Lattice Paths

by

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Author's Declaration

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Abstract

This thesis is a survey of some of the well known results in lattice path theory. Chapter 1 looks into the history of lattice paths. That is, when it began and how it was popularized. Chapter 3 focuses on general lattices and lattice paths. It later looks into different types of properties of some lattice paths. This is divided into two types: quarter-plane and self-avoiding walks. Chapter 4 and 5 explore some of the properties of quarter-plane walks and self-avoiding walks, respectively.

Acknowledgements

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Thanks to my mom and pops for being the best parents ever. Yes, all those stores make #1 Dad and #1 Mom mugs just for you. I also want to put the names of my two best friends: Maheen and Grace. I'm acting like I'm on TV haha. If either of you two do a thesis my name better be up there in the acknowledgements. Lastly (and most importantly), a thank you to someone who cannot even comprehend what a thanks is, my dog Milo (and sure my cat Cisco too).

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Chapter 1

Introduction

Before an overview of the history of lattice paths is given, it is important to note that the information is evidence based. It is taken from the article *A history and a survey of lattice path enumeration* by Katherine Humphreys [28] who emphasized the same idea. That is, what happened and what is believed to have happened may not necessarily align. In other words, every conversation and paper on lattice paths may not have been well documented. As mathematicians it is important to remember this distinction when looking at mathematics history and consider the possibility of lattice paths occurring in earlier work than the one discussed in this section. Consequently, the earliest evidence of lattice path techniques have been dated to 1878 based on the creation of the ballot problem. This problem was introduced in 1877 and solved in 1878. It was a variation to the gambler's ruin problem which has been of interest in probability theory since 1654. The gambler's ruin problem is stated below

A and B take each twelve counters and play with three dice on this condition, that if eleven is thrown A gives a counter to B, and if fourteen is thrown, B gives a counter to A $\cdots$ the winner of the game is one who first obtains all the counters [47, see pg 25]

Even though at the time there was a bijection created with Dyck words [5], which was used by French mathematician Désiré André (best known for his work on Catalan numbers and alternating permutations), the more well-known solution to this problem is through the reflection principle. Based on the evidence, probability theory is believed to have established the study of lattice path enumeration. As we will discuss in this chapter, for many decades, lattice path enumeration was used for probability problems involving games or combinatorial studies. Later, it gained the interest of physicists for the motion of particles. It was also looked at recreationally. Starting in the 1960s, it gained popularity especially as a bijection to young tableaux and compositions. The bijection helped to prove the generalized ballot theorem.

Example 1. ***Ballot Problem***
When counting in a random order the election outcome for the candidates A and B with a total of n votes where A wins with α and B has n-α votes. What is the probability that at each count of the vote A is always ahead?

The problem translates to lattice paths as follows. If $\uparrow$ and $\rightarrow$ represents a vote for A and B, respectively, then the problem is equivalent to a walk from (0,0) to (α, β) such that it stays above the line y=x (except at (0,0) and/or (α, β)) and $\alpha + \beta = n$.

<u>André's Solution</u> [18]: The main idea is to remove the number of bad paths from the total number of paths. The total number of paths is $\binom{\alpha+\beta}{\alpha}$. Let $x = (x_1, \cdots, x_n)$ be a sequence of $\uparrow$ and $\rightarrow$ such that x touches or goes below the line y=x. The total number of bad paths can be broken into two cases.
Case 1: The bad path that begins with a $\rightarrow$. Since the number of $\rightarrow$ exceeds the number of $\uparrow$ after the first step, any choice for $x_2, \cdots, x_n$ will result in x being a bad path. Hence, there are $\binom{\alpha+(\beta-1)}{\alpha}$ bad paths that start with a $\rightarrow$.
Case 2: The bad path that begins with a $\uparrow$. The number of paths for this case is also $\binom{\alpha+(\beta-1)}{\alpha}$ due to the bijection between number of paths with $\alpha \uparrow$ and $\beta - 1 \rightarrow$. To prove this bijection find the smallest $i \in [n]$ such that the number of $\uparrow$ is equal to the number of $\rightarrow$ in $x_1, \cdots, x_i$. In other words, find the first bad $\rightarrow$. Remove x_i and consider the new path $x' = (x_{i+1}, \cdots, x_n, x_1, \cdots, x_{i-1})$. The path x' is a unique path of $\alpha \uparrow$ and $\beta - 1 \rightarrow$. To prove x and x' only map to each other, consider the smallest x_j in $x^{'}$ such that $x_j, \cdots, x_{i-1}$ has one more $\uparrow$ than $\rightarrow$ and let $x^{''} = (x_j, \cdots, x_{i-1}, x_i, x_{i+1}, \cdots, x_n, \cdots, x_{j-1})$. Since i is the smallest index such that $x_1, \cdots, x_i$ has equal number of $\rightarrow$ and $\uparrow$ then j=1. Hence, $x^{''} = x$ and $x, x^{'}$ map only to each other. The proof of this case is illustrated in figure 1.1.
Consequently, there are $\binom{\alpha+\beta}{\alpha} - 2\binom{\alpha+(\beta-1)}{\alpha} = \frac{\alpha-\beta}{\alpha+\beta}\binom{\alpha+\beta}{\alpha}$ good paths and the probability of A always being ahead is $\frac{\alpha-\beta}{\alpha+\beta}$. $\square$

$$
\begin{array}{ccc}
\underbrace{x_1, \cdots, x_i}_{\text{Equal no. of } \uparrow \text{ and } \rightarrow} & & x_{i+1}, \cdots x_n \\
 & \Downarrow & \\
x_1, \cdots, x_{i-1} & x_i & x_{i+1}, \cdots x_n \\
 & \Downarrow & \\
x_{i+1}, \cdots, x_n & & \underbrace{x_1, \cdots, x_{i-1}}_{\text{One more } \uparrow \text{ than } \rightarrow} \\
 & \Downarrow & \\
x_1, \cdots, x_{i-1} & & x_{i+1}, \cdots x_n \\
 & \Downarrow & \\
x_1, \cdots, x_{i-1} & x_i & x_{i+1}, \cdots x_n
\end{array}
$$

Figure 1.1: Bad Paths starting with $\uparrow$

Once probability began the theory of lattice paths, it gained relevance in another subject, namely physics. Humphreys looked into the scientists of the 19th century who aimed to study the nature of matter and arrived on lattice paths. They were interested in the following questions

- *What is Matter?*
- *What is Light?*
- *What is Heat?*
- *How does heat get from here to there?*

These questions were looked at through the behaviour of gases. Marian Von Smoluchowski independently from Einstein, began to develop the theory of Brownian motion in 1905. Brownian motion, which was first observed by a Scottish botanist and paleobotanist Robert Brown, is the study of particles moving in random directions quickly. The speed and random direction leads to many collisions among the particles which lead Brown to believe that the particles were alive. This was later ruled out and Brown's discovery helped indirectly prove the existence of atoms [1]. Being aware of Smoluchowski's work on the Brownian motion, a chemist named Perrin assisted in Brillouin's experiment to measure the diffusion constant of an emulsion by counting the number of molecules adhering to a plate of glass set at an angle into the flask of liquid [43]. Smoluchowski said the number of molecules on the partition in this experiment is the same as gambler's ruin problem [45].

In the early 20$^{\text{th}}$ century, lattice paths were used recreationally. Long after lattice path enumeration started to carry important applications, Mohanty remarked in [27]

> *"As an important note one may be reminded of a series of papers by Grossman on lattice paths, which had been considered as part of recreational mathematics until recently when its impact was seen in various applications"*

This was in reference to the series of papers *Fun with Lattice Points* published from 1946 to 1954. Example 1 encapsulates the type of work presented in these papers. For instance, a problem in [26] was to find the probability of the number of paths in the first octant with step set {(1,0,0), (0,1,0), (0,0,1)} going from the origin to (m, n, p) on the lattice $\mathbb{Z}^3$ that does not touch the planes $-x + y = 1$ or $-y + z = 1$. This was a translation of the ballot problem with m votes for A, n votes for B and p votes for C such that B is always ahead. If $f(a, b, c) = \frac{(m+n+p)!}{[(m-a)!(n-b)!(p-c)!]}$ is the number of lattice paths from (a, b, c) to (m, n, p) then by inclusion-exclusion principle the number of such paths will be $f(0, 0, 0) - f(-1, -1, 0) - f(0, -1, 1) + f(-1, -1, 2) + f(-2, 1, 1) - f(-2, 0, 2)$ [26].

Another remark by Mohanty, which helped understand the attitude towards lattice path enumeration during the recreational period, was made at the 6$^{\text{th}}$ International Conference on Lattice Path Combinatorics and Applications

> *"When I tried to get my book [27] published, my advisor warned not to put anything about lattice points in the title. It was rejected by the mathematics department at Academic Press. Then I sent it to the statistics department; they accepted it."*

After the simultaneous releases of [27] and *Lattice Path Combinatorics with Statistical Application* by T.V. Naryana, in addition to the increased work in applications of lattice paths in other big fields such as computer science, Mohanty realized a lattice path community should be formed. Knowing that the researchers were dispresed around the globe, he aimed to start an international conference. With the support of McMaster university, where Mohanty has been at since 1964, the first conference was held. It took place at McMaster univeristy, Hamilton Canada in 1984.

The success of the first conference and additional publications during the 80s led to the second conference in 1990 at the same location attracting more people to lattice path theory than before. Some of the major publications included Chapter 5 of *Combinatorial Enumeration* by I. Goulden and D. Jackson dedicated to combinatorics of paths and Chapter 1 of *Enumerative Combinatorics - Vol 1* by R. Stanley.

By the second conference, the ambition had increased to the point that some of the attendees showed willingness to start conferences in their residing country. This led to 7 out of the 9 conferences, that have been held so far, to take place in countries other than Canada. Namely, India, Austria, Greece, USA, Italy and France.

The conferences covered a variety of topics related to lattice paths and other combintorial problems. Some topics included q-calculus, orthogonal polynomials, plane partitions, sterling numbers, hypergeometric functions, partial orders, spanning surfaces, generating functions, recurrence relations, bijectivity, algebraic geometry, asymptotics, random walks, nonparametric inference, discrete distributions, urn models, queueing theory, quality control. Some talks were even focused on psychology [51]. The invited talks diversified starting the third conference. This led to the formation of an International Scientific Committee to regulate the accepted talks at future lattice path conferences.

A topic of research that is one of my interests is the collaboration between algorithms and lattice paths. Specifically, using algorithmic methods to prove a lattice path problem. This will be the main basis for Section 4.1. It is also a research interest of Alan Bostan. At the ninth conference of Lattice Paths and Applications, Bostan gave a talk on checking the algebraicity of generating functions using a computer. This question was first raised by Richard Stanley in the 1980 article of d-finite power series

Design an algorithm suitable for computer implementations which decides if a d-finite power series —given by a linear differential equation with polynomial *coefficients and initial conditions— is algebraic, or not.*

Bostan then looked into the best methods that can help in the discovery of such algorithms. This method was to first guess then prove by generating data then making conjectures and eventually proving them. With this idea in mind, Bostan shared the following algorithm during the talk.

- Input: $f(t)\mathbb{Q}[[t]]$, given as the generating function of a binomial sum
- Output: T if $f(t)$ is transcendental, A if $f(t)$ is algebraic

1. Compute an ODE L for $f(t)$
2. Compute L_f^{min}
3. If L_f^{min} has a logarithmic singularity, return T; otherwise return A

Boston conjectured that this algorithm is always correct when it returns A. Further information about the timeline of these conferences as well as Boston's talk can be found on the website for the ninth lattice path conference [2].

Throughout the years many dedications were made at these conferences. A relevant dedication, with respect to this thesis, is the one for G. Kreweras whose contributions played a significant role in the increase in interest for lattice path theory. Kreweras walks will be discussed in Chapter 4, namely, whether its generating function is algebraic.

At the seventh conference Mohanty reminisced about the early days of lattice paths as he said

"The humble beginning, smallness of LP Conferences and their structure have provided a close affinity among those who have been participating. Essentially they have become members of what I call "Lattice Path" family"

Chapter 2

Preliminaries

2.1 Formal Power Series

Since the study of lattice paths is mainly focused on the generating function, it is relevant to introduce some of the notation that will appear in the upcoming chapters.

Definition 1. Polynomial Ring
For a ring $\mathbb{F}$ and an indeterminate x (ie. x is not in the set $\mathbb{F}$, and is not a solution of any algebraic equation with coefficients in $\mathbb{F}$), the polynomial ring is

$$\mathbb{F}[t] = \left\{ \sum_{i=0}^{n} a_i t^i \mid a_i \in \mathbb{F}, n \in \mathbb{N} \right\}$$

Definition 2. Ring of Rational Functions
For $\mathbb{F}$ an integral domain the ring of rational functions in t is

$$\mathbb{F}(t) = \left\{ \frac{f(t)}{g(t)} \mid f(t), g(t) \in \mathbb{F}[t], g(t) \neq 0 \right\}$$

A multivariate polynomial ring is defined similarly to Definition 1. That is, $\mathbb{F}[x,y] = \mathbb{F}[x][y] = \{\sum_i a_i y^i \mid a_i \in \mathbb{F}[x]\} = \left\{\sum_i \left(\sum_j b_{i,j} x^j\right) y^i \mid b_{i,j} \in \mathbb{F}\right\}$. Note that $\mathbb{F}$ may also have indeterminates. For instance, ring of polynomials in x, y with coefficients in the ring of rational functions in t is $\mathbb{F}(t)[x,y] = \left\{\sum_i \left(\sum_j b_{i,j} x^j\right) y^i \mid b_{i,j} \in \mathbb{F}(t)\right\}$.

Definition 3. *Ring of Formal Laurent Series*
The ring of formal laurent series is

$$\mathbb{F}((x)) = \left\{ \sum_{i \geq r} a_i x^i \mid a_i \in \mathbb{F}, r \in \mathbb{Z} \right\}$$

As a consequence of Definition 3, the ring of formal power series denoted $\mathbb{F}[[x]]$ is the subset of the ring of formal laurent series for which $r = 0$. There can also be formal power series for which the coeffiecients are themselves series or polynomials. For instance, the ring of formal power series with coefficients in the ring of formal Laurent series is $\mathbb{F}((x))[[t]] = \left\{ \sum_{i \geq 0} a_i t^i \mid a_i \in \mathbb{F}((x)) \right\}$. Additionally, there can also be multivariate formal power series. Bivariate formal power series form a ring $\mathbb{F}[[x, y]]$ defined similarly as $\mathbb{F}[x, y]$ is defined with respect to Definition 1. More information on formal power series as well as some classical results can be found in [49].

2.2 Real Analysis

The proof of the main results in Chapter 5 relies on some results from real analysis. These results will be introduced and proved in this section.

Lemma 1. *Fekete's Lemma [37] If $(a_n)_{n \in \mathbb{N}}$ is a sequence of non-negative real numbers such that $a_0 = 0$ and $a_{n+m} \leq a_n + a_m$ then the sequence $\left(\frac{a_n}{n}\right)_{n \in \mathbb{N}}$ converges to its lower bound $\alpha < \infty$ where $\alpha = inf_{n \in \mathbb{N}} \frac{a_n}{n}$.*

Proof. Since a_n is non-negative then $0 < \frac{a_n}{n}$. By completeness property of the real numbers, $\alpha < \infty$. By the definition of infimum, for each $\epsilon > 0$ there exists $m \in \mathbb{N}$ such that

$$\frac{a_m}{m} < \alpha + \frac{\epsilon}{2}$$

For each $n \in \mathbb{N}$, let $n = mq + r$ for $q, r \in \mathbb{N}$ and $0 \leq r \leq m - 1$ then

$$a_n = a_{qm+r} \leq a_m + a_m + \ldots a_m + a_r = q a_m + a_r$$

So,

$$\frac{a_n}{n} \leq \frac{q a_m + a_r}{qm + r} = \frac{a_m}{m} \cdot \frac{qm}{n} + \frac{a_r}{n} \leq \left(\alpha + \frac{\epsilon}{2}\right) \cdot \frac{qm}{n} + \frac{a_r}{n}$$

Choose N so that $\frac{max\{a_1,...,a_{m-1}\}}{n} < \frac{\epsilon}{2}$ for all $n \geq N$ then

$$\alpha \leq \frac{a_n}{n} \leq \alpha + \frac{\epsilon}{2} + \frac{max\{a_1, ..., a_{m-1}\}}{n} \leq \alpha + \epsilon.$$

Since ϵ is arbitrary then the proof is concluded by definition of convergence. □

Lemma 2. *The convergence and divergence of a formal power series* $Z(x) = \sum_{n\geq 0} c_n x^n$ *such that* $\mu := \lim_{n\to\infty} c_n^{\frac{1}{n}}$ *is as follows*

- $Z(x) = +\infty$ *for* $|x| > \mu^{-1}$ *and*
- $Z(x) < \infty$ *for* $|x| < \mu^{-1}$

Proof. $Z(x)$ has a radius of convergence, $R \geq 0$. That is, $Z(x) = \infty$ for $|x| > R$ and $Z(x) < \infty$ for $|x| < R$. Let $L = \lim_{n\to\infty} |(c_n x^n)^{\frac{1}{n}}| = \lim_{n\to\infty} |c_n^{\frac{1}{n}} x| = |x| \lim_{n\to\infty} c_n^{\frac{1}{n}}$ then by the root test $Z(x)$ converges for $L < 1$ and diverges for $L > 1$. So, the radius of convergence is $R = \frac{1}{\lim_{n\to\infty} c_n^{\frac{1}{n}}} = \mu^{-1}$ □

Theorem 1. *Monotone Convergence Theorem [46, see thm 2.28]*
Suppose that $\{s_n\}$ *is a monotone sequence. Then* $\{s_n\}$ *is convergent if and only if it is bounded.*

The classical result below will be used to study Gessel walks in Section 4.1.2.

Theorem 2. *Implicit function theorem [17]*
Let $F(T, t, x) \in C^1$ *in a neighbourhood of* (T_0, t_0, x_0) *such that*

- $F(T_0, t_0, x_0) = 0$
- $\frac{\partial F(T_0,t_0,x_0)}{\partial T} \neq 0$

Then there exists a neighbourhood of (T_0, x_0, y_0) *in which there is an implicit function* $T = f(t, x) \in \mathbb{Q}((x))[[t]]$ *such that*

i. $T(t_0, x_0) = T_0$
ii. $F(T, t, x) = 0$ *for every* (t, x) *in the neighbourhood*

In Chapter 4, Gessel walks are introduced. When its generating function was studied by Gessel, he conjectured that the sequence of coefficients in the generating function of Gessel walks ending at the origin must be hypergeomteric.

Definition 4. *[35] Hypergeometric Series*
A hypergeometric series is a $\sum_k c_k$ *for which* $c_0 = 1$ *and the ratio of consecutive*

terms is a rational function of the summation index k. That is,

$$\frac{c_{k+1}}{c_k} = \frac{P(k)}{Q(k)}$$

with $P(k)$ and $Q(k)$ polynomials.

2.3 Probability

To prove the generating function of excursions is not d-finite, Section 4.2 uses elementary probability theory.

Definition 5. *[22] Expected Value*
Consider a random variable $\boldsymbol{X}$ [22] with a finite list $X_1, \cdots, X_k$ of possible outcomes and probability $P(X_i)$ for each outcome X_i, $i \in [k]$. The expected value is

$$\mathbb{E}[\boldsymbol{X}] = \sum_i P(X_i)X_i \tag{2.1}$$

Definition 6. *[22] Covariance*
Let $\boldsymbol{X}$ be a random vector $\boldsymbol{X} = (X_1, \cdots, X_n)^T$ such that X_i is a random variable for $i \in [n]$. Then

$$cov(\boldsymbol{X}) = \begin{bmatrix} 1 & \frac{E[(X_1-\mu_1)(X_2-\mu_2)]}{\sigma(X_1)\sigma(X_2)} & \cdots & \frac{E[(X_1-\mu_1)(X_n-\mu_n)]}{\sigma(X_1)\sigma(X_n)} \\ \frac{E[(X_1-\mu_1)(X_2-\mu_2)]}{\sigma(X_1)\sigma(X_2)} & 1 & \cdots & \frac{E[(X_2-\mu_2)(X_n-\mu_n)]}{\sigma(X_2)\sigma(X_n)} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{E[(X_1-\mu_1)(X_n-\mu_n)]}{\sigma(X_1)\sigma(X_n)} & \frac{E[(X_2-\mu_2)(X_n-\mu_n)]}{\sigma(X_2)\sigma(X_n)} & \cdots & 1 \end{bmatrix}$$

Covariance provides a measure of the strength of the correlation between two or more sets of random variables. For Section 4.2, the random variables will be the steps in an n-step walk. The goal will be to avoid influence between a step and its previous step, hence reducing to excursions with no correlation.

Chapter 3

Lattices and Lattice Walks

The most important element to establish is the lattice on which the lattice walks will lie. In 1 dimension walks are easier to study since there is limited choice for the step set. This is evident in the proof of Proposition 1. Chapters 4 and 5 mainly focus on 2-dimensional lattices. This is because even an increase of 1 degree in the dimension complicates the study of lattice walks. This section, however, will look at lattices in general n^{th} dimension.

Definition 7. *Lattice*
Let $\mathcal{V}$ be a Euclidean vector space $\mathbb{R}^n$ with basis $\{\mathbf{e}_1, ..., \mathbf{e}_n\}$. A lattice *$\Lambda \subset \mathcal{V}$ is a subset such that the usual inner product $\langle \alpha, \beta \rangle \in \mathbb{Z}$ for all $\alpha, \beta \in \Lambda$.*

Definition 8. *Generating Set*
For any finite subset $L \subset \mathcal{V}$, let

$$\mathbb{Z}\, L \;=\; \{\Sigma_{\mathbf{v}\in L} c_{\mathbf{v}} \mathbf{v} \,:\, c_{\mathbf{v}} \in \mathbb{Z} \textit{ for all } \mathbf{v} \in \mathcal{L}\}.$$

If $\mathbb{Z}\, L$ is a lattice, then L is a generating set *for the lattice.*

Below is an example that provides some intuition to the definitions above.

Example 2. *Lattice with its respective generating set*
Consider the integer lattice in d dimension as a subset of $\mathbb{R}^d$. Here $\mathcal{V} = \mathbb{R}^d$ with orthonormal basis $\{\mathbf{e}_1, ..., \mathbf{e}_d\}$, the lattice is $\Lambda = \mathbb{Z}^d$, and a generating ("linking") set is $L = \{-\mathbf{e}_i, \mathbf{e}_i : i \in [d]\}$.

Definition 9. *Lattice Walk*
Given a lattice Λ and a finite set of vectors $\mathcal{S} \subset \mathcal{V}$ such that each $s \in \mathcal{S}$ lies on Λ then a lattice walk ω is an element of $\mathcal{S}^*$. In addition, $\mathcal{S}$ is a step set.

For Chapter 4, the walks at hand have uniform probability for each step set and/or location of the ending point. It will be seen in the chapter how the complexity is still great even with such a condition. However, to have a diverse understanding of lattice walks in general the example below will focus on each move being time dependent.

Example 3. Random Walk
Let $L \subset \mathcal{V}$ be a finite set that generates a lattice $\Lambda = \mathbb{Z}L \subset \mathcal{V}$ in a Euclidean vector space with orthonormal basis $\{\mathbf{e}_1, ..., \mathbf{e}_d\}$. Consider the random walk, with time viewed as a discrete variable, starting at the origin $\mathbf{w}_0 = \mathbf{0}$ at time $t = 0$, and at each time-step $t \in \mathbb{N}$ moves so that the space-step $ß_{i+1} = \mathbf{w}_{t+1} - \mathbf{w}_t \in L$.

- *Case 1: The space-step ß is chosen uniformly at random at every step.*
- *Case 2: The space-step $ß_t$ is chosen according to some (nonuniform) probability distribution that does not depend on time t.*
- *Case 3: The space-step $ß_{t+1} \in \mathcal{L}$ is chosen according to some distribution which depends on the inner product of $ß_{t+1}$ and $ß_t$.*

3.1 Types of Lattices

The study of self-avoiding walks is based on the type of lattice. Consequently, the examples of lattices below are mainly used in Chapter 5. This is because the connective constant, defined in Section 3.3.2, is dependent on the choice of lattice. As a result, learning about these lattices and the relation among them is of benefit.

Definition 10. Honeycomb Lattice
The Honeycomb lattice, denoted $\mathbb{H}$, is a regular tiling of hexagons on the Euclidean plane, in which at most three hexagons meet at each vertex.

Definition 11. Gaussian Integers
The Gaussian integers consists of the set $\mathbb{G} = \{a+bi \mid a,b \in \mathbb{Z}\}$, where $i^2 = -1$.

Definition 12. Square Lattice
The Square lattice consists of Gaussian integers as the vertices on the complex plane.

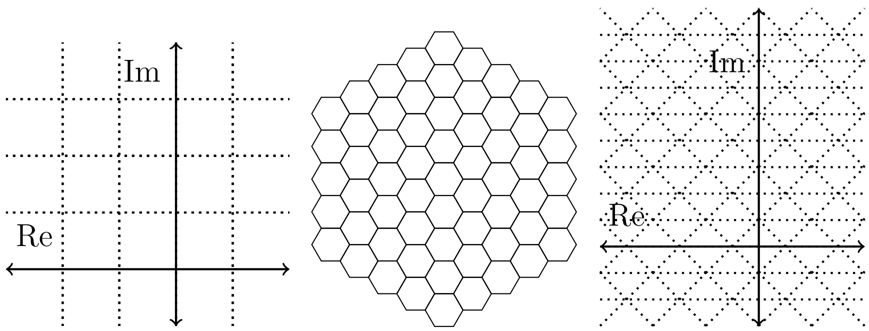


Figure 3.1: Left: Square Lattice, Middle: Honeycomb Lattice, Right: Equilateral Triangular Lattice

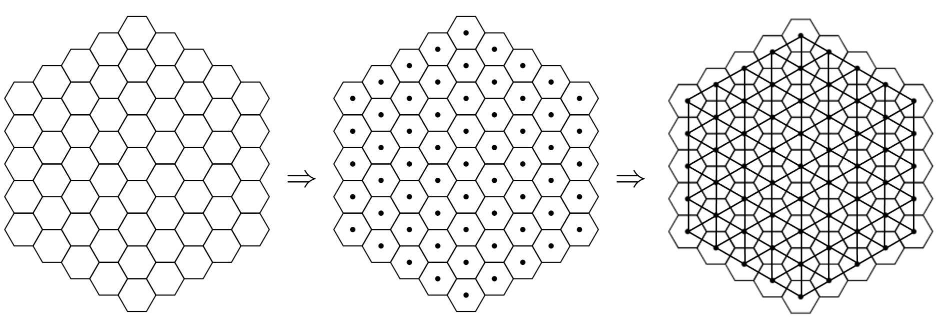
Definition 13. *Eisenstein Integers*
The Eisenstein integers consists of the set $\mathbb{E} = \{a+b\omega \mid a,b\in\mathbb{Z}\}$*, where* $\omega = e^{\frac{-i\pi}{3}}$ *is a primitive cube root of unity.*

Definition 14. *Equilateral Triangular Lattice*
The Triangular lattice consists of Eisenstein integers as the vertices on the complex plane.

Definition 15. *[15] Dual*
The dual lattice of a lattice $\mathcal{G}$ *consists of vertices in the center of each face of* $\mathcal{G}$ *and an edge for each pair of faces in* $\mathcal{G}$ *that are separated from each other by an edge.*

Lemma 3. *The equilateral triangular lattice is the dual of the honeycomb lattice.*

Proof. The proof follows directly from definition 15



□

Despite a connection between the triangular and honeycomb lattice via duality, the study of the connective constant for triangular lattice is not facilitated through the study of honeycomb lattice.

3.2 Types of Lattice Walks

There are many types of lattice walks that are studied. For this section the focus will be directed walks. It is a type of lattice walk for which there is a fixed direction of increase, namely the positive horizontal axis [50, ch 3]. A step set $\mathcal{S}$ for directed walks in $\mathbb{Z}_+ \times \mathbb{Z}$ is a finite set of $\mathbb{Z}_+ \times \mathbb{Z}$ such that for each $(i_1, i_2) \in \mathcal{S}$, $i_j \neq 0$ for some $j \in [2]$. Some classes of directed walks to consider are bridges, meanders and excursions.

Definition 16. *Let $\omega = (\omega_1, \cdots, \omega_n)$ be an n-step directed path on $\mathbb{Z}^2$. Then a*

- *Bridge is a path such that ω_n lies on the x-axis.*
- *Meander is a path that lies in the quarter plane $\mathbb{Z}_+^2$.*
- *Excursion is a bridge and meander.*

The length of ω is n.

Definition 17. *Self Avoiding Walk*
An n-step walk $\omega = (\omega_1, \cdots, \omega_n)$ on the lattice Λ is self avoiding if $\omega_i \neq \omega_j$ $\forall i, j \in [n]$ and $i \neq j$.

Banderier and Flajolet, in [4, fig 1], present the generating functions, in the form of a table, for the walks in Definition 16. The generating function for each walk is given in terms of the characteristic polynomial S and roots of the kernel equation u_i, with respect to a chosen step set. More information on kernel equation and characteristic polynomial is provided in Chapter 3 of [50].

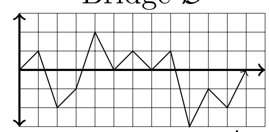
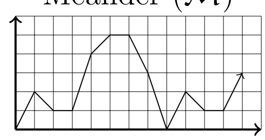
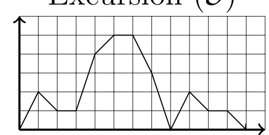
	Ending anywhere	Ending at 0
Unconstrained (on $\mathbb{Z}$)	Walk/Path ($\mathcal{W}$)  $W(z) = \frac{1}{1-zS(1)}$	Bridge $\mathcal{B}$  $B(z) = z\sum_{i=1}^{c} \frac{u_i'(z)}{u_i(z)}$
Constrained (on $\mathbb{Z}_+$)	Meander ($\mathcal{M}$)  $M(z) =$ $\frac{1}{1-zS(1)} \prod_{i=1}^{c}(1 - u_i(z))$	Excursion ($\mathcal{E}$)  $E(z) = \frac{(-1)^{c-1}}{p-c^z} \prod_{i=1}^{c} u_i(z)$

Table 1: Generating functions of the four types of paths in Definition 16

3.3 Properties of Lattice Walks

The most common properties of lattice walks are the holonomy and D-finiteness of their generating functions. As a result, Chapter 4 will look at some examples of lattice walks having or lacking such properties. The definition of self-avoiding walks, however, adds a level of complexity that leaves c_n, the number of n-step self-avoiding walks, unknown. As a result the focused study of self-avoiding walks is on bounds on c_n.

3.3.1 Algebraic and D-finite

The desire for algebraic generating functions is due to its 'nice' structure and certain traits such as closure properties. As well as a connection with formal languages (ie. context free grammar). This helps with many applications of lattice path theory since most are interested in the asymptotic behaviour. Studying them generally is desirable because there is a possibility of finite representation of a class of algebraic functions. If such a possibility exists then it is more likely that efficient algorithms exist that can manipulate its elements. Consequently, they can provide new possibilities to address unsolved problems.

Definition 18. *Complete Generating Function*
The complete generating function for a set $\mathcal{A}$ of walks on a subset of $\mathbb{Z}^2$ starting from a given point ω_0 is the series

$$A(x,y,t) = \sum_{n\geq 0} t^n \sum_{i,j\in\mathbb{Z}} a_{i,j}(n)x^i y^j$$

where $a_{i,j}(n)$ is the number of walks of $\mathcal{A}$ that have length n and end at (i,j).

The definition above is restricted to 3 variables since the walks in Chapter 4 are on 2 dimension lattices and the third variable is for the number of steps. The complete generating function contains all information relevant to a walk. To provide a clear understanding of Definition 18, below is a simple example for the generating function of a walk in the quarter plane.

Example 4. *Continuing Example 1, from Chapter 1, let $\mathcal{A}$ be set of 'good' paths ending at (i,j) and $a_{i,j}(n) = |\mathcal{A}|$. Since the step set consists of $\{(0,1),(1,0)\}$ then $n = i+j$ and the variable t is redundant for this example. So, $a_{i,j}(n) = a_{i,j}(i+j) = \frac{i-j}{i+j}\binom{i+j}{i}$ and the complete generating function of the ballot problem is*

$$\sum_{i>j>0} \frac{i-j}{i+j}\binom{i+j}{i} x^i y^j.$$

Let $i = j+k$ for some $k \in \mathbb{Z}_+$ then the complete generating function can be re-written as

$$\begin{aligned}\sum_{k\geq 1}\sum_{j\geq 0} \frac{k}{k+2j}\binom{k+2j}{j} x^{k+j}y^j &= \sum_{k\geq 1}\left(\sum_{j\geq 0}\frac{k}{k+2j}\binom{k+2j}{j}x^j y^j\right)x^k \\ &= \sum_{k\geq 1}\left(\sum_{n\geq 0}\frac{1}{n+1}\binom{2n}{n}x^n y^n\right)^k x^k \\ &= \sum_{k\geq 1}(C(xy)x)^k = \frac{xC(xy)}{1-xC(xy)},\end{aligned}$$

where $C(xy)$ is the Catalan number generating function.

Definition 19. *[13, see ch 4] Let A be a field of characteristic zero. Let $F(x) = \sum_{n\geq 0} a_n \boldsymbol{x}^n$ be a formal power series in the variables $x_1, \cdots, x_d$ with coefficients in A. The series F is said to be*

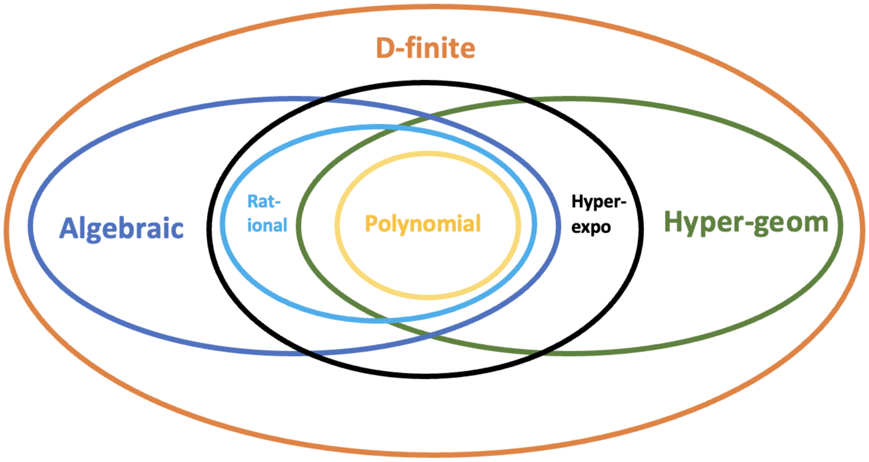


Figure 3.2: This Venn diagram provides information on how different types of generating functions link to one another.

- *Rational: if there exist polynomials P and Q in $A[x_1, \cdots, x_d]$ such that*

$$F(\boldsymbol{x}) = \frac{P(\boldsymbol{x})}{Q(\boldsymbol{x})}$$

- *Algebraic: if there exists a nonzero polynomial P in $d+1$ variables, with coefficients in A, such that*

$$P(x_1, \cdots, x_d, F(\boldsymbol{x})) = 0$$

- *d-finite or holonomic: if the partial derivatives $\frac{\partial^\alpha F}{\partial \boldsymbol{x}^\alpha}$ of F, for $\alpha \in \mathbb{N}^d$, span a finite-dimensional vector space over the field $A(x_1, \cdots, x_d)$ or, equivalently, if for $i \in [d]$, a nontrivial linear differential equation of the form*

$$P_k(\boldsymbol{x})\frac{\partial^k F}{\partial \boldsymbol{x}_i^k} + \cdots + P_1(\boldsymbol{x})\frac{\partial F}{\partial \boldsymbol{x}_i} + P_0(\boldsymbol{x})F = 0$$

holds, where the polynomials P_l have their coefficients in A.

The Venn diagram in figure 3.2 is taken from the paper *Algorithms for d-finite functions* [31, pg. 5].

Theorem 3. *[40, prop 2.14] If $f(z)$ is an algebraic function of degree d over a field $\mathbb{K}$ of characteristic zero then $f(z)$ is D-finite over $\mathbb{K}$, and is annihilated by a linear differential equation of order at most d.*

As mathematicians, finding a set as opposed to an individual element that fulfills a certain property is one of the main goals on the agenda. Stanley raised the question about an algorithm that can check the holonomy of any d-finite series. The mindset between aiming to find the perfect algorithm and aiming to find groups of holonomic walks are almost parallel. Hence, the theorem above leads to an interesting discussion about what are some families of walks that are algebraic. By Section 4.2, algebraic generating functions of quarter plane is not guaranteed. However, it is the case when the restriction is loosened to the right half-plane.

Proposition 1. *Let $\mathcal{S}$ be an arbitrary step set then the complete generating function $Q(x, y, t) \in \mathbb{Q}[[x, y, t]]$ for walks in the right half-plane with step set $\mathcal{S}$ starting at $(i_0, j_0) \in \mathbb{N} \times \mathbb{Z}$ is algebraic.*

Proof. By the definition of formal power series $\mathbb{Q}[[x, y, t]] = \mathbb{Q}[[y]][[x, t]]$. Walks on right half plane can be reduced to one dimensional weighted walks. This can be done by projecting each walk on the right half plane onto the positive x-axis (see figure 3.3) then defining the weight of each step of the 1-dimensional walk by a Laurent polynomial:

$$\sum_{j:(i,j)\in\mathcal{S}} y^j$$

The weight of each 1-dimensional walk is then defined as the product of the weight of its step. That is, for a walk for length n with steps $i_1, \cdots, i_n$ the weight of the walk is

$$\prod_{k=1}^{n} \left(\sum_{j:(i_k,j)\in\mathcal{S}} y^j \right) \in \mathbb{Q}(y)$$

The weight of each walk covers all possible n-step walks on the right half plane for which the x-coordinate is the the sequence $i_1, \cdots, i_n$. Hence $Q(x, y, t) \in \mathbb{Q}(y)[[x, t]]$. Based on the perspecitve of $\mathbb{Q}(y)[[x, t]]$, the formal power series is now of a meander with x marking the final altitude and t marking the size of the walk. Hence, by [4, thm 2], the formal power series $Q(x, y, t)$ is algebraic with respect to x and t. Since it is also algebraic with respect to y then $Q(x, y, t)$ is algebraic. $\square$

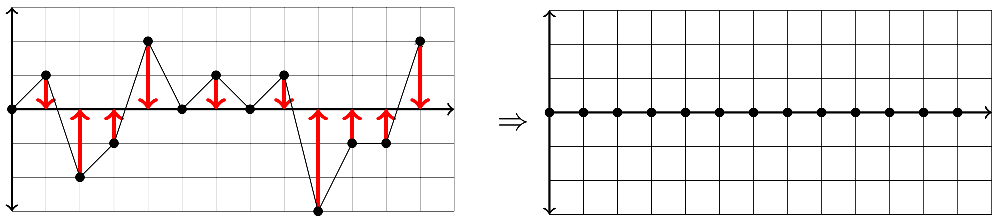


Figure 3.3: Projection of right half-plane walks onto the x-axis.

3.3.2 Connective Constant

Let c_n be the number of n-step self-avoiding walks on $\mathbb{H}$ starting from a fixed point (a, b). A few observations to keep in mind about c_n is the inequality $c_{n+m} \leq c_n c_m$ and its exponential growth. The inequality holds since an $(n+m)$-step self-avoiding walk can be divided into an n-step self-avoiding walk and a parallel m-step self-avoiding walk. The exponential growth is evident from bounds within which c_n lies, such as $\sqrt{2}^n \leq c_n \leq 3 \cdot 2^{n-1}$. This leads to the following limit

$$\mu := \lim_{x \to \infty} c_n^{\frac{1}{n}} \tag{3.1}$$

This value of the limit is referred to as the connective constant. It is studied to understand the universality in two-dimensional statistical physics model [38, Section 1.1]. The connective constant depends on the choice of lattice. However, its study helps determine approaches to other important open problems in self avoiding walks. For instance, the proof by Duminil-Copin and Smirnov for the connective constant of the honeycomb lattice may help with the conjecture that self-avoiding walks converge in the scaling limit to the Schramm–Loewner evolution [20, Section 4].

Chapter 4

Walks Confined to the Quarter Plane

The sections of this chapter will focus on the two main properties that are studied for lattice paths. The first of which is the holonomy of generating functions. This property is looked at with respect to two key examples, namely Kreweras and Gessel walks, defined in [36] and [23] respectively. The Kreweras walk is well known because of its exceptional enumerative and probabilistic properties. This led to bijections with objects from other branches of maths such as cubic maps and depth trees, found in graph theory [7]. Gessel walks, on the other hand, have been used to explore a modern method of proof, namely computer algebra (see [11]). The second property is D-finiteness of generating functions. The key example used is excursions ending at the origin.

4.1 Holonomy of Kreweras and Gessel Walks

The answer to many important questions in lattice path theory involves the cardinality of the set of walks in question. For lattice paths one common question is: Determine the number of walks on the lattice $\mathbb{Z}^2$ that goes from the origin to a specified point $(i, j) \in \mathbb{Z}^2$. If the length of the walk is limited to size $n \in \mathbb{N}$ then finding the total number of such walks is not hard. Hence, the answer to the question above leads to the study of complete generating function. More specifically, whether or not these generating functions are algebraic.

An interesting twist to the question above is when the walk is limited to a certain section of the lattice $\mathbb{Z}^2$. For instance, determine the number of walks with a given step set $\mathcal{S}$ that goes from the origin to $(i, j) \in \mathbb{Z}^2$ and must stay in the quarter-plane $\mathbb{Z}_+ \times \mathbb{Z}_+$. Having this additional condition, the universality of the holonomy of complete generating functions no longer holds in general.

The walks discussed in this section have algebraic generating functions and the proof of this will be the focus of this section. The proof draws from the paper by Alin Bostan and Manual Kauers [11]. Their method provides a way to work with generating functions in an algorithmic approach. This is particularly helpful when trying to determine the algebraic expression of a generating function. This will be shown in Sections 4.1.1 and 4.1.2.

Definition 20. *Kreweras Walks are lattice walks on $\mathbb{Z}^2$ that*

- *Start at the origin (0,0)*
- *Consist of the steps {W, NE, S} = {(-1,0),(1,1), (0,-1)}*
- *Remain in the first quadrant $\mathbb{N}^2$.*

Definition 21. *Gessel Walks are lattice walks on $\mathbb{Z}^2$ that*

- *Start at the origin (0,0)*
- *Consist of the steps {W, NE, SW, E} = {(-1,0),(1,1), (-1,-1), (1,0)}*
- *Remain in the first quadrant $\mathbb{N}^2$.*

Let $f(n, i, j)$ and $g(n, i, j)$ denote the number of Kreweras (resp. Gessel) walks of length n that end at (i, j). The complete generating functions of Kreweras (resp. Gessel) walks are defined as

$$F(t, x, y) = \Sigma_{n\geq 0} \left(\Sigma_{i,j\geq 0} f(n, i, j) x^i y^j\right) t^n \tag{4.1}$$

$$G(t, x, y) = \Sigma_{n\geq 0} \left(\Sigma_{i,j\geq 0} g(n, i, j) x^i y^j\right) t^n \tag{4.2}$$

Both $F(t, x, y)$ and $G(t, x, y)$ are in $\mathbb{Q}[x, y][[t]]$. This is because $f(n, i, j)$ and $g(n, i, j)$ are 0 whenever $i > n$ or $j > n$, hence the inner sum of $F(t, x, y)$ and $G(t, x, y)$ is a polynomial of x and y for each n.

Different values of x and y leads to different interpretations of the generating function. For example, $F(t, 0, 0)$ and $G(t, 0, 0)$ are the generating functions of

Kreweras (resp. Gessel) walks that start and end at the origin. The explicit formulas are

$$F(t,0,0) = \sum_{n=0}^{\infty} \frac{4^n \binom{3n}{n}}{(n+1)(2n+1)} t^{3n} \tag{4.3}$$

$$G(t,0,0) = \sum_{n=0}^{\infty} \frac{(5/6)_n (1/2)_n}{(5/3)_n (2)_n} (4t)^{2n} \tag{4.4}$$

where $(a)_n = a(a+1)...(a+n-1)$. Equation (4.3) was determined in the same paper the walk was introduced [36]. By [32], in a personal communication in 2001 Gessel theorized a similar representation may exist for $g(n,0,0)$. This lead him to suggest equation (4.4) which will be proven in Section 4.1.2.

4.1.1 Kreweras Walks

The simpler result will be proved first. The proof of this classical result will be the basis of the proof of Theorem 5 with slight variations.

Theorem 4. *F(t,x,y) is algebraic*

Before diving into the proof of Theorem 4, lets look at some helpful observations about $F(t,x,y)$.

The recurrence relation of $f(n,i,j)$ is

$$f(n+1,i,j) = f(n,i+1,j) + f(n,i,j+1) + f(n,i-1,j-1) \tag{4.5}$$

By Definition 20, $f(n+1,i,j)$ is the number of all walks from the origin to (i,j) in the first quadrant. Since the step set consists of W, NE and S, $f(n+1,i,j)$ is equivalent to the sum of $f(n,i+1,j), f(n,i,j+1)$ and $f(n,i-1,j-1)$. This is because the walk must reach (i,j) at the $(n+1)$-step hence it must be at $(i+1,j)$ if S is used at the last step. Similarly, the walk must be at $(i,j+1)$ or $(i-1,j-1)$ must be used if the last step is W or NE, respectively.

The boundary conditions of $f(n,i,j)$ are

1. $f(n,-1,0) = 0$ for $n \geq 0$
2. $f(n,0,-1) = 0$ for $n \geq 0$
3. $f(0,i,j) = \delta_{i,j,0}$ for $i,j \geq 0$

where $\delta_{i,j,0}$ is the Kronecker delta of three variables.

By (3), $f(n,0,0) = 0 \ \forall n \geq 1$ and $f(0,0,0) = 1$. So,

$$F(t,x,y) = \Sigma_{n\geq 0}\left(\Sigma_{i,j\geq 0} f(n,i,j)x^i y^j\right)t^n = 1 + \Sigma_{n\geq 0}\left(\Sigma_{i,j\geq 1} f(n,i,j)x^i y^j\right)t^n$$

By equation (4.5),

$$\begin{aligned} F(t,x,y) = 1 + \Sigma_{n\geq 0}\left(\Sigma_{i,j\geq 1} f(n,i+1,j)x^i y^j\right)t^{n+1} + \\ \Sigma_{n\geq 0}\left(\Sigma_{i,j\geq 1} f(n,i,j+1)x^i y^j\right)t^{n+1} + \\ \Sigma_{n\geq 0}\left(\Sigma_{i,j\geq 1} f(n,i-1,j-1)x^i y^j\right)t^{n+1}, \end{aligned} \tag{4.6}$$

This can be rewritten in terms of F as

$$\begin{aligned} F(t,x,y) = 1 &+ \frac{1}{y}t\left(F(t,x,y) - F(t,x,0)\right) \\ &+ \frac{1}{x}t\left(F(t,x,y) - F(t,y,0)\right)) \\ &+ xytF(t,x,y), \end{aligned}$$

Equation (4.1) can now be written as

$$F(t,x,y) = 1 + \left(\frac{1}{x} + \frac{1}{y} + xy\right)tF(t,x,y) - \frac{1}{y}tF(t,x,0) - \frac{1}{x}tF(t,y,0). \tag{4.7}$$

The method that will be used and introduced next is the kernel method. The kernel method can be credited to Knuth [34], in the solution of the Ballot problem. The idea from exercise 2 of Section 2.2.1 of [34] will be used for $F(t,x,y)$. Equation (4.7) can be rearranged to

$$F(t,x,y) = \frac{xtF(t,x,0) + ytF(t,y,0) - xy}{((x+y+x^2y^2)t - xy)} \tag{4.8}$$

Based on the idea from exercise 4 of Section 2.2.1 of [34], we need to find the expression that leads the denomimator to vanish. That is, set the denominator to zero. Afterwards, the number of variables are reduced to two by isolating x or y. Due to the symmetry of the step set about the main diagonal of $\mathbb{N}^2$, isolating x or y will result in the same reduced kernel equation. This is because $F(t,0,y)$ and $F(t,y,0)$ are equal. In this proof, y is isolated.

$$\begin{aligned} y := Y(t,x) &= \frac{x - t - \sqrt{-4t^2x^3 + x^2 - 2tx + t^2}}{2tx^2} \\ &= t + \frac{1}{x}t^2 + \frac{x^3+1}{x^2}t^3 + \frac{3x^3+1}{x^3}t^4 + \ldots \in \mathbb{Q}[x, x^{-1}][[t]] \end{aligned} \tag{4.9}$$

Now the goal is to choose $F(t,x,0)$ so that the numerator vanishes when $y = Y(t,x)$. That is,

$$xtF(t,x,0) + Y(t,x)tF(t,Y(t,x),0) - xY(t,x) = 0$$

Isolating for $F(t,x,0)$ gives the following expression

$$F(t,x,0) = \frac{Y(t,x)}{t} - \frac{Y(t,x)}{x}F(t,Y(t,x))$$

where $F(t,Y(t,x)) = F(t,Y(t,x),0)$. If the above equation is generalized then we obtain the reduced kernel equation

$$U(t,x) = \frac{Y(t,x)}{t} - \frac{Y(t,x)}{x}U(t,Y(t,x)) \tag{4.10}$$

The next lemma will prove that $F(t,x,0)$ is a unique solution of 4.10 in $\mathbb{Q}[[x,t]]$. This will help in the proof of Theorem 4.

Definition 22. *Let $ord_v S$ be the valuation of a power series S w.r.t some variable v occurring in S.*

Lemma 4. *Let A,B,$Y \in \mathbb{Q}[x,x^{-1}][[t]]$ such that $ord_t B > 0$ and $ord_t Y > 0$. Then there exists at most one power series $U \in \mathbb{Q}[[x,t]]$ with*

$$U(t,x) = A(t,x) + B(t,x)U(t,Y(t,x))$$

Proof. By linearity, it is sufficient to show that the only solution for $U(t,x) = B(t,x)U(t,Y(t,x))$ is the trivial solution (ie. $U = 0$), since translating the equation by $A(t,x)$ will result in at most one solution in $\mathbb{Q}[[x,t]]$. Suppose not. Then valuation of $B(t,x)U(t,Y(t,x))$ will be at least $ord_t B + ord_t U$. Since $ord_t B > 0$ then valuation of $B(t,x)U(t,Y(t,x))$ is greater than valuation of U, contradiction. ☐

There are three steps in the proof of Theorem 4. First, guess the algebraic equation $F(t,x,0)$, by observing its initial terms. Let the guess be denoted P. Then prove that P has exactly one solution in $\mathbb{Q}[[x,t]]$, namely $F_{cand}(t,x,0)$. Lastly, prove that the power series $U = F_{cand}(t,x,0)$ satisfies (4.10). Proof of the last statement implies that $F_{cand}(t,x,0)$ and $F(t,x,0)$ coincide. This is due to (4.10) and Lemma 4. Consequently, $F(t,x,0)$ is algebraic by the first and second step. By equation (4.9), $Y(t,x)$ is also algebraic. Since algebraic power series are closed under addition, multiplication and inversion, by equation (4.8), $F(t,x,y)$ is also algebraic.

Guess for Kreweras Walks

If given sufficient number of terms of a power series, a possible algebraic equation can be determined that is satisfied by it. An assumption can be made about the unknown algebraic equation in order to help determine it. Solving a linear system along with the assumption is one method of guessing the algebraic equation. Most commonly Gaussian elimination or an algorithm specifically designed for the power series in question is the method that is used. One approach is based on Hermite-Padé approximation [6] and this method is known as *automated guessing*. It is commonly used in packages such as gfun in Maple.

There is one concern that comes with this method: that a false algebraic equation may be returned. However, if the method is applied properly this is rarely the case. Nevertheless, the next two sections will be an independent proof that the algebraic equation guessed in this section is the correct one.

Another concern Boston and Kauers mention is that the information required to make an accurate guess may be an inordinate amount. This, however, is not an issue for Kreweras walks. In Maple, the computations required to guess the algebraic equation for this walk is feasible.

The following Maple commands compute a guessed algebraic equationor $F(t, x, 0)$:

```
f:=proc(n,i,j)
  option remember;
    if i<0 or j<0 or n<0 then 0
    elif n=0 then if i=0 and j=0 then 1 else 0 fi
    else f(n-1,i-1,j-1)+f(n-1,i,j+1)+f(n-1,i+1,j) fi
end:
S:=series(add(add(f(k,i,0)*x^i,i=0..k)*t^k,k=0..80),t,80):
gfun:-seriestoalgeq(S,Fx(t)):
```

Line 3 ensures the positivity condition of n, i and j. Line 4 is there due to boundary condition (3). Lastly, line 5 is there because of equation (4.5). With the information provided from line 3 to 5, the command on line 7 produces a partial series of $F(t, x, 0)$ for some n. By trial and error, Boston and Kauers found $n = 80$ to be sufficient. In line 8, the guessing function gfun provides a good candidate for an algebraic equation that is satisfied by S. The last command outputs the polynomial which is

$$\begin{aligned}
P(T,t,x) = {} & (16x^3t^4 + 108t^4 - 72xt^3 + 8x^2t^2 - 2t + x) \\
& + (96x^2t^5 - 48x^3t^4 - 144t^4 + 104xt^3 - 16x^2t^2 + 2t - x)T \\
& + (48x^4t^6 + 192xt^6 - 264x^2t^5 + 64x^3t^4 + 32t^4 - 32xt^3 + 9x^2t^2)T^2 \\
& + (192x^3t^7 + 128t^7 - 96x^4t^6 - 192xt^6 + 128x^2t^5 - 32x^3t^4)T^3 \\
& + (48x^5t^8 + 192x^2t^8 - 192x^3t^7 + 56x^4t^6)T^4 \\
& + (96x^4t^9 - 48x^5t^8)T^5 + 16x^6t^{10}T^6,
\end{aligned} \tag{4.11}$$

Running this algorithm on Maple12 on a modern computer requires 80Mb of memory and less than 20 seconds.

Prove Existence and Uniqueness

As mentioned at the end of Section 4.1.1, the next step is to prove that P has exactly one solution in $\mathbb{Q}[[x,t]]$, namely $F_{cand}(t,x,0)$.

By equation (4.11), we know that $P \in C^1$ in the neighbourhood of (1,0). We also know that $P(1,0,x) = 0$ and $\frac{dP}{dT}P(1,0,x) = -x$. By Theorem 2, there exists a unique series $F_{cand}(t,x,0) \in \mathbb{Q}((x))[[t]]$ such that $P(F_{cand}(t,x,0),t,x) = 0$. Consequently, $F_{cand}(t,x,0) \notin \mathbb{Q}[[x,t]]$ or $F_{cand}(t,x,0) \in \mathbb{Q}[[x,t]]$. For the latter, $F_{cand}(t,x,0)$ is unique in $\mathbb{Q}[[x,t]]$.

Proving existence is a little more tricky since Theorem 2 cannot be used. We can look at P from a different perspective, namely $P \in \mathbb{Q}(x)[T,t]$. This helps to show that P is a curve of genus 0 over $\mathbb{Q}(x)$. As a consequence, P can be rationally parametrized. Bostan and Kauers obtain the parametrization with the use of Maple package, namely algcurves. The two rational functions are

$$R_1(u,x) = \frac{u(u+1)(1+2u+u^2+u^2x)^2}{h(u,x)}$$

$$R_2(u,x) = \frac{(u^4x^2 + 2u^2(u+1)^2x + 1 + 4u + 6u^2 + 2u^3 - u^4)h(u,x)}{(1+u)^2(1+2u+u^2+u^2x)^4}$$

where $h(u,x) = u^6x^3+3u^4(u+1)^2x^2+3u^2(u+1)^4x+(u+1)^3(5u^3+3u^2+3u+1)$ with the following properties

1. $P(R_2(u,x), R_1(u,x), x) = 0$
2. there is a unique power series

   $$u_0(t,x) = t + t^2 + (x+1)t^3 + (2x+5)t^4 + (2x^2+3x+9)t^5 + ... \in \mathbb{Q}[[x,t]]$$

   such that $R_1(u_0, x) = t$ and $u_0(0,x) = 0$

The first property is true by equation (4.11). Let $Q(u,t,x) = R_1(t,x) - t$ then $Q \in C^1$ in the neighbourhood of (0,0,0), $Q(0,0,0) = 0$ and $\frac{dQ}{du}P(0,0,0) \neq 0$. By Theorem 2, there exists a unique power series in $\mathbb{Q}[[x,t]]$ for which $Q = 0$. Hence, $Q(u_0,t,x) = R_1(u_0,x) - t = 0$ and u_0 is unique. That is, the second property holds.

Let $F_{cand}(t,x,0) = R_2(u_0(t,x),x)$ then

$$\begin{aligned} P(F_{cand}(t,x,0),t,x) &= P(R_2(u_0(t,x),x), R_1(u_0,x),x) \text{ (by (2))} \\ &= 0 \text{ (by (1))} \end{aligned}$$

This shows that $F_{cand}(t,x,0)$ is a root of $P(T,t,x)$.

Lastly, we need to show that the assignment to F_{cand} is well defined and unique in $\mathbb{Q}[[x,t]]$ Since R_2 does not have a pole at $U = 0$ and $ord_t U_0$ is positive then F_{cand} is well defined in $\mathbb{Q}[[x,t]]$. Since $P(R_2(u_0(t,x),x),t,x) = P(R_2(u_0(t,x),x), R_1(u_0,x),x) = 0$ then F_{cand} is unique in $\mathbb{Q}[[x,t]]$

Prove Compatibility with Reduced Kernel Equation

Now that we have found an algebraic series $F_{cand}(t,x,0)$ the last step is to show that it coincides with $F(t,x,0)$. If we show that the following expression

$$S(t,x) = \frac{Y(t,x)}{t} - \frac{Y(t,x)}{x}F_{cand}(t,Y(t,x))$$

is a root of $P(T,t,x)$ then by uniqueness of $F_{cand}(t,Y(t,x))$ in $\mathbb{Q}[[x,t]]$, $S(t,x) = F_{cand}(t,x,0)$. Consequently, by Lemma 4, $F(t,x,0) = F_{cand}(t,x,0)$. To prove this the following lemma, which provides the classical results of algebraic power series, will be used.

Lemma 5. *Let $\mathbb{K}$ be a field and let $P,Q\in \mathbb{K}[T,t,x]$ be polynomials with algebraic power series $A,B\in\mathbb{K}[x,x^{-1}][[t]]$ as its roots, respectively. Then*

1. *pA is algebraic for every $p\in\mathbb{K}(t,x)$, and it is a root of $p^{deg_T P}P(T/p,t,x)$*
2. *$A\pm B$ is algebraic, and it is a root of $res_z(P(z,t,x),\ Q(\pm(T\text{-}z),t,x))$.*
3. *AB is algebraic, and it is a root of $res_z(P(z,t,x),\ z^{deg_T Q}Q(T/z,t,x))$.*
4. *If $ord_x B>0$, then $A(t,B(t,x))$ is algebraic, and it is a root of $res_z(P(T,t,z),\ Q(z,t,x))$.*

From previous sections it has been established that $Y(t,x)$ and $F_{cand}(t,x,0)$ are algebraic power series in $\mathbb{Q}[x,x^{-1}][[t]]$. In addition, they are the roots of $Q(T,t,x) = (x+T+x^2T^2)t - x$ and $P(T,t,x)$, respectively. By Lemma 5(1), $\frac{Y(t,x)}{t}$ and $\frac{Y(t,x)}{x}$ are algebraic and roots of $p^2Q(T/t,t,x)$ and $p^2Q(T/x,t,x)$, respectively. Since $ord_xY > 0$ then by (4) $F_{cand}(t,Y(t,x))$ is algebraic and root of $res_z(P(T,t,z),Q(z,t,x))$. By (3), $\frac{Y(t,x)}{x}F_{cand}(t,Y(t,x))$ is algebraic and root of

$$res_z(p^2Q(z/x,t,x), z^{deg_T res_z(P(T,t,z),Q(z,t,x))}res_z(P(\frac{T}{z},t,z),Q(z,t,x)))$$

Lastly, by (2) $S(t,x)$ is algebraic and a root of

$$res_z(p^2Q(z/t,t,x), res_z(p^2Q(z/x,t,x), z^{deg_T res_z(P(T,t,z),Q(z,t,x))} \times res_z(P(\frac{-(T-z)}{z},t,z),Q(z,t,x))))$$

This complicated expression can be constructed in Maple through the following commands.

```
ker := (T,t,x) -> (x+T+x^2*T^2)*t-x*T:
pol := unapply(P,T,t,x):
res := resultant(numer(pol(x/z*(z/t-T),t,z)),ker(z,t,x),z):
factor(primpart(res,T));
```

The output is $P(T,t,x)^2$ which means $S(t,x)$ is a root of $P(T,t,x)$. This concludes the proof of Theorem 4.

4.1.2 Gessel Walks

When equation (4.4) was unknown only an observation about the sequence $g(n,0,0)$ was made. The counting sequence $g(n,0,0)$ starts as

$$1, 0, 2, 0, 11, 0, 85, 0, 782, 0, 8004, 0, 88044, 0, 1020162, 0, ..$$

It can be found on the On-line Encyclopedia of Integer Sequences [44]. The sequence in OEIS omits the zeros since they occur at every odd n. Gessel conjectured that this sequence must have a hypergeometric closed form and produced equation (4.4). This expression was then proven in [32] and used as a tool to prove Corollary 1 and consequently Theorem 5 in [11].

Corollary 1. *G(t,0,0) is algebraic*

Proof. Proving algebraicity of a series is equivalent to finding the polynomial for which the series is a root. In this case we will be finding an annihilating polynomial for a series that is slightly different from $G(t,0,0)$ and modifying it so that it becomes the annihilating polynomial of $G(t,0,0)$. Let $g(t) = \sum_{n=0}^{\infty} \frac{(5/6)_n(1/2)_n}{(5/3)_n(2)_n}(16t)^n$ and $P(t,T) \in \mathbb{Q}[t,T]$ be a polynomial for which $g(t)$ is a root. By equation (4.4), $G(t,0,0) = g(t^2)$. Since $g(t)$ is a root of $P(t,T)$ then $P(t,T) = P_1(t,T)(T - g(t))$ for some $P_1(t,T) \in \mathbb{Q}[t,T]$. As a result,

$$P(t,T)\Big|_{(t,T)=(t^2,g(t^2))} = P_1(t^2, g(t^2))(g(t^2) - g(t^2)) = 0.$$

Hence, $G(t,0,0)$ is a root of the polynomial $P(t^2,T)$. Consequently, if $P(t,T)$ exists then $G(t,0,0)$ is algebraic.
The polynomial is guessed as

$$\begin{aligned} P(t,T) = &-1 + 48t - 576t^2 - 256t^3 + (1 - 60t + 912t^2 - 512t^3)T \\ &+ (10t - 312t^2 + 624t^3 - 512t^4)T^2 + (45t^2 - 504t^3 - 576t^4)T^3 \\ &+ (117t^3 - 252t^4 - 288t^5)T^4 + 189t^4T^5 + 189t^5T^6 + 108t^6T^7 + 27t^7T^8 \end{aligned} \tag{4.12}$$

the techniques for guessing are similar to the techniques used in the proof of Theorem 5.
Based on equation (4.12), $P(t,T)$ satisfies the following conditions

- $P(t,T) \in C^1$ in the neighbourhood of (0,1)
- $P(0,1) = 0$ and $\frac{dP}{dT}(0,1) \neq 0$

Therefore, by Theorem 2, there exists $r(t) \in \mathbb{Q}[[t]]$ with $r(0) = 1$ such that $P(t,r(t)) = 0$. Since $P(0,1) = 0$ then $P(0,T) = T - 1$. Since $P(0,T)$ has a single root in $\mathbb{C}$ then $r(t)$ is a unique root of $P(t,T)$ in $\mathbb{C}[[t]]$. By Theorem 3 and $r(t)$ being algebraic, $r(t)$ is also D-finite. So the coefficients of $r(t)$ satisfy a linear recurrence with $P(t,T)$ coefficients.
The following commands in Maple will output the recurrence relation

```
with(gfun):
P:=(t,T) -> -1+48*t-576*t^2-256*t^3+(1-60*t+912*t^2
-512*t^3)*T+(10*t-312*t^2+624*t^3-512*t^4)*T^2+(45*t^2
-504*t^3-576*t^4)*T^3+(117*t^3-252*t^4-288*t^5)*T^4+
189*t^4*T^5+189*t^5*T^6+108*t^6*T^7+27*t^7*T^8:
gfun:-diffeqtorec(gfun:-algeqtodiffeq(P(t,r),r(t)),r(t),
g(n));
```

The linear recurrence is $(n+2)(3n+5)g_{n+1} - 4(6n+5)(2n+1)g_n = 0$, $g_0 = 1$ satisfied by $r(t) = \sum_{n=0}^{\infty} g_n t^n$. Isolating for g_{n+1} results in

$$g_{n+1} = \frac{(\frac{5}{6}+n)+(\frac{1}{2}+n)}{(\frac{5}{3}+n)+(1+n)} 16 g_n$$

Based on the definition of $(a)_n$ and the recursive relation,

$$g_n = \frac{(5/n)_n (1/2)_n}{(5/3)_n (2)_n} 16^n$$

As a result, $r(t) = g(t)$ and therefore is a unique root of P in $\mathbb{C}[[t]]$. $\square$

Theorem 5. *G(t,x,y) is algebraic*

Allowing x and y to remain as parameters adds tremendous amount of work in proving holonomy. This is mostly because a minimal polynomial that admits $G(t,x,y)$ as a root has a size of about 30gb. Therefore, the series will be broken down so that it involves $G(t,x,0), G(t,0,y)$ and $G(t,0,0)$. The core of the proof will be to produce and manipulate $G(t,x,0)$ and $G(t,0,y)$. Since the expressions and computations involved in the proof of this theorem are very large, systems like Maple and Mathematica were not used by Boston and Kauers. Instead, sophisticated algorithms were performed on fast processors with large memory using the Magma computer algebra system.

Before diving into the proof of Theorem 5, lets look at some helpful observations about $G(t,x,y)$.

The recurrence relation of $g(n,i,j)$ is

$$g(n+1,i,j) = g(n,i-1,j-1) + g(n,i+1,j+1) + g(n,i-1,j) + g(n,i+1,j) \tag{4.13}$$

By similar reasoning and boundary conditions as in Section 4.1.1, Equation (4.2) can be written as

$$G(t,x,y) = \frac{(1+y)tG(t,0,y) + tG(t,x,0) - tG(t,0,0) - xy}{(1+y+x^2y+x^2y^2)t - xy} \tag{4.14}$$

Now the goal is to find the reduced kernel equation(s) that will used in the third step of the proof of Theorem 5. Unlike Kreweras walks, there is lack of symmetry w.r.t x and y. This can be seen by looking at the numerator of equation (4.14). The reason for the asymmetry is due to E being part of the step set of Gessel walks. As a result, there will be two reduced kernel equations. One will isolate for x and one for y.

$$y := Y(t,x) = \frac{-(tx^2 - x + t + \sqrt{(tx^2 - x + t)^2 - 4t^2x^2})}{2tx^2} \tag{4.15}$$

$$x := X(t,y) = \frac{y - \sqrt{y(y - 4t^2(y+1)^2)}}{2ty(y+1)} \tag{4.16}$$

The isolation for x and y are done with the help of the quadratic formula. Hence, there are two possibilities for each variable. By similar reasons as for the generating function of Catalan numbers in Example 4, only the negative root is the viable option for x and y. Now to make the numerator vanish wrt $y = Y(t,x)$ and $x = X(t,y)$, we get the following equations

$$G(t,x,0) = \frac{xY(t,x)}{t} + G(t,0,0) - (1+Y(t,x))G(t,0,Y(t,x))$$

$$(1+y)G(t,0,y) = \frac{yX(t,y)}{t} + G(t,0,0) - G(t,X(t,y),0)$$

An important observation about each equations above is that the first one if free of y and the second one is free of x. So if y is renamed to x in the second equation then both expressions belong to $\mathbb{Q}[x,x^{-1}][[t]]$. Now we can say $G(t,x,0) = G(t,0,0) + xU(t,x)$ and $G(t,0,x) = G(t,0,0) + xV(t,x)$ for

$U, V \in \mathbb{Q}[[x,t]]$. So we get the reduced kernel equations

$$
\begin{aligned}
xU(x,t) &= G(t,x,0) - G(t,0,0) \\
&= \left[\frac{xY(t,x)}{t} + G(t,0,0) - (1+Y(t,x))G(t,0,Y(t,x))\right] - G(t,0,0) \\
&= \left[\frac{xY(t,x)}{t} + (1+Y(t,x))(G(t,0,0) - G(t,0,Y(t,x)))\right] \\
&\quad - (1+Y(t,x))G(t,0,0) \\
&= \frac{xY(t,x)}{t} - Y(t,x)(1+Y(t,x))V(t,Y(t,x)) - (1+Y(t,x))G(t,0,0),
\end{aligned}
\tag{4.17}
$$

and

$$
\begin{aligned}
x(1+x)V(x,t) &= (1+x)G(t,0,x) - (1+x)G(t,0,0) \\
&= \left[\frac{xX(t,x)}{t} + G(t,0,0) - G(t,X(t,x),0)\right] - (1+x)G(t,0,0) \\
&= \frac{xX(t,x)}{t} - X(t,x)U(t,X(t,x)) - (1+x)G(t,0,0).
\end{aligned}
\tag{4.18}
$$

The next lemma will prove that $G(t,x,0)$ and $G(t,0,y)$ are a unique pair of solution for (4.17) and (4.18) in $\mathbb{Q}[[x,t]]$.

Lemma 6. *Let $A_1, A_2, B_1, B_2, Y_1, Y_2 \in \mathbb{Q}[x,x^{-1}][[t]]$ such that $ord_t B_1 > 0$, $ord_t B_2 > 0$, $ord_t Y_1 > 0$ and $ord_t Y_2 > 0$. Then there exists at most one pair of power series $(U_1, U_2) \in \mathbb{Q}[[x,t]]^2$ with*

$$U_1(t,x) = A_1(t,x) + B_1(t,x)U_2(t,Y_1(t,x))$$

$$U_2(t,x) = A_2(t,x) + B_2(t,x)U_1(t,Y_2(t,x))$$

Proof. Similar reasoning as Lemma 4, the goal is to show that the only solution for $(U_1,U_2) = (B_1(t,x)U_2(t,Y_1(t,x)), B_2(t,x)U_1(t,Y_2(t,x)))$ is the trivial solution (ie. $(U_1,U_2) = (0,0)$). Suppose not. Then valuation of $B_1(t,x)U_2(t,Y_1(t,x))$ will be at least $ord_t B_1 + ord_t U_2$. Since $ord_t B_1 > 0$ then valuation of $B_1(t,x)U_2(t,Y_1(t,x))$ is greater than valuation of U_2. Similarly, valuation of $B_2(t,x)U_1(t,Y_2(t,x))$ greater than valuation of U_1. So, $ord_t U_1 > ord_t U_2 > ord_t U_1$, contradiction. $\square$

Similar to Theorem 4, the proof of Theorem 5 can be divided into three steps. First, guess the algebraic equations $G(t,x,0)$ and $G(t,0,y)$, by observing their initial terms. Let the guess be denoted P_1 and P_2, respectively. Then prove that P_1 and P_2 have exactly one solution in $\mathbb{Q}[[x,t]]$, namely $U_{cand}(t,x,0)$ and $V_{cand}(t,0,y)$, respectively. Lastly, prove that the power series $U_{cand}(t,x,0)$ and $V_{cand}(t,0,y)$ satisfies (4.17), (4.18). Proof of the last statement implies that $U_{cand}(t,x,0)$ and $V_{cand}(t,0,y)$ are equal to U and V, respectively. This is due to (4.17) and (4.18) and Lemma 6. Consequently, $G(t,x,0)$ and $G(t,0,y)$ are algebraic by the first and second step. Since algebraic power series are closed under addition, multiplication and inversion, by equation (4.14), $G(t,x,y)$ is also algebraic.

Guess for Gessel Walks

The guessing part of the proof is very technical and lengthy in terms of the algorithms that are used. In addition, the polynomial which admits U as a solution is of degree 24, 44, and 32 with respect to T, t, and x. Similarly for V, the polynomial is of degree 24, 46, and 56 with respect to T, t, and y. Together, these polynomials span 30 pages. As a result, only the conceptual part of guessing will be explained in this section.

- Compute the first 1000 terms of $G(t,x,0)$ and $G(t,0,y)$.
- Using Magma construct differential operators $\mathcal{L}^{(i)}_{p,x_0} \in \mathbb{Z}[t]\langle D_t\rangle (i = 1,2,\cdots)$ s.t. $\mathcal{L}^{(i)}_{p,x_0}(t,x_0,0) = O(t^{1000})\text{mod}p$ for various $x_0 = 1,2,\cdots$ and specific primes p. These operators can be regarded as homomorphic images of some operators $\mathcal{L}^{(i)} \in \mathbb{Q}(x)[t]\langle D_t\rangle$ such that $\mathcal{L}^{(i)}G(t,x,0) = 0$
- Using the Grigoriev algorithm [24], for every x_0 and p, compute greatest common right divisor in $\mathbb{Z}[t]\langle D_t\rangle$ of $\mathcal{L}^{(i)}_{p,x_0}$. This will yield a single operator $\mathcal{L}_{p,x_0}$ of order 11 with coefficients of degree at most 96 in t. These operators are homomorphic image of the least order operator $\mathcal{L} \in \mathbb{Q}(x)[t]\langle D_t\rangle$ such that $\mathcal{L}G(t,x,0) = 0$. The reason for using gcrd is because the degree of x drops from more than 1500 to 28 with a reasonable raise in degree of t from 43 to 96.
- From the operators $\mathcal{L}_{p,x_0}$ a good candidate for the preimage $\mathcal{L}$ of order 11 with degree 96 in t and 78 in x is constructed. Similarly, a candidate operator for $G(t,0,y)$ of order 11 with degree 68 in t and 28 in y is constructed. The reason for which the reconstruction algorithms are applied to $\mathcal{L}_{p,x_0}$ rather than $\mathcal{L}^{(i)}_{p,x_0}$ is because the degree of x in their

respective preimage is very large. This would make the computation very expensive.

- The guessed operator $\mathcal{L}$ is checked for correctness through a series of tests. The collection of such tests can be found on [9]. One of these tests is called globally nilpotent [21]. The idea is to check if the order 11 operator $\mathcal{L}$ right divides, for almost all primes p, D_t^{11p} in $\mathbb{Z}_p(x,t)\langle D_t\rangle$. While checking for primes strictly less than 100, the necessary and sufficient condition of Grothendieck's conjecture [25] for the holonomy of linear operators is satisfied. This is an interesting observation because based on the holonomy of $G(t,x,y)$, it either served in favour of or as a counterexample to the conjecture.
- Using fast modular Hermite-Padé approximation, combined with an interpolation scheme, $P_1(T,t,x)$ and $P_2(T,t,x)$ are found such that

$$P_1(\text{U(t,x),t,x}) = 0 \bmod t^{1200}$$
$$P_2(\text{V(t,y),t,y}) = 0 \bmod t^{1200}$$

Prove Existence and Uniqueness

Similar to the Kreweras walks, Theorem 2 cannot be used. In addition, due to the asymmetry in Gessel walks, rational parametrization cannot be used either. This is because the polynomials in question have positive genus. The proof for both polynomials will involve Puiseux's Theorem and an algorithm from [39].

Consider the polynoimal $P_1(T,t,x)$ from the section above. By Puiseux's theorem, there is a full system of solutions in the ring

$$\mathbb{Q}\{t\} := \bigcup_{r\in\mathbb{N}} \bigcup_{\alpha\in\mathbb{Q}} t^{\alpha}\mathbb{Q}[[t^{\frac{1}{r}}]]$$

of the form

$$f(t,x) = \sum_{p,q\in\mathbb{Q}} c_{p,q}x^p t^q \in \mathbb{Q}\{t\}$$

that are annihilated by P_1 such that $c_{p,q} \neq 0$ for an appropriate set of indices. Consequently, $c_{p,q} = 0$ for (p,q) outside of some half-plane $up + vq \leq 0$. The value of u, v depends on the Newton's polytope of P [39]. By Theorem 3.6 in [39] (u,v) can be chosen so that u and v are linearly independent over $\mathbb{Q}$

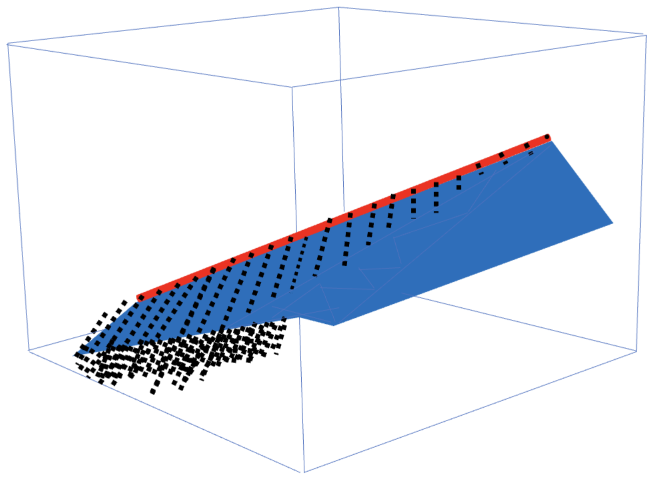


Figure 4.1: e is the red line, two blue half-plane are the boundary of W(e) and the black points are the support of Q

and (u, v) belongs to a 'normal cone' $C^*(e)$ of some admissible edge e in the Newton's polytope of Q. Based on the theorem, e can be $(44, 32, 24)-(4, 12, 4)$. So the barrier wedge, barrier cone and normal cone [39] are

$$W(e) = \{(x, y, z) \in \mathbb{R}^3 : z \leq 12 + x - y \wedge 3z \leq 4 + x + y\}$$

$$C(e) = \{(x, y) \in \mathbb{R}^2 : x + y \geq 0 \wedge x - y \geq 0\}$$

$$C^*(e) = \{(x, y) \in \mathbb{R}^2 : x \leq 0 \wedge x \leq y \geq -x\}$$

So, $(u, v) = (-1, \frac{1}{10}\sqrt{2}) \in C^*(e)$. Figure 4.1 shows e, $W(e)$ and Q. Since $\mathbb{Q}\{t\}$ is a differential ring then we can use gfun to find a system of linear operators $L_1, L_2, L_3, L_4, L_5 \in \mathbb{Q}[t, x]\langle D_t, D_x\rangle$ such that $L_i \cdot f = 0$ whenever $P_1(f(t, x), t, x) = 0$ for $i \in [5]$. Consequently, the difference operators are $R_1, ..., R_5 \in \mathbb{Q}[p, q]\langle S_p, S_q\rangle$ such that $R_i \cdot c_{p,q} = 0$ for any set of coefficients of $f \in \mathbb{Q}\{t\}$. So, there are 5 multivariate recurrence equations

$$a_1(p, q)c_{p,q} = 2(9p^2q + 15p^2 - 9pq^3 - 63pq^2 - 74pq + 18q^4 + 63q^3 + 25q^2 - 51q - 15),$$

$$a_2(p, q)c_{p,q} = 2(p + 1)(3p + 4)(-p + 3q^2 + 6q + 1),$$

$$\begin{aligned} a_3(p, q)c_{p,q} = &- 2(9p^3q + 9p^3 + 36p^2q + 57p^2 - 126pq^2 - 205pq - 9p + 63q^4 \\ &+ 252q^3 + 182q^2 - 120q - 57), \end{aligned}$$

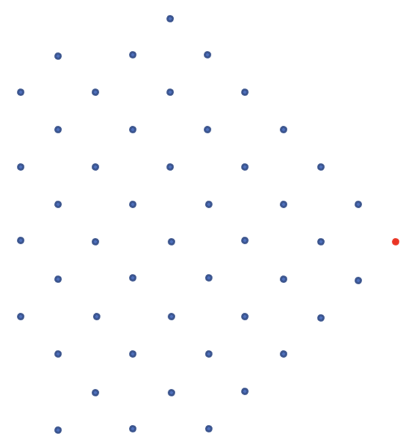


Figure 4.2: Shifts occurring after first recurrence

$$a_4(p,q)c_{p,q} = 2(9p^4 + 54p^3 + 182p^2 - 378pq^2 - 756pq - 54p + 189q^4 + 756q^3 + 546q^2 - 420q - 191),$$

$$a_5(p,q)c_{p,q} = 2(-15p^2q - 30p^2 + 90pq^2 + 140pq + 9q^5 - 100q^3 - 70q^2 + 91q + 30)$$

where the right hand side is a linear combination of some shifted version of $c_{p,q}$. For instance, the red bullet in figure 4.2 is the point (p, q) and a blue bullet at point $(p+i, q+j)$ represents the term $c_{p+i,q+j}$ on the right hand side of the first recurrence.

Due to the recurrence equations for the coefficients of $f \in \mathbb{Q}\{t\}$, $c_{p,q} \neq 0$ only if one of the following is true

1. One of coefficients on the right hand side of one of the recurrence is nonzero. That is, $c_{p+i,q+j} \neq 0$ for some a_i.
2. $a_1 = a_2 = a_3 = a_4 = a_5 = 0$

The second condition is only true if $(p, q) \in$\{(1, 0), (-1,0), ($\frac{-4}{3}$, -1), (-2, -1), (-1, $\frac{-2}{3}$), ($\frac{-5}{3}$, $\frac{-2}{3}$), (-1, $\frac{-4}{3}$), ($\frac{-5}{3}$, $\frac{-4}{3}$), ($\frac{-4}{3}$, $\frac{-5}{3}$), (-1, -2), ($\frac{-4}{3}$, $\frac{-1}{3}$)\}. By the generalized version of Puiseux's algorithm [39], the first few terms of $U_{cand}(t, x)$ are

$$U_{cand}(t, x) = t + x + (5 + x^2)t^3 + (9x + x^3)t^4 + \ldots$$

Suppose there was a term after t^4 with fractional component then some negative fractional terms would appear in U_{cand} above. This is because the shift

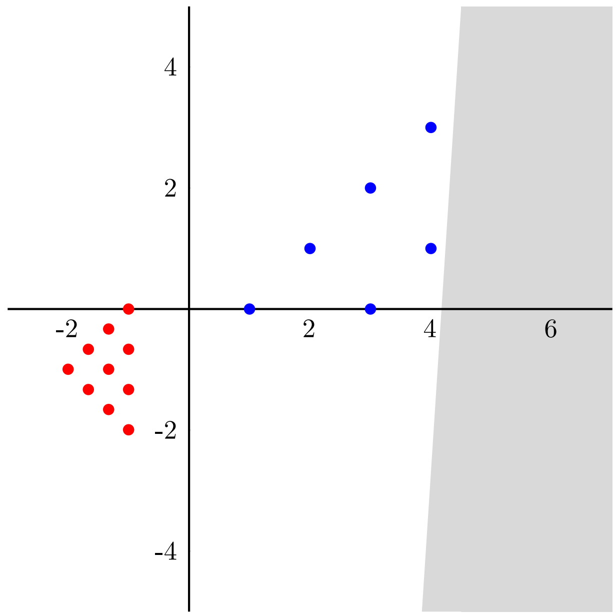


Figure 4.3: Critical Points

distances in the recurrence equations are integers. So for any $f \in \mathbb{Q}\{t\}$, f has fractional components if $t^{-\frac{4}{3}}x^{-1}, t^{-1}x^{\frac{-2}{3}}, t^{\frac{-5}{3}}x^{\frac{-2}{3}}, t^{-1}x^{\frac{-4}{3}}, t^{\frac{-5}{3}}x^{\frac{-4}{3}}, t^{\frac{-4}{3}}x^{\frac{-5}{3}}$ or $t^{\frac{-4}{3}}x^{\frac{-1}{3}}$ has nonzero coefficient. In figure 4.3, these critical points are the red points on the left. To summarize, U_{cand} contains terms $c_{p,q}t^p x^q$ where $(p,q) \in \mathbb{Z}$ such that $p \geq 1$ and $2 - p \leq g \leq p$ and the gray region in figure 4.3 is the only region where the remainder of the terms of U_{cand} lie. For U_{cand} to be in $\mathbb{Q}[x][[t]]$, $c_{p,q} = 0$ for $q < 0$. Let $R_1^{'}, ..., R_7^{'}$ be operators defined as

$$\left(R_1^{'} \dots R_7^{'}\right)^T := M \cdot (R_1 \dots R_7)^T$$

for $M \in \mathbb{Q}[p,q]\langle S_p, S_q\rangle^{7x5}$ such that the support of each $R_i^{'}$ is given by figure 4.4.

The matrix is provided in [10] and it can be seen that $c_{p,q}$ of any solution for P_1 is annihilated by $R_1^{'} \dots R_7^{'}$. Suppose $U_{cand}(t,x)$ contains a term $c_{p,q}t^p x^q$ with nonzero coefficient for some $q < 0$, then by figure 4.4 there will be a left-most line with slope -1 that contains that term. Due to the support of the recurrence equations, $c_{p,q} \neq 0$ only for (p,q) for which the leading coefficients of $R_1, ..., R_5$ vanish simultaneously. This occurs only for (p,q) that satisfy

$$(q-2)(q-1)q(p+q+1)(p+q+3)(3p+3q+5)(3p+3q+7) = 0$$

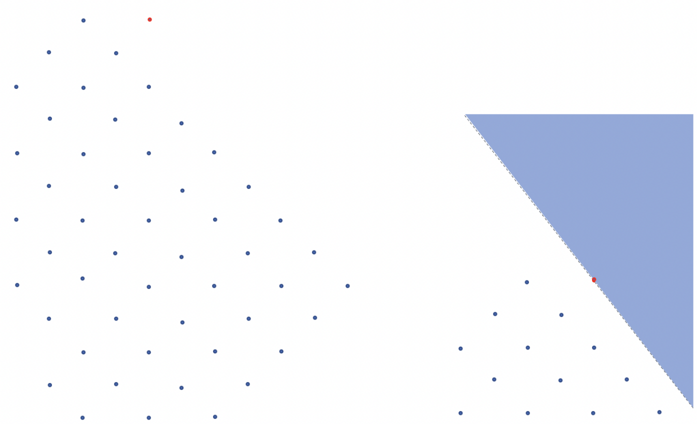


Figure 4.4: Support of the Recurrence Equations

However, this is outside of the critical area shown in figure 4.3. As a result, $c_{p,q} = 0$ for $q < 0$ and $U_{cand} \in \mathbb{Q}[x][[t]]$. By Mcdonald's algorithm [39], the initial terms of all other solutions to P_1 has fractional exponents. Hence, U_{cand} is unique in $\mathbb{Q}[x][[t]]$. The proof for V_{cand} is similar.

Prove Compatibility with Reduced Kernel Equation

Last part of the proof is to show that U_{cand} and V_{cand} satisfy equations (4.17) and (4.18). By equation (4.15) and (4.16), $X(t,y) = X(t, Y(t,x)) = x$. So, if $x = Y(t,x)$ is substituted into equation (4.18) then it becomes (4.17). Consequently, it is sufficient to prove that V_{cand} satisfies (4.18)

$$x(1+x)V_{cand}(x,t) = \frac{xX(t,x)}{t} - X(t,x)U_{cand}(t, X(t,x)) - (1+x)G(t,0,0)$$

Since U and V are defined such that $G(t,x,0) = G(t,0,0) + xU(t,x)$ and $G(t,0,x) = G(t,0,0) + xV(t,x)$ then let $G_1(t,x) = G(t,0,0) + xU_{cand}(t,x)$ and $G_2(t,x) = G(t,0,0) + xV_{cand}(t,x)$. The equation above can be rewritten as

$$(1+x)G_2(t,x) - G(t,0,0) = \frac{xX(t,x)}{t} - G_1(t, X(t,x)).$$

By Corollary 1 and Lemma 5, the left hand side and right hand side of the equation above are algebraic. The rest of the proof carries out in similar fashion as for Kreweras walks. However, they cannot be shown here since P_1 and P_2

are very large and resultant computations can only be tried on software such as Magma. An interesting observation about both power series is that they have identical minimal polynomials.

4.2 D-finite

As seen in figure 3.2, a generating function is more likely to be D-finite than algebraic. However, despite the likelihood there are still some lattice walks for which the generating function is not D-finite either. One of which is for excursions ending at the origin. Alin Boston, Kilian Raschel and Bruno Salvy prove this with the help of some classical results in probability and the asymptotic behaviour of the coefficient of the generating function [12]. This section will discuss their proof.

Let $f_{\mathcal{G}(i,j,n)}$ denote the number of walks that uses the step set $\mathcal{G} \subseteq \{0, \pm 1\}^2 \setminus \{(0,0)\}$, ending at (i,j) after n steps in the first quadrant. Consider the generating function

$$F_{\mathcal{G}}(x,y,t) = \sum_{i,j,n \geq 0} f_{\mathcal{G}(i,j,n)} x^i y^j t^n \in \mathbb{Q}[[x,y,t]]$$

The goal of this section is to show that the generating function of $\mathcal{G}$-excursions, namely $F_{\mathcal{G}}(0,0,t)$, is not D-finite. Let $e_n = e_n^{\mathcal{G}}$ denote the number of $\mathcal{G}$-excursions of length n using only steps in $\mathcal{G}$ then

$$F_{\mathcal{G}}(0,0,t) = \sum_{n \geq 0} e_n^{\mathcal{G}} t^n \in \mathbb{Q}[[t]] \tag{4.19}$$

The total number of different steps are $3^2 - 1 = 8$. That is, $\{0, \pm 1\}^2 \setminus \{(0,0)\} = \{(0,1), (1,0), (0,-1), (-1,0), (1,1), (1,-1), (-1,1), (-1,-1)\}$. So, the total number of possible subsets $\mathcal{G}$ is

$$\sum_{i=0}^{8} \binom{8}{i} = 2^8 = 256.$$

Of these small step sets there are exceptional sets $\mathcal{G}$ for which $F_{\mathcal{G}}(0,0,t) = 1$ which is trivially D-finite. These step sets are called singular. In addition, there are step-sets that are equivalent to models of walks on the half-plane.

It has been discussed in Proposition 1 that walks on half-plane are algebraic and hence D-finite. Eliminating these step-sets along with a few special cases leads to 74 non-singular step sets in $\mathbb{N}^2$, up to some equivalence, for which non D-finite property has not been proven. From these 74 non-singular steps, 51 have a birational transformations group $G_{\mathbb{G}}$ of infinite order. These 51 step sets result in a class of walks, from the set of excursions, for which the generating function is not D-finite. These deductions were made by Mireille Bousquet-Melou and Marni Mishna to reduce 256 step sets 51. These step sets can be found in Table 1 of [12].

Theorem 6. *Let $\mathcal{G} \subseteq \{0, \pm 1\}^2 \setminus \{(0,0)\}$ be any of the 51 nonsingular step sets in $\mathbb{N}^2$ with infinite group $G_{\mathcal{G}}$ then the generating function $F_{\mathcal{G}}(0,0,t)$ of $\mathcal{G}$-excursions is not D-finite. Equivalently, the excursion sequence $(e_n^{\mathcal{G}})_{n\geq 0}$ does not satisfy any nontrivial linear recurrence with polynomial coefficients.*

Before diving into the proof of the theorem above, some important properties will be established.

Theorem 7. *Let $(a_n)_{n\geq 0}$ be an integer-valued sequence whose n-th term a_n behaves asymptotically like $K \cdot \rho^n \cdot n^\alpha$, for some real constant $K > 0$. If the growth constant ρ is transcendental, or if the singular exponent α is irrational, then the generating function $A(t) = \sum_{n\geq 0} a_n t^n$ is not D-finite.*

Theorem 7 is a stronger and less known result from the one that checks transcendental property of generating functions. The noteworthy results in [8] and [48] by Birkhoff-Trjitzinsky and Turrittin, respectively, leads to Theorem 7 as a consequence.

Theorem 8. *Let $\mathcal{G} \subseteq \{0, \pm 1\}^2 \setminus \{(0,0)\}$ be the step set of a walk in the quarter plane $\mathbb{N}^2$, which is not contained in a half-plane. Let $e_n = e_n^{\mathcal{G}}$ and $\mathcal{X} = \mathcal{X}_{\mathcal{G}}$ denote the characteristic polynomial $\sum_{(i,j)\in\mathcal{G}} x^i y^j \in \mathbb{Q}[x, x^{-1}, y, y^{-1}]$ of the step set $\mathcal{G}$. Then, the system*

$$\frac{\partial \mathcal{X}}{\partial x} = \frac{\partial \mathcal{X}}{\partial y} = 0 \tag{4.20}$$

has a unique solution $(x_0, y_0) \in \mathbb{R}^2_{>0}$. Next, define

$$\rho := \mathcal{X}(x_0, y_0), c := \frac{\frac{\partial^2 \mathcal{X}}{\partial x \partial y}}{\sqrt{\frac{\partial^2 \mathcal{X}}{\partial x^2} \cdot \frac{\partial^2 \mathcal{X}}{\partial y^2}}}(x_0, y_0), \alpha := -1 - \frac{\pi}{arccos(-c)} \tag{4.21}$$

Then there exists a constant $K > 0$, which depends only on $\mathcal{G}$, such that

- *If the walk is aperiodic then* $e_n \sim K \cdot \rho^n \cdot n^\alpha$
- *If the walk is periodic (of period 2) then* $e_{2n} \sim K \cdot \rho^{2n} \cdot (2n)^\alpha, e_{2n+1} = 0$

Proof. Let X be a random walk starting at the origin with each step drawn uniformly at random from $\mathcal{G}$. Let τ be the first instance at which the boundary of the translated positive quarter plane ({-1}∪ℕ)^2 is reached by X and $(X_1(k), X_2(k))_{k\geq 1}$ be the coordinates of X after each step. Then the probability of X reaching at (i, j) after n steps while remaining in the first quadrant is

$$\mathbb{P}\left[\sum_{k=1}^{n}(X_1(k), X_2(k)) = (i, j), \tau > n\right] = \frac{f_{\mathcal{G}}(i, j, n)}{|\mathcal{G}|^n} \tag{4.22}$$

The summation ensures that X reached (i, j) after n steps and $\tau > n$ ensures that X remains in the first quadrant. To avoid walks with redundant steps only walks with no drift will be considered. That is, at each point in time, the random walk Y takes a step away from its last recorded position, with steps whose expected value $\mathbb{E}[(Y_1(k), Y_2(k))] = (0, 0)$ for all k. This is achieved by assigning weights to each step

- A weight of $x_0 > 0$ for the East step
- A weight of $\frac{1}{x_0}$ for the West step
- A weight of $y_0 > 0$ for the North step
- A weight of $\frac{1}{y_0}$ for the South step

As a result, the probability for each $(i, j) \in \mathcal{G}$ is $\frac{x_0^j y_0^j}{\mathcal{X}(x_0, y_0)}$. Since the expected value of each position in Y is $(0, 0)$ then x_0, y_0 are fixed and differentiating with respect to x and y gives the expected value of each coordinate as

$$\mathbb{E}[Y_1(k)] = \frac{x_0}{\mathcal{X}(x_0, y_0)}\frac{\partial \mathcal{X}}{\partial x}(x_0, y_0), \mathbb{E}[Y_2(k)] = \frac{y_0}{\mathcal{X}(x_0, y_0)}\frac{\partial \mathcal{X}}{\partial y}(x_0, y_0) \tag{4.23}$$

Since $\mathbb{E}[(Y_1(k), Y_2(k))] = (0, 0)$ then both equations in (4.23) are set to zero. Therefore, the solution (x_0, y_0) for equation (4.23) is equivalent to (4.20). The existence of such a solution is evident from the fact that

$$\lim_{x\to 0^+} = \lim_{y\to 0^+} = \lim_{x\to\infty} = \lim_{y\to\infty} = \infty$$

since $\mathcal{G}$ is not confined to the half-plane. Since $\mathcal{X}$ is a convex Laurent polynomial with positive coefficients then the solution is also unique. By (4.22), Y

relates to X by

$$\mathbb{P}\left[\sum_{k=1}^{n}(Y_1(k),Y_2(k))=(i,j),\tau>n\right]$$
$$=\frac{x_0^j y_0^j|\mathcal{G}|^n}{\mathcal{X}(x_0,y_0)^n}\mathbb{P}\left[\sum_{k=1}^{n}(X_1(k),X_2(k))=(i,j),\tau>n\right]$$

Hence,

$$f_{\mathcal{G}}(i,j,n)=\frac{\rho(x_0,y_0)^n}{x_0^j y_0^j}\mathbb{P}\left[\sum_{k=1}^{n}(Y_1(k),Y_2(k))=(i,j),\tau>n\right] \tag{4.24}$$

To avoid walks in which a step is influenced by previous steps, only walks with no drift and no correlation will be considered. That is, for each walk Z with no drift the covariance matrix $\mathrm{Cov}(Z)=(\mathbb{E}[Z_iZ_j])_{i,j}=I$, so

$$\mathrm{Cov(Y)}\ =\frac{1}{\mathcal{X}(x_0,y_0)}\begin{pmatrix} x_0^2\frac{\partial^2\mathcal{X}}{\partial x^2}(x_0,y_0) & x_0y_0\frac{\partial^2\mathcal{X}}{\partial x\partial y}(x_0,y_0)\\ x_0y_0\frac{\partial^2\mathcal{X}}{\partial x\partial y}(x_0,y_0) & y_0^2\frac{\partial^2\mathcal{X}}{\partial y^2}(x_0,y_0)\end{pmatrix}$$

Walks with no drift and no correlation are achieved by the defining a new walk W in terms of Y then defining Z in terms of W. Let the coordinates of W be $(W_1,W_2)=\left(\frac{Y_1}{\sqrt{\mathbb{E}[Y_1^2]}},\frac{Y_2}{\sqrt{\mathbb{E}[Y_2^2]}}\right)$, so

$$\mathbb{E}[W_1^2]=\mathbb{E}[W_2^2]=1,\mathbb{E}[W_1W_2]=c$$

Thus, by Cauchy–Schwarz inequality $c\in[-1,1]$. Let $M\in\mathbb{R}^{2x2}$ such that $Z=MW$ and $\mathrm{Cov}(Z)=M\mathrm{Cov}(W)M^T=I$. Then

$$Cov(W)=\begin{pmatrix}1 & c\\ c & 1\end{pmatrix},M=\frac{1}{2\sqrt{1-c^2}}\begin{pmatrix}\sqrt{1+c}+\sqrt{1-c} & \sqrt{1-c}-\sqrt{1+c}\\ \sqrt{1-c}-\sqrt{1+c} & 2\sqrt{1+c}\end{pmatrix}$$

Let $\theta\in[0,2\pi]$ such that c $=\frac{\frac{\partial^2\mathcal{X}}{\partial x\partial y}}{\sqrt{\frac{\partial^2\mathcal{X}}{\partial x^2}\cdot\frac{\partial^2\mathcal{X}}{\partial y^2}}}(x_0,y_0)=\sin(2\theta)$ then

$$M=\frac{1}{2\sqrt{1-c^2}}\begin{pmatrix}\cos(\theta) & -\sin(\theta)\\ -\sin(\theta) & \cos(\theta)\end{pmatrix}$$

With respect to probability, Z relates to Y by

$$
\begin{aligned}
&\mathbb{P}\left[\sum_{k=1}^{n}(Y_1(k), Y_2(k)) = (0,0), \tau > n\right] \\
&= \mathbb{P}\left[\sum_{k=1}^{n}(W_1(k), W_2(k)) = (0,0), \tau > n\right] \\
&= \mathbb{P}\left[\sum_{k=1}^{n}(Z_1(k), Z_2(k)) = (0,0), \tau > n\right]
\end{aligned}
$$

The walk Z remains in the cone M($\mathbb{N}^2$) with an angle opening of arccos(-sin(2θ)) = arcccos(-c).

Given these conditions, Denisov and Wachtel showed, in Theorem 6 of [19], that the exit time of Z behaves the same as Brownian motion in the same cone. Consider the probability $g(x, y, t) = \mathbb{P}_{(x,y)}[\tau \geq t]$ of a Brownian motion starting at (x, y) and remaining inside the cone at time t. Then the following diffusion equation holds for g

$$\left(\frac{\partial}{\partial t} - \frac{1}{2}\Delta\right) g(x, y, t) = 0 \tag{4.25}$$

where Δ is the Laplacian with

$$g(x, y, t) = \begin{cases} 1 & \text{inside cone} \\ 0 & \text{for t} \geq 0 \text{ on its border} \end{cases}$$

Since the focus of g is with respect to a cone, the coordinates can switch to polar (r, θ). Through the use of separation of variables for the differential equatiaon (4.25),

$$g(r, \theta, t) = \sum c_k \sin(\mu_k \theta) A_k \left(\frac{t}{r^2}\right) \tag{4.26}$$

where

$$A_k(s) = (2s)^{-\frac{\mu_k}{2}} {}_1F_1(\mu_k/2, \mu_k + 1, 1/(2s)), \lim_{s \to 0^+} A_k(x) = \frac{2^{\mu_k}}{\sqrt{\pi}} \Gamma\left(\frac{\mu_k + 1}{2}\right),$$

$$\mu_k^2 = \left(\frac{k\pi}{\arccos(-c)}\right)^2 \text{ and } k \in \mathbb{N} \setminus \{0\}.$$

For each k, ${}_1F_1$ is a hypergeometric series that is a solution to the left-hand side of μ_k^2. As $t \to \infty$, the leading term of $g(x,y,t)$ is $t^{\frac{-\mu_1}{2}} = t^{\frac{-\pi}{(2arccos(-c))}}$. Consequently, $\mathbb{P}_{(x,y)}[\tau \geq t] \sim \kappa t^{\frac{-\pi}{(2arccos(-c))}}$ for some constant κ depending on x and y. In relation to Z, as $n \to \infty$

$$\begin{aligned} &f_{\mathcal{G}}(i,j,n) \\ &\qquad = \frac{\rho(x_0,y_0)^n}{x_0^j y_0^j} \mathbb{P}\left[\sum_{k=1}^{n} (Y_1(k), Y_2(k)) = (i,j), \tau > n\right] \\ &\qquad = \frac{\rho(x_0,y_0)^n}{x_0^j y_0^j} \mathbb{P}\left[\sum_{k=1}^{n} (Z_1(k), Z_2(k)) = (0,0), \tau > n\right] \\ &\qquad \sim \frac{1}{x_0^j y_0^j} \cdot \rho^n \cdot \kappa n^{\frac{-\pi}{(2arccos(-c))}} \\ &\qquad \approx K \cdot \rho^n \cdot n^{\alpha}. \end{aligned} \tag{4.27}$$

In the case that the walk is periodic then it will be of period 2 [12, see table 1 and 2]. The walk can be reduced to the case above by changing the step set into $\mathcal{G} + \mathcal{G}$ and n to $\frac{n}{2}$. □

Theorem 6 can now be proven.

Proof of Theorem 6. By Theorem 7 and 8, the goal is to show α is irrational. Consequently, if $\frac{arccos(c)}{\pi}$ is irrational then $F_{\mathcal{G}}(0,0,t)$ is not D-finite. The remainder of the proof will be broken into two components

- The minimal polynomial of c will be determined
- Computations on the polynomial will prove irrationality of $\frac{arccos(c)}{\pi}$

Let $\mu_c(t)$ be the minimal polynomial of c. The algorithm is as follows

1. Compute $\mathcal{X}, \mathcal{X}_x =$ numerator$(\frac{\partial \mathcal{X}}{\partial x})$ and $\mathcal{X}_y =$ numerator$(\frac{\partial \mathcal{X}}{\partial y})$
2. Compute the Gröbner basis of the ideal generated in $\mathbb{Q}[x,y,t,u]$ by $(\mathcal{X}_x, \mathcal{X}_y,$ numerator$(t - \mathcal{X}), 1 - uxy)$
   i Extract the polynomial in the basis that depends only on t.
   ii Factor it and identify its factor μ_ρ that annihilates ρ
3. Compute $P(x,y,t) = \text{numerator}\left(t^2 - \frac{\left(\frac{\partial^2 \mathcal{X}}{\partial x \partial y}\right)^2}{\frac{\partial^2 \mathcal{X}}{\partial x^2} \cdot \frac{\partial^2 \mathcal{X}}{\partial y^2}}\right)$
4. Compute the Gröbner basis of the ideal in $\mathbb{Q}[x,y,t,u]$ by $(\mathcal{X}_x, \mathcal{X}_y, P, 1 - uxy)$

i Extract the polynomial in the basis that depends only on t.

ii Factor it and identify its factor μ_c that annihilates c

Suppose $\frac{arccos(c)}{\pi}$ is rational then c will be of the form $\frac{z+\frac{1}{z}}{2} = \frac{z^2+1}{2z}$ with z a root of unity. Consequently, the numerator of $\mu_c(\frac{x^2+1}{2x})$ will contain a root of unity as a root. Hence, $R(x) = x^{deg \mu_c}\mu_c(\frac{x^2+1}{2x})$ would be divisible by a cyclotomic polynomial Φ_N. However, by Table 2 of Appendix 4 in [12] $R(x)$ is irreducible with degree 2deg(μ_c) where deg(μ_c) ≤ 14. Hence degree of $R(x)$ is at most 28. Since Φ_N divides $R(x)$ then deg(Φ_N) ≤ 30. Hence, N is at most 150 and the coefficients of Φ_N are in $\{0, \pm 1, \pm 2\}$. By Table 2 of Appendix 4, for each $\mathcal{G}$ of the 51 cases, $R(x)$ has at least one coefficient of absolute value at least 3. Consequently, $R(x)$ is not divisible by a cyclotomic polynomial, contradiction. □

Example 5. *The algorithm above can be illustrated for the step set* $\mathcal{G} = \{(-1,0),(0,1),(1,0),(1,-1),(0,-1)\}$ *using Maple.*

Step 1:

```
S:=[[-1,0],[0,1],[1,0],[1,-1],[0,-1]]:
chi:=add(x^s[1]*y^s[2], s=S);
chi_x:=numer(diff(chi,x)); chi_y:=numer(diff(chi,y));
```

Step 2:

```
G:=Groebner[Basis]([chi_x, chi\_y, numer(t-chi),1-u*x*y],
lexdeg([x,y,u],[t])):$;
p:=factor(op(remove(has,G,{x,y,u})));
fsolve(p,t,0..infinity);
```

The second command outputs $p := (t+1)(t^3+t^2-18t-43)$ *and since* $\rho > 0$ *then* $\mu_p = t^3+t^2-18t-43$. *This polynomial can be found in the* 6^{th} *row and* 3^{rd} *column of Table 2 of Appendix 4 in [12]. The third command in this step determines the value of* ρ, *namely 4.729031538.*

Step 3:

```
G:=Groebner[Basis]([numer(t^2-diff(chi,x,y)^2/
diff(chi,x,x)/diff(chi,y,y)),  chi_x, chi_y,1-x*y*u],
lexdeg([x,y,u],[t]));
p:=factor(op(remove(has,G,{x,y,u})));
mu_c:=8*t^3+8*t^2+6*t+1:
```

```
evalf(-1-Pi/arccos(-fsolve(mu_c, t)));
```

Similar to the step above, the first command outputs multiple polynomial one of which is free of x and y. This polynomial is determined by the second command, namely $p := (4t^2 + 1)(8t^3 + 8t^2 + 6t + 1)(8t^3 - 8t^2 + 6t - 1)$. Since $c < 0$ then μ_c is set to the second factor of p, in the third command. The last command determines the value of c to be -3.320191962.

Step 4:

```
R:=numer(subs(t=(x^2+1)/x/2, mu_c));
irreduc(R), numtheory[iscyclotomic](R, x);
```

It can then be checked that $R(x)$ is irreducible and not cyclytomic which confirms that the polynomial does not have root of unity as a root.

Based on the different results that are used to prove Theorem 6, concluding remarks can be made about them. For instance, are there more applications for Theorem 7? Bostan, Raschel and Salvy believe this to be the case and wish for more recognition for this theorem. I believe there should be more recognition since a similar statement holds for holonomy and experiments by the authors of [12] have indicated that Theorem 8 allows for efficient determination on the holonomy of excursions.

The proof of Theorem 6 is based on Theorem 7. Theorem 7 is efficiently used in the proof of Theorem 6 due to Theorem 8. As seen at the beginning of the proof of Theorem 8, the proof is dependent on the walk remaining in the quarter plane. Hence, the core of the main argument in this section lies on the walk remaining in the quarter plane. As seen in Section 3, meanders also remain in the quarter plane. Thus, is it reasonable to believe that that argument needed to check D-finite property for meanders may be similar to that of excursions?

Chapter 5

Self Avoiding Walks

A method has not been determined to find the total number of n-step self-avoiding walks on a given lattice. In fact, generating functions of self-avoiding walks cannot currently be determined for all models and the closest research can be on the bounds and limits of the coefficient. The next two sections look at classical results of self-avoiding walks with respect to these studies.

5.1 Connective Constant of Honeycomb Lattice

The honeycomb lattice $\mathbb{H}$ is one of the most commonly used lattice for self-avoiding walks. The only non-trivial known value of the connective constant is of the honeycomb lattice. Theorem 9 will prove this value to be $\sqrt{2+\sqrt{2}}$. This constant was derived by B. Nienhuis in 1982 [41, 42]. The proof addressed in this section was constructed by Hugo Duminil-Copin and Stanislav Smirnov [20].

Since $\log(c_{n+m}) \leq \log(c_n c_m) = \log(c_n) + \log(c_m)$, by Lemma 1 the sequence $\left(\frac{\log(c_n)}{n}\right)_{n\in\mathbb{N}}$ converges. That is,

$$\lim_{n\to\infty} \frac{\log(c_n)}{n} = \lim_{n\to\infty} \log(c_n^{\frac{1}{n}}) = \log\left(\lim_{n\to\infty} c_n^{\frac{1}{n}}\right) < \infty.$$

Hence, the connective constant μ exists. There will be some terminology and

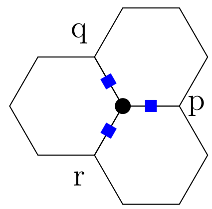


Figure 5.1: Adjacent Mid-Edges

results developed beforehand to help with the proof. For the remainder of this section let $x_c = \frac{1}{\sqrt{2+\sqrt{2}}}$ and j $= e^{\frac{i2\pi}{3}}$.

Theorem 9. *The connective constant of the honeycomb lattice is* $\mu = \sqrt{2+\sqrt{2}}$.

The centers of edges in $\mathbb{H}$ are referred to as mid-edges. The mid-edges will help in the proof of lemmas that are essential in the proof of Theorem 9. For the remainder of section let the set of mid-edges be denoted as H. See figure 5.1 for more clarification.

Definition 23. *Partition Function*
For a walk γ: $a \to E \subset H$, where a is a vertex on $\mathbb{H}$, let the length $\boldsymbol{\ell}(\gamma)$ be the number of vertices visited by γ. The partition function is

$$Z(x) = \sum_{\gamma: a \to H} x^{\boldsymbol{\ell}(\gamma)} \in (0, +\infty]. \tag{5.1}$$

Definition 24. *[20] Hexagonal Lattice Domain*
A hexagonal lattice domain $\Omega \subset H$ is a union of all mid-edges emanating from a given collection of vertices $V(\omega)$ such that a mid-edge z belongs to Ω if at least one end-point of its associated edge is in Ω, it belongs to $\partial\Omega$ if only one of them is in Ω (see Fig. 5.2).

Definition 25. *Winding*
Denoted $W_\gamma(a,b)$, the winding number is the total rotation of the direction (in radians) when γ is traversed from a to b such that counterclockwise is the positive direction of winding (see Fig. 5.3).

Definition 26. *Parafermionic observable*
The parafermionic observable for $a \in \partial\Omega$, $z \in \Omega$, is defined by

$$F(z) = F(a, z, x, \sigma) = \sum_{\gamma \subset \Omega: a \to z} e^{-i\sigma W_\gamma(a,z)} x^{l(\gamma)}. \tag{5.2}$$

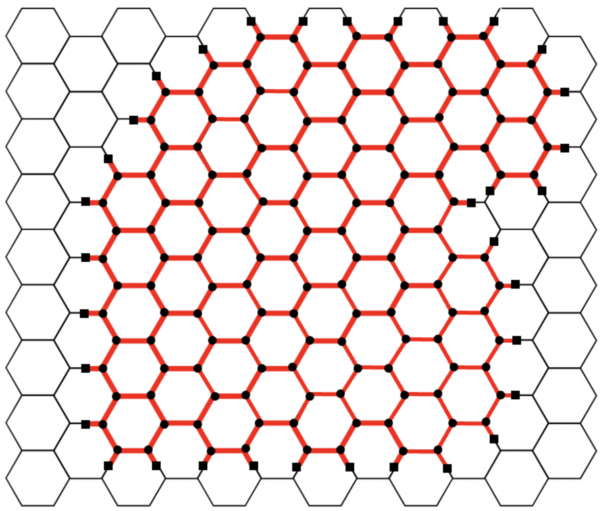


Figure 5.2: Domain Ω with square vertices as boundary mid-edges and circular vertices as the set $\mathrm{V}(\Omega)$

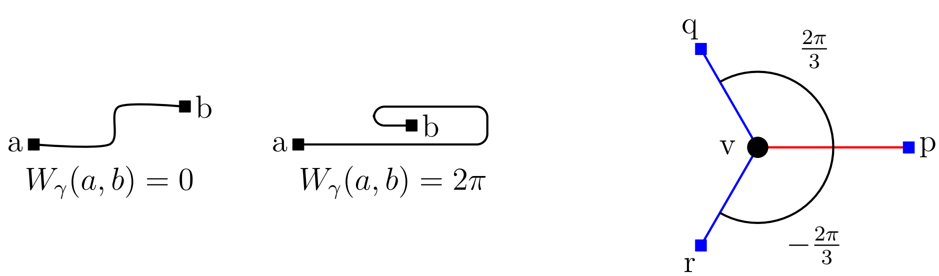


Figure 5.3: **Left:** Winding Number, **Right:** Edge Rotation

Lemma 7. *If $x = x_c$ and $\sigma = \frac{5}{8}$, then F satisfies the following relation for every vertex $v \in V(\Omega)$*

$$(p\text{-}v)F(p) + (q\text{-}v)F(q) + (r\text{-}v)F(r) = 0, \tag{5.3}$$

where p,q,r are the mid-edges of the three edges adjacent to v.

Proof. The left hand side can be written as the sum of contributions $c(\gamma)$ for all possible walks finishing at p, q or r. For instance, the contribution for a walk that ends at p will be

$$c(\gamma) = (p - v)e^{-i\sigma W_\gamma(a,p)} x_c^{l(\gamma)}. \tag{5.4}$$

The goal is to show that for each walk γ there are associated walk(s) such that the sum of the contribution of all associated walk(s) and $c(\gamma)$ is zero. Hence, the left hand side is zero by term by term cancellation. Such a walk γ can be divided into two types of walks.

1. If γ_1 visits all three mid-edges then γ_1 can be broken down to a self-avoiding walk and a loop from v to v. The self-avoiding walk visits two of the mid-edges and the loop visits the last mid-edge. If the loop from v to v is traversed in the opposite direction then another walk is taken into consideration, namely γ_2. See the left hand side of figure 5.4.
2. If γ_1 visits one or two mid-edges. If γ_1 visits one mid-edge then it can be extended to one of the other two mid-edges resulting in two additional walks γ_2 and γ_3. Similarly, a walk visiting two mid-edges can be associated with a walk visiting one mid-edge and extending it. So if one or two mid-edges are visited they can be grouped into triplets where γ_1 visits one mid-edge, γ_2 is broken into γ_1 and visit to a second mid-edge and γ_3 is broken down similarly. See the right hand side of figure 5.4

For (1) there is one associated walk, namely γ_2 on the left hand side of figure 5.4. For (2) there are two associated walks. Hence the goal is to show $c(\gamma_1) + c(\gamma_2) = 0$ and $c(\gamma_1) + c(\gamma_2) + c(\gamma_3) = 0$, respectively.

Without loss of generality let γ_1 and γ_2 end at q, r, respectively where γ_1 and γ_2 are the walks on the left hand side of figure 5.4. Since γ_1 and γ_2 coincide up to the point p then

$$W_{\gamma_1}(a,q) = W_{\gamma_1}(a,p) + W_{\gamma_1}(p,q) = W_{\gamma_1}(a,p) - \frac{4\pi}{3}$$

$$W_{\gamma_2}(a,r) = W_{\gamma_2}(a,p) + W_{\gamma_2}(p,r) = W_{\gamma_1}(a,p) + \frac{4\pi}{3},$$

where a is the starting point of γ_1 and γ_2. In addition, $l(\gamma_1) = l(\gamma_2)$. So,

$$\begin{aligned}
c(\gamma_1) + c(\gamma_2) &= (q-v)x_c^{l(\gamma_1)}e^{-W_{\gamma_1}(a,p)i\sigma + i\sigma\frac{4\pi}{3}} + (r-v)x_c^{l(\gamma_2)}e^{-W_{\gamma_1}(a,p)i\sigma - i\sigma\frac{4\pi}{3}} \\
&= (p-v)x_c^{l(\gamma_1)}e^{-W_{\gamma_1}(a,p)i\sigma}\left((q-v)e^{i\frac{5}{8}\frac{4\pi}{3}} + (r-v)e^{-i\frac{5}{8}\frac{4\pi}{3}}\right) \\
&= (p-v)x_c^{l(\gamma_1)}e^{-W_{\gamma_1}(a,p)i\sigma}\left((q-v)e^{\frac{5\pi i}{6}} + (r-v)e^{\frac{-5\pi i}{6}}\right) \\
&= (p-v)x_c^{l(\gamma_1)}e^{-W_{\gamma_1}(a,p)i\sigma}\Big(((p-v)e^{\frac{2\pi i}{3}})e^{\frac{5\pi i}{6}} \\
&\qquad + ((p-v)e^{\frac{-2\pi i}{3}})e^{-\frac{5\pi i}{6}}\Big) \text{ as in figure 5.3} \\
&= (p-v)x_c^{l(\gamma_1)}e^{-W_{\gamma_1}(a,p)i\sigma}\left(e^{\frac{2\pi i}{3}+\frac{5\pi i}{6}} + e^{\frac{-5\pi i}{6}+\frac{-2\pi i}{3}}\right) \\
&= (p-v)x_c^{l(\gamma_1)}e^{-W_{\gamma_1}(a,p)i\sigma}\left(e^{\frac{3\pi i}{2}} + e^{\frac{-3\pi i}{2}}\right) \\
&= 0
\end{aligned}$$

Without loss of generality let γ_1, γ_2 and γ_3 end at p, q and r, respectively where γ_1, γ_2 and γ_3 are the walks on the right hand side of figure 5.4. Since γ_1, γ_2 and γ_3 coincide up to the point p then

$$W_{\gamma_2}(a,r) = W_{\gamma_2}(a,p) + W_{\gamma_2}(p,q) = W_{\gamma_1}(a,p) - \frac{\pi}{3}$$

$$W_{\gamma_3}(a,r) = W_{\gamma_3}(a,p) + W_{\gamma_3}(p,r) = W_{\gamma_1}(a,p) + \frac{\pi}{3},$$

where a is the starting point of γ_1, γ_2 and γ_3. In addition, $l(\gamma_1) + 1 = l(\gamma_2)$. By similar calculations,

$$\begin{aligned}
c(\gamma_1) + c(\gamma_2) + c(\gamma_3) &= (p-v)x_c^{l(\gamma_1)}e^{-W_{\gamma_1}(a,p)i\sigma}\left(1 + x_c e^{\frac{7\pi i}{8}} + x_c e^{\frac{-7\pi i}{8}}\right) \\
&= (p-v)x_c^{l(\gamma_1)}e^{-W_{\gamma_1}(a,p)i\sigma}\left(1 + x_c\left(-2\cos\left(\frac{\pi}{8}\right)\right)\right) \\
&= (p-v)x_c^{l(\gamma_1)}e^{-W_{\gamma_1}(a,p)i\sigma}(1-1) \\
&= 0
\end{aligned}$$

□

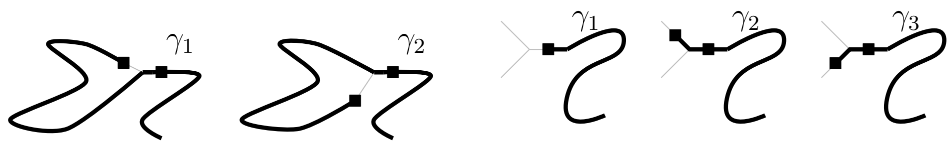


Figure 5.4: Left: Visit all three mid-edges. Right: Visit one or two mid-edges.

$$\begin{aligned}F(t,x,y) = 1 + \Sigma_{n\geq 0}\left(\Sigma_{i,j\geq 1} f(n,i+1,j)x^i y^j\right)t^{n+1} + \\ \Sigma_{n\geq 0}\left(\Sigma_{i,j\geq 1} f(n,i,j+1)x^i y^j\right)t^{n+1} + \\ \Sigma_{n\geq 0}\left(\Sigma_{i,j\geq 1} f(n,i-1,j-1)x^i y^j\right)t^{n+1},\end{aligned}$$

To reduce the complexity of the problem in Theorem 9, the hexagonal lattice is restricted to a finite size. Consider a strip of hexagons as hexagons aligned vertically. Let S_T be T strips of hexagons aligned horizontally. There is a left and right bound on the domain S_T, however, there is no upper or lower bound. This can be introduced in a domain $S_{T,L}$ in which the T strips are cut at height ±L at angle $\pm\frac{\pi}{3}$. The height L is determined in reference to a mid-edge a being 0. The domain $S_{T,L}$ can be seen as the hexagonal lattice, with the diameter of each hexagon being 1, in $\mathbb{C}$ (see figure 5.5). Consequently, the vertices of S_T and $S_{T,L}$ are

$$V(S_T) = \{z \in V(\mathbb{H}) : 0 \leq \text{Re(z)} \leq \frac{3(T+1)}{2}\} \tag{5.5}$$

$$V(S_{T,L}) = \{z \in V(S_T) : |\text{2Im(z) - Re(z)}| \leq \sqrt{3}(2L+1)\} \tag{5.6}$$

Based on the domain $S_{T,L}$, consider the following positive partition functions for the sets of vertices α, β, ϵ and $\bar{\epsilon}$

$$A_{T,L}(x) = \sum_{\gamma \subset S_{T,L}: a \to \alpha \setminus \{a\}} x^{l(\gamma)} \tag{5.7}$$

$$B_{T,L}(x) = \sum_{\gamma \subset S_{T,L}: a \to \beta} x^{l(\gamma)} \tag{5.8}$$

$$E_{T,L}(x) = \sum_{\gamma \subset S_{T,L}: a \to \epsilon \cup \bar{\epsilon}} x^{l(\gamma)} \tag{5.9}$$

The benefit of introducing S_T and $S_{T,L}$ is the ability to use 5.3 with positive weights rather than complex weights.

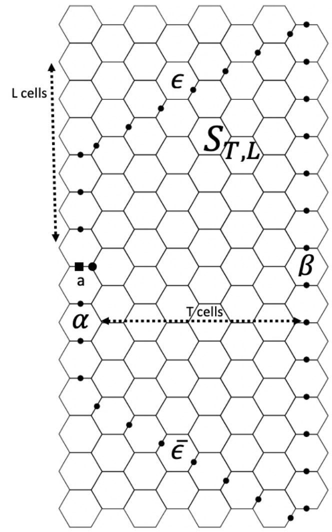


Figure 5.5: $S_{T,L}, \alpha, \beta, \epsilon$ and $\bar{\epsilon}$ are the domain, left/right intervals and upper/lower intervals, respectively

Lemma 8. *$1 = c_\alpha A_{T,L}(x_c) + B_{T,L}(x_c) + c_\epsilon E_{T,L}(x_c)$.*

Proof. If the sum over all vertices in $\mathrm{V}(S_{T,L})$ is taken for equation 5.3 then the result is the following

$$-\sum_{z\in\alpha} F(z) + \sum_{z\in\beta} F(z) + j\sum_{z\in\epsilon} F(z) + \bar{j}\sum_{z\in\bar{\epsilon}} F(z) = 0$$

The sum over the interior vertices vanishes because both ends of the mid-edge are in $\mathrm{V}(S_{T,L})$. Let $p \in H$ such that the vertices v_1, v_2 adjacent to p are in $\mathrm{V}(S_{T,L})$ then $(p - v_1)F(p) + (p - v_2)F(p) = 0$ since $(p - v_1) = -(p - v_2)$. Consider the following observations about the self-avoiding walks starting at a,

1. The winding number from a to the top and bottom part of α is π and $-\pi$, respectively.
2. Due to the symmetry of the domain S_T with respect to the real axis and only self-avoiding walks from a to a having length 0, $F(\bar{z}) = \overline{F(z)}$
3. The winding number from a to any half-edge in β, ϵ and $\bar{\epsilon}$ is 0, $\frac{2\pi}{3}$ and $\frac{-2\pi}{3}$, respectively.

So,

$$\begin{aligned}\sum_{z\,\in\alpha} F(z) &= F(a) + \sum_{z\in\alpha\setminus\{a\}} F(z)\\ &= 1 + \frac{1}{2}\sum_{z\in\alpha\setminus\{a\}} (F(z)+F(\overline{z}))\text{ (by (2))}\\ &= 1 + \frac{e^{-i\sigma\pi}+e^{i\sigma\pi}}{2}A_{T,L}(x)\text{ (by (1))}\\ &= 1+\cos(\sigma\pi)A_{T,L}(x)\\ &= 1-\cos(\frac{3\pi}{8})A_{T,L}(x)\text{ (since }\sigma=\frac{5}{8})\\ &= 1-c_\alpha A_{T,L}(x),\end{aligned} \tag{5.10}$$

while

$$\begin{aligned}\sum_{z\,\in\beta} F(z) &= \frac{1}{2}\sum_{z\in\beta}(F(z)+F(\overline{z}))\text{ (by (2))}\\ &= \frac{e^0+e^0}{2}B_{T,L}(x)\text{ (by (3))}\\ &= B_{T,L}(x),\end{aligned} \tag{5.11}$$

and

$$\begin{aligned}j\sum_{z\,\in\epsilon} F(z)+\overline{j}\sum_{z\,\in\overline{\epsilon}} F(z) &= je^{-\frac{2\pi\sigma i}{3}}\frac{E_{T,L}}{2}+\overline{j}e^{\frac{2\pi\sigma i}{3}}\frac{E_{T,L}}{2}\text{ (by (3) and 5.9)}\\ &= \frac{e^{\frac{2\pi i(1-\sigma)}{3}}+e^{-\frac{2\pi i(1-\sigma)}{3}}}{2}E_{T,L}\\ &= \cos\left(\frac{2\pi(1-\sigma)}{3}\right)E_{T,L}\\ &= \cos(\frac{\pi}{4})E_{T,L}\text{ (since }\sigma=\frac{5}{8})\\ &= c_\epsilon E_{T,L}.\end{aligned} \tag{5.12}$$

The lemma follows by summing 5.10, 5.11 and 5.12. □

Remark 1. *Lemma 8 can also be extended to the domain S_T. By equation (5.7) and (5.8), $(A_{T,L})_{L>0}, (B_{T,L})_{L>0}$ are increasing sequences with respect to their coefficients. By Lemma 8, the sequences are also bounded for $x \le x_c$.*

By the monotone convergence theorem, the limit of both sequences converge to their supremum. That is,

$$A_T(x) = \lim_{L\to\infty} A_{T,L}(x) \tag{5.13}$$

$$B_T(x) = \lim_{L\to\infty} B_{T,L}(x). \tag{5.14}$$

Since the two sequences are increasing and c_α, c_ϵ are positive then, by Lemma 8 and equation 5.9, $(E_{T,L}(x_c))_{L>0}$ decreases and is bounded below by 0. By monotone convergence theorem, the limit of the sequence converges to its supremum. That is,

$$E_T(x_c) = \lim_{L\to\infty} E_{T,L}(x_c). \tag{5.15}$$

Taking the limit of the equation in Lemma 8 results in

$$1 = c_\alpha A_T(x_c) + T(x_c) + c_\epsilon E_T(x_c). \tag{5.16}$$

Definition 27. *[20] Bridge of Width T*
A bridge of width T is a self-avoiding walk in S_T from one side to the opposite side, defined up to vertical translation.

Lemma 9. *Any self-avoiding walks on $\mathbb{H}$ can be uniquely decomposed into a sequence of bridges of widths $T_{-i} < ... < T_{-1}$ and $T_0 > ... > T_j$ such that the starting mid-edge and first vertex visited of the walk are fixed.*

The existence of a bridge decomposition for a self-avoiding walk γ can be proven by induction on the width T_0. The right side of figure 5.6 gives a bridge decomposition for $\overline{\gamma}$. To satisfy the definition of a bridge the only additional parts to $\overline{\gamma}$ are the mid-edges of the horizontal edges not in $\overline{\gamma}$. By using the fact that the starting mid-edge and first vertex are fixed, the uniqueness is proven by showing that the reverse procedure of the bridge decomposition leads to only one walk, namely $\overline{\gamma}$. In other words, there is a 1-1 correspondence.

Proof of Theorem 9 The idea of the proof is to show that $\mu \geq \sqrt{2+\sqrt{2}}$ and $\mu \leq \sqrt{2+\sqrt{2}}$.

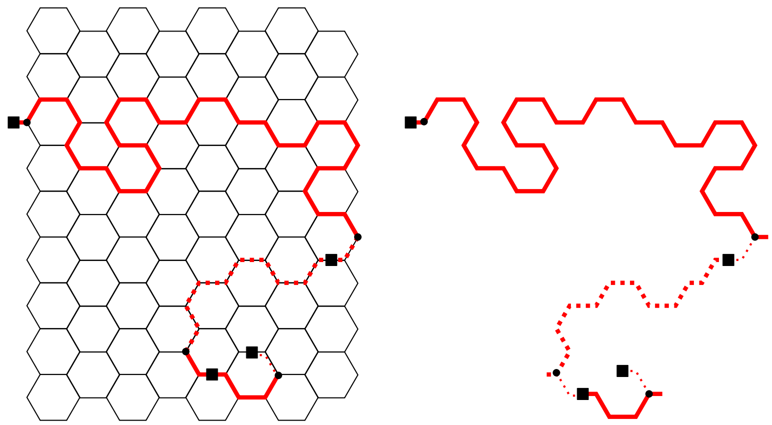


Figure 5.6: Let the walk on the left be denoted $\overline{\gamma}$

If $E_T > 0$ for some T then

$$\begin{aligned} Z(x_c) &\geq \sum_{L>0} E_{T,L}(x_c) \text{ by 5.9} \\ &\geq \sum_{L>0} E_T(x_c) \text{ (by Remark 1)} \\ &= E_T(x_c) \sum_{L>0} 1 \\ &= \infty \text{ (by p-series test)} \end{aligned}$$

If $E_T = 0$ for all T then, by Remark 1, $1 = c_\alpha A_T(x_c) + B_T(x_c)$. A walk γ considered in $A_{T+1}(x_c)$ but not $A_T(x_c)$ must visit a vertex v that is adjacent to an edge e on the right boundary of $S_{T,L}$. The walk can be cut at the first such v into γ_1 and γ_2 with a half-edge added to both walks. This results in a unique decomposition of γ into 2 bridges of width $T+1$ with one step more than γ. As a result,

$$A_{T+1}(x_c) - A_T(x_c) \leq x_c(B_{T+1}(x_c))^2 \text{ (due to extra step and 5.8)}$$

and

$$\begin{aligned} 0 = 1 - 1 &= c_\alpha(A_{T+1}(x_c) - A_T(x_c)) + (B_{T+1}(x_c) - B_T(x_c)) \\ &\leq c_\alpha x_c(B_{T+1}(x_c))^2 + (B_{T+1}(x_c) - B_T(x_c)) \end{aligned}$$

while

$$c_\alpha x_c(B_{T+1}(x_c))^2 + B_{T+1}(x_c) \geq B_T(x_c) \tag{5.17}$$

Consider the following inequality,

$$B_T(x_c) \geq \frac{\min\{B_1(x_c), \frac{1}{c_\alpha x_c}\}}{T}. \tag{5.18}$$

For $T = 1$, the inequality 5.18 holds. Suppose it holds for $T \geq 1$. Then by definition of inductive step, the goal is to show 5.18 holds for $T+1$. By equation 5.17 and the inductive hypothesis

$$c_\alpha x_c (B_{T+1}(x_c))^2 + B_{T+1}(x_c) \geq \frac{\min\{B_1(x_c), \frac{1}{c_\alpha x_c}\}}{T}. \tag{5.19}$$

Let $\delta = c_\alpha$ and $\Delta = \min\{B_1(x_c), \frac{1}{c_\alpha x_c}\}$ then

$$\frac{1}{T} \geq \frac{\delta\Delta}{T+1} \text{ (since } \delta\Delta = \min\{\delta B_1, 1\} \leq 1)$$

$$\frac{1}{T} - \frac{1}{T+1} \geq \frac{\delta\Delta}{(T+1)^2}$$

$$1 + 4\frac{\delta\Delta}{T} \geq 1 + 4\left(\frac{\delta\Delta}{T+1}\right) + 4\left(\frac{\delta\Delta}{T+1}\right)^2 \text{ (by adding 1 and multiplying by } 4\delta\Delta \text{ to both sides)}$$

$$\begin{aligned} B_{T+1} &= \frac{-1 + \sqrt{1 + 4\frac{\delta\Delta}{T}}}{2\delta} \\ &\geq \frac{\delta}{T+1} \text{ (by 5.19 and positivity of } B_{T+1}) \\ &= \frac{\min\{B_1(x_c), \frac{1}{c_\alpha x_c}\}}{T+1}. \end{aligned}$$

By induction, equation 5.18 holds for all $T \geq 1$. As a result,

$$\begin{aligned} Z(x_c) &\geq \sum_{T>0} B_T(x_c) \text{ by 5.8} \\ &\geq \sum_{T>0} B_T(x_c) \text{ (by Remark 1)} \\ &\geq \min\{B_1(x_c), \frac{1}{c_\alpha x_c}\} \sum_{T>0} \frac{1}{T} \text{ (by 5.18)} \\ &= \infty \text{ (by } p\text{-series test)}. \end{aligned}$$

Since $Z(x_c) = \infty$ then, by Lemma 2, $x_c > \mu^{-1}$. Consequently, $\mu \leq x_c^{-1} = \sqrt{2+\sqrt{2}}$.

Since bridge of width T has length at least T then $B_T(x) = \sum_{n \geq T} b_n x^n$. Suppose $x < x_c$ then

$$
\begin{aligned}
\sum_{T>0} B_T(x) &= \sum_{T>0} \sum_{n \geq T} b_n x^n \\
&= \sum_{T>0} x^T \sum_{n \geq T} b_n x^{n-T} \\
&\leq \sum_{T>0} x^T \sum_{n \geq T} b_n x_c^{n-T} \\
&= \sum_{T>0} \frac{x^T}{x_c^T} \sum_{n \geq T} b_n x_c^n \qquad (5.20) \\
&= \sum_{T>0} \left(\frac{x}{x_c}\right)^T B_T(x_c) \\
&\leq \sum_{T>0} \left(\frac{x}{x_c}\right)^T \text{ (by positivity of } \alpha, \epsilon \text{ and eq. 5.16)} \\
&< \infty \text{ (since } x < x_c) .
\end{aligned}
$$

For walks starting at a with the starting mid-edge fixed there are 2 possibilities for the first vertex. So,

$$
\begin{aligned}
Z(x) &\leq 2 \sum_{T_{-i} < \cdots < T_{-1} \text{ and } T_j < \cdots < T_0} \left(\prod_{k=-1}^{j} B_{T_k}(x) \right) \text{ (by Lemma 9)} \\
&= 2 \prod_{T>0} (1 + B_T(x))^2 \\
&< \infty \text{ (by eq. 5.20)} .
\end{aligned}
$$

Since Z(x) $< \infty$ for x $< x_c$ then, by Lemma 2, $x_c < \mu^{-1}$. Consequently, $\mu \geq x_c^{-1} = \sqrt{2 + \sqrt{2}}$. □

Remark 2. *For the other two lattices from Section 3.1, the connective constant has only been estimated. The values are provided in table 2.*

Type of Lattice	***Estimated Connected Constant***
Triangular lattice	*4.15079[30]*
Square Lattice	*2.63815853032790[29]*

The preciseness of the connective constant for the square lattice has been an ongoing process for decades. The contribution has been made by many mathematicians such as Guttmann and Conway in 2001, Clisby and Jensen in 2012. The latest contribution has been made by J. L. Jacobsen, C. R. Scullard, A. J. Guttmann in 2018 using the Topological Transfer Matrix (TTM) method.

5.2 Kesten's Bound on c_n

In this section we will prove Kesten's bound which was derived for the lattice $\mathbb{Z}^d$. The proof in this section uses the ideas by Gordon Slade and Neal Madras from the book The Self-Avoiding Walk. Their proof builds on the proof of the weaker bound, namely Corollary 2, which was provided in Kesten's 1964 paper.

Theorem 10. *For $d \geq 2$ there exists Q depending only on d, such that for every $n \geq 2$*

$$c_n \leq \mu^n exp[Qn^{\frac{2}{d+2}} \log n],$$

Let an n-step self-avoiding walk starting at x be denoted $\omega = (\omega(0), ..., \omega(n)) \in (\mathbb{Z}^d)^{n+1}$. The length of ω be $|\omega|$ and i^{th} component of $\omega(j)$ be $\omega_i(j)$ for i$\in [d]$ and j$\in \{0, \cdots, n\}$. Since $\mathbb{Z}^d$ is in higher dimension than the hexagonal lattice then bridges need to be re-defined. Consider ω to be a bridge if

$$\omega_1(0) < \omega_1(i) \leq \omega_1(n) \text{ for all } i \in [n].$$

Definition 28. *Half-space Walk of n-steps*
An n-step half-space walk is an n-step self-avoiding walk ω with the following condition

$$\omega_1(0) < \omega_1(i) \textit{ for all } i \in [n].$$

Definition 29. *Bridge of n-steps*
An n-step bridge is an n-step self-avoiding walk ω with the following condition

$$\omega_1(0) < \omega_1(i) \leq \omega_1(n) \textit{ for all } i \in [n].$$

Note, these half space walks have a subtle difference from the half space walks discussed in earlier sections. Namely, that these are strict half space walks. For self-avoiding walks on $\mathbb{Z}^d$, let c_n be the number of n-step self-avoiding walks starting at the origin (ie $\omega(0) = 0^d$). Similarly, define h_n and b_n for half-space walks and bridges, respectively. Consequently, $c_0 = h_0 = b_0 = 1$.

Remark 3. *Equivalency to Kesten's bound*
Note that every n-step self-avoiding walk can be decomposed into two half-space walks. For $\omega = (\omega(0), ..., \omega(n))$ *an n-step self-avoiding walk starting at the origin, let m be the last step with the first component being* $min_{1\leq j\leq n}\omega_1(j)$*. In other words,*

$$m = max\{i|\omega_1(i) = min_{1\leq j\leq n}\omega_1(j)\}.$$

Now ω *can be broken into half-space walks* $\omega^{'} = (\omega(m), ..., \omega(n))$ *and* $\omega^{''} = (\omega(m) - e_1, \omega(m), ..., \omega(0))$*. Based on the definition of* $\omega^{'}$*, it can be translated to the origin and become part of (n-m)-step half-space walks. Similarly,* $\omega^{''}$ *is an (m+1)-step half-space walk. Since* ω *is arbitrary then m∈{0,...,n}. So,*

$$c_n \leq \sum_{m=0}^{n} h_{n-m}h_{m+1}. \tag{5.21}$$

By 5.21 and the inequality $x^a + y^a \leq 2^{1-a}(x + y)^a$ *for* $a \in (0, 1), x, y \geq 0$*, it is sufficient to prove the bound for half-space walks. That is,*

$$h_n \leq \mu^n exp[Qn^{\frac{2}{d+2}} \log n].$$

Definition 30. *Span of Self-avoiding Walks of n-steps*
The span of an n-step self-avoiding walk ω*, denoted span(*ω*), is*

$$\max_{1\leq j\leq n} \omega_1(j) - \min_{1\leq j\leq n} \omega_1(j).$$

The number of n-step half-space walks and bridges starting at the origin and having span A are denoted $h_{n,A}$ *and* $b_{n,A}$*, respectively.*

Definition 31. *Let n,s be integers then the total number of half-space walks starting at the origin with span at most s is*

$$h^*_{n,s} = \sum_{i=0}^{s} h_{n,i}.$$

Remark 4. *Now the goal is to try and find a bound for* h_n*. Let* ω *be an n-step half-space walk starting at the origin, K(*ω*) = span(*ω*) and I(*ω*) =* $max\{i|\omega_1(i) = K(\omega))\}$*. Note that* ω *can be decomposed into a bridge* $\omega^{'}$ *and a half-space walk* $\omega^{''}$ *with I(*ω*) and n-I(*ω*) steps, respectively, such that span(*$\omega^{'}$*) = K(*ω*), span(*$\omega^{''}$*)* $\leq$ *K(*ω*). The walk* ω' *consists of the first I(*ω*) steps of* ω*. It is a bridge because it is a half-space walk starting at the origin. Hence,* $\omega'_1(0) < \omega'_1(i) < \omega'_1(I(\omega))\forall n \in [I(\omega)]$*. Since* $(I(\omega), K(\omega)) \in [n] \times [n]$ *and there are* n^2 *possibilities for the tuple then there exist* $i[0], k[0] \in [n] \times [n]$ *such that*

$$h_n \leq n^2 b_{i[0],k[0]} h^*_{n-i[0],k[0]}. \tag{5.22}$$

The following lemma will play a key role in the proof of Kesten's bound.

Lemma 10. *Let k, l, and m be strictly positive integers, $B \in (0,1)$ and $V = (m^{1-B}l)^{\frac{1}{d}}$ then there exists a constant D, depending only on the dimension d, such that*

$$b_{l,k}h^*_{m,k} \leq \mu^{m+l+dV}[D(m+l)]^{12m^B+3dV}.$$

Proof. Let β be an l-step bridge with span k, η be an m-step half-space walk with span at most k and

$$\mathcal{Y} = \{y \in \mathbb{Z}^d : y_1 \geq \eta_1(m) \text{ and } ||y - \eta(m)||_\infty \leq V\}.$$

By definition $\mathcal{Y}$ is a half-cube, hence it contains at least $(\lfloor V \rfloor + 1)(2\lfloor V \rfloor + 1)^{d-1}$ points. So, $|\mathcal{Y}| > (\lfloor V \rfloor + 1)(2\lfloor V \rfloor + 1)^{d-1} > V^d = m^{1-B}l$

For each $y \in \mathbb{Z}^d$, let $J(y)$ be the number of pairs $(i,j) \in \{0, \cdots, m\}X\{0, \cdots, l\}$ such that $\eta(i) - \beta(j) = y$. So the average value of $J(y)$ over $\mathcal{Y}$ is

$$\frac{\sum_{y \in \mathbb{Z}^d} J(y)}{|\mathcal{Y}|} = \frac{(m+1)(l+1)}{|\mathcal{Y}|} \leq \frac{4ml}{m^{1-B}l} = 4m^B.$$

There exist $Y \in \mathcal{Y}$ such that

$$J(Y) \leq 4m^B.$$

Let g $= ||Y - \eta(m)||_1$ then by definition of $\mathcal{Y}$

$$0 \leq g \leq dV.$$

Let ρ be a walk which consists of η followed by a walk of minimal length from η(m) to Y, followed by β. As a result, ρ is a $(m+g+l)$-walk. Consider the following inequality

$$0 < \rho_1(i) \leq Y_1 + k = Y_1 + \beta_1(l) = \rho_1(m+g+l).$$

The inequality can be proven by the following sub-cases

- For $i \in \{0, \cdots, m\}$, $\rho_1(i) = \eta_1(i) \in (0, k]$
- For $i \in \{m, \cdots, m+g\}$, $\eta_1(m) \leq \rho_1(i) \leq Y_1$
- For $i \in \{m+g, \cdots, m+g+l\}$, $\rho_1(i) = (Y_1 + \beta_1(i-m-g)) \in (Y_1, Y_1+k]$

Note that ρ is not required to be a self-avoiding walk. Let T be the number of self-intersections of ρ. Since ρ and β are self-avoiding walks then there are exactly J(Y) intersections in within the first $m+1$ steps and $l+1$ steps. In addition, the minimal walk from $\rho(m)$ to Y has at most $g-1$ intersections

$$T \leq J(Y) + g - 1 \leq 4m^B + dV - 1. \tag{5.23}$$

Consider the first self-intersection of ρ. This loop is a polygon with a distinct self-intersection and orientation with respect to ρ. Remove the loop and traverse through ρ until another self-intersection occurs. Let ϕ be the remainder of ρ then ϕ is a bridge of span $Y_1 + k$. Let r = $|\phi|$, t be the number of polygons removed, a_i the number of steps in the i^{th} polygon and q_n be the total number of n-step polygons then $r + a_1 + \cdots + a_t = |\rho|$. The walk ϕ is a loop erasing walk and the method in which it was determined is one of the well-known methods used to find self-avoiding walk. The number of p-step walks ρ with a combination of ϕ, t and $a_1, \cdots, a_t$ is less than

$$\prod_{j=1}^{t}(2pa_j q_{a_j}),$$

where 2 is for the direction of the polygon $\mathcal{P}$, p is the placement of $\mathcal{P}$ in ρ, a_j is the choice of vertex in $\mathcal{P}$ to be placed in ρ and q_{a_j} is the total number of such $\mathcal{P}$. By equation (3.2.5) and (3.2.9) of [38], for every even $n \geq 2$, $q_n \leq (d-1)\mu^n$. So,

$$\prod_{j=1}^{t}(2pa_j q_{a_j}) \leq [2(d-1)p^2]^t \mu^{a_1 + \cdots a_t} \text{ (since } a_i \leq p).$$

The number of walks ρ is less than

$$\sum_{p=m+l}^{m+l+dV} \sum_{r=1}^{p} b_r \sum_{t=0}^{H} \sum_{\substack{a_1, \cdots, a_t \\ a_1 + \cdots + a_t = p-r}} [2(d-1)p^2]^t \mu^{p-r},$$

where $H = 4m^B + dV - 1$ by (5.23). Since the number of walks ρ is at least

$b_{l,k}h^*_{n,k}$ then

$$\begin{aligned}
b_{l,k}h^*_{n,k} &\leq \sum_{p=m+l}^{m+l+dV}\sum_{r=1}^{p} b_r \sum_{t=0}^{H} \sum_{\substack{a_1,\cdots,a_t \\ a_1+\cdots+a_t=p-r}} [2(d-1)p^2]^t \mu^{p-r} \\
&\leq \sum_{p=m+l}^{m+l+dV}\sum_{r=1}^{p}\sum_{t=0}^{H} \sum_{\substack{a_1,\cdots,a_t \\ a_1+\cdots+a_t=p-r}} [2(d-1)p^2]^H \mu^{p} \\
&\leq \sum_{p=m+l}^{m+l+dV} p(H+1)p^H[2(d-1)p^2]^H\mu^p \\
&\leq \mu^{m+l+dV}4(m+l+dV)^3[2(d-1)(m+l+dV)^3]^{4m^B+dV-1} \\
&\leq \mu^{m+l+dV}[D(m+l+dV)]^{12m^B+3dV} \\
&\leq \mu^{m+l+dV}[D(m+l)]^{12m^B+3dV}\left(\text{since } V \leq (ml)^{\frac{1}{2}} \leq \frac{(m+l)}{2}\right).
\end{aligned}$$

□

An immediate result of this lemma is the weaker bound stated in Corollary 2

Corollary 2. *For $d \geq 2$ there exists Q depending only on d, such that for every $n \geq 2$*

$$c_n \leq \mu^n exp[Qn^{\frac{2}{d+1}}\log n].$$

Proof. By Remark 3, it's sufficient to prove this bound for h_n. This will be shown with the help of Lemma 10 and Remark 4. Let $B = \frac{2}{d+1}$, $l = i[0], m = n - i[0]$, $n \geq 2$ and consider the following two cases
<u>Case 1:</u> $i[0] = n$

$h_n \leq n^2 b_{i[0],k[0]}h^*_{n-i[0],k[0]}$
$= n^2 b_{n,k[0]}h^*_{0,k[0]}$
$= n^2 b_{n,k[0]}$ (since $h^*_{0,k} = 1$ for all k)
$\leq n^2 b_n$
$\leq b_n exp[2\log n]$
$\leq b_n exp[Qn^{\frac{2}{d+1}}\log n]$ (since $n \geq 2, \dfrac{2}{d+1} \in (0,1)$ and $n^a \geq 2^a \geq 1$ for $a \in (0,1)$)
$\leq \mu^n exp[Qn^{\frac{2}{d+1}}\log n]$ (since $b_n \leq \mu^n$).

Case 2: $i[0] < n$
Since

$$\begin{aligned} V &= ((n - i[0])^{1-B} i[0])^{\frac{1}{d}} \\ &\leq (n^{1-B} n)^{\frac{1}{d}} (\text{since } 0 < i[0] < n) \\ &= n^{\frac{(2-B)}{d}} \\ &= n^B \left(\text{since B} = \frac{(2-B)}{d} \right), \end{aligned}$$

then

$$\begin{aligned} &h_n \leq n^2 b_{i[0],k[0]} h^*_{n-i[0],k[0]} \\ &\leq n^2 \mu^{n+dV} [Dn]^{12(n-i[0])^B + 3dV} \\ &\leq n^2 \mu^{n+dV} [Dn]^{12n^B + 3dV} \text{ (since i[0]<n)} \\ &\leq \mu^n (\mu^d)^{n^B} [Dn]^{(12+3d)n^B + 2} \\ &\leq \mu^n (\mu^d)^{n^B} [n^D]^{(12+3d)n^B + 2} \\ &\leq \mu^n (\mu^d)^{n^B} exp[D((12+3d)n^B + 2) \log n] \\ &\leq \mu^n (\mu^d)^{n^B} exp[D((12+3d+2)n^B) \log n] \text{ (since } n \geq 2, B \in (0,1) \text{ and } n^B \geq 2^B \geq 1) \\ &\leq \mu^n (\mu^d)^{n^B} exp[Dn^{\frac{2}{d+1}} \log n] \text{ (since } B = \frac{2}{d+1}) \\ &\leq \mu^n exp[Dn^{\frac{2}{d+1}} \log n]. \end{aligned}$$

□

Proof of Theorem 10 The idea of this proof is to iteratively use Remark 4 until some conditions are satisfied then use Lemma 10. Let $A, B \in (0,1)$ such that $A = \frac{d}{(d+2)}$, $B = \frac{2}{(d+2)}$ and $n \geq 1$ is fixed. In addition, define $i[0], k[0]$ similarly to the ones in Remark 4. Consider the following conditions

1. $n^2 b_{i[0],k[0]} > \mu^{i[0]}$ and $i[0] < n^A$
2. $i[0] = n$

If (1) or (2) is true then set $u = 0$ and stop further iterations. Otherwise, reapply Remark 4 to $(n - i[0])$-step walks with $i[1], k[1] \in \{1, \cdots, n - i[0]\}$ to obtain

$$\begin{aligned} &h^*_{n-i[0],k[0]} \\ &\quad \leq (n - i[0])^2 b_{i[1],k[1]} h^*_{n-i[0]-i[1],k[1]} \text{ (since } h^*_{n-i[0],k[0]} \leq h_{n-i[0]} \text{ by Definition 30),} \end{aligned}$$

By Remark 4,

$$\begin{aligned} h_n &\leq n^2 b_{i[0],k[0]} h^*_{n-i[0],k[0]} \\ &\leq n^2 b_{i[0],k[0]} (n-i[0])^2 b_{i[1],k[1]} h^*_{n-i[0]-i[1],k[1]}. \end{aligned}$$

After j iterations the following actions are taken for the $(j+1)$-iteration. Let $i[0],...,i[j],k[0],...,k[j]$ be the known values from the first j iterations. Consider the following conditions

i. (a) $(n-i[0]-\cdots-i[j-1])^2 b_{i[j],k[j]} > \mu^{i[j]}$ and
   (b) $i[j] < (n-i[0]\cdots i[j-1])^A$

ii. $i[0]+\cdots+i[j]=n$.

If (i) or (ii) is true then set u=j and stop further iterations. Otherwise, reapply Remark 4 to $(n-i[0]-\cdots-i[j])$-step walks with $i[j+1],k[j+1] \in \{1,\cdots,n-i[0]-\cdots-i[j]\}$ to obtain the following

$$h^*_{n-i[0]-\cdots-i[j],k[j]} \leq (n-i[0]-\cdots-i[j])^2 b_{i[j+1],k[j+1]} h^*_{n-i[0]-i[1]-\cdots-i[j+1],k[j+1]}.$$

At the last iteration,

$$\begin{aligned} h_n \leq{}& n^2 b_{i[0],k[0]} (n-i[0])^2 b_{i[1],k[1]} \times \cdots \\ &\times (n-i[0]-\cdots-i[u-1])^2 b_{i[u],k[u]} h^*_{n-i[0]-i[1]-\cdots-i[u],k[u]}. \end{aligned} \tag{5.24}$$

Now consider $\mathcal{I} = \{j \in \{0,...,u-1\} |$ (ia) holds for j$\}$. Then for every $j \in \mathcal{I}$,

$$i[j] \geq (n-i[0]\cdots i[j-1])^A. \tag{5.25}$$

Similarly, for every $j \in \{0,...,u-1\} \setminus \mathcal{I}$,

$$(n-i[0]-\cdots-i[j-1])^2 b_{i[j],k[j]} \leq \mu^{i[j]} \leq n^2 \mu^{i[j]} \text{ (since } n \geq 1).$$

Now 5.24 can be simplified to

$$h_n \leq n^{2|\mathcal{I}|+2} \mu^{i[0]+\cdots+i[u]} h^*_{n-i[0]-i[1]-\cdots-i[u],k[u]}. \tag{5.26}$$

For each integer $a \geq 0$, let $\mathcal{I}_a = \{j \in \mathcal{I} | n2^{-a-1} \leq n-i[0]-\cdots-i[j-1] \leq n2^{-a}\}$. If $|\mathcal{I}_a| \leq 1$ then $|\mathcal{I}| \leq Cn^{1-A}$ for C a constant depending only on A. If If $|\mathcal{I}_a| > 1$, let $f_a = \min \mathcal{I}_a$, $\mathcal{F}_a = \max \mathcal{I}_a$ and $\mathcal{I}^{'}_a = \mathcal{I}_a \setminus \{\mathcal{F}_a\}$. By definition of $\mathcal{I}$,

$$n2^{-a-1} \leq n-i[0]-\cdots-i[\mathcal{F}_a - 1] \leq n-i[0]-\cdots-i[f_a-1] \leq n2^{-a}.$$

Consequently,

$$\sum_{j\in\mathcal{I}'_a} i[j] \leq \sum_{j=f_a}^{\mathcal{F}_a-1} i[j] \leq n2^{-a} - n2^{-a-1} = n2^{-a-1}.$$

By definition of $\mathcal{I}$ and $\mathcal{I}_a$,

$$\begin{aligned}\sum_{j\ \in\mathcal{I}'_a} i[j] &\geq (|\mathcal{I}_a|-1)(n - i[0]\cdots i[j-1])^A \text{ (since } \mathcal{I}'_a \subseteq \mathcal{I}_a \text{ and } 5.25)\\ &\geq (|\mathcal{I}_a|-1)(n2^{-a-1})^A,\end{aligned}$$

so

$$\begin{aligned}(|\mathcal{I}_a|-1)(n2^{-a-1})^A &\leq n2^{-a-1}\\ |\mathcal{I}_a| &\leq 1 + (n2^{-a-1})^{1-A}.\end{aligned} \tag{5.27}$$

Hence,

$$\begin{aligned}|\mathcal{I}| &\leq \sum_{a=0}^{\log_2 n} |\mathcal{I}_a|\\ &\leq 1 + \log_2 n + \sum_{a=0}^{\infty}(n2^{-a-1})^{1-A} \text{ (by } 5.27)\\ &\leq 1 + \log_2 n + Cn^{1-A} \text{ (since the sum converges to constant based on } d)\\ &\leq Cn^{1-A},\end{aligned} \tag{5.28}$$

where C is a constant depending only on A.

After u has been determined, let k=k[u], $l = i[u]$ and $m = n - i[0] - ... - i[u]$ then by 5.26 and 5.28,

$$h_n \leq \mu^{n-m} n^{2Cn^{1-A}+2} h^*_{m,k}. \tag{5.29}$$

If $m = 0$ then

$$\begin{aligned}h_n &\leq \mu^n n^{2Cn^{1-A}+2} h^*_{0,k}\\ &= \mu^n n^{2Cn^{2/(d+2)}+2}\\ &= \mu^n exp(2(Cn^{\frac{2}{d+2}} + 2)\log n)\\ &\leq \mu^n exp(2((C+2)n^{\frac{2}{d+2}})\log n) \text{ (since } n \geq 2 \text{ and } n^a \geq 1 \text{ for } a \in (0,1))\\ &\leq \mu^n exp[Qn^{\frac{2}{d+2}}\log n]\end{aligned}$$

where Q is a constant depending only on d. By definition of u, $m+l \leq n$ and condition (i) the following inequalities hold

- $\mu^l < (m+1)^2 b_{l,k} \leq n^2 b_{l,k}$
- $l < (m+1)^A \leq n^A$

By the conditions above and Lemma 10,

$$n^{-2}\mu^l h^*_{m,k} \leq b_{l,k} h^*_{m,k} \leq \mu^{m+l+dV}[D(m+l)]^{12m^B+3dV}$$

So,

$$h^*_{m,k} \leq \mu^{m+dV} n^2 [D(m+l)]^{12m^B+3dV}$$

By the inequality above and equation (5.29),

$$\begin{aligned} h_n &\leq \mu^{n+dV} n^{2Cn^{1-A}+4}[D(m+l)]^{12m^B+3dV} \\ &\leq \mu^{n+dV} n^{2Cn^{1-A}+4}[Dn]^{12n^B+3dV} \text{ (since } m+l \leq n). \end{aligned}$$

By the second inequality in the bullet points above,

$$V = (m^{1-B}l)^{\frac{1}{d}} \leq (n^{1-B}n^A)^{\frac{1}{d}} = n^{\frac{1-B+A}{d}}.$$

Hence there is a constant Q depending only on A, B and d such that

$$\begin{aligned} h_n &\leq \mu^n (\mu^d)^{n^B} n^{3dn^{((1-B+A)/d)}+2Cn^{1-A}+16n^B} \text{ (since } n^B \geq 1) \\ &\leq \mu^n (\mu^d)^{n^B} exp[(3dn^{((1-B+A)/d)} + 2Cn^{1-A} + 16n^B)\log n] \\ &\leq \mu^n (\mu^d)^{n^B} exp[Q(n^{(1-B+A)/d} + n^{1-A} + n^B)\log n] \\ &= \mu^n (\mu^d)^{n^B} exp[3Qn^{\frac{2}{d+2}} \log n] \text{ (since } \frac{(1-B+A)}{d} = 1-A = B) \\ &= \mu^n exp[Qn^{\frac{2}{d+2}} \log n]. \end{aligned}$$

□

Remark 5. *The Hammersley-Welsh Bound, another well known bound is the strongest currently known bound when $d = 2$. In 3 and 4 dimension the best bound is Kesten's. Though it does not get as much recognition. The Hammersly-Welsh Bound is given as follows:*

> *If $B > \pi\left(\frac{2}{3}\right)^{\frac{1}{2}}$ is fixed then there is an $n_0 = n_0(B)$ independent of the dimension $d \geq 2$, such that*
>
> $$c_n \leq b_{n+1}e^{B\sqrt{n}} \leq \mu^{n+1}e^{B\sqrt{n}} \textit{ for } n \geq n_0.$$

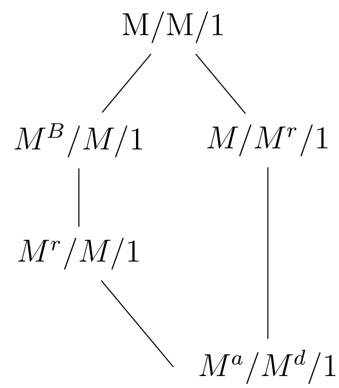


Figure 5.7: Types of Queueing Model

5.3 Current Research and Applications

As mentioned in Chapter 1, lattice paths were studied recreationally up until 1960s. Afterwards, they gained popularity and were considered for various applications. One of these applications is in queuing theory. Queuing theory studies the different components of waiting in a line such as arrival processes, service processes, number of servers, number of system places, and the number of customers, etc [3]. From its definition it can be seen to have immediate real life applications in business, including increasing the efficiency of different forms of customer service, traffic flow, shipments from warehouse, while staying in budget. Queuing theory has become an essential part of almost any type of organization. Hence, advancements in its study can provide immediate benefits. This section will look at how lattice path theory aids in the study of different types of queuing models. W. Böhm provides an overview of a simple model in [16] and builds on it by introducing new models with small characteristics altered. The author then defines lattice paths with respect to step set which translates to each model introduced.

The structure of each model is provided below in Kendall's notation [33]. The notation is written as $A/S/c$, where A denotes the time between arrivals to the queue, S the service time distribution and c the number of service channels open at the node. Consequently, the $M/M/1$ model represents a queue having a single server, with arrivals determined by a Poisson process and job service times having exponential distribution [33].

1. $M/M/1$
   - There is a single server such that the service times are i.i.d exponential random variables with mean $\frac{1}{\mu}$.
   - Customers arrive by Poisson process with rate λ.
   - The system can hold an infinite number of customers.
2. $M^r/M/1$
   - The $M/M/1$ model with the additional feature that customers arrive in bulk of size r.
3. $M/M^r/1$
   - The $M/M/1$ model with the additional feature that customers are served in bulk of size r. If the size of the bulk $< r$ at any time t then service is provided to the customers in line.
4. $M^B/M/1$
   - The $M/M^r/1$ model such that $0 \leq r \leq q$ for some q.
5. $M^a/M^d/1$
   - The $M^r/M/1$ and $M/M^r/1$ models combined

The first model can be translated to a random walk with jumps of magnitude $+1, -1$ occurring with rate λ and μ, respectively. That is, lattice walks in the quarter plane with step set $\mathcal{S} = \{(1,1),(1,-1)\}$. The second model can be translated to the same randomized random walk as model 1 with the exception of the magnitude changing from $+1$ to $+r$. The third model translates to lattice paths similar to the second model with a small change in the step set from $\mathcal{S} = \{(1,r),(1,-1)\}$ to $\mathcal{S} = \{(1,1),(1,-r)\}$. Model 2 covers a relatively more realistic scenario than model 1, however, it is impractical because the number of customers do not always arrive in bulk size of r. In this case, all customers in line must wait for more to arrive in order to be served. This issue is taken care of in model 4. It can be translated to lattice paths with the following step set $\mathcal{S} = \{(1,1),\cdots,(1,r),(1,-1)\}$. Since it is very likely that customers both arrive and depart in bulk then a model concerning that scenario has useful applications. This scenario is covered in model 5 in which the step set is $\mathcal{S} = \{(1,a),(1,-d)\}$.

Based on the description and step set these are self-avoiding and directed walks. To make the translation from models to lattice walks more precise, the walks should return to the x-axis to indicate end of business day. Hence, it is a self-avoiding excursion. Looking at this model from different perspectives may allow for different approaches to study the generating function. For instance, if the complete generating function of walks for some step set $\mathcal{S}$ given above is

$$Q(t,x,y) = \sum_{n\geq 0}\left(\sum_{i,j\geq 0} q(n,i,j)x^i y^j\right)t^n$$

then the generating function of interest will be $Q(t,x,0)$. Here x tallies the total events occurred of customer arrival and service provided/customer departure and y tallies the type of event that occurred. Now, one can study the holonomy or d-finiteness of $Q(t,x,0)$ to understand the queueing models better, perhaps using similar techniques as in Chapter 4.

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