The Edmonds-Giles Conjecture and its Relaxations

by

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Author's Declaration

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

Given a directed graph, a directed cut is a cut with all arcs oriented in the same direction, and a directed join is a set of arcs which intersects every directed cut at least once. Edmonds and Giles [7] conjectured for all weighted directed graphs, the minimum weight of a directed cut is equal to the maximum size of a packing of directed joins. Unfortunately, the conjecture is false; a counterexample was first given by Schrijver [13]. However its "dual" statement, that the minimum weight of a dijoin is equal to the maximum number of dicuts in a packing, was shown to be true by Luchessi and Younger [11].

Various relaxations of the conjecture have been considered; Woodall's conjecture remains open, which asks the same question for unweighted directed graphs, and Edmond-Giles's conjecture was shown to be true in the special case of source-sink connected directed graphs. Following these inquries, this thesis explores different relaxations of the Edmond-Giles's conjecture.

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Chapter 1

Introduction

1.1 The Edmonds-Giles Conjecture

With combinatorial objects, many instances arise when the minimum of some quantity over the object serves as an upperbound for the maximum of another quantity. For instance, this is the relationship displayed by minimum $s-t$ cuts and maximum number of disjoint $s-t$ paths for undirected graphs. It is natural to ask whether these quantities are the same, as is the case with $s-t$ cuts and disjoint $s-t$ paths by Menger's Theorem. In this thesis, we investigate a combinatorial min-max relation where the equality is not always satisfied.

Let $D = (V, A)$ be a directed graph (we will call them *digraphs* for brevity). A *cut* induced by $\emptyset \neq U \subsetneq V$ is the set of arcs $\delta(U) = \{(u, v) \in A : u \in U, v \notin U \text{ or } u \notin U, v \in U\}$ (if the underlying digraph for the cut is ambiguous, we make the digraph explicit by $\delta_D(U)$). A cut $\delta(U)$ can be decomposed into $\delta^+(U) = \{(u, v) \in A : u \in U, v \notin U\}$, the set of outgoing arcs, and $\delta^-(U) = \{(u, v) \in A : u \notin U, v \in U\}$, the set of incoming arcs.

A *directed cut* (or a *dicut*) is a cut induced by $\emptyset \neq U \subsetneq V$ such that all arcs in the cut are either outgoing or incoming, i.e. $\delta(U) = \delta^+(U)$ or $\delta(U) = \delta^-(U)$. For a dicut $\delta(U)$, we call U the *shore* of the dicut. Using the above notation, observe that $\delta^+(U) = \delta^-(V - U)$; we will usually refer to dicuts as $\delta^+(U)$.

A *directed join* (or a *dijoin*) is a set of arcs J such that J intersects every dicut of D at least once. Dijoins are more intuitive when considered under contractions. A digraph D is *strongly connected* if for all pairs of vertices $u, v \in V$, a directed path (or a *dipath*) exists from u to v. It can be shown that D is strongly connected if and only if D does not

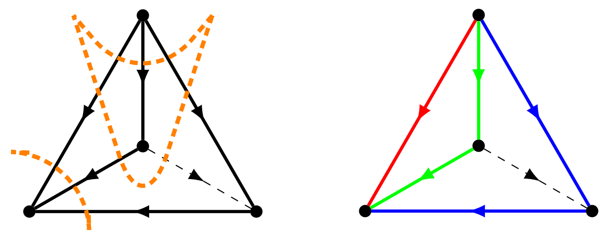

Figure 1.1: A weighted digraph and a weighted packing. Dashed arcs have weight 0, and solid arcs have weight 1. The orange dashed lines show the dicuts of the digraph. The different coloured arcs indicate the dijoins in the weighted packing. Observe that no dijoin in the weighted packing can contain the zero weight arc.

contain a dicut. Hence, a dijoin can be considered to be a set of arcs J such that if all arcs in J are contracted, then D becomes strongly connected.

A *weighted directed graph* is a pair (D, w) where $D = (V, A)$ is a digraph and $w \in \mathbb{Z}_+^A$ is an assignment of non-negative integral weights to every arc in D; in the unweighted case, we simply set $w = 1$. A *w-weighted packing* (or simply a *weighted packing* when the weight is unambiguous) of k dijoins is a collection of k dijoins such that every arc a in D is present in at most $w(a)$ dijoins in the collection (see Figure 1.1).

Let us define $\tau(D, w)$ (or simply τ when the underlying weighted directed graph is clear) as the weight of the minimum weight dicut in (D, w), and $\nu(D, w)$ as the maximum size w-weighted packing of dijoins. We say minimum dicut and minimum weight dicut interchangably.

Dijoins have a non-empty intersection with every dicut, and every $a \in A$ can be in $w(a)$ of the dijoins in the packing. Thus, the size of a w-weighted packing of dijoins is at most the weight of every dicut, leading to the following remark:

Remark 1.1. *For a weighted digraph (D, w), the maximum size of a w-weighted packing of dijoins is at most the minimum weight of a dicut, i.e. $\nu(D, w) \leq \tau(D, w)$.*

It was conjectured [7] that the two quantities are the same.

Conjecture 1.2 (Edmonds and Giles)**.** *For all weighted digraphs, the maximum size of a w-weighted packing of dijoins is equal to the minimum weight of a dicut.*

Unfortunately the conjecture is false; the counter example in Figure 1.2 was given by Schrijver [13]. Before diving into the counter example, we introduce some relevant terms.

A dicut $\delta(U)$ is a *trivial dicut* if $U = \{v\}$ or $U = V - \{v\}$ for a vertex v. A *path* (resp. *cycle*) is a set of arcs which form a path (resp. cycle) in the underlying undirected graph. Given a path P, we partition the arcs into *forward* arcs and *backward* arcs. An *alternating path* (resp. *alternating cycle*) is a path (resp. cycle) where for every pair of adjacent arcs in the path, there is one backward arc and one forward arc.

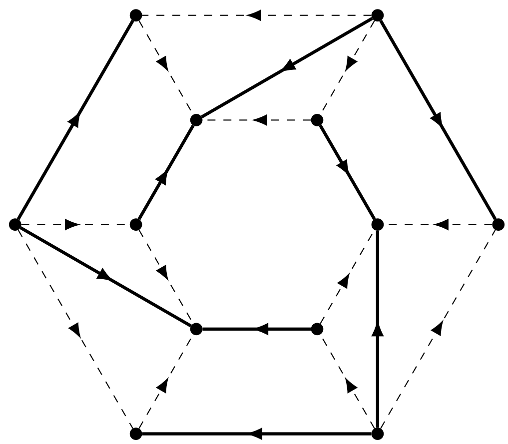

Figure 1.2: Schrijver's counterexample of the Edmonds-Giles conjecture. The dashed arcs are of weight 0, and the solid arcs are of weight 1. The weight of a minimum dicut is 2, but a packing of 2 dijoins does not exist.

Proposition 1.3. *Let (G_s, w_s) be Schrijver's counterexample in Figure 1.2. We have $\nu(G_s, w_s) = 1 < 2 = \tau(G_s, w_s)$.*

Proof. We have $\tau(G_s, w_s) = 2$; it can be readily checked that no dicut of weight 1 exists and a trivial dicut of weight 2 exists.

For an eventual contradiction, suppose (G_s, w_s) contained a w_s-weighted packing of 2 dijoins, labeled J_1 and J_2. Consider the three disjoint alternating paths of length 3 formed by the weight 1 arcs, and observe that the internal vertices of these paths are shores of trivial dicuts. For J_1 and J_2 to intersect these dicuts, each alternating path must be partitioned such that either J_1 or J_2 contains the first and last arc, and the other dijoin must contains the middle arc. Up to symmetry, there are two such partitions of the weight 1 arcs (see Figure 1.4), and in either case J_1 or J_2 has an empty intersection with a dicut in Figure 1.3. $\square$

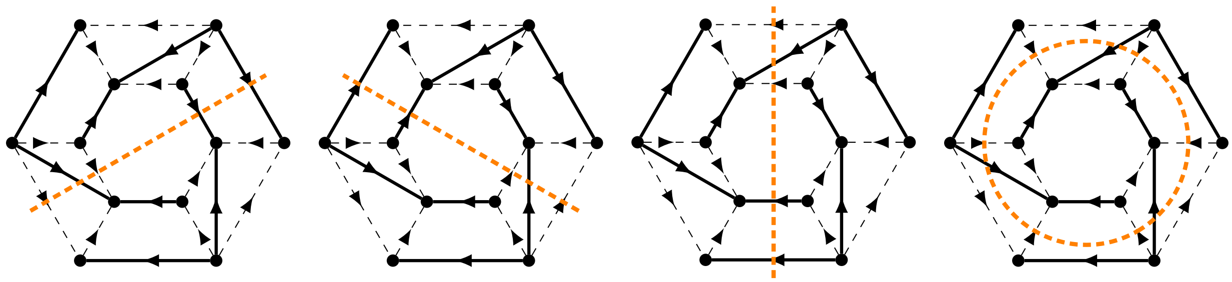

Figure 1.3: The dicuts of interest in Schrijver's counterexample.

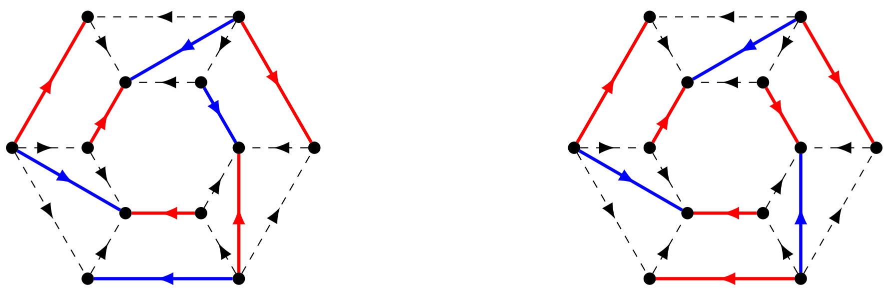

Figure 1.4: Up to symmetry, the partition of weight 1 arcs shown above are the only possible candidates for a packing of 2 dijoins. In the left diagram, the set of blue arcs does not intersect the second cut from Figure 1.3; in the right diagram, the set of red arcs does not intersect the fourth cut from Figure 1.3.

1.2 Swapping Dicuts with Dijoins

The Edmonds-Giles conjecture was particularly enticing because its "dual", derived by swapping the role of dicuts and dijoins, is true.

Theorem 1.4 (Weighted Lucchesi-Younger)**.** *For every weighted digraph, the minimum weight of a dijoin is equal to the maximum size of a w-weighted packing of dicuts.*

In fact, the Weighted Lucchesi-Younger Theorem is implied by the Lucchesi-Younger Theorem [11], which is its unweighted analogue.

Theorem 1.5 (Lucchesi and Younger)**.** *For every directed graph, the minimum cardinality of a dijoin is equal to the maximum number of disjoint dicuts.*

To show the implication, consider the following two-step transformation from a weighted digraph $(D = (V, A), w)$ to an unweighted digraph:

(S1) contract all $a \in A$ such that $w(a) = 0$, and

(S2) for all $a = (u, v) \in A$ with $w(a) > 1$, replace a with a directed path from u to v of length $w(a)$.

Remark 1.6. *Let a be an arc of digraph D. A dicut $\delta_D^+(U)$ is a dicut of D/a, if and only if, $a \notin \delta_D^+(U)$.*

Let (D_{S1}, w_{S1}) be the weighted digraph after applying (S1), and D_{S2} the digraph after applying (S2) to (D_{S1}, w_{S1}). Let us denote A_0 be the set of weight 0 arcs of (D, w). If J is a dijoin of D_{S1}, then $J \cup A_0$ is a dijoin of D. Hence, (S1) preserves the weight of the minimum dijoin. It also preserves the maximum size of a weighted packing of dicuts. By Remark 1.6, the only dicuts of D_{S1} not in D are precisely the dicuts containing weight 0 arcs. However, such dicuts are not in a weighted packing.

Identical statements can be made of (S2). Suppose a single arc $a = (u, v)$ of weight $w(a) > 0$ is replaced by a directed $u-v$ path $a_1, \ldots, a_{w(a)}$ in (D_{S1}, w_{S1}); let this digraph be (D', w'). Observe that if $\delta^+(U)$ is a dicut of D_{S1} with $a \in \delta^+(U)$, then $(\delta^+(U) - \{a\}) \cup \{a_i\}$ for every $i \in [w(a)]$ is a dicut of D'. Hence a minimal dijoin J' of D' either contains all arcs in $a_1, \ldots, a_{w(a)}$ or none of them, preserving the weight of the minimum dijoin. Additionally, a k disjoint dicuts in (D', w') yields a weighted packing of k dijoins in (D_{S1}, w_{S1}); dicuts which contains a_i now contains a.

The arguments above give a natural mapping from a collection of k disjoint dicuts of D_{S2} to a w-weighted packing of k dicuts in (D, w), as well as a minimum cardinality dijoin of D_{S2} to a minimum weight dijoin of (D, w).

However, when packing dijoins, there is no known way of easily reducing the Edmonds-Giles conjecture to an unweighted setting. For arcs a with $w(a) > 1$ we may add $w(a) - 1$ copies of a, fixing all weights (including the orignal arc) to be 1. Every dicut shore before the transformation is still a dicut shore, and no new shores are introduced by adding arcs. In addition, the weight of the dicut for every shore remains the same.

Remark 1.7. *When packing dijoins in weighted digraphs, we may assume that $w \in \{0, 1\}^A$.*

Handling weight 0 arcs are more elusive. The obvious first attempt is to delete such arcs, which would preserve the weight of every dicut which existed prior to the deletion. Unfortunately deleting an arc may introduce new dicuts with weights less than τ (see Figure 1.5), which certifies that a w-weighted packing of size τ does not exist.

Figure 1.5: Deleting a weight 0 arc may introduce new dicuts. The right weighted digraph is produced by deleting the weight 0 arc of the left weighted digraph (arcs are labeled by their weights). The cut indicated by the dashed orange circle is not a dicut in the left weighted digraph, but is a dicut in the right weighted digraph. Furthermore, the value of τ of the right weighted digraph is less than the value of τ of the left weighted digraph.

1.3 Previous Relaxations

We say that a weighted digraph (D, w) *packs* if $\tau(D, w) = \nu(D, w)$. What properties must (D, w) satisfy in order for it to pack? As previously mentioned, unlike for the Weighted Lucchesi-Younger Theorem, we do not know whether the Edmonds-Giles conjecture is equivalent to its unweighted version. Woodall [19] conjectured that for every digraph, $(D, 1)$ packs. Indeed, Edmonds-Giles conjecture is false, and Woodall's conjecture is open.

Conjecture 1.8 (Woodall)**.** *For every digraph, the minimum cardinality of a dicut is equal to the maximum number of disjoint dijoins.*

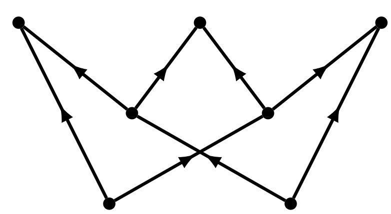

Figure 1.6: A source-sink connected digraph.

Dropping the weights is simply one of the ways to relax the Edmonds-Giles conjecture. In fact, a known instance of when the Edmonds-Giles conjecture is true are for source-sink connected digraphs [14, 8]. A weighted digraph is *source-sink connected* if there exists a directed path from every source to every sink (see Figure 1.3).

Theorem 1.9. *If D is a source-sink connected digraph, then (D, w) packs for every $w \in \mathbb{Z}_+^A$.*

1.4 Outline of the Thesis

The purpose of this thesis is to explore the Edmonds-Giles conjecture and its various relaxations.

In Chapter 2, we take a look at a *fractional packing* of dijoins, which relaxes the Edmonds-Giles conjecture by allowing an assignment of fractional values to dijoins in a packing. We introduce combinatorial objects called *clutters*, which is of interest as the set of dijoins of a digraph forms a clutter. An unpublished conjecture by Seymour ([16], Section 79.3e) about fractional packings for *ideal clutters* is given, and a natural relaxation of the conjecture is proven for dijoins.

Chapter 3 focuses on finding a weighted packing of dijoins of size $k < \tau$. It is an open question whether there exists τ' such that every weighted digraph (D, w) with $\tau(D, w) \geq \tau'$ has a weighted packing of 3 dijoins. We show that if a weighted digraph has bounded packing size, then the weighted digraph is sparse. Furthermore, the notion of ε-balanced digraphs is introduced, and we find a weighted packing of dijoins of size dependent on ε and τ.

Chapter 4 is dedicated to posing a slightly different question to whether a weighted digraph packs. A new object over digraphs called a *kernel* is introduced; the existence of a kernel for a weighted digraph implies that the weighted digraph packs. We give examples of kernels in specific instances.

In Chapter 5, we conclude by reiterating open problems in the field of packing dijoins, and stating natural extentions to the thesis.

Chapter 2

Fractional Packing of Dijoins

In this chapter, we look at fractional packings of dijoins. We introduce combinatorial objects called *clutters*, and packing and covering problems on clutters which generalize packing dijoins. We also introduce the dyadic conjecture by Seymour, and prove a natural relaxation of the conjecture for a fractional packing of dijoins.

For this thesis, we denote by $\mathbb{Z}_+$ as the set of non-negative integers; by $[n]$, we mean $\{1, 2, \ldots, n\}$. All vectors are column vectors.

2.1 A Fractional Packing Always Exists

2.1.1 Clutters

Let V be a finite set of elements. A *clutter* $\mathcal{C}$ is a family of subsets over the ground set V such that no set in $\mathcal{C}$ contains another set in $\mathcal{C}$. A *cover* of a clutter $\mathcal{C}$ over a ground set V is a subset $B \subseteq V$ such that B has a non-empty intersection with all memebers in $\mathcal{C}$.

Various clutters which arise from combinatorial objects have been studied. For digraphs, some examples are the family of minimal $s-t$ directed paths, the family of minimal $s-t$ cuts (by which we mean $\delta^+(U)$ where $s \in U, t \notin U$), as well as the family of dijoins and dicuts. Given an undirected graph $G = (V, E)$ with $T \subseteq V$ where $|T|$ is even, a T-*join* is $J \subseteq E$ such that the set of vertices in (V, J) with odd degree is exactly T; a T-cut is a cut which separates T into two sets of odd size. For undirected graphs, the family of minimal T-joins and T-cuts are clutters.

2.1.2 Linear and Integer Programs

A rational system $Mx \leq b$ is *totally dual integral* (TDI) if for every integral vector w such that $\max\{w^T x : Mx \leq b\}$ exists, its dual $\min\{b^T y : M^T y = w, y \geq 0\}$ has an integral optimal solution.

A polyhedron P is an *integral polyhedron* if every minimal face of P contains an integral point. For a polyhedron $P = \{x : Mx \leq b\}$ to be integral, it is a necessary condition that $\max\{w^T x : x \in P\}$ has an optimal integral value for every integral w such that the linear program admits an optimal solution. Interestingly, it is also a sufficient condition; having an optimal integral value for all integral weights for which the program admits an optimal solution guarantees that P is an integral polyhedron ([4], see Theorem 4.1). The following theorem is a consequence of this fact.

Theorem 2.1 ([4], Theorem 4.26)**.** *If $Mx \leq b$ is TDI and b is integral, then $\{x : Mx \leq b\}$ is an integral polyhedron.*

For a clutter $\mathcal{C}$ over a ground set V, let $M(\mathcal{C})$ be a $|\mathcal{C}| \times |V|$ matrix where the rows are characteristic vectors of $C \in \mathcal{C}$. Consider the following primal-dual pair of linear programs where $w \in \mathbb{Z}_+^V$.

$$\tau^*(\mathcal{C}, w) = \min\{w^T x : M(\mathcal{C})x \geq 1, x \geq 0\} \tag{$P_\mathcal{C}$}$$

$$\nu^*(\mathcal{C}, w) = \max\{1^T y : M(\mathcal{C})^T y \leq w, y \geq 0\} \tag{$D_\mathcal{C}$}$$

Let $(IP_\mathcal{C})$ denote the program derived by adding integrality constraints to $(P_\mathcal{C})$, and $(ID_\mathcal{C})$ the integer program derived from $(D_\mathcal{C})$. Additionally, let $\tau(\mathcal{C}, w)$ denote the optimal value of $(IP_\mathcal{C})$ and $\nu(\mathcal{C}, w)$ denote the optimal value of $(ID_\mathcal{C})$.

The optimal solution $\bar{x}$ of $(IP_\mathcal{C})$ describes a subset of V such that it contains at least one element from each member of $\mathcal{C}$, i.e. a minimum weight cover. Likewise, the optimal solution $\bar{y}$ of $(IP_\mathcal{C})$ describes a collection of members in $\mathcal{C}$ such that $v \in V$ is present in at most $w(v)$ members, i.e. a maximum size weighted packing of members of $\mathcal{C}$. An optimal solution to $(D_\mathcal{C})$ is called a *fractional packing* of the members of the clutter.

A clutter $\mathcal{C}$ is *ideal* if the polyhedron $\{x : M(\mathcal{C})x \geq 1, x \geq 0\}$ is an integral polyhedron. Equivalently, it is ideal if for all integral weights such that $(P_\mathcal{C})$ has an optimal solution, it has an integral optimal solution. The clutter has the *max-flow min-cut* (or *MFMC* for short) property if the system $M(\mathcal{C})x \geq 1, x \geq 0$ is TDI. By Theorem 2.1, we have the following remark.

Remark 2.2. *If a clutter has the MFMC property, then it is ideal.*

The converse does not hold as shown by the clutter

$$Q_6 = \{\{1,2,4\},\{1,3,5\},\{2,3,6\},\{4,5,6\}\}$$

over the ground set [6]. Observe that $\nu(Q_6, 1) = 1$ as no two members in Q_6 are disjoint, while $\tau(Q_6, 1) = 2$ as no single element is contained in all members of Q_6. However, Q_6 is ideal [10].

Remark 2.3. *The clutter Q_6 is ideal, but does not have the MFMC property.*

2.1.3 Blockers

For a clutter $\mathcal{C}$, the *blocker* of $\mathcal{C}$ (denoted $b(\mathcal{C})$) is the clutter of minimal covers of $\mathcal{C}$. A clutter and its blocker form a natural pair, as Edmonds and Fulkerson proved that a blocker of a blocker of a clutter is the clutter itself.

Proposition 2.4 ([5], Theorem 1.15)**.** *For a clutter $\mathcal{C}$, $b(b(\mathcal{C})) = \mathcal{C}$.*

Given a clutter $\mathcal{C}$, we say that $\mathcal{C}$ and $b(\mathcal{C})$ form a *blocking pair*. In fact, Lehman showed that idealness is a property of a blocking pair.

Proposition 2.5 ([5], Theorem 1.17)**.** *A clutter $\mathcal{C}$ is ideal if and only if $b(\mathcal{C})$ is ideal.*

As expected, many of the examples of clutters previously given are blocking pairs. For instance, $s-t$ directed paths and $s-t$ cuts, T-joins and T-cuts, and of course, dijoins and dicuts.

2.1.4 Rational Fractional Packings of Dijoins

For a digraph G, let $\mathcal{J}(G)$ be the clutter of dijoins, and $\mathcal{C}(G)$ the clutter of dicuts of G. We first prove that $\mathcal{J}(G)$ is ideal.

Proposition 2.6. *For all digraphs, the clutter of dijoins is ideal.*

Proof. By the weighted Luchessi-Younger theorem (Theorem 1.4), the system $M(\mathcal{C}(G))x \geq 1, x \geq 0$ is TDI, and therefore $\mathcal{C}(G)$ has the MFMC property. The MFMC property implies idealness of $\mathcal{C}(G)$ (Remark 2.2). The clutters $\mathcal{J}(G)$ and $\mathcal{C}(G)$ form a blocking pair; by Proposition 2.5 idealness is a mutual property of a blocking pair, thus $\mathcal{J}(G)$ is ideal. □

Corollary 2.7. *For every weighted digraph (G, w), $\tau(G, w) = \nu^*(G, w)$; the weight of the minimum dicut is equal to the optimal w-weighted fractional packing of dijoins.*

By Cramer's Rule, we know a fractional solution to $(D_{\mathcal{J}(G)})$ exists which is rational. What more can we say about the dual solution?

2.2 Fractional Packings with Structured Denominators

For $0 < k \in \mathbb{Z}_+$, a fractional packing y is said to be $\frac{1}{k}$-*integral* if $y = \frac{1}{k}\overline{y}$ where $\overline{y}$ is integral. The following remark states a more intuitive way of thinking about fractional packings that are rational; we find a weighted packing where the weights are scaled accordingly.

Remark 2.8. *For a weight w, an optimal $\frac{1}{k}$-integral w-weighted packing exists, if and only if, an optimal kw-weighted packing exists.*

2.2.1 The Dyadic Conjecture

Recall that the clutter $Q_6 = \{\{1, 2, 4\}, \{1, 3, 5\}, \{2, 3, 6\}, \{4, 5, 6\}\}$ is an ideal clutter that does not possess the MFMC property; $\nu(Q_6, 1) = 1 < 2 = \tau(Q_6, 1)$. However, (D_{Q_6}) with $w = 1$ has a $\frac{1}{2}$-integral solution, namely the vector with all $\frac{1}{2}$ entries (every element is present in exactly two members in Q_6). In fact, the statement generalizes to all $w \in Z_+^{[6]}$.

Remark 2.9 ([9], Corollary 1.4)**.** *For all $w \in Z_+^{[6]}$, (D_{Q_6}) has a $\frac{1}{2}$-integral optimal solution.*

However, not every ideal clutter admits an optimal $\frac{1}{2}$-integral dual solution. The following counterexample was given by Seymour [17].

Let $G = (V, E)$ be a graph. The clutter of T-joins of G is ideal; its blocker, the clutter of T-cuts, was shown to be ideal by Edmonds and Johnson ([5], Theorem 2.1). Let G be the Petersen graph with a vertex replaced with a triangle; the three edges incident to the original vertex are incident to distinct vertices of the triangle.

Observe that G is connected, bridgeless, 3-regular and non 3-edge-colourable with an even number of edges and vertices. Let $\mathcal{T}$ be the clutter of V-joins of G. Construct a new graph $H = (V_H, E_H)$ by replacing every edge of G with a path of length two, introducing a "middle" vertex with degree 2 for each edge. Then $|V_H|$ is even. Seymour showed that H does not have an optimal $\frac{1}{2}$-integral packing of V-joins.

Proposition 2.10 ([17], Counterexample 4.3)**.** *The graph H with $w = 1$ does not admit an optimal $\frac{1}{2}$-integral packing of V_H-joins.*

Proof. By assumption, G is connected and bridgeless, thus H is as well; a minimum V_H-cut has at least 2 edges. Additionally, H has vertices of degree two, thus a minimum V_H-cut has exactly 2 edges.

It suffices to show that with $w = 2$, a w-weighted packing of V_H-joins of size $2\tau(\mathcal{T}, 1) = 4$ does not exist (Remark 2.8). For an eventual contradiction, suppose J_1, J_2, J_3, J_4 is such a packing. Since $w = 2$, and every vertex in $V \cap V_H$ is incident to a a vertex of degree 2 in V_H, each edge of H is contained in two of the J_is. Let C_i be the symmetric difference of J_i and J_4 for $i \in [3]$. Observe that each C_i is an edge-disjoint union of cycles, and each path of length 2 which replaced edges in G to form H belong to two of the C_is. Let C_i' be the corresponding edges in G of C_i, and consider their complement $\overline{C_i'}$. Every edge of G belonging to two of the C_is mean $\overline{C_i}$ are edge disjoint. Since C_i are edge-disjoint cycles, and the graph is 3-regular, the $\overline{C_i}$ form edge-disjoint V-joins. However, the $\overline{C_i}$ form a valid 3-edge colouring of G, which is a contradiction. $\square$

Seymour also showed that the above example does not admit an optimal $\frac{1}{3}$-integral packing, but noted that an optimal $\frac{1}{4}$-integral solution exists. Thus, he conjectured the following:

Conjecture 2.11. *For every clutter of T-joins, there exists an optimal fractional packing that is $\frac{1}{4}$-integral.*

Furthermore, Seymour made the following conjecture ([16], see Section 79.3e) for all ideal clutters. A fractional packing is *dyadic* if it is $\frac{1}{2^k}$-integral for some $k \in \mathbb{Z}_+$.

Conjecture 2.12 (Dyadic Conjecture)**.** *Every ideal clutter has an optimal fractional packing that is dyadic.*

Recently, some evidence has been shown in support of the conjecture. Abdi, Cornuéjols and Palion [2] have shown that all graphs and edge weights admit a dyadic packing of T-joins. Additionally, Abdi, Cornuéjols, Guenin and Tunçel [1] showed that for an ideal clutter $\mathcal{C}$ and weights w such that $\tau(\mathcal{C}, w) \geq 2$, a dyadic packing of value 2 always exists; namely, this shows the dyadic conjecture when $\tau(\mathcal{C}, w) = 2$. In fact, they showed that the result holds for a more general class of clutters called *clean clutters*.

2.2.2 Denominators in an Optimal Fractional Packing

The packing and covering linear programs for dijoins is reiterated for convenience below. Recall that by $\mathcal{J}(G)$ being ideal, $\tau^*(G,w) = \tau(G,w)$,

$$\begin{array}{lll} \min & w^T x & \\ \text{s.t.} & \sum(x_a : a \in J) \geq 1 & \forall J \in \mathcal{J}(G) \\ & x_a \geq 0 & \forall a \in A \end{array} \tag{$P_{G,w}$}$$

$$\begin{array}{lll} \max & 1^T y & \\ \text{s.t.} & \sum(y_J : J \in \mathcal{J}(G), a \in J) \leq w(a) & \forall a \in A \\ & y_J \geq 0 & \forall J \in \mathcal{J}(G) \end{array} \tag{$D_{G,w}$}$$

We prove a natural generalization of the dyadic conjecture for the clutter of dijoins. A $[n]$-*adic rational* for $n \in \mathbb{Z}_+, n > 0$ is a rational number of the form $\frac{a}{b}$ where $a, b \in \mathbb{Z}, b > 0$ and b is a product of primes in $[n]$. A vector is a $[n]$-*adic vector* if all its entries are $[n]$-adic rationals. For convenience, we define [1]-adic numbers to be the integers.

We prove the following theorem:

Theorem 2.13. *For a weighted digraph (G,w), an optimal $[\tau(G,w)]$-adic fractional packing of dijoins exist.*

Note that when $\tau = 2$, then a $[\tau]$-adic packing is a dyadic packing.

For a weighted digraph (G,w), let $A_{w>0} = \{a \in A : w(a) > 0\}$. For the remainder of this chapter, we call (G,w) a counterexample if (G,w) does not admit a $[\tau(G,w)]$-adic fractional packing of dijoins. We call (G,w) a minimal counterexample if

1. (G,w) is a counterexample,
2. G minimizes $|V|$ among such digraphs, and subject to that,
3. $|A_{w>0}|$ is minimized, and subject to that,
4. the sum of the weights, $1^T w$, is minimized.

We first prove that minimal counterexamples adhere to a useful structure. Note that if the minimum weight of a dicut is 1, then the set of weight 1 arcs is a dijoin, which is a [1]-adic packing; every minimal counterexample has $\tau \geq 2$.

Lemma 2.14. *If (G, w) is a minimal counterexample with $\tau(G, w) \geq 2$, then every $a \in A_{w>0}$ is contained in a minimum dicut.*

Proof. Clearly an arc a exists with positive weight in order for $\tau(G, w) \geq 2$. For an eventual contradiction, suppose $a' \in A$ is an arc where $w(a') > 0$ and a' does not belong to a minimum dicut of G. Consider a new cost w' where

$$w'(a) = \begin{cases} w(a) & a \in A - \{a'\} \\ w(a) - 1 & a = a'. \end{cases}$$

By assumption every dicut in (G, w) containing a' has value at least $\tau(G, w) + 1$. The weights w and w' agree on every arc except a', where $w'_{a'} = w_{a'} - 1$; every dicut containing a' has weight at least $\tau(G, w)$ in (G, w'). Thus, $\tau(G, w) = \tau(G, w')$.

Since $1^T w' < 1^T w$, by minimality an optimal $[\tau(G, w)]$-adic fractional solution $\overline{y}$ of $(D_{G,w'})$ exists. This $\overline{y}$ is also a feasible solution to $(D_{G,w})$ with value $\tau(G, w)$, which contradicts (G, w) being a minimal counterexample. □

For the next lemma, we require the following remark.

Remark 2.15. *[[3], Remark 4.1] Let $\delta^+(U)$ and $\delta^+(W)$ be dicuts of a digraph $G = (V, A)$. The following statements hold:*

1. *if $U \cup W \neq V$, $\delta^+(U \cup W)$ is a dicut of G, and*
2. *if $U \cap W \neq \emptyset$, $\delta^+(U \cap W)$ is a dicut of G.*

For $U \subseteq V$, let G/U denote the weighted digraph where the vertices in U are contracted to a single vertex v_U, the arcs $(u, v) \in \delta^+(U), (v, u) \in \delta^-(U)$ are replaced with arcs (v_U, v) and (v, v_U) respectively. Let w/U denote the weight of arcs in G/U where the newly added arcs (of the form (v_U, v) or (v, v_U)) have the same weight as the arcs which were replaced. We will also use the shorthand $(G, w)/U = (G/U, w/U)$. The next two lemmas address a counterexample which minimizes the number of vertices, which is not necessarily a minimal counterexample.

Lemma 2.16. *In a counterexample (G, w) minimizing $|V|$, every minimum dicut is a trivial dicut.*

Proof. For an eventual contradiction, suppose a counterexample (G, w) minimizing $|V|$ has a non-trivial minimum dicut $\delta^+_G(U)$. Let $(G_1 = (V_1, A_1), w_1) = (G, w)/U$ and $(G_2 = (V_2, A_2), w_2) = (G, w)/(V - U)$. A dicut of G_i is also a dicut of G with the same weight, thus minimum dicuts of G_i have weight $\tau(G, w)$.

Note that $|V_1|, |V_2| < |V|$; by minimality (G_i, w_i) has an optimal $[\tau(G, w)]$-adic packing $\overline{y}^i$ where $1^T\overline{y}^i = \tau(G, w)$.

Let J_i be a minimal dijoin of G_i. The dicut $\overline{x}$ where

$$\overline{x}_a = \begin{cases} 1 & a \in \delta^+(U) \\ 0 & \text{Otherwise} \end{cases}$$

is a minimum dicut of G_i. By Complementary Slackness, for $a \in \delta^+(U)$, $\sum(\overline{y}^i_{J_i} : a \in J_i) = 1$. In addition, if $\overline{y}^i_{J_i} > 0$, then we have $\sum(x_a : a \in J_i) = 1$, i.e. if $\overline{y}^i_{J_i}$ is positive,

$$|J_i \cap \delta^+(U)| = 1. \tag{2.17}$$

A pair (J_1, J_2) is *valid* if $J_1 \cap \delta^+(U) = J_2 \cap \delta^+(U)$. By (2.17) if (J_1, J_2) are valid, then they share a unique arc in $\delta^+(U)$. We first prove that if (J_1, J_2) are valid then $J_1 \cup J_2$ is a dijoin of G.

Claim 1. If (J_1, J_2) is valid, then $J_1 \cup J_2$ is a dijoin of G.

Proof For an eventual contradiction, suppose $J = J_1 \cup J_2$ is not a dijoin of G; there exists a dicut $\delta^+(W)$, $W \subseteq V$ where $J \cap \delta^+(W) = \emptyset$. Let $a = (u_1, u_2) \in J_1 \cap J_2$, the unique arc in the intersection. Since $a \notin \delta^+(W)$, either $\{u_1, u_2\} \subseteq W$ or $\{u_1, u_2\} \subseteq V - W$.

If $W \subseteq U$ or $W \subseteq V - U$, $\delta^+(W)$ is a dicut of G_1 or G_2 respectively. In either case, $J_i \cap \delta^+(W) = \emptyset$, contradicting that J_i is a dijoin of G_i; we have $W \cap U \neq \emptyset$. Additionally if $W \cup U = V$, then $\delta^+(W \cap U)$ is a dicut of G_2, and $a \notin \delta^+(W \cap U)$ in order for $\delta^+_G(W)$ to be a dicut. This means that $\delta^+(W) \cap J_2 \neq \emptyset$; we have $W \cup U \neq V$.

Suppose $\{u_1, u_2\} \subseteq W$. By Remark 2.15, $\delta^+(U \cup W)$ is a dicut of G. Since $\{u_1, u_2\} \subseteq W$ we have $(u_1, u_2) \notin \delta^+(W \cup U)$. The dicut $\delta^+(W \cup U)$ corresponds to the dicut $\delta^+(\{v_U\} \cup (W \cap (V - U)))$ of G_1. This dicut of G_1 has an empty intersection with J_1, which contradicts J_1 being a dijoin of G_1.

Now suppose $\{u_1, u_2\} \subseteq V - W$. Again, by Remark 2.15 and $U \cap W \neq \emptyset$, $\delta^+(U \cap W)$ is a dicut of G. Similar to above, $\delta^+(U \cap W) \cap J = \emptyset$. The dicut $\delta^+(U \cap W)$ corresponds to a dicut of the digraph G_2, but J_2 does not intersect the dicut, contradicting that J_2 is a dijoin of G_2. $\lozenge$

Since $\overline{y}^1$ and $\overline{y}^2$ are $[\tau(G,w)]$-adic, they can be expressed as $\overline{y}^i = \frac{1}{k_i}\hat{y}^i$ where $\hat{y}^i \in \mathbb{Z}_+^{A_i}$, and k_i is a product of primes in $[\tau(G,w)]$. Let $k = k_1 \times k_2$. Then $k\overline{y}^i$ is an integral solution of (D_{G_i,kw_i}), a kw_i-weighted packing of $k\tau(G,w)$ dijoins. Showing that (G,kw) packs implies that (G,w) admits an optimal $[\tau(G,w)]$-adic fractional packing of dijoins, which yields a contradiction.

For ease of argument, we will work in the setting where the weights are 0 or 1. Consider weighted digraph (G_i', kw_i') derived from $(G_i, w_i), i \in \{1,2\}$, and (G', kw') from (G, kw) by applying Remark 1.7. The G_i's can still formed from G by contracting U.

Observe that a union of different valid pairs form different dijoins of G'. Every dijoin of G_1' or G_2' must include at least one arc in $\delta^+(U)$ and exactly one by 2.17. Thus $\tau(G', kw')$ valid pairs exist (one pair for each weight one arc in $\delta^+_{G'}(U)$), which form $\tau(G,kw) = \tau(G',kw')$ dijoins of G. These dijoins are disjoint, since the packing of dijoins of G_1' and G_2' are disjoint. Hence (G',kw') packs, and thus (G,kw) packs; this yields a contradiction. $\square$

The reduction above (that we may assume minimum dicuts are trivial dicuts) is a routine argument; for instance, it can be found as subclaim (56.2) of Theorem 56.1 in [16].

Lemma 2.18. *A counterexample (G,w) minimizing $|V|$ has no arc with weight $\tau(G,w)$.*

Proof. Consider the case where every arc a that is not a' has weight 0. No dicut of G exists that does not contain a', and no non-trivial dicut exists which contains a' by Lemma 2.16. Let $a' = (v_1, v_2)$. One or both of $\delta^+(v_1)$ and $\delta^+(V - \{v_2\})$ are the only dicuts of G. In either case, $\{a'\}$ is a dijoin, and a collection of $\tau(D,w)$ sets of $\{a'\}$ is an optimal integral packing (which is $[\tau(G,w)]$-adic) of dijoins.

Now suppose a different arc than a' exists with non-zero cost. By Lemma 2.14 a minimum dicut $\delta(v)$ not containing a' exists, which is also a minimum dicut of $(G,w)/a'$ with weight $\tau(G,w)$. Contracting an arc does not decrease the minimum weight of a dicut, thus $\tau(G,w) = \tau(G/a', w/a')$. Consider $(G,w)/a'$, which by minimality has y', an optimal $[\tau(G,w)]$-adic fractional packing of dijoins. Every dijoin of $(G,w)/a'$ with a' is a dijoin of (G,w); for J a dijoin of $(G,w)/a'$, we construct a fractional packing of dijoins y for (G,w) where $y_{J\cup\{a'\}} = y'_J$. The weighted packing y has value $\tau(G,w)$, and is a valid packing since $w(a') = \tau(G,w)$. $\square$

In summary we have proved that for a minimal counterexample (G,w),

1. every minimum dicut of (G,w) is a trivial cut,

2. every arc a with $w(a) > 0$ is contained in a minimum dicut, and
3. $w(a) \in \{0\} \cup [\tau(G, w) - 1]$ for all $a \in A$.

The last two properties are held by every counterexample which minimizes the number of vertices.

The next lemma reveals our plan of attack: using the structure of the minimal counterexample, we derive two weights w^1, w^2 from w where w is a weighted average of w^1 and w^2, and w^1 and w^2 admit $[\tau(G, w)]$-adic fractional packings. We can then find the desired fractional packing for (G, w) by taking a weighted average of the solutions to (G, w^1) and (G, w^2).

Lemma 2.19. *Let $w \in \mathbb{Z}_+^A$. If there exists $w^1, w^2 \in \mathbb{Z}_+^A$ such that*

(1) $w = dw^1 + (1 - d)w^2$ where $0 < d < 1$ is a $[\tau(G, w)]$-adic rational number,

(2) $\tau(G, w) = \tau(G, w^1) = \tau(G, w^2)$, and

(3) (D_{G,w^1}) and (D_{G,w^2}) have an optimal $[\tau(G, w)]$-adic fractional packing,

then $(D_{G,w})$ admits an optimal $[\tau(G, w)]$-adic fractional packing of dijoins.

Proof. By assumptions (2) and (3), (D_{G,w^1}) and (D_{G,w^2}) admit $[\tau(G, w)]$-adic optimal solutions y^1 and y^2 respectively. Consider $y' = dy^1 + (1 - d)y^2$, which is $[\tau(G, w)]$-adic by (1). We show that y' is optimal for $(D_{G,w})$ to prove the lemma. The vector y' is feasible for $(D_{G,w})$ since by (1), for all $a \in A$,

$$\sum(y'_J : a \in J) = \sum(dy^1_J + (1 - d)y^2_J : a \in J) \leq dw^1_a + (1 - d)w^2_a = w(a).$$

In addition, $1^T y' = d1^T y^1 + (1 - d)1^T y^2 = \tau(G, w)$ by (3), which is the optimal value of $(D_{G,w})$ since it is the optimal value of $(P_{G,w})$. $\square$

Let (G, w) be a minimal counterexample. We now construct the weights w^1 and w^2 for (G, w). Let $\sim$ be a relation on $a, a' \in A_{w>0}$ where $a \sim a'$ if a and a' both belong to the same minimum dicut. Clearly $\sim$ is reflexive and symmetric; we also define $\sim$ to be transitive. Let E be an equivalence class of $\sim$.

We call a cycle in a digraph an *odd alternating cycle* if the cycle has odd length and except for a single pair, every pair of adjacent arcs in the cycle contains one forward arc and one backward arc.

Remark 2.20. *Let $T = a_1, a_2, \ldots, a_k$ be an odd alternating cycle where a_1 and a_k are arcs which are either both forward arcs or backward arcs. If a cut $\delta(U)$ contains a_1 and a_k, $\delta(U)$ is not a dicut.*

Proof. If u is the mutual endpoint of a_1 and a_k, one arc leaves u and the other enters u. $\square$

Lemma 2.21. *Let T be either a cycle or a path. If $\delta^+(U)$ is a dicut such that $|T \cap \delta^+(U)| \geq 1$, then the difference between the number of forward arcs and backward arcs in $T \cap \delta^+(U)$ is at most 1.*

Proof. Let $\delta^+(U)$ be such that $|T \cap \delta^+(U)| \geq 2$ (clearly the result holds if $|T \cap \delta^+(U)| = 1$), and

$$T = a_1, a_2, \ldots, a_k$$

where a_1, a_k are the first and last arcs respectively if T is a path. Let T' be a subsequence of T consisting only of arcs in $T \cap \delta^+(U)$, i.e.

$$T' = a_{b_1}, a_{b_2}, \ldots, a_{b_l}$$

where $l \leq k$.

It suffices to show that every pair of arcs of the form $(a_{b_j}, a_{b_{j+1}})$ in T' contains one forward arc and one backward arc.

Without loss of generality, let $a_{b_i}, i \in [l-1]$ be a forward arc. For an eventual contradiction, suppose that $a_{b_{i+1}}$ is also a forward arc. Let v_{j-1}, v_j be the endpoints of the arc a_j. Since $a_{b_i} = (v_{b_i - 1}, v_{b_i}), a_{b_{i+1}} = (v_{b_{i+1}-1}, v_{b_{i+1}})$ are forward arcs in $\delta^+(U)$, $v_{b_i}, v_{b_{i+1}}$ are in $V - U$. Furthermore, T' is a subsequence of T, hence no arc a_j, $b_i < j < b_{i+1}$, is in T'; every vertex between and including v_{b_i} and $v_{b_{i+1}-1}$ in T are in $V - U$. This is a contradiction since both endpoints of $a_{b_{i+1}}$ are in $V - U$, and thus $a_{b_{i+1}} \notin \delta^+(U)$. $\square$

If a dicut and a dicycle have a non-empty intersection, its intersection contains at least two arcs, and the difference between the number of backward and forward arcs is at most one. However, a dicycle only contains forward or backward arcs; dicuts and dicycles are always disjoint. Similarly, dicuts and dipaths have at most one arc in common.

Corollary 2.22. *Dicuts are disjoint from directed cycles.*

Corollary 2.23. *Dicuts and dipaths have at most one arc in common.*

If an (odd) alternating cycle exists in E, let T be such an alternating cycle. Otherwise, we pick a *maximal alternating path* contained in E of length at least 2; an alternating path is maximal if no arcs can be added to create a longer alternating path. If an (odd) alternating cycle does not exist, a maximal alternating path of length at least two exists, since $\tau(D, w) \geq 2$; in all instances, T contains at least 2 arcs. Hence, T is either an (odd) alternating cycle, or a maximal alternating path. We will call T an *alternating structure*. For brevity, we label the following hypotheses:

(H1) T is an alternating structure contained in an equivalence class H under $\sim$, and

$$w_\varepsilon^f(a) = \begin{cases} w(a) + \varepsilon & a \text{ is a forward arc of } T \\ w(a) - \varepsilon & a \text{ if a backward arc of } T \\ w(a) & \text{Otherwise.} \end{cases}$$

(H2) T is an alternating structure contained in an equivalence class H under $\sim$, and

$$w_\varepsilon^b(a) = \begin{cases} w(a) + \varepsilon & a \text{ is a backward arc of } T \\ w(a) - \varepsilon & a \text{ if a forward arc of } T \\ w(a) & \text{Otherwise.} \end{cases}$$

Lemma 2.24. *Let $\delta^+(U)$ be a dicut of (G, w), and $\varepsilon \in \mathbb{Z}_+ - \{0\}$. Suppose w_ε^f, T satisfies (H1) and w_ε^b, T satisfies (H2). Then the following holds:*

1. *if $\delta^+(U) = \delta^+(v)$ is a minimum dicut in (G, w), then $w_\varepsilon^f(\delta^+(v)) = w_\varepsilon^b(\delta^+(v)) = \tau(G, w)$, and*
2. *if $\delta^+(U)$ has a non-empty intersection with T, $w_\varepsilon^f(\delta^+(U))$ is either $w(\delta^+(U)) + \varepsilon$, $w(\delta^+(U)) - \varepsilon$, or $w(\delta^+(U))$, and $w_\varepsilon^b(\delta^+(U))$ is either $w(\delta^+(U)) + \varepsilon$, $w(\delta^+(U)) - \varepsilon$, or $w(\delta^+(U))$.*

Proof. **(1)** Let $\delta^+(v)$ be a minimum dicut in (G, w) such that $\delta^+(v) \cap T \neq \emptyset$. If T is an (odd) alternating cycle, then $|\delta^+(v) \cap T| = 2$. One arc in $\delta^+(v) \cap T$ must be a forward arc and the other a backward arc (guaranteed by Remark 2.20 for odd alternating cycles). Since forward arcs and backward arcs are increased and decreased (or decreased and increased for w_ε^b) by the same amount, the weight is maintained.

If T is an alternating path, it suffices to show that no v exists such that v is an endpoint of T. If such a v existed, then T is not a maximal alternating path; since $w(\delta(v)) \geq \tau(G, w) \geq 2$, another arc in $\delta(v)$ exists which can extend the alternating path.

(2) The intersection $\delta^+(U) \cap T$ contains at most one more forward arc than backward arc (or vice versa) by Lemma 2.21 and Remark 2.20; similar to above, the result holds. $\square$

We want $\alpha, \beta \in \mathbb{Z}_+$ such that w^f_α and w^b_β are non-negative and the weights satisfy at least one of the following conditions:

(C1) a non-trivial dicut becomes a minimum dicut,

(C2) a weight of an arc becomes $\tau(G, w)$,

(C3) an arc $a \in A$ exists with $w(a) > 0$, but has weight zero under the new weight, or

(C4) the sum of the entries for the new weights is less than $1^T w$.

Let α_j (and likewise β_j) denote the minimum positive integer such that $w^f_{\alpha_j}$ satisfies condition (Ci) above (if no such integer exists, then let $\alpha_j = \infty$). Observe that α_2 and α_3 are finite; every arc in T has weight less than $\tau(G, w)$ (Lemma 2.18), and has positive weight since $T \subseteq E \subseteq A_{w>0}$. The choice of $\alpha = \min\{\alpha_i : i \in [4]\}$ (likewise $\beta = \min\{\beta_i : i \in [4]\}$) is well defined. Here are some more hypotheses:

(H3) given (w^f_α, T), the tuple satisfies (H1) and α is the minimum non-negative integer such that w^f_α satisfies (C1)-(C4), and

(H4) given (w^b_β, T), the tuple satisfies (H2) and β is the minimum non-negative integer such that w^b_β satisfies (C1)-(C4).

Remark 2.25. *There exists (w^f_α, T) and (w^b_β, T) such that they satisfy (H3) and (H4) respectively.*

Remark 2.26. *Suppose (w^f_α, T) and (w^b_β, T) satisfies (H3) and (H4) respectively. Then $\tau(G, w^f_\alpha) = \tau(G, w^b_\beta) = \tau(G, w)$.*

The weight of every trivial dicut remains the same, and the weights of non-trivial dicuts may decrease by α or β by Lemma 2.24. However since $\alpha \leq \alpha_1$ (likewise $\beta \leq \beta_1$), weights of non-trivial dicuts do not become less than $\tau(G, w)$.

Lemma 2.27. *Suppose (w^f_α, T) and (w^b_β, T) satisfies (H3) and (H4) respectively. We have $\alpha + \beta \leq \tau(G, w)$.*

Proof. Without loss of generality consider a forward arc $a \in T$. Observe that $w(a) + \alpha \leq \tau(G, w)$, and $w(a) - \beta \geq 0$. Combining the two inequalities, we get

$$\beta \leq w(a) \leq \tau(G, w) - \alpha,$$

i.e. $\alpha + \beta \leq \tau(G, w)$. □

Lemma 2.28. *Suppose (w^f_α, T) and (w^b_β, T) satisfies (H3) and (H4) respectively. The weighted digraphs (G, w^f_α) and (G, w^b_β) admit optimal $[\tau(G, w)]$-adic fractional packing of dijoins.*

Proof. The proof is identical for both weighted digraphs; we consider (G, w^f_α). Recall Remark 2.26; the weight w^f_α preserves the minimum weight of a dicut of (G, w). By our choice of α, one of the following is true.

We have $1^T w^f_\alpha < 1^T w$. Then by minimality of $1^T w$ (and possibly of $|A_{w>0}|$), (G, w^f_α) admits a $[\tau(G, w)]$-adic fractional packing of dijoins.

An arc $a \in A$ exists with $w(a) > 0$, but a has zero weight under w^f_α. Recall that $T \subseteq A_{w>0}$. This implies that $|A_{w>0}|$ strictly decreases; by minimality of $|A_{w>0}|$, (G, w^f_α) admits a $[\tau(G, w)]$-adic fractional packing of dijoins.

There exists $a \in A$ with weight $\tau(G, w)$ under w^f_α. If (G, w^f_α) does not admit an optimal $[\tau(G, w)]$-adic fractional packing, (G, w^f_α) is a counterexample which is minimal in $|V|$; this contradicts Lemma 2.18.

A non-trivial dicut $\delta^+(U)$ of G has weight $\tau(G, w^f_\alpha) = \tau(G, w)$. If (G, w^f_α) does not admit an optimal $[\tau(G, w)]$-adic fractional packing, (G, w^f_α) is a counterexample which is minimal in $|V|$; this contradicts Lemma 2.16, (G, w^f_α).

In all cases (G, w^f_α) has an optimal $[\tau(G, w)]$-adic fractional packing of dijoins as desired. □

Observe that (G, w^f_α) or (G, w^b_β) may not be a minimal counterexample. In particular, if T is an odd alternating cycle or an odd length maximal alternating path, then one of the two weighted digraphs does not minimize the sum of the weights (or possibly the cardinality of the set of positive weight arcs). However, this was of no consequence since both weighted digraphs still minimize the number of vertices; this is why Lemma 2.16 and Lemma 2.18 referred to counterexamples minimizing the number of vertices, rather than a minimal counterexample.

Proof of Theorem 2.13 Let (w^f_α, T) and (w^b_β, T) satisfy (H3) and (H4) respectively; the weights and the alternating structure exist by Remark 2.25. We verify that the weights

w_α^f and w_β^b satisfy the conditions of Lemma 2.19 to prove Theorem 2.13. As stated in Remark 2.26,

$$\tau(G,w) = \tau(G,w_\alpha^f) = \tau(G,w_\beta^b).$$

By Lemma 2.28, both weighted digraphs admit a $[\tau(G,w)]$-adic fractional packing of dijoins.

We claim that for $d = \frac{\beta}{\alpha+\beta}$,

$$w = dw_\alpha^f + (1-d)w_\beta^b,$$

where d is $[\tau(G,w)]$-adic (Lemma 2.27). Clearly this is satisfied for $a \in A - T$; without loss of generality, let $a \in T$ be a forward arc. Then,

$$\frac{\beta}{\alpha+\beta}(w(a)+\alpha) + \frac{\alpha}{\alpha+\beta}(w(a)-\beta) = w(a).$$

All conditions of Lemma 2.19 are satisfied as desired, and (G,w) is not a counterexample. □

Chapter 3

Packing Fewer Than τ Dijoins

The following open problem puts an extremal twist to the Edmonds-Giles conjecture:

Question 3.1. *Is there an increasing function $f(\tau)$ such that for every weighted digraph (D, w) with minimum weight dicut at least τ, there exists a w-weighted packing of dijoins of size $f(\tau)$?*

In fact, the problem is open even for 3 dijoins:

Question 3.2. *Does there exist m such that $\tau(D, w) \geq m$ guarantees that $\nu(D, w) \geq 3$?*

In the unweighted setting, it is known that $\tau \geq 2$ guarantees $\nu \geq 2$ ([16], Theorem 56.3).

For the rest of the chapter, we assume that every weighted digraph (D, w) has $w \in \{0, 1\}^A$, and denote $A_1 = \{a \in A : w_a = 1\}$ as the set of non-zero weight arcs (Remark 1.7). Under this setting, packing dijoins is equivalent to finding disjoint dijoins in A_1.

In Section 3.1, we show that if a weighted digraph (D, w) does not contain k disjoint dijoins in A_1, then the subgraph of D induced by A_1 is "sparse relative to k". In the subsequent sections, the term *ε-balanced* is introduced for weighted digraphs, and we derive a lower bound on the size of a packing of dijoins for weighted digraphs with minimum weight dicut at least τ that is ε-balanced.

3.1 Weighted Digraphs with Bounded Packing Size are Sparse

One difficulty in packing dijoins is showing that a subset of the arcs intersects every dicut at least once. An approach to packing dijoins is to attempt to pack different objects which are easier to identify, but also happen to be dijoins. One such examples are spanning trees of digraphs, a set of arcs which form spanning trees in the underlying undirected graph.

A theorem by Nash-Williams and Tutte characterizes when k disjoint spanning trees exist for a graph:

Theorem 3.3 (Nash-Williams and Tutte)**.** *A graph G has k edge-disjoint spanning trees if and only if for every partition of vertices $\mathcal{P}$, there are at least $k(|\mathcal{P}|-1)$ crossing edges.*

The theorem immediately yields a statement on a packing of spanning trees in weighted digraphs. For later use, we introduce definitions for partitions which certify that a weighted digraph does not contain k arc disjoint spanning trees in A_1. Let $\mathcal{P}$ be a partition of vertices for a weighted digraph (D,w). We denote by $\text{Cross}_{(D,w)}(\mathcal{P})$ as the weight of arcs in D which cross this partition. If $\mathcal{P}$ is a partition such that $\text{Cross}_{(D,w)}(\mathcal{P}) < k(\mathcal{P}-1)$, we say that $\mathcal{P}$ is a *k-deficient partition*, and the *k-deficiency* of $\mathcal{P}$ is $\text{def}_{(D,w)}(\mathcal{P},k) = k(|\mathcal{P}|-1)-\text{Cross}_{(D,w)}(\mathcal{P})$.

Corollary 3.4. *A weighted digraph (D,w) has a w-weighted packing of spanning trees of size k if and only if (D,w) does not have a k-deficient partition.*

A digraph is *weakly connected* if the underlying graph is connected. Given a weighted digraph (D,w) and a subset of the vertices U, we denote by $D[U]=(U,A[U])$ the subgraph induced by U, and $w[U]$ the weights of arcs in D which remain in $D[U]$. We use the shorthand $(D,w)[U]=(D[U],w[U])$. For convenience, we reiterate the contraction notations introduced in the previous chapter. For $U\subseteq V$, let D/U denote the weighted digraph where the vertices in U are contracted to a single vertex v_U, the arcs $(u,v)\in\delta^+(U),(v,u)\in\delta^-(U)$ are replaced with arcs (v_U,v) and (v,v_U) respectively. Let w/U denote the weight of arcs in G/U where the newly added arcs (of the form (v_U,v) or (v,v_U)) have the same weight as the arcs which were replaced. We will also use the shorthand $(G,w)/U=(G/U,w/U)$.

Spanning trees have a non-empty intersection with every cut, a superset of dicuts; spanning trees are "stricter" objects than dijoins. Hence, packing τ spanning trees is unreasonable, and we ask a weaker question: does a function $g(\tau)$ exist such that every weighted digraph with the weight of the minimum dicut at least τ have a packing of spanning trees of size $g(\tau)$? Unfortunately, we cannot do better than $g(\tau)=2$; imposing a

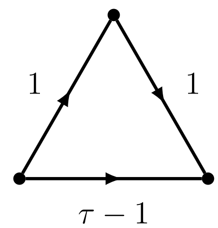

Figure 3.1: A weighted digraph with arcs labeled with its weights. For $\tau \geq 1$, the weight of the minimum dicut is τ, but the maximum size of a weighted packing of spanning trees is 2 for all τ.

lower bound on $\tau(D, w)$ does not guarantee a minimum number of weight 1 arcs incident to every vertex (see Figure 3.1). However, when packing $k < \tau$ dijoins, we may assume that no $U \subseteq V$ exists such that $(D, w)[U]$ has a packing of k spanning trees, as we may instead consider $(D, w)/U$. The next lemma formalizes this idea.

Lemma 3.5. *Let (D, w) be a weighted digraph and let $\emptyset \neq U \subseteq V$ be such that $(U, A_1[U])$ is weakly connected. If $J_U \subseteq A_1[U]$ is a dijoin of $(D, w)[U]$ and $J_{V/U} \subseteq A_1/U$ is a dijoin of $(D, w)/U$, then $J = J_U \cup J_{V/U} \subseteq A_1$ is a dijoin of D.*

Proof. Let $\delta^+(W)$ be a dicut of D. If $U \subseteq W$ or $W \subseteq V - U$, then $\delta^+(W)$ is a dicut of $(D, w)/U$, and has a non-empty intersection with $J_{V/U} \subseteq J$. Now suppose otherwise, i.e. $\emptyset \neq W \cap U \neq U$. The set of arcs $\delta^+_D(W) \cap A_1[U] = \delta^+_{D[U]}(W \cap U) \cap A_1$ is non-empty (since $(U, A_1[U])$ is weakly connected) and $\delta^+_{D[U]}(W \cap U)$ is a dicut of $D[U]$. Hence, $\delta^+(W)$ has a non-empty intersection with $J_U \subseteq J$, and J is a dijoin of D. $\square$

Nash-Williams also proved a result which characterizes when the edges of a graph can be decomposed into t forests:

Theorem 3.6. *A graph G can be partitioned into t edge-disjoint forests if and only if for every $U \subseteq V$, $|E[U]| \leq t(|U| - 1)$.*

The theorem states that the minimum number of forests that a graph can be decomposed into is $t = \left\lceil \max\left\{ \frac{|E[U]|}{|U|-1} : U \subseteq V, |U| \geq 2 \right\} \right\rceil$.

Corollary 3.7. *The weight 1 arcs of (D, w) can be partitioned into t arc-disjoint forests, where $t = \left\lceil \max\left\{ \frac{w(A[U])}{|U|-1} : U \subseteq V, |U| \geq 2 \right\} \right\rceil$*

We call such weighted digraphs to be *t-arboric*. The expression $\frac{w(A[U])}{|U|-1}$ is nearly the ratio between the number of weight one arcs and the number of vertices; it can be interpreted as the density of the graph. Then a weighted digraph that is t-arboric for small t implies that every subgraph is sparse.

The following is the main theorem for this section. It states that if a weighted digraph (D, w) does not contain a w-weighted packing of k dijoins, then we may consider a contraction minor which is sparse (i.e. is k-arboric).

Theorem 3.8. *Suppose (D, w) does not contain a w-weighted packing of k dijoins. Then there exists a weighted directed graph (H, w') which is a contraction minor of (D, w) such that*

1. *(H, w') does not contain a w'-weighted packing of k dijoins, and*
2. *the weight 1 arcs of (H, w') can be decomposed into k forests.*

Proof. **(1)** By assumption (D, w) does not contain a w-weighted packing of k spanning trees; a k-deficient partition $\mathcal{P}$ exists. We pick $\mathcal{P}$ such that

1. $\mathrm{def}_{(D,w)}(\mathcal{P}, k)$ is maximized, and subject to that
2. $|\mathcal{P}|$ is minimized.

Let $W \in \mathcal{P}$ where $|W| \geq 2$. We first show that $(D, w)[W]$ contains a w-weighted packing of k spanning trees. For an eventual contradiction, suppose otherwise: a k-deficient partition $\mathcal{P}_W$ exists for $(D, w)[W]$, and $\mathrm{Cross}_{(D,w)[W]}(\mathcal{P}_W) < k(|\mathcal{P}_W| - 1)$. Now consider $\overline{\mathcal{P}} = (\mathcal{P} - \{W\}) \cup \mathcal{P}_W$, a partition of V. Observe that $\mathrm{Cross}_{(D,w)}(\overline{\mathcal{P}}) = \mathrm{Cross}_{(D,w)}(\mathcal{P}) + \mathrm{Cross}_{(D,w)[W]}(\mathcal{P}_W)$, thus

$$\begin{aligned}\mathrm{def}_{(D,w)}(\overline{\mathcal{P}}) &= k(|\overline{\mathcal{P}}| - 1) - \mathrm{Cross}_{(D,w)}(\overline{\mathcal{P}})\\ &= k(|\mathcal{P}| + |\mathcal{P}_W| - 2) - \mathrm{Cross}_{(D,w)}(\overline{\mathcal{P}})\\ &= \mathrm{def}_{(D,w)}(\mathcal{P}) + \mathrm{def}_{(D,w)[W]}(\mathcal{P}_W).\end{aligned}$$

By assumption, $\mathrm{def}_{(D,w)[W]}(\mathcal{P}_W) > 0$ hence $\mathrm{def}_{(D,w)}(\overline{\mathcal{P}}) > \mathrm{def}_{(D,w)}(\mathcal{P})$. This contradicts the maximality of $\mathrm{def}_{(D,w)}(\mathcal{P}, k)$.

Now let $(H = (V', A'), w')$ be formed from G by contracting each set of $\mathcal{P}$ into a single vertex. We denote by $v_U \in V'$ the vertex resulting from contracting $U \in \mathcal{P}$. If (H, w') has a w'-weighted packing of k dijoins, then pairing each dijoin in the packing with a spanning tree in each $D[W]$, $W \in \mathcal{P}$ forms a collection of k dijoins of D by Lemma 3.5. Hence, (H, w') does not have such a packing.

(2) Let $\emptyset \neq W \subset V'$ with $|W| \geq 2$. We prove that $H[W] = (W, A'_W)$ has strictly less than $k(|W|-1)$ weight 1 arcs. To show a contradiction suppose $w(A'_W) \geq k(|W|-1)$. Let

$$\overline{\mathcal{P}} = (\mathcal{P} - \{U \subseteq V : v_U \in W\}) \cup \left\{ \bigcup_{v_U \in W} U \right\},$$

a partition of V. Note that $\mathrm{Cross}_{(D,w)}(\overline{\mathcal{P}}) = \mathrm{Cross}_{(D,w)}(\mathcal{P}) - w(A'_W)$. Thus,

$$\begin{aligned}
\mathrm{def}_{(D,w)}(\overline{\mathcal{P}}) &= k(|\mathcal{P}| - |W|) - \mathrm{Cross}_{(D,w)}(\overline{\mathcal{P}}) \\
&= k(|\mathcal{P}| - |W|) - \mathrm{Cross}_{(D,w)}(\mathcal{P}) + w(A'_W) \\
&\geq k(|\mathcal{P}| - 1) - k(|W| - 1) - \mathrm{Cross}_{(D,w)}(\mathcal{P}) + k(|W| - 1) \\
&= \mathrm{def}_{(D,w)}(\mathcal{P}),
\end{aligned}$$

and $|\overline{\mathcal{P}}| < |\mathcal{P}|$. If $\mathrm{def}_{(D,w)}(\overline{\mathcal{P}}) > \mathrm{def}_{(D,w)}(\mathcal{P})$, then maximality of $\mathrm{def}_{(D,w)}(\mathcal{P})$ is contradicted. If $\mathrm{def}_{(D,w)}(\overline{\mathcal{P}}) = \mathrm{def}_{(D,w)}(\mathcal{P})$, then minimality of $|\mathcal{P}|$ is contradicted. Thus, for all induced subgraph of H we have that $\frac{w(A'_W)}{|W|-1} < k$; by Theorem 3.7, the weight 1 arcs of G can be decomposed into k forests. $\square$

By Theorem 3.8, to show that a τ exists such that $f(\tau) \geq 3$, we may only consider weighted digraphs with weight 1 arcs which can be decomposed into at most 3 disjoint forests.

In the proof of Theorem 3.8, the contraction minor was derived by contracting induced subgraphs which contained k disjoint spanning trees in A_1. The following results describe when such induced subgraphs can be found; we will use them in the subsequent chapter.

Lemma 3.9. *Let $1 \leq k \in \mathbb{N}$. Either (D, w) has an induced subgraph with a w-weighted packing of k spanning trees, or the partition of vertices which uniquely maximizes k-deficiency is the trivial partition.*

Proof. Suppose that (D, w) does not contain a subgraph which has a packing of k spanning trees. For an eventual contradiction, suppose that a non-trivial partition $\mathcal{P}$ exists which maximizes deficiency. Pick $U \in \mathcal{P}$ that contains more than one vertex, and note that by assumption $(D,w)[U]$ does not contain a packing of k spanning trees; a k-deficient partition $\mathcal{P}_U$ exists for $(D,w)[U]$. Let $\overline{\mathcal{P}} = (\mathcal{P} - W) \cup \mathcal{P}_U$. Then,

$$\operatorname{def}_{(D,w)}(\overline{\mathcal{P}}, k) = \operatorname{def}_{(D,w)}(\mathcal{P}, k) + \operatorname{def}_{(D,w)[U]}(\mathcal{P}_U, k),$$

and since $\mathcal{P}_U$ is k-deficient, $\operatorname{def}_{(D,w)[U]}(\mathcal{P}_U, k) > 0$. The partition $\mathcal{P}$ does not maximize k-deficiency, which is a contradiction. □

Corollary 3.10. *If (D,w) has fractional arboricity at least k, then there exists an induced subgraph of (D,w) which contains a packing of k disjoint spanning trees. In particular, if the average weighted degree of (D,w) is $2k$, then there exists an induced subgraph of (D,w) which contains k disjoint spanning trees.*

Proof. Suppose (D,w) has fractional arboricity at least k; a set $U \subseteq V, |U| \geq 2$ exists with $\frac{w(A[U]))|}{|U|-1} \geq k$. For an eventual contradiction, further suppose that $(D,w)[U]$ does not contain a packing of spanning trees of size k. Then by Lemma 3.9, the trivial partition uniquely maximizes k-deficiency. However, $w(A[U]) \geq k(|U|-1)$, and the k-deficiency of the trivial partition is at most 0. Therefore every partition has negative k-deficiency, and no k-deficient partition exists. This implies that $(D,w)[U]$ contains a packing of k spanning trees, which is a contradiction.

The latter statement follows since if the average weighted degree is at least $2k$, then the fractional arboricity of the graph is at least k. □

3.2 Packing Dijoins in ε-Balanced Weighted Digraphs

Finding a $f(\tau)$ size packing of dijoins seems currently out of reach for a non-constant $f(\tau)$ that is increasing in τ. We introduce a new parameter ε to weighted digraphs, and find a packing of dijoins of size $f(\tau, \varepsilon)$. The central idea on achieving such a packing is similar to the prior section; we attempt to find a packing of spanning trees in an induced subgraph and contract the subgraph. If we can contract the entire digraph to a single vertex, then a packing can be found.

3.2.1 Lifting Operations

A *k-lifting operation* is an operation on weighted digraphs where if a packing of k dijoins exists after the operation, they can be mapped to packing of k dijoins in the weighted digraph before the operation. When such a mapping is applied, we say that the dijoins have been *lifted*. We have already encountered a k-lifting operation; we formally introduce the procedure next.

Let (D, w) be a weighted digraph. If $U \subseteq V$ is such that $(D, w)[U]$ contains a $w[U]$-weighted packing of k spanning trees, contracting U is called a *k-SP contraction* (spanning tree contraction). Lemma 3.5 ensures that k-SP contraction is a k-lifting operation. Corollary 3.10 describe a sufficient condition on when k-SP contraction can be applied.

Recall that dicuts are disjoint from dicycles (Corollary 2.22). The operation to contract a dicycle (*dicycle contraction*) is a k-lifting operation for every $k \geq 1$; the dijoins after the contraction are dijoins before the contraction.

An important aspect of contractions are that the weight of the minimum dicut in the contracted weighted digraph does not decrease; every dicut in the contracted digraph corresponds to a dicut in the original digraph.

Remark 3.11. *Contractions do not decrease the minimum weight of a dicut for a weighted digraph.*

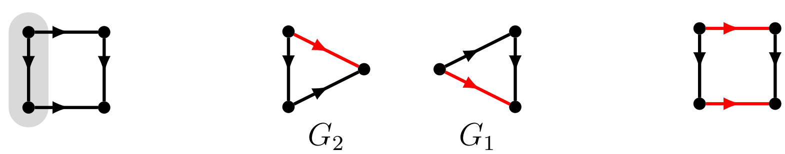

Figure 3.2: When packing less than τ dijoins, how to pair the dijoins of $G_1 = G/U$ and $G_2 = G/(V - U)$. The left digram shows digraph G, and dicut shore U, which is a non-trivial minimum dicut. The red arcs of G_1 and G_2 are dijoins, but their union is not a dijoin of G.

Another contraction operation we have previously used to great effect is seen in the proof of Lemma 2.16; the lemma dealt with optimal $[\tau]$-adic packings, but the proof is virtually identical for optimal integral packings. In the proof, two digraphs G_1 and G_2 were derived from G by contracting U and $V - U$ of a non-trivial minimum dicut $\delta^+(U)$. The union of dijoins of G_1 and G_2 which shared a unique arc across $\delta^+(U)$ was a dijoin of G. Unfortunately, the operation is not always applicable when packing $k < \tau$ dijoins. In general, it is unclear how the dijoins of G_1 and G_2 should be paired to yield a dijoin of G (see Figure 3.2).

What if no induced subgraph exists with a packing of k spanning trees? We introduce a new operation that can remedy such scenarios. A *transitive arc* is an arc (u, v) where the digraph contains a $u - v$ dipath which does not use (u, v).

Remark 3.12. *If U is a shore of a dicut of a digraph, then U is still a shore of a dicut after a transitive arc is added to the digraph.*

For a weighted digraph (D, w) and a $u - v$ dipath P in (D, w) using only the weight 1 arcs, *splitting* P is adding a transitive arc (u, v) of weight 1, and changing the weights of all arcs in P to 0.

Figure 3.3: An illustration of splitting a dipath of length 4. As usual, dashed arcs have weight 0, and solid arcs have weight 1.

Lemma 3.13. *Splitting is a k-lifting operation for all $k \geq 1$. Furthermore, splitting preserves the value of the minimum weight dicut.*

Proof. Let P be the dipath which is split, and let (u, v) be the added transitive arc. Furthermore, let (D, w) be the digraph before splitting, and $(D' = (A', V'), w')$ be the digraph after splitting. We show a mapping from a dijoin $J' \subseteq A'_1$ in the weighted digraph after splitting, to a dijoin $J \subseteq A_1$ before splitting. If $(u, v) \notin J'$, then $J = J' \subseteq A_1$ is a dijoin. If $(u, v) \in J'$, consider $J = (J' - \{(u, v)\}) \cup P$. To show J is a dijoin, it suffices to show that if $(u, v) \in \delta^+(U)$ then $\delta^+(U)$ has a non-empty intersection with P. Indeed this is the case; if $(u, v) \in \delta^+(U)$, then $u \in U$ and $v \notin U$, which are the endpoints of P. Hence an arc in P is in $\delta^+(U)$.

If $\delta^+(U)$ has a non-empty intersection with P, exactly one arc a is in the intersection by Corollary 2.23. Moreover, $(u, v) \in \delta^+(U)$ as $u \in U$ and $v \notin U$. Thus

$$w'(\delta^+_{D'}(U)) = w(\delta^+(U)) - w(a) + w'((u, v)) = w(\delta^+(U)),$$

and the weight of every dicut wht a non-empty intersection with P remains the same; the minimum weight of a dicut is preserved. $\square$

Splitting is useful for when no induced subgraph exists with a packing of k spanning trees. We may split dipaths and introduce new weight 1 arcs to find more spanning trees.

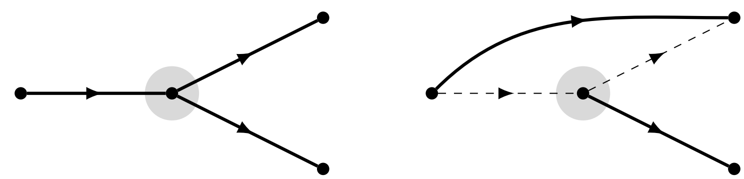

Figure 3.4: Splitting may not preserve ε-balancedness for $\varepsilon > 0$. For the trivial cut indicated in gray, we have that $\frac{|w(\delta^+(v))-w(\delta^-(v))|}{w(\delta(v))}$ is $\frac{1}{3}$ in the left digraph, but becomes 1 after splitting.

3.2.2 Packing $f(\tau, \varepsilon)$ Dijoins in ε-Balanced Digraphs

A weighted digraph (D, w) is *ε-balanced* if for all cuts $\delta(U)$ of D that is not a dicut and has $w(\delta(U)) > 0$,

$$\frac{|w(\delta^+(U)) - w(\delta^-(U))|}{w(\delta(U))} \leq \varepsilon.$$

A useful property of ε-balancedness (with respect to the operations introduced thus far) is that it is closed under contractions. Every cut of the contracted digraph is a cut of the original digraph. Note that ε-balancedness is not closed under splitting; see Figure 3.4

Remark 3.14. *Being ε-balanced is closed under contraction.*

We require the following two results:

Theorem 3.15 (Max-Flow Min-Cut Theorem, [16], Theorem 10.3)**.** *For a weighted digraph, the maximum $s - t$ flow is equal to the weight of the minimum $s - t$ cut.*

Theorem 3.16 (Integral Flow Theorem, [16], Corollary 10.3a)**.** *If the weights of every arc is a non-negative integer, then there exists an integer maximum flow.*

Theorem 3.15 and Theorem 3.16 imply the following:

Corollary 3.17. *The maximum $s - t$ flow of a weighted digraph is equal to the maximum size of a weighted packing of directed $s - t$ paths.*

Furthermore, they show that the clutter of directed $s-t$ paths has the MFMC property (Corollary 3.18). This will be useful to show that enough directed paths exist which can be split to find a dense induced subgraph.

Corollary 3.18. *For a weighted digraph (D, w), then weight of a minimum $s-t$ cut is equal to the maximum size w-weighted packing of directed $s-t$ paths.*

Now we are ready for the main theorem of this section:

Theorem 3.19. *Let $0 \leq \varepsilon < 1$ be a constant, and $1 \leq k \in \mathbb{N}$. If a weighted digraph (G, w) is ε-balanced and $\tau(G, w) \geq \frac{2(1+\varepsilon)}{(1-\varepsilon)}k$, then a w-weighted packing of k dijoins exist.*

Proof. Consider a counterexample (G_1, w_1) minimizing the number of vertices. If G_1 contains a dicycle, then we apply dicycle contraction. The new weighted digraph is ε-balanced, and the value of the minimum weight dicut is at least the value of the original. By minimality, the contracted weighted digraph contains a packing of k dijoins, which can be lifted. Similarly, k-SP contraction cannot be applied to (G_1, w_1).

Since G_1 is a directed acyclic graph, it contains sources and sinks; let S and T respectively be the set of sources and sinks of G_1. Let z be the maximum number of disjoint directed $S-T$ paths in A_1; z is equal to the value of the maximum $S-T$ flow in (G_1, w_1) by Corollary 3.17. We split every such dipath; let (G_2, w_2) be the resulting weighted digraph, and consider induced subgraph $(G_2, w_2)[S \cup T]$. The subgraph has at least z arcs of weight 1; if $z \geq k(|S| + |T|)$, then the average weighted degree of the subgraph is at least $2k$, and by Corollary 3.10, $(G_2, w_2)[S \cup T]$ contains an induced subgraph with k disjoint spanning trees. This is precisely what we prove next.

Claim 1. We have $z \geq k(|S| + |T|)$.

Proof It suffices to show that $z \geq \frac{(1-\varepsilon)}{1+\varepsilon}\tau|S|$ and $z \geq \frac{(1-\varepsilon)}{1+\varepsilon}\tau|T|$. Adding the two inequalities results in

$$2z \geq \frac{(1-\varepsilon)}{(1+\varepsilon)}\tau(|S| + |T|) \geq 2k(|S| + |T|)$$

where the last inequality comes from $\tau \geq \frac{2(1+\varepsilon)}{1-\varepsilon}k$.

Let W be a shore of the minimum $S-T$ cut of (G_1, w_1) with $S \subseteq W$. By Corollary 3.18, $z = w(W)$. We partition V into sets S, T, $\hat{S} = W - S$ and $\hat{T} = V - W - T$ (i.e. $\hat{S}$ and $\hat{T}$ are shores of the minimum $S - T$ cut without the sinks or the sources). If $\hat{S} = \emptyset$, then $w(\delta^+(W)) = w(\delta^+(S))$, and $z = \tau|S| \geq 2k|S|$; similarly, if $\hat{T} = \emptyset$ then $w(\delta^+(W)) = w(\delta^-(T))$ and $z = \tau|T| \geq 2k|T|$. Hence, we may assume that $\hat{S} \neq \emptyset$ and $\hat{T} \neq \emptyset$.

Before proceeding, we give an intuition of why the claim is true. Since S is the set of sources, $\tau|S|$ weight 1 arcs are in $\delta^+(S)$. Every such arc either is in $\delta^-(V - W)$, in which case the arcs contributes to z, or is in $\delta^-(\hat{S})$. The value of z is "small" if many weight 1 arcs in $\delta^+(S)$ enter $\hat{S}$ instead of $V-W$, and not many weight one arcs are in $\delta^+(\hat{S}) \cap \delta^-(V-W)$, i.e. the minimum $S-T$ cut has small weight. In such a case, given the lower bound on τ, $\hat{S}$ has much more weight 1 arcs which enter $\hat{S}$ than which leave $\hat{S}$; the cut $\delta(\hat{S})$ shows that the weighted digraph is not ε-balanced.

For $U_1, U_2 \subseteq V$, let $w_1(U_1, U_2) = w_1(\delta^+(U_1) \cap \delta^-(U_2))$, the sum of the weights of arcs from U_1 to U_2. Observe that $z = w_1(\hat{S}, V-W) - w_1(S, \hat{S}) + w_1(\delta^+(S))$, hence

$$z - w_1(\hat{T}, \hat{S}) = w_1(\hat{S}, V-W) - w_1(S, \hat{S}) + w_1(\delta^+(S)) - w_1(\hat{T}, \hat{S}). \tag{3.20}$$

We have that $w_1(S, \hat{S}) \geq w_1(\hat{S}, V-W)$, as otherwise $\delta^+(S)$ is a $S-T$ cut which has weight at most $w_1(\delta^+(W))$. By ε-balancedness of the cut $\delta(\hat{S})$ and the previous inequality (which allows us to drop the absolute value), $\frac{w_1(\hat{T},\hat{S})+w_1(S,\hat{S})-w_1(\hat{S},V-W)}{w_1(\delta(\hat{S}))} \leq \varepsilon$, i.e.

$$w_1(\hat{S}, V-W) - w_1(S, \hat{S}) - w_1(\hat{T}, \hat{S}) \geq -\varepsilon w_1(\delta(\hat{S})). \tag{3.21}$$

Plugging in 3.21 to 3.20, we get

$$z \geq w_1(\delta^+(S)) - \varepsilon w_1(\delta(\hat{S})) + w_1(\hat{T}, \hat{S}).$$

In fact, $w_1(\delta(\hat{S})) = z + w_1(\hat{T}, \hat{S}) - 2w_1(S, V-W) + w_1(\delta^+(S))$, thus

$$z \geq (1-\varepsilon) w_1(\delta^+(S)) - \varepsilon z + (1-\varepsilon) w_1(\hat{T}, \hat{S}) + 2\varepsilon w_1(S, V-W).$$

Rearranging the inequality,

$$\begin{aligned} z &\geq \frac{1}{1+\varepsilon}((1-\varepsilon) w_1(\delta^+(S)) + (1-\varepsilon) w_1(\hat{T}, \hat{S}) + 2\varepsilon w_1(S, V-W)) \\ &\geq \frac{1-\varepsilon}{1+\varepsilon} w_1(\delta^+(S)) \\ &= \frac{1-\varepsilon}{1+\varepsilon} \tau |S| \end{aligned}$$

as desired.

In the above, by replacing $w_1(\hat{S}, V-W)$ with $w_1(W, \hat{T})$, $w_1(S, \hat{S})$ with $w_1(\hat{T}, T)$, and $w_1(S, V-W)$ with $w_1(W, T)$, ($w_1(\hat{T}, \hat{S})$ is unchanged) the proof follows identically. Namely, it shows that $z \geq \frac{1-\varepsilon}{1+\varepsilon} \tau |T|$. $\Diamond$

Splitting the z disjoint $S-T$ dipaths ensures that k-SP contraction can be applied, but does not ensure that the new graph is ε-balanced. To remedy this, we undo the splitting for an $S-T$ dipath if the transitive arc added when splitting is not part of the k disjoint spanning trees used in the k-SP contraction; equivalently, from graph (G_1, w_1) we selectively split a $S-T$ dipath if the resulting transitive arc is used in the k spanning trees to be contracted. Let (G_3, w_3) be derived from applying k-SP contraction to (G_2, w_2).

Note that G_3 contains dicycles. If a dipath $(s, v_1), (v_1, v_2), \ldots, (v_m, t)$ where $s \in S$, $t \in T$ is split, then the transitive arc (s, t) is used one of the k disjoint spanning trees, i.e. s and t are contracted to a single vertex. Let (G_4, w_4) be the resulting graph after contracting all dicycles of (G_3, w_3).

Claim 2. (G_4, w_4) is ε-balanced.

Proof Let $U \subseteq V$ be the set of vertices contracted in the k-SP contraction to derive G_3 from G_2. We denote by v_U the vertex created by contracting U. Let U be the set of vertices contracted to derive G_4 from G_3. We showed above that $v_U \in U$. Then (G_4, w_4) can be directly derived from (G_1, w_1) by contracting $(U - \{v_U\}) \cup U$. Contraction operations preserve ε-balancedness, thus (G_4, w_4) is ε-balanced. $\diamondsuit$

In addition, $\tau(G_4, w_4) \geq \tau(G_1, w_1)$ (Remark 3.11 and Lemma 3.13). By minimality, (G_4, w_4) has a w_4-weighted packing of k disjoint dijoins. Every operation applied to G_i to derive G_{i+1}, $i \in [3]$ is a lifting operation, therefore (G_1, w_1) also contains k disjoint dijoins, which is a contradiction. $\square$

Corollary 3.22. *Let $0 \leq \varepsilon < 1$ be a constant. Let the function $f(\tau, \varepsilon)$ denote the maximum packing of dijoins that can be achieved in a ε-balanced weighted digraph with the weight of the minimum dicut is at least τ. We have $f(\tau, \varepsilon) \geq \left\lfloor \frac{(1-\varepsilon)}{2(1+\varepsilon)}\tau \right\rfloor$.*

Interestingly when $\varepsilon = 0$, we have that $f(\tau, 0) \geq \left\lfloor \frac{\tau}{2} \right\rfloor$. This is identical to the best lower bound on $f(\tau)$; with Schrijver's counterexample we have $\tau = 2$ but $\nu = 1$.

We also have the following corollary:

Corollary 3.23. *Let $k \in \mathbb{Z}_+, k \geq 3$ be fixed. If a weighted digraph (D, w) with $\tau(D, w) \geq k$ does not admit a w-weighted packing of k dijoins, then the weighted digraph has a cut $\delta(U)$ where*

$$\frac{|w(\delta^+(U)) - w(\delta^-(U))|}{|w(\delta(U))|} > \frac{\tau(D, w) - 2k}{\tau(D, w) + 2k}.$$

Let $k \in \mathbb{Z}_+, k \geq 3$ be fixed, and suppose no m_k exists such that $\tau \geq m_k$ guarantees that $\nu \geq k$. In this case, Corollary 3.23 implies that a weighted digraph with large τ (say,

$\tau > 100k$) without a weighted packing of k-dijoins are "very unbalanced"; such weighted digraphs are *not* ε-balanced for every $\varepsilon \in [0, \frac{\tau-2k}{\tau+2k}]$. Thus, examining weighted digraphs with large τ which are ε-balanced for ε close to 1 may aid in answering Question 3.2.

Chapter 4

Kernels and Equitability

Dijoins have a non-empty intersection with every dicut, but in general there are exponentially many minimal dicuts relative to the number of vertices (Figure 4.1); it seems that descriptions of dijoins (say for an IP which solves maximum weighted packing) requires exponentially many constraints. A natural question arises: can we apply stronger constraints on a smaller number of dicuts to yield dijoins? We study this question under equitability.

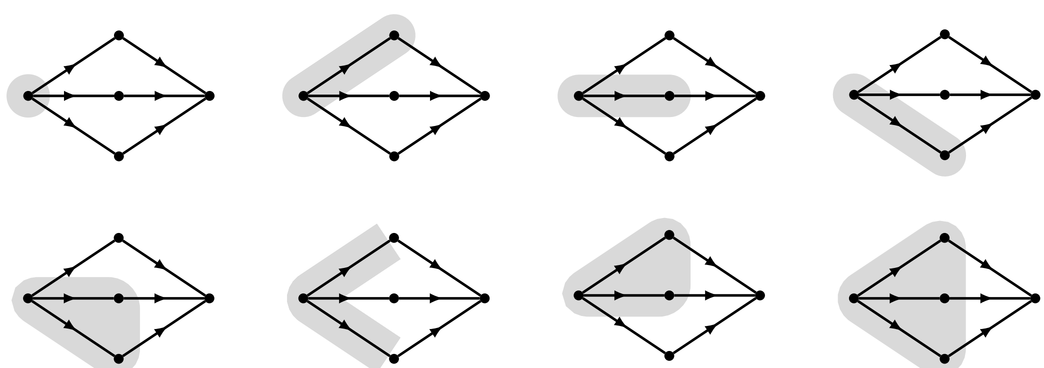

Figure 4.1: In general, digraphs have exponentially many minimal dicuts. The shores of dicuts are indicated in gray. Generalizing the digraph to n dipaths of length two from a source to a sink, the digraph has $n + 2$ vertices but 2^n minimal dicuts.

4.1 The Central Object: Kernels

As in the last chapter, for a weighted digraph (D, w) we assume that $w \in \{0,1\}^A$, and denote by A_1 the weight 1 arcs of (D, w) (or $A_1(D, w)$ if we wish to make its dependence on the weighted digraph explicit). For a set $\mathcal{O}$ of dicut shores of D, $J \subseteq A_1(D, w)$, is *$\mathcal{O}$-equitable* (or *equitable for $\mathcal{O}$*) if for every $U \in \mathcal{O}$,

$$\left\lfloor \frac{1}{\tau(D,w)} w(\delta(U)) \right\rfloor \leq |J \cap \delta(U)| \leq \left\lceil \frac{1}{\tau(D,w)} w(\delta(U)) \right\rceil.$$

Note that $\mathcal{O}$ may include either shore of a dicut.

Equitability is a stronger condition than having a non-empty intersection. The condition is natural in the sense that when partitioning $A_1(D, w)$ into dijoins $J_1, \ldots, J_\tau$, equitability divides the arcs in a fair way along the family; for every dicut $\delta(U)$, $U \in \mathcal{O}$, we have $-1 \leq |J_i \cap \delta(U)| - |J_j \cap \delta(U)| \leq 1$.

For a weighted digraph (D, w), a *kernel* K is a family of dicut shores of D such that

(K1) $A_1(D, w)$ can be partitioned into $\tau(D, w)$ sets which are K-equitable, and

(K2) every K-equitable set is a dijoin.

We will interchangably refer to (K1) as the *partition condition*, and (K2) as the *dijoin condition*.

Proposition 4.1. *If a kernel K exists for a weighted digraph, then it packs.*

Proof. Let $J_1, \ldots J_\tau$ be the partition of A_1 into τ sets which are all equitable for K as guaranteed by the partition condition. Since all J_i are K-equitable, by the dijoin condition, the J_i are dijoins. Thus $J_1, \ldots, J_\tau$ is a weighted packing of τ dijoins. □

Satisfying either of the conditions individually is trivial. Every partition of A_1 is $\emptyset$-equitable. On the other hand, if K is the set of all dicuts of a weighted digraph, then a K-equitable set is a dijoin. These extreme examples suggest a trade-off between achieving the above two conditions: including more dicut shores in the family makes satisfying the dijoin condition more likely, perhaps at the cost of achieving the partition condition and vice versa.

In the subsequent sections, we show examples of kernels. In Section 4.3, we show that if every weight of a dicut multiple of $\tau(D, w)$, then every maximal laminar family of dicut shores is a kernel. Abdi, Cornuéjols and Zlatin [3] introduced a parameter $\rho(D,w) := \frac{1}{\tau(D,w)}\sum_{v\in V}[w(\delta^+(v)) - w(\delta^-(v)) \mod \tau(D,w)]$, and gave a highly technical construction which reduced packing statements on weighted digraphs to packing statements on *weighted* $(\tau,\tau+1)$*-bipartite digraphs* with ρ values at most $\rho(D,w)$. Section 4.4 is dedicated to introducing these bipartite digraphs. Using the tools from [3], we prove in Section 4.5 that when a weighted $(\tau,\tau+1)$-bipartite digraph has ρ value in $\{1,2\}$, a kernel exists. When $\rho = 1$, the family of all singletons suffices; the kernel for $\rho = 2$ is more complex, and we postpone its description.

4.2 Laminarity is Sufficient for the Partition Condition

We say that two sets U and W are *intersecting* if $U \cap W \neq \emptyset$ and $U \not\subseteq W \not\subseteq U$. A family of sets $\mathcal{L}$ is *laminar* if for every $U, W \in \mathcal{L}$, U and W are not intersecting. A sufficient condition when a family of dicut shores $\mathcal{O}$ satisfies (K1) is when $\mathcal{O}$ is laminar. Let $M(\mathcal{O})$ be a $|\mathcal{O}| \times |A_1|$ matrix where its rows are incident vectors of the dicuts in $\mathcal{O}$. Let $b^l(\mathcal{O}), b^u(\mathcal{O}) \in \mathbb{Z}_+^{\mathcal{O}}$ where for $U \in \mathcal{O}$, the entry for $b^l(\mathcal{O})$ is $\left\lfloor \frac{w(\delta(U))}{\tau} \right\rfloor$ and for $b^u(\mathcal{O})$ is $\left\lceil \frac{w(\delta(U))}{\tau} \right\rceil$. Finally, let $M'(\mathcal{O}) = \begin{bmatrix} M(\mathcal{O}) \\ -M(\mathcal{O}) \end{bmatrix}$ and $b'(\mathcal{O}) = \begin{bmatrix} b^u(\mathcal{O}) \\ -b^l(\mathcal{O}) \end{bmatrix}$.

Lemma 4.2. *If there exists $x_i \in \mathbb{Z}_+^{A_1}, i \in [\tau]$ such that $1 = x_1 + \cdots + x_\tau$, and $x_i \in \{x \geq 0 : M'(\mathcal{O})x \leq b'(\mathcal{O})\}$, then $\mathcal{O}$ satisfies the partition condition.*

Proof. Let $J_i \subseteq A_1$ be such that x_i is the characteristic vector of J_i. Since $1 = x_1 + \cdots + x_\tau$, the J_is partition A_1. Vector x_i satisfies $M'x_i \leq b'$ if and only if $b^l(\mathcal{O}) \leq M(\mathcal{O})x_i \leq b^u(\mathcal{O})$. This is precisely the condition that J_i is $\mathcal{O}$-equitable;. □

If $\mathcal{O}$ is laminar, then $M'(\mathcal{O})$ satisfies the above property. We need the following results to prove the claim.

Lemma 4.3 ([15], see proof of Theorem 22.3)**.** *Let $\mathcal{O}$ be a laminar family of dicut shores. The matrix $M(\mathcal{O})$ is a totally unimodular matrix.*

Remark 4.4. *Repeating a row or multiplying a row by -1 maintains total unimodularity.*

Theorem 4.5 ([15], Theorem 19.4)**.** *An integral matrix M is totally unimodular if and only if for all integral vectors b, y and for each natural number $k \geq 1$ with $y \geq 0, My \leq kb$, there are integral vectors $x_1, \ldots, x_k$ in $\{x \geq 0 : Mx \leq b\}$ such that $y = x_1 + \cdots + x_k$.*

Lemma 4.6. *Let $\mathcal{O}$ be a family of dicut shores. If $\mathcal{O}$ is laminar, then there exists a partition of the weight 1 arcs into $J_1, \ldots, J_\tau$ such that the J_i's are $\mathcal{O}$-equitable.*

Proof. The matrix $M(\mathcal{O})$ is totally unimodular by Lemma 4.3. Observe that $M'(\mathcal{O})$ can be constructed from $M(\mathcal{O})$ by repeating rows and multiplying rows by -1; by Remark 4.4 $M'(\mathcal{O})$ is also totally unimodular. For $U \in \mathcal{O}$, let $m_U(\mathcal{O})$ be the row of $M(\mathcal{O})$ corresponding to U. The entry for $b^l(\mathcal{O})$ is $\left\lfloor \frac{1}{\tau} m_U(\mathcal{O})\mathbb{1} \right\rfloor$ and for $b^u(\mathcal{O})$ is $\left\lceil \frac{1}{\tau} m_U(\mathcal{O})\mathbb{1} \right\rceil$. Hence, $\tau b^l(\mathcal{O}) \leq M(\mathcal{O})\mathbb{1} \leq \tau b^u(\mathcal{O})$, and thus $M'(\mathcal{O})\mathbb{1} \leq \tau b'(\mathcal{O})$. Picking $y = \mathbb{1}$, $b = b'(\mathcal{O})$, $k = \tau$ and $M = M'(\mathcal{O})$ in Theorem 4.5, we see that $M'(\mathcal{O})$ satisfies the condition in Lemma 4.2. This completes the proof. $\square$

Let $\mathcal{L}$ be a laminar family over a ground set V. It can be shown that $|\mathcal{L}|$ is at most $2|V| - 1$. Lemma 4.6 fits nicely under our notion of a tradeoff between (K1) and (K2): it is always possible to pick a family with linear size relative to the size of the ground set which guarantees (K1).

4.3 Kernel Example: Every Dicut has Weight Multiple of τ

We show that if weighted digraph (D, w) is such that every dicut has weight that is a multiple of τ, then every laminar family is a kernel of (D, w). Given a family of sets $\mathcal{F}$ and $U \notin \mathcal{F}$ let $\text{inter}(U, \mathcal{F}) = |\{W \in \mathcal{F} : W \text{ and } U \text{ are intersecting }\}|$. We require the following lemma.

Lemma 4.7. *Let $\mathcal{L}$ be a laminar family and suppose $U \in \mathcal{L}$ and $W \notin \mathcal{L}$ where U and W are intersecting. Then* $\text{inter}(W \cap U, \mathcal{L}) < \text{inter}(W, \mathcal{L})$ *and* $\text{inter}(W \cup U, \mathcal{L}) < \text{inter}(W, \mathcal{L})$.

Proof. Given a set $B \in \mathcal{L}$, B and U are not intersecting by $\mathcal{L}$ being laminar. Thus if B contributes to either $\text{inter}(W \cap U, \mathcal{L})$, or $\text{inter}(W \cup U, \mathcal{L})$, then B and W are intersecting. This shows that $\text{inter}(U \cap W, \mathcal{L}) \leq \text{inter}(W, \mathcal{L})$ and $\text{inter}(U \cup W, \mathcal{L}) \leq \text{inter}(W, \mathcal{L})$. The inequality is strict since U and W are intersecting by assumption but not $U \cap W$ nor $U \cup W$. $\square$

Lemma 4.7 is quite standard. For instance, it can be found in [18] in the proof of Lemma 2.3.

A function h is *modular* if $h(W) + h(S) = h(W \cup S) + h(W \cap S)$.

Theorem 4.8. *If (D, w) is a weighted digraph where every dicut has weight that is a multiple of $\tau(D, w)$, then every maximal laminar family of dicut shores is a kernel of (D, w).*

Proof. Let $\mathcal{L}$ be a maximal laminar family of dicut shores. Since $\mathcal{L}$ is laminar, there exists $J_1, \ldots, J_\tau$ in A_1 such that the J_i are all $\mathcal{L}$-equitable (Theorem 4.6). For $U \subseteq V$, let $\phi(U) = w(\delta^+(U)) - w(\delta^-(U))$, and $\phi_{J_i}(U) = w(\delta^+(U) \cap J_i) - w(\delta^-(U) \cap J_i)$. Observe that ϕ and ϕ_{J_i} are modular.

We show that that each J_i is equitable for every dicut. For an eventual contradiction, let $\delta^+(W)$ be a dicut such that a J_i exists which is not equitable for $\delta^+(W)$. We pick $\delta^+(W)$ which minimizes $\text{inter}(W, \mathcal{L})$. Clearly $W \notin \mathcal{L}$, and $\mathcal{L} \cup \{W\}$ is not laminar as otherwise $\mathcal{L}$ is not maximal; there exists $S \in \mathcal{L}$ where S and W are intersecting. Then $\phi_{J_i}(W) = \phi_{J_i}(W \cap S) + \phi_{J_i}(W \cup S) - \phi_{J_i}(S)$. By minimality, and Lemma 4.7, J_i is equitable for $W \cap S$ and $W \cup S$. It is also equitable for S since $S \in \mathcal{L}$. Therefore,

$$\phi_{J_i}(W) = \frac{1}{\tau}(\phi(W \cap S) + \phi(W \cup S) - \phi(S)) = \frac{1}{\tau}\phi(W)$$

Hence J_i is a dijoin which is equitable for W, a contradiction. □

Corollary 4.9. *If a weighted digraph is such that every dicut has weight that is a multiple of τ, then the weighted digraph packs.*

A weighted digraph is said to be *divisible by k* if k divides $\phi(v)$ for every vertex v. Mészáros [12] showed the following lemma:

Lemma 4.10 ([12], Lemma 9)**.** *If a weighted digraph is divisible by k, then there is a partition of weight 1 arcs $J_1, \ldots, J_k$ such that $\phi_{J_i}(v) = \frac{1}{k}\phi(v)$ for all vertex v.*

In particular, the lemma implies that if a weighted digraph is divisible by τ, then a weighted packing of τ dijoins exists. The proof of Theorem 4.8 is similar to the proof of Lemma 4.10. In fact, the two properties are equivalent when D is an acyclic digraph.

Lemma 4.11. *If D is a directed acyclic graph, then every arc belongs to a dicut.*

Proof. Let $a = (v_0, v_1) \in A$ and consider the following procedure:

1. Initialize $U = \{v_0\}$,
2. for every arc $(u, v) \in \delta^-(U)$, add u to U, and

3. repeat step 2 until no such arcs exist.

The procedure halts since D is finite; let U be the set generated by the procedure after halting. Observe that for every $u \in U$, a dipath exists from u to v_0. After halting, $\delta(U)$ is a dicut as long as $U \subsetneq V$, as if an arc enters U, the procedure would not have halted. For an eventual contradiction, suppose $U = V$. Then $v_1 \in U$ and a dipath exists from v_1 to v_0. This path with arc a is a dicycle, a contradiction. $\square$

Lemma 4.12. *For every $v \in V$, if $\delta(v)$ is not a dicut then there exists two dicuts $\delta^+(U), \delta^+(W)$ such that $\phi(v) = \phi(W) - \phi(U)$.*

Proof. Let $v \in V$ such that $\delta(v)$ is not a dicut. For $a = (u, v) \in \delta^-(v)$, let U_a be a dicut shore such that $a \in \delta^+(U_a)$, which exists by Lemma 4.11. Crucially, $v \notin U_a$ for all such a. For $U = \bigcup_{a \in \delta^-(v)} U_a$, $\delta^+(U)$ is a dicut, and $\delta^-(v) \subseteq \delta^+(U)$ since for all $(u, v) \in \delta^-(v)$, $u \in U$.

Now consider $W = U \cup \{v\}$. Observe that for every $(u, v) \in \delta^-(v)$, $u \in U$, hence $\delta^-(W) = \emptyset$, and $\delta^+(W)$ is a dicut. For every $a \in \delta^+(U)$, a does not enter v if and only if $a \in \delta^+(W)$. Hence, $\phi(v) = \phi(W) - \phi(U)$. $\square$

Proposition 4.13. *A weighted acyclic digraph (D, w) is divisible by $\tau(D, w)$, if and only if, the weight of every dicut of (D, w) is a multiple of $\tau(D, w)$.*

Proof. ($\Rightarrow$) Recall that ϕ is modular. Then for every dicut $\delta^+(U)$, $\phi(U) = \sum_{u \in U} \phi(u)$. Clearly $\phi(U)$ is divisible by τ since every $\phi(u)$ is.

($\Leftarrow$) If $v \in V$ is such that $\delta(v)$ is a dicut, then the result is immediate. Otherwise, by Lemma 4.12, there exists dicuts $\delta^+(U), \delta^+(W)$ such that $\phi(v) = \phi(W) - \phi(U)$. Since $\phi(W), \phi(U)$ are divisible by $\tau(D, w)$, $\phi(v)$ is divisible by τ. $\square$

4.4 Weighted $(\tau, \tau + 1)$-Bipartite Digraphs

A *bipartite digraph* is a digraph where every vertex is either a source or a sink. A *weighted $(\tau, \tau+1)$-bipartite digraph* is a weighted bipartite digraph where every dicut has weight at least τ, every vertex has weighted degree τ or $\tau + 1$, and the set of vertices with weighted degree $\tau + 1$ form a stable set. Such weighted digraphs are called *sink-regular* if every sink has weighted degree τ; see Figure 4.2 for an example.

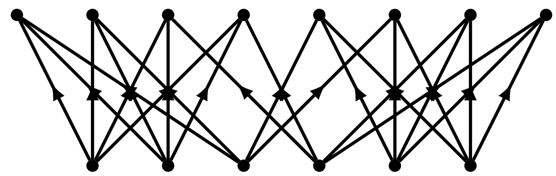

Figure 4.2: A weighted $(3,4)$-bipartite digraph that is sink regular. Every source v has $w(\delta^+(v)) - w(\delta^-(v)) \mod 3 = 1$, thus $\rho = 3$.

Recall that

$$\rho(D,w) := \frac{1}{\tau(D,w)} \sum_{v \in V} [w(\delta^+(v)) - w(\delta^-(v)) \mod \tau(D,w)].$$

Abdi, Cornuéjols and Zlatin [3] introduced the *Decompose-and-Lift* procedure, which decomposes a weighted digraph (D,w) into a finite collection of weighted $(\tau,\tau+1)$-bipartite digraphs (D_i,w_i), $i \in I$. They showed that $\rho(D_i,w_i) \le \rho(D,w)$ for all $i \in [m]$. Furthermore, if J_i is a dijoin of (D_i,w_i) using only the weight 1 arcs, then the collection $J_1,\ldots,J_m$ can be mapped to a dijoin of (D,w).

Theorem 4.14 ([3], Theorem 2.6)**.** *Every weighted digraph (D_1,w_1) with $\rho(D_1,w_1) \le \overline{\rho}$ and $\tau(D_1,w_1) \ge 2$ has a w_1-weighted packing of dijoins of size τ, if and only if, it is true for every sink-regular weighted $(\tau,\tau+1)$-bipartite digraph (D_2,w_2) with $\rho(D_2,w_2) \le \overline{\rho}$ and $\tau(D_2,w_2) \ge 2$.*

In fact, the reduction is more general. A *k-dijoin* is a set of arcs such that it intersects every dicut at least k times. The reduction can map k-dijoins of the weighted $(\tau,\tau+1)$-bipartite digraphs to k-dijoins of original weighted digraphs as well.

The reduction was used in [3] to show that weighted digraphs with $\rho \in \{0,1,2\}$ pack. Prior to their work, Mészáros [12] proved the case for $\rho = 0$; a weighted digraph has $\rho = 0$ precisely when it is divisible by τ. Our goal is to rephrase their results using the language of kernels.

4.5 Kernel Example: Weighted $(\tau, \tau + 1)$-Bipartite Digraphs with $\rho \in \{1, 2\}$

If (D, w) is a weighted $(\tau, \tau + 1)$-bipartite digraph with $\rho(D, w) = 1$, then the family of all singletons is a kernel of (D, w); let

$$K_{\rho=1}(D, w) = \{\{v\} : v \in V\}.$$

For $U \subseteq V$, the *discrepancy* of U (denoted $disc(U)$) is the number of sinks minus the number of sources in U. Note that $disc$ is a modular function. Let $\mathcal{U}^0$ denote the families of dicut shores $\delta^+(U)$ such that $disc(U) = 0$, and let $\mathcal{U}^0_{min}$ denote the minimal sets in $\mathcal{U}^0$. If (D, w) is a weighted $(\tau, \tau + 1)$-bipartite digraph with $\rho(D, w) = 2$, then the family

$$K_{\rho=2}(D, w) = \{\{v\} : v \in V\} \cup \left\{a(V) - U : U \in \mathcal{U}^0_{min}\right\}$$

is a kernel of (D, w). We will first show that $K_{\rho=1}(D, w)$ and $K_{\rho=2}(D, w)$ satisfy the partition condition when $\rho(D, w) = 1$ and 2 respectively. We delay proving the dijoin condition; this allows us to postpone introducing additional the machinery from [3] until it is required.

4.5.1 The Partition Condition is Satisfied

We introduce some terminology for weighted $(\tau, \tau + 1)$-bipartite digraphs. A vertex v is *active* if $w(\delta(v)) = \tau + 1$. For instance, in the weighted $(3, 4)$-bipartite digraph in Figure 4.2, every source is an active vertex. For $U \subseteq V$, let $a(U)$, $s(U)$ and $t(U)$ respectively denote the active vertices, sinks and sources in U. From here on, we say that a weighted digraph (D, w) satisfies *standard assumptions* (or satisfies *(SA)*) if,

1. (D, w) is a weighted $(\tau, \tau + 1)$-bipartite digraph for $\tau \geq 1$ with $w \in \{0, 1\}^A$, and
2. (D, w) is sink regular.

Due to its structure, one advantage of considering weighted $(\tau, \tau+1)$-bipartite digraphs is that it is amenable to counting arguments. This allows us to derive a formula for weights of dicuts easily. The following is from [3] with the proofs paraphrased.

Lemma 4.15 ([3], Lemma 3.7)**.** *For a weighted digraph (D, w) satisfying (SA), the following statements hold:*

1. *$\rho(D,w) = disc(V)$ and $|a(V)| = \tau(D,w) \times disc(V)$,*
2. *For every dicut $\delta^+(U)$ of D, $w(\delta^+(U)) = |a(U)| - \tau(D,w) \times disc(U)$,*
3. *For every dicut $\delta^+(U)$ of D, $disc(U) \leq disc(V) - 1$, and if the equality holds, then $w(\delta^+(U)) = \tau(D,w)$.*

Proof. **(1)** Observe that

$$\begin{aligned}\rho(D,w) &= \frac{1}{\tau}\sum_{v\in V} w(\delta^+(v)) - w(\delta^-(v)) \mod \tau \\ &= \frac{1}{\tau}|a(V)|\end{aligned}$$

since the digraph is sink-regular. Now showing $\rho(D,w) = disc(V)$ proves both claims. Double counting A_1, we have that $\tau|t(V)| = \tau|s(V)| + |a(V)|$, which immediately implies $|a(V)| = \tau|t(V)| - \tau|s(V)| = \tau disc(V)$.

Let $\delta^+(U)$ be a dicut in D. **(2)** Every arc in A_1 incident to a sink in U is incident to a source in U since $\delta^-(U) = \emptyset$. Thus $w(\delta^+(U))$ is the number of arcs in A_1 incident to a source in U, and a sink not in U. The digraph is sink-regular, thus $w(\delta^+(U)) = \tau|s(U)| + |a(U)| - \tau|t(U)| = |a(U)| - \tau disc(U)$.

(3) For an eventual contradiction, suppose $disc(U) \geq disc(V)$. Using (2), we get $w(\delta^+(U)) \leq |a(U)| - \tau disc(V) \leq |a(V)| - \tau disc(V)$. However by (1), $|a(V)| - \tau disc(V) = 0$, which contradicts $\tau > 0$ being the minimum weight of a dicut. Plugging in $disc(U) = disc(V) - 1$ in (2) immediately shows the second claim. ☐

Clearly $K_{\rho=1}(D,w)$ is laminar, and hence satisfies (K1) by Lemma 4.6. Now we proceed to prove that $K_{\rho=2}(D,w)$ satisfies (K1) by also proving it is laminar.

Lemma 4.16. *For a weighted digraph (D,w) satisfying (SA) with $\rho(D,w) = 2$, the union of two distinct sets in $\mathcal{U}^0_{min}$ contains all active vertices of (D,w).*

Proof. Let $U, W \in \mathcal{U}^0_{min}$ be distinct. Recall that $w(\delta^+(U)) = |a(U)| - \tau disc(U)$ ((2) of Lemma 4.15), thus $|a(U)|, |a(W)| \geq \tau$.

Suppose that $a(U) \cap a(W) = \emptyset$. Since $|a(V)| = \rho\tau = 2\tau$, and $a(U)$ and $a(W)$ are disjoint, we have $a(V) = a(U) \cup a(W)$.

Now suppose that $a(U) \cap a(W) \neq \emptyset$, meaning $U \cap W \neq \emptyset$. We may assume that $U \cup W \neq V$ as otherwise the result is trivially true. Then $\delta^+(U \cap W)$ and $\delta^+(U \cup W)$ are dicuts of D, and by the modularity of discrepancy,

$$disc(U \cup W) + disc(U \cap W) = disc(U) + disc(W) = 0.$$

Observe that by minimality $disc(U \cap W) \neq 0$, and thus $disc(U \cup W) \neq 0$. Furthermore, $disc(U \cap W), disc(U \cup W) \in \{-1, 1\}$ since $disc(U) \leq disc(V) - 1$ for any dicut $\delta^+(U)$; exactly one of $disc(U \cap W), disc(U \cup W)$ is 1.

Suppose $disc(U \cap W) = 1$. By (3) of Lemma 4.15, $\delta^+(U \cap W)$ is a minimum weight dicut, thus

$$w(\delta^+(U \cap W)) = |a(U \cap W)| - \tau disc(U \cap W) = |a(U \cap W)| - \tau = \tau,$$

i.e. $|a(U \cap W)| = 2\tau$. Therefore $a(U \cap W)$ contains all active vertices, hence $U \cup W$ contains all active vertices.

Using similar reasoning as above, when $disc(U \cup W) = 1$ we can directly show that $|a(U \cup W)| = 2\tau$. $\square$

Lemma 4.17. *If $\delta^+(U)$ is a dicut of a weighted digraph satisfying (SA) with $disc(U) = \rho - 1$, then $a(V) \subseteq U$.*

Proof. By Lemma 4.15, $w(\delta^+(U)) = |a(U)| - \tau disc(U) = \tau$. Plugging in $disc(U) = \rho - 1$, we get $|a(U)| = \rho\tau = |a(V)|$. $\square$

Lemma 4.18. *If (D, w) satisfies (SA) and $\rho(D, w) = 2$, then $K_{\rho=2}(D, w)$ satisfies (K1).*

Proof. By Lemma 4.6, it suffices to show that $K_{\rho=2}(D, w)$ is laminar. To this end, we prove that the sets in $\{a(V) - a(U) : U \in \mathcal{U}^0_{min}\}$ are pairwise disjoint. For an eventual contradiction, suppose there exists $U, W \in \mathcal{U}^0_{min}$ such that $(a(V) - U) \cap (a(V) - W) \neq \emptyset$. Observe that $a(V) - (U \cup W) = (a(V) - U) \cap (a(V) - W) \neq \emptyset$, and thus $a(V) \not\subseteq U \cup W$. This contradicts Lemma 4.16. $\square$

4.5.2 The Dijoin Condition is Satisfied

We introduce additional terminology from [3]. A set of arcs $J \subseteq A_1$ is a *rounded* 1*-factor* of (D, w) if for every vertex v, $|J \cap \delta(v)| \in \left\{ \left\lfloor \frac{w(\delta(v))}{\tau} \right\rfloor, \left\lceil \frac{w(\delta(v))}{\tau} \right\rceil \right\}$. A *dyad center* of a rounded

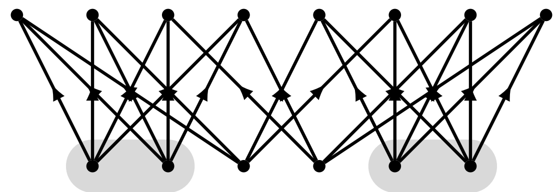

Figure 4.3: A weighted $(3,4)$-bipartite digraph with $\rho = 2$. The shores in $\{a(V) - U : U \in \mathcal{U}^0_{min}\}$ are indicated in gray.

1-factor J is a vertex where J contains two arcs incident to v; dyad centers are necessarily active vertices. The set of dyad centers of J is denoted as $dc(J)$.

Remark 4.19. *For a weighted digraph satisfying (SA), a set of arcs is a rounded 1-factor, if and only if, it is equitable for the family $\{\{v\} : v \in V\}$.*

The next remark shows the utility of considering weighted $(\tau, \tau+1)$-bipartite digraphs when packing dijoins.

Remark 4.20 ([3], Remark 3.2)**.** *If $J_1, \ldots, J_\tau$ are disjoint dijoins in A_1 for a weighted $(\tau, \tau+1)$-bipartite digraph, then the J_i partition A_1 and are rounded 1-factors.*

In addition, A_1 can always be partitioned into τ rounded 1-factors.

Theorem 4.21 ([3], Theorem 3.4, originally from [6])**.** *Let G be a bipartite graph and $k \geq 1$ an integer. The edges of G can be partitioned into $J_1, \ldots, J_k$ such that $|J_i \cap \delta(v)| \in \left\{ \left\lfloor \frac{|\delta(v)|}{k} \right\rfloor, \left\lceil \frac{|\delta(v)|}{k} \right\rceil \right\}$ for each $i \in [k]$.*

The following lemma is from [3] with the proofs paraphrased.

Lemma 4.22 ([3], Lemma 3.8)**.** *Let $J \subseteq A_1$ be a rounded 1-factor of a weighted digraph (D, w) satisfying (SA). The following statements hold:*

1. *$|dc(J)| = \rho(D, w) = disc(V)$,*
2. *for every dicut $\delta^+(U)$ of D, $|J \cap \delta^+(U)| = |dc(J) \cap U| - disc(U)$*
3. *J is a dijoin if and only if $|dc(J) \cap U| \geq 1 + disc(U)$ for every dicut $\delta^+(U)$ of D.*

Proof. **(1)** Sink regularity implies that the dyad centers of J are sources. By J being a rounded 1-factor, J has $|t(V)|$ arcs incident to sinks, and $|s(V)| + |dc(J)|$ arcs incident to sources, i.e. $|dc(J)| = disc(V)$.

(2) There are $|dc(J) \cap U| + |s(U)|$ arcs in J incident to a source in U. Exactly $|t(U)|$ arcs in J are incident to a sink in U since J has precisely one arc incident to every sink, and no arc enters U. Thus

$$|delta^+(U) \cap J| = |dc(J) \cap U| + |s(U)| - |t(U)| = |dc(J) \cap U| - disc(V)$$

(3) is a direct consequences of (2). $\square$

Recall that $K_{\rho=2}(D, w)$ contains $\{a(V) - U : U \in \mathcal{U}^0_{min}\}$. Part (3) of Lemma 4.22 shows that whether a rounded 1-factor is a dijoin is solely a function of its dyad centers. With this lemma in hand, the relevance of $\mathcal{U}^0_{min}$ is clear: to show that a rounded 1-factor J has sufficiently many dyad centers in every $U \in \mathcal{U}^0$, it suffices to consider the minimal such shores. We first show that when $\rho(D, w) = 1$, the family of all singletons satisfies (K2).

Theorem 4.23. *If (D, w) satisfies (SA) and $\rho(D, w) = 1$, then $K_{\rho=1}(D, w)$ is a kernel of (D, w).*

Proof. Let $J_1, \ldots, J_\tau$ be the partition of A_1 such that every J_i is $K_{\rho=1}(D, w)$-equitable. Then, every J_is is a rounded 1-factors (Remark 4.19).

Now it suffices to check that for every dicut $\delta^+(U)$, $|dc(J_i) \cap U| \geq 1 + disc(U)$ holds (Lemma 4.22). By Lemma 4.15, every dicut $\delta^+(U)$ has $disc(U) \leq disc(V) - 1 = \rho - 1 \leq 0$. The inequality is clearly satisfied when $disc(U) < 0$, so suppose $disc(U) = 0$, which is only the case when $\rho = 1$. Then $disc(U) = \rho - 1$, and by Lemma 4.17, $a(V) \subseteq U$ and $|dc(J_i) \cap U| = \rho$; every J_i is a dijoin as desired. $\square$

Corollary 4.24. *Weighted $(\tau, \tau + 1)$-bipartite digraphs satisfying $\rho = 1$ pack.*

Our proof of Theorem 4.23 proceeds similarly to the proof in [3]; they show that every rounded 1-factor is a dijoin. In fact, their proof goes one step further to show that when $\rho = 1$, a w-weighted packing of size τ exists where every dijoin is equitable for every dicut.

When $\rho = 2$, rounded 1-factors do not necessarily intersect dicuts with shores in $\mathcal{U}^0$ (see Figure 4.4); observe that Lemma 4.17 cannot be applied to $\mathcal{U}^0$ when $\rho = 2$. However, Abdi, Cornuéjols and Zlatin [3] showed that it is always possible to make changes to the rounded 1-factors to yield dijoins. Alternating paths between dyad centers of rounded 1-factors were used to swap dyad centers around until every rounded 1-factor satisfied the inequality in Lemma 4.22. In essence, they identify a family (in particular the family of singletons) and derive a partition of A_1 into τ equitable sets, and make local changes until every equitable set is a dijoin. Our method for $\rho = 2$ is stronger in the sense that local changes are no longer necessary; the equitable sets are immediately dijoins.

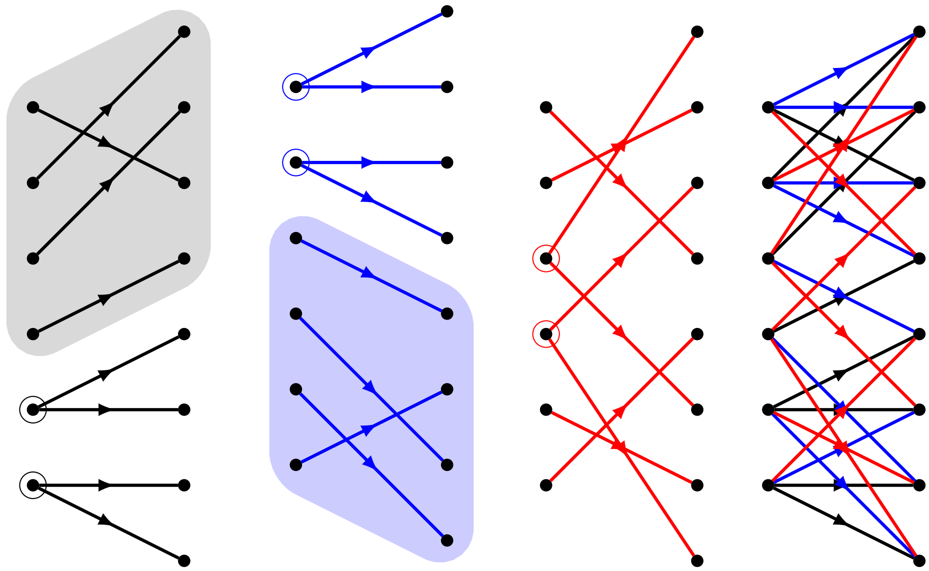

Figure 4.4: A weighted $(3, 4)$-bipartite digraph with $\rho = 2$ where its weight 1 arcs are partitioned into 3 rounded 1-factors. The dyad centers of rounded 1 factors are indicated with circles. Observe that the blue and black rounded 1-factors are not dijoins. Namely, they do not contain dyad centers in dicut shores with discrepancy 0.

Theorem 4.25. *If (D, w) satisfies (SA) and $\rho(D, w) = 2$, then $K_{\rho=2}(D, w)$ is a kernel of (D, w).*

Proof. The set $K_{\rho=2}(D, w)$ is laminar (Lemma 4.18), and let $J_1, \ldots, J_\tau$ be a partition of A_1 into $K_{\rho=2}(D, w)$-equitable sets, which exists by Lemma 4.6. Since the J_is are equitable for every trivial dicut, they are rounded 1-factors.

If $\delta^+(U)$ is a dicut, then $disc(U) \leq 1$. The inequality $|dc(J_i) \cap U| \geq 1 + disc(U)$ is satisfied for $disc(U) < 0$ trivially, and for $disc(U) = \rho - 1 = 1$, by Lemma 4.17. For an eventual contradiction, suppose that a dicut $\delta^+(U)$ where $disc(U) = 0$ and J_i such that $|dc(J_i) \cap U| < 1$. Let $U' \subseteq U$ be such that $U' \in \mathcal{U}^0_{min}$. Recall that $w(\delta^+(U')) =$

$|a(U')| - \tau disc(U') \geq \tau$ (Lemma 4.15), thus $|a(U')| \geq \tau$. Then,

$$\begin{aligned}\left\lceil \frac{1}{\tau} w(\delta^+(a(V) - U')) \right\rceil &= \left\lceil \frac{\tau+1}{\tau} |a(V) - U'|) \right\rceil \\ &= |a(V) - U'| + \left\lceil \frac{1}{\tau} |a(V) - U'| \right\rceil \\ &\leq |a(V) - U'| + 1,\end{aligned}$$

where last inequality comes from $|a(V) - U'| = |a(V)| - |a(U')|$, and $|a(V)| = 2\tau$.

Since $|dc(J_i)| = \rho = 2$, this implies that $|dc(J_i) \cap (a(V) - U)| = |dc(J_i) \cap (a(V) - U')| = 2$; as J_i is a rounded 1-factor, it has one arc incident to every active vertex in $a(V) - U'$, and two additional arcs for each dyad center in $a(V) - U'$. This yields a contradiction as J_i is not equitable for U' since $|J_i \cap \delta(a(V) - U')| = |a(V) - U'| + 2 > \left\lceil \frac{1}{\tau} w(\delta^+(a(V) - U')) \right\rceil$. $\square$

Corollary 4.26. *Weighted $(\tau, \tau+1)$-bipartite digraphs satisfying $\rho = 2$ pack.*

4.6 Beyond $\rho = 2$

Finding a canonical kernel construction for weighted $(\tau, \tau+1)$-bipartite digraphs with $\rho \geq 3$ is out of reach; Schrijver's counterexample has ρ value of 3 (see Figure 4.5). Hence, not every weighted $(\tau, \tau+1)$-bipartite digraph with $\rho \geq 3$ packs.

Given this roadblock, one relaxation of Edmonds-Giles conjecture mentioned in Chapter 3 was to ask what the maximum size weighted packing is for all weighted digraphs with the weight of the minimum dicut τ. In other words, what is the function $f(\tau)$ where every weighted digraph with weight of its minimum dicut τ, we can always find a weighted packing of size $f(\tau)$? Due to Schrijver's counterexample, $f(\tau) = \frac{\tau}{2}$ is the best possible. However, with the notion of k-dijoins (recall that a k-dijoin is a set of arcs which have at least k arcs common with every dicut), another relaxation can be formulated. We say that (D, w) *k-packs* if there exists $J_1, \ldots, J_m \subseteq A_1$ such that $\sum_{i \in [m]} k_i = \tau$, where J_i is a k_i-dijoin with $k_i \leq k$.

Question 4.27. *Given a weighted digraph (G, w), when does (G, w) k-pack?*

It is currently unknown if there is a weighted digraph which 3-packs but does not 2-pack; in Schrijver's counterexample, the set of weight 1 arcs is a 2-dijoin. Thus, a refinement of the above question is:

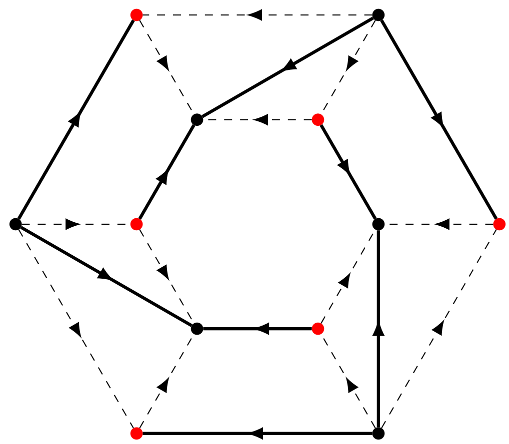

Figure 4.5: The red vertices have value 1, and the black vertices have value 0 under $w(\delta^+(v)) - w(\delta^-(v)) \mod \tau$. Thus $\rho = \frac{6}{2} = 3$ for Schrijver's counterexample.

Question 4.28. *Does every weighted digraph 2-pack?*

We can even extend the notion of kernels to achieve k-packings. For a family of dicut shores $\mathcal{O}$ of (D, w), $J \subseteq A_1(D, w)$ is $(\mathcal{F}, k)$-equitable for $1 \leq k \leq \tau(D, w)$ (or k-equitable for $\mathcal{F}$) if for every dicut shore $U \in \mathcal{O}$,

$$\left\lfloor \frac{k}{\tau(D,w)} w(\delta(U)) \right\rfloor \leq |J \cap \delta(U)| \leq \left\lceil \frac{k}{\tau(D,w)} w(\delta(U)) \right\rceil.$$

For a weighted digraph (D, w), a *k-kernel* $\mathcal{F}$ is a family of dicut shores of D such that

(KK1) $A_1(D, w)$ can be partitioned into $J_1, \ldots, J_n$, $1 \leq n \leq \tau(D, w)$, such that for all $i \in [n]$, J_i is $(\mathcal{F}, k_i)$-equitable where $k_i \leq k$, and

(KK2) a $(\mathcal{F}, m)$-equitable set is a m-dijoin for all $1 \leq m \leq \tau(D, w)$.

Remark 4.29. *If a k-kernel exists for a weighted digraph, then it k-packs.*

Interestingly, we can extend the construction of the kernel for weighted $(\tau, \tau+1)$-bipartite digraphs with $\rho = 2$ to a set of families which satisfies (KK2) for higher ρ values. Suppose (D, w) satisfies (SA). We define $\mathcal{U}^i$ as the set of dicut shores U where $\delta^+(U)$ is a dicut and $disc(U) = i$; let $\mathcal{U}^i_{min}$ be the minimal sets in $\mathcal{U}^i$. Furthermore, let $\mathcal{A}^i$ be the minimal sets in $\{a(U) : U \in \mathcal{U}^i_{min}\}$, and $\overline{\mathcal{A}^i} = \{a(V) - U : U \in \mathcal{A}^i\}$. Let $\mathcal{K}(D, w)$ be a set of families of dicut shores where every family in $\mathcal{K}(D, w)$ can be expressed as

$$\{\{v\} : v \in V\} \cup \bigcup_{0 \leq i \leq \rho(D,w)-2} \mathcal{S}_i$$

where $\mathcal{S}_i$ is either $\mathcal{A}^i$ or $\overline{\mathcal{A}^i}$. When $\rho(D, w) = 2$, picking $\overline{\mathcal{A}^0}$ nearly derives the kernel. Our only concern for rounded 1-factors is whether they have enough dyad centers in each dicut shore; hence we considered only the dicuts minimal in their shores. However, every dyad center is an active vertex; we may further refine the family by considering the dicuts which are minimal in their active vertices among the dicut shores with the same discrepancy. This detail was previously omitted for simplicity.

The flexibility of choosing between $\mathcal{A}^i$ and $\overline{\mathcal{A}^i}$ arises since equitability imposes both a lowerbound and an upperbound. If $\mathcal{A}^i$ is picked, the upperbound imposed on the sets of $\mathcal{A}^i$ will ensure that enough dyad centers are contained in these sets; if $\overline{\mathcal{A}^i}$ is picked, the lowerbound imposed will ensure that not too many dyad centers are picked outside of the sets in $\overline{\mathcal{A}^i}$.

Theorem 4.30. *Suppose (D, w) satisfies (SA). For every $\mathcal{F} \in \mathcal{K}(D, w)$ and $1 \leq k \leq \tau(D, w)$, if J is $(\mathcal{F}, k)$-equitable, then J is a k-dijoin.*

Proof. Suppose J is $(\mathcal{F}, k)$-equitable, where $\mathcal{F} \in \mathcal{K}(D, w)$. Observe that $D[J]$ is a bipartite digraph since D is. Furthermore, since J is $(\mathcal{F}, k)$-equitable, every sink and non-active source has degree k and every active source has degree either k or $k + 1$ in $D[J]$. Using Theorem 4.21, we partition J into $J_1, \ldots, J_k$; the J_i are are rounded 1-factors based on the degrees of vertices in $D[J]$. In other words, J is a union of k disjoint rounded 1-factors. For brevity, we will abuse notation and denote $dc(J) = dc(J_1) \cup \cdots \cup dc(J_k)$. By Lemma 4.22,

$$|J \cap \delta^+(U)| = \sum_{i \in [k]} |J_i \cap \delta^+(U)| = |dc(J) \cap U| - k disc(U).$$

Therefore, we will show that $|dc(J) \cap U| \geq k + k disc(U)$ for all dicut $\delta^+(U)$.

If $disc(U) < 0$, then the above inequality is trivially satisfied; if $disc(U) = \rho - 1$, then Lemma 4.17 implies the above inequality. Assume $0 \leq disc(U) \leq \rho - 2$. Since $w(\delta^+(U)) = |a(U)| - \tau disc(U) \geq \tau$, we have $|a(U)| \geq \tau(1 + disc(U))$.

Suppose that for $i = disc(U)$, $\mathcal{A}^i$ was chosen for $\mathcal{F}$, and let $U' \in \mathcal{U}^0_{min}$ be such that

$a(U') \in \mathcal{A}^i$ and $a(U') \subseteq a(U)$. By J being $(\mathcal{F}, k)$-equitable, where $a(U') \in \mathcal{F}$,

$$\begin{aligned} |J \cap \delta^+(a(U'))| &\geq \left\lfloor \frac{k}{\tau} w(\delta^+(a(U'))) \right\rfloor \\ &= k|a(U')| + \left\lfloor \frac{k}{\tau} |a(U')| \right\rfloor \\ &\geq k|a(U')| + k(1 + disc(U')). \end{aligned}$$

Since every J_i is incident to every vertex in U', and dyad centers of J_i and J_j are disjoint for $i \neq j$, the above inequality implies that $|dc(J) \cap U| \geq |dc(J) \cap U'| \geq k(1 + disc(U'))$ as desired.

Now suppose that $\overline{\mathcal{A}^i}$ was chosen, and pick $a(U')$ to be the same as before. By J being $(\mathcal{F}, k)$-equitable (this time using the upperbound), and since $a(V) - a(U') \in \mathcal{F}$ and $|a(V)| = \rho\tau$,

$$\begin{aligned} |J \cap \delta^+(a(V) - a(U'))| &\leq \left\lceil \frac{1}{\tau} w(\delta^+(a(V) - a(U'))) \right\rceil \\ &= k|a(V) - a(U')| \left\lceil \frac{k}{\tau} |a(V) - a(U')| \right\rceil \\ &\leq k|a(V) - a(U')| + k(\rho - 1 - disc(U)). \end{aligned}$$

Again, this implies that $|dc(J) \cap (a(V) - a(U'))| \leq k(\rho - 1 - disc(U))$. Since $|dc(J)| = k\rho$, we get that $|dc(J) \cap a(U')| \geq k(1 + disc(U))$, as desired. $\square$

Hence every family in $\mathcal{K}(D, w)$ satisfies (KK2). When is there a set in $\mathcal{K}(D, w)$ which satisfies (KK1)?

Question 4.31. *If (D, w) k-packs, does $\mathcal{K}(D, w)$ contain a k-kernel?*

Chapter 5

Future Directions

Let us conclude the thesis by stating some open questions, and natural extentions to the thesis.

We conjecture the following generalization of the Dyadic Conjecture; in Chapter 2, we proved the conjecture for the clutter of dijoins.

Conjecture 5.1. *Every ideal clutter has an optimal fractional packing that is $[\tau]$-adic.*

Finding kernels of weighted digraphs is at least as hard as finding a weighted packing of τ dijoins. This is true for k-kernels and k-packings as well. It is unknown whether finding (k-)kernels and finding (k-)packings is the same question.

Question 5.2. *Suppose a weighted digraph (D, w) k-packs. Does a k-kernel exist for (D, w)?*

One motivation behind finding a (k-)kernel is to apply constraints on a smaller, ideally polynomial, number of dicuts. In the previous chapter, we found a set of families $\mathcal{K}(D, w)$ such that every $\mathcal{F} \in \mathcal{K}(D, w)$ satisfies (KK2) (and hence satisfies (K2)). Unfortunately, the families in $\mathcal{K}(D, w)$ are superpolynomially large in general; we give a construction of a weighted $(\tau, \tau + 1)$-bipartite digraph with such $\mathcal{K}(D, w)$.

For $\tau \geq 4$, we will refer to the following construction of a weighted $(\tau, \tau + 1)$-bipartite digraph (G, w) as $C(\tau)$ (we will say that a digraph was constructed using $C(\tau)$). Let $(D_i = (V_i, A_i), w_i = 1)$ for $i \in [\tau]$ be a weighted $(\tau, \tau + 1)$-bipartite digraph which contains $\tau + 1$ sinks, τ sources (which are all active vertices) and where every source is incident to every sink. Observe that the discrepancy of V_i is 1.

We construct a weighted digraph (G, w) in the following way. The vertex set of digraph G is $\{s, t\} \cup \bigcup_{i \in [\tau]} V_i$, where s is a source and t is a sink. For each (D_i, w_i), we arbitrarily pick an arc (s_i, t_i) and make its weight 0, and add two arcs (s, t_i) and (s_i, t) of weight 1; see Figure 5.1 for an example.

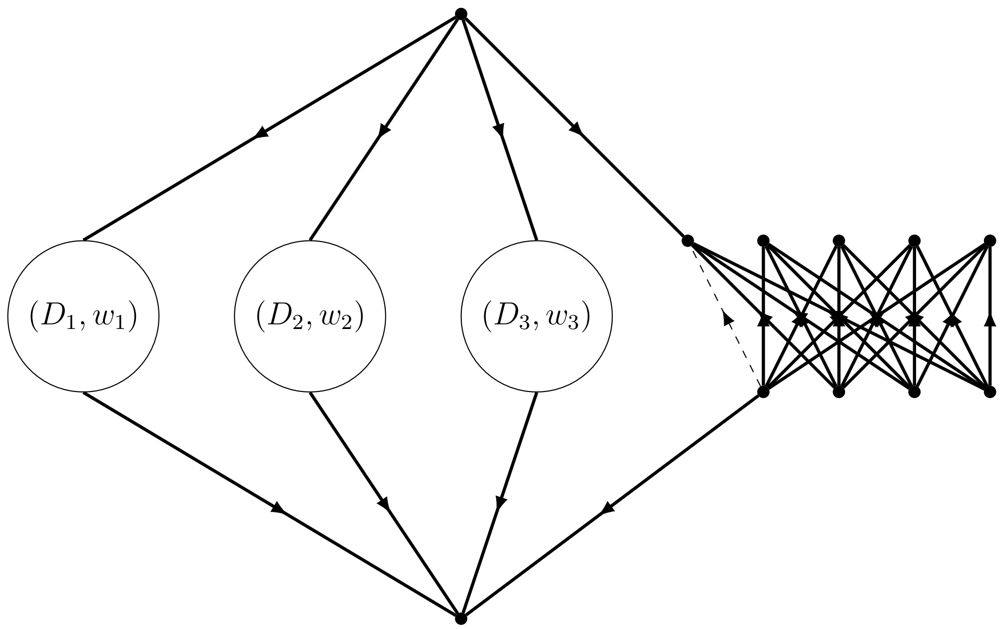

Figure 5.1: Illustration of (G, w) where $\tau = 4$. As usual, solid arcs of weight 1 and dashed arcs are of weight 0. The weighted digraph (D_4, w_4) is explicitly shown; all weighted digraphs (D_i, w_i), $i \in [3]$ can be replaced by the same weighted digraph as (D_4, w_4).

Remark 5.3. *If $\delta^+(U)$ is a dicut and $u \in U$ is a sink, then $\delta^+(U - \{u\})$ is a dicut.*

Lemma 5.4. *If (G, w) is a weighted digraph constructed using $C(\tau)$ where $\tau \geq 4$ and τ is even, then (G, w) a weighted $(\tau, \tau + 1)$-bipartite digraph that is sink regular.*

Proof. Clearly every vertex is either a source or a sink. Every sink has weighted degree τ, and every source except s (which has weighted degree τ) has weighted degree $\tau + 1$. It now suffices to show that no dicut has weight less than τ. Let $\delta^+(U)$ be a dicut of G, and let $U_i = U \cap V_i$. We consider cases on whether s or t are in U.

Claim 1. If $U_i \neq V_i$ and $U_i \neq \emptyset$, then $w(\delta^+_G(U_i)) \geq \tau$.

Proof The set U_i may not be a dicut shore in G due to the arc (s, t_i). However, since U is a dicut in G, and every arc in D_i is present in G, U_i is a dicut shore in D_i. Since (D_i, w_i) is a weighted $(\tau, \tau+1)$-bipartite digraph, $w_i(\delta^+_{D_i}(U_i)) \geq \tau$. The only weight that was decreased when constructing (G, w) is (s_i, t_i). However, if $s_i \in U_i$, then $(s_i, t) \in \delta^+_G(U_i)$. Thus $w(\delta^+_G(U_i)) \geq w_i(\delta^+_{D_i}(U_i)) \geq \tau$. $\Diamond$

No arc exists between V_i and V_j for $i \neq j$. Furthermore, if $s \notin U$, $U_i \neq V_i$ as otherwise (s, t_i) enters U_i. Suppose U contains neither s nor t. Then there exists $U_i \neq \emptyset$, and $\delta^+_G(U_i) \subseteq \delta^+_G(U)$; the above claim implies that $\delta^+_G(U)$ has weight at least τ.

Suppose $s \in U$ but $t \notin U$. No arc enters s, so if there exists a U_i such that $\emptyset \neq U_i \neq V_i$, then $w(\delta^+(U)) \geq w(\delta^+_G(U_i)) \geq \tau$. If every U_i is either empty, or $U_i = V_i$ for all $i \in [\tau]$, then clearly at least τ arcs in $\delta^-(t) \cup \delta^+(s)$ (all of weight 1) are in $\delta^+(U)$.

Suppose $s \notin U$ and $t \in U$. Then $s_i \in U$ for all $i \in [\tau]$, i.e. U_i is non-empty for all $i \in [\tau]$. Moreover, $U_i \neq V_i$ for all $i \in [\tau]$, since otherwise (s, t_i) enters U. Using the above claim, but observing that (s_i, t) is not in $\delta^+(U)$, we have $w(\delta^+(U)) \geq \tau^2 - \tau$ which is at least τ for $\tau \geq 2$.

Finally, suppose $s, t \in U$. Again, U_i is non-empty for all $i \in [\tau]$. At least one U_i exists which is not V_i since $U \neq V$. If $U_i = \{s_i\}$, then there are τ weight 1 arcs in $\{(s, t_i)\} \cup \delta^+(U_i) - \{(s_i, t)\}$, which are all in $\delta^+(U)$. If U_i contains only sources, then clearly the weight of $\delta^+(U)$ is at least τ. If U_i contains a sink, then it must contain all sources in V_i, as every sink is incident to every source. Hence, the contribution to the weight $w(\delta^+(U))$ by arcs incident to V_i when a sink is in U_i is when all but one sink is in U_i. Whatever the excluded sink may be, the contribution of arcs incident to V_i is at least τ. $\square$

Theorem 5.5. *Given (G, w) that was constructed using $C(\tau)$ where $\tau \geq 4$ and τ is even, we have $|\mathcal{A}^{\frac{\tau}{2}-1}| \geq \binom{\tau}{\tau/2}$.*

Proof. Let $N_j \subseteq [\tau]$, $j \in \left[\binom{\tau}{\tau/2}\right]$ be distinct combinations of $[\tau]$ of size $\frac{\tau}{2}$, and let $W_j = \{s\} \cup \bigcup_{i \in N_j} V_i$. Observe that $\delta^+(W_j)$ is a dicut, and every W_j contains $\frac{\tau^2}{2}$ active vertices. The discrepancy of V_i is 1, thus the discrepancy of W_j is $\frac{\tau}{2} - 1 \geq 1$ since $\tau \geq 4$.

We show that the sets $a(W_j)$ are in $\mathcal{A}^{\frac{\tau}{2}-1}$. For an eventual contradiction, suppose that there exists W_j such that $W_j \notin \mathcal{U}^{\frac{\tau}{2}-1}_{min}$, i.e. there exists $\delta^+(U)$ such that $disc(U) = \frac{\tau}{2} - 1$, and $a(U) \subsetneq a(W_j)$. Such a shore must have $|a(U)| < \frac{\tau^2}{2}$, and we pick the shore U which minimizes $|a(U)|$.

Claim 1. For all $i \in [\tau]$, either U contains all or none of the active vertices (i.e. sources) of V_i.

Proof For an eventual contradiction, suppose there exists V_i such that $1 \leq |s(V_i) \cap U| < \tau$. Since every sink is incident to every source, the shore U cannot contain a sink in V_i. Consider U_{new}, which is derived by removing $|V_i \cap U|$ sinks in $U - V_i$. If $t \in U - V_i$ then t is removed; the other $|V_i \cap U| - 1$ sinks are chosen arbitrarily. This is always possible, as the discrepancy of U is positive. The vertices in $U \cap V_i$ are sources which are not incident to sinks outside of V_i except possibly t. Thus $U - V_i - \{t\}$ is a dicut shore, and after removing other sinks it is still a dicut shore (Remark 5.3). Observe that $disc(U_{new}) = \frac{\tau}{2} - 1$, $a(U_{new}) \subsetneq a(U)$ and $\delta^+(U_{new})$ is a dicut, contradicting the minimality of $|a(U)|$. $\Diamond$

Let m be the number of V_i where all its active vertices are contained in U, and possibly after rearranging the digraph, let $[m]$ be the indices. Since all active vertices of D are in one of the V_i, we have $m\frac{\tau}{2} = |a(U)| < \frac{\tau^2}{2}$, i.e. $m < \frac{\tau}{2}$. Suppose there exists $t_i \in U$ for some $i \in [m]$. Then $s \in U$, and the maximum discrepancy of U is $m - 1$, by taking all sinks in each of the $V_i, i \in [m]$. This is a contradiction since $m - 1 < \frac{\tau}{2} - 1$, thus $a(U) \notin \mathcal{A}^i$. If no t_i exists such that $t_i \in U$, then the maximum discrepancy of U is 0, which again yields a contradiction. $\square$

The weighted $(\tau, \tau+1)$-bipartite digraph (G, w) has $2\tau^2 + \tau + 2$ vertices, but since $|\overline{\mathcal{A}^i}| = |\mathcal{A}^i|$, every family in $\mathcal{K}(D, w)$ has at least $\binom{\tau}{\tau/2} \geq 2^{\frac{\tau}{2}}$ sets. Recall that a laminar family of dicut shores guaranteed (K1) (and therefore (KK1)), but at most $2|V| - 1$ shores are in such a family. The above theorem shows the flipside of the tradeoff; families in $\mathcal{K}(D, w)$ guarantee (KK2), but in general its size is superpolynomially large.

Question 5.6. *If a k-kernel exists, does a k-kernel of polynomial size always exist?*

Additionally,

Question 5.7. *When does a weighted $(\tau, \tau+1)$-bipartite digraph (B, w) have a family in $\mathcal{K}(B, w)$ which satisfies (KK1)?*

The set of families $\mathcal{K}(B, w)$ is defined for every value of ρ. For a weighted digraph (D, w), let $(D_i, w_i), i \in [m]$ be the resulting collection of weighted $(\tau, \tau+1)$-bipartite digraph after applying the Decompose-and-Lifting procedure.

Question 5.8. *Is there a natural mapping from $\mathcal{K}(D_i, w_i), i \in [m]$ to a family of dicut shores $\mathcal{F}$ for (D, w) which satisfies (K2)?*

Mészáros [12] proved that every weighted digraph with $\rho = 0$ packs. Abdi, Cornuéjols and Zlatin [3] did the same when $\rho \in \{1, 2\}$ by reducing general packing statements to packing statements for weighted $(\tau, \tau + 1)$-bipartite digraphs.

Question 5.9. *Can we directly show that weighted digraphs with* $\rho \in \{1, 2\}$ *pack without reducing it to weighted* $(\tau, \tau + 1)$*-bipartite digraphs?*

The useful structure of weighted $(\tau, \tau + 1)$-bipartite digraphs can be a double-edged sword; its structure restricts arguments using induction. For instance, we may want to contract a subgraph (e.g. apply k-SP contraction), but in general we have no guarantee that the resulting weighted digraph is a weighted $(\tau, \tau + 1)$-bipartite digraph.

References

[1] Ahmad Abdi, Gérard Cornuéjols, Bertrand Guenin, and Levent Tunçel. Clean clutters and dyadic fractional packings. *SIAM J. Discret. Math.*, 36(2):1012–1037, jan 2022.

[2] Ahmad Abdi, Gérard Cornuéjols, and Zuzanna Palion. On dyadic fractional packings of t-joins. *SIAM Journal on Discrete Mathematics*, 36(3):2445–2451, 2022.

[3] Ahmad Abdi, Gérard Cornuéjols, and Michael Zlatin. On packing dijoins in digraphs and weighted digraphs, 2022.

[4] Michele Conforti, Gérard Cornuéjols, and Giacomo Zambelli. *Integer programming*. Graduate Texts in Mathematics, 271. Springer, Cham, 2014.

[5] Gérard Cornuéjols. *Combinatorial optimization : packing and covering*. CBMS-NSF regional conference series in applied mathematics ; 74. Society for Industrial and Applied Mathematics, Philadelphia, PA, 2001.

[6] D. de Werra. Equitable colorations of graphs. *ESAIM: Mathematical Modelling and Numerical Analysis - Modélisation Mathématique et Analyse Numérique*, 5(R3):3–8, 1971.

[7] Jack Edmonds and Rick Giles. A min-max relation for submodular functions on graphs. In *Studies in Integer Programming*, volume 1 of *Annals of Discrete Mathematics*, pages 185–204. Elsevier, 1977.

[8] Paulo Feofiloff and Daniel H. Younger. Directed cut transversal packing for source-sink connected graphs. *Combinatorica*, 7(3):255–263, oct 1987.

[9] James F. Geelen and Bertrand Guenin. Packing odd circuits in eulerian graphs. *Journal of Combinatorial Theory, Series B*, 86(2):280–295, 2002.

[10] László Lovász. Minimax theorems for hypergraphs. In *Hypergraph Seminar*, Lecture Notes in Mathematics, pages 111–126. Springer Berlin Heidelberg, Berlin, Heidelberg, 2006.

[11] Cláudio L. Lucchesi and Daniel H. Younger. A minimax theorem for directed graphs. *Journal of the London Mathematical Society*, s2-17(3):369–374, 1978.

[12] András Mészáros. Note a note on disjoint dijoins. *Combinatorica*, 38(6):1485–1488, dec 2018.

[13] Alexander Schrijver. A counterexample to a conjecture of Edmonds and Giles. *Discrete Math.*, 32(2):213–214, jan 1980.

[14] Alexander Schrijver. Min-max relations for directed graphs. In *Bonn Workshop on Combinatorial Optimization*, volume 66 of *North-Holland Mathematics Studies*, pages 261–280. North-Holland, 1982.

[15] Alexander Schrijver. Theory of linear and integer programming. In *Wiley-Interscience series in discrete mathematics and optimization*, 1999.

[16] Alexander Schrijver. *Combinatorial Optimization - Polyhedra and Efficiency*. Springer, 2003.

[17] Paul D. Seymour. On Multi-Colourings of Cubic Graphs, and Conjectures of Fulkerson and Tutte. *Proceedings of the London Mathematical Society*, s3-38(3):423–460, 05 1979.

[18] Mohit Singh and Lap Chi Lau. Approximating minimum bounded degree spanning trees to within one of optimal. *Journal of the ACM*, 62(1):1–19, 2015.

[19] Douglas R. Woodall. Menger and könig systems. In *Theory and Applications of Graphs*, volume 642 of *Springer-Verlag Lecture notes in Mathematics*, pages 620–635, 1978.

Glossary of Notation

$\delta(U)$, $\delta_D(U)$	the cut corresponding U, a subset of vertices of D
$\delta^+(U)$	the set of outgoing arcs for a subset of vertices U
$\delta^-(U)$	the set of incoming arcs for a subset of vertices U
$[n]$	the set $\{1, 2, \ldots, n\}$
$\mathbb{Z}_+^S$	a vector of non-negative integers with each entry corresponding to an element in set S
$\mathcal{J}(D)$	the clutter of dijoins of digraph D
$\mathcal{C}(D)$	the clutter of dicuts of digraph D
$\tau(\mathcal{C}, w)$	the minimum weight cover of clutter $\mathcal{C}$
$\tau(D, w)$	the shorthand for $\tau(\mathcal{J}(D), w)$
$\nu(\mathcal{C}, w)$	the maximum size of a w-weighted packing of clutter $\mathcal{C}$
$\nu(D, w)$	the shorthand or $\nu(\mathcal{J}(D), w)$
$A - B$	the set containing elements in A but not in B
A_1, $A_1(D, w)$	the weight 1 arcs of (D, w), used when $w \in \{0, 1\}^A$
$A_{w>0}$	the arcs of a weighted digraph with positive weight
$A[U]$	the arcs of a subgraph induced by subset of vertices U
$D[U]$	the shorthand for $(U, A[U])$
$w[U]$	the weight of remaining arcs in the subgraph induced by U
$(D, w)[U]$	the shorthand for $(D[U], w[U])$

D/U	the digraph after contracting the subset of vertices U into a single vertex
w/U	the weight of arcs remaining after contracting the subgraph induced by U
$(D,w)/U$	the shorthand for $(D/U, w/U)$
$f(\tau)$	the maximum packing number of weighted digraphs with minimum weight dicut at least τ
$f(\tau,\varepsilon)$	the maximum packing number of ε-balanced weighted digraphs with minimum weight dicut at least τ
$\mathrm{Cross}_{(D,w)}(\mathcal{P})$	the weight of crossing arcs of partition $\mathcal{P}$
$\mathrm{def}_{(D,w)}(\mathcal{P},k)$	the k-deficiency of $\mathcal{P}$, i.e. $\mathrm{def}_{(D,w)}(\mathcal{P},k) = \mathrm{Cross}_{(D,w)}(\mathcal{P}) - k(\lvert\mathcal{P}\rvert - 1)$
$\mathrm{inter}(U,\mathcal{F})$	the number of sets in $\mathcal{F}$ with a non-empty intersection with U, but does not contain U and is not contained in U
$\phi(U)$	the value of $w(\delta^+(U)) - w(\delta^-(U))$
$\phi_J(U)$	the value of $w(\delta^+(U)\cap J) - w(\delta^-(U)\cap J)$
$\rho(D,w)$	the value of $\frac{1}{\tau(D,w)}\sum_{v\in V}[w(\delta^+(v)) - w(\delta^-(v)) \mod \tau(D,w)]$
$disc(U)$	the number of sinks minus the number of sources
$a(U)$	the set of active vertices in U
$s(U)$	the set of sources in U
$t(U)$	the set of sinks in U
$\mathcal{U}^i$	the set of shores of dicuts with discrepancy i
$\mathcal{U}^i_{min}$	the set of minimal shores of dicuts with discrepancy i, i.e. the minimal sets of $\mathcal{U}^i$
$\mathcal{A}^i$	the set $\{a(U) : U \in \mathcal{U}^i_{min}\}$
$\overline{\mathcal{A}^i}$	the complement of $\mathcal{A}^i$ in the active vertices, i.e. $\{a(V) - U : U \in \mathcal{A}^i\}$
$K_{\rho=1}(D,w)$	the family of all singletons in D
$K_{\rho=2}(D,w)$	the family $\{\{v\} : v \in V\} \cup \{a(V) - U : U \in \mathcal{U}^0_{min}\}$

$\mathcal{K}(D,w)$	the set of families of dicut shores of (D,w), such that every family can be described by $\{\{v\} : v \in V\} \cup \bigcup_{0\le i\le \rho-2} \mathcal{S}_i$, where $\mathcal{S}_i$ is either $\mathcal{A}^i$ or $\overline{\mathcal{A}^i}$