Exactness and Noncommutative Convexity

by

Nicholas Manor

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Examining Committee Membership

The following served on the Examining Committee for this thesis. The decision of the Examining Committee is by majority vote.

External Examiner	David Blecher Professor Department of Mathematics University of Houston
Supervisors	Matthew Kennedy Associate Professor Department of Pure Mathematics University of Waterloo
	Nico Spronk Professor Department of Pure Mathematics University of Waterloo
Internal Members	Kenneth Davidson University Professor Emeritus Department of Pure Mathematics University of Waterloo
	Vern Paulsen Professor Department of Pure Mathematics University of Waterloo
Internal-External Member	Sophie Spirkl Assistant Professor Department of Combinatorics & Optimization University of Waterloo

Author's Declaration

This thesis consists of material all of which I authored or co-authored: see Statement of Contributions included in the thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Statement of Contributions

Chapter 1 is the author's own work. Chapter 2 is joint work with Matthew Kennedy and Se-Jin Kim. Chapter 3 is joint work with Adam Humeniuk and Matthew Kennedy.

Abstract

This thesis studies two separate topics in connection to operator systems theory: the dynamics of locally compact groups, and noncommutative convex geometry.

In Chapter 1 we study exactness of locally compact groups as it relates to C*-exactness, i.e., the exactness of the reduced C*-algebra. It is known that these two properties coincide for discrete groups. The problem of whether this equivalence holds for general locally compact groups has recently been reduced by Cave and Zacharias to the case of totally disconnected unimodular groups. We prove that the equivalence does hold for the class of locally compact groups whose reduced C*-algebra admits a tracial state.

In Chapter 2 we establish the dual equivalence of the category of generalized (i.e. potentially non-unital) operator systems and the category of pointed compact noncommutative (nc) convex sets, extending a result of Davidson and Kennedy. We then apply this dual equivalence to establish a number of results about generalized operator systems, some of which are new even in the unital setting.

We develop a theory of quotients of generalized operator systems that extends the theory of quotients of unital operator systems. In addition, we extend results of Kennedy and Shamovich relating to nc Choquet simplices. We show that a generalized operator system is a C*-algebra if and only if its nc quasistate space is an nc Bauer simplex with zero as an extreme point, and we show that a second countable locally compact group has Kazhdan's property (T) if and only if for every action of the group on a C*-algebra, the set of invariant quasistates is the quasistate space of a C*-algebra.

In Chapter 3 we expand on recent work of C.K. Ng about duals of operator systems. Call a nonunital operator system S *dualizable* if its dual S^* embeds into $B(H)$ via a complete order embedding and complete norm equivalence. Through the categorical duality of nonunital operator systems to pointed noncommutative convex sets discussed in Chapter 3, we characterize dualizability of S using geometric conditions on the nc quasistate space K in two ways. Firstly, in terms of an nc affine embedding of K into the nc unit ball of a Hilbert space satisfying a bounded positive extension property for nc affine functions. Secondly, we show that Ng's characterization is dual to a normality condition between K and the nc cone $\mathbb{R}_+K$. As applications, we obtain some permanence properties for dualizability and duality of mapping cones in quantum information, and give a new nc convex-geometric proof of Choi's Theorem.

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Chapter 1

Exactness vs C*-exactness for certain non-discrete groups

1.1 Introduction

The property of exactness for locally compact groups has received much attention since it was first introduced and studied by Kirchberg and Wassermann in [33, 35, 36]. In the setting of discrete groups, the theory of exactness is very well developed and connections to dynamics and coarse geometry have long been known [2, 42]. It is also known for a discrete group G that Kirchberg and Wassermann's original definition is equivalent to the apparently weaker condition (here called C*-exactness) that the reduced C*-algebra $C^*_r(G)$ is exact [35, Theorem 5.2]. It was later shown by Anantharaman-Delaroche [2, Theorem 7.3] that the equivalence also holds for inner amenable groups.

More recently, dynamical [2, 7] and coarse geometric [7] characterizations of exactness have been found for locally compact groups. However, it remains a major open problem to determine whether the equivalence of exactness and C*-exactness holds in general, and the problem has recently been reduced to the case of totally disconnected locally compact (tdlc) unimodular groups [9]. In other words, if there is an example of a non-exact locally compact group G with exact reduced C*-algebra $C^*_r(G)$, then there is necessarily a tdlc unimodular such group.

In this chapter we establish some tools to show that exactness and C*-exactness are equivalent for certain classes of groups. In section 1.3 we prove our most general result. In section 2.4 we study examples, and in particular apply our tools to classes of tdlc unimodular groups. In subsection 1.4.1 we study a class of groups first considered by Suzuki [48], where they constructed examples of non-discrete C*-simple groups in this class. In subsection 1.4.2 we study tdlc groups which admit a conjugation invariant neighbourhood of the identity. These are precisely those tdlc groups whose reduced C*-algebras have non-zero center [37, Corollary 1.2]. In subsection 4.3 we show how to produce locally compact groups which are not inner amenable, but to which the methods in this chapter apply. In subsection 4.4 we study a class of non-examples.

1.2 Preliminaries

Recall [35] that a locally compact group G is called *exact* if for every short exact sequence $0 \to I \to A \to B \to 0$ of G-C*-algebras, the corresponding sequence of reduced crossed products is also exact. That is to say,

$$0 \to I \rtimes_r G \to A \rtimes_r G \to B \rtimes_r G \to 0$$

is an exact sequence of C*-algebras. Recall also that a C*-algebra C is called exact if for every short exact sequence $0 \to I \to A \to B \to 0$ of C*-algebras, the corresponding sequence of spatial tensor products

$$0 \to I \otimes C \to A \otimes C \to B \otimes C \to 0$$

is also exact. In the case where I, A and B are all equipped with the trivial action of G, the sequence of crossed products becomes

$$0 \to I \otimes \mathrm{C}^*_{\mathrm{r}}(G) \to A \otimes \mathrm{C}^*_{\mathrm{r}}(G) \to B \otimes \mathrm{C}^*_{\mathrm{r}}(G) \to 0.$$

Hence, exact groups necessarily have exact reduced C*-algebra. We will say G is *C*-exact* if $\mathrm{C}^*_{\mathrm{r}}(G)$ is an exact C*-algebra.

A priori, exactness appears to be strictly stronger than C*-exactness: there is no reason to expect the trivial action to capture all information about the exactness of G. Somewhat surprisingly, the two notions coincide for discrete groups [35, Theorem 5.2].

Theorem 1.2.1. *If G is a C*-exact discrete group, then it is exact.*

1.3 Groups with open amenable radical

In the following sections, we show that the equivalence of exactness and C*-exactness holds for classes of locally compact groups which properly contain all discrete groups. The following observation, while simple, is the key to bootstrapping Theorem 1.2.1 to the classes of groups considered in this chapter.

Proposition 1.3.1. *Let G be a locally compact group with an open normal amenable subgroup. If G is C*-exact then it is exact.*

Proof. Suppose G is C*-exact, and let $N \trianglelefteq G$ be open and amenable. Since N is amenable, the left quasi-regular representation $G \curvearrowright \ell^2(G/N)$ is weakly contained in the left regular representation. To see this, recall that the trivial representation 1_N is weakly contained in the left regular representation λ_N since N is amenable, and then apply continuity of induction of representations to get the desired weak containment (e.g. [4, Theorem F.3.5]).

Hence there is a surjective $*$-homomorphism $\mathrm{C}^*_{\mathrm{r}}(G) \to \mathrm{C}^*_{\mathrm{r}}(G/N)$, and exactness of C*-algebras passes to quotients [50, Corallary 9.3], so G/N is also C*-exact. But N is assumed to be open, meaning the quotient G/N is a discrete group. Theorem 1.2.1 then implies that G/N is an exact group, and we already know that N is exact as it is amenable. Since exactness is preserved by extensions [36, Theorem 5.1], we conclude that G is also exact. □

Remark 1.3.2. Proposition 1.3.1 can be strengthened so that N is not necessarily open, but has the property that G/N is exact if and only if it is C*-exact. Calling a group *admissible* if it satisfies the equivalence, the statement can be strengthened as follows: extensions of amenable groups by admissible groups are admissible.

Locally compact groups admitting a tracial state on the reduced C*-algebra have received recent attention in [20, 31, 39]. The following corollary relates the existence of a trace to the exactness of G, and is in analogy with the implication (2) $\Longrightarrow$ (1) in Ng's characterization of amenability [39, Theorem 8].

Corollary 1.3.3. *If* $C^*_r(G)$ *admits a tracial state and is exact, then* G *is exact.*

Proof. The main theorem of [31] by Kennedy–Raum states that $C^*_r(G)$ admitting a tracial state is equivalent to the existence of an open normal amenable subgroup in G. The result then follows from Proposition 1.3.1. □

Example 1.3.4. Exact groups do not necessarily admit a trace on their reduced C*-algebra. In particular, the converse of this corollary does not hold. Take for example any non-amenable connected group, say $G = SL_2(\mathbb{R})$. Then $C^*_r(G)$ is nuclear as G is connected, but it cannot also have a trace since G is non-amenable [39, Theorem 8]. However, we know by [36, Theorem 6.8] that connected groups are always exact.

The following theorem allows us to extend the class of groups to which our results apply.

Theorem 1.3.5. *Let* G *be a locally compact group, and* $(H_i)_{i\in I}$ *a family of open subgroups with the following conditions.*

- *For every* i*, there is a tracial state on* $C^*_r(H_i)$*.*
- *The union* $\bigcup_{i\in I} H_i$ *is equal to* G*.*

If G *is C*-exact, then it is exact.*

Since exactness is fundamentally a property of the ideal structure in reduced crossed products, we require the following technical fact about ideals in C*-algebras [5, II.8.2.4].

Lemma 1.3.6. *Let* A *be a C*-algebra, and* $(A_i)_{i\in I}$ *a family of C*-subalgebras such that* $\bigcup_{i\in I} A_i$ *is dense in* A*. If* J *is any closed ideal in* A*, then* $J \cap \bigcup_{i\in I} A_i$ *is dense in* J*.*

We also note that if $0 \to I \to A \xrightarrow{q} B \to 0$ is an exact sequence of G-C*-algebras, then it is easy to see $I \rtimes_r G \subseteq \ker q_G$. Indeed, take an arbitrary compactly supported function f from the norm dense subset $C_c(G, I) \subseteq I \rtimes_r G$ and notice $q_G(f)(x) = q(f(x)) = 0$ for all $x \in G$ since $f(x) \in I = \ker q$. The inclusion $I \rtimes_r G \subseteq \ker q_G$ then follows by continuity of q_G. We now give the proof of Theorem 1.3.5.

Proof of Theorem 1.3.5. Let G and $(H_i)_{i\in I}$ be as in the theorem statement. Since each H_i is an open subgroup we have the inclusion $C^*_r(H_i) \subseteq C^*_r(G)$, hence if G is C*-exact then so is each H_i. But each H_i admits a trace, so Corollary 1.3.3 tells us that each H_i is also exact. We show how exactness lifts up to G.

Let $0 \to I \to A \xrightarrow{q} B \to 0$ be an exact sequence of G-C*-algebras. Restricting each G-action to H_i, this is also exact as a sequence of H_i-C*-algebras. Since H_i is an exact group, then

$$0 \to I \rtimes_r H_i \to A \rtimes_r H_i \xrightarrow{q_i} B \rtimes_r H_i \to 0$$

is short exact. Using this fact, we will show that the corresponding sequence

$$0 \to I \rtimes_r G \to A \rtimes_r G \xrightarrow{q_G} B \rtimes_r G \to 0$$

of G-crossed products is short exact, i.e., that $I \rtimes_r G = \ker q_G$.

Since $\bigcup_{i \in I} H_i = G$, we have a norm dense inclusion $\bigcup_{i \in I} A \rtimes_r H_i \subseteq A \rtimes_r G$. We may then apply Lemma 1.3.6 to $\ker q_G$ to get $\overline{\bigcup_{i \in I} \ker q_i} = \ker q_G$. But the H_i were shown to be exact, hence each $I \rtimes_r H_i = \ker q_i$, which gives us the inclusion $\ker q_G \subseteq I \rtimes_r G$ as each $I \rtimes_r H_i \subseteq I \rtimes_r G$. The opposite inclusion $I \rtimes_r G \subseteq \ker q_G$ was discussed before the proof. □

1.4 Examples and non-examples

In subsections 1.4.1 and 1.4.2 we show, without using inner amenability, that C*-exactness implies exactness for certain inner amenable groups. In subsection 1.4.3 we describe a class of non-inner amenable groups to which our results apply. In subsection 1.4.4 we show that certain automorphism groups of trees are non-examples.

1.4.1 Exactness for Suzuki groups

The following result shows C*-exactness implies exactness for a class of tdlc groups considered by Suzuki in the context of C*-simplicity [48].

Proposition 1.4.1. *Let G be a locally compact group with a decreasing neighbourhood base $(K_n)_{n=1}^{\infty}$ of compact open subgroups, and an increasing sequence $(L_n)_{n=1}^{\infty}$ of open subgroups with the following additional properties.*

- *Each K_n is a normal subgroup of L_n.*
- *The union $\bigcup_{n=1}^{\infty} L_n$ is equal to G.*

If G is C-exact, then it is exact.*

Example 1.4.2. Suzuki describes a general construction in [48]. For each $n \in \mathbb{N}$, let Γ_n be a discrete group and let F_n be a finite group acting on Γ_n by automorphisms.

We view the direct sum $\bigoplus_{n=1}^{\infty} \Gamma_n$ as a discrete group, and the direct product $\prod_{n=1}^{\infty} F_n$ as a compact group with the product topology. Defining the action $\prod_{n=1}^{\infty} F_n \curvearrowright \bigoplus_{n=1}^{\infty} \Gamma_n$ component-wise, the semidirect product $G := (\bigoplus_{n=1}^{\infty} \Gamma_n) \rtimes \prod_{n=1}^{\infty} F_n$ satisfies the conditions of Proposition 1.4.1. To see this, for each $n \in \mathbb{N}$ set $K_n := \prod_{k=n+1}^{\infty} F_k$ and $L_n := (\bigoplus_{k=1}^{n} \Gamma_k) \rtimes \prod_{k=1}^{\infty} F_k$.

Proof of Proposition 1.4.1. Since each L_n has a compact (hence amenable) open normal subgroup K_n, it has a trace by [31]. Hence if G is C*-exact then it is exact by Theorem 1.3.5. □

The groups of Proposition 1.4.1 are inner amenable. Indeed, for each n there is a conjugation invariant mean on $L^\infty(L_n)$ given by $\varphi_n(f) = \int_{K_n} f$ since K_n is normal in L_n. Extending the φ_n to $L^\infty(G)$ and picking a weak* cluster point of the sequence $(\varphi_n)_{n\geq 1}$ produces a conjugation invariant mean on $L^\infty(G)$.

Since exactness of groups is preserved by closed subgroups and by extensions, and since exactness of C*-algebras is preserved by quotients, we have G is exact if and only if each L_n/K_n is exact. By Proposition 1.4.1, we know this corresponds also to $C^*_r(G)$ being exact. In the language of Example 1.4.2, this means G is exact if and only if each Γ_n is exact.

1.4.2 Exactness for IN groups

We now study exactness for tdlc groups with an additional topological property: we say a locally compact group G is an *IN group* (invariant neighbourhood group) if there is a compact neighbourhood U of the identity which is invariant under conjugation, i.e., for every $g \in G$ we have $gUg^{-1} = U$. A closely related property is when G admits a neighbourhood base at the identity consisting of conjugation invariant sets. In this case we say that G is a *SIN group* (small invariant neighbourhood group).

Remark 1.4.3. It is easy to show that an IN group (resp. SIN group) G is necessarily unimodular: letting m denote the Haar measure on G, and setting U to be an invariant compact neighbourhood of the identity, we have $gU = Ug$ for all $g \in G$. Hence $\Delta(g)m(U) = m(Ug) = m(gU) = m(U)$ for all g, implying the modular function Δ is constantly equal to 1.

From this it is clear that IN groups, hence SIN groups, are inner amenable. To see this, fix a conjugation invariant compact neighbourhood $U \subseteq G$ of the identity, and define $\varphi : L^\infty(G) \to \mathbb{C}$ by $\varphi(f) = \frac{1}{m(U)} \int_U f$.

We will make use of the following structure theorem [43, 12.1.31], which strongly relates IN groups to SIN groups. It was originally proved by Iwasawa in [26].

Theorem 1.4.4. *Let G be an IN group. Then there is a compact normal subgroup $K \trianglelefteq G$ so that G/K is a SIN group with the quotient topology.*

Remark 1.4.5. Theorem 1.4.4 tells us that IN groups are extensions of compact groups by SIN groups. In fact, the converse holds as well. That is to say, if G has a compact normal subgroup $K \trianglelefteq G$ so that G/K is a SIN group, then G is an IN group.

To see this, note that the map $q : G \to G/K$ is proper as the quotient is by a compact subgroup. So we may fix any invariant compact neighbourhood U of the identity in G/K, and the preimage $q^{-1}(U)$ is an invariant compact neighbourhood of the identity in G.

Lemma 1.4.6. *Let G be a tdlc SIN group. Then G admits a neighbourhood base at the identity consisting of compact open normal subgroups.*

Proof. We will show that every compact open subgroup contains a compact open normal subgroup. Since the compact open subgroups form a neighbourhood this will complete the proof.

Let $K_1 \leq G$ be a compact open subgroup. Then there is a conjugation invariant neighbourhood U of e with $U \subseteq K_1$, and there is in turn a compact open subgroup $K_2 \subseteq U$. Since U is conjugation invariant, then we have the containment

$\bigcup_{g\in G} gK_2g^{-1} \subseteq U \subseteq K_1$, hence the open normal subgroup K generated by the conjugates $\bigcup_{g\in G} gK_2g^{-1}$ is also contained in K_1. As an open, hence closed, subgroup of a compact group, this implies $K \subseteq K_1$ is also compact. □

The following structural fact is probably well known, but we were unable to find a reference.

Proposition 1.4.7. *A topological group G is tdlc and SIN if and only if there is an inverse system $(\Gamma_i, \varphi_{ij})_{i>j\in I}$ of discrete groups so that $|\ker\varphi_{ij}| < \infty$ for all $i > j$, and $G \cong \lim_{\leftarrow} \Gamma_i$.*

Proof. If G is tdlc and SIN, there is a neighbourhood base of compact open normal subgroups by Lemma 1.4.6. Ordering this normal neighbourhood base $(K_i)_{i\in I}$ by reverse inclusion gives us an inverse family $(G/K_i)_{i\in I}$ of discrete groups, with connecting maps $\varphi_{ij} : G/K_i \to G/K_j$ whenever $i > j$. Since each K_i is both compact and open, the φ_{ij} all have finite kernel, i.e., $|\ker\varphi_{ij}| < \infty$. It is routine to check that G is isomorphic to the inverse limit $\lim_{\leftarrow} G/K_i \subseteq \prod_i G/K_i$ with the relative product topology.

Suppose conversely that G is isomorphic to an inverse limit $\lim_{\leftarrow}\Gamma_i$ with respect to the relative product topology, as described in the proposition statement. Viewing G as a subgroup of $\prod_i \Gamma_i$, the subsets

$$K_{j,k} := \{(g_i)_{i\in I} \in \varprojlim \Gamma_i : g_j \in \ker\varphi_{j,k}\}$$

with $j > k$ form a neighbourhood base of compact open normal subgroups. □

Recall that tdlc compact groups are precisely the profinite groups [46, Theorem 2.1.3], hence Remark 1.4.5 and Proposition 1.4.7 taken together tell us that tdlc IN groups are precisely extensions of profinite groups by inverse limits of discrete groups whose connecting maps have finite kernel.

Example 1.4.8. Fix a finite group F and a non-exact discrete group H (interesting examples of non-exact groups may be found in [38, 41]). Then define $G = \prod_H F \rtimes H$, where H acts by left translation on the compact group $\prod_H F$. It is clear that G is an extension of the profinite group $\prod_H F$ by the discrete (hence tdlc SIN) group H, hence G is a tdlc IN group.

However, it is not SIN: take any basic neighbourhood U of the identity e in $\prod_H F$, then some element of U may be left-translated outside of U. This means that U is not invariant under conjugation by elements in H.

Moreover, G is non-exact as H is a non-exact closed subgroup [35, Theorem 4.1]. This gives a tdlc IN group which is not SIN and non-exact.

Proposition 1.3.1 together with Lemma 1.4.6 immediately gives us the equivalence of exactness and C*-exactness for tdlc SIN groups.

Corollary 1.4.9. *Let G be a tdlc SIN group. If G is C^*-exact then it is exact.*

The main result of this section now follows quickly using structure theory of IN groups.

Theorem 1.4.10. *Let G be a tdlc IN group. If G is C^*-exact then it is exact.*

Proof. By Theorem 1.4.4, G has a compact normal subgroup H so that G/H is a SIN group with the quotient topology. Since H is compact it is also amenable, so we get a surjection $C^*_r(G) \to C^*_r(G/H)$ as in the proof of Proposition 1.3.1.

Since exactness passes to quotients [36, Theorem 5.1], G/H is also C*-exact. By Corollary 1.4.9 this implies G/H is exact, hence G is exact as an extension of a compact group H by an exact group G/H. □

An application of [37, Corollary 1.2] makes the hypotheses purely C*-algebraic.

Corollary 1.4.11. *Let G be tdlc. If $C^*_r(G)$ is exact and has non-zero center, then G is exact.*

Although IN groups comprise a rich class of unimodular groups, there are natural examples of tdlc unimodular groups which are not IN.

Example 1.4.12. Let $G := \mathrm{SL}_2(\mathbb{Q}_p)$, where p is any prime and $\mathbb{Q}_p$ denotes the set of p-adic rationals. Then G is generated by its commutators, hence it is necessarily unimodular. However, G is not IN: it is routine to show that all points other than I_2 and $-I_2$ can be conjugated arbitrarily far from I_2. Although this example is not IN, it is a linear group hence exact [23].

1.4.3 Non-inner amenable groups

The result of this section was inspired by [20, Remark 2.6 (ii)]. We outline conditions that allow us to construct a non-inner amenable group G with an open amenable normal subgroup. By Proposition 1.3.1 these groups are exact if and only if they are C*-exact, but since they are not inner amenable we cannot conclude this equivalence from Anantharaman-Delaroche's [2, Theorem 7.3].

Proposition 1.4.13. *Let N be an amenable locally compact group and H a discrete group. If the only conjugation invariant mean on H is evaluation at the identity, and if $\alpha : H \to \mathrm{Aut}(N)$ is an action such that there is no $\alpha(H)$-invariant mean on N, then the semi-direct product $N \rtimes H$ is not inner amenable.*

In the product topology on $N \times H$, N is open, hence it is an open amenable normal subgroup in $N \rtimes H$.

Proof. Suppose φ is a conjugation invariant mean on $N \rtimes H$. Since the only conjugation invariant mean on H is evaluation at the identity, then φ must concentrate on N. Thus we may view φ as a mean on N which is invariant under the action of H, a contradiction. □

Example 1.4.14. It was proved in [20, Remark 2.6 (ii)] that $\mathbb{R}^2$ and $F_6 \subseteq SL_2(\mathbb{R})$ satisfy the hypotheses above, and hence that $\mathbb{R}^2 \rtimes F_6$ is not inner amenable.

To produce a non-exact example, simply let H be any non-exact discrete group and let H act on $\mathbb{R}^2$ trivially. Then the semi-direct product $\mathbb{R}^2 \rtimes (F_6 * H)$ is non-exact, and it is also not inner amenable since by [18, Theorem 1.1] the only conjugation invariant mean on $F_6 * H$ is evaluation at the identity.

1.4.4 Automorphism groups of trees

For $d \geq 3$, we denote by T_d the infinite d-regular tree, e.g., T_4 is the Cayley graph of the free group on two generators $\mathbb{F}_2$. The automorphism group $\mathrm{Aut}(T_d)$ becomes a tdlc group when equipped with the topology of pointwise convergence on the set of vertices $V(T_d)$. For every finite subset $S \subseteq V(T_d)$, the fixator $\mathrm{Fix}_{\mathrm{Aut}(T_d)}(S)$ of S is a compact open subgroup.

Fixing a vertex $b \in V(T_d)$ and an integer $n \geq 1$, we denote by $B_n(b)$ the ball centred at b of radius n in the path metric. The set of fixators $K_n := \mathrm{Fix}_{\mathrm{Aut}(T_d)}(B_n(b))$ forms a sequential neighbourhood base at the identity.

We will show that certain Burger-Mozes groups [8] in $\mathrm{Aut}(T_d)$ are too geometrically dense for our results of section 1.3 to apply. For a good introduction to Burger-Mozes groups see [21, Section 4]. We will denote by $U(F) \leq \mathrm{Aut}(T_d)$ the Burger-Mozes group of $F \leq S_d$.

Definition 1.4.15. We say a subgroup $G \leq \mathrm{Aut}(T_d)$ is *geometrically dense* if it does not fix any proper subtree, and does not fix any end in ∂T_d.

Proposition 1.4.16. *For any subgroup $F \leq S_d$, the amenable radical of $U(F)$ is trivial.*

Proof. Let N be a non-trivial normal subgroup of $U(F)$. Since $U(\{e\})$ is geometrically dense so is $U(F)$, hence N is geometrically dense by [21, Lemma 2.10] as it is non-trivial and normal in $U(F)$. Using [21, Lemma 2.9] and applying the ping pong lemma, one can produce a closed copy of $\mathbb{F}_2$ in N, proving it is non-amenable. □

In particular, the amenable radical is open if and only if $U(F)$ is discrete. This result implies that the only Burger-Mozes groups admitting a tracial state are the discrete ones, which is precisely when the action $F \curvearrowright \{1, \dots, d\}$ is free [21, Proposition 4.6 (v)].

Corollary 1.4.17. *Let $F \leq S_d$ be a subgroup which does not act freely on $\{1, \dots, d\}$, then the reduced C^*-algebra $C_r^*(U(F))$ does not admit a tracial state.*

Proof. By Proposition 1.4.16, the amenable radical of $U(F)$ is trivial, and since $F \curvearrowright \{1, \dots, d\}$ is not free then $U(F)$ is non-discrete. Hence the amenable radical is not open, and by [31] this implies there is no tracial state on $\mathrm{C}_\mathrm{r}^*(U(F))$. □

This means that Corollary 1.3.3 cannot be applied to this class. We would then like to determine whether we can write $U(F)$ as a union $\bigcup L_n$ of open subgroups with open amenable radical so that we may apply Theorem 1.3.5.

Notice that if one of the L_n is geometrically dense, then by the proof of Proposition 1.4.16 it has trivial amenable radical, hence does not have a trace if it is non-discrete. So if we would like to show that the L_n cannot all have open amenable radical, then it suffices to show that at least one is geometrically dense.

Remark 1.4.18. For any subgroup $F \leq S_d$, the group $U(F)$ is compactly generated. Hence, if we write $U(F) = \bigcup L_n$ as an *increasing* union of open subgroups $L_n \leq L_{n+1}$ then the sequence eventually terminates at some L_N. Since $U(F)$ is itself geometrically dense, this says that we can never write $U(F)$ as an increasing union of open subgroups which are not geometrically dense.

The condition on the action $F \curvearrowright \{1,\dots,d\}$ described in the following proposition is a strong converse to freeness.

Proposition 1.4.19. *Let $F \le S_d$ be a subgroup such that for every $l \in \{1,\dots,d\}$, the action of the stabilizer subgroup $\mathrm{St}_F(l) \curvearrowright \{1,\dots,l-1,l+1,\dots,d\}$ is transitive. If $U(F) = \bigcup L_n$ for some sequence $(L_n)_{n\ge0}$ of open subgroups, then there is n so that L_n is geometrically dense.*

We will need the following dynamical lemma.

Lemma 1.4.20. *Let F be as in Proposition 1.4.19, and fix a half-tree $Y \subseteq T_d$. Then the fixator $\mathrm{Fix}_{U(F)}(T_d \smallsetminus Y)$ acts minimally on ∂Y.*

Proof. Let b denote the root of Y, i.e., the unique vertex with degree $d-1$ in Y. Fix a legal labelling [21, Section 4] of T_d so that, without loss of generality, the deleted edge at b has the label d. We define a map $\varphi : \partial Y \to \{(j_n)_{n\ge1} \in \{1,\dots,d\}^{\mathbb{N}} \mid j_1 \neq d,\ j_n \neq j_{n+1} \text{ for all } n \ge 1\}$ by sending x to the sequence $\varphi(x) = (j_n)_{n\ge1}$, where j_n denotes the label of the n^{th} edge along the geodesic ray $[b,x)$ joining b to x.

The map φ is a bijection, and it is a homeomorphism when the codomain is equipped with the topology of point-wise convergence.

Now, given any two ends $x,y \in \partial Y$ with $\varphi(x) = (j_n)_{n\ge1}$ and $\varphi(y) = (i_n)_{n\ge1}$, we show how to produce a sequence $(g_n)_{n\ge1}$ in $\mathrm{Fix}_{U(F)}(T_d \smallsetminus Y)$ such that $g_n \cdot y \to x$.

Since $\mathrm{St}_F(d) \curvearrowright \{1,\dots,d-1\}$ is transitive, there is $h_1 \in \mathrm{Fix}_{U(F)}(T_d \smallsetminus Y)$ so that the first entry of $\varphi(h_1 \cdot y)$ is j_1. Similarly, if e_n is the n^{th} edge along $[b,x)$, and if $\varphi(x)$ and $\varphi((h_n \cdots h_1)\cdot y)$ agree on the first n entries, then by transitivity of $\mathrm{St}_F(j_n) \curvearrowright \{1,\dots,j_n-1,j_n+1,\dots,d\}$ there is $h_{n+1} \in \mathrm{Fix}_{U(F)}(T_d \smallsetminus Y)$ so that $\varphi(x)$ and $\varphi((h_{n+1}\cdots h_1)\cdot y)$ agree on the first $n+1$ entries.

Setting $g_n = h_n \cdots h_1 \in \mathrm{Fix}_{U(F)}(T_d \smallsetminus Y)$, we then have convergence $g_n \cdot y \to x$. This proves that the action is minimal. □

Proof of Proposition 1.4.19. Fix a vertex b. Then for each $n \ge 1$, the complement $T_d \smallsetminus B_n(b)$ is a disjoint union of finitely many (in fact $d(d-1)^{n-1}$) half-trees $Y_1,\dots,Y_k$. Note that $\partial Y_1,\dots,\partial Y_k$ form an open cover of ∂T_d.

Since the union $\bigcup L_n$ is equal to $U(F)$, then there is some N so that L_N contains a hyperbolic element h, and since L_N is assumed to be open, then it must contain the fixator $\mathrm{Fix}_{U(F)}(B_n(b))$ for some $n \ge 1$.

Let $a_h \in \partial T_d$ be the attracting point of h, and without loss of generality assume it is in ∂Y_1. Let $x \in \partial Y_i$ be any other end, then by Lemma 1.4.20 we may assume x is not the repelling point of h. Hence we may find m large enough that $h^m x \in \partial Y_1$ as ∂Y_1 is open, and again by Lemma 1.4.20 we may send this end to a_h.

This proves the action of L_N is transitive on ∂T_d, a similar argument shows that the set of ends arising as attracting points of hyperbolic elements in L_N is all of ∂T_d. Since any L_N-invariant subtree must contain the axes of all hyperbolic elements, then there are no proper L_N-invariant subtrees. This proves L_N is geometrically dense. □

Example 1.4.21. For $d \ge 4$, the alternating subgroup $A_d \le S_d$ satisfies the hypotheses of the previous result. Indeed, fix $l \in \{1,\dots,d\}$ and pick distinct $i,j \in \{1,\dots,l-1,l+1,\dots,d\}$. Then, since d is at least 4, there is $k \neq i,j,l$ hence the 3-cycle (ijk) fixes l and is in A_d. Moreover, (ijk) sends i to j, proving that the point stabilizer of A_d at l is transitive.

Chapter 2

Nonunital operator systems and noncommutative convexity

2.1 Introduction

Werner's abstract notion of a generalized (i.e. potentially nonunital) operator system is an axiomatic, representation-independent characterization of concrete generalized operator systems, which are self-adjoint subspaces of bounded operators acting on a Hilbert space. Werner [52] showed that every concrete generalized operator system satisfies the axioms of an abstract generalized operator system, and conversely that every abstract generalized operator system is isomorphic to a concrete generalized operator system, thereby generalizing an important result of Choi and Effros [11] for unital operator systems.

Recently, Davidson and Kennedy [15] introduced a theory of noncommutative convex sets and noncommutative functions. A key starting point for the theory is the dual equivalence between the category of compact noncommutative convex sets and the category of closed unital operator systems. On the one hand, this equivalence allows the rich theory of unital operator systems and unital C*-algebras to be applied to problems in noncommutative convexity. On the other hand, recent results suggest that that the perspective of noncommutative convexity can also provide new insight on unital operator systems and C*-algebras (see e.g. [14, 16, 32]).

In this chapter we will establish a similar dual equivalence between the category of generalized operator systems in the sense of Werner and a category of objects that we call pointed noncommutative convex sets. These are certain pairs consisting of a compact noncommutative convex set along with a distinguished point in the set. We will then consider a number of applications of this equivalence.

Before stating our results, we will first briefly review some of the basic ideas from the theory of noncommutative convexity.

A compact nc (noncommutative) convex set is a graded set $K = \sqcup K_n$, where each graded component K_n is an ordinary compact convex subset of the set $\mathcal{M}_n(E)$ of $n \times n$ matrices over an operator space E, and the graded components are related by requiring that K is closed under direct sums and compressions. The union is taken over all $n \leq \kappa$ for some sufficiently large infinite cardinal number κ depending on K. The fact that κ is infinite is an essential part of the theory, being necessary for e.g. the existence of extreme points. If E is separable, then it typically suffices to take $\kappa = \aleph_0$.

The conditions on K are equivalent to requiring that K is closed under nc convex combinations, meaning that $\sum \alpha_i^* x_i \alpha_i \in K_n$ for every bounded family of points $\{x_i \in K_{n_i}\}$ and every family of scalar matrices $\{\alpha_i \in \mathcal{M}_{n_i,n}\}$ satisfying $\sum \alpha_i^* \alpha_i = 1_n$.

The prototypical example of a compact nc convex set is the nc state space of a unital operator system S defined by $K = \sqcup K_n$, where $K_n = \mathrm{UCP}(S, \mathcal{M}_n)$ is the set of unital completely positive maps from S into the space $\mathcal{M}_n$ of $n \times n$ matrices. The dual equivalence in [15] implies that S is isomorphic to the unital operator system $\mathrm{A}(K)$ of continuous affine nc functions on K, and that, on the other hand, if K is a compact nc convex set, then K is affinely homeomorphic to the nc state space of the unital operator system $\mathrm{A}(K)$. In particular, every compact nc convex set arises as the nc state space of a unital operator system.

For a generalized operator system S, it is necessary to instead consider the nc quasistate space of S. This is the pair (K, z) consisting of the compact nc convex set $K = \sqcup K_n$, where $K_n = \mathrm{CCP}(S, \mathcal{M}_n)$ is the set of completely contractive and completely positive maps from S into $\mathcal{M}_n$, and $z \in K_1$ is the zero map.

We are therefore led to consider pairs (K, z) consisting of a compact nc convex set K and a distinguished point $z \in K_1$. However, it turns out that not every pair (K, z) arises as the nc quasistate space of a generalized operator system. This is an important point that explains many of the difficulties that arise in the non-unital setting. In order to obtain the desired dual equivalence between generalized operator systems and pointed compact nc convex sets, it is necessary to impose an additional constraint.

Specifically, we say that the pair (K, z) is a pointed compact nc convex set if the generalized operator system $\mathrm{A}(K, z) \subseteq \mathrm{A}(K)$ consisting of continuous affine nc functions on K that vanish at z has nc quasistate space (K, z). Our results will imply that this property is equivalent to (K, z) arising as the state space of a compact nc convex set.

We consider pointed compact nc convex sets and functions on pointed compact nc convex sets in Section 2.3 and Section 2.5 respectively. The following two results establishing the above-mentioned dual equivalence are the main results in Section 2.4.

Theorem 2.1.1. *A generalized operator system S with nc quasistate space (K, z) is isomorphic to the generalized operator system $\mathrm{A}(K, z) \subseteq \mathrm{A}(K)$ of continuous affine nc functions on K that vanish at z. Hence (K, z) is a pointed compact nc convex set if and only if it arises as the nc quasistate space of a generalized operator system.*

Theorem 2.1.1 is the key ingredient in the dual equivalence between the category of generalized operator systems and the category of pointed compact nc convex sets.

Theorem 2.1.2. *The category* GenOpSys *of generalized operator systems is dually equivalent to the category* PoNCConv *of pointed compact nc convex sets.*

An important consequence of Theorem 2.1.1 is that essentially all of the results from [15] about unital operator systems apply to generalized operator systems. For example, in Section 2.6, we establish characterizations of the maximal and minimal C*-covers of a generalized operator system in terms of the C*-algebra of continuous nc functions on its nc quasistate space. As a corollary, we recover results about the minimal C*-cover (i.e. the C*-envelope) recently obtained by Connes and van Suijlekom [13].

Theorem 2.1.3. *Let (K, z) be a pointed compact nc convex set.*

1. *The C*-algebra* $\mathrm{C}(K, z)$ *of pointed continuous nc functions on* (K, z) *is the maximal C*-cover of* $\mathrm{A}(K, z)$.

2. *Let* $I_{\overline{\partial K}}$ *denote the boundary ideal in the C*-algebra* $\mathrm{C}(K)$ *of continuous nc functions on* K *relative to the unital operator system* $\mathrm{A}(K)$*, so that the C*-algebra* $\mathrm{C}(K)/I_{\overline{\partial K}} \cong \mathrm{C}(\overline{\partial K})$ *is the minimal unital C*-cover of* $\mathrm{A}(K)$*, and let* $I_{(\overline{\partial K}, z)} = I_{\overline{\partial K}} \cap \mathrm{C}(K, z)$*. Then the C*-algebra* $\mathrm{C}(K, z)/I_{(\overline{\partial K}, z)}$ *is the minimal C*-cover of* $\mathrm{A}(K, z)$.

In Section 2.8, as another application of the dual equivalence between generalized operator systems and pointed compact nc convex sets, we develop a theory of quotients of generalized operator systems that extends the theory of quotients of unital operator systems developed by Kavruk, Paulsen, Todorov and Tomforde [29].

Theorem 2.1.4. *Let S be a generalized operator system and let $J \subseteq S$ be the kernel of a completely contractive and completely positive map on S. There is a unique pair $(S/J, \varphi)$ consisting of a generalized operator system S/J and a morphism $\varphi : S \to S/J$ with the property that whenever T is a generalized operator system and $\psi : S \to T$ is a completely contractive and completely positive map with $J \subseteq \ker \psi$, then ψ factors through φ. In other words, there is a completely contractive and completely positive map $\omega : S/J \to T$ such that $\psi = \omega \circ \varphi$.*

We also obtain some results that are new even for unital operator systems. In Section 2.9, we establish a characterization of generalized operator systems that are C*-simple, meaning that their minimal C*-cover is simple. We refer to Section 2.6 for the definition of the spectral topology.

Theorem 2.1.5. *A generalized operator system S with nc quasistate space (K, z) is C*-simple if and only if the closed nc convex hull of any nonzero point in the spectral closure of ∂K contains $\partial K \setminus \{z\}$.*

In Section 2.10, we establish a characterization of generalized operator systems that are isomorphic to C*-algebras in terms of their nc quasistate spaces, extending a result for unital operator systems from [32].

Theorem 2.1.6. *Let S be a generalized operator system with nc quasistate space (K, z). Then S is a C*-algebra if and only if K is an nc Bauer simplex and z is an extreme point. The result also holds for unital operator systems with nc quasistate spaces replaced by nc state spaces.*

In Section 2.11, we make another connection to the recent work of Connes and van Suijlekom [13]. They consider generalized operator systems S and T that are stably equivalent in the sense that the minimal tensor product $S \otimes_{\min} \mathcal{K}$ is isomorphic to the minimal tensor product $T \otimes_{\min} \mathcal{K}$, where $\mathcal{K}$ denotes the C*-algebra of compact operators. The next result is a characterization of stable equivalence of generalized operator systems in terms of their nc quasistate spaces.

Theorem 2.1.7. *Let S and T be generalized operator systems with nc quasistate spaces (K, z) and (L, w) respectively. Let $0_{\mathcal{K}}$ and $\mathrm{id}_{\mathcal{K}}$ denote the zero map and the identity representation respectively of $\mathcal{K}$. Then S and T are stably isomorphic if the closed nc convex hulls of $\partial K \otimes \{0_{\mathcal{K}}, \mathrm{id}_{\mathcal{K}}\}$ and $\partial L \otimes \{0_{\mathcal{K}}, \mathrm{id}_{\mathcal{K}}\}$ are pointedly affinely homeomorphic with respect to the points $z \otimes 0_{\mathcal{K}}$ and $w \otimes 0_{\mathcal{K}}$ (see Section 2.11).*

Finally, in Section 2.12 we establish the following characterization of second countable locally compact groups with property (T), extending a result from [32] for discrete groups acting on unital C*-algebras, as well as a result of Glasner and Weiss from [22] for second countable locally compact groups acting on unital commutative C*-algebras.

Theorem 2.1.8. *A second countable locally compact group G has Kazhdan's property (T) if and only if for every action of the group on a C*-algebra, the set of invariant quasistates is the quasistate space of a C*-algebra. The result also holds for unital C*-algebras with quasistate spaces replaced by state spaces.*

2.2 Preliminaries

In this section we will recall the notion of a matrix ordered operator space and introduce the notion of a generalized (i.e. potentially nonunital) operator system. For a reference on operator spaces and unital operator systems, we refer the reader to the books of Paulsen [44] and Pisier [45].

Let E be a self-adjoint operator space, i.e. such that $E = E^*$. We let $E_h = \{x \in E : x = x^*\}$ denote the set of self-adjoint elements in E. For $n \in \mathbb{N}$, we will write $\mathcal{M}_n(E)$ for the operator space of $n \times n$ matrices over E, and we will write $\mathcal{M}_n$ for $\mathcal{M}_n(\mathbb{C})$. A *matrix cone* over E is a family $P = (P_n)_{n\in\mathbb{N}}$ of closed subsets $P_n \subseteq M_n(E)_h$ such that

1. $P_n \cap -P_n = 0$ for all $n \in \mathbb{N}$ and
2. $AP_nA^* \subseteq P_m$ for all $A \in M_{m,n}$ and $m, n \in \mathbb{N}$.

Definition 2.2.1. A *matrix ordered operator space* is a pair (E, P) consisting of a *-vector space E and a matrix cone P over E. For $n \in \mathbb{N}$, an element in $\mathcal{M}_n(E)$ is *positive* if it belongs to P_n.

Remark 2.2.2. When referring to a matrix ordered operator space, we will typically omit the positive cone unless we need to refer to it explicitly. Note that if E is a matrix ordered operator space, then for $m \in \mathbb{N}$, the space $\mathcal{M}_m(E)$ is a matrix ordered operator space in a canonical way. Specifically, letting P denote the matrix cone for E, $(\mathcal{M}_m(E), Q)$ is a matrix ordered operator space, where $Q = (Q_n)_{n\in\mathbb{N}}$ is the matrix cone defined by identifying $\mathcal{M}_m(\mathcal{M}_n(E))$ with $\mathcal{M}_{mn}(E)$ in the obvious way and setting $Q_n = P_{mn}$.

Let E be a matrix ordered operator space. An element $e \in E$ is an *archimedean order unit* for E if for every $x \in E_h$, there is a scalar $\alpha > 0$ such that $-\alpha e \le x \le \alpha e$, and if $x + \alpha e \ge 0$ for all $\alpha > 0$, then $x \ge 0$. It is an *archimedean matrix order unit* for E if for every $n \in \mathbb{N}$, $1_n \otimes e$ is an archimedean order unit for $\mathcal{M}_n(E)$.

If E is a matrix ordered operator space, then an archimedean matrix order unit $e \in E$ induces a norm $\|\cdot\|_e$ on $\mathcal{M}_n(E)$ for each $n \in \mathbb{N}$, defined by

$$\|x\|_e = \inf\left\{\alpha > 0 : \begin{pmatrix} \alpha 1_n \otimes e & x \\ x^* & \alpha 1_n \otimes e \end{pmatrix} \ge 0\right\} \quad \text{for} \quad x \in \mathcal{M}_n(E).$$

The next definition is equivalent to the definition of a unital operator system given by Choi and Effros [11].

Definition 2.2.3. A *unital operator system* S is a complete matrix ordered operator space with an archimedean matrix order unit 1_S that is *distinguished* in the sense that for each n, the norm on $\mathcal{M}_n(S)$ coincides with the norm $\|\cdot\|_{1_S}$ from above.

Remark 2.2.4. Although not strictly necessary, it will be convenient for the purposes of this chapter to assume that unital operator systems are complete. If S is a unital operator system, then the distinguished archimedean order unit 1_S is uniquely determined by the property that for $s \in S$ with $s \geq 0$, $\|s\| \leq 1$ if and only if $s \leq 1_S$.

Let (E, P) and (F, Q) be matrix ordered operator spaces and let $\varphi : E \to F$ be a bounded map. We will write $\varphi_n : \mathcal{M}_n(E) \to \mathcal{M}_n(F)$ for the linear map defined by $\varphi_n = \mathrm{id}_n \otimes \varphi$.

Definition 2.2.5. Let (E, P) and (F, Q) be matrix ordered operator spaces. A linear map $\varphi : E \to F$ is *contractive* if $\|\varphi\| \leq 1$, and *completely contractive* if φ_n is contractive for all $n \in \mathbb{N}$. It is *isometric* if $\|\varphi(x)\| = \|x\|$ for all $x \in E$, and *completely isometric* if $\|\varphi_n(x)\| = \|x\|$ for all $n \in \mathbb{N}$ and all $x \in \mathcal{M}_n(E)$. Similarly, it is *positive* if $\varphi(P_1) \subseteq Q_1$, and *completely positive* if φ_n is positive for each $n \in \mathbb{N}$. The map φ is a *complete order isomorphism* if it is completely positive and invertible with a completely positive inverse. It is a *complete order embedding* if it is completely positive and invertible on its range with a completely positive inverse.

Remark 2.2.6. For unital operator systems, these definitions agree with the usual definitions. Furthermore, because the norm on a unital operator system is completely determined by the matrix order, a unital map between unital operator systems is completely isometric if and only if it is a complete order embedding. However, this is not true for arbitrary matrix ordered operator spaces (see [52]).

We will write UnOpSys for the category of unital operator systems with unital completely positive maps (equivalently, unital complete order homomorphisms) as morphisms. We will refer to unital complete order isomorphisms as *isomorphisms*, and to unital complete order embeddings as *embeddings*.

Choi and Effros [11, Theorem 4.4] showed that every unital operator system is isomorphic to a concrete unital operator system, meaning that there is a unital completely isometric map into some $\mathcal{B}(H)$, where $\mathcal{B}(H)$ denotes the C*-algebra of bounded linear operators acting on a Hilbert space H. We will be interested in matrix ordered operator spaces satisfying an appropriate analogue of this property.

Specifically, we are interested in matrix ordered operator spaces with a completely isometric complete order embedding into some $\mathcal{B}(H)$. It turns out that not every matrix ordered operator space has this property. Following Connes and van Suijlekom [13], we will make use of Werner's [52] characterization of matrix ordered operator spaces with this property in terms of partial unitizations (see below), although other characterizations are also known (see e.g. [47]).

The next definition is [52, Definition 4.1] (see also [13, Definition 2.11]).

Definition 2.2.7. Let E be a matrix ordered operator space. The *partial unitization* of E is the matrix ordered operator space $(E^\sharp, P)$, where $E^\sharp = E \oplus \mathbb{C}$ and the matrix cone $P = (P_n)$ is defined by specifying that for each $n \in \mathbb{N}$, $P_n \subseteq \mathcal{M}_n(E^\sharp)_h = \mathcal{M}_n(E)_h \oplus (\mathcal{M}_n)_h$ consists of all pairs $(x, \alpha) \in \mathcal{M}_n(E)_h \oplus (\mathcal{M}_n)_h$ satisfying

$$\alpha \geq 0 \text{ and } \varphi(\alpha_\epsilon^{-1/2} x \alpha_\epsilon^{-1/2}) \geq -1 \text{ for all } \epsilon > 0 \text{ and } \varphi \in \mathrm{CCP}(E, \mathcal{M}_n),$$

where $\alpha_\epsilon = \alpha + \epsilon 1_n$ and $\mathrm{CCP}(E, \mathcal{M}_n)$ denotes the space of completely contractive and completely positive maps from E to $\mathcal{M}_n$. We will refer to the map $E \to E^\sharp : x \to (x, 0)$ as the *canonical inclusion map*, and we will refer to the map $E^\sharp \to \mathbb{C} : (x, \alpha) \to \alpha$ as the *projection onto the scalar summand.*

The next result is contained in [52, Section 4] (see also [13, Proposition 2.12] and [13, Lemma 2.13]).

Theorem 2.2.8. *Let E be a matrix ordered operator space.*

1. *The partial unitization $E^\sharp$ is a unital operator system.*
2. *Let $\iota : E \to E^\sharp$ denote the canonical inclusion map and let $\tau : E^\sharp \to \mathbb{C}$ denote the projection onto the scalar summand. Then ι is completely contractive and completely positive and τ is unital and positive, and the following sequence is split exact:*
$$0 \longrightarrow E \xrightarrow{\ \iota\ } E^\sharp \xrightarrow{\ \tau\ } \mathbb{C} \longrightarrow 0\,.$$
3. *Let F be a matrix ordered operator space and let $\varphi : E \to F$ be a completely contractive and completely positive map. Then the unitization $\varphi^\sharp : E^\sharp \to F^\sharp$ defined by $\varphi^\sharp((x, \alpha)) = (\varphi(x), \alpha)$ for $(x, \alpha) \in E^\sharp$ is unital and completely positive. Furthermore, if φ is a completely isometric complete order isomorphism then $\varphi^\sharp$ is a unital complete order isomorphism.*

Remark 2.2.9. Note that $E \neq E^\sharp$, even if E is already unital. For a C*-algebra A, the partial unitization $A^\sharp$ coincides with the usual C*-algebraic unitization of A, and hence is a unital C*-algebra.

It follows from the representation theorem of Choi and Effros [11] for unital operator systems that if E is a matrix ordered operator space with partial unitization $E^\sharp$ and the canonical inclusion map $E \to E^\sharp$ is completely isometric, then there is a completely isometric complete order isomorphism of E onto a self-adjoint subspace of bounded operators acting on a Hilbert space. Following [13], this motivates the following definition.

Definition 2.2.10. We will say that a complete matrix ordered operator space S is a *generalized operator system* if the canonical inclusion map $S \to S^\sharp$ is completely isometric, in which case we will refer to $S^\sharp$ as the *unitization* of S.

Remark 2.2.11. As in the unital case, it is not strictly necessary to assume that generalized operator systems are complete. For a generalized operator system S, we will identify S with its image in $S^\sharp$ under the canonical inclusion map. In particular, if T is a generalized operator system and $\varphi : S \to T$ is completely contractive and completely positive, then we will view the unitization $\varphi^\sharp : S^\sharp \to T^\sharp$ as an extension of φ.

Remark 2.2.12. If S is a unital operator system, then it follows from [52, Lemma 4.9] that the identity map on S factors through the canonical inclusion map $S^\sharp$. In particular, this implies that the canonical inclusion map is completely isometric, so S is a generalized operator system in the sense of Definition 2.2.10.

Remark 2.2.13. Let S and T be generalized operator systems and let $\varphi : S \to T$ be a completely contractive completely positive map. If φ is a completely isometric complete order isomorphism, then Theorem 2.2.8 implies that the unitization $\varphi^\sharp : S^\sharp \to T^\sharp$ is a complete order isomorphism. However, if φ is merely a completely isometric complete order embedding, then it is not necessarily true that the unitization $\varphi^\sharp$ is a complete order embedding (see Example 2.2.14). We will need to take this into account when we define embeddings between generalized operator systems below.

In the following example, we construct generalized operator systems S and T and a completely isometric complete order embedding $\varphi : S \to T$ such that the unitization $S^\sharp \to T^\sharp$ is not a complete order embedding. The fundamental issue is that completely contractive completely positive maps on the image of S in T do not necessarily extend to completely contractive completely positive maps on T (see [47, Section 6]).

Example 2.2.14. Define $a, b \in \mathcal{M}_2$ by

$$a = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad b = \begin{bmatrix} 1 & 0 \\ 0 & -1/2 \end{bmatrix}.$$

Let $S = \operatorname{span}\{a\}$ and $T = \operatorname{span}\{1_2, b\} \cong \mathbb{C}^2$. Then S is a non-unital generalized operator system and T is a unital C*-algebra. Define $\varphi : S \to T$ by $\varphi(\alpha a) = \alpha b$ for $\alpha \in \mathbb{C}$. We claim that φ is a completely isometric complete order embedding, but that the unitization $\varphi^\sharp : S^\sharp \to B^\sharp$ is not completely isometric.

Note that $\mathcal{M}_n(S) = \operatorname{span}\{\alpha \otimes a : \alpha \in \mathcal{M}_n\}$. Since

$$\|\varphi(\alpha \otimes a)\| = \|\alpha \otimes b\| = \|\alpha\| = \|\alpha \otimes a\|,$$

φ is completely isometric. Also, $\alpha \otimes a \geq 0$ if and only if $\alpha \otimes b \geq 0$ if and only if $\alpha = 0$, so α is a complete order embedding.

It is not difficult to see that for $\lambda \in [-1, 1]$ the map $\varphi_\lambda : S \to \mathbb{C}$ defined by $\varphi_\lambda(\alpha a) = \lambda\alpha$ for $\alpha \in \mathbb{C}$ is a quasistate, i.e. is completely contractive and completely positive. Furthermore, if $\psi : S \to \mathbb{C}$ is a quasistate, then $\psi = \varphi_\lambda$ for some $\lambda \in [-1, 1]$. Hence the set of quasistates on S can be identified with $[-1, 1]$.

We will see in Section 2.4.4 that this implies that the state space of the unitization $S^\sharp$ is $[-1, 1]$. Since $[-1, 1]$ is a simplex, it follows from a classical result of Bauer that $S^\sharp = \operatorname{span}\{1_2, a\} \cong \mathbb{C}^2$ (see e.g. [32]).

Note that $T^\sharp \cong \mathbb{C}^3$. We can identify $T = \mathbb{C}^2$ with the first two coordinates of $\mathbb{C}^3$. Then $\varphi^\sharp(\alpha 1_2 + \beta a) = \alpha 1_3 + \beta b$. In particular, $\varphi^\sharp(\frac{1}{2} 1_2 + a) = \frac{1}{2} 1_3 + b$. Since $\frac{1}{2} 1_2 + a \ngeq 0$ but $\frac{1}{2} 1_3 + b \geq 0$, it follows that $\varphi^\sharp$ is not a complete order embedding.

We will write GenOpSys for the category of generalized operator systems with completely contractive and completely positive maps as morphisms. We will refer to completely isometric complete order isomorphisms as *isomorphisms*. Motivated by Remark 2.2.13, for generalized operator systems S and T, we will refer to a completely isometric complete order embedding $\varphi : S \to T$ as an *embedding* if the unitization $\varphi^\sharp : S^\sharp \to T^\sharp$ is an embedding in the category of unital operator systems.

Werner was able to isolate the precise obstruction to a matrix ordered operator space being a generalized operator system in the sense of Definition 2.2.10. The next result is [52, Lemma 4.8].

Theorem 2.2.15. *Let E be a matrix ordered operator space with partial unitization $E^\sharp$. For each n, let $\nu_n : \mathcal{M}_n(E) \to \mathbb{R}_{\geq 0}$ denote the map defined by*

$$\nu_n(x) = \sup_{\varphi} \left| \varphi \begin{pmatrix} 0 & x \\ x^* & 0 \end{pmatrix} \right|, \quad \text{for} \quad x \in \mathcal{M}_n(E),$$

where the supremum is taken over all maps $\varphi \in \mathrm{CCP}(\mathcal{M}_{2n}(E), \mathbb{C})$. Then ν_n is a norm on $\mathcal{M}_n(E)$. The inclusion $E \to E^\sharp$ is completely isometric if and only if for each n, the norm on $\mathcal{M}_n(E)$ coincides with ν_n.

2.3 Pointed noncommutative convex sets

A key result from [15] is the dual equivalence between the category of unital operator systems and the category of compact nc convex sets. In this section we will review the definition of a compact nc convex set and introduce the definition of a pointed compact nc convex set. In Section 2.4, we will show that the category of operator systems is dual to the category of pointed compact nc convex sets.

2.3.1 Noncommutative convex sets

Let E be an operator space. For nonzero (potentially infinite) cardinals m and n, let $\mathcal{M}_{m,n}(E)$ denote the operator space of $m \times n$ matrices over X with the property that the set of finite submatrices are uniformly bounded. For brevity, we will write $\mathcal{M}_n(E)$ for $\mathcal{M}_{n,n}(E)$, $\mathcal{M}_{m,n}$ for $\mathcal{M}_{m,n}(\mathbb{C})$ and $\mathcal{M}_n$ for $\mathcal{M}_n(\mathbb{C})$. Restricting to matrices with uniformly bounded finite submatrices ensures that matrices over E can be multiplied on the left and right by scalar matrices of the appropriate size. We identify $\mathcal{M}_n$ with the C*-algebra of bounded operators acting on a Hilbert space H_n of dimension n.

If E is a dual operator space with distinguished predual E_*, then there is a natural operator space isomorphism $\mathcal{M}_n(E) \cong \mathrm{CB}(E_*, \mathcal{M}_n)$, where $\mathrm{CB}(E_*, \mathcal{M}_n)$ denotes the space of completely bounded maps from E_* to $\mathcal{M}_n$. We equip $\mathcal{M}_n(E)$ with the corresponding point-weak* topology.

Let $\mathcal{M}(E) = \sqcup_n \mathcal{M}_n(E)$, where the union is taken over all nonzero cardinal numbers n. Once again, for brevity, we will write $\mathcal{M}$ for $\mathcal{M}(\mathbb{C})$. Although $\mathcal{M}(E)$ is a proper class and not a set, we will only be interested in subsets, so this will not present any set-theoretic difficulties. More generally, we will consider disjoint unions over nonzero cardinal numbers n of subsets of $\mathcal{M}_n(E)$. For a subset $X \subseteq \mathcal{M}(E)$ and a cardinal number n, we will write X_n for the graded component $X_n = X \cap \mathcal{M}_n(E)$.

Definition 2.3.1. Let E be an operator space. An *nc convex set* over E is a graded subset $K = \sqcup_n K_n$ with $K_n \subseteq \mathcal{M}_n(E)$ that is closed under direct sums and compressions, meaning that

1. $\sum \alpha_i x_i \alpha_i^* \in K_n$ for every bounded family of points $x_i \in K_{n_i}$ and every family of isometries $\alpha_i \in \mathcal{M}_{n,n_i}$ satisfying $\sum \alpha_i \alpha_i^* = 1_n$.

2. $\beta^* x \beta \in K_n$ for every $x \in K_m$ and every isometry $\beta \in \mathcal{M}_{m,n}$.

If E is a dual operator space, so that each $\mathcal{M}_n(E)$ is equipped with the weak* topology discussed above, then we will say that K is *closed* if each K_n is closed. Similarly, we will say that K is *compact* if each K_n is compact.

The most important examples of compact nc convex sets are noncommutative state spaces of operator systems. The next definition is [15, Example 2.2.6].

Definition 2.3.2. Let S be a unital operator system. The *nc state space* of S is the set $K = \sqcup_n K_n$ defined by $K_n = \mathrm{UCP}(S, \mathcal{M}_n)$. Here, $\mathrm{UCP}(S, \mathcal{M}_n)$ denotes the space of unital completely positive maps from S to $\mathcal{M}_n$. Elements in K are referred to as *nc states* on S.

Remark 2.3.3. Note that the set K is nc convex and compact since each $\mathrm{UCP}(S, \mathcal{M}_n)$ is compact.

The following characterization of compact nc convex sets as sets that are closed under nc convex combinations is often useful. In particular, it makes the analogy between nc convex sets and ordinary convex sets more explicit. The next result is [15, Proposition 2.2.8].

Proposition 2.3.4. *Let E be a dual operator space and let $K = \sqcup_n K_n$ for closed subsets $K_n \subseteq \mathcal{M}_n(E)$. Then K is nc convex if and only if it is closed under* nc convex combinations, *meaning that $\sum \alpha_i^* x_i \alpha_i \in K_n$ for every bounded family of points $x_i \in K_{n_i}$ and every family $\alpha_i \in \mathcal{M}_{n_i,n}$ satisfying $\sum \alpha_i^* \alpha_i = 1_n$.*

One of the most important justifications for the utility of noncommutative convexity is the fact that there is a robust notion of extreme point for which a noncommutative analogue of the Krein-Milman theorem [15, Theorem 6.4.2] holds, meaning that every compact nc convex set is generated by its extreme points.

Definition 2.3.5. Let K be a compact nc convex set. A point $x \in K_n$ is *extreme* if whenever x is written as a finite nc convex combination $x = \sum \alpha_i^* x_i \alpha_i$ for $\{x_i \in K_{n_i}\}$ and nonzero $\{\alpha_i \in \mathcal{M}_{n_i,n}\}$ satisfying $\sum \alpha_i^* \alpha_i = 1_n$, then each α_i is a positive scalar multiple of an isometry $\beta_i \in \mathcal{M}_{n_i,n}$ satisfying $\beta_i^* x_i \beta_i = x$ and each x_i decomposes with respect to the range of α_i as a direct sum $x_i = y_i \oplus z_i$ for $y_i, z_i \in K$ with y_i unitarily equivalent to x. The set of all extreme points is $\partial K = \sqcup (\partial K)_n$.

The morphism between nc convex sets are the continuous affine noncommutative maps. The next definition is [15, Definition 2.5.1].

Definition 2.3.6. Let K and L be compact nc convex sets. A map $\theta : K \to L$ is an *nc map* if it is graded, respects direct sums and is unitarily equivariant, meaning that

1. $\theta(K_n) \subseteq L_n$ for all n,
2. $\theta(\sum \alpha_i x_i \alpha_i^*) = \sum \alpha_i \theta(x_i) \alpha_i^*$ for every bounded family $\{x_i \in K_{n_i}\}$ and every family of isometries $\{\alpha_i \in \mathcal{M}_{n_i,n}\}$ satisfying $\sum \alpha_i^* \alpha_i = 1_n$,
3. $\theta(\alpha^* x \alpha) = \alpha^* \theta(x) \alpha$ for every $x \in K_m$ and every unitary $\alpha \in \mathcal{M}_n$.

An nc map θ is *affine* if, in addition, it is equivariant with respect to isometries, meaning that

3'. $\theta(\alpha^* x \alpha) = \alpha^* \theta(x) \alpha$ for every $x \in K_m$ and every isometry $\alpha \in \mathcal{M}_{m,n}$.

An affine nc map θ is *continuous* if the restriction $f|_{K_n}$ is continuous with respect to the point-strong* topology on K_n and L_n for each n. It is *bounded* if $\|\theta\|_\infty < \infty$, where $\|\theta\|_\infty$ denotes the uniform norm $\|\theta\|_\infty = \sup_{x\in K} \|\theta(x)\|$. Finally, θ is an *(affine) homeomorphism* and K and L are *(affinely) homeomorphic* if θ is continuous and has a continuous (affine) nc inverse.

Remark 2.3.7. We will consider an appropriate notion of continuity for more general nc maps in Section 2.5.

We will write NCConv for the category of compact nc convex sets with continuous affine nc maps as morphisms. We will refer to affine nc homeomorphisms as *isomorphisms*, and to injective continuous affine nc maps as *embeddings*.

The next definition is [15, Definition 3.2.1].

Definition 2.3.8. Let K be a compact nc convex set. We will write $\mathrm{A}(K)$ for the unital operator system of all continuous affine nc functions from K to $\mathcal{M}$.

Remark 2.3.9. The fact that $\mathrm{A}(K)$ is a unital operator system is discussed in [15, Section 3.2].

For a point $x \in K_n$, the corresponding evaluation map $\mathrm{A}(K) \to \mathcal{M}_n : a \to a(x)$ is an nc state on $\mathrm{A}(K)$. Moreover, by [15, Theorem 3.2.2], every nc state on $\mathrm{A}(K)$ is given by evaluation at some point in K (we will say more about this in Section 2.4). It will be convenient to identify points in K with the corresponding nc state on $\mathrm{A}(K)$.

Following [15, Section 3.2], for each n we will identify the unital operator system $\mathcal{M}_n(\mathrm{A}(K))$ with the space of continuous affine nc maps from K to $\mathcal{M}_n(\mathcal{M})$ in the obvious way.

2.3.2 Pointed noncommutative convex sets

In this section we introduce the notion of a pointed compact nc convex set, of which the most important examples will be nc quasistate spaces of operator systems. Before introducing the definition of a pointed compact nc convex set, we require the definition of a pointed continuous affine nc function.

Definition 2.3.10. Let (K, z) be a pair consisting of a compact nc convex set K and a point $z \in K_1$. We will say that a continuous affine nc function $a \in \mathrm{A}(K)$ is *pointed* if $a(z) = 0$. We let $\mathrm{A}(K, z) \subseteq \mathrm{A}(K)$ denote the space of pointed continuous affine nc functions on K.

Remark 2.3.11. The space $\mathrm{A}(K, z)$ is a matrix ordered operator space with matrix cone $P = \sqcup P_n$ inherited from $\mathrm{A}(K)$. Specifically, for $n \in \mathbb{N}$, the positive cone on P_n consists of the positive functions in $\mathcal{M}_n(\mathrm{A}(K, z))$. Since $\mathrm{A}(K, z)$ is a closed self-adjoint subspace of the unital operator system $\mathrm{A}(K)$, it follows that $\mathrm{A}(K, z)$ is a generalized operator system.

The most important examples of pointed compact nc convex sets will be nc quasistate spaces of generalized operator systems. The idea to utilize nc quasistate spaces in this setting was inspired by the importance of the quasistate space of a non-unital C*-algebra.

Definition 2.3.12. Let S be a generalized operator system. The *nc quasistate space* of S is the pair (K, z), where $K = \sqcup_n K_n$ is defined by $K_n = \mathrm{CCP}(S, \mathcal{M}_n)$ and $z \in K_1$ is the zero map. Here, $\mathrm{CCP}(S, \mathcal{M}_n)$ denotes the space of completely contractive and completely positive maps from S to $\mathcal{M}_n$. We will refer to elements of K as *nc quasistates* on S.

Remark 2.3.13. Note that the set K is nc convex and compact since each $\mathrm{CCP}(S, \mathcal{M}_n)$ is compact.

We are now ready to introduce the definition of a pointed compact nc convex set.

Definition 2.3.14. Let (K, z) be a pair consisting of a compact nc convex set K and a point $z \in K_1$. We will say that (K, z) is a *pointed compact nc convex set* if every nc quasistate on the generalized operator system $\mathrm{A}(K, z)$ belongs to K, i.e. is evaluation at a point in K.

Remark 2.3.15. Since K is the nc state space of the unital operator system $\mathrm{A}(K)$, (K, z) is a pointed compact nc convex set if and only if every nc quasistate on $\mathrm{A}(K, z)$ extends to an nc state on $\mathrm{A}(K)$.

By definition, a pointed compact nc convex set is the nc quasistate space of a generalized operator system. In Section 2.4, we will show that the nc quasistate space of every generalized operator system is a pointed compact nc convex set. The proof of this fact is non-trivial. However, we are now able to give some examples.

Example 2.3.16. Define $K = \sqcup K_n$ by

$$K_n = \{\alpha \in (\mathcal{M}_n)_h : -1_n \leq \alpha \leq 1_n\}, \quad \text{for} \quad n \in \mathbb{N}.$$

Then K is a compact nc convex set (see [15, Example 2.2.4]). Let $z = 0$. We will show that the pair (K, z) is a pointed compact nc convex set.

The unital operator system $\mathrm{A}(K)$ is given by $\mathrm{A}(K) = \mathrm{span}\{1_{\mathrm{A}(K)}, a\}$, where $a \in \mathrm{A}(K, z)$ is the coordinate function $a(\alpha) = \alpha$ for $\alpha \in K$. Hence $\mathrm{A}(K, z) = \mathrm{span}\{a\}$. In fact, $\mathrm{A}(K, z)$ is isomorphic to the non-unital generalized operator system S from Example 2.2.14.

If $\theta : \mathrm{A}(K, z) \to \mathcal{M}_n$ is an nc quasistate, then there is a self-adjoint $\beta \in \mathcal{M}_n$ with $-1_n \leq \beta \leq 1_n$ such that $\theta(\alpha a) = \alpha\beta$ for $\alpha \in \mathbb{C}$. Conversely, it is easy to check that every self-adjoint $\beta \in \mathcal{M}_n$ with $-1_n \leq \beta \leq 1_n$ gives rise to an nc quasistate on $\mathrm{A}(K, z)$ of this form. Hence the nc quasistate space of $\mathrm{A}(K, z)$ is K. Therefore, (K, z) is a pointed compact nc convex set.

Note that $\mathrm{A}(K, z)^\sharp = \mathrm{A}(K)$. In Corollary 2.4.7, we will show that this property characterizes pointed compact nc convex sets.

The next example shows that not every pair (K, z) consisting of a compact nc convex set and a point $z \in K_1$ is a pointed compact nc convex set.

Example 2.3.17. Define $K = \sqcup K_n$ by

$$K_n = \{\alpha \in (\mathcal{M}_n)_h : -\tfrac{1}{2}1_n \leq \alpha \leq 1_n\}, \quad \text{for} \quad n \in \mathbb{N}.$$

Then as in Example 2.3.16, K is a compact nc convex set. Let $z = 0$. We will show that the pair (K, z) is not a pointed compact nc convex set.

The unital operator system $\mathrm{A}(K)$ is given by $\mathrm{A}(K) = \mathrm{span}\{1_{\mathrm{A}(K)}, b\}$, where $b \in \mathrm{A}(K,z)$ is the coordinate function $b(\alpha) = \alpha$ for $\alpha \in K$. Hence $\mathrm{A}(K,z) = \mathrm{span}\{b\}$. In fact, $\mathrm{A}(K)$ is isomorphic to the C*-algebra B from Example 2.2.14.

Define $\theta : \mathrm{A}(K,z) \to \mathbb{C}$ by $\theta(\alpha b) = -\alpha$. Since $\mathrm{A}(K,z)$ does not contain any positive elements, the matrix cone of $\mathrm{A}(K,z)$ is zero, so it is easy to check that θ is an nc quasistate. However, θ does not extend to an nc state on $\mathrm{A}(K)$ since $\frac{1}{2}1_{\mathrm{A}(K)} + b \geq 0$, while $\frac{1}{2} + \theta(b) = -\frac{1}{2} \not\geq 0$. Hence θ does not belong to K and (K,z) is not a pointed compact nc convex set.

We will now establish a geometric characterization of pointed compact nc convex sets.

Proposition 2.3.18. *Let (K,z) be a pair consisting of a compact nc convex set K and a point $z \in K_1$. Then (K,z) is pointed if and only if whenever self-adjoint $a \in \mathcal{M}_m(\mathrm{A}(K,z))$ satisfies $a(x) \leq 1_m \otimes 1_n$ for all $x \in K_n$, then $\theta_m(a) \leq 1_m \otimes 1_p$ for all nc quasistates $\theta : \mathrm{A}(K,z) \to \mathcal{M}_p$.*

Proof. If (K,z) is pointed, then every nc quasistate on $\mathrm{A}(K,z)$ belongs to K, so the condition trivially holds. Conversely, suppose that (K,z) is not pointed. Then there is an nc quasistate $\theta : \mathrm{A}(K,z) \to \mathcal{M}_n$ such that $\theta \notin K$. Identifying K with its image in $\mathcal{M}(\mathrm{A}(K,z)^*)$ and viewing θ as a point in $\mathcal{M}_n(\mathrm{A}(K,z)^*)$, the nc separation theorem [15, Theorem 2.4.1] implies there is a self-adjoint element $a \in \mathcal{M}_n(\mathrm{A}(K,z))$ such that $\theta(a) \not\leq 1_n \otimes 1_n$ but $a(x) \leq 1_n \otimes 1_p$ for all $x \in K_p$. □

Example 2.3.17 is a single instance of a general class of examples.

Corollary 2.3.19. *Let (K,z) be a pair consisting of a compact nc convex set and a point $z \in K_1$ such that the matrix cone for $\mathrm{A}(K,z)$ is zero. Then (K,z) is pointed if and only if whenever $x \in K_n$ satisfies $\alpha z^{(n)} + (1-\alpha)x \in K_n$ for $0 < \alpha < 1$, then $\alpha z^{(n)} - (1-\alpha)x \in K_n$. Here $z^{(n)} \in K_n$ denotes the direct sum of n copies of z.*

Proof. Suppose that (K,z) is pointed and $x \in K_n$ satisfies $\alpha z^{(n)} + (1-\alpha)x \in K_n$ for $0 < \alpha < 1$. Then since the positive cone of $\mathrm{A}(K,z)$ is zero, the map $\theta : \mathrm{A}(K,z) \to \mathcal{M}_n$ defined by $\theta(a) = \alpha a(z^{(n)}) - (1-\alpha)a(x) = -(1-\alpha)a(x)$ for $a \in \mathrm{A}(K,z)$ is an nc quasistate. Hence θ is given by evaluation at a point in K which must be $\alpha z^{(n)} - (1-\alpha)x$. Hence $\alpha z^{(n)} - (1-\alpha)x \in K$.

Conversely, suppose that whenever $x \in K_n$ satisfies $\alpha z^{(n)} + (1-\alpha)x \in K_n$ for $0 < \alpha < 1$, then $\alpha z^{(n)} - (1-\alpha)x \in K_n$. If self-adoint $a \in \mathcal{M}_m(\mathrm{A}(K,z))$ satisfies $a(x) \leq 1_m \otimes 1_n$ for all $x \in K_n$, then for $0 < \alpha < 1$,

$$(1-\alpha)a(x) = a(\alpha z^{(n)} + (1-\alpha)x) \leq 1_m \otimes 1_n$$

and

$$-(1-\alpha)a(x) = a(\alpha z^{(n)} - (1-\alpha)x) \leq 1_m \otimes 1_n.$$

Then taking $\alpha \to 0$ implies

$$-1_m \otimes 1_n \leq a(x) \leq 1_m \otimes 1_n.$$

Hence $\|a\|_\infty \leq 1$. It follows that if $\theta : \mathrm{A}(K,z) \to \mathcal{M}_n$ is an nc quasistate on $\mathrm{A}(K)$, then $\theta(a) \leq 1_m \otimes 1_n$. Therefore, by Proposition 2.3.18, (K,z) is pointed. □

Example 2.3.20. Let K denote the nc state space of $\mathcal{M}_2$ and let $z = \mathrm{tr}$, where $\mathrm{tr} \in K_1$ denotes the normalized trace. Then identifying $\mathcal{M}_2$ with $\mathrm{A}(K)$,

$$\mathrm{A}(K, z) = \left\{ \begin{bmatrix} \alpha & \beta \\ \gamma & -\alpha \end{bmatrix} : \alpha, \beta, \gamma \in \mathbb{C} \right\}.$$

The matrix cone of $\mathrm{A}(K, z)$ is clearly zero. Define $\theta : \mathcal{M}_2 \to \mathbb{C}$ by

$$\theta\left(\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \right) = \alpha \quad \text{for} \quad \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \in \mathcal{M}_2.$$

Then $\frac{1}{2}\mathrm{tr} + \frac{1}{2}\theta \in K_1$. But $\frac{1}{2}\mathrm{tr} - \frac{1}{2}\theta \notin K_1$. Hence by Corollary 2.3.19, the pair (K, z) is not pointed.

We now define the category of pointed compact nc convex sets.

Definition 2.3.21. Let (K, z) and (L, w) be pointed nc convex sets. We will say that an affine nc map $\theta : K \to L$ is *pointed* if $\theta(z) = w$. We will say that (K, z) and (L, w) are *pointedly affinely homeomorphic* if there is a pointed affine homeomorphism from (K, z) to (L, w).

We will write PoNCConv for the category of pointed compact nc convex sets with pointed continuous affine nc maps as morphisms. We will refer to pointed affine nc homeomorphisms as *isomorphisms*, and to pointed injective continuous affine nc maps as *embeddings*.

2.4 Categorical duality

In this section we will prove the dual equivalence between the category GenOpSys of generalized operator systems and the category PoNCConv of pointed compact nc convex sets. We begin by reviewing the details of the dual equivalence between the category UnOpSys of unital operator systems and the category NCConv of compact nc convex sets from [15].

2.4.1 Categorical duality for unital operator systems

The dual equivalence between the category of unital operator systems and the category of compact nc convex sets was developed in [15, Section 3]. It is closely related to a similar dual equivalence established by Webster and Winkler [51].

The next result combines [15, Theorem 3.2.2] and [15, Theorem 3.2.3].

Theorem 2.4.1. *Let K be a compact nc convex set. The nc state space of the unital operator system $\mathrm{A}(K)$ is isomorphic to K. For a unital operator system S with nc state space K, the map $S \to \mathrm{A}(K) : s \to \hat{s}$ defined by*

$$\hat{s}(x) = x(s) \quad \textit{for} \quad s \in S,\ x \in K$$

is an isomorphism.

The dual equivalence between the category UnOpSys of unital operator systems and the category NCConv of compact nc convex sets follows from Theorem 2.4.1. The contravariant functor UnOpSys → NCConv is defined in the following way:

1. A unital operator system S is mapped to its nc state space.
2. For unital operator systems S and T with nc state spaces K and L respectively, a morphism $\varphi : S \to T$ is mapped to the morphism $\varphi^d : L \to K$ defined by

$$\varphi^d(y)(a) = \varphi(a)(y), \quad \text{for} \quad y \in L \text{ and } a \in \mathrm{A}(K).$$

The inverse functor NCConv → UnOpSys is defined in the following way:

1. A compact nc convex set K is mapped to the unital operator system $\mathrm{A}(K)$.
2. If K and L are compact nc convex sets and $\psi : L \to K$ is a morphism, then the corresponding morphism $\psi^d : \mathrm{A}(K) \to \mathrm{A}(L)$ is defined by

$$\psi^d(a)(y) = a(\psi(y)), \quad \text{for} \quad a \in \mathrm{A}(K) \text{ and } y \in L.$$

The next result summarizes this discussion. It is [15, Theorem 3.2.5].

Theorem 2.4.2. *The contravariant functors* UnOpSys → NCConv *and* NCConv → UnOpSys *defined above are inverses. Hence the categories* UnOpSys *and* NCConv *are dually equivalent.*

We will make use of the following result in the next section.

Proposition 2.4.3. *Let K and L be compact nc convex sets. Let $\varphi : \mathrm{A}(K) \to \mathrm{A}(L)$ be a unital completely positive map and let $\varphi^d : L \to K$ denote the continuous affine map obtained by applying Theorem 2.4.2 to φ. Then φ is completely isometric if and only if φ^d is surjective.*

Proof. If φ^d is surjective, then for $a \in \mathcal{M}_n(\mathrm{A}(K))$,

$$\|\varphi(a)\|_\infty = \sup_{y \in L} \|\varphi(a)(y)\|_\infty = \sup_{y \in L} \|a(\varphi^d(y))\|_\infty = \sup_{x \in K} \|a(x)\| = \|a\|_\infty.$$

Hence φ is completely isometric.

Conversely, suppose that φ is completely isometric. Let $S = \varphi(\mathrm{A}(K))$. Then S is a unital operator system. Let M denote the nc state space of S, so that S is isomorphic to $\mathrm{A}(M)$. It follows from Arveson's extension theorem that the restriction map $r : L \to M$ is surjective. Let $\psi : M \to K$ denote the continuous affine nc map obtained by applying Theorem 2.4.2 to S. Then $\varphi^d = \psi \circ r$. Theorem 2.4.2 implies that ψ is an affine homeomorphism. Since r is surjective, it follows that φ^d is surjective. □

2.4.2 Categorical duality for generalized operator systems

Let (K, z) be a pair consisting of a compact nc convex set K and a point $z \in K_1$. Observe that for a point $x \in K$, viewed as a unital completely positive map on $\mathrm{A}(K)$, the restriction $x|_{\mathrm{A}(K,z)}$ is an nc quasistate. For brevity, it will be convenient to simultaneously view points in K as nc states on $\mathrm{A}(K)$ and nc quasistates on $\mathrm{A}(K, z)$. We will take care to ensure that this does not cause any confusion.

If (K, z) is the nc quasistate space of a generalized operator system S, then it follows as in [52, Lemma 4.9] that the extension $x^\sharp : S^\sharp \to \mathcal{M}_n$ defined by $x^\sharp(s, \alpha) = x(s) + \alpha 1_n$ is unital and completely positive, and hence is an nc state on $S^\sharp$. Moreover, it is the unique extension of x to an nc state on $S^\sharp$ with range in $\mathcal{M}_n$. Here we have identified $S^\sharp$ with $S \oplus \mathbb{C}$ as in Definition 2.2.7. Note that $S = \ker z^\sharp$.

Proposition 2.4.4. *Let S be a generalized operator system with nc quasistate space (K, z) and let L denote the nc state space of the unitization $S^\sharp$. For an nc quasistate $x \in K$, let $x^\sharp \in L$ be the nc state defined as above. Then the map $K \to L : x \to x^\sharp$ is an affine homeomorphism with inverse given by the restriction map $L \to K : y \to y|_S$. Hence $S^\sharp$ is isomorphic to $\mathrm{A}(K)$.*

Proof. For $x \in K$, we have already observed that $x^\sharp \in L$. On the other hand, for $y \in L$, the restriction $y|_S$ is completely contractive and completely positive, so $y|_S \in K$. Then by uniqueness, $(y|_S)^\sharp = y$. It follows that the map $K \to L : x \to x^\sharp$ is a bijection with inverse given by the restriction map.

It is clear that the restriction map from L to K is continuous and affine. From above, the restriction to each L_n is a continuous bijection onto K_n. Since L_n is compact, it follows that this restriction is a homeomorphism. Hence the restriction map is a homeomorphism.

The fact that $S^\sharp$ is isomorphic to $\mathrm{A}(K)$ now follows from Theorem 2.4.1. □

Theorem 2.4.5. *Let S be a generalized operator system with nc quasistate space (K, z). Then S is isomorphic to $\mathrm{A}(K, z)$.*

Proof. By Proposition 2.4.4, we can identify the nc state space of the unitization $S^\sharp$ with K. Let $\varphi^\sharp : S^\sharp \to A(K)$ denote the isomorphism from Theorem 2.4.1. Then for $s \in S^\sharp$, $\varphi^\sharp(s) = \hat{s}$, where $\hat{s} : K \to \mathcal{M}$ is the affine nc function defined by $\hat{s}(x) = x^\sharp(s)$ for $x \in K$. In particular, for $s \in S$, $\varphi^\sharp(s)(z) = \hat{s}(z) = z^\sharp(s) = 0$, so $\varphi^\sharp(S) \subseteq \mathrm{A}(K, z)$. Hence restricting $\varphi^\sharp$ to S, we obtain a map $\varphi : S \to \mathrm{A}(K, z)$. Since $\varphi^\sharp$ is an isomorphism, φ is completely positive and completely isometric. It remains to show that φ is a surjective complete order isomorphism.

To see that φ is surjective, choose $a \in \mathrm{A}(K, z)$. By the surjectivity of $\varphi^\sharp$, there is $s \in S^\sharp$ such that $\varphi^\sharp(s) = a$. Then $0 = a(z) = \hat{s}(z) = z^\sharp(s)$. Hence $s \in S$, and we conclude that φ is surjective.

To see that φ is a complete order isomorphism, let $P = \sqcup P_n$ and $Q = \sqcup Q_n$ denote the matrix cones of S and $\mathrm{A}(K, z)$ respectively. If φ is not a complete order isomorphism, then there is $s \in \mathcal{M}_n(S)$ such that $s \notin P_n$ but $\varphi(s) \in Q_n$. Suppose that this is the case. We will apply a separation argument to obtain a contradiction.

Identify S with its image under the canonical embedding into its bidual S^{**} and define $M \subseteq \mathcal{M}(S^{**})$ by $M = \overline{P} = \sqcup \overline{P_m}$, where the closure is taken with respect to the weak* topology. Since P is nc convex, M is nc convex. Hence M is a weak* closed nc convex set. Furthermore, since P_n is convex and uniformly closed, it is weakly closed, implying $s \notin M$. Therefore, by the nc separation theorem [15, Theorem 2.4.1] there is a self-adjoint normal completely bounded linear map $\psi : S^{**} \to \mathcal{M}_n$ such that $\psi(s) \ngeq -1_n \otimes 1_n$ but $\psi(t) \geq -1_n \otimes 1_p$ for all $t \in M_p$.

Since ψ is normal, it can be identified with the unique normal extension of a map $\psi : S \to \mathcal{M}_n$ satisfying $\psi(s) \ngeq -1_n \otimes 1_n$ but $\psi(t) \geq -1_n \otimes 1_p$ for all $t \in P_p$. Then in particular, $\psi(s) \ngeq 0$. However, since P is closed under multiplication by positive scalars, for $t \in P_p$ and $\alpha > 0$, $\psi(t) \geq -\alpha^{-1} 1_n \otimes 1_p$. Taking $\alpha \to \infty$ implies $\psi(t) \geq 0$. Hence $\psi \geq 0$. Multiplying ψ by a sufficiently small positive scalar, we obtain a quasistate $x \in K$ such that $x(s) \ngeq 0$. But then $\hat{s}(x) = x(s) \ngeq 0$, so $\varphi(s) = \hat{s} \ngeq 0$, contradicting the assumption that $\varphi(s) \in Q_n$. □

Corollary 2.4.6. *Let S be a generalized operator system with nc quasistate space (K, z). Then (K, z) is a pointed compact nc convex set.*

Proof. By Theorem 2.4.5, we can identify S with the generalized operator system $\mathrm{A}(K,z)$, and by definition, every nc quasistate on $\mathrm{A}(K,z)$ belongs to K. □

Corollary 2.4.7. *Let (K,z) be a pair consisting of a compact nc convex set K and a point $z \in K_1$. The following are equivalent:*

1. *The pair (K,z) is a pointed compact nc convex set.*

2. *The nc quasistate space of the generalized operator system $\mathrm{A}(K,z)$ is (K,z).*

3. *The generalized operator system $\mathrm{A}(K,z)$ satisfies $\mathrm{A}(K,z)^\sharp = \mathrm{A}(K)$.*

Proof. 1. ⇒ 2. If (K,z) is a pointed compact nc convex set then by definition every nc quasistate on $\mathrm{A}(K,z)$ belongs to K. Since every point in K is an nc quasistate on $\mathrm{A}(K,z)$, it follows that the nc quasistate space of $\mathrm{A}(K,z)$ is (K,z).

2. ⇒ 3. If the nc quasistate space of $\mathrm{A}(K,z)$ is (K,z), then Proposition 2.4.4 implies that the nc state space of $\mathrm{A}(K,z)^\sharp$ is K. It follows from Theorem 2.4.2 that $\mathrm{A}(K,z)^\sharp = \mathrm{A}(K)$.

3. ⇒ 1. If $\mathrm{A}(K,z)^\sharp = \mathrm{A}(K)$, then since every nc quasistate on $\mathrm{A}(K,z)$ extends to an nc state on $\mathrm{A}(K,z)^\sharp$, and since K is nc state space of $\mathrm{A}(K)$, it follows that every nc quasistate of $\mathrm{A}(K,z)$ belongs to K. Hence (K,z) is a pointed compact nc convex set. □

The next result follows immediately from Theorem 2.4.5 and Corollary 2.4.7. It is an analogue of the representation theorem [15, Theorem 3.2.3].

Theorem 2.4.8. *Let S be a generalized operator system with nc quasistate space (K,z). The map $S^\sharp \to \mathrm{A}(K) : s \to \hat{s}$ defined by*

$$\hat{s}(x) = x^\sharp(s) \quad \textit{for} \quad x \in K,$$

is a unital complete order isomorphism that restricts to a completely isometric complete order isomorphism from S to $\mathrm{A}(K,z)$. Hence S is isomorphic to $\mathrm{A}(K,z)$.

Theorem 2.4.5 and Corollary 2.4.7 imply the dual equivalence of the category GenOpSys of generalized operator systems and the category PoNCConv of pointed compact nc convex sets. The contravariant functor GenOpSys → PoNCConv is defined in the following way:

1. A generalized operator system S is mapped to its nc quasistate space.

2. For generalized operator systems S and T with nc quasistate spaces (K,z) and (L,w) respectively, a morphism $\varphi : S \to T$ is mapped to the morphism $\varphi^d : L \to K$ defined by

$$\varphi^d(y)(a) = \varphi(a)(y), \quad \text{for} \quad y \in L \text{ and } a \in \mathrm{A}(K,z).$$

The inverse functor PoNCConv → GenOpSys is defined in the following way:

1. A pointed compact nc convex set (K,z) is mapped to the generalized operator system $\mathrm{A}(K,z)$.

2. If (K,z) and (L,w) are compact nc convex sets and $\psi : L \to K$ is a morphism, then the corresponding morphism $\psi^d : \mathrm{A}(K,z) \to \mathrm{A}(L,w)$ is defined by

$$\psi^d(a)(y) = a(\psi(y)), \quad \text{for} \quad a \in \mathrm{A}(K,z) \text{ and } y \in L.$$

The next result summarizes this discussion.

Theorem 2.4.9. *The contravariant functors* GenOpSys → PoNCConv *and* PoNCConv → GenOpSys *defined above are inverses. Hence the categories* GenOpSys *and* PoNCConv *are dually equivalent.*

The next result characterizing isomorphic generalized operator systems is an analogue of [15, Corollary 3.2.6]. It follows immediately from Theorem 2.4.9.

Corollary 2.4.10. *Let (K,z) and (L,w) be compact pointed nc convex sets. Then $\mathrm{A}(K,z)$ and $\mathrm{A}(L,w)$ are isomorphic if and only if (K,z) and (L,w) are pointedly affinely homeomorphic. Hence two generalized operator systems are isomorphic if and only if their nc quasistate spaces are pointedly affinely homeomorphic.*

We saw in Example 2.2.14 that if S and T are generalized operator systems and $\varphi : S \to T$ is a completely contractive complete order embedding, then it is not necessarily true that the unitization $\varphi^\sharp : S^\sharp \to T^\sharp$ is completely isometric. In other words, φ is not necessarily an embedding. However, we can now state necessary and sufficient conditions for φ to be an embedding.

The following result follows immediately from Theorem 2.4.2, Theorem 2.4.9 and the discussion preceding the statements of these results.

Lemma 2.4.11. *Let (K,z) and (L,w) be pointed compact nc convex sets and let $\varphi : \mathrm{A}(K,z) \to \mathrm{A}(L,w)$ be a completely contractive and completely positive map. Let $\varphi^d : L \to K$ denote the corresponding continuous affine map defined as in Theorem 2.4.9. Then φ^d coincides with the continuous affine map obtained by applying Theorem 2.4.2 to the unitization $\varphi^\sharp : \mathrm{A}(K) \to \mathrm{A}(L)$.*

Corollary 2.4.12. *Let (K,z) and (L,w) be pointed compact nc convex sets. Let $\varphi : \mathrm{A}(K,z) \to \mathrm{A}(L,w)$ be a completely contractive and completely positive map and let $\varphi^d : L \to K$ denote the pointed continuous affine map given by applying Theorem 2.4.9 to φ. Then φ is an embedding if and only if φ^d is surjective.*

Proof. By Lemma 2.4.11, the map φ^d coincides with the map obtained by applying Theorem 2.4.2 to the unitization $\varphi^\sharp : \mathrm{A}(K) \to \mathrm{A}(L)$. By Proposition 2.4.3, $\varphi^\sharp$ is completely isometric if and only if φ^d is surjective. □

2.5 Pointed noncommutative functions

2.5.1 Noncommutative functions

In order to define a more general notion of continuous nc function, it is necessary to introduce the *point-strong* topology* on a compact nc convex set K. This is the weakest topology on each K_n making the maps $x \to a(x)\eta$ and $x \to a(x)^*\xi$ continuous for all $a \in \mathrm{A}(K)$ and all vectors $\xi, \eta \in H_n$.

The following definition is essentially [15, Definition 4.2.1].

Definition 2.5.1. Let K be a compact nc convex set. An *nc function* on K is an nc map $f : K \to \mathcal{M}$ in the sense of Definition 2.3.6. An nc function f is *continuous* if it is continuous with respect to the point-strong* topology on K from above. We will write $\mathrm{B}(K)$ and $\mathrm{C}(K)$ for the unital C*-algebras of bounded and continuous nc functions on K respectively.

Remark 2.5.2. It is clear that $\mathrm{A}(K) \subseteq \mathrm{C}(K) \subseteq \mathrm{B}(K)$. The product on $\mathrm{B}(K)$ is the pointwise product, meaning that for $f, g \in \mathrm{B}(K)$ and $x \in K$, $(fg)(x) = f(x)g(x)$. The adjoint is defined by $f^*(x) = f(x)^*$ for $f \in \mathrm{B}(K)$ and $x \in K$. By [15, Theorem 4.4.3], $\mathrm{C}(K) = \mathrm{C}^*(\mathrm{A}(K))$, i.e. $\mathrm{C}(K)$ is the C*-algebra generated by $\mathrm{A}(K)$. We will say more about $\mathrm{C}(K)$ in Section 2.6.

For $x \in K_n$, we will write $\delta_x : \mathrm{B}(K) \to \mathcal{M}_n$ for the point evaluation *-homomorphism defined by $\delta_x(f) = f(x)$ for $f \in \mathrm{B}(K)$. This is a noncommutative analogue of an evaluation functional, since for $f \in \mathrm{C}(K)$, $\delta_x(f) = f(x)$.

Elements in the enveloping von Neumann algebra $\mathrm{C}(K)^{**}$ can naturally be identified with bounded nc functions on K. Specifically, for $x \in K_n$, it follows from the universal property of $\mathrm{C}(K)^{**}$ as the enveloping von Neumann algebra of $\mathrm{C}(K)$ that the *-homomorphism $\delta_x : \mathrm{C}(K) \to \mathcal{M}_n$ has a unique extension to a normal *-homomorphism $\delta_x^{**} : \mathrm{C}(K)^{**} \to \mathcal{M}_n$. For $f \in \mathrm{C}(K)^{**}$, the function $\tilde{f} : K \to \mathcal{M}$ defined by $\tilde{f}(x) = \delta_x^{**}(f)$ for $x \in K$ is a bounded nc function and hence belongs to $\mathrm{B}(K)$. In fact, much more can be said.

The following result is contained in [15, Theorem 4.4.3] and [15, Corollary 4.4.4].

Theorem 2.5.3. *Let K be a compact nc convex set. The map $\sigma : \mathrm{C}(K)^{**} \to \mathrm{B}(K)$ defined as above is a normal *-isomorphism that restricts to a normal unital complete order isomorphism from $\mathrm{A}(K)^{**}$ onto the unital operator system $\mathrm{A}_b(K)$ of bounded affine nc functions.*

2.5.2 Pointed noncommutative functions

Definition 2.5.4. Let (K, z) be a pointed compact nc convex set. We will say that an nc function $f : K \to \mathcal{M}$ is *pointed* if $f(z) = 0$. We let $\mathrm{B}(K, z)$ denote the space of pointed bounded nc functions on K. Similarly, we let $\mathrm{C}(K, z) = \mathrm{C}(K) \cap \mathrm{B}(K, z)$ denote the space of pointed continuous nc functions on K.

Remark 2.5.5. It is clear that $\mathrm{B}(K, z)$ is a closed two-sided ideal of $\mathrm{B}(K)$ and that $\mathrm{C}(K, z)$ is a closed two-sided ideal of $\mathrm{C}(K)$. In particular, $\mathrm{B}(K, z)$ and $\mathrm{C}(K, z)$ are C*-algebras. Furthermore, it follows from the identification $\mathrm{C}(K)^{**} = \mathrm{B}(K)$ that the representation δ_z^{**} is normal on $\mathrm{B}(K)$. Hence $\mathrm{B}(K, z)$ is a weak*-closed ideal of $\mathrm{B}(K)$.

Proposition 2.5.6. *Let (K, z) be a pointed compact nc convex set. Then $\mathrm{C}(K, z)^\sharp = \mathrm{C}(K)$ and $\mathrm{C}(K, z) = \mathrm{C}^*(\mathrm{A}(K, z))$.*

Proof. By Corollary 2.4.7, $\mathrm{A}(K, z)^\sharp = \mathrm{A}(K)$. Hence $\mathrm{A}(K) = \mathrm{A}(K, z) + \mathbb{C}1_{\mathrm{A}(K)}$. Since $\mathrm{C}(K) = \mathrm{C}^*(\mathrm{A}(K))$, it follows that $\mathrm{C}(K) = \mathrm{C}(K, z) + \mathbb{C}$. Hence $\mathrm{C}(K, z) = \mathrm{C}^*(\mathrm{A}(K, z))$.

To see that $\mathrm{C}(K, z)^\sharp = \mathrm{C}(K)$, it suffices to show that for any *-homomorphism $\pi : \mathrm{C}(K, z) \to \mathcal{M}_n$, there is a unital *-homomorphism $\tilde{\pi} : \mathrm{C}(K) \to \mathcal{M}_n$ extending π. The restriction $\pi|_{\mathrm{A}(K,z)}$ is an nc quasistate, so by the assumption that (K, z) is pointed, it is given by evaluation at a point $x \in K_n$. Then the unital *-homomorphism $\delta_x : \mathrm{C}(K) \to \mathcal{M}_n$ extends $\pi|_{\mathrm{A}(K,z)}$. Since $\mathrm{A}(K,z)$ generates $\mathrm{C}(K,z)$, it follows that $\delta_x|_{\mathrm{C}(K,z)} = \pi$. □

The next result follows from restricting the *-isomorphism in the statement of Theorem 2.5.3.

Theorem 2.5.7. *Let (K,z) be a compact pointed nc convex set. Then the map $\mathrm{C}(K,z)^{**} \to \mathrm{B}(K,z) : f \to \tilde{f}$ defined by*

$$\tilde{f}(x) = \delta_x^{**}(f) \quad \text{for} \quad f \in \mathrm{C}(K,z)^{**} \text{ and } x \in K,$$

*is a normal *-isomorphism of von Neumann algebras that restricts to a normal completely isometric complete order isomorphism from $\mathrm{A}(K,z)^{**}$ onto the generalized operator system $\mathrm{A}_b(K,z)$ of pointed bounded affine nc functions.*

2.6 Minimal and maximal C*-covers

The deepest results in [15] arise from the interplay between unital operator systems of continuous affine nc functions on compact nc convex sets and unital C*-covers of nc functions on the sets. Connes and van Suijlekom [13] introduced an analogous notion of C*-cover for operator systems. In this section we will review the notion of a unital C*-cover of a unital operator system before considering the more general notion of a C*-cover of a generalized operator system.

2.6.1 Minimal and maximal unital C*-covers

Let S be a unital operator system.

1. A pair (A,ι) consisting of a unital C*-algebra A and an embedding $\iota : S \to A$ is a *unital C*-cover* of S if $A = \mathrm{C}^*(\iota(S))$.

2. If (A',ι') is another unital C*-cover of S, then we will say that (A,ι) and (A,ι') are *equivalent* if there is a unital *-isomorphism $\pi : A \to A'$ such that $\pi \circ \iota = \iota'$.

3. We will say that a unital C*-cover (A,ι) of S is *maximal* if for any unital C*-cover (B,φ) of S, there is a surjective unital *-homomorphism $\sigma : A \to B$ such that $\varphi = \sigma \circ \iota$.

$$\begin{array}{ccc} S & \xrightarrow{\ \iota\ } & A = \mathrm{C}^*(\iota(S)) \\ & \searrow^{\varphi} & \Big\downarrow \sigma \\ & & B = \mathrm{C}^*(\varphi(S)) \end{array}$$

4. We will say that a unital C*-cover (A,ι) of S is *minimal* if for any unital C*-cover (B,φ) of S, there is a surjective unital *-homomorphism $\pi : B \to A$ such that $\pi \circ \varphi = \iota$.

$$\begin{array}{ccc} S & \xrightarrow{\ \iota\ } & A = \mathrm{C}^*(\iota(S)) \\ & \searrow^{\varphi} & \Big\uparrow \pi \\ & & B = \mathrm{C}^*(\varphi(S)) \end{array}$$

The existence and uniqueness of the maximal unital C*-cover of a unital operator system was established by Kirchberg and Wassermann [34]. The following result is non-trivial. It is implied by [15, Theorem 4.4.3].

Theorem 2.6.1. *Let K be a compact nc convex set. The maximal unital C*-cover for the unital operator system* $\mathrm{A}(K)$ *is the C*-algebra* $\mathrm{C}(K)$ *of continuous nc functions on K.*

The existence and uniqueness of the minimal unital C*-cover of a unital operator system was established by Hamana [24]. The results in [15] and [32] imply a description in terms of the nc state space of the unital operator system, which we will now describe.

Let K be a compact nc convex set. It follows from Theorem 2.6.1 that there is a surjective *-homomorphism π from $\mathrm{C}(K)$ onto the minimal unital C*-cover of $\mathrm{A}(K)$. A result of Dritschel and McCullough [17] implies that $\ker \pi$ is the *boundary ideal* in $\mathrm{C}(K)$ relative to $\mathrm{A}(K)$, i.e. the unique largest ideal in $\mathrm{C}(K)$ with the property that the restriction of the corresponding quotient *-homomorphism to $\mathrm{A}(K)$ is completely isometric.

Let $I_{\overline{\partial K}} = \ker \pi$ and let $\mathrm{C}(\overline{\partial K}) = \mathrm{C}(K)/I_{\overline{\partial K}}$. We will refer to $\mathrm{C}(\overline{\partial K})$ as the minimal unital C*-cover of $\mathrm{A}(K)$. In order to explain this choice of notation and give a description of $\mathrm{C}(\overline{\partial K})$ in terms of K, we require the spectral topology from [32, Section 9].

Definition 2.6.2. Let K be a compact nc convex set. We will say that a point $x \in K$ is *reducible* if x is unitarily equivalent to a direct sum $x \simeq y \oplus z$ for points $y, z \in K$. We will say that x is *irreducible* if it is not reducible, and we will write $\mathrm{Irr}(K)$ for the set of irreducible points in K.

Remark 2.6.3. Note that a point $x \in K$ is irreducible if and only if the corresponding *-homomorphism δ_x is. In particular, $\partial K \subseteq \mathrm{Irr}(K)$, where ∂K denotes the set of all extreme points as in Definition 2.3.5.

Let K be a compact nc convex set. Let $\mathrm{Spec}(\mathrm{C}(K))$ denote the C*-algebraic spectrum of $\mathrm{C}(K)$, i.e. the set of unitary equivalence classes of irreducible representations of $\mathrm{C}(K)$ equipped with the hull-kernel topology. For a point $x \in \mathrm{Irr}(K)$, we have already observed that the *-homomorphism δ_x is irreducible. Hence letting $[\delta_x]$ denote the unitary equivalence class of δ_x, $[\delta_x] \in \mathrm{Spec}(\mathrm{C}(K))$. Note that the map $\mathrm{Irr}(K) \to \mathrm{Spec}(\mathrm{C}(K)) : x \to [\delta_x]$ is surjective.

Definition 2.6.4. The *spectral topology* on $\mathrm{Irr}(K)$ is the pullback of the hull-kernel topology on $\mathrm{Spec}(\mathrm{C}(K))$. Specifically, the open subsets of $\mathrm{Irr}(K)$ are the preimages of open subsets of $\mathrm{Spec}(\mathrm{C}(K))$ under the map $\mathrm{Irr}(K) \to \mathrm{Spec}(\mathrm{C}(K)) : x \to [\delta_x]$.

The results in [32, Section 9] imply that

$$I_{\overline{\partial K}} = \{f \in \mathrm{C}(K) : f(x) = 0 \text{ for all } x \in \overline{\partial K}\},$$

where $\overline{\partial K}$ denotes the closure of ∂K with respect to the spectral topology on $\mathrm{Irr}(K)$.

The minimal unital C*-cover $\mathrm{C}(\overline{\partial K}) = \mathrm{C}(K)/I_{\overline{\partial K}}$ can be viewed as a noncommutative generalization of the classical Shilov boundary. For this reason, the ideal $\ker \pi$ is sometimes referred to as the Shilov ideal (see e.g. [32]).

2.6.2 Minimal and maximal C*-covers

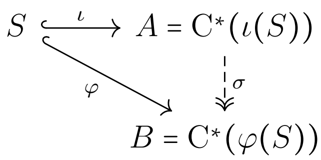
Connes and van Suijlekom [13] introduced an analogue for generalized operator systems of a unital C*-cover of a unital operator system from Section 2.6, which they refer to as a C$^\sharp$-cover. We will instead refer to C*-covers.

Definition 2.6.5. Let S be a generalized operator system.

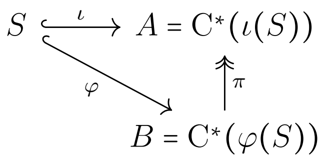
1. We will say that a pair (A, ι) consisting of a C*-algebra A and an embedding $\iota : S \to A$ is a *C*-cover* of S if $A = \mathrm{C}^*(\iota(S))$.

2. If (A', ι') is another C*-cover of S, then we will say that (A, ι) and (A, ι') are *equivalent* if there is a *-isomorphism $\pi : A \to A'$ such that $\pi \circ \iota = \iota'$.

3. We will say that a C*-cover (A, ι) of S is *maximal* if for any C*-cover (B, φ) of S there is a surjective *-homomorphism $\sigma : A \to B$ such that $\varphi = \sigma \circ \iota$.



4. We will say that a C*-cover (A, ι) of S is *minimal* if for any C*-cover (B, φ) of S, there is a surjective *-homomorphism $\pi : B \to A$ such that $\pi \circ \varphi = \iota$.



Remark 2.6.6. Let (K, z) be a pointed compact nc convex set. If (A, ι) is a C*-cover for $\mathrm{A}(K, z)$, then since φ is an embedding, the unitization $\varphi^\sharp : \mathrm{A}(K) \to A^\sharp$ is an embedding. Hence $(A^\sharp, \iota^\sharp)$ is a unital C*-cover of $\mathrm{A}(K)$.

The existence and uniqueness of the minimal C*-cover of a generalized operator system was established in [13, Theorem 2.2.5] under the name C$^\sharp$-envelope. In this section we will prove the existence and uniqueness of the maximal C*-cover, and we will describe the maximal and minimal C*-covers of a generalized operator system in terms of the maximal and minimal unital C*-covers of its unitization.

Proposition 2.6.7. *Let S be a generalized operator system. If the maximal and minimal C*-covers of S exist, then they are unique up to equivalence.*

Proof. Let (A, ι) and (A', ι') be maximal C*-covers for S. Then by definition there are surjective homomorphisms $\sigma : A \to A'$ and $\sigma' : A' \to A$ such that $\iota' = \sigma \circ \iota$ and $\iota = \sigma' \circ \iota'$. Hence $\sigma^{-1} = \sigma'$, so σ is a *-isomorphism and hence (A, ι) and (A, ι') are equivalent. The proof for the minimal C*-cover is similar. $\square$

Theorem 2.6.8. *Let (K, z) be a compact pointed nc convex set.*

1. *The C*-algebra $\mathrm{C}(K, z)$ is a maximal C*-cover for $\mathrm{A}(K, z)$ with respect to the canonical inclusion.*

2. *Let $I_{\overline{\partial K}}$ denote the boundary ideal in the C*-algebra* $\mathrm{C}(K)$ *of continuous nc functions on K relative to* $\mathrm{A}(K)$*, so that the quotient* $\mathrm{C}(K)/I_{\overline{\partial K}}$ *is the minimal unital C*-cover of* $\mathrm{A}(K)$*, and let* $I_{(\overline{\partial K},z)} = I_{\overline{\partial K}} \cap \mathrm{C}(K,z)$*. Then the C*-algebra* $\mathrm{C}(K,z)/I_{(\overline{\partial K},z)}$ *is the minimal C*-cover of* $\mathrm{A}(K,z)$ *with respect to the quotient *-homomorphism. In particular, the C*-algebra generated by the image of* $\mathrm{A}(K,z)$ *under the canonical embedding of* $\mathrm{A}(K)$ *into* $\mathrm{C}(\overline{\partial K})$ *is isomorphic to* $\mathrm{C}(\overline{\partial K},z)$*.*

Proof. 1. Let (B,φ) be a C*-cover for $\mathrm{A}(K,z)$. We can assume that $B \subseteq \mathcal{M}_n$ for some n, so that $\varphi = x$ and $B = \delta_x(\mathrm{C}(K,z))$ for some $x \in K_n$. It follows that $\mathrm{C}(K,z)$ is a maximal C*-cover for $\mathrm{A}(K,z)$ with respect to the canonical inclusion.

2. Since $\mathrm{C}(K)/I_{\overline{\partial K}}$ is a unital C*-cover for $\mathrm{A}(K)$, $\mathrm{C}(K,z)/I_{(\overline{\partial K},z)}$ is a C*-cover for $\mathrm{A}(K,z)$. To see that it is minimal, it suffices to show that if (B,φ) is any C*-cover for $\mathrm{A}(K,z)$, then $\ker\sigma \subseteq I_{(\overline{\partial K},z)}$.

By 1., there is a surjective unital *-homomorphism $\sigma : \mathrm{C}(K,z) \to B$ such that $\sigma|_{\mathrm{A}(K,z)} = \varphi$. The unitization $\sigma^\sharp : \mathrm{C}(K) \to B^\sharp$ is a unital *-homomorphism satisfying $\sigma^\sharp|_{\mathrm{A}(K)} = \varphi^\sharp$. Since φ is an embedding, $\varphi^\sharp$ is completely isometric, so $\ker\sigma^\sharp \subseteq I_{\overline{\partial K}}$. Hence $\ker\sigma \subseteq I_{(\overline{\partial K},z)}$. ☐

Definition 2.6.9. Let (K,z) be a pointed compact nc convex set. Let $I_{(\overline{\partial K},z)}$ denote the ideal in $\mathrm{C}(K,z)$ from Theorem 2.6.8 and let $\mathrm{C}(\overline{\partial K},z) = \mathrm{C}(K,z)/I_{(\overline{\partial K},z)}$. We will refer to $\mathrm{C}(\overline{\partial K},z)$ as the *minimal C*-cover* of $\mathrm{A}(K,z)$, and we will refer to the corresponding quotient *-homomorphism as the *canonical embedding* of $\mathrm{A}(K,z)$ into $\mathrm{C}(\overline{\partial K},z)$.

Remark 2.6.10. The ideal $I_{(\overline{\partial K},z)} = \ker\pi$ is a pointed analogue of the boundary ideal from Section 2.6.1. It is the largest ideal in $\mathrm{C}(K,z)$ such that the corresponding quotient *-homomorphism restricts to an embedding of $\mathrm{A}(K,z)$.

Example 2.6.11. Define $a,b \in \mathcal{M}_2$ by

$$a = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad b = \begin{bmatrix} 1 & 0 \\ 0 & -1/2 \end{bmatrix}.$$

Let $S = \mathrm{span}\{a\}$ and $T = \mathrm{span}\{b\}$. Then S and T are non-unital generalized operator systems, and it is not difficult to verify that S and T are isomorphic to the non-unital generalized operator systems considered in Example 2.3.16 and Example 2.3.17 respectively.

Let (K,z) denote the nc quasistate space of S. Note that this is the same (K,z) from Example 2.3.16. Since $K_1 = [-1,1]$ is a simplex, the results in [32] imply that $\partial K = \partial K_1 = \{-1,1\}$. Hence identifying S with $\mathrm{A}(K,z)$, the minimal C*-cover of $S^\sharp = \mathrm{A}(K)$ is $\mathrm{C}(\overline{\partial K}) = \mathrm{C}(\{-1,1\}) \cong \mathbb{C}^2$.

Let $\iota : \mathrm{A}(K) \to \mathrm{C}(\overline{\partial K})$ denote the canonical embedding. Then $\iota(\mathrm{A}(K,z)) \cong \{(-\alpha,\alpha) : \alpha \in \mathbb{C}\} \cong \mathbb{C}$. Hence $\mathrm{C}(\overline{\partial K},z) \cong \mathbb{C}$. Note that $\mathrm{C}(\overline{\partial K},z)$ is unital even though $\mathrm{A}(K,z)$ is nonunital.

Define $\theta : S \to T$ defined by $\theta(\alpha a) = \alpha b$ for $\alpha \in \mathbb{C}$. Then arguing as in Example 2.3.17, θ is an isomorphism. Hence the minimal C*-cover of T is also isomorphic to $\mathbb{C}^2$.

Example 2.6.12. Let A be a C*-algebra with nc quasistate space (K,z). Then A is clearly a C*-cover of itself with respect to the identity map. By definition, there is a surjective *-homomorphism $\pi : A \to \mathrm{C}(\overline{\partial K}, z)$ that is completely isometric on A. Therefore, π is a *-isomorphism, implying $A = \mathrm{C}(\overline{\partial K}, z)$.

Let K be a compact nc convex set. A useful fact implied by [14, Theorem 3.4] and [15, Proposition 5.2.4] is that the direct sum of the points in ∂K extends to a faithful representation of the minimal unital C*-cover $\mathrm{C}(\overline{\partial K})$. Specifically, define $y \in K$ by $y = \oplus_{x \in \partial k} x$. Then the *-homomorphism δ_y satisfies $\ker \delta_y = \ker I_{\overline{\partial K}}$. Hence $\delta_y(\mathrm{C}(K)) \cong \mathrm{C}(\overline{\partial K})$. The following result is an analogue of this fact for the minimal C*-cover of a generalized operator system.

Proposition 2.6.13. *Let (K, z) be a pointed compact nc convex set. Define $y \in K$ by $y = \oplus_{x \in \partial K \setminus \{z\}} x$. Then the *-homomorphism δ_y satisfies $\ker \delta_y = I_{(\overline{\partial K}, z)}$, where $I_{(\overline{\partial K}, z)}$ is the ideal from Theorem 2.6.8. Hence $\delta_y(\mathrm{C}(K, z)) \cong \mathrm{C}(\overline{\partial K}, z)$.*

Proof. From above, $(\delta_y \oplus \delta_z)(\mathrm{C}(K)) \cong \mathrm{C}(\overline{\partial K})$. So considered as a *-representation of $\mathrm{C}(K)$, $\ker(\delta_y \oplus \delta_z) = I_{\overline{\partial K}}$. By Theorem 2.6.8, $\ker(\delta_y \oplus \delta_z) \cap \mathrm{C}(K, z) = I_{(\overline{\partial K}, z)}$. Since δ_y is zero on $\mathrm{A}(k, z)$ and so also on $\mathrm{C}(K, z)$, it follows that $\ker \delta_y \cap \mathrm{C}(K, z) = I_{(\overline{\partial K}, z)}$. Hence by Theorem 2.6.8, $\delta_y(\mathrm{C}(K, z)) \cong \mathrm{C}(\overline{\partial K}, z)$. □

We will say more about the minimal C*-cover in Section 2.7.

2.7 Characterization of unital operator systems

In this section we will apply the results from Section 2.6 to establish a characterization of unital generalized operator systems in terms of their nc quasistate space. We note that a closely related problem, of characterizing operator spaces that are unital operator systems, has been considered by Blecher and Neal [6].

Theorem 2.7.1. *Let (K, z) be a pointed compact nc convex set. The following are equivalent for a pointed continuous affine nc function $e \in \mathrm{A}(K, z)$:*

1. *The function e is a distinguished archimedean matrix order unit for $\mathrm{A}(K, z)$.*
2. *The image of e under the canonical embedding of $\mathrm{A}(K, z)$ into $\mathrm{C}(\overline{\partial K}, z)$ is the identity.*
3. *For every n and every $x \in (\partial K \setminus \{z\})_n$, $e(x) = 1_n$.*

Proof. (1) ⇒ (2) Suppose that e is a distinguished archimedean matrix order unit for $\mathrm{A}(K, z)$. Then $\mathrm{A}(K, z)$ is a unital operator system, so it follows from [11, Theorem 4.4] that there is $y \in K_n$ such that y is a unital complete isometry on $\mathrm{A}(K, z)$ with $e(y) = 1_n$. Then $\ker \delta_y$ is contained in the boundary ideal $I_{\overline{\partial K}, z}$ from Remark 2.6.10. It follows that the canonical embedding of $\mathrm{A}(K, z)$ into $\mathrm{C}(\overline{\partial K}, z)$ factors through y, and hence maps e to the identity.

(2) ⇒ (3) Suppose that the image of e under the canonical embedding of $\mathrm{A}(K, z)$ into $\mathrm{C}(\overline{\partial K}, z)$ is the identity. Proposition 2.6.13 implies that the restriction to $\mathrm{A}(K, z)$ of every nc quasistate in $\partial K \setminus \{z\}$ factors through $\mathrm{C}(\overline{\partial K}, z)$. It follows that for $x \in (\partial K \setminus \{z\})_n$, $e(x) = 1$.

(3) ⇒ (1) Suppose that for every n and every $x \in (\partial K \setminus \{z\})_n$, $e(x) = 1_n$. Then it follows from Proposition 2.6.13 that the image of e under the canonical embedding of $\mathrm{A}(K, z)$ into $\mathrm{C}(\overline{\partial K}, z)$ is the identity. It follows that e is a distinguished archimedean matrix order unit for $\mathrm{C}(\overline{\partial K}, z)$, and hence also for $\mathrm{A}(K, z)$. □

Corollary 2.7.2. *Let S be a generalized operator system with nc quasistate space (K,z). The following are equivalent:*

1. *The generalized operator system S is unital.*

2. *There is $e \in S$ such that for every n and every $x \in (\partial K \setminus \{z\})_n$, $e(x) = 1_n$*

The next result is [13, Theorem 2.25 (ii)].

Corollary 2.7.3. *Let S be a unital operator system. Then the minimal unital C*-cover of S and the minimal C*-cover of S coincide.*

Proof. Let (A,ι) and (B,κ) denote the minimal unital C*-cover of S and the minimal C*-cover of S respectively. It follows from Theorem 2.7.1 and Proposition 2.6.13 that B is unital. Hence by the universal property of A, there is a surjective *-homomorphism $\pi : A \to B$ such that $\pi \circ \iota = \kappa$. On the other hand, by the universal property of B, there is a surjective *-homomorphism $\sigma : B \to A$ such that $\sigma \circ \kappa = \iota$. Hence A and B are isomorphic. □

2.8 Quotients of generalized operator systems

In this section we will utilize the dual equivalence between the category GenOpSys of generalized operator systems and the category PoNCConv of pointed compact nc convex sets to develop a theory of quotients for generalized operator systems. We will show that the theory developed here extends the theory of quotients for unital operator systems developed by Kavruk, Paulsen, Todorov and Tomforde [29]. We note that the theory of quotients for unital operator systems can be developed in a similar way using the dual equivalence between the category of unital operator systems and the category of compact nc convex sets from [15, Section 3].

Definition 2.8.1. Let S be a generalized operator system and let (K,z) denote the nc quasistate space of S. We will say that a subset $J \subseteq S$ is a *kernel* if there is an nc quasistate $x \in K$ such that $J = \ker x$.

Remark 2.8.2. For $x \in K_n$, the closure of the image $x(S) \subseteq \mathcal{M}_n$ is a generalized operator system. Hence J is a kernel if and only if there is a generalized operator system T and a completely contractive and completely positive map $\varphi : S \to T$ with $\ker \varphi = J$.

Let S be a generalized operator system and let (K,z) denote the nc quasistate space of S. For a subset $Q \subseteq S$, the *annihilator* of Q is $Q^\perp = \{x \in K : a(x) = 0 \text{ for all } a \in Q\}$. Note that $Q^\perp$ is a closed nc convex set. Similarly, for a subset $X \subseteq K$, the *annihilator* of X is $X^\perp = \{a \in S : a(x) = 0 \text{ for all } x \in X\}$.

The next result is a noncommutative analogue of [1, II.5.3].

Lemma 2.8.3. *Let K be a compact nc convex set and let $X \subseteq K$ be a subset. Then $X^{\perp\perp} = Y \cap K$, where $Y \subseteq \mathcal{M}(\mathrm{A}(K)^*)$ denotes the closed nc convex hull generated by $\sqcup \operatorname{span} X_n$, where $\operatorname{span} X_n$ is taken in $\mathcal{M}_n(\mathrm{A}(K)^*)$.*

Proof. It is clear that $Y \cap K \subseteq X^{\perp\perp}$. For the other inclusion, suppose for the sake of contradiction there is $z \in (X^{\perp\perp})_n \setminus (Y \cap K)$. Then $z \notin Y$. Hence by the nc separation theorem [15, 2.4.1], there is a self-adjoint element $a \in \mathrm{A}(K)$ satisfying $a(z) \nleq 1_n \otimes 1_n$ but $a(y) \leq 1_n \otimes 1_p$ for all $y \in Y_p$. Since each Y_p is a subspace, this forces $a(y) = 0$ for all $y \in Y_p$. Hence viewing a as an $n \times n$ matrix $a = (a_{ij})$ over $\mathrm{A}(K)$, $a_{ij}(y) = 0$ for all $y \in Y$. In particular, $a_{ij}(x) = 0$ for all $x \in X$. Hence $a_{ij} \in X^{\perp}$ for all i, j. Since $z \in X^{\perp\perp}$, it follows that $a_{ij}(z) = 0$ for all i, j. Therefore, $a(z) = 0$, giving a contradiction. □

Proposition 2.8.4. *Let (K, z) be a pointed compact nc convex set. A subset $J \subseteq \mathrm{A}(K, z)$ is a kernel if and only if $J = J^{\perp\perp}$. If J is a kernel and $M = J^{\perp}$, then the completely contractive completely positive restriction map $\mathrm{A}(K, z) \to \mathrm{A}(M, z)$ has kernel J. Moreover, $z \in M$ and the pair (M, z) is a pointed compact nc convex set.*

Proof. Suppose that $J = J^{\perp\perp}$. Let $M = J^{\perp}$. Then $J = M^{\perp}$. Let $r : \mathrm{A}(K, z) \to \mathrm{A}(M, z)$ denote the restriction map. Then r is completely contractive and completely positive and $\ker r = M^{\perp} = J$. Hence J is a kernel.

Conversely, suppose that J is a kernel. It is clear that $J \subseteq J^{\perp\perp}$. For the other inclusion, choose $x \in K$ such that $J = \ker x$. Let T denote the closure of the image $\mathrm{A}(K, z)(x) \subseteq \mathcal{M}_n$. Then T is a generalized operator system. Letting (L, w) denote the nc quasistate space of T, we can identify T with $\mathrm{A}(L, w)$. Let $\psi : L \to K$ denote the continuous affine map obtained by applying Theorem 2.4.9 to x. Then for $a \in J$ and $y \in L$, $0 = a(x)(y) = a(\psi(y))$. Hence $\psi(L) \subseteq J^{\perp}$, so for $a \in J^{\perp\perp}$ and $y \in L$, $0 = a(\psi(y)) = a(x)(y)$, i.e. $a(x) = 0$. Hence $J^{\perp\perp} \subseteq J$, so $J = J^{\perp\perp}$.

If J is a kernel and $M = J^{\perp}$, then clearly $z \in M$. To see that (M, z) is a pointed compact nc convex set, let $\theta : \mathrm{A}(M, z) \to \mathcal{M}_n$ be an nc quasistate. Let $r : \mathrm{A}(K, z) \to \mathrm{A}(M, z)$ denote the restriction map from above. Then the composition $\theta \circ r$ is an nc quasistate on $\mathrm{A}(K, z)$. Since (K, z) is a pointed compact nc convex set, by definition there is $x \in K$ such that $\theta \circ r = x$. Since x factors through r, $x \in J^{\perp} = M$. □

Definition 2.8.5. Let S be a generalized operator system and let (K, z) denote the nc quasistate space of S. For a kernel $J \subseteq S$, we let S/J denote the generalized operator system $\mathrm{A}(M, z)$, where $M = J^{\perp}$. We will refer to S/J as the *quotient* of S by J, and we will refer to the restriction map $S \to \mathrm{A}(M, z)$ obtained by identifying S with $\mathrm{A}(K, z)$ as the *canonical quotient map*.

Remark 2.8.6. Note that we have applied Theorem 2.4.8 to identify S with $\mathrm{A}(K, z)$. It is clear that the canonical quotient map $\mathrm{A}(K, z) \to \mathrm{A}(M, z)$ is completely contractive and completely positive.

The next result characterizes generalized operator system quotients in terms of a natural universal property. It is an analogue of [29, Proposition 3.6].

Theorem 2.8.7. *Let S be a generalized operator system and let $J \subseteq S$ be a kernel. The quotient S/J is the unique generalized operator system up to isomorphism satisfying the following universal property: there is a completely contractive and completely positive map $\varphi : S \to S/J$, and whenever T is a generalized operator system and $\psi : S \to T$ is a completely contractive and completely positive map with*

$J \subseteq \ker\psi$, then ψ factors through φ. In other words, there is a completely contractive and completely positive map $\omega : S/J \to T$ such that $\psi = \omega \circ \varphi$.

$$\begin{array}{ccccc} J & \hookrightarrow & S & \xrightarrow{\varphi} & S/J \\ & & & \searrow^{\psi} & \downarrow{\scriptstyle \omega} \\ & & & & T \end{array}$$

Proof. To see that S/J satisfies this universal property, first note that the canonical quotient map $\varphi : S \to S/J$ is completely contractive and completely positive. Let T be a generalized operator system and let $\psi : S \to T$ be a completely contractive and completely positive map with $J \subseteq \ker\psi$. Letting (K, z) and (L, w) denote the nc quasistate spaces of S and T respectively, we can assume that $S = \mathrm{A}(K, z)$ and $T = \mathrm{A}(L, w)$. Let $\psi^d : L \to K$ denote the continuous affine nc map obtained by applying Theorem 2.4.8 to ψ.

Let $M = J^\perp$. For $a \in J$ and $y \in L$, the fact that $J \subseteq \ker\psi$ implies that $0 = \psi(a)(y) = a(\psi^d(y))$. Hence $\psi^d(L) \subseteq J^\perp = M$. Restricting the codomain of ψ^d to M and applying Theorem 2.4.8 to ψ^d, we obtain a completely contractive and completely positive map $\omega : \mathrm{A}(M, z) \to \mathrm{A}(L, w)$ such that $\omega \circ \varphi = \psi$.

To see that S/J is the unique generalized operator system with this universal property, suppose that R is another generalized operator system that satisfies the property from the statement of the theorem, then there are surjective completely contractive and completely positive maps $S/J \to R$ and $R \to S/J$ such that the composition is the identity map on S/J. It follows that each of the individual maps must be a completely isometric complete order isomorphism. Hence R is isomorphic to S/J. □

In order to relate our theory of quotients of generalized operator systems to the theory of quotients of unital operator systems from [29], we require the following result.

Lemma 2.8.8. *Let (K, z) be a pointed compact nc convex set such that $\mathrm{A}(K, z)$ is a unital operator system and let $e \in \mathrm{A}(K, z)$ denote the distinguished archimedean matrix order unit. Let $J \subseteq \mathrm{A}(K, z)$ be a kernel and let $M = J^\perp$. Then for $x \in \partial M \setminus \{z\}$, $e(x) = 1$.*

Proof. Let K^0 and K^1 denote the closed nc convex hulls of $\{z\}$ and $\partial K \setminus \{z\}$ respectively. By Theorem 2.7.1, $e(x) = 1$ for $x \in \partial K \setminus \{z\}$. Hence by the continuity of e, $e(x) = 1$ for all $x \in K^1$. Since $e(z) = 0$, in particular this implies that $K^0 \cap K^1 = \emptyset$.

For $x \in \partial K$, either $e(x) = 1$ or $e(x) = 0$. It follows from [14, Theorem 3.4] and [15, Proposition 5.2.4] that the image of e under the canonical embedding of $\mathrm{A}(K)$ into its minimal unital C*-cover $\mathrm{C}(\overline{\partial K})$ is a projection in the center of $\mathrm{C}(\overline{\partial K})$.

Choose $x \in K_m$ and let $y \in K_n$ be a maximal dilation of x. Then there is an isometry $\alpha \in \mathcal{M}_{n,m}$ such that $x = \alpha^* y \alpha$. By [15, Proposition 5.2.4], the *-homomorphism δ_y factors through $\mathrm{C}(\overline{\partial K})$. Hence from above, y decomposes as a direct sum $y = y_0 \oplus y_1$ for $y_0 \in K^0_{n_0}$ and $y_1 \in K^1_{n_1}$. This implies that x can be written as an nc convex combination $x = \alpha_0^* y_0 \alpha_0 + \alpha_1^* y_1 \alpha_1$ for $\alpha_0 \in \mathcal{M}_{n_0,m}$ and $\alpha_1 \in \mathcal{M}_{n_1,m}$ satisfying $\alpha_0^*\alpha_0 + \alpha_1^*\alpha_1 = 1_m$.

Suppose $x \in (\partial M)_m$ and write x as an nc convex combination $x = \alpha_0^* y_0 \alpha_0 + \alpha_1^* y_1 \alpha_1$ as above. Then for $a \in J$,

$$0 = a(x) = \alpha_0^* a(y_0) \alpha_0 + \alpha_1^* a(y_1) \alpha_1.$$

Since y_0 is a direct sum of copies of z, $a(y_0) = 0$. Hence $\alpha_1^* a(y_1)\alpha_1 = 0$.

From above, we can decompose y_1 with respect to the range of α as

$$y_1 = \begin{bmatrix} u_1 & * \\ * & * \end{bmatrix}$$

for $u_1 \in M_k$, and there is $\beta_1 \in \mathcal{M}_{k,m}$ such that $\beta_1^* u_1 \beta_1 = \alpha_1^* y_1 \alpha_1$ and $\alpha_0^*\alpha_0 + \beta_1^*\beta_1 = 1_m$.

Now since $y_0, u_1 \in M$ and $x \in (\partial M)_m$, it follows that either $\alpha_0 = 0$ or $\beta_1 = 0$. Hence either $x \in K^0$ or $x \in K^1$. In the former case, $x = z$, while in the latter case, $e(x) = 1$. □

Proposition 2.8.9. *Let S be a unital operator system. Then for every kernel $J \subseteq S$, the quotient generalized operator system S/J is unital.*

Proof. Letting (K, z) denote the nc quasistate space of S, we can assume that $S = \mathrm{A}(K, z)$. Let $M = J^\perp$, so that $S/J = \mathrm{A}(M, z)$, and let $\varphi : \mathrm{A}(K, z) \to \mathrm{A}(M, z)$ denote the canonical quotient map. Let $e \in \mathrm{A}(K, z)$ denote the distinguished archimedean matrix order unit. Then for $x \in \partial M \setminus \{z\}$, Corollary 2.8.8 implies that $e(x) = 1$. Hence by Theorem 2.7.1, $\varphi(e)$ is an archimedean matrix order unit. □

Remark 2.8.10. If S is a unital operator system and J is the kernel of a unital completely positive map, then the quotient S/J from Definition 2.8.5 coincides with the definition of quotient in [29]. Indeed, the quotient T of S by J that they consider in their paper is the unique unital operator system satisfying a universal property analogous to the property in Theorem 2.8.7 for unital completely positive maps into unital operator systems. By Proposition 2.8.9, S/J is a unital operator system, so it follows from Theorem 2.8.7 that $S/J = T$.

Lemma 2.8.11. *Let (K, z) be a pointed compact nc convex set and let $J \subseteq \mathrm{A}(K, z)$ be a kernel. Let $M = J^\perp$. Then the closed two-sided ideal I of $\mathrm{C}(K, z)$ generated by J is $I = \{f \in \mathrm{C}(K, z) : f|_M = 0\}$. Hence letting $\pi : \mathrm{C}(K, z) \to \mathrm{C}(M, z)$ denote the restriction *-homomorphism, $I = \ker \pi$.*

Proof. Let $I' = \ker \pi$. Then $I' = \{f \in \mathrm{C}(K, z) : f|_M = 0\}$. By Proposition 2.8.4, the restriction $\pi|_{\mathrm{A}(K,z)}$ satisfies $\ker \pi|_{\mathrm{A}(K,z)} = J$, so it is clear that $I \subseteq I'$.

For the other inclusion, first note that $J = I' \cap \mathrm{A}(K, z)$. Hence by the definition of I and the fact from above that $I \subseteq I'$,

$$J \subseteq I \cap \mathrm{A}(K, z) \subseteq I' \cap \mathrm{A}(K, z) = J,$$

implying $J = I \cap \mathrm{A}(K, z)$. Let $\rho : \mathrm{C}(K, z) \to \mathrm{C}(K, z)/I$ denote the quotient *-homomorphism. Since the restriction $\rho|_{\mathrm{A}(K,z)}$ has kernel J, Theorem 2.8.7 implies that $\rho|_{\mathrm{A}(K,z)}$ factors through $\mathrm{A}(M, z)$. It follows that there is a completely contractive and completely positive map $\omega : \mathrm{A}(M, z) \to \mathrm{C}(K, z)/I$ such that $\omega \circ \pi|_{\mathrm{A}(K,z)} = \rho|_{\mathrm{A}(K,z)}$.

By the universal property of $\mathrm{C}(M, z)$, ω extends to a *-homomorphism $\sigma : \mathrm{C}(M, z) \to \mathrm{C}(K, z)/I$. Hence $I' \subseteq I$, and we conclude that $I' = I$. □

Proposition 2.8.12. *Let (K, z) be a pointed compact nc convex set. Let $J \subseteq \mathrm{A}(K, z)$ be a subset and let I denote the closed two-sided ideal of $\mathrm{C}(K, z)$ generated by J. Then J is a kernel if and only if $I \cap \mathrm{A}(K, z) = J$.*

Proof. If J is a kernel, then letting $M = J^\perp$, Proposition 2.8.4 and Lemma 2.8.11 imply that $I \cap \mathrm{A}(K, z) = \{a \in \mathrm{A}(K, z) : a|_M = 0\} = M^\perp = J$. Conversely, if $I \cap \mathrm{A}(K, z) = J$, then letting $\pi : \mathrm{C}(K, z) \to \mathrm{C}(K, z)/I$ denote the quotient *-homomorphism, $J = \ker \pi|_{\mathrm{A}(K,z)}$. Hence J is a kernel. □

2.9 C*-simplicity

In this section we will establish a characterization of generalized operator systems with the property that their minimal C*-cover (i.e. their C*-envelope) is simple. The characterization will be in terms of the nc quasistate space of a generalized operator system.

Definition 2.9.1. We will say that a generalized operator system S is *C*-simple* if its minimal C*-cover $\mathrm{C}^*_{\min}(S)$ is simple.

We will require the *spectral topology* on the irreducible points in a compact nc convex set from Section 2.6.1), which was introduced in [32, Section 9]. Recall that for a compact nc convex set K, the spectral topology on the set $\mathrm{Irr}(K)$ of irreducible points in K is defined in terms of the hull-kernel topology on the spectrum of the C*-algebra $\mathrm{C}(K)$.

By Proposition 2.6.13, letting $y = \oplus_{x \in \partial K \setminus \{z\}} x$, the kernel of the *-homomorphism δ_y on $\mathrm{C}(K,z)$ is the boundary ideal $I_{(\overline{\partial K},z)}$ from Theorem 2.6.8. In particular, the quotient $\mathrm{C}(K,z)/I_{(\overline{\partial K},z)}$ is isomorphic to the minimal C*-cover $\mathrm{C}(\overline{\partial K},z)$ of $\mathrm{A}(K,z)$. The proof of the following result now follows exactly as in the proof of [32, Proposition 9.4].

Proposition 2.9.2. *Let (K,z) be a pointed compact nc convex set. A point $x \in \mathrm{Irr}(K)$ belongs to the closure of $\partial K \setminus \{z\}$ with respect to the spectral topology if and only if the corresponding representation $\delta_x : \mathrm{C}(K,z) \to \mathcal{M}_n$ factors through the minimal C*-cover $\mathrm{C}(\overline{\partial K},z)$ of $\mathrm{A}(K,z)$.*

Theorem 2.9.3. *Let (K,z) be a pointed compact nc convex set. The generalized operator system $\mathrm{A}(K,z)$ is C*-simple if and only if the closed nc convex hull of any nonzero point in the spectral closure of ∂K contains $\partial K \setminus \{z\}$.*

Proof. Suppose that $\mathrm{A}(K,z)$ is C*-simple, so that its minimal C*-cover $\mathrm{C}(\overline{\partial K},z)$ is simple. Choose nonzero $x \in K_m$ in the spectral closure of $\partial K \setminus \{z\}$ and let $M \subseteq K$ denote the closed nc convex hull of x. Suppose for the sake of contradiction there is $y \in (\partial K)_n \setminus \{z\}$ such that $y \notin M$.

By Proposition 2.9.2, the corresponding representation $\delta_x : \mathrm{C}(K,z) \to \mathcal{M}_n$ factors through the minimal C*-cover $\mathrm{C}(\overline{\partial K},z)$ of $\mathrm{A}(K,z)$. Since $\mathrm{C}(\overline{\partial K},z)$ is simple, it follows that the kernel of δ_x is the boundary ideal $I_{(\overline{\partial K},z)}$ from Theorem 2.6.8, so the range of δ_x is isomorphic to the minimal C*-cover $\mathrm{C}(K,z)/I_{(\overline{\partial K},z)} \cong \mathrm{C}(\overline{\partial K},z)$. In particular, x is an embedding. Similarly, y is an embedding.

By the nc separation theorem [15, Corollary 2.4.2], there is self-adjoint $a \in \mathcal{M}_n(\mathrm{A}(K,z))$ and self-adjoint $\gamma \in \mathcal{M}_n$ such that $a(y) \nleq \gamma \otimes 1_n$ but $a(u) \leq \gamma \otimes 1_p$ for $u \in M_p$. In particular, $a(x) \leq \gamma \otimes 1_m$ but $a(y) \nleq \gamma \otimes 1_n$. However, from above x and y are embeddings, meaning that they are complete order embeddings on $\mathrm{A}(K)$, giving a contradiction.

Conversely, suppose that the closed nc convex hull of any nonzero point in the spectral closure of ∂K contains $\partial K \setminus \{z\}$. Let I be a proper ideal in $\mathrm{C}(\overline{\partial K},z)$ and choose nonzero irreducible $y \in K_n$ such that the *-homomorphism δ_y on $\mathrm{C}(K,z)$ factors through $\mathrm{C}(\overline{\partial K},z)/I$. Then by Proposition 2.9.2, y is in the spectral closure of ∂K. Hence by assumption the closed nc convex hull of y contains $\partial K \setminus \{z\}$.

By [15, Theorem 6.4.3], every point in ∂K is a limit of compressions of y. Hence, replacing y with a sufficiently large amplification, there are isometries $\alpha_i \in \mathcal{M}_{p,n}$ such that $\lim \alpha_i^* y \alpha_i = \oplus_{x \in \partial K \setminus \{z\}} x$. By passing to a subnet we can assume that there is an nc state μ on $\mathrm{C}(K)$ such that the *-homomorphism δ_y satisfies $\lim \alpha_i^* \delta_y \alpha_i = \mu$ in the nc state space of $\mathrm{C}(K)$. Then since $\mu|_{\mathrm{A}(K)} = \oplus_{x \in \partial K \setminus \{z\}} x$, and since extreme points in K have unique extensions to nc states on $\mathrm{C}(K)$, μ is the *-homomorphism $\mu = \oplus_{x \in \partial K \setminus \{z\}} \delta_x$ (see [15, Theorem 6.1.9]).

By Proposition 2.6.13, the image of $\mathrm{C}(K,z)$ under this *-homomorphism is isomorphic to $\mathrm{C}(\overline{\partial K}, z)$. It follows that the canonical *-homomorphism from $\mathrm{C}(K,z)$ onto $\mathrm{C}(\overline{\partial K}, z)$ factors through δ_y. Hence $I = 0$. Since I was arbitrary, we conclude that $\mathrm{C}(\overline{\partial K}, z)$ is simple. □

The following corollary applies when the set ∂K of extreme points of K is closed in the spectral topology. This is equivalent to the statement that every nonzero irreducible representation of $\mathrm{C}(\overline{\partial K}, z)$ restricts to an extreme point of K.

Corollary 2.9.4. *Let (K,z) be a pointed nc convex set such that ∂K is closed in the spectral topology. Then $\mathrm{A}(K,z)$ is C*-simple if and only if for every nonzero compact nc convex subset $M \subseteq K$, either $M \cap \partial K = \varnothing$ or $M \cap \partial K = \partial K$.*

Proof. Suppose that $\mathrm{A}(K,z)$ is C*-simple. If $M \cap \partial K \neq \varnothing$ then Theorem 2.9.3 implies that $\partial K \subseteq M$. Conversely, suppose that for every nonzero compact nc convex subset $M \subseteq K$, either $M \cap \partial K = \varnothing$ or $M \cap \partial K = \partial K$. By assumption, ∂K is spectrally closed, and for any point $x \in \partial K$, the closed nc convex hull M generated by x trivially satisfies $M \cap \partial K \neq \varnothing$. Hence by assumption $\partial K \subseteq M$, so Theorem 2.9.3 implies that $\mathrm{A}(K,z)$ is C*-simple. □

2.10 Characterization of C*-algebras

A classical result of Bauer characterizes unital function systems that are unital commutative C*-algebras in terms of their state space. Specifically, he showed that if C is a compact convex set, then the unital function system $\mathrm{A}(C)$ of continuous affine functions on C is a unital commutative C*-algebra if and only if C is a Bauer simplex (see e.g. [1, Theorem II.4.3]).

Kennedy and Shamovich [32, Theorem 10.5] introduced a definition of noncommutative simplex that generalizes the classical definition and established a generalization of Bauer's result for unital operator systems. Specifically, they showed that if K is a compact nc convex set, then the unital operator system $\mathrm{A}(K)$ of continuous affine nc functions on K is a unital C*-algebra if and only if K is an nc Bauer simplex.

In this section we will extend this result by showing that a generalized operator system is a C*-algebra if and only if its nc quasistate space is a Bauer simplex with zero as an extreme point. Before introducing the notion of a Bauer simplex, we need to recall some preliminary definitions.

Let K be a compact nc convex set. For a point $x \in K$, viewed as an nc state on the unital operator system $\mathrm{A}(K)$, the *-homomorphism δ_x is an extension of x. We will be interested in other nc states on $\mathrm{C}(K)$ that extend x. Specifically, we will be interested in nc states that are maximal in a certain precise sense. The following definition is [15, Definition 4.5.1].

Definition 2.10.1. Let K be a compact nc convex set and let $\mu : \mathrm{C}(K) \to \mathcal{M}_n$ be an nc state on $\mathrm{C}(K)$. The *barycenter* of μ is the restriction $\mu|_{\mathrm{A}(K)} \in K_n$. The nc state μ is said to be a *representing map* for its barycenter. We will say that a point $x \in K$ has a *unique representing map* if the *-homomorphism δ_x is the unique nc state on $\mathrm{C}(K)$ with barycenter x.

We will also require the notion of a convex nc function. The following definition is [15, Definition 3.12].

Definition 2.10.2. Let K be a compact nc convex set. For a bounded self-adjoint nc function $f \in \mathcal{M}_n(\mathrm{B}(K))_h$, the *epigraph* of f is the set $\mathrm{Epi}(f) \subseteq \sqcup K_m \times \mathcal{M}_n(\mathcal{M}_m)$ defined by

$$\mathrm{Epi}(f)_m = \{(x, \alpha) \in K_m \times \mathcal{M}_n(\mathcal{M}_m) : x \in K_m \text{ and } \alpha \geq f(x)\}.$$

The function f is *convex* if $\mathrm{Epi}(f)$ is an nc convex set.

Davidson and Kennedy introduced a notion of nc Choquet order on the set of representing maps of a point in a compact nc convex set that plays a key role in noncommutative Choquet theory. The following definition is [15, Definition 8.2.1].

Definition 2.10.3. Let K be a compact nc convex set and let $\mu, \nu : \mathrm{C}(K) \to \mathcal{M}_n$ be nc states. We say that μ is dominated by ν in the *nc Choquet order* and write $\mu \prec_c \nu$ if $\mu(f) \leq \nu(f)$ for every n and every continuous convex nc function $f \in \mathcal{M}_n(\mathrm{C}(K))$. We will say that μ is a *maximal representing map* for its barycenter if it is maximal in the nc Choquet order.

Remark 2.10.4. Several equivalent characterizations of the nc Choquet order were established in [15]. These are among the deepest results in that paper.

We are finally ready to state the definition of an nc simplex. The following definitions are [32, Definition 4.1] and [32, Definition 10.1] respectively.

Definition 2.10.5.

1. A compact nc convex set K is an *nc simplex* if every point in K has a unique maximal representing map on $\mathrm{C}(K)$.

2. An nc simplex K is an *nc Bauer simplex* if the extreme boundary ∂K is a closed subset of the set $\mathrm{Irr}(K)$ of irreducible points in K with respect to the spectral topology.

Remark 2.10.6. It was shown in [32] that these definitions generalize the classical definitions. Specifically, if C is a classical simplex then there is a unique nc simplex K with $K_1 = C$. Furthermore, if C is a Bauer simplex then K is an nc Bauer simplex.

It was shown in [32, Theorem 10.5] that if K is a compact nc convex set, then the unital operator system $\mathrm{A}(K)$ is a C*-algebra if and only if K is an nc Bauer simplex. The next example shows that the obvious generalization of this statement for operator systems does not hold.

Example 2.10.7. Let (K, z) be the pointed compact nc convex set from Example 2.3.16, so $K = \sqcup K_n$ is defined by

$$K_n = \{\alpha \in (\mathcal{M}_n)_h : -1_n \le \alpha \le 1_n\}, \quad \text{for} \quad n \in \mathbb{N},$$

and $z = 0$. Since $K_1 = [-1, 1]$ is a Bauer simplex, it follows from the above discussion that K is the unique compact nc convex set with this property and K is an nc Bauer simplex. However, $\mathrm{A}(K, z)$ is not a C*-algebra. Note that $z \notin \partial K_1$ and hence $z \notin \partial K$.

Lemma 2.10.8. *Let S be a generalized operator system with nc quasistate space (K, z). Then S is a C*-algebra if and only if its unitization $S^\sharp$ is a C*-algebra and $z \in \partial K$.*

Proof. If S is a C*-algebra, say A, then its unitization $A^\sharp$ is the C*-algebraic unitization $A^\sharp$ of A, and hence is also a C*-algebra. Furthermore, A is an ideal in $A^\sharp$ and z is an irreducible *-representation of $A^\sharp$ satisfying $\ker z = A$. Hence by [15, Example 6.1.8], $z \in \partial K$.

Conversely, suppose that $S^\sharp$ is a C*-algebra, say B, and $z \in \partial K$. Then by [15, Example 6.1.8], z is an irreducible representation of B. Since $\mathrm{A}(K, z) = \ker z$, $\mathrm{A}(K, z)$ is an ideal in B, and in particular is a C*-algebra. □

The next result extends [32, Theorem 10.5].

Theorem 2.10.9. *Let S be a generalized operator system with nc quasistate space (K, z). Then S is a C*-algebra if and only if K is an nc Bauer simplex and $z \in \partial K$. The result also holds for unital operator systems with nc quasistate spaces replaced by nc state spaces.*

Proof. By Lemma 2.10.8, S is isomorphic to a C*-algebra if and only if $S^\sharp$ is isomorphic to a C*-algebra and $z \in \partial K$. By [32, Theorem 10.5], the former property is equivalent to K being a Bauer simplex. □

2.11 Stable equivalence

Connes and van Suijlekom [13, Section 2.6] considered stable equivalence for generalized operator systems. Generalized operator systems S and T are said to be *stably equivalent* if the generalized operator systems $S \otimes_{\min} \mathcal{K}$ and $T \otimes_{\min} \mathcal{K}$ are isomorphic. Here, $\mathcal{K} = \mathcal{K}(H_{\aleph_0})$ denotes the C*-algebra of compact operators on $H_{\aleph_0}$ and the minimal tensor products $S \otimes_{\min} \mathcal{K}$ and $T \otimes_{\min} \mathcal{K}$ are defined as in [28], i.e. $S \otimes_{\min} \mathcal{K}$ is the closed generalized operator system generated by the algebraic tensor product of S and $\mathcal{K}$ in $\mathrm{C}^*_{\min}(S) \otimes_{\min} \mathrm{C}^*_{\min}(\mathcal{K}) = \mathrm{C}^*_{\min}(S) \otimes_{\min} \mathcal{K}$ and similarly for $T \otimes_{\min} \mathcal{K}$.

In this section we will describe the nc quasistate space of the stabilization of a generalized operator system. This will yield a characterization of stable equivalence in terms of nc quasistate spaces.

Let S be a generalized operator system and let (K, z) denote the nc quasistate space of S. Let (L, w) denote the nc quasistate space of $\mathcal{K}$. For $x \in K$ and $u \in L$, we obtain a completely contractive map $x \otimes u$ on $S \otimes_{\min} \mathcal{K}$ from the theory of tensor products of operator spaces (see e.g. [45]). However, it is not immediately obvious that $x \otimes u$ is an nc quasistate. The next result implies that it is, and moreover, that every nc quasistate on $S \otimes_{\min} \mathcal{K}$ arises in this way.

Theorem 2.11.1. *Let S be a generalized operator system with nc quasistate space (K, z) and let (L, w) denote the nc quasistate space of $\mathcal{K}$. The nc quasistate space of $S \otimes_{\min} \mathcal{K}$ is $(K \otimes L, z \otimes w)$, where $K \otimes L$ denotes the closed nc convex hull of $\{x \otimes u : x \in K \text{ and } u \in L\}$ and $x \otimes u$ is defined as in the above discussion. Furthermore, letting (M, t) denote the nc quasistate space of $S \otimes_{\min} \mathcal{K}$, $\partial M \subseteq \partial K \otimes \partial L$.*

Proof. We can identify S with $\mathrm{A}(K, z)$ and identify $\mathrm{A}(K, z)$ with its image under the canonical embedding into its minimal C*-cover $\mathrm{C}(\overline{\partial K}, z)$. By [13, Proposition 2.37], the minimal C*-cover of $\mathrm{A}(K, z) \otimes \mathcal{K}$ is $\mathrm{C}(\overline{\partial M}, t) = \mathrm{C}(\overline{\partial K}, z) \otimes \mathcal{K}$. Every point $x \in K_m$ extends to an nc quasistate $\tilde{x} : \mathrm{C}(\overline{\partial K}, z) \to \mathcal{M}_m$ (see Section 2.6). Then for $u \in L_n$, we obtain an nc quasistate $\tilde{x} \otimes y : \mathrm{C}(\overline{\partial K}, z) \otimes \mathcal{K} \to \mathcal{M}_m \otimes \mathcal{M}_n$. The restriction $\tilde{x} \otimes y|_{\mathrm{A}(K,z) \otimes \mathcal{K}} = x \otimes y$ is therefore an nc quasistate on $\mathrm{A}(K, z) \otimes_{\min} \mathcal{K}$. Hence $K \otimes L \subseteq M$. It is clear that $z \otimes w$ is the zero map.

For the reverse inclusion, let $r \in \partial M$ be an extreme point. Then by Proposition 2.6.13, the *-homomorphism δ_r on $\mathrm{C}(M, t)$ factors through $\mathrm{C}(\overline{\partial K}, z) \otimes \mathcal{K}$. Since r is extreme, [15, Theorem 6.1.9] implies that δ_r is irreducible. Hence there is an irreducible representation $\pi : \mathrm{C}(\overline{\partial K}, z) \to \mathcal{M}_m$ such that δ_r is unitarily equivalent to $\pi \otimes u$, where $u \in L$ is either the identity representation of $\mathcal{K}$ or $u = w$. Letting $x = \pi|_{\mathrm{A}(K,z)} \in K_m$, $r|_{\mathrm{A}(K,z) \otimes_{\min} \mathcal{K}} = x \otimes u$. In fact, it is easy to verify that since $r \in \partial L$, $x \in \partial K$. It follows from the nc Krein-Milman theorem [15, Theorem 6.4.2] that $M \subseteq K \otimes L$. □

Corollary 2.11.2. *Let S_1 and S_2 be generalized operator systems with nc quasistate spaces (K_1, z_1) and (K_2, z_2) respectively. Let $0_{\mathcal{K}}$ and $\mathrm{id}_{\mathcal{K}}$ denote the zero map and the identity representation respectively of $\mathcal{K}$. Then S and T are stably isomorphic if and only if the closed nc convex hulls of the sets $\partial K \otimes \{0_{\mathcal{K}}, \mathrm{id}_{\mathcal{K}}\}$ and $\partial L \otimes \{0_{\mathcal{K}}, \mathrm{id}_{\mathcal{K}}\}$ are pointedly affinely homeomorphic with respect to the points $z_1 \otimes 0_{\mathcal{K}}$ and $z_2 \otimes 0_{\mathcal{K}}$.*

Proof. Let $(L, 0_{\mathcal{K}})$ denote the nc quasistate space of $\mathcal{K}$. Then it follows from Theorem 2.11.1 and Corollary 2.4.10 that S and T are stably isomorphic if and only if $(K_1 \otimes L, z_1 \otimes 0_{\mathcal{K}})$ and $(K_2 \otimes L, z_2 \otimes 0_{\mathcal{K}})$ are pointedly affinely homeomorphic.

By Proposition 2.4.4, the unitization $\mathcal{K}^\sharp$ is a unital C*-algebra with nc state space L. Since every irreducible *-representation of $\mathcal{K}^\sharp$ is unitarily equivalent to $0_{\mathcal{K}}$ or $\mathrm{id}_{\mathcal{K}}$, [15, Example 6.1.8] implies that L is the closed nc convex hull of $\{0_{\mathcal{K}}, \mathrm{id}_{\mathcal{K}}\}$. The result now follows from Theorem 2.11.1 and the nc Krein-Milman theorem [15, Theorem 6.4.2]. □

2.12 Dynamics and Kazhdan's property (T)

The fact that simplices arise as fixed point sets of affine actions of groups on spaces of probability measures has a number of important applications in classical dynamics. Glasner and Weiss showed that a second countable locally compact group has Kazhdan's property (T) if and only if the simplices that arise from this result are always Bauer simplices [22].

Kennedy and Shamovich extended these results to actions of discrete groups on nc state spaces of unital C*-algebras. Specifically, it was shown that nc simplices arise as fixed point sets of affine actions of discrete groups on nc state spaces of unital C*-algebras [32, Theorem 12.12]. It was further shown that a discrete group has property (T) if and only if the nc simplices that arise from this result are always nc Bauer simplices [32, Theorem 14.2]. Consequently, a discrete group has property (T) if and only if whenever it acts on a unital C*-algebra, the set of invariant states is the state space of a unital C*-algebra [32, Corollary 14.3].

In this section we will extend these results to actions of locally compact groups on (potentially nonunital) C*-algebras. In fact, we will see that the hard work was already accomplished in earlier sections of this chapter. After introducing appropriate definitions and applying the dual equivalence between the category GenOpSys of generalized operator systems and the category PoNCConv of pointed compact nc convex sets, the proofs in [32] will apply essentially verbatim.

The next definition is a slight generalization of [32, Definition 12.1] and [32, Definition 12.2].

Definition 2.12.1.

1. An *nc dynamical system* is a triple (S, G, σ) consisting of a generalized operator system S, a locally compact group G and a group homomorphism $\sigma : G \to \operatorname{Aut}(S)$ with the property that the orbit map $G \to S : g \to \sigma_g(s)$ is continuous for all $s \in S$.

2. A *affine nc dynamical system* is a triple (K, G, κ) consisting of a compact nc convex set K, a locally compact group G and a group homomorphism $\kappa : G \to \operatorname{Aut}(K)$ with the property that for each n, the orbit map $G \to K_n : g \to \kappa_g(x)$ is continuous for all $x \in K_n$.

Remark 2.12.2. Unless we need to refer to σ, we will write (S, G) for (S, G, σ) and gs for $\sigma_g(s)$. Similarly, unless we need to refer to κ, we will write (K, G) for (K, G, κ) and gx for $\kappa_g(x)$. If S is a C*-algebra, say A, then we will refer to (A, G) as a *C*-dynamical system.*

We will utilize the fact that if (K, z) is a pointed compact nc convex set and $(\mathrm{A}(K, z), G)$ is an nc dynamical system, then the dual equivalence from Theorem 2.4.9 gives rise to an affine nc dynamical system (K, G), determined by

$$a(\kappa_g(x)) = \sigma_{g^{-1}}(a)(x), \quad \text{for} \quad a \in \mathrm{A}(K),\ g \in G \text{ and } x \in K.$$

It seems worth pointing out that an nc dynamical system over a generalized operator system lifts to an nc dynamical system on its unitization.

Lemma 2.12.3. *Let (S, G, σ) be an nc dynamical system. Define $\sigma^\sharp : G \to \operatorname{Aut}(S)$ by $(\sigma^\sharp)_g = (\sigma_g)^\sharp$. Then $(S^\sharp, G, \sigma^\sharp)$ is an nc dynamical system.*

Proof. For $g \in G$, $(\sigma_g)^\sharp(s, \alpha) = (\sigma_g(s), \alpha)$ for $s \in S^\sharp$. It follows immediately that $\sigma^\sharp : G \to \operatorname{Aut}(S^\sharp)$ is a group homomorphism and that the corresponding orbit maps are continuous. □

Let G be a locally compact group. Recall that a *continuous* unitary representation of G on a Hilbert space H is a group homomorphism $\rho : G \to \mathcal{U}(H)$ such that the orbit map $G \to H : g \to \rho(g)\xi$ is continuous for every $\xi \in H$. Here $\mathcal{U}(H)$ denotes the set of unitary operators on H.

The next result follows immediately from [32, Theorem 12.12], since we can view the action of a non-discrete locally compact group as an action by its discretization.

Theorem 2.12.4. *Let (K, G) be an affine nc dynamical system such that K is an nc simplex. Then the fixed point set*

$$K^G = \{x \in K : gx = x \text{ for all } g \in G\}$$

is an nc simplex.

Corollary 2.12.5. *Let (A, G) be a C*-algebra and let (K, z) denote the nc quasistate space of A. Then the fixed point set K^G is an nc simplex.*

Proof. By Theorem 2.10.9, K is an nc Bauer simplex, so the result follows immediately from Theorem 2.12.4. □

Definition 2.12.6. Let G be a second countable locally compact group.

1. A continuous unitary representation $\rho : G \to \mathcal{U}(H)$ is said to have *almost invariant vectors* if there is a net of unit vectors $\{\xi_i \in H\}$ such that for every compact subset $C \subseteq G$,

$$\lim_i \sup_{g \in C} \|\rho(g)\xi_i - \xi_i\| = 0.$$

2. The group G is said to have *Kazhdan's property (T)* if every unitary representation of G with almost invariant vectors has a nonzero invariant vector.

The next result is a generalization for (potentially nonunital) C*-algebras and second countable locally compact groups of [32, Theorem 14.2].

Theorem 2.12.7. *Let A be a C*-algebra with nc quasistate space (K, z) and let G be a second countable locally compact group with Kazhdan's property (T) such that (A, G) is a C*-dynamical system. The set K^G of invariant nc quasistates on A is an nc Bauer simplex. If A is unital, then the result also holds for the nc state space of A instead of its nc quasistate space.*

Proof. The proof of [32, Theorem 14.2] works essentially verbatim here. If G is nondiscrete, then it is necessary to verify that the unitary representation constructed in the proof of the dilation theorem for invariant nc states [32, Lemma 12.6] is continuous. However, this is an easy consequence of the continuity of the orbit maps. □

The following corollary extends a result of Glasner and Weiss for commutative C*-algebras (see [22, Theorem 1] and [22, Theorem 2]).

Corollary 2.12.8. *Let G be a second countable locally compact group. Then G has Kazhdan's property (T) if and only if whenever A is a C*-algebra with nc quasistate space (K, z) and (A, G) is a C*-dynamical system, then the set K_1^G of invariant quasistates is pointedly affinely homeomorphic to the quasistate space of a C*-algebra. If A is unital, then the result also holds with the quasistate space of A replaced by its state space.*

Proof. If G has Kazhdan's property (T), then Theorem 2.12.7 implies that K^G is an nc Bauer simplex. By Lemma 2.10.8, $z \in \partial K$. Hence by Theorem 2.10.9, (K^G, z) is pointedly affinely homeomorphic to the nc quasistate space of a C*-algebra. In particular, the set K_1^G of invariant quasistates of A is pointedly affinely homeomorphic to the quasistate space of a C*-algebra.

Conversely, if G does not have Kazhdan's property (T), then it follows from [22, Theorem 2] that there is a compact Hausdorff space X and a commutative C*-dynamical system $(\mathrm{C}(X), G)$ such that the space $\mathrm{Prob}(X)^G$ of invariant probability measures on X is a Poulsen simplex. In particular, the set $\partial(\mathrm{Prob}(X)^G)$ of extreme points of $\mathrm{Prob}(X)^G$ is not closed.

We need to translate this to a statement about the quasistate space Q of $\mathrm{C}(X)$. Since Q is a compact convex set, the set Q^G of invariant quasistates is a simplex (see e.g. [32, Corollary 12.13]). Note that $\mathrm{Prob}(X) \subseteq Q$. In fact, Q is the closed convex hull of $\mathrm{Prob}(X) \cup \{z\}$, where z denotes the zero map on $\mathrm{C}(X)$. For nonzero $\mu \in \partial(Q^G)$, since $\mu(X)^{-1}\mu \in Q^G$, it follows that $\mu(X) = 1$. Hence $\mu \in \partial(\mathrm{Prob}(X)^G)$. On the other hand, it is clear that $\partial(\mathrm{Prob}(X)^G) \subseteq \partial(Q^G)$. Hence $\partial(Q^G) \subseteq \partial(\mathrm{Prob}(X)^G) \cup \{z\}$. Since $\partial(\mathrm{Prob}(X)^G)$ is not closed and z is isolated from $\mathrm{Prob}(X)$, it follows that $\partial(Q^G)$ is not closed. Therefore, Q^G is not a Bauer simplex.

The result now follows from the fact that if the quasistate space of a C*-algebra (equivalently, the state space of its unitization) is a simplex, then the C*-algebra is commutative and its quasistate space is a Bauer simplex (see e.g. Theorem 2.10.9). □

Chapter 3

Operator system duals and noncommutative convexity

3.1 Introduction

A unital operator system S is a $*$-closed unital subpsace of the bounded operators $B(H)$ on a Hilbert space H. In this chapter, we assume that all operator spaces and operator systems are norm-complete. Choi and Effros [12] gave an abstract characterization of unital operator systems as matrix ordered $*$-vector spaces which contain an archimedean matrix order unit. Using this characterization, it is natural to ask if the dual space S^* is itself a unital operator system.

The dual S^* is at least a complete operator space, and inherits a $*$-operation and matrix ordering from S. One says that S^* is a *matrix ordered operator space*. However, S^* typically fails to have an order unit in infinite dimensions. For instance, if $S = C(X)$ is a commutative C*-algebra, then the dual $C(X)^*$ is the space of Radon measures on the compact Hausdorff space X, which never has an order unit if X is uncountable. So, one requires a theory of *nonunital* operator systems if S^* is to be an operator system. Werner [52] defined nonunital operator systems–which we hereafter refer to as simply "operator systems", as matrix ordered operator spaces which embed completely isometrically and completely order isomorphically into $B(H)$. Werner gave an abstract characterization that extends the Choi-Effros axioms in the unital setting. One would hope that S^* is such an operator system, but it turns out that this is too much to ask. For instance, we have the standard duality $M_n^* \cong M_n$, but this duality is not completely isometric. In fact, an embedding $M_n^* \hookrightarrow B(H)$ cannot be completely isometric and completely order isomorphic at the same time. However, the isomorphism $M_n^* \cong M_n$ is a *complete isomorphism*, inducing completely equivalent matrix norms.

Call an operator system S *dualizable* if the dual matrix ordered operator space S^* embeds into $B(H)$ via a map which is a complete order isomorphism and is *completely bounded below*. That is, S^* can be re-normed with completely equivalent matrix norms in a way that makes it an operator system. Recently, C.K. Ng [40] obtained an intrinsic characterization of dualizability. The operator system S is dualizable if and only if it satisfies the following completely bounded positive decomposition property: There is a constant $C > 0$, such that for every $n \geq 1$ and every selfadjoint $x \in M_n(S)^{\mathrm{sa}}$, there are positives $y, z \in M_n(S)^+$ with $x = y - z$ and $\|y\| + \|z\| \leq C\|x\|$. Using the order unit, every unital operator system S is dualizable with $C = 1$. Similarly, the continuous functional calculus implies that every (possibly nonunital) C*-algebra is dualizable with $C = 1$. So, the dualizable systems form a large class. However, not every operator system is dualizable. For instance, the operator systems

$$S = \{a : [-1,1] \to \mathbb{R} \mid a \text{ is affine and } a(0) = 0\} \subseteq C([-1,1]) \quad \text{and} \\ T = \operatorname{span}\{E_{12}, E_{21}\} \subseteq M_2$$

contain no nonzero positive elements. So, the matrix cones in S^* and T^* are not proper, and these cannot be re-normed into operator systems.

Chapter 2 showed that the study of nonunital operator systems is categorically dual to studying *pointed* noncommutative compact convex sets. This is a nonunital, quantized version of classical Kadison duality for function systems [27]. A noncommutative (nc) convex set is graded into matrix levels

$$K = \coprod_{n \geq 1} K_n \subseteq \coprod_{n \geq 1} M_n(E)$$

over an operator space E, which is closed under direct sums and compression by scalar isometries [15]. Nc convex sets are essentially equivalent to the *matrix* convex sets of Wittstock [53], with the distinction that an nc convex set contains infinite matrix levels up to some infinite cardinal α depending on E. (In separable settings, usually one takes $\alpha = \aleph_0$.) While nc convex sets are determined by their finite levels, one needs the infinite levels to find all nc extreme points. Here when n is an infinite cardinal, we use the convention $M_n = B(H)$, where $\dim H = n$. We say that K is closed/compact if each level K_n is closed/compact.

The canonical example of an nc convex set is the nc state space

$$\mathcal{S}(S) = \coprod_{n \geq 1} \{\varphi : S \to M_n \mid \varphi \text{ is unital and completely positive}\}$$

of a unital operator system S. The unital noncommutative Kadison duality of Webster-Winkler [51] and Davidson-Kennedy [15] asserts that $\mathcal{S}(S)$ completely determines S. If S is a *nonunital* operator system, the appropriate replacement for the nc state space is the *nc quasistate space*

$$\mathcal{QS}(S) = \coprod_{n \geq 1} \{\varphi : S \to M_n \mid \varphi \text{ is completely contractive and positive}\}.$$

If K is a compact nc convex set, and $z \in K_1$ is a prescribed basepoint, we form the nonunital operator system $A(K, z)$ of *pointed* nc affine functions

$$a : (K, z) \to (\mathcal{M}, 0),$$

where $\mathcal{M} = \coprod_{n \geq 1} M_n$. The pair (K, z) is a *pointed* nc convex set if every nc quasistate on $A(K, z)$ is a point evaluation in K. In [30], it was shown that the functor

$$S \mapsto (\mathcal{QS}(S), 0)$$

is a contravariant equivalence of categories between the category of operator systems and the category of pointed compact nc convex sets, with essential inverse $(K, z) \mapsto A(K, z)$.

Via this equivalence, operator systems can be completely described by the nc convex geometry of the nc quasistate space $(K, z) = (\mathcal{QS}(S), 0)$. Our main question is: What geometric condition on $(K, 0)$ detects dualizability of S? We obtain two geometric answers to this question. The first is extrinsic, and the second is intrinsic to K.

Firstly, in Theorem 3.4.9, we show that S is dualizable if and only if there is a Hilbert space H and a pointed embedding

$$(K, 0) \hookrightarrow (L, 0)$$

into the positive nc unit ball

$$\coprod_{n\geq 1} B_1(M_n(B(H)))^+$$

satisfying the following extension property: Every nc affine function on K extends to an nc affine on L with a complete norm bound, and every positive nc affine function $a \in M_n(A(K, 0))^+$ extends to a positive nc affine on L. Equivalently, the restriction map $A(L, 0) \to A(K, 0)$ is an operator space quotient map that maps the positives onto the positives at all matrix levels.

Secondly, in Theorem 3.5.7, we show that Ng's bounded positive decomposition property for S is equivalent to a complete normality condition for S^* in the sense of [3, Section 2.1]. This is equivalent to the geometric condition

$$(K - \mathbb{R}_+ K) \cap \mathbb{R}_+ K \subseteq CK$$

in $\coprod_{n\geq 1} M_n(S^*)$, for some constant $C > 0$. It is equivalent to simply require that the set $(K - \mathbb{R}_+ K) \cap \mathbb{R}_+ K$ is norm-bounded.

The structure of this chapter is as follows. After some preliminaries in Section 3.2, we discuss quotients of matrix ordered operator spaces in Section 3.3. In Section 3.4, given an inclusion $0 \in K \subseteq L$ of pointed compact nc convex sets, we discuss the problem of extending nc affine functions from K to L with norm bounds or while preserving positivity, which characterizes when the restriction map $A(L, 0) \to A(K, 0)$ is a quotient. In Section 3.5, we prove our main results, characterizing dualizability of S via geometric conditions on the nc quasistate space $\mathcal{QS}(S)$. In Section 3.6, we discuss positive generation for a nonunital system S, and show that–in contrast to the classical case, a matrix ordered operator space may be positively generated but not satisfy Ng's condition of bounded positive generation. In Section 3.7, we give some examples and applications. We obtain some permanence properties, showing that quotients and pushouts of dualizable operator systems are again dualizable. Using the nc quasistate space, we obtain a new proof of Choi's Theorem.

3.2 Background

3.2.1 Nonunital operator systems

All vector spaces in this chapter are over $\mathbb{C}$, unless stated otherwise. If V is a vector space and $n \in \mathbb{N}$, we let $M_n(V)$ be the vector space of $n \times n$-matrices with entries in V. Frequently we naturally identify $M_n(V)$ with $M_n \otimes V$, and write for instance

$$1_2 \otimes x = \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}$$

where $x \in V$ and $1_2 \in M_2$ is the identity matrix. We will also use the notation

$$\mathcal{M}(V) := \coprod_{n \geq 1} M_n(V)$$

to denote the matrix universe over V.

If V is any normed vector space and $r \geq 0$, we will frequently use $B_r(V)$ to denote the closed ball in V with radius r and center $0 \in V$.

Following Ng [40], we fix the following definitions. An **operator space** E is a vector space equipped with a complete family of L^∞-matrix norms, which we will denote either by $\|\cdot\|$, $\|\cdot\|_E$, or $\|\cdot\|_{M_n(E)}$ as appropriate.

Definition 3.2.1. A **semi-matrix ordered operator space** (X, P) consists of an operator space X equipped with a conjugate-linear completely isometric involution $x \mapsto x^*$, and a distinguished selfadjoint matrix convex cone $P = \coprod_{n \geq 1} P_n \subseteq \coprod_d M_n(X)^{\mathrm{sa}}$ such that each P_n is norm-closed in $M_n(X)$. Usually we omit the symbol P and write $M_n(X)^+ := P_n$. If in addition each cone $M_n(X)^+$ satisfies $M_n(X)^+ \cap (-M_n(X)^+) = \{0\}$, then we say X is a **matrix ordered operator space**. If X is in addition a dual space $X = (X_*)^*$, we say X is a **dual matrix ordered operator space** if the positive cones $M_n(X)^+$ are weak-$*$ closed.

Definition 3.2.2. A semi-matrix ordered operator space X is **positively generated** if

$$M_n(X)^{\mathrm{sa}} = M_n(X)^+ - M_n(X)^+$$

for all $n \geq 1$.

Example 3.2.3. If X is a positively generated matrix ordered operator space, then X^* is naturally a dual matrix ordered operator space with the standard norm and order structure that identifies

$$\begin{aligned} M_n(X^*) &\cong \mathrm{CB}(X, M_n) \quad \text{isometrically, and} \\ M_n(X^*)^+ &\cong \mathrm{CP}(X, M_n). \end{aligned}$$

Definition 3.2.4. Let X and Y be matrix ordered operator spaces, and let $\varphi : X \to Y$ be a linear map. For any $n \geq 1$, φ induces a linear map $\varphi_n : M_n(X) \to M_n(Y)$. We say that φ is completely bounded, contractive, bounded below, isometric, positive, or a complete order isomorphism when each induced map φ_n satisfies the same property uniformly in n. If φ is completely bounded below and positive, we say φ is a **complete embedding**. If φ is completely isometric and positive, we say φ is a **completely isometric embedding**. If φ is also a linear isomorphism, we call φ a **complete isomorphism** or **completely isometric isomorphism** as appropriate.

The class of all matrix ordered operator spaces forms a category, where one usually chooses the morphisms to be completely contractive and completely positive (ccp) maps, or completely bounded and completely positive (cbp) maps. In the interest of readability, we hereafter adopt the convention that "**completely contractive and positive**" always means "completely contractive and completely positive", and similarly for "completely bounded and positive". That is, "completely" modifies both the words "contractive" and "positive". Since we have no need to consider maps which are positive but not completely positive, there is hopefully no risk of confusion.

Example 3.2.5. Let S be a unital operator system, i.e. an $*$-matrix ordered space with archimedian matrix order unit 1_S. Then S is a matrix ordered operator space with norm

$$\|x\|_{M_n(S)} = \inf\left\{t \geq 0 \mid \begin{pmatrix} t(1_n \otimes 1_s) & x \\ x^* & t(1_n \otimes 1_s) \end{pmatrix} \geq 0 \text{ in } M_{2n}(S)^{\text{sa}}\right\}.$$

This norm agrees with the induced norm from any unital complete order embedding $S \subseteq B(H)$. In particular, for any Hilbert space H, the space $B(H)$ is a unital operator system.

Definition 3.2.6. Let S be a matrix ordered operator space. We say that S is a **quasi-operator system** if there is a complete embedding $S \to B(H)$ for some Hilbert space H, and that S is a **operator space** if there is a completely isometric embedding $S \to B(H)$. If S is in addition a *dual* matrix ordered operator space, then we say S is a dual (quasi-)operator system if there is a weak-$*$ homeomorphic (complete embedding) completely isometric embedding into some $B(H)$.

That is, a quasi-operator system S is a matrix ordered operator space which is completely isomorphic to an operator system. Put another way, one can choose a completely equivalent system of norms on S, for which S embeds completely isometrically and order isomorphically into $B(H)$.

3.2.2 Pointed noncommutative convex sets

Suppose that $E = (E_*)^*$ is a dual operator space. Let

$$\mathcal{M}(E) := \coprod_{n \geq 1} M_n(E),$$

where the union is taken over all cardinals $n \geq 1$ up to some fixed cardinal α at least as large as the density character of E. (In practice we suppress α.) When n is infinite, we take the convention $M_n := B(H_n)$, where H_n is a Hilbert space of dimension n. By naturally identifying

$$M_n(E) = \text{CB}(E_*, E),$$

we may equip each $M_n(E)$ with its corresponding point-weak-$*$ topology. Note that if $E = M_k$, this is the just the usual weak-$*$ topology on $M_n(M_k) \cong M_{nk}$.

Definition 3.2.7. We say that a graded subset

$$K = \coprod_{n \geq 1} K_n \subseteq \mathcal{M}(E)$$

is an **nc convex set** if for every norm-bounded family $(x_i) \in K_{n_i}$ and every family of matrices $\alpha_i \in M_{n_i,n}$ which satisfies

$$\sum_i \alpha_i^* \alpha_i = 1_n, \tag{3.1}$$

we have

$$\sum_i \alpha_i^* x_i \alpha_i \in K_n. \tag{3.2}$$

Here the sums (3.1) and (3.2) are required to converge in the point-weak-$*$ topologies on M_n and $M_n(E)$, respectively. We say in addition that K is a **compact nc convex set** if each matrix level K_n is point-weak-$*$ compact in $M_n(E)$.

Usually we refer to the sum in (3.2) as an **nc convex combination** of the points x_i. Succinctly, an nc convex set is one that is *closed under nc convex combinations*. It is equivalent to require only that K is closed under direct sums (3.1) in which the α_i's are co-isometries with orthogonal domain projections, and *compressions* (3.2) when there is only one α_i, which must be an isometry.

Definition 3.2.8. Let K and L be nc convex sets. A function $a : K \to L$ is an **nc affine function** if it is graded

$$a(K_n) \subseteq L_n, \qquad \text{for all } n \geq 1,$$

and respects nc convex combinations, i.e. whenever $x_i \in K$ are bounded and α_i are scalar matrices of appropriate sizes such that $\sum_i \alpha_i^* x_i \alpha_i$, then

$$a\Big(\sum_i \alpha_i^* x_i \alpha_i\Big) = \sum_i \alpha_i^* a(x_i) \alpha_i.$$

We say a is **continuous** if each restriction $a|_{K_n}$ is point-weak-$*$ continuous.

Classical Kadison duality [27] asserts that the category of **function systems**–partially ordered Banach spaces with an archimedean order unit, is equivalent to the category of compact convex sets with continuous affine functions as morphisms. Noncommutative Kadison duality asserts a similar equivalence for unital operator systems.

Theorem 3.2.9. [51, Proposition 3.5][15, Theorem 3.2.5] *The category of unital operator systems with ucp maps as morphisms is contravariantly equivalent to the category of compact nc convex sets with continuous nc affine functions as morphisms. On objects, the essential inverse functors send an operator system S to its nc state space*

$$\mathcal{S}(S) = \coprod_{n \geq 1} \{\varphi : S \to M_n \mid \varphi \text{ is unital and completely positive}\},$$

and send a compact nc convex set K to the operator system

$$A(K) = \{a : K \to \mathcal{M} = \mathcal{M}(\mathbb{C}) \mid a \text{ continuous nc affine}\}.$$

The operator system structure and norm on $A(K)$ is pointwise, i.e. one identifies $M_n(A(K)) \cong A(K, \mathcal{M}(M_n))$, and declares a matrix valued nc affine function if it takes positive values at every point. The order unit is the "constant function" $x \in K_n \mapsto 1_n \in M_n$. Both essential inverse functors act on morphisms by precomposition. That is, if $\pi : S \to T$ is a ucp map between operator systems, then the corresponding map on state spaces sends $\rho : T \to M_n$ to $\rho\pi : S \to M_n$. Likewise, if $a : K \to L$ is nc affine, then $f \mapsto f \circ a : A(L) \to A(K)$ is nc affine.

Chapter 2 settled the question of Kadison duality for nonunital operator systems. The key challenge is that in the absence of order units, if S is a nonunital operator system then one must remember the whole **nc quasistate space**

$$\mathcal{QS}(S) = \coprod_{n \geq 1} \{\varphi : S \to M_n \mid \varphi \text{ is contractive and completely positive}\}$$

and consider **pointed** nc affine functions which fix the zero quasistates.

Definition 3.2.10. Let K be a compact nc convex set and fix a distinguished point z. We let

$$A(K,z) = \{a \in A(K) \mid a(z) = 0\}$$

denote the operator system of nc affine functions which vanish at z. We say that the pair (K,z) is a **pointed nc convex set** if the natural evaluation map

$$K \to \mathcal{QS}(A(K,z))$$
$$x \mapsto (a \mapsto a(x))$$

is surjective (and hence bijective).

The main subtlety in nonunital Kadison duality is that while the correspondence $S \mapsto (\mathcal{QS}(S), 0)$ is a full and faithful functor, it is only essentially surjective onto the *pointed* compact nc convex sets.

Theorem 3.2.11. [30, Theorem 4.9] *The category of operator systems with ccp maps as morphisms is contravariantly equivalent to the category of pointed compact nc convex sets with pointed continuous nc affine functions as morphism. On objects, the essential inverse functors send an operator system S to is pointed nc quasistate space $(\mathcal{QS}(S), 0)$, and send a pointed compact nc convex set K to the operator system $A(K,z)$ of pointed continuous nc affine functions on (K,z).*

Again, on morphisms the essential inverse functors in Theorem 3.2.11 act in the natural way by precomposition on either nc affine functions or on nc quasistates.

3.3 Quotients of matrix ordered spaces

3.3.1 Operator space quotients

Here, we recall the basic theory of quotients for operator spaces. If E is an operator space, and $F \subseteq E$ is a closed subspace, then the quotient vector space E/F is an operator space where the matrix norms isometrically identify $M_n(E/F)$ with the standard Banach space quotient $M_n(E)/M_n(F)$.

Definition 3.3.1. Let $\varphi : E \to F$ be a completely bounded map between operator spaces E and F. We will say that $\varphi : E \to F$ is a **operator space quotient map with constant** $C > 0$ if any of the following equivalent conditions hold

(1) $B_1(M_n(F)) \subseteq \overline{\varphi_n(B_C(M_n(E)))} = C\overline{\varphi_n(B_1(M_n(E)))}$ for all $n \in \mathbb{N}$.

(2) $B_1(M_n(F)) \subseteq (C+\epsilon)\cdot\varphi_n(B_1(M_n(E)))$ for all $n \in \mathbb{N}$ and every $\epsilon > 0$.

(3) The induced map $\tilde{\varphi} : E/\ker\varphi \to F$ is an isomorphism and satisfies $\|\tilde{\varphi}^{-1}\|_{\text{cb}} \leq C$.

The equivalence of (1) and (2) follows from a standard series argument using completeness of E. We will simply say **operator space quotient map** if we have no need to refer to C explicitly.

The following fact is standard in operator space theory, but we provide a proof for completeness.

Proposition 3.3.2. *Let $\varphi : E \to F$ be a completely bounded map between operator spaces E and F. The map φ is a quotient map with constant $C > 0$ if and only if the dual map $\varphi^* : F^* \to E^*$ is completely bounded below by $1/C$. Moreover, in this case, φ^* is weak-$*$ homeomorphism onto its range.*

Proof. Suppose that $C\varphi_n(B_1(M_n(E)))$ is dense in $B_1(M_n(F))$ for every n. Given $f \in M_m(F^*) \cong \mathrm{CB}(F, M_m)$, approximating unit vector $y \in B_1(M_n(F))$ with vectors of the form $\varphi(x)$ for $x \in B_C(E)$ shows that $\|f\|_{\mathrm{cb}} \le C\|\varphi_m^*(f)\|_{\mathrm{cb}}$.

Conversely, suppose that

$$\coprod_{n\ge 1} B_1(M_n(F)) \notin C \coprod_{n\ge 1} \varphi_n(B_1(M_n(E))).$$

By the Effros-Winkler nc Bipolar theorem [19], there are $m, n \ge 1$, an $x \in CB_1(M_n(E))$, and an $f \in M_m(F^*) \cong \mathrm{CB}(F, M_m)$, such that

$$\operatorname{Re} f_k(y) \le 1_{mk} \quad \text{for all } k \ge 1, y \in B_1(M_k(F)),$$

and yet $\operatorname{Re} f_n(x) \nleq 1_{mn}$. It follows that $\|f\| \le 1$, but $\|x\| \le C$ and $\|f_n(\varphi_n(x))\| > 1$, so $\|\varphi_m^*(f)\| > \|f\|_{\mathrm{cb}}/C$. This shows φ^* is not completely bounded below by $1/C$.

Finally, if φ is an operator space quotient map, it is bounded and surjective, and so its dual map φ^* is weak-$*$ homeomorphic onto its range. □

3.3.2 Matrix ordered operator space quotients

Definition 3.3.3. Let X be a matrix ordered operator space. We call a closed subspace $J \subseteq X$ a **kernel** if it is the kernel of a ccp map $\varphi : X \to Y$ for some matrix ordered operator space Y. In this case, we define an matrix ordered operator space structure on the operator space X/J with involution

$$(x + J)^* := x^* + J$$

and matrix order

$$M_n(X/J)^+ := \overline{\{x + M_n(J) \mid x \in M_n(X)^+\}},$$

where the closure is taken in the quotient norm topology on $M_n(X/J) \cong M_n(X)/M_n(J)$.

Proposition 3.3.4. *If X is a matrix ordered operator space, and $J = \ker\varphi$ is a kernel, then X/J is a matrix ordered operator space.*

Proof. Since the involution on X is completely isometric and J is selfadjoint, it follows that the involution on $M_n(X/J)$ is completely isometric. It is straightforward to check that X/J is a matrix ordered operator space. To prove that it is a matrix ordered operator space, suppose $x + J \in M_n(X/J)^+ \cap (-M_n(X/J)^+)$. Then for any ϵ, there are $y, z \in M_n(X)^+$ with $\|x - y + M_n(J)\|, \|x + z + M_n(J)\| < \epsilon$. Hence

$$\|\varphi_n(x) - \varphi_n(y)\| \le \|x - y + M_n(J)\| < \epsilon$$

and similarly $\|\varphi_n(x) + \varphi_n(z)\| < \epsilon$. Since φ is cp, $\varphi_n(y), \varphi_n(z) \ge 0$. As ϵ is arbitrary and Y is a matrix ordered operator space, this shows

$$\varphi_n(x) \in \overline{M_n(Y)^+} \cap (-\overline{M_n(Y)^+}) = M_n(Y)^+ \cap (-M_n(Y)^+) = \{0\}.$$

Therefore $x \in M_n(\ker \varphi) = M_n(J)$, and so $x + M_n(J) = 0$. This shows

$$M_n(X/J)^+ \cap (-M_n(X/J)^+) = \{0\},$$

so X/J is a matrix ordered operator space. □

One can form a category of matrix ordered operator spaces with morphisms as either completely contractive and positive (ccp) or completely bounded and positive (cbp) maps. If V is a normed vector space and $r > 0$, then we let $B_r(V)$ denote the closed ball in V with radius r and center 0.

Definition 3.3.5. Let X and Y be matrix ordered operator spaces, and let $\varphi : X \to Y$ be a cbp map. We say that φ is a **matrix ordered operator space quotient map with constant** $C > 0$ if for all $n \in \mathbb{N}$ we have both

(1) $B_1(M_n(Y)) \subseteq C\overline{\varphi_n(B_1(M_n(X)))}$, and

(2) $M_n(Y)^+ = \overline{\varphi_n(M_n(X))^+}$.

For brevity, we will usually simply refer to φ as a **quotient map**, whenever it is clear that we are speaking only in the context of matrix ordered operator spaces.

That is, a matrix ordered operator space quotient map is just an operator space quotient map that maps the positives (densely) onto the positives at each matrix level. Comparing to Definition 3.3.1.(2), a quotient map is surjective. Each map $\varphi_n : M_n(X) \to M_n(Y)$ is therefore open and closed, and since the positive cones $M_n(X)^+$ and $M_n(Y)^+$ are norm-closed, it follows that $\varphi_n(M_n(X)^+)$ is closed and $\varphi_n(M_n(X)^+) = M_n(Y)^+$ for all n. That is, the closure in condition (2) is redundant. The first thing to show is that such maps are in fact categorical quotients in the category of matrix ordered operator spaces.

Proposition 3.3.6. *Let $\varphi : X \to Y$ be a cbp map between matrix ordered operator spaces. The following are equivalent.*

(1) The map φ is a quotient map with constant $C > 0$.

(2) The dual map $\varphi^* : Y^* \to X^*$ is completely bounded below and a complete order injection.

(3) With $J = \ker \varphi$, the induced map $\tilde{\varphi} : X/J \to Y$ such that

$$\begin{array}{ccc} X & \xrightarrow{\ \varphi\ } & Y \\ \big\downarrow{\scriptstyle q} & \nearrow_{\tilde{\varphi}} & \\ X/J & & \end{array}$$

commutes is an isomorphism with cbp inverse satisfying $\|\tilde{\varphi}^{-1}\|_{cb} \leq C$.

(4) For every matrix ordered operator space Z and cbp map $\psi : X \to Z$ with $\ker \varphi \subseteq \ker \psi$, there is a unique cbp map $\tilde{\psi} : Y \to Z$ making the diagram

$$\begin{array}{ccc} X & \xrightarrow{\ \psi\ } & Z \\ \big\downarrow{\scriptstyle \varphi} & \nearrow_{\tilde{\psi}} & \\ Y & & \end{array}$$

commute, with $\|\tilde{\psi}\|_{cb} \leq C\|\psi\|_{cb}$.

In this case, φ^* is weak-$*$ homeomorphic onto its range.

Proof. To prove (1) and (2) are equivalent, after invoking Proposition 3.3.2, it suffices to show that φ^* is a complete order injection if and only if Condition (2) in Definition 3.3.5 holds. Note that because φ is completely positive, so is φ^*. Suppose $\varphi_n(M_n(X)^+)$ is dense in $M_n(Y)^+$ for every $n \geq 0$. Let $f \in M_m(Y^*)$ with $\varphi_m^*(f) \geq 0$. Given $n \geq 1$ and $y \in M_n(Y)^+$, approximating y with points of the form $\varphi_n(x_i)$ for $x_i \in M_n(X)^+$ shows that

$$f_n(y) = \lim_i f_n(\varphi_n(x_i)) = \lim_i (\varphi_m^*(f))_n(x_i) \geq 0.$$

This shows $f \geq 0$.

Conversely, suppose that $\varphi_n(M_n(X)^+)$ is not dense in $M_n(Y)^+$ for some $n \geq 1$. By the Effros-Winkler nc Bipolar Theorem [19] applied to the closed nc convex sets

$$\coprod_{k\geq 1} \overline{\varphi_k(M_n(X)^+)} \quad \not\supseteq \quad \coprod_{k\geq 1} M_k(Y)^+,$$

there is a selfadjoint matrix functional $f \in M_m(Y^*)^{\text{sa}}$ such that $f_k(y) \geq -1_{mk}$ for every k and every $y \in M_k(Y)^+$, but

$$f_n(z) \not\geq -1_{mn}$$

for some $z \in \overline{\varphi_n(M_n(X)^+)} \setminus M_k(Y)^+$. A rescaling argument shows that $f \geq 0$ in $M_k(Y)$. However, approximating x by points of the form $\varphi_n(x)$, $x \in M_n(X)^+$ shows that $\varphi_m^*(f)$ cannot be positive. Hence, φ^* is not a complete order isomorphism.

If φ is a quotient map with constant $C > 0$, then it follows immediately from the definition of the matrix order and matrix norms on X/J that $\tilde{\varphi} : X/J \to Y$ is a complete order and norm isomorphism with $\|\tilde{\varphi}^{-1}\|_{\text{cb}} \leq C$. Conversely, note that by definition the quotient map $q : X \to X/J$ is a quotient map with constant 1. Hence, if $\tilde{\varphi}$ is a complete order isomorphism with $\|\tilde{\varphi}^{-1}\|_{\text{cb}} \leq C$, it follows that $\varphi = \tilde{\varphi} \circ q$ is a quotient map with constant C. This proves (1) and (3) are equivalent.

To show (3) and (4) are equivalent, it is enough to note that the quotient map $q : X \to X/J$ satisfies the universal property (4) with constant $C = 1$. In detail, if (3) holds, composing the universal map from (4) applied to $q : X \to X/J$ with $\tilde{\varphi}^{-1}$ shows that (4) holds for φ with constant C. Conversely, if (4) holds, then it holds for both φ and q, and there are induced maps $\tilde{\varphi} : X/J \to Y$ and $\tilde{q} : Y \to X/J$ with $\|\tilde{q}\| \leq \|\tilde{\varphi}\|_{\text{cb}}$ and $\|\tilde{q}\| \leq C\|q\|_{\text{cb}} = C$. Comparing diagrams shows $\tilde{q} = \tilde{\varphi}^{-1}$, and $\tilde{\varphi}$ is an isomorphism. □

Condition (4) in Proposition 3.3.6 shows that a matrix ordered operator space quotient map is a categorical quotient in the category of matrix ordered operator spaces with cbp maps as morphisms. Moreover, the norm bound shows that a quotient map with constant $C = 1$ is a categorical quotient in the subcategory of matrix ordered operator spaces with ccp maps as morphisms.

Remark 3.3.7. Every unital operator system is a matrix ordered operator space, and so if $\varphi : S \to T$ is a ucp map between operator systems with $J = \ker S$, we may form the quotient matrix ordered operator space $S/\ker\varphi$, but there is no a priori guarantee that this quotient is again an operator system. The matrix ordered operator space quotient is generally *not* isomorphic to the unital operator system quotient defined by Kavruk, Paulsen, Todorov, and Tomforde [29]. For example, they show in [29, Example 4.4] that the order norm on the unital operator system quotient need not be completely equivalent to the quotient operator space norm.

3.4 Extension property for compact nc convex sets

If $K = \coprod_n K_n$ is a compact nc convex set, we will define

$$\operatorname{span}_{\mathbb{R}} K := \coprod_{n\geq 1} \operatorname{span}_{\mathbb{R}} K_n \subseteq \mathcal{M}(E).$$

The set $\operatorname{span}_{\mathbb{R}} K$ is also nc convex, but need not be closed in E.

Lemma 3.4.1. *Let $0 \in K \subseteq \mathcal{M}(E)$ be a compact nc convex set containing 0. Let $K - K$ denote the levelwise Minkowski difference of K with itself. Then we have inclusions*

$$\frac{K-K}{2} \subseteq \overline{\operatorname{ncconv}}(K \cup (-K)) \subseteq K - K.$$

Consequently, $\overline{\operatorname{ncconv}}(K \cup (-K)) \subseteq \operatorname{span}_{\mathbb{R}} K$.

Proof. It is immediate that $(K-K)/2 \subseteq \operatorname{ncconv}(K \cup (-K)) \subseteq \overline{\operatorname{ncconv}}(K \cup (-K))$. Given $z \in \operatorname{ncconv}(K \cup (-K))_n$, we can write

$$z = \sum_i \alpha_i^* x_i \alpha_i - \sum_j \beta_j^* y_j \beta_j$$

for uniformly bounded families $\{x_i\}, \{y_i\}$ in K and matrix coefficients satisfying $\sum_i \alpha_i^* \alpha_i + \sum_j \beta_j^* \beta_j = 1_n$. Since $0 \in K$ and $\sum_i \alpha_i^* \alpha_i \leq 1$, we have $x := \sum_i \alpha_i^* x_i \alpha_i \in K_n$. Similarly $y := \sum_j \beta_j^* y_j \beta_j \in K$, and so $z = x - y$ is in $(K-K)_n = K_n - K_n$. Therefore

$$\operatorname{ncconv}(K \cup (-K)) \subseteq K - K,$$

and since the latter is compact, $\overline{\operatorname{ncconv}}(K \cup (-K)) \subseteq K - K$. □

When $0 \in K$, by extending the inclusion map $K \subseteq \coprod_n M_n(A(K,0)^*)$ linearly at each level, we will think of elements in $(\operatorname{span}_{\mathbb{R}} K)_n$ as nc functionals in

$$M_n(A(K,0)^*) = \operatorname{CB}(A(K,0), M_n).$$

Proposition 3.4.2. *Let $0 \in K \subseteq \mathcal{M}(E)$ be a compact nc convex set in a dual operator space $E = (E_*)^*$. For each $n \in \mathbb{N}$, the inclusion $K \to QS(A(K,0))$ extends uniquely to a well-defined nc affine isomorphism*

$$\eta : \coprod_{n\geq 1} \operatorname{span}_{\mathbb{R}} K_n \to \coprod_{n\geq 1} M_n(A(K,0)^*)^{sa}$$

which is levelwise linear. The norm unit ball in $M_n(A(K,0)^*)^{sa}$ is

$$B_1(M_n(A(K,0)^*)) = CC(A(K,0), M_n) = \overline{\operatorname{ncconv}}(\eta(K) \cup (-\eta(K)))_n,$$

and for each n, η is homeomorphic on $K - K$.

Proof. Since K_n is convex, we have $\operatorname{span}_{\mathbb{R}} K_n = \{sx - ty \mid x, y \in K_n, s, t \geq 0\}$. Given $sx - ty \in \operatorname{span}_{\mathbb{R}} K_n$, we define

$$\eta(sx - ty)(a) = sa(x) - ta(y)$$

for $a \in A(K, 0)$. Since such functions a are affine and satisfy $a(0) = 0$, it follows that $\eta|_{K_n}$ is well-defined and linear, and that η is nc affine. Since E_* contains a separating family of functionals, which restrict to nc affine functions in $A(K, 0)$, the map η is injective.

Next we will show the closed unit ball is

$$B_1(M_n(A(K,0)^*)^{\text{sa}}) = \overline{\text{ncconv}}(\eta(K) \cup (-\eta(K)))_n$$

for every n. That is, if L is the compact nc convex set

$$L = \coprod_{n \geq 1} L_n = \coprod_{n \geq 1} B_1(M_n(A(K,0)^*)^{\text{sa}}),$$

we want to show $L = \overline{\text{ncconv}}(\eta(K) \cup (-\eta(K)))$. Since $\eta(K)$ consists of nc quasistates on $A(K, 0)$, it is clear that $L \supseteq \overline{\text{ncconv}}(\eta(K) \cup (-\eta(K)))$. To prove the reverse inclusion, by the nc Bipolar theorem of Effros and Winkler [19], it suffices to suppose that for some $n \in \mathbb{N}$ and $a \in M_n(A(K,0))^{\text{sa}}$ that we have

$$\varphi_n(a) \leq 1_k \otimes 1_n = 1_{kn}$$

for all $k \in \mathbb{N}$ and all $\varphi \in \overline{\text{ncconv}}(\eta(K) \cup (-\eta(K)))$, and then show that $\psi_n(a) \leq 1_k \otimes 1_n$ for all k and all $\psi \in L_k$. Because $\overline{\text{ncconv}}(\eta(K) \cup (-\eta(K)))$ contains both $\eta(K)$ and $-\eta(K)$, we have

$$-1_{kn} \leq a(x) \leq 1_{kn}$$

for all k and all $x \in K_k$. Hence $\|a\|_{M_n(A(K,0))} \leq 1$, and so $\psi_n(a) \leq \|a\| 1_{kn} \leq 1_{kn}$ for every $\psi \in L$. This proves $L = \overline{\text{ncconv}}(\eta(K) \cup (-\eta(K)))$, and consequently η is also surjective. Since η is homeomorphic on K and $K - K$ is (levelwise) compact, it is easy to check that η is continuous on each $K_n - K_n$. Being a continuous injection on a compact Hausdorff space, the map $\eta|_{K_n - K_n}$ is automatically a homeomorphism onto its range. □

Recall that the pair $(K, 0)$ in Proposition 3.4.2 is a **pointed nc convex set** exactly when we have

$$\coprod_{n \geq 1} B_1(M_n(A(K,0)^*)^+) = \text{QS}(A(K,0)) = \eta(K).$$

In practice, we will often identify $M_n(A(K,0)^*)^{\text{sa}}$ with $\operatorname{span}_{\mathbb{R}} K_n$ and so omit the symbol η. Note that since η is homeomorphic on $K - K \supseteq \overline{\text{ncconv}}(K \cup (-K))$ (Lemma 3.4.1), we are free to identify

$$\overline{\text{ncconv}}(\eta(K) \cup (-\eta(K))) = \eta(\overline{\text{ncconv}}(K \cup (-K))).$$

That is, when we identify $M_n(A(K,0)^*)^{\text{sa}} = \operatorname{span}_{\mathbb{R}} K_n$, the unit ball of $M_n(A(K,0)^*)^{\text{sa}}$ is $\overline{\text{ncconv}}(K \cup (-K))_n$.

For a closed convex set X in a vector space V containing 0, we use the usual Minkowski functional

$$\gamma_X(v) := \inf\{t \geq 0 \mid v \in tX\}, \qquad v \in V.$$

If $0 \in K = \bigcup_n K_n$ is a compact nc convex set over a dual operator space E, we will use the shorthand

$$\gamma_K(x) = \gamma_{K_n}(x)$$

when $x \in M_n(E)$.

Definition 3.4.3. [49] If X is a closed convex set in some vector space V, then for $d \in V$, we define the **width of** V **(with respect to** d**)** or the d**-width of** V as

$$\begin{aligned} |X|_d &:= \sup\{t \geq 0 \mid td \in X - X\} \\ &= \frac{1}{\gamma_{X-X}(d)}. \end{aligned}$$

Definition 3.4.4. If $K = \coprod_n K_n \subseteq \mathcal{M}(E)$ is a closed nc convex set over a dual operator space E, then for any n and any $d \in M_n(E)$ we define the **width**

$$|K|_d := |K_n|_d = \frac{1}{\gamma_{K-K}(d)},$$

Lemma 3.4.5. *If $0 \in K \subseteq \mathcal{M}(E)$ is a compact nc convex set containing 0, then for $d \in M_n(E)$, we have $|K|_d > 0$ if and only if $d \in \operatorname{span}_{\mathbb{R}} K$. Moreover, for $d \in \operatorname{span}_{\mathbb{R}} K$, we have*

$$\frac{1}{|K|_d} \leq \|\eta(d)\|_{M_n(A(K,0)^*)} \leq \frac{2}{|K|_d}.$$

That is, $d \mapsto 1/|K|_d = 1/|K_n|_d$ defines a norm on $\operatorname{span}_{\mathbb{R}} K_n$ that is equivalent to the norm induced by the isomorphism $\eta : \operatorname{span}_{\mathbb{R}} K_n \to M_n(A(K,0)^*)^{sa}$.

Proof. By Lemma 3.4.1, we have inclusions

$$\frac{K-K}{2} \subseteq \overline{\operatorname{ncconv}}(K \cup (-K)) \subseteq K - K.$$

It follows that for $d \in \operatorname{span}_{\mathbb{R}} K$, we have

$$2\gamma_{K-K}(d) \geq \gamma_{\overline{\operatorname{ncconv}}(K\cup(-K))}(d) \geq \gamma_{K-K}(d).$$

By definition, $\gamma_{K-K} = 1/|K|_d$. By Proposition 3.4.2, the norm unit ball of $M_n(A(K,0)^*)^{\mathrm{sa}}$ is

$$\overline{\operatorname{ncconv}}(\eta(K) \cup (-\eta(K))) = \eta(\overline{\operatorname{ncconv}}(K \cup (-K))),$$

and hence $\gamma_{\overline{\operatorname{ncconv}}(K\cup(-K))}(d) = \gamma_{\overline{\operatorname{ncconv}}(\eta(K)\cup(-\eta(K)))}(\eta(d)) = \|\eta(d)\|$. □

Given compact nc convex sets $0 \in L \subseteq K$. The restriction map $\rho : A(K,0) \to A(L,0)$ is always completely contractive and positive, and has dense range. When is this map an operator space quotient map? Equivalently, this means there is a constant $C > 0$ so that any nc affine function $g \in M_n(A(L,0))$ extends to an nc affine function f on all of K with

$$f|_L = g \quad \text{and} \quad \|f\|_{M_n(A(K,0))} \leq C\|g\|_{M_n(A(L,0))}.$$

Here is a noncommutative version of [49, Theorem 1].

Proposition 3.4.6. *Let $0 \in L \subseteq K \subseteq \mathcal{M}(E)$ be compact nc convex sets containing 0. The following are equivalent*

(1) *The restriction map* $A(K) \to A(L)$ *is an operator space quotient map.*

(2) *The restriction map* $\rho : A(K,0) \to A(L,0)$ *is an operator space quotient map.*

(3) *The dual map* $\rho^* : A(L,0)^* \to A(K,0)^*$ *is completely bounded below.*

(4) *There is a constant* $c > 0$ *such that for all* $n \geq 1$ *and all* $d \in M_n(E)$ *with* $|L|_d > 0$*, we have*
$$|L|_d \geq c|K|_d.$$

(5) *There is a constant* $C > 0$ *such that*
$$(K - K) \cap \operatorname{span}_{\mathbb{R}} L \subseteq C(L - L).$$

Proof. Clearly (1) implies (2). Suppose $\rho : A(K,0) \to A(L,0)$ is an operator space quotient map with constant $C \geq 0$. Given $a \in A(L)$, we have $a - a(0) \otimes 1_{A(L)} \in A(L,0)$. Thus there is a $b \in A(K,0)$ with $b|_L = a - a(0) \otimes 1_{A(L)}$ and $\|b\| \leq C\|a - a(0) \otimes 1_{A(L)}\| \leq 2C\|a\|$. Then, $b + a(0) \otimes 1_{A(K)} \in A(K)$ restricts to a on L and satisfies $\|b + a(0) \otimes 1_{A(K)}\| \leq \|b\| + \|a\| \leq (2C+1)\|a\|$. This proves $A(K) \to A(L)$ is an operator space quotient map with constant $2C + 1$, so (2) implies (1).

The equivalence of (2) and (3) is Proposition 3.3.2. To prove (3) is equivalent to (4), first note by taking real and imaginary parts that (3) occurs if and only if the restrictions $\rho_n^* : M_n(A(L,0)^*)^{\mathrm{sa}} \to M_n(A(K,0)^*)^{\mathrm{sa}}$ are bounded below by a universal constant. By Proposition 3.4.2, we may identify
$$\operatorname{span}_{\mathbb{R}} L_n = M_n(A(L,0)^*)^{\mathrm{sa}} \quad \text{and} \quad \operatorname{span}_{\mathbb{R}} K_n = M_n(A(K,0)^*)^{\mathrm{sa}}.$$
With this identification, ρ^* is just the inclusion map $\operatorname{span}_{\mathbb{R}} L_n \to \operatorname{span}_{\mathbb{R}} K_n$. By Lemma 3.4.5, the induced norms on $\operatorname{span}_{\mathbb{R}} L$ and $\operatorname{span}_{\mathbb{R}} K$ are completely equivalent to $d \mapsto 1/|L|_d$ and $d \mapsto 1/|K|_d$. Thus the dual map ρ^* is completely bounded below if and only if for some constant $c > 0$, we have
$$\frac{1}{c|K|_d} \leq \frac{1}{|L|_d} \iff |L|_d \geq c|K|_d$$
whenever $d \in \operatorname{span}_{\mathbb{R}} L = \{d \in \mathcal{M}(E) \mid |L|_d > 0\}$, by Lemma 3.4.5.

For $d \in \mathcal{M}(E)$, recall that $|K|_d = \frac{1}{\gamma_{K-K}(d)}$ and $|L|_d = \frac{1}{\gamma_{L-L}(d)}$. Hence condition (3) holds if and only if
$$\gamma_{L-L}|_{\operatorname{span}_{\mathbb{R}} L} \leq \frac{1}{c}\gamma_{K-K}|_{\operatorname{span}_{\mathbb{R}} L} = \gamma_{c(K-K)}|_{\operatorname{span}_{\mathbb{R}} L}.$$
Using only the definition of the Minkowski gauges γ_{K-K} and γ_{L-L}, this holds if and only if
$$c(K - K) \cap \operatorname{span}_{\mathbb{R}} L \subseteq L - L.$$
Hence condition (4) holds with constant $c > 0$ if and only if condition (5) holds with constant $C = 1/c > 0$. ☐

Note that for any general inclusion $L \subseteq K$ of compact nc convex sets, we can freely translate to assume $0 \in L$ and apply Proposition 3.4.6. Thus conditions (1), (4), and (5) are equivalent in total generality. Note also that we do not require in 3.4.6 that $(L,0)$ and $(K,0)$ are *pointed* nc convex sets.

Example 3.4.7. It is possible that the restriction map $A(K,0) \to A(L,0)$ in Proposition 3.4.6 is surjective but not an operator space quotient. For instance, let E be an infinite dimensional Banach space. Let $\max(E)$ and $\min(E)$ denote E equipped with its maximal and minimal operator space norms which restrict to the usual norm on E [19, Section 3.3]. There are standard operator space dualities $\max(E)^* \cong \min(E^*)$ and $\min(E)^* \cong \max(E^*)$. As E is infinite dimensional, the maximal and minimal matrix norms on E are not completely equivalent [44, Theorem 14.3]. So, the identity map $\max(E) \to \min(E)$ is surjective and not an operator space quotient map. Consider the minimal and maximal nc unit balls

$$K = \coprod_{n\geq 1} B_1(M_n(\min(E^*))) \quad \text{and} \quad L = \coprod_{n\geq 1} B_1(M_n(\max(E^*)))$$

in $\mathcal{M}(E^*)$. By the dualities $\max(E)^* \cong \min(E^*)$ and $\min(E)^* \cong \max(E^*)$, we have

$$A(K,0) \cong \max(E) \quad \text{and} \quad A(L,0) \cong \min(E)$$

completely isometrically. The restriction map $A(K,0) \to A(L,0)$ is just the identity map $\max(E) \to \min(E)$, which is surjective, but not an operator space quotient map.

Proposition 3.4.6 provides a guarantee that every matrix-valued nc affine function on L lifts to an nc affine function on K with a complete norm bound. However, there is no guarantee that we can lift a *positive* affine function to one that is positive. For instance, the restriction map of function systems

$$A([-1,1],0) \to A([0,1],0)$$

is an operator space quotient map with constant $c = 1$, but does not map the positives onto the positives because $A([-1,1],0)^+ = \{0\}$.

Proposition 3.4.8. *Let $0 \in L \subseteq K \subseteq \mathcal{M}(E)$ be compact nc convex sets such that $(0,L)$ and $(0,K)$ are pointed compact nc convex sets. Let $\rho : A(K,0) \to A(L,0)$ be the restriction map. The following are equivalent*

1. *For all $n \geq 1$, $\overline{\rho_n(M_n(A(K,0)^+))} = M_n(A(L,0))^+$.*
2. *The dual map $\rho^* : A(L,0)^* \to A(K,0)^*$ is a complete order embedding.*
3. *$K \cap \operatorname{span}_{\mathbb{R}} L \subseteq \mathbb{R}_+ L$.*
4. *$K \cap \overline{\operatorname{ncconv}}(L \cup (-L)) = L$.*

Proof. To prove (1) $\iff$ (2), consider the closed nc convex sets

$$P = \coprod_{n\geq 1} M_n(A(L,0))^+ \quad \text{and} \quad Q = \coprod_{n\geq 1} \overline{\rho_n(M_n(A(K,0))^+)}.$$

By the nc Bipolar theorem of Effros and Winkler [19], we have $Q = P$ if and only if their nc polars Q^π and P^π are equal. But by scaling, we have

$$\begin{aligned} P^\pi &= \{\varphi \in M_k(A(L,0)^*) \mid k \in \mathbb{N}, \operatorname{Re} \varphi_n(b) \leq 1_{nk} \text{ for all } n \geq 1, b \in P_n\} \\ &= \{\varphi \in M_k(A(L,0)^*) \mid k \in \mathbb{N}, \operatorname{Re} \varphi \leq 0\} \end{aligned}$$

and similarly

$$\begin{aligned}Q^\pi &= \{\varphi \in M_k(A(L,0)^*) \mid k \in \mathbb{N}, \operatorname{Re} \varphi_n(\rho_n(a)) \le 1_{nk} \text{ for all } n \ge 1, a \in M_n(A(K,0))^+\}\\ &= \{\varphi \in M_k(A(L,0)^*) \mid k \in \mathbb{N}, \operatorname{Re} \rho_k^*(\varphi) \le 0\}.\end{aligned}$$

Thus $P = Q$ if and only if ρ^* is a complete order injection.

When we identify $A(K,0)^* = \operatorname{span}_{\mathbb{R}} K_1$ and $A(L,0)^* = \operatorname{span}_{\mathbb{R}} L_1$ as in Proposition 3.4.2, the dual map $\rho^* : \operatorname{span}_{\mathbb{R}} L \to \operatorname{span}_{\mathbb{R}} K$ is just the inclusion map. Since $(K,0)$ and $(L,0)$ are pointed, the positive cones in $M_n(A(K,0)^*) = \operatorname{span}_{\mathbb{R}} K_n$ and $M_n(A(L,0)^*) = \operatorname{span}_{\mathbb{R}} L_n$ are just $\mathbb{R}_+ K_n$ and $\mathbb{R}_+ L_n$, respectively. Hence the inclusion map is a complete order injection if and only if we have

$$\mathbb{R}_+ K \cap \operatorname{span}_{\mathbb{R}} L = \mathbb{R}_+ L.$$

A rescaling argument shows that this is equivalent to

$$K \cap \operatorname{span}_{\mathbb{R}} L \subseteq \mathbb{R}_+ L,$$

and so (2) and (3) are equivalent.

If $K \cap \overline{\operatorname{ncconv}}(L \cup (-L)) = L$, then scaling gives

$$\mathbb{R}_+(K \cap \operatorname{span}_{\mathbb{R}} L) = \mathbb{R}_+ K \cap \operatorname{span}_{\mathbb{R}} L = \mathbb{R}_+ L,$$

which is again equivalent to (3), so (4) implies (3). Now suppose that $K \cap \operatorname{span}_{\mathbb{R}} L \subseteq \mathbb{R}_+ L$. Clearly $L \subseteq K \cap \overline{\operatorname{ncconv}}(L \cup (-L))$. Conversely, if $x \in K \cap \overline{\operatorname{ncconv}}(L \cup (-L))$, then by Lemma 3.4.1, we also have $x \in K \cap \operatorname{span}_{\mathbb{R}} L = \mathbb{R}_+ L$. Hence

$$x \in \overline{\operatorname{ncconv}}(L \cup (-L)) \cap \mathbb{R}_+ L.$$

Because $(L,0)$ is pointed, this implies $x \in L$, proving that (3) implies (4). □

Combining Propositions 3.4.6 and 3.4.8 yields

Theorem 3.4.9. *Let $(L,0)$ and $(K,0)$ be pointed compact nc convex sets with $L \subseteq K \subseteq \mathcal{M}(E)$. The following are equivalent.*

(1) The restriction map $A(K,0) \to A(L,0)$ is a matrix ordered operator space quotient map.

(2) There is a constant $C > 0$ such that

(i) $(K - K) \cap \operatorname{span}_{\mathbb{R}} L \subseteq C(L - L)$, and

(ii) $K \cap \operatorname{span}_{\mathbb{R}} L \subseteq \mathbb{R}_+ L$.

3.5 Dualizability via nc quasistate spaces

Recall that the trace class operators $\mathcal{T}(H) = B(H)_*$ inherit a matrix ordered operator space structure via the embedding $\mathcal{T}(H) = B(H)_* \subseteq B(H)^*$, where $B(H) \cong (B(H)_*)^*$ completely isometrically and order isomorphically. By Ng's [40] results, since $B(H)$ is a C*-algebra, $B(H)^*$ is an operator system, and so $\mathcal{T}(H) = B(H)_* \subseteq B(H)^*$ is also an operator system. The nc quasistate space of $\mathcal{T}(H)$ is the compact nc convex set

$$\mathcal{P}(H) := \coprod_n M_n(B(H))_1^+ = \coprod_n \{x \in M_n(B(H)) | x \geq 0, \|x\| \leq 1\}.$$

Applying Theorem 3.4.9 and Proposition 3.3.6 yields the following extrinsic geometric characterization of dualizability for an operator system.

Corollary 3.5.1. *Let S be an operator system with pointed nc quasistate space $(K, 0)$, and let H be a Hilbert space. The following are equivalent.*

(1) There is a weak-$*$ homeorphic complete embedding $S^* \to B(H)$.

(2) There is a matrix ordered operator space quotient map $\mathcal{T}(H) \to S$.

(3) There is a pointed continuous nc affine injection $\varphi : (K, 0) \to \mathcal{P}(H)$ such that

(i) $(\mathcal{P}(H) - \mathcal{P}(H)) \cap \operatorname{span}_{\mathbb{R}} \varphi(K) \subseteq C(\varphi(K) - \varphi(K))$ for some constant $C > 0$, and

(ii) $\mathcal{P}(H) \cap \operatorname{span}_{\mathbb{R}} \varphi(K) \subseteq \mathbb{R}_+ \varphi(K)$.

Definition 3.5.2. Let E be an ordered $*$-Banach space with closed positive cone E^+. We say E is **α-generated** for a constant $\alpha > 0$ if for each $x \in E^{sa}$, we can write

$$x = y - z$$

for $y, z \in E^+$ satisfying $\|y\| + \|z\| \leq \|x\|$. Or, equivalently,

$$B_1(E) = \alpha \operatorname{conv}(B_1(E^+) \cup (-B_1(E^+))).$$

If X is a matrix ordered operator space, then we say X is **completely α-generated** if each matrix level $M_n(X)$ is α-generated.

In [40, Theorem 3.9], Ng proved that an operator system S is dualizable if and only if it is completely α-generated for some $\alpha > 0$. The following definition is the dual property of α-generation.

Definition 3.5.3. An ordered $*$-Banach space E is **α-normal** for some $\alpha > 0$ if for all $x, y, z \in E^{sa}$,

$$x \leq y \leq z \implies \|y\| \leq \alpha \max\{\|x\|, \|y\|\}. \tag{3.3}$$

If X is a matrix ordered operator space, then X is **completely α-normal** if each matrix level $M_n(X)$ is α-normal.

The condition of α-normality can be viewed as a strict requirement about how the norm and order structure on E interact. Normality means that "order bounds" $x \leq y \leq z$ should imply "norm bounds" $\|x\| \leq \alpha \max\{\|y\|, \|z\|\}$. If one does not care about the exact value of α, it is enough to check the normality identity (3.3) on positive elements in the special case $x = 0$.

Proposition 3.5.4. *If E is an ordered $*$-Banach space, then E is α-normal for some $\alpha > 0$ if and only if there is a constant $\beta > 0$ such that*

$$0 \le x \le y \implies \|x\| \le \beta\|y\| \tag{3.4}$$

for $x, y \in E^+$.

Proof. If E is α-normal, then (3.4) holds with $\beta = \alpha$. Conversely, suppose (3.4) holds, and let $x \le y \le z$ in E^{sa}. Then $0 \le y - x \le z - x$, and so $\|y - x\| \le \beta\|z - x\|$. Then, we get the bound

$$\begin{aligned} \|y\| &\le \|y - x\| + \|x\| \\ &\le \beta\|z - x\| + \|x\| \\ &\le \beta(\|z\| + \|x\|) + \|x\| \\ &\le (2\beta + 1)\max\{\|x\|, \|z\|\}, \end{aligned}$$

proving E is $(2\beta + 1)$-normal. □

Proposition 3.5.5. *Let X be a matrix ordered operator space, with dual matrix ordered operator space X^*, and let $\alpha > 0$. If X is completely α-generated, then X^* is completely 2α-normal. Conversely, if X^* is completely α-normal, then X is completely 2α-generated for all $\epsilon > 0$.*

Proof. Suppose that X is completely α-generated. Let $k \in \mathbb{N}$ and suppose $x, y, z \in M_k(X^*)^{\text{sa}}$ satisfy $x \le y \le z$ in the dual matrix ordering on X^*. By definition of the dual norm, we have

$$\|y\|_{M_k(X^*)} = \sup\{\|\langle\langle a, x\rangle\rangle\| \mid n \ge 1, a \in M_n(X)^{\text{sa}}\},$$

where $\langle\langle \cdot, \cdot \rangle\rangle$ denotes the usual matrix pairing between $\mathcal{M}(X)$ and $\mathcal{M}(X^*)$. Given $n \in \mathbb{N}$ and $a \in M_n(X)^{\text{sa}}$, we can write $a = b - c$ where $b, c \in M_n(X)^+$ satisfy $\|b\| + \|c\| \le \alpha\|a\|$. Then, we have the operator inequality

$$\begin{aligned} \langle\langle a, y\rangle\rangle &= \langle\langle b, y\rangle\rangle - \langle\langle c, y\rangle\rangle \\ &\le \langle\langle b, z\rangle\rangle - \langle\langle c, x\rangle\rangle \\ &\le (\|z\|\|b\| + \|x\|\|c\|)1_{nk} \\ &\le (\|x\| + \|z\|)\alpha\|a\|1_{nk}. \end{aligned}$$

Symmetrically,

$$\begin{aligned} \langle\langle a, y\rangle\rangle &\ge \langle\langle b, x\rangle\rangle - \langle\langle c, z\rangle \\ &\ge -(\|x\|\|b\| + \|z\|\|c\|)1_{nk} \\ &\ge -(\|x\| + \|z\|)\alpha\|a\|1_{nk}. \end{aligned}$$

It follows that

$$\|\langle\langle a, y\rangle\rangle\| \le (\|x\| + \|z\|)\alpha\|a\|.$$

Since a was arbitrary, this shows $\|y\| \le \alpha(\|x\| + \|z\|) \le 2\alpha\max\{\|x\|, \|z\|\}$, proving X^* is completely 2α-normal.

Now suppose X^* is completely 2α-normal. Consider the closed matrix convex subsets

$$\begin{aligned} K &:= \coprod_{n\geq 1} B_1(M_n(X)^{\mathrm{sa}}) = B_1(\mathcal{M}(X)^{\mathrm{sa}}), \\ K^+ &:= \coprod_{n\geq 1} B_1(M_n(X)^+) = K \cap \mathcal{M}(X)^+, \\ L &:= \overline{\mathrm{ncconv}}\,(K^+ \cup (-K^+)) \end{aligned}$$

of $\mathcal{M}(X)$. We will show that $K \subseteq \alpha L$.

To prove $K \subseteq 2\alpha L$, by the selfadjoint version of the nc separation Theorem of Effros and Winkler [15, Theorem 2.4.1], it suffices to show that the selfadjoint nc polars

$$K^\rho := \coprod_{n\geq 1} \{x \in M_n(X)^{\mathrm{sa}} \mid \langle\!\langle a, x\rangle\!\rangle \leq 1_{nk} \text{ for all } k \geq 1, x \in K_k\}$$

and L^ρ (defined similarly) satisfy $L^\rho \subseteq 2\alpha K^\rho$. The relevant selfadjoint polars are

$$\begin{aligned} K^\rho &= \coprod_{k\geq 1} B_1(M_k(X^*)), \\ (K^+)^\rho &= K^\rho - \mathcal{M}(X^*)^+ \\ &= \coprod_{k\geq 1} \{x \in M_k(X^*)^{\mathrm{sa}} \mid x \leq y \text{ for some } y \in K^\rho\}, \qquad \text{and} \\ L^\rho &= (K^+)^\rho \cap (-K^+)^\rho \\ &= (K^\rho - \mathcal{M}(X^*)^+) \cap (K^\rho + \mathcal{M}(X^*)^+) \\ &= \coprod_{k\geq 1} \{y \in M_k(X^*)^{\mathrm{sa}} \mid x \leq y \leq z \text{ for some } x, z \in K^\rho\}. \end{aligned}$$

Hence, if $y \in L_k^\rho$, then y satisfies $x \leq y \leq z$ for some $x, z \in M_k(X^*)^+$ with $\|x\|, \|z\| \leq 1$. By complete α-normality, this implies $\|y\| \leq \alpha$, so $y \in \alpha K^\rho$. This proves $L^\rho \subseteq \alpha K^\rho$, so $K \subseteq \alpha L$.

Hence $K \subseteq \alpha L = \overline{\mathrm{ncconv}}(K^+ \cup (-K^+))$. Using Lemma 3.4.1, we have

$$\overline{\mathrm{ncconv}}(K^+ \cup (-K^+)) \subseteq K^+ - K^+.$$

Hence $K \subseteq \alpha(K^+ - K^+)$, and by rescaling every element $x \in \mathcal{M}(X)^{\mathrm{sa}}$ can be decomposed as $x = y - z$ with $y, z \geq 0$ and $\|y\|, \|z\| \leq \alpha\|x\|$, and so $\|y\| + \|z\| \leq 2\alpha\|x\|$. This shows X is completely 2α-normal. □

Remark 3.5.6. If H is a Hilbert space, then $B(H)$ is completely 1-normal. Consequently, if X is a matrix ordered operator space which is completely norm and order isomorphic to a subspace of $B(H)$ (a quasi-operator system), then X must be α-normal for some $\alpha > 0$.

Because complete α-normality is dual to complete α-generation, [40, Theorem 3.9] can be viewed as a partial converse to Remark 3.5.6. If $X = S^*$ is the dual of an operator space, then if X is completely α-normal, it is a dual quasi-operator system. Translating the normality condition into a condition on the nc quasistate space gives the following intrinsic characterization of dualizability.

Theorem 3.5.7. *Let $(K, 0)$ be a pointed compact nc convex set, with associated operator space $S = A(K, 0)$. The following are equivalent.*

(1) *S^* is a dual quasi-operator system.*

(2) *S is completely α-generated for some $\alpha > 0$.*

(3) *S^* is completely α-normal for some $\alpha > 0$.*

(4) *There is a constant $C > 0$ such that*

$$(K - \mathbb{R}_+ K) \cap \mathbb{R}_+ K \subseteq CK,$$

where $K - \mathbb{R}_+ K$ denotes the levelwise Minkowski difference.

(5) *The closed nc convex set $(K - \mathbb{R}_+ K) \cap \mathbb{R}_+ K$ is bounded.*

Proof. The equivalence of (1) and (2) was proved by Ng in [40, Theorem 3.9]. Proposition 3.5.5 shows that (2) and (3) are equivalent. To prove that (3) and (4) are equivalent, we may use Proposition 3.4.2 to identify $(S^*)^{\mathrm{sa}} = \operatorname{span}_{\mathbb{R}} K$. After doing so, the positive elements in S^* correspond to the closed nc convex set $\mathbb{R}_+ K$, and for $d \in \mathbb{R}_+ K_n$, we have $\|d\|_{M_n(S^*)} = \gamma_K(d)$. Consequently,

$$\begin{aligned}(K - \mathbb{R}_+ K) \cap \mathbb{R}_+ K &= \{d \in \operatorname{span}_{\mathbb{R}} K \mid 0 \le d \le x \text{ for some } x \in K\} \\ &= \{d \in \coprod_n M_n(S^*)^{\mathrm{sa}} \mid 0 \le d \le x \text{ for some } x > 0 \text{ in } K_n \text{ with } \|x\| \le 1\}.\end{aligned}$$

Thus (4) holds if and only if

$$0 \le x \le y \text{ and } \|y\| \le 1 \implies \|x\| \le C,$$

in $M_n(S^*)^{\mathrm{sa}}$ for all $n \in \mathbb{N}$. By rescaling, this is equivalent to asserting that

$$0 \le x \le y \implies \|x\| \le C\|y\|$$

in $M_n(S^*)^{\mathrm{sa}}$. Then, Proposition 3.5.4 shows that if (3) holds, then (4) holds with $C = \alpha$, and if (4) holds, then (3) holds with $\alpha = 2C+1$. Finally, because $(K - \mathbb{R}_+ K) \cap \mathbb{R}_+ K$ is a subset of $\mathbb{R}_+ K$, on which the matrix norms from S^* agree with the Minkowski gauge γ_K, (4) holds if and only if $(K - \mathbb{R}_+ K) \cap \mathbb{R}_+ K$ is bounded by $C > 0$, i.e. if and only if (5) holds. □

Remark 3.5.8. The analogous version of Theorem 3.5.7 holds in the classical case: If $(K, 0)$ is a pointed compact convex set, then the nonunital function system $A(K, 0)$ is α-generated for some $\alpha > 0$ if and only if $(K - \mathbb{R}_+) \cap \mathbb{R}_+ K$ is bounded.

Corollary 3.5.9. *Let $z \in K \subseteq L$ be compact nc convex sets such that (K, z) and (L, z) are pointed. If $A(L, z)$ is dualizable, then so is $A(K, z)$.*

Proof. By translating, it suffices to consider this when $z = 0$. This follows by noting that

$$(K - \mathbb{R}_+ K) \cap \mathbb{R}_+ K \subseteq (L - \mathbb{R}_+ L) \cap \mathbb{R}_+ L,$$

and using condition (5) in Theorem 3.5.7. □

In [30, Section 8], quotients of (nonunital) operator systems were defined. There, a quotient of operator systems $S \to S/J$ corresponds dually to a restriction map $A(K, z) \to A(M, z)$ between pointed compact nc convex sets, where $M \subseteq K$ is the annihilator of the kernel $J \subseteq K$. Applying Corollary 3.5.9 gives

Corollary 3.5.10. *If S is a dualizable operator system, then every quotient of S is dualizable.*

3.6 Positive generation versus completely bounded positive generation

Classically, if an ordered Banach space E is **positively generated** in the sense that $E^{sa} = E^+ - E^+$, then E is in fact α-generated for some $\alpha > 0$. This is a consequence of the Baire category theorem [3, Theorem 2.1.2]. In the special case where $E = A(K,0)$ is the nonunital function system of continuous affine functions on a pointed compact convex set K containing 0 which vanish at 0, the following classical analogue of Theorem 3.5.7 holds: The function system $A(K,0)$ is α-generated if and only if the classical convex set $(K - \mathbb{R}_+ K) \cap \mathbb{R}_+ K \subseteq A(K,0)^*$ is bounded. If $A(K,0)$ is positively generated, it is a consequence of the Banach-Steinhaus Principle of Uniform Boundedness that $(K - \mathbb{R}_+ K) \cap \mathbb{R}_+ K$ is bounded, and so $A(K,0)$ automatically has bounded positive generation. This proof is essentially the dual version of the proof of [3, Theorem 2.1.2].

In this section, we discuss the noncommutative situation. First, we show that an operator system S has **complete positive generation**, meaning $M_n(S)^{sa} = M_n(S)^+ - M_n(S)^+$ for all $n \geq 1$, if and only if S is positively generated at the first level. In contrast to the classical situation, complete positive generation need not imply complete α-generation. In Example 3.6.6, we give an example of a matrix ordered operator space which is positively generated but not completely α-generated for any $\alpha > 0$.

One might also consider the following weaker property. Call an ordered Banach space E **approximately positively generated** if $E^+ - E^+$ is dense in E. Note that even though the postiive cone E^+ is closed, it need not be the case that $E^+ - E^+$ is closed, even when E is an operator space, as the following example shows.

Example 3.6.1. Let $S = C([0,1])$, and define S^+ to be the closed cone of functions which are both positive and convex. Then $S^+ - S^+$ is dense in $S = C([0,1])$, because it contains all C^2 functions, but $S^+ - S^+ \neq S$, because the convex functions in S^+ are automatically differentiable on the interior $(0,1)$. So, S is an ordered Banach space which is approximately positively generated, but not positively generated. In fact, S is an operator system. Indeed, if we let

$$K = \{\varphi \in S^* \mid \|\varphi\| \leq 1 \text{ and } \varphi(S^+) \subseteq [0,\infty)\}$$

be the classical quasistate space of K, then since every probability measure on $[0,1]$ lies in K, the natural map

$$S \to A(K)$$

into the continuous affine functions on K is isometric and order isomorphic. That is, S is isometrically order isomorphic to a nonunital function system, and so inherits an operator system structure.

There are many examples of the same kind as Example 3.6.1. It suffices to take any function system S, and equip it with a new closed positive cone $P \subseteq S^+$ for which $P - P$ is not closed. In a private correspondence, Ken Davidson suggested another example in which $S = \mathbb{C} \oplus c_0$ is equipped with the new positive cone

$$P = \Big\{(t, (x_n)_{n\geq 1}) \in \mathbb{C} \oplus c_0 \mid t \geq 0, (x_n)_{n\geq 1} \geq 0, \text{ and } \sum_{n=1}^{\infty} x_n \leq t\Big\}.$$

Here, again $P - P$ is dense and not closed in S.

Proposition 3.6.2. *Let S be an operator system with quasistate space $K \subseteq S^*$. Then S is approximately positively generated if and only if S^+ separates points in K.*

Proof. If S is densely spanned by its positives, then the positives must separate points in K. Conversely, suppose that S is not positively generated. Then there exists an element $x \in S^{\text{sa}} \setminus \overline{(S^+ - S^+)}$. By the Hahn-Banach Separation Theorem, there is a self-adjoint linear functional $\varphi \in S^*$ so that for all $y \in S^+ - S^+$ we have

$$\varphi(x) < \varphi(y).$$

But since $S^+ - S^+$ is a real vector space, this implies that φ is identically zero on $S^+ - S^+$. Moreover, by the Hahn-Jordan decomposition theorem there are positive functionals $\varphi^+, \varphi^- \in E^d$ with $\varphi = \varphi^+ - \varphi^-$. Since $\varphi(x) < 0$, the functionals φ^+ and φ^- are necessarily distinct, but they are equal on $S^+ - S^+$ and hence on S^+. Normalizing $\varphi^\pm$ to obtain quasistates shows that S^+ does not separate quasistates. □

Remark 3.6.3. The Hahn-Jordan decomposition theorem ensures that, as an ordered vector space, the dual space S^* is always positively generated.

By the following result, if S is positively generated then so are each of its matrix levels $M_n(S)$. Again by [3, Chapter 2, Theorem 1.2], each $M_n(S)$ is α_n-generated for some α_n. In order for S to be dualizable, we would need the sequence (α_n) to be bounded.

Proposition 3.6.4. *If S is positively generated, then so is $M_n(S)$ for each n.*

Before proving this, we will need a technical lemma which proves a much stronger statement in the finite dimensional setting.

Lemma 3.6.5. *If S is a finite dimensional and positively generated operator system, then it contains a matrix order unit.*

Proof. Since S is positively generated, then it admits a basis $B = \{p_1, \dots, p_m\}$ consisting of positive elements. We claim that $e := \sum_{i=1}^m p_i$ is an order unit. For any x in S^{sa}, we can write x uniquely as a real linear combination

$$x = \sum_{i=1}^m \alpha_i p_i,$$

and we define $\lambda_x := \max\{1, |\alpha_1|, \dots, |\alpha_m|\}$. It is clear that $\lambda_x e \pm x$ are positive in S, so e is an order unit.

Next we let $n \geq 0$ and show that $e_n := e \otimes I_n$ is an order unit for $M_n(S)$, so fix an $X = (x_{ij})_{i,j=1}^n \in M_n(S)^{\text{sa}}$. Since E is positively generated, for every $i \leq j$ we can decompose the corresponding entries of X as

$$x_{ij} = \text{Re } x_{ij}^+ - \text{Re } x_{ij}^- + i(\text{Im } x_{ij}^+ - \text{Im } x_{ij}^-).$$

To find a large enough coefficient of e_n to dominate X, we let

$$\lambda_X := \lambda_d + \lambda_{\text{Re}} + \lambda_{\text{Im}}.$$

Where $\lambda_d := \max\{\lambda_{x_{ii}}\}_{i=1}^n$, $\lambda_{\mathrm{Re}} := \sum_{i<j} \lambda_{\mathrm{Re}\ x_{ij}^+ + \mathrm{Re}\ x_{ij}^-}$, and $\lambda_{\mathrm{Im}} := \sum_{i<j} \lambda_{\mathrm{Im}\ x_{ij}^+ + \mathrm{Im}\ x_{ij}^-}$. Note that it makes sense to write $x_{ii}^{\pm}$ since the x_{ii} must all be self-adjoint, as they lie on the diagonal of $X = X^*$.

Fix a concrete representation $S \hookrightarrow B(H)$ of S as a norm closed and $*$-closed subspace of the bounded operators on a Hilbert space. We'll show that $\lambda_X e_n + X \geq 0$ concretely using inner products. Take an arbitrary vector $a = (a_i)_{i=1}^n \in H^n = \bigoplus_{i=1}^n H$, and compute

$$\begin{aligned}
\langle(\lambda_X e_n + X)a, a\rangle =&\lambda_X \langle e_n a, a\rangle + \langle Xa, a\rangle \\
=&\lambda_X \sum_{i=1}^n \langle ea_i, a_i\rangle + \sum_{i=1}^n \langle x_{ii}a_i, a_i\rangle + \sum_{i<j} \langle x_{ij}a_j, a_i\rangle + \langle x_{ji}a_i, a_j\rangle \\
=&\lambda_X \sum_{i=1}^n \langle ea_i, a_i\rangle + \sum_{i=1}^n \langle x_{ii}a_i, a_i\rangle + \sum_{i<j} 2\mathrm{Re}\ \langle x_{ij}a_j, a_i\rangle \\
=&\left(\lambda_d \sum_{i=1}^n \langle ea_i, a_i\rangle + \sum_{i=1}^n \langle x_{ii}a_i, a_i\rangle\right) \\
&+\left((\lambda_{\mathrm{Re}} + \lambda_{\mathrm{Im}}) \sum_{i=1}^n \langle ea_i, a_i\rangle + \sum_{i<j} 2\mathrm{Re}\ \langle x_{ij}a_j, a_i\rangle\right) \\
=&\left(\lambda_d \sum_{i=1}^n \langle ea_i, a_i\rangle + \sum_{i=1}^n \langle x_{ii}a_i, a_i\rangle\right) \\
&+\left(\lambda_{\mathrm{Re}} \sum_{i=1}^n \langle ea_i, a_i\rangle + 2\sum_{i<j} \mathrm{Re}\ \langle \mathrm{Re}\ x_{ij}a_j, a_i\rangle\right) \\
&+\left(\lambda_{\mathrm{Im}} \sum_{i=1}^n \langle ea_i, a_i\rangle - 2\sum_{i<j} \mathrm{Im}\ \langle \mathrm{Im}\ x_{ij}a_j, a_i\rangle\right).
\end{aligned}$$

For the remainder of the proof, we will show that each of the three terms above is non-negative. Starting with the first term,

$$\begin{aligned}
\lambda_d \sum_{i=1}^n \langle ea_i, a_i\rangle + \sum_{i=1}^n \langle x_{ii}a_i, a_i\rangle &= \sum_{i=1}^n \langle(\lambda_d e + x_{ii})a_i, a_i\rangle \\
&\geq \sum_{i=1}^n \langle(\lambda_{x_{ii}} e + x_{ii})a_i, a_i\rangle \\
&\geq 0,
\end{aligned}$$

where the last inequality follows from the first paragraph of the proof.

To prove that the second term is non-negative, note

$$\begin{aligned}
&\lambda_{\mathrm{Re}} \sum_{k=1}^n \langle ea_k, a_k\rangle + 2\sum_{i<j} \mathrm{Re}\ \langle \mathrm{Re}\ x_{ij}a_j, a_i\rangle \\
&= \sum_{i<j} (\lambda_{\mathrm{Re}\ x_{ij}^+ + \mathrm{Re}\ x_{ij}^-}) \sum_{k=1}^n \langle ea_k, a_k\rangle + 2\sum_{i<j} \mathrm{Re}\ \langle \mathrm{Re}\ x_{ij}a_j, a_i\rangle \\
&= \sum_{i<j} (\lambda_{\mathrm{Re}\ x_{ij}^+ + \mathrm{Re}\ x_{ij}^-}) \sum_{k=1}^n \langle ea_k, a_k\rangle + 2\mathrm{Re}\ \langle \mathrm{Re}\ x_{ij}a_j, a_i\rangle.
\end{aligned}$$

We now show that for each pair $i < j$, the corresponding summand is non-negative:

$$
\begin{aligned}
&(\lambda_{\text{Re } x_{ij}^+ + \text{Re } x_{ij}^-}) \sum_{k=1}^{n} \langle e a_k, a_k \rangle + 2\text{Re } \langle \text{Re } x_{ij} a_j, a_i \rangle \\
=&(\lambda_{\text{Re } x_{ij}^+ + \text{Re } x_{ij}^-}) \sum_{k=1}^{n} \langle e a_k, a_k \rangle + 2\text{Re } \langle (\text{Re } x_{ij}^+ - \text{Re } x_{ij}^-) a_j, a_i \rangle \\
\geq&(\lambda_{\text{Re } x_{ij}^+ + \text{Re } x_{ij}^-}) \langle e a_i, a_i \rangle + (\lambda_{\text{Re } x_{ij}^+ + \text{Re } x_{ij}^-}) \langle e a_j, a_j \rangle + 2\text{Re } \langle (\text{Re } x_{ij}^+ - \text{Re } x_{ij}^-) a_j, a_i \rangle \\
\geq& \big(\langle \text{Re } x_{ij}^+ a_i, a_i \rangle + \langle \text{Re } x_{ij}^+ a_j, a_j \rangle + 2\text{Re } \langle \text{Re } x_{ij}^+ a_j, a_i \rangle \big) \\
+& \big(\langle \text{Re } x_{ij}^- a_i, a_i \rangle + \langle \text{Re } x_{ij}^- a_j, a_j \rangle - 2\text{Re } \langle \text{Re } x_{ij}^- a_j, a_i \rangle \big) \\
=&\langle \text{Re } x_{ij}^+ (a_i + a_j), a_i + a_j \rangle \\
+&\langle \text{Re } x_{ij}^- (a_i - a_j), a_i - a_j \rangle \\
\geq&0.
\end{aligned}
$$

The last inequality follows since each Re $x_{ij}^{\pm}$ is a positive operator. The proof that the third term is non-negative is similar. □

We now prove Proposition 3.6.4

Proof of Proposition 3.6.4. To show $M_n(S)$ is positively generated, fix $X = (x_{ij})_{i,j=1}^n \in M_n(S)^{\text{sa}}$. Since S is positively generated, each x_{ij} can be written as a linear combination of four positives Re x_{ij}^+, Re x_{ij}^-, Im x_{ij}^+, and Im x_{ij}^-. Let S_X denote the linear span of these positives, as i and j range from 1 to n. Since S_X is a finite dimensional operator system, by the previous lemma there is a matrix order unit $e_X \in S_X$ and in particular there is a constant $\lambda > 0$ so that both $\lambda 1_n \otimes e_X \pm X \geq 0$. Since $X = (\lambda 1_n \otimes e_X + X)/2 - (\lambda 1_n \otimes e_X - X)/2$ and all entries are ultimately in S, this shows $M_n(S)$ is positively generated. □

So, complete positive generation coincides with positive generation at the first level. However, the following example shows that for matrix ordered operator spaces, positive generation at all matrix levels does not imply complete α-generation for any α.

Example 3.6.6. Any Banach space E has a unique maximal and minimal system of L^∞-matrix norms which give E an operator space structure and restrict to the norm on E at the first matrix level. We denote the resultant operator spaces by $\max(E)$ and $\min(E)$, respectively. There are natural operator space dualities $\max(E)^* = \min(E^*)$ and $\min(E)^* = \max(E)^*$ [19, Section 3.3].

We will consider the Banach space ℓ^1 and its dual ℓ^∞. Because ℓ^∞ is a commutative C^*-algebra, we have $\ell^\infty = \min(\ell^\infty)$ [19, Proposition 3.3.1]. The embedding $\ell^1 \subseteq (\ell^\infty)^*$ gives a matrix ordered operator space structure on ℓ^1, which coincides with the max norm $\ell^1 = \max(\ell^1)$. Using the natural linear identifications

$$M_n(\ell^\infty) = \ell^\infty(\mathbb{N}, M_n) \quad \text{and} \quad M_n(\ell^1) = \ell^1(\mathbb{N}, M_n),$$

the resultant positive cones in ℓ^∞ and ℓ^1 consist of those sequences of matrices which are positive in each entry.

We will consider the minimal operator space $\min(\ell^1)$ equipped with the same matrix ordering as $\ell^1 = \max(\ell^1)$. Because the matrix cones $M_n(\ell^1)^+ = \ell^1(\mathbb{N}, M_n^+)$ are closed in the topology of pointwise weak-$*$ convergence, which is weaker than the topology induced by either the minimal or maximal norms on $M_n(\ell^1)$, the matrix cones $M_n(\ell^1)^+$ are closed in the minimal norm topology. Thus $\min(\ell^1)$ has the structure of a matrix ordered operator space. Because M_n is 1-generated, it follows that each $M_n(\min(\ell^1)) = \ell^1(\mathbb{N}, M_n)$ is positively generated, so $\min(\ell^1)$ is completely positively generated.

However, we will show that $\min(\ell^1)$ is not completely α-generated for any $\alpha > 0$. We will do so using Proposition 3.5.5, by proving the dual matrix ordered operator space $\min(\ell^1)^* = \max(\ell^\infty)$ (equipped with the usual matrix ordering on ℓ^∞) is not completely α-normal for any $\alpha > 0$. Since ℓ^∞ is infinite dimensional, the minimal and maximal matrix norms on ℓ^∞ are not completely equivalent [44, Theorem 14.3]. Thus there is a sequence $x_k \in M_{n_k}(\ell^\infty)$ for which

$$\|x_k\|_{\min} \le 1 \quad \text{and} \quad \|x_k\|_{\max} \ge k.$$

In the C*-algebras $M_{n_k}(\ell^\infty)$, we can write each x_k as a linear combination

$$x_k = (\operatorname{Re} x_k)^+ - (\operatorname{Re} x_k)^- + i(\operatorname{Im} x_k)^+ - i(\operatorname{Im} x_k)^-$$

of positive elements $(\operatorname{Re} x_k)^\pm, (\operatorname{Im} x_k)^\pm$ of min-norm at most 1. Since $\|x_k\|_{\max} > k$, by suitably choosing $y_k \in \{(\operatorname{Re} x_k)^\pm, (\operatorname{Im} x_k)^\pm\}$, we can obtain a sequence of positive elements $y_k \in M_{n_k}(\ell^\infty)^+$ with

$$\|y_k\|_{\min} \le 1 \quad \text{and} \quad \|y_k\|_{\max} > k/4.$$

Since the minimal norm on $M_{n_k}(\ell^\infty)$ is just the usual C*-algebra norm, we have $0 \le y_k \le 1_{M_{n_k}(\ell^\infty)}$. Because the maximal norms satisfy the L^∞-matrix identity, we have $\|1_{M_{n_k}(\ell^\infty)}\|_{\max} = 1$. Thus

$$0 \le y_k \le 1_{M_{n_k}(\ell^\infty)}, \quad \|1_{M_{n_k}(\ell^\infty)}\|_{\max} \le 1, \quad \text{and} \quad \|y_k\|_{\max} > k/4$$

for all $k \in \mathbb{N}$. So, ℓ^∞ is not completely $k/4$-normal, and taking $k \to \infty$ shows that ℓ^∞ cannot be completely α-normal for any $\alpha > 0$.

Example 3.6.6 is a minimal example of this kind. One cannot restrict to the finite dimensional spaces ℓ^1_d and $\ell^\infty_d = (\ell^1_d)^*$ because the maximal and minimal norms on a finite dimensional Banach space are completely equivalent [44, Theorem 14.3], and so $\max(\ell^1_d) \cong \min(\ell^1_d)$ is a dualizable quasi-operator system.

3.7 Examples and applications

3.7.1 Nonunital operator system pushouts and coproducts

If $K = \coprod_{n\ge1} K_n$ and $L = \coprod_{n\ge1} L_n$ are compact nc convex sets, we denote by

$$K \times L := \coprod_{n\ge1} K_n \times L_n$$

their levelwise cartesian product. In [25], it was shown that $A(K \times L)$ is the categorical coproduct of the unital operator systems $A(K)$ and $A(L)$ in the category of unital operator systems with ucp maps as morphisms. The following result will let us assert a similar result in the pointed context, for nonunital operator systems.

Proposition 3.7.1. *Let (K,z) and (L,w) be pointed compact nc convex sets. Then $(K\times L,(z,w))$ is pointed, and there is a vector space isomorphism*

$$A(K\times L,(z,w))\cong A(K,z)\oplus A(L,w).$$

Proof. We will prove the result in the special case when $z=0$ and $w=0$ in the ambient spaces containing K and L. The general case follows by translation. Define a linear map $A(K,z)\oplus A(L,w)\to A(K\times L,(z,w))$ by $(a,b)\mapsto a\oplus b$, where $(a\oplus b)(x,y):=a(x)+b(y)$ for $x\in K$, $y\in L$. Since $a(z)=0=b(w)$, it is easy to see that this map is injective. Given $c\in A(K\times L,(0,0))$, let $a(x)=c(x,0)$ and $b(y)=c(0,y)$ for $x\in K$, $y\in L$. Then since $c(0,0)=0$,

$$\begin{aligned}c(x,y)&=2c\left(\frac{x}{2},\frac{y}{2}\right)\\&=2\left(\frac{c(x,0)}{2}+\frac{c(0,y)}{2}\right)\\&=a(x)+b(y)=(a\oplus b)(x,y).\end{aligned}$$

This proves that $A(K,0)\oplus A(L,0)\to A(K\times L,(0,0))$ is a linear isomorphism.

Now, it will follow from this isomorphism that $(K\times L,(z,w))$ is pointed. Let $\rho:A(K\times L,(z,w))\to M_n$ be any nc quasistate. Then

$$\varphi(a)=\rho(a\oplus 0)\quad\text{and}\quad\psi(b)=\rho(0\oplus b)$$

define nc quasistates on $A(K,0)$ and $A(L,0)$, respectively. Because $(K,0)$ and $(L,0)$ are pointed, all nc quasistates are point evaluations, so we have $\varphi(a)=a(x)$ and $\varphi(b)=b(y)$ for some $(x,y)\in(K\times L)_n$ and all $a\in A(K,0)$, $b\in A(L,0)$. From linearity, it follows that ρ is just point evaluation at (x,y), so $(K\times L),(0,0))$ is pointed. ☐

Definition 3.7.2. Let S and T be operator systems with respective nc quasistate spaces $(K,0)$ and $(L,0)$. We define the **operator system coproduct** to be the vector space $S\oplus T$ equipped with the operator system structure such that

$$S\oplus T\cong A(K,0)\oplus A(L,0)\cong A(K\times L,(0,0))$$

is a completely isometric complete order isomorphism.

Explicitly, the matrix norms on $S\oplus T$ satisfy

$$\|(x,y)\|_{M_n(S\oplus T)}=\sup\{\|\varphi_n(x)+\psi_n(y)\|\mid\varphi\in K,\psi\in L\}$$

for $(x,y)\in M_n(S\oplus T)=M_n(S)\oplus M_n(T)$. The matrix cones just identify $M_n(S\oplus T)^+=M_n(S)^+\oplus M_n(T)^+$.

Proposition 3.7.3. *The bifunctor $(S,T)\mapsto S\oplus T$ is the categorical coproduct in the category of operator systems with ccp maps as morphisms. That is, given any operator system R and ccp maps $\varphi:S\to R$ and $\psi:T\to R$, the linear map $\varphi\oplus\psi:S\oplus T\to R$ is ccp.*

Proof. This follows either by the explicit description of the matrix norms and order on $S\oplus T$, or by showing that $(K\times L,(0,0))$ is the categorical *product* of $(K,0)$ and $(L,0)$ in the category of pointed compact nc convex sets, and using Theorem 3.2.11. ☐

Remark 3.7.4. The operator space norm on $S \oplus T$ is neither the usual ℓ^∞-product nor the ℓ^1-product of the operator spaces S and T. For example, if

$$K = L = \coprod_{n \geq 1} \{x \in M_n^+ \mid 0 \leq x \leq 1_n\}$$

is the nc simplex generated by $[0,1]$, and $a \in A(K,0)$ is the coordinate function $a(x) = x$, then

$$\|a \oplus a\|_{A(K^2,(0,0)} = 2 > \|a \oplus a\|_\infty \quad \text{and}$$
$$\|a \oplus (-a)\|_{A(K^2,(0,0))} = 1 < \|a \oplus a\|_1.$$

Proposition 3.7.5. *Let S and T be operator systems. If S and T are dualizable, then $S \oplus T$ is dualizable.*

Proof 1. We will use Theorem 3.5.7. Let the nc quasistate spaces of S and T be $(K,0)$ and $(L,0)$, respectively. Then $(K - \mathbb{R}_+ K) \cap \mathbb{R}_+ K$ and $(L - \mathbb{R}_+ L) \cap \mathbb{R}_+ L$ are norm bounded. Checking that

$$(K \times L - \mathbb{R}_+(K \times L)) \cap \mathbb{R}_+(K \times L) \subseteq ((K - \mathbb{R}_+ K) \cap \mathbb{R}_+ K) \times ((L - \mathbb{R}_+ L) \cap \mathbb{R}_+ L)$$

shows that $(K \times L - \mathbb{R}_+(K \times L)) \cap \mathbb{R}_+(K \times L)$ is bounded, so $S \oplus T \cong A(K \times L, (0,0))$ is dualizable. □

It is also possible to give a proof of Proposition 3.7.5 using only Ng's bounded decomposition property, which appears in 3.5.7.(2).

More generally, we can form finite pushouts in the operator system category by taking pullbacks in the category of pointed compact nc convex sets.

Definition 3.7.6. Let

$$\begin{array}{ccc} R & \xrightarrow{\varphi} & S \\ \downarrow{\scriptstyle\psi} & & \\ T & & \end{array}$$

be a diagram of operator systems with ccp maps as morphisms. Let S, T, and R, have respective quasistate spaces $(K,0)$, $(L,0)$, and $(M,0)$. We define the **pushout** $S \oplus_{R,\varphi,\psi} T$ as the operator system

$$A(K \times_{M,\varphi^*,\psi^*} L, (0,0)),$$

where

$$K \times_{M,\varphi^*,\psi^*} L = \{(x,y) \in K \times L \mid \varphi^*(x) = \psi^*(y)\} \subseteq K \times L,$$

equipped with the natural maps

$$\iota_S : S \to S \oplus T \to S \oplus_{R,\varphi,\psi} T \quad \text{and}$$
$$\iota_T : T \to S \oplus T \to S \oplus_{R,\varphi,\psi} T$$

which make the diagram

$$\begin{array}{ccc} R & \xrightarrow{\varphi} & S \\ {\scriptstyle\psi}\downarrow & & \downarrow{\scriptstyle\iota_S} \\ T & \xrightarrow{\iota_T} & S \oplus_{R,\varphi,\psi} T \end{array} \tag{3.5}$$

commute.

When the morphisms φ and ψ are understood, we will usually just write $S \oplus_R T$ and $K \times_M L$. Note that the coproduct $S \oplus T$ coincides with the pushout $S \oplus_0 T$ of the diagram

$$\begin{array}{ccc} 0 & \xrightarrow{0} & S \\ {\scriptstyle 0}\big\downarrow & & \\ T & & \end{array}$$

as expected, where 0 denotes the 0 operator system.

To verify that $A(K \times_M L, (0,0))$ is an operator system, we need to show that:

Proposition 3.7.7. *$(K \times_M L, (0,0))$ is pointed.*

Proof. Let $\rho : A(K \times_M L, (0,0)) \to M_n$ be an nc quasistate. Pulling ρ back to $A(K \times L, (0,0))$ gives a point evaluation at some point $(x,y) \in K \times L$. It will suffice to show that $(x,y) \in K \times_M L$, in which case ρ must be point evaluation at (x,y).

We must show that $\varphi^*(x) = \psi^*(y)$ in M. Given $a \in R \cong A(M,0)$. Since the diagram (3.5) commutes, upon pulling back to $S \oplus T$, we have

$$\rho(\iota_S \varphi(a)) = (\varphi(a) \oplus 0)(x,y) = (0 \oplus \psi(a))(x,y) = \rho(\iota_T \psi(a)),$$

that is, $\varphi(a)(x) = a(\varphi^*(x)) = \psi(a)(y) = a(\psi^*(y))$. Since $a \in R = A(M,0)$ was arbitrary, this proves $\varphi^*(x) = \psi^*(y)$, so $(x,y) \in K \times_M L$. □

Proposition 3.7.8. *The diagram 3.5 is a pushout in the category of operator systems with ccp maps as morphisms.*

Proof. It is easiest to verify that the diagram

$$\begin{array}{ccc} (K \times_M L, (0,0)) & \longrightarrow & (K, 0) \\ \big\downarrow & & \big\downarrow{\scriptstyle \varphi^*} \\ (L, 0) & \xrightarrow{\psi^*} & (M, 0) \end{array}$$

is a pullback in the category of pointed compact nc convex sets with pointed continuous nc affine functions as morphisms, where the unlabeled maps are just the coordinate projections. Checking this is fairly immediate, using the fact that the point-weak-$*$ topology on $K \times_M L \subseteq K \times L$ coincides with the restriction of the product topology. By the contravariant equivalence of categories Theorem 3.2.11, it follows that (3.5) is a pushout. □

Proposition 3.7.9. *If S and T are dualizable operator systems, then any pushout $S \oplus_{R,\varphi,\psi} T$ is also dualizable.*

Proof. This follows from Proposition 3.7.5 combined with Corollary 3.5.9 used with the inclusion $(0,0) \subseteq K \times_M L \subseteq K \times L$. □

It follows by induction that any finite pushout of dualizable operator systems is again dualizable.

3.7.2 A new proof of Choi's theorem

In [40], Ng showed that if S is a dualizable operator system, then there is a canonical choice of completely equivalent matrix norm on the dual S^* for which S^d is an operator system, embedding completely isometrically into some $B(H)$. This canonical dual matrix norm is

$$\|f\|_d = \sup\{\|f_n(x)\| \mid n \geq 1, x \in M_n(S)^+, \|x\| \leq 1\}, \quad m \geq 1, f \in M_m(S^*)$$

where the key difference is that the supremum is taken only over *positive* elements x. Ng denotes by S^d the operator system S^* renormed with the matrix norms $\|\cdot\|_d$.

Theorem 3.7.10. *The nc quasistate space (K,z) of M_n is pointedly affinely homeomorphic to $(\coprod_{k=1}^{\infty} M_k(M_n)_1^+, 0)$, and its nc extreme points consist of unitary conjugates of the Choi matrix $\sum_{i,j=1}^n E_{ij} \otimes E_{ij}$ together with the zero scalar.*

Proof. Note that the canonical map $\Phi : M_n \to M_n^d$ given by $\Phi(E_{ij}) = \delta_{ij}$ is a complete order isomorphism, where $\delta_{ij}(E_{kl}) = 1$ when $(i,j) = (k,l)$ and 0 otherwise.

In particular, we can write $M_n^d = A(K,z)$ and view K as lying in the ambient space $\coprod_{k=1}^{\infty} M_k(M_n^{dd}) = \coprod_{k=1}^{\infty} M_k(M_n)$. The last equality follows from M_n being finite dimensional. Under this identification, and with the matrix norms $\|\cdot\|_d$ on M_n^d, it is clear that the completely contractive and completely positive maps on M_n^d are precisely the elements of $(\coprod_{k=1}^{\infty} M_k(M_n)_1^+, 0)$.

This proves that M_n is isomorphic as an operator system to $A(\coprod_{k=1}^{\infty} M_k(M_n)_1^+, 0)$. To describe the boundary of the nc quasistate space we note that M_n is a C*-algebra, and so the boundary consists of its irreducible representations. These are precisely the unitary conjugates of the identity map on M_n together with the zero map.

Using the same notation as above, the identity map on M_n can be written as $\sum_{i,j=1}^n E_{ij} \otimes \delta_{ij}$. Indeed, if $(x_{kl}) \in M_n$ then

$$\begin{aligned}
\left(\sum_{i,j=1}^n E_{ij} \otimes \delta_{ij}\right)(x_{kl}) &= \sum_{i,j=1}^n E_{ij} \otimes \delta_{ij}((x_{kl})) \\
&= \sum_{i,j=1}^n E_{ij} \otimes x_{ij} \\
&= \sum_{i,j=1}^n x_{ij} E_{ij} \\
&= (x_{kl}).
\end{aligned}$$

This shows that $\Phi_n^{-1}(\mathrm{id}_{M_n}) = \sum_{i,j=1}^n E_{ij} \otimes E_{ij}$, where Φ_n denotes the n^{th} amplification of Φ. Hence, as the unitary orbit of id_{M_n} together with the zero map are the extreme boundary of K, the Choi matrix together with the zero scalar are the extreme boundary of $\coprod_{k=1}^{\infty} M_k(M_n)_1^+$ under the identification $\coprod_{k=1}^{\infty} M_k(M_n)_1^+ = K$ given above. □

As a corollary, we obtain a celebrated result of Choi [10].

Corollary 3.7.11. *A map $\Phi : M_n \to M_k$ is completely positive if and only if $\sum_{i,j=1}^n E_{ij} \otimes \Phi(E_{ij})$ is positive in $M_n(M_k)$.*

Proof. By identifying M_k to its vector space dual, and applying the standard operation of uncurrying Φ, we obtain a new map $\tilde{\Phi} : M_k(M_n) \to \mathbb{C}$ defined by

$$\tilde{\Phi}(E_{ij} \otimes E_{kl}) = \langle \Phi(E_{ij}), E_{kl} \rangle,$$

where $\langle \cdot, \cdot \rangle$ denotes the Hilbert-Schmidt inner product on M_k. It is a well known fact that $\tilde{\Phi}$ is a positive functional if and only if Φ is completely positive.

In this way, we may view $\tilde{\Phi}$ as an element of the dual $M_k(M_n)^d = M_k(M_n^d)$. Using the identification $M_k(M_n^d) = M_k(A(\coprod_{m=1}^{\infty} M_m(M_n)_1^+, 0))$, and the further identification that $M_k(A(\coprod_{m=1}^{\infty} M_m(M_n)_1^+, 0)) = A((\coprod_{m=1}^{\infty} M_m(M_n)_1^+, 0), (\mathcal{M}_k, 0))$, we obtain that $\tilde{\Phi}$ is a positive functional if and only if it takes positive values on the extreme boundary of $\coprod_{m=1}^{\infty} M_m(M_n)_1^+$ when viewed as an element of $A((\coprod_{m=1}^{\infty} M_m(M_n)_1^+, 0), (\mathcal{M}_k, 0))$. By the previous result, this happens precisely when its evaluation at the Choi matrix is positive. $\square$

Corollary 3.7.12. *For any contractive positive matrix $A \in M_n$, there are k matrices $X_1, \ldots, X_k \in M_{n,n^2}$ with $X_1 X_1^* + \cdots + X_k X_k^* = 1_n$ so that*

$$A = X_1 C X_1 + \cdots + X_k C X_k,$$

where $C = \sum_{i,j=1}^{n} E_{ij} \otimes E_{ij}$ denotes the Choi matrix. Moreover, k is a polynomial in n.

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