure, it suffices to prove that K contains $\delta_x \circ \mathbb{E}$ for any single point $x \in X$.

To this end, let $I_G(C(X)) = C(\partial_F(G, X))$ be the G-injective envelope of $C(X)$. Extend the state ϕ to a state $\widetilde{\phi}$ on $C(\partial_F(G, X)) \rtimes_\lambda G$. By Lemma 6.3.6, we can find a net $(\mu_\lambda) \subseteq P_f(G, C(X))$ with the property that $\widetilde{\phi}|_{C(X)}\mu_\lambda \to \delta_x$ for some $x \in X$. Dropping to a subnet if necessary, we have that $\widetilde{\phi}\mu_\lambda \to \psi \in S(C(\partial_F(G, X)) \rtimes_\lambda G)$ with the property that $\psi|_{C(X)} = \delta_x$. Observe that $\psi|_{C(X) \rtimes_\lambda G} \in K$.

Minimality tells us that $\psi|_{C(X)}$ is contractible, and so $\psi|_{C(\partial_F(G,X))}$ is contractible as well by Theorem 6.3.3. This tells us that there is a net (g_i) with $\psi|_{C(\partial_F(G,X))}g_i \to \delta_y$ for some

$y \in \partial_F(G,X)$. Again dropping to a subnet yields a state $\eta \in S(C(\partial_F(G,X)) \rtimes_\lambda G)$ with the property that $\eta|_{C(\partial_F(G,X))} = \delta_y$. Observe once more that $\eta|_{C(X)\rtimes_\lambda G} \in K$.

We claim that $\eta|_{C(X)\rtimes_\lambda G}$ is the state we are looking for. Simplicity of $C(X) \rtimes_\lambda G$ implies that the action of G on $\partial_F(G,X)$ is free by Theorem 6.3.5. From here, the rest is a common argument. We know that $C(\partial_F(G,X))$ lies in the multiplicative domain of η. Thus, it suffices to show that $\eta(\lambda_t) = 0$ for $t \neq e$, as this will imply that for any $f \in C(\partial_F(G,X))$, we have

$$\eta(f\lambda_t) = f(y)\eta(\lambda_t) = 0.$$

Now let $f \in C(\partial_F(G,X))$ be such that $f(y) = 1$ and $f(ty) = 0$. This is possible because $ty \neq y$. We have

$$f(y)\eta(\lambda_t) = \eta(f\lambda_t) = \eta(\lambda_t(t^{-1}\cdot f)) = \eta(\lambda_t)f(ty).$$

This forces $\eta(\lambda_t) = 0$, as desired. ■

The fact that arbitrary functionals on a C*-algebra are a finite linear combination of states gives us a similar result on the entire dual space $(C(X) \rtimes_\lambda G)^*$.

Proposition 6.3.8. *Let X be a minimal G-space, and assume that the crossed product is simple. Then given any $\omega \in (C(X) \rtimes_\lambda G)^*$, we have*

$$\{\omega(1)\nu \circ \mathbb{E} \mid \nu \in P(X)\} \subseteq \overline{\{\omega\mu \mid \mu \in P_f(G, C(X))\}}^{\mathrm{w}^*}.$$

Proof. Again, for convenience, denote this latter set by K. Write $\omega = \sum_{i=1}^4 c_i\phi_i$, a linear combination of four states. By Proposition 6.3.7, we can find a net $(\mu_\lambda) \subseteq P_f(G, C(X))$ with the property that $\phi_1\mu_\lambda \to \nu_1 \circ \mathbb{E}$. Dropping to a subnet if necessary, we may also assume that $(\phi_i\mu_\lambda)$ are all convergent to some ϕ_i' for $i \geq 2$. In particular, $(\omega\mu_\lambda)$ converges to some $\omega' \in K$ with the property that $\omega' = c_1\nu_1 \circ \mathbb{E} + \sum_{i=2}^4 c_i\phi_i'$. Noting that the set $\{\nu \circ \mathbb{E} \mid \nu \in P(X)\}$ is weak*-closed and closed under the right action of $P_f(G, C(X))$, repeating this averaging trick three more times nets us (no pun intended) that there is some element in K of the form $\eta \circ \mathbb{E}$ satisfying $\eta(1) = \omega(1)$.

Writing $\eta = \sum_{i=1}^4 d_i\psi_i$ as a linear combination of four states on $C(X)$, fixing $x \in X$, and letting (μ_j) be as in Lemma 6.3.6, we have that $(\eta \circ \mathbb{E})\mu_j = (\eta\mu_j) \circ \mathbb{E}$ converges to $\omega(1)\delta_x \circ \mathbb{E}$, which must lie in K. Minimality of X and G-invariance, weak*-closure, and convexity of K yield that every $\omega(1)\nu \circ \mathbb{E}$ lies in K as well. ■

From here, it is an application of the Hahn-Banach separation argument that gives us the strong generalized Powers' averaging property. Conversely, lack of nontrivial ideals can be directly deduced even from just being able to average elements $a \in C(X) \rtimes_\lambda G$ satisfying $\mathbb{E}(a) = 0$.

Proof of Theorem 6.1.2. (1) $\implies$ (4) Given that the extreme points of $P(X)$ are the Dirac masses δ_x, it suffices to prove the following: if $a \in C(X) \rtimes_\lambda G$ and $x \in X$, then

$$\mathbb{E}(a)(x) \in \overline{\{\mu a \mid \mu \in P_f(G, C(X))\}}.$$

Assume otherwise, so that there is some $a \in C(X) \rtimes_\lambda G$ and $x \in X$ for which this doesn't hold. Then there is some functional $\omega \in (C(X) \rtimes_\lambda G)^*$ and $\alpha \in \mathbb{R}$ with the property that

$$\operatorname{Re} \omega(\mathbb{E}(a)(x)) < \alpha \leq \operatorname{Re} \omega(\mu a) \quad \forall \mu \in P_f(G, C(X)).$$

However, given that $\omega(\mathbb{E}(a)(x)) = \omega(1)\mathbb{E}(a)(x)$, and by Proposition 6.3.8, $\omega(\mu a)$ can be made arbitrarily close to $\omega(1)(\delta_x \circ \mathbb{E})(a)$, this cannot happen.

(4) $\implies$ (3) Our aim is to show that we may approximate $\mathbb{E}(a)$ by $C(X)$-convex combinations of $\mathbb{E}(a)(x)$, where $x \in X$. Let $a \in C(X) \rtimes_\lambda G$, and let $\varepsilon > 0$. Given any $x \in X$, by continuity of $\mathbb{E}(a) \in C(X)$, there is some open neighbourhood U_x of x for which $|\mathbb{E}(a)(x) - \mathbb{E}(a)(y)| < \varepsilon$ for all $y \in U_x$. By compactness, there is some finite subcover $U_{x_1}, \ldots, U_{x_n}$ of X. Let F_i be a partition of unity subordinate to the open cover, and let $f_i = F_i^{1/2}$. Observe that, given any $x \in X$, we have

$$\begin{aligned}
&\left| \sum_{i=1}^n f_i(x)(\mathbb{E}(a)(x_i)) f_i(x) - \mathbb{E}(a)(x) \right| \\
&= \left| \sum_{i=1}^n f_i(x)(\mathbb{E}(a)(x_i) - \mathbb{E}(a)(x)) f_i(x) \right| \\
&\leq \sum_{i=1}^n f_i(x)^2 \left| \mathbb{E}(a)(x_i) - \mathbb{E}(a)(x) \right| \\
&< \sum_{i=1}^n f_i(x)^2 \cdot \varepsilon \\
&= \varepsilon
\end{aligned}$$

Thus, we have $\|\sum_{i=1}^n f_i \mathbb{E}(a)(x_i) f_i - \mathbb{E}(a)\| < \varepsilon$. By Remark 6.2.10, we have that $\mathbb{E}(a) \in \overline{\{\mu a \mid \mu \in P_f(G, C(X))\}}$, as this set is closed under $C(X)$-convex combinations.

(3) $\implies$ (2) This direction is clear.

(2) $\implies$ (1) Let I be any nontrivial ideal of $C(X) \rtimes_\lambda G$, and let a be any nonzero element of I. Replacing a by a^*a, we may assume without loss of generality that a is a nonzero positive element. Faithfulness of the canonical expectation tells us that $\mathbb{E}(a)$ is also a nonzero positive element, and so there is some $\varepsilon > 0$ and open subset $U \subseteq X$ with the property that $\mathbb{E}(a)(x) > \varepsilon$ for all $x \in U$. Using the same trick as in the proof of Lemma 6.3.6, minimality of X gives us that $X = s_1 U \cup \cdots \cup s_n U$ for finitely many $s_i \in G$. Hence, replacing a by $s_1 a + \cdots + s_n a$, we may assume without loss of generality that $\mathbb{E}(a) > \varepsilon$. If we choose $\mu \in P(G, C(X))$ so that $\|\mu(a - \mathbb{E}(a))\| < \frac{\varepsilon}{2}$, then as this value is in particular self-adjoint, we have that $\mu(a - \mathbb{E}(a)) \geq -\frac{\varepsilon}{2}$. Consequently,

$$\mu a = \mu(\mathbb{E}(a)) + \mu(a - \mathbb{E}(a)) \geq \varepsilon - \frac{\varepsilon}{2} = \frac{\varepsilon}{2}.$$

In particular, $\mu a \in I$ is invertible, which gives us that I is the entire crossed product $C(X) \rtimes_\lambda G$. ■

6.4 Unique stationarity and applications

This section generalizes the various results in [HK17] on equivalence between C*-simplicity and unique stationarity of the canonical trace in $C^*_\lambda(G)$, along with its consequences.

It is worth noting that one cannot expect simplicity of $C(X) \rtimes_\lambda G$ to be equivalent to unique stationarity of an element of $S(C(X) \rtimes_\lambda G)$, even with respect to a generalized measure $\mu \in P(G, C(X))$. This is because of the fact that there may not exist a uniquely stationary state on $C(X)$, and any μ-stationary state on $C(X)$ will extend to one on the whole crossed product. The natural fix is to instead expect that the μ-stationary states on $C(X) \rtimes_\lambda G$ all be of the form $\nu \circ \mathbb{E}$, where ν ranges over the μ-stationary measures $\nu \in P(X)$. It is also worth noting that one cannot expect to work with the usual notion of measure $\mu \in P(G)$, as this would again imply unique stationarity of $\tau_\lambda \in S(C^*_\lambda(G))$. However, this is equivalent to C*-simplicity of G [HK17, Theorem 5.2], which is by no means necessary for the crossed product $C(X) \rtimes_\lambda G$ to be simple - take, for example, $C(\mathbb{T}) \rtimes_\lambda \mathbb{Z}$, where $\mathbb{Z}$ acts on the circle $\mathbb{T}$ by an irrational rotation.

We begin with the observation that averaging elements in the reduced group C*-algebra $C^*_\lambda(G)$, even with respect to a generalized measure $\mu \in P(G, C(X))$, is enough to average elements in the crossed product $C(X) \rtimes_\lambda G$.

Lemma 6.4.1. *Let X be a minimal G-space, and let $\mu \in P(G, C(X))$. Then given any $t \in G$ and $f \in C(X)$, we have*

$$\|\mu(f\lambda_t)\| \leq \|f\| \, \|\mu\lambda_t\| .$$

Proof. It is well-known that the crossed product $\ell^\infty(G) \rtimes_\lambda G$ (the uniform Roe algebra), can canonically be viewed as a C*-subalgebra of $B(\ell^2(G))$. Fixing $x_0 \in X$ gives us a unital G-equivariant injective *-homomorphism $\iota : C(X) \hookrightarrow \ell^\infty(G)$, given by $\iota(f)(t) = f(tx_0)$. This lets us view $C(X) \rtimes_\lambda G$ as a C*-subalgebra of $B(\ell^2(G))$ as well.

Write $\mu = \sum_{i \in I} g_i s_i g_i$. Given $\xi \in \ell^2(G)$ and $r \in G$, we have

$$\begin{aligned} ((\mu(f\lambda_t))\xi)(r) &= \left(\sum_{i \in I} g_i (s_i f)(s_i t s_i^{-1} g_i) \lambda_{s_i t s_i^{-1}} \xi \right)(r) \\ &= \sum_{i \in I} g_i(rx_0) f(s_i^{-1} r x_0) g_i(s_i t^{-1} s_i^{-1} r x_0) \xi(s_i t^{-1} s_i^{-1} r). \end{aligned}$$

Now letting $|\xi| \in \ell^2(G)$ be given by $|\xi|(r) = |\xi(r)|$, we note that $\||\xi|\| = \|\xi\|$. Moreover, we have

$$\begin{aligned} \|(\mu(f\lambda_t))\xi\|^2 &= \sum_{r \in G} \left| \sum_{i \in I} g_i(rx_0) f(s_i^{-1} r x_0) g_i(s_i t^{-1} s_i^{-1} r x_0) \xi(s_i t^{-1} s_i^{-1} r) \right|^2 \\ &\leq \|f\|^2 \sum_{r \in G} \left(\sum_{i \in I} g_i(rx_0) g_i(s_i t^{-1} s_i^{-1} r x_0) \left| \xi(s_i t^{-1} s_i^{-1} r) \right| \right)^2 \\ &= \|f\|^2 \, \|(\mu\lambda_t) \, |\xi|\|^2 . \end{aligned}$$

It follows that $\|\mu(f\lambda_t)\| \leq \|f\| \, \|\mu\lambda_t\|$. ∎

It is also an easy remark that Powers' averaging property can be made to work with finitely many elements at once.

Lemma 6.4.2. *Assume $C(X) \rtimes_\lambda G$ has Powers' averaging property. Then given any $a_1, \ldots, a_n \in C(X) \rtimes_\lambda G$ satisfying $\mathbb{E}(a_i) = 0$, and $\varepsilon > 0$, there is some $\mu \in P(G, C(X))$ with the property that $\|\mu a_i\| < \varepsilon$ for all $i = 1, \ldots, n$.*

Proof. Let $\mu_1 \in P(G, C(X))$ be such that $\|\mu_1 a_1\| < \varepsilon$. Choosing $\mu_{k+1} \in P(G, C(X))$ inductively by letting μ_{k+1} be such that $\|\mu_{k+1}(\mu_k \ldots \mu_1 a_{k+1})\| < \varepsilon$, we see that $\mu = \mu_n \ldots \mu_1$ is the generalized measure we are looking for. ∎

Proof of Theorem 6.1.3. First, we claim that there is such a measure that works for all elements $a \in C^*_\lambda(G) \subseteq C(X) \rtimes_\lambda G$ satisfying $\tau_\lambda(a) = 0$. This is a near-verbatim repeat of the proof of [HK17, Theorem 5.1]. We repeat the construction of μ here, along with the appropriate modifications.

Let (n_k) be an increasing sequence of positive integers satisfying $\left(\sum_{i=1}^{k} \frac{1}{2^i}\right)^{n_k} < \frac{1}{2^k}$, and let (a_i) be any dense sequence in the unit ball of $\ker \tau_\lambda \subseteq C^*_\lambda(G)$. Let $\mu_1 \in P(G, C(X))$ be anything. Using Lemma 6.4.2, we may inductively build μ_l for $l \geq 2$ so that

$$\|\mu_l \mu_{k_r} \dots \mu_{k_1} a_s\| < \frac{1}{2^l}$$

for all $1 \leq s, k_1, \dots, k_r < l$ and $0 \leq r < n_l$. Here, by $r = 0$, we mean that $\mu_l \mu_{k_r} \dots \mu_{k_1} a_s$ becomes $\mu_l a_s$. A tedious computation then shows that $\mu = \sum_{l=1}^{\infty} \frac{1}{2^l} \mu_l$ will satisfy $\mu^n a \to 0$ for any $a \in \ker \tau_\lambda$.

To force μ to have full support, observe that if $(s_n)_{n \in \mathbb{N}}$ is an enumeration of G, then the measure

$$\nu = \sum_{n=1}^{\infty} \frac{1}{2^{n+1}} s_n \frac{1}{2^{n+1}} \in P(G, C(X))$$

has full support. Then fixing any l and letting $\alpha > 0$ be sufficiently small, we may replace μ_l by $\alpha\nu + (1 - \alpha)\mu_l$ and still satisfy the required approximation properties above. Thus, without loss of generality, some μ_l has full support, and hence so does μ.

Now, to see that $\mu^n a \to 0$ whenever $a \in C(X) \rtimes_\lambda G$ satisfies $\mathbb{E}(a) = 0$, we first prove this for elements $a_0 = f_1 \lambda_{t_1} + \cdots + f_n \lambda_{t_n}$, where $t_i \neq e$. Note that by Lemma 6.4.1, we have

$$\|\mu^n a_0\| \leq \sum_{i=1}^{n} \|\mu(f_i \lambda_{t_i})\| \leq \sum_{i=1}^{n} \|f_i\| \, \|\mu \lambda_{t_i}\| \to 0.$$

Now given an arbitrary a with $\mathbb{E}(a) = 0$, and $\varepsilon > 0$, we may choose a_0 as before with $\|a - a_0\| < \varepsilon$. Choosing N such that, given $n \geq N$, we have $\|\mu^n a_0\| < \varepsilon$, we also have

$$\|\mu^n a\| \leq \|\mu^n a_0\| + \|\mu^n (a - a_0)\| < \varepsilon + \varepsilon = 2\varepsilon.$$

■

Remark 6.4.3. In the above proof, if we instead wanted to directly construct a generalized measure $\mu \in P(G, C(X))$ with the property that $\mu^n a \to 0$ for all $a \in C(X) \rtimes_\lambda G$ with $\mathbb{E}(a) = 0$, as opposed to $a \in C^*_\lambda(G)$ with $\tau_\lambda(a) = 0$, we would have required separability of $\ker \mathbb{E} \subseteq C(X) \rtimes_\lambda G$, which requires separability of $C(X)$ (metrizability of X). Proceeding with $\ker \tau_\lambda \subseteq C^*_\lambda(G)$ first and then lifting the averaging to the entire crossed product avoids this extra assumption. There are natural examples of spaces on which G acts that are not metrizable. For example, if G is not amenable, then the Furstenberg boundary $\partial_F G$ is such a space [KK17, Corollary 3.17].

For a minimal G-space X, it is well known that if $\mu \in P(G)$ has full support, then any μ-stationary state on $C(X)$ is faithful. A similar result holds for generalized probability measures (with the definition of full support given in Definition 6.2.15).

Lemma 6.4.4. *Let X be a minimal G-space. Let $\mu \in P(G, C(X))$ be a generalized probability measure with full support. Then every μ-stationary state on $C(X)$ is faithful.*

Proof. Let $f \in C(X)$ be such that $f \geq 0$ and $f \neq 0$. Let $\mu = \sum_{s\in G}\sum_{i \in I_s} f_i s f_i$ be a generalized probability measure with full support. Since f is nonzero, there exists $x_0 \in X$ such that $f(x_0) > 0$. It follows from X being minimal that, for every $x \in X$, there exists $s_x \in G$ such that $f(s_x^{-1}x) > 0$. Moreover, since μ has full support, there exists $i_x \in I_{s_x}$ such that $f_i(x) > 0$. Therefore,

$$\mu(f)(x) = \sum_{s\in G}\sum_{i\in I_s} f_i(x) f(s^{-1}x) f_i(x) > f_{i_x}(x) f(s_x^{-1}x) f_{i_x}(x) > 0.$$

By compactness of X, it follows that there exists a $\delta > 0$ such that $\mu(f) \geq \delta$. Consequently, for any μ-stationary state τ on $C(X)$, we see that

$$\tau(f) = \tau(\mu(f)) \geq \delta.$$

Hence, τ is faithful. ∎

Proof of Corollary 6.1.4. Let X be a minimal G-space. Suppose that $C(X) \rtimes_\lambda G$ is simple. Let $\mu \in P(G, C(X))$ be the generalized measure obtained from Theorem 6.1.3 and let τ be a μ-stationary state on $C(X) \rtimes_\lambda G$. Then, for any $a \in C(X) \rtimes_\lambda G$ with $\mathbb{E}(a) = 0$, we have that

$$\tau(a) = \tau(\mu^n a) \to \tau(0) = 0,$$

and so for general $a \in C(X) \rtimes_\lambda G$, we have

$$\tau(a) = \tau(\mathbb{E}(a)) + \tau(a - \mathbb{E}(a)) = \tau(\mathbb{E}(a)).$$

In other words, $\tau = \tau|_{C(X)} \circ \mathbb{E}$.

On the other hand, suppose that there exists a generalized probability measure $\mu \in P(G, C(X))$ with full support along with the property that every μ-stationary state on $C(X) \rtimes_\lambda G$ is of the form $\nu \circ \mathbb{E}$ for some μ-stationary $\nu \in P(X)$. By faithfulness of $\mathbb{E}$ and Lemma 6.4.4, every μ-stationary state on $C(X) \rtimes_\lambda G$ is faithful. This is enough to guarantee that the $C(X) \rtimes_\lambda G$ is simple - the proof is similar to [HK17, Proposition 4.9].

Assume that there was a nontrivial ideal $I \subseteq C(X) \rtimes_\lambda G$. Observe that the quotient map $\pi : C(X) \rtimes_\lambda G \to (C(X) \rtimes_\lambda G)/I$ is nonfaithful, as nontrivial ideals always contain nonzero positive elements. Moreover, the quotient $(C(X) \rtimes_\lambda G)/I$ is canonically a G-C*-algebra, with $t \in G$ acting by $\operatorname{Ad} \pi(\lambda_t)$, and the quotient map π is G-equivariant. In particular, we still canonically have $C(X) \subseteq (C(X) \rtimes_\lambda G)/I$ (under the quotient map π) by minimality of X, and so there is at least one μ-stationary state $\phi \in S_\mu((C(X) \rtimes_\lambda G)/I)$ by Proposition 6.2.13. The composition $\phi \circ \pi$ is a μ-stationary state on $C(X) \rtimes_\lambda G$ that is not faithful, contradicting our earlier conclusion. ∎

One should notice that G-simplicity doesn't necessarily pass to sub-algebras and therefore, simplicity for invariant sub-algebras of simple crossed products shouldn't be expected to hold in general. Consider, for example, any simple C*-algebra A, any C*-simple group G acting on A trivially, and any abelian C*-subalgebra $B \subseteq A$. However, given an inclusion of unital G-C*-algebras $C(Y) \subset C(X)$ (via a factor map $\pi : X \to Y$), since any G-invariant C*-subalgebra A, $C(Y) \subset A \subset C(X)$ is of the form $C(Z)$ where Z is an equivariant factor of X, and minimality passes to factors, it follows from the characterization of Kawabe [Kaw17, Theorem 6.1] that $C(Z) \rtimes_\lambda G$ is simple. We follow arguments similar to the proof of [AK20, Theorem 1.3] to deal with general intermediate C*-subalgebras between $C(Y) \rtimes_\lambda G$ and $C(X) \rtimes_\lambda G$, not necessarily of the above form.

Proof of Theorem 6.1.5. By Theorem 6.1.3, there exists a generalized measure $\mu \in P(G, C(Y))$ with full support and the property that $\mu^n a \to 0$ whenever $a \in C(Y) \rtimes_\lambda G$ is such that $\mathbb{E}(a) = 0$. Observe that we canonically have $P(G, C(Y)) \subseteq P(G, C(X))$. Since X is minimal and μ has full support, it follows from Lemma 6.4.4 that every μ-stationary state ν on $C(X)$ is faithful, and since $\mathbb{E}$ is also faithful, it follows that every μ-stationary state on $C(X) \rtimes_\lambda G$ of the form $\nu \circ \mathbb{E}$ is faithful. We claim that the proof is complete once we establish that every μ-stationary state τ on $C(X) \rtimes_\lambda G$ is of the form $\nu \circ \mathbb{E}$.

Indeed, if this is the case, let A be any intermediate C*-algebra of the form $C(Y) \rtimes_\lambda G \subseteq A \subseteq C(X) \rtimes_\lambda G$. Suppose that I is a proper closed two-sided ideal of A. Then the action of G on A induces an action of G on A/I (as I is necessarily G-invariant). Moreover, by minimality of Y, we also canonically have $C(Y) \subseteq A/I$. By Proposition 6.2.13, there exists a μ-stationary state φ on A/I. Upon composing φ with the canonical quotient map $A \to$

A/I, we obtain a μ-stationary state $\widetilde{\varphi}$ on A which vanishes on I. Using Proposition 6.2.13 again, extend $\widetilde{\varphi}$ to a μ-stationary state τ on $C(X) \rtimes_\lambda G$. By our assumption, τ being of the form $\nu \circ \mathbb{E}$, is faithful. But $\tau|_I = 0$, which cannot occur if I is nontrivial.

We return to the question of showing that every μ-stationary state on $C(X) \rtimes_\lambda G$ is indeed of the form $\nu \circ \mathbb{E}$ for some μ-stationary $\nu \in P(X)$. We claim that $\mu^n a \to 0$ whenever $a \in C(X) \rtimes_\lambda G$ (not just $C(Y) \rtimes_\lambda G$) satisfies $\mathbb{E}(a) = 0$. To see this, first let $f \in C(X)$ and $t \neq e$. Lemma 6.4.1 tells us that

$$\|\mu^n(f\lambda_t)\| \leq \|f\| \, \|\mu^n \lambda_t\| \to 0.$$

It follows that for finite linear combinations $a_0 = f_1\lambda_{t_1} + \cdots + f_n\lambda_{t_n}$, where $f_i \in C(X)$ and $t_i \neq e$, we have $\mu^n a_0 \to 0$ as well. Finally, let $a \in C(X) \rtimes_\lambda G$ with $\mathbb{E}(a) = 0$, $\varepsilon > 0$, and a_0 as before with the additional property that $\|a - a_0\| < \varepsilon$. Then given $N \in \mathbb{N}$ such that $\|\mu^n a\| < \varepsilon$ for any $n \geq N$, we have

$$\|\mu^n a\| \leq \|\mu^n a_0\| + \|\mu^n(a - a_0)\| < \varepsilon + \varepsilon = 2\varepsilon.$$

It follows that $\mu^n a \to 0$. The proof of Corollary 6.1.4 shows that any μ-stationary state on $C(X) \rtimes_\lambda G$ is of the form we want. ■

With Theorem 6.1.3 in hand, we generalize Hartman and Kalantar's result on C*-simplicity being equivalent to unique stationarity of the action of G on the space of amenable subgroups $\mathrm{Sub}_a(G)$. Recall that Kawabe [Kaw17, Theorem 5.2] introduced the G-space $\mathrm{Sub}_a(X, G)$ of pairs (x, H), where $x \in X$ and H is an amenable subgroup of the stabilizer group G_x. Observe that the canonical projection $\mathrm{Sub}_a(X, G) \twoheadrightarrow X$ induces an inclusion $C(X) \subseteq C(\mathrm{Sub}_a(X, G))$.

Proof of Corollary 6.1.6. There is a G-equivariant, unital and completely positive map $\theta : C(X) \rtimes_\lambda G \to C(\mathrm{Sub}_a(X, G))$ given by $\theta(f\lambda_t)(x, H) = f(x)1_H(t)$. A similar map can be found used in the proof of [Kaw17, Theorem 5.2], but a proof of the existence of such a map is not given. We briefly argue existence here. Given any $(x, H) \in \mathrm{Sub}_a(X, G)$, it is not hard to show that there is a state $\phi \in S(C(X) \rtimes_\lambda G)$ given by $\phi(f\lambda_t) = f(x)1_H(t)$. This gives us a continuous map from $\mathrm{Sub}_a(X, G)$ to $S(C(X) \rtimes_\lambda G)$, and $\theta : C(X) \rtimes_\lambda G \to C(\mathrm{Sub}_a(X, G))$ is dual to this map.

Choose a generalized measure $\mu \in P(G, C(X))$ as in Corollary 6.1.4. Now given a μ-stationary η in $P(\mathrm{Sub}_a(X, G))$, we have that $\eta \circ \theta : C(X) \rtimes_\lambda G \to \mathbb{C}$ is necessarily of the form $\nu \circ \mathbb{E}$. In particular, we note that for $t \neq e$,

$$\eta(\{(x, H) | t \in H\}) = \eta(\theta(\lambda_t)) = 0.$$

Countability of G gives us that $\bigcup_{t \neq e} \{(x, H) | t \in H\}$ is also a null set, or in other words, its complement $X \times \{\{e\}\}$ has measure 1.

Conversely, assume that the crossed product $C(X) \rtimes_\lambda G$ is not simple. Then by [Kaw17, Theorem 6.1], there must exist a closed G-invariant subset of $Z \subseteq \mathrm{Sub}_a(X, G)$ that does not intersect $X \times \{\{e\}\}$. Observe that we still canonically have $C(X) \subseteq C(Z)$ by minimality of X. Thus, for any $\mu \in P(G, C(X))$, if we choose any μ-stationary state on $C(Z)$ (such a state always exists by Proposition 6.2.13), composing with the canonical quotient $C(\mathrm{Sub}_a(X, G)) \twoheadrightarrow C(Z)$ gives us a μ-stationary state on $C(\mathrm{Sub}_a(X, G))$ with support disjoint from $X \times \{\{e\}\}$. ■

Chapter 7

The ideal intersection property for essential groupoid C*-algebras

7.1 Introduction

Groupoids provide a framework encompassing both groups and topological spaces, and provide an abstraction of the notion of a quotient space in these settings, much like stacks in algebraic geometry. They naturally arise as transformation groupoids encoding the topological dynamical structure of discrete groups, and as transversal groupoids of foliations in differential geometry. Groupoids arising from these and many other examples are étale, which is a notion suitably abstracting the topological properties of groupoids arising from actions of discrete groups. Renault [Ren80] and Connes [Con82], by introducing various C*-algebraic completions of an appropriate convolution algebra, discovered that étale groupoids give rise to an extraordinarily rich class of operator algebras. On the other hand, since the structure of these algebras encodes much of the structure of the underlying groupoid, they are capable of serving as a proxy for the study of structures relevant to other areas of mathematics, such as semigroup C*-algebras [CELY17] and Bost-Connes systems for arbitrary number fields [LLN09]. Furthermore, étale groupoids also naturally arise from within the theory of operator algebras, where they encode important structural features through the notion of Cartan subalgebras [Ren08, Li20].

Groupoids appearing in applications are frequently étale or, after choosing a transversal, Morita equivalent to an étale groupoid. However, they may be non-Hausdorff, and there is no way to remedy this fact. Nevertheless, even in the non-Hausdorff case, the unit space of an étale groupoid is typically locally compact and Hausdorff. Therefore, understanding the structural properties of the C*-algebras associated to étale groupoids with locally compact Hausdorff unit spaces is an important challenge at the intersection of operator algebras and many other areas of mathematics. We meet this challenge in the present article in four ways. First, we provide a complete characterisation of étale Hausdorff groupoids whose reduced groupoid C*-algebra is simple. Second, for non-Hausdorff groupoids satisfying the mild countability assumption of σ-compactness, we characterise the simplicity of their essential groupoid C*-algebra. It has recently become clear that this C*-algebra, which agrees with the reduced groupoid C*-algebra in the Hausdorff setting, is the correct replacement for the reduced groupoid C*-algebra in the non-Hausdorff setting, when searching for an algebraic-dynamical description of the ideal space of a groupoid C*-algebra. It comes as a surprise that a careful development of the necessary techniques and notions leads to a clean characterisation of C*-simplicity also in the non-Hausdorff case. Third, we obtain appropriate analogues in the setting of groupoids of the averaging results considered by Powers and many others for reduced group C*-algebras. This allows us to strengthen previously known results on C*-irreducibility of inclusions arising from groupoids of germs. Fourth, behind these results lies the development of a novel category of groupoid C*-dynamical systems, including a theory of boundaries for groupoids that culminates in a dynamical construction of the Furstenberg boundary of a groupoid.

The first main result of this article completely solves the problem of characterising the simplicity of reduced groupoid C*-algebras of étale Hausdorff groupoids, and the simplicity of essential groupoid C*-algebras of σ-compact étale groupoids.

Theorem 7.1.1 (See Theorem 7.7.13)**.** *Let $\mathcal{G}$ be an étale groupoid with locally compact Hausdorff space of units. Assume that $\mathcal{G}$ is Hausdorff, $\mathcal{G}$ is σ-compact or $\mathcal{G}$ has a compact space of units. Then the essential groupoid C*-algebra $C^*_{\mathrm{ess}}(\mathcal{G})$ is simple if and only if $\mathcal{G}$ is minimal and has no essentially confined amenable sections of isotropy groups.*

Let us explain the notation we use and put our results into the context of previous work. Theorem 7.1.1 generalises the breakthrough results characterising C*-simple discrete groups [KK17, BKKO17, Ken20, Haa16], and completes a sequence of partial results obtained for dynamical systems [Kaw17, KS21] and special classes of groupoids [FS82, Ren91, KS02, BCFS14, Bor20, Bor19, KM21]. The notion of confined sections of isotropy groups emerged from work on group rings [HZ97] and C*-simple groups [Ken20] and was subsequently generalised to dynamical systems [Kaw17] and Hausdorff groupoids [Bor19]. In this setting, a section of isotropy groups is confined if, roughly speaking, it cannot be approximately conjugated into the groupoid's unit space. The terminology stems from the special case of groups, where a confined subgroup is separated in a strong sense from the trivial subgroup. For non-Hausdorff groupoids, it is necessary to consider a more general notion of essential confinedness, taking into account the closure of the unit space. Definition 7.7.1 captures this notion rigorously.

While the essential groupoid C*-algebras of a Hausdorff groupoid is identified with its reduced groupoid C*-algebra, for non-Hausdorff groupoids, it is necessary to adapt the construction of the reduced groupoid C*-algebra, if one aims to relate the ideal structure of a groupoid C*-algebra to the algebraic and dynamical structure of the groupoid. The need to consider a modification of the reduced groupoid C*-algebra became apparent beginning with work of Khoshkam-Skandalis [KS02], followed by instructive examples of Exel [Exe11], and work of Exel-Pitts [EP19] and Clark-Exel-Pardo-Sims-Starling [CEP+19] and Kwaśniewski-Meyer [KM21]. The essential groupoid C*-algebra is the quotient of the reduced groupoid C*-algebra by an ideal of singular elements [CEP+19, KM21]. This explains the need for some kind of countability assumption, such as σ-compactness, in order to control the singular elements in a suitable way. By adopting this perspective, [CEP+19, KM21] obtained a characterisation of certain amenable groupoids whose essential groupoid C*-algebra is simple. In particular [KM21] showed that topologically free groupoids have simple essential groupoid C*-algebras.

Let us now explain the technical core of the present work. In the context of groupoid C*-algebras, the problem of proving simplicity is subdivided by considering the ideal intersection property [Tom92, ST09] and the study of orbits. The inclusion $C_0(\mathcal{G}^{(0)}) \subseteq C^*_{\mathrm{ess}}(\mathcal{G})$ has the ideal intersection property if every nonzero ideal of $C^*_{\mathrm{ess}}(\mathcal{G})$ has nonzero intersection with $C_0(\mathcal{G}^{(0)})$. We abuse terminology slightly and refer to $C^*_{\mathrm{ess}}(\mathcal{G})$ having the ideal intersection property. This C*-algebra is simple if and only if it has the ideal intersection property and $\mathcal{G}$ is minimal. While it is straightforward to determine the minimality of a groupoid, it has been a major open problem to characterise the ideal intersection property for essential groupoid C*-algebras.

Existing work on simplicity and the ideal intersection property for groupoid C*-algebras has generally proceeded in two different directions. First, for amenable groupoids, the reduced and the full groupoid C*-algebras agree, making characterisations of simplicity more accessible, as is also visible from our main results, where the amenability condition on sections of isotropy groups is automatically satisfied. From an operator algebraic perspective this is reflected in the abundance of *-homomorphisms defined on the full groupoid C*-algebra. Previous work on amenable groupoids includes the work of Archbold-Spielberg on transformation groups associated with actions of abelian groups [AS94], the work of Kumjian-Pask [KP00] and Robertson-Sims [RS07] characterising simplicity of groupoid C*-algebras arising from higher-rank graphs, and finally the work of Brown-Clark-Farthing-Sims [BCFS14] resulting in a complete characterisation of étale Hausdorff groupoids with simple full groupoid C*-algebra. As explained above, characterisations of the simplicity of essential groupoid C*-algebras of amenable non-Hausdorff groupoids can be deduced from recent work by Clark-Exel-Pardo-Sims-Starling [CEP+19] and by Kwaśniewski-Meyer [KM21].

Second, a completely different approach to simplicity and the ideal intersection property for groupoid C*-algebras emerged from the breakthrough results about C*-simple discrete groups obtained by Kalantar-Kennedy [KK17] and Breuillard-Kalantar-Kennedy-Ozawa [BKKO17]. The crucial insight of [KK17] was that the simplicity of reduced group C*-algebras has to be coupled to and understood through the group's action on its Furstenberg boundary. Combined with Le Boudec's work [LB17], along with further characterisations obtained by Kennedy [Ken20] and Haagerup [Haa16], this resolved the long-standing problem of characterising discrete groups whose reduced group C*-algebra is simple, as reported in the Séminaire Bourbaki [Rau20].

The success of these methods in understanding the C*-simplicity of groups triggered subsequent work on simplicity and the ideal intersection property for reduced crossed product C*-algebras associated with dynamical systems of discrete groups [BK16, Bry22, Kaw17, KS19]. In all of this work, the C*-algebra being acted on is unital, which in the setting of a dynamical system translates to the assumption that the underlying topological space is compact. The next advance in this direction was obtained by Borys [Bor20, Bor19], who introduced an analogue of the Furstenberg boundary for étale Hausdorff groupoids with compact unit space that enabled him to prove the ideal intersection property for groupoids with no confined amenable sections of isotropy groups. This necessary condition was a partial analogue of results obtained by Kennedy [Ken20] for groups and by Kawabe [Kaw17] for topological dynamical systems. The most recent advance was achieved by Kalantar-Scarparo [KS21], who utilised the Alexandrov one-point compactification to extend results of Kawabe to group actions on locally compact spaces.

Our proof of Theorem 7.1.1 begins with the study of the ideal intersection property. We first treat in Theorem 7.7.2 groupoids with compact space of units, and subsequently obtain a result for groupoids with locally compact space of units in Section 7.7.2. For groupoids that do not necessarily arise from a group action, a one-point compactification of the unit space leads to the notion of Alexandrov groupoid $\mathcal{G}^+$, which allows us to obtain the following theorem from the corresponding result for groupoids with compact space of units.

Theorem 7.1.2 (See Theorems 7.7.2 and 7.7.10)**.** *Let $\mathcal{G}$ be an étale groupoid with locally compact Hausdorff space of units. Assume that $\mathcal{G}$ is Hausdorff, that $\mathcal{G}^+$ is σ-compact, or that $\mathcal{G}$ is minimal and has a compact space of units. Then $C^*_{\mathrm{ess}}(\mathcal{G})$ has the ideal intersection property if and only if $\mathcal{G}$ has no essentially confined amenable sections of isotropy groups.*

Further, $\mathcal{G}^+$ is σ-compact if $\mathcal{G}$ is σ-compact.

In Corollary 7.7.12, we combine work of Bönicke-Li [BL20] with our results to obtain a characterisation of étale Hausdorff groupoids whose reduced groupoid C*-algebra has the ideal separation property, that is its ideals can be completely described in terms of the groupoid's dynamics.

Following the paradigm established by [KK17, BKKO17], an important intermediate step in establishing our elementary characterisation of the ideal intersection property is the control of equivariant ucp maps defined on the essential groupoid C*-algebras of $\mathcal{G}$ by means of a suitable boundary, which implicitly requires a $\mathcal{G}$-action on the groupoid C*-algebra. However, the lack of such an action has been recognised as a major obstruction to extending the previous-mentioned work on groups and dynamical systems to the present setting. In particular, the C*-algebra $C^*_{\text{ess}}(\mathcal{G})$ is not a $\mathcal{G}$-C*-algebra in the sense of [Ren87], which is also the definition employed in [Bor20, Bor19], and requires a $\mathcal{G}$-C*-algebra to be fibered over the unit space of the groupoid. Therefore, an important step in developing the results of the present article is the introduction of a new notion of groupoid action on C*-algebras. For this, we replace elements of the groupoid with elements from the pseudogroups of open bisections [LL13]. Groupoid C*-algebras are then defined in terms of families of hereditary C*-subalgebras with *-isomorphisms associated to open bisections of the groupoid. See Section 7.3 for details.

With respect to the above definition, we prove that $C^*_{\text{ess}}(\mathcal{G})$ is naturally a $\mathcal{G}$-C*-algebra. The following result is an important first step towards Theorem 7.1.2.

Theorem 7.1.3 (See Proposition 7.4.7 and Theorem 7.4.9)**.** *Let $\mathcal{G}$ be a étale groupoid with compact Hausdorff unit space. Then $\ell^\infty(\mathcal{G})$ is an injective object in the category of $\mathcal{G}$-C*-algebras. Further, there is a unique injective envelope of $C(\mathcal{G}^{(0)})$ in the category of $\mathcal{G}$-C*-algebras, which is commutative.*

The spectrum of the $\mathcal{G}$-injective envelope of $C(\mathcal{G}^{(0)})$ is, by definition, the Hamana boundary $\partial_H\mathcal{G}$ of $\mathcal{G}$. Expressed using this terminology, Borys actually constructed the Hamana boundary of an étale Hausdorff groupoid with compact space of units, considered within the category of classical $\mathcal{G}$-C*-algebras from [Ren87]. We show in Theorem 7.4.9 that, in the setting considered by Borys, his construction of the Hamana boundary agrees with our construction of the Hamana boundary in the larger category of $\mathcal{G}$-C*-algebras. We also develop a dynamical approach to boundary theory for an étale groupoid $\mathcal{G}$ with compact Hausdorff unit space and construct the Furstenberg boundary $\partial_F\mathcal{G}$ of $\mathcal{G}$ within this framework. Specifically, we consider the category of $\mathcal{G}$-flows, which are compact Hausdorff spaces equipped with a $\mathcal{G}$-action. We single out the $\mathcal{G}$-boundaries, which are the $\mathcal{G}$-flows that are both minimal and strongly proximal in an appropriate sense. The Furstenberg boundary $\partial_F \mathcal{G}$ is the universal $\mathcal{G}$-boundary, meaning that every $\mathcal{G}$-boundary is the image of $\partial_F \mathcal{G}$ under a morphism of $\mathcal{G}$-flows. Using the universal properties satisfied by the Hamana boundary and the Furstenberg boundary, we establish in Theorem 7.4.19 that they coincide. This identification is an important ingredient in the study of Powers averaging property, leading to Theorem 7.1.5 below.

Having developed a boundary theory for groupoids and, specifically, having constructed the Hamana boundary, we are able to introduce an equivariant analogue of Pitts' pseudo-expectations [Pit17], in particular obtaining a natural $\mathcal{G}$-pseudo-expectation $C^*_{\text{ess}}(\mathcal{G}) \to C(\partial_F \mathcal{G})$ in Section 7.5.1. This provides a new perspective on work of Kwaśniewski-Meyer on local conditional expectations [KM21] in terms of C*-simplicity theory. We are able to prove the following characterisation of the ideal intersection property, which is the foundation for all of our subsequent results.

Theorem 7.1.4 (See Theorem 7.6.1)**.** *Let $\mathcal{G}$ be an étale groupoid with compact Hausdorff space of units and denote by $\partial_F \mathcal{G}$ the Furstenberg boundary of $\mathcal{G}$. Assume that $\mathcal{G}$ is Hausdorff, that $\mathcal{G}$ is minimal or $\mathcal{G}$ is σ-compact. Then the following statements are equivalent.*

- *$C^*_{\text{ess}}(\mathcal{G})$ has the ideal intersection property.*
- *$C^*_{\text{ess}}(\mathcal{G} \ltimes \partial_F \mathcal{G})$ has the ideal intersection property.*
- *There is a unique $\mathcal{G}$-pseudo expectation $C^*_{\text{ess}}(\mathcal{G}) \to C(\partial_F \mathcal{G})$.*

Let us mention Remark 7.6.9, which shows that the intersection property for $C^*_r(\mathcal{G} \ltimes \partial_F \mathcal{G})$ implies that $\mathcal{G}$ is Hausdorff. Hence, for non-Hausdorff groupoids, there is no possible variant of Theorem 7.1.4 that solely relies on the reduced groupoid C*-algebra.

Research on C*-simplicity began with an averaging argument devised by Powers [Pow75], based on the Dixmier averaging theorem for von Neumann algebras. Following the work on C*-simplicity in [KK17, BKKO17], it was subsequently shown independently by Haagerup [Haa16] and Kennedy [Ken20] that Powers' averaging property characterises C*-simplicity. That is, G is a C*-simple group if and only if the norm-closed convex hull of the set of G-conjugates of an element of the reduced group C*-algebra contains the trace of the element.

This is not only of aesthetic value, but is the basis for transferring simplicity results to related operator algebras, as happened for example with Hecke C*-algebras in [CKL21, Kli21] and in Phillips' work on L^p-simplicity [Phi19].

Very recently, the correct analogue of Powers averaging for crossed product C*-algebras associated with group actions on compact spaces was developed in [AU22], using the concept of a generalised probability measure. We introduce a suitable notion of generalised probability measure along with a suitable variant of the Powers averaging property for étale groupoids in Section 7.8. We then obtain a corresponding characterisation of simple essential C*-algebras generalising the results in [AU22], as well as the results about relative versions of Powers averaging property in [AK20, Amr21, Urs22].

An inclusion $A \subseteq B$ of unital C*-algebras was termed C*-irreducible in [Rør21] if every intermediate C*-algebra is simple. Such inclusions are an important ingredient in an emerging C*-algebraic analogue of Jones' subfactor theory [Jon83]. In particular, C*-irreducible inclusions arising from group C*-algebras, crossed products and groupoids of germs have recently received a great deal of interest [AK20, AU22, Rør21, KS21]. The relative Powers averaging property implies C*-irreducibility of the associated inclusion of C*-algebras.

Theorem 7.1.5 (See Theorem 7.8.14)**.** *Let $\mathcal{G}$ be a minimal étale groupoid with compact Hausdorff space of units. Then the following statements are equivalent.*

- $C^*_{\mathrm{ess}}(\mathcal{G})$ *is simple*
- $C^*_{\mathrm{ess}}(\mathcal{G})$ *satisfies the relative Powers averaging property with respect to any covering and contracting semigroup of generalised probability measures.*
- $A \subseteq C^*_{\mathrm{ess}}(\mathcal{G})$ *is C*-irreducible for every C*-subalgebra $A \subseteq C^*_{\mathrm{ess}}(\mathcal{G})$ supporting a covering and contracting semigroup of generalised probability measures.*

Applying Theorem 7.1.5 to suitable subgroups of the topological full group, we obtain many examples of groups of unitaries satisfying the relative Powers averaging property. We remark that in view of the degeneration phenomena described in [BS19], it is important to allow for the consideration of proper subgroups of the topological full group.

Corollary 7.1.6 (See Corollary 7.9.1)**.** *Let $\mathcal{G}$ be an étale groupoid with compact Hausdorff space of units. Assume that there is a subgroup of the topological full group $G \leq \mathbf{F}(\mathcal{G})$ that covers $\mathcal{G}$ and such that $G \curvearrowright \mathcal{G}^{(0)}$ is a G-boundary. Denote by $\pi : G \to C^*_{\mathrm{ess}}(\mathcal{G})$ the unitary representation of G in the essential groupoid C*-algebra of $\mathcal{G}$. If $C^*_{\mathrm{ess}}(\mathcal{G})$ is simple, then $C^*_{\mathrm{ess}}(\mathcal{G})$ satisfies Powers averaging property relative to $\pi(G)$.*

Considering groupoids of germs, simplicity of the associated essential groupoid C*-algebra can be guaranteed thanks to topological freeness. Applied to this situation, our

Theorem 7.1.6 yields the following result, extending work of Kalantar-Scarparo [KS22, KS21].

Theorem 7.1.7 (See Theorem 7.9.2)**.** *Let G be a countable discrete group and $G \curvearrowright X$ a boundary action. Denote by $\mathcal{G}$ its groupoid of germs and by $\pi : G \to C^*_{\mathrm{ess}}(\mathcal{G})$ the associated unitary representation. Then $\pi(G) \subseteq C^*_{\mathrm{ess}}(\mathcal{G})$ satisfies the relative Powers averaging property.*

Let us explain the terminology in the statement of the previous theorem, referring to Section 7.9 for more details. An action of a discrete group $G \curvearrowright X$ is a boundary action if X is compact and the action is minimal and strongly proximal. These actions were introduced by Furstenberg [Fur63] and further developed by Glasner [Gla76], and are of fundamental importance in topological dynamics. Given an action of a discrete group $G \curvearrowright X$, its groupoid of germs is the quotient groupoid $G \ltimes X/\mathrm{Iso}(G \ltimes X)^\circ$, dividing out the interior of its isotropy from the associated transformation groupoid.

Recall that Thompson's group F consists of the piecewise linear transformations of $[0, 1]$ with slopes and breakpoints in $\mathbb{Z}[\frac{1}{2}]$. The amenability of F is a major open problem in group theory. It is a subgroup of Thompson's group T, which consists of the piecewise $\mathrm{PSL}_2(\mathbb{Z}[\frac{1}{2}])$ transformations of S^1 with breakpoints in $e^{2\pi i \mathbb{Z}[\frac{1}{2}]}$. The quasi-regular representation of T with respect to F has received a great deal of attention since Haagerup-Olesen and Le Boudec-Matte Bon considered it within the context of the amenability of F [HO17, LM18]. The amenability of F is equivalent to the statement that this quasi-regular representation is weakly contained in the regular representation of T. This fact motivated Kalantar-Scarparo to establish the simplicity of the C*-algebra generated by this representation in [KS22]. Thanks to Theorem 7.1.7, we obtain a significant strengthening of their result.

Example 7.1.8 (See Example 7.9.8)**.** Denote by K the totally disconnected cover of S^1, doubling dyadic integer points. Let $\mathcal{G}$ be the groupoid of germs of the action $\mathrm{T} \curvearrowright K$ and $\pi : \mathrm{T} \to C^*_r(\mathcal{G})$ the associated unitary representation. Then $\pi(\mathrm{T}) \subseteq C^*_r(\mathcal{G})$ satisfies the relative Powers averaging property.

Since it is known that $C^*_r(\mathcal{G}) \cong \mathcal{O}_2$, in the setting of Examples 7.1.8, the above result shows that if F is amenable, then there is a unitary representation of T into the Cuntz algebra $\mathcal{O}_2$ that enjoys the relative Powers averaging property. Many more examples of unitary representations satisfying the relative Powers averaging property can be obtained from groups of homeomorphisms of the circle, as we explain in Remark 7.9.7.

Organisation of the article

This article has 9 sections. After this introduction, Section 7.2 describes preliminary results and fixes notation concerning groupoids and inverse semigroups. In Section 7.3, we introduce our new notion of groupoid C*-algebras. In Section 7.4, we prove Theorem 7.1.3 and develop the dynamical approach to boundary theory for étale groupoids and prove the identification of the Furstenberg and the Hamana boundary. In Section 7.5, we study essential groupoid C*-algebra of groupoids with compact space of units from the point of view of the Furstenberg boundary and show that there is a natural inclusion of C*-algebras $C^*_{\mathrm{ess}}(\mathcal{G}) \subseteq C^*_{\mathrm{ess}}(\mathcal{G} \ltimes \partial_F \mathcal{G})$ if $\mathcal{G}$ is either minimal or σ-compact. In Section 7.6 we prove some fundamental characterisations of the ideal intersection property for groupoids with compact space of units, eventually leading to Theorem 7.1.4. In Section 7.7, we introduce the notion of essentially confined sections of isotropy groups, as well as the Alexandrov groupoid, leading to a proof of Theorem 7.1.2. Theorem 7.1.1 is a special case of this. In Section 7.8, we single out the appropriate notion of relative Powers averaging property for essential groupoid C*-algebras and prove Theorem 7.1.5 as well as Corollary 7.1.6. In Section 7.9, we apply the previous results and prove Theorem 7.1.7. We also describe Example 7.1.8.

7.2 Preliminaries

7.2.1 Groupoids and their C*-algebras

For basics on étale groupoids and their C*-algebras, we refer the reader to Renault's book [Ren80] and Sims' lecture notes [SSW20].

A groupoid is a small category whose morphisms are invertible. We denote by r and s the range and source map of a groupoid and we adopt the convention that $g \cdot h$ is defined if $\mathrm{r}(h) = \mathrm{s}(g)$. We denote by $\mathcal{G}^{(0)} = \{gg^{-1} \mid g \in \mathcal{G}\} \subseteq \mathcal{G}$ the set of units of a groupoid $\mathcal{G}$ and by $\mathrm{Iso}(\mathcal{G}) = \{g \in \mathcal{G} \mid \mathrm{s}(g) = \mathrm{r}(g)\}$ its isotropy bundle. For $x, y \in \mathcal{G}^{(0)}$, we denote by $\mathcal{G}_x = \mathrm{s}^{-1}(x)$ and $\mathcal{G}^x = \mathrm{r}^{-1}(x)$ the fibres of range and source map. We also write $\mathcal{G}_x^x = \mathcal{G}_x \cap \mathcal{G}^x$ for the isotropy group at x.

A topological groupoid is a groupoid equipped with a topology such that multiplication and inversion become continuous and the range and source maps are open. An étale groupoid is a topological groupoid whose range and source maps are local homeomorphisms. Every étale groupoid $\mathcal{G}$ has a basis of its topology consisting of open bisections, that is open subsets $U \subseteq \mathcal{G}$ such that $s|_U$ and $r|_U$ are homeomorphisms onto their image. A topological groupoid $\mathcal{G}$ is effective if it satisfies $\mathrm{Iso}(\mathcal{G})^\circ = \mathcal{G}^{(0)}$, that is the interior of its isotropy bundle equals its space of units. We will need the following proposition, which generalises [Bor20, Proposition 4.2.19] to arbitrary étale groupoids with extremally disconnect space of units.

Proposition 7.2.1. *Let $\mathcal{G}$ be an étale groupoid with extremally disconnected, locally compact space of units. Then the isotropy bundle of $\mathcal{G}$ is clopen.*

Proof. It follows from continuity of range and source map that $\mathrm{Iso}(\mathcal{G})$ is closed. Let $g \in \mathrm{Iso}(\mathcal{G})$ and let U be a compact open bisection containing g. Since range and source map are continuous and open, U defines a partial homeomorphism $\varphi : \mathrm{s}(U) \to \mathrm{r}(U)$ between clopen subsets of $\mathcal{G}^{(0)}$. So [Pit17, Proposition 2.11] (see also [Arh00, Proof of Theorem 1]) implies that $\mathrm{Fix}(\varphi) \subseteq \mathcal{G}^{(0)}$ is clopen. Now $(s|_U)^{-1}(\mathrm{Fix}(\varphi)) \subseteq U \subseteq \mathcal{G}$ is an open neighbourhood of g inside $\mathrm{Iso}(\mathcal{G})$. So $\mathrm{Iso}(\mathcal{G})$ is open. ■

Given a topological groupoid $\mathcal{G}$, a $\mathcal{G}$-space is a topological space with a surjection $p : X \to \mathcal{G}^{(0)}$ and a continuous action map $\mathcal{G}\,{}_{\mathrm{s}}\!\times_p X \to X$ satisfying the natural associativity condition. Here

$$\mathcal{G}\,{}_{\mathrm{s}}\!\times_p X = \{(g, x) \in \mathcal{G} \times X \mid \mathrm{s}(g) = p(x)\}$$

denotes the fibre product with respect to s and p. A $\mathcal{G}$-space $p : X \to \mathcal{G}^{(0)}$ is called irreducible, if every closed $\mathcal{G}$-invariant subset $A \subseteq X$ satisfying $p(A) = \mathcal{G}^{(0)}$ must equal X.

Given a $\mathcal{G}$-space X, one defines the transformation groupoid $\mathcal{G} \ltimes X$ as the topological space $\mathcal{G}\,{}_{\mathrm{s}}\!\times_p X$ equipped with the range and source maps $\mathrm{s}(g, x) = x$ and $\mathrm{r}(g, x) = gx$, respectively, and the multiplication $(g, x)(h, y) = (gh, y)$.

A subgroupoid $\mathcal{H} \subseteq \mathrm{Iso}(\mathcal{G})$ is normal if $ghg^{-1} \in \mathcal{H}$ holds for all $h \in \mathcal{H}$ and all $g \in \mathcal{G}_{\mathrm{r}(h)}$. Assuming that $\mathcal{H} \subseteq \mathrm{Iso}(\mathcal{G})$ is an open normal subgroupoid of an étale groupoid, a straightforward verification shows that the quotient space $\mathcal{G}/\mathcal{H}$ carries a natural structure of an étale groupoid too. It does not need to be Hausdorff.

The next definition of groupoid C*-algebras for not-necessarily Hausdorff groupoids goes back to Connes' work on foliations [Con82]. We also refer to the article of Khoshkam-Skandalis for the construction of the regular representation for non-Hausdorff groupoids [KS02].

Definition 7.2.2. Let $\mathcal{G}$ be an étale groupoid with locally compact Hausdorff space of units. Denote by $\mathcal{C}(\mathcal{G})$ the linear span inside $\ell^\infty(\mathcal{G})$ of all subspaces $C_c(U)$, where $U \subseteq \mathcal{G}$ runs through open bisections. For $x \in \mathcal{G}^{(0)}$, denote by $\lambda_x : \mathcal{C}(\mathcal{G}) \to \mathcal{B}(\ell^2(\mathcal{G}_x))$ the convolution representation satisfying $\lambda_x(f)\delta_g = \sum_{h \in \mathcal{G}_{\mathrm{r}(g)}} f(h)\delta_{hg}$. We adapt the following notation:

- The maximal groupoid C*-algebra $C^*(\mathcal{G})$ is the universal enveloping C*-algebra of $\mathcal{C}(\mathcal{G})$.
- The reduced groupoid C*-algebra $C^*_r(\mathcal{G})$ is the C*-completion of $\mathcal{C}(\mathcal{G})$ with respect to the family of *-representations $(\lambda_x)_{x\in\mathcal{G}^{(0)}}$.
- The universal enveloping von Neumann algebra of the maximal groupoid C*-algebra is denoted by $W^*(\mathcal{G}) = C^*(\mathcal{G})^{**}$.

To each element $a \in C^*_r(\mathcal{G})$ we associate the function

$$g \mapsto \hat{a}(g) = \langle \lambda_{\mathrm{s}(g)}(a)\delta_{\mathrm{s}(g)}, \delta_g \rangle \,.$$

If $f \in \mathcal{C}(\mathcal{G})$, then $\hat{f} = f$ holds. We remark that unless $\mathcal{G}$ is Hausdorff, there are always non-continuous functions in $\mathcal{C}(\mathcal{G})$ and a fortiori elements $a \in C^*_r(\mathcal{G})$ such that $\hat{a}$ is non-continuous.

While the correct definition of the maximal and reduced groupoid C*-algebra for non-Hausdorff groupoids was already clarified in the 1980's by Connes, it is only much more recently that a clear picture of the essential groupoid C*-algebra has emerged. After work of Khoshkam-Skandalis [KS02], Exel [Exe11], Exel-Pitts [EP19] and Clark-Exel-Pardo-Sims-Starling [CEP+19], the abstract framework of local conditional expectations described in [KM21] allowed to define the essential crossed product of semigroup actions on Fell bundles of bimodules. Even in the setup of groupoid C*-algebras, removing any action from the picture, the work of Kwaśniewski-Meyer provided the first complete and systematic account on this problem (see [KM21, End of Section 4]). It slightly differs from the treatment given in [CEP+19] in as far as the reference to functions with meager strict support allows for a more systematic framework than considering functions whose strict support has empty interior.

For a locally compact Hausdorff space X denote by $\mathcal{B}^\infty(X)$ the algebra of all bounded Borel functions on X and by $\mathcal{M}^\infty(X)$ its ideal of functions with meager support. We denote by $\mathrm{Dix}(X) = \mathcal{B}^\infty(X)/\mathcal{M}^\infty(X)$ the Dixmier algebra of X. Then by a result of Gonshor, $\mathrm{Dix}(X)$ is isomorphic with the local multiplier algebra $\mathrm{M}_{\mathrm{loc}}(C_0(X))$ of $C_0(X)$ and the injective envelope of $C_0(X)$ in the category of C*-algebras with *-homomorphisms as morphisms [Gon70]. Recall that $\mathrm{M}_{\mathrm{loc}}(C_0(X)) = \varinjlim C_b(U)$, where U runs through dense open subsets of X.

Let $\mathcal{G}$ be an étale groupoid with locally compact Hausdorff space of units. The local conditional expectation $\mathrm{E}_{\mathrm{red}} : C^*_r(\mathcal{G}) \to \mathrm{M}_{\mathrm{loc}}(C_0(\mathcal{G}^{(0)}))$ defined by Kwaśniewski-Meyer is characterised by the formula $\mathrm{E}_{\mathrm{red}}(f) = f|_U$ for every $f \in \mathcal{C}(\mathcal{G})$, where $U \subseteq \mathcal{G}^{(0)}$ is a dense open subset on which f is continuous. Thanks to [KM21, Proposition 4.3], it can be identified with the continuous extension of the natural map $\mathcal{C}(\mathcal{G}) \to \mathrm{Dix}(\mathcal{G}^{(0)})$ induced by restriction.

There is a representation of $C_r^*(\mathcal{G})$ into the adjointable operators on a Hilbert-$\mathrm{Dix}(\mathcal{G}^{(0)})$-module associated with $\mathrm{E}_{\mathrm{red}}$ by the KSGNS-construction [Lan95, Theorem 5.6].

Definition 7.2.3. Let $\mathcal{G}$ be an étale groupoid with locally compact Hausdorff space of units. The *essential groupoid C*-algebra* of $\mathcal{G}$ is the image $C_{\mathrm{ess}}^*(\mathcal{G})$ of $C_r^*(\mathcal{G})$ in the Hilbert-$\mathrm{Dix}(\mathcal{G}^{(0)})$-module associated with $\mathrm{E}_{\mathrm{red}}$.

By construction, the essential groupoid C*-algebra comes with a generalised conditional expectation $\mathrm{E}_{\mathrm{ess}} : C_{\mathrm{ess}}^*(\mathcal{G}) \to \mathrm{Dix}(\mathcal{G}^{(0)})$, which is faithful by [KM21, Theorem 4.11]. We follow Kwaśniewski-Meyer's development and call the kernel of the map $C_r^*(\mathcal{G}) \to C_{\mathrm{ess}}^*(\mathcal{G})$ its ideal of *singular elements* and denote it by J_{sing}. By faithfulness of $\mathrm{E}_{\mathrm{ess}}$, we have

$$J_{\mathrm{sing}} = \{a \in C_r^*(\mathcal{G}) \mid \mathrm{E}_{\mathrm{red}}(a^*a) = 0\}\,.$$

For groupoids covered by countably many open bisections, [KM21, Proposition 7.18] shows that J_{sing} consists exactly of those elements such that $\mathrm{s}(\operatorname{supp}\hat{a}) \subseteq \mathcal{G}^{(0)}$ is meager, thereby connecting to the treatment in [CEP+19]. For general groupoids the following description of elements vanishing under the local conditional expectation, extracted from [KM21, Section 4], is useful. We give a proof for the reader's convenience.

Proposition 7.2.4. *Let $\mathcal{G}$ be an étale groupoid with locally compact Hausdorff space of units. Assume that $a \in \ker(\mathrm{E}_{\mathrm{red}})$. Then there is a dense subset $U \subseteq \mathcal{G}^{(0)}$ such that $\hat{a}|_U = 0$.*

Proof. Let $a \in \ker(\mathrm{E}_{\mathrm{red}})$ and let $(a_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{C}(\mathcal{G})$ converging to a in $C_r^*(\mathcal{G})$. Then also $a_n = \hat{a}_n \to \hat{a}$ in $\|\cdot\|_\infty$. For every $n \in \mathbb{N}$ there is a dense open subset $U_n \subseteq \mathcal{G}^{(0)}$ such that $a_n|_{U_n}$ is continuous and $\mathrm{E}_{\mathrm{red}}(a_n) = a_n|_{U_n} \in C_b(U_n) \subseteq \mathrm{M}_{\mathrm{loc}}(C_0(\mathcal{G}^{(0)}))$. Since $\mathrm{E}_{\mathrm{red}}(a_n) \to \mathrm{E}_{\mathrm{red}}(a) = 0$, it follows that $\|a_n|_{U_n}\|_\infty \to 0$. Let $U = \bigcap_n U_n$, which is a comeager subset of $\mathcal{G}^{(0)}$. Since $\mathcal{G}^{(0)}$ is locally compact, we infer that $U \subseteq \mathcal{G}^{(0)}$ is dense. We also have $\hat{a}|_U = \lim \hat{a}_n|_U = 0$. ■

We will need the following observation made in [KM21, Lemma 7.15], which follows directly from the fact that for every open bisection U, the intersection $U \cap (\overline{\mathcal{G}^{(0)}} \setminus \mathcal{G}^{(0)})$ is meager.

Proposition 7.2.5. *Let $\mathcal{G}$ be an étale groupoid with locally compact Hausdorff space of units. Assume that $\mathcal{G}$ is covered by countably many open bisections. Then the set*

$$\{x \in \mathcal{G} \mid \overline{\mathcal{G}^{(0)}}_x = \{x\}\} \subseteq \mathcal{G}^{(0)}$$

is dense.

We say that an inclusion of C*-algebras $A \subseteq B$ has the *ideal intersection property* if zero is the only ideal $I \trianglelefteq B$ satisfying $I \cap A = 0$. We will in particular consider the intersection property for $C_0(\mathcal{G}^{(0)}) \subseteq C^*_r(\mathcal{G})$ and $C_0(\mathcal{G}^{(0)}) \subseteq C^*_{\mathrm{ess}}(\mathcal{G})$. In these cases we will write that $C^*_r(\mathcal{G})$ and $C^*_{\mathrm{ess}}(\mathcal{G})$, respectively, have the ideal intersection property.

We will need several times the fact that if $\mathcal{N} \subseteq \mathrm{Iso}(\mathcal{G})$ is a normal open subgroupoid whose isotropy groups are amenable, then the quotient map $\mathcal{G} \to \mathcal{G}/\mathcal{N}$ extends to a quotient map $C^*_r(\mathcal{G}) \to C^*_r(\mathcal{G}/\mathcal{N})$. While this seems to be a folklore result, we are not aware of any written account. We therefore provide a full proof for the convenience of the reader.

Proposition 7.2.6. *Let $\mathcal{G}$ be an étale groupoid with locally compact Hausdorff space. Let $\mathcal{N} \subseteq \mathrm{Iso}(\mathcal{G})$ be a normal open subgroupoid with amenable isotropy groups and denote by $p : \mathcal{G} \to \mathcal{G}/\mathcal{N}$ be the quotient map. Then there is a unique *-homomorphism $\pi : C^*_r(\mathcal{G}) \to C^*_r(\mathcal{G}/\mathcal{N})$ which restricts to the natural *-isomorphism $p_* : C_c(U) \to C_c(p(U))$ on every open bisection $U \subseteq \mathcal{G}$.*

Proof. Write $\mathcal{H} = \mathcal{G}/\mathcal{N}$ for the quotient and observe that the quotient map $p : \mathcal{G} \to \mathcal{H}$ restricts to a homeomorphism on every open bisection of $\mathcal{G}$. So there is a well-defined *-homomorphism $\pi_{\mathrm{alg}} : \mathcal{C}(\mathcal{G}) \to \mathcal{C}(\mathcal{H})$. Let us show that π_{alg} extends to a *-homomorphism $\pi : C^*_r(\mathcal{G}) \to C^*_r(\mathcal{H})$. Fix $x \in \mathcal{H}^{(0)} \cong \mathcal{G}^{(0)}$ and we will show that $\lambda^{\mathcal{H}}_x \circ \pi_{\mathrm{alg}}$ extends to $C^*_r(\mathcal{G})$, where $\lambda^{\mathcal{H}}_x$ denotes the left-regular representation $\lambda^{\mathcal{H}}_x : C^*_r(\mathcal{H}) \to \mathcal{B}(\ell^2(\mathcal{H}_x))$. Since the group $\mathcal{N}^x_x$ is amenable, we can find a Følner net $(F_i)_i$ in there and consider the states

$$\varphi_i = \frac{1}{|F_i|} \sum_{g \in F_i} \langle \lambda^{\mathcal{G}}_x(\cdot)\delta_g, \delta_g \rangle_{\ell^2(\mathcal{G}_x)} \, .$$

Passing to a subnet, we may assume that $\varphi_i \to \varphi \in (C^*_r(\mathcal{G}))^*$ in the weak-*-topology. Then an elementary computation shows that for all open bisections $U \subseteq \mathcal{G}$ and all $f \in C_c(U)$ we have

$$\varphi(f) = \begin{cases} f(g) & \text{if } U \cap \mathcal{N}^x_x = \{g\} \, , \\ 0 & \text{if } U \cap \mathcal{N}^x_x = \emptyset \, . \end{cases}$$

So $\varphi|_{\mathcal{C}(\mathcal{G})} = \langle \lambda_x^{\mathcal{H}} \circ \pi_{\mathrm{alg}}(\cdot)\delta_x, \delta_x \rangle$. For $a, b \in \mathcal{C}(\mathcal{G})$ we find that

$$\begin{aligned}\langle \lambda_x^{\mathcal{H}} \circ \pi_{\mathrm{alg}}(a^*a)(\lambda_x^{\mathcal{H}} \circ \pi_{\mathrm{alg}}(b)\delta_x), \lambda_x^{\mathcal{H}} \circ \pi_{\mathrm{alg}}(b)\delta_x \rangle &= \varphi(b^*a^*ab) \\ &\leq \|a\|^2_{C_r^*(\mathcal{G})} \varphi(b^*b) \\ &= \|a\|^2_{C_r^*(\mathcal{G})} \|\lambda_x^{\mathcal{H}} \circ \pi_{\mathrm{alg}}(b)\delta_x\|^2 .\end{aligned}$$

Since $\pi_{\mathrm{alg}}(\mathcal{C}(\mathcal{G})) = \mathcal{C}(\mathcal{H})$ and $\mathcal{C}(\mathcal{H})$ acts cyclically on $\ell^2(\mathcal{H}_x)$, this shows that $\|\lambda_x^{\mathcal{H}} \circ \pi_{\mathrm{alg}}(a)\| \leq \|a\|_{C_r^*(\mathcal{G})}$. So $\lambda_x^{\mathcal{H}} \circ \pi_{\mathrm{alg}}$ extends to $C_r^*(\mathcal{G})$. Since $x \in \mathcal{G}^{(0)}$ was arbitrary, this proves that π_{alg} extends to a *-homomorphism $\pi_{\mathrm{red}} : C_r^*(\mathcal{G}) \to C_r^*(\mathcal{H})$. ■

7.2.2 Inverse semigroups and pseudogroups of open bisections

We refer the reader to Lawson's book [Law98] for a comprehensive introduction to inverse semigroups. Recall that an inverse semigroup is a semigroup S such that for every $s \in S$ there is a unique $s^* \in S$ such that $ss^*s = s$ and $s^*ss^* = s^*$. The following notion of pseudogroups goes back to Resende [Res07] and Lawson-Lenz [LL13].

Definition 7.2.7. Let S be an inverse semigroup. Elements $s, t \in S$ are *compatible* if s^*t and t^*s are idempotent. Given a family of compatible elements $(s_i)_i$ in S its join is the minimal element $s \in S$ such that $s_i \leq s$ for all i. We say that S has *infinite compatible joins* if every if any compatible family in S admits a join. A *pseudogroup* is an inverse semigroup with infinite compatible joins in which multiplication distributes over arbitrary joins.

Example 7.2.8 (Proposition 2.1 of [LL13])**.** Let $\mathcal{G}$ be an étale groupoid with compact Hausdorff space of units. Then the semigroup of open bisections $\Gamma(\mathcal{G})$ is a pseudogroup when equipped with the multiplication $U \cdot V = \{gh \mid g \in U, h \in V, \mathrm{s}(g) = \mathrm{r}(h)\}$. Then $U^* = \{g^{-1} \mid g \in U\}$ follows.

Notation 7.2.9. In this article, we will refer to open bisections of an étale groupoid $\mathcal{G}$ either as subsets, usually denoted by $U, V \subseteq \mathcal{G}$ or alternatively as elements of the pseudogroup of $\mathcal{G}$, usually denoted by $\gamma \in \Gamma(\mathcal{G})$. Both kinds of notation make sense in different contexts. We denote by $\operatorname{supp} \gamma = \gamma^*\gamma = \mathrm{s}(\gamma)$ the support and by $\operatorname{im} \gamma = \gamma\gamma^* = \mathrm{r}(\gamma)$ the image of $\gamma \in \Gamma(\mathcal{G})$.

7.3 A new notion of groupoid C*-algebras

In this section we introduce a new notion of groupoid C*-algebras that will allow us to apply methods from the toolbox of C*-simplicity developed over the past years. The key point is that the reduced groupoid C*-algebra $C_r^*(\mathcal{G})$, as well as the essential groupoid C*-algebra $C_{\mathrm{ess}}^*(\mathcal{G})$, become $\mathcal{G}$-C*-algebras, which is not the case for existing notions of groupoid C*-algebras used in [Bor19, Bor20].

The next example frames the action of a groupoid on its base space in a way that is compatible with the perspective of operator algebras and pseudogroups. It will serve as a building block for the subsequent definition of groupoid C*-algebras.

Example 7.3.1. Let $\mathcal{G}$ be an étale groupoid with locally compact Hausdorff space of units $\mathcal{G}^{(0)}$. Every open bisection $\gamma \in \Gamma(\mathcal{G})$ defines a partial homeomorphism $\psi_\gamma = \mathrm{r}|_\gamma \circ (\mathrm{s}|_\gamma)^{-1} : \operatorname{supp}\gamma \to \operatorname{im}\gamma$. Dually, we obtain a *-homomorphism $\alpha_\gamma : C_0(\operatorname{supp}\gamma) \to C_0(\operatorname{im}\gamma)$ by the assignment $\alpha_\gamma(f) = f \circ \psi_{\gamma^*} = f \circ \psi_\gamma^{-1}$. Associativity of the multiplication in $\Gamma(\mathcal{G})$ implies that $\psi_{\gamma_1\gamma_2} = \psi_{\gamma_1} \circ \psi_{\gamma_2}$ on $\psi_{\gamma_2^*}(\operatorname{supp}\gamma_1 \cap \operatorname{im}\gamma_2)$ and $\alpha_{\gamma_1} \circ \alpha_{\gamma_2} = \alpha_{\gamma_1\gamma_2}$ on $C_0(\psi_{\gamma_2^*}(\operatorname{supp}\gamma_1 \cap \operatorname{im}\gamma_2))$.

Let us also introduce the following notation for hereditary C*-subalgebras.

Notation 7.3.2. Let X be compact Hausdorff space, A a unital C*-algebra and $C(X) \subseteq A$ a unital inclusion of C*-algebras. For an open subset $U \subseteq X$ we denote by

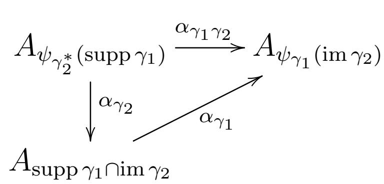
$$A_U = \overline{C_0(U)AC_0(U)}$$

the hereditary C*-subalgebra of A associated with U.

We are now ready to formulate our definition of groupoid C*-algebras. In Proposition 7.3.11 we will compare this new definition with the classical definition of $\mathcal{G}$-C*-bundles introduced by Renault [Ren87], showing that our definition is a suitable generalisation. We will only require this definition for groupoids with compact space of units, fitting the needs of Sections 7.4 and 7.6.

Definition 7.3.3. Let $\mathcal{G}$ be an étale groupoid with compact Hausdorff space of units. A unital $\mathcal{G}$-C*-algebra is a unital C*-algebra A with an injective unital *-homomorphism $\iota : C(\mathcal{G}^{(0)}) \to A$ and a family of *-isomorphisms $\alpha_\gamma : A_{\operatorname{supp}\gamma} \to A_{\operatorname{im}\gamma}$ indexed by $\gamma \in \Gamma(\mathcal{G})$ such that

- for all $\gamma \in \Gamma(\mathcal{G})$ and all $f \in C_0(\operatorname{supp}\gamma)$ we have $\iota(f \circ \psi_{\gamma^*}) = \alpha_\gamma \circ \iota(f)$, and
- for all $\gamma_1, \gamma_2 \in \Gamma(\mathcal{G})$ the following diagram commutes.



When working with a unital $\mathcal{G}$-C*-algebra A, we will frequently identify $C(\mathcal{G}^{(0)})$ with its image in A under ι and suppress the explicit map ι.

Remark 7.3.4. In the setting of Fell bundles over semigroups as used in [KM21], it is possible to replace actions of the pseudogroup of bisections by actions of wide subsemigroups [KM21, Definition 2.1 and Proposition 2.2]. The analogue of this fact for $\mathcal{G}$-C*-algebras as introduced here does not hold, since this notion is genuinely noncommutative. An example can be found by considering the discrete groupoid $\mathcal{G} = \{0,1\} \times \mathbb{Z}$ with unit space $\{0,1\}$. Its group of global bisections can be identified with $\mathbb{Z} \oplus \mathbb{Z}$. We identify bisections supported on 0 and 1, respectively, with $\{\emptyset\} \times \mathbb{Z}$ and $\mathbb{Z} \times \{\emptyset\}$. Then the pseudogroup of bisections can be described as

$$\Gamma(\mathcal{G}) = (\mathbb{Z} \oplus \mathbb{Z}) \sqcup (\{\emptyset\} \times \mathbb{Z}) \sqcup (\mathbb{Z} \times \{\emptyset\}) \sqcup \emptyset\,.$$

The subsemigroup $S = (\{\emptyset\} \times \mathbb{Z}) \sqcup (\mathbb{Z} \times \{\emptyset\}) \sqcup \emptyset \subseteq \Gamma(\mathcal{G})$ is wide in the sense of [KM21, Definition 2.1]. Consider the non-central, diagonal, unitary matrix $\mathrm{diag}(1,-1)$ and the embedding as diagonal matrices $C(\mathcal{G}^{(0)}) \cong \mathbb{C}^2 \subseteq \mathrm{M}_2(\mathbb{C})$. The action of $\Gamma(\mathcal{G})$ on $\mathrm{M}_2(\mathbb{C})$ for which the global bisection $(1,1)$ acts by conjugation with $\mathrm{diag}(1,-1)$ is non-trivial. However, its restriction to S is trivial.

In Proposition 7.3.6 we will see that for sufficiently large subsemigroups $S \subseteq \Gamma(\mathcal{G})$, there is a well-behaved correspondence between $\mathcal{G}$-C*-algebras and their obvious generalisations to S-C*-algebras.

A key role in the theory of C*-simplicity is played by unital completely positive maps. Let us fix the corresponding notion of $\mathcal{G}$-ucp maps and define the category of unital $\mathcal{G}$-C*-algebras considered in subsequent sections.

Definition 7.3.5. Let $\mathcal{G}$ be an étale groupoid with compact Hausdorff space of units.

- A $\mathcal{G}$-ucp map between unital $\mathcal{G}$-C*-algebras (A, ι_A, α) and (B, ι_B, β) is a unital completely positive map $\varphi : A \to B$ such that $\varphi \circ \iota_A = \iota_B$ holds and the diagram

$$\begin{array}{ccc} A_{\mathrm{supp}\,\gamma} & \xrightarrow{\alpha_\gamma} & A_{\mathrm{im}\,\gamma} \\ \downarrow{\scriptstyle \varphi|_{\mathrm{supp}\,\gamma}} & & \downarrow{\scriptstyle \varphi|_{\mathrm{im}\,\gamma}} \\ B_{\mathrm{supp}\,\gamma} & \xrightarrow{\beta_\gamma} & B_{\mathrm{im}\,\gamma} \end{array}$$

  commutes for every $\gamma \in \Gamma(\mathcal{G})$, where $\varphi|_U : A_U \to B_U$ is the restriction of φ.

- The category of unital $\mathcal{G}$-C*-algebras has as its objects unital $\mathcal{G}$-C*-algebras and as morphisms $\mathcal{G}$-ucp maps.
- For unital $\mathcal{G}$-C*-algebras A and B, a $\mathcal{G}$-ucp map $\phi : A \to B$ is an embedding if it is a complete order embedding.

In Remark 7.3.4, we saw that there are wide subsemigroups $S \subseteq \Gamma(\mathcal{G})$ that do not contain enough elements to remember a $\mathcal{G}$-C*-algebra. However, for sufficiently large semigroups it is desirable to know that there is a well-behaved correspondence. The prime examples should be the Boolean inverse semigroup of all compact open bisections of an ample groupoid [Law10, Law12] and the Boolean inverse semigroup of all locally regular open bisections. Modifying Definition 7.3.3, one obtains the notion of a unital S-C*-algebra A, which is equipped with a unital embedding $C(\mathcal{G}^{(0)}) \to A$ and a compatible family of *-isomorphism $\alpha_s : A_{\operatorname{supp} s} \to A_{\operatorname{im} s}$, $s \in S$.

Proposition 7.3.6. *Let $\mathcal{G}$ be an étale groupoid with compact Hausdorff space of units. Let $S \subseteq \Gamma(\mathcal{G})$ be a subsemigroup such that for all all $\gamma \in \Gamma(\mathcal{G})$ and all compact subsets $K \subseteq \operatorname{supp} \gamma$ there is some open set $K \subseteq U \subseteq \operatorname{supp} \gamma$ and an element $s \in S$ such that $s|_U = \gamma|_U$. Denote by $\mathcal{G}$-C^*-alg the category of unital $\mathcal{G}$-C*-algebras and by S-C^*-alg the category unital S-C*-algebras. Then the forgetful functor from $\mathcal{G}$-C^*-alg to S-C^*-alg is an isomorphism of categories.*

Proof. We will show that every S-C*-algebra carries a unique compatible structure of a $\mathcal{G}$-C*-algebra. Let (A, α) be a unital S-C*-algebra, let $\gamma \in \Gamma(\mathcal{G})$ and $a \in A_{\operatorname{supp} \gamma}$ be positive. There is an ascending sequence of positive functions $f_n \in C_c(\operatorname{supp} \gamma)$ such that $f_n a f_n \to a$. Since $\operatorname{supp} f_n$ is compact for all n, there are open subsets $\operatorname{supp} f_n \subseteq U_n \subseteq \operatorname{supp} \gamma$ and elements $s_n \in S$ satisfying $s_n|_{U_n} = \gamma|_{U_n}$. Note that in particular $s_n|_{U_m} = s_m|_{U_m}$ holds for all $m < n$. So if $m < n$ satisfy $\|f_n a f_n - f_m a f_m\| < \varepsilon$, then we find that

$$\|\alpha_{s_m}(f_m a f_m) - \alpha_{s_n}(f_n a f_n)\| \leq \|\alpha_{s_m}(f_m a f_m) - \alpha_{s_n}(f_m a f_m)\| + \varepsilon = \varepsilon\,.$$

Consequently $(\alpha_{s_n}(f_n a f_n))_n$ is a Cauchy sequence in $A_{\operatorname{supp} \gamma}$. We then define $\alpha_\gamma(a) = \lim_n \alpha_{s_n}(f_n a f_n)$. Straightforward calculations show that this defines a $\Gamma(\mathcal{G})$-action on A and it is clear that it is the unique such action that extends the action of S. ■

The main motivation to pass to the greater generality of Definition 7.3.3 is the fact that the maximal groupoid C*-algebra $C^*(\mathcal{G})$ becomes a $\mathcal{G}$-C*-algebra in a natural way. Indeed, our approach to $\mathcal{G}$-C*-algebras via a pseudogroup action allows for a straightforward definition of inner actions, which we will now present. We need the following preparatory lemma.

Lemma 7.3.7. *Let $\mathcal{G}$ be an étale groupoid with a compact Hausdorff space of units. Denote by S the inverse semigroup of partial isometries in the universal enveloping von Neumann algebra $W^*(\mathcal{G})$. There is a semigroup homomorphism $\Gamma(\mathcal{G}) \to S : \gamma \mapsto u_\gamma$ such that for all $\gamma \in \Gamma(\mathcal{G})$ the following statements hold.*

1. *The net $(f)_{0 \leq f \leq 1_\gamma}$ in $C_c(\gamma) \subseteq \mathcal{C}(\mathcal{G})$ converges to u_γ in the weak-*-topology.*
2. *We have $u_\gamma u_\gamma^* = 1_{\operatorname{im}\gamma}$ and $u_\gamma^* u_\gamma = 1_{\operatorname{supp}\gamma}$.*
3. *If $g \in C_c(\operatorname{supp}\gamma)$, then we have $u_\gamma g = g \circ \mathrm{m}_{\gamma^*}$, where $m_\gamma : h \mapsto \gamma h$ denotes left multiplication with γ.*

Proof. Fix $\gamma \in \Gamma(\mathcal{G})$ and consider the subspace $C_c(\gamma) \subseteq C^*(\mathcal{G})$. For every $g \in C_c(\gamma)$, the convolution product satisfies $g * g^* \in C(\mathcal{G}^{(0)})$. So the C*-identity implies that there is an isometric isomorphism of Banach spaces $C_0(\gamma) \cong \overline{C_c(\gamma)} \subseteq C^*(\mathcal{G})$. Thus, we obtain an isometric embedding

$$\mathcal{B}^\infty(\gamma) \subseteq C_0(\gamma)^{**} \subseteq C^*(\mathcal{G})^{**} = W^*(\mathcal{G}) \,,$$

where $\mathcal{B}^\infty(\gamma)$ denotes the space of bounded Borel functions on γ. The net $(f)_{0 \leq f \leq 1_\gamma}$ is monotone and converges pointwise to the indicator function 1_γ in $\mathcal{B}^\infty(\gamma)$. By the monotone convergence theorem, it also converges in the weak-*-topology. Since $C_0(\gamma)^{**}$ is weak-*-closed in $W^*(\mathcal{G})$, the net $(f)_{0 \leq f \leq 1_\gamma}$ converges also in $W^*(\mathcal{G})$. We denote its limit by u_γ.

Next take $g \in C_c(V)$ for some open bisection V satisfying $\mathrm{s}(V) \subseteq \operatorname{supp}\gamma$. Then for every $0 \leq f \leq 1_\gamma$ in $C_c(\gamma)$, we have

$$f * g(x) = \begin{cases} f(y) g(y^{-1}x) & \text{for } y \in \gamma \text{ and } y^{-1}x \in V \,, \\ 0 & \text{otherwise.} \end{cases}$$

Considering the map $\mathcal{B}^\infty(\gamma) \to \mathcal{B}^\infty(\gamma V) : \ f \mapsto f * g$, the definition of the convolution product shows continuity with respect to the topology of pointwise convergence in $\mathcal{B}^\infty(\gamma)$ and $\mathcal{B}^\infty(\gamma V)$. This implies that $u_\gamma g = g(\gamma^* \cdot) = g \circ \mathrm{m}_{\gamma^*}$ holds.

Let us now show that $\gamma \mapsto u_\gamma$ is a semigroup homomorphism. We have

$$u_{\gamma_1} u_{\gamma_2} = \lim_{0 \leq f \leq 1_{\gamma_2}} u_{\gamma_1} f = \lim_{0 \leq f \leq 1_{\gamma_2}} f \circ \mathrm{m}_{\gamma_1^*} = u_{\gamma_1 \gamma_2}$$

where the convergence claimed by the last equality holds because of the first part of this proof. It similarly follows that

$$u_\gamma^* = \lim_{0 \leq f \leq 1_\gamma} f^* = u_{\gamma^*} \,,$$

and

$$u_\gamma u_\gamma^* = \lim_{0 \leq f \leq 1_{\gamma^*}} u_\gamma f = \lim_{0 \leq f \leq 1_{\gamma^*}} f \circ \mathrm{m}_{\gamma^*} = 1_{\operatorname{im}\gamma} .$$

Combining the previous statements, we also find that

$$u_\gamma^* u_\gamma = u_{\gamma^*} u_{\gamma^*}^* = 1_{\operatorname{im}\gamma^*} = 1_{\operatorname{supp}\gamma} .$$

■

We can now conclude that there is a natural structure of a $\mathcal{G}$-C*-algebra on the maximal groupoid C*-algebra of an étale groupoid with compact unit space. Applied to the quotient maps $C^*(\mathcal{G}) \to C^*_r(\mathcal{G})$ and $C^*(\mathcal{G}) \to C^*_{\mathrm{ess}}(\mathcal{G})$ it also exhibits natural $\mathcal{G}$-C*-algebra structures on the reduced and the essential groupoid C*-algebra, respectively.

Proposition 7.3.8. *Let $\mathcal{G}$ be an étale groupoid with compact Hausdorff space of units. Consider the standard embedding $C(\mathcal{G}^{(0)}) \subseteq C^*(\mathcal{G})$. Then there is a unique structure $(\alpha_\gamma)_{\gamma \in \Gamma(\mathcal{G})}$ of a unital $\mathcal{G}$-C*-algebra on $C^*(\mathcal{G})$ satisfying*

$$\alpha_\gamma(f) = f \circ \mathrm{c}_{\gamma^*}$$

for all $\gamma \in \Gamma(\mathcal{G})$, all open bisections $U \subseteq \mathcal{G}$ satisfying $\mathrm{s}(U), \mathrm{r}(U) \subseteq \operatorname{supp}\gamma$ and all $f \in C_c(U)$. Here $\mathrm{c}_\gamma : h \mapsto \gamma h \gamma^*$ denotes conjugation with γ.

*Further, if $\pi : C^*(\mathcal{G}) \to A$ is a unital *-homomorphism that is injective on $C(\mathcal{G}^{(0)})$, there is the structure of a unital $\mathcal{G}$-C*-algebra on A so that π is $\mathcal{G}$-equivariant. If π is surjective, this structure is unique.*

Proof. Consider the partial isometries u_γ, $\gamma \in \Gamma(\mathcal{G})$ provided by Lemma 7.3.7. It follows from this lemma that the inclusion $C(\mathcal{G}^{(0)}) \subseteq C^*(\mathcal{G})$ and the maps $\operatorname{Ad} u_\gamma : W^*(\mathcal{G})_{\operatorname{supp}\gamma} \to W^*(\mathcal{G})_{\operatorname{im}\gamma}$ turn the enveloping von Neumann algebra into a $\mathcal{G}$-C*-algebra in such a way that the formula $\operatorname{Ad} u_\gamma(f) = f \circ \mathrm{c}_{\gamma^*}$ holds for all $\gamma \in \Gamma(\mathcal{G})$, all open bisections $U \subseteq \mathcal{G}$ satisfying $\mathrm{s}(U), \mathrm{r}(U) \subseteq \operatorname{supp}\gamma$ and all $f \in C_c(U)$. In order to show that $C^*(\mathcal{G})$ is a unital $\mathcal{G}$-algebra satisfying the conditions of the proposition, it suffices to fix $\gamma \in \Gamma(\mathcal{G})$ and show that $\operatorname{Ad} u_\gamma$ maps $C^*(\mathcal{G})_{\operatorname{supp}\gamma}$ to $C^*(\mathcal{G})$. To this end, recall that $C^*(\mathcal{G})_{\operatorname{supp}\gamma} = \overline{C_0(\operatorname{supp}\gamma) C^*(\mathcal{G}) C_0(\operatorname{supp}\gamma)}$. Invoking item 3 of Lemma 7.3.7 we thus find that

$$\begin{aligned} u_\gamma C^*(\mathcal{G})_{\operatorname{supp}\gamma} u_\gamma^* &= u_\gamma \overline{C_c(\operatorname{supp}\gamma) C^*(\mathcal{G}) C_c(\operatorname{supp}\gamma)} u_\gamma^* \\ &\subset \overline{\mathcal{C}(\mathcal{G}) C^*(\mathcal{G}) \mathcal{C}(\mathcal{G})} \\ &= C^*(\mathcal{G}) . \end{aligned}$$

Note that this is the unique $\mathcal{G}$-C*-algebra structure on $C^*(\mathcal{G})$ satisfying the conditions of the proposition, since

$$\operatorname*{span}_{s(U)\subseteq \operatorname{supp}\gamma} C_c(U) = C_0(\operatorname{supp}\gamma)\mathcal{C}(\mathcal{G})C_0(\operatorname{supp}\gamma) \subseteq C^*(\mathcal{G})_{\operatorname{supp}\gamma}$$

is dense.

Assume now that $\pi : C^*(\mathcal{G}) \to A$ is a unital *-homomorphism that is injective on $C(\mathcal{G}^{(0)})$. We identify $C(\mathcal{G}^{(0)})$ with a C*-subalgebra of A, that is $\pi|_{C(\mathcal{G}^{(0)})} = \mathrm{id}$. For $\gamma \in \Gamma(\mathcal{G})$, we start by defining a contractive *-homomorphism

$$\beta_{\gamma,\mathrm{c}} : \operatorname{span} C_c(\operatorname{supp}\gamma)AC_c(\operatorname{supp}\gamma) \to \operatorname{span} C_c(\operatorname{im}\gamma)AC_c(\operatorname{im}\gamma)\,.$$

Let $a = \sum_{i=1}^n f_{1,i}a_i f_{2,i} \in \operatorname{span} C_c(\operatorname{supp}\gamma)AC_c(\operatorname{supp}\gamma)$. Put $K = \bigcup_{i=1}^n \operatorname{supp} f_{1,i} \cup \operatorname{supp} f_{2,i}$. Then $K \subseteq \operatorname{supp}\gamma$ is compact, so that there is some function $g \in C_c(\operatorname{supp}\gamma)$ satisfying $0 \leq g \leq 1$ and $g|_K \equiv 1$. Observe that $u_\gamma g \in \mathcal{C}(\mathcal{G})$. We claim that the expression $\pi(u_\gamma g)a\pi(u_\gamma g)^*$ does not depend on the choice of g. Indeed, if $h \in C_c(\operatorname{supp}\gamma)$ is another function satisfying $0 \leq h \leq 1$ and $h|_K \equiv 1$, then

$$\begin{aligned}
\pi(u_\gamma gh)a\pi(u_\gamma gh)^* &= \sum_{i=1}^n \pi(u_\gamma gh) f_{1,i}a_i f_{2,i}\pi(hgu_{\gamma^*}) \\
&= \sum_{i=1}^n \pi(u_\gamma f_{1,i})a_i\pi(f_{2,i}u_{\gamma^*}) \\
&= \pi(u_\gamma g)a\pi(u_\gamma g)^*\,.
\end{aligned}$$

Also observe that

$$\pi(u_\gamma g)a\pi(u_\gamma g)^* = \sum_{i=1}^n \alpha_\gamma(f_{1,i})\pi(u_\gamma)a_i\pi(u_{\gamma^*})\alpha_\gamma(f_{2,i})$$

So we can put $\beta_{\gamma,\mathrm{c}}(a) = \pi(u_\gamma g)a\pi(u_\gamma g)^*$. Since $\beta_{\gamma,\mathrm{c}}$ is contractive it extends to a *-homomorphism $\beta_\gamma : A_{\operatorname{supp}\gamma} \to A_{\operatorname{im}\gamma}$.

If $\gamma_1, \gamma_2 \in \Gamma(\mathcal{G})$ satisfy $\operatorname{im}\gamma_2 \cap \operatorname{supp}\gamma_1 \neq \emptyset$, we want to show that $\beta_{\gamma_1\gamma_2} = \beta_{\gamma_1} \circ \beta_{\gamma_2}|_{A_{\operatorname{supp}\gamma_1\gamma_2}}$. To this end let $K \subseteq \operatorname{supp}\gamma_1\gamma_2$ be a compact subset, let $g_2 \in C_c(\operatorname{supp}\gamma_1\gamma_2)$ satisfy $0 \leq g_2 \leq 1$ and $g_2|_K \equiv 1$ and let $g_1 \in C_c(\operatorname{im}\gamma_2 \cap \operatorname{supp}\gamma_1)$ satisfy $g_1|_{\varphi_{\gamma_2}(K)} \equiv 1$. Then Lemma 7.3.7 implies that

$$u_{\gamma_1} f_1 u_{\gamma_2} f_2 = u_{\gamma_1} u_{\gamma_2} \alpha_{\gamma_2^*}(f_1) f_2 = u_{\gamma_1\gamma_2}\alpha_{\gamma_2^*}(f_1) f_2\,.$$

Since $\alpha_{\gamma_2^*}(f_1)f_2|_K \equiv 1$, this implies

$$\beta_{\gamma_1\gamma_2,\mathrm{c}} = \beta_{\gamma_1,\mathrm{c}} \circ \beta_{\gamma_2,\mathrm{c}}|_{\operatorname{span} C_c(\operatorname{supp}\gamma_1\gamma_2)AC_c(\operatorname{supp}\gamma_1\gamma_2)}\,.$$

By continuity the desired equality follows.

Assume now that $\pi : C^*(\mathcal{G}) \twoheadrightarrow A$ is surjective *-homomorphism and let $\beta_\gamma : A_{\operatorname{supp}\gamma} \to A_{\operatorname{im}\gamma}$ define some $\mathcal{G}$-action on A making π equivariant. Fix $\gamma \in \Gamma(\mathcal{G})$ and $a \in A_{\operatorname{supp}\gamma}$. By approximation, we may assume that $a = \pi(f)a_0\pi(f)$ for some $f \in C_0(\operatorname{supp}\gamma)$ and $a_0 \in A$. Let $b_0 \in C^*(\mathcal{G})$ be a preimage of a_0 and put $b = fb_0f$. Then $\pi(b) = a$ and thus

$$\beta_\gamma(a) = \beta_\gamma(\pi(b)) = \pi(\operatorname{Ad} u_\gamma(b)) = \pi(u_\gamma)\pi(b)\pi(u_\gamma^*) = \operatorname{Ad}\pi(u_\gamma)(a)\,.$$

This shows uniqueness of the $\mathcal{G}$-C*-algebra structure on A. ■

Next we will show that the classical notion of $\mathcal{G}$-C*-algebras introduced in [Ren87] is in a precise sense subsumed by our notion. Recall that given a compact Hausdorff space X, a unital $C(X)$-algebra is a unital C*-algebra A with a unital inclusion $C(X) \to \mathcal{Z}(A)$ into the centre of A. Given such algebra, the evaluation maps $\mathrm{ev}_x : C(X) \to \mathbb{C}$ extend to quotient maps $A \to A_x$ onto the fibres of a C*-bundle.

Definition 7.3.9. Let $\mathcal{G}$ be an étale groupoid with compact Hausdorff space of units. A unital $\mathcal{G}$-C*-bundle is a unital $C(\mathcal{G}^{(0)})$-algebra A with an associative and continuous map $\alpha : \mathcal{G}\,{}_{\mathrm{s}}\!\times \bigsqcup_{x\in\mathcal{G}^{(0)}} A_x \to \bigsqcup_{x\in\mathcal{G}^{(0)}} A_x$ such that $\alpha_x = \mathrm{id}$ for all $x \in \mathcal{G}^{(0)}$ and $\alpha_g \circ \alpha_h = \alpha_{gh}$ whenever $\mathrm{s}(g) = \mathrm{r}(h)$.

A $\mathcal{G}$-ucp map between unital $\mathcal{G}$-C*-bundles (A,α) and (B,β) is a unital $C(\mathcal{G}^{(0)})$-modular unital completely positive map $\varphi : A \to B$ satisfying $\varphi_{\mathrm{r}(g)} \circ \alpha_g = \beta_g \circ \varphi_{\mathrm{s}(g)}$ for all $g \in \mathcal{G}$.

Let us make the following ad-hoc definition.

Notation 7.3.10. Let A be a C*-algebra. Writing groupoid actions implicitly, we call a structure of a $\mathcal{G}$-C*-bundle on A and the structure of a $\mathcal{G}$-C*-algebra on A compatible, if they define the same inclusion $C(\mathcal{G}^{(0)}) \subseteq A$ and the equality

$$(\gamma a)_{\mathrm{r}(g)} = g a_{\mathrm{s}(g)}$$

is satisfied for every $\gamma \in \Gamma(\mathcal{G})$, every $g \in \gamma$ and every $a \in A_{\operatorname{supp}\gamma}$.

Proposition 7.3.11. *Let $\mathcal{G}$ be an étale groupoid with compact Hausdorff space of units.*

- *If A is a unital $\mathcal{G}$-C^*-bundle, then there is a unique compatible structure of a unital $\mathcal{G}$-C^*-algebra on A.*

- *If A is a unital $\mathcal{G}$-C^*-algebra such that $C(\mathcal{G}^{(0)}) \subseteq A$ is central, then there is a unique compatible structure of a unital $\mathcal{G}$-C^*-bundle on A.*

- *If A and B are unital $\mathcal{G}$-C^*-algebras such that $C(\mathcal{G}^{(0)})$ is central in A and B and $\varphi : A \to B$ is a unital completely positive map, then φ is $\mathcal{G}$-equivariant in the sense of $\mathcal{G}$-C^*-algebras if and only if it is $\mathcal{G}$-equivariant in the sense of $\mathcal{G}$-C^*-bundles.*

Proof. Let A be a unital $\mathcal{G}$-C*-bundle. Let $\gamma \in \Gamma(\mathcal{G})$ be an open bisection, $f_1, f_2 \in C_0(\operatorname{supp}\gamma)$ and $a \in A$. Then $f_1 a f_1 \in A_{\operatorname{supp}\gamma}$ and

$$\operatorname{supp}\gamma \to \mathcal{G} \,{}_s\!\times \bigsqcup_{x \in \mathcal{G}^{(0)}} A_x : \, x \mapsto ((s|_\gamma)^{-1}(x), (f_1 a f_2)_{\mathrm{s}(g)})$$

extends by zero to a continuous section of $\mathcal{G} \,{}_s\!\times \bigsqcup_{x \in \mathcal{G}^{(0)}} A_x$. Its image under the action of $\mathcal{G}$ defines a continuous section in $\bigsqcup_{x \in \mathcal{G}^{(0)}} A_x$ whose support lies in $\operatorname{im}\gamma$. We denote this element by $\alpha_\gamma(f_1 a f_2) \in A_{\operatorname{im}\gamma} \subseteq A$. The map $f_1 a f_2 \mapsto \alpha_\gamma(f_1 a f_2)$ is bounded on $C_0(\operatorname{supp}\gamma) A C_0(\operatorname{supp}\gamma)$ and thus extends to a bounded map $A_{\operatorname{supp}\gamma} \to A_{\operatorname{im}\gamma}$. It is straightforward to check that this defines a compatible structure of a unital $\mathcal{G}$-C*-algebra on A.

Vice versa, assume that (A, α) is a unital $\mathcal{G}$-C*-algebra such that $C(\mathcal{G}^{(0)}) \subseteq A$ is central. Let $g \in \mathcal{G}$ and $a_{\mathrm{s}(g)} \in A_{\mathrm{s}(g)}$. Let $a \in A$ be a lift of $a_{\mathrm{s}(g)}$ and $\gamma \in \Gamma(\mathcal{G})$ an open bisection containing g. Let $f_1, f_2 \in C_c(\operatorname{supp}\gamma)$ such that $0 \leq f_1, f_2 \leq 1$ and $f_1(\mathrm{s}(g)) = 1 = f_2(\mathrm{s}(g))$. Then

$$\alpha_\gamma(f_1 f_2 a f_2 f_1)|_{\mathrm{r}(g)} = f_1(\gamma^* \mathrm{r}(g))^2 \alpha_\gamma(f_2 a f_2)_{\mathrm{r}(g)} = \alpha_\gamma(f_2 a f_2)_{\mathrm{r}(g)} \, .$$

This shows well-definedness of a map $\mathcal{G} \,{}_{\mathrm{s}}\!\times \bigsqcup_{x \in \mathcal{G}^{(0)}} A_x \to \bigsqcup_{x \in \mathcal{G}^{(0)}} A_x$ satisfying the compatibility condition of Notation 7.3.10. A straightforward calculation shows that this defines a groupoid action, and it remains to check continuity. We denote by p the map from the bundle $\bigsqcup_{x \in \mathcal{G}^{(0)}} A_x$ to its base space. Let $(g, a_{\mathrm{s}(g)}) \in \mathcal{G} \,{}_{\mathrm{s}}\!\times \bigsqcup_{x \in \mathcal{G}^{(0)}} A_x$ and take a basic neighbourhood of $g a_{\mathrm{s}(g)}$, given by

$$N(\tilde{b}, U, \varepsilon) = \{ b \in \bigsqcup_{x \in \mathcal{G}^{(0)}} A_x \mid p(b) \in U \text{ and } \| b - \tilde{b}_{p(b)} \| < \varepsilon \}$$

for $\tilde{b} \in A$, $U \subseteq \mathcal{G}^{(0)}$ open and $\varepsilon > 0$. Without loss of generality, we may reduce the size of U in order to assume that there is an open bisection $\gamma \in \Gamma(\mathcal{G})$ that contains g and satisfies

$\overline{U} \subseteq \operatorname{im}\gamma$. Further, replacing $\tilde{b}$ by $f\tilde{b}f$ for some $f \in C_0(\operatorname{im}\gamma)$ satisfying $f|_{\overline{U}} \equiv 1$, we may assume that $\tilde{b} \in A_{\operatorname{im}\gamma}$. Put $\tilde{a} = \gamma^*\tilde{b}$. Then by construction $N(\tilde{a}, U, \varepsilon)$ maps into $N(\tilde{b}, U, \varepsilon)$, proving continuity of the action at $(g, a_{\mathrm{s}(g)})$.

A straightforward calculation shows the notions of $\mathcal{G}$-equivariance for ucp maps agree. ■

7.4 Boundary theory for étale groupoids

Boundary theory for discrete groups has played an important role in the analysis of the algebraic structure of the reduced C*-algebra of the group. A key result established in [KK17], is that the Furstenberg boundary and the Hamana boundary of a discrete group coincide, that is the C*-algebra of continuous functions on the Furstenberg boundary is the unique essential and injective object in the category of C*-dynamical systems. In this section, we will construct the Hamana boundary and the Furstenberg boundary of any étale groupoid $\mathcal{G}$ with compact Hausdorff unit space and identify the two.

A Hamana boundary of an étale Hausdorff groupoid $\mathcal{G}$ with compact unit space was constructed by Borys in [Bor19, Bor20] and directly termed Furstenberg boundary. In this section, we denote the object he constructed by $\partial_B\mathcal{G}$. He constructed $C(\partial_B\mathcal{G})$ as the injective envelope of $C(\mathcal{G}^{(0)})$ in the category of concrete $\mathcal{G}$-operator systems, which agrees with the injective envelope in the category of $\mathcal{G}$-C*-bundles.

We will construct the Hamana boundary $\partial_H\mathcal{G}$ of $\mathcal{G}$, for groupoids that are not necessarily Hausdorff. The Hamana boundary is the spectrum of the injective envelope of $C(\mathcal{G}^{(0)})$ in the category of unital $\mathcal{G}$-C*-algebras as introduced in Section 7.3. This terminology highlights that the identification with the Furstenberg boundary is an à posteriori result and honours Hamana's development of the theory of injective envelopes [Ham79, Ham85]. Our proof of the existence of injective envelopes in the category of unital $\mathcal{G}$-C*-algebras builds on Sinclair's proof from [Sin15] of the existence of injective envelopes in the category of operator systems. The key technical device utilised in his proof is the existence of idempotents in compact right topological semigroups. Utilising the convexity of the specific semigroup under consideration allows us to further deduce the rigidity and essentiality of the Hamana boundary.

We also develop a dynamical approach to boundary theory for an étale groupoid $\mathcal{G}$ with compact Hausdorff unit space and construct the Furstenberg boundary $\partial_F\mathcal{G}$ of $\mathcal{G}$ within this framework. Specifically, consider the category of $\mathcal{G}$-flows, which are compact Hausdorff spaces equipped with a $\mathcal{G}$-action. We single out the $\mathcal{G}$-boundaries, which are the $\mathcal{G}$-flows that are both minimal and strongly proximal in a sense made precise in Definition 7.4.11. The Furstenberg boundary $\partial_F\mathcal{G}$ is the universal $\mathcal{G}$-boundary, meaning that every $\mathcal{G}$-boundary is the image of $\partial_F\mathcal{G}$ under a morphism of $\mathcal{G}$-flows.

Once we have established the existence and uniqueness of the Furstenberg boundary $\partial_F\mathcal{G}$, we will prove that it coincides with the Hamana boundary $\partial_H\mathcal{G}$. Moreover, we will prove that if $\mathcal{G}$ is Hausdorff, then $\partial_F\mathcal{G}$ also coincides with the boundary $\partial_B\mathcal{G}$ constructed by Borys.

Both perspectives on the Furstenberg boundary developed here are of critical importance to our results. We will utilise the operator algebraic approach to the Furstenberg boundary throughout this paper, and in particular when we introduce pseudo-expectations in Section 7.6. We will utilise the dynamical approach to the Furstenberg boundary when we consider Powers averaging property in Section 7.8.

7.4.1 Groupoid actions on states and probability measures

Let $\mathcal{G}$ be an étale groupoid with compact Hausdorff unit space and let A be a unital $\mathcal{G}$-C*-algebra. Let $\iota_* : \mathcal{S}(A) \to \mathcal{P}(\mathcal{G}^{(0)})$ denote the restriction map, where $\mathcal{S}(A)$ denotes the state space of A equipped with the weak* topology and $\mathcal{P}(\mathcal{G}^{(0)})$ denotes the space of probability measures on $\mathcal{G}^{(0)}$ equipped with the weak* topology. Although the $\mathcal{G}$-C*-algebra structure on A does not necessarily induce a $\mathcal{G}$-space structure on $\mathcal{S}(A)$, it does induce a $\mathcal{G}$-space structure on a canonical closed subspace of $\mathcal{S}(A)$.

Definition 7.4.1. Let $\mathcal{G}$ be an étale groupoid with compact Hausdorff unit space. For a unital $\mathcal{G}$-C*-algebra A, we denote by $\mathcal{S}_{\mathcal{G}^{(0)}}(A) \subseteq \mathcal{S}(A)$ the closed subspace defined by

$$\mathcal{S}_{\mathcal{G}^{(0)}}(A) = \{\varphi \in \mathcal{S}(A) \mid \iota_*\varphi = \delta_x \text{ for some } x \in \mathcal{G}^{(0)}\}.$$

The next lemma shows how the $\mathcal{G}$-C*-algebras structure on A induces a $\mathcal{G}$-space structure on $\mathcal{S}_{\mathcal{G}^{(0)}}(A)$.

Lemma 7.4.2. *Let $\mathcal{G}$ be an étale groupoid with compact Hausdorff space of units. Let (A, α) be a unital $\mathcal{G}$-C*-algebra, let $\varphi \in \mathcal{S}(A)$ be such that $\varphi|_{C(\mathcal{G}^{(0)})} = \mathrm{ev}_x$ for some $x \in \mathcal{G}^{(0)}$ and let $g \in \mathcal{G}_x$. Let γ be an open bisection of $\mathcal{G}$ containing g and let $f \in C_c(\operatorname{im}\gamma)$ be a positive function satisfying $f(\mathrm{r}(g)) = 1$. The formula*

$$g\varphi(a) = \varphi(\alpha_{\gamma^*}(faf))$$

defines a state on A satisfying $(g\varphi)|_{C(\mathcal{G}^{(0)})} = \mathrm{ev}_{\mathrm{r}(g)}$. It does not depend on the choice of γ and f.

Proof. Since the intersection of any pair of open bisections containing g is another open bisection containing g, it suffices to observe that for any pair of functions $f, h \in C_c(\operatorname{im}\gamma)$ satisfying $0 \leq f, h \leq 1$ and $f(\mathrm{r}(g)) = h(\mathrm{r}(g)) = 1$, the equality

$$\varphi(\alpha_{\gamma^*}(hfafh)) = \mathrm{ev}_x(\alpha_{\gamma^*}(h))^2\varphi(\alpha_{\gamma^*}(faf)) = h(\mathrm{r}(g))^2\varphi(\alpha_{\gamma^*}(faf)) = \varphi(\alpha_{\gamma^*}(faf))$$

holds. It is clear that $(g\varphi)|_{C(\mathcal{G}^{(0)})} = \mathrm{ev}_{\mathrm{r}(g)}$, showing in particular that $g\varphi$ is a state. ∎

Let us consider the special case of commutative $\mathcal{G}$-C*-algebras separately. We observe that there is a correspondence between commutative unital $\mathcal{G}$-C*-algebras $C(X)$ and $\mathcal{G}$-spaces $p : X \to \mathcal{G}^{(0)}$. This leads to the following reformulation of Definition 7.4.1.

Definition 7.4.3. Let $\mathcal{G}$ be an étale groupoid with compact Hausdorff unit space. For a compact $\mathcal{G}$-space $p : Z \to \mathcal{G}^{(0)}$, we define

$$\mathcal{P}_{\mathcal{G}^{(0)}}(Z) = \{\nu \in \mathcal{P}(Z) \mid p_*\nu = \delta_x \text{ for some } x \in \mathcal{G}^{(0)}\}\,.$$

Remark 7.4.4. It is clear that $\mathcal{P}_{\mathcal{G}^{(0)}}(Z)$ is closed in the relative weak* topology, because $\mathcal{G}^{(0)} \subseteq \mathcal{P}(\mathcal{G}^{(0)})$ is weak-*-closed. The $\mathcal{G}$-space structure on Z induces a canonical $\mathcal{G}$-space structure on $\mathcal{P}_{\mathcal{G}^{(0)}}(Z)$. Namely, for $\nu \in \mathcal{P}_{\mathcal{G}^{(0)}}(Z)$ and $g \in \mathcal{G}$ with $\delta_{\mathrm{s}(g)} = p_*\nu$, we observe that $\operatorname{supp}(\nu) \subseteq p^{-1}(p_*\nu) \subseteq Z$ and we can define the probability measure $g\nu \in \mathcal{P}(Z)$ by

$$\int_Z f \,\mathrm{d}(g\nu) = \int_{p^{-1}(\mathrm{s}(g))} f(gz)\,\mathrm{d}\nu(z) \qquad \text{for } f \in C(Z)\,.$$

This is the dual of the action of $\mathcal{G}$ on $\mathcal{S}_{\mathcal{G}^{(0)}}(C(Z))$ described by Lemma 7.4.2.

7.4.2 The Hamana boundary

Let $\mathcal{G}$ be an étale groupoid with compact Hausdorff unit space. In this section we will prove the existence of injective envelopes in the category of unital $\mathcal{G}$-C*-algebras, as defined in Definition 7.4.6. We will begin by establishing the injectivity of the unital $\mathcal{G}$-C*-algebra $\ell^\infty(\mathcal{G})$.

We first observe that $\ell^\infty(\mathcal{G})$ is a $\mathcal{G}$-C*-algebra. Since $\mathcal{G}^{(0)}$ is compact, the canonical embedding of $C(\mathcal{G}^{(0)})$ into $\ell^\infty(\mathcal{G})$ given by $f \mapsto f \circ \mathrm{r}$ is a unital *-homomorphism. Recall from Notation 7.3.2 that for an open subset $U \subseteq \mathcal{G}^{(0)}$, we write $\ell^\infty(\mathcal{G})_U = \overline{C_0(U)\ell^\infty(\mathcal{G})}^{\|\cdot\|}$ for the hereditary C*-subalgebra associated with U. Observing that $C(\mathcal{G}^{(0)}) \subseteq \ell^\infty(\mathcal{G})$ is unital, it is straightforward to see that

$$\ell^\infty(\mathcal{G})_U \subseteq \{f \in \ell^\infty(\mathcal{G}) \mid \operatorname{supp}(f) \subseteq \mathrm{r}^{-1}(U)\} = \overline{\ell^\infty(\mathcal{G})_U}^{\mathrm{w}^*}\,.$$

For an open bisection $\gamma \in \Gamma(\mathcal{G})$, the partial homeomorphism of γ on $\mathcal{G}$ defines a *-isomorphism $\alpha_\gamma : \ell^\infty(\mathcal{G})_{\mathrm{supp}(\gamma)} \to \ell^\infty(\mathcal{G})_{\mathrm{im}(\gamma)}$ satisfying

$$\alpha_\gamma(f)(g) = \begin{cases} f(\gamma^* g) & \text{if } r(g) \in \mathrm{im}(\gamma) \\ 0 & \text{otherwise.} \end{cases}$$

The next result characterises $\mathcal{G}$-C*-algebra morphisms into $\ell^\infty(\mathcal{G})$.

Proposition 7.4.5. *Let $\mathcal{G}$ be an étale groupoid with compact Hausdorff unit space. Let (A, α) be a unital $\mathcal{G}$-C*-algebra. There is a bijective correspondence between the following objects:*

1. *$\mathcal{G}$-ucp maps $\phi : A \to \ell^\infty(\mathcal{G})$.*
2. *Unital completely positive maps $\psi : A \to \ell^\infty(\mathcal{G}^{(0)})$ satisfying $\psi|_{C(\mathcal{G}^{(0)})} = \mathrm{id}_{C(\mathcal{G}^{(0)})}$.*
3. *Families of states $\{\mu_x\}_{x\in\mathcal{G}^{(0)}}$ on A satisfying $\mu_x|_{C(\mathcal{G}^{(0)})} = \delta_x$.*

For a map ϕ as in 1, the map ψ is defined by $\psi(f) = \phi(f)|_{\mathcal{G}^{(0)}}$. For a map ψ as in 2, the family $\{\mu_x\}_{x\in\mathcal{G}^{(0)}}$ is defined by setting $\mu_x = \delta_x \circ \psi$ for each $x \in \mathcal{G}^{(0)}$. For a family $\{\mu_x\}_{x\in\mathcal{G}^{(0)}}$ as in 3, the map ϕ is defined by $\phi(a)(g) = (g\mu_{s(g)})(a)$ for $a \in A$.

Proof. It is easy to verify the bijective correspondence between maps as in 2 and families of states as in 3. We will prove the bijective correspondence between maps as in 1 and families of states as in 3.

Denote by $(\beta_\gamma)_{\gamma\in\Gamma(\mathcal{G})}$ the $\mathcal{G}$-action on $\ell^\infty(\mathcal{G})$. Let $\{\mu_x\}_{x\in\mathcal{G}^{(0)}}$ be a family of states on A as in 3. Then defining $\phi : A \to \ell^\infty(\mathcal{G})$ by $\phi(a)(g) = (g\mu_{\mathrm{s}(g)})(a)$ for $a \in A$ yields a unital completely positive map $\phi : A \to \ell^\infty(\mathcal{G})$. Furthermore, ϕ is the identity on $C(\mathcal{G}^{(0)})$ since for $f \in C(\mathcal{G}^{(0)})$ and $g \in \mathcal{G}$,

$$\phi(f)(g) = (g\mu_{\mathrm{s}(g)})(f) = \delta_{\mathrm{r}(g)}(f) = (f \circ \mathrm{r})(g)\,.$$

In particular $\phi(\alpha_\gamma(a)) \in \ell^\infty(\mathcal{G})_{\mathrm{im}\,\gamma}$ for all $a \in A_{\mathrm{supp}\,\gamma}$. To see that ϕ is $\mathcal{G}$-equivariant, choose $\gamma \in \Gamma(\mathcal{G})$ and $a \in A_{\mathrm{supp}(\gamma)}$. We must show that $\phi(\alpha_\gamma(a))(g) = \beta_\gamma(\phi(a))(g)$ for all $g \in \mathrm{im}\,\gamma$.

Suppose $g \in \mathcal{G}$ satisfies $\mathrm{r}(g) \in \mathrm{im}\,\gamma$. Choose $\eta \in \Gamma(\mathcal{G})$ with $g \in \eta$. Then $\mathrm{r}(g) \in \mathrm{im}\,\gamma \cap \mathrm{im}\,\eta$, so in particular $\mathrm{im}\,\gamma \cap \mathrm{im}\,\eta \neq \emptyset$. Choose $f \in C_c(\mathrm{im}\,\gamma \cap \mathrm{im}\,\eta)$ such that

$f(\mathrm{r}(g)) = 1$. Then

$$\begin{aligned}\phi(\alpha_\gamma(a))(g) &= (g\mu_{\mathrm{s}(g)})(\alpha_\gamma(a))\\ &= \mu_{\mathrm{s}(g)}(\alpha_{\eta^*}(f\alpha_\gamma(a)f))\\ &= \mu_{\mathrm{s}(g)}(\alpha_{(\gamma^*\eta)^*}(\alpha_{\gamma^*}(f)a\alpha_{\gamma^*}(f)))\\ &= \mu_{\mathrm{s}(\gamma^* g)}(\alpha_{(\gamma^*\eta)^*}(\alpha_{\gamma^*}(f)a\alpha_{\gamma^*}(f)))\\ &= ((\gamma^* g)\mu_{\mathrm{s}(\gamma^* g)})(a)\\ &= \phi(a)(\gamma^* g)\\ &= \beta_\gamma(\phi(a))(g)\,.\end{aligned}$$

Hence ϕ is equivariant. Moreover, it is clear that ϕ satisfies $\delta_x \circ \phi = \mu_x$ for all $x \in \mathcal{G}^{(0)}$. Therefore, the map from families of states as in 3 to $\mathcal{G}$-ucp maps as in 1 is injective.

It remains to show that the map is surjective. For this, let $\phi : A \to \ell^\infty(\mathcal{G})$ be a $\mathcal{G}$-ucp map as in 1. Define a family of states $\{\mu_x\}_{x\in\mathcal{G}^{(0)}}$ on A by $\mu_x = \delta_x \circ \phi$ and let $\phi' : A \to \ell^\infty(\mathcal{G})$ be the corresponding $\mathcal{G}$-ucp map constructed as above. For $g \in \mathcal{G}$, let $\gamma \in \Gamma(\mathcal{G})$ be an open bisection with $g \in \gamma$ and let $f \in C_c(\operatorname{im}\gamma)$ be a function satisfying $f(\mathrm{r}(g)) = 1$. Then by the equivariance of ϕ, we have

$$\begin{aligned}\phi'(a)(g) &= (g\mu_{\mathrm{s}(g)})(a)\\ &= \mu_{\mathrm{s}(g)}(\alpha_{\gamma^*}(faf))\\ &= (\delta_{\mathrm{s}(g)} \circ \phi)(\alpha_{\gamma^*}(faf))\\ &= \delta_{\mathrm{s}(g)}(\beta_{\gamma^*} \circ \phi(faf))\\ &= \phi(faf)(\gamma \mathrm{s}(g))\\ &= \phi(faf)(g).\end{aligned}$$

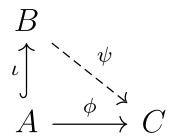
Now using the fact that ϕ is a $C(\mathcal{G}^{(0)})$-bimodule map gives

$$\phi(faf)(g) = f(\mathrm{r}(g))\phi(a)(g)f(\mathrm{r}(g)) = \phi(a)(g)\,.$$

Hence $\phi'(a)(g) = \phi(a)(g)$ for all $g \in \mathcal{G}$, and we conclude that $\phi' = \phi$. It follows that the map from families of states as in 3 to $\mathcal{G}$-ucp maps as in 1 is surjective. ■

We now define injectivity in the category of unital $\mathcal{G}$-C*-algebras. For unital $\mathcal{G}$-C*-algebras A and B, recall that a $\mathcal{G}$-ucp map $\phi : A \to B$ is an embedding if it is a complete order embedding and note that if either of A or B is commutative, then ϕ is an embedding if and only if it is isometric.

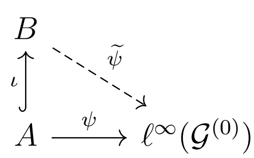
Definition 7.4.6. Let $\mathcal{G}$ be an étale groupoid with compact Hausdorff unit space. We say that a unital $\mathcal{G}$-C*-algebra C is injective in the category of unital $\mathcal{G}$-C*-algebras if whenever A and B are unital $\mathcal{G}$-C*-algebras with an embedding $\iota : A \to B$ and a $\mathcal{G}$-ucp map $\phi : A \to C$, there is a $\mathcal{G}$-ucp map $\psi : B \to C$ extending ϕ, that is the following diagram commutes.



With the correspondence from Proposition 7.4.5, we are now able to prove the injectivity of $\ell^\infty(\mathcal{G})$ in the category of unital $\mathcal{G}$-C*-algebras.

Proposition 7.4.7. *Let $\mathcal{G}$ be an étale groupoid with compact Hausdorff unit space. The C*-algebra $\ell^\infty(\mathcal{G})$ is injective in the category of unital $\mathcal{G}$-C*-algebras.*

Proof. Let $\iota : A \to B$ be a unital $\mathcal{G}$-C*-algebra embedding and let $\phi : A \to \ell^\infty(\mathcal{G})$ be a $\mathcal{G}$-ucp map. Let $\psi : A \to \ell^\infty(\mathcal{G}^{(0)})$ be the corresponding unital completely positive map satisfying $\psi|_{C(\mathcal{G}^{(0)})} = \mathrm{id}_{C(\mathcal{G}^{(0)})}$ as in Proposition 7.4.5. Since $\ell^\infty(\mathcal{G}^{(0)})$ is a commutative von Neumann algebra, it is injective in the category of operator systems. Hence there is a unital positive map $\tilde{\psi} : B \to \ell^\infty(\mathcal{G}^{(0)})$ extending ψ, that is the following diagram commutes.



Applying Proposition 7.4.5 again, we obtain a $\mathcal{G}$-ucp map $\tilde{\phi} : B \to \ell^\infty(\mathcal{G})$. For $a \in A$, we have $\tilde{\phi}(a)|_{\mathcal{G}^{(0)}} = \tilde{\psi}(a) = \psi(a) = \phi(a)|_{\mathcal{G}^{(0)}}$, so it follows from the correspondence in Proposition 7.4.5 that $\tilde{\phi}|_A = \phi$. We conclude that $\ell^\infty(\mathcal{G})$ is injective. ■

Now that we have established the existence of a von Neumann algebra in the category of unital $\mathcal{G}$-C*-algebra, we will be able to establish the existence an injective envelope.

Definition 7.4.8. Let $\mathcal{G}$ be an étale groupoid with compact Hausdorff unit space and let $A \subseteq B$ be a $\mathcal{G}$-ucp embedding of unital $\mathcal{G}$-C*-algebras.

1. We say that B is a rigid extension of A if any $\mathcal{G}$-ucp map $\phi : B \to B$ with $\phi|_A = \mathrm{id}_A$ satisfies $\phi = \mathrm{id}_B$.

2. We say that B is an essential extension of A if whenever C is a unital $\mathcal{G}$-C*-algebra and $\phi : B \to C$ is a $\mathcal{G}$-ucp map such that the restriction $\phi|_A$ is an embedding, then ϕ is an embedding.

3. A $\mathcal{G}$-injective envelope of A is an injective unital $\mathcal{G}$-C*-algebra C with an essential $\mathcal{G}$-ucp embedding $A \hookrightarrow C$.

The next result establishes the existence and uniqueness of the injective envelope of $C(\mathcal{G}^{(0)})$ in the category of unital $\mathcal{G}$-C*-algebras, along with its rigidity property. The first part of the proof closely follows Sinclair's proof from [Sin15] of the existence of the injective envelope of an operator system using the existence of minimal idempotents in compact right topological semigroups. By utilising the convexity of the semigroup under consideration, we then deduce rigidity and essentiality. The proof will make use of basic facts about compact right topological semigroups as presented for example in [HS12].

Theorem 7.4.9. *Let $\mathcal{G}$ be an étale groupoid with compact Hausdorff unit space. The $\mathcal{G}$-C*-algebra $C(\mathcal{G}^{(0)})$ admits an injective envelope in the category of unital $\mathcal{G}$-C*-algebras. It is a rigid extension of $C(\mathcal{G}^{(0)})$. It is also a commutative C*-algebra and unique up to *-isomorphism.*

Proof. By Proposition 7.4.7, the commutative von Neumann algebra $\ell^\infty(\mathcal{G})$ is injective in the category of unital $\mathcal{G}$-C*-algebras. Let S denote the set of $\mathcal{G}$-C*-ucp maps $\phi : \ell^\infty(\mathcal{G}) \to \ell^\infty(\mathcal{G})$ satisfying $\phi|_{C(\mathcal{G}^{(0)})} = \mathrm{id}_{C(\mathcal{G}^{(0)})}$, equipped with the relative point-weak* topology. Then S is a compact convex right topological semigroup under composition, meaning that for fixed $\psi \in S$ and a net (ϕ_i) in S converging to $\phi \in S$, we have $\lim \phi_i \circ \psi = \phi \circ \psi$.

Since S is a compact right topological semigroup, it contains a minimal left ideal $L \subseteq S$. Note that L is necessarily closed. We claim that L is also a left zero semigroup, in the sense that $\phi \circ \psi = \phi$ for all $\phi, \psi \in L$. To see this, fix $\psi \in L$ and observe that $S\psi$ is a left ideal of S contained in L, so the minimality of L implies $L = S\psi$. In particular, since S is convex, this implies that L is convex. The map $L \to L : \phi \mapsto \phi \circ \psi$ is thus a continuous affine map, so it admits a fixed point. Hence $L_0 = \{\phi \in L : \phi \circ \psi = \phi\}$ is a left ideal of S contained in L. Applying the minimality of L again implies that $L_0 = L$. Hence L is a left zero semigroup. This implies that every element of L, and in particular ψ, is idempotent.

Let $A = \operatorname{im} \psi$. Since ψ is idempotent, A is a C*-algebra under the Choi-Effros product defined by $a \circ b = \psi(ab)$. Since $\ell^\infty(\mathcal{G})$ is commutative, A is commutative. Also, since $\psi|_{C(\mathcal{G}^{(0)})} = \mathrm{id}_{C(\mathcal{G}^{(0)})}$, it follows that $C(\mathcal{G}^{(0)})$ is a C*-subalgebra of A and belongs to the multiplicative domain of ψ. Thanks to $\mathcal{G}$-equivariance of ψ, the $\mathcal{G}$-C*-algebra structure on $\ell^\infty(\mathcal{G})$ restricts to a $\mathcal{G}$-C*-algebra structure on A. Furthermore, since A is the range of an idempotent $\mathcal{G}$-ucp map from the injective $\mathcal{G}$-C*-algebra $\ell^\infty(\mathcal{G})$, it follows that A is injective in the category of $\mathcal{G}$-C*-algebras.

Before proving that A is actually the injective envelope of $C(\mathcal{G}^{(0)})$, it will be convenient to first prove that it is a rigid extension of $C(\mathcal{G}^{(0)})$. To see this, let $\phi : A \to A$ be a $\mathcal{G}$-ucp map. Let $\iota : A \to \ell^\infty(\mathcal{G})$ denote the $\mathcal{G}$-embedding as a subset. Then $\iota \circ \phi \circ \psi \in S$ and even $\iota \circ \phi = \iota \circ \phi \circ \psi^2 \in L$. Since L is a left zero semigroup, $\psi = \psi \circ \iota \circ \phi \circ \psi$, which implies that $\phi = \mathrm{id}_A$. Hence A is a rigid extension of $C(\mathcal{G}^{(0)})$.

To see that A is an essential extension of $C(\mathcal{G}^{(0)})$, let C be a unital $\mathcal{G}$-C*-algebra and let $\phi : A \to C$ be a $\mathcal{G}$-ucp map. Note that $\phi|_{C(\mathcal{G}^{(0)})} = \mathrm{id}_{C(\mathcal{G}^{(0)})}$, so in particular $\phi|_{C(\mathcal{G}^{(0)})}$ is an embedding. By the injectivity of A, there is a $\mathcal{G}$-ucp map $\eta : C \to A$ satisfying $(\eta \circ \phi)|_{C(\mathcal{G}^{(0)})} = \mathrm{id}_{C(\mathcal{G}^{(0)})}$. Hence by the rigidity of A, we have $\eta \circ \phi = \mathrm{id}_A$, implying that ϕ is an embedding.

Finally, to see that A is unique, let D be an injective envelope of $C(\mathcal{G}^{(0)})$ in the category of unital $\mathcal{G}$-C*-algebras. By the injectivity of D, there is a $\mathcal{G}$-ucp map $\phi : A \to D$ such that $\phi|_{C(\mathcal{G}^{(0)})}$ is an embedding. So the essentiality of A implies that ϕ is an embedding. Symmetrically, we obtain an embedding $\psi : D \to A$. The composition $\psi \circ \phi$, must be the identity map, since A is rigid, which implies that ϕ is surjective. Hence ϕ is an isometric complete order isomorphism between C*-algebras, and therefore is a *-isomorphism. ■

We are now able to define the Hamana boundary of $\mathcal{G}$.

Definition 7.4.10. Let $\mathcal{G}$ be an étale groupoid with compact Hausdorff unit space. The Hamana boundary $\partial_H \mathcal{G}$ of $\mathcal{G}$ is the spectrum of the injective envelope of $C(\mathcal{G}^{(0)})$ in the category of unital $\mathcal{G}$-C*-algebras.

We observe that $C(\partial_H \mathcal{G})$ is also injective in the category of unital C*-algebras. In particular, this implies that $\partial_H \mathcal{G}$ is extremally disconnected. We will make use of this fact below.

7.4.3 The Furstenberg boundary

Let $\mathcal{G}$ be an étale groupoid with compact Hausdorff unit space. In this section we will construct the Furstenberg boundary of $\mathcal{G}$ by developing an analogue for groupoids of Furstenberg and Glasner's theory of topological dynamical boundaries for groups (see e.g. [Gla76]).

In the next definition, we make use of the notation $\mathcal{G} \cdot A = \{ga \mid \mathrm{s}(g) = p(a)\}$ for a subset $A \subseteq X$ of a $\mathcal{G}$-space $p : X \to \mathcal{G}^{(0)}$.

Definition 7.4.11. Let $\mathcal{G}$ be an étale groupoid with compact Hausdorff space of units and let $p : Y \to \mathcal{G}^{(0)}$ be a compact $\mathcal{G}$-space.

- We will say that Y is irreducible if whenever $Z \subseteq Y$ is a closed $\mathcal{G}$-invariant subspace satisfying $p(Z) = \mathcal{G}^{(0)}$, then $Z = Y$.
- We will say that Y is strongly proximal if whenever $(\mu_x)_{x\in\mathcal{G}^{(0)}}$ is a family of probability measures on Y satisfying $p_*\mu_x = \delta_x$ for all $x \in \mathcal{G}^{(0)}$, then

$$\mathcal{G}^{(0)} \subseteq p(Y \cap \overline{\mathcal{G} \cdot \{\mu_x \mid x \in \mathcal{G}^{(0)}\}})\,.$$

- We will say that Y is a $\mathcal{G}$-boundary if it is both irreducible and strongly proximal.

Remark 7.4.12. We will use the fact that a compact $\mathcal{G}$-space $p : Y \to \mathcal{G}^{(0)}$ is a $\mathcal{G}$-boundary if and only if for every family $(\mu_x)_{x\in\mathcal{G}^{(0)}}$ of probability measures $\mu_x \in \mathcal{P}(Y)$ satisfying $p_*\mu_x = \delta_x$, we have $Y \subseteq \overline{\mathcal{G} \cdot \{\mu_x \mid x \in \mathcal{G}^{(0)}\}}$.

The next proposition provides a characterisation of strong proximality in the minimal setting that will be useful for the arguments in Section 7.8.

Proposition 7.4.13. *Let $\mathcal{G}$ be a minimal étale groupoid with compact Hausdorff unit space. Let $p : Y \to \mathcal{G}^{(0)}$ be an irreducible $\mathcal{G}$-space. Then Y is strongly proximal if and only if the following condition holds: for every $x \in \mathcal{G}^{(0)}$ and every probability measure $\mu \in \mathcal{P}(Y)$ satisfying $p_*\mu = \delta_x$, there is $y \in p^{-1}(x)$ and a net (g_i) in $\mathcal{G}_x$ such that $g_i\mu \to \delta_y$.*

Proof. If Y satisfies the condition of the proposition, then it is clear that Y is strongly proximal. For the converse, suppose that Y is strongly proximal and that $\mu \in \mathcal{P}(Y)$ satisfies $p(\mu) = \delta_{x_0}$ for some $x_0 \in \mathcal{G}^{(0)}$. By the minimality of $\mathcal{G}$,

$$p_*(\overline{\mathcal{G} \cdot \mu}) = \overline{\mathcal{G} \cdot p_*\mu} = \mathcal{G}^{(0)}\,.$$

So we can find a family $(\mu_x)_{x\in\mathcal{G}^{(0)}}$ in $\overline{\mathcal{G} \cdot \mu}$ satisfying $p(\mu_x) = \delta_x$ for all $x \in \mathcal{G}^{(0)}$. From the strong proximality of Y, we infer that

$$p(Y \cap \overline{\mathcal{G} \cdot \mu}) \supseteq p(Y \cap \overline{\mathcal{G} \cdot \{\mu_x \mid x \in \mathcal{G}^{(0)}\}}) = \mathcal{G}^{(0)}\,,$$

which finishes the proof of the proposition. ■

In Furstenberg and Glasner's theory of topological dynamical boundaries for groups, the affine flow of probability measures on a compact flow plays an important role. In the present setting, a complication arises from the fact that a $\mathcal{G}$-flow Z does not necessarily induce a $\mathcal{G}$-flow structure on the entire space $\mathcal{P}(Z)$ of probability measures on Z. Instead, it is necessary to work with the subspace $\mathcal{P}_{\mathcal{G}^{(0)}}(Z) \subseteq \mathcal{P}(Z)$ introduced in Section 7.4.1.

Proposition 7.4.14. *Let $\mathcal{G}$ be an étale groupoid with compact Hausdorff unit space. Let $p_Z : Z \to \mathcal{G}^{(0)}$ be a $\mathcal{G}$-boundary and let $p_Y : Y \to \mathcal{G}^{(0)}$ be any compact $\mathcal{G}$-space.*

1. *The image of every $\mathcal{G}$-map $Y \to \mathcal{P}_{\mathcal{G}^{(0)}}(Z)$ contains Z.*
2. *If Y is irreducible, then every $\mathcal{G}$-map $Y \to \mathcal{P}_{\mathcal{G}^{(0)}}(Z)$ maps onto Z, and there is at most one $\mathcal{G}$-map $Y \to Z$.*

Proof. Let $\phi : Y \to \mathcal{P}_{\mathcal{G}^{(0)}}(Z)$ be a $\mathcal{G}$-map. Then $(p_X)_*\phi(y) = \delta_{p_Y(y)}$ holds for all $y \in Y$. Since Z is a $\mathcal{G}$-boundary, we have $Z \subseteq \overline{\mathcal{G} \cdot \phi(Y)}$ by Remark 7.4.12. Because ϕ is $\mathcal{G}$-equivariant and Y is compact, we infer that $Z \subseteq \phi(Y)$.

Now assume in addition that that Y is irreducible. The subset $\phi^{-1}(Z) \subseteq Y$ satisfies $p_Y(\phi^{-1}(Z)) = \mathcal{G}^{(0)}$, so by the irreducibility of Y, we have $\phi^{-1}(Z) = Y$. Combined with the previous paragraph, this implies that $\phi(Y) = Z$. If $\psi : Y \to Z$ is another $\mathcal{G}$-map, then $\frac{1}{2}(\phi + \psi) : Y \to \mathcal{P}_{\mathcal{G}^{(0)}}(Z)$ is also a $\mathcal{G}$-map that, from above, must take values in Z. It follows that $\phi = \psi$. ■

The next theorem establishes the existence of a Furstenberg boundary in analogy with the classical argument for groups.

Theorem 7.4.15. *Let $\mathcal{G}$ be an étale groupoid with compact Hausdorff space of units. There is a $\mathcal{G}$-boundary $\partial_F \mathcal{G}$ that is universal in the sense that for every $\mathcal{G}$-boundary Y there is a (necessarily surjective) $\mathcal{G}$-map $\partial_F \mathcal{G} \to Y$. Furthermore, $\partial_F \mathcal{G}$ is the unique $\mathcal{G}$-boundary with this property up to isomorphism of $\mathcal{G}$-spaces.*

Proof. We will prove the existence of $\partial_F \mathcal{G}$, whereupon uniqueness will follow from Proposition 7.4.14. First observe that by irreducibility, the density character of every $\mathcal{G}$-boundary does not exceed $|\mathcal{G}|$. So there is a family of representatives for isomorphism classes of $\mathcal{G}$-boundaries $q_i : Y_i \to \mathcal{G}^{(0)}$, $i \in I$. Consider the fibre product

$$Y = \prod_I (Y_i, q_i) = \varprojlim_{i_1, \dots, i_n \in I} Y_{i_1} \times_{\mathcal{G}^{(0)}} Y_{i_2} \times_{\mathcal{G}^{(0)}} \cdots \times_{\mathcal{G}^{(0)}} Y_{i_n}$$

and note that Y is compact, being the projective limit of compact Hausdorff spaces. Let $p : Y \to \mathcal{G}^{(0)}$ and $p_i : Y \to Y_i$ denote the natural projections.

We now show that Y is strongly proximal. Let $(\mu_x)_{x \in \mathcal{G}^{(0)}}$ be a family of probability measures on Y satisfying $p_* \mu_x = \delta_x$ for all $x \in \mathcal{G}^{(0)}$. For a finite subset $F \subseteq I$, write $Y_F = \prod_F (Y_i, q_i)$ and denote by $p_F : Y \to Y_F$ and $q_F : Y_F \to \mathcal{G}^{(0)}$ the natural projections. We will show by induction on the size of F that

$$\mathcal{G}^{(0)} \subseteq q_F(Y_F \cap \overline{\mathcal{G} \cdot (p_F)_* \{\mu_x \mid x \in \mathcal{G}^{(0)}\}}) .$$

The case $|F| = 1$ follows from the assumed strong proximality for all Y_i. Fix a finite set $F \subseteq I$ and assume that the statement is proven for all strictly smaller sets than F. Let $i \in F$. Then by induction hypothesis, we have

$$\mathcal{G}^{(0)} \subseteq q_{F \setminus \{i\}}(Y_{F \setminus \{i\}} \cap \overline{\mathcal{G} \cdot (p_{F \setminus \{i\}})_* \{\mu_x \mid x \in \mathcal{G}^{(0)}\}}) .$$

By the compactness of $\mathcal{P}(Y)$ this means that for every $x \in \mathcal{G}^{(0)}$ there is $\nu_x \in \overline{\mathcal{G} \cdot \{\mu_x \mid x \in \mathcal{G}^{(0)}\}}$ such that $(p_{F \setminus \{i\}})_* \nu_x \in Y_{F \setminus \{i\}}$ and $p_* \nu_x = \delta_x$. By the strong proximality of Y_i, we find that

$$\mathcal{G}^{(0)} \subseteq q_i(Y_i \cap \overline{\mathcal{G} \cdot (p_i)_* \{\nu_x \mid x \in \mathcal{G}^{(0)}\}}) .$$

Choose probability measures σ_x, $x \in \mathcal{G}^{(0)}$ in $\overline{\mathcal{G} \cdot \{\nu_x \mid x \in \mathcal{G}^{(0)}\}}$ such that $(p_i)_* \sigma_x \in Y_i$ and $p_* \sigma_x = \delta_x$. We observe that

$$(p_{F \setminus \{i\}})_* \sigma_x \in (p_{F \setminus \{i\}})_* (\overline{\mathcal{G} \cdot \{\nu_x \mid x \in \mathcal{G}^{(0)}\}}) \subseteq Y_{F \setminus \{i\}}$$

so that $(p_F)_* \sigma_x \in Y_F$ follows for all $x \in \mathcal{G}^{(0)}$. This finishes the induction.

In summary, we have found for every $x \in \mathcal{G}^{(0)}$ a net of probability measures $(\mu_{x,F})_{F \subseteq I \text{ finite}}$ in $\overline{\mathcal{G} \cdot \{\mu_x \mid x \in \mathcal{G}^{(0)}\}}$ such that $p_*(\mu_{x,F}) = \delta_x$ and $(p_F)_*(\mu_{x,F}) \in Y_F$ for all $x \in \mathcal{G}^{(0)}$ and all finite subsets $F \subseteq I$. By compactness, there are probability measures ν_x, $x \in \mathcal{G}^{(0)}$ on Y satisfying $p_*(\nu_x) = \delta_x$ and $(p_F)_*(\nu_x) \in Y_F$ for all $x \in \mathcal{G}^{(0)}$ and all finite subsets $F \subseteq I$. Since cylinder sets generate the Σ-algebra of Y, this implies that $\nu_x \in Y$ for all $x \in \mathcal{G}^{(0)}$, which finishes the proof of strong proximality.

Consider now the family of closed $\mathcal{G}$-invariant subsets $A \subseteq Y$ that satisfy $p(A) = \mathcal{G}^{(0)}$. This family is ordered by inclusion and by compactness satisfies the descending chain condition. Hence it contains a minimal element, which necessarily will be an irreducible $\mathcal{G}$-space. Since it inherits strong proximality from Y, this proves the existence of $\partial_F \mathcal{G}$. ■

We are now able to define the Furstenberg boundary of $\mathcal{G}$.

Definition 7.4.16. Let $\mathcal{G}$ be an étale groupoid with compact Hausdorff unit space. The Furstenberg boundary $\partial_F \mathcal{G}$ of $\mathcal{G}$ is the $\mathcal{G}$-boundary constructed in Theorem 7.4.15.

7.4.4 Equivalence of boundaries

In this section we will prove that the Hamana boundary and the Furstenberg boundary coincide.

Let us start by the following lemma, which leads to a characterisation of $\mathcal{G}$-boundaries in Proposition 7.4.18. Its proof is based on Milman's partial converse to the Krein-Milman theorem, which states that if Y is a closed subset of a compact convex set K with the property that the closed convex hull of Y is equal to K, then Y contains all of the extreme points of K.

Lemma 7.4.17. *Let $\mathcal{G}$ be an étale groupoid with compact Hausdorff unit space. Let Y be a compact Hausdorff space identified with the closed subset of Dirac measures in $\mathcal{P}(Y)$. Let $(\mu_i)_{i\in I}$ be a family of probability measures on Y. Then $Y \subseteq \overline{\{\mu_i\}_{i\in I}}^{\mathrm{w}^*}$ if and only if the map $\bigoplus_{i\in I}\mu_i : C(Y) \to \ell^\infty(I)$ is isometric.*

Proof. Suppose that the map $\bigoplus_{i\in I}\mu_i : C(Y) \to \ell^\infty(I)$ is isometric but that there is $y \in Y$ such that $y \notin \overline{\{\mu_i\}_{i\in I}}^{\mathrm{w}^*}$. Then letting $K \subseteq \mathcal{P}(Y)$ denote the closed convex hull of the set $\{\mu_i\}_{i\in I}$, we infer that K is a proper subset of $\mathcal{P}(Y)$. Hence by the Hahn-Banach separation theorem there is positive $f \in C(Y)$ and $\alpha \geq 0$ such that

$$\sup_{i\in I}\mu_i(f) \leq \alpha < f(y)\,,$$

contradicting the fact that the map $\bigoplus_{i\in I}\mu_i$ is isometric.

Conversely, suppose that $Y \subseteq \overline{\{\mu_i\}_{i\in I}}^{\mathrm{w}^*}$. The map $\bigoplus_{i\in I}\mu_i$ induces a continuous map $\phi : \mathcal{P}(\beta I) \to \mathcal{P}(Y)$ satisfying $\phi(\delta_i) = \mu_i$ for $i \in I$. Hence $\overline{\phi(\beta I)}^{\mathrm{w}^*} \supseteq Y$. Since $\phi(\mathcal{P}(\beta I))$ is compact and weak*-closed, the Krein-Milman theorem implies that $\phi(\mathcal{P}(\beta I)) = \mathcal{P}(Y)$, that is ϕ is surjective. It follows that the map $\bigoplus_{i\in I}\mu_i$ is isometric. ■

Recall the bijective correspondence between commutative unital $\mathcal{G}$-C*-algebras and compact $\mathcal{G}$-spaces from Section 7.4.1. The next result implies that this correspondence restricts to a bijective correspondence between essential commutative unital $\mathcal{G}$-C*-algebra extensions of $C(\mathcal{G}^{(0)})$ and $\mathcal{G}$-boundaries.

Proposition 7.4.18. *Let $\mathcal{G}$ be an étale groupoid with compact Hausdorff unit space. Let $C(Y)$ be a commutative unital $\mathcal{G}$-C*-algebra. Then $C(Y)$ is an essential extension of $C(\mathcal{G}^{(0)})$ if and only if Y is a $\mathcal{G}$-boundary.*

Proof. First recall from Remark 7.4.12 that Y is a $\mathcal{G}$-boundary if and only if for any family of probability measures $(\mu_x)_{x\in\mathcal{G}^{(0)}}$ on Y with the property that $\mu_x|_{C(\mathcal{G}^{(0)})} = \delta_x$ for each $x \in \mathcal{G}^{(0)}$, we have

$$Y \subseteq \overline{\{g\mu_{\mathrm{s}(g)} \mid g \in \mathcal{G}\}}^{\mathrm{w}^*}.$$

Suppose that Y is a $\mathcal{G}$-boundary. Let B be a $\mathcal{G}$-C*-algebra and let $\phi : C(Y) \to B$ be a $\mathcal{G}$-ucp map. We must show that ϕ is an embedding. Equivalently, since $C(Y)$ is commutative, we must show that ϕ is isometric. By Proposition 7.4.7, we know that $\ell^\infty(\mathcal{G})$ is an injective $\mathcal{G}$-C*-algebra. So there is a $\mathcal{G}$-ucp map $B \to \ell^\infty(\mathcal{G})$. It suffices to show that the composition $\psi \circ \phi$ is isometric. Hence without loss of generality, we can assume that $B = \ell^\infty(\mathcal{G})$. For $x \in \mathcal{G}^{(0)}$, let $\mu_x = \delta_x \circ \phi$. Then by Proposition 7.4.7, we have $\phi = \bigoplus_{g\in\mathcal{G}} g\mu_{\mathrm{s}(g)}$. It now follows from Lemma 7.4.17 and the characterisation of $\mathcal{G}$-boundaries from the beginning of the proof that ϕ is isometric.

Conversely, suppose that $C(Y)$ is an essential extension of $C(\mathcal{G}^{(0)})$. Let $(\mu_x)_{x\in\mathcal{G}^{(0)}}$ be a family of probability measures on Y satisfying $p_*\mu_x = \delta_x$ for all $x \in \mathcal{G}^{(0)}$. By Proposition 7.4.7, we obtain a $\mathcal{G}$-ucp map $\phi = \bigoplus_{g\in\mathcal{G}} g\mu_{\mathrm{s}(g)}$. By essentiality, ϕ is isometric. Hence by Lemma 7.4.17, we have $Y \subseteq \overline{\{g\mu_{\mathrm{s}(g)} \mid g \in \mathcal{G}\}}^{\mathrm{w}^*}$, so by the characterisation of $\mathcal{G}$-boundaries from Remark 7.4.12, Y is a $\mathcal{G}$-boundary. ∎

We now deduce the equality of the Hamana boundary $\partial_H\mathcal{G}$ and the Furstenberg boundary $\partial_F\mathcal{G}$.

Theorem 7.4.19. *Let $\mathcal{G}$ be an étale groupoid with compact Hausdorff unit space. The Hamana boundary $\partial_H\mathcal{G}$ and the Furstenberg boundary $\partial_F\mathcal{G}$ are isomorphic as $\mathcal{G}$-spaces.*

Proof. We will prove that $C(\partial_H\mathcal{G})$ and $C(\partial_F\mathcal{G})$ are isomorphic as $\mathcal{G}$-C*-algebras. The result will then follow from the discussion in Section 7.4.1.

Theorem 7.4.9 and Proposition 7.4.18 imply that $\partial_H\mathcal{G}$ is a $\mathcal{G}$-boundary. Hence by the universal property of $\partial_F\mathcal{G}$ there is a (necessarily surjective) $\mathcal{G}$-map $\pi : \partial_F\mathcal{G} \to \partial_H\mathcal{G}$. This map corresponds to a $\mathcal{G}$-ucp embedding $\phi : C(\partial_H\mathcal{G}) \to C(\partial_F\mathcal{G})$. By the injectivity of $C(\partial_H\mathcal{G})$, there is a $\mathcal{G}$-ucp map $\psi : C(\partial_F\mathcal{G}) \to C(\partial_H\mathcal{G})$ such that the following diagram commutes:

$$\begin{array}{ccc} C(\partial_F\mathcal{G}) & & \\ {\scriptstyle\phi}\uparrow & \overset{\psi}{\searrow} & \\ C(\partial_H\mathcal{G}) & \xrightarrow{\mathrm{id}} & C(\partial_H\mathcal{G}) \end{array}$$

Applying Proposition 7.4.18 again, $C(\partial_F \mathcal{G})$ is an essential extension of $C(\mathcal{G}^{(0)})$, so ψ must be an embedding, forcing all of the maps in the above diagram to be *-isomorphisms. In particular, ϕ is a *-isomorphism. ■

Finally, in the Hausdorff setting, we deduce the equality of the Furstenberg boundary as constructed in this section with the Furstenberg boundary constructed by Borys, which we denote by $\partial_B \mathcal{G}$. Recall that $C(\partial_B \mathcal{G})$ is the injective envelope of $C(\mathcal{G}^{(0)})$ in the category of concrete $\mathcal{G}$-operator systems in the terminology of [Bor19, Bor20]. In particular, it is the injective envelope of $C(\mathcal{G}^{(0)})$ in the category of $\mathcal{G}$-C*-bundles.

Theorem 7.4.20. *Suppose that $\mathcal{G}$ is an étale Hausdorff groupoid with compact space of units. Then the Furstenberg boundary $\partial_F \mathcal{G}$ and the boundary $\partial_B \mathcal{G}$ constructed by Borys are isomorphic as $\mathcal{G}$-spaces.*

Proof. We will prove that $C(\partial_F \mathcal{G})$ and $C(\partial_B \mathcal{G})$ are isomorphic as $\mathcal{G}$-C*-algebras. The result will then follow from the discussion in Section 7.4.1.

Since $C(\partial_F \mathcal{G})$ is a commutative unital $\mathcal{G}$-C*-algebra, Proposition 7.3.11 says that it has a unique compatible structure of a unital $\mathcal{G}$-C*-bundle. By Proposition 7.3.11 and injectivity of $C(\partial_B \mathcal{G})$ in this category, there is a $\mathcal{G}$-ucp map $\phi : C(\partial_F \mathcal{G}) \to C(\partial_B \mathcal{G})$. Since $C(\partial_B \mathcal{G})$ is a unital $\mathcal{G}$-C*-algebras, there is also a $\mathcal{G}$-ucp map $\psi : C(\partial_B \mathcal{G}) \to C(\partial_F \mathcal{G})$.

By the rigidity of $C(\partial_F \mathcal{G})$, we infer that $\psi \circ \phi$ is the identity map. Similarly, by the rigidity of $C(\partial_B \mathcal{G})$, we have that $\phi \circ \psi$ is the identity map. Hence ϕ and ψ are both *-isomorphisms. ■

7.5 Essential groupoid C*-algebras

In this section, we develop some understanding of essential groupoid C*-algebras, which will be necessary to adapt methods from the theory of C*-simplicity to non-Hausdorff groupoids. While there is always an inclusion $C^*_r(\mathcal{G}) \subseteq C^*_r(\mathcal{G} \ltimes \partial_F \mathcal{G})$, it is à priori not clear that there is a similar inclusion on the level of essential groupoid C*-algebras. In Section 7.5.2, we will show that this is the case for minimal groupoids and σ-compact groupoids. For the proof of this fact, we reformulate the local conditional expectation of Kwaśniewski-Meyer in terms of continuous extensions of functions on extremally disconnected spaces, such as the Furstenberg boundary $\partial_F \mathcal{G}$. This will be done in Section 7.5.1. Finally, the point of view developed here, naturally leads to the question whether the local conditional expectation of the Furstenberg groupoid is actually related to a reduced groupoid C*-algebra, which appears in the background. This is indeed the case, as we will show in Section 7.5.3, when introducing the Hausdorffification of a groupoid with extremally disconnected unit space. Also this concept will be useful in our further discussion on the intersection property in Sections 7.6 and 7.7.

7.5.1 The local conditional expectation for groupoid C*-algebras via continuous extensions on extremally disconnected spaces

In this section we describe an alternative point of view on the local conditional expectation introduced by of Kwaśniwski-Meyer [KM21], by means of continuous extensions of functions.

The next lemma is an adaption of [BKKO17, Lemma 3.2] (see also [Urs22, Lemma 3.3]) to the setting of groupoid dynamical systems that are not necessarily minimal.

Lemma 7.5.1. *Let $\mathcal{G}$ be an étale groupoid with compact Hausdorff space of units and assume that $\mathcal{G}$ is minimal or σ-compact. Let $p : X \to \mathcal{G}^{(0)}$ be a totally disconnected, irreducible, compact $\mathcal{G}$-space. Then*

- *for every open subset $U \subseteq X$ the projection $p(U) \subseteq \mathcal{G}^{(0)}$ has non-empty interior.*
- *for every dense subset $D \subseteq \mathcal{G}^{(0)}$, the inverse image $p^{-1}(D) \subseteq X$ is dense.*

Proof. Denote by $\pi : \mathcal{G} \ltimes X \to \mathcal{G}$ the natural extension of p. Let $U \subseteq X$ be an open subset. Denote by φ_γ the partial homeomorphism of X associated to an open bisection γ of $\mathcal{G} \ltimes X$ and ψ_γ the partial homeomorphism of $\mathcal{G}^{(0)}$ associated with an open bisection of $\mathcal{G}$.

If $\mathcal{G}$ is minimal, then $\mathcal{G} \curvearrowright X$ is minimal too. By compactness of X we find finitely many compact open bisections $(\gamma_n)_n$ of $\mathcal{G} \ltimes X$ such that $(\operatorname{im} \varphi_{\gamma_n})_n$ covers X and for every n, we have $\operatorname{supp} \varphi_{\gamma_n} \subseteq U$ and there is an open bisection β_n of $\mathcal{G}$ such that $\gamma_n \subseteq \pi^{-1}(\beta_n)$. Since each of the sets $p(\operatorname{im} \varphi_{\gamma_n})$ is closed and X is a Baire space, it follows that there is some n such that $p(\operatorname{im} \varphi_{\gamma_n})$ has non-empty interior. Then also $\psi_{\beta_n^*} \circ p(\operatorname{im} \varphi_{\gamma_n}) = p(\operatorname{supp} \varphi_{\gamma_n})$ has non-empty interior.

If $\mathcal{G}$ is σ-compact, observe that $\pi : \mathcal{G} \ltimes X \to \mathcal{G}$ is proper, so that $\mathcal{G} \ltimes X$ is σ-compact and as such covered by countably many compact open bisections $(\gamma_n)_{n \in \mathbb{N}}$, each contained in $\pi^{-1}(\beta_n)$ for a some open bisection β_n of $\mathcal{G}$. Consider the open subset $O = \bigcup_{n \in \mathbb{N}} \varphi_{\gamma_n}(U) \subseteq X$. Then $X \setminus O \subseteq X$ is a proper closed $\mathcal{G}$-invariant subset. Thus $p(X \setminus O) \subseteq \mathcal{G}^{(0)}$ is a proper subset by irreducibility of X. Further, since X is compact and $\mathcal{G}^{(0)}$ is Hausdorff,

$p(X \setminus O)$ is also closed. It follows that $p(O)$ has non-empty interior. Since it is the countable union of the closed subsets $(p(\varphi_{\gamma_n}(U)))_{n\in\mathbb{N}}$ and X is a Baire space, there is $n \in \mathbb{N}$ such that $p(\varphi_{\gamma_n}(U))$ has non-empty interior. As before, one concludes that $\psi_{\beta_n^*} \circ p(\varphi_{\gamma_n}(U)) = p(U \cap \operatorname{supp} \varphi_{\gamma_n})$ has non-empty interior.

Let now $D \subseteq \mathcal{G}^{(0)}$ be a dense subset. Given an open subset $U \subseteq X$, we know by the first part that $p(U)$ has non-empty interior and thus intersects D non-trivially. Thus $U \cap p^{-1}(D)$ is non-empty either, which proves density of $p^{-1}(D) \subseteq X$. ■

Remark 7.5.2. A topological space X is extremally disconnected if and only if for every open subset $U \subseteq X$ and every continuous function $f \in C_b(U)$ there is a continuous function $g \in C_b(X)$ satisfying $g|_U = f$. We refer to [GJ60, Exercise 1.H.6, p.23], which uses a suitable version of Urysohn's extension theorem (see [GJ60, 1.17, p.18]). One may choose g to be supported in the clopen subset $\overline{U} \subseteq X$.

Let us now describe our perspective on the local conditional expectation. We adopt the original perspective of the local multiplier algebra as explained in Section 7.2.1. Recall the terminology of [KM21, Section 3.1] that for an inclusion $A \subseteq B$ of C*-algebras, a generalised conditional expectation is given by another inclusion $A \subseteq C$ and a completely positive, contractive map $\mathrm{F} : B \to C$ that restricts to the identity on A. In the next proposition, the role of A is played by $C(\mathcal{G}^{(0)})$.

Proposition 7.5.3. *Let $\mathcal{G}$ be an étale groupoid with compact Hausdorff space of units and assume that $\mathcal{G}$ is minimal or σ-compact. Let $p : X \to \mathcal{G}^{(0)}$ be an extremally disconnected, irreducible, compact $\mathcal{G}$-space and denote by $\pi : \mathcal{G} \ltimes X \to \mathcal{G}$ its natural extension. Then there is a unique generalised conditional expectation $\mathrm{F} : C_r^*(\mathcal{G}) \to C(X)$ such that $\operatorname{supp}(\mathrm{F}(f)) \subseteq \overline{p^{-1}(U \cap \mathcal{G}^{(0)})}$ and*

$$\mathrm{F}(f)|_{p^{-1}(U \cap \mathcal{G}^{(0)})} = (f \circ \pi)|_{p^{-1}(U \cap \mathcal{G}^{(0)})}$$

for all open bisections $U \subseteq \mathcal{G}$ and all $f \in C_c(U)$. If $\mathrm{E}_{\mathrm{red}} : C_r^*(\mathcal{G}) \to \mathrm{M}_{\mathrm{loc}}(C(\mathcal{G}^{(0)}))$ denotes the local conditional expectation, then there is a unique $\mathcal{G}$-equivariant embedding $\mathrm{M}_{\mathrm{loc}}(C(\mathcal{G}^{(0)})) \to C(X)$ such that the following diagram commutes

$$\begin{array}{ccc} C_r^*(\mathcal{G}) & \xrightarrow{\ \mathrm{F}\ } & C(X) \\ & \searrow^{\mathrm{E}_{\mathrm{red}}} & \uparrow \\ & & \mathrm{M}_{\mathrm{loc}}(C(\mathcal{G}^{(0)})) \end{array}$$

Proof. Let $U \subseteq \mathcal{G}$ be an open bisection and $f \in C_c(U)$. Write $V = p^{-1}(U \cap \mathcal{G}^{(0)}) \subseteq X$ and denote by $g \in C(X)$ the unique continuous extension of $(f \circ \pi)|_V$ having support in $\overline{V}$. This defines a map $\mathrm{F} : \bigoplus_U C_c(U) \to C(X)$. Let $f = \sum_{i=1}^n f_i \in \mathcal{C}(\mathcal{G})$ for open bisections $U_i \subseteq \mathcal{G}$ and $f_i \in C_c(U_i)$. Write $V_i = p^{-1}(U_i \cap \mathcal{G}^{(0)})$ and observe that $|\sum_i \mathrm{F}(f_i)(x)| = |\sum_i f_i \circ \pi(x)| \leq \|\sum_i f_i\|_{C_r^*}$ for all $x \in X \setminus \bigcup_i \partial V_i$, where f_i is considered as a Borel function defined on all of $\mathcal{G}$ and having support in U_i. So continuity of each $\mathrm{F}(f_i)$ implies that

$$\begin{aligned}
\|\sum_i F(f_i)\| &= \sup\{|\sum_i F(f_i)(x)| \mid x \in X\} \\
&= \sup\{|\sum_i F(f_i)(x)| \mid x \in X \setminus \bigcup_i \partial V_i\} \\
&\leq \|f\|_{C_r^*}
\end{aligned}$$

So F factors through $\mathcal{C}(\mathcal{G})$ and extends to a well-defined map $C_r^*(\mathcal{G}) \to C(X)$.

Given a dense open subset $O \subseteq \mathcal{G}^{(0)}$ and $f \in C_b(O)$, Lemma 7.5.1 says that $p^{-1}(O) \subseteq X$ is dense, so that there is a unique continuous extension of $f \circ \pi$ to X, which defines an element in $C(X)$. We thus obtain a well-defined map $C_b(O) \to C(X)$. Varying O, these embeddings define an injective *-homomorphism $\iota : \mathrm{M}_{\mathrm{loc}}(C(\mathcal{G}^{(0)})) \to C(X)$.

Given an open bisection $U \subseteq \mathcal{G}$ and $f \in C_c(U)$, fix $O = \mathcal{G}^{(0)} \setminus \partial U$. Then $\mathrm{E}_{\mathrm{red}}(f) \in C_b(O) \subseteq \mathrm{M}_{\mathrm{loc}}(C(\mathcal{G}^{(0)}))$ by definition. Further, $\iota(\mathrm{E}_{\mathrm{red}}(f))(x) = \mathrm{F}(f)(x)$ for all $x \in X \setminus p^{-1}(\mathcal{G}^{(0)} \cap \partial U)$. By Lemma 7.5.1 the latter set is dense in X and we can infer $\iota(\mathrm{E}_{\mathrm{red}}(f)) = \mathrm{F}(f)$ from continuity. ∎

7.5.2 Inclusions of essential groupoid C*-algebras

Using the skyscraper groupoid as in [KS02, Example 2.5], it was observed in [KM21, Remark 4.8] that the essential groupoid construction is not functorial. The present section demonstrates that in the situations arising from the study of C*-simplicity, we do see a natural inclusion of essential groupoid C*-algebras arising from suitable surjections of groupoids. We make use of the Dixmier algebra picture of the local conditional expectation, as explained in Section 7.2.1.

Theorem 7.5.4. *Let $\mathcal{G}$ and $\mathcal{H}$ be étale groupoids with compact Hausdorff space of units and that $\mathcal{H}$ is minimal or σ-compact. Assume that $\pi : \mathcal{G} \to \mathcal{H}$ is a fibrewise bijective, proper surjection of groupoids. Then precomposition with π defines a *-homomorphism*

$\mathcal{C}(\mathcal{H}) \to \mathcal{C}(\mathcal{G})$ *giving rise to the following commuting diagram of inclusions.*

$$\begin{array}{ccccc} \mathcal{C}(\mathcal{H}) & \hookrightarrow & C^*_r(\mathcal{H}) & \hookrightarrow & C^*_{\mathrm{ess}}(\mathcal{H}) \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{C}(\mathcal{G}) & \hookrightarrow & C^*_r(\mathcal{G}) & \hookrightarrow & C^*_{\mathrm{ess}}(\mathcal{G}) \end{array}$$

Proof. Whenever $U \subseteq \mathcal{H}$ is an open bisection, then $\pi^{-1}(U) \subseteq \mathcal{G}$ is an open bisection, since π is fibrewise bijective. Since π is proper, precomposition with π defines a *-isomorphism $C_c(U) \cong C_c(\pi^{-1}(U))$. This proves that $\pi_* : \mathcal{C}(\mathcal{H}) \to \mathcal{C}(\mathcal{G})$ is well-defined. It is injective, since π is surjective.

For $x \in \mathcal{G}^{(0)}$, we have $\lambda^{\mathcal{G}}_x \circ \pi_* = \lambda^{\mathcal{H}}_{\pi(x)}$ as a straightforward calculation on the subspaces $C_c(U) \subseteq \mathcal{C}(\mathcal{H})$ for open bisections $U \subseteq \mathcal{H}$ shows. It follows that π_* extends continuously to an embedding $C^*_r(\mathcal{H}) \hookrightarrow C^*_r(\mathcal{G})$.

Write now $X = \mathrm{spec}(\mathrm{Dix}(\mathcal{G}^{(0)}))$ and observe that X is an extremally disconnected $\mathcal{G}$-space. Denote its unit space projection by $p : X \to \mathcal{G}^{(0)}$. Since π is fibrewise bijective, X also becomes an $\mathcal{H}$-space. For an element $h \in \mathcal{H}$ and an element $x \in X$ satisfying $\pi \circ p(x) = \mathrm{s}(h)$, there is a unique $g \in \pi^{-1}(h)$ satisfying $\mathrm{s}(g) = p(x)$. Then $hx = gx$ holds by definition. Denote by $\mathrm{E}^{\mathcal{G}}_{\mathrm{red}} : C^*_r(\mathcal{G}) \to C(X)$ the local conditional expectation. Then

$$\mathrm{E}^{\mathcal{G}}_{\mathrm{red}} \circ \pi_* : C^*_r(\mathcal{H}) \to C(X)$$

falls in the scope of Proposition 7.5.3. Hence we find that $\mathrm{E}^{\mathcal{G}}_{\mathrm{red}} \circ \pi_* = \iota \circ \mathrm{E}^{\mathcal{H}}_{\mathrm{red}}$, where we denote by $\iota : \mathrm{Dix}(\mathcal{H}^{(0)}) \to C(X)$ the natural inclusion and by $\mathrm{E}^{\mathcal{H}}_{\mathrm{red}} : C^*_r(\mathcal{H}) \to \mathrm{Dix}(\mathcal{H}^{(0)})$ the local conditional expectation of $\mathcal{H}$.

$$\begin{aligned} \ker(C^*_r(\mathcal{H}) \to C^*_{\mathrm{ess}}(\mathcal{H})) &= \{a \in C^*_r(\mathcal{H}) \mid \mathrm{E}^{\mathcal{H}}_{\mathrm{red}}(a^*a) = 0\} \\ &= \{a \in C^*_r(\mathcal{H}) \mid \mathrm{E}^{\mathcal{G}}_{\mathrm{red}} \circ \pi_*(a^*a) = 0\} \\ &= \ker(C^*_r(\mathcal{H}) \to C^*_{\mathrm{ess}}(\mathcal{G}))\,. \end{aligned}$$

■

Remark 7.5.5. Our proof of Theorem 7.5.4 makes direct use of the definition of the essential groupoid. It should be remarked that Lemma 7.5.1 applied to the map $\mathcal{G} \ltimes X \to \mathcal{G}$ also implies that preimages of meager sets remain meager. So functions with meager support are pulled back to functions with meager support. Recall from Section 7.2.1 that for $a \in C^*_r(\mathcal{G})$, one defines a function on $\mathcal{G}$ by $\hat{a}(g) = \langle \lambda_{\mathrm{s}(g)}(a)\delta_{\mathrm{s}(g)}, \delta_g\rangle$. It follows from [KM21, Proposition 7.18] and its proof that for étale groupoids $\mathcal{G}$, the kernel of $C^*_r(\mathcal{G}) \to C^*_{\mathrm{ess}}(\mathcal{G})$ consists exactly of elements $a \in C^*_r(\mathcal{G})$ such that $\widehat{a^*a}$ has meager support. This provides an alternative approach to Theorem 7.5.4 for σ-compact groupoids and minimal groupoids.

Remark 7.5.6. Despite a long list of characterisations obtained in [KM21, Proposition 7.18] an elementary description of singular elements has not yet been obtained. We observe the following restriction on such elements, strengthening [KM21, Lemma 7.15]. It follows from Theorem 7.5.4 that an element $a \in C^*_r(\mathcal{G})$ is singular if and only if it is singular in $C^*_r(\mathcal{G} \ltimes \partial_F \mathcal{G})$. Let $D = \mathrm{s}(\overline{\mathcal{G}^{(0)}} \setminus \mathcal{G}^{(0)})$ and $D_\partial = \mathrm{s}(\overline{\partial_F \mathcal{G}} \setminus \partial_F \mathcal{G})$. Then $\hat{a} \circ \pi$ is supported on $\mathrm{s}^{-1}(D_\partial)$ by [KM21, Proposition 7.18], so that $\hat{a}$ is supported in

$$\mathrm{s}^{-1}(D) \cap \{g \in \mathcal{G} \mid \pi^{-1}(g) \subset \mathrm{s}^{-1}(D_\partial)\} = \mathrm{s}^{-1}(D) \setminus \pi(\mathcal{G} \setminus \mathrm{s}^{-1}(D_\partial)) \,.$$

In the terminology of [KM21], the preimage of $\mathrm{s}(g)$ under π must consists entirely of dangerous points.

Remark 7.5.7. We currently do not know whether the assumption of minimality or σ-compactness is necessary in Theorem 7.5.4. It is a natural problem to either prove a result or provide a counterexample in this generality.

continue here

7.5.3 Hausdorffification of groupoids with extremally disconnected space of units

In this section we describe the essential groupoid C*-algebra of a non-Hausdorff groupoid with an extremally disconnected space of units as the reduced groupoid C*-algebra of a suitable Hausdorffification. In doing so, the main object we have in mind is of course the Furstenberg groupoid $\mathcal{G} \ltimes \partial_F \mathcal{G}$.

Recall from Proposition 7.2.1 that the isotropy groupoid of an étale groupoid with an extremally disconnected space of units is clopen. The analogue statement holds true for the closure of its space of units, which will be the starting point of defining a Hausdorffification.

Lemma 7.5.8. *Let $\mathcal{G}$ be an étale groupoid whose space of units is an extremally disconnected, locally compact Hausdorff space. Then $\mathcal{G}$ is extremally disconnected. In particular, $\overline{\mathcal{G}^{(0)}} \subseteq \mathcal{G}$ is a clopen normal subgroupoid.*

Proof. Let $O \subseteq \mathcal{G}$ be an open subset and $g \in \overline{O}$. Let $U \subseteq \mathcal{G}$ be an open bisection containing g. Then $U \cap O$ contains a net converging to g, so that g lies in the relative closure $\overline{U \cap O}^U \subseteq U$. Since U is an open bisection and as such homeomorphic to an open subset of $\mathcal{G}^{(0)}$, it is extremally disconnected. So $\overline{U \cap O}^U \subseteq \overline{O}$ is an open neighbourhood of g. ■

Recall that quotients of étale groupoids by open normal subgroupoids remain étale. In the setting of Lemma 7.5.8, the extended unit space is even clopen, so that we obtain a Hausdorff quotient.

Definition 7.5.9. Let $\mathcal{G}$ be an étale groupoid whose space of units is an extremally disconnected, locally compact Hausdorff space. The *extended unit space* of $\mathcal{G}$ is $\overline{\mathcal{G}^{(0)}} \subseteq \mathcal{G}$. The *Hausdorffification* of $\mathcal{G}$ is $\mathcal{G}_{\mathrm{Haus}} = \mathcal{G}/\overline{\mathcal{G}^{(0)}}$.

Before we proceed to the next theorem, let us remark that if $\mathcal{G}$ is an étale groupoid with extremally disconnected compact Hausdorff space of units, then Proposition 7.5.3 can be applied to the trivial $\mathcal{G}$-space $\mathcal{G}^{(0)}$. It follows that there is a conditional expectation $C_r^*(\mathcal{G}) \to C(\mathcal{G}^{(0)})$.

Theorem 7.5.10. *Let $\mathcal{G}$ be an étale groupoid whose space of units is an extremally disconnected, compact Hausdorff space and whose extended unit space has amenable isotropy groups. Denote by $p : \mathcal{G} \to \mathcal{G}_{\mathrm{Haus}}$ the quotient map to its Hausdorffification.*

- *There is a unique *-isomorphism $\pi : C_{\mathrm{ess}}^*(\mathcal{G}) \to C_r^*(\mathcal{G}_{\mathrm{Haus}})$ which restricts to the natural *-isomorphism $p_* : C_c(U) \to C_c(p(U))$ for every open bisection $U \subseteq \mathcal{G}$.*
- *Denote by $\mathrm{E} : C_r^*(\mathcal{G}_{\mathrm{Haus}}) \to C(\mathcal{G}^{(0)})$ and $\mathrm{E}_{\mathrm{ess}} : C_{\mathrm{ess}}^*(\mathcal{G}) \to C(\mathcal{G}^{(0)})$ the natural conditional expectations. Then $\mathrm{E} \circ \pi = \mathrm{E}_{\mathrm{ess}}$.*

Proof. By assumption every isotropy group of $\overline{\mathcal{G}^{(0)}}$ is amenable, so that Proposition 7.2.6 implies that there is a *-homomorphism $\pi_{\mathrm{red}} : C_r^*(\mathcal{G}) \to C_r^*(\mathcal{G}_{\mathrm{Haus}})$ restricting to the natural *-isomorphism $p_* : C_c(U) \to C_c(p(U))$ for every open bisection $U \subseteq \mathcal{G}$. We will show that π_{red} factors through the essential groupoid C*-algebra of $\mathcal{G}$ to a *-homomorphism $\pi : C_{\mathrm{ess}}^*(\mathcal{G}) \to C_r^*(\mathcal{G}_{\mathrm{Haus}})$ satisfying $\mathrm{E} \circ \pi = \mathrm{E}_{\mathrm{ess}}$. Since E is faithful, it suffices to show that $\mathrm{E} \circ \pi_{\mathrm{red}}$ is the local conditional expectation of $C_r^*(\mathcal{G})$. To this end, we verify the conditions of Proposition 7.5.3 for $X = \mathcal{G}^{(0)}$. For every open bisection $U \subseteq \mathcal{G}$ and every $f \in C_c(U)$ the function satisfying

$$\mathrm{E} \circ \pi_{\mathrm{red}}(f)(x) = \sum_{g \in \overline{\mathcal{G}^{(0)}}_x} f(g)$$

is continuous, supported in $\overline{U \cap \mathcal{G}^{(0)}}$ and satisfies $\mathrm{E} \circ \pi_{\mathrm{red}}(f)|_{U \cap \mathcal{G}^{(0)}} = f|_{U \cap \mathcal{G}^{(0)}}$. So $\mathrm{E} \circ \pi_{\mathrm{red}}(f) = \mathrm{E}_{\mathrm{ess}}(f)$ follows. ∎

7.6 Fundamental characterisations of the ideal intersection property

In this section we establish characterisations of the ideal intersection property that will be of fundamental importance for further reasoning both about the existence of essentially confined amenable sections of the isotropy group in Section 7.7, and about Powers averaging property in Section 7.8.

The characterisations we will consider here come in three different flavours. First, the ideal intersection property of $C(\mathcal{G}^{(0)}) \subseteq C^*_{\mathrm{ess}}(\mathcal{G})$ is equivalent to the ideal intersection property of $C(\partial_F \mathcal{G}) \subseteq C^*_{\mathrm{ess}}(\mathcal{G} \ltimes \partial_F \mathcal{G})$, making it possible to exploit the extremal disconnectedness of the Furstenberg boundary. Second, on the level of $\mathcal{G} \ltimes \partial_F \mathcal{G}$, it is possible to express the ideal intersection property as a simple statement about principality, which must, however, take into account the possible non-Hausdorffness of the groupoid. This is expressed in the definition of essentially principal groupoids in Definition 7.6.3. Third, the ideal intersection property can be expressed in terms of a uniqueness statement for $\mathcal{G}$-pseudo expectations. It is this last characterisation that will be most frequently applied in further results. Recall that, following Pitts [Pit17], a $\mathcal{G}$-pseudo expectation on $C^*_{\mathrm{ess}}(\mathcal{G})$ is a $\mathcal{G}$-equivariant generalised conditional expectation with values in $C(\partial_F \mathcal{G})$.

Theorem 7.6.1. *Let $\mathcal{G}$ be an étale groupoid with compact Hausdorff space of units. Assume that $\mathcal{G}$ is Hausdorff, that $\mathcal{G}$ is minimal or that $\mathcal{G}$ is σ-compact. Then the following statements are equivalent.*

1. *$C(\mathcal{G}^{(0)}) \subseteq C^*_{\mathrm{ess}}(\mathcal{G})$ has the ideal intersection property.*
2. *$C(\partial_F \mathcal{G}) \subseteq C^*_{\mathrm{ess}}(\mathcal{G} \ltimes \partial_F \mathcal{G})$ has the ideal intersection property.*
3. $\mathrm{Iso}(\mathcal{G} \ltimes \partial_F \mathcal{G}) = \overline{\partial_F \mathcal{G}}^{\mathcal{G} \ltimes \partial_F \mathcal{G}}$.
4. *There is a unique $\mathcal{G}$-pseudo expectation $C^*_{\mathrm{ess}}(\mathcal{G}) \to C(\partial_F \mathcal{G})$.*

Remark 7.6.2. It is worth commenting on the necessity of the assumption of minimality or σ-compactness for non-Hausdorff groupoids in Theorem 7.6.1. Most C*-simplicity results for groups hold without any assumption on countability of the group, and this is also reflected in the Hausdorff case for étale groupoids. However, the essential groupoid C*-algebra, which replaces the reduced groupoid C*-algebra in the non-Hausdorff case, is by its very definition governed by the interaction between open dense subsets of the groupoid. As a consequence, it is not even clear that there is an inclusion of $C^*_{\mathrm{ess}}(\mathcal{G})$ into $C^*_{\mathrm{ess}}(\mathcal{G} \ltimes \partial_F \mathcal{G})$.

Only under the additional assumption of Theorem 7.6.1 have we been able to prove the existence of such an inclusion in Section 7.5.2. This inclusion is therefore vital for arguments pertaining to the theory of C*-simplicity.

We will adopt the following terminology in the remainder of this article.

Definition 7.6.3. Let $\mathcal{G}$ be an étale groupoid with locally compact Hausdorff space of units. We will say that $\mathcal{G}$ is *essentially principal* if $\mathrm{Iso}(\mathcal{G}) = \overline{\mathcal{G}^{(0)}}$. It is *essentially effective* if $\mathrm{Iso}(\mathcal{G})^\circ \subseteq \overline{\mathcal{G}^{(0)}}$.

We remark that, by Proposition 7.2.1, an essentially effective groupoid with extremally disconnected unit space is automatically essentially principal.

Let us begin with the following result, proving one implication of Theorem 7.6.1.

Proposition 7.6.4. *Let $\mathcal{G}$ be an étale groupoid with extremally disconnected, compact Hausdorff space of units. Further assume that all isotropy groups of $\mathcal{G}$ are amenable. If $C(\mathcal{G}^{(0)}) \subseteq C^*_{\mathrm{ess}}(\mathcal{G})$ has the ideal intersection property, then $\mathcal{G}$ is essentially principal.*

Proof. First assume that $\mathcal{G}$ is Hausdorff. Since the unit space of $\mathcal{G}$ is extremally disconnected, Proposition 7.2.1 says that the isotropy groupoid of $\mathcal{G}$ is clopen. So there is a well-defined conditional expectation $C^*_r(\mathcal{G}) \to C^*_r(\mathrm{Iso}(\mathcal{G}))$. Further, consider the trivial representation $C^*_r(\mathrm{Iso}(\mathcal{G})) \to C(\mathcal{G}^{(0)})$, which exists by Proposition 7.2.6 thanks to amenability of isotropy groups. We denote by $F : C^*_r(\mathcal{G}) \to C(\mathcal{G}^{(0)})$ the composition of these two maps. Observe that F is tracial and that $C^*_r(\mathrm{Iso}(\mathcal{G}))$ lies in the multiplicative domain of F. Denote by H the Hilbert-$C(\mathcal{G}^{(0)})$-module obtained from separation-completion coming with a map $\Lambda : C^*_r(\mathcal{G}) \to H$ satisfying

$$\langle \Lambda(a), \Lambda(b) \rangle = F(ab^*)$$

for all $a, b \in C^*_r(\mathcal{G})$. Denote by π the associated KSGNS-representation of $C^*_r(\mathcal{G})$ [Lan95], which is faithful on $C(\mathcal{G}^{(0)})$. Since $C_0(\mathcal{G}^{(0)}) \subseteq C^*_r(\mathcal{G})$ has the ideal intersection property, this implies that π is faithful on $C^*_r(\mathcal{G})$.

Let $U \subseteq \mathrm{Iso}(\mathcal{G})$ be an open bisection and $f \in C_c(U)$. Write $g = f \circ (\mathrm{s}|_U)^{-1} \in C_c(\mathrm{s}(U))$. Then $F(f - g) = 0$. Combined with the fact that $C^*_r(\mathrm{Iso}(\mathcal{G}))$ lies in the multiplicative domain of F, this leads to the calculation

$$\begin{aligned}
\langle \Lambda(a), \pi(f - g)\Lambda(b) \rangle &= F(a^*(f - g)b) \\
&= F(ba^*(f - g)) \\
&= F(ba^*)F(f - g) \\
&= 0\,,
\end{aligned}$$

for all $a, b \in C^*_r(\mathcal{G})$. This shows that $f - g$ is zero. Since $f \in C_c(U)$ was arbitrary, this shows that $U = \mathrm{s}(U) \subseteq \mathcal{G}^{(0)}$. Since $\mathrm{Iso}(\mathcal{G})$ is open by Proposition 7.2.1, this proves the proposition for Hausdorff groupoids.

If $\mathcal{G}$ is a possibly non-Hausdorff groupoid satisfying the assumptions of the lemma, we observe that $C(\mathcal{G}^{(0)}) \subseteq C^*_{\mathrm{ess}}(\mathcal{G}) \cong C^*_r(\mathcal{G}_{\mathrm{Haus}})$ has the ideal intersection property. So the Hausdorffification $\mathcal{G}_{\mathrm{Haus}}$ introduced in Section 7.5.3 is principal. This implies that $\mathrm{Iso}(\mathcal{G}) = \overline{\mathcal{G}^{(0)}}$. ■

In order to apply Proposition 7.6.4 to Furstenberg groupoids, we need to verify that their isotropy groups are amenable. This was shown by Borys for Hausdorff groupoids. We present a different proof, which is a suitable adaption of the standard argument for groups.

Proposition 7.6.5. *Let $\mathcal{G}$ be an étale groupoid with compact Hausdorff space of units and let $x \in \partial_F \mathcal{G}$. Then $(\mathcal{G} \ltimes \partial_F \mathcal{G})^x_x$ is an amenable group.*

Proof. Write $\mathcal{H} = \mathcal{G} \ltimes \partial_F \mathcal{G}$ for the Furstenberg groupoid and fix $x \in \partial_F \mathcal{G}$. Consider the unital $\mathcal{G}$-C*-algebra $\ell^\infty(\mathcal{H})$. By $\mathcal{G}$-injectivity, the inclusion $C(\partial_F \mathcal{G}) \subseteq \ell^\infty(\mathcal{H})$ gives rise to a $\mathcal{G}$-ucp map $\ell^\infty(\mathcal{H}) \to C(\partial_F \mathcal{G})$. Passing to the fibre at x of both $C(\partial_F \mathcal{G})$-algebras, we obtain an $\mathcal{H}^x_x$-invariant state $\ell^\infty(\mathcal{H}^x) \to \mathbb{C}$. After choice of any $\mathcal{H}^x_x$-equivariant map $\mathcal{H}^x \to \mathcal{H}^x_x$, we obtain an $\mathcal{H}^x_x$-equivariant *-homomorphism $\ell^\infty(\mathcal{H}^x_x) \to \ell^\infty(\mathcal{H}^x)$. The composition of these maps gives an $\mathcal{H}^x_x$-invariant state on $\ell^\infty(\mathcal{H}^x_x)$, which proves amenability of $\mathcal{H}^x_x$. ■

Proposition 7.6.6. *Let $\mathcal{G}$ be an étale groupoid with compact Hausdorff space of units. Assume that $\mathcal{G}$ is Hausdorff, that $\mathcal{G}$ is minimal or that $\mathcal{G}$ is σ-compact. If $\mathcal{G} \ltimes \partial_F \mathcal{G}$ is essentially principal then there is a unique $\mathcal{G}$-pseudo expectation $C^*_{\mathrm{ess}}(\mathcal{G}) \to C(\partial_F G)$.*

Proof. Existence of a $\mathcal{G}$-pseudo expectation $C^*_{\mathrm{ess}}(\mathcal{G}) \to C(\partial_F G)$ follows from Proposition 7.5.3 and the fact that the local conditional expectation $\mathrm{E}_{\mathrm{red}} : C^*_r(\mathcal{G}) \to \mathrm{Dix}(\mathcal{G}^{(0)})$ factors through $C^*_{\mathrm{ess}}(\mathcal{G})$. We will prove uniqueness. Let $\varphi : C^*_{\mathrm{ess}}(\mathcal{G}) \to C(\partial_F \mathcal{G})$ be a $\mathcal{G}$-pseudo expectation. Let us write $\mathcal{H} = \mathcal{G} \ltimes \partial_F \mathcal{G}$ throughout the rest of the proof. If $\mathcal{G}$ is Hausdorff, then $C^*_{\mathrm{ess}}(\mathcal{G}) = C^*_r(\mathcal{G}) \subseteq C^*_r(\mathcal{H}) = C^*_{\mathrm{ess}}(\mathcal{H})$. If $\mathcal{G}$ is minimal or σ-compact, then by Theorem 7.5.4 there is an inclusion $C^*_{\mathrm{ess}}(\mathcal{G}) \subseteq C^*_{\mathrm{ess}}(\mathcal{H})$. Applying $\mathcal{G}$-injectivity of $C(\partial_F \mathcal{G})$ we can extend φ to $C^*_{\mathrm{ess}}(\mathcal{H})$. Theorem 7.5.10 provides the natural identification $C^*_{\mathrm{ess}}(\mathcal{H}) \cong C^*_r(\mathcal{H}_{\mathrm{Haus}})$, and we denote the resulting $\mathcal{G}$-pseudo expectation by $\psi : C^*_r(\mathcal{H}_{\mathrm{Haus}}) \to C(\partial_F \mathcal{G})$. By $\mathcal{G}$-rigidity we find that ψ restricted to $C(\mathcal{H}^{(0)}_{\mathrm{Haus}}) = C(\partial_F \mathcal{G})$ is the identity. Consequently, $C(\partial_F \mathcal{G})$ lies in the multiplicative domain of ψ. Let now

$\gamma \subseteq \mathcal{H}_{\mathrm{Haus}} \setminus \partial_F \mathcal{G}$ be some open bisection and $f \in C_c(\gamma)$. Since $\mathcal{H}$ is essentially principal, we have $\mathrm{Iso}(\mathcal{H}_{\mathrm{Haus}}) = \partial_F \mathcal{G}$. So if $f \neq 0$, then for every non-empty open subset $V_0 \subseteq \operatorname{supp} \psi(f)$ there is some nonempty open subset $V \subseteq V_0$ such that $\gamma V \cap V = \emptyset$. This leads to

$$\psi(f)1_V = \psi(f1_V) = \psi(1_{\gamma V} f) = 1_{\gamma V}\psi(f)\,.$$

This is a contradiction, showing that $\psi(f) = 0$. We showed that ψ is the natural conditional expectation of $C_r^*(\mathcal{H}_{\mathrm{Haus}})$. So the composition

$$\varphi : C^*_{\mathrm{ess}}(\mathcal{G}) \hookrightarrow C^*_{\mathrm{ess}}(\mathcal{H}) \xrightarrow{\cong} C_r^*(\mathcal{H}_{\mathrm{Haus}}) \to C(\partial_F \mathcal{G})$$

equals the natural $\mathcal{G}$-pseudo expectation by Theorem 7.5.10. ■

Proposition 7.6.7. *Let $\mathcal{G}$ be an étale groupoid with compact Hausdorff space of units. Assume that there is a unique $\mathcal{G}$-pseudo expectation $C^*_{\mathrm{ess}}(\mathcal{G}) \to C(\partial_F \mathcal{G})$. Then $C(\mathcal{G}^{(0)}) \subseteq C^*_{\mathrm{ess}}(\mathcal{G})$ has the ideal intersection property.*

Proof. Let $\pi : C^*_{\mathrm{ess}}(\mathcal{G}) \twoheadrightarrow A$ be a surjective *-homomorphism that is faithful on $C(\mathcal{G}^{(0)})$. Proposition 7.3.8 shows that A carries a unital $\mathcal{G}$-C*-algebra structure such that π is $\mathcal{G}$-equivariant. Then $(\pi|_{C(\mathcal{G}^{(0)})})^{-1} : \pi(C(\mathcal{G}^{(0)})) \to C(\mathcal{G}^{(0)})$ extends to a $\mathcal{G}$-ucp map $\varphi : A \to C(\partial_F \mathcal{G})$. The composition $\varphi \circ \pi : C^*_{\mathrm{ess}}(\mathcal{G}) \to C(\partial_F \mathcal{G})$ is a $\mathcal{G}$-pseudo expectation of $\mathcal{G}$. By assumption of the lemma, it must thus be the natural $\mathcal{G}$-pseudo expectation, which is faithful. So also π must be faithful. ■

Proposition 7.6.8. *Let $\mathcal{G}$ be an étale groupoid with a compact Hausdorff space of units. Assume that $\mathcal{G}$ is Hausdorff, that $\mathcal{G}$ is minimal or that $\mathcal{G}$ is σ-compact. If $C^*_{\mathrm{ess}}(\mathcal{G})$ has the ideal intersection property, then also $C^*_{\mathrm{ess}}(\mathcal{G} \ltimes \partial_F \mathcal{G})$ has the ideal intersection property.*

Proof. We write $\mathcal{H} = \mathcal{G} \ltimes \partial_F \mathcal{G}$ throughout the proof. Let $\pi : C^*_{\mathrm{ess}}(\mathcal{H}) \twoheadrightarrow A$ be a surjective *-homomorphism that is faithful on $C(\partial_F \mathcal{G})$. Consider the inclusion $C^*_{\mathrm{ess}}(\mathcal{G}) \subseteq C^*_{\mathrm{ess}}(\mathcal{H})$ provided by Theorem 7.5.4 in case $\mathcal{G}$ is minimal or σ-compact. If $\mathcal{G}$ is Hausdorff, then also $\mathcal{H}$ is Hausdorff so that their essential groupoid C*-algebras are equal to their reduced groupoid C*-algebras, and we also obtain the inclusion $C^*_{\mathrm{ess}}(\mathcal{G}) \subseteq C^*_{\mathrm{ess}}(\mathcal{H})$. Since $C(\mathcal{G}^{(0)}) \subseteq C^*_{\mathrm{ess}}(\mathcal{G})$ has the ideal intersection property, it follows that π is faithful on $C^*_{\mathrm{ess}}(\mathcal{G})$. Denote by $\mathrm{E}_{\mathrm{ess}} : C^*_{\mathrm{ess}}(\mathcal{G}) \to C(\partial_F \mathcal{G})$ the natural $\mathcal{G}$-pseudo expectation. We observe that A becomes a unital $\mathcal{G}$-C*-algebra by Proposition 7.3.8. So $\mathrm{E}_{\mathrm{ess}} \circ (\pi|_{C^*_{\mathrm{ess}}(\mathcal{G})})^{-1} : \pi(C^*_{\mathrm{ess}}(\mathcal{G})) \to C(\partial_F \mathcal{G})$ extends to a $\mathcal{G}$-ucp map $\varphi : A \to C(\partial_F \mathcal{G})$. Write $\psi = \varphi \circ \pi : C^*_{\mathrm{ess}}(\mathcal{H}) \to C(\partial_F \mathcal{G})$. Then $\mathcal{G}$-rigidity implies that $\psi|_{C(\partial_F \mathcal{G})}$ is the identity map. In particular, $C(\partial_F \mathcal{G})$ lies in the multiplicative domain of ψ. Since $C^*_{\mathrm{ess}}(\mathcal{H}) = \overline{\mathrm{span}}\, C^*_{\mathrm{ess}}(\mathcal{G})C(\partial_F\mathcal{G})$ and $\psi|_{C^*_{\mathrm{ess}}(\mathcal{G})} = \mathrm{E}_{\mathrm{ess}}$, we find that ψ is the natural conditional expectation of $C^*_{\mathrm{ess}}(\mathcal{H})$. In particular ψ, and thus also π, is faithful. ■

Let us summarise how the principal result of this section follows.

Proof of Theorem 7.6.1. If $C(\mathcal{G}^{(0)}) \subseteq C^*_{\mathrm{ess}}(\mathcal{G})$ has the ideal intersection property, then also $C(\partial_F\mathcal{G}) \subseteq C^*_{\mathrm{ess}}(\mathcal{G} \ltimes \partial_F\mathcal{G})$ has it by Proposition 7.6.8 since $\mathcal{G}$ is assumed to be Hausdorff, to be minimal or to be σ-compact. Since all isotropy groups of $\mathcal{G} \ltimes \partial_F\mathcal{G}$ are amenable by Proposition 7.6.5, we can invoke Proposition 7.6.4 to infer that the ideal intersection property for $C(\partial_F\mathcal{G}) \subseteq C^*_{\mathrm{ess}}(\mathcal{G} \ltimes \partial_F\mathcal{G})$ implies that $\mathcal{G} \ltimes \partial_F\mathcal{G}$ must be essentially principal. In turn, Proposition 7.6.6 says that essential principality of $\mathcal{G} \ltimes \partial_F\mathcal{G}$ implies that there is a unique $\mathcal{G}$-pseudo expectation $C^*_{\mathrm{ess}}(\mathcal{G}) \to C(\partial_F\mathcal{G})$, using another time the assumption that $\mathcal{G}$ is Hausdorff, minimal or σ-compact. Finally, Proposition 7.6.7 shows that uniqueness of the $\mathcal{G}$-pseudo expectation $C^*_{\mathrm{ess}}(\mathcal{G}) \to C(\partial_F\mathcal{G})$ implies the ideal intersection property for $C(\mathcal{G}^{(0)}) \subseteq C^*_{\mathrm{ess}}(\mathcal{G})$. ■

Remark 7.6.9. We made it clear that it is necessary to consider the essential groupoid C*-algebra in order to obtain a relation between the ideal structure and the algebraic-dynamical structure of a groupoid. This can be underpinned by the fact the ideal intersection property for $C^*_r(\mathcal{G} \ltimes \partial_F\mathcal{G})$ implies that $\mathcal{G}$ is Hausdorff. Indeed, assume that $C^*_r(\mathcal{G} \ltimes \partial_F\mathcal{G})$ has the ideal intersection property. Since the map $C^*_r(\mathcal{G} \ltimes \partial_F\mathcal{G}) \to C^*_{\mathrm{ess}}(\mathcal{G} \ltimes \partial_F\mathcal{G})$ is injective on $C(\partial_F\mathcal{G})$, it follows that the ideal of singular functions in $C^*_r(\mathcal{G} \ltimes \partial_F\mathcal{G})$ is trivial. Since $\overline{\partial_F\mathcal{G}} \subseteq \mathcal{G} \ltimes \partial_F\mathcal{G}$ is clopen by Lemma 7.5.8, this implies that $\mathcal{G} \ltimes \partial_F\mathcal{G}$ is Hausdorff. Then also $\mathcal{G}$ is Hausdorff, since the quotient map $\mathcal{G} \ltimes \partial_F\mathcal{G} \to \mathcal{G}$ maps $\overline{\partial_F\mathcal{G}}$ onto $\overline{\mathcal{G}^{(0)}}$.

7.7 Essentially confined subgroupoids

An important point in our understanding of C*-simplicity of discrete groups has been the characterisation in terms of confined subgroups from [Ken20], since this provides criteria that can be checked in terms of the group itself. Note that the terminology of “confined” subgroups does not appear in [Ken20], although it does appear in prior work on group algebras of locally finite groups [HZ97].

In this section we obtain results about the ideal intersection property for groupoids. The key achievement is Theorem 7.7.2, which establishes a characterisation in terms of confined amenable sections of isotropy groups for groupoids with a compact Hausdorff space of units that satisfy a regularity property such as Hausdorffness, minimality or σ-compactness. Subsequently, we will be able to remove the compactness assumption on the unit space by considering a notion of Alexandrov groupoid, whose unit space is the one-point compactification of the unit space of the original groupoid. In this we way, we will obtain complete results for Hausdorff groupoids and σ-compact groupoids in Section 7.7.2.

Given an étale groupoid $\mathcal{G}$ with locally compact Hausdorff space of units, as in [Bor20, § 4.2.4, p. 78], we define $\mathrm{Sub}(\mathcal{G})$ as the space of all subgroups of the isotropy groups of $\mathcal{G}$, viewed as a subspace of the space $\mathfrak{C}(\mathcal{G})$ of all closed subsets of $\mathcal{G}$ equipped with the Chabauty topology (see for instance [Fel62]). We will denote elements of $\mathrm{Sub}(\mathcal{G})$ by (x, H), where $x \in \mathcal{G}^{(0)}$ and H is a subgroup of $\mathcal{G}_x^x$. Observe that the first factor projection $\mathrm{Sub}(\mathcal{G}) \to \mathcal{G}^{(0)}$ together with the conjugation action of $\mathcal{G}$ turns $\mathrm{Sub}(\mathcal{G})$ into a $\mathcal{G}$-space.

The next definition extends [Bor20, Definition 4.2.24] to the non-Hausdorff case.

Definition 7.7.1. Let $\mathcal{G}$ be an étale groupoid with locally compact Hausdorff space of units. A section of isotropy subgroups is a collection $\{(x, H_x) \mid x \in \mathcal{G}^{(0)}\}$ of isotropy subgroups $H_x \subseteq \mathcal{G}_x^x$ for all $x \in \mathcal{G}^{(0)}$. Such a section is called amenable if all H_x are amenable. A section of isotropy subgroups $\Lambda = \{(x, H_x) \mid x \in \mathcal{G}^{(0)}\}$ is called essentially confined if there exists $x \in \mathcal{G}^{(0)}$ such that $H \not\subseteq \overline{\mathcal{G}^{(0)}}$ for all $(x, H) \in \overline{\mathcal{G} \cdot \Lambda}$.

We recall that the terminology of confinedness, first arose in the study of groups, where it qualifies subgroups that are isolated from the trivial group in a strong sense.

7.7.1 The ideal intersection property for groupoids with compact space of units

Combining arguments similar to those used by Kawabe [Kaw17] and Borys [Bor20] with the techniques developed in the present article yields the following characterisation of the intersection property for étale groupoids with compact space of units, which is new in the non-Hausdorff case and completes Borys's sufficient criterion in the Hausdorff case.

Theorem 7.7.2. *Let $\mathcal{G}$ be an étale groupoid with compact Hausdorff space of units. Assume that $\mathcal{G}$ is Hausdorff, that $\mathcal{G}$ is minimal or that $\mathcal{G}$ is σ-compact. Then $C(\mathcal{G}^{(0)}) \subseteq C^*_{\mathrm{ess}}(\mathcal{G})$ has the intersection property if and only if $\mathcal{G}$ has no essentially confined amenable sections of isotropy subgroups.*

Remark 7.7.3. Previous work on crossed product C*-algebras and groupoid C*-algebras obtained partial results that can be recovered from Theorem 7.7.2.

Kawabe considered in [Kaw17] crossed product C*-algebras $C(X) \rtimes_r \Gamma$ for a compact Hausdorff space X and a discrete group Γ. Note that $C(X) \rtimes_r \Gamma \cong C^*_r(\Gamma \ltimes X)$ is the groupoid C*-algebra associated with the transformation groupoid $\Gamma \ltimes X$. Kawabe's main result [Kaw17, Theorem 1.6] characterised the ideal intersection property for $C(X) \subseteq C(X) \rtimes_r \Gamma$ by the following property: For every point $x \in X$ and every amenable subgroup $\Lambda \leq \Gamma_x$, there is a net (g_i) in Γ such that $(g_i x)$ converges to x and $(g_i \Lambda g_i^{-1})$ converges to $\{e\}$ in the Chabauty topology. We observe that $\mathrm{Sub}(X \rtimes \Gamma) = \{(x, \Lambda) \in X \times \mathrm{Sub}(\Gamma) \mid \Lambda \leq \Gamma_x\}$ as topological spaces. This shows how Kawabe's result is generalised by our Theorem 7.7.2.

Borys considered in [Bor20, Bor19] étale Hausdorff groupoids with compact space of units and proved that the absence of confined amenable sections in the isotropy is a sufficient criterion for simplicity of $C^*_r(\mathcal{G})$. Since for any such Hausdorff groupoid, we have a *-isomorphism $C^*_r(\mathcal{G}) \cong C^*_{\mathrm{ess}}(\mathcal{G})$ and the notions of essentially confined sections of isotropy groups agrees with the notion of confined sections of isotropy groups, our Theorem 7.7.2 recovers Borys sufficient criterion and shows its necessity.

Additionally, we will comment on Kwaśniewski-Meyer's result [KM21, Theorem 7.29] in Remark 7.7.15, since groupoids with locally compact space of units are considered in this work.

We begin with the following observation, which is probably well known.

Lemma 7.7.4. *Let $\mathcal{G}$ be an étale groupoid with locally compact Hausdorff space of units.* $\mathrm{Sub}(\mathcal{G})$ *is closed in* $\mathfrak{C}(\mathcal{G})$.

Proof. Suppose that $(x_i, H_i) \in \mathrm{Sub}(\mathcal{G})$ converges to C in $\mathfrak{C}(\mathcal{G})$. As explained in [Fel62], the subset C consists precisely of the elements of $g \in \mathcal{G}$ with the following property: For every open subset $U \subseteq \mathcal{G}$ with $g \in U$, we have $H_i \cap U \neq \emptyset$ eventually. First of all, we claim that $C \subseteq \mathcal{G}^x_x$ for some $x \in \mathcal{G}^{(0)}$. Indeed, take $g, h \in C$ with $\mathrm{s}(g) = x$ and $\mathrm{s}(h) = y$. If $x \neq y$, then we can find disjoint open subsets U and V of $\mathcal{G}^{(0)}$ containing x and y, respectively. Then we must have $H_i \cap \mathrm{s}^{-1}(U) \neq \emptyset$ eventually, which implies $x_i \in U$ eventually. Similarly, $x_i \in V$ eventually. This shows that $x = y$. The same argument, applied to the range map, shows that $C \subseteq \mathcal{G}^x_x$. This also shows that we must have $\lim_i x_i = x$. It remains to show that C is a subgroup of $\mathcal{G}^x_x$. Take $g \in C$ and an open subset $V \subseteq \mathcal{G}$ with $g^{-1} \in V$. Then $g \in V^{-1}$, so that $H_i \cap V^{-1} \neq \emptyset$ eventually. Applying the inverse map, we deduce that $H_i \cap V \neq \emptyset$ eventually. Hence $g^{-1} \in C$. Now take $g, h \in C$ and an open subset $W \in \mathcal{G}$ with $gh \in W$. By continuity of multiplication, we find open subsets $U, V \subseteq \mathcal{G}$ with $g \in U$, $h \in V$ such that $UV \subseteq W$. Then $g, h \in C$ implies $H_i \cap U \neq \emptyset$ and $H_i \cap V \neq \emptyset$ eventually. Thus $H_i \cap UV \neq \emptyset$ eventually. We conclude that $gh \in C$. So $C \in \mathrm{Sub}(\mathcal{G})$, as desired. ∎

The next proposition establishes the reverse implication of Theorem 7.7.2. Portions of the proof will closely follow the presentation in [Bor20] for Hausdorff groupoids. Nevertheless, we include a complete proof for the convenience of the reader.

Proposition 7.7.5. *Let $\mathcal{G}$ be an étale groupoid with compact Hausdorff space of units. Suppose that $\mathcal{G}$ is Hausdorff, $\mathcal{G}$ is minimal or $\mathcal{G}$ is covered by countably many bisections. If $\mathcal{G}$ has no essentially confined amenable sections of isotropy subgroups, then $\mathcal{G} \ltimes \partial_F \mathcal{G}$ is essentially principal.*

Proof. The theorem is proven in [Bor20, Theorem 4.2.25] for the case of Hausdorff groupoids, in which case $C^*_{\mathrm{ess}}(\mathcal{G} \ltimes \partial_F \mathcal{G}) = C^*_r(\mathcal{G} \ltimes \partial_F \mathcal{G})$ holds. So we need to consider the cases where $\mathcal{G}$ is minimal and where $\mathcal{G}$ is covered by countably many bisections. Let $\pi : \mathcal{G} \ltimes \partial_F \mathcal{G} \to \mathcal{G}$ be the canonical projection and consider the set of isotropy subgroups $\Lambda = \{\pi((\mathcal{G} \ltimes \partial_F \mathcal{G})^y_y) \mid y \in \partial_F \mathcal{G}\}$. The same argument as in [Bor20, Proof of Theorem 4.2.25], which does not use the assumption that $\mathcal{G}$ is Hausdorff, shows that Λ is a closed invariant subspace of $\mathrm{Sub}(\mathcal{G})$. So our assumption that $\mathcal{G}$ has no essentially confined amenable section of isotropy subgroups implies that for all $x \in \mathcal{G}^{(0)}$, there exists $y \in \pi^{-1}(x)$ such that $\pi((\mathcal{G} \ltimes \partial_F \mathcal{G})^y_y) \subseteq \overline{\mathcal{G}^{(0)}}_x$.

Let us consider the case when $\mathcal{G}$ is minimal. Then we define

$$Z = \{y \in \partial_F \mathcal{G} \mid \pi((\mathcal{G} \ltimes \partial_F \mathcal{G})^y_y) \subseteq \overline{\mathcal{G}^{(0)}}_{\pi(y)}\}$$

and observe that $\pi(Z) = \mathcal{G}^{(0)}$ and Z is $\mathcal{G}$-invariant. If $y \in \overline{Z}$ and $g \in (\mathcal{G} \ltimes \partial_F \mathcal{G})^y_y$, we take an open bisection $U \subseteq \mathrm{Iso}(\mathcal{G} \ltimes \partial_F \mathcal{G})$ that contains g. Let $(y_i)_i$ be a net in $\mathrm{s}(U) \cap Z$ converging to y. Then $(s|_U)^{-1}(y_i) \to g$ and thus $\pi(g) = \lim_i \pi((s|_U)^{-1}(y_i)) \in \overline{\mathcal{G}^{(0)}}$. This proves that $y \in Z$. So Z is closed, which implies $Z = \partial_F \mathcal{G}$ by irreducibility. Equivalently, $\pi(\mathrm{Iso}(\mathcal{G} \ltimes \partial_F \mathcal{G})) \subseteq \overline{\mathcal{G}^{(0)}}$. Put $A = \mathrm{Iso}(\mathcal{G} \ltimes \partial_F \mathcal{G}) \setminus \overline{\partial_F \mathcal{G}}$. We will conclude the proof in the minimal case by assuming that $A \neq \emptyset$ and deducing a contradiction. If $A \neq \emptyset$, we can find an open bisection $U \subseteq \mathcal{G}$ such that $U \cap \pi(A) \neq \emptyset$. Then $V = \pi^{-1}(U) \cap A \subseteq \mathcal{G} \ltimes \partial_F \mathcal{G}$ is a non-empty open bisection. Since $\mathcal{G}$ is minimal, Lemma 7.5.1 says that $\mathrm{s}(\pi(V)) = \pi(\mathrm{s}(V))$ contains a non-empty open subset. Since $\pi(V) \subseteq U$ is contained in an open bisection, this implies that $\emptyset \neq \pi(V)^\circ \subseteq \pi(A) \subseteq \overline{\mathcal{G}^{(0)}} \setminus \mathcal{G}^{(0)}$. This is the desired contradiction. Hence $\mathcal{G} \ltimes \partial_F \mathcal{G}$ is essentially principal.

Assume now that $\mathcal{G}$ is covered by countably many open bisections, we define

$$Z = \{y \in \partial_F \mathcal{G} \mid (\mathcal{G} \ltimes \partial_F \mathcal{G})^y_y \subseteq \overline{\partial_F \mathcal{G}}_y\}.$$

As the extended unit space $\overline{\partial_F \mathcal{G}}$ is normal in $\mathcal{G} \ltimes \partial_F \mathcal{G}$, it follows that Z is $\mathcal{G}$-invariant. Moreover, the identity $\partial_F \mathcal{G} \setminus Z = \mathrm{s}\big(\mathrm{Iso}(\mathcal{G} \ltimes \partial_F \mathcal{G}) \setminus \overline{\partial_F \mathcal{G}}\big)$ implies that Z is closed. Hence

$\pi(Z)$ is a closed subset of $\mathcal{G}^{(0)}$. We want to show that $\pi(Z) = \mathcal{G}^{(0)}$ and to this end it will suffice to prove that $\pi(Z) \subseteq \mathcal{G}^{(0)}$ is dense. Since $\mathcal{G}$ is covered by countably many open bisections, Proposition 7.2.5 says that the set $\{x \in \mathcal{G}^{(0)} \mid \overline{\mathcal{G}^{(0)}}_x = \{x\}\}$ is dense in $\mathcal{G}^{(0)}$. Now for all x in this set, there exists $y \in \pi^{-1}(x)$ with $\pi((\mathcal{G} \ltimes \partial_F \mathcal{G})_y^y) \subseteq \overline{\mathcal{G}^{(0)}}_x = \{x\}$, that is $\pi((\mathcal{G} \ltimes \partial_F \mathcal{G})_y^y) = \{x\}$ and thus $(\mathcal{G} \ltimes \partial_F \mathcal{G})_y^y = \{y\}$. This shows that $\pi(Z)$ is dense in $\mathcal{G}^{(0)}$. Since $\pi(Z)$ is also closed, we conclude that $\pi(Z) = \mathcal{G}^{(0)}$, as desired. Since $\partial_F \mathcal{G}$ is $\mathcal{G}$-irreducible, it follows that $Z = \partial_F \mathcal{G}$, that is $\mathrm{Iso}(\mathcal{G} \ltimes \partial_F \mathcal{G}) \subseteq \overline{\partial_F \mathcal{G}}$. ■

Let us now explain the proof of the forward implication of Theorem 7.7.2. The following proof is an adaptation of Kawabe's argument for dynamical systems of groups [Kaw17], as presented in [Bor20, Proof of Theorem 3.3.6]. Note that this proposition is new even in the Hausdorff case.

Proposition 7.7.6. *Let $\mathcal{G}$ be an étale groupoid with compact Hausdorff space of units. Assume that $\mathcal{G}$ is Hausdorff, that $\mathcal{G}$ is minimal or that $\mathcal{G}$ is σ-compact. If there is a unique $\mathcal{G}$-pseudo-expectation $C^*_{\mathrm{ess}}(\mathcal{G}) \to C(\partial_F \mathcal{G})$, then $\mathcal{G}$ has no essentially confined amenable sections of isotropy subgroups.*

Proof. Let $\Lambda = \{(x, H_x) \mid x \in \mathcal{G}^{(0)}\}$ be an amenable section of isotropy subgroups and Y its orbit closure, that is $Y = \overline{\mathcal{G} \cdot \Lambda} \subseteq \mathrm{Sub}(\mathcal{G})$. Define $\theta : C^*_r(\mathcal{G}) \to C(Y)$ as follows. Given $a \in C^*_r(\mathcal{G})$ and $(x, H) \in Y$, let $\theta(a)(x, H)$ be the image of a under the composition

$$C^*_r(\mathcal{G}) \xrightarrow{\mathrm{E}_x} C^*_r(\mathcal{G}_x^x) \xrightarrow{\mathrm{E}_H} C^*_r(H) \xrightarrow{\chi} \mathbb{C}$$

where E_x and E_H are the natural conditional expectations and χ is the character corresponding to the trivial representation. It is clear from the construction that θ is unital and completely positive, once we showed that $\theta(a)$ is continuous on Y. It suffices to prove this for $a \in C_c(U)$, where $U \subseteq \mathcal{G}$ is an open bisection. For such a, our construction yields

$$\theta(a)(x, H) = \begin{cases} a(Ux) 1_H(Ux) & \text{if } x \in \mathrm{s}(U) \\ 0 & \text{otherwise.} \end{cases}$$

Now let $O = \{g \in U \mid a(g) \neq 0\}$, which is open, and let $K \subseteq U$ be compact such that $O \subseteq K$. The sets $\mathcal{O}_O = \{(x, H) \in \mathrm{Sub}(\mathcal{G}) \mid H \cap O \neq \emptyset\}$ and $\mathcal{O}'_K = \{(x, H) \in \mathrm{Sub}(\mathcal{G}) \mid H \cap K = \emptyset\}$ are open in $\mathrm{Sub}(\mathcal{G})$. We have $\theta(a)|_{(Y \cap \mathcal{O}_O)^c} \equiv 0$, and $\theta(a)|_{(Y \cap \mathcal{O}'_K)^c}$ is given as the composition

$$(Y \cap \mathcal{O}'_K)^c \to \mathrm{s}(K) \to \mathbb{C} : (x, H) \mapsto x \mapsto a(Ux)\,.$$

It follows that both restrictions $\theta(a)|_{(Y\cap\mathcal{O}_O)^c}$ and $\theta(a)|_{(Y\cap\mathcal{O}'_K)^c}$ are continuous. Moreover, $O \subseteq K$ implies that $(Y\cap\mathcal{O}_O)\cap(Y\cap\mathcal{O}'_K)=\emptyset$, so that we obtain a decomposition into two closed sets $Y=(Y\cap\mathcal{O}_O)^c\cup(Y\cap\mathcal{O}'_K)^c$. Since $\theta(a)$ is continuous on both of these closed subsets, it follows that $\theta(a)$ is continuous on Y.

Now we show that θ is $\mathcal{G}$-equivariant. Take an open bisection γ, an open bisection U with $\mathrm{r}(U),\mathrm{s}(U)\subseteq\mathrm{s}(\gamma)$ and $a\in C_c(U)$. Then $\alpha_\gamma(a)\in C_c(\gamma U\gamma^*)$. For all $(x,H)\in Y$, we have $\theta(\alpha_\gamma(a))(x,H)=0$ if $x\notin\mathrm{s}(\gamma U\gamma^*)$ and $\theta(a)(\psi_{\gamma^*}(x),\gamma^*H\gamma)=0$ if $\psi_{\gamma^*}(x)\notin\mathrm{s}(U)$. The conditions $x\notin\mathrm{s}(\gamma U\gamma^*)=\psi_\gamma\circ\mathrm{s}(U)$ and $\psi_{\gamma^*}(x)\notin\mathrm{s}(U)$ are equivalent. Using the fact that $\psi_{\gamma^*}(x)=\gamma^*x\gamma$, for $x\in\mathrm{s}(\gamma U\gamma^*)$, we have

$$\begin{aligned}\theta(\alpha_\gamma(a))(x,H)&=\alpha_\gamma(a)(\gamma U\gamma^*x)1_H(\gamma U\gamma^*x)\\&=a(U\gamma^*x\gamma)1_H(\gamma U\gamma^*x)\\&=a(U\gamma^*x\gamma)1_{\gamma^*H\gamma}(U\gamma^*x\gamma)\\&=\theta(a)(\psi_{\gamma^*}(x),\gamma^*H\gamma)\,.\end{aligned}$$

The projection onto the first coordinate $Y\to\mathcal{G}^{(0)}$ induces a $\mathcal{G}$-equivariant embedding $C(\mathcal{G}^{(0)})\to C(Y)$. Hence $\mathcal{G}$-injectivity of $C(\partial_F\mathcal{G})$ provides us with a $\mathcal{G}$-equivariant ucp map $\varphi: C(Y)\to C(\partial_F\mathcal{G})$. It follows that $\varphi\circ\theta: C^*_r(\mathcal{G})\to C(\partial_F\mathcal{G})$ is a $\mathcal{G}$-ucp map with $(\varphi\circ\theta)|_{C(\mathcal{G}^{(0)})}=\mathrm{id}$. We claim that $\varphi\circ\theta$ factors as

$$C^*_r(\mathcal{G})\xrightarrow{\ \mathfrak{q}\ }C^*_{\mathrm{ess}}(\mathcal{G})\xrightarrow{\ \Psi\ }C(\partial_F\mathcal{G}),$$

or equivalently, that $(\varphi\circ\theta)(J_{\mathrm{sing}})=0$. If $\mathcal{G}$ is Hausdorff, this is tautological. Otherwise, given $a\in J_{\mathrm{sing}}$ we have $\mathrm{E}_{\mathrm{red}}(a^*a)=0$. So by Proposition 7.2.4, there is a dense subset $U\subseteq\mathcal{G}^{(0)}$ such that $\widehat{a^*a}|_U=0$. Since

$$\widehat{a^*a}(x)=\sum_{g\in\mathcal{G}_x}|\hat{a}(g)|^2$$

for all $x\in\mathcal{G}^{(0)}$, it follows that $\hat{a}|_{\mathcal{G}_x}=0$ for all $x\in U$. Thus $\theta(a)|_{\{x\}\times\mathrm{Sub}(\mathcal{G}^x_x)}=0$. By $C(\mathcal{G}^{(0)})$-modularity, it follows that $(\varphi\circ\theta)(a)|_{\pi^{-1}(x)}=0$ for all $x\in U$, where $\pi:\partial_F\mathcal{G}\to\mathcal{G}^{(0)}$ is the natural projection. By Lemma 7.5.1 the preimage of U in $\partial_F\mathcal{G}$ is dense. So $(\varphi\circ\theta)(a)=0$ follows. Hence, indeed, $(\varphi\circ\theta)(J_{\mathrm{sing}})=0$.

Now it follows that Ψ is a $\mathcal{G}$-pseudo-expectation $C^*_{\mathrm{ess}}(\mathcal{G})\to C(\partial_F\mathcal{G})$. By assumption, there is only one such $\mathcal{G}$-pseudo-expectation. It follows that $\Psi\circ\mathfrak{q}=\mathrm{E}\circ\pi_*$, where $\mathrm{E}: C^*_r(\mathcal{G}\ltimes\partial_F\mathcal{G})\to C(\partial_F\mathcal{G})$ is the (unique) conditional expectation described in Proposition 7.5.3. Thus $\varphi\circ\theta=\mathrm{E}\circ\pi_*$.

Now take $y \in \partial_F \mathcal{G}$ and set $x = \pi(y)$. The composition $\mathrm{ev}_y \circ \varphi$ defines a state on $C(Y)$, hence it corresponds to a probability measure $\mu_y = \varphi_*(\delta_y)$ on Y. The commutative diagram

$$\begin{array}{ccc} \partial_F \mathcal{G} & \xrightarrow{\quad \varphi_* \quad} & \mathcal{P}(Y) \\ & \searrow \quad \swarrow & \\ & \mathcal{P}(\mathcal{G}^{(0)}) & \end{array}$$

shows that $\mathrm{supp}(\mu_y) \subseteq \{x\} \times \mathrm{Sub}(\mathcal{G}_x^x)$. Now assume, for the sake of contradiction, that Λ is essentially confined. Then there exists $x \in \mathcal{G}^{(0)}$ such that $H \not\subseteq \overline{\mathcal{G}^{(0)}}_x$ for all $(x, H) \in Y = \overline{\mathcal{G} \cdot \Lambda}$. Choose $y \in \partial_F \mathcal{G}$ with $\pi(y) = x$. We claim that $\mu_y(\{(x, H) \in Y \mid g \in H\}) = 0$ for all $g \in \mathcal{G}_x^x$ with $g \notin \overline{\mathcal{G}^{(0)}}$. Indeed, take an open subset $U \subseteq \mathcal{G} \setminus \overline{\mathcal{G}^{(0)}}$ with $g \in U$. Choose $a \in C_c(U)$ satisfying $a(g) = 1$. Then $U \cap \overline{\mathcal{G}^{(0)}} = \emptyset$ implies that $\pi^{-1}(U) \cap \overline{\partial_F \mathcal{G}} = \emptyset$. Hence it follows that $\mathrm{E}(a \circ \pi) = 0$. Now, using $\mathrm{supp}(\mu_y) \subseteq \{x\} \times \mathrm{Sub}(\mathcal{G}_x^x)$, we obtain

$$\begin{aligned} \mu_y(\{(x, H) \in Y \mid g \in H\}) &= \int_Y 1_H(g) \mathrm{d}\mu_y(x, H) \\ &= \int_Y \theta(a)(x, H) \mathrm{d}\mu_y(x, H) \\ &= (\mathrm{ev}_y \circ \varphi \circ \theta)(a) \\ &= \mathrm{ev}_y((\mathrm{E} \circ \pi^*)(a)) \\ &= 0. \end{aligned}$$

To finish the proof, let us observe that $H \not\subseteq \overline{\mathcal{G}^{(0)}}_x$ for all $(x, H) \in Y$ implies that

$$(\{x\} \times \mathrm{Sub}(\mathcal{G}_x^x)) \cap Y = \bigcup_{g \in \mathcal{G}_x^x \setminus \overline{\mathcal{G}^{(0)}}_x} \{(x, H) \mid g \in H\}.$$

The latter set is exhausted by compact subsets of the form $\{x\} \times K_F$, where $K_F = \{H \in \mathrm{Sub}(\mathcal{G}_x^x) \mid F \cap H \neq \emptyset\}$ and $F \subseteq \mathcal{G}_x^x \setminus \overline{\mathcal{G}^{(0)}}$ is finite. Since μ_y is supported on $\{x\} \times \mathrm{Sub}(\mathcal{G}_x^x)$, inner regularity implies that there is a finite subset $F \subseteq \mathcal{G}_x^x \setminus \overline{\mathcal{G}^{(0)}}_x$ such that $0 < \mu_y(\{x\} \times K_F)$. We find that

$$0 < \mu_y(\{x\} \times K_F) \leq \sum_{g \in F} \mu_y(\{(x, H) \mid g \in H\}) = 0.$$

This is a contradiction. ■

Let us finish by proving the main theorem in this section.

Proof of Theorem 7.7.2. By Theorem 7.6.1, we know that the ideal intersection property for $C^*_{\mathrm{ess}}(\mathcal{G})$ is equivalent to essential principality of $\mathcal{G} \ltimes \partial_F \mathcal{G}$ and to the uniqueness of a $\mathcal{G}$-pseudo expectation $C^*_{\mathrm{ess}}(\mathcal{G}) \to C(\partial_F \mathcal{G})$. Also observe that every σ-compact étale groupoid is covered by countably many open bisections. So the present theorem follows from Propositions 7.7.5 and 7.7.6. ■

7.7.2 The Alexandrov groupoid

In this section we provide a full characterisation of the ideal intersection property for étale groupoids with locally compact space of units which are either Hausdorff or σ-compact. To this end, we will combine Theorem 7.7.2 with the study of a suitable notion of Alexandrov groupoid.

Recall that if X is a locally compact Hausdorff space, its Alexandrov or one-point compactification is the compact Hausdorff space $X^+ = X \sqcup \{\infty\}$ whose topology is determined by specifying that the inclusion $X \hookrightarrow X^+$ is a homeomorphism onto its image and a neighbourhood basis of ∞ is provided by the set $\{\infty\} \cup (X \setminus K)$, where $K \subseteq X$ runs through compact subsets of X. In particular, $X \subseteq X^+$ is dense if X is non-compact and ∞ is an isolated point in X^+ if X is compact.

The next definition provides a suitable notion of Alexandrov compactification for groupoids whose unit space is not necessarily compact.

Definition 7.7.7. Let $\mathcal{G}$ be an étale groupoid with locally compact Hausdorff space of units. Then the Alexandrov groupoid $\mathcal{G}^+$ is the set $\mathcal{G} \cup (\mathcal{G}^{(0)})^+$ with the topology determined by specifying that the inclusions $\mathcal{G}, (\mathcal{G}^{(0)})^+ \subseteq \mathcal{G}^+$ are open, and the groupoid structure extending the groupoid structure of $\mathcal{G}$ and making ∞ a unit.

We directly observe that $\mathcal{G}^+$ is an étale groupoid whose unit space is, by construction, the compact Hausdorff space $(\mathcal{G}^{(0)})^+$.

We identify the essential groupoid C*-algebra of the Alexandrov groupoid.

Proposition 7.7.8. *Let $\mathcal{G}$ be an étale groupoid with locally compact Hausdorff space of units. Then the inclusion $\mathcal{C}(\mathcal{G}) \subseteq \mathcal{C}(\mathcal{G}^+)$ extends to an inclusion $C^*_{\mathrm{ess}}(\mathcal{G}) \trianglelefteq C^*_{\mathrm{ess}}(\mathcal{G}^+)$ isomorphic with the unitisation $C^*_{\mathrm{ess}}(\mathcal{G}) \trianglelefteq C^*_{\mathrm{ess}}(\mathcal{G})^+$.*

Proof. Let us first observe that there is indeed an inclusion $\mathcal{C}(\mathcal{G}) \subseteq \mathcal{C}(\mathcal{G}^+)$, since $\mathcal{G} \subseteq \mathcal{G}^+$ is open. It moreover defines a *-isomorphism $\mathcal{C}(\mathcal{G})^+ \cong \mathcal{C}(\mathcal{G}^+)$ mapping $1 \in \mathcal{C}(\mathcal{G})^+$ to $1_{(\mathcal{G}^{(0)})^+}$. Let us identify the essential C*-algebra norm on $\mathcal{C}(\mathcal{G})$ and $\mathcal{C}(\mathcal{G}^+)$. To this end, we show that for all $f \in \mathcal{C}(\mathcal{G}^+)$ and $h_1, \ldots, h_n \in \mathcal{C}(\mathcal{G})$ there is $\tilde{f} \in \mathcal{C}(\mathcal{G})$ such that $f * h_i = \tilde{f} * h_i$ for all $i \in \{1, \ldots, n\}$. For each i, we know that $\operatorname{supp} h_i \subseteq \mathcal{G}$ is compact. So also $K = \bigcup_{i=1}^n \mathrm{r}(\operatorname{supp} h_i) \subseteq \mathcal{G}^{(0)}$ is compact. Let $g \in C_c(\mathcal{G}^{(0)})$ be a function satisfying $0 \leq g \leq 1$ and $g|_K \equiv 1$. Put $\tilde{f} = f \cdot (g \circ \mathrm{s})$ and observe that $\tilde{f} \in \mathcal{C}(\mathcal{G})$. Then for $x \in \mathcal{G}$ and $i \in \{1, \ldots, n\}$, we have

$$\tilde{f} * h_i(x) = \sum_{x=yz} \tilde{f}(y) h_i(z) = \sum_{x=yz} f(y) g(\mathrm{r}(z)) h_i(z) = \sum_{x=yz} f(y) h_i(z) = f * h_i(x) .$$

Denote by $\mathrm{E}_{\mathrm{ess}} : C^*_{\mathrm{ess}}(\mathcal{G}) \to \mathrm{Dix}(\mathcal{G}^{(0)})$ the local condition expectation and let $f \in \mathcal{C}(\mathcal{G})$. Recall that $\mathcal{G}^{(0)} \subseteq (\mathcal{G}^{(0)})^+$ is dense if $\mathcal{G}^{(0)}$ is non-compact and ∞ is an isolated point in $(\mathcal{G}^{(0)})^+$ otherwise. Combining this with [KM21, Proposition 4.10] and the previous paragraph, we find that for $f \in C_c(\mathcal{G})$,

$$\begin{aligned}
\|f\|^2_{C^*_{\mathrm{ess}}(\mathcal{G})} &= \sup\{\|\mathrm{E}(g^* * f^* * f * g)\| \mid g \in \mathcal{C}(\mathcal{G})\} \\
&= \sup_{g \in \mathcal{C}(\mathcal{G})} \inf_{U \subseteq \mathcal{G}^{(0)} \text{ dense open}} \sup_{x \in U} |g^* * f^* * f * g|(x) \\
&= \sup_{g \in \mathcal{C}(\mathcal{H})} \inf_{U \subseteq (\mathcal{G}^{(0)})^+ \text{ dense open}} \sup_{x \in U} |g^* * f^* * f * g|(x) \\
&= \|f\|^2_{C^*_{\mathrm{ess}}(\mathcal{G}^+)} .
\end{aligned}$$

This shows that $\mathcal{C}(\mathcal{G}) \subseteq \mathcal{C}(\mathcal{G}^+)$ extends to an inclusion $C^*_{\mathrm{ess}}(\mathcal{G}) \subseteq C^*_{\mathrm{ess}}(\mathcal{G}^+)$. The universal property of the unitisation provides an injective extension to a *-homomorphism $C^*_{\mathrm{ess}}(\mathcal{G})^+ \hookrightarrow C^*_{\mathrm{ess}}(\mathcal{G}^+)$, which must also be surjective since it restricts to the *-isomorphism $\mathcal{C}(\mathcal{G})^+ \cong \mathcal{C}(\mathcal{G}^+)$. ■

Let us now translate the condition about the absence of essentially confined amenable sections of isotropy.

Lemma 7.7.9. *Let $\mathcal{G}$ be an étale groupoid with locally compact Hausdorff space of units. Then $\mathcal{G}$ has no essentially confined amenable sections of isotropy groups if and only if $\mathcal{G}^+$ has no essentially confined amenable sections of isotropy groups.*

Proof. Assume that $\mathcal{G}^+$ has no essentially confined amenable sections of isotropy groups and let $(\Lambda_x)_{x \in \mathcal{G}^{(0)}}$ be an amenable section of isotropy groups of $\mathcal{G}$. Putting $\Lambda_\infty = \{\infty\}$, we obtain an amenable section of isotropy groups of $\mathcal{G}^+$. Put $Y = \overline{\mathcal{G}^+ \cdot \{(x, \Lambda_x) \mid x \in (\mathcal{G}^{(0)})^+\}} \subseteq \mathrm{Sub}(\mathcal{G}^+)$. By assumption, for every $x \in (\mathcal{G}^{(0)})^+$ there is $(x, H) \in Y$ such that $H \subseteq \overline{\mathcal{G}^{(0)}}^+{}_x$. Since ∞ is $\mathcal{G}^+$ invariant, this shows that $(\Lambda_x)_{x \in \mathcal{G}^{(0)}}$ is not essentially confined.

Assume that $\mathcal{G}$ has no essentially confined amenable sections of isotropy groups and let $(\Lambda_x)_{x \in (\mathcal{G}^{(0)})^+}$ be an amenable section of isotropy groups of $\mathcal{G}^+$, and then put $Y = \overline{\mathcal{G} \cdot \{(x, \Lambda_x) \mid x \in \mathcal{G}^{(0)}\}} \subseteq \mathrm{Sub}(\mathcal{G})$. By assumption, for every $x \in \mathcal{G}^{(0)}$, there is $(x, H) \in Y$ such that $H \subseteq \overline{\mathcal{G}^{(0)}}_x$. Since $\mathcal{G}^+_\infty = \{\infty\}$, this implies that $(\Lambda_x)_{x \in (\mathcal{G}^{(0)})^+}$ is not confined. ■

We are now able to combine the discussion in this section with our results from Section 7.7.1. This completes the proof of Theorem 7.1.2.

Theorem 7.7.10. *Let $\mathcal{G}$ be an étale groupoid with locally compact Hausdorff space of units. Assume that $\mathcal{G}$ is Hausdorff or $\mathcal{G}^+$ is σ-compact. Then $C^*_{\mathrm{ess}}(\mathcal{G})$ has the ideal intersection property if and only if $\mathcal{G}$ has no essentially confined amenable sections of isotropy groups. Further, $\mathcal{G}^+$ is σ-compact if $\mathcal{G}$ is σ-compact.*

Proof. Consider the Alexandrov groupoid $\mathcal{G}^+$. If $\mathcal{G}$ is Hausdorff, then $\mathcal{G}^{(0)} \subseteq \mathcal{G}$ is closed, so that also $(\mathcal{G}^+)^{(0)} = \mathcal{G}^{(0)} \cup \{\infty\} \subseteq \mathcal{G}^+$ is closed. So also $\mathcal{G}^+$ is Hausdorff. If $\mathcal{G}$ is σ-compact and $(K_n)_n$ is a sequence of compact subsets exhausting $\mathcal{G}$, then $(K_n \cup \{\infty\})_n$ exhausts $\mathcal{G}^+$. So the Alexandrov groupoid is also σ-compact.

Since $C^*_{\mathrm{ess}}(\mathcal{G}) \subseteq C^*_{\mathrm{ess}}(\mathcal{G}^+)$ is isomorphic with the inclusion $C^*_{\mathrm{ess}}(\mathcal{G}) \subseteq C^*_{\mathrm{ess}}(\mathcal{G})^+$ by Proposition 7.7.8, it follows that $C_0(\mathcal{G}^{(0)}) \subseteq C^*_r(\mathcal{G})$ has the ideal intersection property if and only if $C((\mathcal{G}^{(0)})^+) \subseteq C^*_r(\mathcal{G}^+)$ has the ideal intersection property. We can now combine Theorem 7.7.2 and Lemma 7.7.9 to conclude the proof. ■

Remark 7.7.11. As stated in Theorem 7.7.10, the Alexandrov groupoid $\mathcal{G}^+$ is always σ-compact if $\mathcal{G}$ is so. The converse holds if $\{x \in \mathcal{G}^{(0)} \mid \mathcal{G}_x = \{x\}\}$ is σ-compact, in particular if $\mathcal{G}_x \neq \{x\}$ holds for every $x \in \mathcal{G}^{(0)}$. Indeed, if $(K_n)_n$ is a sequence of compact subsets exhausting $\mathcal{G}^+$, then the subsets $C_n = K_n \setminus (\mathcal{G}^{(0)})^+ \subset \mathcal{G}$ are compact and exhaust $\mathcal{G} \setminus \mathcal{G}^{(0)}$. Then $\mathcal{G}^{(0)} = \{x \in \mathcal{G}^{(0)} \mid \mathcal{G}_x = \{x\}\} \cup \bigcup_n \mathrm{s}(C_n)$, showing that $\mathcal{G}$ is σ-compact.

A significant strengthening of the ideal intersection property is the ideal separation property. Given an étale Hausdorff groupoid with locally compact Hausdorff space of units, the inclusion $C_0(\mathcal{G}^{(0)}) \subseteq C^*_r(\mathcal{G})$ separates ideals if and only if every ideal in $C^*_r(\mathcal{G})$ is induced from $C_0(\mathcal{G}^{(0)})$ (see [BL20, Theorem 3.10]). The following result provides a characterisation of the ideal separation property of $C^*_r(\mathcal{G})$ in terms of the groupoid $\mathcal{G}$ itself, solving a problem described in [BL20, Paragraph after the proof of Theorem 3.10]. Its formulation employs the notion of inner exactness. Recall that a locally compact étale Hausdorff groupoid $\mathcal{G}$ is inner exact if and only if for every closed $\mathcal{G}$-invariant subset $X \subseteq \mathcal{G}^{(0)}$ the sequence

$$0 \to C_r^*(\mathcal{G}|_{\mathcal{G}^{(0)} \setminus X}) \to C_r^*(\mathcal{G}) \to C_r^*(\mathcal{G}|_X) \to 0 .$$

is exact.

Corollary 7.7.12. *Let $\mathcal{G}$ be a locally compact étale Hausdorff groupoid. Then $C_r^*(\mathcal{G})$ has the ideal separation property if and only if $\mathcal{G}$ is inner exact and for every $\mathcal{G}$-invariant closed subset $X \subseteq \mathcal{G}^{(0)}$ the restricted groupoid $\mathcal{G}|_X$ has no confined amenable sections of isotropy.*

Proof. By [BL20], $C_r^*(\mathcal{G})$ has the ideal separation property if and only if $\mathcal{G}$ is inner exact and has the residual ideal intersection property. The latter means by definition that $\mathcal{G}|_X$ has the ideal intersection property for every $\mathcal{G}$-invariant closed subset $X \subseteq \mathcal{G}^{(0)}$. So the result follows directly from Theorem 7.1.2. ∎

Let us now give a prove of Theorem 7.1.1, combining the results of this section.

Theorem 7.7.13. *Let $\mathcal{G}$ be an étale groupoid with locally compact Hausdorff space of units. Assume that $\mathcal{G}$ is Hausdorff, $\mathcal{G}$ is σ-compact or $\mathcal{G}$ has a compact space of units. Then the essential groupoid C*-algebra $C_{\text{ess}}^*(\mathcal{G})$ is simple if and only if $\mathcal{G}$ is minimal and has no essentially confined amenable sections of isotropy groups.*

Proof. In view of Theorems 7.7.2 and 7.7.10, we only have to show that $\mathcal{G}$ is minimal if $C_{\text{ess}}^*(\mathcal{G})$ is simple. Assume that $\mathcal{G}$ is not minimal and let $U \subseteq \mathcal{G}^{(0)}$ be a non-empty, proper, open $\mathcal{G}$-invariant subset of its unit space. Consider the ideal $I = \overline{C_{\text{ess}}^*(\mathcal{G}) C_0(U) C_{\text{ess}}^*(\mathcal{G})}$ generated by $C_0(U)$ inside the essential groupoid C*-algebra. It is non-zero since U is non-empty and it is proper since its preimage in $C_r^*(G)$ is the proper ideal $\overline{C_r^*(\mathcal{G}) C_0(U) C_r^*(\mathcal{G})}$. So $C_{\text{ess}}^*(\mathcal{G})$ is not simple. ∎

Let us next compare our Theorem 7.7.10 with the work of Kwaśniewski-Meyer [KM21] and Kalantar-Scarparo [KS21]. In order to do so, we need to introduce their notion of topologically free groupoids, which is a strengthening of essentially effective groupoids as introduced in Definition 7.6.3, as we will observe below.

Definition 7.7.14 (See [KM21, Definition 2.20])**.** Let $\mathcal{G}$ be an étale groupoid with locally compact Hausdorff space of units. Then $\mathcal{G}$ is topologically free, if for every bisection $U \subset \mathcal{G} \setminus \mathcal{G}^{(0)}$ the set $\{x \in \mathcal{G}^{(0)} \mid \mathcal{G}_x^x \cap U \neq \emptyset\}$ has empty interior.

Remark 7.7.15. In their recent work [KS21, Theorem 6.3], Kalantar-Scarparo obtained a characterisation of simplicity for crossed product C*-algebra. Their result is implied by our Theorem 7.7.10 as applied to transformation groupoids arising from minimal group actions.

Kwaśniewski-Meyer [KM21, Theorem 7.29] characterised étale groupoids with locally compact Hausdorff space of units for which the kernel of $C^*(\mathcal{G}) \to C^*_{\mathrm{ess}}(\mathcal{G})$ is the unique maximal ideal of $C^*(\mathcal{G})$ that intersects $C_0(\mathcal{G}^{(0)})$ trivially. These are precisely the topologically free groupoids in the sense of Definition 7.7.14. In particular, the essential groupoid C*-algebra of a topologically free groupoid is simple. Let us explain how simplicity of $C^*_{\mathrm{ess}}(\mathcal{G})$ can be proven by means of our Theorem 7.7.10, and we refer to Corollary 7.7.16 for a related characterisation of simplicity. We assume that $\mathcal{G}$ is topologically free and observe that the bisection U in Definition 7.7.14 may be assumed to be open and a subset of $\mathcal{G} \setminus \overline{\mathcal{G}^{(0)}}$. Fix a compact subset $C \subset \mathcal{G}^{(0)}$ with non-empty interior. Given a section of isotropy groups $(\Lambda_x)_{x \in \mathcal{G}^{(0)}}$ (amenable or not), we will exhibit a net $(x_K)_K$ in C indexed by compact subsets of $\overline{\mathcal{G}^{(0)}}$ such that $\Lambda_{x_K} \cap K = \emptyset$. Compactness of the Chabauty space $\mathrm{Sub}(\mathcal{G})$, which is a non-trivial fact for non-Hausdorff groupoids [Fel62], then implies that $(\Lambda_{x_K})_K$ has a cluster point, which is a subgroup of $\overline{\mathcal{G}^{(0)}}_x$ for some $x \in C$. Let $K \subseteq \mathcal{G} \setminus \overline{\mathcal{G}^{(0)}}$ be a compact subset and let $U_1, \ldots, U_n$ be open bisections of $\mathcal{G}$ covering K. Then $\{x \in \mathcal{G}^{(0)} \mid \bigcup_{i=1}^n U_i \cap \mathcal{G}_x^x \neq \emptyset\}$ has empty interior. In particular, there is $x \in C$ such that $\Lambda_x \cap K = \emptyset$. We can put $x_K = x$ and finish the argument, showing that $(\Lambda_x)_x$ is not essentially confined.

As explained in the introduction, characterisations of simplicity and a suitable generalised ideal intersection property of groupoid C*-algebras were limited to the maximal groupoid C*-algebra, as in [BCFS14, Theorem 5.1] and [KM21, Theorem 7.29]. For amenable groupoids, this leads to characterisations of simplicity of the essential groupoid C*-algebra. The next corollary clarifies the relation of our Theorem 7.7.10 to such characterisations for amenable groupoids.

Corollary 7.7.16. *Let $\mathcal{G}$ be a σ-compact étale groupoid with locally compact Hausdorff space of units and amenable isotropy groups. Then $C^*_{\mathrm{ess}}(\mathcal{G})$ has the ideal intersection property if and only if $\mathcal{G}$ is topologically free.*

Proof. If $\mathcal{G}$ is topologically free, it follows from [KM21, Theorem 7.29] (see also Remark 7.7.15) that $C^*_{\mathrm{ess}}(\mathcal{G})$ has the ideal intersection property. Assume that $\mathcal{G}$ is not topologically free. Then there is a bisection $U \subseteq \mathcal{G} \setminus \mathcal{G}^{(0)}$ such that $\{x \in \mathcal{G}^{(0)} \mid \mathcal{G}_x^x \cap U \neq \emptyset\}$ has non-empty interior. We may assume that U is open and that $U \subset \mathrm{Iso}(\mathcal{G})$. Putting $\Lambda_x = \mathcal{G}_x^x$, we obtain an amenable section of isotropy groups of $\mathcal{G}$. We will show that it is confined. For every $g \in \mathcal{G}_x$, the equality $g\Lambda_x g^{-1} = \Lambda_{\mathrm{r}(g)}$ holds, so that $\overline{\mathcal{G} \cdot \{\Lambda_x\}_x} = \overline{\{\Lambda_x\}_x}$. Let us denote this set by Y. Since for all $x \in \mathrm{s}(U)$, we have $\overline{\Lambda_x \cap U \neq \emptyset}$, also every element $(H, x) \in Y$ with $x \in \mathrm{s}(U)$ satisfies $H \cap U \neq \emptyset$. So $H \not\subseteq \overline{\mathcal{G}^{(0)}}$. We can now apply Theorem 7.7.10 and conclude that $C^*_{\mathrm{ess}}(\mathcal{G})$ does not have the ideal intersection property. ■

7.8 Powers averaging for minimal groupoids

In this section, we generalise the work done in [AU22], and derive the Powers averaging property for simple essential groupoid C*-algebras based on the notion of generalised probability measures. We also prove the relative Powers averaging property for certain semigroups of generalised probability measures, leading to natural C*-irreducible inclusions into essential groupoid C*-algebras and applications to unitary representations in Section 7.9.

7.8.1 Generalised probability measures

In [AU22], a notion of generalised probability measure on a compact space X was introduced. It combines the action of a group $G \curvearrowright X$ with the action of positive elements in $C(X)$ by elementary operators. This idea naturally fits with our approach to the ideal intersection property for groupoid C*-algebras by means of the pseudogroup of open bisections.

Definition 7.8.1. Let $\mathcal{G}$ be an étale groupoid with compact Hausdorff space of units. A (finite) generalised $\mathcal{G}$-probability measure on $\mathcal{G}^{(0)}$ is a finite formal sum of pairs $\sum_{i \in I}(\gamma_i, f_i)$ such that $\gamma_i \in \Gamma(\mathcal{G})$ and $f_i \in C_c(\mathrm{supp}(\gamma_i), \mathbb{R}_{\geq 0})$ is a positive, continuous, compactly supported function on the support of γ_i satisfying $\sum_i f_i \circ \psi_{\gamma_i^*} = 1_{\mathcal{G}^{(0)}}$. We denote the set of (finite) generalised $\mathcal{G}$-probability measures on $\mathcal{G}^{(0)}$ by $\mathcal{CP}_{\mathcal{G}}(\mathcal{G}^{(0)})$.

We point out that the concept of generalised probability measures allows for the repetition of bisections. Therefore, it is not necessarily true that the formal sum in the definition can be indexed by a subset of $\Gamma(\mathcal{G})$.

Remark 7.8.2. In some situations it is preferable to think of a pair (γ, f) as in Definition 7.8.1 and being represented as $(f \circ \psi_{\gamma^*}, \gamma)$. This is for example the case when interpreting the unitality condition $\sum_i f_i \circ \psi_{\gamma_i^*} = 1_{\mathcal{G}^{(0)}}$ and later in Definition 7.8.5, where covering semigroups of generalised probability measures are introduced.

Generalised probability measures naturally act both on groupoid C*-algebras and on their state spaces. To prove this, we will first exhibit their natural semigroup structure.

Proposition 7.8.3. *Let $\mathcal{G}$ be an étale groupoid with compact Hausdorff space of units. Given two generalised probability measures $\mu_1 = \sum_i (\gamma_{1,i}, f_{1,i})$ and $\mu_2 = \sum_j (\gamma_{2,j}, f_{2,j})$ from $\mathcal{CP}_{\mathcal{G}}(\mathcal{G}^{(0)})$, we define their product by the following formula.*

$$\mu_1\mu_2 = \sum_{i,j} (\gamma_{1,i}\gamma_{2,j}, (f_{1,i} \circ \psi_{\gamma_{2,j}}) f_{2,j}) \, .$$

Then $\mathcal{CP}_{\mathcal{G}}(\mathcal{G}^{(0)})$ becomes a semigroup with this product.

Proof. Once the product is well-defined, its associativity follows from associativity of the product in $\Gamma(\mathcal{G})$ and the fact that $\Gamma(\mathcal{G})$ acts on $C(\mathcal{G}^{(0)})$. Thus, we only have to prove that $\mu_1\mu_2$ as defined above is again an element of $\mathcal{CP}_{\mathcal{G}}(\mathcal{G}^{(0)})$.

First observe that for all i, j the product $(f_{1,i} \circ \psi_{\gamma_{2,j}}) f_{2,j}$ is a continuous function whose support lies in $\psi_{\gamma_{2,j}^*}(\operatorname{supp} \gamma_{1,i} \cap \operatorname{im} \gamma_{2,j}) = \operatorname{supp} \gamma_{1,i}\gamma_{2,j}$. Further,

$$(f_{1,i} \circ \psi_{\gamma_{2,j}}) f_{2,j} = (f_{1,i}(f_{2,j} \circ \psi_{\gamma_{2,j}^*})) \circ \psi_{\gamma_{2,j}} \in C_c(\operatorname{supp} \gamma_{1,i}\gamma_{2,j}) \, .$$

Evaluating the following sum in the space of Borel functions on $\mathcal{G}^{(0)}$, we find that

$$\begin{aligned} \sum_{i,j} ((f_{1,i} \circ \psi_{\gamma_{2,j}}) f_{2,j}) \circ \psi_{(\gamma_{1,i}\gamma_{2,j})^*} &= \sum_i f_{1,i} \circ \psi_{\gamma_{1,i}^*} \cdot \left(\sum_j f_{2,j} \circ \psi_{\gamma_{2,j}^*} \right) \circ \psi_{\gamma_{1,i}^*} \\ &= \sum_i (f_{1,i} \circ \psi_{\gamma_{1,i}^*}) \cdot 1_{\operatorname{im} \gamma_{1,i}} \\ &= 1_{\mathcal{G}^{(0)}} \, . \end{aligned}$$

This finishes the proof of the proposition. ■

Generalised probability measures were introduced in [AU22] to provide a notion of contractibility of measures on spaces admitting a minimal group action. In order to generalise this to groupoids in the context of C*-simplicity, we require suitable actions of $\mathcal{CP}_{\mathcal{G}}(\mathcal{G}^{(0)})$ on $C(\mathcal{G}^{(0)})$ and on $C^*_{\mathrm{ess}}(\mathcal{G})$.

Proposition 7.8.4. *Let $\mathcal{G}$ be an étale groupoid with compact Hausdorff space of units and let (A, α) be a unital $\mathcal{G}$-C*-algebra. There is a semigroup action of $\mathcal{CP}_{\mathcal{G}}(\mathcal{G}^{(0)})$ on A by completely positive, completely contractive and non-degenerate maps defined by*

$$\mu a = \sum_i \alpha_{\gamma_i}(f_i^{1/2} a f_i^{1/2})$$

for $a \in A$ and $\mu = \sum_i (\gamma_i, f_i)$. Consequently, there is a right semigroup action on the state space $\mathcal{S}(A)$ of A defined by

$$(\varphi\mu)(a) = \varphi(\mu a)$$

for $\varphi \in \mathcal{S}(A)$, $\mu \in \mathcal{CP}_{\mathcal{G}}(\mathcal{G}^{(0)})$ and $a \in A$.

Proof. Let us first check that the formula in the statement defines an action. Given two generalised $\mathcal{G}$-probability measures $\mu_1 = \sum_i (\gamma_{1,i}, f_{1,i})$ and $\mu_2 = \sum_j (\gamma_{2,j}, f_{2,j})$ and an element $a \in A$, we find

$$\begin{aligned}
\mu_1(\mu_2 a) &= \mu_1(\sum_j \alpha_{\gamma_{2,j}}(f_{2,j}^{\frac{1}{2}} a f_{2,j}^{\frac{1}{2}})) \\
&= \sum_i \alpha_{\gamma_{1,i}}(f_{1,i}^{\frac{1}{2}} \sum_j \alpha_{\gamma_{2,j}}(f_{2,j}^{\frac{1}{2}} a f_{2,j}^{\frac{1}{2}}) f_{1,i}^{\frac{1}{2}}) \\
&= \sum_i \alpha_{\gamma_{1,i}}(\sum_j \alpha_{\gamma_{2,j}}((f_{1,i} \circ \psi_{\gamma_{2,j}})^{\frac{1}{2}} f_{2,j}^{\frac{1}{2}} a f_{2,j}^{\frac{1}{2}} (f_{1,i} \circ \psi_{\gamma_{2,j}})^{\frac{1}{2}})) \\
&= (\mu_1 \mu_2)(a)\,.
\end{aligned}$$

For every $\mu \in \mathcal{CP}_{\mathcal{G}}(\mathcal{G}^{(0)})$ the map $a \mapsto \mu a$ is completely positive as a sum of a composition of completely positive maps. Further, it is unital on A, since

$$\mu(1_A) = \sum_i \alpha_{\gamma_i}(f_i^{\frac{1}{2}} 1_A f_i^{\frac{1}{2}}) = \sum_i \alpha_{\gamma_i}(f_i^{\frac{1}{2}} f_i^{\frac{1}{2}}) = \sum_i \alpha_{\gamma_i}(f_i) = 1_{\mathcal{G}^{(0)}} = 1_A\,.$$

This shows that $a \mapsto \mu a$ is completely contractive and non-degenerate on A. It follows directly that $\varphi\mu(a) = \varphi(\mu a)$ defines an action on the state space of $\mathcal{S}(A)$. ∎

7.8.2 Contractive and covering semigroups

When we consider Powers averaging property, we will require the following definition, which provides a suitable generalisation of boundary actions of groups.

Definition 7.8.5. Let $\mathcal{G}$ be an étale groupoid with compact Hausdorff space of units. We will say that a subsemigroup $S \subseteq \mathcal{CP}_{\mathcal{G}}(\mathcal{G}^{(0)})$ is covering if for every $g \in \mathcal{G}$ there is $\mu = \sum_i (\gamma_i, f_i) \in S$ such that $f_i \circ \psi_{\gamma_i^*}(\mathrm{r}(g)) \neq 0$ implies $g \in \gamma_i$. If $\mathcal{G}$ is minimal, we will say that S is contractive if for any $\nu \in \mathcal{P}(\mathcal{G}^{(0)})$ and any $x \in \mathcal{G}^{(0)}$ there is a net $(\mu_i)_i$ in S such that $\nu\mu_i \xrightarrow{\mathrm{w}^*} \delta_x$.

The following result was established for the special case of crossed products by minimal group actions in [AU22, Lemma 3.6].

Proposition 7.8.6. *Let $\mathcal{G}$ be an étale groupoid with compact Hausdorff space of units. Then $\mathcal{CP}_{\mathcal{G}}(\mathcal{G}^{(0)})$ is a covering semigroup. If $\mathcal{G}$ is minimal, then $\mathcal{CP}_{\mathcal{G}}(\mathcal{G}^{(0)})$ is contractive.*

Proof. Let us first prove that $\mathcal{CP}_{\mathcal{G}}(\mathcal{G}^{(0)})$ is covering. To this end take $g \in \mathcal{G}$, let γ be an open bisection of $\mathcal{G}$ containing g and let $f \in C_c(\operatorname{im}\gamma)$ with $0 \leq f \leq 1$ and $f(\mathrm{r}(g)) = 1$. Denoting by $e \in G$ the neutral element, define

$$\mu_g = (\gamma, f \circ \psi_\gamma) + (e, 1 - f)\,.$$

Then μ_g satisfies the conditions of Definition 7.8.5 and hence witnesses that $\mathcal{CP}_{\mathcal{G}}(\mathcal{G}^{(0)})$ is covering.

Assume that $\mathcal{G}$ is minimal. Fix any $x \in \mathcal{G}^{(0)}$ and an open neighbourhood V of x. We will find $\mu_V \in \mathcal{CP}_{\mathcal{G}}(\mathcal{G}^{(0)})$ such that $\nu\mu_V$ is supported in $\overline{V}$ for all $\nu \in \mathcal{P}(\mathcal{G}^{(0)})$. Given any $y \in \mathcal{G}^{(0)}$, by minimality there exists some $\gamma_y \in \Gamma(\mathcal{G})$ with $y \in \operatorname{supp}\gamma_y$ and $\psi_{\gamma_y}(y) \in V$. In other words, $y \in \psi_{\gamma_y^*}(V)$. It follows that the family $(\psi_{\gamma^*}(V))_{\gamma \in \Gamma(\mathcal{G})}$ is an open cover of $\mathcal{G}^{(0)}$. So by compactness, there is a finite number of open bisections $\gamma_1, \ldots, \gamma_n$ such that $(\psi_{\gamma_i^*}(V))_{i=1}^n$ covers $\mathcal{G}^{(0)}$. Let g_i be a partition of unity subordinate to this open cover and put $f_i = g_i \circ \psi_{\gamma_i}$ as well as $\mu_V = \sum_{i=1}^n (\gamma_i, f_i)$. Then μ_V is a generalised probability measure.

Let $\nu \in \mathcal{P}(\mathcal{G}^{(0)})$ and let $h \in C(\mathcal{G}^{(0)})$ be such that $h \geq 0$ and $h|_{\overline{V}} = 0$. We have

$$(\nu\mu_V)(h) = \nu(\mu_V h) = \sum_i \nu(\alpha_{\gamma_i}(f_i^{\frac{1}{2}} h f_i^{\frac{1}{2}})) = 0\,,$$

since all functions f_i are supported in V. ■

Recall from Section 7.4.1 the action of $\mathcal{G}$ on states in $\mathcal{S}_{\mathcal{G}^{(0)}}(A)$ for a $\mathcal{G}$-C*-algebra A. That is, if $\varphi \in \mathcal{S}(A)$ satisfies $\varphi|_{C(\mathcal{G}^{(0)})} = \mathrm{ev}_x$, then $g\varphi = \varphi(\alpha_{\gamma^*}(f \cdot f))$ for any $g \in \mathcal{G}_x$, any open bisection γ containing g and any $f \in C_c(\operatorname{im}\gamma)$ satisfying $f(\mathrm{r}(g)) = 1$. The next lemmas shows how covering semigroups can be used to implement this action.

Lemma 7.8.7. *Let $\mathcal{G}$ be an étale groupoid with compact Hausdorff space of units and let (A, α) be a unital $\mathcal{G}$-C*-algebra. Assume that $S \subseteq \mathcal{CP}_{\mathcal{G}}(\mathcal{G}^{(0)})$ is a semigroup covering $\mathcal{G}$ and let $\varphi \in \mathcal{S}(A)$ be a state satisfying $\varphi|_{C(\mathcal{G}^{(0)})} = \mathrm{ev}_x$ for some $x \in \mathcal{G}^{(0)}$. Then for every $g \in \mathcal{G}_x$ there is $\mu_g \in S$ such that $\varphi\mu_g = g\varphi$.*

Proof. Since S is covering, we can choose $\mu_g = \sum_i (\gamma_i^*, f_i) \in S$ such that $f_i \circ \psi_{\gamma_i}(\mathrm{s}(g)) \neq 0$ implies $g \in \gamma_i$. The intersection of all γ_i containing g is an open bisection γ containing g. Since all functions f_i have compact support and $\mathcal{G}^{(0)}$ is Hausdorff, we can choose $h \in C_c(\operatorname{im}\gamma)$ with $0 \leq h \leq 1$ and $h(x) = 1$ such that $\operatorname{supp} h \cap \operatorname{supp} f_i \circ \psi_{\gamma_i} = \emptyset$ whenever $g \notin \gamma_i$. Then we can calculate

$$
\begin{aligned}
\varphi\mu_g &= \sum_i \varphi(\alpha_{\gamma_i^*}(f_i^{\frac{1}{2}} \cdot f_i^{\frac{1}{2}})) \\
&= \sum_i h(x)\varphi(\alpha_{\gamma_i^*}(f_i^{\frac{1}{2}} \cdot f_i^{\frac{1}{2}})) \\
&= \sum_i \varphi(h\alpha_{\gamma_i^*}(f_i^{\frac{1}{2}} \cdot f_i^{\frac{1}{2}})) \\
&= \sum_i \varphi(\alpha_{\gamma^*}((\alpha_\gamma(h)f_i)^{\frac{1}{2}} \cdot (\alpha_{\gamma^*}(h)f_i)^{\frac{1}{2}})) \\
&= g\varphi \cdot \left(\sum_i (\alpha_\gamma(h)f_i)(\mathrm{r}(g))\right) \\
&= g\varphi \cdot \left(\sum_i (h(x) f_i \circ \psi_{\gamma_i}(\mathrm{s}(g))\right) \\
&= g\varphi\,.
\end{aligned}
$$

This finishes the proof. ■

We will need to know that the contractivity of certain semigroups of generalised probability measures is preserved by passing to the Furstenberg groupoid.

Lemma 7.8.8. *Let $\mathcal{G}$ be a minimal étale groupoid with compact Hausdorff space of units and let $S \subseteq \mathcal{CP}_{\mathcal{G}}(\mathcal{G}^{(0)})$ be a contractive and covering semigroup. Denote by $\pi : \partial_F\mathcal{G} \to \mathcal{G}^{(0)}$ the projection map. For $\mu = \sum_i (\gamma_i, f_i) \in \mathcal{CP}_{\mathcal{G}}(\mathcal{G}^{(0)})$ write $\pi^*(\mu) = \sum_i (\pi^{-1}(\gamma_i), f_i \circ \pi) \in \mathcal{CP}_{\mathcal{G} \ltimes \partial_F\mathcal{G}}(\partial_F\mathcal{G})$. Then $\pi^*(S) \subseteq \mathcal{CP}_{\mathcal{G} \ltimes \partial_F\mathcal{G}}(\partial_F\mathcal{G})$ is contractive and covering.*

Proof. It is clear that $\pi^*(S)$ is covering, so we have to show that it is contractive. Note that considering $C(\partial_F\mathcal{G})$ as a $\mathcal{G}$-C*-algebra, we have $\nu\pi^*(\mu) = \nu \cdot \mu$, for all $\nu \in \mathcal{P}(\partial_F\mathcal{G})$ and all $\mu \in \mathcal{CP}_{\mathcal{G}}(\mathcal{G}^{(0)})$. We will use the simple notation $\nu\mu$. Let $\nu \in \mathcal{P}(\partial_F\mathcal{G})$ and $y \in \partial_F\mathcal{G}$. Put $x = \pi(y) \in \mathcal{G}^{(0)}$. Since S is contractive, we can find a net $(\mu_i)_i$ in S such that $\pi_*(\nu)\mu_i \to \delta_x$. Passing to a subnet, we may assume that $\nu\mu_i \to \tilde{\nu}$ for some $\tilde{\nu} \in \mathcal{P}(\partial_F\mathcal{G})$. Then $\pi_*(\tilde{\nu}) = \delta_x$ holds. Since $\mathcal{G}$ is minimal, Proposition 7.4.13 and strong proximality of $\partial_F\mathcal{G}$ imply that there is a net $(g_i)_i$ in $\mathcal{G}_x$ such that $g_i\tilde{\nu} \to \delta_y$ for some $y \in \pi^{-1}(x)$. Lemma 7.8.7 shows that there is a net $(\tilde{\mu}_i)_i$ in S such that $\tilde{\nu}\tilde{\mu}_i = g_i\tilde{\nu} \to \delta_y$. Summarising, we found that $\delta_y \in \overline{\nu \cdot S} = \overline{\nu \cdot \pi^*(S)}$, which proves contractivity of $\pi^*(S)$. ■

7.8.3 Powers averaging

We are ready to establish Powers averaging property for simple essential groupoid C*-algebras. We begin by considering the state space of $C^*_{\mathrm{ess}}(\mathcal{G})$. To put the statement of the next proposition into context, let us recall that the natural conditional expectation $\mathrm{E}_{\mathrm{ess}} : C^*_{\mathrm{ess}}(\mathcal{G}) \to C(\partial_F\mathcal{G})$ does not take values in $C(\mathcal{G}^{(0)})$ if $\mathcal{G}$ is not Hausdorff. Also recall from Theorem 7.7.13 that if $C^*_{\mathrm{ess}}(\mathcal{G})$ is simple, then $\mathcal{G}$ is necessarily minimal, so that Theorem 7.5.4 implies there is a natural inclusion $C^*_{\mathrm{ess}}(\mathcal{G}) \subseteq C^*_{\mathrm{ess}}(\mathcal{G} \ltimes \partial_F\mathcal{G})$. In the remainder of this section, we let $\mathrm{E} : C^*_{\mathrm{ess}}(\mathcal{G} \ltimes \partial_F\mathcal{G}) \to C(\partial_F\mathcal{G})$ denote the conditional expectation described in Proposition 7.5.3.

In order to formulate averaging results conveniently, it is useful to introduce a notion of convex combinations of generalised probability measures.

Definition 7.8.9. Let $\mathcal{G}$ be an étale groupoid with compact Hausdorff space of units. For generalised probability measures indexed over disjoint sets I and J, a formal convex combination is an expression of the form

$$c\sum_{i\in I}(\gamma_i, f_i) + (1-c)\sum_{j\in J}(\gamma_j, f_j) = \sum_{i\in I\sqcup J}(\gamma_i, 1_I(i)cf_i + 1_J(i)(1-c)f_i).,$$

We call a subset of $\mathcal{CP}_{\mathcal{G}}(\mathcal{G}^{(0)})$ convex if it is closed under convex combinations.

Proposition 7.8.10. *Let $\mathcal{G}$ be an étale groupoid with compact Hausdorff space of units. Let $S \subseteq \mathcal{CP}_{\mathcal{G}}(\mathcal{G}^{(0)})$ be a contractive and covering convex semigroup. Assume that $C^*_{\mathrm{ess}}(\mathcal{G})$ is simple. Then given any $\varphi \in \mathcal{S}(C^*_{\mathrm{ess}}(\mathcal{G} \ltimes \partial_F\mathcal{G}))$, we have*

$$\{\nu \circ \mathrm{E} \mid \nu \in \mathcal{P}(\partial_F\mathcal{G})\} \subseteq \overline{\{\varphi\mu \mid \mu \in S\}}^{\mathrm{w}^*}$$

Proof. Write $K = \overline{\{\varphi\mu \mid \mu \in S\}}^{\mathrm{w}^*}$ and observe that K is weak-* closed and convex. Hence, it suffices to prove that $\mathrm{ev}_y \circ \mathrm{E} \in K$ for all $y \in \partial_F\mathcal{G}$. Fix $y \in \partial_F\mathcal{G}$ and let $x = \pi(y) \in \mathcal{G}^{(0)}$. By contractivity of S, there is $\sigma \in K$ satisfying $\sigma|_{C(\mathcal{G}^{(0)})} = \mathrm{ev}_x$. Since S is both contractive and covering, Lemma 7.8.8 shows that there is a net $(\mu_i)_i$ in S such that $\sigma|_{C(\partial_F\mathcal{G})}\mu_i \to \mathrm{ev}_y$. Dropping to a subnet, we may assume that $\sigma\mu_i \to \omega \in \mathcal{S}(C^*_{\mathrm{ess}}(\mathcal{G} \ltimes \partial_F\mathcal{G}))$. Observe that $\omega|_{C^*_{\mathrm{ess}}(\mathcal{G})} \in K$.

By Theorem 7.5.10 we can identify $C^*_{\mathrm{ess}}(\mathcal{G} \ltimes \partial_F\mathcal{G}) \cong C^*_r(\mathcal{H})$ for the Hausdorffification $\mathcal{H} = (\mathcal{G} \ltimes \partial_F\mathcal{G})_{\mathrm{Haus}}$. Then E is identified with the natural conditional expectation of $C^*_r(\mathcal{H})$. We use the fact that $C^*_r(\mathcal{H})$ is simple to infer by Theorem 7.6.1 that $\mathcal{H}$ is principal. So it suffices to show that $\omega(u_\gamma f) = 0$ for all open bisections $\gamma \subseteq \mathcal{H} \setminus \partial_F\mathcal{G}$ and all positive functions $f \in C_c(\operatorname{supp}\gamma)$. If $y \notin \operatorname{supp}\gamma$, then we see that

$$\omega(u_\gamma f) = \omega(u_\gamma f^{\frac{1}{2}})\omega(f^{\frac{1}{2}}) = \omega(u_\gamma f^{\frac{1}{2}})f^{\frac{1}{2}}(y) = 0\,.$$

If $y \in \operatorname{supp}\gamma$, the fact that $\psi_\gamma(y) \neq y$ allows us to choose a function $h \in C_c(\operatorname{supp}\gamma)$ such that $h(y) = 1$ and $h(\psi_\gamma(y)) = 0$. It follows that

$$\omega(u_\gamma f) = h(y)\omega(u_\gamma f) = \omega(hu_\gamma f) = \omega(u_\gamma f(\alpha_{\gamma^*}(h))) = \omega(u_\gamma f)h(\psi_\gamma(y)) = 0\,.$$

This finishes the proof. ■

We now extend the previous result to the entire dual space of $C^*_{\mathrm{ess}}(\mathcal{G})$.

Corollary 7.8.11. *Let $\mathcal{G}$ be an étale groupoid with compact Hausdorff space of units. Assume that $C^*_{\mathrm{ess}}(\mathcal{G})$ is simple, let $S \subseteq \mathcal{CP}_{\mathcal{G}}(\mathcal{G}^{(0)})$ be a contractive and covering convex semigroup and let $\omega \in C^*_{\mathrm{ess}}(\mathcal{G} \ltimes \partial_F\mathcal{G})^*$. Then*

$$\{\omega(1)\nu \circ E \mid \nu \in P(\partial_F\mathcal{G})\} \subseteq \overline{\{\omega\mu \mid \mu \in S\}}^{\mathrm{w}^*}\,.$$

Proof. We write $K = \overline{\{\omega\mu \mid \mu \in S\}}^{\mathrm{w}^*}$. Since K is convex and weak-* closed, it suffices to shows that $\omega(1)\mathrm{ev}_y \circ \mathrm{E} \in K$ for any $y \in \partial_F\mathcal{G}$. We decompose $\omega = \sum_{k=1}^4 c_k\varphi_k$ as a convex combination with four states $\varphi_i \in \mathcal{S}(C^*_{\mathrm{ess}}(\mathcal{G} \ltimes \partial_F\mathcal{G}))$. By Proposition 7.8.10, we may find a net (μ_i) in S with $\varphi_1\mu_i \xrightarrow{\mathrm{w}^*} \nu_1 \circ \mathrm{E}$ for some $\nu_1 \in \mathcal{P}(\partial_F\mathcal{G})$. Dropping to a subset, we may assume that $\omega\mu_i \xrightarrow{\mathrm{w}^*} \nu_1 \circ \mathrm{E} + \sum_{k=2}^4 c_k\varphi'_k$ for some new states $\varphi'_k \in \mathcal{S}(C^*_{\mathrm{ess}}(\mathcal{G}))$. Repeating this process three more times, and noting that the set $\{\nu \circ \mathrm{E} \mid \nu \in \mathcal{P}(\partial_F\mathcal{G})\}$ is invariant under the right action of $\mathcal{CP}_{\mathcal{G}}(\mathcal{G}^{(0)})$, we see that there is some element of K of the form $\omega(1)\nu \circ \mathrm{E}$ for some finite complex measure ν on $\partial_F\mathcal{G}$. Thanks to Lemma 7.8.8 we find that $\omega(1)\mathrm{ev}_y \circ \mathrm{E} \in K$ for every $y \in \partial_F\mathcal{G}$. ■

We are now able to dualise Corollary 7.8.11 in order to obtain a version of Powers averaging property for essential groupoid C*-algebras. The next definition provides a notion of Powers averaging that subsumes the classical notion for groups and dynamical systems.

Definition 7.8.12. Let $\mathcal{G}$ be an étale groupoid with compact Hausdorff space of units and let $S \subset \mathcal{CP}_{\mathcal{G}}(\mathcal{G}^{(0)})$ be a subsemigroup. We say that $C^*_{\text{ess}}(\mathcal{G})$ satisfies relative Powers averaging property with respect to S if $0 \in \overline{\text{conv}}\{\mu a \mid \mu \in S\}$ for all $a \in C^*_{\text{ess}}(\mathcal{G})$ satisfying $\mathrm{E}_{\text{ess}}(a) = 0$.

Recall that in [Rør21] a unital inclusion of C*-algebras $A \subseteq B$ is said to be C*-irreducible if every intermediate C*-algebra is simple. This notion will be linked to the relative Powers averaging property by means the following definition, relating generalised probability measures to C*-subalgebras of $C^*_{\text{ess}}(\mathcal{G})$.

Definition 7.8.13. Let $\mathcal{G}$ be an étale groupoid with compact Hausdorff space of units and let $\mu = \sum_{i=1}^{n}(\gamma_i, f_i) \in \mathcal{CP}_{\mathcal{G}}(\mathcal{G}^{(0)})$ be a generalised probability measure. Let $A \subseteq C^*_{\text{ess}}(\mathcal{G})$ be a C*-subalgebra. Then A is said to *support* μ, if $u_{\gamma_i} f_i \in A$ for all $i \in \{1, \dots, n\}$. Further, if $S \subseteq \mathcal{CP}_{\mathcal{G}}(\mathcal{G}^{(0)})$ is a subsemigroup, then A is said to support S if it supports every element of S.

We observe that $C^*_{\text{ess}}(\mathcal{G})$ supports $\mathcal{CP}_{\mathcal{G}}(\mathcal{G}^{(0)})$.

We are now ready to state our main result on Powers averaging for essential groupoid C*-algebras.

Theorem 7.8.14. *Let $\mathcal{G}$ be a minimal étale groupoid with compact Hausdorff space of units. Then the following statements are equivalent.*

1. *$C^*_{\text{ess}}(\mathcal{G})$ is simple.*
2. *$C^*_{\text{ess}}(\mathcal{G} \ltimes \partial_F \mathcal{G})$ satisfies relative Powers averaging property with respect to any covering and contractive semigroup of generalised probability measures on $\mathcal{G}$.*
3. *Given any $a \in C^*_{\text{ess}}(\mathcal{G} \ltimes \partial_F \mathcal{G})$ and any $\nu \in \mathcal{P}(\partial_F \mathcal{G})$, we have $\nu \circ \mathrm{E}(a) \in \overline{\text{conv}}\{\mu a \mid \mu \in S\}$ for any covering and contractive semigroup S of generalised probability measures on $\mathcal{G}$.*
4. *$A \subseteq C^*_{\text{ess}}(\mathcal{G})$ is C*-irreducible for every C*-subalgebra A supporting a covering and contractive semigroup of generalised probability measures on $\mathcal{G}$.*
5. *$A \subseteq C^*_{\text{ess}}(\mathcal{G} \ltimes \partial_F \mathcal{G})$ is C*-irreducible for every C*-subalgebra A supporting a covering and contractive semigroup of generalised probability measures on $\mathcal{G}$.*

If $\mathcal{G}$ is Hausdorff, then all these conditions are equivalent to the following statement.

6. *Given any $a \in C^*_r(\mathcal{G})$, we have $\mathrm{E}(a) \in \overline{\mathrm{conv}}\{\mu a \mid \mu \in S\}$ for every covering and contractive semigroup S of generalised probability measures on $\mathcal{G}$.*

Remark 7.8.15. The assumptions on S being covering and contractive are satisfied for $S = \mathcal{CP}_{\mathcal{G}}(\mathcal{G}^{(0)})$ by Proposition 7.8.6, since $\mathcal{G}$ is assumed to be minimal.

Proof of 7.8.14. The implication from 5 to 4 is clear. It is also clear that 4 implies 1.

For the rest of the proof we fix a covering and contractive semigroup S of generalised probability probability measures on $\mathcal{G}$. Without loss of generality, we may assume that S is convex.

Let us show that 1 implies 3. For a contradiction, assume there were some $\nu \in \mathcal{P}(\partial_F \mathcal{G})$ and some $a \in C^*_{\mathrm{ess}}(\mathcal{G} \ltimes \partial_F \mathcal{G})$ for which the conclusion of 3 does not hold. By the Hahn-Banach separation theorem, there is some functional $\omega \in C^*_{\mathrm{ess}}(\mathcal{G} \ltimes \partial_F \mathcal{G})^*$ and some $\alpha \in \mathbb{R}$ with

$$\mathrm{Re}\, \omega(1)\nu(\mathrm{E}(a)) < \alpha \leq \mathrm{Re}\, \omega(\mu a)$$

for all $\mu \in S$. This contradicts Corollary 7.8.11.

Let us next assume that 3 holds. Given $a \in C^*_{\mathrm{ess}}(\mathcal{G} \ltimes \partial_F \mathcal{G})$ such that $\mathrm{E}(a) = 0$, we have $0 = \nu(\mathrm{E}(a)) \in \overline{\{\mu a \mid \mu \in S\}}$ for any auxiliary $\nu \in \mathcal{P}(\partial_F \mathcal{G})$. This proves 2.

We will now show that 2 implies 5. Assume that $I \subseteq C^*_{\mathrm{ess}}(\mathcal{G} \ltimes \partial_F \mathcal{G})$ is a nonzero C*-subalgebra which is invariant under multiplication with elements from A. Let $a \in I$ be nonzero. Replacing a by a^*a, we may assume without loss of generality that a is positive. Consider the nonzero positive function $f = \mathrm{E}(a) \in C(\partial_F \mathcal{G})$. Fix $x \in \partial_F \mathcal{G}$ such that $f(x) \neq 0$. Denote by $\pi : \mathcal{G} \ltimes \partial_F \mathcal{G} \to \mathcal{G}$ the natural projection. Since S is covering $\mathcal{G}$, it also covers $\mathcal{G} \ltimes \partial_F \mathcal{G}$ by Lemma 7.8.8. So for every $g \in (\mathcal{G} \ltimes \partial_F \mathcal{G})_x$ there is some $\mu = \sum_i (\gamma_i, f_i)$ such that $f_i \circ \psi_{\gamma_i^*}(\mathrm{r}(g)) \neq 0$ implies $g \in \gamma_i$. Hence,

$$\mu f(\mathrm{r}(g)) = \sum_i \alpha_{\gamma_i}(f_i^{\frac{1}{2}} f f_i^{\frac{1}{2}})(\mathrm{r}(g)) = \sum_i (f_i f)(\psi_{\gamma_i^*}(\mathrm{r}(g))) = \sum_i (f_i f)(x) = f(x) \neq 0\,.$$

So $gx \in \mathrm{supp}\, \mu f$ follows. By minimality of $\mathcal{G} \ltimes \partial_F \mathcal{G}$ and compactness of $\partial_F \mathcal{G}$ there are finitely many elements $\mu_1, \dots, \mu_n \in S$ such that $\frac{1}{n}\sum_i \mu_i f$ is nowhere zero. By compactness of $\partial_F \mathcal{G}$, there is $\delta > 0$ such that $\frac{1}{n}\sum_i \mu_i f \geq \delta$. Since E is a $\mathcal{G}$-expectation, it follows that $\mathrm{E}(\frac{1}{n}\sum_i \mu_i a) = \frac{1}{n}\sum_i \mu_i \mathrm{E}(a) \geq \delta$. So we found a positive element $b \in I$ such that $\mathrm{E}(b) \geq \delta > 0$. Then for arbitrary $\mu \in S$, we have

$$\mu b = \mu(b - E(b)) + \mu(E(b)) \geq \mu(b - E(b)) + \delta\,.$$

Since S is convex, our assumption allows to choose $\mu \in S$ such that $\|\mu(b - E(b))\| \leq \frac{\delta}{2}$. Then we infer that $\mu b \geq \frac{\delta}{2}$, and hence $\mu b \in I$ is invertible. This shows that $I = C^*_{\mathrm{ess}}(\mathcal{G} \ltimes \partial_F \mathcal{G})$, and thus 5.

Assuming additionally that $\mathcal{G}$ is Hausdorff, we observe that the proof that 6 implies 1 follows from an obvious simplification of the argument that 2 implies 5, making use of the fact that $\mathrm{E}_{\mathrm{red}}(a) \in C(\mathcal{G}^{(0)})$ holds for all $a \in C^*_r(\mathcal{G})$. The implication from 3 to 6 follows from contractivity of S, since every function $f \in C(\mathcal{G}^{(0)})$ lies in the S closure the functions $f(x)1_{\mathcal{G}^{(0)}}$. ∎

7.9 From boundary actions to Powers averaging

In this section, we will apply Theorem 7.8.14 in order to obtain concrete examples of unitary group representations into C*-algebras satisfying relative Powers averaging property. Let us recall that for a discrete group G, a G-boundary is a compact Hausdorff space with a minimal and strongly proximal action of G. Also recall that the topological full group $\mathbf{F}(\mathcal{G})$ of a groupoid is the group of its global bisections. We can now formulate the following corollary of Theorem 7.8.14.

Corollary 7.9.1. *Let $\mathcal{G}$ be an étale groupoid with compact Hausdorff space of units. Assume that there is a subgroup of the topological full group $G \leq \mathbf{F}(\mathcal{G})$ that covers $\mathcal{G}$ and such that $G \curvearrowright \mathcal{G}^{(0)}$ is a G-boundary. Denote by $\pi : G \to C^*_{\mathrm{ess}}(\mathcal{G})$ the unitary representation of G in the essential groupoid C*-algebra of $\mathcal{G}$. If $C^*_{\mathrm{ess}}(\mathcal{G})$ is simple, then $C^*_{\mathrm{ess}}(\mathcal{G})$ satisfies Powers averaging relative to $\pi(G)$.*

Proof. We define S to be the convex subsemigroup generated by G inside $\mathcal{CP}_{\mathcal{G}}(\mathcal{G}^{(0)})$. Then it suffices to note that S is contractive and covering to apply Theorem 7.8.14. ∎

Recall that the groupoid of germs associated with an action of a discrete group $G \curvearrowright X$ is the quotient $(G \ltimes X)/\mathrm{Iso}(G \ltimes X)^\circ$. We will apply Corollary 7.9.1 to groupoids of germs, thereby obtaining concrete examples of groupoid C*-algebras satisfying relative Power averaging with respect to a natural group of unitaries. C*-irreducibility of the associated inclusions has been studied in [KS21].

Theorem 7.9.2. *Let G be a countable discrete group and $G \curvearrowright X$ a boundary action. Denote by $\mathcal{G}$ its groupoid of germs and by $\pi : G \to C^*_{\mathrm{ess}}(\mathcal{G})$ the associated unitary representation. Then $\pi(G) \subseteq C^*_{\mathrm{ess}}(\mathcal{G})$ satisfies the relative Powers averaging property.*

Proof. Let us first show that $C^*_{\mathrm{ess}}(\mathcal{G})$ is simple. To this end, we observe that $\mathcal{G}$ is minimal, since it has the same orbits as $G \curvearrowright X$. Further, since G is countable, the set $\bigcup_{g \in G} \partial \operatorname{Fix}(g)$ is meager in X. Hence, its complement is dense in X, and it follows that $\mathcal{G}$ is topologically principal. It follows from Theorem 7.1.2 (and already from [KM21, Theorem 7.26]) that $C^*_{\mathrm{ess}}(\mathcal{G})$ is simple. By Corollary 7.9.1 applied to $G/\ker(G \curvearrowright X)$, it now follows that $\pi(G) \subseteq C^*_{\mathrm{ess}}(\mathcal{G})$ satisfies the relative Powers averaging property. ∎

Remark 7.9.3. The representation of π appearing in Theorem 7.9.2 can be identified with a quasi-regular representation as employed in [KS22, KS21]. To this end recall the following notation, given an action of a discrete group $G \curvearrowright X$. The open stabiliser at $x \in X$ is defined as

$$G_x^\circ = \{g \in G \mid \text{ there is a neighbourhood } x \in U \text{ such that } g|_U = \mathrm{id}\},$$

and, for $g \in G$, we denote by $\operatorname{Fix}(g) = \{x \in X \mid gx = x\}$ the fixed point set of g.

Now if $x \in X$ is such that

- the subquotient G_x/G_x^0 is amenable, and
- $x \notin \partial(\operatorname{Fix}(g)^\circ)$ for every $g \in G$,

then the inclusion $\pi(G) \subseteq C^*_{\mathrm{ess}}(\mathcal{G})$ is isomorphic with an inclusion $\lambda_{G/G_x}(G) \subseteq C^*_{\mathrm{ess}}(\mathcal{G})$ arising from the regular representation of $\mathcal{G}$ associated with x.

Indeed, consider the regular representation λ_x of $C^*_r(\mathcal{G})$ on $\ell^2(\mathcal{G}_x)$. We recall from Section 7.2.1 that if $a \in C^*_r(\mathcal{G})$ is singular, then $\mathrm{s}(\operatorname{supp} \hat{a}) \subseteq \mathrm{s}(\overline{\mathcal{G}^{(0)}} \setminus \mathcal{G}^{(0)}) = \bigcup_{g \in G} \partial(\operatorname{Fix}(g)^\circ)$. So by the assumption on x, it follows that λ_x factors through a *-homomorphism $C^*_{\mathrm{ess}}(\mathcal{G}) \to \mathcal{B}(\ell^2(\mathcal{G}_x))$. Since $\mathcal{G}_x^x = G_x/G_x^0$ is amenable, it factors further to a *-homomorphism $C^*_{\mathrm{ess}}(\mathcal{G}) \to \mathcal{B}(\ell^2(\mathcal{G}_x/\mathcal{G}_x^x))$. Observe that this map is injective, because $C^*_{\mathrm{ess}}(\mathcal{G})$ is simple. The associated representation of G on $\ell^2(\mathcal{G}_x/\mathcal{G}_x^x) \cong \ell^2(G/G_x)$ is the quasi-regular representation λ_{G,G_x}.

Remark 7.9.4. Previous results about relative Powers averaging arising from group actions were obtained by Amrutam-Kalantar [AK20], who showed that if G is a C*-simple discrete group and X is any minimal, compact G-space, then the inclusion $G \subseteq C(X) \rtimes_r G = C^*_r(X \rtimes G)$ satisfies relative Powers averaging. In these examples, C*-simplicity of the group is the source of Powers averaging. In contrast, our Theorem 7.9.2 makes no assumption on the acting group G, but relies instead on the assumption that X is a G-boundary. Nevertheless, there are some situations where the two results overlap such as hyperbolic groups G with trivial finite radical acting on their Gromov boundary ∂G. Since ∂G is a topologically free boundary action of a C*-simple group, both [AK20, Theorem 1.3] and our Theorem 7.9.2 imply the relative Powers averaging property for the inclusion $G \subseteq C^*_r(\partial G \rtimes G)$.

Remark 7.9.5. In their work [KS22], Kalantar-Scarparo considered boundary actions of a discrete group $G \curvearrowright X$, and obtained a characterisation of C*-simplicity for quasi-regular representations arising from point stabilisers G_x with $x \in X \setminus \bigcup_{g \in G} \partial(\mathrm{Fix}(g)^\circ)$ [KS22, Corollary 5.3]. Our Theorem 7.9.2 and Remark 7.9.3 consider the associated groupoid of germs $\mathcal{G}$ and say that there the inclusion $\lambda_{G,G_x}(G) \subseteq C^*_{\mathrm{ess}}(\mathcal{G})$ satisfies relative Powers averaging. This is a considerably stronger conclusion. No prediction about C*-simplicity of the associated quasi-regular representations for G_x is made for $x \in \bigcup_{g \in G} \partial(\mathrm{Fix}(g)^\circ)$, and [KS22, Example 6.4] even shows that in general λ_{G,G_x} it will not necessarily be C*-simple in this case. Theorem 7.9.2 provides a conceptual explanation of this phenomenon, by drawing attention to the difference between the reduced and the essential groupoid C*-algebra.

More recently in [KS21, Theorem 5.6] simplicity results for the groupoid of germs associated with a minimal group action on a locally compact space $G \curvearrowright X$ were obtained. This result is implied by Theorem 7.7.10. However, in this situation there is no natural map from G to the essential groupoid C*-algebra of the groupoid of germs, so that Powers averaging and thus an analogue of Theorem 7.9.2 needs further care to be even formulated.

We now consider the concrete case of Thompson's group T acting on the circle as well as Thompson's group V acting on a totally disconnected cover of the circle. The latter action was previously considered in [KS22], in order to reprove the simplicity of the Cuntz algebra $\mathcal{O}_2$ using techniques from the theory of C*-simplicity.

Example 7.9.6. Consider Thompson's group $\mathrm{T} \subseteq \mathrm{Homeo}(\mathrm{S}^1)$. It is the group of piecewise linear transformations of $\mathrm{S}^1 \cong \mathbb{RP}^1$ with breakpoints in $\exp(2\pi i \mathbb{Z}[\frac{1}{2}])$ and derivatives in $\mathbb{Z}[\frac{1}{2}]$. It acts transitively on non-trivial intervals of the circle, whose end points lie in $\exp(2\pi i \mathbb{Z}[\frac{1}{2}])$. In particular, $\mathrm{T} \curvearrowright \mathrm{S}^1$ is a boundary action. Let $\mathcal{G}$ be the groupoid of germs for $\mathrm{T} \curvearrowright \mathrm{S}^1$. We observe that $\mathcal{G}$ is non-Hausdorff, since T does not act topologically freely while S^1 is connected. Denote by $\pi : \mathrm{T} \to C^*_{\mathrm{ess}}(\mathcal{G})$ the associated unitary representation of T. By Theorem 7.9.2 the inclusion $\pi(\mathrm{T}) \subseteq C^*_{\mathrm{ess}}(\mathcal{G})$ satisfies relative Powers averaging. If $x \in \mathrm{S}^1 \setminus \exp(2\pi i \mathbb{Z}[\frac{1}{2}])$ then Remark 7.9.3 further identifies this inclusion with $\lambda_{\mathrm{T},\mathrm{T}_x}(T) \subseteq C^*_{\mathrm{ess}}(\mathcal{G})$. Considering the action of T_x on $\mathrm{S}^1 \setminus \{x\} \cong \mathbb{R}$, one sees that $\mathrm{T}_x \cong [F, F]$ is isomorphic to the subgroup of $\mathrm{F} \subseteq \mathrm{Homeo}(\mathbb{R})$ acting with trivial germs at infinity. In contrast, it was shown in [KS22, Example 6.4] that the quasi-regular representation associated with the standard inclusion $[F,F] \cong \mathrm{T}_1^0 \subseteq \mathrm{T}$ does not even generate a simple C*-algebra.

Remark 7.9.7. The example of Thompson's group T acting on the circle should be considered in the more general context of groups of homeomorphisms of the circle and the real line, which provides many examples of boundary actions that are not topologically free. We mention several concrete examples. First, Monod considered in [Mon13] groups of piecewise projective homeomorphisms of the real line, arising as point stabiliser of $\infty \in \mathbb{RP}^1 \cong \mathrm{S}^1$ of $\mathrm{PSL}_2(A)$ for arbitrary countable subrings $A \subseteq \mathbb{R}$. We have $\mathrm{PSL}_2(\mathbb{Z}[\frac{1}{2}]) = \mathrm{T}$ as subgroups of $\mathrm{Homeo}^+(\mathrm{S}^1)$. Second, we like to point out recent work of Hyde-Lodha [HL19] producing finitely generated, simple groups of homeomorphisms of the real line, which are constructed as variations of Thompson's group T. Finally, Navas' survey [Nav18, p. 2056ff] and the recent book of Kim and Koberda [KK21] contain concrete questions about and provide examples of groups of homeomorphism of the circle, and include consideration of their contraction properties. It should be pointed out that a group G acting by homeomorphisms on a one-dimensional manifold $M \in \{\mathbb{R}, \mathrm{S}^1\}$ is strongly proximal if and only if it is extremally proximal, in the sense that for every pair of non-trivial open intervals $I, J \subseteq M$ there is $g \in G$ such that $gI \subseteq J$. This is because an open interval can be described by its endpoints together with a point in its interior. Such actions are also called CO-transitive in the dynamics community.

The next example considers the action of T on a suitable totally disconnected cover of the circle, and it yields a Hausdorff groupoid of germs. A particularly interesting feature is that it produces a unitary representation of T into the Cuntz algebra $\mathcal{O}_2$ satisfying the relative Powers averaging property.

Example 7.9.8. Consider the following cover of the circle,

$$K = \big(\mathrm{S}^1 \setminus \exp(2\pi i \mathbb{Z}[\frac{1}{2}])\big) \cup \big(\{+,-\} \times \exp(2\pi i \mathbb{Z}[\frac{1}{2}])\big)$$

equipped with the natural topology arising from the cyclic order. We write z_+ and z_- for the elements $(+, z)$ and $(-, z)$, respectively. The action of T lifts uniquely to an action on K preserving the cyclic order. By definition, V is the topological full group of this action. It follows directly from the definitions that $\mathrm{V} \curvearrowright K$ is a boundary action, so that Theorem 7.9.2 applies to the groupoid of germs $\mathcal{G}(\mathrm{V} \curvearrowright K) = \mathcal{G}(\mathrm{T} \curvearrowright K) = \mathcal{G}$. This groupoid is Hausdorff, since $\mathrm{Fix}(g)^\circ$ is clopen for every $g \in \mathrm{T}$. Considering the stabiliser $\mathrm{F} \cong \mathrm{T}_{1_+} \leq \mathrm{T}$, and employing Remark 7.9.3 we obtain an inclusion $\lambda_{\mathrm{T},\mathrm{F}}(\mathrm{T}) \subseteq C^*_r(\mathcal{G})$ satisfying the relative Powers averaging property. By [BS19] we know that $C^*_r(\mathcal{G})$ is generated by the image of Thompson's group V, so that [HO17, Proposition 5.3] allows to make the identification with the Cuntz algebra $C^*_r(\mathcal{G}) \cong \mathcal{O}_2$.

Remark 7.9.9. It is known that $\mathbf{F}(\mathcal{G}) \curvearrowright \mathcal{G}^{(0)}$ is extremally proximal for every purely infinite Hausdorff groupoid $\mathcal{G}$. But the work in [BS19] shows that $\mathbf{F}(\mathcal{G})$ generates $C_r^*(\mathcal{G})$ in this case. In view of Theorem 7.9.2 and Example 7.9.8, it is natural to ask the following question. Is there a systematic approach to constructing subgroups $G \leq \mathbf{F}(\mathcal{G})$ such that the C*-algebra inclusion generated by G inside $C_r^*(\mathcal{G})$ is proper?

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A generalized Powers averaging property for commutative crossed products

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APPENDICES

Appendix A

The simplest proof of when reduced group C*-algebras are simple

It is often the case in math that the first discovered proof of a result is quite roundabout and overcomplicated, and the proof gradually becomes simpler as time goes on. The question of characterizing when the reduced group C*-algebra $C^*_\lambda(G)$ of a discrete group G is simple is no exception, with the results having been worked out by Breuillard, Kalantar, Kennedy, and Ozawa in the following papers: [KK17], [BKKO17], [Ken20] (in chronological order). Easier arguments are also occasionally given in [KS19], a paper of Kennedy and Schafhauser which deals with noncommutative crossed products $A \rtimes_\lambda G$.

This appendix serves the purpose of picking out the easiest proofs in all of these, arranging them in just the right way, and perhaps also adding a pinch of originality, all in order to give the easiest possible path to deducing when $C^*_\lambda(G)$ is simple. To my knowledge, no such work exists anywhere. In terms of simplification, much of the work in the aforementioned papers was done from the perspective of topological spaces (in the context of group boundaries), but this can be almost completely avoided, and it is possible to mostly stick with the category of C*-algebras. Moreover, it is possible to avoid any annoying ε-estimates, and also a particularly difficult proof of the fact that if the Furstenberg boundary crossed product $C(\partial_F G) \rtimes_\lambda G$ is simple, then the reduced group C*-algebra is.

It is very much worth mentioning that such a compilation will skip much of the important intuition that can still be obtained by reading the original works. In particular, we can avoid most of the topological properties of the Furstenberg boundary $\partial_F G$, including the fact that it is the universal minimal strongly proximal G-space. All this is to say, there is still plenty to be gained by reading the original papers.

It is also worth noting that in the unpublished paper [Kaw17], Kawabe generalizes this to asking when the reduced crossed product $C(X) \rtimes_\lambda G$ has the ideal intersection property. For the purpose of elegance, we will stick to the group C*-algebra setting, and it is worth noting that most of the proofs in the crossed product setting will be almost identical anyways (with perhaps a little bit of care being needed in the non-minimal setting).

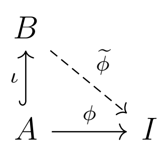
A.1 A brief word on injective envelopes

One of the key ideas underpinning the simplicity proofs is the use of an *injective envelope* of a certain object in the appropriate category, and the original constructions relating to C*-algebras and operator systems were given by Hamana in [Ham79], and in [Ham85] in the equivariant case. It is possible to simplify Hamana's original proofs as is done by Sinclair in [Sin15], using the theory of compact semigroups. However, it is worth noting that Sinclair's proof skips out on proving an extremely useful property known as *rigidity* of the injective envelope.

Given that the injective envelope is constructed in the context of groupoid C*-algebras in Section 7.4.2, and in particular in Theorem 7.4.9 (including proving rigidity), we will not repeat the construction here. However, we will at least repeat the appropriate definitions and results for completeness.

Of interest to us is the category whose objects are unital G-C*-algebras, morphisms are G-equivariant unital and completely positive maps, and embeddings are G-equivariant unital complete order embeddings. Note that we do not require an embedding to be the same as the usual category-theoretic definition of a monomorphism.

Definition A.1.1. Consider a category, and let I be an object in this category. We say that I is *injective* if whenever A and B are objects in this category, $\iota : A \to B$ is an embedding, and $\phi : A \to I$ is a morphism, then there is a map $\tilde{\phi} : B \to I$ making the following diagram commute:



Definition A.1.2. Assume that A and B are objects in our category with $A \subseteq B$.

1. We say that B is an *injective envelope* of A if B is injective, and it is also *minimal*, in the sense that for any other injective object C together with an embedding $\kappa : A \to C$, if $\phi : C \to B$ is an embedding that makes the following diagram commute:

$$\begin{array}{ccc} C & \xrightarrow{\phi} & B \\ {\scriptstyle\kappa}\uparrow & & \uparrow \\ A & \xrightarrow{\text{id}} & A \end{array}$$

then $\phi(C) = B$, or equivalently, ϕ is an isomorphism.

2. We say that B is a *rigid* extension of A if whenever $\phi : B \to B$ is a morphism in our category with $\phi|_A = \text{id}_A$, we have that $\phi = \text{id}_B$.

3. We say that B is an *essential extension* of A if whenever we have another object C, and a morphism $\phi : B \to C$, if $\phi|_A$ is an embedding, then ϕ is an embedding.

Theorem A.1.3. *Consider the category of G-C^*-algebras. Every object A admits a G-injective envelope $I_G(A)$, such that $A \subseteq I_G(A)$ as a unital G-C^*-subalgebra. It is unique up to isomorphism, in the sense that if C is any other such object, there is a G-equivariant *-isomorphism $\pi : I_G(A) \to C$ making the following diagram commute:*

$$\begin{array}{ccc} I_G(A) & \xrightarrow{\pi} & C \\ \uparrow & & \uparrow \\ A & \xrightarrow{\text{id}} & A \end{array}$$

The object $I_G(A)$ is also G-rigid and G-essential. If A is commutative, then so is $I_G(A)$.

A.2 Dynamical characterization of C*-simplicity

In this section, G will denote a discrete group. While the assumption of countability is often added when these results are cited, it is in fact not necessary for the simplicity results mentioned here. (Other things, like the unique stationarity results of Hartman and Kalantar [HK17] might indeed require countability).

The main ideas of Breuillard, Kalantar, Kennedy, and Ozawa for proving simplicity of $C^*_\lambda(G)$ can be summarized as follows. First, consider the G-injective envelope of $\mathbb{C}$, i.e.

$I_G(\mathbb{C})$, which we know is commutative and can be written as $C(\partial_F G)$. Due to historical reasons, this spectrum is often called the Furstenberg boundary of G, due to Furstenberg originally studying it mostly from a topological and measure-theoretic perspective. See for example [Fur73] (and also [Fur63]). It is also sometimes called the Hamana boundary, due to Hamana's original construction of the injective and G-injective envelopes of operator systems and C*-algebras. See the discussion in Section A.1.

As it turns out, the crossed product $C(\partial_F G) \rtimes_\lambda G$ shares much of the simplicity properties of $C^*_\lambda(G)$, as one is simple if and only if the other is. However, the larger crossed product $C(\partial_F G) \rtimes_\lambda G$ is much nicer to work with in general, as explicitly writing down ideals in this C*-algebra is much easier. Dropping down to ideals on $C^*_\lambda(G)$ is an injectivity (hence, axiom of choice) argument, explaining the difficulty.

Simplicity is linked to the dynamics of G on $\partial_F G$. Given the fact that $\partial_F G$ shows up as an injective envelope construction (hence, axiom of choice), it is extremely difficult to describe in practice. The only description to my knowledge is given in [KRS22]. It is therefore much nicer to drop down to dynamics on a space more easily understood, and this turns out to be the space of amenable subgroups of G.

We begin with the first half of the simplicity results. One of the tools we will use are something called *pseudoexpectations* $E : C^*_\lambda(G) \to C(\partial_F G)$ (or more precisely, *equivariant pseudoexpectations*, given that the term *pseudoexpectations* shows up in a non-equviariant sense in papers of Pitts and Zarikian. See for example [Pit17]). The following definition explains the terminology.

Definition A.2.1. Let $A \subseteq B$ be an inclusion of unital G-C*-algebras. A *pseudoexpectation* is a G-equivariant unital and completely positive map $E : B \to I_G(A)$ with the property that $E|_A = \mathrm{id}_A$.

Moreover, one of the more convenient topological facts that comes into play here is the fact that the fixed point sets $\mathrm{Fix}(s) \subseteq \partial_F G$ are clopen for every $s \in G$. It is almost certainly possible to get around this, and work purely operator algebraically by using Kallman's inner/properly outer decomposition of von Neumann algebras, which also applies to monotone complete C*-algebras. For reference, this sort of decomposition is used in the traces results of Chapter 3, and Kallman's results are stated in Theorem 3.2.8 and the discussion around it. However, for the purpose of C*-simplicity of discrete groups, this is one of the few instances where, in my opinion, a slight detour into topology makes things easier.

In essence, we will show that the G-injective envelope is non-equivariantly injective, and it is known that injective commutative C*-algebras have extremally disconnected spectrum due to a result of Gleason and others. Afterwards, it is a result of Frolík that fixed point sets of homeomorphisms of these spaces are clopen.

Proposition A.2.2. *The G-injective envelope $C(\partial_F G)$ is non-equivariantly injective as well.*

Proof. It is well-known that the space $\ell^\infty(G)$ is G-injective and non-equivariantly injective as well. By G-injectivity of $\ell^\infty(G)$, there is a G-equivariant map $\iota : C(\partial_F G) \to \ell^\infty(G)$. By G-essentiality of $C(\partial_F G)$, this map is a complete order embedding. By G-injectivity of $C(\partial_F G)$, there is a G-equivariant map $\phi : \ell^\infty(G) \to C(\partial_F G)$ making the following diagram commute:

$$\begin{array}{ccc} \ell^\infty(G) & & \\ \iota \uparrow & \overset{\phi}{\searrow} & \\ C(\partial_F G) & \xrightarrow{\text{id}} & C(\partial_F G) \end{array}$$

Given that we can just forget that all of the above maps are G-equivariant, it follows from non-equivariant injectivity of $\ell^\infty(G)$ that $C(\partial_F G)$ is non-equivariantly injective as well. ■

Definition A.2.3. A compact Hausdorff space is called *extremally disconnected* if the closure of any open set is still open.

Theorem A.2.4. *Consider a commutative unital C*-algebra $C(X)$. The following are equivalent:*

1. *$C(X)$ is non-equivariantly injective.*

2. *X is extremally disconnected.*

Theorem A.2.5 (Frolík's theorem, [Fro71, Theorem 3.1])**.** *Assume X is an extremally disconnected compact Hausdorff space, and $\alpha : X \to X$ is a homeomorphism. Then the set of fixed points* $\mathrm{Fix}(\alpha) = \{x \in X \mid \alpha(x) = x\}$ *is clopen.*

With the above in hand, we obtain the topological result we were after:

Corollary A.2.6. *Given any $s \in G$, the fixed point set* $\mathrm{Fix}(s) \subseteq \partial_F G$*, i.e.* $\mathrm{Fix}(s) = \{x \in \partial_F G \mid sx = x\}$*, is clopen.*

Furthermore, another very convenient fact is the following, as it will allow us to use the fact that the universal and reduced group C*-algebras coincide for amenable groups:

Lemma A.2.7. *Let $H \leq G$ be a containment of discrete groups. The group H is amenable if and only if $\ell^\infty(G)$ admits an H-invariant state.*

Proof. First, recall that for any subgroup $H \leq G$, there is always a (very non-canonical) H-equivariant injective *-isomorphism $\iota : \ell^\infty(H) \to \ell^\infty(G)$, given as follows: let $T \subseteq G$ be a right-transversal (axiom of choice) of the right coset space $H\backslash G$. That is, $G = \sqcup_{r \in T} Hr$. We can then let the map ι be given by

$$\iota(f)(hr) = f(h).$$

As a consequence, if we can obtain an H-invariant state on $\ell^\infty(G)$, then H is amenable. Conversely, if H is amenable, then any compact convex H-space admits a fixed point, and in particular this applies to the state space of $\ell^\infty(G)$. ■

Proposition A.2.8. *Given any $x \in \partial_F G$, the point stabilizer $G_x = \{s \in G \mid sx = x\}$ is amenable.*

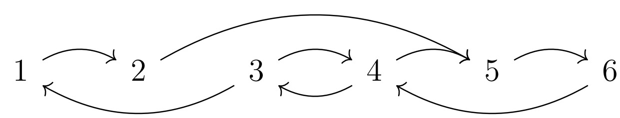
Proof. We will construct a G_x-invariant state on $\ell^\infty(G)$, and consequently we will be done by Lemma A.2.7. Let $x \in \partial_F G$, let $\phi : \ell^\infty(G) \to C(\partial_F G)$ be a G-equivariant morphism (existence guaranteed by G-injectivity), and observe that the composition $\delta_x \circ \phi : \ell^\infty(G) \to \mathbb{C}$ is indeed a G_x-invariant state. ■

Now we are ready to prove the first half of the main result:

Theorem A.2.9. *The following are equivalent:*

1. *$C^*_\lambda(G)$ is simple.*
2. *$C(\partial_F G) \rtimes_\lambda G$ is simple, where $C(\partial_F G) = I_G(\mathbb{C})$.*
3. *There is a unique pseudoexpectation $E : C^*_\lambda(G) \to C(\partial_F G)$.*
4. *There is a unique G-equivariant conditional expectation $E : C(\partial_F G) \rtimes_\lambda G \to C(\partial_F G)$.*
5. *All G-equivariant conditional expectations $E : C(\partial_F G) \rtimes_\lambda G \to C(\partial_F G)$ are faithful.*
6. *The action of G on $\partial_F G$ is free.*

Proof. For convenience, the following is the chain of implications that will be proven:

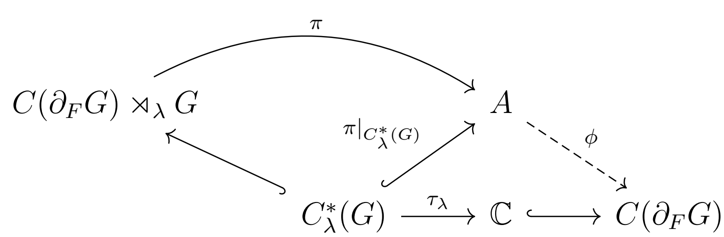


There may be slight redundancy in the above implications, but it is used to highlight some important connections.

First, we show that items (3) and (4) are indeed equivalent, by showing there is a correspondence between the two sets of expectations. Given any conditional expectation $E : C(\partial_F G) \rtimes_\lambda G \to C(\partial_F G)$, it clearly restricts to a pseudoexpectation on $C^*_\lambda(G)$. We claim that this restriction map is bijective. First, to see that it is injective, note that because any such E is the identity on $C(\partial_F G)$, we have that $C(\partial_F G)$ is in the multiplicative domain of E, and therefore E is in fact uniquely determined by the values it takes on $C^*_\lambda(G)$. To see that the restriction map is also surjective, start with any pseudoexpectation $E : C^*_\lambda(G) \to C(\partial_F G)$. It extends to a G-equivariant map $\widetilde{E} : C(\partial_F G) \rtimes_\lambda G \to C(\partial_F G)$ by injectivity. By rigidity, $\widetilde{E}$ must in fact be the identity on $C(\partial_F G)$, and therefore a conditional expectation.

It is clear that (4) $\implies$ (5), as the canonical conditional expectation is faithful. What is certainly not as clear is the converse, and this will end up being proven in a more roundabout way through the rest of the implications.

Now we show (1) $\implies$ (2). Assume $\pi : C(\partial_F G) \rtimes_\lambda G \to A$ is a *-homomorphism. Our aim is to show that it is injective. Note that there is a G-action on A, given by conjugation by the unitaries $\pi(g) \in A$, and with respect to this action, π is G-equivariant. Furthermore, given that $C^*_\lambda(G)$ is simple, we have that $\pi|_{C^*_\lambda(G)}$ is an embedding. Thus, letting $\tau_\lambda : C^*_r(G) \to \mathbb{C}$ be the canonical trace, by injectivity, there is a map $\phi : A \to C(\partial_F G)$ making the following diagram commute:



The composition $\phi \circ \pi : C(\partial_F G) \rtimes_\lambda G \to C(\partial_F G)$ is the identity on $C(\partial_F G)$ by rigidity, and also coincides with the canonical trace on $C^*_\lambda(G)$. By a multiplicative domain argument, it must be the case that $\phi \circ \pi$ is the canonical conditional expectation, which is faithful. This forces π to be faithful, or equivalently, injective.

Showing (3) $\implies$ (1) follows a similar idea, but is a bit shorter. Assume that $\pi : C^*_\lambda(G) \to A$ is a *-homomorphism. Again, there is a G-action on A given by conjugation by the unitaries $\pi(g) \in A$, and this makes the map π G-equivariant. By injectivity, we have that there is a G-equivariant map $\phi : A \to C(\partial_F G)$. The composition $\phi \circ \pi : C^*_\lambda(G) \to C(\partial_F G)$ is a pseudoexpectation, and therefore must coincide with the canonical trace on $C^*_\lambda(G)$, which is faithful. This forces π to be faithful, and thus surjective.

Now we show (6) $\implies$ (4). To see this, let $x \in \partial_F G$, and let $s \in G$ be such that $s \neq e$. Our aim is to show that $E(\lambda_s)(x) = 0$. As the action of G on $\partial_F G$ is free, we have that $sx \neq x$. By Urysohn's lemma, we have that there is some $f \in C(\partial_F G)$ with $f(x) = 1$ and $f(sx) = 0$. Thus, using multiplicative domain again, we have:

$$fE(\lambda_s) = E(f\lambda_s) = E(\lambda_s(s^{-1}f)) = E(\lambda_s)(s^{-1}f).$$

Evaluating both sides at x gives us that

$$E(\lambda_s)(x) = f(x)E(\lambda_s)(x) = E(\lambda_s)(x)f(sx) = 0,$$

and since x was arbitrary, this says that $E(\lambda_s) = 0$ for any $s \neq e$. It follows, once more, from multiplicative domain that E must be the canonical conditional expectation.

The implication (2) $\implies$ (5) is next. We argue by contrapositive. Assume that $E : C(\partial_F G) \rtimes_\lambda G \to C(\partial_F G)$ is a G-equivariant conditional expectation that is not faithful. It is a well-known fact of unital and completely positive maps that the set

$$I = \{a \in C(\partial_F G) \rtimes_\lambda G \mid E(a^*a) = 0\}$$

is a closed left-ideal of $C(\partial_F G) \rtimes_\lambda G$, which must also be nonempty as E is not faithful. Magically, this set actually turns out to be a two-sided ideal. To see this, let $f \in C(\partial_F G)$, and let $a \in I$. Then by multiplicative domain, we have

$$E((af)^*(af)) = E(f^*a^*af) = f^*E(a^*a)f = 0,$$

and so $af \in I$. Similarly, if $s \in G$, then

$$E((a\lambda_s)^*(a\lambda_s)) = E(\lambda_s^*(a^*a)\lambda_s) = E(s \cdot (a^*a)) = s \cdot E(a^*a) = 0,$$

and so $a\lambda_s \in I$ as well. It follows that I is a two-sided ideal, which as remarked before is nonzero, and does not contain the identity either. Thus, the crossed product cannot be simple.

Finally, it remains to show (5) $\implies$ (6). We argue by contrapositive. Assume that the action of G on $\partial_F G$ is not free. We know that the collection of fixed point sets

Fix(s) are clopen by Corollary A.2.6, and therefore we can talk about the projections $p_s = 1_{\mathrm{Fix}(s)} \in C(\partial_F G)$, at least one of which is nonzero for some $s \neq e$. Observe that these projections satisfy $s \cdot p_t = p_{sts^{-1}}$ and $p_{s^{-1}} = p_s$. We claim that there is a pseudoexpectation $E : C^*_\lambda(G) \to C(\partial_F G)$ given by $E(\lambda_s) = p_s$.

To see this, fix any $x \in \partial_F G$, and consider the state $\phi_x : C^*_\lambda(G) \to \mathbb{C}$ given by the composition

$$C^*_\lambda(G) \xrightarrow{E_{G_x}} C^*_\lambda(G_x) = C^*(G_x) \xrightarrow{1_{G_x}} \mathbb{C},$$

where the map E_{G_x} is the canonical conditional expectation onto the smaller group C*-algebra, and 1_{G_x} is the canonical *-homomorphism corresponding to the trivial representation. Note that we are using amenability of G_x from Proposition A.2.8 to say that the reduced and universal group C*-algebras coincide. In short, $\phi_x : C^*_\lambda(G) \to \mathbb{C}$ maps λ_s to 1 if $s \in G_x$ and 0 if $s \notin G_x$. Now consider the giant direct sum

$$E = \bigoplus_{x \in \partial_F G} \phi_x : C^*_\lambda(G) \to \ell^\infty(\partial_F G).$$

We claim that it is the pseudoexpectation we are looking for. First, note that we indeed have $E(\lambda_s) = p_s$. Given that all of the projections p_s lie in $C(\partial_F G)$, it follows that the range of the entire map also lies in $C(\partial_F G)$. Moreover, the map E is in fact equivariant, as

$$E(s \cdot \lambda_t) = E(\lambda_{sts^{-1}}) = p_{sts^{-1}} = s \cdot p_t = s \cdot E(\lambda_t).$$

Now recall the correspondence given in the proof of (3) $\iff$ (4). This map extends to a G-equivariant conditional expectation $\widetilde{E} : C(\partial_F G) \rtimes_\lambda G \to C(\partial_F G)$ determined by $E(f\lambda_s) = fp_s$. This map is not faithful, as if we consider an $s \neq e$ with $p_s \neq 0$, we have a nonzero element $a = p_s - \lambda_s p_s \in C(\partial_F G) \rtimes_\lambda G$ satisfying

$$E(a^*a) = E(p_s(1-\lambda_s)^*(1-\lambda_s)p_s) = p_s E(2 - \lambda_s - \lambda_s^*)p_s = p_s(2 - p_s - p_s)p_s = 0.$$

■

A.3 Intrinsic characterization of C*-simplicity

As mentioned earlier, the space $\partial_F G$ turns out to be extremely difficult to describe in practice, and so it would be nice to have a space that is easier to get a handle on. This turns out to be the space of amenable subgroups of G.

Definition A.3.1. The space of subgroups of G, denoted $\mathrm{Sub}(G)$, viewed as a subset of the compact Hausdorff space $2^G = \prod_G \{0, 1\}$, admits an induced topology called the *Chabauty topology.* Similarly, the space of amenable subgroups $\mathrm{Sub}_a(G)$ also admits such a topology.

Proposition A.3.2. *Both the spaces* $\mathrm{Sub}(G)$ *and* $\mathrm{Sub}_a(G)$ *are compact under the Chabauty topology.*

Proof. Essentially, a net of subsets (A_λ) converges to A in 2^G if and only if whenever $a \in A$, we eventually have $a \in A_\lambda$, and when $a \notin A$, we eventually have $a \notin A_\lambda$. From this, it is extremely easy to check that $\mathrm{Sub}(G)$ is closed, hence compact.

The space of amenable subgroups is only slightly more challenging, and involves applying Lemma A.2.7. Assume (H_λ) is a net of amenable subgroups converging to some $H \leq G$. We know there are H_λ-invariant states $\phi_\lambda \in S(\ell^\infty(G))$. Dropping to a subnet, we may assume that $\phi_\lambda \to \phi \in S(\ell^\infty(G))$ as well in the weak*-topology. Thus, if $h \in H$, we eventually have $h \in H_\lambda$, and so

$$\phi(hf) = \lim_\lambda \phi_\lambda(hf) = \lim_\lambda \phi_\lambda(f) = \phi(f),$$

or in other words, ϕ is H-invariant. ∎

Our main theorem will be in terms of existence of amenable confined subgroups, with the appropriate definition given below:

Definition A.3.3. We say that a subgroup $H \leq G$ is *confined* if the closure of the conjugation orbit $\{gHg^{-1} \mid g \in G\}$ in $\mathrm{Sub}(G)$ does not contain the trivial subgroup $\{e\}$.

We also need minimality of the Furstenberg boundary $\partial_F G$. The original C*-simplicity proofs take a large detour into a discussion of the fact that this is indeed the universal minimal and strongly proximal space, but this can for the most part be skipped for our purposes.

Proposition A.3.4. *The space* $\partial_F G$ *is* minimal*, in the sense that there are no nonempty proper* G*-invariant closed subsets.*

Proof. If there were such a subset $Y \subseteq \partial_F G$, then the restriction map $C(\partial_F G) \to C(Y)$ would not be a complete order embedding, which would contradict G-essentiality. ∎

Theorem A.3.5. *The following are equivalent:*

1. *G has no amenable confined subgroups.*

2. *$C^*_\lambda(G)$ is simple.*

Proof. Assume $C^*_\lambda(G)$ is not simple, so that the action of G on $\partial_F G$ is not free by Theorem A.2.9. Consider the map $\omega : \partial_F G \to \mathrm{Sub}_a(G)$ given by mapping x to G_x. It is well-defined by Proposition A.2.8.

Moreover, we will use Corollary A.2.6 to show that the map is continuous. Assume $x_\lambda \to x$. We wish to show that $G_{x_\lambda} \to G_x$. Assume $s \in G_x$, so that $x \in \mathrm{Fix}(s)$. Given that $\mathrm{Fix}(s)$ is clopen, we eventually have $x_\lambda \in \mathrm{Fix}(s)$ as well, $s \in G_{x_\lambda}$. Similarly, if $s \notin G_x$, i.e. $x \notin \mathrm{Fix}(s)$, then we eventually have $s \notin G_{x_\lambda}$.

It is clear that ω is G-equivariant, as $sG_xs^{-1} = G_{sx}$. Using this, continuity, and the fact that $\partial_F G$ is minimal, we have that the image $\omega(\partial_F G)$ is a minimal compact Hausdorff G-space as well. Thus, given that we know one of the stabilizers G_x has to be nontrivial, it follows that its orbit cannot contain the trivial subgroup, or in other words, G_x is confined (and we already know it is amenable).

To show the converse, assume that H is an amenable confined subgroup. It is an easy exercise in topology that because of the fact that $\{e\} \notin \overline{\{gHg^{-1} \mid g \in G\}}$, there must exist some nonempty finite $F \subseteq G \setminus \{e\}$ with the property that for every $g \in G$, we have $gHg^{-1} \cap F \neq \emptyset$.

Now recall that for any amenable $K \leq G$, there is a map $\phi_K : C^*_\lambda(G) \to \mathbb{C}$ mapping λ_g to 1 if $g \in K$, and 0 if $g \notin K$. It is obtained through the following composition:

$$C^*_\lambda(G) \xrightarrow{E_K} C^*_\lambda(K) = C^*(K) \xrightarrow{1_K} \mathbb{C}.$$

Letting

$$P = \bigoplus_{g \in G} \phi_{gHg^{-1}} : C^*_\lambda(G) \to \ell^\infty(G)$$

be a giant direct sum over all these maps (easily checked to be G-equivariant), we observe that the element $a = \sum_{g \in F} \lambda_g$ maps to some element $P(a) \geq 1$. Thus, letting $\psi : \ell^\infty(G) \to C(\partial_F G)$ be obtained through G-injectivity, we have that the composition $\psi \circ P$ maps a to a nontrivial element. In other words, $C^*_\lambda(G)$ admits a nontrivial pseudoexpectation, and by Theorem A.2.9, we are done. ■