Higher Gauge Theory and Discrete Geometry

by

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A thesis
presented to the University of Waterloo
in fulfillment of the
thesis requirement for the degree of
Doctor of Philosophy
in
Applied Mathematics

Waterloo, Ontario, Canada, 2023

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Author's Declaration

This thesis consists of material all of which I authored or co-authored: see Statement of Contributions included in the thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Statement of Contributions

This thesis contains material adapted from [5, 61, 58, 55, 59].

Chapter 2 is based on [55], co-authored by Marc Geiller, Christophe Goeller, Florian Girelli, and Peter Tsimiklis. The chapter also contains some work, including figures, from [59] which is co-authored by Florian Girelli, Matteo Laudonio, and Peter Tsimiklis.

Chapter 3 is based off of [61] and [59] the first of which is coauthored by Florian Girelli and Peter Tsimiklis.

Chapter 4 contains work presented in [5], co-authored by Seth Asante, Bianca Dittrich, Florian Girelli, Aldo Riello, and Peter Tsimiklis, and work presented in [58], which is co-authored by Florian Girelli, Matteo Laundonio, Adrian Tanasa, and Peter Tsimiklis.

Abstract

In four dimensions, gravity can be seen as a constrained topological model. This provides a natural way to construct quantum gravity models, since topological models are relatively straightforward to quantize. Difficulty arises in the implementation of the constraints at the quantum level. Different procedures have generated so-called spin foam models.

Following the dimensional/categorical ladder, the natural structure to quantize 4d topological models are 2-categories, augmenting the gauge group symmetries of the model into 2-group symmetries. One can study these models classically by examining their phase space. At the quantum level, one attempts to construct a partition function. As there are no local degrees of freedom in topological theories, it is convenient to characterize its phase space in terms of a discretization, providing insights to the quantum theory. A key question is understanding how these topological models defined in terms of 2-categories can be related to gravity.

A first hint that 2-categories are relevant to describe quantum gravity models comes when we introduce a cosmological constant in the theory. As we will recall, this can be done at the classical (discrete) level in a consistent manner, only if we use 2-group symmetries.

This thesis focuses on understanding the symmetry aspects, the different possible discretizations and the quantization of four dimensional topological theories for some skeletal 2-group symmetries.When we discuss the quantum aspects using some field theory techniques to generate the quantum amplitudes, we extend the construction to non-skeletal 2-groups.

Acknowledgements

I would like to thank my supervisor Florian Girelli for his support. I am grateful not only for his scientific guidance and perspective, but for his enthusiasm and humour as well. This thesis could not have been written without his patience.

I'd also like to thank Adrian Tanasa for the many meetings and his insight.

I am grateful to have been part of the University of Waterloo community for as long as I have. I am happy to have had the chance to both learn and teach here.

I am also lucky to have been able to attend many meetings, seminars, and winter retreats with the quantum gravity group at the Perimeter Institute, where I learned more than I knew there was.

I am especially glad to have met so many wonderful people in these communities.

I am thankful to all the collaborators who made the works presented in this thesis possible.

I'd like to thank Costa, for being my brother (even though that wasn't up to him).

And finally, I'd like to thank my parents (and Hunter) for putting up with me for most of a global pandemic.

Dedication

To my family

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Chapter 1

Introduction

Spacetime: The final frontier.

The two most impressive and unassailable breakthroughs in physics of the 20th century were the theory of relativity and quantum theory. Both these theories provide myriad applications. Relativity has given us predictions for the existence of black holes, GPS technology, the theory of gravitational waves with applications in astronomy, and more. Quantum mechanics on the other hand has allowed for advances in such varied fields as materials physics and condensed matter, computing, the standard model and particle physics, to name just a few.

Aside from their many applications, these two theories have sparked the imaginations of everyone from physicists to philosophers to science-fiction writers to quantum-snake-oil salespeople. The reason for this ubiquity lies in the fact that once the technical aspects and the unreasonably effective mathematical descriptions are removed, the theory of relativity and the theory of quantum mechanics reveal to us a description of reality so far from our experience. At the core, general relativity strives to define the nature of space and time while quantum theory describes the building blocks of matter, while raising questions about determinism. Perhaps more striking than all these esoteric questions of space and time is the fact that both these theories have withstood generations of experimental tests.

There is some tension between the two theories, however. While each theory is robust and falsifiable on their own and in their domain of applicability, the assumptions of one appears to be in contradiction with the results of the other. What's worse is that these contradictions appear at scales for which there are no direct experiments available.

For example, the assumption that space and time are part of one continuous manifold is central to general relativity. On that manifold, the dynamical field is the metric g, which determines lengths of trajectories, volumes of regions, as well as other geometrical features of the manifold such as its curvature. The geometry of the manifold as described by g is responsible for what one observes as gravitation. Quantum mechanics on the other hand insists that all dynamical variables must be probabilistic. It might be that the metric g ought to be replaced by some quantum operator which has some expectation value. Such an operator would have fluctuations and physics at the order of these fluctuations would necessarily depend on the quantum nature of gravity. It is presumed that these quantum fluctuations become important at the order of the Planck length, $\ell_P = \sqrt{\frac{G\hbar}{c^3}} \approx 10^{-35}\,\mathrm{m}$. What's certain is the quantum effects are not directly perceptible on scales which have been measured [68, 103].

In a theory where the metric is replaced by some operator whose dynamics is governed by probabilities, the question of what happens to the underlying manifold arises. What replaces the mantra of "spacetime is a manifold and its geometry is what we call gravity" when the thing that describes geometry is a quantum object?

General relativity predicts its own incompleteness in a way. The starting assumption is that spacetime should be a smooth manifold but solutions to Einstein's equations, which govern the dynamics of the metric, might include geometries which are not consistent with smooth manifolds. Examples include the singularity in the Schwartzchild black hole or the singularity in expanding universe solutions in cosmology. These singularities are in a sense point-like and therefore may be resolved by considering very small scales – which would require quantum gravity.

Quantum theory also seems inconsistent with general relativity. There are several problems with treating gravity as one would treat any other classical theory to be quantized. In standard quantum theory (without gravity), the quantum fields are free to interact on a fixed spacetime. One cannot treat a quantum gravity field in the same way, since that field *is* the spacetime! Standard quantum theory also relies on things like time-ordering of operators or a specified time coordinate in order to normalize quantum states. This poses a problem since the choice of a time coordinate is already a spacetime dependent notion. More generally, quantum fields are typically local in the sense that they are functions of points in spacetime. General relativity is diffeomorphism invariant meaning that all observables should be non local [99].

Finally, we cannot treat gravity as a perturbation of some flat background manifold either because general relativity is nonrenormalizable [62]. The usual methods for managing the divergences of a quantum theory do not hold for gravity when the energy scale is large, equivalently when the spatial scale is small.

All these apparent contradictions tell us there is some work to do to before coming to a consistent theory of quantum gravity. Some of this work is conceptual and some is technical. Work on quantum gravity started almost as soon as quantum theory and general relativity were invented (see [91] for a timeline and historical references).

This work's focus is not on quantum gravity, except motivationally. The focus is on a family of theories which are called topological. As we will see soon, the reason for this focus comes from the fact that general relativity is *almost* topological – but not quite (at least not in our four dimensional spacetime). These theories have the property of being diffeomorphism invariant while also having no local observables making the quantization process more manageable.

As there are no local observables in these theories, the physically meaningful quantities are so called topological invariants: Quantities defined by the spacetime manifold that do not change under continuous transformations of the manifold. The construction of invariants of three dimensional manifolds may be algebraically described in terms of the category of group representations. For four dimensional manifolds, the invariants are suspected to be given by 2-categories, according to the categorical/dimensional ladder [37, 71]. As such, the theories studied in this work will exhibit 2-group symmetries, which are a higher categorification of the groups (which we describe in detail in chapter 2).

The topological theories with 2-group symmetries serve as a tool for constructing *discrete geometries*. These geometries act as replacements for the smooth manifold of space(time) and are given by graphs or 2-complexes which have algebraic data imposed on each edge and faces.

Topological Models and Gravity

We started this chapter by discussing gravity, quantum theory, and why a quantum theory of gravity may be necessary. But this thesis works with topological theories, which do not include gravity. At this stage I should give some details and technical information about why topological theories are worth studying and how they are used in quantum gravity models.

Let's start by writing the Lagrangian for general relativity as well as the equations of motion in the vacuum without a cosmological constant

$$L = \int \sqrt{-g}\, R \, \mathrm{d}^d x \tag{1.1}$$

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 0 \tag{1.2}$$

where $R_{\mu\nu}$ is the Ricci curvature tensor and R is its trace (and the overall constants are set to 1). g is the determinant of the metric. Taking the trace of Einstein's equations tells us that solutions to these equations are metrics which have $R_{\mu\nu} = 0$. Three dimensional general relativity has a special property: the Riemann tensor is linear in $R_{\mu\nu}$, meaning that when $R_{\mu\nu}$ vanishes, the manifold is locally flat as well (the Riemann tensor also vanishes). In the presence of a cosmological constant, the local curvature is constant. In either case, there are no local degrees of freedom and two particles in the manifold would not feel a gravitational attraction at all [38]. The lack of local degrees of freedom is the typical feature of topological theories.

The quantization of this model is most naturally accomplished in the first order formalism [104, 32, 7, 1]. Instead of taking the metric $g_{\mu\nu}$ as the dynamic variable of the theory, we introduce new fields called triads which define local frames and satisfy

$$e^I_\mu e^J_\nu \eta_{IJ} = g_{\mu\nu} \tag{1.3}$$

where I and J are indices which take values $0, 1, 2$. The second ingredient is the spin connection ω which is assumed to be independent of e for the sake of the variational principle, and is a Lie valued differential form. The action for three dimensional gravity becomes

$$L = \int e^I \wedge F_I \tag{1.4}$$

where $F_I = \mathrm{d}\omega_I + \frac{1}{2}\epsilon_I{}^{JK}\omega_J \wedge \omega_K$ is the curvature of the spin connection.

Theories with Lagrangians that look like (1.4) are called BF theories [64]. BF theories and their generalizations are the focus of much of this work. One can write a BF theory in any dimension by defining the action

$$S = \int_M \langle B \wedge F \rangle \tag{1.5}$$

where M is a d-dimensional manifold, B is a $(d-2)$-form, and F is the curvature of a connection A,

$$F = \mathrm{d}A + \frac{1}{2}[A, A]. \tag{1.6}$$

The basic fields A and B are forms which take value the Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$ respectively and the $\langle \cdot, \cdot \rangle : \mathfrak{g}_1 \otimes \mathfrak{g}_2 \to \mathbb{R}$ is a bilinear form that is invariant under some Lie algebra action.

In four dimensions (where we apparently live) gravity is not topological. This fact does not render BF theories in four dimensions useless. There are several ways to obtain gravity in four (or higher) dimensions by constraining or modifying topological BF-type actions in order to recover gravity [33, 50]. In the tetrad (the four dimensional version of triads) formulation, the action of four dimensional general relativity is

$$S = \int \epsilon_{IJKL}(e^I \wedge e^J \wedge F^{KL} + \Lambda e^I \wedge e^J \wedge e^K \wedge e^L) \tag{1.7}$$

where Λ is the cosmological constant. The action (1.7) would almost be a BF theory (when $\Lambda = 0$) if only we could treat $\epsilon_{IJKL}e^I \wedge e^J$ as a single field B_{KL}. One therefore recovers gravity by starting with a BF theory and constraining B to be related to some 1-forms e in this way.

For instance the Plebanski action [83] is a BF-type action based on the group $SL(2, \mathbb{C})$ which is equivalent to general relativity. It is given by the action, B is now a 2-form,

$$\int B_i \wedge F^i - \frac{1}{2}(\Psi_{ij} + \Lambda\delta_{ij})B^i \wedge B^j - \rho \mathrm{Tr}\Psi \tag{1.8}$$

where Λ is the cosmological constant, Ψ is a symmetric matrix and ρ is a 4-form. These auxiliary fields act as Lagrange multipliers. Eliminating ρ via the equations of motion implies that B can be written in terms of a set of 1-forms e^I,

$$B^i = ie^0 \wedge e^i - \frac{1}{2}\epsilon^i{}_{jk}e^j \wedge e^k. \tag{1.9}$$

If some additional reality conditions on B (in the Lorentzian case) are met, then one can construct the so-called Urbantke metric [31, 101]

$$\sqrt{|g|}g_{\mu\nu} \propto \epsilon_{ijk}\epsilon^{\alpha\beta\gamma\delta}B^i_{\mu\alpha}B^j_{\beta\gamma}B^k_{\delta\nu}. \tag{1.10}$$

Putting (1.9) into the Urbantke metric gives that e^I can be interpreted as tetrad fields (up to a conformal factor), $g_{\mu\nu} = \eta_{IJ}e^I_\mu e^J_\nu$. Eliminating the Lagrange multipliers in the action recovers Einstein's gravity.

This connection between BF theory and gravity offers a promising path towards quantum gravity. Spin foam models, such as the Barrett-Crane model and the EPRL/FK models are examples of quantum gravity theories where a BF theory is quantized and then constrained [54, 49, 24]. In spin foams, the topological theory is quantized via the path integral formulation [30], which may be regularized using familiar methods of lattice gauge theory or controlled using effective field theory [6, 27]. The constraints are then imposed on the quantum level.

Higher Categories

As previously mentioned, the quantum picture of the topological theory with gauge group G involves decorated discrete geometries described in terms of some simplicial complex embedded in space(time). The spin foam can be interpreted as a sort of Feynman diagram for a quantum geometry, where the boundaries of the triangulated manifold are seen as the initial and final quantum states. This is a picture developed in the group field theory approach [28, 80].

In four dimensions, classical discrete geometry of these boundary states, representing a three dimensional slice is specified by holonomies on the 1-skeleton of the simplex and their flux variables which are associated to two dimensional surfaces dual to the 1-skeleton. The holonomy and flux variables act as the configuration and momentum variables of the classical theory. The holonomy variables in G serve the purpose of defining a parallel transport for objects which move along the links on which they are defined and are therefore valued in the gauge group G. The fluxes on the other hand are elementsin the dual (as a vector space) Lie algebra $\mathfrak{g}^*$ of G. The phase space is then the natural phase space associated to $T^*G \cong G \ltimes \mathfrak{g}^*$.

The quantum theory is constructed following the Dirac prescription for constrained systems [39]. The constraints of the theory represent geometrical information about the states. For example, the Gauss constraint imposes that the fluxes associated to surfaces which form a closed polyhedron sum to zero. The resulting states of the quantum theory are represented by graphs which are eigenvectors for some geometric operators such as the area operator. This is the spin network basis. For example if $G = SU(2)$, in three dimensions the spin-network states can be drawn as graphs where edges are labelled by positive half-integers j_e, encoding the discrete spectrum of possible areas of the (dual) surfaces.

To get a full picture of the quantum amplitudes, one needs to sum over all the possible labels in the spin foam. Doing so for a Lie group, such as $G = SU(2)$ will generally lead to a divergent sum and a regularisation process must be devised. In this specific example, a natural regularization occurs when using the quantum group $U_q(\mathfrak{su}(2))$ where q is a root of unity. Since such $U_q(SU(2)$ has a finite number of representations, considering this quantum group introduces a natural cutoff for the theory. This gauge quantum group typically appears in the presence of a nonzero positive cosmological constant in the Euclidean signature [78, 47].

In order to construct quantum gravity spin foams for homogeneously curved spaces, ie in the presence of a non-zero cosmological constant, one deforms the given spin foam model using a gauge quantum group [76, 63]. At the classical level, the introduction of a quantum group means that we should no longer consider T^*G as the phase space associated to the discrete geometry. Instead we deal with a Heisenberg double [2].

When the cosmological constant is zero, the fluxes, associated to the surfaces, are quantized as elements of $U(\mathfrak{su}(2))$, the enveloping algebra of $\mathfrak{su}(2)$. Such enveloping algebra can be seen as the quantization of the Poisson algebra of the functions over $\mathfrak{su}^*(2) \cong \mathbb{R}^3$. Indeed, in the classical discrete picture, we have elements of an abelian group $\mathbb{R}^3$, equipped with a non-trivial Poisson bracket, decorating the surfaces.

The deformation $U(\mathfrak{su}(2)) \to U_q(\mathfrak{su}(2))$ is equivalent to replacing the classical Poisson algebra of functions on $\mathbb{R}^3$ by the Poisson algebra of functions on the non-abelian group AN_2 [73].This means that to have quantum fluxes in $U_q(\mathfrak{su}(2))$ we must also have classical fluxes as group elements of AN_2 decorating the surfaces.

This poses an immediate problem, as there is no way to naturally compose such non-abelian surface decorations as clearly stated in the Eckmann-Hilton argument [48].

To explain the argument, consider what a holonomy does. If there is a path in the manifold M parameterized as $\gamma : [0, 1] \to M$, then the holonomy h_γ acts on tensors by transporting them from a tensor space at the point $\gamma(0)$ to a tensor space at $\gamma(1)$ in a consistent way. If there are two paths γ_1 and γ_2 with $\gamma_1(1) = \gamma_2(0)$ one can define a curve $\gamma_{1,2}$ which is the concatenation of the two, and one can define the holonomy $h_{\gamma_{1,2}} = h_{\gamma_1} h_{\gamma_2}$ in a consistent way. The crucial ingredient here is the group structure of the holonomies and the groupoid structure of the paths.

We can also compose variables on surfaces in a similar way if the corresponding surfaces share part of their boundary, at least when the variables are part of a commutative group (such as $\mathfrak{g}^*$ where the group operation is addition).Consider the situation in Fig. 1.1. The different surface fusion combinations only lead to a consistent result when the decorations are valued in an abelian group. If it is not the case, there is some arbitrariness in the way surfaces are composed.

A solution to the issue of including nonabelian surface variables is the notion of a 2-group. In essence, one must introduce additional holonomy variables on the boundary of the surface (not necessarily belonging to the same group as the holonomies already mentioned). To make things fully consistent, there must be some consistency relations between the surface variables and the boundary holonomy. In chapter 2, we will give details on the type of consistency relations are required.

The added decorations on the boundary of the faces allow for nonabelian flux variables, which might therefore correspond to spin foams which include cosmological constant. They

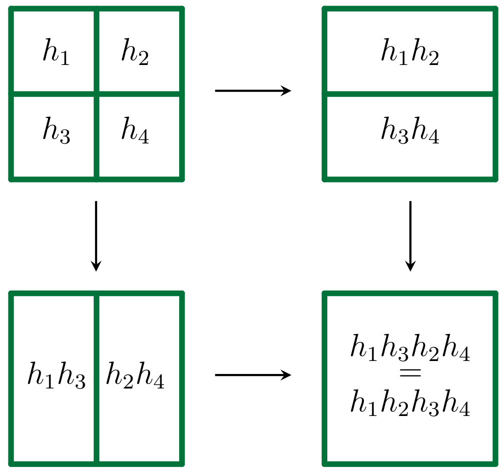

Figure 1.1: The Eckmann-Hilton argument shows that in order to compose faces so that the above diagram commutes, the face decorations must commute. This can be avoided by decorating *both* edges and faces by the considering 2-group elements.

are an extension (a vertical categorification) of gauge theory as well. In addition to the usual lattice gauge theory defined by holonomies, we think of the surface variables as 2-holonomies, which transport paths the same way the more familiar holonomies transport tensors.

In the same way that spin foams coming from Lie groups or quantum groups can be given as an expression involving the category of representations, it is suspected that spin foams coming from 2-groups can be given in terms of the 2-category of their 2-representations. 2-categories themselves have been proposed as tools for constructing topological invariants in four dimensions [15, 37, 71]. The higher categorification approach can be extended a step further by introducing 3-gauge theories which could provide a way to include matter coupled with gravity [85, 86].

In this spin foam picture, the fact that some of the fundamental variables decorate surfaces hints that 2-groups structures may be used in their description. The introduction of a quantum group structure for such face decoration to encode (at the quantum level) the presence of a non-zero cosmological constant indicates that 2-groups are actually needed to have a proper description of a homogeneously curved discrete geometry with face decorations, following the Eckmann-Hilton argument.

The goal in this thesis is to initiate an exploration of topological theories relying on

2-group symmetries by on one hand exploring their discretization and on the other hand by defining the associated quantum theory.

We will mostly focus on a specific class of 2-groups, namely the skeletal ones, where the face decorations are actually independent of the edge decorations. While it is not directly obvious that this class of 2-groups will be most relevant for quantum gravity, our work aims at developing techniques that, may be used to define some improved quantum gravity models.

Summary of Results and Outline

In chapter 2, we introduce a family of topological theories and study symmetries associated with them. This will lead into the definition of 2-groups and their use in defining 2-gauge theories. We will stress how these algebraic objects may be used to decorate 2-complexes.

Using the geometric interpretation of 2-groups as objects which decorate both surfaces and their boundary, in chapter 3, we construct the phase space for a three dimensional triangulation associated to a topological action. First we directly use the symplectic potential coming from the Euclidean 2-group BFCG action– a particularly simple 2-gauge theory – to define the discretized phase space. Then, by using the Heisenberg double of a group, a generalisation of the cotangent phase space T^*G, we are able to construct the phase space of a three dimensional triangulation for a more general class of (Poisson) 2-group than what we previously considered.

In chapter 4, we deal with the quantization of theories based on 2-groups. We use the group field theory formalism, which allows us to use Feynman diagrams to build spin foam transition amplitudes. These amplitudes act as projections which impose that the holonomy around a closed loop is given in terms of the face decoration and the composition of surfaces which make up a closed polyhedron is also trivial. Schematically the amplitude looks like

$$\prod_{\substack{l:\text{ loop}\\ f:\text{ face},\partial f=l}} \delta(H_l t(Y_f)) \prod_{p:\text{ polyhedron}} \delta(\prod_{\substack{f:face\\ f\in\partial p}} Y_f) \tag{1.11}$$

where H_l are holonomies around loops, Y_f are surface variables, and t is the boundary function or t-map, a critical component of the definition of a 2-group. The first product of deltas imposes that the holonomy H_l about the loop l is equal to $t(Y_f)$, the surface variable enclosed by the loop. The second product of delta functions are over the closed polyhedra.

The product of Y_f's have to be carefully considered: They must be arranged in such a way that they share boundaries so they can be composed. We are once again only able to do this consistently by using the properties of 2-groups.

Finally, a Hilbert space of states, up which these geometric constraints act, is constructed for the case of Euclidean 2-group BFCG. Upon these states which represent boundaries of the spin foam, we impose some geometric constraints, including part of the 1-flatness constraint (the first set of deltas in (1.11)) which allows us to connect the BFCG theory partition function with an existing topological invariant which uses 2-representations of the Euclidean 2-group [22], despite the absence of a Plancherel formula/Peter-Weyl theorem for 2-groups.

Chapter 2

Higher gauge and BF type theories

This chapter introduces two of the main tools used in this thesis: BF-type topological theories and 2-groups.

As mentioned in the introduction, topological theories such as the BF theory have been used to construct models of quantum gravity. They are also useful as tools for gaining insight into diffeomorphism invariant theories in general. Since there are no local degrees of freedom, the Noether charges of the theory, coming from the gauge transformations are concentrated on codimension-2 boundaries. From these boundary charges we reconstruct not only the diffeomorphism charges, but new independent "dual diffeomorphism charges". The construction of these charges hinges on a particular decomposition of the the gauge group.

At the end of section 2.1, we introduce a particular member of the family of BF-type theories, the BFCG model. This leads us to the definitions and tools used in 2-gauge theory in 2.2. These 2-gauge theories are generalizations of gauge theories defined on a lattice, as they involve holonomies along paths, and 2-holonomies on surfaces. The 2-holonomies act as a transport operator for extended objects like paths themselves.

2.1 Boundary charges in BF type theories

2.1.1 Boundary charges in BF theory

One uses boundaries in general relativity as a way to define physical charges such as energy and momenta. For example, the energy of an asymptotically flat spacetime may be defined as a boundary integral [66]. Asymptotic boundaries also allow one to consider approximate symmetries which only become exact at infinity giving rise to the BMS group in asymptotically flat spaces for example [94, 25]. The boundary charges in the classical theory might also give insight into the quantization of the theory as is the case in AdS/CFT or celestial holography [29]. Recently, there has been interest in studying charges which have support on co-dimension 2 boundaries which are not located at asymptotic infinity. Such local holographic theories may grant insight into quantizing diffeomorphism invariant theories by considering the entanglement areas between subsystems [43, 44, 98].

The family of theories studied in this section is meant to be a(n incomplete) generalisation of the Mielke-Baekler action, which in three dimensions is the most general diffeomorphism and Lorentz invariant theory in a triad and a connection variable [10, 77]. The analogous analysis for the Mielke-Baekler action has been done in [56].

The prototypical action is the four dimensional BF-theory with quadratic potential

$$S = \int_M \langle \mathcal{B} \wedge \mathcal{F} \rangle - \kappa \langle \mathcal{B} \wedge \mathcal{B} \rangle \tag{2.1}$$

where here the pairing $\langle \cdot, \cdot \rangle$ is taken to be the Killing form on the Lie algebra $\mathfrak{g}$ of some Lie group G, to which $\mathcal{B}$ and the connection $\mathcal{A}$ belong, $\kappa \in \mathbb{R}$ is a coupling parameter and $\mathcal{F} = \mathrm{d}\mathcal{A} + \frac{1}{2}[\mathcal{A}, \mathcal{A}]$. Our analysis follows that of [55]. The equations of motion for the theory, obtained by varying $\mathcal{A}$ and $\mathcal{B}$ are

$$\mathrm{d}_{\mathcal{A}}\mathcal{B} \equiv \mathrm{d}\mathcal{B} + [\mathcal{A} \wedge \mathcal{B}] \approx 0, \qquad \mathcal{F} - 2\kappa\mathcal{B} \approx 0, \tag{2.2}$$

where $\approx$ will be used to indicate the equality holds on-shell. Furthermore, the variation in the connection leads to a boundary term

$$\delta S \approx \int_{\partial M} \langle \mathcal{B} \wedge \delta\mathcal{A} \rangle. \tag{2.3}$$

The boundary term $\theta = \langle \mathcal{B} \wedge \delta\mathcal{A} \rangle$ is called the symplectic potential. From θ the symplectic form can be defined on a three dimensional submanifold M_3,

$$\Omega = \int_{M_3} \delta\theta = \int_{M_3} \langle \delta\mathcal{B} \wedge \delta\mathcal{A} \rangle. \tag{2.4}$$

The action exhibits two types of symmetries: First is the gauge transformations which are parameterized by $g \in G$:

$$\mathcal{A} \to g^{-1}\mathcal{A}g + g^{-1}\mathrm{d}g, \qquad \mathcal{B} \to g^{-1}\mathcal{B}g. \tag{2.5}$$

(written above in the adjoint representation). The second symmetry is the shift symmetry parameterized by a $\mathfrak{g}$-valued 1-form ϕ:

$$\mathcal{A} \to \mathcal{A} + 2\kappa\phi, \qquad \mathcal{B} \to \mathcal{B} + \mathrm{d}_{\mathcal{A}}\phi. \tag{2.6}$$

These symmetries can be linearized to obtain the infinitesimal transformations:

$$\begin{aligned} \delta^{\mathcal{J}}_{\alpha}\mathcal{B} &= [\mathcal{B}, \alpha], \qquad & \delta^{\mathcal{J}}_{\alpha}\mathcal{A} &= \mathrm{d}_{\mathcal{A}}\alpha \\ \delta^{\mathcal{T}}_{\phi}\mathcal{B} &= \mathrm{d}_{\mathcal{A}}\phi, \qquad & \delta^{\mathcal{T}}_{\phi}\mathcal{A} &= 2\kappa\phi \end{aligned} \tag{2.7}$$

where the infiniesimal version of the gauge transformations are given by a $\mathfrak{g}$ valued function α. These symmetries form the following algebra

$$\left[\delta^{\mathcal{J}}_{\alpha}, \delta^{\mathcal{T}}_{\phi}\right] = \delta^{\mathcal{T}}_{[\alpha,\phi]}, \qquad \left[\delta^{\mathcal{J}}_{\alpha}, \delta^{\mathcal{J}}_{\beta}\right] = \delta^{\mathcal{J}}_{[\alpha,\beta]}, \qquad \left[\delta^{\mathcal{T}}_{\phi}, \delta^{\mathcal{T}}_{\chi}\right] = 0. \tag{2.8}$$

Note that since $\delta^{\mathcal{T}}_{\phi}\mathcal{A}$ contains neither $\mathcal{A}$ nor $\mathcal{B}$, it is not a surprise that the shift sector of the algebra commutes for any value of κ.

With these symmetries and the symplectic potential defined, we can compute the charges corresponding to these symmetries or the corresponding Hamiltonian functions. In general the charge Q corresponding to a symmetry is defined by $\delta^{Q} \lrcorner \Omega = -\not{\delta} Q$ where $\not{\delta}$ indicates that in general, the right hand side is not a total variation, as will be the case when we look at field dependent transformations. The corresponding charges of the transformations in (2.7) are

$$\begin{aligned} \not{\delta}\mathcal{J}(\alpha) &= -\int_{M_3} \langle \delta^{\mathcal{J}}_{\alpha}\mathcal{B} \wedge \delta\mathcal{A}\rangle - \langle \delta\mathcal{B} \wedge \delta^{\mathcal{J}}_{\alpha}\mathcal{A}\rangle \\ &= -\int_{M_3} \langle \alpha \wedge \delta \mathrm{d}_{\mathcal{A}}\mathcal{B}\rangle + \oint_S \langle \delta\mathcal{B} \wedge \alpha\rangle \\ &\approx \oint_S \langle \delta\mathcal{B} \wedge \alpha\rangle \end{aligned} \tag{2.9}$$

and similarly

$$\not{\delta}\mathcal{T}(\phi) \approx -\oint_S \langle \phi \wedge \delta\mathcal{A}\rangle, \tag{2.10}$$

where S is the boundary of the 3-manifold M_3. If α and ϕ are field independent (if $\delta\alpha = \delta\phi = 0$), then we can read off

$$\mathcal{J}(\alpha) = \oint_S \langle \mathcal{B} \wedge \alpha\rangle \qquad \mathcal{T}(\phi) = -\oint_S \langle \phi \wedge \mathcal{A}\rangle. \tag{2.11}$$

The Poisson algebra of these charges again comes from the symplectic form and is defined by $\{Q_1, Q_2\} = -\delta^{Q_1} \lrcorner \delta^{Q_2} \lrcorner \Omega$. The algebra arising from the charges in (2.10) is then

$$\begin{aligned}
\{\mathcal{J}(\alpha), \mathcal{T}(\phi)\} &= \mathcal{T}([\alpha, \phi]) - \oint_S \langle \alpha, \mathrm{d}\phi \rangle, \\
\{\mathcal{J}(\alpha), \mathcal{J}(\beta)\} &= \mathcal{J}([\alpha, \beta]), \\
\{\mathcal{T}(\phi), \mathcal{T}(\chi)\} &= 2\kappa \oint_S \langle \phi \wedge \chi \rangle.
\end{aligned} \tag{2.12}$$

These brackets are similar to those in (2.8), with central extensions.

This theory, being topological, is diffeomorphism invariant. In fact diffeomorphisms, which at the infinitesimal level are given by Lie derivatives, can be expressed as a combination of the symmetries of (2.7), *on shell*. The slight complication is that the parameters (α and ϕ in (2.7)) must be field dependant. The diffeomorphism generated by the vector field ξ on the basic fields may be written as

$$\begin{aligned}
\delta^{\mathcal{D}}_\xi \mathcal{A} =& \delta^{\mathcal{T}}_{\xi \lrcorner \mathcal{B}} \mathcal{A} + \delta^{\mathcal{J}}_{\xi \lrcorner \mathcal{A}} \mathcal{A} = \mathrm{d}_{\mathcal{A}}(\xi \lrcorner \mathcal{A}) + 2\kappa \xi \lrcorner \mathcal{B} \\
=& \mathcal{L}_\xi \mathcal{A} - \xi \lrcorner (\mathcal{F} - 2\kappa \mathcal{B}) \approx \mathcal{L}_\xi \mathcal{A} \\
\delta^{\mathcal{D}}_\xi \mathcal{B} =& \delta^{\mathcal{T}}_{\xi \lrcorner \mathcal{B}} \mathcal{B} + \delta^{\mathcal{J}}_{xi \lrcorner \mathcal{A}} \mathcal{B} = [\mathcal{B}, \xi \lrcorner \mathcal{A}] + \mathrm{d}_{\mathcal{A}}(\xi \lrcorner \mathcal{B}) \\
=& \mathcal{L}_\xi \mathcal{B} - \xi \lrcorner \mathrm{d}_{\mathcal{A}} \mathcal{B} \approx \mathcal{L}_\xi \mathcal{B}
\end{aligned} \tag{2.13}$$

where the identity $\mathcal{L}_\xi X = \xi \lrcorner \mathrm{d}X + \mathrm{d}(\xi \lrcorner X)$ is used. If we take ξ to be tangential to M_3, the charge associated to the corresponding diffeomorphism symmetry is

$$\mathcal{D}(\xi) = \mathcal{T}(\xi \lrcorner \mathcal{B}) + \mathcal{J}(\xi \lrcorner \mathcal{A}). \tag{2.14}$$

If we are able to decompose the Lie algebra in a certain way, one may find a new set of diffeomorphism charges which we call dual diffeomorphisms. As an example, consider the Lie algebra with generators P_i and J_i with $i \in \{1, 2, 3\}$ satisfying the brackets

$$[J_i, P_j] = \epsilon_{ij}{}^k P_k, \qquad [J_i, J_j] = \epsilon_{ij}{}^k J_k, \qquad [P_i, P_j] = \Lambda \epsilon_{ij}{}^k J_k. \tag{2.15}$$

The basic fields of the theory $\mathcal{A}$ and $\mathcal{B}$ could then be decomposed as

$$\mathcal{A} = A^i J_i + C^i P_i \qquad \mathcal{B} = B^i J_i + \Sigma^i P_i, \tag{2.16}$$

where A, C, B, and Σ are the projection of $\mathcal{A}$ or $\mathcal{B}$ into the relevant subspace. For such algebras, we consider two possible invariant forms

$$\begin{aligned}
&\langle P_i, J_j \rangle_1 = 0, \qquad &\langle P_i, P_j \rangle_1 = \Lambda \delta_{ij}, \qquad &\langle J_i, J_j \rangle_1 = \delta_{ij}, \\
&\langle P_i, J_j \rangle_2 = \delta_{ij}, \qquad &\langle P_i, P_j \rangle_2 = 0, \qquad &\langle J_i, J_j \rangle_2 = 0.
\end{aligned} \tag{2.17}$$

Writing the Lagrangian (2.1) using the first pairing gives

$$L_+ = B^i \wedge F[A]_i + \frac{1}{2}B^i \wedge [C,C]_i + \Lambda\Sigma^i \wedge \mathrm{d}_A C_i - \kappa B^i \wedge B_i - \kappa\Lambda\Sigma^i \wedge \Sigma_i \tag{2.18}$$

while using the second pairing gives

$$L_- = B^i \wedge \mathrm{d}_A C_i + \Sigma^i \wedge F[A]_i - 2\kappa B^i \wedge \Sigma_i + \frac{1}{2}\Sigma^i \wedge [C,C]_i. \tag{2.19}$$

One can also decompose the generators of charges in the same way. For example, if we write $\alpha = \alpha_1^i J_i + \alpha_2^i P_i$ then in the first pairing, the charge $\mathcal{J}(\alpha)$ may be decomposed as

$$\mathcal{J}(\alpha) = \oint_S \alpha_1^i B_i + \Lambda\alpha_2^i \Sigma_i \equiv \mathcal{J}_1(\alpha_1) + \mathcal{J}_2(\alpha_2). \tag{2.20}$$

We can see the J and P components of α as associated with their own charges. The transformations corresponding to these charges can be obtained by restricting the transformations of (2.7) to either the J or P components only. There is also the possibility of swapping the role of the J and P components. At the level of the pairings, this means that regardless of which action we choose in the Lagrangian, using either $\langle\cdot,\cdot\rangle_1$ or $\langle\cdot,\cdot\rangle_2$ in the definition of the charges will both be valid. The freedom to swap the roles of the J and P components and to arrive at valid charges is what allows us to define a new type of diffeomorphism-like charge as we see in the next section.

2.1.2 A more general Lagrangian

To study L_+, L_-, and combinations of the two we introduce a new Lagrangian which is parametrized by nine parameters σ_i:

$$\begin{aligned} L =&(\sigma_1 B^i + \sigma_2\Sigma^i) \wedge F_i + (\sigma_3 B^i + \sigma_4\Sigma^i) \wedge \mathrm{d}_A C_i + \frac{\sigma_5}{2}B^i \wedge B_i + \frac{\sigma_6}{2}\Sigma^i \wedge \Sigma_i \\ &+ \sigma_7 B^i \wedge \Sigma_i + (\frac{\sigma_8}{2}B^i + \frac{\sigma_9}{2}\Sigma^i) \wedge [C,C]_i. \end{aligned} \tag{2.21}$$

Setting $\sigma_1 = \sigma_8 = 1$, $\sigma_4 = \Lambda$, $\sigma_5 = -2\kappa$, and $\sigma_6 = -2\kappa\Lambda$ (and all others to zero) recovers L_+ while setting $\sigma_2 = \sigma_3 = \sigma_9 = 1$ and $\sigma_7 = -2\kappa$ (with the rest set to zero) recovers L_-.

In order to save space, we introduce notation for common combinations of the σ's,

$$[ijkl] = \frac{\sigma_i\sigma_j - \sigma_k\sigma_l}{\sigma_1\sigma_4 - \sigma_2\sigma_3} \tag{2.22}$$

and set[1]

$$\begin{array}{lllll} p_1 = [3745], & p_2 = [2517], & p_3 = [3428], & p_4 = [1833] & r_1 = [4839], \\ q_1 = [3647], & q_2 = [2716], & q_3 = [4429], & q_4 = [1934] & r_2 = [1928]. \end{array} \tag{2.23}$$

In order for these to be well defined, we demand that $\sigma_1\sigma_4 - \sigma_2\sigma_3 \neq 0$. With this in mind, the equations of motion are

$$F^i = p_1 B^i + q_1 \Sigma^i - \frac{r_1}{2}[C, C]^i \tag{2.24a}$$

$$\mathrm{d}_A C^i = p_2 B^i + q_2 \Sigma^i - \frac{r_2}{2}[C, C]^i \tag{2.24b}$$

$$\mathrm{d}_A B^i = p_3 [B, C]^i + q_3 [\Sigma, C]^i \tag{2.24c}$$

$$\mathrm{d}_A \Sigma^i = p_4 [B, C]^i + q_3 [\Sigma, C]^i \ , \tag{2.24d}$$

Once again, the symplectic potential comes from the boundary term of the variation,

$$\theta = (\sigma_1 B^i + \sigma_2 \Sigma^i) \wedge \delta A_i + (\sigma_3 B^i + \sigma_4 \Sigma^i) \wedge \delta C_i \ . \tag{2.25}$$

If we assume the additional constraint,

$$\sigma_1[7968] + \sigma_2[7859] + \sigma_3[3647] + \sigma_4[4537] = 0 \ , \tag{2.26}$$

the Lagrangian (2.21) is also invariant under two additional symmetries, generated by a Lie algebra value 0-form and 1-form. Denoting the Lie algebra value 0-forms in the J and P decomposition by (α, χ) and 1-form by (ϕ, τ), the fields transform according to Table 2.1.

As before we can compute the charges of the above transformations using the symplectic form. These are

$$\delta\mathcal{J}_1(\alpha) = -\int_{M_3} \alpha_i \ \delta\mathrm{EOM}_A{}^i + \oint_S \alpha^i(\sigma_1 \delta B_i + \sigma_2 \delta\Sigma_i) \ , \tag{2.27a}$$

$$\delta\mathcal{J}_2(\chi) = -\int_{M_3} \chi_i \ \delta\mathrm{EOM}_C{}^i + \oint_S \chi^i(\sigma_3 \delta B_i + \sigma_4 \delta\Sigma_i) \ , \tag{2.27b}$$

$$\delta\mathcal{T}_1(\phi) = -\int_{M_3} \phi_i \wedge \delta\mathrm{EOM}_B{}^i + \oint_S (\sigma_1 \delta A_i + \sigma_3 \delta C_i) \wedge \phi^i \ , \tag{2.27c}$$

$$\delta\mathcal{T}_2(\tau) = -\int_{M_3} \tau_i \wedge \delta\mathrm{EOM}_\Sigma{}^i + \oint_S (\sigma_2 \delta A_i + \sigma_4 \delta C_i) \wedge \tau^i \ . \tag{2.27d}$$

[1] For reference, to recover L_+, we have $p_1 = q_2 = 2\kappa$, $p_4 = \frac{1}{q_3} = \Lambda$ and $r_1 = 1$. The parameters for L_- are the same except for $p_4 = \frac{1}{q_3} = -1$.

0-form transformations in J:	0-form transformation in P:
$\delta^{J_1}_\alpha A = \mathrm{d}_A\alpha$ $\delta^{J_1}_\alpha C = [C,\alpha]$ $\delta^{J_1}_\alpha B = [B,\alpha]$ $\delta^{J_1}_\alpha \Sigma = [\Sigma,\alpha]$	$\delta^{J_2}_\chi A = r_1[C,\chi]$ $\delta^{J_2}_\chi C = \mathrm{d}_A\chi + r_2[C,\chi]$ $\delta^{J_2}_\chi B = p_3[B,\chi] + q_3[\Sigma,\chi]$ $\delta^{J_2}_\chi \Sigma = p_4[B,\chi] + q_4[\Sigma,\chi]$
1-form shift in J:	1-form shift in P:
$\delta^{T_1}_\phi A = p_1\phi$ $\delta^{T_1}_\phi C = p_2\phi$ $\delta^{T_1}_\phi B = \mathrm{d}_A\phi + p_3[C,\phi]$ $\delta^{T_1}_\phi \Sigma = p_4[C,\phi]$	$\delta^{T_2}_\tau A = q_1\tau$ $\delta^{T_2}_\tau C = q_2\tau$ $\delta^{T_2}_\tau B = q_3[C,\tau]$ $\delta^{T_2}_\tau \Sigma = \mathrm{d}_A\tau + q_4[C,\tau]$

Table 2.1: The infinitesimal symmetries of the Lagrangian (2.21) sorted by the gauge symmetry (top two blocks) and the shift symmetry (bottom two blocks), as well as generators' projections into the J subspace (left two blocks) or the P subspace (right two blocks).

where EOM_X refers to the equations of motion obtained by varying the field X, hence $\mathrm{EOM}_X \approx 0$. As expected for gauge symmetries, on-shell, the charges have support on the boundary S. The Poisson algebra of these charges is

$$\{\mathcal{J}_1(\alpha),\mathcal{J}_1(\alpha')\} = \mathcal{J}_1([\alpha,\alpha']) \; , \tag{2.28a}$$
$$\{\mathcal{J}_1(\alpha),\mathcal{J}_2(\chi)\} = \mathcal{J}_2([\alpha,\chi]) \; , \tag{2.28b}$$
$$\{\mathcal{J}_2(\chi),\mathcal{J}_2(\chi')\} = r_1\mathcal{J}_1([\chi,\chi']) + r_2\mathcal{J}_2([\chi,\chi']) \; , \tag{2.28c}$$
$$\{\mathcal{J}_1(\alpha),\mathcal{T}_1(\phi)\} = \mathcal{T}_1([\alpha,\phi]) - \sigma_1\oint_S \alpha\mathrm{d}\phi \; , \tag{2.28d}$$
$$\{\mathcal{J}_1(\alpha),\mathcal{T}_2(\tau)\} = \mathcal{T}_2([\alpha,\tau]) - \sigma_2\oint_S \alpha\mathrm{d}\tau \; , \tag{2.28e}$$
$$\{\mathcal{J}_2(\chi),\mathcal{T}_1(\phi)\} = p_3\mathcal{T}_1([\chi,\phi]) + p_4\mathcal{T}_2([\chi,\phi]) - \sigma_3\oint_S \chi\mathrm{d}\phi \; , \tag{2.28f}$$
$$\{\mathcal{J}_2(\chi),\mathcal{T}_2(\tau)\} = q_3\mathcal{T}_1([\chi,\tau]) + q_4\mathcal{T}_2([\chi,\tau]) - \sigma_4\oint_S \chi\mathrm{d}\tau \; , \tag{2.28g}$$
$$\{\mathcal{T}_1(\phi),\mathcal{T}_1(\phi')\} = -\sigma_5\oint_S \phi\wedge\phi', \tag{2.28h}$$
$$\{\mathcal{T}_2(\tau),\mathcal{T}_2(\tau')\} = -\sigma_6\oint_S \tau\wedge\tau' \; , \tag{2.28i}$$
$$\{\mathcal{T}_1(\phi),\mathcal{T}_2(\tau)\} = -\sigma_7\oint_S \phi\wedge\tau. \tag{2.28j}$$

The generators $(\mathcal{T}_1, \mathcal{T}_2)$ span a centrally extended abelian current algebra while the generators $(\mathcal{J}_1, \mathcal{J}_2)$ span a non-centrally extended $\mathfrak{so}$-like current algebra. Indeed, defining the new generator $\tilde{\mathcal{J}}_2 = \mathcal{J}_2 - \frac{r_2}{2}\mathcal{J}_1$ leads to

$$\{\mathcal{J}_1(\alpha), \mathcal{J}_1(\alpha')\} = \mathcal{J}_1([\alpha, \alpha']) \ , \tag{2.29a}$$

$$\{\mathcal{J}_1(\alpha), \tilde{\mathcal{J}}_2(\chi)\} = \tilde{\mathcal{J}}_2([\alpha, \chi]) \ , \tag{2.29b}$$

$$\{\tilde{\mathcal{J}}_2(\chi), \tilde{\mathcal{J}}_2(\chi')\} = -\lambda \mathcal{J}_1([\chi, \chi']) \ , \tag{2.29c}$$

with $\lambda = -(r_1 + \frac{r_2^2}{4})$.

Just as the BF action, (2.21) is diffeomorphism invariant. Defining the diffeomorphism symmetry, generated by a parameter vector field ξ, as the action of the Lie derivative $\delta^d_\xi(\cdot) = \mathcal{L}_\xi(\cdot)$, we find that the transformation can be expressed in terms of gauge transformations with field dependent gauge transformation (up to the equations of motion)

$$\delta^d_\xi A = \left(\delta^{J_1}_{\xi\lrcorner A} + \delta^{J_2}_{\xi\lrcorner C} + \delta^{T_1}_{\xi\lrcorner B} + \delta^{T_2}_{\xi\lrcorner\Sigma}\right) A + \xi\lrcorner(F + \frac{r_1}{2}[C, C] - p_1 B - q_1 \Sigma) \ , \tag{2.30a}$$

$$\delta^d_\xi C = \left(\delta^{J_1}_{\xi\lrcorner A} + \delta^{J_2}_{\xi\lrcorner C} + \delta^{T_1}_{\xi\lrcorner B} + \delta^{T_2}_{\xi\lrcorner\Sigma}\right) C + \xi\lrcorner(d_A C + \frac{r_2}{2}[C, C] - p_2 B - q_2 \Sigma) \ , \tag{2.30b}$$

$$\delta^d_\xi B = \left(\delta^{J_1}_{\xi\lrcorner A} + \delta^{J_2}_{\xi\lrcorner C} + \delta^{T_1}_{\xi\lrcorner B} + \delta^{T_2}_{\xi\lrcorner\Sigma}\right) B + \xi\lrcorner(d_A B - p_3[B, C] - q_3[\Sigma, C]) \ , \tag{2.30c}$$

$$\delta^d_\xi \Sigma = \left(\delta^{J_1}_{\xi\lrcorner A} + \delta^{J_2}_{\xi\lrcorner C} + \delta^{T_1}_{\xi\lrcorner B} + \delta^{T_2}_{\xi\lrcorner\Sigma}\right) \Sigma + \xi\lrcorner(d_A \Sigma - p_1[B, C] - q_1[\Sigma, C]) \ . \tag{2.30d}$$

When taken on shell, the second half of the above expressions (the part that is contracted with ξ) vanishes. There are at least two ways of determining the associated charges. We can of course go back to the symplectic potential, and contract it with the δ^d_ξ or we can use the fact that the diffeomorphism are just field dependent gauge transformation and so the charges will be a linear combination of the other gauge transformations. Using this, we obtain

$$\not{\delta}\mathcal{D}(\xi) = \not{\delta}\mathcal{J}_1(\xi\lrcorner A) + \not{\delta}\mathcal{J}_2(\xi\lrcorner C) + \not{\delta}\mathcal{T}_1(\xi\lrcorner B) + \not{\delta}\mathcal{T}_2(\xi\lrcorner\Sigma)$$

where again the notation $\not{\delta}$ emphasizes the fact that the field-space form obtained is not necessarily integrable since the gauge charges are only integrable for some field independent gauge parameter

$$\not{\delta}\mathcal{J}_1(\xi\lrcorner A) = \oint_S \xi\lrcorner A \wedge (\sigma_1 \delta B + \sigma_2 \delta\Sigma) \not\Rightarrow \mathcal{J}_1(\xi\lrcorner A) = \oint_S \xi\lrcorner A \wedge (\sigma_1 B + \sigma_2 \Sigma) \ . \tag{2.31}$$

In order to actually obtain a charge associated to the diffeomorphism, we could either impose boundary conditions to the field to render the field-space form integrable, or we could consider ξ to be tangent to both M_3 and its boundary S. In that case, the non-integrable part does not contribute, and we are left with the diffeomorphism charge

$$\mathcal{D}(\xi) = \mathcal{J}_1(\xi \lrcorner A) + \mathcal{J}_2(\xi \lrcorner C) + \mathcal{T}_1(\xi \lrcorner B) + \mathcal{T}_2(\xi \lrcorner \Sigma) \ . \tag{2.32}$$

Similarly to what was done in three dimensions in [56], it happens that the diffeomorphism is not the only combination of field dependent symmetries that is exact. There is an additional quadratic charge that is integrable with tangent parameter ξ that forms a stable algebra with the gauge charges, which we denote $\mathcal{C}(\xi)$

$$\mathcal{C}(\xi) = \mathcal{J}_1(\xi \lrcorner (r_1 C - p_3 A)) + \mathcal{J}_2(\xi \lrcorner (q_4 C + A)) + q_3 \mathcal{T}_1(\xi \lrcorner \Sigma) + \mathcal{T}_2(\xi \lrcorner (p_4 B + (q_4 - p_3)\Sigma)) \ . \tag{2.33}$$

We refer the reader to the Appendix A for the derivation of the charge.

For completeness, note that the symmetry associated to the charge $\mathcal{C}(\xi)$ can be written as

$$\delta^{\mathcal{C}}_\xi = \delta^{\mathcal{J}_1}_{\xi \lrcorner (-p_3 A + r_1 C)} + \delta^{\mathcal{J}_2}_{\xi \lrcorner (A + q_4 C)} + q_3 \delta^{\mathcal{T}_1}_{\xi \lrcorner \Sigma} + \delta^{\mathcal{T}_2}_{\xi \lrcorner (p_4 B + (q_4 - p_3)\Sigma)} \tag{2.34}$$

giving the infinitesimal transformations

$$\delta^{\mathcal{C}}_\xi A^i = - p_3 \mathcal{L}_\xi A^i + r_1 \mathcal{L}_\xi C^i \tag{2.35}$$
$$\delta^{\mathcal{C}}_\xi C^i = \mathcal{L}_\xi A^i + q_4 \mathcal{L}_\xi C^i \tag{2.36}$$
$$\delta^{\mathcal{C}}_\xi B^i = q_3 \mathcal{L}_\xi \Sigma^i \tag{2.37}$$
$$\delta^{\mathcal{C}}_\xi \Sigma^i = p_4 \mathcal{L}_\xi B^i + (q_4 - p_3) \mathcal{L}_\xi \Sigma^i \tag{2.38}$$

With these transformations, we can determine the Poisson algebra of the charges. The brackets are

$$\{\mathcal{D}(\xi), \mathcal{D}(\zeta)\} = - \mathcal{D}([\xi, \zeta]) \tag{2.39}$$
$$\{\mathcal{D}(\xi), \mathcal{C}(\zeta)\} = - \mathcal{C}([\xi, \zeta]) \tag{2.40}$$
$$\{\mathcal{C}(\xi), \mathcal{C}(\zeta)\} = (p_3 - q_4)\mathcal{C}([\xi, \zeta]) - p_4 q_3 \mathcal{D}([\xi, \zeta]). \tag{2.41}$$

We do a change of variables to have a nice algebra, which also allows us to define the "dual diffeomorphisms" as

$$\mathcal{D}^*(\xi) = C(\xi) + \frac{1}{2}(p_3 - q_4)\mathcal{D}(\xi). \tag{2.42}$$

The algebra now reads

$$\{\mathcal{D}^*(\xi), \mathcal{D}^*(\zeta)\} = \lambda\mathcal{D}([\xi,\zeta]) \tag{2.43}$$
$$\{\mathcal{D}(\xi), \mathcal{D}^*(\zeta)\} = -\mathcal{D}^*([\xi,\zeta]) \tag{2.44}$$
$$\{\mathcal{D}(\xi), \mathcal{D}(\zeta)\} = -\mathcal{D}([\xi,\zeta]). \tag{2.45}$$

where again $\lambda = -(r_1 + \frac{1}{4}r_2^2)$. This algebra is similar to the one met in the 3d case [56]. In the special case where parameters are chosen so that (2.21) is L_+, we have that $\mathcal{D}^*(\xi) = \mathcal{C}(\xi) = \mathcal{J}_1(\xi \lrcorner C) + \mathcal{J}_2(\xi \lrcorner A) + \frac{1}{\lambda}\mathcal{T}_1(\xi \lrcorner \Sigma) + \lambda\mathcal{T}_2(\xi \lrcorner B)$. The diffeomorphism and dual diffeomorphism charges are related by swapping the roles of A with C and B with $\frac{1}{\Lambda}\Sigma$.

2.1.3 Self-dual formulation

In the previous subsection, it was shown that there is a second set of diffeomorphism-like charges which arise in the general Lagrangian (2.21). To get a grip on what these two types of charges are, we take a specific algebra, $\mathfrak{so}(4)$ and use the direct sum basis, writing $\mathfrak{so}(4) = \mathfrak{su}(2) \oplus \mathfrak{su}(2)$ (a similar construction can be made for $\mathfrak{so}(3,1)$ where the decomposition would then be $\mathfrak{su}(2) \oplus i\,\mathfrak{su}(2)$). We denote the generators by σ_i^A, where $A = 1, 2$ denotes the term in the direct product and use the pairing $\langle \sigma_i^A \sigma_j^B \rangle = 2\delta^{AB}\delta_{ij}$. Explicitly, the basis vectors are

$$\sigma_i^1 = J_i + \sqrt{\lambda}P_i \qquad \sigma_i^2 = J_i - \sqrt{\lambda}P_i \qquad [\sigma_i^a, \sigma_j^a] = 2\delta^{ab}\epsilon_{ij}{}^k\sigma_k^a. \tag{2.46}$$

The fields can be decomposed as

$$\mathcal{A} = A_1^i\sigma_i^1 + A_2^i\sigma_i^2, \tag{2.47a}$$

$$\mathcal{B} = B_1^i\sigma_i^1 + B_2^i\sigma_i^2. \tag{2.47b}$$

The BF Lagrangian in the first pairing of (2.17) becomes

$$L_+ = \langle \mathcal{B} \wedge \mathcal{F} \rangle_1 - \kappa \langle \mathcal{B} \wedge \mathcal{B} \rangle_1 = 2(L_1 + L_2) \tag{2.48}$$
$$L_A \equiv \langle B_A \wedge F_A \rangle - \kappa \langle B_A \wedge B_A \rangle, \qquad A = 1, 2, \tag{2.49}$$

which is essentially a pair of uncoupled BF actions. If the other pairing is chosen we obtain

$$L_- = \langle \mathcal{B} \wedge \mathcal{F} \rangle_2 - \kappa \langle \mathcal{B} \wedge \mathcal{B} \rangle_2 = 2(L_1 - L_2). \tag{2.50}$$

$\tau \in \mathfrak{so}(4)$	$\tau_1 \in \mathfrak{su}^1(2)$	$\tau_2 \in \mathfrak{su}^2(2)$
$\delta^{\mathcal{J}}_{\tau}\mathcal{A} = \mathrm{d}_{\mathcal{A}}\tau$	$\delta^{\mathcal{J}_1}_{\tau_1}A_1 = \mathrm{d}_{A_1}\tau_1$ $\delta^{\mathcal{J}_1}_{\tau_1}A_2 = 0$	$\delta^{\mathcal{J}_2}_{\tau_2}A_2 = 0$ $\delta^{\mathcal{J}_2}_{\tau_2}A_2 = \mathrm{d}_{A_2}\tau_2$
$\delta^{\mathcal{J}}_{\tau}\mathcal{B} = [\mathcal{B}, \tau]$	$\delta^{\mathcal{J}_1}_{\tau_1}B_1 = [B_1, \tau_1]$ $\delta^{\mathcal{J}_1}_{\tau_1}B_2 = 0$	$\delta^{\mathcal{J}_2}_{\tau_2}B_1 = 0$ $\delta^{\mathcal{J}_2}_{\tau_2}B_2 = [B_2, \tau_2]$

$\psi \in \mathfrak{so}(4)$	$\psi^1 \in \mathfrak{su}^1(2)$	$\psi^2 \in \mathfrak{su}^2(2)$
$\delta^{\mathcal{T}}_{\psi}\mathcal{A} = 2\alpha\psi$	$\delta^{\mathcal{T}_1}_{\psi_1}A_1 = 2\alpha\psi_1$ $\delta^{\mathcal{T}_1}_{\psi_1}A_2 = 0$	$\delta^{\mathcal{T}_2}_{\psi_2}A_1 = 0$ $\delta^{\mathcal{T}_2}_{\psi_2}A_2 = 2\alpha\psi_2$
$\delta^{\mathcal{T}}_{\psi}\mathcal{B} = \mathrm{d}_{\mathcal{A}}\psi$	$\delta^{\mathcal{T}_1}_{\psi_1}B_1 = \mathrm{d}_{A_1}\psi_1$ $\delta^{\mathcal{T}_1}_{\psi_1}B_2 = 0$	$\delta^{\mathcal{T}_2}_{\psi_2}B_1 = 0$ $\delta^{\mathcal{T}_2}_{\psi_2}B_2 = \mathrm{d}_{A_2}\psi_2$

Table 2.2: Symmetries in the self-dual basis. The Lie algebra function and 1-form τ and ψ are broken into their components τ^i and ϕ^i, each in their own subspace.

The symmetries for these Lagrangians parameterized by 1-form ψ and function τ are given in Table 2.2 as well as the transformations arising by restricting the parameters to a single term of the direct sum.

Together with the usual charges,

$$\begin{aligned} \oint\mathcal{J}^1(\tau_1) = -\delta^{\mathcal{J}_1}_{\tau_1}\lrcorner\Omega = \int_S \langle \delta B_1 \wedge \tau_1\rangle, \quad \oint\mathcal{J}^2(\tau_2) = -\delta^{\mathcal{J}_2}_{\tau_2}\lrcorner\Omega = \int_S \langle \delta B_2 \wedge \tau_2\rangle \\ \oint\mathcal{T}^1(\psi_1) = -\delta^{\mathcal{T}_1}_{\psi_1}\lrcorner\Omega = \int_S \langle \delta A_1 \wedge \psi_1\rangle, \quad \oint\mathcal{T}^2(\psi_2) = -\delta^{\mathcal{T}_2}_{\psi_2}\lrcorner\Omega = \int_S \langle \delta A_2 \wedge \psi_2\rangle, \end{aligned} \tag{2.51}$$

when the generators are field independent, the charges form an algebra, obtained from the symplectic structure, as before. The algebra is

$$
\begin{aligned}
\{\mathcal{J}_1(\tau_1), \mathcal{J}_1(\tau_1')\} =& \mathcal{J}_1([\tau_1, \tau_1']),\\
\{\mathcal{J}_1(\tau_1), \mathcal{T}_1(\psi_1)\} =& \mathcal{T}_1([\tau_1, \psi_1]) - \oint_S \langle d\tau_1 \wedge \psi^1 \rangle,\\
\{\mathcal{J}_2(\tau_2), \mathcal{J}_2(\tau_2')\} =& \mathcal{J}_2([\tau_2, \tau_2']),\\
\{\mathcal{J}_2(\tau_2), \mathcal{T}_2(\psi_2)\} =& \mathcal{T}_1([\tau_2, \psi_2]) - \oint_S \langle d\tau_2 \wedge \psi_2 \rangle,\\
\{\mathcal{T}_1(\psi_1), \mathcal{T}_1(\psi_1')\} =& -\alpha_0 \oint_S 2\langle \psi_1 \wedge \psi_1' \rangle\\
\{\mathcal{T}_2(\psi_2), \mathcal{T}_2(\psi_2')\} =& -\alpha_0 \oint_S 2\langle \psi_2 \wedge \psi_2' \rangle
\end{aligned}
\tag{2.52}
$$

Each sector of the algebra comes with its own diffeomorphism symmetry,

$$
\oint \mathcal{D}_a(\xi) = -\delta_\xi^{\mathcal{D}_a} \lrcorner \Omega = \oint \mathcal{J}_a(\xi \lrcorner A_a) + \oint \mathcal{T}_a(\xi \lrcorner B_a). \tag{2.53}
$$

The two diffeomorphism charges give the algebra of two commuting diff(S^2) algebras,

$$
\{\mathcal{D}_1(\xi), \mathcal{D}_1(\zeta)\} = -\mathcal{D}_1([\xi, \zeta]), \tag{2.54a}
$$
$$
\{\mathcal{D}_2(\xi), \mathcal{D}_2(\zeta)\} = -\mathcal{D}_2([\xi, \zeta]), \tag{2.54b}
$$
$$
\{\mathcal{D}_1(\xi), \mathcal{D}_2(\zeta)\} = 0. \tag{2.54c}
$$

Of course, the sum of the two diffeomorphism charges will be the charge which generates diffeomorphisms on the total $\mathfrak{so}(4)$ fields $\mathcal{B}$ and $\mathcal{A}$. We can also consider the difference of the two diffeomorphism charges, which we can identify as the dual diffeomorphism charges. If we denote

$$
\mathcal{D}^+(\xi) = \mathcal{D}_1(\xi) + \mathcal{D}_2(\xi), \quad \mathcal{D}^-(\xi) = \mathcal{D}_1(\xi) - \mathcal{D}_2(\xi), \tag{2.55}
$$

the closed algebra is

$$
\{\mathcal{D}^+(\xi), \mathcal{D}^+(\zeta)\} = -\mathcal{D}^+([\xi, \zeta]) \ , \tag{2.56a}
$$
$$
\{\mathcal{D}^-(\xi), \mathcal{D}^-(\zeta)\} = -\mathcal{D}^+([\xi, \zeta]) \ , \tag{2.56b}
$$
$$
\{\mathcal{D}^+(\xi), \mathcal{D}^-(\zeta)\} = -\mathcal{D}^-([\xi, \zeta]) \ . \tag{2.56c}
$$

Looking at the corresponding transformations we have

$$\delta_\xi^{\mathcal{D}^+} A_1 = \mathcal{L}_\xi A_1, \quad \delta_\xi^{\mathcal{D}^+} A_2 = \mathcal{L}_\xi A_2, \tag{2.57}$$

$$\delta_\xi^{\mathcal{D}^+} B_1 = \mathcal{L}_\xi B_1, \quad \delta_\xi^{\mathcal{D}^+} B_2 = \mathcal{L}_\xi B_2, \tag{2.58}$$

$$\delta_\xi^{\mathcal{D}^-} A_1 = \mathcal{L}_\xi A_1, \quad \delta_\xi^{\mathcal{D}^-} A_2 = -\mathcal{L}_\xi A_2 \tag{2.59}$$

$$\delta_\xi^{\mathcal{D}^-} B_1 = \mathcal{L}_\xi B_1, \quad \delta_\xi^{\mathcal{D}^-} B_2 = -\mathcal{L}_\xi B_2 \tag{2.60}$$

That is, the charge $\mathcal{D}^-(\xi)$ corresponds to a diffeomorphism in the first sector generated by ξ and a diffeomorphism generated by $-\xi$ in the second sector.

We can take this construction further by taking the algebra $\mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(2)$ for example. The decomposition of the connection and $\mathcal{B}$ field is similar to the $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ picture; $\mathcal{A} = \sum_{I=1}^3 A^{Ii}\sigma_i^I$ and $\mathcal{B} = \sum_{I=1}^3 B^{Ii}\sigma_i^I$. We can consider three independent pairings

$$\langle \sigma_i^1, \sigma_j^1 \rangle_{+,+} = \langle \sigma_i^2, \sigma_j^2 \rangle_{+,+} = \langle \sigma_i^3, \sigma_j^3 \rangle_{+,+} = \delta_{ij} \ , \tag{2.61a}$$

$$\langle \sigma_i^1, \sigma_j^1 \rangle_{-,+} = \langle \sigma_i^2, \sigma_j^2 \rangle_{-,+} = \delta_{ij} = -\langle \sigma_i^3, \sigma_j^3 \rangle_{-,+} \ , \tag{2.61b}$$

$$\langle \sigma_i^1, \sigma_j^1 \rangle_{+,-} = \langle \sigma_i^3, \sigma_j^3 \rangle_{+,-} = \delta_{ij} = -\langle \sigma_i^2, \sigma_j^2 \rangle_{+,-} \tag{2.61c}$$

and construct Lagrangians using linear combinations thereof.

If we denote diffeomorphism in the first, second, or third term in the direct sum by $\mathcal{D}^a$, we can form linear combinations which once again make for a closed Poisson algebra:

$$\mathcal{D}^{\pm,\pm}(\xi) = \mathcal{D}^1(\xi) \pm \mathcal{D}^2(\xi) \pm \mathcal{D}^3(\xi). \tag{2.62}$$

(The notation here is that the first (resp. second) superscript $\pm$ on the left hand side corresponds to the first (resp. second) $\pm$ on the right hand side.) These charges once again form a closed algebra,

$$\{\mathcal{D}^{\epsilon_1,\epsilon_2}(\xi), \mathcal{D}^{\epsilon_3,\epsilon_4}(\zeta)\} = -\mathcal{D}^{\epsilon_1\epsilon_3,\epsilon_2\epsilon_4}([\xi,\zeta]), \quad \epsilon_i = \pm 1. \tag{2.63}$$

The associated transformations are

$$\delta_\xi^{\mathcal{D}^{\epsilon_1,\epsilon_2}} A_1 = \mathcal{L}_\xi A_1, \quad \delta_\xi^{\mathcal{D}^{\epsilon_1,\epsilon_2}} A_2 = \mathcal{L}_{\epsilon_1\xi} A_a, \quad \delta_\xi^{\mathcal{D}^{\epsilon_1,\epsilon_2}} A_3 = \mathcal{L}_{\epsilon_2\xi} A_3, \tag{2.64}$$

$$\delta_\xi^{\mathcal{D}^{\epsilon_1,\epsilon_2}} B_1 = \mathcal{L}_\xi B_1, \quad \delta_\xi^{\mathcal{D}^{\epsilon_1,\epsilon_2}} B_2 = \mathcal{L}_{\epsilon_1\xi} B_a, \quad \delta_\xi^{\mathcal{D}^{\epsilon_1,\epsilon_2}} B_3 = \epsilon_2\mathcal{L}_{\epsilon_2\xi} B_3. \tag{2.65}$$

Once again, the "dual" charges act as generators for transformations which act differently in each sector of the algebra, either flowing in the direction of ξ or opposite ξ.

0-form transformations in J	0-form transformations in P	1-form shifts in J	1-form shifts in P
$\delta^{J_1}_\alpha A = \mathrm{d}_A \alpha$	$\delta^{J_2}_\chi A = 0$	$\delta^{T_1}_\phi A = 0$	$\delta^{T_2}_\tau A = 0$
$\delta^{J_1}_\alpha C = [C, \alpha]$	$\delta^{J_2}_\chi C = \mathrm{d}_A \chi$	$\delta^{T_1}_\phi C = 0$	$\delta^{T_2}_\tau C = 0$
$\delta^{J_1}_\alpha B = [B, \alpha]$	$\delta^{J_2}_\chi B = [\Sigma, \chi]$	$\delta^{T_1}_\phi B = \mathrm{d}_A \phi$	$\delta^{T_2}_\tau B = [C, \tau]$
$\delta^{J_1}_\alpha \Sigma = [\Sigma, \alpha]$	$\delta^{J_2}_\chi \Sigma = 0$	$\delta^{T_1}_\phi \Sigma = 0$	$\delta^{T_2}_\tau \Sigma = \mathrm{d}_A \tau$

Table 2.3: The symmetries corresponding to the action of (2.66), obtained by setting $\sigma_1 = \sigma_4 = 1$ and all others to zero in (2.21).

2.1.4 BFCG Theory

We now consider a specific type of action that arises in (2.21). With some choice of parameters in that action, we can recover a so-called 2-gauge theory.

Before looking into what 2-gauge theories are in general, let's consider an example. Taking $\sigma_1 = \sigma_4 = 1$ and all others to be zero in (2.21) gives

$$L = B \wedge F + \Sigma \wedge \mathrm{d}_A C. \tag{2.66}$$

For these specific values of σ_i, the symmetries are given in Table 2.3

In this example, we must assume that the subspaces containing C and Σ are subalgebras ($\Lambda = 0$). For example, in a later example, the Lie algebra will be the Poincaré algebra and C and Σ will be the translation subalgebra. It will be useful to rewrite the action (2.66) using a integration by parts to emphasize some symmetry. We can write

$$L = B \wedge F + C \wedge \mathrm{d}_A \Sigma. \tag{2.67}$$

The reason for this rewriting is that the quantity $\mathrm{d}_A \Sigma$ will play a role similar to a curvature, and so this writing is more symmetric. In addition, the integration by parts also introduces a boundary term, which contributes to the symplectic potential and alters what we think of as momentum and configuration variables. The fields B and C now appear as Lagrange multipliers enforcing flatness $F = 0$ and the the constraint $\mathrm{d}_A \Sigma = 0$. The transformations $\delta^{J_1}_\alpha$ appear as what is known as a 1-gauge transformation. In 2-gauge theory, Σ plays the role of a so-called 2-connection. We will expand on this later on, but for now we mean that as a connection transports an object from one point to another on a path, a 2-connection transports a path along a surface. The infinitesimal transformation $\delta^{T_2}_\tau$, is then called a 2-gauge transformation as it plays the role of making the transport properly gauge invariant.

This alternate way of viewing the symmetries in this example show how A and Σ can be made to relate to one another in a way that we will show is geometrically appealing. The 0- and 1-form translations $\delta_{\chi}^{J_2}$ and $\delta_{\phi}^{T_1}$ then play a similar role to the "shift" symmetry in BF theory. We are also able to see C and B themselves as being 1- and 2-connections. The view that what we have here is two related 2-gauge theories will be used in the discretization process.

In order to make what is discussed in this section more precise, we will describe 2-groups which will formalise the 2-holonomy picture in the next section. Then we will make clear the 2-gauge theory which comes out of an infinitesimal view of 2-groups, just as a Lie algebra can be derived from a Lie group.

2.2 2-Groups and 2-gauge theory

In this section, we will give the definition of a 2-group and describe the notion of a 2-gauge theory. These structures give meaning to the notion of 2-holonomies which transport extended objects just as the usual (1-)holonomies transport objects which are defined at a point. 2-holonomies can be used to extend usual concepts from lattice gauge theory.

Of critical importance for us is an algebraic structure called a strict 2-group, also known as a crossed module [16]. The rules which define the crossed module give us rules for composing surfaces and paths alike. In the next section, we give the definition of the infinitesimal version 2-groups, Lie 2-algebras which are used in the construction of 2-gauge theory.

The notion of a 2-group comes from the higher categorification of a group. We will not dwell for very long on the category theory point of view but will mention the basics for some perspective on where these objects come from, motivated by [16, 14].

First, a category consists of a collection of of objects which are typically drawn as dots, and a collection of morphisms or arrows between the objects. The morphisms are composable: If f is a morphism from objects A and B and g is a morphism from B and an object C, then there is a third morphism between A and C, the composition of f and g, which we denote $g \circ f$, or simply gf. The composition of morphisms should also be associative. In addition we demand that for each object A there is an identity morphism 1_A for which $f 1_A = f$ for any morphism f from A.

An important example of a category is a groupoid, which occurs if each of the morphisms have inverses. Another example would be a group, which is a groupoid where every pair of morphisms can be composed, therefore satisfying the axioms of group multiplication. A group, seen as a category is thought of as having a single object.

In gauge theory, the important category is the groupoid of paths on a manifold, in which each point is an object and the morphisms are paths between points. Concretely, one assigns a group element to each morphism. We often call these assignments “decorations”. These are the holonomies. The composition of paths relates to the multiplication of the group elements decorating the paths.

The holonomy on a path gives information on how a tensor changes as it travels along the path. The 2-holonomies we will define will give information about how an extended object changes as it sweeps out some two dimensional area.

From the categorical point of view, 2-categories are categories (objects and arrows between them) with arrows between arrows, which we call 2-morphisms. The rules for composing these arrows in a consistent way will be given shortly. Just as Lie groups allow us to decorate the paths in the category of paths on a manifold, 2-groups or crossed modules will let us do the same for the 2-category of paths and surfaces between paths on a manifold [18, 12, 13, 15].

2.2.1 Crossed modules

A crossed module consists of a pair of groups, G and H with a homomorphism $t : H \to G$ and an action $\alpha : G \times H \to H$ which we denote by $\alpha(g, h) = g \triangleright h$ for $g \in G$ and $h \in H$. The action and homomorphism must be satisfy compatibility relations,

$$t(g \triangleright h) = gt(h)g^{-1} \tag{2.68}$$

$$t(h) \triangleright h' = hh'h^{-1} \tag{2.69}$$

for all $h, h' \in H$ and $g \in G$. As we will see these compatibility conditions have geometric applications as well.

We can represent an element of the crossed module (h, g_1) by a bigon – a surface decorated by $h \in H$ having its boundary divided into two paths, both starting and ending at the same point. One path is decorated by $g_1 \in G$ and the other by an element given by $g_2 = t(h)g_1 \in G$. This element of the crossed module is shown diagrammatically in Fig. 2.1.

The crossed module can be equipped with two products which correspond to two ways of composing surfaces. The first product is called vertical multiplication. If $t(h_1) = g_2 g_1^{-1}$,

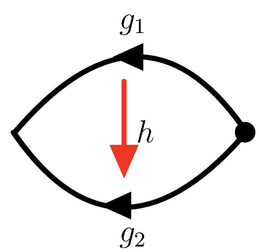

Figure 2.1: Graphical representation of (h, g_1) in the 2-group given by $(G, H, t, \triangleright)$, with $t(h) = g_2 g_1^{-1}$.

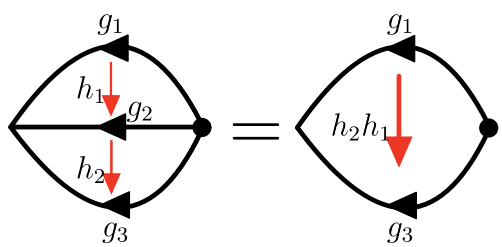

Figure 2.2: The diagrammatic depiction of the vertical composition $(h_2, g_2) \diamond (h_1, g_1) = (h_2 h_1, g_1)$. The condition $t(h_1) = g_2 g_1^{-1}$ is expressed graphically by requiring that the two surfaces being composed share an edge.

then we can define the vertical product as $(h_2, g_2) \diamond (h_1, g_1) = (h_2 h_1, g_1)$. Figure 2.2 gives a graphical representation in terms of paths and surfaces. The condition $t(h_1) = g_2 g_1^{-1}$ represents the condition that the two surfaces being composed share a boundary path. Now is a good time to acknowledge the fact that in defining the two paths on the boundary, we are specifying two points: a starting point for the paths, which we call the source, and an ending point which we call the target. We are free to change these points by a process called *whiskering*. This is where the the group action of G on H comes in. Let the original source point be at s on the boundary, and say that we would like to move the source to a new point s' also on the boundary. If the path along the boundary from s to s' is decorated by g, then we have the relation $(h, g_1)_s = (h', g_1 g'^{-1})_{s'}$, where the subscripts indicate the base point. The new surface variable, h' satisfies $t(h') = t(h)$, as we can see from Fig. 2.3.

In a similar vein, we can change the target point, the point on the boundary where the two paths g_1 and g_2 meet. Let the original target point be denoted by τ and the new

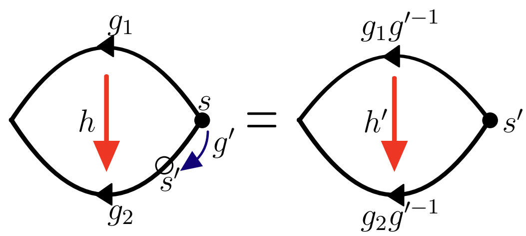

Figure 2.3: Diagrammatic representation of changing the source point of a surface from s to s', where the two sources are separated by g'.

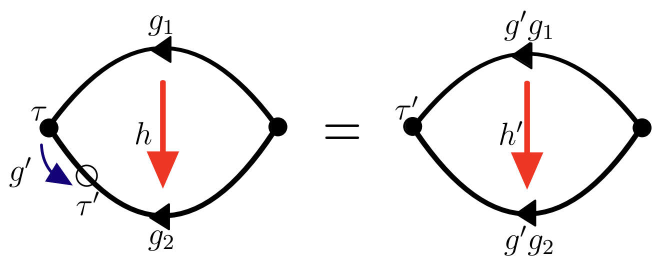

Figure 2.4: Changing the target point by a path g'. We have that $h' = g' \triangleright h$.

target point by τ', and say the path connecting τ to τ' is given by g'. Then the new crossed module element is given by $(h, g_1) \to (g' \triangleright h, g' g_1)$. This is shown in Fig. 2.4. Since the map t is calculated by taking the path around the loop starting at the target point, it is no surprise that the new surface variable is given by the action of g'.

In fact, the new target or source point need not be on the boundary of the surface. We can choose a new target or source which lies off the boundary altogether without changing the calculation. The whiskering process is shown in Fig. 2.5.

The second of the two products we consider is horizontal multiplication. If the source point on one surface is the target point of a second boundary, we may compose surfaces according to Fig. 2.6. The product is given by $(h_1, g_1) \bullet (h_2, g_2) = (h_1(g_1 \triangleright h_2), g_1 g_2)$. From this, we can see that whiskering is simply the horizontal product $(1, g') \bullet (h, g)$.

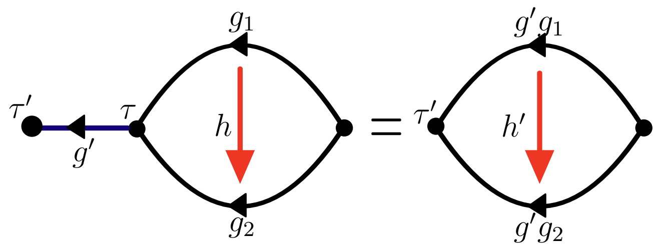

Figure 2.5: Whiskering on the target of the surface. Once again we have that $h' = g' \triangleright h$.

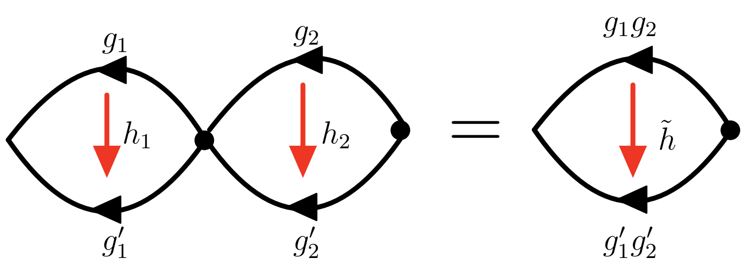

Figure 2.6: Horizontal product $(h_1, g_1) \bullet (h_2, g_2) = (\tilde{h}, g_1g_2)$, where $\tilde{h} = h_1(g_1 \triangleright h_2)$.

2.2.2 Skeletal crossed modules

A simple case which comes up often is the case where $t(h) = 1$ for all $h \in H$. We call these skeletal crossed modules. The compatibility equations for skeletal crossed modules imply that H is abelian. The geometric picture then implies that $t(h) = g_1 g_2^{-1} = 1$, meaning that the holonomy around the boundary is trivial. Let's now consider some simple examples to illustrate skeletal crossed modules.

- *Trivial crossed modules:* Consider the crossed module where $H = 1$ is the trivial group and G any group. In this case, both the action and map t are trivial. This 2-group is essentially a group. This is the usual framework for lattice gauge theory with G decorating paths and no information on the faces (or trivial group elements on faces).

  Alternatively, we may consider G to be the trivial group. Once again, this 2-group is simply a group H. Geometrically, in analogy with the lattice picture before, we have H decorations on plaquettes, but not on paths. The properties of the action and the map t in this case make it so that H has to be abelian in this case.

- *Poincaré and Euclidean 2-group:* The Poincaré (or Euclidean) group can be seen as examples of 2-groups, with H the group of translations and G the Lorentz (or rotation) group.

- *Co-adjoint 2-group:* Let G be a Lie group and $H = \mathfrak{g}^*$ be its dual Lie algebra, which we see as an abelian group. Then, G acts on $\mathfrak{g}^*$ via the co-adjoint action. With the t-map being trivial, $\mathfrak{g}^* \ltimes G$ can be viewed as a skeletal crossed module.

2.2.3 Gauge theory

Gauge theories arise in several physical contexts. As a tool to do calculations them, one can introduce a lattice. The fields of the theory are discretized – assigned value only on the vertices of the lattice, while the gauge variables, the connection, are concentrated on the paths between vertices. Regularizing the theory in this way allows for simplified computational methods and elucidates some geometric features of the theory [90].

Just as gauge theories are built from connections, which are Lie algebra valued and are used to define the holonomies which describe how fields are transported within the lattice, a 2-gauge theory must involve a 2-connection. The gauge symmetries of the 1-

and 2-connection are derived from geometrical considerations which we will review now [60, 57].

We start by introducing Lie 2-algebras [11] which we think of as infinitesimal versions of 2-groups.

A strict Lie 2-algebra is crossed module of Lie algebras, given in terms of a pair of Lie algebras with a Lie algebra homomorphism $\tau : \mathfrak{h} \to \mathfrak{g}$ and an action $\triangleright : \mathfrak{g} \times \mathfrak{h} \to \mathfrak{h}$ satisfying

$$[x_1, x_2] \triangleright y = x_1 \triangleright (x_2 \triangleright y) - x_2 \triangleright (x_1 \triangleright y) \qquad x \triangleright [y_1, y_2] = [x \triangleright y_1, y_2] + [y_1, x \triangleright y_2] \tag{2.70}$$

for all $x_i \in \mathfrak{g}$ and $y_i \in \mathfrak{h}$. The homomorphism and the action must be compatible in the sense that

$$\tau(x \triangleright y) = [x, \tau(y)] \qquad\qquad \tau(y_1) \triangleright y_2 = [y_1, y_2]. \tag{2.71}$$

A skeletal strict Lie 2-algebra is an infinitesimal crossed module $\mathfrak{g} \ltimes \mathfrak{h}$ with a trivial τ-map, $\tau = 0$, so that $\mathfrak{h}$ is abelian.

If G and H are Lie groups which together can form a 2-group, then their algebras, respectively $\mathfrak{g}$ and $\mathfrak{h}$ form a Lie 2-algebra. In that case, the Lie algebra homomorphism τ is the derivative of the t-map of the 2-group, $\tau = \mathrm{d}t$. The action $\triangleright$ is also the infinitesimal version of the 2-group action (which we also denoted $\triangleright$) in the sense that if the action is given by $\alpha : G \to \mathrm{Aut}(H)$, with $\alpha(g)(h) = g \triangleright h$, then $\mathrm{d}\alpha : \mathfrak{g} \to \mathrm{Der}(\mathfrak{h})$ is given by $\mathrm{d}\alpha(x)(y) = x \triangleright y$ [11].

An example of this follows from the Poincaré or Euclidean case we saw for 2-groups. Seeing $ISO(3)$ as a 2-group for example, ($G = SO(3)$ and $H = \mathbb{R}^3$), the corresponding Lie 2-algebra has $\mathfrak{g} = \mathfrak{so}(3)$ and $\mathfrak{h} = \mathbb{R}^3$. The t-map was the constant $t(g) = 1$, so the derivative is $\tau(x) = 0$ for all $x \in \mathfrak{so}(3)$. The action on $\mathbb{R}^3$ by $\mathfrak{so}(3)$ is given by the Lie bracket in $\mathfrak{iso}(3)$.

We can now introduce connections and 2-connections in this context. We recall that for non-abelian gauge theory on a lattice, when one performs a gauge transformation, the generators of the transformation are assigned to nodes of the lattice. For example, if parallel transport on a link between points x and y is given in a certain gauge by $g \in G$. It can be expressed in another gauge by the transformation

$$g' = \eta_y^{-1} g \eta_x \tag{2.72}$$

where η_x and η_y are elements of the gauge group and are associated to x and y respectively. The gauge transformation is depicted in the commuting diagram Fig. 2.7.

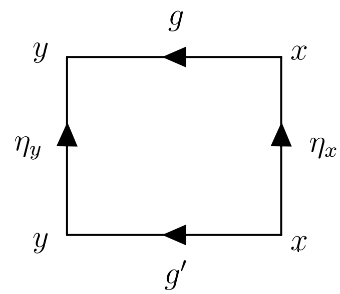

Figure 2.7: Commuting diagram showing the gauge transformations in a lattice gauge theory. The vertical lines represent gauge transformations at points x and y in the lattice.

As emphasized already, in 2-gauge lattice theories, the two-dimensional plaquettes are adorned with group elements as well. How do these plaquette decorations transform under the gauge transformations generated by η_x and η_y? For simplicity, assume that there are two paths between x and y in the lattice (x and y are corners of a bigon) which we denote by g and $\tilde{g}$. We denote the plaquette variable by $h \in H$. The gauge-transformed g and $\tilde{g}$ are given by (2.72). The situation is drawn in the "tin can" diagram in Fig. 2.8 with h' being the gauge transformed h variable.

We can view of Fig. 2.8 as a bunch of 2-group elements which may be simplified according to the whiskering and composition rules outlined in Section 2.2. To determine h' we demand that this simplification 2-commutes. In other words, we compose all the surfaces on the tin can and the result should be the identity.

Before proceeding with the simplification of the diagram, we note that so far there are two unlabelled faces on the tin can. This suggests a generalization to Fig. 2.7 and (2.72). Instead of demanding that the diagram commute, we can place $\rho \in H$ inside the plaquette instead, and impose that the bigon encodes a 2-group element in the sense of Section 2.2:

$$g' = \eta_y t(\rho) g \eta_x^{-1}. \tag{2.73}$$

The adjusted diagram for this generalized transformation is given in Fig. 2.9. We also add surface variables ρ and $\tilde{\rho}$ to the two faces on the tin can, where ρ is defined as having its source g and target $\eta_y g' \eta_x^{-1}$ (and similarly for $\tilde{\rho}$) as shown in Fig. 2.10. These labels ρ and $\tilde{\rho}$ will parameterize what we call *2-gauge transformations*.

Now that all surfaces of the tin can are properly labelled, we proceed to use the composition rules and whiskering to simplify the diagram. Explicitly the four bigons are written as (g, h), (g, ρ), (g', h'), and $(\tilde{g}, \tilde{\rho})$. To compose the first two bigons listed vertically, we

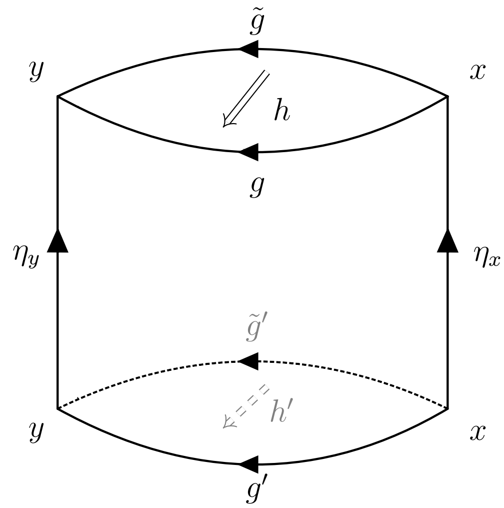

Figure 2.8: Tin can diagram showing how the 1-holonomies g and $\tilde{g}$ transform under gauge transformations and how the 2-holonomy h transforms under a gauge transformation.

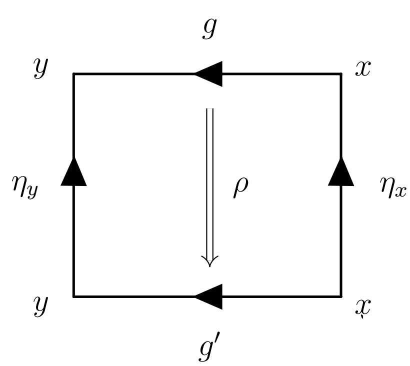

Figure 2.9: Adjusted diagram for gauge transformations including a 2-gauge variable ρ. The 2-gauge transformed variable is determined by setting the path around the loop equal to $t(\rho)$.

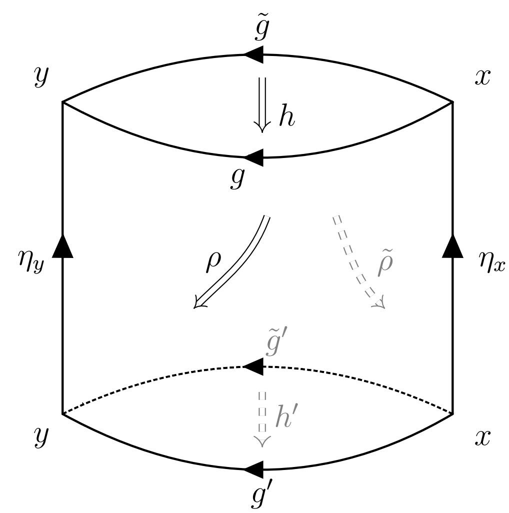

Figure 2.10: The tin can diagram showing both 1- and 2- gauge transformations. The top and bottom surfaces are decorated with h and h' respectively and the front and back with ρ and $\tilde{\rho}$. The primed variables are determined by ensuring that travelling along any loop gives t applied to the surface bounded by the loop and by demanding that composing all the surfaces gives the identity.

invert (g, ρ) to obtain $(\eta_y^{-1} g' \eta_x, \rho^{-1})$, which has g as the target and can therefore be combined with (g, h). The composed bigon has $\tilde{g}$ as the target, which fits with the source of $(\tilde{g}, \tilde{\rho})$ allowing another vertical composition. Finally, the (g', h') bigon must be whiskered and inverted before composing, giving $(\eta_y^{-1} \tilde{g}' \eta_x, \eta_y \triangleright h'^{-1})$. Putting all this together and demanding that the resulting surface variable be the identity we find

$$\rho^{-1} h \tilde{\rho} (\eta_y \triangleright^{-1} h'^{-1}) = 1 \tag{2.74}$$

or

$$h' = \eta_y \triangleright (\rho^{-1} h \tilde{\rho}). \tag{2.75}$$

Together, (2.73) and (2.75) provide the combined 1- and 2-gauge transformations in the lattice theory.

As we have discussed in the previous section, the action for gauge theories is given in terms of connections and their curvature, not in terms of holonomies. The same is true for 2-gauge theory. Let's now proceed to understand how the gauge and 2-gauge transformations look at the level of the connection and 2-connection.

The connection is an element of the Lie algebra of G defined by its properties under a gauge transformation. A holonomy along a path is given by the path ordered exponential $\mathcal{P} \exp \left(- \int A \right)$ where the integral is over the path and A is the connection. In order for the holonomy to transform according to (2.72) one requires that A transform according to

$$A' = \eta^{-1} A \eta + \eta^{-1} \mathrm{d}\eta. \tag{2.76}$$

It will be worth explaining how one gets this result, as the corresponding transformations of the 2-gauge theory are obtained in the same way. We proceed by assuming that the holonomy is over a path of length a, which is small. The holonomy can therefore be written as $\mathcal{P} \exp \left(e^{-aA} \right) = 1 - aA + \mathcal{O}(a^2)$. The gauge transformation (2.72) becomes then

$$\eta(0) \mathcal{P} \exp \left(- \int A \right) \eta(a)^{-1} = \mathcal{P} \exp \left(- \int A' \right) = \eta(0)\eta(a)^{-1} + a\eta(0) A \eta(a)^{-1}. \tag{2.77}$$

Here instead of writing the lattice site as a subscript, we denote by $\eta(0)$ the gauge transformation at the starting point of the path, and by $\eta(a)$ the ending point of the path. We assume that the function η itself is continuous. In particular, we may write $\eta(a) = \eta(0) + ad\eta(0)$, omitting the higher powers of a. So we arrive at the expression we wanted to show,

$$\mathcal{P} \exp \left(\int A' \right) = 1 - aA' = 1 - a(\eta \mathrm{d}\eta^{-1} + \eta A \eta^{-1}). \tag{2.78}$$

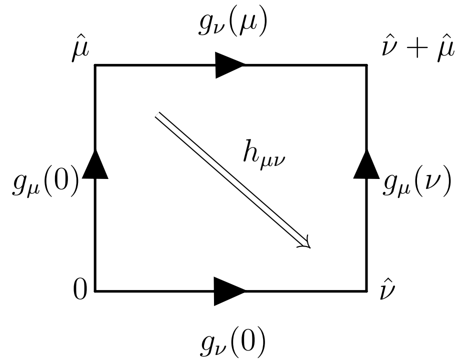

Figure 2.11: A square loop with holonomies in the $\hat{\mu}$ and $\hat{\nu}$ directions. An infinitesimal loop like this is used to derive the curvature in 2-gauge theory.

We can go further to look at infinitesimal transformations, generated at the Lie algebra level, instead of by η. We note that as an element of a Lie group which we assume to be simply connected, it may be written as $\eta = \exp(-sX)$ with X in the Lie algebra and s some parameter. If s is small, as we assume now, it is reasonable to use the approximation $\eta = 1 - sX$ and omit higher order terms. This gives the infinitesimal transformations familiar from the analysis of BF type theories:

$$A' = A + \delta_X A = A + \mathrm{d}X + [A, X]. \tag{2.79}$$

The same procedure applies when finding the infinitesimal version of (2.73). Once again, we can write $\rho = \exp(-s'Y)$ for some $Y \in \mathfrak{h}$ where s' is again a small parameter. The homomorphism t and the Lie algebra homomorphism τ are related by a derivative. Explicitly, this means that $t(\rho) = \exp(-s'\tau(Y))$. Expanding (2.73) to leading order gives

$$\eta(a)^{-1}t(\rho)\exp(-aA)\,\eta(a) = \exp(-aA') = 1 - aA' = 1 - a(\mathrm{d}_A X + \tau(Y)) \tag{2.80}$$

$$\implies A' = A + \mathrm{d}_A X + \tau(Y). \tag{2.81}$$

Before looking at the transformations of the 2-holonomies, let's consider the curvature of the 1-connection A. Consider a rectangular loop in the lattice with holonomies $g_\mu(0)$, $g_\mu(\nu)$, $g_\nu(\mu)$, and $g_\nu(0)$ where μ and ν indicate the direction of the holonomy as is shown in Fig. 2.11. In the limit where the loop becomes vanishingly small, the curvature is defined as the Lie algebra element F such that $e^{-\int_\Box F} = g_\nu(0)^{-1}g_\mu(\nu)^{-1}g_\nu(0)g_\mu(0)$ where the integral is over the interior of the plaquette. Approximating the left hand side by $1 - a^2F$ and the right hand side up to $\mathcal{O}(a^2)$ recovers the familiar expression we have already used several times $F = \mathrm{d}A + \frac{1}{2}[A, A]$.

We can now include the surface into this calculation. We define the curvature of the connection in 2-gauge theory as the leading order of

$$\exp\left(-\int F\right) = g_\nu(0)^{-1} g_\mu(\nu)^{-1} t(h_{\mu\nu}(0)) g_\nu(\mu) g_\mu(0),$$

where $h_{\mu\nu}$ is the surface variable. Just as for the holonomies g, we can write the 2-holonomy $h = \exp\left(-\int \Sigma\right)$, where Σ is a 2-form which we call the 2-holonomy. Thus we find the curvature to be

$$F = \mathrm{d}A + \frac{1}{2}[A, A] + \tau(\Sigma) = \tilde{F} + \tau(\Sigma) \tag{2.82}$$

with $\tilde{F}$ being the usual curvature in the 1-gauge theory. We note that in order for the loop in Fig. 2.11 to really be represent 2-group element, we need the holonomy on the boundary to equal $t(h_{\mu\nu})$. This is equivalent the condition

$$g_\nu(0)^{-1} g_\mu(\nu)^{-1} t(h_{\mu\nu}(0)) g_\nu(\mu) g_\mu(0) = 1$$

or equivalently, $\exp\left(-\int F\right) = 1$. This implies that $F = 0$ in any 2-gauge theory. In other words, the curvature of the 1-connection must be related to the 2-connection.

We have seen how the 2-gaugedness of the theory puts a constraint on the curvature F. The 2-connection also comes with its own notion of curvature. Consider a cube in the lattice. Similarly to how the curvature was defined in terms of the composition of boundary holonomies, we define the 2-curvature G in terms of the composition of the faces of this cube. The decorated cube is shown in Fig. 2.12. The 2-curvature is defined by

$$\exp\left(-a^3 G_{\mu\nu\sigma}\right) = \left(g_\sigma(0) \triangleright h_{\mu\nu}(\sigma)\right) h_{\mu\nu}(0) \left(g_\mu(0) \triangleright h_{\nu\sigma}(\mu)\right) h_{\sigma\mu}(0) \left(g_\nu(0) \triangleright h_{\sigma\mu}(\nu)\right) h_{\sigma\nu}(0) \tag{2.83}$$

where the right hand side is the result of composing surfaces of the cube, which we expand to third order in the lattice spacing. To do the expansion, we should remind ourselves that the action of the Lie groups corresponds to the derivative in the Lie algebra. That means that if $g_\mu(0) = \exp(-aA)$ and $h_{\nu\sigma}(\mu) = \exp(-a^2\Sigma_{\nu\sigma}(\mu)) = \exp(-a^2\Sigma_{\nu\sigma}(0) - a^3\partial_\mu\Sigma_{\nu\sigma}(0))$ (always assuming that a is small), the action of g on h, up to third order in a, is given by

$$g_\mu(0) \triangleright h_{\nu\sigma}(\mu) = \exp\left(-a^2\Sigma_{\nu\sigma}(0) - a^3\partial_\mu\Sigma_{\nu\sigma}(0) + a^3 \mathrm{d}\alpha(A_\mu)(\Sigma_{\nu\sigma}(0))\right), \tag{2.84}$$

where again $\alpha : G \to \mathrm{Aut}(H)$ is the action of G on H. Expanding each of the terms then gives

$$\begin{aligned} G_{\mu\nu\sigma} =& (\mathrm{d}\Sigma)_{\mu\nu\sigma} + \mathrm{d}\alpha(A_\nu)(\Sigma_{\sigma\mu}) + \mathrm{d}\alpha(A_\sigma)(\Sigma_{\mu\nu}) + \mathrm{d}\alpha(A_\mu)(\Sigma_{\nu\sigma}) \\ =& (\mathrm{d}\Sigma + A \triangleright \Sigma)_{\mu\nu\sigma}. \end{aligned} \tag{2.85}$$

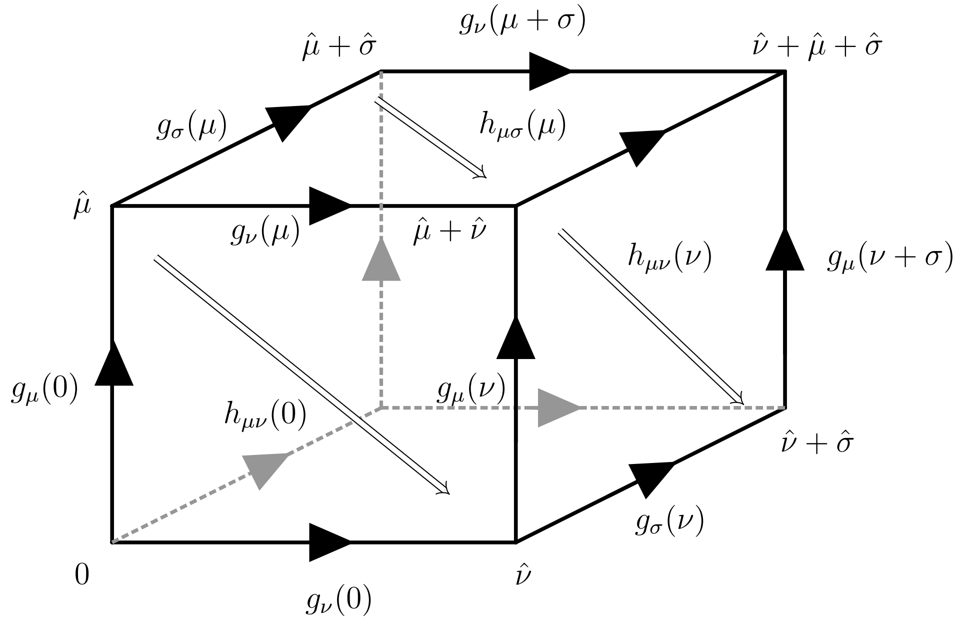

Figure 2.12: A cube used to define the 2-curvature by linearizing the 1- and 2-holonomies g and h.

Finally we turn to the transformation properties of Σ the 2-connection. We proceed by looking at the infinitesimal version of (2.75). The method we have taken in deriving the 1-gauge transformations and the 1- and 2-curvatures made use of a cubic lattice in order to assign local coordinates and indices to the expressions. This made it simpler to express things like $A_\mu(\nu)$ and $A_\nu(0)$ at the same point by linearizing. For this reason, we choose to rewrite (2.75) using a cube instead of the tin can. The result is

$$h'_{\mu\nu} = \exp\left(-a^2\Sigma'\right) = \eta_\sigma(0)^{-1} \triangleright \left(\rho_{\nu\sigma}(0)(g_\nu(0) \triangleright \rho_{\mu\sigma}(0))h_{\mu\nu}(0)(g_\mu(0) \triangleright \rho^{-1}_{\nu\sigma}(\mu))\rho^{-1}_{\mu\sigma}(0)\right). \tag{2.86}$$

Expanding this to second order in a, and first order in the vertical distance, one obtains

$$\delta_{Y,X}\Sigma = -d\Sigma - A \triangleright \Sigma - X \triangleright \Sigma. \tag{2.87}$$

BFCG Redux We can now be precise in describing how the example brought up at the end of Sec. 2.1.4 is indeed a 2-gauge theory. Consider the Euclidean 2-group and the Lagrangian

$$L = \langle B \wedge F\rangle + \langle \Sigma \wedge \mathrm{d}_A C\rangle. \tag{2.88}$$

Since the Euclidean 2-group is skeletal, at the algebra level we have $\tau = 0$. In addition the action of the rotation subalgebra on the translation subalgebra is given by $X \triangleright Y = [X, Y]$.

Now recall the transformation rules derived in Chapter 2, and in particular in Section 2.1.4. The connection A does indeed transform as a connection, $A \to A + \mathrm{d}_A X + \tau(Y) = A + \mathrm{d}_A X$, where in the lingo of Section 2.1.4, X is the generator of 0-form Lorentz transformations and Y is the generator of 1-form Lorentz transformations. In the language developed in this chapter, X is the generator of 1-gauge transformations and Y the generator of 2-gauge transformations. The 2-form Σ, is not surprisingly the 2-connection. It transforms according to $\Sigma \to \Sigma + \mathrm{d}_A Y + [\Sigma, X]$, as it should.

This example is important for what comes in the following chapters. The lattice version, which is necessary for the discretization, and eventual quantization of the theory will be familiar based on what we have already seen in this section.

Chapter 3

Discretization

Now that we have defined 2-groups and 2-gauge theory, we can begin to analyse some examples. The goal in this chapter is to study classical topological theories with 2-gauge symmetries primarily by looking at or constructing phase spaces. The topological nature of the theories are used to concentrate all information onto a triangulation/dual complex of the spacetime manifold.

We start with Euclidean 2-group BFCG theory and introduce a triangulation on the underlying manifold. Doing so allows us to write the theory in terms of discrete variables which are concentrated on the edges and faces of the triangulation. The theory in these new variables can be seen as a truncation of the continuous theory.

Then in section 3.4, we apply a method of constructing phase spaces to include more general types of skeletal 2-gauge theories, instead of solely BFCG in the Euclidean 2-group. There, we will construct a phase space for a 2-complex by utilising a pair of 2-Lie algebras which are dual to one another. A phase space for an "atom" of the triangulation: a pair of one dimensional segments and a pair of surfaces which are their dual. Once the phase space for these atoms are established we will be able to glue them together in a way which preserves the phase space structure order to create a triangulation of the three dimensional manifold appropriately decorated as a lattice 2-gauge theory. The method used is developped in [59].

3.1 Naive discretization of BFCG

We start by considering the BF action defined in terms of the Euclidean group[1].

$$S_{BF} = \int_M \langle B \wedge F[A] \rangle_{SO(4)} + \langle \Sigma \wedge \mathrm{d}_A C \rangle_{\mathbb{R}^4} \tag{3.1}$$

$$= \int_M \frac{1}{2} B^{IJ} \wedge F[A]_{IJ} + \Sigma^I \wedge (\mathrm{d}_A C)_I. \tag{3.2}$$

where $\mathrm{d}_A C = \mathrm{d}C + [A, C]$. In light of the discussion of 2-groups, we will expect Σ to play the role of a 2-connection. For this reason, perform integration by parts in the $\mathbb{R}^4$ sector and add a term to offset the extra boundary term that arises.

$$S_{BFCG} = \int_M \langle B \wedge F[A] \rangle_{SO(4)} + \langle C \wedge G \rangle_{\mathbb{R}^4} \tag{3.3}$$

where $G = \mathrm{d}_A \Sigma$ is the 2-curvature. The equations of motion for the BFCG action are

$$\begin{aligned} \mathrm{d}_A C = 0 \qquad & F[A] = 0 \\ \mathrm{d}_A \Sigma = 0 \qquad & \mathrm{d}_A B = C \wedge \Sigma, \end{aligned} \tag{3.4}$$

where $C \wedge \Sigma \in SO(4)$ has components $(C \wedge \Sigma)^{IJ} = C^{[I} \wedge \Sigma^{J]}$.

The equations of motions give rise to constraints which generate the different types of (2-)gauge transformations

$$\text{1-Gauss: } {}^1\mathcal{G} := \mathrm{d}_A B - C \wedge \Sigma \approx 0 \qquad \text{1-Lorentz} \tag{3.5a}$$

$$\text{2-Gauss: } {}^2\mathcal{G} := T[C, A] \equiv \mathrm{d}_A C \approx 0 \qquad \text{2-Lorentz} \tag{3.5b}$$

$$\text{1-flatness: } {}^1\mathcal{F} := F[A] \equiv \mathrm{d}A + A \wedge A \approx 0 \qquad \text{1-shift} \tag{3.5c}$$

$$\text{2-flatness: } {}^2\mathcal{F} := G[\Sigma, A] \equiv \mathrm{d}\Sigma + A \triangleright \Sigma \approx 0 \qquad \text{2-shift} \tag{3.5d}$$

To avoid having operator valued fields in the Poisson theory, one smears the constraints over a three dimensional surface. We set

$$\begin{aligned} {}^1\mathcal{G}[\xi] = \int \langle {}^1\mathcal{G} \wedge \xi \rangle \qquad & {}^2\mathcal{G}[\eta] = \int \langle {}^2\mathcal{G} \wedge \eta \rangle \\ {}^1\mathcal{F}[\lambda] = \int \langle {}^1\mathcal{F} \wedge \lambda \rangle \qquad & {}^1\mathcal{G}[\xi] = \int \langle {}^2\mathcal{F} \wedge \mu \rangle \end{aligned} \tag{3.6}$$

[1] A similar construction would apply for the Poincaré group.

where λ and ξ are 0- and 1-forms respectively with value in $\mathfrak{so}(4)$, and μ and η are 0- and 1-forms with value in $\mathbb{R}^4$. The corresponding constraint algebra can be computed:

$$
\begin{aligned}
\{{}^1\mathcal{G}[\xi], {}^1\mathcal{G}[\xi']\} &= {}^1\mathcal{G}[[\xi,\xi']], & \{{}^1\mathcal{G}[\xi], {}^2\mathcal{F}[\mu]\} &= {}^2\mathcal{F}[\xi \triangleright \mu], \\
\{{}^1\mathcal{G}[\xi], {}^2\mathcal{G}[\eta]\} &= {}^2\mathcal{G}[\xi \triangleright \eta], & \{{}^1\mathcal{F}[\lambda], {}^1\mathcal{G}[\xi]\} &= {}^1\mathcal{F}[[\lambda,\xi]], \\
\{{}^2\mathcal{F}[\mu], {}^2\mathcal{G}[\eta]\} &= {}^1\mathcal{F}[\mu \wedge \eta], & &
\end{aligned} \tag{3.7}
$$

In the traditional BF theory, to recover the action of general relativity, one needs to impose that B is a function of some tetrad fields for which the connection is compatible. The BFCG action defined in terms of the (four dimensional) Euclidian group already has a 1-form which which plays the role of the tetrad. In particular, the torsion free condition for the tetrad is precisely the equation of motion/constraint $\mathrm{d}_A C = 0$.

Smearing variables The truncation step of the discretization procedure is to smear the variables. To do this we introduce a triangulation of the manifold M. The triangulation is seen as a simplicial complex consisting of 4-simplices σ_i whose union is M. Two adjacent 4-simplices will meet at a shared tetrahedron, τ_i. Intersections of tetrahedra are triangles t_i which are 2-simplices shared. Finally the 1-simplices are called edges e_i and the 0-simplices are vertices.

As we are interested in performing a discretization on a spacial slice of the manifold in order to study the Poisson structure, we introduce a similar three dimensional triangulation $\triangle$ and a corresponding dual complex $\triangle^*$. The relation between $\triangle$ and $\triangle^*$ is as follows: The 0-simplices of $\triangle^*$, called nodes, are situated in the center of each 3-simplex (tetrahedron) of $\triangle$. The 1-simplices of $\triangle^*$, which we call links, are dual to triangles in $\triangle$. A link between two nodes is dual to the triangle shared by the tetrahedra dual to each node. The 2-dimensional objects of $\triangle^*$, are dual to the edges of $\triangle$. As a matter of notation, the simplices dual to one another are indicated by a *. For example, if $e \in \triangle$ is an edge then $e^* \in \triangle^*$ is the two dimensional object dual to e.

In addition, we will sometimes specify particular links and edges by their endpoints. For example, a link l oriented from nodes c to c' may be written as (cc').

With this notation in mind, we define the smeared 1- and 2-connections over a link $l \in \triangle^*$ and wedge $f \in \triangle^*$

$$
H_l = \mathcal{P} \exp\left(-\int_l A\right) \qquad\qquad X_f(n) = \exp\left(\int_f \Sigma\right), \tag{3.8}
$$

where n is the node at which the face f is rooted (recall that the root of the face variable can be changed by whiskering).

The 1-form C is smeared over the edges of the triangulation

$$L_e = \int_e C. \tag{3.9}$$

The constraints (3.5) will also have a discretized version in terms of H, L, and X. As we have seen the constraint $F = 0$ corresponds to the closure of loops in a lattice on which the holonomies are defined, in this case the loops in the dual complex $\triangle^*$. This translates to

$$H_f \equiv \prod_{l \in \partial f} H_l = 1 \tag{3.10}$$

where the product is over links on the boundary of some face f and the product is taken in a proper order. The 2-flatness constraint also is related to the 2-holonomies as the closure of the 3-simplices in $\triangle^*$:

$$\sum_{f \in \partial e^*} X_f(n) = 0 \tag{3.11}$$

where e^* is a 3-simplex bounded by faces f and n is a node.

As a consequence of the constraint $\mathrm{d}_A C = 0$ the edge variables which bound a triangle in $\triangle$ will sum to zero. To see this consider integrating the constraint over a triangle,

$$\int_t \mathrm{d}_A C = \sum_{e \in \partial t} \int_e C = \sum_{e \in \partial t} L_e. \tag{3.12}$$

Beside these constraints we see that $0 = \mathrm{d}_A \mathrm{d}_A C = [F, C]$. This allows for a weaker constraint on F than imposing flatness. The discrete version of this is

$$H_f \triangleright L_e = L_e \tag{3.13}$$

which can be verified by going to the differential picture. This weaker-than-flatness constraint is called edge simplicity [40, 41, 42] and will allow for us to transport edge variables unambiguously.

The discretization presented here does not provide directly a phase space structure. In order to recover Poisson brackets we will instead consider the discretization of the symplectic potential. Recall that the symplectic form is determined by integrating the potential over a three dimensional surface. The three dimensional surface we integrate over will be made into a cellular decomposition. Within each cell, we will determine solutions to the equations of motion. Finally, imposing that the fields are continuous between cells, we introduce new discrete variables representing the change of frames between adjacent cells. The symplectic potential can be written entirely in terms of these discrete variables.

The discrete variables will be supported on 1- and 2-simplicies of the cellular decomposition, as well as the one and two dimensional objects in the dual complex. The links and edges of the complex and its dual are taken as holonomies and are elements of some group. The two dimensional surfaces dual to the links and edges will be decorated with elements of the "dual group" which we define below.

3.2 Algebraic structures for BFCG

3.2.1 Lie bialgebras and classical doubles

This subsection deals with formulating the BFCG action in terms of a Lie algebra $\mathfrak{g}$ and its dual $\mathfrak{g}^*$. As a vector space, $\mathfrak{g}^*$ is nothing more than the dual space of $\mathfrak{g}$, the space of linear functions on $\mathfrak{g}$. In order to endow $\mathfrak{g}^*$ with a Lie algebra structure, one introduces a bialgebra structure on $\mathfrak{g}$.

A Lie bialgebra is a Lie algebra $\mathfrak{g}$ with an antisymmetric bilinear map $\delta : \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$ called the cocommutator or cobracket [73, 74]. We sometimes use the shorthand notation $\delta x = \sum_i x_{1i} \otimes x_{2i} = x_1 \otimes x_2$, omitting the summation altogether. The cobracket must satisfy the co-Jacobi identity, meaning the cyclic permutations of $(\mathrm{id} \otimes \delta) \otimes \delta$ sum to zero. It must also satisfy consistency conditions with the Lie bracket:

$$\delta([x, y]) = \mathrm{ad}_x(\delta y) - \mathrm{ad}_y(\delta x). \tag{3.14}$$

The dual space of a Lie bialgebra is also a Lie bialgebra. If e_i is a basis of $\mathfrak{g}$, we define the dual basis e^{*i} by

$$e^{*i}(e_j) := \langle e^{*i}, e_j \rangle = \delta^i_j \tag{3.15}$$

where we introduced the dual pairing $\langle \cdot, \cdot \rangle$. The duality pairing allows us to define a Lie bracket $[\cdot, \cdot]_*$ and cobracket δ_* on $\mathfrak{g}^*$ by:

$$\langle [\phi, \psi]_*, x \rangle = \langle \phi \otimes \psi, \delta x \rangle \qquad \langle \delta_* \phi, x \otimes y \rangle = \langle \phi, [x, y] \rangle \qquad \phi, \psi \in \mathfrak{g}^*, \ x, y \in \mathfrak{g}. \tag{3.16}$$

Given a dual pair of bialgebras $\mathfrak{g}$ and $\mathfrak{g}^*$, we introduce the double $\mathfrak{d}(\mathfrak{g})$ which, as a vector space, is defined by $\mathfrak{d}(\mathfrak{g}) = \mathfrak{g} \oplus \mathfrak{g}^*$. This space can iteself be made into a bialgebra by constructing the Lie (co)bracket defined on $\mathfrak{g}$ and $\mathfrak{g}^*$:

$$\begin{aligned} [x, y]_{\mathfrak{d}} = [x, y], \qquad [\phi, \psi]_{\mathfrak{d}} = -[\phi, \psi]_* \\ [x, \phi]_{\mathfrak{d}} = \phi_1 \langle \phi_2, x \rangle + x_1 \langle \phi, x_2 \rangle \\ \delta_{\mathfrak{d}} x = \delta x, \qquad \delta_{\mathfrak{d}} \phi = \delta_* \phi \end{aligned} \tag{3.17}$$

for $x, y \in \mathfrak{g}$ and $\phi, \psi \in \mathfrak{g}^*$.

If e_i and e_i^* are dual bases of $\mathfrak{g}$ and $\mathfrak{g}^*$ then we can define $r = \sum_i e_i \otimes e_i^*$ which gives $\delta_{\mathfrak{d}} \xi = ad_\xi r$ for all $\xi \in \mathfrak{d}$. In this case one would call $\mathfrak{d}(\mathfrak{g})$ a coboundary Lie bialgebra.)

Upon exponentiation, the bi-algebra structure can give rise a notion of phase space, called the Heisenberg double, which generalizes the usual notion of cotangent bundle of a Lie group. At the same time, it gives rise to the notion of Drinfeld double which is the natural symmetry structure for the Heisenberg double. We will discuss how the notion of Heisenberg double is the natural tool to construct the phase space for a 2d triangulation in Section 3.4.1. In fact, such phase space can also be used in the context of skeletal 2-bialgebras.

3.2.2 2-Bialgebras

Just as one can introduce a cobracket on a Lie algebra to make it a Lie bialgebra, one can dualize the τ-map and the action $\triangleright$ of a Lie 2-algebra given by $(\mathfrak{g} \xrightarrow{\tau} \mathfrak{h}, \triangleright)$ [20, 35]. In particular we assume that the cocycle δ is a map with components

$$\delta : \mathfrak{h} \to \mathfrak{h} \otimes \mathfrak{h} \qquad \delta : \mathfrak{g} \to (\mathfrak{h} \otimes \mathfrak{g}) \oplus (\mathfrak{g} \otimes \mathfrak{h}). \tag{3.18}$$

By duality, the cobracket δ defines a Lie bracket on $\mathfrak{h}$ and an action $\triangleright^* : \mathfrak{h}^* \times \mathfrak{g}^* \to \mathfrak{g}^*$:

$$\begin{aligned} \langle \delta Y, f_1 \otimes f_2 \rangle = \langle Y, [f_1, f_2] \rangle \qquad Y \in \mathfrak{h} \quad f_1, f_2 \in \mathfrak{h}^* \\ \langle \delta X, f \otimes \phi \rangle = \langle X, f \triangleright^* \phi \rangle \qquad X \in \mathfrak{g}, \quad f \in \mathfrak{h}^*, \quad \phi \in \mathfrak{g}^*. \end{aligned} \tag{3.19}$$

Because of the identity for all $x_i \in \mathfrak{g}$ and $y_i \in \mathfrak{h}$,

$$\tau(x \triangleright y) = [x, \tau(y)] \qquad\qquad \tau(y_1) \triangleright y_2 = [y_1, y_2], \tag{3.20}$$

the Lie bracket on $\mathfrak{h}$ can be uniquely determined by the map τ and the action $\triangleright$. Similarly, the bracket on $\mathfrak{g}^*$ will can be uniquely determined by the dual homomorphism $\tau^* : \mathfrak{h}^* \to \mathfrak{g}^*$ defined by

$$\langle \tau^* \phi, X \rangle = \langle \phi, \tau X \rangle, \quad X \in \mathfrak{g} \quad \phi \in \mathfrak{h}^*. \tag{3.21}$$

The τ-map for the Lie 2-algebra $(\mathfrak{h}^* \to \mathfrak{g}^*, \triangleright^*)$, which we call $\tilde{\tau} = -\tau^*$ uniquely determines the Lie bracket on $\mathfrak{g}^*$ by

$$\tilde{\tau}(\psi) \triangleright^* \phi = [\psi, \phi] \tag{3.22}$$

or equivalently,

$$\langle [\psi, \phi], X \rangle = -\langle \phi \otimes \psi, \delta\tau X \rangle \tag{3.23}$$

for all $\phi, \psi \in \mathfrak{g}^*$ and $X \in \mathfrak{g}$. Lie-bialgebras appear as the infinitesimal version of Poisson-Lie groups [73], while the 2-bialgebras defined here may be thought of as being the infinitesimal version of Poisson 2-groups [20, 35].

3.2.3 Matched pairs and semi-dualization

A final algebraic construction we will consider is that of a matched pair [73]. Consider two Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$ with actions $\triangleleft : \mathfrak{g}_2 \otimes \mathfrak{g}_1 \to \mathfrak{g}_2$ and $\triangleright : \mathfrak{g}_2 \otimes \mathfrak{g}_1 \to \mathfrak{g}_1$ satisfying

$$\begin{aligned} \phi \triangleright [\xi, \eta] &= [\phi \triangleright \xi, \eta] + (\phi \triangleleft \xi) \triangleright \eta - (\xi \leftrightarrow \eta) \\ [\phi, \psi] \triangleleft \xi &= [\phi \triangleleft \xi, \psi] + \phi \triangleleft (\psi \triangleright \xi) - (\phi \leftrightarrow \psi) \end{aligned} \tag{3.24}$$

for all $\phi, \psi \in \mathfrak{g}_2$ and $\xi, \eta \in \mathfrak{g}_1$. Such Lie algebras are called a matched pair.

If $(\mathfrak{g}_1, \mathfrak{g}_2)$ form a matched pair, we can define a double crossed sum Lie algebra denoted by $\mathfrak{g}_1 \bowtie \mathfrak{g}_2$. The underlying vector space is $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ with the Lie bracket

$$[\xi \oplus \phi, \eta \oplus \psi] = ([\xi, \eta] + \phi \triangleright \eta - \psi \triangleright \xi) \oplus ([\phi, \psi] + \phi \triangleleft \eta - \psi \triangleleft \xi). \tag{3.25}$$

The mutual actions between $\mathfrak{g}_1$ and $\mathfrak{g}_2$ give rise to coactions $\tilde{\alpha} : \mathfrak{g}_2^* \to \mathfrak{g}_2^* \otimes \mathfrak{g}_1^*$ and $\tilde{\beta} : \mathfrak{g}_1^* \to \mathfrak{g}_2^* \otimes \mathfrak{g}_1^*$ in the dual coalgebra by

$$\langle \psi \triangleleft \xi, f \rangle = \langle \psi \otimes \xi, \tilde{\alpha}(f) \rangle \tag{3.26}$$

$$\langle \psi \triangleright \xi, g \rangle = \langle \psi \otimes \xi, \tilde{\beta}(g) \rangle \tag{3.27}$$

for all $f \in \mathfrak{g}_2^*$ and $g \in \mathfrak{g}_1^*$. The dual coalgebra in this case will be denoted $\mathfrak{g}_1^* \blacktriangleright\!\!\blacktriangleleft \mathfrak{g}_2^*$ (the black triangles are used for coactions).

We now have the main ingredients to construct a bicross sum Lie bialgebra. These will be used to create a "self-dual" cross module which will be used in creating the phase space associated to the three dimensional triangulation (and its dual complex). In essence, the bicross sum is halfway between $\mathfrak{g}_1 \bowtie \mathfrak{g}_2$ and $\mathfrak{g}_1^* \blacktriangleright\!\!\blacktriangleleft \mathfrak{g}_2^*$: Let $\mathfrak{g}_1$ and $\mathfrak{g}_2$ be a matched pair. A bialgebra structure is added to the vector space $\mathfrak{g}_2^* \oplus \mathfrak{g}_1$ by defining

$$[f \oplus \xi, h \oplus \eta] = (\xi \triangleright h - \eta \triangleright f) \oplus [\xi, \eta] \tag{3.28}$$

where $\triangleright$ here denotes to adjoint of the action $\mathfrak{g}_2 \otimes \mathfrak{g}_1 \to \mathfrak{g}_2$ explicitly given by $\langle \phi \triangleleft \xi, f \rangle = \langle \phi, \xi \triangleright f \rangle$. The cocommutator on $\mathfrak{g}_2^*$ is obtained by dualisation of $\mathfrak{g}_2$. If f^i form a basis of $\mathfrak{g}_2$ and f_i^* the dual basis of $\mathfrak{g}_2^*$ then the cocommutator on $\mathfrak{g}_1$ is defined to be

$$\delta\xi = \sum_i f^i \triangleright \xi \otimes f_i^* - f_i^* \otimes f^i \triangleright \xi. \tag{3.29}$$

The bialgebra structure is called the semidualisation of $\mathfrak{g}_1 \bowtie \mathfrak{g}_2$ and is denoted by $\mathfrak{g}_2^* \blacktriangleright\!\!\triangleleft \mathfrak{g}_1$. A similar construction applies to the bialgebra $\mathfrak{g}_2 \triangleright\!\!\blacktriangleleft \mathfrak{g}_1^*$. In fact, the bialgebras $\mathfrak{g}_2^* \blacktriangleright\!\!\triangleleft \mathfrak{g}_1$ and $\mathfrak{g}_2 \triangleright\!\!\blacktriangleleft \mathfrak{g}_1^*$ are dual to one another and so we can construct the classical double $\mathfrak{d} = (\mathfrak{g}_2^* \blacktriangleright\!\!\triangleleft \mathfrak{g}_1) \bowtie (\mathfrak{g}_2 \triangleright\!\!\blacktriangleleft \mathfrak{g}_1^*)$.

The formalism described can now be applied to Lie 2-algebras. In particular if $\mathfrak{g}_2 \rtimes \mathfrak{g}_1^*$ exponentiates to a skeletal crossed module, so too does its dual.

By many aspects, when the τ-map is trivial, the 2-bialgebra essentially behaves as a bialgebra. We are going to discuss in the next section how the change of polarization in the Euclidean group BF theory can be seen as a semi-dualization giving rise to a 2-bialgebra with trivial τ map. In section 3.4.2, we will discuss how the associated Heisenberg double can be used to construct the phase space for a 3d triangulation for a general class of skeletal (Poisson) 2-group. This phase space will coincide in particular with the one derived explicitly from the action Euclidean group BFCG action.

3.2.4 Euclidean BFCG theory

We can now revisit the change from a the Poincaré/Euclidean BF model to a 2-Poincaré BFCG model in terms of bialgebras and doubles. Let $\mathfrak{g} = \mathfrak{iso}(4) \cong \mathfrak{so}(4) \ltimes \mathbb{R}^4$. The subalgebra $\mathbb{R}^4$ is generated by P^μ and the rotation algebra $\mathfrak{so}(4)$ is generated by $J^{\mu\nu}$ with $J^{\mu\nu} = -J^{\nu\mu}$. Greek indices range from 0 to 3. The Lie brackets are

$$[J^{\mu\nu}, J^{\sigma\rho}] = \eta^{\mu\rho} J^{\nu\sigma} + \eta^{\nu\sigma} J^{\mu\rho} - \eta^{\mu\sigma} J^{\nu\rho} - \eta^{\nu\rho} J^{\mu\sigma} \tag{3.30}$$

$$[P^\mu, P^\nu] = 0, \quad [J^{\mu\nu}, P^\sigma] = \eta^{\mu\sigma} P^\nu - \eta^{\nu\sigma} P^\mu, \tag{3.31}$$

where η is the Euclidean metric. We will sometimes write $[J^{\mu\nu}, J^{\sigma\rho}] = f^{\mu\nu\sigma\rho}{}_{\alpha\beta}J^{\alpha\beta}$ where f is the structure constant of $\mathfrak{so}(4)$. The dual space, denoted $\mathfrak{iso}(4)^*$ has generators J^* and P^* defined by the bilinear pairing

$$\langle P^*_\nu, P^\mu \rangle = \delta^\mu_\nu, \quad \langle J^*_{\mu\nu}, J^{\sigma\rho} \rangle = \delta^\sigma_\mu \delta^\rho_\nu - \delta^\sigma_\nu \delta^\rho_\mu. \tag{3.32}$$

The dual space is made into a Lie algebra by imposing the trivial brackets, or equivalently, $\mathfrak{iso}(4)$ is seen as a bialgebra with a trivial cocycle $\delta = 0$:

$$[P^*_\mu, P^*_\nu] = [J^*_{\mu\nu}, J^*_{\sigma\rho}] = [J^*_{\mu\nu}, P^*_\sigma] = [J^*_{\mu\nu}, P^\rho] = 0. \tag{3.33}$$

Finally, the algebra on which the theory is defined is defined on the double $\mathfrak{d}(\mathfrak{iso}(4))$ where the cross brackets are

$$\begin{aligned}
[P^\mu, P^*_\nu] =& \eta^{\mu\sigma} J^*_{\sigma\nu} \\
[J^{\sigma\rho}, P^*_\nu] =& (\eta^{\rho\alpha}\delta^\sigma_\nu - \eta^{\sigma\alpha}\delta^\rho_\nu) P^*_\alpha \\
[J^{\mu\nu}, J^*_{\sigma\rho}] =& (\eta^{\alpha\nu}\delta^\mu_\rho - \eta^{\alpha\mu}\delta^\nu_\rho) J^*_{\alpha\sigma} + (\eta^{\alpha\mu}\delta^\nu_\sigma - \eta^{\alpha\nu}\delta^\mu_\sigma) J^*_{\alpha\rho}
\end{aligned} \tag{3.34}$$

are defined so that the pairing, extended to $\mathfrak{d}$, is invariant.

The brackets between the $\mathfrak{iso}(4)$ and $\mathfrak{iso}(4)^*$ sectors define an action as well:

$$\begin{aligned}
[P^\mu, P^*_\nu] :=& P^\mu \triangleright P^*_\nu \\
[J^{\sigma\rho}, P^*_\nu] :=& J^{\sigma\rho} \triangleright P^*_\nu \\
[J^{\mu\nu}, J^*_{\sigma\rho}] :=& J^{\mu\nu} \triangleright J^*_{\sigma\rho}.
\end{aligned} \tag{3.35}$$

We can now introduce the $\mathfrak{iso}(4)$ BF action.

$$S_{ISO(4)} = \int_{\mathcal{M}} \langle \mathcal{B} \wedge \mathcal{F} \rangle \tag{3.36}$$

The B-field is taken to be in $\mathfrak{iso}(4)^*$, while the connection and its curvature are in $\mathfrak{iso}(4)$ The fields may be decomposed according to

$$\begin{aligned}
\mathcal{B} = B + \Sigma, \quad & B \in \mathfrak{so}(4)^* \qquad \Sigma \in \mathbb{R}^{4*} \\
\mathcal{A} = A + C, \quad & A \in \mathfrak{so}(4) \qquad C \in \mathbb{R}^4
\end{aligned} \tag{3.37}$$

The BF action then becomes

$$S_{ISO(4)} = \int_{\mathcal{M}} \langle B \wedge F \rangle + \int_{\mathcal{M}} \langle \Sigma \wedge \mathrm{d}_A C \rangle = \int_{\mathcal{M}} \langle B \wedge F \rangle - \int_{\mathcal{M}} \langle \mathrm{d}_A \Sigma \wedge C \rangle + \int_{\mathcal{M}} \mathrm{d} \langle \Sigma \wedge C \rangle. \tag{3.38}$$

We define the 2-curvature of Σ by $G = \mathrm{d}_A\Sigma$. Adding a term to counter the boundary contribution in (3.38) allows us to define the BFCG action,

$$S_{BFCG} = \int_M \langle B \wedge F \rangle + \int_M \langle C \wedge G \rangle = S_{ISO(4)} - \int_{\mathcal{M}} \mathrm{d}\langle \Sigma \wedge C \rangle. \tag{3.39}$$

The equations of motion and symmetries follow according to chapter 2, but we include them here as well for completeness. The equations of motion are

$$\begin{aligned} F &= 0 \qquad d_A C = 0 \\ G &= 0 \qquad d_A B = -[C \wedge \Sigma]. \end{aligned}$$

We can establish a pair of dual 2-Lie algebras here. First we have the skeletal 2-Lie algebra $(\mathfrak{so}(4) \to \mathbb{R}^{4*}, \triangleright)$ where the action is the commutator. Second is the skeletal 2-Lie algebra $(\mathbb{R}^4 \to \mathfrak{so}(4)^*, \triangleright)$ where the action is trivial ($[P, J^*] = 0$).

Hence, the change of polarization we considered moving from the BF theory to the BFCG theory can be interpreted as the semi-dualization procedure we described in the previous section.

The symmetries, generated by functions and 1-forms in $\mathfrak{iso}(4)$ and $\mathfrak{iso}(4)^*$ respectively are the gauge transformations

$$\text{Gauge}: \begin{cases} \delta_\chi \mathcal{A} = d_\mathcal{A}\chi \\ \delta_\chi \mathcal{B} = [\mathcal{B}, \chi] \end{cases} \to \begin{array}{l} \chi = \alpha + X \\ \alpha \in \mathfrak{so}(4),\ X \in \mathbb{R}^4 \end{array} \tag{3.40}$$

$$\tag{3.41}$$

which give the two sets of transformations:

$$\text{1-gauge transformation} \begin{cases} \delta_\alpha A = d_A\alpha \\ \delta_\alpha B = [B, \alpha] \\ \delta_\alpha C = [C, \alpha] \\ \delta_\alpha \Sigma = [\Sigma, \alpha] \end{cases}$$
$$\text{2-shift} \begin{cases} \delta_X A = 0 \\ \delta_X C = d_A X \\ \delta_X B = [\Sigma, X] \\ \delta_X \Sigma = 0 \end{cases} \tag{3.42}$$

and the shift transformations

$$\text{Shift}: \begin{cases} \delta_\beta \mathcal{A} = 0 \\ \delta_\beta \mathcal{B} = d_\mathcal{A}\beta \end{cases} \to \begin{array}{l} \beta = \zeta + Y, \\ \zeta \in \mathbb{R}^{4*},\ Y \in \mathfrak{so}(4)^* \end{array} \tag{3.43}$$

which generate the transformations

$$
\begin{aligned}
&\text{2-gauge transformation} \begin{cases} \delta_\zeta A = 0 \\ \delta_\zeta C = 0 \\ \delta_\zeta B = [C \wedge \zeta] \\ \delta_\zeta \Sigma = d_A \zeta \end{cases} \\
&\text{1-shift} \begin{cases} \delta_Y A = 0 \\ \delta_Y C = 0 \\ \delta_Y B = d_A Y \\ \delta_Y \Sigma = 0 \end{cases}
\end{aligned} \tag{3.44}
$$

These are the familiar 2-gauge symmetries as mentioned in chapter 2.

3.3 Discretization of the symplectic potentials

3.3.1 The BF symplectic potential

Let us now go into detail on the procedure for obtaining a discretized theory from the continuum action. The method used in this section appear in [61] using methods from [52, 97].

First we recall the notation. We divide the spatial slice M into subregions forming a cellular decomposition. The three dimensional cells will be tetrahedra for simplicity, but more general decompositions could be considered. Within each tetrahedron we identify a center point c, which we refer to as a node. The tetrahedron dual to c is denoted c^*. We use the ordered pair (cc') to denote a link which starts at c and ends at c'. The vertices $\overline{v}$ of tetrahedra are denoted with an overline to distinguish them from the nodes. The edges between two vertices, $\overline{v}$ and $\overline{v}'$ is then denoted $[\overline{v}\overline{v}']$ with square brackets to distinguish them from the links. For each vertex $\overline{v}$, there is a set of tetrahedra which share $\overline{v}$. The nodes at the centers of these tetrahedra form the vertices of a polyhedron in the dual cellular complex. This polyhedron will be denoted $\overline{v}^*$. The faces of these polyhedron, called dual faces or wedges are labelled by the edge it intersects, $[\overline{v}\overline{v}']^*$ or by the set of nodes forming its corners. Similarly, the triangles making up the surface of the tetrahedra are either labelled by the three vertices they contain such as $[\overline{v}_1\overline{v}_2\overline{v}_3]$ or by the link intersecting it, $(cc')^*$. Some of the structures are shown in figure 3.1.

Before tackling the BFCG theory directly we illustrate the method which has been used already in [47, 46, 51]. The symplectic potential can be expressed as a sum over the

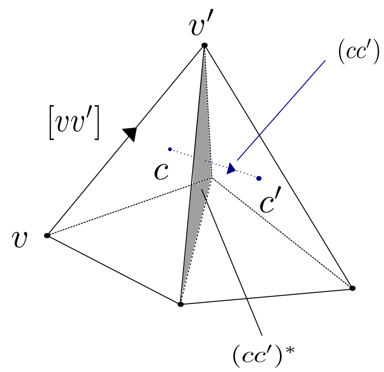

Figure 3.1: A small piece of the cellular decomposition, showing how we label structures. The centers of the tetrahedron are labelled by c and c' with the connecting link labelled (cc'). The triangle which the link passes through is labelled $(cc')^*$. The vertices shown are labelled by v and v' and the connecting edge by $[vv']$. There are arrows on the links and edges indicating the orientation.

individual cells,

$$\Theta_{BF} = \int_{M_3} \langle \mathcal{B} \wedge \delta \mathcal{A} \rangle = \sum_c \int_{c^*} \langle \mathcal{B}_c \wedge \delta \mathcal{A}_c \rangle. \tag{3.45}$$

The subscript on the fields indicate that they are defined within the interior of the cell c^*.

The equations of motion (3.40) imply that $\mathcal{A}$ is a flat connection and therefore pure gauge. For an $ISO(4)$ group element $\mathcal{H}_c(x)$, interpreted as an ISO(4) holonomy connecting c to a point x in the cell,

$$\mathcal{A}_c = \mathcal{H}_c^{-1} d\mathcal{H}_c \tag{3.46}$$

is a solution for $\mathcal{F} = 0$. Then, for a $\mathfrak{iso}(4)^*$ valued 1-form χ,

$$\mathcal{B}_c = \mathcal{H}_c^{-1} d\chi_c \mathcal{H}_c, \tag{3.47}$$

is a solution to $d_{\mathcal{A}}\mathcal{B} = 0$. Hence both $\mathcal{A}_c$ and $\mathcal{B}_c$ are pure gauge.

As mentioned above, the fields inside each cell c may be considered separately so long as we impose continuity between cells. This gives some conditions which $\mathcal{H}$ and χ must satisfy on the cell boundary.

The continuity relations for $\mathcal{A}$ and $\mathcal{B}$ on the (interior of the) triangle shared by c^* and c'^* are expressed as

$$\begin{aligned} \mathcal{A}_c(x) = \mathcal{A}_{c'}(x), &\implies \mathcal{H}_{c'}(x) = \mathcal{G}_{c'c}\mathcal{H}_c(x) \\ \mathcal{B}_c(x) = \mathcal{B}_{c'}(x) &\implies d\chi_{c'}(x) = \mathcal{G}_{c'c} d\chi_c(x) \mathcal{G}_{cc'}, \end{aligned} \tag{3.48}$$

where $\mathcal{G}_{cc'} \in ISO(4)$ does not depend on x and is strictly a function of the triangle. We denote the inverse $\mathcal{G}_{cc'}^{-1} = \mathcal{G}_{c'c}$. The above continuity equations are valid on the interior of the triangle. If there is no curvature concentrated on the edges (we will assume this later on), the equations would be valid there as well.

We can also use continuity between several cells by using the identity $[\mathcal{F}, \mathcal{B}] = 0$. In the truncated picture this is realized by

$$d\chi_c = \Big(\prod_{(c_i c_{i+1}) \in \partial e^*} \mathcal{G}_{c_i c_{i+1}}\Big)^{-1} d\chi_c \Big(\prod_{(c_i c_{i+1}) \in \partial e^*} \mathcal{G}_{c_i c_{i+1}}\Big), \tag{3.49}$$

where the product of holonomies are around the edge e.

In order to write the potential in terms of $\mathcal{H}$ and χ, we should express the variation $\delta\mathcal{A}$ in terms of $\mathcal{H}$:

$$\delta\mathcal{A}_c = \mathcal{H}_c^{-1}(d\Delta\mathcal{H}_c)\mathcal{H}_c \tag{3.50}$$

where $\Delta\mathcal{H}_c = \delta\mathcal{H}_c\mathcal{H}_c^{-1}$. The potential evaluated on-shell in the cells reads

$$\Theta_{BF} \approx \sum_c \int_{c^*} \langle d\chi_c \wedge d\Delta\mathcal{H}_c \rangle, \tag{3.51}$$

where again, $\approx$ means we in on-shell *in* the interior of c^*. We can then use the continuity equations (3.48) to simplify its expression and recover the well-known results.

We note that the integrand in (3.51) is a total derivative so we can use Stokes theorem to recast it as an integral over triangles bounding each tetrahedron. However, there is a choice to be made as to which variable keeps the derivative when dealing with the integral on the boundary. A similar choice arises when dealing with (3d) gravity, and we have the LQG or dual LQG picture [47]. For now we will deal with the case where the derivative is kept on the 1-form χ, writing $d\chi_c \wedge d\Delta\mathcal{H}_c = d(d\chi_c \wedge \Delta\mathcal{H}_c)$. Note that this relies on the assumption that the cells are topologically balls since we use $d^2 = 0$.

The standard discretization of the 4d BF theory is summarized in the following proposition.

Proposition 1. *The symplectic potential is given as a sum of symplectic potentials associated to the phase space $T^*ISO(4)$.*

$$\Theta_{BF} = \int_M \langle \mathcal{B} \wedge \delta\mathcal{A} \rangle \approx \sum_{(cc')} \langle \beta_{(cc')^*}, \Delta\mathcal{G}_c^{c'} \rangle, \tag{3.52}$$

which is constructed the continuity equations (3.48),

$$\chi_{c'} = \mathcal{G}_{c'c}(\chi_c + d\mathcal{Z}_c^{c'})\mathcal{G}_{cc'}, \quad \mathcal{H}_{c'} = \mathcal{G}_{c'c}\mathcal{H}_c, \tag{3.53}$$

and

$$\beta_{(cc')^*} = \int_{(cc')^*} d\chi_c \qquad \Delta\mathcal{G}_c^{c'} = \delta G_{cc'} G_{c'c} \tag{3.54}$$

Table 3.1 provides the geometric structure which the discretized fields are attached to.

Link (cc')	Dual face e^*	Edges e	Triangles $(cc')^*$
$\mathcal{G}_{cc'} \in ISO(4)$	–	–	$\beta_{(cc')^*} \in \mathfrak{iso}^*(4)$

Table 3.1: Localization of the discrete variables.

These discrete variables satisfy by definition the Gauss constraint:

$$\sum_{(cc)^* \in \partial c^*} \beta_{(cc)^*} = 0 \tag{3.55}$$

Furthermore, if we assume there is no curvature, then we have the flatness constraint and the discretized Bianchi identity.

$$\mathcal{G}_e = \prod_{(c_i c_{i+1}) \in \partial e^*} \mathcal{G}_{c_i c_{i+1}} = 1, \qquad \prod_{e^* \in \partial v^*} \mathcal{G}_e = 1. \tag{3.56}$$

We note that the flatness constraint implies the face simplicity as well as the discretized Bianchi identity (as it should).

$$\beta_{(cc)^*} = \Big(\prod_{(c_i c_{i+1}) \in \partial e^*} \mathcal{G}_{c_i c_{i+1}} \Big)^{-1} \beta_{(cc)^*} \Big(\prod_{(c_i c_{i+1}) \in \partial e^*} \mathcal{G}_{c_i c_{i+1}} \Big). \tag{3.57}$$

Proof. Let us evaluate the symplectic potential with the given choice of application of Stokes theorem.

$$\Theta_{BF} \approx -\sum_c \int_{\partial c^*} \langle d\chi_c \, , \, \Delta\mathcal{H}_c \rangle \tag{3.58}$$

The boundary of the tetrahedra c^* is made up of four triangles. Since each triangle is shared by two tetrahedra, the contribution to the potential from each triangle contains two terms with a relative minus sign to account for the opposite orientation:

$$\Theta = \sum_{(cc')} \int_{(cc')^*} \Theta_{(cc')} \tag{3.59}$$

$$\begin{aligned}\Theta_{(cc')} &:= \langle d\chi_c \,,\, \Delta\mathcal{H}_c\rangle - \langle d\chi_{c'} \,,\, \Delta\mathcal{H}_{c'}\rangle \\ &= \langle \Delta\mathcal{G}_c^{c'} \,,\, d\chi_c\rangle \end{aligned}\tag{3.60}$$

The last equality is obtained by using the continuity equations and defining $\Delta\mathcal{G}_c^{c'} = \delta\mathcal{G}_{cc'}\mathcal{G}_{c'c}$. We can identify the factors with structures of the cellular decomposition and its dual graph. We define discrete variables $\mathcal{G}_{(cc')} = \mathcal{G}_{cc'}$ to be associated with the link (cc') and $\beta_{(cc')^*} = \int_{(cc')^*} d\chi_c$ is the discrete variable associated with the triangle $(cc')^*$. Thus, as a function of discrete variables, the potential is therefore

$$\Theta_{BF} \approx \sum_{(cc')} \langle \beta_{(cc')^*}, \Delta\mathcal{G}_c^{c'}\rangle. \tag{3.61}$$

From this potential, we can determine the Poisson brackets, which are the canonical ones associated with the cotangent bundle $T^*ISO(4)$. We will review this in the following section.

Gauss constraint. *By construction* the phase space variables satisfy some constraints. For a given tetrahedron, if we perform the sum over the triangles

$$\sum_{(cc_i)^* \in \partial c^*} \beta_{(cc_i)^*} = \sum_{(cc_i)^* \in \partial c^*} \int_{(cc_i)^*} d\chi_c = 0, \tag{3.62}$$

by Stokes theorem while again assuming that the topology of the interior of c^* is a ball. This constraint is the discretization of the (pull-back of the) continuum constraint $d_{\mathcal{A}}\mathcal{B} = 0$.

In order to accommodate the different possible orientations of the links connecting c^* to its neighbours, we point out that the base point of the variable β can be changed according to

$$\beta_{(c'c)^*} = \int_{(c'c)^*} d\chi_{c'} = -\int_{(cc')^*} \mathcal{G}_{c'c} d\chi_c \mathcal{G}_{cc'} = -\mathcal{G}_{c'c}\beta_{(cc')^*}\mathcal{G}_{cc'}. \tag{3.63}$$

□

Additional constraints not implied by the definitions of the discrete variables are the flatness constraint, the Bianchi identity, and the "face simplicity" constraint.

Flatness constraint. The definition of the discretized fields does not imply that the holonomies $\mathcal{G}_{c_i c_{i+1}}$ should be flat, so we implement it by hand.

$$\prod_{(c_i c_{i+1}) \in \partial e^*} \mathcal{G}_{c_i c_{i+1}} = 1 \tag{3.64}$$

This constraint is the discretization of the (pull-back of the) continuum constraint $\mathcal{F} = 0$. One can check that they generate the discretized version of the BF symmetries. We note that this is a non-abelian group valued momentum map [3]. Momentum maps in general will be reviewed in a later section of this chapter.

Face simplicity. This constraint is weaker than the flatness constraint.

$$d\chi_c = \Big(\prod_{(c_i c_{i+1}) \in \partial e^*} \mathcal{G}_{c_i c_{i+1}} \Big)^{-1} d\chi_c \Big(\prod_{(c_i c_{i+1}) \in \partial e^*} \mathcal{G}_{c_i c_{i+1}} \Big), \tag{3.65}$$

where the product of links begins and ends on the node c, we can just perform the integration over $(cc')^*$ and get the face simplicity constraint.

$$\beta_{(cc)^*} = \int_{(cc)^*} d\chi_c = \Big(\prod_{(c_i c_{i+1}) \in \partial e^*} \mathcal{G}_{c_i c_{i+1}} \Big)^{-1} \left(\int_{(cc)^*} d\chi_c \right) \Big(\prod_{(c_i c_{i+1}) \in \partial e^*} \mathcal{G}_{c_i c_{i+1}} \Big). \tag{3.66}$$

The form of this constraint will be necessary for consistency when we introduce edge variables to the picture. This constraint is the discrete form of $\mathrm{d}_{\mathcal{A}} \mathrm{d}_{\mathcal{A}} \mathcal{B} = [cF, \mathrm{d}_{\mathcal{A}} \mathcal{B}] = 0$ which follows from the equations of motion.

Bianchi identity. This condition is naturally discretized by demanding that concatenating all the holonomies on the dual faces of a (dual) polyhedron v^* (as dual to a vertex v) gives the identity. This is automatically satisfied if each face is flat. The constraint then reads for every dual polyhedron v^e with faces e^*,

$$\prod_{e^* \in \partial v^*} \mathcal{G}_{e^*} = 1. \tag{3.67}$$

3.3.2 The BFCG symplectic potential

Now we can go through the procedure once more starting from the BFCG action. We decompose the potential by breaking up the integral into a sum over cells,

$$\Theta_{BFCG} = \sum_c \int_{c^*} \langle B_c \wedge \delta A_c \rangle - \sum_c \int_{c^*} \langle C_c \wedge \delta \Sigma_c \rangle. \tag{3.68}$$

Continuity eqs	Solutions to continuity eqs	Fields
$g_c^{-1}dg_c = g_{c'}^{-1}dg_{c'}$	$g_c = h_{cc'}g_{c'}$	g_c function in SO(4), $h_{cc'}$ a constant in SO(4)
$dc_{c'} = h_{c'c}dc_c h_{cc'}$	$c_{c'} = h_{c'c}(c_c + x_c^{c'})h_{cc'}$	c_c function in $\mathbb{R}^4$, $x_c^{c'}$ constant in $\mathbb{R}^4$
$d\sigma_{c'} = h_{c'c}d\sigma_c h_{cc'}$	$\sigma_{c'} = h_{c'c}(\sigma_c + d\varsigma_c^{c'})h_{cc'}$	σ_c 1-form in $\mathbb{R}^{4*}$, $\varsigma_c^{c'}$ function in $\mathbb{R}^{4*}$
$db_{c'} = h_{c'c}(db_c - [d\sigma_c, x_c^{c'}])h_{cc'}$	$b_{c'} = h_{c'c}(b_c - [\sigma_c, x_c^{c'}] + dy_c^{c'})h_{cc'}$	b_c 1-form in $\mathfrak{so}(4)^*$, $y_c^{c'}$ function in $\mathfrak{so}(4)^*$

Table 3.2: Continuity equations and their solution.

We then decompose the solutions

$$\mathcal{A}_c = \mathcal{H}_c^{-1}d\mathcal{H}_c, \quad \mathcal{B}_c = \mathcal{H}_c^{-1}d\chi_c\mathcal{H}_c. \tag{3.69}$$

according into subalgebra components. We write $\mathcal{H} = e^c g$, where g is a rotation and e^c is a translation. χ is decomposed into $\chi = b + \sigma$ for $b \in \mathfrak{so}(4)^*$ and $\sigma \in \mathbb{R}^{4*}$. Thus, (still using the convenient representation $e^c = 1 + c$)

$$\mathcal{A}_c = A_c + C_c = g_c^{-1}dg_c + g_c^{-1}dc_c g_c \quad \mathcal{B}_c = B_c + \Sigma_c = g_c^{-1}(db_c + [d\sigma_c, c_c] + d\sigma_c)g_c \tag{3.70}$$

giving

$$A_c = g_c^{-1}dg_c \qquad C_c = g_c^{-1}dc_c g_c \tag{3.71}$$

$$\Sigma_c = g_c^{-1}d\sigma_c g_c \qquad B_c = g_c^{-1}(db_c + [d\sigma_c, c_c])g_c, \tag{3.72}$$

where the different fields are defined in Table 3.2. If we apply the continuity equations consecutively around a loop ∂e^*, we also get the equations

$$dc_c = h_e^{-1}dc_c h_e, \tag{3.73}$$

$$db_c = h_e^{-1}db_c h_e, \quad d\sigma_c = h_e^{-1}d\sigma_c h_e, \quad h_e \equiv \prod_{c_i c_{i+1} \in \partial e^*} h_{c_i c_{i+1}}, \tag{3.74}$$

where again, the product along the loop of links begins and ends at node c. The condition (3.73) can be seen as the discretization of $[F, C] = 0$, while the second line (3.74) come from $d_{\mathcal{A}}\mathcal{B} = 0$. The solutions of the continuity equations are given in the Table 3.2. Anticipating a bit, we will see that in the BFCG discretization, even if we allow for some curvature on the edges, we will still need to assume that on the edges of the triangulation

$$\sigma_c = h_e^{-1}\sigma_c h_e. \tag{3.75}$$

This extra condition will ensure that we can integrate the symplectic potential. In order to express the potential in terms of the fields g, σ, b, and c we need to express the variation of the fields A and Σ in these variables. We define

$$\Delta g_c \coloneqq \delta g_c g_c^{-1} \tag{3.76}$$

then,

$$\delta A_c = g_c^{-1} d\Delta g_c g_c \qquad \delta\Sigma_c = g^{-1}(\delta d\sigma_c + [d\sigma_c, \Delta g_c])g_c \tag{3.77}$$

Using these expressions, the potential in a cell is

$$\Theta_c \approx \langle db_c \wedge d\Delta g_c\rangle + d\langle [d\sigma_c, c_c], \Delta g_c\rangle - \langle dc_c \wedge d\delta\sigma_c\rangle, \tag{3.78}$$

We see that Θ_c is a total derivative and can be written as an integral over the boundary ∂c^* by Stokes theorem. As in the previous section, there is a choice to make regarding which variable keeps the derivative when we perform Stokes' theorem.

Recovering BF discretization

The first choice will recover the picture we had in the previous section. We write

$$\Theta_{BFCG} \approx \sum_c \int_{\partial c*} (\langle db_c\,,\, \Delta g_c\rangle + \langle [d\sigma_c, c_c]\,,\, \Delta g_c\rangle - \langle c_c\,,\, d\delta\sigma_c\rangle), \tag{3.79}$$

or

$$\Theta_{BFCG} \approx \sum_{(cc')^*} \int_{(cc')^*} \Theta_{(cc')^*} \tag{3.80}$$

$$\begin{aligned}\Theta_{(cc')^*} = &\langle db_c\,,\, \Delta g_c\rangle - \langle db_{c'}\,,\, \Delta g_{c'}\rangle + \langle [d\sigma_c, c_c]\,,\, \Delta g_c\rangle - \langle [d\sigma_{c'}, c_{c'}]\,,\, \Delta g_{c'}\rangle \\ &- \langle c_c\,,\, d\delta\sigma_c\rangle + \langle c_{c'}\,,\, d\delta\sigma_{c'}\rangle.\end{aligned} \tag{3.81}$$

Utilizing the continuity equations gives

$$\Theta_{BFCG} \approx \sum_{(cc')^*} \left\langle \Delta h_c^{c'}\,,\, \int_{(cc')^*} db_c \right\rangle + \left\langle [\Delta h_c^{c'}, x_c^{c'}]\,,\, \int_{(cc')^*} d\sigma_c \right\rangle + \left\langle x_c^{c'}\,,\, \delta \int_{(cc')^*} d\sigma_c \right\rangle \tag{3.82}$$

$$\approx \sum_{(cc')} \langle \Delta h_{(cc')}\,,\, b_{(cc')^*}\rangle + \langle [\Delta h_{(cc')}, x_{(cc')}]\,,\, V_{(cc')^*}\rangle + \langle x_{(cc')}\,,\, \delta V_{(cc')^*}\rangle. \tag{3.83}$$

The factors in Θ_{BFCG} can be associated to structures in the cellular decomposition. We already saw that $h_{cc'}$ is related to the the links in the dual cellular decomposition. Similarly $x_c^{c'}$ is also associated to the links. The factors involving integrals over triangles are associated to the triangles in the cellular decomposition, which are dual to the links.

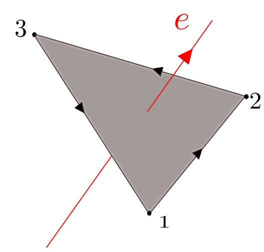

Figure 3.2: An example of the type of edge used for illustrative calculations. The edge is shown in red labelled by e with surrounding nodes forming a triangle.

Alternate polarization

We can make an alternative choice in how we apply Stoke's theorem in (3.78) by writing $\langle dc_c \wedge d\delta\sigma_c\rangle = -d\langle dc_c \wedge \delta\sigma_c\rangle$. Once again this leaves us with integrals over shared boundaries of cells. Writing contributions for each cell on the boundary using the continuity equations gives

$$\int_{(cc')^*} \Theta_{(cc')^*} = \int_{(cc')^*} \langle d(b_c + [c_c, \sigma_c])\,,\, \Delta h_c^{c'}\rangle - \int_{(cc')^*} d\langle (dc_c\,,\, [\Delta h_c^{c'}, \varsigma_c^{c'}])\rangle + \int_{(cc')^*} d\langle dc_c\,,\, \delta\varsigma_c^{c'}\rangle. \tag{3.84}$$

The last term is a problematic one. In contrast to the previous section, neither dc_c nor $\delta\varsigma_c^{c'}$ are constant and so we must do some work to perform this integration. We shall once again use Stokes' theorem on each triangle and deal with integrals over edges bounding triangles instead.

$$\sum_{(cc')^*} \int_{(cc')^*} d\langle dc_c\,,\, \delta\varsigma_c^{c'}\rangle = \sum_{(cc')} \int_{\partial(cc')^*} \langle dc_c\,,\, \delta\varsigma_c^{c'}\rangle \tag{3.85}$$

$$= \sum_{e} \int_{e} \sum_{(cc')\in e^*} \epsilon^e_{(cc')} \langle dc_c\,,\, \delta\varsigma_c^{c'}\rangle \tag{3.86}$$

The first sum and the integral are over edges e. The second sum is over the links (cc') which make up the polygon e^* dual to the e. The factor $\epsilon^e_{(cc')}$ is either 1 or -1, depending on whether the orientation of $(cc')^*$ is aligned with e or not.

To illustrate we take the example edge we have in Fig. 3.2 which is dual to a triangle (three links). The contribution of this edge to the potential is

$$\int_e \sum_{(cc')^* \in e^*} \langle dc_c \, , \, \delta\varsigma_c^{c'} \rangle = \int_e \langle dc_1 \, , \, \delta\varsigma_1^2 \rangle + \int_e \langle dc_2 \, , \, \delta\varsigma_2^3 \rangle + \int_e \langle dc_3 \, , \, \delta\varsigma_3^1 \rangle \tag{3.87}$$

$$= \int_e \langle dc_1 \, , \, (\delta\varsigma_1^2 + h_{12}\delta\varsigma_2^3 h_{21} + h_{13}\delta\varsigma_3^1 h_{31}) \rangle \tag{3.88}$$

$$= \int_e \langle dc_1 \, , \, \delta(\varsigma_1^2 + h_{12}\varsigma_2^3 h_{21} + h_{13}\varsigma_3^1 h_{31}) \rangle + \int_e \langle [dc_1, h_{12}\varsigma_2^3 h_{21}] \, , \, \Delta h_1^2 \rangle + \int_e \langle [dc_1, h_{13}\varsigma_3^1 h_{31}] \, , \, \Delta h_1^3 \rangle \tag{3.89}$$

In the second line, we were able to use the continuity equation in the variable c to base each term at the center 1 (an arbitrary choice). The second and third term involve something proportional to Δh_1^2 and Δh_1^3 (the value of the superscript and subscript are a result of the arbitrary choice made to base everything at 1) and can therefore be absorbed in the first term of (3.84) (since the total potential involves summing over the links). This will be the source of a non-trivial closure constraint for the tetrahedron.

The first term involves a combination of the continuity variables ς. We now show that this combination turns out to be a constant. Consider the three continuity equations for σ which are satisfied on $e = (12)^* \cap (23)^* \cap (31)^*$:

$$\sigma_2 = h_{21}(\sigma_1 + d\varsigma_1^2)h_{21} \qquad \sigma_3 = h_{32}(\sigma_2 + d\varsigma_2^3)h_{23} \qquad \sigma_1 = h_{13}(\sigma_3 + d\varsigma_3^1)h_{31}. \tag{3.90}$$

Putting these together we have

$$\sigma_3 = h_{32}h_{21}h_{13}\sigma_3 h_{31}h_{12}h_{23} + h_{32}h_{21}h_{13}d\varsigma_3^1 h_{31}h_{12}h_{23} + h_{32}h_{21}d\varsigma_1^2 h_{21}h_{23} + h_{32}d\varsigma_2^3 h_{23}. \tag{3.91}$$

Assuming[2] that

$$\sigma_3 = h_{32}h_{21}h_{13}\sigma_3 h_{31}h_{12}h_{23}. \tag{3.92}$$

and putting together (3.91) and (3.92), we get

$$d\varsigma_1^2 + h_{13}d\varsigma_3^1 h_{31} + h_{12}d\varsigma_2^3 h_{21} = 0. \tag{3.93}$$

[2]If we enforce the fact there is *no* curvature on the edges, this assumption is obviously true. If, we do not impose flatness right away, we have to make this assumption to get to the relevant result. Hence we can get our result, even though there is no flatness, but such that our assumption is satisfied.

And so we have that

$$\varsigma_1^2 + h_{13}\varsigma_3^1 h_{31} + h_{12}\varsigma_2^3 h_{21} = V_1^{e^*}, \tag{3.94}$$

for some constant $V_1^{e^*}$, which then decorates the dual face e^*. This is exactly the expression which appears in the first term of (3.89). Since V^{e^*} is a constant, we are able to consider $\int_e dc_1$ as our discrete variable associated to e and $V_1^{e^*}$ as the discrete variable associated to the polygon e^*. The potential due to edge e is then

$$\int_e \sum_{(cc')^* \in e^*} \langle dc_c \,,\, \delta\varsigma_c^{c'} \rangle = \langle \delta V_1^{e^*} \,,\, \int_e dc_1 \rangle + \int_e \langle [dc_1, h_{12}\varsigma_2^3 h_{21}] \,,\, \Delta h_1^2 \rangle + \int_e \langle [dc_1, h_{13}\varsigma_3^1 h_{31}] \,,\, \Delta h_1^3 \rangle. \tag{3.95}$$

Summarizing, the symplectic potential takes now the shape

$$\Theta_{BFCG} \approx \sum_{(cc')} \left\langle \int_{(cc')^*} \tilde{b}_c^{c'} \,,\, \Delta h_c^{c'} \right\rangle + \sum_e \left\langle \int_e dc_{c_e} \,,\, \delta V_{c_e}^{e^*} \right\rangle \tag{3.96}$$

A lot has been concealed in writing equation (3.96). The label c_e is the choice of base point in the polygon dual to e (in the example edge we took c_e to be the node 1). We also introduced $\tilde{b}_c^{c'}$. Simply put, this is shorthand notation for everything which appears in Θ next to $\Delta h_c^{c'}$. The explicit form of such a term depends on the choices of c_e and so we won't write it out in general. We will define $\tilde{b}_c^{c'}$ for an explicit example shortly.

We can now determine the discrete variables. The discrete variables are $b_{(cc')^*} = \int_{(cc')^*} \tilde{b}_c^{c'} \in \mathfrak{so}(4)^*$ on triangles, $h_{cc'} \in \mathrm{SO}(4)$ on links dual to triangles, $\ell_e = \int_e dc_{c_e} \in \mathbb{R}^4$ on edges, and $V_{e^*} = V_{c_e}^{e^*}$ on polygons dual to an edge. The variables are summarized in table 3.3.

The Poisson brackets coming from the symplectic potential are

$$\begin{aligned} \{\ell_\alpha, V^\beta\} &= -\delta_\alpha^\beta \\ \{h^\alpha{}_\beta, b^{\sigma\rho}\} &= (J^{\sigma\rho}h)^\alpha{}_\beta \\ \{b^{\sigma\rho}, b^{\alpha\beta}\} &= \eta^{\sigma\alpha}b^{\rho\beta} + \eta^{\rho\beta}b^{\sigma\alpha} - \eta^{\sigma\beta}b^{\rho\alpha} - \eta^{\rho\alpha}b^{\sigma\beta}. \end{aligned} \tag{3.97}$$

To determine the kinematic constraints, it is helpful to have an explicit expression of $\tilde{b}_c^{c'}$. To this end, we consider the specific triangulation of a 4-simplex. The space is divided into 5 tetrahedra the centers of which will be labelled by integers $\{1, 2, \ldots, 5\}$. The vertices of the tetrahedra will be labelled by overlined integers $\{\overline{1}, \overline{2}, \ldots, \overline{5}\}$. The tetrahedron i^*

Discrete variable	Definition in terms of continuous variables	Home in cellular complex
V_{c_e}	linear combination of ς's around an edge	Polygon e^*, a face in the dual complex
ℓ_{c_e}	$\int_e dc_{c_e}$	Edge e of tetrahedron
$h_{(cc')}$	$h_{cc'}$	Links
$b_{(cc')^*}$	$\int_{(cc')^*} \tilde{b}_c^{c'}$	Triangles

Table 3.3: Summary of discretization of BFCG theory. The key result is that $b_{(cc')^*}$ depends on many variables, namely c, ς and b, as illustrated in (3.99). In particular, we have integrations both on the triangle and some of the edges forming its boundary.

will have vertices $\{\bar{1}, \ldots, \bar{5}\} \backslash \bar{i}$. A diagram indicating the orientation of the links and edges is shown in Fig. 3.3.

The calculation on the edges goes just like in the example we did to obtain (3.94). The resulting dual face variables are

$$
\begin{aligned}
V_3^{[\overline{12}]^*} &= \varsigma_3^4 + h_{34}\varsigma_4^5 + h_{35}\varsigma_5^3 & \qquad V_2^{[\overline{31}]^*} &= -h_{24}\varsigma_4^2 + h_{24}\varsigma_4^5 - \varsigma_2^5 \\
V_2^{[\overline{14}]^*} &= \varsigma_2^3 - \varsigma_2^5 - h_{25}\varsigma_5^3 & V_2^{[\overline{51}]^*} &= h_{24}\varsigma_4^2 + h_{23}\varsigma_3^4 + \varsigma_2^3 \\
V_1^{[\overline{23}]^*} &= h_{14}\varsigma_4^5 + h_{15}\varsigma_5^1 + \varsigma_1^4 & V_3^{[\overline{42}]^*} &= -h_{35}\varsigma_5^3 - \varsigma_3^1 + h_{35}\varsigma_5^1 \\
V_1^{[\overline{25}]^*} &= h_{13}\varsigma_3^4 - h_{13}\varsigma_3^1 - \varsigma_1^4 & V_1^{[\overline{34}]^*} &= \varsigma_1^2 + h_{15}\varsigma_5^1 + h_{12}\varsigma_2^5 \\
V_1^{[\overline{53}]^*} &= -h_{14}\varsigma_4^2 - \varsigma_1^4 + \varsigma_1^2 & V_1^{[\overline{45}]^*} &= \varsigma_1^2 + h_{12}\varsigma_2^3 + h_{13}\varsigma_3^1
\end{aligned} \tag{3.98}
$$

In the above, we have always based the variables at the lowest node in numerical order. This is a choice made arbitrarily, any node which is a vertex of $[\overline{ij}]^*$ would be equally valid.

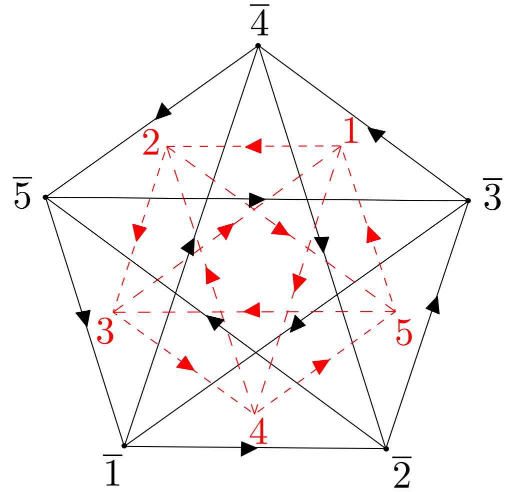

Figure 3.3: The edges of the complex are shown with solid black lines and the dual complex is shown with dotted red lines. The arrows indicate the orientation chosen.

The resulting expressions for the $\mathfrak{so}(4)^*$ variables are:

$$
\begin{aligned}
b_{(12)^*} &= \int_{[345]} (db_1 - d[c_1, \sigma_1] - [dc_1, d\varsigma_1^2]) + \int_{[45]} [dc_1, h_{12}\varsigma_2^3 h_{21}] + \int_{[34]} [dc_1, h_{12}\varsigma_2^5 h_{21}] \\
b_{(23)^*} &= \int_{[145]} (db_2 - d[c_2, \sigma_2] - [dc_2, d\varsigma_2^3]) + \int_{[51]} [dc_2, h_{23}\varsigma_3^4 h_{32}] \\
b_{(34)^*} &= \int_{[125]} (db_3 - d[c_3, \sigma_3] - [dc_3, d\varsigma_3^4]) + \int_{[12]} [dc_3, h_{34}\varsigma_4^5 h_{43}] \\
b_{(45)^*} &= \int_{[123]} (db_4 - d[c_4, \sigma_4] - [dc_4, d\varsigma_4^5]) \\
b_{(51)^*} &= \int_{[234]} (db_5 - d[c_4, \sigma_4] - [dc_5, d\varsigma_5^1]) - \int_{[34]} [dc_5, \varsigma_5^1] - \int_{[23]} [dc_5, \varsigma_5^1] \\
b_{(14)^*} &= \int_{[235]} (db_1 - d[c_1, \sigma_1] - [dc_1, d\varsigma_1^4]) + \int_{[23]} [dc_1, h_{14}\varsigma_4^5 h_{41}] - \int_{[53]} [dc_1, h_{14}\varsigma_4^2 h_{41}] \\
b_{(25)^*} &= \int_{[134]} (db_2 - d[c_2, \sigma_2] - [dc_2, d\varsigma_2^5]) - \int_{[14]} [dc_2, h_{25}\varsigma_5^3 h_{52}] \\
b_{(31)^*} &= \int_{[245]} (db_3 - d[c_3, \sigma_3] - [dc_3, d\varsigma_3^1]) + \int_{[25]} [dc_3, \varsigma_3^1] - \int_{[25]} [dc_3, \varsigma_3^4] - \int_{[45]} [dc_3, \varsigma_3^1] \\
b_{(42)^*} &= \int_{[135]} (db_4 - d[c_4, \sigma_4] - [dc_4, d\varsigma_4^2]) - \int_{[51]} [dc_4, \varsigma_4^2] - \int_{[31]} [dc_4, \varsigma_4^5] + \int_{[31]} [dc_4, \varsigma_4^2] \\
b_{(53)^*} &= \int_{[124]} (db_5 - d[c_5, \sigma_5] - [dc_5, d\varsigma_5^3]) - \int_{[42]} [dc_5, \varsigma_5^1] + \int_{[42]} [dc_5, \varsigma_5^3] - \int_{[12]} [dc_5, \varsigma_5^3]
\end{aligned}
\tag{3.99}
$$

Clearly there is a lack of symmetry due to the orientation choices for the links.

Compared to the BF case, there are two sets of new constraints, the 2-Gauss constraints encoding the triangles are closed and the 2-flatness encoding that the dual polyhedra close. We then have the more usual set of constraints, the 1-Gauss constraints and the 1-flatness constraints. The latter encodes that the holonomies along the links forming a closed loop should be trivial. The former encodes that the triangle decoration should be equal to a specific quantity. Finally, there is also the edge simplicity constraints. This constraint is actually implied if we have flatness, but can hold without having flatness.

The 2-Gauss constraint states that the triangles in the complex is closed. In the example considered this amounts the statement that the following are zero:

$$\begin{aligned}
\mathcal{G}_4^{[123]} =& h_{43}\ell_3^{[12]} + h_{41}\ell_1^{[23]} + h_{42}\ell_2^{[31]} \\
\mathcal{G}_5^{[124]} =& h_{53}\ell_3^{[12]} + h_{51}\ell_1^{[25]} + h_{52}\ell_2^{[51]} \\
\mathcal{G}_3^{[125]} =& \ell_3^{[12]} - \ell_3^{[42]} - h_{32}\ell_2^{[14]} \\
\mathcal{G}_2^{[134]} =& -\ell_2^{[31]} - \ell_2^{[14]} + h_{21}\ell_1^{[34]} \\
\mathcal{G}_4^{[135]} =& -h_{42}\ell_2^{[31]} - h_{41}\ell_1^{[53]} + h_{42}\ell_2^{[51]} \\
\mathcal{G}_2^{[145]} =& \ell_2^{[14]} + h_{21}\ell_1^{[45]} + \ell_2^{[51]} \\
\mathcal{G}_5^{[234]} =& h_{51}\ell_1^{[23]} + h_{53}\ell_3^{[42]} + h_{51}\ell_1^{[34]} \\
\mathcal{G}_1^{[235]} =& \ell_1^{[23]} - \ell_1^{[53]} + \ell_1^{[25]} \\
\mathcal{G}_3^{[245]} =& -h_{31}\ell_1^{[25]} - \ell_3^{[42]} + h_{31}\ell_1^{[45]} \\
\mathcal{G}_1^{[345]} =& \ell_1^{[34]} + \ell_1^{[45]} + \ell_1^{[53]}.
\end{aligned} \tag{3.100}$$

These amount to a discretized version of $\mathrm{d}_A C = 0$.

The 2-flatness constraints indicate the closure of the dual polyhedra, which are tetrahedra in our example. Geometrically this is can be interpreted as the area vectors of a polyhedron sum to zero,

$$\begin{aligned}
\mathcal{P}^{\bar{1}} =& h_{23}V_3^{[12]^*}h_{32} + V_2^{[14]^*} - V_2^{[31]^*} - V_2^{[51]^*} \\
\mathcal{P}^{\bar{2}} =& V_1^{[23]^*} + V_1^{[25]^*} - h_{13}V_3^{[12]^*}h_{31} - h_{13}V_3^{[24]^*}h_{31} \\
\mathcal{P}^{\bar{3}} =& h_{13}V_3^{[31]^*}h_{31} + V_1^{[34]^*} - V_2^{[53]^*} - V_2^{[23]^*} \\
\mathcal{P}^{\bar{4}} =& V_1^{[45]^*} + V_1^{[42]^*} - h_{12}V_2^{[14]^*}h_{21} - V_1^{[34]^*} \\
\mathcal{P}^{\bar{5}} =& V^{[53]^*} + h_{12}V_2^{[51]^*}h_{21} - V_1^{[45]^*} - V_1^{[25]^*}.
\end{aligned} \tag{3.101}$$

The 1-Gauss constraints are the closure of the $\mathfrak{so}(4)$ variables,

$$
\begin{aligned}
\mathcal{T}^5 =& b_{(51)^*} + b_{(53)^*} - h_{54} b_{(45)^*} h_{45} - h_{52} b_{(35)^*} h_{25} \\
\mathcal{T}^4 =& b_{(45)^*} + b_{(42)^*} - h_{41} b_{(14)^*} h_{41} - h_{43} b_{(34)^*} h_{34} \\
\mathcal{T}^3 =& b_{(34)^*} + b_{(31)^*} - h_{32} b_{(23)^*} h_{32} - h_{35} b_{(53)^*} h_{53} - [\ell_3^{[12]}, V_3^{[12]^*}] - [\ell_3^{[42]}, V_3^{[42]^*}] \\
\mathcal{T}^2 =& b_{(23)^*} + b_{(25)^*} - h_{24} b_{(42)^*} h_{42} - h_{21} b_{(12)^*} h_{21} - [\ell_2^{[14]}, V_2^{[14]^*}] - [\ell_2^{[31]}, V_2^{[31]^*}] - [\ell_2^{[51]}, V_2^{[51]^*}] \\
\mathcal{T}^1 =& b_{(12)^*} + b_{(14)^*} - h_{13} b_{(31)^*} h_{31} - h_{15} b_{(51)^*} h_{51} - [\ell_1^{[45]}, V_1^{[45]^*}] - [\ell_1^{[23]}, V_1^{[23]^*}] \\
& - [\ell_1^{[53]}, V_1^{[53]^*}] - [\ell_1^{[34]}, V_1^{[34]^*}] - [\ell_1^{[25]}, V_1^{[25]^*}]
\end{aligned} \tag{3.102}
$$

The extra terms involving the brackets come up because of the choice of nodes which we at which we base our variables.

3.4 Phase space structure for triangulations

In the previous section, we started with a potential current, for a specific topological 2-gauge theory (based on the Euclidean (skeletal) 2-group), which is integrated over a 3-manifold. The integration is carried out by introducing a specific simplicial complex and solving the equations of motion within the interior of each cell.The result is a symplectic potential in terms of discrete variables which were introduced by demanding the fields of the theory are continuous across cell boundaries. These discrete variables belong to a pair of (skeletal) 2-groups, the Euclidean 2-group and its dual, and decorate the triangulation and its dual complex.

The configuration variable in the BF theory was the 1-form $\mathcal{A} = A + C$ with value in $(\mathfrak{g}_1 \ltimes \mathfrak{g}_2)$ and the momentum variable was the 2-form $\mathcal{B} = B + \Sigma$ with value in $\mathfrak{g}_1^* \times \mathfrak{g}_2^*$, where both $\mathfrak{g}_1^*$ and $\mathfrak{g}_2^*$, as well as $\mathfrak{g}_2$ are abelian Lie algebras. The change of polarization leads to consider a BFCG theory with configuration variables a 1-form A with value in $\mathfrak{g}_1$ and a 2-form Σ with value in $\mathfrak{g}_2^*$, while the momentum variables are a 1-form C with value in $\mathfrak{g}_2$ and a 2-form B with value in $\mathfrak{g}_1^*$. This is an example of the semi-dualization we considered in section 3.2.3, which is particularly simple since $\mathfrak{g}_2$ is abelian. This enabled us to perform explicitly the discretization of the symplectic form.

One could try to derive the discrete picture when $\mathfrak{g}_2$ is non-abelian. The key example would then be the $\mathfrak{so}(3,1)$ BF theory where we can write $\mathfrak{so}(3,1) = \mathfrak{su}(2) \bowtie \mathfrak{an}_2 \equiv \mathfrak{g}_1 \bowtie \mathfrak{g}_2$. Due to the back action of $\mathfrak{g}_2 = \mathfrak{an}_2$ on $\mathfrak{g}_1 = \mathfrak{su}(2)$, the explicit discretization in the style of the previous section is much harder to do. Nevertheless, we know that at the continuum level, there is a BFCG formulation defined in terms of a pair of skeletal Lie 2-algebra $\mathfrak{g}_1 \ltimes \mathfrak{g}_2^*$ and $\mathfrak{g}_2 \ltimes \mathfrak{g}_1^*$, where still both $\mathfrak{g}_1^*$ and $\mathfrak{g}_2^*$ are abelian Lie algebras (and where we omitted the coactions) [34]. As we will discuss in section 3.4.2, such semi-dualization for the $\mathfrak{so}(3,1)$ case gives rise naturally to a κ-Poincaré structure [106].

Since the explicit discretization is hard we will build the phase space for a three dimensional triangulation by hand based on the building blocks $\mathfrak{g}_1 \ltimes \mathfrak{g}_2^*$ and $\mathfrak{g}_2 \ltimes \mathfrak{g}_1^*$, which exponentiation will provide us an Heisenberg double.

We begin directly with a symplectic space (a Heisenberg double) for small pieces of a complex: A dual pair of links and faces and a dual pair of edges and wedges, which are decorated by a pair of dual skeletal groups. We then construct a larger complex by gluing these small atomic pieces together in such a way that the complex also has a Poisson structure defined on it. This construction will be useful to understand the generalization of the group field theory to a 2-group which we will discuss in a later chapter.

As a warm up, let's review the phase space construction as it is done for polygons, which will allow us to introduce the notion of Heisenberg double in a simpler context, before moving onto the case where we construct three dimensional complexes.

3.4.1 Phase space for two dimensional triangulations

Let's begin by introducing a Poisson structure on the space of triangles. Considering triangles embedded in three dimensional Euclidean space, one may describe any point by the three vectors ℓ_i which are the edges of the triangle. Three edges is more than necessary: we reduce the degrees of freedom by imposing that $C := \sum_i \ell_i = 0$. There is a natural non-trivial Poisson structure on this space [65].

We let ℓ_i^a be the a-component of ℓ_i and set the Poisson brackets to

$$\{\ell_i^a, \ell_j^b\} = \sum_c \delta_{ij} \epsilon^{abc} \ell_i^c, \tag{3.103}$$

so that the constraint C generates rotations of the vectors,

$$\{C^a, \ell_i^b\} = \epsilon^{abc} \ell_i^c. \tag{3.104}$$

This Poisson structure on $\mathbb{R}^3$ arises naturally if one views it as the dual of $\mathfrak{su}(2)$.

In general, if f_c^{ab} are the structure constants of a Lie algebra $\mathfrak{g}$ with generators $\{e_a\}$, then one can define a Poisson bracket on $\mathfrak{g}^*$ by

$$\{e^{*a}, e^{*b}\} = f_c^{ab} e^{*c} \tag{3.105}$$

where here e^{*a} is thought of as the function $\mathfrak{g}^* \to \mathbb{R}$ which gives the a^{th}. If one restricts to the coadjoint orbits of the group, this Poisson structure is symplectic.

We can go further by considering a Poisson bracket on $(T^*SU(2))^3 \cong (SU(2) \ltimes \mathfrak{su}(2)^*)^3$:

$$\{\ell_i^a, \ell_j^b\} = \delta_{ij}\epsilon^{ab}{}_c\ell_i^c \ , \quad \{h_i{}^a{}_b, h_j{}^c{}_d\} = 0 \ , \quad \{\ell_i^a, h_j{}^b{}_c\} = \delta_{ij}(h_i\sigma^a)^b{}_c, \tag{3.106}$$

where σ are the Pauli matrices. The functions $h_i{}^a{}_b$, with i going from 1 to 3 representing which edge of the triangle we are considering, are the matrix entries of h_i in the fundamental representation. The holonomy h are again interpreted as being perpendicular to the edges. This is a special case of a Heisenberg double [2, 96, 45].

The phase space $T^*SU(2)$ is not yet the phase space we associate with a triangle. We also demand that the rotational invariance is imposed. To obtain the physical variables which are invariant under rotations, we should reduce the phase space by a process known as symplectic reduction [75].

The idea of having the phase space invariant under rotations is realized as an action of $SU(2)$ on $(T^*SU(2))^3$, given explicitly by $g \triangleright (h_i, \ell_i) = (gh_i, g \triangleright \ell_i)$. The action of $g \in SU(2)$ on $\ell_i \in R^3$ is a by the coadjoint action (specifically, one may see ℓ_i as a function on R^3 with an action induced by the coadjoint action of $SU(2)$ on $\mathfrak{su}(2)^* \cong \mathbb{R}^3$). The action itself commutes with the Poisson brackets (3.106), so we call it a Poisson action.

The action of $SU(2)$ on functions of $(T^*SU(2))^3$ induces an action of $\mathfrak{su}(2)$ on the same space. This is defined by

$$X \triangleright f(x) = \frac{\mathrm{d}}{\mathrm{d}x} f(\exp(tX) \triangleright x)|_{t=0} \tag{3.107}$$

for any X in $\mathfrak{su}(2)$. In particular this tells us

$$\begin{aligned} X \triangleright \ell^a &= X^i \ell^j \epsilon^a_{ij} \\ X \triangleright h^i{}_j &= -X^a (h\sigma_a)^i{}_j. \end{aligned} \tag{3.108}$$

The action is diagonal, meaning that X acts on all three factors of $T^*SU(2)$ in the same way.

This action is generated by $H_X(\ell_I) := -\sum_I \langle \ell_I, X\rangle = -\sum_I \ell_I^a X_a$ where $\langle \cdot, \cdot \rangle$ is the canonical pairing between $\mathfrak{su}(2)$ and $\mathfrak{su}(2)^*$. The map $(\Phi : T^*SU(2))^3 \to (\mathfrak{su}(2)^*)^3$ given by $(\ell_I) \to H_.(\ell)$ is called the moment map associated to the action of $SU(2)$ [9].

The phase space we consider is

$$\mathcal{P}_{\text{triangle}} = \Phi^{-1}(0)/SU(2). \tag{3.109}$$

This general approach of determining moment maps and defining the phase space from that will be used in the following sections. The closure constraint of the triangles appears in $\Phi^{-1}(0) = \{\ell_I \in R^3 : \Phi(\ell_I) = \sum_I \ell_I = 0\}$. A general feature which we will mention again later is the interplay between a constraint in $\mathfrak{g}^*$ corresponding to an action of G.

In more general cases we will consider later, the phase spaces we will reduce will not be cotangent bundles and the Hamiltonians H_X will have to be defined differently.

To explain how one goes from this phase space for a triangle to the phase space of other polygons, we will introduce Heisenberg and Drinfeld doubles.

Heisenberg and Drinfeld Doubles Let G and G^* are Lie groups of same dimension with $\mathfrak{g}$ and $\mathfrak{g}^*$ as Lie algebras, which form a matched pair. The group with lie algebra $\mathfrak{g} \bowtie \mathfrak{g}^*$ is denoted $\mathcal{D}$. Locally, one can factor the group as $\mathcal{D} \cong G \bowtie G^* \cong G^* \bowtie G$. For simplicity we will always assume that this decomposition is global or that we are working in a set neighbourhood where such a decomposition is possible. The a group element $d \in \mathcal{D}$ can be factored in two ways:

$$d = \ell h = \tilde{h}\tilde{\ell}, \qquad\qquad h, \tilde{h} \in G, \quad \ell, \tilde{\ell} \in G^*. \tag{3.110}$$

The two factorizations are related via the actions

$$\begin{aligned} \ell &= \tilde{h} \triangleright \tilde{\ell} \qquad & h &= \tilde{h} \triangleleft \tilde{\ell} \\ \tilde{\ell} &= \ell \triangleleft h \qquad & \tilde{h} &= \ell \triangleright h. \end{aligned} \tag{3.111}$$

We will assume that the cobracket of $\mathfrak{g} \bowtie \mathfrak{g}^*$ is given by an r-matrix: $\delta\xi = ad_\xi r$. Concretely, we may write

$$r := r_{12} = e^i \otimes e^*_i \qquad r_{21} = e^*_i \otimes e^i \tag{3.112}$$

where e and e^* are generators of $\mathfrak{g}$ and $\mathfrak{g}^*$. On $\mathcal{D}$, one may define a Poisson bivector π_+.

$$\pi_+(d) = -[d \otimes d, r_-]_+ = -(d \otimes d)r_- - r_-(d \otimes d) \tag{3.113}$$

where $2r_- = r_{12} - r_{21}$ is the antisymmetrization of r and $[\cdot,\cdot]_+$ stands for the anticommutator. The Poisson bivector in (3.113) turns out to be invertible and so it defines a symplectic space. The symplectic space defined by π_+ and the group $\mathcal{D}$ is called the Heisenberg double (of G) [2, 96, 45]. The Poisson brackets between matrix entries can be determined by

$$\{d^i{}_j, d^k{}_l\}_+ = -d^i{}_a d^k{}_b r_-{}^a{}_j{}^b{}_l - r_-{}^i{}_a{}^k{}_b d^a{}_j d^b{}_l. \tag{3.114}$$

Starting from (3.113), one can determine the Poisson brackets between the factors h, ℓ, $\tilde{h}$, and $\tilde{\ell}$. We will use the convenient notation $u_1 = u \otimes 1$ and $u_2 = 1 \otimes u$, for $u = \ell, \tilde{\ell}, h, \tilde{h}$,

$$\begin{aligned} \{\tilde{h}_1, \tilde{h}_2\} &= -[r_-, \tilde{h} \otimes \tilde{h}] & \{\tilde{h}_1, \tilde{\ell}_2\} &= -(1 \otimes \tilde{h}) r_{12} (1 \otimes \tilde{\ell}) \\ \{\tilde{\ell}_1, \tilde{\ell}_2\} &= -[\tilde{\ell} \otimes \tilde{\ell}, r_-] & \{\tilde{\ell}_1, \tilde{h}_2\} &= (1 \otimes \tilde{h}) r_{21} (\tilde{\ell} \otimes 1) \\ \{\ell_1, \ell_2\} &= -[r_-, \ell_1 \otimes \ell_2] & \{h_1, h_2\} &= [r_-, h \otimes h] \\ \{h_1, \ell_2\} &= -(1 \otimes \ell) r_{12} (h \otimes 1) & \{\ell_1, h_2\} &= (\ell \otimes 1) r_{21} (1 \otimes h) \\ \{\tilde{h}_1, \ell_2\} &= -r_{12} \tilde{h} \otimes \ell & \{\tilde{\ell}_1, h_2\} &= \tilde{\ell} \otimes h r_{21}. \end{aligned} \tag{3.115}$$

The Poisson bivector is invertible and so the phase space is symplectic. The symplectic form at d, written in terms of factors of d, is

$$\Omega(d) = \frac{1}{2}\big(\langle \Delta\ell \wedge \Delta\tilde{h}\rangle + \langle \underline{\Delta}\tilde{\ell} \wedge \underline{\Delta} h\rangle\big) \tag{3.116}$$

where $\Delta u = \delta u u^{-1}$ and $\underline{\Delta} u = u^{-1}\delta u$ are respectively the right and left Maurer Cartan forms.

There is an additional Poisson structure which can be added onto $\mathcal{D}$ which is not symplectic. The group $\mathcal{D}$ together with the Poisson bivector

$$\pi_-(d) = [d \otimes d, r_-] \tag{3.117}$$

is the Drinfeld double [2, 96, 45]. Unlike the Heisenberg double, the Drinfel'd double is a Poisson-Lie group meaning that the group multiplication is a Poisson map. In particular the Poisson brackets are degenerate at the origin. Furthermore, the Poisson bivector is related to the cobracket of the Lie bialgebra, $\delta X = \frac{\mathrm{d}}{\mathrm{d}t}\pi_-(e^{tX})_{t=0}$.

The phase space of the polygon in the previous section is a Heisenberg double with $\mathfrak{g} = \mathfrak{su}(2)$ with a trivial cobracket $\delta = 0$. The dual algebra in this case is isomorphic to $\mathbb{R}^3$. One may choose as a basis of $\mathfrak{su}(2)$ in the adjoint representation

$$J_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad J_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad J_3 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{3.118}$$

and the basis of $\mathfrak{so}(3)^* \cong \mathbb{R}^3$ as

$$e_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad e_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad e_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{3.119}$$

Explicitly computing the r-matrix and then using (3.113), we recover the brackets (3.106).

Now we can consider the action of the group $\mathcal{D}$ (seen as the Poisson Lie Group with π_-) on $\mathcal{D}$ (seen as a symplectic manifold with π_+). This is the action we will reduce by when we wish to glue triangles together. To do this we should point out some generalizations to when the group which acts on a symplectic manifold has a Poisson structure itself.

If H is a Poisson Lie group which acts on a symplectic manifold M then the action is a Poisson-Lie action if [95]

$$\begin{aligned}\{f(h\triangleright x), g(h\triangleright x)\}_{H\times M} &= \{f(h\triangleright \cdot), g(h\triangleright \cdot)\}_M(x) + \{f(\cdot\triangleright x), g(\cdot\triangleright x)\}_H(h)\\ &= \{f,g\}_M(h\triangleright x)\end{aligned} \tag{3.120}$$

The first equality is simply the definition of the cartesian product of Poisson manifolds, while the second is that the action is covariant. For our purposes $M = \mathcal{D}_+$ and $H = \mathcal{D}_-$ and the action is multiplication from the left or from the right.

Once again, the moment map will be the generator of the infinitesimal transformation coming from the group multiplication, which we find by restricting our attention to multiplication by the subgroups G and G^* separately. Using the factorisation for $d = \ell h = \tilde{h}\tilde{\ell}$, the multiplication on the left by an element $h' \in G$ can be written as

$$d \to h'd = h'\ell h = (h'\triangleright \ell)(h'\triangleleft \ell)h \tag{3.121}$$

and

$$d \to h'd = h'\tilde{h}\tilde{\ell}. \tag{3.122}$$

The infinitesimal version of the left multiplication can be inferred by writing $h' \approx 1+\alpha$ where $\alpha \in \mathfrak{g}$. The resulting infinitesimal transformations, denoted by δ^L_α are

$$\begin{aligned}\delta^L_\alpha h &= (\alpha\triangleleft \ell)h\\ \delta^L_\alpha \ell &= \alpha\ell - \ell(\ell^{-1}\triangleright \alpha)\end{aligned} \tag{3.123}$$

and

$$\begin{aligned}\delta^L_\alpha \tilde{h} &= \alpha\tilde{h}\\ \delta^L_\alpha \tilde{\ell} &= 0\end{aligned} \tag{3.124}$$

The relations $h\ell = (h\triangleright \ell)(h\triangleleft \ell)$ are used to derive these expressions, as well as the compatibility relations of a matched pair of groups [73]

$$\begin{aligned}\ell\triangleright(h_1h_2) &= (\ell\triangleright h_1)((\ell\triangleleft h_1)\triangleright h_2),\\ (\ell_1\ell_2)\triangleleft h &= (\ell_1\triangleleft(\ell_2\triangleright h))(\ell_2\triangleleft h)\\ (\ell\triangleright h)^{-1} &= h^{-1}\triangleleft \ell^{-1}.\end{aligned} \tag{3.125}$$

Similarly the multiplication by $\ell' \in G^*$ with $\ell' \approx 1 + \phi$, $\phi \in \mathfrak{g}^*$ gives

$$\begin{aligned} d \to \ell' d &= \ell' \ell h \\ \Longrightarrow \delta^L_\phi \ell &= \phi \ell \\ \delta^L_\phi h &= 0 \end{aligned} \tag{3.126}$$

and

$$\begin{aligned} d \to \ell' d &= \ell' \tilde{h} \tilde{\ell} \\ \Longrightarrow \delta^L_\phi \tilde{h} &= \phi \tilde{h} - \tilde{h}(\tilde{h}^{-1} \triangleright \phi) \\ \delta^L_\phi \tilde{\ell} &= (\phi \triangleleft \tilde{h}) \tilde{\ell}. \end{aligned} \tag{3.127}$$

The infinitesimal multiplication from the right can be found similarly.

We can now try to determine the generalization of the Hamiltonian which generates these infinitesimal transformations to the Heisenberg double [8]. We can then determine the moment maps and hence perform the necessary reduction.

One can check that $\{\ell_1, \cdot\}(\ell^{-1} \otimes 1)$ generates left multiplications by G:

$$\begin{aligned} \langle \{\ell_1, h_2\}(\ell^{-1} \otimes 1), \alpha \otimes 1 \rangle_1 &= \langle (\ell \otimes 1) r_{21} (1 \otimes h)(\ell^{-1} \otimes 1), \alpha \otimes 1 \rangle \\ &= \langle \ell e^*_I \ell^{-1}, \alpha \rangle e^I h \\ &= \langle e^*_I, \ell^{-1} \alpha \ell \rangle e^I h \\ &= (\ell^{-1} \alpha \ell)_{\mathfrak{g}} h = \delta^L_\alpha h \end{aligned} \tag{3.128}$$

where the subscript 1 on the angle brackets indicates the pairing is between the first factor in the tensor problem. In the last line the subscript $\mathfrak{g}$ indicates the projection onto the $\mathfrak{g}$ algebra. This is equivalent to the action $\alpha \triangleleft \ell$.

Similarly we can check that $\{\ell_1, \cdot\}(\ell^{-1} \otimes 1)$ generates the multiplication on the other functions, ℓ, $\tilde{\ell}$ and $\tilde{h}$ in the same way. For this reason we define the map $\Phi^L_G : \mathcal{D} \to G^*$ given by $\Phi^L_G(d = \ell h) = \ell$. This is the generalization of the moment map in the non abelian case.

Similarly the moment map for multiplication by G^* (parametrized by ϕ) is the projection onto $\tilde{h}$: $\delta^L_\phi \cdot = \langle \{\tilde{h}_1, \cdot\}(\tilde{h}^{-1} \otimes 1), \phi \otimes 1 \rangle_1$.

The right transformations are given by $\delta^R_\phi \cdot = \langle (1 \otimes h^{-1})\{\cdot, h_2\}, \phi \rangle_2$ and $\delta^R_\alpha = \langle (1 \otimes \ell^{-1})\{\cdot, \tilde{\ell}_2\}, \alpha \rangle_2$.

Thus we have that the moment maps for multiplication by G is the projection map onto G^* and vice versa.

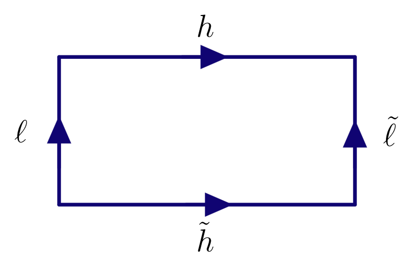

Figure 3.4: The two decompositions of an element in the Heisenberg double are depicted by a ribbon diagram

To glue two phase spaces $\mathcal{D}^1$ and $\mathcal{D}^2$ along an edge (which we label with an element of G^*), we consider a group action which is generated by $\ell_1\ell_2$, where the subscripts indicate which phase space they belong to. One can check that this is the generator of G acting diagonally on $\mathcal{D}^1$ and $\mathcal{D}^2$.

Graphically one can express the idea of identifying the edges of two phase spaces by ribbon diagrams. To account for the two possible factorisations $d = \ell h = \tilde{h}\tilde{\ell}$, we can replace the link/edge pairs by rectangular commuting diagrams as in Fig. 3.4. The gluing procedure is evidently results in a new larger ribbon. To see the G portion of the new ribbon one can consider the sum of symplectic potentials,

$$\begin{aligned} 2\Omega_{1,2} &:= \Omega_1 + \Omega_2 && (3.129)\\ &= \langle \Delta\ell_1 \wedge \Delta\tilde{h}_1\rangle + \langle \underline{\Delta}\tilde{\ell}_1 \wedge \underline{\Delta}h_1\rangle + \langle \Delta\ell_2 \wedge \Delta\tilde{h}_2\rangle + \langle \underline{\Delta}\tilde{\ell}_2 \wedge \underline{\Delta}h_2\rangle. && (3.130)\end{aligned}$$

Defining the variables

$$h_{12} = h_1\tilde{h}_2^{-1} \qquad \tilde{h}_{12} = \tilde{h}_1 h_2^{-1} \tag{3.131}$$

implies

$$\Delta\tilde{h}_1 = \Delta\tilde{h}_{12} + \tilde{h}_{12}\Delta h_2\tilde{h}_{12}^{-1} \qquad \underline{\Delta}h_2 = \tilde{h}_1^{-1}\Delta\tilde{h}_{12}\tilde{h}_1 + \underline{\Delta}\tilde{h}_1. \tag{3.132}$$

These relations can be used to eliminate $\underline{\Delta}\tilde{h}_1$ and $\Delta\tilde{h}_2$ in the first and third term in (3.130). Furthermore, using $\ell h = \tilde{h}\tilde{\ell}$, we can write for $i = 1, 2$

$$\Delta\ell_i = \Delta\tilde{h}_i + \tilde{h}_i\Delta\tilde{\ell}_i\tilde{h}_i^{-1} - \tilde{h}_i\tilde{\ell}_i\underline{\Delta}h_i\tilde{\ell}_i^{-1}\tilde{h}_2^{-1} \tag{3.133}$$

Doing so gives

$$\begin{aligned} 2\Omega_{1,2} =&\langle \Delta\ell_1 \wedge \Delta\tilde{h}_{12}\rangle + \langle \underline{\Delta}\ell_2^{-1} \wedge \underline{\Delta}h_{12}\rangle\\ &+ \langle (\tilde{h}_1^{-1}\Delta\ell_1\tilde{h}_1 + \underline{\Delta}\tilde{\ell}_2) \wedge \underline{\Delta}h_2\rangle + \langle (\tilde{h}_2^{-1}\Delta\ell_2\tilde{h}_2 + \underline{\Delta}\tilde{\ell}_1) \wedge \underline{\Delta}h_1\rangle. && (3.134)\end{aligned}$$

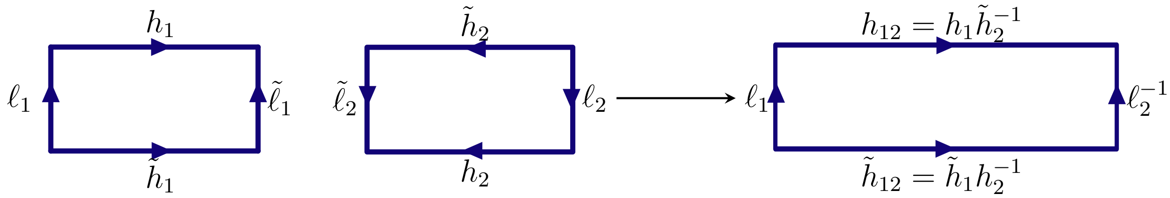

Figure 3.5: Each phase space is represented by a ribbon diagram. Two ribbons are glued along an edge to create a single ribbon. The geometric gluing is the same as demanding the constrain $\tilde{\ell}_1\tilde{\ell}_2 = 1$.

Finally we can impose the constraint which comes from the moment map: $\tilde{\ell}_1 = \tilde{\ell}_2^{-1}$ which implies $\underline{\Delta}\tilde{\ell}_2 = -\underline{\Delta}\ell_1$, giving

$$\Omega_{1,2} = \frac{1}{2}\big(\langle \Delta\ell_1 \wedge \Delta\tilde{h}_{12}\rangle + \langle \underline{\Delta}\ell_2^{-1} \wedge \underline{\Delta}h_{12}\rangle\big) \tag{3.135}$$

Thus the imposition of the constraint in G^* naturally gives rise to new fields in G, here h_{12} and $\tilde{h}_{12}$. The gluing of phase spaces can be seen by gluing two ribbons as shown in Fig. 3.5.

Heisenberg doubles and two dimensional triangulation. We can construct the phase space of a triangle (or of any polygon) using several copies of a Heisenberg double $\mathcal{D}_+ = G \bowtie G^*$ and by extension the phase space of a triangulation. Without loss of generality, we will assume that the G^* decorations are associated with the triangle edges, while the G decorations will be associated to the dual complex.

The phase space of a triangle is obtained by considering three copies of the Heisenberg double. We impose then that the product of elements in G^* is the identity, see Fig. 3.6. This is the closure constraint $\mathcal{C}$ encoding that the triangle closes. It is easy to see that it is a momentum map implementing a global G transformation on each copy of the Heisenberg double. The triangle phase space is therefore obtained as the symplectic reduction of the three Heisenberg doubles through the momentum map implementing a diagonal G transformation.

$$\mathcal{P}_{triangle} = \big(\mathcal{D}_+\big)^{\times 3}//G. \tag{3.136}$$

Note that this constraint is geometrically equivalent to gluing three ribbons in terms of a triangle. This can be seen as the fattening of the dual complex to the triangle - making the links dual to the edges into ribbons.

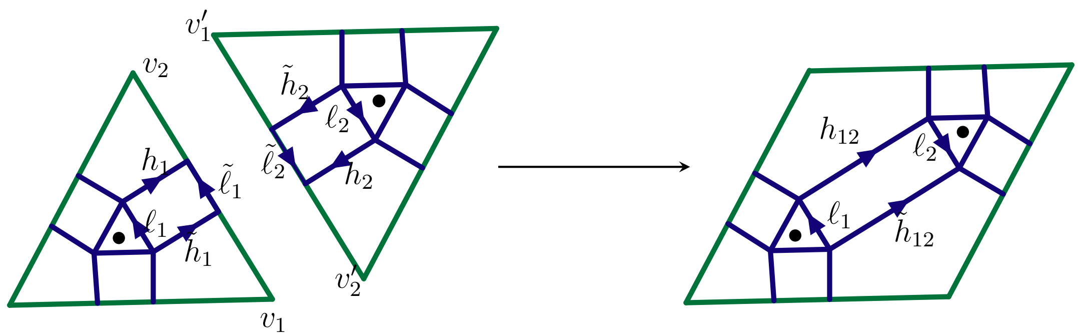

Figure 3.6: The phase space of a triangle is given by a set of three ribbons subject to the constraint some of their G^* decorated sides form a triangle as illustrated on the left hand side. The phase space of two triangles (and by extension of a triangulation) is obtained by gluing ribbons associated to the shared edge.

The extension to a triangulation follows directly. If the triangulation is made by n triangles, we consider n copies of the triangle phase space $\mathcal{P}_{triangle}$ and identify the G^* decorations of the ribbon appropriately, see Fig. 3.6. In terms of the ribbon picture, this corresponds to gluing the ribbons as discussed above by symplectic reduction. This provides a well defined phase space construction for a two dimensional triangulation. This construction extends to an arbitrary cellular decomposition, using polygons of any kind.

3.4.2 Phase spaces for three dimensional triangulations

The above describes the procedure for creating phase spaces that we can identify with polygons and, by gluing, phase spaces identified with triangulations of 2-manifolds. This is useful for constructing states in (2+1) - dimensional theories [26]. For theories in (3+1)-dimensions we will introduce phase spaces with data on the links, faces subtended by links, and their respective 2- and 1-dimensional duals. In order to make the 1- and 2-dimensional pieces appear in a consistent way, we will work with skeletal 2-Lie algebras and their double. To illustrate this generalisation we can consider an example:

Example: κ-Poincaré Consider the Lorentz algebra once again, with generators $\mathcal{J}_{\mu\nu}$ satisfying the Lie brackets relation

$$[J^{\mu\nu}, J^{\rho\sigma}] = \eta^{\mu\rho} J^{\nu\sigma} + \eta^{\nu\sigma} J^{\mu\rho} - \eta^{\mu\sigma} J^{\nu\rho} - \eta^{\nu\rho} J^{\mu\sigma}. \tag{3.137}$$

This can be made into a Lie bialgebra trivially by introducing the cobracket

$$\delta_{\mathfrak{so}}(J^{\mu\nu}) = 0. \tag{3.138}$$

In addition, one considers the $\mathfrak{an}$ algebra with the four generators P^μ satisfying

$$[P^\mu, P^\nu] = \kappa^{-1}(\eta^{\mu 0}P^\nu - \eta^{\nu 0}P^\mu) \ , \tag{3.139}$$

which can also be made into a bialgebra by setting

$$\delta_{\mathfrak{an}}(P^\mu) = 0. \tag{3.140}$$

The parameter κ is a real number not equal to zero. These two algebras form a matched pair if we define the actions according to

$$J^{\mu\nu} \triangleleft P^\rho = \kappa^{-1}(\eta^{\nu 0}J^{\mu\rho} - \eta^{\mu 0}J^{\nu\rho}) \qquad J^{\mu\nu} \triangleright P^\rho = \eta^{\mu\rho}P^\nu - \eta^{\nu\rho}P^\mu. \tag{3.141}$$

We note that in the limit when $\kappa \to 0$, the matched pair is simply the traditional Poincaré algebra where P^μ are the commuting translation generators. According to (3.2.3) these actions imply the crossed bracket

$$[P^\rho, J^{\mu\nu}] = \kappa^{-1}(\eta^{\nu 0}J^{\mu\rho} - \eta^{\mu 0}J^{\nu\rho}) + \eta^{\mu\rho}P^\nu - \eta^{\nu\rho}P^\mu. \tag{3.142}$$

The dual Lie bialgebra $\mathfrak{su}^* \blacktriangleright\!\!\blacktriangleleft \mathfrak{an}^*$ is defined as being generated by $J^*_{\mu\nu}$ and P^*_μ and the pairing

$$\langle P^\mu, P^*_\nu \rangle = \delta^\mu_\nu \ , \quad \langle J^{\mu\nu}, J^*_{\rho\sigma} \rangle = \delta^\mu_\rho \delta^\nu_\sigma - \delta^\nu_\rho \delta^\mu_\sigma. \tag{3.143}$$

The subspaces $\mathfrak{so}^*$ and $\mathfrak{an}^*$ each have a bialgebra structure determined by dualising both the Lie brackets and the (trivial) cocommutator.

The semidualisations of $\mathfrak{so} \bowtie \mathfrak{an}$ and its dual are therefore Lie 2-bialgebras. The phase space corresponding to the bialgebra $\mathfrak{b} = (\mathfrak{so} \rtimes\!\!\blacktriangleleft \mathfrak{an}^*) \bowtie (\mathfrak{so}^* \blacktriangleright\!\!\ltimes \mathfrak{an})$ will be used as the main example in the following.

The classical double $\mathfrak{b}$ is a coboundary Lie bialgebra and so we can define the classical r-matrix

$$r = P^\mu \otimes P^*_\mu + \frac{1}{2}J^{\mu\nu} \otimes J^*_{\mu\nu}. \tag{3.144}$$

The brackets for $\mathfrak{b}$ are, according to (3.28) and (3.29)

$$\begin{aligned}[J^{\mu\nu}, P^*_\sigma] &= \delta^\nu_\sigma \eta^{\mu\alpha} P^*_\alpha - \delta^\mu_\sigma \eta^{\nu\alpha} P^*_\alpha \\ \delta P^*_\rho &= \kappa^{-1} \eta^{\sigma 0} (P^*_\sigma \otimes P^*_\rho - P^*_\rho \otimes P^*_\sigma) \\ \delta J^{\mu\nu} &= \kappa^{-1} (\eta^{\nu 0} J^{\mu\rho} - \eta^{\mu 0} J^{\nu\rho}) \otimes P^*_\rho - \kappa^{-1} P^*_\rho \otimes (\eta^{\nu 0} J^{\mu\rho} - \eta^{\mu 0} J^{\nu\rho}) \\ [P^\sigma, J^*_{\mu\nu}] &= \kappa^{-1} \eta^{\beta 0} (\delta^\alpha_\mu \delta^\sigma_\nu - \delta^\sigma_\mu \delta^\alpha_\nu) J^*_{\alpha\beta} \\ \delta J^*_{\mu\nu} &= \eta^{\rho\sigma} J^*_{\mu\rho} \otimes J^*_{\nu\sigma} - \eta^{\rho\sigma} J^*_{\nu\sigma} \otimes J^*_{\mu\rho} \\ \delta P^\mu &= \eta^{\sigma\mu} (P^\rho \otimes J^*_{\sigma\rho} - J^*_{\sigma\rho} \otimes P^\rho)\end{aligned} \tag{3.145}$$

as well as the Lie brackets in (3.138) and (3.139). The cross brackets can be determined by using (3.17),

$$\begin{aligned}[J^{\mu\nu}, J^*_{\sigma\rho}] =& (\eta^{\alpha\nu} \delta^\mu_\rho - \eta^{\alpha\mu} \delta^\nu_\rho) J^*_{\sigma\alpha} - \kappa^{-1} (\delta^\mu_\sigma \eta^{\nu 0} - \eta^{\mu 0} \delta^\nu_\sigma) P^*_\rho - (\sigma \leftrightarrow \rho) \\ [P^\mu, P^*_\nu] =& \kappa^{-1} \eta^{\sigma 0} (\delta^\mu_\nu P^*_\sigma - \delta^\mu_\sigma P^*_\nu) - \eta^{\sigma\mu} J^*_{\sigma\nu} \\ [P^\rho, J^{\mu\nu}] =& \kappa^{-1} (\eta^{\nu 0} J^{\mu\rho} - \eta^{\mu 0} J^{\nu\rho}) + \eta^{\mu\rho} P^\nu - \eta^{\nu\rho} P^\mu.\end{aligned} \tag{3.146}$$

This example gives a non trivial Poisson structure to $AN \ltimes SO^* \cong AN \ltimes \mathbb{R}^n$ which, upon quantization, is the κ-Poincaré algebra, a deformation of the Poincaré Lie algebra [69, 70, 106].

We can interpret $SO \ltimes \mathbb{R}^m$ and $AN \ltimes \mathbb{R}^n$, where n and m are the respective dimensions of the groups SO and AN, as a pair of skeletal Poisson 2-groups. This pair then forms a Heisenberg double which we can associate to a basic building geometric block to generate the triangulation.

This κ-Poincaré structure is therefore an example of the type of phase space we will consider in our discretization.

General skeletal Poisson 2-group In more general terms we let $\mathcal{G} = G_1 \ltimes G_2^*$ be a skeletal 2-group. The Heisenberg double is then $\mathcal{B} = \mathcal{G} \bowtie \mathcal{G}^*$.

The analogue of the link-edge pair in this higher dimensional piece is given by the "atomic phase space" defined by

- a link
- a two dimensional surface perpendicular to the link

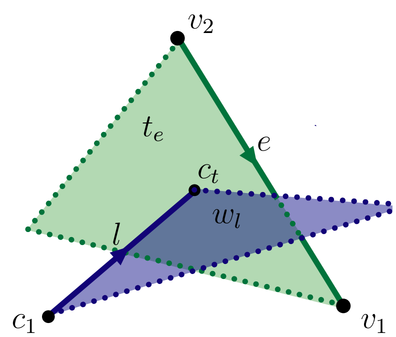

Figure 3.7: The building block we will glue to reconstruct a full 3d triangulation. The (half) link and wedge will be decorated by $\mathcal{G}$ elements, while the edge and face will be decorated by elements in $\mathcal{G}^*$.

- an edge on the boundary of that surface
- a two dimensional surface perpendicular to the edge which includes the link as part of its boundary.

The situation is drawn in Fig. 3.7.

We will now detail the gluing procedure for this slightly more complicated set up. There are now more ways to factor an element of $\mathcal{B}$ leading to more factors to identify in the gluing procedure.

For any $d \in \mathcal{B}$, we have the factorization $d = \ell h = \tilde{h}\tilde{\ell}$. In addition since $\mathcal{G}$ and $\mathcal{G}^*$ are crossed modules, we can write $h \in \mathcal{G}$ as $h = uy = \overline{y}u$ with $u \in G_1$ and $y, \overline{y} \in G_2^*$ where y and $\overline{y}$ are related by the action of the crossed module. We can also write $\ell = \beta\lambda = \lambda\overline{\beta}$ for $\lambda \in G_1^*$ and β, $\tilde{\beta} \in G_2^*$. There are similar factorizations for $\tilde{h}$ and $\tilde{\ell}$ as well.

There are several ways that such atomic pieces can be glued together.

- G_1 gluing: Links are decorated by G_1 holonomies. To glue two links together, the dual variables on the triangles valued in G_1^* should be identified.
- G_1^* gluing: Triangles decorated with G_1^* variables can be composed by identifying the dual G_1 variables.
- G_2 gluing: Edges decorated with G_2 variables can be glued together by identifying the surfaces dual to the edges which are decorated by G_2^* variables.

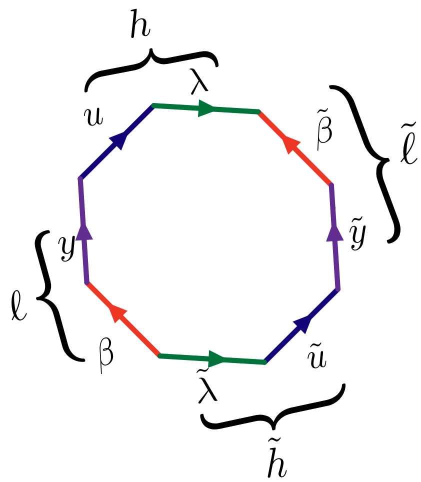

Figure 3.8: The ribbon diagram representing the division of G and G^* into subgroups.

- G_2^* gluing: Identifying edges gives the composition of their respective dual two dimensional surfaces which are decorated in G_2^*.

To guide the somewhat cumbersome notation we introduce a analogue to the ribbon diagrams in Fig. 3.4 and Fig. 3.5. In the polyhedron picture there are four factors, G_1, G_1^*, G_2, G_2^* which means that instead of rectangular ribbons, we can consider octogonal ribbons as in Fig. 3.8. Note that the order in which factors appear in the figure can be changed by utilizing the action between subgroups. As in the rectangular ribbon diagrams, identifying geometric structures implies gluing sides of the ribbon.

We can now consider the gluing of the various subgroups of the phase space. We will show in some detail how to glue together a pair of links in some detail.

G_1 **gluing:** Using Fig. 3.9 gluing the links together implies that the triangle decorations are to be identified. From the figure, we set $c_t = c_t'$ and $v_2 = v_2'$. The first edge-link pair is given an element $d_1 = \tilde{h}_1\tilde{\ell}_1 \in (G_2^* \ltimes G_1) \bowtie (G_2 \rtimes G_1^*)$. The $\tilde{\ell}_1$ variable is further factored into $\tilde{\ell}_1 = \tilde{\lambda}_1\tilde{\beta}_1$, where now $\tilde{\beta}_1$ decorates the face and $\tilde{\lambda}$ decorates the edge. The identification of the faces is therefore accomplished by imposing $\bar{\tilde{\beta}}_1 = -\bar{\tilde{\beta}}_2$. The ribbon diagram for this gluing is shown in Fig. 3.10.

We begin by isolating the G_1^* elements of two phase spaces

$$\ell_i h_i = \tilde{h}_i\tilde{\ell}_i = \tilde{h}_i(\tilde{\lambda}_i\bar{\tilde{\beta}}_i) \implies \bar{\tilde{\beta}}_i = \tilde{\lambda}_i^{-1}\tilde{h}_i^{-1}\ell_i h_i \tag{3.147}$$

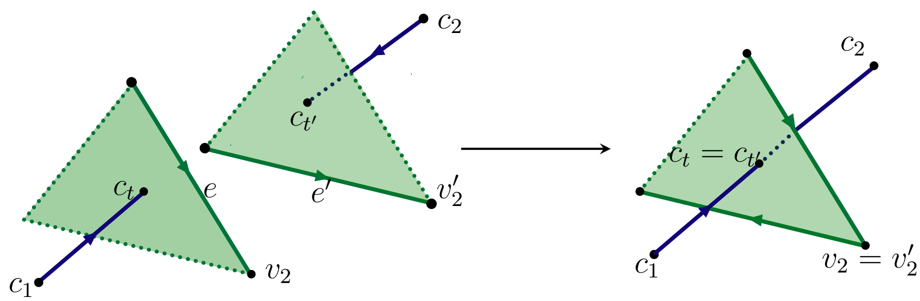

Figure 3.9: We fuse two links by identifying the dual faces t_e and $t_{e'}$. We require that the points c_t and $c_{t'}$ in the respective face to match. As we discussed before, face decorations are rooted at a point. Matching the face decorations imposes that they have the same root. Picking v_2 and v_2' as respectively root of t_e and $t_{e'}$, we therefore must have $v_2 = v_2'$.

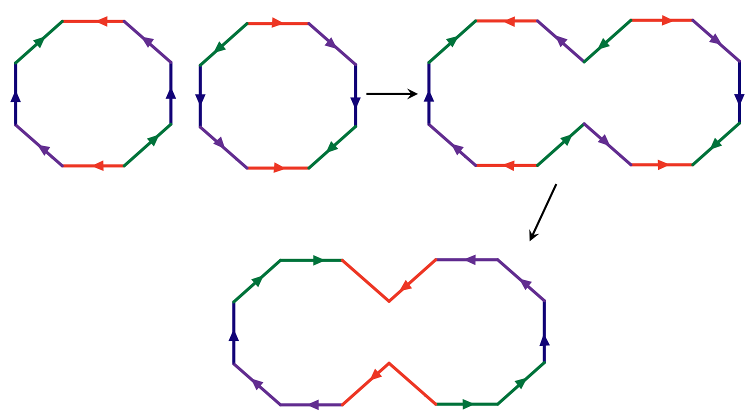

Figure 3.10: We glue two octagons. We then re-arrange the sides belonging to the same groups. This rearranging, involving either actions, back actions and conjugations can induce non-trivial contributions, especially from the conjugations. The choice of ordering is a priori arbitrary and for our concerns, will depend on the choice of frame we intend to express our variables in.

and then imposing that $\tilde{\bar{\beta}}_1 = -\tilde{\bar{\beta}}_2$,

$$\tilde{\lambda}_1^{-1}(\tilde{u}_1\tilde{y}_1)^{-1}(\lambda_1\bar{\beta}_1)(\bar{y}_1u_1) = (\bar{y}_2u_2)^{-1}(\lambda_2\bar{\beta}_2)^{-1}(\tilde{u}_2\tilde{y}_2)\tilde{\lambda}_2. \tag{3.148}$$

The factors can be commuted with one another by using the mutual action defined by the Heisenberg double or the action defining the crossed module. Additionally, the brackets in the Heisenberg double imply that the conjugation of a G_2^* (resp. G_1^*) element by a G_1 (resp. G_2) element will have a component in G_1^* and G_2^*. Explicitly, we write $(u_1 \triangleright \tilde{\lambda}_2^{-1})^{-1}\, \bar{y}_1\, (u_1 \triangleright \tilde{\lambda}_2^{-1}) = y'\beta'$ and $(\tilde{\lambda}_1 \triangleleft u_2^{-1})\, \bar{y}_2\, (\tilde{\lambda}_1 \triangleleft u_2^{-1})^{-1} = y''\beta''$ for some $\beta', \beta'' \in G_2^*$ and $y', y'' \in G_1^*$. Reorganizing (3.148) yields

$$\begin{aligned}&\big(\lambda_1(u_1 \triangleright \tilde{\lambda}_2^{-1})\big)\big(\bar{\beta}_1 \triangleleft (u_1 \triangleright \tilde{\lambda}_2^{-1})\beta'\big)\big(y'((u_1 \triangleleft \tilde{\lambda}_2^{-1}) \triangleright \tilde{y}_2^{-1})\big)\big((u_1 \triangleleft \tilde{\lambda}_2^{-1})\tilde{u}_2^{-1}\big)\\ &= \big(\tilde{u}_1(\tilde{\lambda}_1 \triangleright u_2^{-1})\big)\big((\tilde{y}_1 \triangleleft (\tilde{\lambda}_1 \triangleright u_2^{-1}))y''\big)\big(\beta''(\tilde{\lambda}_1 \triangleleft u_2^{-1}) \triangleright \bar{\beta}_2^{-1}\big)\big((\tilde{\lambda}_1 \triangleleft u_2^{-1})\lambda_2^{-1}\big).\end{aligned} \tag{3.149}$$

This gives a similar ribbon condition to the original phase space. The factors in the final equation gives the "fused" variables, similar to the h_{12} variable in the rectangular ribbon. This new ribbon structure can then be used to define the symplectic form for the new phase space.

The other gluings are done in a similar fashion.

Examples

Knowing how to glue together atomic phase spaces now lets us construct more general stuctures.

For example, to construct a triangle, we start with three phase spaces $\mathcal{B}_i$. Since the triangle is dual to a link, the moment map should have the effect of gluing the links of each of the atoms, $l_1 = l_2 = l_3$. As such, the holonomy decorations u must all be identified $u_1 = \tilde{u}_2$, $u_2 = \tilde{u}_3$. Applying this gives a ribbon constraint. In addition one needs to impose the constraint that the edge variables close, $\lambda_1\lambda_2\lambda_3 = 1$.

The identification of the G_1 variables (the u's) defines a new triangle variable,

$$\bar{\beta}_t = \bar{\beta}_1 \triangleleft (\lambda_2\lambda_3) + \bar{\beta}_2 \triangleleft \lambda_3 + \bar{\beta}_3 + \big(\lambda_3^{-1}\bar{y}_2\lambda_3\big)\big|_{G_1^*} + \big((\lambda_2\lambda_3)^{-1}\,\bar{y}_1\,(\lambda_2\lambda_3)\big)\big|_{G_1^*}. \tag{3.150}$$

The phase space would then be written as

$$\mathcal{P}_t = (\mathcal{B}_{l_1,e_1} \times \mathcal{B}_{l_2,e_2} \times \mathcal{B}_{l_3,e_3})//\big((G_1^* \times G_1^*) \times G_2^*\big), \tag{3.151}$$

As another example, one can consider combining atomic phase spaces in such a way that we have a tetrahedron. One can do so by taking twelve copies of the Heisenberg double, $\mathcal{B}_{l_{a;i},e_{a;i}}$ for $a = 1,2,3,4$ and $i = 1,2,3$, associated to full links $l_{a;i} = (c_{a;i}c'_{a;i})$ and edges $e_{a;i} = (v_{a;i}v'_{a;i})$. Later on, the nodes $c_{a;1}, c_{a;2}, c_{a;3}$ will be identified as the source of the link dual to triangle labelled by a on the boundary of the tetrahedron and denoted c_a; then, the sources of the four links c_1, c_2, c_3, c_4 will be further identified as the center of the tetrahedron and be denoted as c.

To construct the tetrahedron phase space, we first construct four tetrahedra as previously described. The edges of each triangle is then glued to the other three triangles eg by

$$\lambda_{1;1} = \lambda_{3;3}^{-1} \ , \quad \lambda_{1;2} = \lambda_{4;3}^{-1} \ , \quad \lambda_{1;3} = \lambda_{2;1}^{-1} \ , \quad \lambda_{2;2} = \lambda_{4;2}^{-1} \ , \quad \lambda_{2;3} = \lambda_{3;1}^{-1} \ , \quad \lambda_{3;2} = \lambda_{4;1}^{-1}. \tag{3.152}$$

Gluing the phase spaces along a link to get the triangle introduces new variables akin to (3.150):

$$\begin{aligned}
\bar{\beta}_{t_1} &= \beta_{1;1} + \bar{\beta}_{1;2} \lhd \lambda_{1;3} + \bar{\beta}_{1;3} + \big(\lambda_{1;3}^{-1}\bar{y}_{1;2}\lambda_{1;3}\big)\big|_{G_1^*} + \big((\lambda_{1;2}\lambda_{1;3})^{-1}\,\bar{y}_{1;1}\,(\lambda_{1;2}\lambda_{1;3})\big)\big|_{G_1^*},\\
\bar{\beta}_{t_2} &= \beta_{2;1} + \bar{\beta}_{2;2} \lhd \lambda_{2;3} + \bar{\beta}_{2;3} + \big(\lambda_{2;3}^{-1}\bar{y}_{2;2}\lambda_{2;3}\big)\big|_{G_1^*} + \big((\lambda_{2;2}\lambda_{2;3})^{-1}\,\bar{y}_{2;1}\,(\lambda_{2;2}\lambda_{2;3})\big)\big|_{G_1^*},\\
\bar{\beta}_{t_3} &= \beta_{3;1} + \bar{\beta}_{3;2} \lhd \lambda_{3;3} + \bar{\beta}_{3;3} + \big(\lambda_{3;3}^{-1}\bar{y}_{3;2}\lambda_{3;3}\big)\big|_{G_1^*} + \big((\lambda_{3;2}\lambda_{3;3})^{-1}\,\bar{y}_{3;1}\,(\lambda_{3;2}\lambda_{3;3})\big)\big|_{G_1^*},\\
\bar{\beta}_{t_4} &= \beta_{4;1} + \bar{\beta}_{4;2} \lhd \lambda_{4;3} + \bar{\beta}_{4;3} + \big(\lambda_{4;3}^{-1}\bar{y}_{4;2}\lambda_{4;3}\big)\big|_{G_1^*} + \big((\lambda_{4;2}\lambda_{4;3})^{-1}\,\bar{y}_{4;1}\,(\lambda_{4;2}\lambda_{4;3})\big)\big|_{G_1^*}.
\end{aligned} \tag{3.153}$$

Furthermore gluing edges pairwise will result in new wedge variables:

$$\begin{aligned}
&\bar{y}_{12} = -\bar{y}_{2;1} + (\lambda_{1;3}\,\bar{y}_{1;3}\,\lambda_{1;3}^{-1})|_{G_2^*} \ , && \bar{y}_{23} = -\bar{y}_{3;1} + (\lambda_{2;3}\,\bar{y}_{2;3}\,\lambda_{2;3}^{-1})|_{G_2^*} \ ,\\
&\bar{y}_{13} = \bar{y}_{1;1} - (\lambda_{3;3}\,\bar{y}_{3;3}\,\lambda_{3;3}^{-1})|_{G_2^*} \ , && \bar{y}_{24} = \bar{y}_{2;2} - (\lambda_{4;2}\,\bar{y}_{4;2}\,\lambda_{4;2}^{-1})|_{G_2^*} \ ,\\
&\bar{y}_{14} = \bar{y}_{1;2} - (\lambda_{4;3}\,\bar{y}_{4;3}\,\lambda_{4;3}^{-1})|_{G_2^*} \ , && \bar{y}_{34} = \bar{y}_{3;2} - (\lambda_{4;1}\,\bar{y}_{4;1}\,\lambda_{4;1}^{-1})|_{G_2^*}
\end{aligned} \tag{3.154}$$

where we denoted $\bar{y}_{ab}$ the fused wedge decoration shared by triangles a and b.

Finally, the closure of the tetrahedron is imposed. This is the constraint which comes by demanding that the composition of the face variables (using the 2-group technology developed) gives the identity. After whiskering one obtains

$$\bar{\beta}_{t_1} + \bar{\beta}_{t_2} + \bar{\beta}_{t_3} + \lambda_{1;1} \rhd \bar{\beta}_{t_4} + (\lambda_{1;1}\,\bar{y}_{t_4}\,\lambda_{1;1}^{-1})|_{G_1^*} = 0, \tag{3.155}$$

where

$$\bar{y}_{t_4} = \bar{y}_{4:3} + (\lambda_{4;3}^{-1}\,\bar{y}_{4;2}\,\lambda_{4;3})|_{G_2^*} + \big((\lambda_{4;2}\lambda_{4;3})^{-1}\,\bar{y}_{4;1}\,(\lambda_{4;2}\lambda_{4;3})\big)\big|_{G_2^*}. \tag{3.156}$$

Finally, we can look to create the phase space corresponding to the three dimensional boundary. This is achieved through gluing five tetrahedra phase spaces together along the triangles. The triangle gluing is done via

$$\begin{aligned}
\bar\beta_{12} \equiv& \bar\beta_{t_{1;1}} = -(u_{12}\,\bar\beta_{t_{2;4}}\,u_{12}^{-1})|_{G_1^*}\ , & \bar\beta_{31} \equiv \bar\beta_{t_{3;3}} &= -(u_{13}^{-1}\,\bar\beta_{t_{1;2}}\,u_{13})|_{G_1^*}\ , \\
\bar\beta_{14} \equiv& \bar\beta_{t_{1;3}} = -(u_{14}\,\bar\beta_{t_{4;2}}\,u_{14}^{-1})|_{G_1^*}\ , & \bar\beta_{51} \equiv \bar\beta_{t_{5;1}} &= -(u_{15}^{-1}\,\bar\beta_{t_{1;4}}\,u_{15})|_{G_1^*}\ , \\
\bar\beta_{23} \equiv& \bar\beta_{t_{2;1}} = -(u_{23}\,\bar\beta_{t_{3;4}}\,u_{23}^{-1})|_{G_1^*}\ , & \bar\beta_{42} \equiv \bar\beta_{t_{4;3}} &= -(u_{24}^{-1}\,\bar\beta_{t_{2;2}}\,u_{24})|_{G_1^*}\ , \\
\bar\beta_{25} \equiv& \bar\beta_{t_{2;3}} = -(u_{25}\,\bar\beta_{t_{5;2}}\,u_{25}^{-1})|_{G_1^*}\ , & \bar\beta_{34} \equiv \bar\beta_{t_{3;1}} &= -(u_{34}\,\bar\beta_{t_{4;4}}\,u_{34}^{-1})|_{G_1^*}\ , \\
\bar\beta_{53} \equiv& \bar\beta_{t_{5;3}} = -(u_{35}^{-1}\,\bar\beta_{t_{3;2}}\,u_{35})|_{G_1^*}\ , & \bar\beta_{45} \equiv \bar\beta_{t_{4;1}} &= -(u_{45}\,\bar\beta_{t_{5;4}}\,u_{45}^{-1})|_{G_1^*},
\end{aligned} \tag{3.157}$$

where the first index specifies to which tetrahedron $\bar\beta$ belongs and the second indicates on which triangle of the tetrahedron the variables is defined.

As always, gluing triangles introduces new variables on the dual structures, the links,

$$\begin{aligned}
&u_{12} = u_{1;1}u_{2;4}^{-1}\ , \quad u_{13} = u_{1;2}u_{3;3}^{-1}\ , \quad u_{14} = u_{1;3}u_{4;2}^{-1}\ , \quad u_{15} = u_{1;4}u_{5;1}^{-1}\ , \quad u_{23} = u_{2;1}u_{3;4}^{-1}\ , \\
&u_{24} = u_{2;2}u_{4;3}^{-1}\ , \quad u_{25} = u_{2;3}u_{5;2}^{-1}\ , \quad u_{34} = u_{3;1}u_{4;4}^{-1}\ , \quad u_{35} = u_{3;2}u_{5;3}^{-1}\ , \quad u_{45} = u_{4;1}u_{5;4}^{-1}\ ,
\end{aligned} \tag{3.158}$$

There is also the gluing of edges, which we rename λ^a_{bcd} where the subscripts indicate the three tetrahedra sharing the edge and the superscript indicates where the variable is rooted (which we need to consider when we are whiskering):

$$\begin{aligned}
&\lambda^1_{123} \equiv \lambda_{1;12} = u_{12} \rhd \lambda_{2;14} = u_{13} \rhd \lambda_{3;34}\ , \quad &&\lambda^1_{124} \equiv \lambda_{1;13} = u_{12} \rhd \lambda_{2;24} = u_{14} \rhd \lambda_{4;23}\ , \\
&\lambda^1_{125} \equiv \lambda_{1;14} = u_{12} \rhd \lambda_{2;34} = u_{15} \rhd \lambda_{5;12}\ , \quad &&\lambda^1_{134} \equiv \lambda_{1;23} = u_{13} \rhd \lambda_{3;13} = u_{14} \rhd \lambda_{4;24}\ , \\
&\lambda^1_{135} \equiv \lambda_{1;24} = u_{13} \rhd \lambda_{3;23} = u_{15} \rhd \lambda_{5;13}\ , \quad &&\lambda^1_{145} \equiv \lambda_{1;34} = u_{14} \rhd \lambda_{4;12} = u_{15} \rhd \lambda_{5;14}\ , \\
&\lambda^2_{234} \equiv \lambda_{2;12} = u_{23} \rhd \lambda_{3;14} = u_{24} \rhd \lambda_{4;34}\ , \quad &&\lambda^2_{235} \equiv \lambda_{2;13} = u_{23} \rhd \lambda_{3;24} = u_{25} \rhd \lambda_{5;23}\ , \\
&\lambda^2_{245} \equiv \lambda_{2;23} = u_{24} \rhd \lambda_{4;13} = u_{25} \rhd \lambda_{5;24}\ , \quad &&\lambda^3_{345} \equiv \lambda_{3;12} = u_{34} \rhd \lambda_{4;14} = u_{35} \rhd \lambda_{5;34}\ .
\end{aligned} \tag{3.159}$$

Finally, these identifications imply a new set of wedge variables dual to the edges. These are

$$
\begin{aligned}
\bar{y}^1_{123} &= \bar{y}_{1;12} + u_{12} \rhd \bar{y}_{2;14} + u_{13} \rhd \bar{y}_{3;34} + (u_{12}\,\bar{\beta}_{23}\,u_{12}^{-1})|_{G_2^*} + (u_{13}\,\bar{\beta}_{31}\,u_{13}^{-1})|_{G_2^*}\,,\\
\bar{y}^1_{124} &= \bar{y}_{1;13} + u_{12} \rhd \bar{y}_{2;24} + u_{14} \rhd \bar{y}_{4;23} + (u_{14}\,\bar{\beta}_{42}\,u_{14}^{-1})|_{G_2^*}\,,\\
\bar{y}^1_{125} &= \bar{y}_{1;14} + u_{12} \rhd \bar{y}_{2;34} + u_{15} \rhd \bar{y}_{5;12} + (u_{12}\,\bar{\beta}_{25}\,u_{12}^{-1})|_{G_2^*} + (u_{15}\,\bar{\beta}_{51}\,u_{15}^{-1})|_{G_2^*}\,,\\
\bar{y}^1_{134} &= \bar{y}_{1;23} + u_{13} \rhd \bar{y}_{3;14} + u_{14} \rhd \bar{y}_{4;24} + (u_{13}\,\bar{\beta}_{34}\,u_{13}^{-1})|_{G_2^*} + (u_{13}\,\bar{\beta}_{31}\,u_{13}^{-1})|_{G_2^*}\,,\\
\bar{y}^3_{135} &= \bar{y}_{1;24} + u_{13} \rhd \bar{y}_{3;23} + u_{15} \rhd \bar{y}_{5;13} + (u_{35}\,\bar{\beta}_{51}\,u_{35}^{-1})|_{G_2^*} + (u_{35}\,\bar{\beta}_{53}\,u_{35}^{-1})|_{G_2^*}\,,\\
\bar{y}^1_{145} &= \bar{y}_{1;34} + u_{14} \rhd \bar{y}_{4;12} + u_{15} \rhd \bar{y}_{5;14} + (u_{14}\,\bar{\beta}_{45}\,u_{14}^{-1})|_{G_2^*} + (u_{15}\,\bar{\beta}_{51}\,u_{15}^{-1})|_{G_2^*}\,,\\
\bar{y}^2_{235} &= \bar{y}_{2;13} + u_{23} \rhd \bar{y}_{3;24} + u_{25} \rhd \bar{y}_{5;23} + (u_{25}\,\bar{\beta}_{53}\,u_{25}^{-1})|_{G_2^*}\,,\\
\bar{y}^2_{245} &= \bar{y}_{2;23} + u_{24} \rhd \bar{y}_{4;13} + u_{25} \rhd \bar{y}_{5;24} + (u_{24}\,\bar{\beta}_{45}\,u_{24}^{-1})|_{G_2^*} + (u_{24}\,\bar{\beta}_{42}\,u_{24}^{-1})|_{G_2^*}\,,\\
\bar{y}^2_{234} &= \bar{y}_{2;12} + u_{23} \rhd \bar{y}_{3;14} + u_{24} \rhd \bar{y}_{4;34} + (u_{23}\,\bar{\beta}_{34}\,u_{23}^{-1})|_{G_2^*} + (u_{24}\,\bar{\beta}_{42}\,u_{24}^{-1})|_{G_2^*}\,,\\
\bar{y}^3_{345} &= \bar{y}_{3;12} + u_{34} \rhd \bar{y}_{4;14} + u_{35} \rhd \bar{y}_{5;34} + (u_{34}\,\bar{\beta}_{45}\,u_{34}^{-1})|_{G_2^*} + (u_{35}\,\bar{\beta}_{53}\,u_{35}^{-1})|_{G_2^*},
\end{aligned}
\tag{3.160}
$$

The 1-Gauss constraints, the closure of each boundary tetrahedron, are given by

$$\begin{aligned}\tau_1: \quad & b_{12} - (u_{13}\, b_{31}\, u_{13}^{-1})|_{G_1^*} + \lambda^1_{135} \triangleright b_{14} - (u_{15}\, b_{51}\, u_{15}^{-1})|_{G_1^*} \\ & + (\lambda^1_{123}\, \bar{y}_{1;12}\, (\lambda^1_{123})^{-1})|_{G_1^*} + \lambda^1_{125} \triangleright (\lambda^1_{124}\, \bar{y}_{1;13}\, (\lambda^1_{124})^{-1})|_{G_1^*} + (\lambda^1_{125}\, \bar{y}_{1;14}\, (\lambda^1_{125})^{-1})|_{G_1^*} \\ & + \lambda^1_{135} \triangleright (\lambda^1_{134}\, \bar{y}_{1;23}\, (\lambda^1_{234})^{-1})|_{G_1^*} + (\lambda^1_{135}\, \bar{y}_{1;24}\, (\lambda^1_{135})^{-1})|_{G_1^*} \\ & + \lambda^1_{135} \triangleright (\lambda^1_{145}\, \bar{y}_{1;34}\, (\lambda^1_{145})^{-1})|_{G_1^*} = 0, \end{aligned} \tag{3.161}$$

$$\begin{aligned}\tau_2: \quad & b_{23} - \lambda^2_{125} \triangleright (u_{24}\, b_{42}\, u_{24}^{-1})|_{G_1^*} + b_{25} - (u_{12}^{-1}\, b_{12}\, u_{12})|_{G_1^*} \\ & + \lambda^2_{235} \triangleright (\lambda^2_{234}\, \bar{y}_{2;12}\, (\lambda^2_{234})^{-1})|_{G_1^*} + (\lambda^2_{235}\, \bar{y}_{2;13}\, (\lambda^2_{235})^{-1})|_{G_1^*} + (\lambda^2_{123}\, \bar{y}_{2;14}\, (\lambda^2_{123})^{-1})|_{G_1^*} \\ & + \lambda^2_{125} \triangleright (\lambda^2_{245}\, \bar{y}_{2;23}\, (\lambda^2_{245})^{-1})|_{G_1^*} + \lambda^2_{125} \triangleright (\lambda^2_{124}\, \bar{y}_{2;24}\, (\lambda^2_{124})^{-1})|_{G_1^*} \\ & + (\lambda^2_{125}\, \bar{y}_{2;34}\, (\lambda^2_{125})^{-1})|_{G_1^*} = 0, \end{aligned} \tag{3.162}$$

$$\begin{aligned}\tau_3: \quad & \lambda^3_{135} \triangleright b_{34} - (u_{35}\, b_{53}\, u_{35}^{-1})|_{G_1^*} + b_{31} - (u_{23}^{-1}\, b_{23}\, u_{23})|_{G_1^*} \\ & + \lambda^3_{135} \triangleright (\lambda^3_{345}\, \bar{y}_{3;12}\, (\lambda^3_{345})^{-1})|_{G_1^*} + \lambda^3_{135} \triangleright (\lambda^3_{134}\, \bar{y}_{3;13}\, (\lambda^3_{134})^{-1})|_{G_1^*} \\ & + \lambda^3_{235} \triangleright (\lambda^3_{234}\, \bar{y}_{3;14}\, (\lambda^3_{234})^{-1})|_{G_1^*} \quad + (\lambda^3_{135}\, \bar{y}_{3;23}\, (\lambda^3_{135})^{-1})|_{G_1^*} \\ & + (\lambda^3_{235}\, \bar{y}_{3;24}\, (\lambda^3_{235})^{-1})|_{G_1^*} + (\lambda^3_{123}\, \bar{y}_{3;34}\, (\lambda^3_{123})^{-1})|_{G_1^*} = 0, \end{aligned} \tag{3.163}$$

$$\begin{aligned}\tau_4: \quad & \lambda^4_{135} \triangleright \big(b_{45} - (u_{14}^{-1}\, b_{14}\, u_{14})|_{G_1^*} + \lambda^4_{145} \triangleright b_{42} - (u_{34}^{-1}\, b_{34}\, u_{34})|_{G_1^*}\big) \\ & + \lambda^4_{135} \triangleright (\lambda^4_{145}\, \bar{y}_{4;12}\, (\lambda^4_{145})^{-1})|_{G_1^*} + \lambda^4_{125} \triangleright (\lambda^4_{245}\, \bar{y}_{4;13}\, (\lambda^4_{245})^{-1})|_{G_1^*} \\ & + \lambda^4_{135} \triangleright (\lambda^4_{345}\, \bar{y}_{4;14}\, (\lambda^4_{345})^{-1})|_{G_1^*} \quad + \lambda^4_{125} \triangleright (\lambda^4_{124}\, \bar{y}_{4;23}\, (\lambda^4_{124})^{-1})|_{G_1^*} \\ & + \lambda^4_{135} \triangleright (\lambda^4_{134}\, \bar{y}_{4;24}\, (\lambda^4_{134})^{-1})|_{G_1^*} + \lambda^4_{235} \triangleright (\lambda^4_{234}\, \bar{y}_{4;34}\, (\lambda^4_{234})^{-1})|_{G_1^*} = 0, \end{aligned} \tag{3.164}$$

$$\begin{aligned}\tau_5: \quad & b_{51} - (u_{25}^{-1}\, b_{25}\, u_{25})|_{G_1^*} + b_{53} - \lambda^5_{135} \triangleright (u_{45}^{-1}\, b_{45}\, u_{45})|_{G_1^*} + (\lambda^5_{125}\, \bar{y}_{5;12}\, (\lambda^5_{125})^{-1})|_{G_1^*} \\ & + (\lambda^5_{135}\, \bar{y}_{5;13}\, (\lambda^5_{135})^{-1})|_{G_1^*} + \lambda^5_{135} \triangleright (\lambda^5_{145}\, \bar{y}_{5;14}\, (\lambda^5_{145})^{-1})|_{G_1^*} \\ & + (\lambda^5_{235}\, \bar{y}_{5;23}\, (\lambda^5_{235})^{-1})|_{G_1^*} + \lambda^5_{125} \triangleright (\lambda^5_{245}\, \bar{y}_{5;24}\, (\lambda^5_{245})^{-1})|_{G_1^*} \\ & + \lambda^5_{135} \triangleright (\lambda^5_{345}\, \bar{y}_{5;34}\, (\lambda^5_{345})^{-1})|_{G_1^*} = 0. \end{aligned} \tag{3.165}$$

This has been a lot of variable definitions that is hard to parse. Let's now argue that the variables at least make sense in a familiar setting.

Let $\mathcal{G} = ISO(4) \cong SO(4) \ltimes \mathbb{R}^4 \cong \mathbb{R}^4 \rtimes SO(4)$ and $\mathcal{G}^* = ISO(4)^* \cong SO^*(4) \times \mathbb{R}^4 \cong \mathbb{R}^4 \times SO^*(4)$. Let $\mathcal{T}$ be a triangulation with edges and faces decorated by elements of

$\mathbb{R}^4$ and $SO^*(4)$ respectively. The dual 2-complex $\mathcal{T}^*$ has links and wedges decorated by holonomies of $SO(4)$ and $\mathbb{R}^{4*} \cong \mathbb{R}^4$ respectively. Specifically, we have

$$G_1 = SO(4) \ni u\,, \quad G_2^* = \mathbb{R}^4 \ni y, \bar{y}\,, \quad G_2 = \mathbb{R}^4 \ni \lambda, \tilde{\lambda}\,, \quad G_1^* = SO^*(4) \cong \mathbb{R}^6 \ni \beta, \tilde{\beta}. \tag{3.166}$$

Note that there is no action of $\mathbb{R}^4$ on $SO(4)$ nor on $SO^*(4)$ and the conjugations/projections take a simple shape

$$\begin{aligned} u\beta u^{-1} = \beta' y' \quad &\Leftrightarrow \left\{ \begin{array}{l} \beta' = (u\beta u^{-1})|_{SO^*(4)} \\ y' = (u\beta u^{-1})|_{\mathbb{R}^4} = 1 \end{array} \right. \\ \lambda y \lambda^{-1} = y'' \beta'' \quad &\Leftrightarrow \left\{ \begin{array}{l} \beta' = (\lambda y \lambda^{-1})|_{SO^*(4)} \equiv [\lambda, y] \\ y' = (\lambda y \lambda^{-1})|_{\mathbb{R}^4} = y, \end{array} \right. \end{aligned} \tag{3.167}$$

where for the second conjugation we used a convenient representation[3] of $\mathbb{R}^4$.

First, to construct the triangle phase space we impose the momentum maps $u_1 = \tilde{u}_2$ and $u_2 = \tilde{u}_3$, plus the closure constraint that here takes the form $\lambda_1 + \lambda_2 + \lambda_3 = 0$. The fused face decoration (3.150) is thus

$$\beta_t = \beta_1 + \beta_2 + \beta_3 + \beta_4 - [\lambda_2\,,\, \bar{y}_1] - [\lambda_3\,,\, \bar{y}_1] - [\lambda_3\,,\, \bar{y}_2]. \tag{3.168}$$

Similarly, to construct the tetrahedron phase space we first consider four triangle phase spaces and fuse them imposing the six momentum maps (3.152),

$$\lambda_{1;1} = -\lambda_{3;3}\;, \quad \lambda_{1;2} = -\lambda_{4;3}\;, \quad \lambda_{1;3} = -\lambda_{2;1}\;, \quad \lambda_{2;2} = -\lambda_{4;2}\;, \quad \lambda_{2;3} = -\lambda_{3;1}\;, \quad \lambda_{3;2} = -\lambda_{4;1}. \tag{3.169}$$

From these momentum maps we then derive the six fused wedge decorations (3.154) represented at the center of the tetrahedron:

$$\begin{aligned} \bar{y}_{12} &= -\bar{y}_{2;1} + \bar{y}_{1;3}\;, \quad \bar{y}_{13} = \bar{y}_{1;1} - \bar{y}_{3;3}\;, \quad \bar{y}_{14} = \bar{y}_{1;2} - \bar{y}_{4;3}\;, \\ \bar{y}_{23} &= -\bar{y}_{3;1} + \bar{y}_{2;3}\;, \quad \bar{y}_{24} = \bar{y}_{2;2} - \bar{y}_{4;2}\;, \quad \bar{y}_{34} = \bar{y}_{3;2} - \bar{y}_{4;1}. \end{aligned} \tag{3.170}$$

The last momentum map that we impose for the construction of the tetrahedron is the 1-Gauss constraint (3.168), that now reduces to

$$b_1 + b_2 + b_3 + b_4 + [\lambda_{12}\,,\, \bar{y}_{12}] + [\lambda_{13}\,,\, \bar{y}_{13}] + [\lambda_{14}\,,\, \bar{y}_{14}] + [\lambda_{23}\,,\, \bar{y}_{23}] + [\lambda_{24}\,,\, \bar{y}_{24}] + [\lambda_{34}\,,\, \bar{y}_{34}] = 0, \tag{3.171}$$

[3] If X_μ is a Lie algebra generator of $\mathbb{R}^4$, we use a representation such that $X_\mu X_\nu = 0$. This implements that $e^X = 1 + X$. If Y_ν are the generators of another $\mathbb{R}^4$, then $e^Y e^X e^{-Y} = 1 + X + [Y, X]$.

where

$$
\begin{aligned}
b_1 &= \beta_{1;1} + \beta_{1;2} + \beta_{1;3} - [\lambda_{21}\,,\,\bar{y}_{1;3}] + [\lambda_{13}\,,\,\bar{y}_{1;2}],\\
b_2 &= -\beta_{2;1} + \beta_{2;2} + \beta_{2;3} - [\lambda_{32}\,,\,\bar{y}_{2;3}] + [\lambda_{12}\,,\,\bar{y}_{2;2}],\\
b_3 &= -\beta_{3;1} + \beta_{3;2} - \beta_{3;3} - [\lambda_{13}\,,\,\bar{y}_{3;3}] + [\lambda_{23}\,,\,\bar{y}_{3;2}],\\
b_4 &= -\beta_{4;1} - \beta_{4;2} - \beta_{4;3} - [(\lambda_{13} - \lambda_{14})\,,\,\bar{y}_{4;3}] - [(\lambda_{12} - \lambda_{24})\,,\,\bar{y}_{4;2}] - [(\lambda_{23} - \lambda_{34})\,,\,\bar{y}_{4;1}].
\end{aligned}
\tag{3.172}
$$

Once we derived the tetrahedron phase space, we can use it to build the 4-simplex boundary phase space.

Performing a change of variables,

$$
\begin{aligned}
b'_{12} &= b_{12} - u_{12}\,[\lambda^2_{123}\,,\,\bar{y}_{2;14}]\,u_{12}^{-1} - u_{12}\,[\lambda^2_{124}\,,\,\bar{y}_{2;24}]\,u_{12}^{-1} - u_{12}\,[\lambda^2_{125}\,,\,\bar{y}_{2;34}]\,u_{12}^{-1}\,,\\
b'_{31} &= b_{31} + [\lambda^3_{134}\,,\,\bar{y}_{3;13}] + [\lambda^3_{123}\,,\,\bar{y}_{3;34}] - u_{13}^{-1}\,[\lambda^1_{135}\,,\,\bar{y}_{1;24}]\,u_{13}\,,\\
b'_{14} &= b_{14} - u_{14}\,[\lambda^4_{145}\,,\,\bar{y}_{4;12}]\,u_{14}^{-1} - u_{14}\,[\lambda^4_{124}\,,\,\bar{y}_{4;23}]\,u_{14}^{-1} - u_{14}\,[\lambda^4_{134}\,,\,\bar{y}_{4;24}]\,u_{14}^{-1}\,,\\
b'_{51} &= b_{51} + [\lambda^5_{125}\,,\,\bar{y}_{5;12}] + [\lambda^5_{145}\,,\,\bar{y}_{5;14}]\,,\\
b'_{23} &= b_{23} - u_{23}\,[\lambda^3_{234}\,,\,\bar{y}_{3;14}]\,u_{23}^{-1} - u_{23}\,[\lambda^3_{235}\,,\,\bar{y}_{3;24}]\,u_{23}^{-1}\,,\\
b'_{42} &= b_{42} + [\lambda^4_{245}\,,\,\bar{y}_{4;13}] + [\lambda^4_{234}\,,\,\bar{y}_{4;34}]\,,\\
b'_{25} &= b_{25} - u_{25}\,[\lambda^5_{235}\,,\,\bar{y}_{5;23}]\,u_{25}^{-1} - u_{25}\,[\lambda^5_{245}\,,\,\bar{y}_{5;24}\,u_{25}^{-1}\,,\\
b'_{34} &= b_{34} - u_{34}\,[\lambda^4_{345}\,,\,\bar{y}_{4;14}]\,u_{34}^{-1}\,,\\
b'_{53} &= b_{53} + [\lambda^5_{135}\,,\,\bar{y}_{5;13}] + [\lambda^5_{345}\,,\,\bar{y}_{5;34}]
\end{aligned}
\tag{3.173}
$$

in the five 1-Gauss constraints (3.161)-(3.165), in order to express the closure constraints in terms of the closed faces (wedges) variables (3.160).

$$
\begin{aligned}
\tau_1:\quad & b'_{12} - (u_{13}\,b'_{31}\,u_{13}^{-1}) + b'_{14} - (u_{15}\,b'_{51}\,u_{15}^{-1}) + [\lambda^1_{123}\,,\,\bar{y}^1_{123}] + [\lambda^1_{124}\,,\,\bar{y}^1_{124}] + [\lambda^1_{125}\,,\,\bar{y}^1_{125}]\\
& \qquad + [\lambda^1_{134}\,,\,\bar{y}^1_{134}] + [\lambda^1_{145}\,,\,\bar{y}^1_{145}] = 0, \qquad (3.174)\\
\tau_2:\quad & b'_{23} - (u_{24}\,b'_{42}\,u_{24}^{-1}) + b'_{25} - (u_{12}^{-1}\,b'_{12}\,u_{12}) + [\lambda^2_{235}\,,\,\bar{y}^2_{235}] + [\lambda^2_{245}\,,\,\bar{y}^2_{245}] + [\lambda^2_{234}\,,\,\bar{y}^2_{234}] = 0, \qquad (3.175)\\
\tau_3:\quad & b'_{34} - (u_{35}\,b'_{53}\,u_{35}^{-1}) + b'_{31} - (u_{23}^{-1}\,b'_{23}\,u_{23}) + [\lambda^3_{135}\,,\,\bar{y}^3_{135}] + [\lambda^3_{345}\,,\,\bar{y}^3_{345}] = 0, \qquad (3.176)\\
\tau_4:\quad & b_{45} - (u_{14}^{-1}\,b'_{14}\,u_{14}) + b'_{42} - (u_{34}^{-1}\,b'_{34}\,u_{34}) = 0, \qquad (3.177)\\
\tau_5:\quad & b'_{51} - (u_{25}^{-1}\,b'_{25}\,u_{25}) + b'_{53} - (u_{45}^{-1}\,b_{45}\,u_{45}) = 0. \qquad (3.178)
\end{aligned}
$$

These are precisely the constraints derived when we were considering the BFCG model and its symplectic potential. The glued variables are thus the same as those introduced by smearing the continuous fields over the simplicies.

We can then extend this phase space construction to a full three dimensional triangulation. If the triangulation is made of n tetrahedra, we consider n copies of the tetrahedron phase space and implement a symplectic reduction induced by the appropriate identification of the edge decorations, which induce a gluing of the dual variables.

The highlight of this construction is that we are able to use non-trivial Poisson 2-groups to construct a triangulation phase space. In particular, the κ-Poincaré deformation appears as a natural example of deformation of the Poincaré 2-group. Furthermore it provides a candidate new discretization of the $\mathfrak{so}(3,1)$ BF theory. To prove it is actually the case, one would either discretize explicitly the $\mathfrak{so}(3,1)$ BF theory as a $\mathfrak{su}(2) \ltimes \mathfrak{an}_2^*$ 2-gauge BFCG theory, or compare the quantum amplitude for the $\mathfrak{so}(3,1)$ BF theory to the quantum amplitude for the $\mathfrak{su}(2) \ltimes \mathfrak{an}_2^*$ BFCG theory, which would amount to know the (2-)representations theory of the quantum 2-group given by the κ-Poincaré 2-group.

Chapter 4

Quantization

In the previous chapters, we studied classical topological theories, the BF and 2-BF (BFCG) models and their phase spaces. In particular, emphasis was put on obtaining discrete variables which satisfy particular Poisson brackets. The present section deals with expanding on the discrete geometries introduced, and attempting to infer a quantum theory for these geometries.

Discrete geometries have long been used in theories of quantum gravity. In the 1970s, Penrose introduced spin-networks as trivalent graphs with edges labelled by $j = 0, 1/2, 1, \ldots$ [82]. Three edges meeting at a vertex were required to have their half-integer labels satisfying the triangle inequalities for the addition of angular momentum: If j_1, j_2, j_3 are the labels of edges meeting at a vertex, then they must satisfy $|j_1 - j_2| \leq j_3 \leq j_1 + j_2$. That is to say, the labels of the edges are really the dimensions of irreducible representations of $SU(2)$ and edges only meet if the collision of three particles with spin j_i can combine to create a $j = 0$ state. From each admissible graph using the angular momentum addition, Penrose was able to interpret the spin networks as a reconstruction of space.

More recently, spin networks are used to construct a basis of states in a Hilbert space upon which geometrical operators such as volume and area operators act diagonally. The area and volumes are quantized since their operators have a discrete spectrum [92, 93].

A few years prior to the introduction of spin networks, another connection between geometry and representations of the angular momentum group was observed. The first state-sum model – a systematic method of (formally) assigning a value to a graph based on local data – was found by Ponzano and Regge [84]. Through generalizing the $6j$-symbol (to be explained later) they recovered what could be interpreted as a sum over three dimensional Euclidean discrete geometries weighted by an approximation to the Einstein-Hilbert action [87]. This suggests a path integral type interpretation, where each term in the sum is given by the value of a state sum associated to a graph embedded in some topological 3-manifold. Since the sum includes all possible labelling of the edges by positive half integers, there are infinitely many terms and the sum may diverge. Various regularization schemes have been introduced ensure that the sums remain finite, for example by using representations of the quantum group $U_q(\mathfrak{su}(2))$, with q root of unity, which has a natural cutoff in the dimension of the representations allowed [100].

These state sum models are topological invariants of three dimensional manifolds and can be generalized to four dimensional manifolds [100, 36]. In particular we can show that they are invariant under an appropriate set of transformations on the 2-complex.

In this chapter, then we are interested in 1) spin-network-like states corresponding to graphs which are 2-complexes decorated with 2-group elements or representations and 2) topological invariants defined in terms of those 2-group decorated complexes.

For each triangulation and dual 2-complex in a manifold, the set of decorations on the links, edges, faces, and triangles, there needs to be a set of constraints which describe the closure of polygons and polyhedra as we saw in chapter 3. Upon quantization, these constraints act as projectors on the space of states in order to define a physical inner product. As a tool for generating such constraints, we introduce a 2-group field theoretic model (2-GFT). This field theory has as variables functions defined on 2-groups. The Feynman diagrams associated to these field theories are dual to a 2-complex, and the amplitudes are the set of constraints that define the closure of polygons and polyhedra.

4.1 1-GFT

4.1.1 Group field theory in 3 dimensions

Let's start by reviewing how state sums are assignments of numbers to graphs. One starts by considering a manifold M with an associated triangulation $\triangle$, where for each edge e, one assigns a colouring $\phi(e)$. Every value in the image of ϕ is given a weight, so that the weight associated to edge e is given by $w_{\phi(e)} \in \mathbb{C}$ (the weight function will be different depending on whether the coloured edge is on the boundary of M or not). Then for each coloring, one assigns the number

$$A(M, \triangle)_\phi = \prod_{v=\text{vertices}} w_v \prod_{e=\text{edges}} w_{\phi(e)} \prod_{t=\text{3-cells}} T_t. \tag{4.1}$$

The product on the vertices w_v, at least in the Ponzano-Regge model, which we base this review on, gives an overall factor of $(-1)^V$, where V is a number depending on the quantity of vertices in the triangulation. The number associated to the tetrahedron, T_t is a function of the six colourings of its edges. The state sum of M is defined as the sum over all possible colourings,

$$Z(M, \triangle) = \sum_{\phi} A(M)_{\phi} \tag{4.2}$$

The weight of the colouring and the function associated to the 3-cells must be chosen in a consistent way. For the state sum to be topologically invariant, any two triangulations which coincide on the boundary of M, must yield the same value for (4.2).

The simplest state sum model considered is the Ponzano-Regge model where the colouring ϕ is a function from edges to positive half integers $j = 0, \frac{1}{2}, 1, \ldots$. The weight of the edges are chosen to be the dimension of the spin $\phi(e)$ representation, $w_j = 2j+1$. The weight of the teterahedron are the $6j$-symbols: Suppose the six edges of the tetrahedron are labelled by j_i with $i = 1, \ldots, 6$, where the four boundary triangles are given by (j_1, j_2, j_3), (j_1, j_5, j_6), (j_2, j_4, j_5) and (j_3, j_4, j_6). The $6j$-symbol is written as

$$\begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix}.$$

The value of the $6j$ symbol is obtained by representing a system of three particles with spins j_1, j_2, and j_4 in two ways: First, by considering the tensor product of the j_1 and j_2 particles obtaining a spin j_3 particle to obtain a final j_6 particle, and then adding the j_4 spin to that. Second, the j_2 and j_4 states can be added to obtain a j_5 state, to which one adds j_1 to once again obtain the j_6 state. The matrix elements of the isomorphism between the two j_6 representations are the $6j$-symbols.

It was shown that the state sum with the above weights is topologically invariant, and when the spins j_i are large, the weight of a manifold (4.1) will approach $e^{iS_{\text{Regge}}}$, where S_{Regge} is an approximation to the three dimensional Einstein-Hilbert action [84, 87]. The state sum is then thought of as a sum over such exponentials, which is interpreted as a path integral.

The path integral for three dimensional gravity can be formally integrated as well. We write the partition function as $Z = \int \mathcal{D}e\mathcal{D}A \exp\left(-i \int e \wedge F\right)$. The integration over the triad field gives $Z = \int \mathcal{D}A\delta(F)$. As usual, one can introduce a lattice with which to regulate this integral as we have described before schematically giving

$$\mathcal{Z}(\triangle^*) = \int \mathrm{d}h \prod_e \delta(h_e). \tag{4.3}$$

where h_e are the holonomies around the face dual to edge e, $h_e = \prod_{l=\text{ link } \in \partial e^*} h_l$

The expression (4.3) can be seen as a product of constraints. Roughly, the delta's are interpreted as projectors which constrain the loops of the dual triangulation to be trivial reflecting local flatness.

The path integral invites us to consider a diagramatic way of evaluating. This is what group field theories provide.

Group field theories (GFTs) are field theories where the fields are functions over a group G. These theories will be non-local: The action of the theories will involve the field evaluated at different group variables. Through studying the interaction terms of the theory, we will interpret the Feynman rules as being the projectors akin to those that appear in (4.3). As such, every Feynman diagram corresponds to a triangulation, and its amplitude is related to a transition amplitude.

The goal then is to devise an action whose propagator and interaction kernel generate the amplitudes (4.3) as Feynman diagrams. In the next subsections, we outline the procedure in three dimensions (in order to be able to draw pictures). Then, we detail the reasoning for the construction of an action which assigns to each Feynman diagram, a 2-complex decorated by 2-group elements.

4.1.2 Construction of the action

Let's now go into more detail about the construction of the Boulatov GFT action [28] as a preview of what will come. Similar constructions have been done in four dimensions, but we will stick to three for now [79].

Our guiding principle for constructing the action of the GFT will be geometrical. The fields themselves will be given the interpretation of three-valent nodes in the dual complex and the variables of the fields will be interpreted as holonomy which share that node. We therefore take the fields to be functions on $SU(2)^{\times 3}$. In order to enforce that the links share a node, we impose a symmetry on the field:

$$\phi(g_1, g_2, g_3) = \phi(g_1 h, g_2 h, g_3 h) \tag{4.4}$$

for all $h \in SU(2)$. This can be enforced by defining $\phi(g_1, g_2, g_3) = \int \mathrm{d}h \tilde{\phi}(g_1 h, g_2 h, g_3 h)$, where $\tilde{\phi}$ has no symmetry imposed. (One should also consider permutations of the g_i's and do an average over those as well. Expressions get muddied, so we will keep that averaging implicit.)

The action will consist of two terms: the kinetic term which will give the propagator and a potential interaction term. Geometrically, the propagator glues two triangles together. We write the kinetmatic term of the action as

$$\begin{aligned} S_{\mathcal{K}} &= \int \prod_{i=1}^{3} \mathrm{d}g_i \phi(g_1, g_2, g_3)\phi(g_3, g_2, g_1) \\ &= \int \prod_{i=1}^{6} \mathrm{d}g_i \phi(g_1, g_2, g_3)\mathcal{K}(g_1, g_2, g_3; g_4, g_5, g_6)\phi(g_4, g_5, g_6). \end{aligned} \tag{4.5}$$

The kernel $\mathcal{K}$ acts as a operation which sets $g_4 = g_1$ and so on. However, since ϕ is invariant as in (4.4), so too is $\mathcal{K}$. So the integration kernel can be written explicitly as

$$\mathcal{K}(g_1, g_2, g_3; g_4, g_5, g_6) = \int \mathrm{d}h \delta(g_1 h g_6^{-1})\delta(g_2 h g_5^{-1})\delta(g_3 h g_4^{-1}). \tag{4.6}$$

Note that one can obtain this by taking a naive approach. You can imagine a product of deltas $\tilde{\mathcal{K}}(g_1, g_2, g_3; g_4, g_5, g_6) = \delta(g_1 g_6^{-1})\delta(g_2 g_5^{-1})\delta(g_3 g_4^{-1})$ as gluing holonomies g_1, g_2, g_3 emanating from one vertex to holonomies g_4, g_5, g_6 emanating from a second vertex. To get $\mathcal{K}$, one then averages over gauge transformations by setting $g_i \to g_i h_1$ for $i = 1, 2, 3$ and $g_j \to g_j h_2$ for $j = 4, 5, 6$. (Note that in (4.6) one of h_1 and h_2 has been eliminated using the invariance of the Haar measure.)

The Feynman diagrams themselves are dual to triangulations, which means that the vertex terms should be quartic in ϕ since the vertex term is to be dual to a tetrahedron. The interaction term in the action is then

$$\begin{aligned} S_{\mathcal{V}} &= \int \prod_{i=1}^{6} \mathrm{d}g_i \phi(g_{1,2,3})\phi(g_{3,4,5})\phi(g_{4,2,6})\phi(g_{5,6,1}) \\ &= \int \prod_{i=1}^{12} \mathrm{d}g_i \mathcal{V}(g_{1,2,3}; g_{4,5,6}; g_{7,8,9}; g_{10,11,12})\phi(g_{1,2,3})\phi(g_{4,5,6})\phi(g_{7,8,9})\phi(g_{10,11,12}) \end{aligned} \tag{4.7}$$

Where the notation $g_{a,b,c} = g_a, g_b, g_c$ is introduced for brevity. The interaction kernel $\mathcal{V}$ serves the purpose of gluing four triangles such that they form a tetrahedron. Once again, we also insist that $\mathcal{V}$ is invariant under the transformation $g_i \to g_i h$. In particular, each g_i belonging to the same triangle is multiplied by the same h. Explicitly,

$$\begin{aligned} \mathcal{V}(g_{1,2,3}; g_{4,5,6}; g_{7,8,9}; g_{10,11,12}) = \int \prod_{i=1}^{4} \mathrm{d}h_i &\delta(g_1 h_1 h_4^{-1} g_{12}^{-1})\delta(g_2 h_1 h_3^{-1} g_8^{-1})\delta(g_3 h_1 h_2^{-1} g_4^{-1}) \\ &\delta(g_5 h_2 h_3^{-1} g_7^{-1})\delta(g_g h_2 h_4^{-1} g_{10}^{-1})\delta(g_9 h_3 h_4^{-1} g_{11}^{-1}). \end{aligned} \tag{4.8}$$

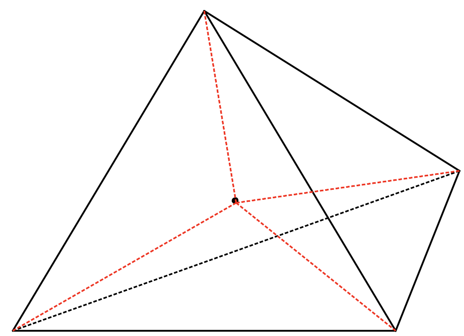

Figure 4.1: The 1-4 Pachner move which takes a tetrahedron (shown in black) and replaces it with four tetrahedra which meet at a vertex in the center of the old tetrahedron

An important feature of this action is that it will generate topologically invariant Feynman amplitudes. Showing the invariance means showing that the amplitude of the Feynman diagram dual to a triangulation, $\triangle_1$, is equal to the amplitude of the Feynman diagram dual to another triangulation $\triangle_2$ if $\triangle_1$ and $\triangle_2$ differ by a series of local transformations called Pachner moves [4, 81].

In three dimensions, there are two such Pachner moves. First, there is the 1-4 move which replaces a single tetrahedron by four tetrahedra sharing a vertex. This is done by adding a vertex the the centre of the tetrahedron and then adding edges from the vertices of the tetrahedron to the new point at the centre as shown in Fig. 4.1. The amplitude associated with the single tetrahedron and the four tetrahedra should be the same, up to some regularization.

In terms of Feynman diagrams representing the triangulations shown in Fig. 4.1, we consider each tetrahedron as a vertex and the triangles as edges. Boundary triangles appear as external legs of the diagram while internal, or bulk triangles are edges connecting vertices. The amplitude of the diagram is calculated according to the Feynman rules in terms with the propagator and interaction term given. The two diagrams associated to the triangulations are shown in Fig. 4.2.

The second Pachner move is the 2-3 move which transforms two tetrahedra sharing a triangle into three tetrahedra sharing an edge.

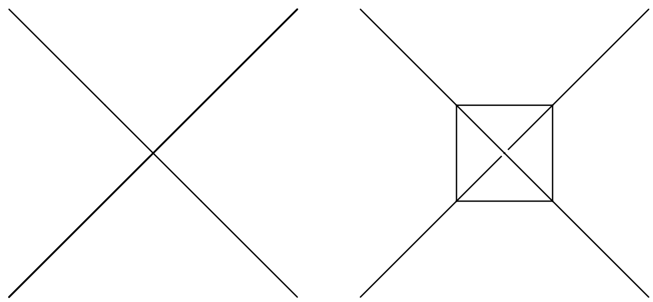

Figure 4.2: The two diagrams corresponding to a 1-4 Pachner move. These two diagrams must have equal amplitudes. Each of the external lines indicate boundary triangles/fields ϕ and will therefore have external variables g_i on them. The vertices are tetrahedra.

4.2 2-GFT in 3 dimensions

Now we will attempt to generalize the previous construction to include fields which have as variables elements of 2-groups. More precisely the fields will be functions of the group elements of $(\mathcal{G}, H)$ with the action and t-map that make the pair a 2-group. We wish to introduce 2-groups in the GFT setting in order to make contact with work by Yetter and Mackaay who have used 2-categories to create topological invariants for four dimensional manifolds [72, 71]. Higher category theory also has applications to topological quantum field theories in four dimensions, which share some similarities to gravity [15, 7]. For simplicity we focus on three dimensions here and address the four dimensional case in the next subsection.

In the preceding section, the idea was to write an action from which the Feynman diagrams can be interpreted as amplitudes for triangulations.

For each diagram we wanted to identify a triangulation, and for each link in the dual of the triangulation, we wanted to have a $SU(2)$ holonomy, as well as the constraint that the closed paths are trivial. Now, we want to include decorations on the surfaces between links as well. Surfaces and their boundaries are decorated consistently using 2-groups, and so the fields of the theory will be functions of two groups which form a 2-group.

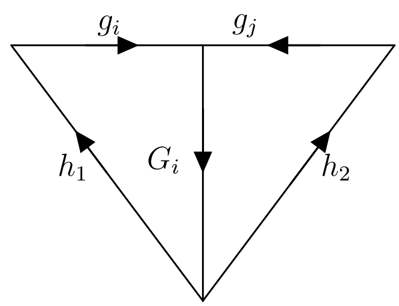

Figure 4.3: The graphical interpretation of one of the delta's appearing in the kinematic term of the 1-GFT in (4.9).

4.2.1 Definition of the action

Our basic fields will maintain the same interpretation they had for standard GFT (1-GFT). Each field will depend on three group variables $g_i \in \mathcal{G}$. Additionally, it will also depend on variables $Y_{i,j}$ in a group H where $(\mathcal{G}, H)$ form a 2-group, $\phi(g_1, g_2, g_3; Y_{1,2}, Y_{2,3}, Y_{3,1})$. The $Y_{i,j}$ will be interpreted (after deducing the symmetries and defining the appropriate interaction term in the action) as the surface variable on the surface subtended by g_i and g_j.

To guide the formulation of the action of the GFT based on 2-groups, let's interpret the previous section's action geometrically.

The kinetic term of the action (4.6) may be written as

$$\begin{aligned}\mathcal{K}(g_1, g_2, g_3; g_4, g_5, g_6) = \int & \mathrm{d}h_1 \mathrm{d}h_2 \mathrm{d}G_1 \mathrm{d}G_2 \mathrm{d}G_3 \\ & \delta(g_1 h_1 G_1^{-1}) \delta(G_1 h_2^{-1} g_6^{-1}) \delta(g_2 h_1 G_2^{-1}) \\ & \delta(G_2 h_2^{-1} g_5^{-1}) \delta(g_3 h_1 G_3^{-1}) \delta(G_3 h_2^{-1} g_4^{-1}),\end{aligned} \tag{4.9}$$

Where G_i have been introduced and may be integrated out trivially to recover (4.6). Writing in this way allows for an interpretation that is more easily generalisable. We can think of each of the delta's above as loops in a graph as shown in Fig. 4.3. Writing it in this way let's us view $\mathcal{K}$ as gluing the G_i's of different loops in order to create a larger loop which has a boundary curve of the form $g_i g_j^{-1}$.

In the 2-GFT, we will augment this picture by including $X_i \in H$ on the surfaces enclosed by g_i, h, and G_i, which we can imagine as having its source at $h^{-1} g_i^{-1}$. We demand that

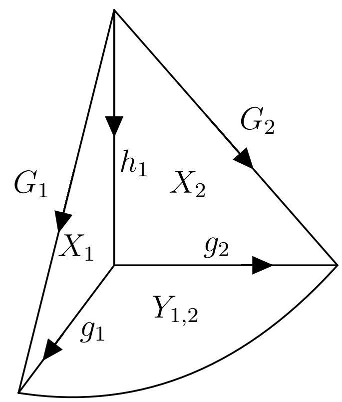

Figure 4.4: The closed polyhedron which is encoded in the delta's of the 2-group GFT kinematic term.

the holonomy around the surface is equal to $t(X_i)$, as is necessary for a bigon representing an element of a 2-group. Just as h plays the role of enforcing gauge invariance, we will see that X will play a role in enforcing 2-gauge invariance of the kinematic propagator.

Introducing the surface variable X in the delta functions is the first generalisation needed for 2-GFT. Next, one should consider how to incorporate the new $Y_{i,j}$ variables. We impose, via delta functions that X_i, X_j, and $Y_{i,j}$ are all sides of a tetrahedron. The fourth side we label $\beta_{i,j}$, and is only used as an intermediate step like the G_i's were: They are the faces on which we glue different tetrahedra. The kinematic term in the action should then be a product of delta functions imposing the closure of the polyhedra with boundary tetrahedra X_i, X_j, Y_{ij} and β_{ij}. The picture is shown in 4.4. By "closure", we mean the composition of the surface variables should be trivial.

This, together with the constraint on the bigons gives:

$$
\begin{aligned}
\mathcal{K}(g_{1,2,3};Y_{1,2,3};g_{4,5,6};Y_{4,5,6}) = \int \mathrm{d}h_1\mathrm{d}h_2 \prod_{a=1}^{6}\mathrm{d}X_a \prod \mathrm{d}\beta \\
\prod_{i=1}^{3}\mathrm{d}G_i\delta(G_i g_i h_1 t(X_i^{-1})) \prod_{j=4}^{6}\mathrm{d}G_j\delta(G_j g_j h_2 t(X_j^{-1})) \\
\delta(X_1X_2^{-1}(G_2^{-1}\triangleright Y_{2,1})(G_2^{-1}\triangleright \beta_{2,1}))\delta(X_2X_3^{-1}(G_3^{-1}\triangleright Y_{3,2})(G_3^{-1}\triangleright \beta_{3,2})) \\
\delta(X_3X_1^{-1}(G_1^{-1}\triangleright Y_{1,3})(G_1^{-1}\triangleright \beta_{1,3}))\delta(X_4X_5^{-1}(G_5^{-1}\triangleright Y_{5,4})(G_5^{-1}\triangleright \beta_{5,4})) \\
\delta(X_5X_6^{-1}(G_6^{-1}\triangleright Y_{6,5})(G_6^{-1}\triangleright \beta_{6,5}))\delta(X_6X_4^{-1}(G_4^{-1}\triangleright Y_{4,6})(G_4^{-1}\triangleright \beta_{4,6})) \\
\delta(G_1G_4^{-1})\delta(G_2G_5^{-1})\delta(G_3G_6^{-1})\delta(\beta_{1,2}\beta_{4,5}^{-1})\delta(\beta_{2,3}\beta_{4,6}^{-1})\delta(\beta_{3,1}\beta_{6,4}^{-1})
\end{aligned}
\tag{4.10}
$$

Geometrically, this long expression provides the following information: The first line of delta functions imposes that the X's are surface variables with the correct combination of g, G, and h on its boundary. The next six deltas impose that, say, X_1, X_2, $Y_{1,2}$ and β_{ij} form a closed polyhedron. The $G's$ that appear in these six deltas are to ensure that the variables are sourced at the correct place (this is not strictly needed, but will turn out to give a cleaner interpretation). The final six deltas are the identification of different edges G and surfaces β which results in the interpretation of $\mathcal{K}$ being the object which acts as glue between two fields.

As the G_i's and $\beta_{i,j}$'s were introduced strictly for illustrative purposes, we can eliminate them altogether to get a slightly nicer expression. Some of the X's can also be eliminated as well giving:

$$
\begin{aligned}
\mathcal{K}(g_{1,2,3};Y_{1,2,3};g_{4,5,6};Y_{4,5,6}) = \int \mathrm{d}h \prod_{i=1}^{3}\mathrm{d}X_i \\
\delta(g_1t(X_1^{-1})h^{-1}g_4^{-1})\delta(g_2t(X_2^{-1})h^{-1}g_5^{-1})\delta(g_3t(X_3^{-1})h^{-1}g_6^{-1}) \\
\delta(X_2(g_2^{-1}\triangleright (Y_{5,4}Y_{1,2}))X_1^{-1})\delta(X_3(g_3^{-1}\triangleright (Y_{6,5}Y_{2,3}))X_2^{-1}) \\
\delta(X_1(g_1^{-1}\triangleright (Y_{4,5}Y_{3,1}))X_3^{-1})
\end{aligned}
\tag{4.11}
$$

where the notation $Y_{i,j,k} = Y_{i,j}, Y_{j,k}, Y_{k,j}$ is introduced and $Y_{i,j}^{-1} = Y_{j,i}$.

For the interaction term, one can consider twelve tetrahedra as described before with deltas combining them appropriately. Instead of going through the procedure as before, we skip to the interaction kernel immediately:

$$
\begin{aligned}
\mathcal{V}(g_{1,2,3};Y_{1,2,3};g_{4,5,6};Y_{4,5,6};g_{7,8,9};Y_{7,8,};g_{10,11,12};Y_{10,11,12}) = \int \prod_{i=1}^{12} \mathrm{d}\, X_i \prod_{j=1}^{4} \mathrm{d}h_j \\
\delta(g_1 h_1 t(X_1^{-1}X_{12})h_4^{-1}g_{12}^{-1})\,\delta(g_2 h_1 t(X_2^{-1}X_8)h_3^{-1}g_8^{-1}) \\
\delta(g_3 h_1 t(X_3^{-1}X_4)h_2^{-1}g_4^{-1})\,\delta(g_5 h_2 t(X_5^{-1}X_7)h_3^{-1}g_7^{-1}) \\
\delta(g_6 h_2 t(X_6^{-1}X_{10})h_4^{-1}g_{10}^{-1})\,\delta(g_9 h_3 t(X_9^{-1}X_{11})h_4^{-1}g_{11}^{-1}) \\
\delta(X_1(h_1^{-1}g_2^{-1}\triangleright Y_{2,1})X_2^{-1}X_8(h_3^{-1}g_9^{-1}\triangleright Y_{9,8})X_9^{-1}X_{11}(h_4^{-1}g_{12}^{-1}\triangleright Y_{12,11})X_{12}^{-1}) \\
\delta(X_2(h_1^{-1}g_3^{-1}\triangleright Y_{3,2})X_3^{-1}X_4(h_2^{-1}g_5^{-1}\triangleright Y_{5,4})X_5^{-1}X_7(h_3^{-1}g_8^{-1}\triangleright Y_{8,7})X_8^{-1}) \\
\delta(X_3(h_1^{-1}g_1^{-1}\triangleright Y_{1,3})X_1^{-1}X_{12}(h_4^{-1}g_{10}^{-1}\triangleright Y_{10,12})X_{10}^{-1}X_6(h_2^{-1}g_4^{-1}\triangleright Y_{4,6})X_4^{-1}) \\
\delta(X_5(h_2^{-1}g_6^{-1}\triangleright Y_{6,5})X_6^{-1}X_{10}(h_4^{-1}g_{11}^{-1}\triangleright Y_{11,10})X_{11}^{-1}X_9(h_3^{-1}g_7^{-1}\triangleright Y_{7,9})X_7^{-1})
\end{aligned}
\tag{4.12}
$$

As a short cut to arriving at (4.12) we can take the approach of writing the "non-gauge invariant" constraints and performing a group averaging. For example, starting with $\delta(g_1 g_{12}^{-1})$ and then averaging over 2-gauge transformations on g_1 given by parameters (h_1, X_1) and transformations on g_{12} given by (h_4, X_{12}), we arrive at the result. The same logic holds for the last four deltas. The h's and X's appear as gauge parameters for 2-gauge transformations of $g_i^{-1}\triangleright Y_{ji}$.

4.2.2 Topological invariance and consistency checks

We have now defined a 2-GFT. The action is given by

$$
\begin{aligned}
S = \int \mathrm{d}g_i\, \mathrm{d}Y_{ij}\mathcal{K}(g_{1,2,3};Y_{1,2,3};g_{4,5,6};Y_{4,5,6})\phi(g_{1,2,3};Y_{1,2,3})\phi(g_{4,5,6},Y_{4,5,6})+ \\
\mathcal{V}(g_{1,2,3};Y_{1,2,3};g_{4,5,6};Y_{4,5,6};g_{7,8,9};Y_{7,8,};g_{10,11,12};Y_{10,11,12}) \\
\phi(g_{1,2,3};Y_{1,2,3})\phi(g_{4,5,6};Y_{4,5,6})\phi(g_{7,8,9};Y_{7,8,9})\phi(g_{10,11,12};Y_{10,11,12})
\end{aligned}
\tag{4.13}
$$

We now need to verify that this action provides Feynman graphs which are dual to triangulations of 3-manifolds and the amplitude of those graphs are projectors (products of delta functions) which impose the correct closure of loops and two-dimensional surfaces. We shall also check that the action is topologically invariant.

In the following calculations, we assume that the Haar measures dg_i and dY_{ij} are invariant under multiplication by the left and the right (the measures are unimodular). We also assume that dY_{ij} is invariant under the action of the group G. These assumptions make everything much simpler.

The first thing to check is that $\mathcal{K}$ defines a meaningful propagator. So we check

$$\int \mathrm{d}u_i \, \mathrm{d}Z_{ij} \mathcal{K}(g_{1,2,3}; Y_{1,2,3}; u_{1,2,3}; Z_{1,2,3}) \mathcal{K}(u_{1,2,3}; Z_{1,2,3}; g_{4,5,6}; Y_{4,5,6}) = \mathcal{K}(g_{1,2,3;Y_{1,2,3}}; g_{4,5,6}; Y_{4,5,6}). \tag{4.14}$$

We begin writing the left hand side

$$\begin{aligned}
&\int \mathrm{d}u_i \, \mathrm{d}Z_{ij} \mathcal{K}(g_{1,2,3}; Y_{1,2,3}; u_{1,2,3}; Z_{1,2,3}) \mathcal{K}(u_{1,2,3}; Z_{1,2,3}; g_{4,5,6}; Y_{4,5,6}) \\
&= \int \prod_{i=1}^{4} \mathrm{d}h_i \prod_{i=1}^{12} \mathrm{d}X_i \, \mathrm{d}u_i \, \mathrm{d}Z_{ij} \delta(g_1 h_1 t(X_1^{-1} X_4) h_2^{-1} u_1^{-1}) \, \delta(g_2 h_1 t(X_2^{-1} X_5) h_2^{-1} u_2^{-1}) \\
&\delta(g_3 h_1 t(X_3^{-1} X_6) h_2^{-1} u_3^{-1}) \delta(X_1 (h_1^{-1} \triangleright Y_{2,1}) X_2^{-1} X_4 (h_2 \triangleright Z_{2,1}) X_5^{-1}) \\
&\delta(X_2 (h_1^{-1} \triangleright Y_{3,2}) X_3^{-1} X_5 (h_2 \triangleright Z_{3,2}) X_6^{-1}) \delta(X_3 (h_1^{-1} \triangleright Y_{3,2}) X_1^{-1} X_6 (h_2 \triangleright Z_{1,3}) X_6^{-1}) \\
&\delta(u_1 h_3 t(X_1^{-1} X_4) h_4^{-1} g_4^{-1}) \, \delta(u_2 h_3 t(X_2^{-1} X_5) h_4^{-1} g_5^{-1}) \, \delta(u_3 h_3 t(X_3^{-1} X_6) h_4^{-1} g_6^{-1}) \\
&\delta(X_7 (h_3^{-1} \triangleright Z_{2,1}) X_8^{-1} X_{10} (h_4 \triangleright Y_{5,4}) X_{11}^{-1}) \\
&\delta(X_8 (h_3^{-1} \triangleright Z_{3,2}) X_9^{-1} X_{11} (h_4 \triangleright Y_{6,5}) X_{12}^{-1}) \\
&\delta(X_9 (h_3^{-1} \triangleright Z_{3,2}) X_7^{-1} X_{12} (h_4 \triangleright Y_{4,6}) X_{12}^{-1})
\end{aligned} \tag{4.15}$$

One can use the first three deltas to eliminate the u_i's (this uses the invariance of the Haar measure on G). The Z's may be integrated out as well, taking with them six superfluous X dummy variables. All this yields the right hand side of (4.14).

As a consistency check, we might as well integrate out the gauge variables (h's and X's) in $\mathcal{K}(g_{123}; Y_{123}; g_{456}; Y_{456})$ as a way to illustrate why the Feynman diagrams and the amplitudes are related. A Feynman diagram with amplitude $\mathcal{K}$ is one with no interaction vertices. We will consider external lines in a Feynman diagram to be triangles on the boundary of the 3-manifold. In this case, we have two external legs, hence our geometric picture is that of a triangulation of a 2-sphere with two triangles.

After integrating h and X's in the propagator (4.11) we find

$$\begin{aligned}
\mathcal{K}(g_{123}; Y_{123}; g_{456}; Y_{456}) &= \delta(t(g_2^{-1} \triangleright (Y_{5,4} Y_{1,2})) g_1^{-1} g_4 g_5^{-1} g_2) \\
&\delta(t(g_3^{-1} \triangleright (Y_{6,5} Y_{2,3})) g_2^{-1} g_5 g_6^{-1} g_3) \\
&\delta((g_3^{-1} \triangleright (Y_{5,4} Y_{1,2}))(g_2^{-1} \triangleright (Y_{5,4} Y_{1,2}))(g_1^{-1} \triangleright (Y_{4,6} Y_{3,1})))
\end{aligned} \tag{4.16}$$

Reading this as a series of constraints, the first tells us that the surface decorated with $g_2^{-1} \triangleright (Y_{5,4} Y_{1,2})$ is enclosed by holonomy $g_1^{-1} g_4 g_5^{-1} g_2$. The second constraint is similarly telling us that the boundary of the surface $g_3^{-1} \triangleright (Y_{6,5}Y_{2,3})$ is given by $g_2^{-1}g_5g_6^{-1}g_3$. The final piece of the puzzle is that the three surfaces mentioned combine to make a closed two dimensional surface $(g_3^{-1} \triangleright (Y_{5,4}Y_{1,2}))(g_2^{-1} \triangleright (Y_{5,4}Y_{1,2}))(g_1^{-1} \triangleright (Y_{4,6}Y_{3,1})) = 1$. This is precisely the triangulation on the sphere as expected. One can consider the g_i's as half links, and combinations like $g_1^{-1}g_4$, $g_2^{-1}g_5$ and $g_3^{-1}g_6$ are therefore the links dual to edges. The $Y_{i,j}$'s are then interpreted as half surfaces. The role of the gauge variables h and X is then to glue these half-links and half-surfaces together.

In a similar fashion we can integrate the gauge variables out of the interaction term. Since this is a four-valent vertex, the corresponding triangulated manifold is simply a tetrahedron. The four external legs indicate the four faces.

Finally, we need to consider what makes such a theory topologically invariant. That the theory be topologically invariant means that the amplitude associated to any particular triangulation is the same as the amplitude of another triangulation of a homeomorphic manifold which has the same boundary triangulation.

Triangulations of homeomorphic manifolds are related by a series of Pachner moves. In three dimensions, there are two such moves: the 1-4 move and the 2-3 move, as well as their inverses as described previously.

At the level of the Feynman diagrams, the 1-4 move relates to taking a single vertex and replacing it with four vertices as is shown in Fig. 4.2.

The action we defined with (4.12) and (4.11) satisfy the following property, up to some divergent factor:

$$\begin{aligned}
V(g_{1,2,3}; Y_{1,2,3}; g_{4,5,6}; Y_{4,5,6}; g_{7,8,9}; Y_{7,8,9}; g_{10,11,12}; Y_{10,11,12}) = \\
\int (\prod \mathrm{d}u)(\prod \mathrm{d}w)(\prod \mathrm{d}z)(\prod \mathrm{d}y)(\prod \mathrm{d}Z)(\prod dC)(\prod \mathrm{d}B)(\prod \mathrm{d}A) \\
V(g_{1,2,3}; Y_{1,2,3}; u_{1,2,3}; Z_{1,2,3}; u_{4,5,6}; Z_{4,5,6}; u_{7,8,9}; Z_{7,8,9}) \\
V(g_{4,5,6}; Y_{4,5,6}; w_{1,2,3}; A_{1,2,3}; w_{4,5,6}; A_{4,5,6}; w_{7,8,9}; A_{7,8,9}) \\
V(g_{7,8,9}; Y_{7,8,9}; z_{1,2,3}; B_{1,2,3}; z_{4,5,6}; B_{4,5,6}; z_{7,8,9}; B_{7,8,9}) \\
V(g_{10,11,12}; Y_{10,11,12}; y_{1,2,3}; C_{1,2,3}; y_{4,5,6}; C_{4,5,6}; y_{7,8,9}; C_{7,8,9}) \\
K(u_{1,2,3}; Z_{1,2,3}; w_{1,2,3}; A_{1,2,3})\, K(w_{4,5,6}; A_{4,5,6}; z_{1,2,3}; B_{1,2,3}) \\
K(z_{7,8,9}; B_{7,8,9}; y_{7,8,9}; C_{7,8,9})\, K(y_{4,5,6}; C_{4,5,6}; w_{4,5,6}; A_{4,5,6}) \\
K(u_{7,8,9}; Z_{7,8,9}; y_{1,2,3}; C_{1,2,3})\, K(u_{4,5,6}; Z_{4,5,6}; z_{4,5,6}; B_{4,5,6}),
\end{aligned} \tag{4.17}$$

The divergent factor that appears is due to the internal loop of (4.2) corresponding to the internal vertex. The same occurs in the Ponzano-Regge model as well and one needs to introduce some regularization. Showing that (4.17) is true requires the use of properties such as $\int \mathrm{d}u\mathrm{d}Z\mathcal{K}(u_{a,b,c}, Z_{a,b,c}; w_{a,b,c}, A_{a,b,c})\mathcal{V}(u_{a,b,c}; Z_{a,b,c}; \dots) = \mathcal{V}(w_{a,b,c}; A_{a,b,c}; \dots)$ which can be derived after a calculation. The details are left out as a mercy to the reader.

The property (4.17) tells us that the model described is invariant under the 1-4 move. One can check the other move in a similar way to conclude that the theory is indeed topologically invariant.

4.3 2-GFT in 4 dimensions

The previous construction works similarly in four dimensions. In three dimensions, the arguments of the field were interpreted as encoding a triangle, more specifically its dual picture: Three links g_i emanating from a shared node and the three surfaces $Y_{i,j}$ subtended by pairs of links. In four dimensions then, the arguments of the field should be four group elements of G and six decorations in H each living on the surface between any two pairs of links. Since each vertex should be dual to a 4-simplex (which has five tetrahedra on its boundary) they should be 5-valent and the interaction term should be quintic in ϕ.

Using the same rational as in the three dimensional case, the kinematic term in the action is

$$\mathcal{K}(g_{1,2,3,4}; Y_{1,2,3,4}; g_{5,6,7,8}; Y_{5,6,7,8}) = \int \prod_{i=1}^{8} \mathrm{d}\, X_i \prod_{j=1}^{2} \mathrm{d}h_j \delta(g'_1 {g'_8}^{-1})\delta(g'_2 {g'_7}^{-1})\delta(g'_3 {g'_6}^{-1})\delta(g'_4 {g'_5}^{-1})$$
$$\delta(Y'_{1,2} {Y'_{8,7}}^{-1})\delta(Y'_{2,3} {Y'_{7,6}}^{-1})\delta(Y'_{3,4} {Y'_{6,5}}^{-1})\delta(Y'_{4,1} {Y'_{5,8}}^{-1})\delta(Y'_{1,3} {Y'_{8,6}}^{-1})\delta(Y'_{2,4} {Y'_{7,5}}^{-1}) \tag{4.18}$$

where the primes indicate the same 2-gauge transformations as in the previous section under corresponding X and h parameters, for example $g'_1 = g_1 h_1 t(X_1)^{-1}$ and $Y'_{ij} = X_j (h^{-1} g_j^{-1} \triangleright Y_{ij}) X_i^{-1}$. The interaction term is

$$\mathcal{V}(g_{1,2,3,4}; Y_{1,2,3,4}; g_{5,6,7,8}; Y_{5,6,7,8}; g_{9,10,11,12}; Y_{9,10,11,12}; g_{13,14,15,16}; Y_{13,14,15,16}; g_{17,18,19,20}; Y_{17,18,19,20}) =$$
$$\int dh_i dX_j \delta(g'_1 {g'_{20}}^{-1})\, \delta(g'_2 {g'_{14}}^{-1})\, \delta(g'_3 {g'_{10}}^{-1})\, \delta(g'_4 {g'_5}^{-1})\, \delta(g'_6 {g'_{19}}^{-1})$$
$$\delta(g'_7 {g'_{15}}^{-1})\, \delta(g'_8 {g'_9}^{-1})\, \delta(g'_{11} {g'_{18}}^{-1})\, \delta(g'_{12} {g'_{13}}^{-1})\, \delta(g'_{16} {g'_{17}}^{-1})$$
$$\delta(Y'_{1,2} Y'_{17,20} Y'_{14,16})\, \delta(Y'_{2,3} Y'_{13,14} Y'_{10,12})\, \delta(Y'_{3,4} Y'_{9,10} Y'_{5,8})\, \delta(Y'_{4,1} Y'_{6,5} Y'_{20,19})\, \delta(Y'_{1,3} Y'_{18,20} Y'_{10,11})$$
$$\delta(Y'_{2,4} Y'_{15,14} Y'_{5,7})\, \delta(Y'_{6,7} Y'_{17,19} Y'_{15,16})\, \delta(Y'_{7,8} Y'_{13,15} Y'_{9,12})\, \delta(Y'_{8,6} Y'_{11,9} Y'_{19,18})\, \delta(Y'_{11,12} Y'_{17,18} Y'_{13,16}) \tag{4.19}$$

Once again this action is invariant under the Pachner moves in four dimensions. This was proven in [58].

The general shape of the amplitude of the 2-GFT which can be seen as the dual of a 4d triangulation $\mathcal{T}$ (with no boundary) is given by

$$\mathcal{Z}_{\mathcal{T}} = \int \mathrm{d}X\mathrm{d}h \prod_t \delta_G(h_t t(X_t)) \prod_e \delta_H(X_e), \tag{4.20}$$

where $X_t \in H$ decorates the face dual to the triangle $t \in \mathcal{T}$, while $h_t \in G$ decorates the boundary of this face. $X_e \in G$ decorates the closed surface dual to an edge $e \in \mathcal{T}$.

4.4 Quantization of BFCG

Finally, we make contact with an existing state sum model, the KBF model [23, 21] and try to connect it to the BFCG model introduced previously. The follows the procedure for the derivation of the Ponzano-Regge state sum, at least formally. We consider the partition function of the Euclidean 2-group ($G = SO(4)$, $H = \mathbb{R}$) in four dimensions and formally integrate out the B field (as before) and the C field to get,

$$\begin{aligned} Z_{BFCG} &= \int \mathcal{D}B\mathcal{D}A\mathcal{D}C\mathcal{D}\Sigma \exp\left(i\int_{M_4} \langle B \wedge F\rangle + \langle C \wedge G\rangle\right) \\ &= \int \mathcal{D}A\,\mathcal{D}\Sigma\,\delta(F)\delta(G), \end{aligned} \tag{4.21}$$

from which we see that the 2-gauge theory is flat ($F = 0$) and 2-flat ($G = 0$).

To regularize the integral, we introduce a triangulation $\triangle$ of the manifold and its corresponding Poincaré dual $\triangle^*$ as usual. On the cells of the complex $\triangle^*$ we recall the smeared 1- and 2-connections

$$H_l = \mathcal{P}\exp\int_l A \qquad\qquad X_f(n) = \exp\int_f \Sigma \tag{4.22}$$

where $l, f \in \triangle^*$ are links and faces of the dual complex. The node $n \in \triangle^*$ is where the surface variable is rooted as an element of the Euclidean 2-group. As such, in the integral, Σ must be rooted at n as well. The regularized partition becomes (in accordance with (4.20))

$$Z = \int \mathcal{D}H_l\mathcal{D}X_f \prod_{f\in\triangle^*} \delta_{\mathrm{SO}(4)}\left(\prod_{l\in\partial f} H_l^{\epsilon(l|f)}\right) \prod_{b\in\triangle^*} \delta_{\mathbb{R}^4}\left(\sum_{f\in\partial b} \epsilon(f|b) X'_f(n)\right) \tag{4.23}$$

where b are the bubbles (three dimensional cells in $\triangle^*$) and $\epsilon(f|b)$ is ± 1 depending on the orientation. The prime on the X_f's indicates that appropriate whiskering has been done in order to compose the face variables. Such a 2-group state sum model in the finite case is the Yetter-model [105]. The connection between the Euclidean 2-group BFCG theory and the Yetter-type model came up in [57].

Another state sum similar to the Ponzano-Regge model is the KBF model [67, 23, 21], which arises from the representation theory of the Euclidean 2-group. In general representations of 2-groups appear as objects in 2-categories [14]. The KBF model uses concrete representation of the Euclidean 2-group to construct state sums.

The KBF model is defined on a triangulated 4-manifold. The edges e are labelled with positive real numbers l_e and the triangles t are labelled with integers s_t. The partition function is

$$Z_{KBF} = \int \mathrm{d}l_e \, \mathrm{d}s_t \prod_{t\in\triangle} 2A_t(\{l_e\}) \prod_{\tau\in\triangle} (-1)^{\sum_{t\in\tau} s_t} \prod_{\sigma\in\triangle} \frac{\cos\left(S_{KBF}\right)}{4!\mathrm{Vol}_\sigma\left(\{l_e\}\right)} \tag{4.24}$$

where σ are the 4-simplicies of the triangulation, A_t is the area of triangle t as a function of the lengths, and the KBF action is $S_{KBF} = \sum_{\ell\in\sigma} s_t \Theta_t(\{l_e\})$ with Θ being the dihedral angle at which two tetrahedra meet at t. The functional measure s_t here is simply the counting measure $\mathrm{s}_t = \sum_{s_t\in\mathbb{Z}}$. The quantity $\mathrm{Vol}_\sigma\{l_e\}$ is the volume of the 4-simplex σ which has edges labelled by ℓ_e. Note the similarity between the KBF action and the action of the Regge calculus approximation of the Einstein action in three dimensions - $S_R = \sum j_e\theta_e$, where j_e is the spin number on edge e and θ is the angle between the two triangles meeting at e [87].

The goal is to connect the two pictures in (4.24) and (4.23). This is a non-trivial step as there is no analogue of the Peter-Weyl theorem for any 2-groups. Nevertheless, guided by (discrete) geometrical insights and due to the nature of the 2-group being skeletal, we can recover by hand a (2-)representation picture (ie (4.24)) of the partition function (4.23).

To do so we once again take inspiration from the Ponzano-Regge model in three dimensions, seeing the partition function as a projection operator acting on a boundary state ψ [88]

$$\begin{aligned} Z_{BF}(\psi) &= \int \mathrm{d}h_e \, Z_{BF}(\triangle)\psi(h_e) \\ &= \int \mathrm{d}h_e \prod_{f\in\text{faces}} \delta\left(\prod_{l\in\partial f} h_l\right)\psi(h_e) \end{aligned} \tag{4.25}$$

One then has to give details regarding these boundary states. They must belong to a Hilbert space of spin-network states, meaning that ψ is a function of the holonomies,

$\psi \in \mathcal{C}(SU(2)^V)$, where V is the number of vertices of the boundary graph. Furthermore, the space of states also must be invariant under gauge transformations, $\psi(h_e) = \psi(h_{t(e)}h_e h_{s(e)})$, where $h_{s(e)}$ and $h_{t(e)}$ are gauge transformations occurring at the source and edge of the edge e. To impose this invariance, the Gauss constraint, which generates the transformation, is promoted to an operator and we demand that ψ is in its kernel. The Ponzano-Regge state sum can then be recovered by writing ψ in its Fourier components, using the Peter-Weyl theorem.

In order to do a similar thing here, we need to define the Hilbert space on which the projector (4.23) acts. To do that, we need to impose some kinematic constraints: The 1- and 2-Gauss constraints, as well as the edge simplicity constraints, which we recall as a weaker condition than flatness.

As a first step, we consider functions of the boundary holonomies and 2-holonomies which we denote by h_l and x_f. Since we are aiming for something akin to the KBF model, we would prefer to work in terms of edge variables. For this reason we perform a Fourier transform in the $\mathbb{R}^4$ sector:

$$\psi_\Gamma(h_l, x_f) =: \int \mathrm{d}\ell_f \, e^{i\sum_f \ell_f \cdot x_f} \tilde{\psi}_\Gamma(h_l, \ell_f), \tag{4.26}$$

where Γ is the boundary complex of $\triangle^*$, the dual triangulation of the manifold. Here we introduce the $\mathbb{R}^4$ variables ℓ_f. Similarly in the partition function (4.23), we can write the delta function on the R^4 sector as

$$\delta_{\mathbb{R}^4}\!\left(\sum_{f:f\in\partial b} \epsilon(f|b) X'_f(n)\right) = \frac{1}{(2\pi)^4}\int_{\mathbb{R}^4} \mathrm{d}^4 L_b \exp\left(i\, L_b \cdot \sum_{f:f\in\partial b} \epsilon(f|b) X'_f\right). \tag{4.27}$$

Integrating out X_f will yield the partition function in terms of the new L variables only,

$$Z = \int \mathcal{D}H_l \mathcal{D}L_b \prod_{f\in\triangle^*} \delta_{\mathrm{SO}(4)}\!\left(\prod_{l\in\partial f} H_l^{\epsilon(l|f)}\right) \prod_{f\in\triangle^*} \delta_{\mathbb{R}^4}\left(\sum_{b:f\in\partial b} \epsilon(f|b) L'_b.\right) \tag{4.28}$$

As the new L variable are dual to the 2-holonomies, we can gather from the discussion in section 3.3.2 that these discrete variables my be defined by

$$L_b := L_e = \int_e C$$

where we make explicit the connection between the 3-cell b in $\triangle^*$ and the edge $e = b^*$ in $\triangle$. The second delta function in (4.28) therefore is the closure of the triangle edges, corresponding to the constraint $\mathrm{d}_A C = 0$.

For concreteness going forward, we stick to the case where the manifold in question is triangulated by a single 4-simplex, which we call σ and the simplicies are labelled according to the conventions described in section 3.3.2. The boundary amplitude is formally given by

$$Z(\sigma|\tilde{\psi}_\Gamma) = \int \mathrm{d}h_l \mathrm{d}\ell_e \prod_{f\in\Gamma} \delta_{\mathrm{SO}(4)}\left(\prod_{l:l\in\partial f} h_l^{\epsilon(l|f)}\right) \prod_{t:t\in\partial\sigma} \delta_{\mathbb{R}^4}\left(\sum_{e:e\in t} \epsilon(e|t)\ell'_e\right)\tilde{\psi}_\Gamma(h_l,\ell_e). \tag{4.29}$$

As mentioned, we will need to impose edge simplicity in a similar way we did in section 3.3.2. Each vector ℓ_e, is defined implicitly with respect to a reference frame and each reference frame is defined by one of the five tetrahedra making up σ. The holonomies h_l act on the ℓ_e's by changing their reference frame. In order to refer to the edge variables ℓ_e in an unambiguous way, we impose

$$h_{e^*} \triangleright \ell_e = \ell_e \tag{4.30}$$

where $h_{e^*} = \prod_{l\in\partial e^*} h_l$. This is the same edge simplicity constraint that was introduced when we discretized the symplectic potential for BFCG theory. It is necessary since one edge is shared by multiple triangles, in order to write the closure of triangles consistently we need to be able to express the edges in multiple frames. This constraint is the minimum necessary for the closure of the triangles to be independent of the frame we choose to express the constraint in [41, 102].

The edge simplicity constraint actually implies that h_{e^*} has only one free parameter for each link. To see this we note that if τ and τ' are two tetrahedra sharing triangle t, then edge simplicity implies

$$\ell_e[\tau'] = h_{t^*} \triangleright \ell_e[\tau] \tag{4.31}$$

where the square brackets indicate the frame in which the variable is represented. Taking any two edges on the boundary of t, say e_1 and e_2, we can factor h_{t^*} into one piece which fixes the e_1-e_2 plane, and one which does not,

$$h_{t^*} = \mathcal{B}_{t^*}\mathcal{R}_{t^*} \tag{4.32}$$

The $SO(4)$ element $\mathcal{R}_{t^*}$ is defined uniquely by the requirements

$$\ell_{e_i}[\tau'] = R_{t^*} \triangleright \ell_{e_i}[\tau] \qquad n'_\tau[\tau'] = R_{t^*} \triangleright n_\tau(\tau) \tag{4.33}$$

where n_τ is the unique, up to a sign, normal vector to the tetrahedron τ ie it is normal to the hyperplane containing τ. $\mathcal{B}_{t^*}$ stabilizes the t:

$$\ell_{e_1}[\tau'] = \mathcal{B}_{t^*} \triangleright \ell_{e_2}[\tau'] \quad \text{and} \quad \ell_{e_2}[\tau'] = \mathcal{B}_{t^*} \triangleright \ell_{e_2}[\tau']. \tag{4.34}$$

Any rotation leaving the e_1-e_2 unrotated can be parameterized by a single angle θ_t. In a specific representation, where J are the generators of $\mathfrak{so}(4)$, we can write

$$\mathcal{B}_{t^*} = \exp\left(\theta_t \, \frac{\star(\ell_{e_1}[\tau'] \wedge \ell_{e_2}[\tau']) \cdot J}{2A_t}\right), \tag{4.35}$$

where the $\star$ operator is given by $\star(\ell \wedge \ell')^{ab} := \frac{1}{2}\epsilon^{abcd}\ell_c \, \ell'_d$ and again, A_t is the area of the triangle t.

4.4.1 4-simplex phase space: fixing notations

In this section we will reiterate some of the features of section 3.3.2, where we did a similar construction. The goal of this section is to fix the labels/decorations of the 4-simplex and its dual to prepare for the quantization procedure. This will allow us in particular to get the explicit shape of the relevant constraints to consider.

Again, we only consider a triangulation given by a 4-simplex, which has five tetrahedra on its boundary labelled by $1, 2, 3, 4, 5$. We define the discrete variables by smearing the continuous fields over the simplicies of the triangulation and its dual.

$$\begin{aligned} h_{ji} &= \exp \int_{(ji)} A & \qquad x_{kji} &= \exp \int_{(kji)} \Sigma \\ b_{(ij)^*} &= \int_{(ij)^*} B & \qquad \ell_{kji} &= \int_{(kji)^*} C \end{aligned} \tag{4.36}$$

The constraints in these variables are the following. First there is the 1-Gauss constraint

$$^1\mathcal{G}_i^{ab} = \sum_{j:j\neq i} b_{ji}^{ab} - 2 \sum_{k,j:k>j>i} \ell_{kji}^{[a} x_{kji}^{b]} \tag{4.37}$$

which generates the action of $SO(4)$ on $\mathfrak{so}(4)$, $\mathbb{R}^4$ and $\mathbb{R}^{4*}$. Then there is the 2-Gauss constraint

$$^2\mathcal{G}_{\overline{kji}} = \begin{cases} -\ell_{\overline{ki}} + \ell_{\overline{kj}} + \ell_{\overline{ji}} & \text{if } \bar{i}, \bar{j}, \bar{k} \neq \bar{1} \\ -\ell_{\overline{k1}} + h_{21} \triangleright \ell_{\overline{kj}} + \ell_{\overline{j1}} & \text{if } \bar{i} = \bar{1};\ \bar{j}, \bar{k} \neq \bar{2} \\ -h_{32} \triangleright \ell_{\overline{k1}} + h_{31} \triangleright \ell_{\overline{k2}} + \ell_{\overline{21}} & \text{if } (\bar{i}, \bar{j}) = (\bar{1}, \bar{2});\ \bar{k} \neq \bar{3} \\ -h_{42} \triangleright \ell_{\overline{31}} + h_{41} \triangleright \ell_{\overline{32}} + h_{43} \triangleright \ell_{\overline{21}} & \text{if } (\bar{i}, \bar{j}, \bar{k}) = (\bar{1}, \bar{2}, \bar{3}) \end{cases} \tag{4.38}$$

Recall that the overline refers to the vertices of the triangulation and not the nodes in the dual triangulation. In (4.38) we have chosen a particular frame in which to base these constraints. The requirement that the constraints are consistent with closed triangles in a tetrahedron is satisfied by the edge simplicity which we will mention again shortly. These constraints generate one sector of the shift symmetry of the BFCG action.

The other sector of the shift symmetry is given by the 1-flatness constraint

$$ {}^1\mathcal{F}_{kji} \;=\; h_{ik}h_{kj}h_{ji} - \mathbb{I}_{SO(4)} \qquad (i<k<j). \tag{4.39} $$

This constraint evidently tells us that the holonomy around a closed loop is the identity, hence the usual notion of flatness in a gauge theory.

Finally, the 2-flatness comes as the closure of the face variables

$$ {}^2\mathcal{F}_{mkji} \;=\; x_{kji} + x_{mki} - x_{mji} - h_{ji}^{-1} \triangleright x_{mkj} \qquad (i<j<k<m). \tag{4.40} $$

Again there was a choice made in writing (4.40) as to which frame things are expressed. Again, the edge simplicity constraint ensures consistency no matter which choice we make.

Speaking of edge simplicity, we would like to impose those constraints on our kinematic Hilbert space which ensure the boundary data is geometric. By this we mean, if we are saying the ℓ variables are edges of a triangle, then the sum of the ℓ's belonging to the edges of the triangle had better be zero. This is given to us by the 2-Gauss constraint. We would also like that these triangles are part of tetrahedra. In particular, looking at the tetrahedron labelled by (2) the four boundary triangle constraints are obtained by reading those entries in (4.38) which don't contain the index $\overline{2}$. They are

$$ \begin{aligned} h_{21} \triangleright \ell_{\overline{45}} + h_{21} \triangleright \ell_{\overline{53}} + h_{21} \triangleright \ell_{\overline{34}} = 0; \qquad & h_{21} \triangleright \ell_{\overline{34}} + \ell_{\overline{41}} + \ell_{\overline{13}} = 0; \\ h_{21} \triangleright \ell_{\overline{53}} + \ell_{\overline{15}} + \ell_{\overline{31}} = 0; \qquad & h_{21} \triangleright \ell_{\overline{45}} + \ell_{\overline{51}} + \ell_{\overline{14}} = 0. \end{aligned} \tag{4.41} $$

From these, one can construct a tetrahedron which has side lengths $\{h_{21} \triangleright \ell_{\overline{45}}, h_{21} \triangleright \ell_{\overline{53}}, h_{21} \triangleright \ell_{\overline{34}}, \ell_{\overline{51}}, \ell_{\overline{41}}, \ell_{\overline{31}}\}$, which is consistent with the geometric picture we are describing.

On the other hand, if we consider the tetrahedron labelled by (5), we get the following triangle constraints

$$ \begin{aligned} h_{43} \triangleright \ell_{\overline{12}} + h_{41} \triangleright \ell_{\overline{23}} + h_{42} \triangleright \ell_{\overline{31}} = 0; \qquad & h_{31} \triangleright \ell_{\overline{42}} + h_{32} \triangleright \ell_{\overline{14}} + \ell_{\overline{21}} = 0; \\ h_{21} \triangleright \ell_{\overline{34}} + \ell_{\overline{41}} + \ell_{\overline{13}} = 0; \qquad & \ell_{\overline{43}} + \ell_{\overline{24}} + \ell_{\overline{32}} = 0. \end{aligned} \tag{4.42} $$

From here it is not possible to identify six edges which make up a tetraherdon, or equivalently, we cannot say that these four constraints describe the boundary triangles of a shared tetrahedron. To do so, we need to identify some connection between something like $\ell_{\overline{32}}$ and $h_{41} \triangleright \ell_{\overline{32}}$ which appear. This is precisely what edge simplicity does:

$$ (\mathcal{E}_{kji})_b \;:=\; (\ell_{kji})_a({}^1\mathcal{F}_{kji})^a{}_b \;=\; (\ell_{kji})_a(h_{ik}h_{kj}h_{ji})^a{}_b - (\ell_{kji})_b \quad , \qquad (i<k<j). \tag{4.43} $$

4.4.2 Quantization: G-networks

The quantization method formalized in the Dirac method [39]. The classical variables are promoted to operators, the Poisson brackets are replaced by commutators, and the constraints are lifted to operators which annihilate the physical states, which are vectors in a representation of the operator algebra.

The Hilbert space chosen here are square integral functions,

$$\mathcal{H} = \mathcal{L}^2 \left(\bigotimes_{i<j} SO(4) \otimes \bigotimes_{i<j<k} \mathbb{R}^4 \right) \tag{4.44}$$

with states $\tilde{\psi}(h_{ji}, \ell_{kji}) \in \mathcal{H}$. The operator $\hat{h}_{jiab}$ acts multiplicatively on $\tilde{\psi}$

$$\hat{h}_{jiab}\tilde{\psi}((h_{ji}, \ell_{kji}) = h_{jiab}\tilde{\psi}((h_{ji}, \ell_{kji}) \tag{4.45}$$

picking out the ab component of h_{ij}. The states are similarly diagonal for the ℓ operators. The dual variables, the b's and the x's will be defined by derivatives,

$$x^a_{kji} \to -i\frac{\partial}{\partial \ell^a_{kji}} \tag{4.46}$$

and b_{ji} is represented by iL_{ji}, the left invariant derivative defined by

$$L^{ab}_{ji}\,\tilde{\psi}(h_{ji}, \ell_{kji}) := \frac{\mathrm{d}}{\mathrm{d}t}_{|t=0}\tilde{\psi}(h_{ji}e^{tJ^{ab}}, h_{j'i'}, \ell_{kji}). \tag{4.47}$$

(On the right hand side, only the variable h_{ji} whose indices match those of L_{ji} is multiplied by $e^{tJ^{ab}}$.)

The constraints are written in terms of these operators as well. For example

$${}^2\mathcal{F}^a_{mkji}\,\tilde{\psi} = -i\left(\frac{\partial}{\partial \ell^a_{kji}} + \frac{\partial}{\partial \ell^a_{mki}} - \frac{\partial}{\partial \ell^a_{mji}} - (h_{ij})_{ab}\frac{\partial}{\partial \ell^b_{mkj}}\right)\tilde{\psi}. \tag{4.48}$$

The boundary data, the variables (h_l, ℓ_e) are given geometric meaning by the 1- and 2-Gauss constraints, and by the edge simplicity constraint. We therefore will impose that these constraints annihilate $\tilde{\psi}$:

$${}^1\mathcal{G}_i\tilde{\psi} = 0, \qquad {}^2\mathcal{G}_{ji}\tilde{\psi} = 0 \qquad \mathcal{E}_{kji}\tilde{\psi} = 0 \tag{4.49}$$

Recall that we started by wanting to recover the KBF model, which requires positive numbers on the edges and integers on faces. The imposition of the 1-Gauss constraint reduces the degrees of freedom of the ℓ variables so that only their lengths, $l_e := \sqrt{\ell_e \cdot \ell_e} > 0$ are left. These will be the positive numbers decorating the edges.

To introduce the integer variable on faces, we recall that the edge simplicity reduces each holonomy h_t to a single angle θ_t and we can write $\tilde{\psi}(h_t, \ell_e) = \tilde{\psi}(\theta_t(\ell_e, h_t), l_e(\ell_e))$. The states $\tilde{\psi}$ are therefore periodic functions and we can perform a Fourier transform,

$$\tilde{\psi}(\theta_t, l_e) = \sum_{z_t \in \mathbb{Z}} \prod_{t \in \sigma} \exp(i s_t \theta_t) \tilde{\Psi}(s_t, l_e). \tag{4.50}$$

We now see that the integers s_t of the KBF model arise since $\mathbb{Z}$ is dual to $U(1)$, the gauge group post-edge simplicity.

The partition function

$$Z(\sigma|\tilde{\psi}_\Gamma) = \int \mathrm{h}_l \ell_e \prod_{f \in \Gamma} \delta_{\mathrm{SO}(4)} \left(\prod_{l: l \in \partial f} h_l^{\epsilon(l|f)} \right) \prod_{t: t \in \partial\sigma} \delta_{\mathbb{R}^4} \left(\sum_{e: e \in t} \epsilon(e|t) \ell'_e \right) \tilde{\psi}(h_l, \ell_e). \tag{4.51}$$

can now be simplified dramatically. The delta in the $\mathbb{R}^4$ sector is now redundant since we already assume that $\tilde{\psi}$ satisfies the constraint. The $SO(4)$ delta function imposes the 1-flatness constraint. Since we have already imposed the edge simplicity constraint, this delta can be simplified. From (4.35), we see that theta is the angle relating the normal n_τ viewed from two different tetrahedra. In the case where the tetrahedra are embedded in a flat manifold (as in the case where the 1-flatnesss constraint holds) it is the dihedral angle between the two tetrahedra sharing t, up to an overall sign. We denote the dihedral angle $\Theta(l)$ to remind us that it is in fact determined by the edge lengths. Therefore, the only remaining constraint on the $SO(4)$ sector must impose that the variables θ_t are the dihedral angles,

$$\sum_{\epsilon = \pm 1} \delta(\exp(i\theta_t - i\epsilon\Theta_t(l)) = \sum_{\epsilon = \pm 1} \sum_{s_t \in \mathbb{Z}} \exp(-i s_t(\theta_t - \epsilon\Theta_t(l))). \tag{4.52}$$

The sum over ± 1 will bring about the $\cos(S_{KBF})$ appearing in (4.24). All together the partition function can be written as

$$Z(\sigma|\tilde{\Psi}) = \int \mathrm{d}l_e \mathrm{s}_t \prod_t \sqrt{2A_t} \frac{\cos(S_{KBF})}{4! \mathrm{Vol}_\sigma} \tilde{\Psi}(s_t, l_e) \tag{4.53}$$

The $\sqrt{2A_t}$ factor arise by normalization of the states in the Hilbert space.

We emphasize the importance of imposing some part of the flatness constraint in order to achieve this reformulation. We have therefore related the partition function (2.66) to the KBF partition function (4.24) which was shown to be a state sum model in terms of the 2-representations of the Poincaré 2-group [23].

The highlight of this construction is that we were able to connect the Euclidean 2-group BFCG partition function to the KBF model defined in terms of the 2-representations of the Euclidian 2-group, without knowledge of a Plancherel formula or a Peter-Weyl theorem for the 2-representations of the Euclidian 2-group.

Chapter 5

Conclusion

We have analysed a family of topological theories which generalise BF theory. Through analysis of boundary charges we found to independent diffeomorphism charges which are dual to one another. These charges are obtained from considering the gauge transformations and their respective charges on codimension-2 surfaces. To recover these charges it is necessary to choose a particular decomposition for the underlying gauge group.

A useful example is the $SO(4) \ltimes \mathbb{R}^4$ BF theory, where the connection field belongs to the Euclidean group and the B-field is in the commutative dual Lie algebra. One can discretize this theory and obtain the standard cotangent bundle phase space $T^*(SO(4) \ltimes \mathbb{R}^4)$. On the other hand, by introducing a boundary term, one can write the BF theory as a BFCG theory instead, which is better seen as a 2-gauge theory with Euclidean 2-group symmetries. This change of polarisation lends itself to a different discretization which is better described by a pair of 2-gauge theories with 2-groups $SO(4) \ltimes \mathbb{R}^4$ and $\mathfrak{so}^*(4) \times R^4 \cong \mathbb{R}^6 \times \mathbb{R}^4$. Through this semi-dualisation we can discretize the BFCG action not only on triangles in a triangulation $\triangle$ and links in the dual complex $\triangle^*$, but also on edges in $\triangle$ and two dimensional faces in $\triangle^*$.

These extra decorations on edges in the triangulation and faces in the dual complex are precisely the type of construction necessary to accommodate non abelian flux variables in spin foam models. A first step in this direction is to replace the 2-group $\mathbb{R}^4 \times \mathbb{R}^6$ by the nonabelian $AN(3) \ltimes \mathbb{R}^6$, so that the triangulation edges are decorated by nonabelian elements. The face decorations are still abelian, as this is still a skeletal 2-group. In terms of phase space, the Heisenberg double is a more appropriate structure than the cotangent bundle when the momentum space is not an abelian group. Geometrically, discrete geometries decorated by skeletal Poisson 2-group elements are constructed by identifying elements of the Heisenberg double with an edge, a surface, and their respective duals. The gluing of these atomic pieces together in the geometric picture provides a phase space structure for the full triangulation of the considered manifold. Algebraically the gluings correspond to a symplectic reduction.

The quantization of the theory is considered by applying a group field theoretic approach wherein the desired transition amplitudes, coming from Feynman graphs, reflect the geometric constraints necessary in a topological 2-gauge theory. This is done by considering appropriately 2-gauge invariant fields. The resulting partition function for such a GFT may be seen as a projection operator in order to construct transition amplitudes between two boundary states ψ_1, ψ_2, by $\langle\psi_1|Z(\triangle)\psi_2\rangle$.

Applying these constraints to the relevant states, which we called G-networks, we recover the KBF state sum model for the Euclidean 2-group. To define these G-networks one has to impose, on a kinematical level, the Gauss constraints as is done for spin networks, as well as 2-Gauss constraints which impose the closure of edges of the triangulation. For consistency, we need to ensure that the 2-Gauss constraints defining triangles are consistent as triangles making up the 1-skeleton of a tetrahedron. To ensure this consistency, the additional edge simplicity constraint is imposed.

Outlook

Let us highlight some future directions of research.

Regarding the dual diffeomorphism charges we found, it would be interesting to understand the algebraic structure which dictates when such dual diffeomorphisms can be constructed.

Our construction in 3 is based on the fact that *skeletal* Poisson 2-groups share many similarities with Poisson Lie groups. It would be worth considering what new structures need to be considered in order to work with more general 2-groups. This would allow us to have *non-abelian decorations on faces* as well.

During our discussion, we did not consider excitations of any kind. We could have decorated the vertices with 2-curvature excitation or the edges with 1-curvature excitation,which would be string like topological excitations [17, 19]. It would be interesting to see how these are expressed when using the non-trivial 2-gauge picture, ie the BFCG formulation.

The simplicity constraints which project the topological theory into a gravitational one are implemented by imposing that the face variables are related to the tetrads, $B = \star e \wedge e$.

In the BFCG model the tetrads are naturally associated to C, which lives on the edges. In order to impose the simplicity then, one needs to relate the edge variables to the face variables perhaps by considering non-trivial t-maps. It would be interesting to see how the implementation of the simplicity constraints in the 2-gauge theory relates to a discretized theory of gravity. It might be that the skeletal 2-group is not the appropriate structure, and one might look instead at the identity 2-group, where $H = G$. We leave this for further investigations.

In the 2-GFT picture, some assumptions about the measures involved in the definition of the 2-GFT action were made. It would be interesting to see where these assumptions could be relaxed. There was also the issue of divergences in the 2-GFT model. Similar issues appear in three dimensional GFT as well [53, 89]. It is not clear whether similar discussions can be had for the 2-GFT theory.

Starting from G-networks, it was shown how we could recover the KBF amplitude/state sum model defined in terms of the Euclidean 2-group 2-representations. Interestingly, there was no need to use a 2-representation version of the Peter-Weyl theorem to arrive at the KBF model. This is key since in general the 2-category of 2-representations is not well understood outside of the skeletal case [14]. It would be interesting to see whether we can avoid relying on such theorems and focus on a geometric point of view in order to determine the deformed version of the KBF model.

The benefit of using skeletal 2-groups is how they can be manipulated similarly to groups. This makes for ripe analogies between gauge theories and 2-gauge theories. For example, some BF action might only differ from a BFCG action by a boundary term. It would be worth investigating in what way the respective discretizations and quantizations as spin networks are related. In particular one should expect some interesting relations between the nj-symbol for the usual representations of the gauge group and some other nj-symbols for 2-representations [21] of the 2-gauge group, which would be a skeletal (quantum) 2-group. The main example to consider would be the standard $\mathfrak{so}(3,1)$ BF theory which could be discretized and quantized as a 2-gauge theory based on the κ-Poincaré deformation. We leave this interesting question for later investigations.

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APPENDIX

Appendix A

Derivation of the dual diffeomorphism charge

In this appendix we give the detailed derivation of the set of quadratic charges built from field-dependent generators of the current algebra. This contents of this appendix are also shown in [55]. We require these quadratic charges to be integrable for tangential vector fields and to form a closed algebra with themselves and with the charges of the current algebra.

In order to perform this construction, let us consider the most general quadratic field-dependent combination of the charges $(\mathcal{J}_1, \mathcal{J}_2, \mathcal{T}_1, \mathcal{T}_2)$. This is given by

$$\oint \mathcal{G}(\xi) = \oint \mathcal{J}_1\big(\xi \lrcorner (aA + cC)\big) + \oint \mathcal{J}_2\big(\xi \lrcorner (bC + dA)\big) + \oint \mathcal{T}_1\big(\xi \lrcorner (eB + g\Sigma)\big) + \oint \mathcal{T}_2\big(\xi \lrcorner (f\Sigma + hB)\big), \tag{A.1}$$

where the parameters (a, b, c, d, e, f, g, h) are to be determined and the vector field ξ is field-independent. We now impose the following two requirements on this general quadratic charge:

1) The charges must be integrable when the vector fields ξ are tangent to the codimension-2 surface S.

2) The charges must form a closed algebra (up to possible central extensions) with the gauge charges (2.27) of the initial current algebra.

Let us start with the condition of integrability. Using the explicit form of the gauge charges, we find that $\oint\!\!\!\!/\,\mathcal{G}(\xi)$ takes the form

$$\begin{aligned}\oint\!\!\!\!/\,\mathcal{G}(\xi) &= \xi\lrcorner(aA+cC)(\sigma_1\delta B+\sigma_2\delta\Sigma)+\xi\lrcorner(bC+dA)(\sigma_3\delta B+\sigma_4\delta\Sigma)\\ &\quad+(\sigma_1\delta A+\sigma_3\delta C)\wedge\xi\lrcorner(eB+g\Sigma)+(\sigma_2\delta A+\sigma_4\delta C)\wedge\xi\lrcorner(f\Sigma+hB) \qquad \text{(A.2a)}\\ &=\delta\Big(\xi\lrcorner(aA+cC)\wedge(\sigma_1B+\sigma_2\Sigma)+\xi\lrcorner(bC+dA)\wedge(\sigma_3B+\sigma_4\Sigma)\Big)\\ &\quad-\xi\lrcorner\Big((a\delta A+c\delta C)\wedge(\sigma_1B+\sigma_2\Sigma)+(b\delta C+d\delta A)\wedge(\sigma_3B+\sigma_4\Sigma)\Big)\\ &\quad-(\sigma_1a+\sigma_3d-\sigma_2h-\sigma_1e)\delta A\wedge\xi\lrcorner B-(\sigma_2a+\sigma_4d-\sigma_1g-\sigma_2f)\delta A\wedge\xi\lrcorner\Sigma\\ &\quad-(\sigma_1c+\sigma_3b-\sigma_3e-\sigma_4h)\delta C\wedge\xi\lrcorner B-(\sigma_2c+\sigma_4b-\sigma_3g-\sigma_4f)\delta C\wedge\xi\lrcorner\Sigma \qquad \text{(A.2b)}\end{aligned}$$

Equation (A.2a) is the explicit expression for $\oint\!\!\!\!/\,\mathcal{G}(\xi)$ in terms of the gauge charges (2.27) of the current algebra. Equation (A.2b) is then a rewriting of $\oint\!\!\!\!/\,\mathcal{G}(\xi)$ which isolates the integrable part and the part which vanishes when ξ is tangent to S. The last two lines are four independent and generically non-integrable contributions to $\oint\!\!\!\!/\,\mathcal{G}(\xi)$. To obtain (A.2b), we performed a δ integration by part over the first line of (A.2a), followed by an "integration by parts" on $\xi\lrcorner$ in the non-exact term arising from the δ integration by part. Note that this procedure is of course not unique. Instead of using the first line of (A.2a), we could have used the second one. While this choice does affect the intermediate results, the final result of this appendix is of course independent of it.

Without additional conditions on the fields and/or on the vector field ξ to render the last two lines of (A.2b) integrable, we need to enforce the following four conditions on the parameters in order for $\oint\!\!\!\!/\,\mathcal{G}(\xi)$ to be integrable:

$$\sigma_1a+\sigma_3d-\sigma_2h-\sigma_1e=0, \qquad \text{(A.3a)}$$
$$\sigma_2a+\sigma_4d-\sigma_1g-\sigma_2f=0, \qquad \text{(A.3b)}$$
$$\sigma_1c+\sigma_3b-\sigma_4h-\sigma_3e=0, \qquad \text{(A.3c)}$$
$$\sigma_2c+\sigma_4b-\sigma_3g-\sigma_4f=0. \qquad \text{(A.3d)}$$

We now assume that (a,b,c,d,e,f,g) are such that the constraints (A.3) hold, and we therefore consider the 4-dimensional space (spanned by (a,b,c,d,e,f,g) respecting (A.3)) of integrable charges of the form

$$\mathcal{G}(\xi)=\mathcal{J}_1\big(\xi\lrcorner(aA+cC)\big)+\mathcal{J}_2\big(\xi\lrcorner(bC+dA)\big)+\mathcal{T}_1\big(\xi\lrcorner(eB+g\Sigma)\big)+\mathcal{T}_2\big(\xi\lrcorner(f\Sigma+hB)\big). \qquad \text{(A.4)}$$

We now look at the Poisson brackets of these charges with the charges $(\mathcal{J}_1, \mathcal{J}_2, \mathcal{T}_1, \mathcal{T}_2)$ of the initial current algebra. We first consider the bracket with $\mathcal{T}_1(\phi)$, which reads

$$\begin{aligned}\{\mathcal{G}(\xi), \mathcal{T}_1(\phi)\} = &\, \mathcal{T}_1\big([\xi\lrcorner(aA+cC), \phi]\big) - \sigma_1 \oint_S \xi\lrcorner(aA+cC)\wedge \mathrm{d}\phi \\ &+ p_3\mathcal{T}_1\big([\xi\lrcorner(dA+bC),\phi]\big) + p_4\mathcal{T}_2\big([\xi\lrcorner(dA+bC),\phi]\big) \\ &- \sigma_3 \oint_S \xi\lrcorner(dA+bC)\wedge\mathrm{d}\phi - \sigma_5\oint_S \xi\lrcorner(eB+h\Sigma)\wedge\phi \\ &- \sigma_7\oint_S \xi\lrcorner(gB+f\Sigma)\wedge\phi. \end{aligned} \tag{A.5}$$

Here the first line corresponds to the Poisson bracket between $\mathcal{J}_1\big(\xi\lrcorner(aA+cC)\big)$ and $\mathcal{T}_1(\phi)$, the second between $\mathcal{J}_2\big(\xi\lrcorner(bC+dA)\big)$ and $\mathcal{T}_1(\phi)$, the third between $\mathcal{T}_1\big(\xi\lrcorner(eB+g\Sigma)\big)$ and $\mathcal{T}_1(\phi)$ and the fourth line between $\mathcal{T}_2\big(\xi\lrcorner(f\Sigma+hB)\big)$ and $\mathcal{T}_1(\phi)$. To go further, we now use the explicit expressions for the charges $\mathcal{T}_1$ and $\mathcal{T}_2$ to write

$$\begin{aligned}&\{\mathcal{G}(\xi), \mathcal{T}_1(\phi)\} = \\ &\quad \oint_S \big(\sigma_1 a + (p_3\sigma_1+p_4\sigma_2)d\big)[A,\xi\lrcorner A]\wedge\phi + \big(\sigma_3 c + (p_3\sigma_3+p_4\sigma_4)b\big)[C,\xi\lrcorner C]\wedge\phi \\ &\quad + \oint_S \big(\sigma_1 c + (p_3\sigma_1+p_4\sigma_2)b\big)[A,\xi\lrcorner C]\wedge\phi + \big(\sigma_3 a + (p_3\sigma_3+p_4\sigma_4)d)\big)[C,\xi\lrcorner A]\wedge\phi \\ &\quad - \sigma_1\oint_S \big(\xi\lrcorner(a\mathrm{d}A+c\mathrm{d}C)\big)\wedge\phi - \sigma_3\oint_S\big(\xi\lrcorner(d\mathrm{d}A+b\mathrm{d}C)\big)\wedge\phi \\ &\quad - \sigma_5\oint_S\big(\xi\lrcorner(eB+h\Sigma)\big)\wedge\phi - \sigma_7\oint_S\big(\xi\lrcorner(gB+f\Sigma)\big)\wedge\phi, \end{aligned} \tag{A.6}$$

where we have integrated by parts on $\mathrm{d}\phi$ and used the relation $[P\wedge Q]\wedge R = (-1)^{(p+q)r}[R\wedge P]\wedge Q$ in order to isolate ϕ. Now, noting that the couplings satisfy the relations

$$p_3\sigma_1 + p_4\sigma_2 = \sigma_3, \qquad p_3\sigma_3+p_4\sigma_4 = \sigma_8, \tag{A.7}$$

one can rewrite the bracket as

$$\begin{aligned}\{\mathcal{G}(\xi), \mathcal{T}_1(\phi)\} = &\oint_S \big(\sigma_1 a+\sigma_3 d\big)[A,\xi\lrcorner A]\wedge\phi + \big(\sigma_3 c+\sigma_8 b\big)[C,\xi\lrcorner C]\wedge\phi \\ &+ \oint_S \big(\sigma_1 c+\sigma_3 b\big)[A,\xi\lrcorner C]\wedge\phi + \big(\sigma_3 a+\sigma_8 d)\big)[C,\xi\lrcorner A]\wedge\phi \\ &- \sigma_1\oint_S\big(\xi\lrcorner(a\mathrm{d}A+c\mathrm{d}C)\big)\wedge\phi - \sigma_3\oint_S\big(\xi\lrcorner(d\mathrm{d}A+b\mathrm{d}C)\big)\wedge\phi \\ &- \sigma_5\oint_S\big(\xi\lrcorner(eB+h\Sigma)\big)\wedge\phi - \sigma_7\oint_S\big(\xi\lrcorner(gB+f\Sigma)\big)\wedge\phi. \end{aligned} \tag{A.8}$$

Finally, using Cartan's magic formula, the fact that the vector field ξ is tangential, and massaging the various terms, the bracket becomes

$$
\begin{aligned}
\{\mathcal{G}(\xi),\mathcal{T}_1(\phi)\} = b_1\mathcal{T}_1(\mathcal{L}_\xi\phi)+b_2\mathcal{T}_2(\mathcal{L}_\xi\phi)\\
-\oint_S \xi\lrcorner\Big((\sigma_1 a+\sigma_3 d)F+\frac{1}{2}(\sigma_3 c+\sigma_8 b)[C\wedge C]+(\sigma_1 c+\sigma_3 b)\mathrm{d}_A C\\
+(\sigma_5 e+\sigma_7 g)B+(\sigma_5 h+\sigma_7 f)\Sigma\Big)\wedge\phi\\
-\oint_S(\sigma_1 c+\sigma_3 b-\sigma_3 a-\sigma_8 d)[\xi\lrcorner A,C]\wedge\phi,
\end{aligned}
\tag{A.9}
$$

where

$$
b_1=\frac{\sigma_1\sigma_2 c+\sigma_2\sigma_3 b-\sigma_1\sigma_4 a-\sigma_3\sigma_4 d}{\sigma_1\sigma_4-\sigma_2\sigma_3},\qquad b_2=\frac{\sigma_1\sigma_3 a+\sigma_3^2 d-\sigma_1^2 c-\sigma_1\sigma_3 b}{\sigma_1\sigma_4-\sigma_2\sigma_3}. \tag{A.10a}
$$

One can see in (A.9) that the Poisson bracket of $\mathcal{G}$ with $\mathcal{T}_1$ has two types of contributions. The terms on the first line are once again charges of the current algebra. The last three lines however are neither expressible in terms of gauge charges nor central extensions. Therefore, the only possibility for the algebra to close is that somehow these terms cancel or vanish on-shell. More precisely, we would like to find a condition on the parameters such that the bracket takes the form

$$
\{\mathcal{G}(\xi),\mathcal{T}_1(\phi)\}\stackrel{?}{=}b_1\mathcal{T}_1(\mathcal{L}_\xi\phi)+b_2\mathcal{T}_2(\mathcal{L}_\xi\phi)-\oint_S\xi\lrcorner\Big(\alpha\mathrm{EOM}_B+\beta p_4\mathrm{EOM}_\Sigma\Big)\wedge\phi, \tag{A.11}
$$

where (α,β) are arbitrary parameters and where the rescaling by p_4 has been introduced for later convenience. The reason for which only the equations of motion enforced by B and Σ appear here is because the terms which need to be cancelled in the bracket (A.9) only involve contributions from these equations of motion. It is clear that the last line in (A.9) never appears in the equations of motion. We therefore need to impose that this term is vanishing, which amounts to the condition

$$
\sigma_1 c+\sigma_3 b=\sigma_3 a+\sigma_8 d. \tag{A.12}
$$

Then a comparison with the equations of motion (2.24) tells us that the remaining terms in (A.9) can be written in the form appearing in (A.11) provided we have the following relations:

$$\alpha\sigma_1 + \beta p_4\sigma_2 = \sigma_1 a + \sigma_3 d, \tag{A.13a}$$
$$\alpha\sigma_8 + \beta p_4\sigma_9 = \sigma_3 c + \sigma_8 b, \tag{A.13b}$$
$$\alpha\sigma_3 + \beta p_4\sigma_4 = \sigma_1 c + \sigma_3 b, \tag{A.13c}$$
$$\alpha\sigma_5 + \beta p_4\sigma_7 = \sigma_5 e + \sigma_7 g, \tag{A.13d}$$
$$\alpha\sigma_7 + \beta p_4\sigma_6 = \sigma_5 h + \sigma_7 f. \tag{A.13e}$$

In summary, the coupling therefore have to satisfy the 10 equations (A.3), (A.12), and (A.13). This system is overdetermined, but luckily some of these equations are redundant. The system can be solved and the space of solutions is in fact 2-dimensional. It can be easily parametrized by two parameters (x, y), in terms of which we get

$$a = x - yp_3, \tag{A.14a}$$
$$b = x + yq_4, \tag{A.14b}$$
$$c = yr_1, \tag{A.14c}$$
$$d = y, \tag{A.14d}$$
$$e = x, \tag{A.14e}$$
$$f = x + y(q_4 - p_3), \tag{A.14f}$$
$$g = yq_3, \tag{A.14g}$$
$$h = yp_4, \tag{A.14h}$$

provided that the σ's satisfy

$$\sigma_5 q_3 + \sigma_7 q_4 - \sigma_7 p_3 - \sigma_6 p_4 = 0, \tag{A.15}$$

which is in fact equivalent to the condition (2.26) which needs to hold in order for the theory to be topological.

We have therefore found a 2-dimensional space of parameters within (a, b, c, d, e, f, g, h) which gives integrable charges $\mathcal{G}(\xi)$ satisfying a closed algebra with $\mathcal{T}_1(\phi)$. Remarkably, the conditions on the parameters also guarantee that the other Poisson brackets are closed. If we denote a quadratic charge which solve the above conditions by $\mathcal{G}_{x,y}(\xi)$, then the Poisson brackets with the gauge charges are

$$\{\mathcal{G}_{x,y}(\xi), \mathcal{T}_1(\phi)\} = -x\mathcal{T}_1(\mathcal{L}_\xi\phi) - yp_4\mathcal{T}_2(\mathcal{L}_\xi\phi), \tag{A.16a}$$
$$\{\mathcal{G}_{x,y}(\xi), \mathcal{T}_2(\tau)\} = -yq_3\mathcal{T}_1(\mathcal{L}_\xi\tau) - \big(x + y(q_4 - p_3)\big)\mathcal{T}_2(\mathcal{L}_\xi\tau), \tag{A.16b}$$
$$\{\mathcal{G}_{x,y}(\xi), \mathcal{J}_1(\alpha)\} = -\left(x - yp_3\right)\mathcal{J}_1(\mathcal{L}_\xi\alpha) - y\mathcal{J}_2(\mathcal{L}_\xi\alpha), \tag{A.16c}$$
$$\{\mathcal{G}_{x,y}(\xi), \mathcal{J}_2(\chi)\} = -yr_1\mathcal{J}_1(\mathcal{L}_\xi\chi) - (x + yq_4)\mathcal{J}_2(\mathcal{L}_\xi\chi). \tag{A.16d}$$

We note that the brackets with $\mathcal{J}_1$ and $\mathcal{J}_2$ are closed regardless of the constraint on the parameters.

We can now pick two convenient representatives in the 2-parameter family $\mathcal{G}_{x,y}$ of quadratic charges. The usual diffeomorphism corresponds to taking $(x, y) = (1, 0)$, for which we find

$$\mathcal{G}_{1,0}(\xi) = \mathcal{D}(\xi) = \mathcal{J}_1(\xi\lrcorner A) + \mathcal{J}_2(\xi\lrcorner C) + \mathcal{T}_1(\xi\lrcorner B) + \mathcal{T}_2(\xi\lrcorner\Sigma). \tag{A.17}$$

An obvious other independent quadratic charge is found by taking $(x, y) = (0, 1)$. In this case we obtain the charge (2.34) mentioned in the main text, i.e.

$$\begin{aligned} &\mathcal{G}_{0,1}(\xi) = \mathcal{C}(\xi) \\ &= \mathcal{J}_1\big(\xi\lrcorner(r_1C - p_3A)\big) + \mathcal{J}_2\big(\xi\lrcorner(A + q_4C)\big) + q_3\mathcal{T}_1(\xi\lrcorner\Sigma) + \mathcal{T}_2\big(\xi\lrcorner(p_4B + q_4\Sigma - p_3\Sigma)\big). \end{aligned} \tag{A.18}$$

These two charges $(\mathcal{D}, \mathcal{C})$ form a basis of the integrable quadratic charges for tangent vector fields, and we have

$$\mathcal{G}_{x,y}(\xi) = x\mathcal{D}(\xi) + y\mathcal{C}(\xi). \tag{A.19}$$

Finally, we want to compute the bracket between these quadratic charges themselves. To do so, we can either use the definition of the quadratic charges and the elementary brackets of the current algebra, or alternatively use the action of the quadratic generators on the fields. By construction, the diffeomorphism acts by the Lie derivative. On the other hand, $\mathcal{C}$ acts as

$$\delta^{\mathcal{C}}_\xi A = \mathcal{L}_\xi\big(r_1C - p_3A\big), \tag{A.20a}$$
$$\delta^{\mathcal{C}}_\xi C = \mathcal{L}_\xi\big(A + q_4C\big), \tag{A.20b}$$
$$\delta^{\mathcal{C}}_\xi B = q_3\mathcal{L}_\xi\Sigma, \tag{A.20c}$$
$$\delta^{\mathcal{C}}_\xi \Sigma = \mathcal{L}_\xi\big(p_4B + q_4\Sigma - p_3\Sigma\big). \tag{A.20d}$$

Using these expressions in the covariant phase space formula for the Poisson brackets, it can be shown that the algebra between $(\mathcal{D}, \mathcal{C})$ is closed and takes the form

$$\{\mathcal{D}(\xi), \mathcal{C}(\zeta)\} = -\mathcal{C}([\xi, \zeta]), \tag{A.21a}$$
$$\{\mathcal{D}(\xi), \mathcal{D}(\zeta)\} = -\mathcal{D}([\xi, \zeta]), \tag{A.21b}$$
$$\{\mathcal{C}(\xi), \mathcal{C}(\zeta)\} = (p_3 - q_4)\mathcal{C}([\xi, \zeta]) - p_4q_3\mathcal{D}([\xi, \zeta]), \tag{A.21c}$$

which can further be rewritten as (2.43) upon redefining the new generator (2.42).