A Linear Algebraic Method on the Chromatic Symmetric Function

by

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Author's Declaration

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Abstract

The Stanley-Stembridge conjecture is a longstanding conjecture that has evaded proof for nearly 30 years. Concerned with the e-basis expansions of the chromatic symmetric functions of unit-interval graphs, this conjecture has served as a significant motivator of research in algebraic graph theory in recent years. We summarize a great deal of the existing work done in favor of this conjecture, giving an overview of the various techniques that have previously been used in the study of this problem. Moreover, we develop a novel technique using methods from linear algebra and use it to obtain an e-basis expansion of graphs known as single clique-blowups of paths. Using this same method, we use this result to prove the e-positivity of double clique-blowups of paths, a previously unknown result.

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Chapter 1

Introduction

Graph theory is a rich and varied branch of mathematics that has many applications in math, computer science, and physics. A branch of graph theory that is of particular interest today is algebraic graph theory, an area that applies algebraic methods to graphs and examines certain invariants related to different graph structures. The chromatic polynomial is a central invariant in algebraic graph theory, dating back to 1912 where it was introduced by Birkhoff in an attempt to prove the 4-color theorem [6]. Since then, it has been generalized and seen applications in many different research areas such as algebraic geometry [24, 28] and mathematical physics [8, 32] to name a few.

On the other hand, symmetric function theory is a branch of algebra closely related to combinatorics. It considers functions that are invariant under the order of their arguments, which, as a result, have a nice combinatorial structure. First introduced in 1815 by Cauchy [7], symmetric functions have been studied for many years, and still remain relevant to this day, with applications in quantum physics [8] and statistics [22], among others.

The focus of this thesis is a combination of these two major areas of study, the chromatic symmetric function; introduced in 1995 by Stanley, the chromatic symmetric function has spawned a large area of research in algebraic graph theory. In particular, the introduction of the Stanley-Stembridge conjecture in [31] (which was later rephrased in the context of chromatic symmetric functions in [30]) has inspired a wealth of research in algebraic graph theory and remains a prominent open problem in the field to this day. This conjecture seeks to characterize the positivity of the e-basis expansion of the chromatic symmetric functions of certain graphs. While there has been significant progress made towards a proof of this problem [4, 21, 27, 28], a complete proof has evaded discovery for decades.

In this thesis, we present a new method that can be used to prove the e-positivity of certain classes of graphs. This method works by using the triple-deletion property from [27] to recursively add or remove edges from a graph until a known e-positive graph is obtained. This method allows us to convert a graph theoretical problem into a purely algebraic problem. While in some cases this can result in a very tedious algebraic argument, it always gives one or more e-basis expansions for the given graph and its "intermediary graphs" (that is, the graphs that can be obtained by adding or removing edges from the original graphs) in terms of graphs with known e-basis expansions. Unfortunately though, there are some limits on the graphs that this method is applicable to.

In Chapter 1, we begin with a discussion of the preliminary information necessary for the thesis and then dive into the research done in this particular topic. Section 1.1 includes a background on graph theory, posets, and symmetric functions. This thesis is meant to be self-contained for those that have a bit of background knowledge in graph theory, so it may not be exhaustive in all the information needed in certain sections, particularly section 1.2. The reader looking for a more in-depth background can refer to [15].

In section 1.2, we begin by providing some background and a bit of history on chromatic symmetric functions and the Stanley-Stembridge conjecture. We then examine the common methods and theorems used by people to prove similar problems and build a better understanding of how we can work on this conjecture. We then end this section by providing a look at the state of the research in this area today. Section 1.2.3 is meant to serve as a starting resource for those interested in this field of research to gain an understanding of what has been done and what else can be done. It aims to give a broad understanding of a large area of research rather than a deep examination of one or a few areas.

In Chapter 2, we present this new method by using it to prove the e-positivity of paths, a class of graphs that is already known to be e-positive. In doing so, we also reprove the e-positivity of lollipop graphs and melting lollipop graphs, showing the power that this method holds. Furthermore, in Chapter 3, we use a modified version of this method to prove the e-positivity of what we call single clique-blowups of paths and their melting variety. While these classes of graphs are known to be e-positive, their exact e-basis expansions were not previously known. We use this method to obtain an e-basis expansion of these graphs. Using this expansion, we determine the e-positivity of double clique-blowups of paths, a result that was previously unknown.

1.1 Preliminaries

1.1.1 Graph Theory

A **graph** G is a pair (V, E) where V is a set and E is a set of pairs of elements of V. The elements of V are called **vertices** while the elements of E are called **edges**. Given a graph $G = (V, E)$, we define $V(G) = V$ and $E(G) = E$. We considers only **simple graphs**, that is, E is not a multiset. We say that $U \subseteq V(G)$ is a **stable set** in G if for all vertices $v, u \in U$, $(v, u) \notin E(G)$. Let G be graph and n be a positive integer. A function $\kappa : V(G) \to [n]$ is called an n**-coloring** of G. κ is said to be a **proper coloring** if for all $i \in [n]$, $\kappa^{-1}(i)$ is a stable set. Let $\chi_G(n)$ denote the number of n-colorings of G. It is a classic result in graph theory that $\chi_G(n)$ is a polynomial function on n, which is called the **chromatic polynomial** of G [15].

We define a **labelled graph** on d vertices to be a graph with vertex set $[d]$. While it is often useful to have a labelling scheme for graphs to define them and examine them in greater detail, we don't always want the graphs we work with to behave like labelled graphs. As such, throughout this paper we will often define a labelling scheme for convenience. However, we consider graphs to be unlabelled unless explicitly stated otherwise.

For graphs G, H such that $V(G) \cap V(G) = \emptyset$, the **disjoint union** of G and H (denoted $G \sqcup H$) is defined as the graph where $V(G \sqcup H) = V(G) \cup V(H)$ and $E(G \sqcup H) = E(G) \cup E(H)$. For labelled graphs G, H on m, n vertices, respectively, we label the vertices of $G \sqcup H$ such that each vertex originally appearing in G has the same label, while each vertex i originally appearing in H is labelled with $i + m$. For these same labelled G, H let $G + H$ denote the **concatenation** of G and H, the labelled graph on $[m + n - 1]$ obtained from $G \sqcup H$ by formally identifying vertices m and $m + 1$ together and appropriately shifting labels such that they have the same relative order.

For clarity, we will now define a few important classes of graphs that are relevant to this research. A **complete graph** on $m \geq 1$ vertices, denoted K_m, is the graph where each pair of vertices is adjacent. When an induced subgraph H of a graph G is a complete graph, we call H a **clique**. Note that the term clique is often used interchangeably with complete graph. A **path** on $n \geq 1$ vertices, denoted P_n, is the tree on that contains 2 vertices of degree 1 and $n - 2$ vertices of degree 2 when $n \geq 2$; when $n = 1$, say $P_1 = K_1$. By convention, we label paths in increasing order; that is to say in P_n, the vertices of degree 1 are labelled with either 1 or n and each remaining vertex $i \in [n]$ is adjacent to $i - 1$ and $i + 1$.

1.1.2 Lollipops and Blowups

We now turn our attention to some of the graphs we study in greater detail throughout the thesis. We first examine lollipop graphs. A **lollipop graph** is the concatenation of a complete graph and a path, denoted by $L_{m,n} = K_m + P_{n+1}$. We can also define a generalization of this with **melting lollipop graphs**.

Definition 1. *For $0 \leq k \leq m-1$, let $L_{m,n}^{(k)}$ denote the graph obtained from $L_{m,n}$ by removing edges $(m, 1), (m, 2), \ldots, (m, k)$. Any graph that can be obtained in such a way is known as a **melting lollipop graph**.*

For an example of these two kinds of graphs, Figure 1.1 shows the lollipop graph $L_{6,3}$ and the melting lollipop graph $L_{6,3}^{(2)}$. Note that $L_{m,n}^{(0)} = L_{m,n}$, $L_{m,n}^{(m-2)} = L_{m-1,n+1}$, and that $L_{m,n}^{(m-1)}$ is the disjoint union of K_{m-1} and P_{n+1}.

We will now explore a method of constructing graphs known as blowups. Given a graph G, we say that G' is a **blowup** of G if G' can be obtained by replacing each vertex v in G with a graph H_v such that vertices u and v are adjacent in G if and only if every vertex of H_u is adjacent to every vertex of H_v in G'. For the purposes of this paper, we look specifically at **clique-blowups**; that is, a blowup of G where each vertex is replaced with a complete graph. We are particularly concerned with clique-blowups where the original graph G is a path.

Let G' be a blowup of a graph G where each $v \in V(G)$ is replaced with the graph H_v. We define the **blasted vertices of** H_v to be the set of vertices $B(H_v) = \{v' \in V(G') :$

$v' \in H_u \forall u \in N(v)\}$, where $N(v)$ is the open neighborhood of v. In other words, the blasted vertices of H_v are the vertices in G' that are adjacent to everything in H_v, but are not in H_v themselves. As the focus of this paper is on clique-blowups, it is important to make the distinction between the cliques and their blasted vertices. For example, note that the subgraph induced by a clique K and one of its blasted vertices makes a larger clique; however, if there are multiple, disconnected blasted vertices, it becomes difficult to determine how large the clique actually is without explicit notation.

Definition 2. *Let $\mathbf{b} = (b_1, b_2, \ldots, b_n)$ be a list of positive integers. Let $P_\mathbf{b}$ denote the **clique-blowup of the path** P_n obtained by replacing vertex i in P_n with K_{b_i}.*

*When there is only one $i \in [n]$ such that $b_i > 1$, we say that $P_\mathbf{b}$ is a **single clique-blowup of the path** P_n (abbreviated SCBP). Furthermore, when there exists distinct $i, j \in [n]$ such that $b_i, b_j > 1$, we say that $P_\mathbf{b}$ is a **double clique-blowup of the path** P_n (abbreviated DCBP). This can be generalized for any integer $1 \leq k \leq n$; when there exists distinct $i_1, i_2, \ldots, i_k \in [n]$ such that $b_{i_1}, b_{i_2}, \ldots, b_{i_k} > 1$, we say that $P_\mathbf{b}$ is a **k clique-blowup of the path** P_n (abbreviated kCBP).*

Our work will focus mostly on SCBPs and DCBPs. As such, it is helpful to make the notation for these cases a bit more compact. In general, given integers $n_1, n_2 \geq 0$, $m > 1$ we denote the SCBP as

$$P_\mathbf{b} = P_{1^{n_1},m,1^{n_2}} \text{ for } \mathbf{b} = (\underbrace{1, 1, \ldots, 1}_{n_1}, m, \underbrace{1, 1, \ldots, 1}_{n_2}).$$

removing the parentheses for ease of notation. For the DCBP case, we can define a similar notation for the general case; given integers $n_1, n_2, n_3 \geq 0$, $m_1, m_2 > 1$ we denote the DCBP as

$$P_\mathbf{b} = P_{1^{n_1},m_1,1^{n_2},m_2,1^{n_3}} \text{ for } \mathbf{b} = (\underbrace{1, 1, \ldots, 1}_{n_1}, m_1, \underbrace{1, 1, \ldots, 1}_{n_2}, m_2, \underbrace{1, 1, \ldots, 1}_{n_3}).$$

However, for the purposes of this thesis, it is helpful to make a distinction for the case where $n_2 = 0$ (that is, where the blown-up vertices are adjacent to one another). As such, we generally denote the DCBP with adjacent blown-up vertices using

$$P_\mathbf{b} = P_{1^{n_1},m_1,m_2,1^{n_2}} \text{ for } \mathbf{b} = (\underbrace{1, 1, \ldots, 1}_{n_1}, m_1, m_2, \underbrace{1, 1, \ldots, 1}_{n_2}).$$

We consider SCBPs and DCBPs to be a sort of generalization of lollipop graphs; note that $L_{m,n} = P_{1^0,m,1^n}$. As such, we also define a melting version of SCBPs and DCBPs.

Definition 3. *For $0 \leq k \leq m-1$, let $P^{(k)}_{1^{n_1},m,1^{n_2}}$ denote the graph obtained from $P_{1^{n_1},m,1^{n_2}}$ by removing edges $(n_1, n_1+m), (n_1, n_1+m-1), \ldots, (n_1, n_1+m-k+1)$. Any graph that can be obtained in such a way is known as a **melting SCBP**. Moreover, for $0 \leq k \leq m_1 - 1$, let $P^{(k)}_{1^{n_1},m_1,m_2,1^{n_2}}$ denote the graph obtained from $P_{1^{n_1},m_1,m_2,1^{n_2}}$ by removing edges $(n_1, n_1+m_1), (n_1, n_1+m_1-1), \ldots, (n_1, n_1+m_1-k+1)$ from $P_{1^{n_1},m_1,m_2,1^{n_2}}$. Any graph that can be obtained in such a way is known as a **melting DCBP**.*

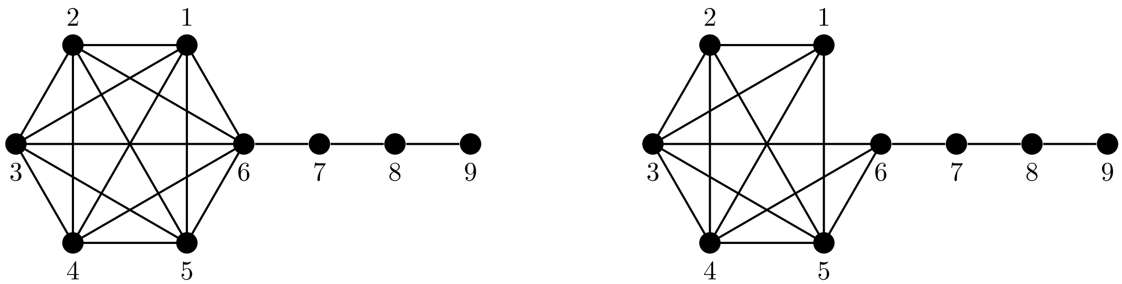

Figure 1.1: The lollipop graph $L_{6,3}$ (left) and the melting lollipop graph $L_{6,3}^{(2)}$ (right). These graphs are the incomparability graphs of natural unit-interval orders $P(6,6,6,6,6,7,8,9)$ and $P(5,5,6,6,6,7,8,9)$, respectively.

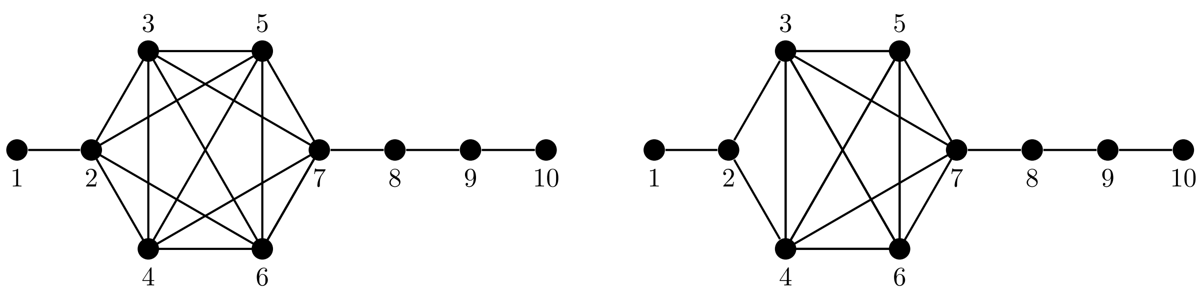

Figure 1.2: The SCBP $P_{1^2,4,1^4}$ (left) and the melting SCBP $P^{(2)}_{1^2,4,1^4}$ (right). Note that vertices $\{3,4,5,6\}$ form a clique corresponding to one vertex of a path, and vertices 2 and 7 are its blasted vertices.

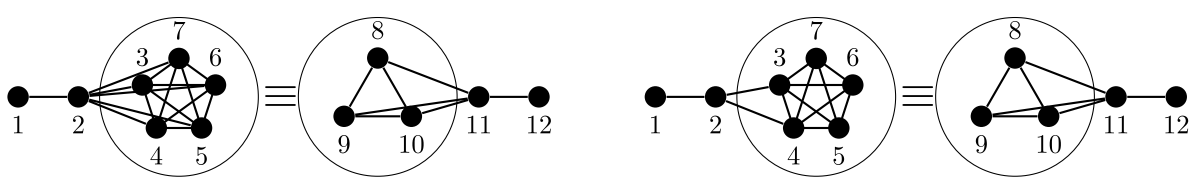

Figure 1.3: The DCBP $P_{1^2,5,3,1^2}$ (left) and the melting DCBP $P^{(3)}_{1^2,5,3,1^2}$ (right). We use three lines between the cliques to represent the edges between them without compromising legibility.

First, note that our definition of melting DCBPs uses only the case where the blown-up cliques are adjacent; while this can be generalized to all DCBPs, it is not necessary for the work done here and is omitted to avoid confusion. Also note that in both the single and double cases, edges are only removed from the first path in the clique-blowup, i.e. the "leftmost" path. While it may seem natural to define melting SCBPs and DCBPs such that we can remove edges from either path, doing so complicates the notation as it can become unclear when certain vertices in the clique are missing one, both, or neither edge to the path. Additionally, proofs in later chapters only require that we remove edges between one path and one cliques. However, we often switch the placement of the paths to denote the removal of edges between P_{n_2} and the cliques (that is, we will write $P^{(k)}_{1^{n_2},m,1^{n_1}}$ or $P^{(k)}_{1^{n_2},m_2,m_1,1^{n_1}}$ to denote this removal of edges).

1.1.3 Posets

The work of this thesis takes particular interest in incomparability graphs, graphs that are obtained from algebraic structures known as posets. We define a **partially-ordered set** $(P, <_P)$, commonly known as a **poset**, to be a set P together with a binary relation $<_P$ placing some partial ordering on the elements of the set. We say that two elements $p, q \in P$ are **comparable** if $p <_P q$ or $q <_P p$; otherwise, they are **incomparable**. We represent posets using **Hasse diagrams**, where each element is represented by a vertex in the plane and p has a line segment going upward to q when $p <_P q$. Lastly, let $inc(P)$ denote the **incomparability graph** of P, the graph obtained from P by letting $V(inc(P)) = P$ and $E(inc(P)) = \{(p, q) : p, q \text{ are incomparable}\}$. Any graph that can be obtained in this way is said to be an incomparability graph.

We are particularly concerned with posets that avoid certain induced subposets. For a positive integer a, suppose we have a subset $\{p_1, p_2, \ldots, p_a\} \subset P$ where $p_1 <_P p_2 <_P \cdots <_P p_a$; we call this an **$a$-element chain** of P. For $a, b \in \mathbb{N}_+$, we say that $a+b$ is the poset made up of a disjoint union of an a-element chain and a b-element chain. Then, a poset is said to be **$(a + b)$-free** if it contains no induced subposet isomorphic to $a + b$.

There are also special types of posets that are of great interest for this thesis, which we define here. We say that a poset is a **unit-interval order** if it is isomorphic to the poset formed by a finite collection $\mathcal{I}$ of intervals of the form $[a, a + 1]$ on the real line, partially-ordered by the relation $[a, a + 1] <_{\mathcal{I}} [b, b + 1]$ if $a + 1 < b$ (see Figure 1.4). It is well-known that a poset is a unit-interval order if and only if it is $(3+1)$-free and $(2+2)$-free [29]. Furthermore, if a graph G is isomorphic to the incomparability graph of a unit-interval order, we say G is a **unit-interval graph**.

Now, let $\mathbf{l} = (l_1, l_2, \ldots, l_{n-1})$ be a list of non-decreasing positive integers with $i \leq l_i \leq n$ for each $i \in [n-1]$. We say that the **natural unit-interval order** with respect to $\mathbf{l}$ is the poset $P(\mathbf{l})$ on $[n]$ that is partially-ordered by the relation $i <_{P(\mathbf{l})} j$ if $i < n$ and $l_i < j \leq n$, for $i, j \in [n]$. It was shown in [28] that the incomparability graph of any natural unit-interval order is a unit-interval graph; furthermore, every unit-interval graph is isomorphic to the incomparability graph of a unique natural unit-interval order. See Figure 1.4 for an example of how these relate to unit-interval orders.

Figure 1.4: The Hasse diagram of the unit-interval order on intervals $\mathcal{I} = \{[0,1], [0.5,1.5], [1,2], [2,3], [3,4]\}$ (left) and the incomparability graph of this poset (right). This corresponds to the natural unit-interval order $P(3,3,4,5)$.

For the purposes of this thesis, we are greatly concerned with the incomparability graphs of these natural unit-interval orders. As such, it is important to understand the relationship between these posets and their incomparability graphs. Note for example that the path graph P_n is the incomparability graph of $P(2,3,\ldots,n)$. We now make some observations about the posets related to some of the important graphs defined in Section 1.1.1. Firstly, note that the lollipop graph $L_{m,n}$ is the incomparability graph of $P(\mathbf{l})$ where for each $l_i \in \mathbf{l}$, $l_i = m$ when $i < m$ and $l_i = i+1$ otherwise. Moreover, the melting lollipop graph $L_{m,n}^{(k)}$ is the incomparability graph of the natural unit-interval order obtained from the poset of $L_{m,n}$ by subtracting 1 from the first k elements of $\mathbf{l}$. For an example of these properties, the graphs $L_{6,3}$ and $L_{6,3}^{(2)}$ (see Figure 1.1) are the incomparability graphs of natural unit-interval orders $P(6,6,6,6,6,7,8,9)$ and $P(5,5,6,6,6,7,8,9)$, respectively. Furthermore, observe that the (melting) SCBP $P_{1^{n_1},m,1^{n_2}}^{(k)}$ is the incomparability graph of $P(\mathbf{l})$ where for each $l_i \in \mathbf{l}$,

$$l_i = \begin{cases} i+1, & \text{if } i < n_1 \text{ or } i > n_1 + m \\ n_1 + m - k, & \text{if } i = n_1 \\ n_1 + m + 1, & \text{otherwise.} \end{cases}$$

Lastly, note that the (melting) DCBP $P_{1^{n_1},m_1,m_2,1^{n_2}}^{(k)}$ is the incomparability graph of $P(\mathbf{l})$ where for each $l_i \in \mathbf{l}$

$$l_i = \begin{cases} i+1, & \text{if } i < n_1 \text{ or } i > n_1 + m \\ n_1 + m_1 - k, & \text{if } i = n_1 \\ n_1 + m_1 + m_2, & \text{if } n_1 < i \leq n_1 + m_1 \\ n_1 + m_1 + m_2 + 1, & \text{otherwise.} \end{cases}$$

Note that since each of these graph classes have a corresponding natural unit-interval order, they are therefore unit-interval graphs.

1.1.4 Symmetric Functions

In addition to graph theory, this thesis focuses a great deal on symmetric functions. We will provide a rigorous definition of symmetric functions and their bases throughout this section, but they can generally be thought of as functions that are invariant under the order of their arguments. We rely on much of the definitions from [1] and [16] for this background.

We begin with a basic definition of symmetric polynomials in finite variables. Let S_n denote the symmetric group of size n. Consider a polynomial f on a finite set of variables $X_n = \{x_1, x_2, \dots, x_n\}$; we say that f is a **symmetric polynomial** if for any permutation $\sigma \in S_n$

$$f(x_1, x_2, \dots, x_n) = f(x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(n)}).$$

Then let $\Lambda(X_n)$ denote the set of symmetric polynomials on n variables with coefficients in $\mathbb{Q}$. Furthermore, we say that a symmetric polynomial is **homogeneous of degree** k if every term in the polynomial has total degree k; we use $\Lambda_k(X_n)$ to denote the set of symmetric polynomials with coefficients in $\mathbb{Q}$ that are homogeneous of degree k on n variables.

We now extend the idea of symmetric polynomials to infinite variables using formal power series. Let $\mathbb{N}$ denote the set of nonnegative integers and let $\mathbb{N}^\infty = \mathbb{N} \times \mathbb{N} \times \cdots$. Then a **formal power series** with coefficients in $\mathbb{Q}$ is a function $f : \mathbb{N}^\infty \to \mathbb{Q}$ such that if $f(n_1, n_2, \dots) \neq 0$, then only finitely many of $n_1, n_2, \dots$ are nonzero [16].

Definition 4. *Suppose $X = \{x_i\}_{i=1}^\infty$ be a set of variables and f a formal power series in X. Moreover, let $\mathbb{Z}^+$ denote the set of positive integers. We say f is a **symmetric function** in X if for any automorphism $\pi : \mathbb{Z}^+ \to \mathbb{Z}^+$, $f(x_1, x_2, \dots) = f(x_{\pi(1)}, x_{\pi(2)}, \dots)$. Let $\Lambda(X)$ denote the set of all symmetric functions in X and let $\Lambda_k(X)$ denote the set of all symmetric functions in X that are homogeneous of degree k.*

Throughout this thesis, we look only at symmetric functions on infinitely many variables; as such, we drop the X and write Λ and Λ_k for brevity. Furthermore, note that we can provide Λ and Λ_k with a ring structure using the conventional addition and multiplication rules. As such Λ is often called the **ring of symmetric functions**.[1]

We now provide some of the background and notation used to define the algebraic bases of Λ. An **integer partition** $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_l)$ is a weakly decreasing sequence of positive integers (for the sake of brevity, we often refer to them just as "partitions"). That is to say, $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_l > 0$. We call each λ_i a **part** of the partition, and we say $l = l(\lambda)$ is the **length** of the partition and that $|\lambda| = \lambda_1 + \lambda_2 + \cdots + \lambda_l$ is the **size** of λ. When $|\lambda| = n$ for some positive integer n, we write $\lambda \vdash n$.

Partitions can be represented combinatorially using objects called Young diagrams (sometimes called Ferrer's diagrams). The **Young diagram** of a partition λ is a left-justified drawing of empty boxes where the topmost row has λ_1 boxes, the row below it has λ_2 boxes, and so on (see Figure 1.5). We call λ the **shape** of the Young diagram. Given another partition μ with $\mu_i \leq \lambda_i$ for all i, a Young diagram can also be defined by a **skew shape** λ/μ. In this case, we get a **skew Young diagram** which can be obtained by drawing a Young diagram of shape λ and removing the left-justified set of boxes of shape μ.

[1] In fact, Λ admits the conventional scalar multiplication in $\mathbb{Q}$, so it is more accurate to call it the algebra of symmetric functions. This scalar multiplication is extremely important for the idea of e-positivity that we introduce later on.

Figure 1.5: A Young diagram of the partition $\lambda = (6, 4, 4, 2, 1)$ (left) and a semi-standard Young Tableaux of the same shape with content $C(T) = (1, 2, 2, 2, 5, 3, 0, 2)$ (right).

A **tableau** T of shape λ is a Young diagram with shape λ where the boxes are filled with positive integers. We say that a tableau of shape λ is a **semi-standard Young tableau** if the boxes are filled such that they are weakly increasing across the rows from left to right and strongly increasing along the columns from top to bottom. We let $SSYT(\lambda)$ denote the set of all semi-standard Young tableau of shape λ. We let $C(T) = (c_1(T), c_2(T), \dots, c_{|\lambda|(T)})$ denote the **content** of T, where $c_i(T) :=$ the number of time i appears in the tableau T.

We now present a number of commonly used bases of Λ. These bases give us a way to rewrite symmetric functions, which is fundamental to the work done in this thesis. Firstly, we define the **monomial symmetric functions** of a partition λ as

$$m_\lambda = \sum_{\alpha \sim \lambda} x_1^{\alpha_1} x_2^{\alpha_2} \dots,$$

where $\alpha \sim \lambda$ if α is a permutation of the partition λ. Note that $\{m_\lambda | \lambda \vdash k\}$ forms a basis for Λ_k. Next, for an integer k and partition λ, let the **elementary symmetric functions** be defined as

$$e_k = m_{1^k} = \sum_{0 < i_1 < i_2 < \dots < i_k} x_{i_1} x_{i_2} \dots x_{i_k}, \quad e_\lambda = \prod_{i>0} e_{\lambda_i}$$

for any set of positive integers $i_1, i_2, \dots, i_k$. Note that the elementary symmetric function $\{e_n\}_{n=1}^\infty$ form an algebraic basis of Λ; that is to say, every element of Λ can be written as a polynomial in $\{e_n\}_{n=1}^\infty$ [16]. Similar to the elementary symmetric functions, we define the **homogenous symmetric functions** for an integer k and a partition λ to be

$$h_k = \sum_{\lambda \vdash k} m_\lambda = \sum_{0 < i_1 \le i_2 \le \dots \le i_k} x_{i_1} x_{i_2} \dots x_{i_k}, \quad h_\lambda = \prod_{i>0} h_{\lambda_i}.$$

We also define the **power-sum symmetric functions** for an integer k and partition λ as

$$p_k = m_k = x_1^k + x_2^k + \dots, \quad p_\lambda = \prod_{i>0} p_{\lambda_i}.$$

Finally, we define the **Schur symmetric functions** for a partition λ as

$$s_\lambda = \sum_{T \in SSYT(\lambda)} \prod_{i>0} x_i^{c_i(T)}$$

Each of the common types of symmetric functions listed above act as a basis for Λ. Note additionally that there are commonly known transition matrices between these bases. Furthermore, we say that a symmetric function is **e-positive** if it can be written in the e-basis using only nonnegative coefficients. There are similar positivity definitions for the m-, p-, h-, and Schur-bases, but the main focus of this paper is on e-positivity.

1.2 The State of the Research

1.2.1 History of CSFs and the Stanley-Stembridge Conjecture

The work of this thesis is focused on the **chromatic symmetric functions** (CSFs) of certain kinds of graphs. The idea of the CSF was first introduced by Stanley in [30] as a generalization of the chromatic polynomial. He defined it as follows:

Definition 5. *Let $x_1, x_2, \dots$ be commuting indeterminates. For a simple graph $G = (V, E)$, the* ***chromatic symmetric function*** *of G, denoted $\mathbf{X}_G$, is defined as*

$$\mathbf{X}_G = \mathbf{X}_G(\mathbf{x}) = \mathbf{X}_G(x_1, x_2, \dots) = \sum_{\kappa} \prod_{v \in V} x_{\kappa(v)}$$

where the sum ranges over all proper colorings κ of G.

Since their introduction, these functions have been of great interest in the field of algebraic graph theory. In particular, there has been one open problem of great interest regarding the e-positivity of these functions, known as the Stanley-Stembridge conjecture, that has evaded proof for nearly three decades. In [30], Stanley shows that the net and claw graphs (Figure 1.6) are not e-positive, and he notes the following conjecture:

Conjecture 6 (Stanley-Stembridge Conjecture)**.** *Let P be a (3+1)-free poset. Then, $\mathbf{X}_{inc(P)}$ is e-positive.*

First introduced in [31] and then reworked into the context of CSFs in [30], there have been many results in favor of this conjecture since its initial statement. We will examine many of these later in this chapter. Note that we say a graph G is e-positive to mean that the CSF of G is e-positive.

There have been several reductions and refinements to this conjecture. One of the more significant findings came from Guay-Paquet in 2013. In unpublished work [21], he found that it was sufficient to prove that the statement holds for (3+1)- and (2+2)-free posets, also known as unit-interval orders. This is equivalent to proving that (claw, net, anti-net, and C_n for $n \geq 4$)-free graphs are e-positive.

Another major refinement of this conjecture followed the introduction of **chromatic quasisymmetric functions** (CQSFs) by Shareshian and Wachs in [28]. This serves as a q-analogue of the typical CSF definition. q-analogues are a common type of generalization in combinatorics, wherein a variable q is added such that the original expression is obtained as q approaches 1, so the discovery of CQSFs is a very natural generalization of CSFs. Shareshian and Wachs define CQSFs as follows:

Figure 1.6: The net graph (left) and the claw graph (right).

Definition 7. *Let $x_1, x_2, \ldots$ be commuting indeterminates. For a simple graph $G = (V, E)$ with $V \subset \mathbb{N}_+$, the **chromatic quasisymmetric function** of G is*

$$\mathbf{X}_G(\mathbf{x}; q) = \sum_{\kappa} \prod_{v \in V} q^{asc(\kappa)} x_{\kappa(v)}$$

where $asc(\kappa) = |\{(i,j) \in E : i < j \text{ and } \kappa(i) < \kappa(j)\}|$.

Note that when $q = 1$, CQSFs are CSFs. So, while the focus here is on CSFs, we are able to use many of the results about CQSFs in our methods. Note the commonly used notation $[n]_q = 1 + q + \cdots + q^{n-1}$ when writing out e-basis expansions for CQSFs; while we do not make use of this notation often here, it appears very often in the literature. Additionally, in [28], Shareshian and Wachs generalize the Stanley-Stembridge conjecture in terms of CQSFs using the results from Guay-Paquet:

Conjecture 8 (Shareshian-Wachs Conjecture)**.** *Let G be the incomparability graph of a unit-interval order. Then, $\mathbf{X}_G(\mathbf{x}; q)$ is e-positive.*

1.2.2 Tools

We will use this section to illustrate the common theorems and properties that are used in the modern study of the Stanley-Stembridge and Shareshian-Wachs conjectures. Additionally, we will describe some of the ideas from linear algebra and matrix theory that we make use of later in our proofs.

Firstly, we describe the **deletion-contraction property**. A function f on graphs is said to have the deletion-contraction property if for any graph G and any edge e of G, we can rewrite $f(G)$ as a function of $f(G-e)$ and $f(G/e)$ where $G-e$ represents the deletion of an edge and G/e represents the contraction of an edge. We call the formula itself the **deletion-contraction relation** of a function f. In the most cases, f will have a deletion-contraction relation similar to the following expression:

$$f(G) = f(G-e) + f(G/e).$$

This property is typically very useful when studying graph functions, as it allows functions of complicated graphs to be broken up into linear combinations of functions of smaller, easier

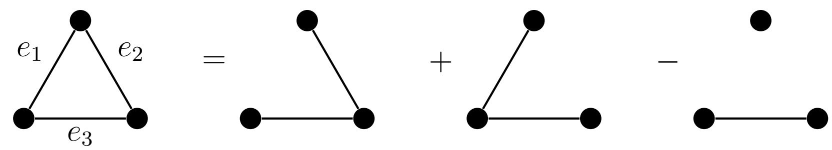

Figure 1.7: A visual representation of the triple deletion property.

to compute graphs. Additionally, this relationship is common for many graph functions; in fact, a whole class of functions derived from Tutte polynomials are known to have this property. Furthermore, with regard to the focus of this thesis, the chromatic polynomial has the deletion-contraction property.

Theorem 9. *Let G be a graph and $\chi_G(n)$ be the chromatic polynomial of G. Then,*

$$\chi_G(n) = \chi_{G-e}(n) - \chi_{G/e}(n).$$

However, CSFs and CQSFs do not satisfy the deletion-contraction property [10]. This is because each monomial in $\mathbf{X}_G$ has degree $|V(G)|$; since contraction reduces the number of vertices, its CSF is made up of monomials with a smaller degree than $\mathbf{X}_G$. As such, much of the focus of research in CSFs and CQSFs has been on either finding a similar property or modifying the current definitions of CSFs such that they admit a deletion-contraction property. Perhaps the most important example of a similar kind of property comes from [27], where Orellana and Scott prove the following property:

Theorem 10. *[27, Theorem 3.1] Let G be a graph where $e_1, e_2, e_3 \in E(G)$ form a triangle. Then,*

$$\mathbf{X}_G = \mathbf{X}_{G-e_1} + \mathbf{X}_{G-e_2} - \mathbf{X}_{G-\{e_1,e_2\}}$$

See Figure 1.7. Known as the **triple deletion property** (and sometimes called the Orellana-Scott modular relation), this property has motivated many of the algebraic methods used in other e-positivity results. Another small point to note in regards to the algebra we can do with CSFs is that the CSF of a disjoint union of graphs is the product of those graphs, i.e. $\mathbf{X}_{G \sqcup H} = \mathbf{X}_G \mathbf{X}_H$.

Additionally, as an aside, there are several modifications to the definitions of CSFs and CQSFs that do admit a deletion-contraction property. One such example is the development of $\mathbf{Y}_G$ and (e)-positivity–a reformulation of the $\mathbf{X}_G$ and e-positivity definitions in a different basis–in [20], which we discuss in the next section. Another example of modifications to CSFs comes from [10], where Crew and Spirkl consider CSFs of graphs with a vertex-weight function. They are then able to redefine the CSF of the pair (G, w), where G is a graph and $w : V(G) \to \mathcal{N}$ is a weight function, as

$$\mathbf{X}_{(G,w)} = \mathbf{X}_{(G,w)}(\mathbf{x}) = \mathbf{X}_{(G,w)}(x_1, x_2, \dots) = \sum_{\kappa} \prod_{v \in V(G)} x_{\kappa(v)}^{w(v)}$$

which admits a deletion-contraction relation of the form

$$\mathbf{X}_{(G,w)} = \mathbf{X}_{(G-e,w)} - \mathbf{X}_{(G/e,w/e)}$$

where w/e represents the function obtained from w by mapping the vertex obtained by contracting e to the sum of the weights of the endpoints of e, and mapping every other vertex to its original weight.

The goal of the method outlined in this paper is to convert a problem in symmetric functions and graph theory into a purely algebraic problem. So, on top of the numerous graph theory and symmetric functions tools we use, there are also several ideas that come from linear algebra used here. Firstly, as many of the matrices can be very large, it is helpful to reduce them as much as possible. As such, we often interpret our matrices as **block matrices**, which is just a way of breaking a matrix up into its submatrices. Block matrices have some nice properties, including having a formula for calculating their inversion.

Lemma 11. *Given a block matrix divided into four submatrices, we can calculate the inverse of that matrix as follows:*

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} = \begin{bmatrix} (A - BD^{-1}C)^{-1} & \mathbf{0} \\ \mathbf{0} & (D - CA^{-1}B)^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{I} & -BD^{-1} \\ -CA^{-1} & \mathbf{I} \end{bmatrix}$$

where A and D are invertible square matrices.

It will also be important to know a few types of matrices so we can use their properties later on. First, we say a **tridiagonal Toeplitz matrix** [26], T_a, is a square matrix that has the form

$$T_a = \begin{bmatrix} a & -1 & & & \\ -1 & a & -1 & & \\ & & \ddots & & \\ & & -1 & a & -1 \\ & & & -1 & a \end{bmatrix}$$

where all other entries are 0. These matrices in particular are so important because in [26], Meurant determines the exact values of their inverses, which becomes extremely useful in later proofs.

A real matrix M is a **Z-matrix** if $M = (m_{ij})$ and $m_{ij} \leq 0$ for all $i \neq j$. Furthermore, M is an **L-matrix** if M is a Z-matrix and $m_{ij} > 0$ for all $i = j$ [5]. M is said to be **weakly diagonally dominant** (w.d.d.) if $|m_{ii}| \geq \sum_{j \neq i} |m_{ij}|$.

Let M be a square matrix. Then the **spectral radius** of M, denoted $\rho(M)$, is the maximum of the absolute values of its eigenvalues. M is called **singular** if M does not have an inverse. M is called an **M-matrix** if it can be expressed in the form $M = sI - B$, where B is a non-negative square matrix and $s \geq \rho(B)$. While M-matrices are interesting in their own right, it is not so important here what they are. The main reason we utilize them is because the inverse of a non-singular M-matrix is non-negative, a fact that will help us prove e-positivity [17].

Figure 1.8: A pyramid graph (left) and a bull graph (right). Clique-blowups of the unfilled vertices give us generalized pyramids and generalized bulls, respectively. Note that the pyramid is the complement of the net graph (see Figure 1.6).

1.2.3 Summary of e-Positivity Results

The main endeavor of this thesis is to resolve the Stanley-Stembridge conjecture for some clique-blowups of paths. As such, it is important to understand the current state of the research, so this section will be devoted to summarizing many of the major results made towards proving this conjecture. We summarize a great deal of the results at the end in Tables 1.1 and 1.2 and provide explanations of the methods used throughout this section. We hope that this will act as a guide to those who are becoming newly acquainted with the subject so they understand what has been done and where they can begin their work.

Much of the work in this thesis is related to the work done by Dahlberg and van Willigenburg in [14], the paper that originally proved the e-positivity of lollipop graphs and gave an e-basis expansion of their CSFs. As these methods are so similar, we refer to the proof of e-positivity of paths and lollipop graphs in Chapter 2 for an understanding of how this works. The work in this paper was very influential, and was used in particular in [23] to prove the Shareshian-Wachs conjecture for melting lollipop graphs as well. By using a family of symmetric functions called LLT polynomials (which we discuss later in this section, see Definition 22), Huh, Nam, and Yoo are able to convert the e-basis formula of lollipop graphs found in [14] to a CQSF and extend this to an e-basis expansion for CQSFs of melting lollipop graphs.

In [18] and [19], Foley, Hoàng, and Merkel summarize and prove some smaller known e-positivity results, which we list in the tables below. In particular, [19] resolves the Stanley-Stembridge conjecture for (claw, H)-free graphs, for all graphs H with 4 vertices except the co-diamond. As one might predict from the study of induced subgraphs, these proofs rely mostly on case work and techniques similar to those found in structural graph theory. The rest of the paper is then dedicated to making progress toward a proof for the (claw, co-diamond)-free case, ending with conjectures that graphs called generalized pyramids and (claw, co-diamond)-free graphs are e-positive. A **generalized pyramid** is a graph obtained from taking a clique-blowup of the unfilled vertices of the pyramid graph in Figure 1.8. While the latter of these still remains open, the former was proven in [25].

Furthermore, in [18], Foley, Hoàng, and Merkel show that unit-interval graphs whose complements are also unit-interval are e-positive. This was discovered initially by examining the case of H-free unit-interval graphs, where H is a four-vertex graph, proving the case

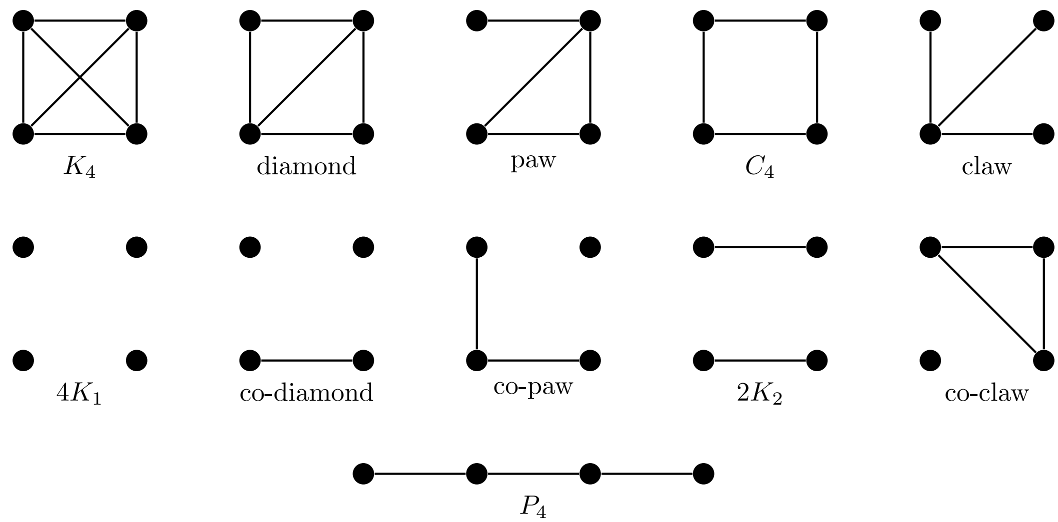

Figure 1.9: All 4 vertex graphs.

for co-claw-free unit-interval graphs with exhaustive case work, and then showing that these are exactly the same class of graphs. They also prove numerous properties related to the e-positivity of unit-interval graphs, with particular interest given to $2K_2$-free unit-interval graphs (see Figure 1.9). While no proof was given for $2K_2$-free unit-interval graphs here, their e-positivity was also shown in [25].

In [25], Li and Yang resolve two conjectures from [18] and [19]. Firstly, they consider generalized pyramids. Using results from Stanley, they obtain the monomial basis expansion of the CSF of generalized pyramids. They are then able to use transition matrices to convert this CSF into the elementary basis and explicitly determine its coefficients to prove their non-negativity. They then consider $2K_2$-free unit-interval graphs. Using a characterization of these graphs from a different paper, Li and Yang are able to show that $2K_2$-free unit-interval graphs are either $3K_1$-free or generalized bull graphs, both of which are known to be e-positive (see [18]).

In addition to the study of induced subgraphs as in the previous three papers, there is also some interest in the contractability of certain graphs. We say a graph G is **contractible** to another graph H if there is a series of contractions we can perform on G to obtain H. Even in the original paper by Stanley defining CSFs, there were questions about contractability; he observed that there was no known graph that is not contractible to the claw that is not e-positive [30]. In other words, there was an implied conjecture that any graph that is not contractible to the claw is e-positive. This, however, was proven false in [12], where Dahlberg, Foley, and van Willigenburg give multiple infinite graph classes that are not contractible to the claw and not e-positive. In particular, they show that a class of graphs known as triangular towers, which are not contractible to the claw and are claw-free, are not e-positive.

In [20], Gebhard and Sagan generalize the idea of symmetric functions, CSFs, and e-positivity to variables that don't commute. This can essentially be thought of as e-positivity on labelled graphs. In this algebra (called NCSym), rather than being based on integer partitions like the classical symmetric function, the functions are based on **set partitions** π of $[d]$.

Definition 12. *Let $\pi \vdash d$ denote a **set partition** of $[d]$, which is a collection of disjoint, nonempty sets $B_1, B_2, \ldots, B_{l(\pi)}$ called **blocks** whose union is $[d]$. Moreover, for an integer $i \in [d]$, let $B_{\pi,i}$ denote the block of π containing i.*

These set partitions allow us to label the elements of the symmetric function so that we ensure they don't commute. By removing the requirement of commutability, Gebhard and Sagan discovered a generalization of the CSF that they denote with $\mathbf{Y}_G$, and then they considered the problem of e-positivity on these functions. However, as this work is done on a different algebra, the e-basis functions are slightly different. Given a set partition π,

$$e_\pi = \sum_{(i_1, i_2 \ldots, i_d)} x_{i_1} x_{i_2} \cdots x_{i_d}$$

where we sum over all tuples $(i_1, i_2 \ldots, i_d)$ such that $i_j \neq i_k$ when $B_{\pi,j} = B_{\pi,k}$. While this may seem to over complicate things a bit, the motivation comes from the ability to apply a deletion-contraction relation on $\mathbf{Y}_G$, something that is not possible on $\mathbf{X}_G$. Without any modification to the definition, the problem of e-positivity is extremely simple for $\mathbf{Y}_G$, as most graphs are not e-positive. In fact, a graph is e-positive on non-commuting variables if and only if $G = K_\pi$ for some set partition π [4]. Here, K_π is the graph on $[d]$ where i, j share an edge when $B_{\pi,i} = B_{\pi,j}$. However, a modification to the classical definition of e-positivity was defined using the equivalence relation

$$e_{\pi_1} \equiv_i e_{\pi_2} \text{ if and only if } \lambda(\pi_1) = \lambda(\pi_2) \text{ and } |B_{\pi_1,i}| = |B_{\pi_2,i}|$$

where π_1, π_2 are set partitions and $\lambda(\pi)$ denotes the integer partition whose parts are $|B_1|, |B_2|, \ldots, |B_l(\pi)|$. This relation is called **congruence modulo** i by Gebhard and Sagan. Using this congruence relation, they found numerous graphs that they called **(e)-positive**.

Definition 13. *A graph G is said to be **(e)-positive** if $\mathbf{Y}_G$ is congruent to an e-positive function modulo $|V(G)|$.*

To provide some more motivation to this definition, we provide the example used in [20] to exhibit the difference between $\mathbf{Y}_G$ before and after applying this relation. As mentioned above, most graphs are not e-positive in this basis; for example, consider Y_{P_3}. We can compute the e-basis expansion of this in NCSym, finding that $Y_{P_3} = \frac{1}{2}e_{12/3} - \frac{1}{2}e_{13/2} + \frac{1}{2}e_{1/23} + \frac{1}{2}e_{123}$, where adjacent numbers represent blocks, which are separated by a diagonal line. However, applying the congruence relation to this same function, we find that $Y_{P_3} \equiv_3 \frac{1}{2}e_{12/3} + \frac{1}{2}e_{123}$, and thus P_3 is (e)-positive.

The pattern here is not unique to paths though. In fact, if a graph is (e)-positive in NCSym, then it is also e-positive in Λ [20]. Making use of the deletion-contraction law that NCSym admits, Gebhard and Sagan end their paper by using this notion of (e)-positivity to prove the e-positivity of $\mathbf{X}_G$ when G is a K_α-chain.

Definition 14. *[20] First, define an **indifference graph** as a graph G with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set*

$$E(G) = \{(v_i, v_j) : i, j \in [k, l] \in C\}$$

where C is a collection of discrete intervals $[k, l] = \{k, k+1, \ldots, l\} \subset [n]$.

*Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ be a composition of n and let $\tilde{\alpha}_i = \sum_{j \leq i} \alpha_j$. A K_α**-chain** is the indifference graph using the collection of intervals $C = \{[1, \tilde{\alpha}_1], [\tilde{\alpha}_1, \tilde{\alpha}_2], \ldots, [\tilde{\alpha}_{k-1}, \tilde{\alpha}_k]\}$.*

In other words, a K_α-chain can be thought of as the graph obtained from a series of concatenations of complete graphs $(((K_{\alpha_1} + K_{\alpha_2}) + K_{\alpha_3}) + \cdots + K_{\alpha_k})$. Note that K_α-chains are not a type of clique-blowup of a path, as the blasted vertices of a clique-blowup of a path are not adjacent.

In [4], Aliniaeifard, Wang, and van Willigenburg expand on the ideas of NCSym and $\mathbf{Y}_G$ with the definition of a new space UBCSym, which works by projecting the e-basis vectors of NCSym to an e-basis with a single distinguished point. This leads to the development of a different generalization of $\mathbf{Y}_G$ in $y_{G:v}$, which distinguishes a single vertex v in the graph G; this requires a slightly modified version of (e)-positivity that takes the distinguished vertex into account, but is consistent with the definition found in [20]. Using this modified (e)-positivity, the authors explore the concatenations of graphs and define a new concept that they call appendable (e)-positivity.

Definition 15. *[4, Definition 4.3] We say that a labelled graph H is **appendable** (e)**-positive** if and only if $G + H$ is (e)-positive for all (e)-positive labelled graphs G. Equivalently, H is appendable (e)-positive if and only if $K_d + H$ is (e)-positive for all $d \in \mathbb{Z}^+$.*

Note that all graphs that are appendable (e)-positive are (e)-positive themselves (and hence e-positive). Appendable (e)-positivity is extremely useful in proving the e-positivity of large classes of graphs, as it allows one to combine different types of graphs together while maintaining positivity. In many ways, these concatenations of graphs behave similarly to blowups of paths, which we will explore in greater detail in a later chapter. After proving the appendable (e)-positivity of certain graphs, such as complete graphs and cycles, Aliniaeifard, Wang, and van Willigenburg summarize how one can combine graphs in the following theorem:

Theorem 16. *[4, Corollary 6.6] For a labelled graph G on d vertices, let G^r denote the reverse graph, where vertex i is relabelled with $d + 1 - i$. Furthermore, for a sequence of graphs $(H_i)_{i=1}^k$, let $\sum_{i=1}^k H_i$ denote the sequential concatenation of graphs $H_1 + H_2 + \cdots + H_k$. If G, G' are (e)-positive and $(H_i)_{i=1}^k$ are appendable (e)-positive, then*

1. *$\sum_{i=1}^k H_i$ is appendable (e)-positive,*

2. *$G + \sum_{i=1}^k H_i$ is (e)-positive, and*

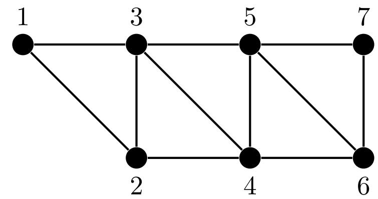

Figure 1.10: The Triangular ladder TL_7.

3. $G + \sum_{i=1}^{k} H_i + G'^r$ is e-positive.

Finally, after their exploration of appendable (e)-positivity, they prove a general property about their version of e-positivity that allows for even more results.

Theorem 17. *[4, Theorem 6.8] For a graph G, if G is (e)-positive at v, i.e. if $y_{G:v}$ is (e)-positive, then $G - v$ is e-positive.*

In [11], the ideas of Gebhard and Sagan are even further expanded to prove the e-positivity of **triangular ladders**, sometimes also known as zigzag graphs.

Definition 18. *Let $P_{n,k}$ denote the class of unit-interval graphs defined by the intervals $[1, 1+k], [2, 2+k], \ldots, [n-k, n]$. When $k = 2$, we call these graphs* ***triangular ladders*** *and denote them by TL_n.*

Another way we can think of the triangular ladder TL_n is as the incomparability graphs of natural unit-interval order $P(3, 4, \ldots, n-1, n, n)$ (for example, the graph in Figure 1.10 is $inc(P(3, 4, 5, 6, 7, 7))$). Dahlberg proves the e-positivity of these graphs by examining the arc diagrams of unit-interval graphs, which are a way of encoding the intervals in a more condensed manner. Using these diagrams and the deletion-contraction property of $\mathbf{Y}_G$, she obtains a combinatorial formula for $\mathbf{Y}_G$ when G is a unit-interval graph. She then proves a few properties about the arc diagrams of triangular ladders and is able to use this formula to obtain a formula for them, allowing her to show their e-positivity.

In [13], the e-positivity of trees is examined in great detail. While they may seem simple, the e-positivity of trees is a fairly complicated topic. In fact, the CSF of trees are of great interest in algebraic combinatorics, and are in fact the center of their own unsolved conjecture by Stanley in [30]:

Conjecture 19. *For any two non-isomorphic trees S, T, $X_S \neq X_T$.*

Commonly known as **Stanley's isomorphism conjecture**, this conjecture has inspired a lot of work in the study of the CSFs of trees. In particular, Dahlberg, She, and van Willigenburg combine the ideas used in studying the Stanley-Stembridge conjecture with ideas that arose from Stanley's isomorphism conjecture. Through this investigation, they discovered that n-vertex connected graphs with no perfect matching (when n even) and no almost-perfect matching (when n odd) are not e-positive. They then used this to show that

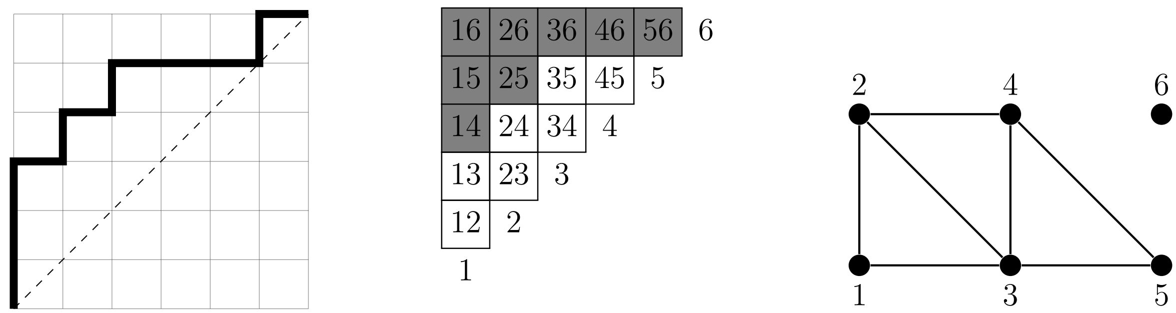

Figure 1.11: (from left to right) A Dyck path, it's corresponding Dyck diagram, and the corresponding graph. Note that in the Dyck diagram, the grey squares represent the outer shape while the white squares represent the inner shape.

a number of other very large classes of claw-free graphs are not e-positive, including n-vertex trees (for $n \geq 4$) with a degree d vertex such that $d \geq \log_2 n + 1$. They then generalized this even further in a proof showing that any n-vertex connected graph with a cut vertex whose removal disconnects the graph into $d \geq \log_2 n + 1$ components is not e-positive. This is expanded upon in [33], where Zheng shows that any tree with a vertex of degree at least 6 is not e-positive.

In [2], Alexandersson and Panova explore e-positivity in a more directly combinatorial light. They study a class of graphs called **circular unit arc digraphs**, which serve as a generalization of unit-interval graphs. They are able to examine these graphs by presenting them using **(circular) Dyck diagrams** and **Dyck paths**.

Definition 20. *Given an $n \times n$ lattice of square cells, a* ***Dyck path*** *is a lattice path from $(0, 0)$ to (n, n) using only* ***north steps*** *$n = (0, 1)$ and* ***east steps*** *$e = (1, 0)$ such that the walk never crosses below the diagonal $y = x$. Given a Dyck path P, the squares above P are known as the* ***outer shape*** *while the squares between P and the diagonal are known as the* ***inner shape****. These two shapes together form the* ***Dyck diagram*** *of P.*

Dyck paths and diagrams can then be placed in bijection with unit-interval graphs. Let each cell in a Dyck diagram correspond to an edge in an n-vertex graph as in Figure1.11. Then we obtain a graph Γ_P with vertex set $[n]$ and edge set $E(G)$ corresponding to the edges in the inner shape of the Dyck diagram of P. Thusly, we have encoded unit-interval graphs using Dyck paths. Circular Dyck diagrams are a slightly more general version of the figures and explanations here; we refer the interested reader to Section 1 of [2] for a deeper understanding of these diagrams. These graphs are then given an **orientation** to make them directed graphs:

Definition 21. *An* ***orientation*** *of a unit-interval graph $\Gamma = (V, E)$ is a function $\theta : E \to V^2$ that assigns each edge $uv \in E$ a directed edge $\overrightarrow{uv}$ or $\overrightarrow{vu}$. We call an orientation the* ***natural orientation*** *if θ assigns each edge uv with the directed edge $\overrightarrow{uv}$ where $u < v$.*

This then leads to an examination of a kind of symmetric function called **LLT polynomials**, which are closely related to chromatic quasisymmetric functions and share many of the same properties.

Definition 22. *[2, Definition 10] Let ν be a k-tuple of skew Young diagrams. Given such a tuple, let $SSYT(\nu) = SSYT(\nu^1) \times SSYT(\nu^2) \times \cdots \times SSYT(\nu^k)$, where $SSYT(\lambda)$ is the set of skew semi-standard Young tableaux of shape λ. Given $T = (T^1, T^2, \ldots, T^k) \in SSYT(\nu)$, let $\mathbf{x}^T$ denote the product $\mathbf{x}^{T^1}\mathbf{x}^{T^2}\ldots\mathbf{x}^{T^k}$ where $\mathbf{x}^{T^i}$ is the usual weight of the semi-standard Young tableau T^i. For cells u, v, entries $T^i(u) > T^j(v)$ form an **inversion** if either*

- *$i < j$ and $c(u) = c(v)$, or*
- *$i > j$ and $c(u) = c(v) + 1$*

*where $c(u)$ denotes the **content** of u and the content of a cell (i, j) in a skew diagram is $i - j$. Finally, we can define the **LLT polynomial***

$$G_\nu(\mathbf{x}; q) = \sum_{T \in SSYT(\nu)} q^{inv(T)} \mathbf{x}_T$$

where $inv(T)$ is the total number of inversions appearing in T. In particular, when $q = 1$, G_ν is a product of the Schur functions of $\nu^1, \nu^2, \ldots, \nu^k$.

Using these ideas, Alexandersson and Panova are able to (among other things) give a natural combinatorial interpretation of the e-coefficients of line graphs and cycle graphs. While the e-coefficients of these graphs were well-known at the time, most methods relied heavily on algebraic techniques. Hence, the combinatorial approach of this method is unique and inspired some of the initial work done in this thesis.

Additionally, and perhaps most importantly, Alexandersson and Panova present an analogue of the Stanley-Stembridge conjecture for LLT polynomials.

Conjecture 23. *[2, Conjecture 25] Let ν be a (circular) Dyck diagram. Then $G_\nu(\mathbf{x}; q + 1)$ is e-positive.*

A generalization of this conjecture was later proven in [3], which we now discuss. Here, Alexandersson and Sulzgruber expand on the study of the relations between Dyck diagrams and CQSFs, giving a completely new characterization of CQSFs of unit-interval graphs in terms of Dyck paths. They do this by studying a generalization of Dyck paths called **Schröder paths**, that are defined to be Dyck paths that also allow a **diagonal step** $d = (1, 1)$. From here, they explore the idea of **bounce paths** and **bounce decompositions**.

Definition 24. *[3, Section 2.4] Let P be a Schröder path and let $(u_1, u_0) \in \mathbb{Z}^2$ be a point on P. Define the **bounce path** of P starting at point (u_1, u_0) with the following steps:*

1. *From the starting point (u_i, u_{i-1}), move south until you reach the point (u_i, u_i) on the main diagonal.*
2. *Now move west until you reach the point (u_{i+1}, u_i) on P.*
3. *If (u_{i+1}, u_i) lies between two diagonal steps, repeat the previous two steps with (u_{i+1}, u_i) as the starting point. Otherwise, stop.*

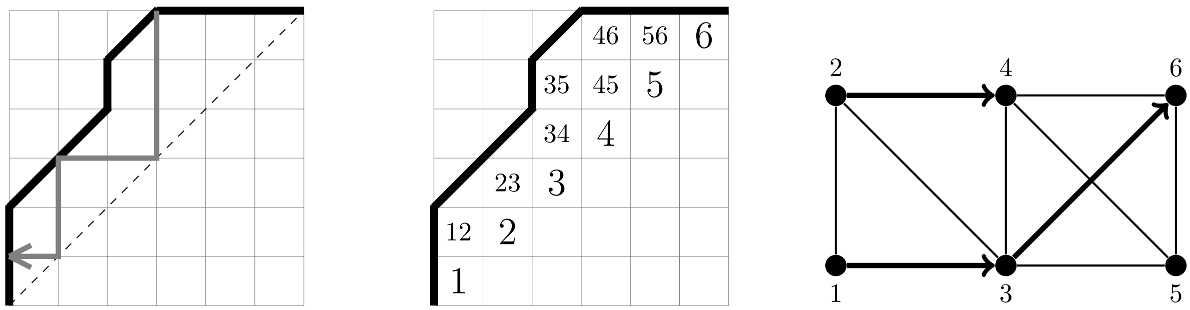

Figure 1.12: (left) The black path represents a Schröder path and the grey path represents the bounce path starting at point $(3, 6)$. In this example, the bounce points are $(3, 3)$ and $(1, 1)$ while the bounce decomposition of this Schröder path is $Un.nVd.eW$, where $U = \emptyset$, $V = ddn$, and $W = ee$. The Schröder path in the middle lists the edges that appear in the decorated unit-interval graph on the right (excluding the strict edges, which have a diagonal through them). The edges with arrows in the graph are the strict edges.

The points (u_i, u_i) for all $i \in [k]$ where the bounce path touches the main diagonal are called ***bounce points*** *(of the path).*

Lastly, we define the ***bounce decomposition*** *of P to be the unique decomposition*

$$P = Us_1.s_2Vs_3.s_4W$$

where $s_1, s_2, s_3, s_4 \in \{n, d, e\}$ and $U, V, W \in \{n, d, e\}^*$ such that Us_1 is a path from $(0, 0)$ to the endpoint (u_{k+1}, u_k) of the bounce path, and s_4W is a path from the starting point (u_1, u_0) of the bounce path to (n, n). Note that dots are used to indicate the start and end points of the bounce path.

For an example of Schröder paths and bounce paths, see the left image in Figure 1.12. We associate each Schröder path P with a **decorated unit-interval graph** $\Gamma_P = (V, E, S)$, where $S \subset E$ is called the set of **strict edges**. Γ_P is defined to have $V = [n]$ with the edge set and strict edge set defined as follows: for each $u, v \in [n]$ with $u < v$, there is a non-strict edge uv (in $E \setminus S$) for every cell in column u and row v below the Schröder path P and above the diagonal. Furthermore, if (u, v) is the endpoint of a diagonal step in P, then uv is a strict edge (in S) in Γ_P.

These strict edges are used as a way to fix the direction of an edge when we apply an orientation to the graph Γ_P. Let $\mathcal{O}(P)$ denote the set of orientations θ of Γ_P where for any edge $uv \in S$, $\theta(uv) = \overrightarrow{uv}$ if and only if $u < v$. We furthermore define the **ascending edges** of Γ_P with orientation θ to be the edges $uv \notin S$ such that $u < v$ and $\theta(uv) = \overrightarrow{uv}$. Then let $\operatorname{asc}(\theta)$ denote the number of ascending edges in Γ_P with orientation θ.

They furthermore define set and integer partitions that arise from this orientation by using the concept of the **highest reachable vertex** of Γ_P.

Definition 25. *Let P be a Schröder path and $\theta \in \mathcal{O}(P)$ an orientation of P. For a vertex $u \in [n]$, the* ***highest reachable vertex*** *is the maximal vertex $v \in [n]$ such that there is a* *directed path from u to v using only strict and ascending edges. Then θ defines a set partition $\pi(\theta)$ of the vertices of Γ_P where two vertices are in the same block if and only if they have the same highest reachable vertex. We then let $\lambda(\theta)$ denote the integer partition given by the sizes of the blocks of $\pi(\theta)$.*

To achieve the main results of their paper, Alexandersson and Sulzgruber realized that when bounce paths followed certain patterns, they were able to prove several nice properties that were helpful in an e-positivity proof (to see these, refer to Section 2.4 and Figure 3 of [3]. To begin using them, they define a set of initial conditions that allow them to use a set of modular relations on bounce paths and associate the Schröder paths with certain LLT polynomials (see Theorem 2.1 of [3]). Finally, they discover an e-basis expansion that leads to a proof of Conjecture 23:

Theorem 26. *[3, Theorem 2.9, Corollaries 2.10, 6.19] Given a Schröder path P of size n, $G_P(\mathbf{x}; q+1)$ expands positively into elementary symmetric functions. In particular, we obtain an explicit expansion of the vertical-strip LLT polynomials in terms of elementary symmetric functions as follows:*

$$G_P(\mathbf{x}; q+1) = \sum_{\theta \in \mathcal{O}(P)} q^{asc(\theta)} e_{\lambda(\theta)}(\mathbf{x}).$$

In particular, when P is a Dyck path, we obtain the following positive e-basis expansion:

$$X_P(\mathbf{x}; q) = \sum_{\theta \in \mathcal{O}(P)} (q-1)^{asc(\theta)-n} e_{\lambda(\theta)}[\mathbf{x}(q-1)].$$

At the time of writing, this is the most recent significant progress that has been made towards a proof of the Stanley-Stembridge conjecture.

The idea of bounce paths have also been useful in proving other e-positivity results. In [9], Cho and Hong examine Dyck paths that have 2 bounce points.[2] They are then able to show that the unit-interval graph corresponding to a Dyck path with n bounce points has independence number $n+1$. Thusly, by examining Dyck paths with 2 bounce points, Cho and Hong show that unit-interval graphs with independence number 3 are e-positive. Furthermore, they note that independence number 2 follows directly from their proof, even though this was known before.

[2]Note from Definition 24 that a Dyck path can have at most 1 bounce point. [9] uses a slightly different definition of bounce paths that relies on Hessenberg functions, something we will not cover here.

Set H	e-Positive?	Reference
P_3	yes	[18]
$3K_1$	yes	[18]
claw, co-P_3	yes	[18]
claw, K_3	yes	[18]
claw, P_4	yes	[18]
claw, paw	yes	[19]
claw, co-paw	yes	[19]
claw, co-claw (excluding net)	yes	[18]
claw, co-diamond, diamond	yes	[19]
claw, co-diamond, co-claw	yes	[19]
claw, co-diamond	conjectured yes	[18]
claw	not necessarily	[18]
claw, diamond	not necessarily	[19]
claw, K_4	not necessarily	[19]
claw, $4K_1$	not necessarily	[19]
claw, C_4	not necessarily	[19]
claw, $2K_2$	not necessarily	[19]

Table 1.1: Table of known e-positivity results for H-free graphs.

Graph Class	e-Positive?	Reference
paths	yes	[30]
cycles	yes	[30]
complete graphs	yes	[30]
K_α-chains	yes	[20]
generalized bulls	yes	[18]
generalized pyramids	yes	[25]
lollipops	yes	[14], Cor. 32
melting lollipops	yes	[23], Cor. 32
triangular ladders	yes	[11]
(melting) SCBPs	yes	[4], Thm. 37, Cor. 38
DCBPs	yes	Thm. 40
graphs contractible to the claw	not necessarily	[30]
the claw	no	[30]
the net	no	[30]
trees with a vertex of degree $d \geq \log_2 n + 1$	no	[13]
trees with a vertex of degree ≥ 6	no	[33]
graphs with no (almost) perfect matching	no	[13]

Table 1.2: Table of graph classes that are known to be e-positive.

Chapter 2

Proof of Concept: Paths and (Melting) Lollipops

In this chapter, we provide a novel proof of the e-positivity of paths and (melting) lollipop graphs. Recall that a lollipop graph is the concatenation of a complete graph and a path, denoted $L_{m,n} = K_m + P_{n+1}$. Furthermore, recall from Definition 1 that melting lollipop graphs are any graph that can be obtained by removing edges between the path and the clique in a lollipop graph. Our proof finds the explicit formula for the CSF of (melting) lollipop graphs in terms of the elementary symmetric functions. This chapter is meant to serve as a simple introduction to the methodology used later in the paper while also showing some of the inspiration behind its development.

2.1 Definitions and Preliminaries

In this proof, we use a diagram to represent the adjacencies of a labelled graph.

Definition 27. *Let G be a graph with $|V(G)| = n$ and labelling function $l : V(G) \to [n]$. Let A be a Ferrer's diagram of shape $(n-1, n-2, \ldots, 1)$. Label the columns from left to right with $1, 2, \ldots, n-1$ and the rows from the bottom up with $2, 3, \ldots, n$. Denote by $A_{i,j}$ the box in row i and column j of A. We say A is an* ***adjacency diagram of*** *G if when $v_1v_2 \in E(G)$, $l(v_1) < l(v_2)$, the box $A_{l(v_1),l(v_2)}$ is filled.*

Let A be an adjacency diagram. We define a total ordering $<$ to the boxes of A as follows: if $i_1 < i_2$, then $A_{i_1,j_1} < A_{i_2,j_2}$. Otherwise when $i_1 = i_2$, if $j_1 > j_2$ then $A_{i_1,j_1} < A_{i_2,j_2}$. Additionally, we denote by $A + A_{i,j}$ and $A - A_{i,j}$ the adjacency diagram obtained from A by filling or emptying, respectively, box $A_{i,j}$. In other words, the boxes are ordered right-to-left, bottom-to-top.

Additionally, recall the definitions of L-matrices, M-matrices, and weakly diagonally dominant (w.d.d) from Section 1.2.2. We will use the properties of these matrices to prove e-positivity, particularly that the inverse of a non-singular M-matrix is non-negative [17] and the following Lemma:

Figure 2.1: A graph and its adjacency diagram.

Lemma 28. *[5, Theorem 2.24] A matrix M is a non-singular w.d.d. L-matrix if and only if it is a non-singular w.d.d. M-matrix.*

2.2 The Proof

We prove the e-positivity of paths by setting up a system of linear equations for each path, which we obtain by iteratively applying the Orellana-Scott modular relation. This creates a system of equations on the chromatic symmetric functions of melting lollipop graphs, which we also show are e-positive. Let T denote the graph that we update throughout this process and let L denote the set of linear equations involving the chromatic symmetric functions. We define this process as follows:

Algorithm 1 Obtaining a set of linear equations of CSFs given an adjacency diagram.

Require: A is a path
    $T \leftarrow A$
    $L \leftarrow \emptyset$
    while T contains an unfilled box **do**
        $(i, j) \leftarrow$ location of smallest unfilled box
        Add $\mathbf{X}_T + \mathbf{X}_{T+T_{i,j}-T_{i,j+1}} = \mathbf{X}_{T+T_{i,j}} + \mathbf{X}_{T-T_{i,j+1}}$ to L
        $T \leftarrow T + T_{i,j}$
    end while

Note that we use adjacency diagrams and the graphs they represent interchangeably. Additionally, note that each step of the algorithm fills an additional box in the adjacency diagram, so it will always terminate, doing so when the complete graph is obtained. Furthermore, due to the reverse lexicographic ordering, this algorithm fills one row of the adjacency diagram at a time from the bottom up, with each row being filled from right to left. However, the symmetry of the diagram means that instead the columns can be filled one at a time from right to left to obtain the same result. This method of filling ensures that every graph obtained during this process is a melting lollipop graph. We now prove two short lemmas about this algorithm.

Lemma 29. *In Algorithm 1, the graphs produced by T and $T + T_{i,j} - T_{i,j+1}$ are isomorphic.*

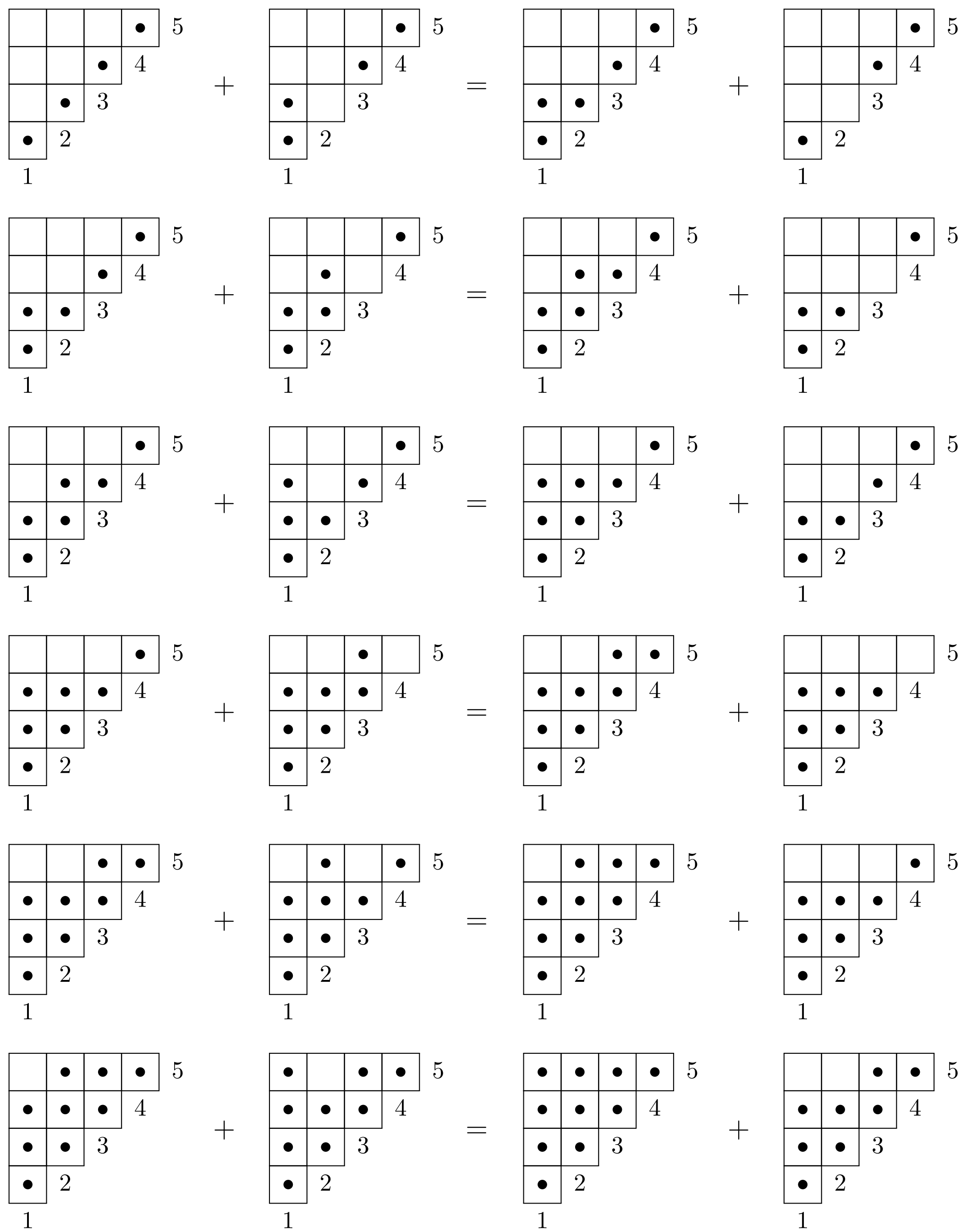

Figure 2.2: A visual representation of the process described in Algorithm 1 using tableau.

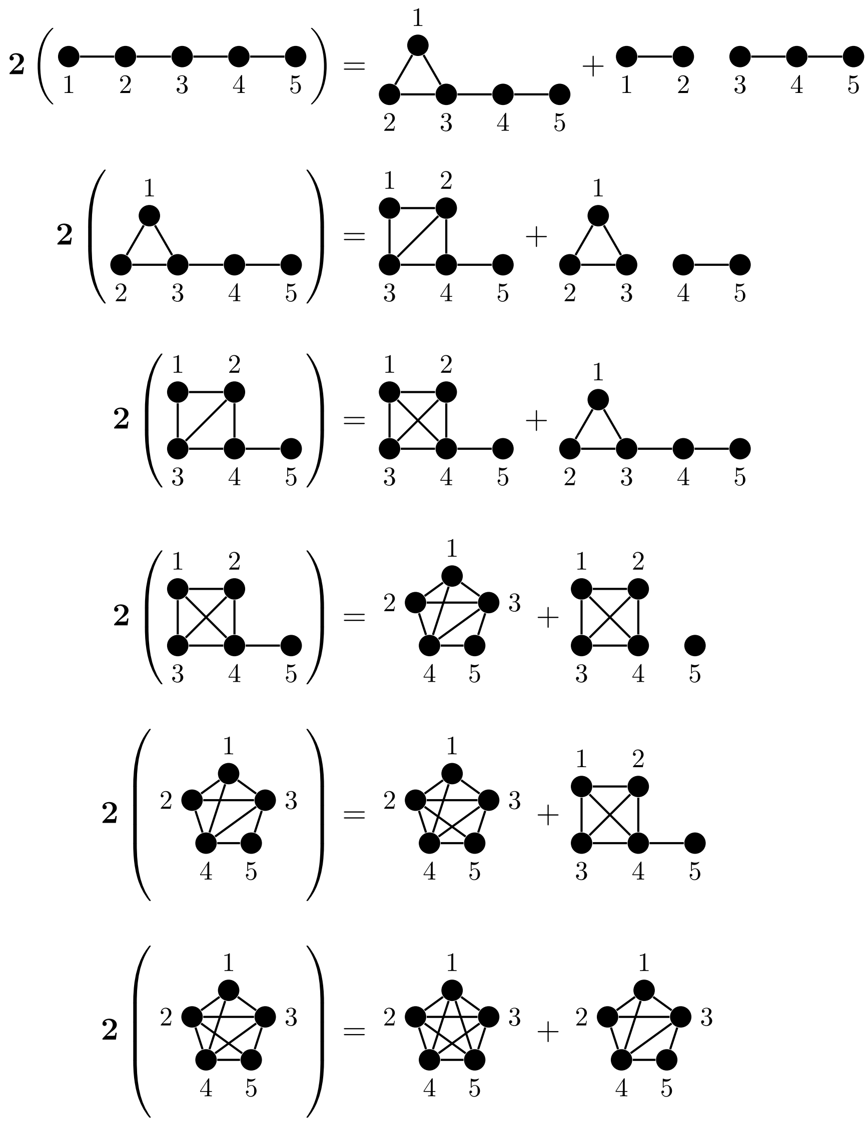

Figure 2.3: A visual representation of the process described in Algorithm 1 using graphs.

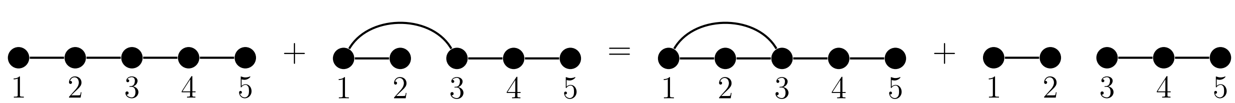

Figure 2.4: The result of the first step of Algorithm 1 on P_5. Note that the two graphs on the left-hand side of the equation are isomorphic.

Proof. Let T be the adjacency diagram at step n in the algorithm. Let i be the row with the smallest unfilled box. Then for all $i' > i$, the box (i', j) is filled for all $j \le i'$. These filled rows form a clique, with row i determining the adjacencies of vertices in this clique with a single vertex. But then, T and $T + T_{i,j} - T_{i,j+1}$ are both melting lollipop graphs with the same clique size, path length, and number of edges adjacent to the clique, and are hence isomorphic. □

Lemma 30. *In Algorithm 1, the graph produced by $T - T_{i,j+1}$ is either the disjoint union of a clique and a smaller path or a graph that has already been obtained earlier in the algorithm*

Proof. Let T be the adjacency diagram at step s in the algorithm, let (i, j) be the location of the smallest unfilled box at step s, and let n be the number of vertices in the path.

If $n - i = j + 1$, then $T_{i,j+1}$ is the box on the diagonal. As the algorithm fills the smallest boxes first, row i and every row above it is completely unfilled other than the box on the diagonal. Furthermore, every row below row i is completely filled, forming a clique. Hence, $T - T_{i,j+1}$ is the disjoint union of a clique and a smaller path

Otherwise, consider the diagram T' at step $s - 1$ in the algorithm and let (i', j') be the location of the smallest box. Clearly, $i' = i$. Further, since T' has one fewer box filled and the boxes are filled from right to left, $j' = j + 1$, so $T - T_{i,j+1} = T'$. □

For an example of this process with initial graph P_5, see Figure 2.3. This figure serves as a visual representation of the following system of questions:

$$\begin{aligned}
2(P_5) &= L_{3,2} + P_2P_3 \\
2(L_{3,2}) &= L^{(1)}_{4,1} + K_3P_2 \\
2(L^{(1)}_{4,1}) &= L_{4,1} + L_{3,2} \\
2(L_{4,1}) &= L^{(2)}_{5,0} + K_4P_1 \\
2(L^{(2)}_{5,0}) &= L^{(1)}_{5,0} + L_{4,1} \\
2(L^{(1)}_{5,0}) &= K_5 + L^{(2)}_{5,0}
\end{aligned}
\qquad
\begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 \end{bmatrix}
\begin{bmatrix} P_5 \\ L_{3,2} \\ L^{(1)}_{4,1} \\ L_{4,1} \\ L^{(2)}_{5,0} \\ L^{(1)}_{5,0} \end{bmatrix}
=
\begin{bmatrix} K_2P_3 \\ K_3P_2 \\ 0 \\ K_4P_1 \\ 0 \\ K_5 \end{bmatrix}.$$

Note that both the left and right hand side are the same system of equations, just represented in different ways.

Theorem 31. *Paths are e-positive.*

Proof. We prove this by induction on the length of the path. As a base case, we know that P_2 and K_2 are isomorphic, so $\mathbf{X}_{P_2} = e_2$.

For the inductive step, suppose this holds for all P_{n-1}, $n \ge 3$ and consider P_n with vertices $p_1, \ldots, p_n$. Label the vertices of P_n such that $l(p_i) = i$ for each vertex in the path.

Then, the adjacency diagram A of P_n is filled only on the outermost diagonal. Let $L\mathbf{x} = \mathbf{e}$ be the matrix representation of the system of linear equations obtained by applying Algorithm 1 to A, where $\mathbf{x}$ is a vector of CSFs and $\mathbf{e}$ is a vector of linear combinations of elementary symmetric functions.

From Lemma 29, we know that L can be written up to rearrangement such that the diagonal is only 2's. Further, by induction, we know how to write the CSF of a path in terms of the e-basis. Hence from Lemma 30, we can fill $\mathbf{e}$ with the CSFs of a clique and a smaller path. Additionally, we know that the entries that do not have such known CSFs are found elsewhere in the process, so we can write L as follows:

$$L = \begin{bmatrix} 2 & -1 & & & & & \\ & 2 & -1 & & & & \\ & -1 & 2 & -1 & & & \\ & & & 2 & -1 & & \\ & & & -1 & 2 & -1 & \\ & & & & -1 & 2 & -1 \\ & & & & & & \ddots \end{bmatrix}$$

More explicitly, if we let D_i denote the $i \times (i+1)$ matrix of the form

$$D_i = \begin{bmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & & \ddots & & & \\ & & -1 & 2 & -1 & \\ & & & -1 & 2 & -1 \end{bmatrix}$$

and D_i' denote the $i \times i$ matrix of the form

$$D_i' = \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & & \ddots & & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{bmatrix},$$

we can instead write L as follows:

$$L = \begin{bmatrix} D_1 & & & & \\ & D_2 & & & \\ & & \ddots & & \\ & & & D_{n-3} & \\ & & & & D_{n-2}' \end{bmatrix}$$

such that the diagonal is just made up of 2's

This matrix is clearly a w.d.d. L-matrix, so by Lemma 28 it is a w.d.d. M-matrix. Since non-singular M-matrices have non-negative inverses, it suffices to show that L is non-singular, i.e. that it is invertible. To do this, we show that $\mathbf{det}(L) = (n-1)!$.

Note for a path of length n, $\mathbf{det}(L) = \prod_{i=1}^{n-2} \mathbf{det}(D_i')$. Moreover, note that $\mathbf{det}(D_i') = 2\mathbf{det}(D_{i-1}') - \mathbf{det}(D_{i-2}')$ and $\mathbf{det}(D_1') = 2$, so $\mathbf{det}(D_i') = i+1$. Hence, $\mathbf{det}(L) = (n-1)!$ and thus L is non-singular. So L is a M-matrix and therefore has a non-negative inverse. $\square$

Note that while this clearly proves the positivity of paths, it also proves the positivity of all the graphs obtained by each step of the algorithm as the entire matrix is shown to have a non-negative inverse. As such, we can conclude the following:

Corollary 32. *Lollipop and melting lollipop graphs are e-positive.*

Furthermore, this process allows us to obtain e-basis expansions for paths, lollipops and melting lollipops. These expansions are already well known, so we will not show the process of extracting them here. However, we provide the expansions of lollipop graphs and melting lollipop graphs here as they become much more useful later:

Theorem 33. *[23, Proposition 4.4, 4.8] For $m \geq 2$ and $n \geq 0$, the e-basis expansion of the CSF of a lollipop graph is*

$$\mathbf{X}_{L_{m,n}} = (m-1)! \left((m+n)e_{m+n} + \sum_{i=0}^{n-1} (m+i-1)\mathbf{X}_{P_{n-i}} e_{m+i} \right)$$

Furthermore, the CSF of melting lollipop graphs have the following relation:

$$\mathbf{X}_{L_{m,n}^{(k)}} = \frac{m-k-1}{m-1} \mathbf{X}_{L_{m,n}} + k(m-2)! \mathbf{X}_{P_{n+1}} e_{m-1}$$

Note here that we leave the CSF of paths unexpanded as it is a very well-known expansion. In general, these expansions get fairly large and difficult to read, so when possible we leave CSFs with known expansions unexpanded when writing them out.

This alternate proof of the e-positivity of paths really shows the power of this method. It allows us to swiftly prove e-positivity and find e-basis expansions of multiple large classes of graphs all at once. As we will see in the following chapter, though, it is not always quite as simple as taking an inverse and this method sometimes requires a lot of computation to arrive at a usable answer.

Chapter 3

Clique-Blowups of Paths

In this chapter, we prove the e-positivity of single clique-blowups of paths (SCBPs) and their melting variety (recall Definition 2). Both of these results were implicitly proven in [4], however the method used here allows us to obtain an e-basis expansion of these graphs, something that was previously unknown. Moreover, we prove the e-positivity of double clique-blowups of paths (DCBPs), another previously unknown result. These results suggest a potential to extend this proof to more general clique-blowups of paths, which we elaborate on in the final section of this chapter.

We will begin this chapter with a brief explanation of the methods used to prove the positivity of (melting) SCBPs and DCBPs in [4]. We will then use our new method to expand these results, obtaining some e-basis expansions and new e-positivity results along the way.

3.1 Previous Methods

Here we will explain the methods used by Aliniaeifard, Wang, and van Willigenburg in [4] for proving the positivity of certain clique-blowups of paths. Recall from Definition 15 and Theorem 16 the usefulness of appendable (e)-positivity. Proving that a graph is appendable (e)-positive allows one to concatenate graphs together while maintaining (e)-positivity. While not explicitly stated, Aliniaeifard, Wang, and van Willigenburg implicitly prove the (e)-positivity of certain clique-blowups of paths with their examples of appendable (e)-positive graphs.

Firstly, they note that complete graphs are appendable (e)-positive. This is significant in particular because of K_2; the appendability of K_2 combined with the continuous concatenation found in Theorem 16 allows one to prove the appendable (e)-positivity of all paths and lollipops. However, note that this is not yet sufficient for a proof of clique-blowups of paths, as the blasted vertices of a clique are not adjacent in a clique-blowup. In other words, when placed between two paths, cliques have an extra edge (e.g. $P_{n_1} + K_m + P_{n_2}$ is not a clique-blowup).

They are able to remedy this by proving the appendable (e)-positivity of what they call **twin peaks graphs** on $n \geq 3$ vertices, where TP_n is obtained by removing the edge $(1, n)$ from K_n. The proof of this is reduced down to an expansion into the (e)-basis and a positivity check. Furthermore, they prove the appendable (e)-positivity of what they call **melting ice cream scoop graphs** and their reverse graphs on $n \geq 3$ vertices; these graphs are denoted with $IC_n^{(k)}$, which is obtained by deleting edges $(1, n+1), (2, n+1), \ldots, (k, n+1)$ in K_n. The proof for this comes from a clever use of the known results about twin peaks graphs and a corollary obtained earlier in their paper.

These proofs definitively prove the e-positivity of a large number of clique-blowups of paths and their melting varieties, including (melting) SCBPs and DCBPs with non-adjacent cliques. However, proving appendable (e)-positivity does not directly lead to the discovery of an e-basis expansion for these graphs as there is not yet a method to derive an expansion from the concatenated parts of the graph. Furthermore, there is no appendable graph given that allows one to create a DCBP with adjacent blown-up vertices. Throughout the rest of this chapter, we will resolve these issues for (melting) SCBPs and DCBPs.

3.2 Single Clique-Blowups of a Path

This proof hinges on using the Orellana-Scott relation in two ways to perform an inductive step. Namely, we use this modular relation to first remove and then add edges so that the position of the blown-up vertex changes. This allows us to obtain a system of linear equations and prove inverse positivity similarly to the proof for paths. We begin by proving what the system of linear equations is.

Corollary 34. *We can obtain a linear equation including $P_{1^{n_1},m,1^{n_2}}$ of the form*

$$\begin{bmatrix} A_{m-1} & B_{m-1} \\ \mathbf{0}_{m-1} & D_{m-1} \end{bmatrix} \begin{bmatrix} \mathbf{X}_{P_{1^{n_1},m,1^{n_2}}} \\ \mathbf{X}_{P^{(1)}_{1^{n_2},m,1^{n_1}}} \\ \vdots \\ \mathbf{X}_{P^{(m-2)}_{1^{n_2},m,1^{n_1}}} \\ \mathbf{X}_{P^{(m-1)}_{1^{n_1-1},m,1^{n_2+1}}} \\ \mathbf{X}_{P^{(m-2)}_{1^{n_1-1},m,1^{n_2+1}}} \\ \vdots \\ \mathbf{X}_{P^{(1)}_{1^{n_1-1},m,1^{n_2+1}}} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ -\mathbf{X}_{L_{m+1,n_1-1}}\mathbf{X}_{P_{n_2}} \\ \mathbf{X}_{L_{m+1,n_2}}\mathbf{X}_{P_{n_1-1}} \\ 0 \\ \vdots \\ 0 \\ \mathbf{X}_{P_{1^{n_1-1},m,1^{n_2+1}}} \end{bmatrix} \begin{matrix} \left.\vphantom{\begin{matrix}0\\ \vdots \\ 0\end{matrix}}\right\} m-2 \\ \vphantom{\mathbf{X}} \\ \vphantom{\mathbf{X}} \\ \left.\vphantom{\begin{matrix}0\\ \vdots \\ 0\end{matrix}}\right\} m-3 \\ \vphantom{\mathbf{X}} \end{matrix} \tag{3.1}$$

where $A_m, B_m, D_m, \mathbf{0}_m$ are $m \times m$ square matrices such that $\mathbf{0}_m$ is the matrix of all zeros and

$$A_m = \begin{bmatrix} 1 & -2 & 1 & & & \\ & 1 & -2 & 1 & & \\ & & & \ddots & & \\ & & & 1 & -2 & 1 \\ & & & & 1 & -2 \\ & & & & & 1 \end{bmatrix}, B_m = \begin{bmatrix} 0 & 0 & \ldots & 0 \\ \vdots & & \ddots & \\ 0 & 0 & \ldots & 0 \\ 1 & 0 & \ldots & 0 \\ -2 & 0 & \ldots & 0 \end{bmatrix}, D_m = \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & & \ddots & & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{bmatrix}.$$

Proof. Let $G = P_{1^{n_1},m,1^{n_2}}$. Without loss of generality, assume that $n_1 \le n_2$. We begin by providing an alternate labeling of some notable vertices for convenience.

Label the blasted vertices in P_{n_1} and P_{n_2} with b_1 and b_2, respectively. Furthermore, label the vertex in P_{n_1} adjacent to b_1 with b_1'; note that in our conventional labelling scheme, b_1, b_1', and b_2 correspond to n_1, $n_1 - 1$, and $n_1 + m + 1$, respectively. Finally, we arbitrarily label the vertices of K_m with $k_1, k_2, \ldots, k_m$; note that in our conventional labelling scheme, these vertices correspond to $n_1 + 1, n_1 + 2, \ldots, n_1 + m$.

We now describe the process of removing edges in Algorithm 2. Where the proof for paths started by looking at induced P_2 subgraphs and adding an edge to it, this process starts with an induced K_3 subgraph and removes an edge.

Algorithm 2 Removing edges from a single clique-blowup.

Require: $H = G$
- $L_1 \leftarrow \emptyset$
- **for** $i = (1, \ldots, m-1)$ **do**
  - $e_1 \leftarrow (b_2, k_i)$
  - $e_2 \leftarrow (b_2, k_{i+1})$
  - Add $\mathbf{X}_H + \mathbf{X}_{H-e_1-e_2} = \mathbf{X}_{H-e_1} + \mathbf{X}_{H-e_2}$ to L_1
  - $H \leftarrow H - e_1$
- **end for**

First, note that in each step of Algorithm 2, $H - e_1$ is isomorphic to $H - e_2$, as e_1 and e_2 are edges adjacent to a clique. Furthermore, note that when $i > 1$, $H - e_1$ in step i is isomorphic to $H - e_1 - e_2$ in step $i - 1$, as both are missing the same number of edges from b_2 to the clique. Lastly, note that in the final step of the **for** loop, $H - e_1 - e_2$ is the disjoint union of a lollipop and a path, both of which are known to be e-positive (see the first three lines of Figure 3.1).

Combining all of this, we find L_1 to be the following system of linear equations:

$$A'_{m-1}\mathbf{x} = \mathbf{b} \iff \begin{bmatrix} 1 & -2 & 1 & & & \\ & 1 & -2 & 1 & & \\ & & \ddots & & & \\ & & & 1 & -2 & 1 \\ & & & & 1 & -2 \end{bmatrix} \begin{bmatrix} \mathbf{X}_{P_{1^{n_1},m,1^{n_2}}} \\ \mathbf{X}_{P^{(1)}_{1^{n_2},m,1^{n_1}}} \\ \vdots \\ \mathbf{X}_{P^{(m-1)}_{1^{n_2},m,1^{n_1}}} \end{bmatrix} = \left.\begin{bmatrix} 0 \\ \vdots \\ 0 \\ -\mathbf{X}_{L_{m+1,n_1-1}}\mathbf{X}_{P_{n_2}} \end{bmatrix}\right\} m-2$$

where A'_{m-1} is a $(m-1)\times m$ matrix. Note that $\mathbf{X}_{L_{m+1,n_1-1} \sqcup P_{n_2}} = \mathbf{X}_{L_{m+1,n_1-1}}\mathbf{X}_{P_{n_2}}$.

After removing edges, we then add edges from b_1' to K_m using a nearly identical method to that used in the argument for paths. Let k_m be the vertex adjacent to b_2 at the end of Algorithm 2. Furthermore, note that the subgraph induced by $\{b_1, k_1, \ldots, k_{m-1}\}$ is a clique. Then, continuing from the end of Algorithm 2, we apply Algorithm 3:

Algorithm 3 Adding edges in a single clique-blowup of a path.

Require: $H = P^{(m-1)}_{1^{n_1},m,1^{n_2}}$

$L_2 \leftarrow \emptyset$

for $i = 1, \ldots, m-1$ **do**

$e_1 \leftarrow (b'_1, b_1)$

$e_2 \leftarrow (b'_1, k_i)$

Add $\mathbf{X}_H + \mathbf{X}_{H-e_1+e_2} = \mathbf{X}_{H-e_1} + \mathbf{X}_{H+e_2}$ to L_1

$H \leftarrow H + e_2$

end for

We now note some important facts about Algorithm 3. Firstly, since $\{b_1, k_1, \ldots, k_{m-1}\}$ induces a clique, H is isomorphic to $H - e_1 + e_2$ at each step in the algorithm. Furthermore, when $i > 1$, $H - e_1$ in step i is isomorphic to H in step $i-1$, as both are missing the same number of edges from b'_1 to the clique. Additionally, note that in the first step of the algorithm, $H - e_1$ is the disjoint union of a lollipop and a path. Finally, note that in the last step of the algorithm, $H + e_2$ is a single clique-blowup of the same path with the clique in a different place, namely $P_{1^{n_1-1},m,1^{n_2+1}}$ (see the last three lines of Figure 3.1).

Let $G = P^{(m-1)}_{1^{n_2},m,1^{n_1}}$. Combining all the information above, we find L_2 to be the following system of linear equations:

$$D_{m-1}\mathbf{x} = \mathbf{b} \iff \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & & \ddots & & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{bmatrix} \begin{bmatrix} \mathbf{X}_{P^{(m-1)}_{1^{n_1-1},m,1^{n_2+1}}} \\ \mathbf{X}_{P^{(m-2)}_{1^{n_1-1},m,1^{n_2+1}}} \\ \mathbf{X}_{P^{(m-3)}_{1^{n_1-1},m,1^{n_2+1}}} \\ \vdots \\ \mathbf{X}_{P^{(1)}_{1^{n_1-1},m,1^{n_2+1}}} \end{bmatrix} = \begin{bmatrix} \mathbf{X}_{L_{m+1,n_2}}\mathbf{X}_{P_{n_1-1}} \\ 0 \\ \vdots \\ 0 \\ \mathbf{X}_{P_{1^{n_1-1},m,1^{n_2+1}}} \end{bmatrix} \left.\vphantom{\begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}}\right\} m-3$$

Note that $\mathbf{X}_{P^{(m-1)}_{1^{n_2},m,1^{n_1}}} = \mathbf{X}_{P^{(m-1)}_{1^{n_1-1},m,1^{n_2+1}}}$. Then, by combining L_1 and L_2 we obtain the desired result. $\square$

For an example of the process explained in the previous proof, see Figure 3.1. This figure visually represents the following system of equations obtained from this process with initial graph $P_{1^2,4,1^3}$:

$$\begin{aligned}
\mathbf{X}_{P_{1^2,4,1^3}} + \mathbf{X}_{P^{(2)}_{1^3,4,1^2}} &= 2\mathbf{X}_{P^{(1)}_{1^3,4,1^2}} \\
\mathbf{X}_{P^{(1)}_{1^3,4,1^2}} + \mathbf{X}_{P^{(3)}_{1^1,4,1^4}} &= 2\mathbf{X}_{P^{(2)}_{1^3,4,1^2}} \\
\mathbf{X}_{P^{(2)}_{1^3,4,1^2}} + \mathbf{X}_{L_{5,1}}\mathbf{X}_{P_3} &= 2\mathbf{X}_{P^{(3)}_{1^1,4,1^4}} \\
2\mathbf{X}_{P^{(3)}_{1^1,4,1^4}} &= \mathbf{X}_{P^{(2)}_{1^1,4,1^4}} + \mathbf{X}_{L_{5,3}}\mathbf{X}_{P_1} \\
2\mathbf{X}_{P^{(2)}_{1^1,4,1^4}} &= \mathbf{X}_{P^{(1)}_{1^1,4,1^4}} + \mathbf{X}_{P^{(3)}_{1^1,4,1^4}} \\
2\mathbf{X}_{P^{(1)}_{1^1,4,1^4}} &= \mathbf{X}_{L^{(1)}_{6,3}} + \mathbf{X}_{P^{(2)}_{1^1,4,1^4}}
\end{aligned}$$

Note that $P^{(3)}_{1^1,4,1^4}$ is isomorphic to $P^{(3)}_{1^3,4,1^2}$; for consistency, we use only the former both here and in Figure 3.1, but in practice we use them interchangeably.

Now that we have obtained a system of equations for SCBPs, we need to extract their e-basis expansion. To do so, we begin by obtaining some interesting intermediary results using some ideas from linear algebra. Recall the definition of tridiagonal Toeplitz matrices from Section 1.2.2. In [26], Meurant describes an equation for the values of the inverse of tridiagonal Toeplitz matrices. In particular, he proves the following:

Lemma 35. *[26, Theorem 2.8] For a tridiagonal Toeplitz matrix T_a with $a = 2$, we have that*

$$(T_a)^{-1}_{i,j} = i\frac{n-j+1}{n+1}.$$

Using this lemma, we prove the following:

Corollary 36. *Let $n_1, n_2 \geq 1$, be integers. For $P_{1^{n_1},m,1^{n_2}}$,*

$$\mathbf{X}_{P_{1^{n_1},m,1^{n_2}}} = (m-1)\mathbf{X}_{L_{m+1,n_2}}\mathbf{X}_{P_{n_1-1}} - (m-1)\mathbf{X}_{L_{m+1,n_1-1}}\mathbf{X}_{P_{n_2}} + \mathbf{X}_{P_{1^{n_1-1},m,1^{n_2+1}}} \tag{3.2}$$

Furthermore, for $P^{(k)}_{1^{n_1},m,1^{n_2}}$ with $1 \leq k \leq m-2$,

$$\mathbf{X}_{P^{(k)}_{1^{n_1},m,1^{n_2}}} = \frac{m-k}{m}\mathbf{X}_{P_{1^{n_1},m,1^{n_2}}} + \frac{k}{m}\mathbf{X}_{L_{m+1,n_2-1}}\mathbf{X}_{P_{n_1}} \tag{3.3}$$

Proof. We begin by finding M_{m-1}, the inverse of the matrix from Corollary 34:

$$M_{m-1} = \begin{bmatrix} A_{m-1} & B_{m-1} \\ \mathbf{0}_{m-1} & D_{m-1} \end{bmatrix}^{-1}.$$

Using the block matrix inversion formula from Lemma 11, we rewrite M_{m-1} as

$$M_{m-1} = \begin{bmatrix} A^{-1}_{m-1} & \mathbf{0} \\ \mathbf{0} & D^{-1}_{m-1} \end{bmatrix} \begin{bmatrix} \mathbf{I} & -B_{m-1}D^{-1}_{m-1} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}.$$

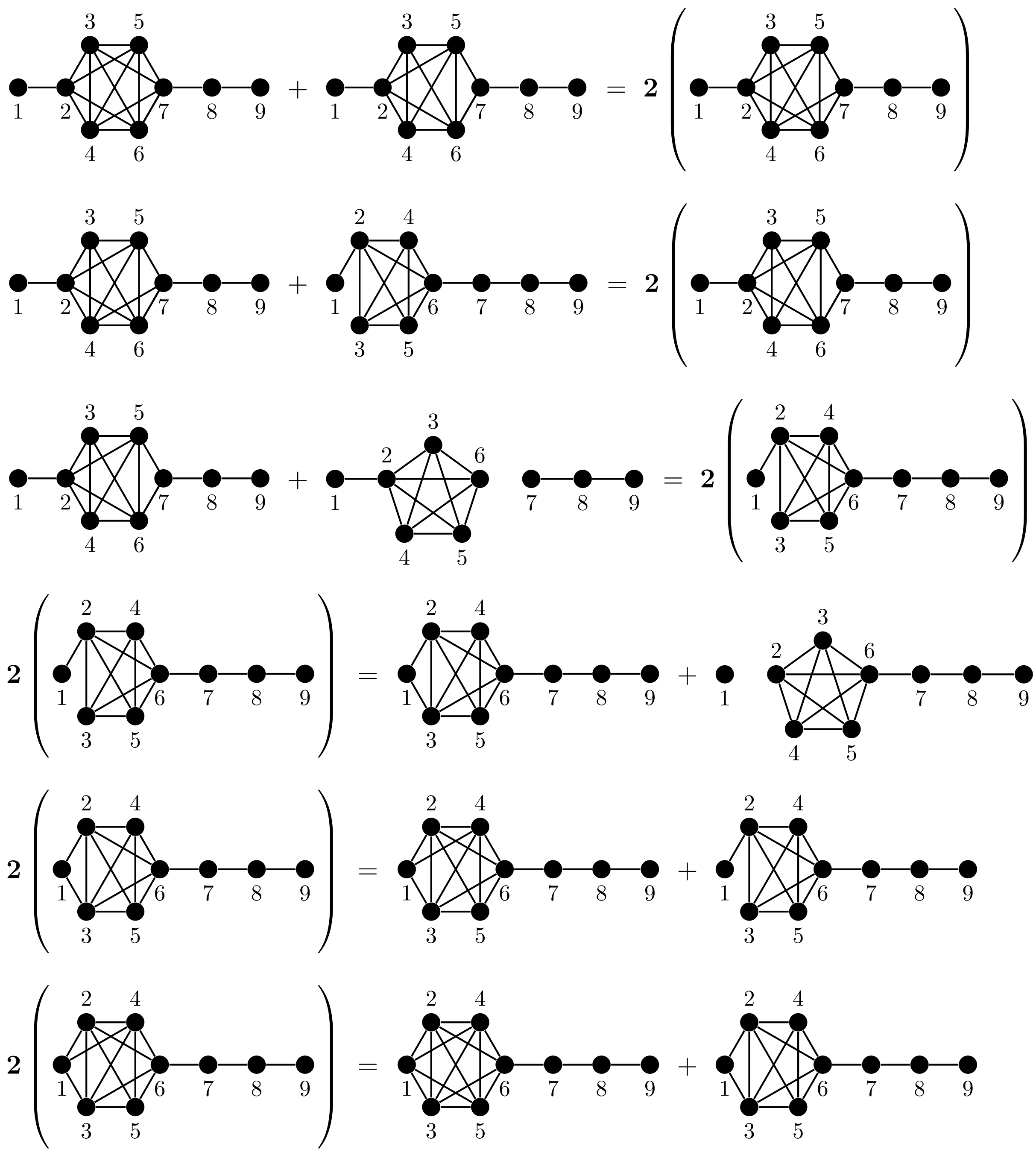

Figure 3.1: A visual representation of the process described in the proof of Corollary 34 with initial graph $P_{1^2,4,1^3}$. The first three lines show Algorithm 2 and the last three lines show Algorithm 3.

Now, note that the inverse of A_{m-1} is the matrix

$$A_{m-1}^{-1} = \begin{bmatrix} 1 & 2 & 3 & \dots & m-1 \\ 0 & 1 & 2 & \dots & m-2 \\ & & & \ddots & \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}.$$

We can verify this by considering $A_{m-1}^{-1}A_{m-1}$. Clearly, the resulting matrix is upper triangular with 1's along the diagonal. Now note that for any positive integer x, $(x-1)-2x+(x+1)=0$, so $A'_{m-1}A_{m-1}$ has only 0's above the diagonal.

Moreover, note that D_{m-1} is the $(m-1)\times(m-1)$ tridiagonal Toeplitz matrix with $a=2$. So, by Lemma 35, when $j \geq i$, $(D_{m-1})_{i,j}^{-1} = \frac{i(m-j)}{m}$; otherwise when $j<i$, $\frac{j(m-i)}{m}$. Hence,

$$-B_{m-1}D_{m-1}^{-1} = \begin{bmatrix} 0 & 0 & \dots & 0 \\ & & \ddots & \\ -1 & 0 & \dots & 0 \\ 2 & 0 & \dots & 0 \end{bmatrix} \begin{bmatrix} \frac{m-1}{m} & \frac{m-2}{m} & \dots & \frac{1}{m} \\ \frac{m-2}{m} & 2\frac{m-2}{m} & \dots & \frac{2}{m} \\ & & \ddots & \\ \frac{1}{m} & \frac{2}{m} & \dots & \frac{m-1}{m} \end{bmatrix} = \begin{bmatrix} 0 & 0 & \dots & 0 \\ & & \ddots & \\ 0 & 0 & \dots & 0 \\ -\frac{(m-1)}{m} & -\frac{(m-2)}{m} & \dots & -\frac{1}{m} \\ 2\frac{(m-1)}{m} & 2\frac{(m-2)}{m} & \dots & 2\frac{1}{m} \end{bmatrix}$$

Thus,

$$M_{m-1} = \begin{bmatrix} A_{m-1}^{-1} & \mathbf{0} \\ \mathbf{0} & D_{m-1}^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{I} & -B_{m-1}D_{m-1}^{-1} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} = \begin{bmatrix} A_{m-1}^{-1} & E_{m-1} \\ \mathbf{0} & D_{m-1}^{-1} \end{bmatrix},$$

where

$$E_{m-1} = \begin{bmatrix} \frac{m-1}{m}(2(m-1)-(m-2)) & \frac{m-2}{m}(2(m-1)-(m-2)) & \dots & \frac{1}{m}(2(m-1)-(m-2)) \\ \frac{m-1}{m}(2(m-2)-(m-3)) & \frac{m-2}{m}(2(m-2)-(m-3)) & \dots & \frac{1}{m}(2(m-2)-(m-3)) \\ & & \ddots & \\ \frac{m-1}{m}(4-1) & \frac{m-2}{m}(4-1) & \dots & \frac{1}{m}(4-1) \\ \frac{m-1}{m}(2) & \frac{m-2}{m}(2) & \dots & \frac{1}{m}(2) \end{bmatrix}$$

$$= \begin{bmatrix} m-1 & m-2 & \dots & 1 \\ (m-1)\frac{m-1}{m} & (m-1)\frac{m-2}{m} & \dots & (m-1)\frac{1}{m} \\ & & \ddots & \\ 3\frac{m-1}{m} & 3\frac{m-2}{m} & \dots & 3\frac{1}{m} \\ 2\frac{m-1}{m} & 2\frac{m-2}{m} & \dots & 2\frac{1}{m} \end{bmatrix}.$$

Multiplying M_{m-1} on the left to both sides of (3.1), we obtain (3.2). To obtain (3.3), notice that after this multiplication, we obtain the equation

$$\mathbf{X}_{P^{(k)}_{1^{n_2},m,1^{n_1}}} = (m-k)\frac{(m-1)}{m}\mathbf{X}_{L_{m+1,n_2}}\mathbf{X}_{P_{n_1-1}} - (m-k-1)\mathbf{X}_{L_{m+1,n_1-1}}\mathbf{X}_{P_{n_2}} + \frac{m-k}{m}\mathbf{X}_{P_{1^{n_1-1},m,1^{n_2+1}}}$$

But notice that $\mathbf{X}_{P_{1^{n_1-1},m,1^{n_2+1}}} = \mathbf{X}_{P_{1^{n_2+1},m,1^{n_1-1}}}$. Applying (3.2) and relabelling, we obtain (3.3). $\square$

While this matrix is inverse positive, that alone does not suffice in proving the e-positivity of single clique-blowups of paths. This is because the linear equation found in Theorem 34 contains a negative on the right hand side of the equation. Therefore, we must directly examine the formulae we obtained to deduce the desired e-basis expansion.

Theorem 37. *The chromatic symmetric functions of single clique-blowups of a path are e-positive. Furthermore, the e-basis expansion of the CSF of a SCBP is*

$$\begin{aligned}\mathbf{X}_{P_{1^{n_1},m,1^{n_2}}} = m!\Bigg[&\frac{m}{(m+n_1)!}\mathbf{X}_{L_{m+n_1+1,n_2-1}} + \mathbf{X}_{P_{n_1+n_2-1}}e_{m+1}\\ &+\sum_{k=1}^{n_1-1}\frac{m-1}{(m+k)!}\mathbf{X}_{P_{n_1-k}}\mathbf{X}_{L_{m+k+1,n_2-1}} + (n_1-k+1)\mathbf{X}_{P_{n_2+k-1}}e_{m+n_1-k+1}\Bigg]\end{aligned} \tag{3.4}$$

Proof. Let $G = P_{1^{n_1},m,1^{n_2}}$ and assume without loss of generality that $n_1 \le n_2$. First, note that when $n_1 < 2$, $P_{1^{n_1},m,1^{n_2}}$ is either a lollipop graph ($n_1 = 0$) or a melting lollipop graph ($n_1 = 1$), which we know to be e-positive.

To begin, let $G_k^+ = L_{m+1,n_2+k-1} \sqcup P_{n_1-k}$ and $G_k^- = L_{m+1,n_1-k} \sqcup P_{n_2+k-1}$. Consider first G where $n_1 = 2$. Applying Corollary 36 to such a G, we obtain the equation

$$\mathbf{X}_G = (m-1)\mathbf{X}_{G_1^+} - (m-1)\mathbf{X}_{G_1^-} + \mathbf{X}_{P_{1^1,m,1^{n_2+1}}}.$$

Notice that $P_{1^1,m,1^{n_2+1}} = L^{(1)}_{m+2,n_2}$, a melting lollipop graph. Therefore, for G with $n_1 \ge 2$, recursively applying Corollary 36 will terminate, and we obtain the following equation:

$$\mathbf{X}_G = \mathbf{X}_{L^{(1)}_{m+2,n_1+n_2-2}} + (m-1)\sum_{k=1}^{n_1-1}\mathbf{X}_{G_k^+} - \mathbf{X}_{G_k^-} \tag{3.5}$$

Note that $1 \le k \le n_1 - 1$, since we remove a vertex from the shorter side each time, meaning $k = n_1 - 1$ corresponds to only one vertex left. From here, it suffices to show that each instance of G_k^- cancels with something in the positive terms.

To begin, we write out the chromatic symmetric functions in more explicit terms. Using Theorem 33, we find that

$$\mathbf{X}_{G_k^-} = m!\mathbf{X}_{P_{n_2+k-1}}\left[(m+n_1-k+1)e_{m+n_1+1-k} + \sum_{i=0}^{n_1-k-1}(m+i)\mathbf{X}_{P_{n_1-k-i}}e_{m+i+1}\right] \tag{3.6}$$

$$\mathbf{X}_{G_k^+} = m!\mathbf{X}_{P_{n_1-k}}\left[(m+n_2+k)e_{m+n_2+k} + \sum_{i=0}^{n_2+k-2}(m+i)\mathbf{X}_{P_{n_2+k-i-1}}e_{m+i+1}\right] \tag{3.7}$$

$$\mathbf{X}_{L^{(1)}_{m+2,n_1+n_2-2}} = m(m!)\left[(m+n_1+n_2)e_{m+n_1+n_2} + \sum_{i=0}^{n_1+n_2-3}(m+i+1)\mathbf{X}_{P_{n_1+n_2-2-i}}e_{m+i+2}\right] + m!\mathbf{X}_{P_{n_1+n_2-1}}e_{m+1} \tag{3.8}$$

From here, it is just a matter of ensuring that every term in each instance of $\mathbf{X}_{G_k^-}$ cancels out. First, consider the $\mathbf{X}_{P_{n_2+k-1}}e_{m+n_1+1-k}$ term from (3.6). In (3.5), each $\mathbf{X}_{G_k^-}$ will contain this term with coefficient $-m!(m-1)(m+n_1+1-k) = -m!(m^2+(m-1)n_1-(m-1)k-1)$. Now consider the sum in (3.8) and note that when $i = n_1-k-1$, we obtain this same term with coefficient $m!m(m+n_1-k) = m!(m^2+mn_1-mk)$ in (3.5). Additionally, note that since i is dependant on k, this term in each $\mathbf{X}_{G_k^-}$ corresponds to a unique term in the sum from (3.8). Adding these two coefficients together, we obtain $m!(n_1-k+1)$. But since $k \leq n_1-1$, $n_1-k+1 \geq 2$ and hence the negative occurrences of this term in $\mathbf{X}_{G_k^-}$ cancel out. This leaves the following from $\mathbf{X}_{L^{(1)}_{m+2,n_1+n_2-2}}$ in the e-basis expansion:

$$m!\left[m\left((m+n_1+n_2)e_{m+n_1+n_2} + \sum_{i=n_1-1}^{n_1+n_2-3}(m+i+1)\mathbf{X}_{P_{n_1+n_2-2-i}}e_{m+i+2}\right)\right.$$
$$\left.+\ \mathbf{X}_{P_{n_1+n_2-1}}e_{m+1} + \sum_{k=1}^{n_1-1}(n_1-k+1)\mathbf{X}_{P_{n_2+k-1}}e_{m+n_1-k+1}\right]$$

Reindexing the sum across i in this equation, we find that

$$\sum_{i=n_1-1}^{n_1+n_2-3}(m+i+1)\mathbf{X}_{P_{n_1+n_2-2-i}}e_{m+i+2} = \sum_{j=0}^{n_2-2}(m+n_1+j)\mathbf{X}_{P_{n_2-j-1}}e_{m+n_1+j+1}$$

Notice that this reindexed sum combined with the $e_{m+n_1+n_2}$ term produces the e-basis expansion of a lollipop graph. Hence, we find that after cancelling the corresponding negative terms from each $\mathbf{X}_{G_k^-}$, the e-basis expansion of $\mathbf{X}_{L^{(1)}_{m+2,n_1+n_2-2}}$ is

$$m!\left[\frac{m}{(m+n_1)!}\mathbf{X}_{L_{m+n_1+1,n_2-1}} + \mathbf{X}_{P_{n_1+n_2-1}}e_{m+1} + \sum_{k=1}^{n_1-1}(n_1-k+1)\mathbf{X}_{P_{n_2+k-1}}e_{m+n_1-k+1}\right] \tag{3.9}$$

Now consider the sum in (3.6). Note that the i^{th} term of this sum in each $\mathbf{X}_{G_k^-}$ is $\mathbf{X}_{P_{n_2+k-1}}\mathbf{X}_{P_{n_1-k-i}}e_{m+i+1}$ with coefficient $-m!(m-1)(m+i)$ in (3.5). Now, from (3.7), we see that the i^{th} term of the sum in $\mathbf{X}_{G_{k+i}^+}$ is $\mathbf{X}_{P_{n_2+k-1}}\mathbf{X}_{P_{n_1-k-i}}e_{m+i+1}$ with coefficient $m!(m-1)(m+i)$ in (3.5). As such these terms cancel out and we need only confirm that each term appearing with a negative coefficient in $\mathbf{X}_{G_k^-}$ also appears with a positive coefficient in $\mathbf{X}_{G_{k+i}^+}$. First note that since $k+i \leq k+(n_1-k-1) = n_1-1$, the needed $\mathbf{X}_{G_{k+i}^+}$ will always appear in (3.5). Moreover, note that the terms required for cancelling out in each $\mathbf{X}_{G_{k+i}^+}$ are the first k terms of the sum. From (3.7), the sum has n_2+k-1 terms, so since $n_2 \geq n_1 \geq 2$, the sum will always have at least $k+1$ terms. Hence, for each $\mathbf{X}_{G_k^-}$, the terms in the sum will always cancel out. This leaves the following from the sum of all $\mathbf{X}_{G_k^+}$ in the e-basis expansion:

$$m!(m-1)\sum_{k=1}^{n_1-1}P_{n_1-k}\left((m+n_2+k)e_{m+n_2+k} + \sum_{i=k}^{n_2+k-2}(m+i)\mathbf{X}_{P_{n_2+k-i-1}}e_{m+i+1}\right)$$

Reindexing the sum across i in this equation, we find that

$$\sum_{i=k}^{n_2+k-2}(m+i)\mathbf{X}_{P_{n_2+k-i-1}}e_{m+i+1} = \sum_{j=0}^{n_2-2}(m+k+j)\mathbf{X}_{P_{n_2-j-1}}e_{m+k+j+1}$$

Notice that this reindexed sum combined with the e_{m+n_2+k} term produces the e-basis expansion of a lollipop graph. Hence, we find that after cancelling the corresponding negative terms from each $\mathbf{X}_{G_k^-}$, the e-basis expansion of each $\mathbf{X}_{G_k^+}$ is

$$m!\sum_{k=1}^{n_1-1}\frac{m-1}{(m+k)!}P_{n_1-k}L_{m+k+1,n_2-1} \tag{3.10}$$

Hence, all terms with negative coefficients in $\mathbf{X}_G$ cancel out and SCBPs are e-positive. Moreover, we obtain the e-basis expansion for SCBPs by taking the sum of (3.9) and (3.10). □

Note that the e-basis expansion of SCBPs can nearly be expressed in one sum from $k=0$ to n_1, but there is a slight difference in coefficients that prevents this.

While this result is not obvious from (3.2) alone, the proof relies solely on algebra and known e-basis expansions for the required graphs. In summary, once a linear equation has been obtained, this method allows us to prove the e-positivity of the graph of interest using algebra alone. Furthermore, it allows us to obtain an equation for the intermediary graphs, in this case being the melting SCBPs, which were not previously known to be e-positive.

Corollary 38. *All melting SCBPs are e-positive.*

Proof. Follows directly from Corollary 36 and Theorem 37. □

3.3 Double Clique-Blowups of Paths

As we explained in Section 3.1, a large number of clique-blowups of paths are known to be e-positive. It remains to extend this result to the case where blown-up vertices are adjacent. As such, we focus the rest of this thesis on finding results on adjacent clique-blowups, beginning here with double clique-blowups of paths.

Theorem 39. *Let $m_1, m_2 \geq 2$ and $n_1, n_2 \geq 1$ be positive integers. We can obtain a linear equation including $P_{1^{n_1},m_1,m_2,1^{n_2}}$ of the form*

$$\begin{bmatrix}\mathbf{X}_{P_{1^{n_1},m_1,m_2,1^{n_2}}}\\ \mathbf{X}_{P^{(1)}_{1^{n_1},m_1,m_2,1^{n_2}}}\\ \vdots\\ \mathbf{X}_{P^{(m_1-2)}_{1^{n_1},m_1,m_2,1^{n_2}}}\end{bmatrix} = \begin{bmatrix}1 & 2 & 3 & \cdots & m_1-1\\ 0 & 1 & 2 & \cdots & m_1-2\\ & & & \ddots & \\ 0 & 0 & 0 & \cdots & 1\end{bmatrix}\begin{bmatrix}0\\ 0\\ \vdots\\ 0\\ -\mathbf{X}_{P^{(m_1-1)}_{1^{n_2},m_1+m_2-1,1^{n_1+1}}}\\ 2\mathbf{X}_{P^{(m_1-1)}_{1^{n_2},m_1+m_2-1,1^{n_1+1}}}-\mathbf{X}_{L^{(m_1)}_{m_1+m_2+1,n_2-1}}\mathbf{X}_{P_{n_1}}\end{bmatrix}$$

Therefore,

$$\mathbf{X}_{P_{1^{n_1},m_1,m_2,1^{n_2}}} = m_1\mathbf{X}_{P^{(m_1-1)}_{1^{n_2},m_1+m_2-1,1^{n_1+1}}} - (m_1-1)\mathbf{X}_{L^{(m_1)}_{m_1+m_2+1,n_2-1}}\mathbf{X}_{P_{n_1}} \tag{3.11}$$

and

$$\begin{aligned}\mathbf{X}_{P^{(k)}_{1^{n_1},m_1,m_2,1^{n_2}}} &= (m_1 - k\mathbf{X}_{P^{(m_1-1)}_{1^{n_2},m_1+m_2-1,1^{n_1+1}}} - (m_1-k-1)\mathbf{X}_{L^{(m_1)}_{m_1+m_2+1,n_2-1}}\mathbf{X}_{P_{n_1}} &(3.12)\\ &= P_{1^{n_1},m_1,m_2,1^{n_2}} - k\left(\mathbf{X}_{P^{(m_1-1)}_{1^{n_2},m_1+m_2-1,1^{n_1+1}}} - \mathbf{X}_{L^{(m_1)}_{m_1+m_2+1,n_2-1}}\mathbf{X}_{P_{n_1}}\right) &(3.13)\end{aligned}$$

Proof. We begin by describing the process used to obtain this linear equation in Algorithm 4.

Algorithm 4 Removing edges in a double clique-blowup of a path.

Require: $G = P_{1^{n_1},m_1,m_2,1^{n_2}}$
  $L \leftarrow \emptyset$
  for $i = 1, \dots, m_1 - 1$ **do**
    $e_1 \leftarrow (n_1, n_1 + i)$
    $e_2 \leftarrow (n_1, n_1 + i + 1)$
    Add $\mathbf{X}_G + \mathbf{X}_{G-e_1-e_2} = \mathbf{X}_{G-e_1} + \mathbf{X}_{G-e_2}$ to L_1
    $G \leftarrow G - e_1$
  end for

In this algorithm, we obtain a linear equation by recursively applying the triple deletion property (Theorem 10) with a fixed edge $(i, i+1)$ (and hence swapping edges (n_1, i) and $(n_1, i+1)$ with each other) where $n_1 < i < n_1 + m_1$ and $i = n_1 + 1$ initially. In less precise terms, we use the triple deletion property to remove edges from the leftmost path to the clique K_{m_1} until there are no more edges left. Now, notice that the graph obtained by removing all but one of these edges is a melting SCBP and furthermore that the graph obtained by removing all of these edges is the disjoint union of a path and a melting lollipop graph (see Figure 3.2) for an example of this). Doing so, we obtain the following linear equation:

$$\begin{bmatrix} 1 & -2 & 1 & & \\ & 1 & -2 & 1 & \\ & & & \ddots & \\ & & & 1 & -2 \\ & & & & 1 \end{bmatrix} \begin{bmatrix} \mathbf{X}_{P_{1^{n_1},m_1,m_2,1^{n_2}}} \\ \mathbf{X}_{P^{(1)}_{1^{n_1},m_1,m_2,1^{n_2}}} \\ \vdots \\ \mathbf{X}_{P^{(m_1-2)}_{1^{n_1},m_1,m_2,1^{n_2}}} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ -\mathbf{X}_{P^{(m_1-1)}_{1^{n_2},m_1+m_2-1,1^{n_1+1}}} \\ 2\mathbf{X}_{P^{(m_1-1)}_{1^{n_2},m_1+m_2-1,1^{n_1+1}}} - \mathbf{X}_{L^{(m_1)}_{m_1+m_2+1,n_2-1}}\mathbf{X}_{P_{n_1}} \end{bmatrix}$$

From here, recall that we found the inverse of the leftmost matrix in the proof of Corollary 36. Multiplying both sides of this linear equation by this inverse, we obtain the desired result. □

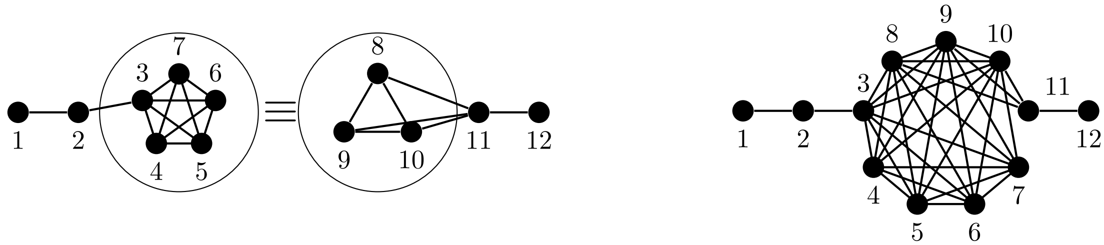

Figure 3.2: $P^{(4)}_{1^2,5,3,1^2}$ (left) and $P^{(4)}_{1^2,7,1^3}$ (right). Since the two cliques in the melting DCBP are complete to one another, we can combine them into one clique to obtain a melting SCBP. Furthermore, notice that when we remove the edge $(2,3)$ from both figures, we find that $P^{(5)}_{1^2,5,3,1^2}$ and $L_{9,1} \sqcup P_2$ are isomorphic, as desired.

For an example of the process described in Algorithm 4, see Figure 3.3. This process visually represents the following system of equations obtained by applying this Algorithm to the graph $P_{1^2,5,3,1^2}$:

$$\begin{aligned}
\mathbf{X}_{P_{1^2,5,3,1^2}} + \mathbf{X}_{P^{(2)}_{1^2,5,3,1^2}} &= 2\mathbf{X}_{P^{(1)}_{1^2,5,3,1^2}} \\
\mathbf{X}_{P^{(1)}_{1^2,5,3,1^2}} + \mathbf{X}_{P^{(3)}_{1^2,5,3,1^2}} &= 2\mathbf{X}_{P^{(2)}_{1^2,5,3,1^2}} \\
\mathbf{X}_{P^{(2)}_{1^2,5,3,1^2}} + \mathbf{X}_{P^{(4)}_{1^2,7,1^3}} &= 2\mathbf{X}_{P^{(3)}_{1^2,5,3,1^2}} \\
\mathbf{X}_{P^{(3)}_{1^2,5,3,1^2}} + \mathbf{X}_{L^{(5)}_{9,1}}\mathbf{X}_{P_2} &= 2\mathbf{X}_{P^{(4)}_{1^2,7,1^3}}
\end{aligned}$$

Now, combining Theorems 37 and 39, we are able to prove that DCBPs are e-positive.

Theorem 40. *DCBPs are e-positive.*

Proof. First note that from [4], non-adjacent clique-blowups of paths are e-positive, so we need only consider the case where cliques are adjacent. As such, we consider cases of the form $P_{1^{n_1},m_1,m_2,1^{n_2}}$. First note that if either $n_1, n_2 = 0$, $P_{1^{n_1},m_1,m_2,1^{n_2}}$ is a melting lollipop and thus e-positive. Additionally, if either $m_1, m_2 = 1$, $P_{1^{n_1},m_1,m_2,1^{n_2}}$ is a SCBP and thus e-positive. Hence, consider $P_{1^{n_1},m_1,m_2,1^{n_2}}$ for $n_1, n_2 \geq 1$ and $m_1, m_2 \geq 2$.

From Theorem 39, we know that

$$\mathbf{X}_{P_{1^{n_1},m_1,m_2,1^{n_2}}} = m_1\mathbf{X}_{P^{(m_1-1)}_{1^{n_2},m_1+m_2-1,1^{n_1+1}}} - (m_1-1)\mathbf{X}_{L^{(m_1)}_{m_1+m_2+1,n_2-1}}\mathbf{X}_{P_{n_1}}$$

As we know that SCBPs, melting lollipop graphs, and paths are e-positive, we need only ensure that the negative term in this process cancels. Applying Theorem 33, we find that the negative term can be written as

$$\begin{aligned}
&(m_1-1)\mathbf{X}_{L^{(m_1)}_{m_1+m_2+1,n_2-1}}\mathbf{X}_{P_{n_1}} \\
&\quad = (m_1-1)\mathbf{X}_{P_{n_1}}\left(\frac{m_2}{m_1+m_2}\mathbf{X}_{L_{m_1+m_2+1,n_2-1}} + m_1(m_1+m_2-1)!\mathbf{X}_{P_{n_2}}e_{m_1+m_2}\right) \quad (3.14)
\end{aligned}$$

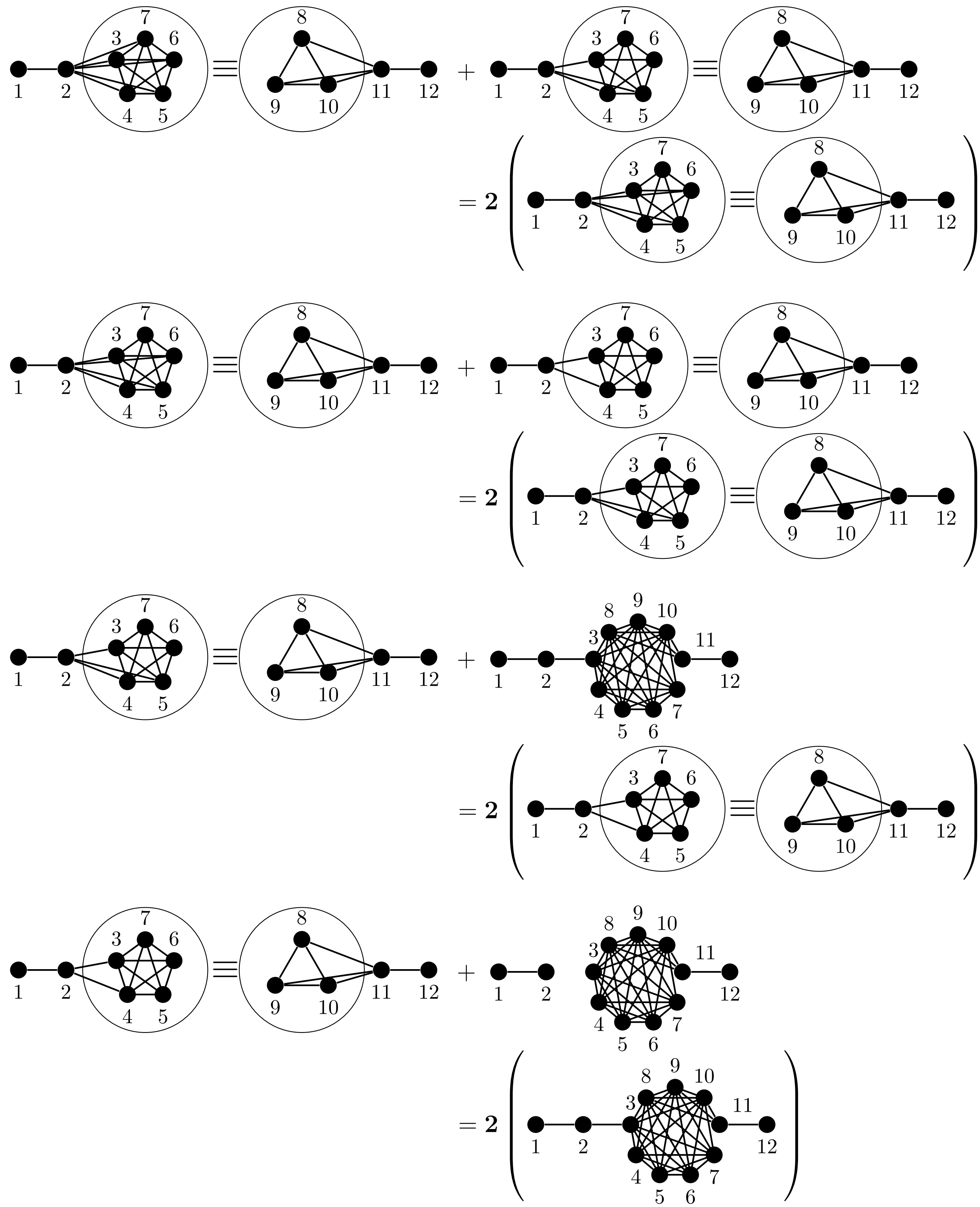

Figure 3.3: A visual representation of the process described in Algorithm 4 starting with the graph $P_{1^2,5,3,1^2}$.

We can also use Corollary 36 to rewrite the positive term of this equation:

$$m_1\mathbf{X}_{P^{(m_1-1)}_{1^{n_2},m_1+m_2-1,1^{n_1+1}}} = \frac{m_1}{m_1+m_2-1}\left(m_2\mathbf{X}_{P_{1^{n_2},m_1+m_2-1,1^{n_1+1}}} + (m_1-1)\mathbf{X}_{L_{m_1+m_2,n_1}}\mathbf{X}_{P_{n_2}}\right) \tag{3.15}$$

We first consider the term $\mathbf{X}_{P_{n_1}}\mathbf{X}_{P_{n_2}}e_{m_1+m_2}$ from 3.14; note that this term has coefficient $(m_1-1)m_1(m_1+m_2-1)!$. From 3.15, note that we have the term

$$\begin{aligned}\mathbf{X}_{L_{m_1+m_2,n_1}}\mathbf{X}_{P_{n_2}} &= \mathbf{X}_{P_{n_2}}(m_1+m_2-1)!(m_1+m_2+n_1)e_{m_1+m_2+n_1}\\ &\quad + \mathbf{X}_{P_{n_2}}(m_1+m_2-1)!\sum_{i=0}^{n_1-1}(m_1+m_2+i-1)\mathbf{X}_{P_{n_1-i}}e_{m_1+m_2+i}\end{aligned}$$

after applying Theorem 33. When $i=0$ in the sum of this expansion, we get the term $\mathbf{X}_{P_{n_2}}\mathbf{X}_{P_{n_1}}e_{m_1+m_2}$ with coefficient $(m_1-1)m_1(m_1+m_2-1)!$. Hence this term with a negative coefficient cancels out.

Now consider the term $\mathbf{X}_{P_{n_1}}\mathbf{X}_{L_{m_1+m_2+1,n_2-1}}$ from 3.15; note that this term has coefficient $\frac{(m_1-1)m_2}{m_1+m_2}$. Now consider SCBP term from 3.15. Note that we can swap the placement of n_2 and n_1+1 and obtain a graph that is isomorphic. We expand out this term after swapping using Theorem 37:

$$\begin{aligned}\mathbf{X}_{P_{1^{n_1+1},m_1+m_2-1,1^{n_2}}} &= (m_1+m_2-1)!\left(\frac{m_1+m_2-1}{(m_1+m_2+n_1)!}\mathbf{X}_{L_{m_1+m_2+n_1+1,n_2-1}} + \mathbf{X}_{P_{n_1+n_2}}e_{m_1+m_2}\right.\\ &\left. + \sum_{k=1}^{n_1}\frac{m_1+m_2-2}{(m_1+m_2+k-1)!}\mathbf{X}_{P_{n_1-k+1}}\mathbf{X}_{L_{m_1+m_2+k,n_2-1}} + (n_1-k+2)\mathbf{X}_{P_{n_2+k-1}}e_{m_1+m_2+n_1-k+1}\right)\end{aligned}$$

When $k=1$, notice that we get the term $\mathbf{X}_{P_{n_1}}\mathbf{X}_{L_{m_1+m_2+1,n_2-1}}$ with coefficient $\frac{m_1m_2(m_1+m_2-2)}{(m_1+m_2-1)(m_1+m_2)}$. Finally, note that

$$\begin{aligned}\frac{m_1m_2(m_1+m_2-2)}{(m_1+m_2-1)(m_1+m_2)} - \frac{(m_1-1)m_2}{m_1+m_2} &= \frac{m_2}{m_1+m_2}\left(\frac{m_1^2+m_1m_2-2m_1}{m_1+m_2-1} - (m_1-1)\right)\\ &= \frac{m_2(m_2-1)}{(m_1+m_2)(m_1+m_2-1)}\end{aligned}$$

Thus, the negative instance of this term cancels out and $P_{1^{n_1},m_1,m_2,1^{n_2}}$ is e-positive. □

Chapter 4

Conclusion and Future Works

There are a few aspects of this method that would seem to be very useful in determining e-positivity and e-basis expansions of certain graphs. Firstly, adjacency diagrams are very useful in finding the algorithms used throughout this paper. It would be useful to generalize a few of the properties that these diagrams hold to expand their usefulness in these cases–in particular, determining what different graph structures look like in adjacency diagrams. A more clear determination of how the triple-deletion property applies to these diagrams would make them much more usable as well.

In general, this method is held back a bit by the arithmetic required to find e-basis expansions and the difficulty in determining how and where to apply the triple deletion property. As such, a generalization of the ways and places one can add and remove edges and develop an algorithm would make this method a lot faster. Furthermore, a characterization of the graphs and structures that this method is applicable to would help with scaling up the method to larger classes of graphs. In particular, determining the structures that result in isomorphic graphs when swapping two edges.

The results about DCBPs point to many potential future directions. Firstly, the most obvious thing to do is to find the precise formula for DCBPs with adjacent cliques and then expand this to a general e-basis expansion for all DCBPs. While there is an e-basis expansion implicitly found from the process we describe, a more general formula would more difficult to develop. There is most likely a different formula for DCBPs with non-adjacent clique-blowups than the one found from DCBPs with adjacent cliques and a new process would need to be found to discover the expansion for the non-adjacent case. This could likely be achieved by applying modified versions of Algorithms 2 and 3 to the non-adjacent cases and inducting on the length of one of the paths.

Once these expansions have been discovered, this process could most likely be expanded to 3-clique-blowups of paths (3CBPs) and beyond. We predict that a proof for 3CBPs could be discovered in the following way: as we know from [4], 3CBPs with non-adjacent cliques are e-positive, we need only consider the cases with adjacent cliques. First, consider the case where the blown-up vertices are an endpoint and two adjacent vertices and use induction on the size of the endpoint clique to prove e-positivity. Then, consider the case where the blown-up vertices are two adjacent vertices and one vertex not adjacent to either of these vertices. Applying an adjusted version of Algorithms 2 and 3 to this non-adjacent clique and performing induction on the length of the path to the endpoint to produce this result. The case where all three blown-up vertices are adjacent will likely require a different method all together, but the most probably case is performing induction on one of the outer cliques and removing edges from one of the vertices to the inner clique. A similar kind of process could be used to expand this to 4-clique-blowups of paths, which would hopefully expand to a more general version of this process.

Another potential method that would expand these findings would be to prove the appendable (e)-positivity of adjacent DCBPs. In particular, a proof for the appendable (e)-positivity of $P_{1,m_1,m_2,1}$ (that is, where $n_1 = n_2 = 1$) would allow one to prove the e-positivity of a great deal of clique-blowups of paths. It is unlikely that this process itself would allow for such a proof, but it would likely generalize to the appendable (e)-positivity of larger adjacent clique-blowups which would, in general, help the proof for clique-blowups of paths quite a bit.

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