# Distance-Biregular Graphs and Orthogonal Polynomials

by

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## Author's Declaration

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

## Abstract

This thesis is about distance-biregular graphs– when they exist, what algebraic and structural properties they have, and how they arise in extremal problems.

We develop a set of necessary conditions for a distance-biregular graph to exist. Using these conditions and a computer, we develop tables of possible parameter sets for distance-biregular graphs. We extend results of Fiol, Garriga, and Yebra characterizing distance-regular graphs to characterizations of distance-biregular graphs, and highlight some new results using these characterizations. We also extend the spectral Moore bounds of Cioabă et al. to semiregular bipartite graphs, and show that distance-biregular graphs arise as extremal examples of graphs meeting the spectral Moore bound.

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Last, but certainly not least, I would like to thank my family, especially Katherine Lato and Barry Glicklich. They have inspired me towards mathematics and higher education, and I couldn't have done it without their support.

## Dedication

To my family.

# Table of Contents

# List of Symbols

$R_r$ restriction of $E_r$ in a bipartite graph to vertices indexed by $\beta$. 29

$V(n,q)$ $n$-dimensional vector space over $GF(q)$. 55

$\Delta_{rs}(t)$ Krein parameter for $\beta$. 110

$\Phi_u$ eigenvalue support of $u$. 60

$X_i$ $i$-th distance adjacency matrix of halved graph induced by $\beta$. 43

$Y_i$ $i$-th distance adjacency matrix of halved graph induced by $\gamma$. 43

$\beta$ cell of a partition of a bipartite graph. 13

$\gamma$ cell of a partition of a bipartite graph. 13

$\circ$ Schur product. 108

$\langle f,g\rangle_G$ inner product with respect to graph $G$. 24

$\langle f,g\rangle_\pi$ inner product with respect to cell $\pi$. 24

$\langle f,g\rangle_u$ inner product with respect to vertex $u$. 24

$\lambda_{rs}(t)$ Krein parameter for $\beta$. 110

$\mathbf{1}_{i,j}$ $i\times j$ matrix of all ones. 31

$\mathcal{A}$ distance basis of coherent configuration for distance-biregular graph. 105

$\mathcal{A}$ spectral basis of coherent configuration for distance-biregular graph. 106

$\mathrm{pg}(s,t,\alpha)$ partial geometry. 46

$\mathbf{p}_i$ restriction of perron vector. 75

$\mathbf{p}$ eigenvector of a connected graph with all positive entries and norm one. 28

$\pi(u)$ cell of a partition containing vertex $u$. 46

$[i]_q$ number of $(i-1)$-dimensional subspaces in a given $i$-dimensional subspace. 47

$\rho_{rs}(t)$ Krein parameter for $\beta$. 110

$\rho$ largest eigenvalue of connected graph. 28

$\theta_r$ eigenvalue of graph. 27

$\mathbf{e}_u$ characteristic vector of vertex $u$.. 11

$\mathbf{E}_\pi$ characteristic matrix of cell $\pi$.. 13

$a_i^*$ intersection coefficient of distance polynomial for graph $G$. 25

$a_i$ coefficient of $p_i$ in three-term recurrence. 10

$b_i^\pi$ intersection coefficient of distance polynomial for cell $\pi$. 25

$b_{i-1}$ coefficient of $p_{i-1}$ in three-term recurrence. 10

$b_i^*$ intersection coefficient of distance polynomial for graph $G$. 25

$c_i^*$ intersection coefficient of distance polynomial for graph $G$. 25

$c_i^\pi$ intersection coefficient of distance polynomial for cell $\pi$. 25

$c_{i+1}$ coefficient of $p_{i+1}$ in three-term recurrence. 10

$d_\pi$ covering radius of cell $\pi$. 25

$f_i^u(x)$ distance polynomial for vertex $u$. 62

$g_i^u(x)$ local $i$-excess polynomial for $u$. 75

$k_i$ number of vertices at distance $i$ from a locally distance-regular vertex $u$. 26

$m_r^\pi$ bipartite multiplicity of eigenvalue $\theta_r$. 32

$m_r$ multiplicity of eigenvalue $\theta_r$. 27

$p^\pi(i, j; h)$ intersection number. 92

$q \circ M$ Schur polynomial of $q$ evaluated at matrix $M$. 115

# Chapter 1

# Introduction

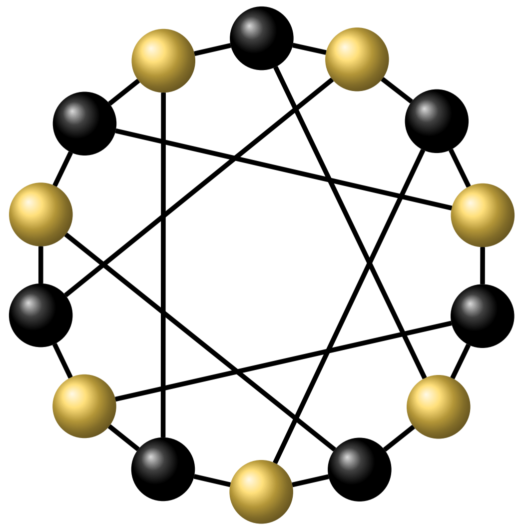

Figure 1.1: Example of a generalized polygon

Distance-biregular graphs are the subject of this thesis, and orthogonal polynomials are the primary tool we use to study them. Distance-biregular graphs are a class of bipartite graphs that contain interesting examples, have a nice algebraic and combinatorial structure, and arise as extremal examples of graphs meeting certain bounds. This thesis is structured around those three different facets of distance-biregular graphs.

Chapter 2 is about orthogonal polynomials and the other tools we will use in the thesis. Chapter 3 explores some constructions and necessary conditions for distance-biregular graphs to exist, Chapter 4 uses some of the algebraic properties to give alternate characterizations of distance-biregular graphs, and Chapter 5 motivates distance-biregular graphs from an extremal perspective. Distance-biregular graphs are an interesting and understudied class of graphs, so Chapter 6 details more of the algebraic structure and lists a number of problems for future research.

## 1.1 Precursors to Distance-Biregular Graphs

Distance-biregular graphs are bipartite graphs, and as such can be used to represent the relationship of incidence structures. Many examples of incidence structures coming from algebra, design theory, and finite geometry have a high degree of regularity, which is reflected in the incidence graphs being distance-biregular. Thus a first notion of distance-biregular graphs is that they are a class of bipartite graphs that contain:

- Generalized polygons. Generalized polygons are bipartite graphs with diameter $d$ and girth $2d$. They were introduced by Tits [152] in the study of groups of Lie type, and many of the constructions come from algebra. Generalized polygons recur throughout this thesis, and additional information can be found in the surveys by Payne [126], Payne and Thas [129], Thas [149], and Van Maldeghem [153]. An explicit example is shown in Figure 1.1.

- Certain 2-designs. A 2-design is an incidence structure where every point is incident to $\ell$ blocks, every block contains $k$ points, and every pair of points appear in exactly $\lambda$ common blocks. The systematic study of designs began with the work of Fisher [75] on biological statistics, though examples of 2-designs such as Steiner triple systems [107, 139] or Latin Squares [64] appear much earlier. In this thesis, 2-designs are discussed further in Section 3.1. More information can also be found in references such as the *Handbook of Combinatorial Designs* [39] or the books of Beth, Jungnickel, and Lenz [19] or Hughes and Piper [102].

- Partial geometries. A partial geometry is a point- and block-regular incidence structure where, for every point $u$ and block $x$, there are exactly $\alpha$ incident pairs $(v, y)$ such that $u$ is incident to $y$ and $v$ is incident to $x$. Partial geometries were introduced by Bose [25] in connection to designs and strongly regular graphs, and include Steiner triple systems and generalized polygons of diameter four, as well as infinite proper families. Partial geometries are a particularly important example in Section 5.7. More information can be found in the surveys of Brouwer and Van Lint [30], De Clerck and Van Maldeghem [54], or Thas [150].

Distance-biregular graphs can also be seen as an extension of the class of distance-regular graphs.

Biggs and Smith [23] introduced the notion of distance-transitive graphs. A graph $G$ is *distance-transitive* if, for any pair of vertices at some distance $i$ from each other, there is an automorphism mapping that pair to another pair at distance $i$. This symmetry forces a large amount of structure onto the graph. Distance-transitive graphs are *distance-regular*, meaning that for all vertices $u, v$ and all non-negative integers $i, j$ the number of vertices at distance $i$ from $u$ and distance $j$ from $v$ depends only on the distance between $u$ and $v$, and not on the specific choice of $u$ and $v$. Adel'son-Velskii, Maksimovich, Weisfeiler, Leman, and Faradzhev [4] constructed a distance-regular graph which is not distance-transitive, proving distance-regularity is distinct from distance-transitivity.

The *adjacency matrix* $A$ of a graph is the matrix indexed by vertices where the corresponding entry is one if two vertices are adjacent and zero otherwise. This can be extended to $i$-th *distance adjacency matrices* $A_i$ where the entry indexed by a pair of vertices is one if the vertices at distance $i$ and zero otherwise. The adjacency algebra of a distance-regular graph is closely related to association schemes, which arose in a different context around the same time as distance-regular graphs. Working in design theory, Bose and Shimamoto [27] introduced sets of matrices to represent relations and a list of axioms that such sets must satisfy to form an *association scheme*. In particular, the set of distance adjacency matrices of a distance-regular graph form an association scheme. The algebra of association schemes was subsequently developed by Bose and Mesner [26]. Ideas similar to association schemes and the adjacency matrices of a distance-regular graph can also be found in the paper of Higman [94] on finite permutation groups. The paper of Higman [94] contains ideas related not just to the subsequent theory of distance-regular graphs, but also connections to the work of Feit and Higman [66] on generalized polygons.

More information about distance-regular graphs can be found in the book of Brouwer, Cohen, and Neumaier [28] or the more recent survey of Van Dam, Koolen, and Tanaka [47].

Distance-biregular graphs also arise as extremal examples of graphs meeting certain spectral or structural bounds.

Let $G$ be a connected bipartite graph with cells of the partition $\beta$ and $\gamma$. Suppose every vertex in $\beta$ has valency $k$ and every vertex in $\gamma$ has valency $\ell$. If we fix a vertex $u \in \gamma$, then $u$ has $\ell$ neighbours. Each neighbour has at most $k-1$ neighbours that are at distance two from $u$, each of those neighbours has at most $\ell - 1$ neighbours at distance three from $u$, and so on. The number of vertices of $\beta$ is the number of vertices that are at an odd distance from $u$, so if $G$ has diameter $d$, we can sum over all the possible distances from $u$ to see that

$$|\beta| \leq \ell \sum_{i=0}^{\lfloor \frac{d-1}{2} \rfloor} (k-1)^i (\ell - 1)^i . \tag{1.1.1}$$

If equality holds, then we know that for any vertex $v \in \beta$, there is exactly one vertex that is adjacent to $v$ and at distance $d(u, v) - 1$ from $u$. It follows that there are $k - 1$ neighbours of $v$ at distance $d(u, v) + 1$ from $u$. If instead $v \in \gamma$, there is still only one neighbour of $v$ at distance $d(u, v) - 1$ from $u$, and there are $\ell - 1$ neighbours of $v$ at distance $d(u, v) + 1$ from $u$. If we instead choose $u \in \gamma$, we can rewrite Equation 1.1.1 to bound $|\gamma|$, and we get similar characterizations when that bound is tight.

A connected bipartite graph is *distance-biregular* if for any vertices $u, v$ and $i, j \geq 0$, the number of vertices at distance $i$ from $v$ and at distance $j$ from $u$ depends only on the distance between $u$ and $v$ and the cell of the partition that $u$ lies in. The extremal examples where Equation 1.1.1 is tight for both cells of the partition are distance-biregular. We will consider a different perspective on this example in Section 2.2, and consider more variations of this bound in Section 5.1.

## 1.2 History of Distance-Biregular Graphs

Distance-biregular graphs were introduced by Delorme [56], and were the subject of Shawe-Taylor's thesis [134], papers [132, 133], and joint work with Godsil [85] and Mohar [118]. Distance-biregular graphs have also been the subject of work by other authors.

Distance-biregular graphs arise in a natural way from extending the theory of distance-regular graphs. Recall that a graph is distance-regular if, for any vertices $u, v$ the number of vertices at distance $i$ from $u$ and distance $j$ from $v$ depends only on the distance between $u$ and $v$ and not the choice of $u$ and $v$. A bipartite graph is distance-biregular if this number also depends on the choice of cell that $u$ lies in. A further weakening would be that the number of vertices at distance $i$ from $u$ and distance $j$ from $v$ only depends on the choice of $u$ and the distance $v$ is from $u$. However, Godsil and Shawe-Taylor [85] proved that all such graphs are distance-regular or distance-biregular. Fiol [71] gave a new proof of a stronger result.

The definitions of bipartite distance-regular graphs, generalized polygons, 2-designs, and partial geometries predate the introduction of distance-biregular graphs. However, there have also been numerous constructions specific to the context of distance-biregular graphs. Mohar and Shawe-Taylor [118] characterized which subdivision graphs can occur as distance-biregular graphs. Delorme [55, 56] and Shawe-Taylor [134] gave several constructions of distance-biregular graphs coming from distance-regular graphs and finite geometry. Shawe-Taylor [133] used covers of complete bipartite graphs to construct examples of distance-biregular graphs. Some of these examples also came up in other contexts, such as graphs satisfying a dual version of Pasch's axiom [43] or uniformly geodetic graphs [108].

The high degree of regularity of distance-biregular graphs imposes a large number of numerical constraints on the possible constructions of distance-biregular graphs. Delorme [55, 56] listed some conditions. Godsil and Shawe-Taylor [85] gave a list of conditions for a distance-biregular graph to be *feasible*, meaning the parameters satisfy some set of necessary, but not sufficient, numerical conditions. Shawe-Taylor [134] expanded on this notion of feasibility. Other necessary conditions for distance-biregular graphs with girth divisible by four were given by Nomura [124] and Suzuki [143], and Bang [13] gave conditions on distance-biregular graphs with girth at least eight.

One of the major motivations of distance-biregular graphs is that they can be used to naturally extend the class of distance-regular graphs. Distance-biregular graphs can also be motivated directly in the same way that distance-regular graphs were.

Shawe-Taylor [132, 134] considered distance-bitransitive graphs. This bipartite extension of distance-transitivity was extended further by Devilllers, Giudici, Li, and Praeger [61]. As in the general case, distance-bitransitive graphs are a subclass of distance-biregular graphs.

The adjacency algebra has also been extended from distance-regular graphs to distance-biregular graphs. Delorme [55, 56] first set up some algebraic formulations of distance-biregular graphs. More recently, Fernández and Miklavič [68] considered the Terwilliger algebra of distance-biregular graphs, and Fernández and Penjić [69] considered 2-homogeneous distance-biregular graphs. Other matrix formulations use the Laplacian matrix [2, 103] and the matrix of distances [101].

There is another close connection between distance-regular and distance-biregular graphs. Let $G$ be a graph, and let $G'$ be the graph where two vertices are adjacent in $G'$ if and only if they are at distance two in $G$. If $G$ is distance-biregular, then the connected components of $G'$ are distance-regular. Shawe-Taylor [134] and Yamazaki [155] investigated which distance-regular graphs could arise from distance-biregular graphs in this way. More generally, distance-regular and distance-biregular graphs have highly regular subgraphs, which motivated work of Suzuki [144] and Hiraki [95, 96] on strongly closed subgraphs of distance-regular and distance-biregular graphs.

Distance-regular graphs have been considered as extremal examples for certain bounds. Fiol [72] proved a spectral excess theorem for distance-biregular graphs, which can be interpreted as a spectral bound for which distance-biregular graphs are the extremal examples. However, most of the results on distance-regular graphs have been motivated more from the subgraph structure or related algebras, and so the perspective of distance-biregular graphs as extremal examples has generally been lacking.

## 1.3 Main Results

In this thesis, we will primarily use orthogonal polynomials to extend results on distance-regular graphs to distance-biregular graphs. The major ideas and notation that we use throughout the thesis are described in Chapter 2.

In Chapter 3, we describe famous families of distance-biregular graphs. We also give a list of necessary conditions that distance-biregular graphs must satisfy. These numerical conditions make it possible for a computer to generate a list of possible parameters for a distance-biregular graphs. We include annotated tables in Appendix A and Appendix B. Appendix A mainly focuses on the smaller examples, including bipartite distance-regular graphs and partial geometries, that have been studied under other contexts. Appendix B is focused on the graphs that have less literature about them.

Much of the nice structure of distance-regular graphs applies equally well to distance-biregular graphs. In Chapter 4, we illustrate this by extending two characterizations of distance-graphs to distance-biregular analogues. We improve and extend a result of Fiol, Garriga, and Yebra [74] to get the following characterization of distance-biregular graphs.

**4.4.3 Theorem.** *Let $G$ be a connected bipartite graph with diameter $d$, adjacency matrix $A$, $d$-th distance adjacency matrix $A_d$, and cells of the partition $\beta, \gamma$. Then $G$ is distance-biregular if and only if $G$ has $d+1$ distinct eigenvalues and there exist polynomials $f^\beta$ and $f^\gamma$ of degree $d$ such that for $\pi \in \{\beta, \gamma\}$*

$$f^\pi(A)\,\mathbf{E}_\pi = A_d \mathbf{E}_\pi,$$

*where $\mathbf{E}_\pi$ is the matrix whose columns are characteristic vectors for vertices in $\pi$.*

Building on the work of Fiol, Garriga, and Yebra [74], Fiol and Garriga [73] proved a spectral excess theorem for distance-regular graphs. We similarly build on Theorem 4.4.3 to get a distance-biregular spectral excess theorem.

**4.6.4 Theorem.** *Let $G$ be a semiregular bipartite connected graph with diameter $d$ and largest eigenvalue $\rho$. Then $G$ is distance-biregular if and only if it has $d+1$ distinct eigenvalues and there exist polynomials $f^\beta, f^\gamma$ of degree $d$ such that for $\pi \in \{\beta, \gamma\}$ and all vertices $u \in \pi$, we have*

$$\frac{1}{|\pi|} \sum_{w \in \pi} \mathbf{e}_w^T f^\pi(A)^2 \, \mathbf{e}_w = f^\pi(\rho) = \left| \{v \in V(G) : d(u,v) = d\} \right|,$$

*where $\mathbf{e}_w$ is the characteristic vector of $w$.*

Using these characterizations, we extend a result of Abiad, Van Dam, and Fiol [3] for bipartite distance-regular graphs of large girth to distance-biregular graphs.

**4.7.4 Theorem.** *Let $G$ be a connected semiregular bipartite graph with diameter $d$ and $d+1$ distinct eigenvalues. If the girth $g \geq 2d-2$, then $G$ is distance-biregular.*

We further consider the adjacency algebra of distance-biregular graphs in Chapter 6.

In Chapter 5, we explore distance-biregular graphs as extremal examples. We mainly focus on semiregular bipartite extensions of the bounds of Nozaki [125], Cioabă, Koolen, Nozaki, and Vermette [37] and Cioabă, Koolen, and Nozaki [36]. In particular, amalgamating several results in Section 5.6, we get the following result.

**1.3.1 Theorem.** *Let $G$ be a bipartite semiregular graph where every vertex in cell $\beta$ has valency $k$, and every vertex in $\gamma$ has valency $\ell$. Let $t, c > 0$ be constants determined by the choice of $k, \ell$, and $\lambda$. Then*

$$|\beta| \leq \sum_{i=0}^{t-2} (\ell-1)^i (k-1)^i + \frac{\ell(\ell-1)^{t-1}(k-1)^{t-1}}{c}$$

*or*

$$|\beta| \leq 1 + k \sum_{i=1}^{t-2} (\ell-1)^i (k-1)^{i-1} + \frac{k(\ell-1)^{t-1}(k-1)^{t-2}}{c}.$$

*If either bound is tight, then $G$ is distance-biregular.*

# Chapter 2

# Background

In this thesis, we are primarily interested in using sequences of orthogonal polynomials to study distance-biregular graphs, especially with respect to extremal problems. Orthogonal polynomials have led to some powerful results for distance-regular graphs, which we extend to distance-biregular graphs. In this chapter, we present the ideas and notation that we will use throughout the thesis. Our approach to orthogonal polynomials is non-standard, but equivalent to the basic results that can be found in references such as Nikiforov, Suslov, and Uvarov [121] or Szegö [145]. We also set up the spectral decomposition for graphs and bipartite graphs which will be fundamental to later results.

## 2.1 Moore Graphs

Let $G$ be a connected graph, and let $A_i$ be the matrix indexed by the vertices of $G$ where the $(u, v)$-entry is one if $u$ and $v$ are at distance $i$ and zero otherwise. We call $A_i$ the $i$-th *distance adjacency matrix*. The *adjacency matrix* $A_1$ is denoted by $A$. The *distance graph* $G_i$ of $G$ is the graph with adjacency matrix $A_i$. If $i$ is greater than the diameter of the graph, $A_i$ is the all-zero matrix. We are particularly interested in graphs where the distance adjacency matrices can be written as polynomials evaluated at the adjacency matrix, because they have strong algebraic and combinatorial properties.

Let $\mathcal{P}$ be the vector space of real polynomials. Given a sequence of polynomials $p_0, p_1, \ldots$ we say the polynomials satisfy a *three-term recurrence* if for all positive integers $i$, the polynomial $p_i(x)$ has degree $i$ and there exist coefficients $b_{i-1}, a_i$, and $c_{i+1}$ with $b_i c_{i+1} > 0$ satisfying

$$xp_i(x) = b_{i-1}p_{i-1}(x) + a_i p_i(x) + c_{i+1}p_{i+1}(x)\,. \tag{2.1.1}$$

Similarly, for some positive integer $n$, we can define a finite sequence of polynomials $p_0, \ldots, p_n$ satisfying a three-term recurrence so long as $b_0, \ldots, b_{n-2}, a_0, \ldots, a_{n-1}$, and $c_1, \ldots, c_n$ satisfy $b_i c_{i+1} > 0$ and Equation 2.1.1.

By convention, we define $p_{-1}(x) = 0$. Then we can let $b_{-1} = 1$ and extend the three-term recurrence relation to hold for all non-negative integers $i$. Also by convention, we will assume that $p_0(x) = 1$.

*2.1.1 Example.* Let $k \geq 2$. We define a sequence of polynomials $F_i^k(x)_{i\geq 0}$ satisfying the three-term recurrence with $a_i = 0$ for all $i \geq 0$, $b_0 = k$, $b_i = k-1$ for all $i \geq 1$, and $c_i = 1$ for all $i \geq 1$. Then we have $F_0^k(x) = 1, F_1^k(x) = x$,

$$F_2^k(x) = xF_1^k(x) - kF_0^k(x)\,,$$

and, for $i \geq 1$, we have

$$F_{i+1}^k(x) = xF_i^k(x) - (k-1)\,F_{i-1}^k(x)\,.$$

Working out a few small examples, we see that

$$F_2^k(x) = x^2 - k,$$

$$F_3^k(x) = x^3 - (2k-1)\,x,$$

and

$$F_4^k(x) = x^4 - (3k-2)\,x^2 + k(k-1)\,.$$

This example will be useful for its relation to the the $k$-regular tree. The *$k$-regular tree* is an infinite tree where every vertex has valency $k$. We extend the notion of distance adjacency matrix to distance adjacency operators.

**2.1.2 Proposition.** *Let $(A_i)_{i\geq 0}$ be the distance adjacency operators of the $k$-regular tree for $k \geq 2$, and let $\left(F_i^k\right)_{i\geq 0}$ be defined as in Example 2.1.1. Then for $i \geq 0$, we have*

$$F_i^k(A_1) = A_i.$$

*Proof.* This is clearly true if $i = 0, 1$. Suppose by induction that $F_i^k(A) = A_i$ for some $i \geq 1$. We have

$$F_{i+1}^k(A) = AF_i^k(A) - b_{i-1}F_{i-1}^k(A) = AA_i - b_{i-1}A_{i-1}.$$

Let $u$ and $v$ be vertices with characteristic vectors $\mathbf{e}_u, \mathbf{e}_v$, respectively. Then

$$\mathbf{e}_u^T AA_i\mathbf{e}_v = \sum_{w\in V(G)} \mathbf{e}_u^T A\mathbf{e}_w\mathbf{e}_w^T A_i\mathbf{e}_v = \left|\{w \sim u : d(w,v) = i\}\right|,$$

is clearly zero unless $d(u,v) = i-1$ or $i+1$. If $d(u,v) = i+1$ then this quantity is equal to one, if $d(u,v) = i-1 \geq 1$, it equals $k-1$, and if $d(u,v) = 0 = i-1$, it is $k$. Thus

$$A_{i+1} = AA_i - b_{i-1}A_{i-1} = F_{i+1}(A)\,. \qquad \square$$

We will refer to $F_i^k(x)$ as the *sequence of polynomials associated to the $k$-regular tree.*

A connected graph $G$ of diameter $d$ with distance adjacency matrices $A_0, \ldots, A_d$ is *distance-regular* if there exists a sequence of polynomials $F_0^G(x), F_1^G(x), \ldots, F_{d+1}^G(x)$ such that, for $0 \leq i \leq d+1$, the polynomial $F_i^G(x)$ has degree $i$ and $F_i^G(A) = A_i$.

*2.1.3 Remark.* Distance-regular graphs are often defined, as they were in Chapter 1, as connected graphs where, for any pair of vertices $u$ and $v$, the number of vertices at distance $i$ from $u$ and distance $j$ from $v$ depends only on the distance between $u$ and $v$, and not on the specific choice of vertices $u$ and $v$.

Another common definition is that a connected graph of diameter $d$ is distance-regular if, for any $0 \leq i \leq d$ and any pair of vertices $u$ and $v$ at distance $i$, the number of vertices adjacent to $v$ and distance $i-1, i$ and $i+1$ does not depend on the choice of vertices $u$ and $v$.

These three definitions of distance-regular are equivalent and can be found, for instance, in Section 4. 1 of Brouwer, Cohen, and Neumaier [28].

We call the sequence $F_0^G, F_1^G, \dots, F_{d+1}^G$ the *distance polynomials* for $G$. The $k$-regular tree is an infinite distance-regular graph. Terwilliger [146] proved that $k$-regular trees are the only infinite distance-regular graphs, though the associated polynomials are still useful for finite graphs.

The sequence of polynomials associated to a $k$-regular tree was introduced by Singleton [138], though he did not refer to them that way. Rather, he used them to count non-backtracking walks. A walk on the vertices of the graph is *non-backtracking* if the sequence $uvu$ does not appear for any vertices $u, v$.

**2.1.4 Lemma** (Singleton [138]). *Let $G$ be a $k$-regular graph with adjacency matrix $A$. Then $\mathbf{e}_v^T F_i^k(A)\,\mathbf{e}_u$ counts the of non-backtracking walks of length $i$ from $u$ to $v$.* □

We will apply this result and use the sequence of polynomials associated to the $k$-regular tree to describe a family of finite distance-regular graphs.

**2.1.5 Proposition.** *If $G$ is a regular graph with diameter $d$ and girth $2d+1$, then $G$ is distance-regular.*

*Proof.* Let $k$ be the valency of $G$, and let $A_0, A_1, \dots, A_d$ be the distance adjacency matrices.

For $0 \le i \le d$ and vertices $u, v$, we have that $\mathbf{e}_u^T F_i^k(A)\,\mathbf{e}_v$ is zero unless $d(u,v) \le i$. Suppose otherwise, then if $d(u,v) = j < i$, then there exists a path from $v$ to $u$ of length $j$ and a non-backtracking walk from length $u$ to $v$ of length $i$. Concatenating them gives us a nontrivial closed walk of length $i+j < 2d+1$, which is a contradiction because cycles must have length at least $2d+1$. Similarly, if $d(u,v) = i$ and $\mathbf{e}_u^T F_i(A)\,\mathbf{e}_v \ge 2$, we can concatenate two non-backtracking walks from $u$ to $v$ to get a non-trivial closed walk of length $2i < 2d+1$, leading to another contradiction.

For $0 \le i \le d$, let $F_i^G(x) = F_i^k(x)$ and let

$$F_{d+1}^G(x) = (x-k+1)\,F_d^G(x) - (k-1)\,F_{d-1}^G(x)\,.$$

Then

$$F_{d+1}^G(A) = AA_d - (k-1)\,A_d - (k-1)\,A_{d-1}.$$

If $u, v$ are vertices at distance $d-1$ from each other, then we have

$$\begin{aligned} k &= |\{w : w \sim u\}| \\ &= \mathbf{e}_v^T AA_{d-2}\mathbf{e}_u + \mathbf{e}_v^T AA_{d-1}\mathbf{e}_u + \mathbf{e}_v^T AA_d\mathbf{e}_u \\ &= 1 + \mathbf{e}_v^T AA_d\mathbf{e}_u. \end{aligned}$$

Similarly, if $u, v$ are vertices at distance $d$ from each other, then we have

$$k = 1 + \mathbf{e}_v^T AA_d\mathbf{e}_u.$$

Thus

$$F_{d+1}^G(A) = AA_d - (k-1)\,A_d - (k-1)\,A_{d-1} = \mathbf{0} = A_{d+1},$$

and so $G$ is distance-regular. □

A graph with diameter $d$ and girth $2d+1$ is called a *Moore graph*. A counting argument, such as the one that can be found in Section 5.8 of the book by Godsil and Royle [84] shows that all Moore graphs are regular, and therefore distance-regular.

*2.1.6 Remark.* Let $G$ be a $k$-regular graph with diameter $d$. Then a simple counting argument gives us the *Moore bound*

$$|V(G)| \leq 1 + \sum_{i=0}^{d-1} k(k-1)^i .$$

By fixing an arbitrary vertex $u$ in a Moore graph and counting the number of vertices at distance $i$ from $u$, we see that Moore graphs meet this bound. Further, any graph that meets this bound must have girth at least $2d+1$, and so Moore graphs are precisely the graphs meeting the Moore bound. The Moore bound, and a spectral variant, will be considered more thoroughly in Chapter 5.

## 2.2 Generalized Polygons

Let $G = (\beta \cup \gamma, E)$ denote a bipartite graph with cells of the partition $\beta$ and $\gamma$. It is $(k,\ell)$-*semiregular* if all vertices in $\beta$ have valency $k$, and all vertices in $\gamma$ have valency $\ell$. Let $\pi \in \{\beta, \gamma\}$. The *characteristic matrix* $\mathbf{E}_\pi$ of $\pi$ is the $|V(G)| \times |\pi|$ matrix whose columns are the characteristic vectors of the vertices in $\pi$.

Analogously to the regular tree, we define the $(k,\ell)$-semiregular tree to be the infinite tree where every vertex has valency $k$ or $\ell$, a vertex with valency $k$ is only adjacent to vertices of valency $\ell$, and a vertex of valency $\ell$ is only adjacent to vertices of valency $k$. As we did with the regular tree, we can define an associated sequence of polynomials satisfying a three-term recurrence.

*2.2.1 Example.* Let $k, \ell \geq 2$. For all $i \geq 0$, let $a_i = 0$. Let $b_0 = k$, and for $i \geq 0$ let $b_{2i+1} = \ell - 1$ and $b_{2i+2} = \ell - 1$. Let $c_i = 1$ for all $i \geq 0$. We define $\left(F_i^{k,\ell}(x)\right)_{i \geq 0}$ to be the sequence of polynomials satisfying the three-term recurrence with these coefficients. In other words, this is the sequence of polynomials defined by $F_0^{k,\ell}(x) = 1, F_1^{k,\ell}(x) = x, F_2^{k,\ell}(x) = x^2 - k$ and, for $i \geq 0$,

$$F_{2i+1}^{k,\ell}(x) = xF_{2i}^{k,\ell}(x) - (\ell - 1)\, F_{2i-1}^{k,\ell}(x)$$

$$F_{2i+2}^{k,\ell}(x) = xF_{2i+1}^{k,\ell}(x) - (k-1)\, F_{2i}^{k,\ell}(x)\,.$$

We can compute that

$$F_3^{k}(x) = x^3 - (k + \ell - 1)\, x,$$

and

$$F_4^{k}(x) = x^4 - (2k + \ell - 2)\, x^2 + k(k-1)\,.$$

Using these polynomials, we can extend some of our results from the previous section to bipartite semiregular graphs.

**2.2.2 Lemma.** *Let $G = (\beta \cup \gamma, E)$ be the $(k, \ell)$-semiregular tree with distance adjacency operators $(A_i)_{i\geq 0}$. Let $\left(F_i^k\right)_{i\geq 0}$ be defined as in Example 2.1.1. For all $i \geq 0$, we have*

$$F_i^{k,\ell}(A)\,\mathbf{E}_\beta = A_i\mathbf{E}_\beta.$$

*Proof.* Suppose by induction $F_i^{k,\ell}(A)\,\mathbf{E}_\beta = A_i\mathbf{E}_\beta$ for some $i \geq 1$. We have

$$AA_i\mathbf{E}_\beta = AF_i^{k,\ell}(A)\,\mathbf{E}_\beta = b_{i-1}F_{i-1}^{k,\ell}(A)\,\mathbf{E}_\beta + F_{i+1}^{k,\ell}(A)\,\mathbf{E}_\beta.$$

If $u \in \beta$ and $v$ is a vertex, then

$$\mathbf{e}_v^T AA_i\mathbf{e}_u = \sum_{w\in V(G)} \mathbf{e}_v^T A\mathbf{e}_w\mathbf{e}_w^T A_i\mathbf{e}_v = \left|\{w \sim v : d(u,w) = i\}\right|,$$

from which we see that

$$AA_i\mathbf{E}_\beta = b_{i-1}A_{i-1}\mathbf{E}_\beta + A_{i+1}\mathbf{E}_\beta,$$

so

$$F_{i+1}^{k,\ell}(A)\,\mathbf{E}_\beta = A_{i+1}\mathbf{E}_\beta.$$

A similar inductive argument gives us

$$F_i^{\ell,k}(A)\,\mathbf{E}_\gamma = A_i\mathbf{E}_\gamma.$$

□

A connected bipartite graph $G = (\beta \cup \gamma, E)$ of diameter $d$ with distance adjacency matrices $A_0, \ldots, A_d$ is *distance-biregular* if there exists two sequences of polynomials $F_0^\beta, \ldots, F_{d+1}^\beta$ and $F_0^\gamma, \ldots, F_{d+1}^\gamma$ such that for $0 \leq i \leq d+1$, and $\pi \in \{\beta, \gamma\}$ the polynomial $F_i^\pi(x)$ has degree $i$ and satisfies

$$F_i^\pi(A)\,\mathbf{E}_\pi = A_i\mathbf{E}_\pi.$$

The sequences $F_0^\beta, \ldots, F_{d+1}^\beta$ and $F_0^\gamma, \ldots, F_{d+1}^\gamma$ are the *distance polynomials* for $\beta$ and $\gamma$.

*2.2.3 Remark.* As with distance-regular graphs, distance-biregular graphs can be defined in several different ways. Some of these equivalences can be found in [110].

The definition of distance-regular graphs used by Delorme [55] and Godsil and Shawe-Taylor [85] is that a connected bipartite graph of diameter $d$ is distance-regular if, for any $0 \leq i \leq d$ and any pair of vertices $u$ and $v$ at distance $i$, the number of vertices adjacent to $v$ and distance $i-1$ and $i+1$ depends only on the cell of the partition that $u$ lies in. In Section 2.4, we will justify why that is equivalent to the definition in terms of distance polynomials given here.

Analogously to distance-regular graphs, we could also consider the connected bipartite graphs where, for any pair of vertices $u$ and $v$ and any $i, j \geq 0$, the number of vertices at distance $i$ from $u$ and $v$ depends only on the distance between $u$ and $v$ and the cell of the partition that $u$ lies in. We will expand on why this is equivalent to our definition of distance-biregular graphs in Section 6.2.

Building on the work of Terwilliger [146], Delorme [55, 56] and Shawe-Taylor [134] proved that semiregular trees are the only infinite distance-biregular graphs. As with the distance-regular case, the polynomials associated to a semiregular tree are related to finite graphs with large girth, since when evaluated at the adjacency matrix of a finite semiregular bipartite graph $G$, the entries count the number of non-backtracking walks in $G$.

**2.2.4 Lemma.** *Let $G = (\beta \cup \gamma, E)$ be a $(k, \ell)$-semiregular bipartite graph with adjacency matrix $A$. Then if $u, v$ are vertices with $u \in \beta$, the entry $\mathbf{e}_v^T F_i^{k,\ell}(A)\, \mathbf{e}_u$ is the number of non-backtracking walks of length $i$ from $u$ to $v$.*

*Proof.* The statement is trivial for $i = 0, 1$. For $i = 2$ we note that

$$\mathbf{e}_v^T F_2^{k,\ell}(A)\, \mathbf{e}_u = \mathbf{e}_v^T \left(A^2 - kI\right) \mathbf{e}_u$$

is zero unless $v = u$ or vertices $u$ and $v$ are at distance two. Any walk of length two between vertices at distance two is non-backtracking, and the number of closed backtracking walks of length two from $u$ to itself is the valency of $u$, which is $k$. Thus we may assume by induction that the statement holds for some $2i \geq 2$, some $u \in \beta$, and all vertices $v$.

Let $v$ be a vertex. We have

$$\begin{aligned} \mathbf{e}_v^T F_{2i+1}^{k,\ell}(A)\, \mathbf{e}_u &= \mathbf{e}_v^T A F_{2i}^{k,\ell}(A)\, \mathbf{e}_u - (\ell - 1)\, \mathbf{e}_v^T F_{2i-1}^{k,\ell}(A)\, \mathbf{e}_u \\ &= \sum_{w \sim v} \mathbf{e}_w^T F_{2i}^{k,\ell}(A)\, \mathbf{e}_u - (\ell - 1)\, \mathbf{e}_v^T F_{2i-1}^{k,\ell}(A)\, \mathbf{e}_u. \end{aligned}$$

By the inductive hypothesis, $\mathbf{e}_v^T F_{2i-1}^{k,\ell}(A)\, \mathbf{e}_u$ counts the number of non-backtracking walks of length $2i - 1$ from $u$ to $v$. Let $W$ be such a non-backtracking walk. Each of the $\ell - 1$ neighbours of $v$ that is not the penultimate vertex in $W$ can be appended to the end of $W$, followed by $v$, to create a walk of length $2i + 1$ from $u$ to $v$ that backtracks in precisely the last step. Since

$$\sum_{w \sim v} \mathbf{e}_w^T F_{2i}^{k,\ell}(A)\, \mathbf{e}_u$$

counts the walks that backtrack in at most the last step, we have that $\mathbf{e}_v^T F_{2i+1}^{k,\ell}(A)\, \mathbf{e}_u$ counts the non-backtracking walks of length $2i + 1$ from $u$ to $v$. A similar computation holds to compute $F_{2i+2}^{k,\ell}$, so by induction we get the desired result. $\square$

**2.2.5 Corollary.** *Let $G = (\beta \cup \gamma, E)$ be a $(k, \ell)$-semiregular graph with adjacency matrix $A$. Then if $u, v$ are vertices with $u \in \gamma$, the entry $\mathbf{e}_v^T F_i^{\ell,k}(A)\, \mathbf{e}_u$ is the number of non-backtracking walks of length $i$ from $u$ to $v$.*

*2.2.6 Remark.* Let $G$ be a graph with adjacency matrix $A$, and let $D$ be the diagonal matrix indexed by the vertices of $G$, with $\mathbf{e}_u^T D \mathbf{e}_u$ equal to the valency of $u$. For $r \geq 0$, let $M_i$ be the matrix indexed by vertices with $\mathbf{e}_v^T M_i \mathbf{e}_u$ equal to the number of non-backtracking walks of length $i$ from $u$ to $v$. Chan and Godsil [31] proved that $M_i$ can be written as a polynomial in $A$ and $D$, where $A$ has degree $i$. This generalizes the result of Singleton [138] in Lemma 2.1.4.

When $G$ is regular, then $D$ is a polynomial of degree 0 in $A$, and so the result of Chan and Godsil [31] simplifies to a polynomial in the adjacency matrix of degree $i$. A similar simplification happens when considering a semiregular bipartite graph and restricting to a single cell of the partition. Thus Lemma 2.2.4 and Corollary 2.2.5 are special cases of the result of Chan and Godsil [31], rewritten in a form more analogous to the result of Singleton [138].

As we did in the regular case, we use these sequences of polynomials to obtain finite distance-biregular graphs.

*2.2.7 Example.* Let $G = (\beta \cup \gamma, E)$ be a bipartite $(k, \ell)$-semiregular graph with diameter $d$ and girth $2d$. Let $A_0, A_1, \ldots, A_d$ be the distance adjacency matrices.

Let $u \in \beta$. For $0 \leq i \leq d$ and vertex $v$, we have that $\mathbf{e}_v^T F_i^{k,\ell}(A)\, \mathbf{e}_u$ is one if $d(u, v) = i$ and zero otherwise, since anything else would create a cycle of length strictly less than $2d$. An analogous argument holds for $u \in \gamma$ and $F_i^{\ell,k}$. As such, for $0 \leq i \leq d$, we define

$$F_i^{\beta}(x) = F_i^{k,\ell}(x)$$

and

$$F_i^{\gamma}(x) = F_i^{\ell,k}(x)\,.$$

Let $b_d^{\beta} = k$ if $d$ is even and $\ell$ if $d$ is odd, and let $b_d^{\gamma} = \ell$ if $d$ is even and $k$ if $d$ is odd. Then for $\pi \in \{\beta, \gamma\}$, we define

$$F_{d+1}^{\pi}(x) := x F_d^{\pi}(x) - b_d^{\pi} F_{d-1}^{\pi}(x)\,.$$

If $u$ and $v$ are at distance $d$ from each other, then every neighbour of $u$ must be at distance $d-1$ from $v$. This tells us that

$$F_{d+1}^{\pi}(A)\, \mathbf{E}_{\pi} = \mathbf{0} = A_{d+1} \mathbf{E}_{\pi},$$

so $G$ is distance-biregular.

Bipartite graphs with girth twice the diameter are called *generalized polygons*, and were introduced by Tits [152] in the study of groups of Lie type.

Even cycles are trivial examples of generalized polygons. Given a graph $G$, the *$k$-fold subdivision* $G'$ is the graph obtained by replacing every edge in $G$ with a path of length $k$. If $G$ is a generalized polygon with diameter $d$, then the $k$-fold subdivision $G'$ has diameter $kd$ and girth $2kd$, and is therefore also a generalized polygon. Even cycles and $k$-fold subdivisions all have minimum degree two. We exclude these constructions by considering *thick* generalized polygons, with minimum degree at least three.

Yanushka [156] proved that a generalized polygon that is not thick is the $k$-fold subdivision of a multiple edge or the $k$-fold subdivision of a thick generalized polygon, and further, thick generalized polygons are semiregular. Feit and Higman [66] proved that any thick generalized polygon has diameter $d = 2, 3, 4, 6$ or 8. Infinite families of thick generalized polygons exist for all these viable options for $d$. More information can be found in the surveys of Payne [126], Payne and Thas [129], Thas [149], and Van Maldeghem [153].

Moore graphs and generalized polygons both arose from families of polynomials satisfying a three-term recurrence, but the definition of distance-regular and distance-biregular graphs did not require that. In fact, the distance polynomials will always satisfy a three-term recurrence, as we can see through further development of the theory.

## 2.3 Orthogonal Polynomials

In this section, we wish to prove an equivalence between sequences of polynomials satisfying a three-term recurrence described earlier, and the orthogonal polynomials in the title of this section and this thesis. In particular, we wish to prove the following:

**2.3.1 Theorem.** *Let $(p_i)_{i\geq 0}$ be a sequence of polynomials such that $p_i$ has degree $i$ for all non-negative integers $i$. The following are equivalent:*

*(a) For any non-negative integer $i$, there exist coefficients $b_{i-1}, a_i, c_{i+1}$ with $b_i c_{i+1}$ positive such that*

$$xp_i(x) = b_{i-1}p_{i-1}(x) + a_i p_i(x) + c_{i+1}p_{i+1}(x)\,;$$

*(b) There exists a non-decreasing function $\alpha(x)$ which is not constant on the interval $[a, b]$ such that*

$$\int_a^b p_i(x)\,p_j(x)\,d\alpha(x) = 0$$

*for any non-negative integers $i, j$ with $i \neq j$; and*

*(c) There exists an inner product on the vector space of polynomials $\langle , \rangle$ with the additional property that*

$$\langle xf, g\rangle = \langle f, xg\rangle$$

*such that*

$$\langle p_i, p_j\rangle = 0$$

*for any non-negative integers $i, j$ with $i \neq j$.* □

We say that a sequence of polynomials is a sequence of *orthogonal polynomials* if they satisfy (a), (b), or (c). Most often, we will define orthogonal polynomials in terms of the three-term recurrence, though sometimes it is will be easier to define orthogonal polynomials by giving an inner product satisfying (c).

Standard treatments of orthogonal polynomials such as Szegö or Nikiforov, Suslov, and Uvarov [121] begin by defining a sequence of polynomials satisfying (b) and developing other properties from there. Godsil [80] defines orthogonal polynomials as a sequence satisfying (c) and shows the equivalence to the other definitions from there. We begin with the three-term recurrence of (a), and working from there develop properties to show the equivalence of this definition to (b) and (c).

A key result for polynomials satisfying a three-term recurrence is the Christoffel-Darboux identity. Christoffel [32] was working with the Legendre polynomials, which satisfy the three-term recurrence

$$xp_n(x) = \frac{n}{2n+1}p_{n-1}(x) + \frac{n+1}{2n+1}p_{n+1}(x)\,,$$

and Darboux [51] proved their namesake identity in a more general context. For some $n \geq 0$ and $0 \leq i \leq n$, we define

$$\hat{b}_{i,n} = \begin{cases} \prod_{j=i+1}^{n} \frac{b_j}{c_{j+1}} & i \leq n-1 \\ 1 & i = n \end{cases}.$$

**2.3.2 Theorem** (Christoffel-Darboux)**.** *Let $(p_i)_{i\geq 0}$ be a sequence of polynomials satisfying a three-term recurrence. Then*

$$\frac{p_n(x)\,p_{n+1}(y) - p_n(y)\,p_{n+1}(x)}{y-x} = \frac{1}{c_{n+1}}\sum_{i=0}^{n}\hat{b}_{i,n}p_i(x)\,p_i(y)\,.$$

*Proof.* We have

$$p_1(x) = \frac{1}{c_1}(x - a_0)\,,$$

so

$$\frac{p_0(x)\,p_1(y) - p_0(y)\,p_1(x)}{y-x} = \frac{1}{c_1}p_0(x)\,p_0(y)\,.$$

We can inductively assume that for $n \geq 1$ we have

$$\frac{p_{n-1}(x)\,p_n(y) - p_{n-1}(y)\,p_n(x)}{y-x} = \frac{1}{c_n}\sum_{i=0}^{n-1}\hat{b}_{i,n-1}p_i(x)\,p_i(y)\,.$$

Using the three-term recurrence and the inductive hypothesis, we compute

$$\begin{aligned}
\frac{p_n(x)\,p_{n+1}(y) - p_n(y)\,p_{n+1}(x)}{y-x} &= \frac{p_n(x)}{c_{n+1}(y-x)}\big((y-a_n)\,p_n(y) - b_{n-1}p_{n-1}(y)\big) \\
&\quad - \frac{p_n(y)}{c_{n+1}(y-x)}\big((x-a_n)\,p_n(x) - b_{n-1}p_{n-1}(x)\big) \\
&= \frac{1}{c_{n+1}} p_n(x)\,p_n(y) + \frac{b_{n-1}}{c_{n+1}}\left(\frac{1}{c_n}\sum_{i=0}^{n-1}\hat{b}_{i,n-1}p_i(x)\,p_i(y)\right) \\
&= \frac{1}{c_{n+1}} p_n(x)\,p_n(y) + \frac{1}{c_{n+1}}\sum_{i=0}^{n-1}\hat{b}_{i,n}p_i(x)\,p_i(y) \\
&= \frac{1}{c_{n+1}}\sum_{i=0}^{n}\hat{b}_{i,n}p_i(x)\,p_i(y)\,.
\end{aligned}$$

□

Taking the limit of both sides as $y$ approaches $x$ gives us a well-known corollary.

**2.3.3 Corollary.** *Let $(p_i)_{i\geq 0}$ be a sequence of polynomials satisfying a three-term recurrence. Then*

$$p_n(x)\,p'_{n+1}(x) - p'_n(x)\,p_{n+1}(x) = \frac{1}{c_{n+1}}\sum_{i=0}^{n}\hat{b}_{i,n}p_i(x)^2\,.$$

**2.3.4 Corollary.** *Let $(p_i)_{i\geq 0}$ be a sequence of polynomials satisfying a three-term recurrence. Then for $n \geq 1$, the roots of $p_{n+1}$ are simple, and distinct from the roots of $p_n$.*

*Proof.* Suppose that $\theta$ were a multiple root of $p_{n+1}(x)$. Then $\theta$ is also a root of $p'_{n+1}(x)$, and so by Corollary 2.3.3, we have

$$\begin{aligned}
0 &= c_{n+1}\big(p_n(\theta)\,p'_{n+1}(\theta) - p'_n(\theta)\,p_{n+1}(\theta)\big) \\
&= \sum_{i=0}^{n}\hat{b}_{i,n}p_i(\theta)^2 \\
&\geq p_0(\theta)^2 \\
&> 0.
\end{aligned}$$

The same contradiction holds if $\theta$ is a root of $p_n$ and $p_{n+1}$. □

To prove his namesake identity for Legendre polynomials, Christoffel [32] developed a version of discrete orthogonality, though he remarked that the identity also followed directly from the three-term recurrence. Following the approach of Godsil [80] in Section 8. 4, we use our direct proof of the Christoffel-Darboux formula of Theorem 2.3.2 to establish a version of discrete orthogonality for polynomials satisfying a three-term recurrence.

Let $\theta_1 > \cdots > \theta_n$ be the zeros of $p_n(x)$. For $k = 1, \ldots, n$, we define the *Christoffel numbers*

$$\alpha_{n,k} := \frac{1}{c_n} \hat{b}_{0,n-1} \frac{1}{p_{n-1}(\theta_k)\, p_n'(\theta_k)}.$$

By Corollary 2.3.4, we know that they are well-defined, and by Corollary 2.3.3, we know that they are positive.

**2.3.5 Theorem** (Discrete Orthogonality [80])**.** *Let $(p_i)_{i\geq 0}$ be a sequence of polynomials satisfying a three-term recurrence. Let $\theta_1 > \cdots > \theta_n$ be the zeros of $p_n(x)$, and for $k = 1, \ldots, n$, let $\alpha_{n,k}$ be the Christoffel number. If $0 \leq i, j < n$, then*

$$\frac{1}{\sqrt{\hat{b}_{0,i}\hat{b}_{0,j}}} \sum_{m=0}^{n-1} \alpha_{n,m} p_i(\theta_m)\, p_j(\theta_m) = c_n \delta_{i,j}.$$

*Proof.* Let $U$ be the $n \times n$ matrix with entries defined by

$$U_{i,j} = \sqrt{\alpha_{n,j}} \frac{p_i(\theta_j)}{\sqrt{\hat{b}_{0,i}}}.$$

We wish to show that $U$ is orthogonal.

For $0 \leq i, j \leq n-1$, we compute

$$\begin{aligned}
\left(U^T U\right)_{i,j} &= \sum_{k=0}^{n-1} U_{k,i} U_{k,j} \\
&= \sum_{k=0}^{n-1} \sqrt{\alpha_{n,i}} \frac{p_k(\theta_i)}{\sqrt{\hat{b}_{0,k}}} \sqrt{\alpha_{n,j}} \frac{p_k(\theta_j)}{\sqrt{\hat{b}_{0,k}}} \\
&= \sqrt{\alpha_{n,i}\alpha_{n,j}} \sum_{k=0}^{n-1} \frac{1}{\hat{b}_{0,k}} p_k(\theta_i)\, p_k(\theta_j) \\
&= \frac{1}{\sqrt{p_{n-1}(\theta_i)\, p_n'(\theta_i)\, p_{n-1}(\theta_j)\, p_n'(\theta_j)}} \frac{1}{c_n} \sum_{k=0}^{n-1} \hat{b}_{k,n-1} p_k(\theta_i)\, p_k(\theta_j).
\end{aligned}$$

If $i \neq j$, then Theorem 2.3.2 tells us that

$$\left(U^T U\right)_{i,j} = \frac{p_{n-1}(\theta_i)\, p_n(\theta_j) - p_{n-1}(\theta_j)\, p_n(\theta_i)}{\sqrt{p_{n-1}(\theta_i)\, p_n'(\theta_i)\, p_{n-1}(\theta_j)\, p_n'(\theta_j)}(\theta_i - \theta_j)} = 0,$$

and Corollary 2.3.3 tells us

$$\left(U^T U\right)_{i,i} = \frac{p_{n-1}(\theta_i)\, p_n'(\theta_i) - p_{n-1}'(\theta_i)\, p_n(\theta_i)}{p_{n-1}(\theta_i)\, p_n'(\theta_i)} = 1.$$

Thus

$$U^T U = I.$$

Since $UU^T = I$, it follows that

$$\begin{aligned}\delta_{ij} &= \left(UU^T\right)_{ij} \\ &= \sum_{k=0}^{n-1} \sqrt{\alpha_{n_k}} \frac{p_i(\theta_k)}{\sqrt{\hat{b}_{0,i}}} \sqrt{\alpha_{n_k}} \frac{p_j(\theta_k)}{\sqrt{\hat{b}_{0,j}}} \\ &= \frac{1}{\sqrt{\hat{b}_{0,i}\hat{b}_{0,j}}} \sum_{k=0}^{n-1} \alpha_{n,k} p_i(\theta_k)\, p_j(\theta_k)\,.\end{aligned}$$

□

We are now ready to sketch a proof of Theorem 2.3.1.

*Proof.* We define a discrete measure using the Christoffel numbers. Specifically, if $S \subseteq \mathbb{R}$, we define

$$\mu_n(S) := \sum_{\theta_k \in S} \alpha_{n,k}.$$

If $0 \le i, j \le n-1$, we have

$$\int p_i p_j \mu_n = \sum_{k=0}^{n-1} \alpha_{n,k} p_i(\theta_k)\, p_j(\theta_k) = \hat{b}_{0,i} \delta_{ij}.$$

This gives us an interval and a discrete measure for any finite sequence of polynomials satisfying a three-term recurrence. Given an infinite sequence satisfying a three-term recurrence, the measures converge, and therefore the sequence is an infinite sequence of orthogonal polynomials. This result is often attributed to Favard [65], although he was far from the first to prove it. A number of mathematicians working in different contexts proved some version of this claim, beginning with Stieltjes [140] in the context of continued fractions. Independent proofs of "Favards's Theorem" were developed by, among others, Natanson [120], Shohat [135], and Stone [141].

For a fixed non-decreasing function $\alpha(x)$ which is not constant on the interval $[a, b]$, we can define an inner product by

$$\langle f, g \rangle_\alpha = \int_a^b f(x)\, g(x)\, d\alpha(x)\,.$$

This satisfies the additional property that

$$\langle xf, g \rangle_\alpha = \langle f, xg \rangle_\alpha. \tag{2.3.1}$$

Given an inner product with the additional property of Equation 2.3.1, we can use the Gram-Schmidt process to define a sequence of polynomials $p_0, p_1, \ldots$ orthogonal with respect to this inner product such that for $i \geq 0$, the polynomial $p_i$ has degree $i$. Note that $p_0, \ldots, p_i$ is a basis for the vector space of polynomials of degree at most $i$. Then since $xp_i(x)$ is a polynomial of degree $i+1$, we can write it as a linear combination of $p_0, \ldots, p_{i+1}$. If $0 \leq j \leq i+1$, we take the inner product to see that

$$\langle xp_i, p_j \rangle = \langle p_i, xp_j \rangle$$

must be zero unless $j + 1 \geq i$. Thus there exist coefficients $b_{i-1}, a_i$, and $c_{i+1}$ such that

$$xp_i(x) = b_{i-1}p_{i-1}(x) + a_i p_i(x) + c_{i+1}(x)\,.$$

Further,

$$b_{i-1}\|p_{i-1}\|^2 = \langle xp_i, p_{i-1} \rangle = \langle p_i, xp_{i-1} \rangle = c_i \|p_i\|^2,$$

so

$$\frac{b_{i-1}}{c_i} = \frac{\|p_i\|^2}{\|p_{i-1}\|^2} > 0$$

and $p_0, p_1, \ldots$ satisfies a three-term recurrence. □

*2.3.6 Remark.* Additional restrictions can be placed on the sequence of orthogonal polynomials. For instance, Szegö [145] assumes that the sequence is orthonormal, and Godsil [80] assumes that the sequence is monic. We place no such restrictions. Further, we wish to emphasize the coefficients of the three-term recurrence, since we will be working with classes of graphs and inner products where they have a nice combinatorial interpretation. This leads to a different formulation of results than is found in the standard texts, though the proofs are largely the same.

*2.3.7 Example.* The sequence of polynomials $\left(F_i^k(x)\right)_{i \geq 0}$ associated to the $k$-regular tree is orthogonal with respect to the inner product defined on the interval

$$\left[-2\sqrt{k-1}, 2\sqrt{k-1}\right]$$

with weight function

$$\alpha(x) = \frac{k\sqrt{4(k-1) - x^2}}{2\pi(k^2 - x^2)}.$$

This was computed explicitly in Chapter 4 of Hora and Obata [99], building off work of Kesten [106] studying walks on groups. McKay [115] derived equivalent results to study the eigenvalues of large regular graphs, and Section 4.5 of Karlin [105] obtained a similar expression using stochastic walks.

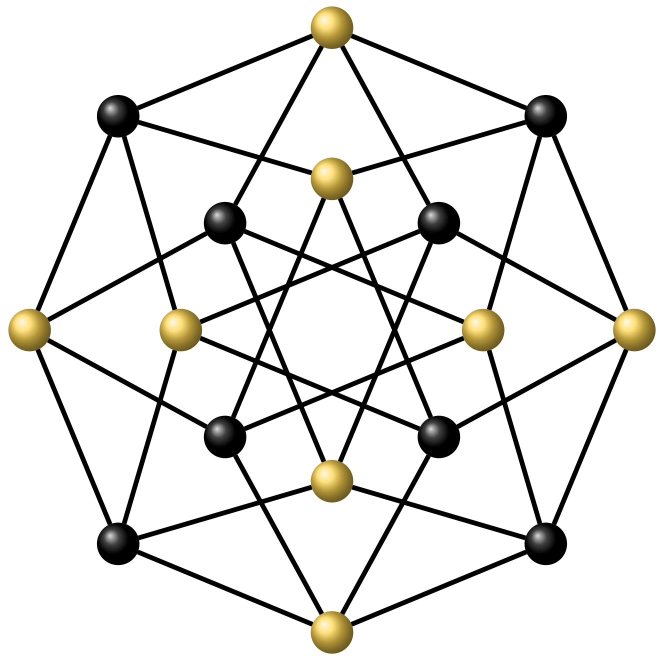

Figure 2.1: Hypercube graph $Q_4$

## 2.4 Locally Distance-Regular

We wish to apply the theory of orthogonal polynomials to distance-regular and distance-biregular graphs. It will be useful to work with a local version of distance-regularity, since this unifies the two classes. Note that a graph can be both distance-regular and distance-biregular, such as the hypercube graph in Figure 2.1.

Recall that the *eccentricity* of a vertex is the maximum distance from that vertex to the other vertices in the graph. Let $G$ be a graph, and let $u$ be a vertex of $G$ with eccentricity $e$. We say that $u$ is *locally distance-regular* if there exists a sequence of polynomials $F_0^u, \ldots, F_{e+1}^u$ such that for all $0 \le i \le e+1$, the polynomial $F_i^u(x)$ has degree $i$ and satisfies

$$F_i^u(A)\,\mathbf{e}_u = A_i \mathbf{e}_u.$$

If a graph is distance-regular, then every vertex is locally distance-regular with the same sequence of local distance polynomials. If a graph is distance-biregular, then every vertex in the same cell of the partition has the same sequence of local distance polynomials. It seems like this definition could extend further to an arbitrary partition of the vertices into sets of locally distance-regular vertices with the same sequence of local distance polynomials. However, it turns out that if every vertex in a graph is distance-regular, this forces the graph to be regular or bipartite semiregular. This leads to the following result of Godsil and Shawe-Taylor [85].

**2.4.1 Theorem** (Godsil and Shawe-Taylor [85]). *Let $G$ be a graph such that every vertex is locally distance-regular. Then $G$ is either distance-regular or distance-biregular.*

Distance-biregular graphs are the subject of this thesis, and Theorem 2.4.1 is one of the motivations for studying them, since it shows they naturally extend the well-studied class of distance-regular graphs. We will give a new proof of a stronger version of Theorem 2.4.1 in Section 4.3.

Results about locally distance-regular vertices translate to both distance-regular and distance-biregular graphs.

Let $G$ be a graph with adjacency matrix $A$ and vertex $u$. We define a local inner product relative to $u$ by

$$\langle f, g\rangle_u = \mathbf{e}_u^T f(A)\, g(A)\, \mathbf{e}_u.$$

Note that since $A$ is symmetric, polynomials of $A$ commute, and therefore this satisfies the additional property of Equation 2.3.1.

**2.4.2 Lemma.** *If $u$ is locally distance-regular, the local distance polynomials are orthogonal with respect to the $u$-inner product.*

*Proof.* Let $e$ be the eccentricity of $u$, and let $F_0, \ldots, F_{e+1}$ be the local distance polynomials with respect to $u$. For $0 \le i, j \le e$ we have

$$\begin{aligned}
\langle F_i, F_j\rangle_u &= \mathbf{e}_u^T F_i(A)\, F_j(A)\, \mathbf{e}_u \\
&= \mathbf{e}_u A_i A_j \mathbf{e}_u \\
&= \begin{cases} |\{v \in V(G) : d(u,v) = i\}| & i = j \\ 0 & i \neq j \end{cases}.
\end{aligned}$$

Thus $F_0, \ldots, F_{e+1}$ are a sequence of orthogonal polynomials with respect to the given inner product. $\square$

We can use this inner product to define a graph inner product and bipartite inner product. For a graph $G$, we define

$$\langle f, g\rangle_G = \frac{1}{|V(G)|} \sum_{u \in V(G)} \langle f, g\rangle_u = \frac{1}{|V(G)|} \operatorname{tr}(f(A)\, g(A)).$$

If $G = (\beta \cup \gamma, E)$ is bipartite and $\pi \in \{\beta, \gamma\}$, we define

$$\langle f, g\rangle_\pi = \frac{1}{|\pi|} \sum_{u \in \pi} \langle f, g\rangle_u.$$

Since the distance polynomials for a distance-regular or distance-biregular graph are orthogonal, we can define them entirely in terms of the three-term recurrence.

If $G$ is a distance-regular graph with diameter $d$, then for any $0 \le i \le d$, there exist coefficients $b^*_{i-1}, a^*_i, c^*_{i+1}$ such that

$$AF^G_i(A) = b^*_{i-1}F^G_{i-1}(A) + a^*_i F^G_i(A) + c^*_{i+1}F^G_{i+1}(A)\,.$$

We are not too concerned with the specific values of $b^*_{-1}$ and $c^*_{d+1}$ as long as they are positive. The coefficients $b^*_0, \ldots, b^*_{d-1}, a^*_0, \ldots, a^*_d, c^*_1, \ldots, c^*_d$ are the *intersection coefficients* of the distance-regular graph $G$. They have a combinatorial interpretation, since if $u, v$ are at distance $i$, we have

$$b^*_i = b^*_i \mathbf{e}_u^T A_i \mathbf{e}_v = \mathbf{e}_u^T A A_{i+1} \mathbf{e}_v = |\{w \sim u : d(u,v) = i+1\}|\,,$$

and similarly for $a^*_i$ and $c^*_i$.

Thus

$$b^*_i + a^*_i + c^*_i = |\{w : w \sim v\}| = b^*_0.$$

Since we can derive $a^*_i$ from $b^*_0, b^*_i$, and $c^*_i$, this allows us to compactly write the intersection coefficients in the *intersection array* $\left(b^*_0, \ldots, b^*_{d-1}; c^*_1, \ldots, c^*_d\right)$.

Let $G = (\beta \cup \gamma, E)$ be a distance-biregular graph with $\pi \in \{\beta, \gamma\}$. The *covering radius* $d_\pi$ of $\pi$ is the maximum eccentricity of vertices of $\pi$. Then for any $0 \le i \le d_\pi$, there exist coefficients $b^\pi_{i-1}, c^\pi_{i+1}$ such that

$$AF^\pi_i(A)\,\mathbf{E}_\pi = b^\pi_{i-1}F^\pi_{i-1}(A)\,\mathbf{E}_\pi + c^\pi_{i+1}F^\pi_{i+1}(A)\,\mathbf{E}_\pi.$$

*2.4.3 Remark.* These coefficients for distance-biregular graphs also have a combinatorial interpretation. Let $G = (\beta \cup \gamma, E)$ be a distance-biregular graph. If $u \in \beta$ and $v$ is a vertex at some distance $i$ from $u$, then the number of vertices adjacent to vertex $v$ and at distance $i+1$ from $u$ is $b^\beta_i$, and is thus independent of the choice of vertices $u, v$. Similar arguments hold for $c^\beta_i, b^\gamma_i$, and $c^\gamma_i$.

Conversely, suppose that for any vertices $u$ and $v$ at distance $i$, the numbers of vertices adjacent to $v$ and at distance $i-1$ and $i+1$ from $u$ depends only on $i$ and the cell of the partition that $u$ lies in. Then for $\pi \in \{\beta, \gamma\}$ and $0 \le i \le d_\pi$, there exist coefficients $b^\pi_{i-1}, c^\pi_{i+1}$ such that

$$AA_i\mathbf{E}_\pi = b^\pi_{i-1}A_{i-1}\mathbf{E}_\pi + c^\pi_{i+1}A_{i+1}\mathbf{E}_\pi.$$

We can use this to inductively define sequences of distance polynomials for each cell of the partition. This establishes the equivalence of some of the notions of distance-biregular graphs discussed in Remark 2.2.3.

We can derive $b^\pi_i$ from $b^\beta_0, b^\gamma_0$, and $c^\pi_i$. Following the notation of Delorme [55, 56], we represent these coefficients in the *intersection array* for the distance-biregular graph

$$\begin{vmatrix} k; & c^\beta_1, & , \ldots, & c^\beta_{d_\beta} \\ \ell; & c^\gamma_1, & , \ldots, & c^\gamma_{d_\gamma} \end{vmatrix}. \tag{2.4.1}$$

This is a slight abuse of notation– more accurately the expression in Equation 2.4.1 represents the two intersection arrays of a distance-biregular graph. However, it is more convenient to refer to them collectively as the intersection array, and when we wish to speak of one sequence of intersection coefficients in particular, we will refer to it as the line of the intersection array.

**2.4.4 Lemma.** *Let $u$ be a locally distance-regular vertex of eccentricity $e$, and let $F_0, \dots, F_{e+1}$ be the sequence of local distance polynomials satisfying*

$$xF_i(x) = b_{i-1}F_{i-1}(x) + a_iF_i(x) + c_{i+1}F_{i+1}(x)$$

*for $0 \leq i \leq e$. Let $k_i$ be the number of vertices at distance $i$ from $u$. Then $k_0 = 1$ and for $0 \leq i \leq e-1$, we have*

$$k_{i+1} = \frac{b_i k_i}{c_{i+1}}. \tag{2.4.2}$$

*Proof.* Let $S_i$ be the set of vertices at distance $i$ from $u$. Consider the edges between $S_i$ and $S_{i+1}$. Since $u$ is locally distance-regular, then every vertex in $S_i$ is adjacent to $b_i$ vertices at $S_{i+1}$, and every vertex in $S_{i+1}$ is adjacent to $c_{i+1}$ vertices in $S_i$. Thus we have

$$b_i \left|S_i\right| = c_{i+1} \left|S_{i+1}\right|$$

or

$$k_{i+1} = \frac{b_i k_i}{c_{i+1}}.$$

□

## 2.5 Spectral Decomposition

For distance-regular graphs, the distance adjacency matrices are polynomials of the adjacency matrix, so the eigenvalues of the distance adjacency matrix are determined by the eigenvalues of the adjacency matrix. Similar relationships exist for locally distance-regular vertices and distance-biregular graphs. It will be useful to work with the spectrum of the adjacency matrix, and one of the primary tools to do that is the spectral decomposition.

Let $G$ be a graph with adjacency matrix $A$. Since $A$ is symmetric, it is diagonalizable, so there exists a diagonal matrix $D$ and an invertible matrix $L$ such that

$$A = LDL^{-1}.$$

The diagonal entries of $D$ are the eigenvalues of $A$. Let $\theta_0 > \theta_1 > \cdots > \theta_t$ be the distinct eigenvalues of $A$. Then there exist 01-diagonal matrices $D_0, \dots, D_t$ such that

$$\sum_{r=0}^{t} D_r = I$$

and

$$\sum_{r=0}^{t} \theta_r D_r = D.$$

For $0 \le r \le t$, let

$$E_r = L D_r L^{-1}.$$

Then we have

$$A = LDL^{-1} = \sum_{r=0}^{t} \theta_r L D_r L^{-1} = \sum_{r=0}^{t} \theta_r E_r.$$

Further,

$$\sum_{r=0}^{t} E_r = L\left(\sum_{r=0}^{t} E_r\right) L^{-1} = LL^{-1} = I.$$

For $0 \le r, s \le t$, we have

$$E_r E_s = L D_r L^{-1} L D_s L^{-1} = L D_r D_s L^{-1} = \begin{cases} E_r & r = s \\ 0 & r \ne s \end{cases}.$$

Thus $E_0, \dots, E_t$ represent orthogonal projections. These are in fact orthogonal projections into the eigenspaces. Recall that we have $AL = LD$, and therefore

$$AE_r = ALD_r L^{-1} = LDD_r L^{-1} = L\theta_r D_r L^{-1} = \theta_r E_r.$$

We will refer to $E_r$ as the *spectral idempotent* for $\theta_r$. Since the spectral idempotents are pairwise orthogonal idempotent matrices, for any polynomial $f$ we have

$$f(A) = \sum_{r=0}^{t} f(\theta_r)\, E_r,$$

so the spectral decomposition allows us to evaluate polynomials of a matrix by evaluating the function at the eigenvalues of the matrix.

There are several particularly important consequences of the spectral decomposition. The *multiplicity* $m_r$ of $\theta_r$ is the trace of the spectral idempotent for $\theta_r$, since

$$\operatorname{tr}(E_r) = \operatorname{tr}\big(L D_r L^{-1}\big) = \operatorname{tr}\big(D_r L L^{-1}\big) = \operatorname{tr}(D_r) = m_r.$$

We also have that the spectral idempotents are polynomials of $A$, since for a fixed eigenvalue $\theta_r$, the polynomial

$$f(x) := \prod_{s \ne r} \frac{x - \theta_s}{\theta_r - \theta_s}$$

satisfies $f(A) = E_r$.

Spectral decomposition is a key tool in algebraic graph theory, and more information can be found in references such as Section 8. 12 of Godsil and Royle [84]. With a slight abuse of notation, when we refer to the *eigenvalues* of a graph $G$, we mean the eigenvalues of the adjacency matrix of $G$.

The *spectrum* of a graph $G$ is the multi-set of eigenvalues with their multiplicities. If $\theta_0 > \cdots > \theta_t$ are the eigenvalues of $G$ with multiplicities $m_1, \ldots, m_t$, we write the spectrum

$$\left\{\theta_0^{(m_0)}, \ldots, \theta_t^{(m_t)}\right\}.$$

Of particular importance is the largest eigenvalue and its corresponding idempotent. Key to its importance is the following result.

**2.5.1 Theorem** (Perron-Frobenius)**.** *Let $G$ be a connected graph with largest eigenvalue $\rho$. Then $\rho$ has multiplicity one, and there is an eigenvector with all positive entries. Further, $G$ is bipartite if and only if $-\rho$ is an eigenvalue.*

*2.5.2 Remark.* Perron [130] and Frobenius [78] were working in the more general context of non-negative matrices. Since we are only considering the adjacency matrices of graphs, we restrict our statement of the Perron-Frobenius theorem to this context. The work of Perron [130] can be interpreted as saying that the largest eigenvalue of a graph is simple if there exists some $i$ such that there is a walk of length $i$ between any two vertices of the graph. Frobenius [78] extended Perron's result to all connected graphs, characterizing bipartite graphs in the process.

For a connected graph $G$, we will let $\rho$ be the largest eigenvalue and the *Perron vector* $\mathbf{p}$ be an eigenvector for $\rho$ with norm one and all positive entries.

## 2.6 Spectral Decomposition of a Bipartite Graph

We will be working primarily with bipartite graphs in this thesis, so it is worth establishing some of the spectral structure of bipartite graphs.

We can think of a bipartite graph as an *incidence graph* of some incidence structure. If $G = (\beta \cup \gamma, E)$ is a bipartite graph, then $\beta$ are the *points* and $\gamma$ are the *blocks*. It will sometimes be convenient to think of the blocks as subsets of the points, and the incidence relation as containment.

We can represent the incidence relation by the *biadjacency matrix* $N$ from $\beta$ to $\gamma$. This is the $|\beta| \times |\gamma|$ matrix where the $(u, v)$ entry is one if $u$ is contained in $v$ and zero otherwise. We can write the adjacency matrix of $G$ as a block matrix of the form

$$A = \begin{pmatrix} 0 & N \\ N^T & 0 \end{pmatrix}.$$

*2.6.1 Example.* An incidence structure is a $2-(v,k,\lambda)$-*design* if it has $v$ points, every block contains $k$ points, and any pair of points are contained in exactly $\lambda$ blocks. We can fix a point $u$ and let $r_u$ be the number of blocks incident to $u$. Then by counting the collinear points to $u$ and the blocks containing both points, we have

$$r_u = \frac{\lambda(v-1)}{k-1},$$

and so the number of points incident to a block is independent of the choice of block. Thus the incidence graph of a 2-design is semiregular.

We will denote the valency of the blocks by $r$, and the number of blocks by $b$. It is not always true that every pair of blocks intersect in the same number of points, so the incidence structure obtained by flipping points and blocks is not necessarily a design. Trivial designs are the 2-designs with incidence graph $K_{v,b}$.

We have already seen that

$$v(v-1)\,\lambda = v\ell(k-1)\,, \tag{2.6.1}$$

and by counting the number of incident point/blocks in two different ways, we see

$$bk = vr.$$

Let $N$ be the biadjacency matrix of a bipartite graph $G$. Then $G$ is the incidence graph of a 2-design if and only if

$$NN^T = (r-\lambda)\,I + \lambda J. \tag{2.6.2}$$

To see this, we consider points $u, v \in \beta$ and compute

$$\mathbf{e}_u^T NN^T \mathbf{e}_u = \left|\{w \in \gamma : u \sim w, w \sim v\}\right|.$$

The block structure of the adjacency matrix for a bipartite graph extends to the spectral decomposition. That is, if $\theta_r$ is an eigenvalue with spectral idempotent $E_r$, we can write

$$E_r = \begin{pmatrix} R_r & D_r \\ D_r^T & L_r \end{pmatrix}$$

for matrices $R_r, D_r$, and $L_r$.

Let $\theta_r$ be a nonzero eigenvalue with spectral idempotent

$$E_r = \begin{pmatrix} R_r & D_r \\ D_r^T & L_r \end{pmatrix}.$$

We have

$$\begin{pmatrix} \theta R_r & \theta D_r \\ \theta D_r^T & \theta L_r \end{pmatrix} = AE_r = \begin{pmatrix} 0 & N \\ N^T & 0 \end{pmatrix} \begin{pmatrix} R_r & D_r \\ D_r^T & L_r \end{pmatrix} = \begin{pmatrix} ND_r^T & NL_r \\ N^T R_r & N^T D_r \end{pmatrix}.$$

Then
$$\begin{pmatrix} 0 & N \\ N^T & 0 \end{pmatrix} \begin{pmatrix} R_r & -D_r \\ -D_r^T & L_r \end{pmatrix} = \begin{pmatrix} -ND_r^T & NL_r \\ N^TR_r & -N^TD_r \end{pmatrix} = \begin{pmatrix} -\theta R_r & \theta D_r \\ \theta D_r^T & -\theta L_r \end{pmatrix},$$
so the columns of
$$E_{-r} := \begin{pmatrix} R_r & -D_r \\ -D_r^T & L_r \end{pmatrix}$$
are eigenvectors for $-\theta_r$. We also have
$$\begin{pmatrix} R_r & D_r \\ D_r^T & L_r \end{pmatrix} = E_r = E_r^2 = \begin{pmatrix} R_r^2 + D_rD_r^T & R_rD_r + D_rL_r \\ D_r^TR_r + L_rD_r^T & D_r^TD_r + L_r^2 \end{pmatrix},$$
so
$$E_{-r}^2 = \begin{pmatrix} R_r^2 + D_rD_r^T & -R_rD_r - D_rL_r \\ -D_r^TR_r - L_rD_r^T & D_r^TD_r + L_r^2 \end{pmatrix} = \begin{pmatrix} R_r & -D_r \\ -D_r^T & L_r \end{pmatrix} = E_{-r}.$$
This tells us that $E_{-r}$ is the spectral idempotent of $-\theta_r$.

The following linear algebraic result will be useful. It can be found, for instance, in Section 10.3 of Godsil [80].

**2.6.2 Lemma.** *For any matrix $N$, the nonzero eigenvalues of $NN^T$ and $N^TN$ are the same, with the same multiplicity.* □

Suppose that $N$ is a $v \times b$ matrix. Note that $NN^T$ has $v$ eigenvalues and $N^TN$ has $b$ eigenvalues. By Lemma 2.6.2 we know that $NN^T$ and $N^TN$ share nonzero eigenvalues with multiplicities. In particular, if $b \geq v$, then 0 must be an eigenvalue of $N^TN$ with multiplicity at least $b - v$. This leads us to the following result, which generalizes results that can be found in Delorme [55, 56].

**2.6.3 Lemma.** *Let $G = (\beta \cup \gamma, E)$ be a semiregular bipartite graph with diameter $d$ and $d+1$ eigenvalues. If $d$ is odd, then $G$ is regular and the covering radii of $\beta$ and $\gamma$ are $d$.*

*Proof.* If $d$ is odd, then we know there are vertices $u \in \beta, v \in \gamma$ with $d(u, v) = d$, so the covering radii of both $\beta$ and $\gamma$ are $d$. Further, the number of distinct eigenvalues is even. Since the eigenvalues of a bipartite graph are symmetric about the real axis, the number of eigenvalues can only be even if 0 is not an eigenvalue. By Lemma 2.6.2, we have that 0 is an eigenvalue of $G$ with multiplicity at least $||\beta| - |\gamma||$, from which we see $|\beta| = |\gamma|$. Since $G$ is semiregular and both cells of the partition have the same valency, we conclude that $G$ must be regular. □

*2.6.4 Example.* We compute the spectrum of a 2-design. Let $G = (\beta \cup \gamma, E)$ be the incidence graph of a 2-design with biadjacency matrix $N$. We have
$$A^2 = \begin{pmatrix} NN^T & 0 \\ 0 & N^TN \end{pmatrix},$$

so we can compute the eigenvalues of $A$ by computing the eigenvalues of $NN^T$.

Equation 2.6.2 relates the eigenvalues of $NN^T$ to the eigenvalues of $J$ and $I$. The eigenvalues of $I$ are 1 with multiplicity $v$, and the eigenvalues of $J$ are $v$ with multiplicity one and 0 with multiplicity $v-1$. Thus $NN^T$ has $r-\lambda$ as an eigenvalue with multiplicity $v-1$ and $r-\lambda+\lambda v$ with multiplicity one. From Equation 2.6.1, we have

$$r-\lambda+\lambda v = r-\lambda(v-1) = r-r(k-1) = \ell k.$$

From Lemma 2.6.2 we know that $N^TN$ must have 0 as an eigenvalue with multiplicity $|\gamma|-|\beta|$. Thus the spectrum of $A^2$ is

$$\left\{(kr)^{(2)}, (r-\lambda)^{(2(v-1))}, 0^{(|\gamma|-|\beta|)}\right\}.$$

Since the eigenvalues of a bipartite graph are symmetric about the real axis, we conclude that $A$ has spectrum

$$\left\{\sqrt{kr}^{(1)}, \sqrt{r-\lambda}^{(v-1)}, 0^{(|\gamma|-|\beta|)}, -\sqrt{r-\lambda}^{(v-1)}, -\sqrt{r\ell}^{(1)}\right\}.$$

The incidence graph of a 2-design is a $(k,r)$-semiregular graph, and the largest eigenvalue is $\sqrt{kr}$. This is true more generally. Let $G=(\beta\cup\gamma,E)$ be a bipartite $(k,\ell)$-semiregular graph. Then $\sqrt{k\ell}$ is the largest eigenvalue with Perron vector

$$\mathbf{p} = \frac{1}{\sqrt{2\,|\beta|}}\begin{pmatrix}1\\ \vdots\\ 1\\ \sqrt{\frac{\ell}{k}}\\ \vdots\\ \sqrt{\frac{\ell}{k}}\end{pmatrix} = \frac{1}{\sqrt{2\,|\gamma|}}\begin{pmatrix}\sqrt{\frac{k}{\ell}}\\ \vdots\\ \sqrt{\frac{k}{\ell}}\\ 1\\ \vdots\\ 1\end{pmatrix}.$$

For positive integers $i,j$ let $\mathbf{1}_{i,j}$ denote the $i\times j$ matrix of all ones, and let $\mathbf{1}_i$ denote the $i\times i$ matrix of all ones. Then the spectral idempotent for $\sqrt{k\ell}$ is

$$\mathbf{p}\mathbf{p}^T = \begin{pmatrix}\frac{1}{2|\beta|}\mathbf{1}_{|\beta|} & \frac{\sqrt{\ell}}{2|\beta|\sqrt{k}}\mathbf{1}_{|\beta|,|\gamma|}\\ \frac{\sqrt{k}}{2|\gamma|\sqrt{\ell}}\mathbf{1}_{|\gamma|,|\beta|} & \frac{1}{2|\gamma|}\mathbf{1}_{|\gamma|}\end{pmatrix}.$$

**2.6.5 Lemma.** *If $E_r$ and $E_s$ are distinct spectral idempotents such that $E_r+E_s$ is zero on the off-diagonal blocks, then $\theta_s=-\theta_r$.*

*Proof.* Since $\theta_r$ and $\theta_s$ are not both zero, we may assume without loss of generality that $\theta_r\neq 0$, so $E_{-r}$ is distinct from $E_r$. Suppose it is also distinct from $E_s$. Since $E_r$ is orthogonal to $E_{-r}-E_s$, we have

$$\begin{pmatrix}R_r(R_r-R_s) & D_r(L_r-L_s)\\ D_r^T(R_r-R_s) & L_r(L_r-L_s)\end{pmatrix} = \begin{pmatrix}R_r & D_r\\ D_r^T & L_r\end{pmatrix}\begin{pmatrix}R_r-R_s & \mathbf{0}\\ \mathbf{0} & L_r-L_s\end{pmatrix} = \begin{pmatrix}\mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{pmatrix}.$$

Since $E_{-r}$ and $E_s$ are distinct, they are orthogonal, so

$$\begin{pmatrix} R_rR_s + D_rD_r^T & -R_rD_r - D_rL_s \\ -D_r^TR_s - L_rD_r^T & D_r^TD_r + L_rL_s \end{pmatrix} = \begin{pmatrix} R_r & -D_r \\ -D_r^T & L_r \end{pmatrix} \begin{pmatrix} R_s & -D_r \\ -D_r^T & L_s \end{pmatrix} = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}.$$

Since $E_r$ is idempotent, we have

$$\begin{pmatrix} R_r & D_r \\ D_r^T & L_r \end{pmatrix} = \begin{pmatrix} R_r^2 + D_rD_r^T & R_rD_r + D_rL_r \\ D_r^TR_r + L_rD_r^T & D_r^TD_r + L_r^2 \end{pmatrix}.$$

Comparing the top left block gives us

$$R_r = R_r^2 + D_rD_r^T = R_r^2 - R_rR_s = R_r(R_r - R_s) = \mathbf{0}.$$

Similarly, comparing the bottom right block gives us

$$L_r = L_r^2 + D_r^TD_r = L_r(L_r - L_s) = \mathbf{0}.$$

This implies that $E_r = \mathbf{0}$, which is impossible, and therefore $E_s = E_{-r}$. □

## 2.7 Multiplicities

For a distance-regular graph, the spectrum is particularly important. Biggs [21] proved that the intersection array determines the spectrum of a distance-regular graph, and Van Dam and Haemers [45] gave the first written proof that for a distance-regular graph, the spectrum determines the intersection array. In this section, we extend their results to distance-biregular graphs.

Godsil and Shawe-Taylor [85] worked out the multiplicities of a distance-biregular graph using a different method than the one used in this section. Their description was considerably expanded by Shawe-Taylor [134] in Chapter 3 of his thesis. Since we are already working with the distance polynomials of distance-biregular graphs, the adaptation here of the proof of Biggs [21] is more convenient for our purposes, and the adaptation of the result of Van Dam and Haemers [45] is new.

Let $G = (\beta \cup \gamma, E)$ be a bipartite graph and let $\theta_r$ be an eigenvalue. Let

$$m_r^\beta = \operatorname{tr}(R_r)$$

and

$$m_r^\gamma = \operatorname{tr}(L_r)\,.$$

We will refer to $m_r^\pi$ as the *bipartite multiplicities*, because they can be interpreted as the multiplicity relative to cells of the partition.

**2.7.1 Lemma.** *Let $G = (\beta \cup \gamma, E)$ be a bipartite graph with adjacency matrix $A$. If $\theta$ is a nonzero eigenvalue of $A$ with multiplicity $m_\theta$, then*

$$m_\theta^\beta = m_\theta^\gamma = \frac{1}{2} m_\theta.$$

*Further,*

$$m_0^\beta = \frac{1}{2}(|\beta| - |\gamma| + m_0)$$

*and*

$$m_0^\gamma = \frac{1}{2}(|\gamma| - |\beta| + m_0)\,.$$

*Proof.* Let $\theta$ be a nonzero eigenvalue of $A$. Note that $\theta^2$ is an eigenvalue of

$$A^2 = \begin{pmatrix} NN^T & 0 \\ 0 & N^T N \end{pmatrix}$$

with multiplicity $2m_\theta$, since the $-\theta$ eigenvalue of $A$ also contributes multiplicity $m_\theta$ to the multiplicity of $\theta^2$ in $A^2$. Using the spectral decomposition, we have

$$\begin{pmatrix} NN^T & 0 \\ 0 & NN^T \end{pmatrix} = \sum_{\theta_r} \theta_r^2 E_r = \sum_{\substack{\theta_r \\ \theta_r \geq 0}} \theta_r^2 (E_r + E_{-r}) = \sum_{\theta_r > 0} \theta_r^2 \begin{pmatrix} 2R_r & \mathbf{0} \\ \mathbf{0} & 2L_r \end{pmatrix}.$$

By Lemma 2.6.2 we know that $NN^T$ and $N^T N$ share nonzero eigenvalues with multiplicity. Thus

$$m_\theta^\beta = \operatorname{tr}(R_\theta) = \operatorname{tr}(L_\theta) = m_\theta^\gamma,$$

giving us

$$m_\theta^\beta = m_\theta^\gamma = \frac{1}{2} m_\theta.$$

If 0 is an eigenvalue, we have

$$m_0^\beta = |\beta| - \sum_{\theta \neq 0} m_\theta^\beta$$

and similarly for $m_0^\gamma$. Then we have

$$m_0^\beta - m_0^\gamma = |\beta| - |\gamma|$$

and

$$m_0^\beta + m_0^\gamma = m_0,$$

so

$$m_0^\beta = \frac{1}{2}(|\beta| - |\gamma| + m_0)\,,$$

and similarly for $m_0^\gamma$. □

**2.7.2 Theorem.** *In a distance-biregular graph $G$, the intersection array determines and is determined by the spectrum and the valencies.*

*Proof.* Let $G = (\beta \cup \gamma, E)$ be a distance-biregular graph with valencies $k, \ell$.

Let $f$ and $g$ be polynomials. We have that

$$\langle f, g\rangle_\beta = \frac{1}{|\beta|}\sum_{u\in\beta} \mathbf{e}_u^T f(A)\, g(A)\, \mathbf{e}_u = \sum_{r=0}^{d} f(\theta_r)\, g(\theta_r) \sum_{u\in\beta} \mathbf{e}_u^T E_r \mathbf{e}_u = \sum_{r=0}^{d} m_r^\beta f(\theta_r)\, g(\theta_r)\, .$$

An analogous result holds for $\gamma$. This shows that the inner product is determined by the bipartite multiplicities. These are determined by the multiplicities of the eigenvalues and the sizes of $\beta, \gamma$. Since $G$ is $(k, \ell)$-semiregular, we have

$$k\,|\beta| = \ell\,|\gamma|$$

and

$$|\beta| + |\gamma| = \sum_{r=0}^{d} m_r,$$

and so the sizes of $\beta, \gamma$ are also determined by the spectrum and valencies. Thus the spectrum and valencies determine the inner product.

Now consider the inner product $\langle f, g\rangle_\beta$. We can use the Gram-Schmidt process to obtain a unique sequence of orthogonal polynomials $p_0, \ldots, p_{d_\beta}$ normalized such that

$$\langle p_{2i}, p_{2i}\rangle_\beta = p_{2i}\left(\sqrt{k\ell}\right)$$

and

$$\langle p_{2i+1}, p_{2i+1}\rangle_\beta = \frac{\sqrt{k}}{\sqrt{\ell}} p_{2i+1}\left(\sqrt{k\ell}\right)$$

Since $G$ is distance-biregular, we know that there is also a sequence of distance polynomials $F_0, \ldots, F_{d_\beta}$ which are orthogonal with respect to $\langle , \rangle_\beta$. For any vertex $u \in \beta$, the distance polynomials satisfy

$$\langle F_i, F_i\rangle_\beta = \frac{1}{|\beta|}\sum_{u\in\beta} |\{v : d(u,v) = i\}| = \frac{1}{|\beta|}\sum_{u\in\beta} (A_i J)_{u,u}\, .$$

Since the spectral idempotent for $\sqrt{k\ell}$ is a polynomial of $A$, there exists some polynomial $p$ such that

$$p(A) = \begin{pmatrix} \mathbf{1}_{|\beta|,|\beta|} & \frac{\sqrt{\ell}}{\sqrt{k}}\mathbf{1}_{|\beta|,|\gamma|} \\ \frac{\sqrt{\ell}}{\sqrt{k}}\mathbf{1}_{|\gamma|,|\beta|} & \frac{2|\beta|}{2|\gamma|}\mathbf{1}_{|\gamma|,|\gamma|} \end{pmatrix}.$$

Then

$$\langle F_{2i}, F_{2i}\rangle_\beta = \langle F_{2i}, p\rangle_\beta = F_{2i}\left(\sqrt{k\ell}\right)$$

and
$$\langle F_{2i+1}, F_{2i+1}\rangle_\beta = \frac{\sqrt{k}}{\sqrt{\ell}}\langle F_{2i+1}, p\rangle_\beta = \frac{\sqrt{k}}{\sqrt{\ell}} F_{2i+1}\left(\sqrt{k\ell}\right),$$
and so the orthogonal polynomials defined using the spectrum are the distance polynomials. The same argument holds for $\gamma$.

Suppose conversely we have the intersection array. We compute the spectrum for one cell of the partition from the distance polynomials $F_0, \dots, F_{d+1}$ for that cell. First, note that the eigenvalues are the roots of $F_{d+1}$.

Let $\theta$ be an eigenvalue, and let $k_i$ be defined recursively as in Equation 2.4.2. We define
$$\psi(x) = \sum_{i=0}^{d} \frac{F_i(\theta)}{k_i} F_i(x)\,.$$

By the three-term recurrence, we have
$$\begin{aligned}
A\psi(A) &= \sum_{i=0}^{d} \frac{F_i(\theta)}{k_i} A F_i(A) \\
&= \sum_{i=0}^{d} \frac{F_i(\theta)}{k_i} (c_{i+1}F_{i+1}(A) + b_{i-1}F_{i-1}(A)) \\
&= \sum_{i=1}^{d+1} \frac{c_i F_{i-1}(\theta)}{k_{i-1}} F_i(A) + \sum_{i=-1}^{d-1} \frac{b_i F_{i+1}(\theta)}{k_{i+1}} F_i(A) \\
&= \sum_{i=0}^{d} \frac{(b_{i-1}F_{i-1}(\theta) + c_{i+1}F_{i+1}(\theta))}{k_i} F_i(A) \\
&= \sum_{i=0}^{d} \frac{\theta F_i(\theta)}{k_i} F_i(A) \\
&= \theta\psi(A)\,.
\end{aligned}$$

The columns of $\psi(A)$ are eigenvectors for $\theta$, and therefore they must be orthogonal to eigenvectors for any eigenvalue $\tau \neq \theta$. Then using the spectral decomposition, we get
$$\sum_{u\in\pi} \mathbf{e}_u^T \psi(A)\,\mathbf{e}_u = m_\theta^\pi \psi(\theta)\,.$$

On the other hand, we also have
$$\sum_{u\in\pi} \mathbf{e}_u^T \psi(A)\,\mathbf{e}_u = \sum_{u\in\pi}\sum_{i=0}^{d} \frac{F_i(\theta)}{k_i} \mathbf{e}_u^T F_i(A)\,\mathbf{e}_u = \sum_{u\in\pi}\sum_{i=0}^{d} \frac{F_i(\theta)}{k_i} \mathbf{e}_u^T A_i \mathbf{e}_u = \frac{F_0(\theta)}{k_i}\,|\pi|\,.$$

Therefore,

$$m_\theta^\pi = \frac{|\pi|}{\psi(\theta)},$$

which is determined by the sequence of distance polynomials. □

*2.7.3 Remark.* Theorem 2.7.2 needs the hypothesis that $G$ is distance-biregular.

If a graph $G$ is not distance-biregular, then the distance polynomials and intersection array are not well-defined. This is possible even if $G$ has the same spectrum as a distance-biregular graph. We consider this further in Section 4.7.

Theorem 2.7.2 can also be used to prove that a distance-biregular graph with a particular intersection array does not exist. For instance, from an intersection array, we can define sequences of orthogonal polynomials, and use that to determine the spectrum of a putative distance-biregular graph $G$. If $G$ exists, then the multiplicity of every eigenvalue must be a positive integer. This forms a strong necessary condition for the intersection array of a distance-biregular graph. We will look closer at such conditions in the next chapter.

# Chapter 3

# Distance-Biregular Graphs

Any three sequences of positive real coefficients define a sequence of polynomials satisfying a three-term recurrence. If this sequence of polynomials is the sequence of distance polynomials for some graph, then there are additional combinatorial properties the coefficients must satisfy. This is true for distance-regular graphs, and doubly true for distance-biregular graphs. This leads us to the motivating question for this chapter.

**3.0.1 Question.** *Given a potential intersection array, when does a distance-biregular graph with this intersection array exist?*

We are not expecting a full answer to this question, since the analogous question for distance-regular graphs is open, even for the low diameter case of strongly regular graphs. In general, the intersection array does not give us enough information to construct a distance-regular or distance-biregular graph. However, any construction of distance-biregular graphs, or any necessary properties that the intersection array must satisfy, gives a partial answer to Question 3.0.1.

In this chapter, we will look at some of the constructions of distance-biregular graphs coming from Delorme [55, 56] and Shawe-Taylor [134]. We will also consider *feasibility conditions*, a set of bare-minimum conditions that the intersection array must satisfy to avoid any obvious obstructions to a distance-biregular graph existing. We have already seen several feasibility condition for distance-biregular graphs, since Lemma 2.4.4 and Theorem 2.7.2 define quantities $k_i^\pi$ and $m_r^\pi$ which must be positive integers. We study further feasibility conditions, which we use to compute tables of feasible intersection arrays of low diameter and valency found in Appendix A and Appendix B.

## 3.1 Symmetric and Quasi-Symmetric 2-Designs

We begin by considering a well-known result in design theory, first shown by Fisher [76]

**3.1.1 Lemma** (Fisher's Inequality)**.** *Let $D$ be a $2-(v,k,\lambda)$ design with $b$ blocks. If $v > k$, then*

$$b \geq v.$$

*Proof.* Consider a $2-(v,k,\lambda)$ design with $b$ blocks where every point is incident to $r$ blocks. If $\lambda \neq r$, then 0 is not an eigenvalue of $NN^T$, so by Lemma 2.6.2, we must have that $b \geq v$. By Equation 2.6.1, we know that $\lambda = r$ precisely when $k = v$. □

Consider a $2-(v,k,\lambda)$ design with $b$ blocks where every point is incident to $r$ blocks. If any two blocks intersect in some number $\lambda'$ points, then the dual incidence structure obtained by flipping points and blocks is a $2-(b,r,\lambda')$ design. One consequence of Fisher's inequality is that if the dual of a design is a design, then $b = v$, and hence $\lambda' = \lambda$. We say that a 2-design is a *symmetric design* if $b = v$, or equivalently, if any two blocks have exactly $\lambda$ common points.

Cvetković, Doob, and Sachs [44] proved that a bipartite graph of diameter three is distance-regular if and only if it is the incidence graph of a symmetric 2-design. They used this to prove that a connected bipartite regular graph with four distinct eigenvalues is distance-regular. This extends the well-known result that can be found in, for instance, section 10.2 of Godsil and Royle [84] that a connected regular graph with diameter two is distance-regular if it has three distinct eigenvalues. However, for $d \geq 4$, there exist bipartite regular graphs with diameter $d$ and $d+1$ distinct eigenvalues which are not distance-regular.

By Lemma 2.6.3, a distance-biregular graph with odd diameter is regular, and thus the characterization of Cvetković, Doob, and Sachs [44] is also a characterization of distance-biregular graphs of diameter three. There is a similar equivalence between distance-biregular graphs of diameter four where one cell of the partition has covering radius three and a different class of 2-designs.

A weakening of the notion of symmetric 2-design gives *quasi-symmetric 2-designs*, 2-designs where any two blocks intersect in either $s$ or $t$ points. We are particularly interested in the case of quasi-symmetric 2-designs where $t = 0$, that is, any two blocks are either disjoint, or they intersect in $s$ points.

*3.1.2 Example.* Let $G$ be a distance-biregular graph with intersection array

$$\begin{vmatrix} r; & 1, & \lambda, & k & \\ k; & 1, & s, & c_3, & k \end{vmatrix}.$$

Then

$$\begin{pmatrix} \mathbf{1}_v \\ \mathbf{0} \end{pmatrix} = F_2^{\beta}(A)\,\mathbf{E}_\beta = \frac{1}{\lambda}\big(A^2 - rI\big)\,\mathbf{E}_\beta = \frac{1}{\lambda}\begin{pmatrix} NN^T - rI_v \\ \mathbf{0} \end{pmatrix},$$

so the incidence structure is a 2-design. There also exists some 01-matrix $Y_1$ such that

$$\begin{pmatrix} \mathbf{0} \\ Y_1 \end{pmatrix} = F_2^{\gamma}(A)\,\mathbf{E}_\gamma = \frac{1}{s}\big(A^2 - kI\big)\,\mathbf{E}_\gamma = \frac{1}{s}\begin{pmatrix} \mathbf{0} \\ N^TN - kI_b \end{pmatrix}$$

or

$$N^TN = sY_1 + kI_b.$$

This tells us that any two distinct blocks either share no common points, or they share $s$ common points, so $G$ must be the incidence graph of a quasi-symmetric 2-design.

The converse is also true.

**3.1.3 Lemma.** *The incidence graph of a quasi-symmetric $2-(v,k,\lambda)$ design with block intersection numbers $s, 0$ is distance-biregular.*

*Proof.* Let $G$ be the incidence graph, let $b$ be the number of blocks, $r$ the valency of the points, and $s$ the size of the intersection of two non-disjoint blocks. We define

$$F_2^{\beta}(x) = \frac{1}{\lambda}\big(x^2 - r\big),$$

and

$$F_2^{\gamma}(x) = \frac{1}{s}\left(x^2 - k\right).$$

We can see from the definition of our quasi-symmetric 2-design that for $\pi \in \{\beta, \gamma\}$ we have

$$F_2^{\pi}(A)\,\mathbf{E}_{\pi} = A_2\mathbf{E}_{\pi}.$$

Let

$$Y_1 = \frac{1}{s}\left(N^T N - kI\right),$$

and note that it is the adjacency matrix of some graph $H$ with vertex set $\gamma$.

Let

$$F_3^{\beta}(x) = \frac{1}{k\lambda}\left(x^3 - (r + k\lambda)\,x\right) = F_3^{\gamma}(x).$$

Note that

$$F_3^{\gamma}(A)\,\mathbf{E}_{\gamma} = \frac{1}{k}AF_2^{\beta}(A)\,\mathbf{E}_{\gamma} - A\mathbf{E}_{\gamma} = \frac{1}{k}\begin{pmatrix} N^T\mathbf{1}_b \\ \mathbf{0} \end{pmatrix} - \begin{pmatrix} N^T \\ \mathbf{0} \end{pmatrix} = A_3\mathbf{E}_{\gamma}.$$

Further, since $A$ and $A_3$ are symmetric, we must have

$$F_3^{\gamma}(A) = A_3,$$

so $F_3^{\gamma}$ is indeed the distance-three polynomial.

We define

$$F_4^{\beta}(x) = (x - kr)(x - r + \lambda).$$

Recall the spectrum of $NN^T$ is

$$\left\{(kr)^{(1)}, (r-\lambda)^{(v-1)}\right\},$$

from which we see

$$F_4^{\beta}(A)\,\mathbf{E}_{\beta} = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \end{pmatrix} = A_4\mathbf{E}_{\beta}.$$

The graph $H$ is a connected graph of diameter two with three distinct eigenvalues. This means $H$ is distance-regular. Letting $Y_2$ be the distance-two graph of $H$, we know there exist constants $k^*, a^*, c^*$ with $c^*$ positive such that

$$Y_1^2 = k^* I + a^* Y_1 + c^* Y_2.$$

We define

$$F_4^{\gamma} = \frac{1}{c^*}\left(F_2^{\gamma}(x)^2 - a^* F_2^{\gamma}(x) - k^*\right),$$

and note it satisfies

$$F_4^{\gamma}(A)\,\mathbf{e}_{\gamma} = \begin{pmatrix} Y_2 \\ \mathbf{0} \end{pmatrix} = A_4\mathbf{E}_{\gamma}.$$

Finally, $G$ has five distinct eigenvalues, so we can define $F_5^{\beta} = F_5^{\gamma}$ to be the minimal polynomial of $G$. $\square$

The connection between distance-biregular graphs and quasi-symmetric 2-designs was previously shown by Delorme [55, 56] and Shawe-Taylor [134] using a counting argument. This new proof using orthogonal polynomials foreshadows ideas which we expand on in Section 3.2 and Section 3.5.

*3.1.4 Example.* A *Steiner system* $S(2,k,v)$ is a 2-design with $v$ points where every block is incident to $k$ points and every pair of points intersect in a unique block. Note that the size of the intersection of two blocks must be at most one, and therefore the incidence graph of a Steiner system is distance-biregular with intersection array

$$\begin{vmatrix} \frac{v-1}{k-1}; & 1, & 1, & k & \\ k; & 1, & 1, & k, & k \end{vmatrix}.$$

Steiner systems are a well-studied class of combinatorial designs, and more information can be found, for instance, in the overviews by Beth, Jungnickel, and Lenz [19], Colbourn and Mathon [40], and Colbourn and Rosa [41]. Of particular note are *Steiner triple systems*, $S(2,3,v)$, which Kirkman [107] proved exist if and only if $v \cong 1,3 \pmod 6$.

A *Hadamard matrix* $H$ of order $n$ is an $n \times n$ matrix with entries in $\{-1,1\}$ such that $HH^T = nI$. It is in *standardized form* if the first row and column are all positive.

A common construction gives a bipartite distance-regular graph of diameter four from a Hadamard matrix. However, there is a second construction of distance-biregular graphs, as noted by Delorme [55, 56].

*3.1.5 Example.* Let $H$ be a Hadamard matrix of order $4n$ in standardized form. Deleting the first row and column and replacing every $-1$ entry with 0, we get the incidence matrix of a symmetric $2-(4n-1, 2n-1, n-1)$ design. If instead we replace every 1 with a 0, we get the incidence matrix of a symmetric $2-(4n-1, 2n, n)$ design.

Let $N$ be the incidence matrix of the $2-(4n-1, 2n-1, n-1)$-design, so $J-N$ is the incidence matrix of the complement 2-design. Wallis [154] observed that

$$\begin{pmatrix} N & J-N \\ \mathbf{1} & \mathbf{0} \end{pmatrix}$$

is the incidence matrix of a $2-(4n, 4n-1, 2n-1)$-design, since

$$\begin{pmatrix} N & J-N \\ \mathbf{1} & \mathbf{0} \end{pmatrix} \begin{pmatrix} N^T & \mathbf{1}^T \\ J-N^T & \mathbf{0}^T \end{pmatrix} = \begin{pmatrix} 2nI + (2n-1)\,J & (2n-1)\,\mathbf{1}^T \\ (2n-1)\,\mathbf{1} & 4n-1 \end{pmatrix}.$$

Further, for a block $b$, there is a unique block that is disjoint from it, and every other block shares $n$ points. Thus it is a quasi-symmetric 2-design, which gives us a distance-biregular graph with the array

$$\begin{vmatrix} 4n-1; & 1, & 2n-1, & 2n & \\ 2n; & 1, & n, & 4n-2, & 2n \end{vmatrix}.$$

The graph for the Hadamard matrix of order eight is shown in Figure 3.1.

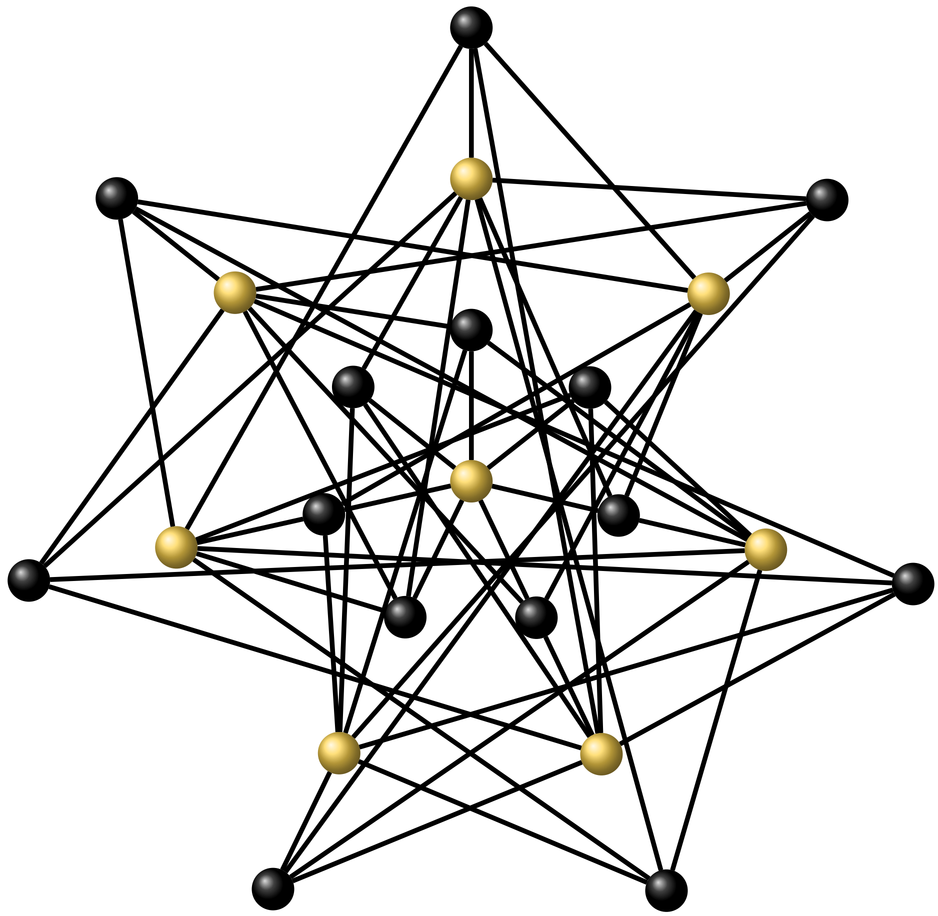

Figure 3.1: Quasi-symmetric 2-design for $8 \times 8$ Hadamard matrix

More information on quasi-symmetric 2-designs can be found in the overviews written by Shrikhande [136] or Shrikhande and Sane [137], and and more information about the connection to distance-biregular graphs can be found in Chapter 5 of Shawe-Taylor's thesis [134].

## 3.2 Even Distance Polynomials

Let $G = (\beta \cup \gamma, E)$ be a bipartite graph of diameter $d$. Since $G$ is bipartite, we can write the distance two graph $G_2$ as the disjoint union of the subgraphs of $G_2$ induced by the cells of the partition, called the *halved graphs*. Then for any $i$, we see that $G_{2i}$ is the disjoint union of the $i$th-distance graph of the halved graphs. If $X_0, \ldots, X_{\lfloor \frac{d}{2} \rfloor}$ are the distance matrices of the halved graph induced by $\beta$, and $Y_0, \ldots, Y_{\lfloor \frac{d}{2} \rfloor}$ are the distance matrices of the halved graph induced by $\gamma$, we have

$$A_{2i} = \begin{pmatrix} X_i & \mathbf{0} \\ \mathbf{0} & Y_i \end{pmatrix}.$$

*3.2.1 Remark.* Shawe-Taylor [118, 134] called the connected components of the distance-two graph the *derived graphs*. Brouwer, Cohen, and Neumaier [28] referred to these graphs as the halved graphs, because in the case of regular bipartite graphs each component has half the vertices of the original graph. Although the name is less accurate for semiregular graphs, we will maintain consistency with the now standard terminology of Brouwer, Cohen, and Neumaier. For an incidence structure, the halved graph induced by the point set is the *point graph*, and the halved graph induced by the set of blocks is the *block graph*.

In the proof of Lemma 3.1.3, we used the fact that the block graph of a quasi-symmetric 2-design is strongly regular. More generally, it is true that the halved graphs of a distance-biregular graph are distance-regular. This was shown by Delorme [55, 56] and Mohar and Shawe-Taylor [118]. We prove it in a new way through orthogonal polynomials.

Let $G = (\beta \cup \gamma, E)$ be an infinite distance-biregular graph. Fix a cell of the partition $\pi$ and consider the associated distance polynomials $(F_i^{\pi})_{i \geq 0}$.

*3.2.2 Remark.* In fact, the only infinite distance-biregular graphs are the biregular trees, since Delorme [55, 56] and Shawe-Taylor [134] generalized the proof of Terwilliger [146] to bound the diameter of a distance-biregular graph by its girth. The distance polynomials of biregular trees are key to the results in Chapter 5, and by working with an arbitrary distance-biregular graph of infinite diameter, we lay the groundwork for both those results and the finite-diameter case.

For $i \geq 0$, we define the *even distance polynomials* associated to the $(k, \ell)$-semiregular tree $P_i^{k,\ell}(x)$ by

$$P_i^{k,\ell}(x^2) = F_{2i}^{k,\ell}(x).$$

Since $G$ is distance-biregular, we may let $Z_i$ be the distance matrix of the halved graph induced by vertices of valency $k$ and write

$$P_i(NN^T) = Z_i. \tag{3.2.1}$$

Letting $P_{-1}(x) = 0$, for $i \geq 0$ we have

$$\begin{aligned} x^2 P_i(x^2) &= x(xF_{2i}(x)) \\ &= xb_{2i-1}F_{2i-1}(x) + xc_{2i+1}F_{2i+1} \\ &= b_{2i-1}b_{2i-2}F_{2i-2}(x) + (b_{2i-1}c_{2i} + c_{2i+1}b_{2i})\, F_{2i}(x) + c_{2i+1}c_{2i+2}F_{2i+2}(x) \\ &= b_{2i-1}b_{2i-2}P_{i-1}(x^2) + (b_{2i-1}c_{2i} + c_{2i+1}b_{2i})\, P_i(x^2) + c_{2i+1}c_{2i+2}P_{i+1}(x^2) \end{aligned}$$

and so $(P_i)_{i\geq 0}$ is a sequence of orthogonal polynomials.

Let $H$ be the halved graph induced by $\pi$. By Equation 3.2.1 when $i = 1$, we have constants $c, k$ such that

$$NN^T = cA(H) + k. \tag{3.2.2}$$

For $i \geq 0$, we define

$$F_i^*(x) = P_i\left(\frac{1}{c}(x-k)\right),$$

and note that

$$F_i^*(A(H)) = P_i\big(NN^T\big) = Z_i,$$

and so $H$ is distance-regular.

This extends to finite distance-biregular graphs as well. If $G$ is a distance-biregular graph and $F_1^\pi, \ldots, F_d^\pi$ are the distance polynomials for a cell of the partition $\pi$, we can define

$$P_i^\pi(x) := F_{2i}^\pi\big(x^2\big).$$

The caveat is that if $G$ is distance-biregular with diameter $d$, we cannot define

$$P_{d+1}(x) = F_{2d+2}(x).$$

However, with a bit of caution, we can define $P_{d+1}$ differently to give us the required polynomial.

**3.2.3 Theorem.** *Let $G$ be a distance-biregular graph with cell $\pi$ of covering radius $d$ and intersection coefficients $b_0, \ldots, b_{d-1}, c_0, \ldots, c_d$. Let $d^* = \lfloor \frac{d}{2} \rfloor$. Then the halved graph induced by $\pi$ is distance-regular with intersection array*

$$\left(\frac{b_0 b_1}{c_2}, \frac{b_1 b_2}{c_2}, \ldots, \frac{b_{2d^*-2} b_{2d^*-1}}{c_2}; \frac{c_1 c_2}{c_2}, \frac{c_3 c_4}{c_2}, \ldots, \frac{c_{2d^*-1} c_{2d^*}}{c_2}\right).$$

*Proof.* Let $F_0, \ldots, F_{d+1}$ be the sequence of polynomials associated to $\pi$, and let $d' = \lfloor \frac{d+1}{2} \rfloor$. For $0 \leq i \leq d'$, we define

$$P_i\big(x^2\big) = F_{2i}(x).$$

For $0 \leq i \leq d' - 1$, we define

$$b_{i-1}^* = \frac{b_{2i-1} b_{2i-2}}{c_2},$$

$$a_i^* = \frac{b_{2i-1} c_{2i} + c_{2i+1} b_{2i} - b_0}{c_2},$$

and

$$c_{i+1}^* = \frac{c_{2i+1} c_{2i+2}}{c_2}.$$

For $0 \le i \le d' - 1$, the polynomial $F_{2i+2}(x)$ is well-defined, so we use the three-term recurrence to compute

$$\begin{aligned} P_1(x)\,P_i(x) &= \frac{1}{c_2}(x - b_0)\,P_i(x) \\ &= \frac{1}{c_2}\left(b_{2i-1}b_{2i-2}P_{i-1}(x) + (b_{2i-1}c_{2i} + c_{2i+1}b_{2i} - b_0)\,P_i(x) + c_{2i+1}c_{2i+2}P_{i+1}(x)\right) \\ &= b^*_{i-1}P_{i-1}(x) + a^*_i P_i(x) + c^*_{i+1}P_{i+1}(x)\,. \end{aligned}$$

For $0 \le i \le d^*$, the matrix $P_i\big(NN^T\big)$ is the $i$-th distance adjacency matrix of the halved graph. Note that the diameter of the halved graph is $d^*$. If $d^* \le d' - 1$, we have found all the distance polynomials for the halved graph, so we are done. Otherwise, $d$ is even so if we fix a vertex $u \in \pi$, the number of vertices at distance $d$ from $u$ in $G$ is the same as the number of vertices at distance $\frac{d}{2}$ in the halved graph. Then by Lemma 2.4.4, we have

$$\frac{b_0 \cdots b_{d-1}}{c_1 \cdots c_d} = \frac{b^*_0 \cdots b^*_{d^*-1}}{c^*_1 \cdots c^*_{d^*}} = \frac{b^*_{d^*-1}}{c^*_{d^*}} \frac{b_0 b_1 \cdots b_{d-4} b_{d-3}}{c_1 c_2 \cdots c_{d-3} c_{d-2}}.$$

Note that $c^*_d = \frac{c_{d-1}c_d}{c_2}$, so

$$b^*_{d^*-1} = \frac{b_{d-2}b_{d-1}}{c_{d-1}c_d} c^*_d = \frac{b_{d-2}b_{d-1}}{c_2}.$$

□

Delorme [55, 56] and Mohar and Shawe-Taylor [118] also proved that the halved graph of a distance-biregular graph is distance-regular. Mohar and Shawe-Taylor explicitly computed the parameters of the halved graph, although there is an indexing error in their computation.

## 3.3 Distance-Regular Halved Graphs

Theorem 3.2.3 tells us that we can take a distance-biregular graph and obtain two distance-regular graphs from it. It is natural to ask when we can go the other direction.

**3.3.1 Problem.** *Given a distance-regular graph $H$, is $H$ the halved graph of a distance-biregular graph?*

*3.3.2 Remark.* A related problem is the square root of a graph, studied by Mukhopadhyay [119]. The *square* of a graph $G$ is the union of the $G$ and the distance-two graph $G_2$. The *square root* of $G$ is a subgraph $H$ whose square is $G$. Mukhopadhyay [119] characterized which graphs $G$ have a square root in terms of the complete subgraphs of $G$.

Every complete graph is the halved graph of a complete bipartite graph.

If $H$ is strongly regular, then one special case of this problem is when a strongly regular graph is the block graph of a quasi-symmetric 2-design. This problem was considered at length in Chapter 5 of Shawe-Taylor's thesis [134]. Another specific case of Problem 3.3.1 for strongly regular graphs predates the definition of distance-biregular graphs.

A *partial geometry* $\mathrm{pg}(s,t,\alpha)$ is an incidence structure in which each block is incident to $s+1$ points, each point is incident to $t+1$ blocks, any two points are in at most one block, and if a point $u$ and a block $x$ are not incident, then there are $\alpha$ coincident pairs $(v,y)$ such that $v$ is incident to $x$ and $y$ is incident to $u$.

Bose [25] introduced the notion of partial geometries to study strongly regular graphs. They can also be thought of as the distance-biregular graphs with intersection array

$$\begin{vmatrix} s+1; & 1, & 1, & \alpha, & s+1 \\ t+1; & 1, & 1, & \alpha, & t+1 \end{vmatrix}.$$

A $\mathrm{pg}(s,t,s+1)$ is a Steiner system, and $\mathrm{pg}(s,t,1)$ is a generalized quadrangle. Between these two extremes are *proper partial geometries*. There are constructions of infinite families given by De Clerck and Thas [151], De Clerck, Dye, and Thas [53], and Mathon [113], as well as sporadic examples by De Clerck [52], Haemers [89, 90], Van Lint and Schrijver [111], and Mathon [114]. More information on partial geometries can be found in the surveys of Brouwer and Van Lint [30], De Clerck and Van Maldeghem [54], or Thas [150].

Leaving the small diameter examples, we can also ask Question 3.3.1 about infinite diameter families of distance-biregular graphs. Examples that come up in the early work on distance-biregular graphs [55, 56, 85, 118, 134] are bipartite analogues to the Johnson and Grassmann graphs.

*3.3.3 Example.* Let $n \geq 2k+2$. The *BiJohnson graph* $BJ(n,k)$ is the bipartite graph $G=(\beta \cup \gamma, E)$ where $\beta$ is the set of $k$-element subsets of $[n]$, $\gamma$ is the set of $(k+1)$-element subsets of $[n]$, and $v \in \beta$ is adjacent to $w \in \gamma$ when $v \subset w$. The halved graphs of $G$ are the Johnson graph $J(n,k)$ and the Johnson graph $J(n,k+1)$.

Two vertices $u,v \in \beta$ are at distance $2i$ when $|u \cap v| = k-i$, and similarly, two vertices $u,v \in \gamma$ are at distance $2i$ when $|u \cap v| = k+1-i$. Combined with the adjacency rule, this tells us that $u \in \beta$ and $v \in \gamma$ are at distance $2i+1$ from each other when $|u \cap v| = k-i$. Thus $\beta$ has covering radius $2k+1$ and $\gamma$ has covering radius $2k+2$.

Fix a vertex $u$ and a vertex $v$ at distance $i$ from $u$. Let $\pi(u)$ denote the cell of the partition containing $u$. Using the distances just derived, we have

$$c_i^{\pi(u)} = |\{w \sim v : d(u,w) = i-1\}| = |u \setminus (u \cap v)| = \left\lfloor \frac{i}{2} \right\rfloor.$$

This gives us the intersection array

$$\begin{vmatrix} n-k; & 1, & 1, & 2, & \ldots, & p, & k, & k+1 & \\ k+1; & 1, & 1, & 2, & \ldots, & k, & k, & k+1 & k+1 \end{vmatrix}.$$

We can extend the definition of the BiJohnson graphs to $n = 2k+1$, and the resulting graphs are regular graphs of diameter $2k+1$. We are primarily interested in examples of distance-biregular graphs that are not distance-regular, and so we will omit a more detailed description.

*3.3.4 Example.* The BiGrassmann graphs are to the Grassmann graphs what the BiJohnson graphs are to the Johnson graphs. Formally, let $\mathbb{F}_q$ be a field with $q$ elements, and for $n \geq 2k+2$, let $V$ be the $n$-dimensional vector space over $\mathbb{F}_q$. The *BiGrassmann graph* $J_q(n,k)$ is a bipartite graph with $\beta$ the $k$-dimensional subspaces of $V$, $\gamma$ the $(k+1)$-dimensional subspaces of $V$, and $u \in \beta$ adjacent to $v \in \gamma$ when $u$ is a subspace of $v$. The halved graphs are Grassmann graphs $J_q(n,k)$ and $J_q(n,k+1)$.

Let

$$[i]_q = \frac{q^i - 1}{q-1},$$

and note that this counts the number of $(i-1)$-dimensional subspaces contained in a given $i$-dimensional subspace. Then an analogous counting argument used for the BiJohnson graphs gives us the intersection array

$$\left|\begin{matrix} [n-k]_q; & [1]_q, & [1]_q, & [2]_q, & \ldots, & [k]_q, & [k]_q, & [k+1]_q & \\ [k+1]_q; & [1]_q, & [1]_q, & [2]_q, & \ldots, & [k]_q, & [k]_q, & [k+1]_q, & [k+1]_q \end{matrix}\right|.$$

As before, we can extend the definition of BiGrassmann graphs to $n = 2k+1$, and the resulting graphs are regular with diameter $2k+1$.

In Chapter 6 of his thesis, Shawe-Taylor [134] considered Problem 3.3.1 for other families of distance-regular graphs. In particular, he proved that the Hamming graphs and dual-space polar forms graphs do not generally occur as halved graphs of distance-biregular graphs.

## 3.4 First Attempts at Feasibility

A working definition of feasibility for distance-regular graphs is as follows.

**3.4.1 Definition.** *An intersection array* $\left(b_0^*, b_1^*, \ldots, b_{d-1}^*; c_1^*, c_2^*, \ldots, c_d^*\right)$ *is feasible if:*

*(i) The intersection coefficients satisfy*

$$b_0^* > b_1^* \geq b_2^* \geq \cdots \geq b_{d-1}^* \geq b_d^*$$

*and*

$$1 = c_1^* \leq c_2^* \leq \cdots \leq c_d^*;$$

*(ii) The numbers $k^*_0, \ldots, k^*_d$ defined recursively by*

$$k^*_{i+1} = \frac{b^*_i k^*_i}{c^*_{i+1}}.$$

*are positive integers; and*

*(iii) The multiplicities of the eigenvalues, which can be computed from the intersection array, are positive integers.*

*3.4.2 Remark.* An intersection array being feasible for a distance-regular graph does not mean that a graph with that intersection array exists. Rather, the feasibility conditions are a set of necessary, but not sufficient, conditions an intersection array must satisfy. Accordingly, the definition of feasibility criteria vary a fair amount in the literature. The feasibility conditions described here are the intersection of the conditions in Chapter 21 of Biggs [20] and Section 4. 1 of Brouwer, Cohen, and Neumaier [28]. These conditions are effective at eliminating impossible parameter sets while still being straightforward to program.

Our first notion of feasibility for a distance-biregular graph is that an intersection array is feasible if the halved intersection arrays satisfy Definition 3.4.1. However, we lose a lot of information in only considering the halved graphs.

Let $c_1, \ldots, c_d$ be one line of the intersection array. Then the halved graph satisfies (i) if for all $1 \leq i \leq \lfloor \frac{d-1}{2} \rfloor$ we have

$$c_{2i-1}c_{2i} \leq c_{2i+1}c_{2i+2}.$$

However, a stronger result is true for distance-biregular graphs, as observed by Delorme [55, 56] and Shawe-Taylor [85, 134].

**3.4.3 Lemma.** *Let $G = (\beta \cup \gamma, E)$ be a distance-biregular graph with intersection array*

$$\begin{vmatrix} k; & c^\beta_1, & \ldots, & c^\beta_{d_\beta} \\ \ell; & c^\gamma_1, & \ldots, & c^\gamma_{d_\gamma} \end{vmatrix}.$$

*For any $0 \leq i \leq d_\beta - 1$, we have*

$$c^\beta_i \leq c^\gamma_{i+i}$$

*and for any $0 \leq i \leq d_\gamma - 1$, we have*

$$c^\gamma_i \leq c^\beta_{i+1}.$$

*Proof.* Let $u$ be a vertex and let $v$ be at distance $i+1$ from $u$. Let $w$ be adjacent to $u$ and at distance $i$ from $v$. We will let $\pi(u)$ denote the cell of the partition that $u$ lies in, and similarly for $w$. Note that $\pi(u) \neq \pi(w)$.

If $x$ is adjacent to $v$ and at distance $i-1$ from $w$, it must be at distance $i$ from $u$. Thus we have

$$c_i^{\pi(u)} = |\{x \sim v : d(w,x) = i-1\}| \leq |\{x \sim v : d(u,x) = i\}| = c_{i+1}^{\pi(w)}.$$

□

*3.4.4 Example.* Consider the intersection array

$$\begin{vmatrix} 8; & 1, & 1, & 2, & 4, & 1, & 8 \\ 9; & & & & & & \end{vmatrix}.$$

We clearly see that $c_3 > c_5$, contradicting Lemma 3.4.3.

The halved intersection array is $(64, 49, 32; 1, 8, 8)$, which does have the desired property that $c_1^* \leq c_2^* \leq c_3^*$ and $b_0^* \geq b_1^* \geq b_2^*$. In fact, this intersection array satisfies all the criteria in Definition 3.4.1, even though a distance-regular graph with these parameters does not exist.

Condition (ii) is also weaker when computing $k_i^*$ for the halved graph rather than $k_i^\pi$ for the corresponding cell of the partition.

*3.4.5 Example.* Consider the intersection array

$$\begin{vmatrix} 2; & 1, & 1, & 3, & 2 \\ 6; & & & & \end{vmatrix}.$$

We can see that

$$k_3 = \frac{2 \cdot 5 \cdot 1}{1 \cdot 1 \cdot 3} = \frac{10}{3}$$

is not an integer.

On the other hand, the halved graph has intersection array $(10, 3; 1, 6)$, which is the intersection array of the Clebsch graph.

Condition (iii) is equivalent whether we compute the multiplicities of the halved graphs or the original distance-biregular graph.

Let $G = (\beta \cup \gamma, E)$ be a distance-biregular graph, let $d$ be the covering radius of $\beta$, and let $d^* = \lfloor \frac{d}{2} \rfloor$. Recall that

$$m_\theta^\pi = \frac{|\beta|}{\psi(\theta)}$$

where

$$\psi(x) = \sum_{i=0}^{d} \frac{F_i(\theta)}{k_i} F_i(x)\,.$$

If $\theta$ is a non-zero eigenvalue of $G$, then so is $-\theta$, and $F_2(\theta) = F_2(-\theta)$ is an eigenvalue of the halved graph. We can compute the multiplicity of $F_2(\theta)$ in the halved graph by defining

$$\phi(x) = \sum_{i=0}^{d^*} \frac{F_{2i}(\theta)}{k_{2i}} F_{2i}(x)$$

and proceeding the same way in the proof of Theorem 2.7.2 to get

$$m_\theta^* = \frac{|\beta|}{\phi(\theta)} + \frac{|\beta|}{\phi(-\theta)} = \frac{2\,|\beta|}{\phi(\theta)}$$

since $F_2(-\theta)$ is the same eigenvalue of the halved graph with the same multiplicity. Note that

$$\psi(\theta) + \psi(-\theta) = 2\phi(\theta)\,,$$

so the multiplicity of $F_2(\theta)$ is an integer in the halved graph if and only if the bipartite multiplicity of $\theta$ is an integer in $G$.

There is an additional feasibility condition coming from the halved graphs that

$$b_{i-1}^* = \frac{b_{2i-1}b_{2i-2}}{c_2}$$

and

$$c_{i+1}^* = \frac{c_{2i+1}c_{2i+2}}{c_2}$$

are positive integers.

However, most of these criteria only consider a single cell of the partition at a time. To have a better definition of feasible intersection arrays for distance-biregular graphs, we want to know when the two lines of an intersection array are compatible.

## 3.5 Odd Distance Polynomials

The proof of Lemma 3.1.3 defined $F_3^\gamma$ to be equal to $F_3^\beta$. This can be done more generally, motivating the introduction of another families of polynomials.

Let $G = (\beta \cup \gamma, E)$ be a distance-biregular graph of diameter $d$ with $d$ possibly infinite. Let $d^* = \lfloor \frac{d}{2} \rfloor$. Consider the sequence of distance polynomials $F_0^\pi, \ldots, F_{d+1}^\pi$ associated to one cell of the partition $\pi$. For $0 \leq i \leq d^*$, we define $I_i(x)$, the *odd distance polynomials* associated to $\pi$, by

$$I_i^\pi\left(x^2\right) x = F_{2i+1}^\pi(x)\,.$$

For $0 \leq i \leq d^* - 1$, we have

$$\begin{aligned}
x^2 I_i\left(x^2\right) x &= x(xF_{2i+1}(x)) \\
&= b_{2i}xF_{2i}(x) + c_{2i+2}xF_{2i+2}(x) \\
&= b_{2i}b_{2i-1}F_{2i-1}^\pi(x) + (b_{2i}c_{2i+1} + c_{2i+2}b_{2i+1})\,F_{2i+1}(x) + c_{2i+2}c_{2i+3}F_{2i+3}(x) \\
&= b_{2i}b_{2i-1}I_{i-1}\left(x^2\right) x + (b_{2i}c_{2i+1} + c_{2i+2}b_{2i+1})\,I_i\left(x^2\right) x + c_{2i+2}c_{2i+3}I_{i+1}\left(x^2\right) x.
\end{aligned}$$

The *distance biadjacency matrix* $N_{2i+1}$ is the $|\beta| \times |\gamma|$ matrix where the $(u, v)$-entry is one if $u$ and $v$ are at distance $2i+1$ and zero otherwise. Then

$$A_{2i+1} = \begin{pmatrix} 0 & N_{2i+1} \\ N_{2i+1}^T & 0 \end{pmatrix}.$$

Let $I_0^\beta, \ldots, I_{d^*}^\beta$ be the odd distance polynomials associated to $\beta$, and $I_0^\gamma, \ldots, I_{d^*}^\gamma$ be the odd distance polynomials associated to $\gamma$. For $0 \leq i \leq d^*$, we have

$$I_i^\beta\left(NN^T\right) N = N_{2i+1}$$

and

$$I_i^\gamma\left(N^T N\right) N^T = N_{2i+1}^T.$$

But since $I_i^\beta$ and $I_i^\gamma$ are polynomials of $A$, and $A$ is a symmetric matrix, they must be symmetric too. Thus we have

$$I_i^\beta\left(A^2\right) A = A_{2i+1} = I_i^\gamma\left(A^2\right) A.$$

This suggests a relationship between the intersection coefficients from the two cells of the partition. This relationship can be proven combinatorially, as done by Delorme [55, 56] and Shawe-Taylor [134].

**3.5.1 Proposition** (Delorme [55, 56], Shawe-Taylor [134])**.** *Let $G = (\beta \cup \gamma, E)$ be a distance-biregular graph of diameter $d$ with intersection array*

$$\begin{vmatrix} k; & 1, & c_2^\beta, & \ldots, & c_{d_\beta}^\beta \\ \ell; & 1, & c_2^\gamma, & \ldots, & c_{d_\gamma}^\gamma \end{vmatrix}.$$

*For all $1 \leq i \leq \lfloor \frac{d-1}{2} \rfloor$, we have*

$$c_{2i}^\beta c_{2i+1}^\beta = c_{2i}^\gamma c_{2i+1}^\gamma$$

*and*

$$b_{2i-1}^\beta b_{2i}^\beta = b_{2i-1}^\gamma b_{2i}^\gamma.$$

*Proof.* Let $u$ and $v$ be at distance $2i+1$. They lie in different cells of the partition, so counting the number of paths of length $2i+1$ between them gives us

$$c_2^\beta c_3^\beta \cdots c_{2i}^\beta c_{2i+1}^\beta = c_2^\gamma c_3^\gamma \cdots c_{2i}^\gamma c_{2i+1}^\gamma,$$

which we can apply inductively to see that $c_{2i}^\beta c_{2i+1}^\beta = c_{2i}^\gamma c_{2i+1}^\gamma$.

Similarly, using Lemma 2.4.4 to count the number of pairs of vertices at distance $2i+1$, we have

$$|\beta| \frac{k b_1^\beta b_2^\beta \cdots b_{2i-1}^\beta b_{2i}^\beta}{c_2^\beta c_3^\beta \cdots c_{2i}^\beta c_{2i+1}^\beta} = |\beta|\, k_i^\beta = |\gamma|\, k_i^\gamma = |\gamma| \frac{\ell b_1^\gamma b_2^\gamma \cdots b_{2i-1}^\gamma b_{2i}^\gamma}{c_2^\gamma c_3^\gamma \cdots c_{2i}^\gamma c_{2i+1}^\gamma}.$$

Since $k\,|\beta| = \ell\,|\gamma|$, by our previous work we can use induction to establish that $b_{2i-1}^\beta b_{2i}^\beta = b_{2i-1}^\gamma b_{2i}^\gamma$. □

As noted by Shawe-Taylor [134] in Chapter 3 of his thesis, this allows us to compute one set of intersection coefficients from the other.

**3.5.2 Theorem.** *Let $G=(\beta\cup\gamma, E)$ be a distance-biregular graph with intersection array*

$$\begin{vmatrix} k; & 1, & c_2^\beta, & \ldots, & c_{d_\beta}^\beta \\ \ell; & 1, & c_2^\gamma, & \ldots, & c_{d_\gamma}^\gamma \end{vmatrix}.$$

*Then $c_2^\gamma, \ldots, c_{d_\gamma}^\gamma$ can be expressed in terms of $k, \ell$, and $c_2^\beta, \ldots, c_{d_\beta}^\beta$.*

*Proof.* We have that $b_0^\gamma = \ell, c_1^\gamma = 1$, and $b_1^\gamma = k-1$, so we may suppose by induction that $b_0^\gamma, \ldots, b_{2i-1}^\gamma, c_1^\gamma, \ldots, c_{2i-1}^\gamma$ are determined by $k, \ell$, and $c_1^\beta, \ldots, c_{2i-1}^\beta$. If $i \leq \lfloor \frac{d_\beta - 1}{2} \rfloor$, we use Proposition 3.5.1 to compute that

$$c_{2i}^\gamma = \ell - b_{2i}^\gamma = \ell - \frac{b_{2i}^\beta b_{2i-1}^\beta}{b_{2i-1}^\gamma}$$

and

$$c_{2i+1}^\gamma = \frac{c_{2i}^\beta c_{2i+1}^\beta}{c_{2i}^\gamma}$$

are determined by $k, \ell$, and $c_1^\beta, \ldots, c_{2i+1}^\beta$.

It remains to show this applies to the entire sequence $c_1^\gamma, \ldots, c_{d_\gamma}^\gamma$. We have

$$2\left\lfloor \frac{d_\beta + 1}{2} \right\rfloor + 1 \geq d_\beta - 1 + 1 \geq d_\gamma - 1,$$

and we can define $c_{d_\gamma}$ to be $k$ if $d_\gamma$ is odd and $\ell$ if $d_\gamma$ is even. □

## 3.6 Feasibility Conditions

We are now ready to give a working definition of feasible intersection array for distance-biregular graphs.

**3.6.1 Definition.** *The intersection coefficients $0 = c_0, c_1, c_2, \ldots, c_d$ are feasible for a $(k,\ell)$-distance-biregular graph if:*

*(i) The numbers $k_0, \ldots, k_d$ defined recursively by*

$$k_{2i+1} = \frac{(k - c_{2i})\, k_{2i}}{c_{2i+1}}$$

*and*

$$k_{2i+2} = \frac{(\ell - c_{2i+1})\, k_{2i+1}}{c_{2i+2}}$$

*are positive integers;*

*(ii)* *For any* $0 \leq i \leq \lfloor \frac{d}{2} \rfloor$, *the numbers*

$$\frac{(k - c_{2i})(\ell - c_{2i+1})}{c_2}$$

*and*

$$\frac{c_{2i+1}c_{2i+2}}{c_2}$$

*are positive integers;*

*(iii)* *The multiplicities as defined in Theorem 2.7.2 are positive integers;*

*(iv)* *The second array* $(c'_1, c'_2, \ldots, c'_d)$ *as defined in Theorem 3.5.2 has positive integer values* $c'_1, c'_2, \ldots, c'_d$;

*(v)* *Swapping the place of* $k$ *and* $\ell$, *the values of* $(c'_1, c'_2, \ldots, c'_d)$ *satisfy (i), (ii), and (iii); and*

*(vi)* *For any* $1 \leq i \leq d-1$, *we have*

$$c_i \leq c'_{i+1}$$

*and*

$$c'_i \leq c_{i+1}.$$

This is comparable to the definition of feasible intersection arrays given by Godsil and Shawe-Taylor [85] and Shawe-Taylor [134]. Condition (ii) is not included in either of those feasibility criteria, and Chapter 3 of Shawe-Taylor's thesis [134] includes an additional spectral condition.

The feasibility conditions we've given here are relatively minimal, but they are powerful. We demonstrate this by using Definition 3.4.1 to characterize distance-biregular graphs where one cell of the partition has valency two. This characterization was previously given by Mohar and Shawe-Taylor [118] using combinatorial arguments about the halved graphs. We show it more directly using our definition of feasibility.

**3.6.2 Theorem.** *Let* $G$ *be a distance-biregular graph. If* $G$ *has vertices of valency two, then* $G$ *is* $K_{2,k}$ *or the subdivision graph of either a Moore graph or a generalized polygon.*

*Proof.* Let $G = (\beta \cup \gamma, E)$ be a distance-biregular graph, and suppose without loss of generality that vertices in $\gamma$ have valency two. Then $c_2^\gamma$ is either one or two.

If $c_2^\gamma = 2$, then the intersection array must have the form

$$\begin{vmatrix} k; & 1, & k \\ 2; & 1, & 2 \end{vmatrix},$$

so $G = K_{2,k}$.

Otherwise let $H$ be the halved graph of $G$ induced by the vertex set of $\beta$. We can think of elements of $\gamma$ as two-element subsets of $\beta$, where two vertices are adjacent in $H$ if and only if they belong to the same element of $\gamma$. In other words, $\gamma$ is the set of edges of $H$. Then $G$, the incidence graph of vertices and edges of $H$ is equivalent to the subdivision graph of $H$.

We can write the intersection array for $G$ as either

$$\begin{vmatrix} k; & 1, & 1, & c_3^\beta, & \ldots, & c_{2d}^\beta \\ 2; & 1, & 1, & c_3^\gamma, & \ldots, & c_{2d}^\gamma \end{vmatrix} \tag{3.6.1}$$

or

$$\begin{vmatrix} k; & 1, & 1, & c_3^\beta, & \ldots, & c_{2d-1}^\beta & \\ 2; & 1, & 1, & c_3^\gamma, & \ldots, & c_{2d-1}^\gamma & c_{2d}^\gamma \end{vmatrix}. \tag{3.6.2}$$

By Theorem 3.2.3 we know $H$ is distance-regular, and we let its intersection array be $\left(b_0^*, b_1^*, \ldots, b_{d-1}^*; c_1^*, \ldots, c_d^*\right)$. For $1 \leq i \leq d$, since $b_{2i-1}^\beta \neq 0$, we see that

$$b_{2i-1}^\beta = 1 = c_{2i-1}^\beta.$$

By Theorem 3.2.3 we have

$$c_i^* = \frac{c_{2i}^\beta c_{2i-1}^\beta}{c_2^\beta} = c_{2i}.$$

Then for $0 \leq i \leq d-1$, we have

$$b_i^* = \frac{b_{2i}^\beta b_{2i+1}^\beta}{c_2^\beta} = b_{2i}^\beta = k - c_{2i}^\beta = k - c_i^*.$$

This tells us that $a_i^* = 0$ for $0 \leq i \leq d-1$.

Now, for $1 \leq i \leq d$, since $b_{2i}^\gamma \neq 0$, we know that

$$b_{2i}^\gamma = 1 = c_{2i}^\gamma.$$

By Proposition 3.5.1, we have

$$c_{2i+1}^\gamma = c_{2i}^\gamma c_{2i+1}^\gamma = c_{2i}^\beta c_{2i+1}^\beta = c_{2i}^\beta$$

and

$$c_{2i+1}^\gamma = k - b_{2i+1}^\gamma = k - b_{2i+1}^\gamma b_{2i+2}^\gamma = k - b_{2i+1}^\beta b_{2i+2}^\beta = k - b_{2i+2}^\beta = c_{2i+2}^\beta.$$

This gives us

$$1 = c_2^\beta = c_4^\beta = \cdots = c_{2d-2}^\beta.$$

If the intersection array is in the form of Equation 3.6.2, we can apply Proposition 3.5.1 once more to get $c_{2d}^\beta = c_{2d-2}^\beta = 1$. Then we have

$$1 = c_1^* = \cdots = c_d^*,$$

and since

$$0 = a_0^* = \cdots = a_{d-1}^*,$$

we see that $G$ is a Moore graph.

If the intersection array is in the form of Equation 3.6.1, we have

$$c_d^* = c_{2d}^\beta c_{2d-1}^\beta = k.$$

Then since

$$0 = a_0^* = \cdots = a_{d-1}^* = a_d^*,$$

we know $G$ is bipartite, and since

$$1 = c_1^* = \cdots = c_{d-1}^*,$$

we conclude that $G$ is a generalized polygon. □

This proof takes repeated advantage of the fact that, since $b_i \neq 0$ and one of the valencies is two, we know that half the intersection coefficients must be one. For larger valencies, we would not have that information. However, since Biggs, Boshier, and Shawe-Taylor [22] have characterized distance-regular graphs with valency three, there is hope for a similar characterization for distance-biregular graphs.

**3.6.3 Problem.** *Characterize the distance-biregular graphs where one cell of the partition has valency three.*

Besides allowing us to characterize special cases of distance-biregular graphs, the other main use for a notion of feasibility is to let us generate tables of feasible intersection arrays. Specifically, given valencies $k$ and $\ell$ and a diameter $d$, we can use a computer to generate all the possible intersection arrays that satisfy the conditions of Definition 3.6.1. A table of feasible intersection arrays with $d = 4$ and $3 \leq k, \ell \leq 12$ is found in Appendix A.

Of particular interest are the intersection arrays for distance-biregular graphs that are not partial geometries or distance-regular, since they have been considered far less. A table of feasible intersection arrays with $d = 4$, $c_2^\beta \geq 2$, and $3 \leq k < \ell \leq 36$ is found in Appendix B

## 3.7 Constructions from Finite Geometry

Several of the parameter sets in Appendix B belong to infinite families of distance-biregular graphs arising from finite geometry. The definitions and results here are standard, and can be found in references such as Ball and Weiner [12] or Van Lint and Wilson [112].

For a prime power $q$ and a positive integer $n$, let $V(n, q)$ be a vector space of dimension $n$ over $GF(q)$. The *projective geometry* $PG(n-1, q)$ is the geometry where the points are 1-dimensional subspaces of $V(n, q)$, the lines are 2-dimensional subspaces, the planes are 3-dimensional subspaces, etc.

The *affine geometry* $AG(n,q)$ is the geometry where the points are the cosets of $V(n,q)$ of dimension 0, the lines are cosets of dimension 1, the planes are cosets of dimension 2, and so on to the hyperplanes, which are cosets of dimension $n-1$.

Affine geometries lead to another well-known family of quasi-symmetric 2-designs.

*3.7.1 Example.* Let $q$ be a prime power and consider the affine geometry $AG(n,q)$. Let $\beta$ be the set of the points, and let $\gamma$ be the set of hyperplanes, with the obvious incidence relation of inclusion. Every hyperplane contains $q^{n-1}$ points, and the number of hyperplanes incident to a point is equal the number of $(n-1)$-dimensional subspaces of $V$, which is $[n]_q$.

The number of hyperplanes containing a pair of points is the number of hyperplanes containing the unique line between them, which is $[n-1]_q$. Additionally, any non-disjoint hyperplanes intersect each other in an $(n-2)$-dimensional affine space, so there are $q^{n-2}$ common points.

Let $u$ be a point and let $v$ be a point at distance four from $u$. There is a unique line containing $u$ and $v$, and there is a unique $(n-1)$-dimensional subspace $U$ that does not contain this line. This gives us a unique coset that contains $v$ but not $u$. It follows there are

$$[n]_q - 1 = \frac{q^n-1}{q-1} - \frac{q-1}{q-1} = q\left(\frac{q^{n-1}-1}{q-1}\right) = q\,[n-1]_q$$

affine hyperplanes incident to $v$ and at distance three from $u$.

From this, we get the intersection array

$$\left|\begin{matrix} [n]_q; & 1, & [n-1]_q, & q^{n-1} & \\ q^{n-1}; & 1, & q^{n-2}, & q\,[n-1]_q, & q^{n-1} \end{matrix}\right|.$$

An *arc* $\mathcal{A}$ of degree $r$ is a set of $r+1$ points in $PG(2,q)$ with the property that every line is incident with at most $r$ points of $\mathcal{A}$. If $\mathcal{A}$ is an arc, we can fix a point $p$ in $\mathcal{A}$. There are $q+1$ lines through $p$, each of which is incident to at most $r-1$ other points of $\mathcal{A}$. Thus an arc can have at most

$$1+(q+1)(r-1) = rq - q + r$$

points in it. If equality holds, we call $\mathcal{A}$ a *maximal arc*, and every line meets $\mathcal{A}$ at 0 or $r$ points.

Maximal arcs lead to a construction of distance-biregular graphs defined by Delorme [55]. The description here has been been considerably expanded.

*3.7.2 Example.* Consider $GF(q)$. Let $V$ be a three-dimensional vector space over $GF(q)$, let $\mathcal{A}$ be a maximal arc in $PG(2,q)$ of degree $r$, and let $s = rq - q + r$. Let $\hat{\mathcal{A}}$ be the dual of the maximal arc, that is, $\hat{\mathcal{A}}$ is a set of $s$ two-dimensional subspaces of $V$ such that any one-dimensional subspace of $V$ is incident with 0 or $r$ elements of $\hat{\mathcal{A}}$.

We define a bipartite graph where the points are the $q^3$ points of $V$ and the blocks are the $qs$ elements formed by the cosets of the two-dimensional subspaces of $\hat{\mathcal{A}}$. Note that each block contains $q^2$ points, and each point lies in $s$ blocks.

Fix a block $x_0$. Without loss of generality, we can assume it is an element of $\hat{\mathcal{A}}$. Suppose that $x_2$ is at distance two from $x_0$, so $x_0$ and $x_2$ share a point $u_1$. Note that $x_0 - u_1$ and $x_2 - u_1$ are two-dimensional subspaces, so they intersect in a one-dimensional vector space, and therefore $x_0$ and $x_2$ have $q$ points in common. Let $u_3$ be a point not on $x_0$. The blocks at distance four from $x_0$ are the cosets of $x_0$, and $u_3$ lies in the unique coset $x_0 + u_3$. This gives us the line in the intersection array

$$\begin{vmatrix} q^2; & 1, & q, & s-1, & q^2 \end{vmatrix}.$$

Fix a point $u_0$. Without loss of generality, we can assume it is the origin. Then the blocks at distance one from $u_0$ are the elements of $\hat{\mathcal{A}}$ and the blocks at distance three are the cosets. If $u_2$ is a point at distance two from $u_0$, then $u_0$ and $u_2$ define a one-dimensional subspace of $V$, so by definition, we know that both points lie in $r$ common elements of $\hat{\mathcal{A}}$.

Let $x_3$ be at distance three from $u_0$. Since $x_3$ is a coset of $\hat{\mathcal{A}}$, we may write it $x_1 + u$, where $x_1 \in \hat{\mathcal{A}}$ and $u$ lies in the one-dimensional subspace dual to $x_1$. Then if $y_1 \neq x_1$ is another element of $\hat{\mathcal{A}}$, we must have $u \in y_1$, so $y_1 - u$ and $x_3 - u$ are both two-dimensional subspaces, and thus they intersect in a line. On the other hand, each line that meets elements of $\hat{\mathcal{A}}$ meet them $r$ times. Therefore, $x_3 - u$, and by extension $x_3$, are incident to $\frac{q(s-1)}{r}$ points incident to elements of $\hat{\mathcal{A}}$.

Pulling this together, we get the intersection array

$$\begin{vmatrix} s; & 1, & r, & q(s-1)/r, & s \\ q^2; & 1, & q, & s-1, & q^2 \end{vmatrix}.$$

Clearly, a requirement for a maximal arc to exist is that $r$ divides $q$. Denniston [60] proved that a maximal arc exists whenever $q$ and $r$ are both powers of two. Ball, Blokhuis, and Mazzocca [11, 10] showed that these were in fact the only maximal arcs.

Chapter 4

# Characterizations of Distance-Biregular Graphs

Any characterization of locally distance-regular vertices applied globally gives a characterization of distance-regular or distance-biregular graphs. The problem with this approach is that local distance-regularity is a strong property, and it can be difficult to come up with alternate characterizations. One solution is to come up with a weaker notion of local distance-regularity that, applied globally, is still strong enough to force the graph to be distance-regular or distance-biregular.

Fiol, Garriga, and Yebra [74] introduced the notion of *pseudo-distance-regular vertices*, and proved that if every vertex in a graph is pseudo-distance-regular, then every vertex is locally distance-regular. Thus Theorem 2.4.1 of Godsil and Shawe-Taylor [85] applies to prove that the graph is distance-regular or distance-biregular. More recently, Fiol [71] gave a direct proof that if every vertex is pseudo-distance-regular, the graph is distance-regular or distance-biregular. Fiol, Garriga, and Yebra [74] gave an alternate characterization of pseudo-distance-regular vertices to characterize distance-regular graphs, and Fiol and Garriga [73] built on that work to give another characterization of pseudo-distance-regular vertices, and another characterization of distance-regular graphs.

In this chapter, we set up pseudo-distance-regular vertices in a different way, which we use to derive new proofs of the results in Fiol [71], Fiol, Garriga and Yebra [74], and Fiol and Garriga [73]. We further extend their results to obtain new characterizations of distance-biregular graphs.

Fiol [70] asked for a distance-biregular analogue to the characterization of Fiol, Garriga, and Yebra [74]. We provide that with Theorem 4.4.3. Using this characterization, we investigate the problem of when a bipartite graph with distance-regular halved graphs is distance-biregular. We also observe that a counterexample of Delorme [55] gives a negative answer to another question of Fiol [70].

Fiol [72] previously extended the characterization of Fiol and Garriga [73] to distance-biregular graphs. We improve on that characterization in Theorem 4.6.4. Further, we use our characterization to consider the problem of when a graph with the spectrum of a distance-biregular graph is itself distance-regular, an analogue of a problem studied by, among others, Abiad, Van Dam, and Fiol [3], Van Dam and Haemers [45, 46], Haemers [88], and Haemers and Spence [91].

The results in this chapter can also be found in the author's paper [109].

## 4.1 Spectrally Extremal Vertices

Let $G$ be a graph with a vertex $u$. The *eigenvalue support* of $u$, denoted $\Phi_u$, is the set

$$\{\theta_r : E_r \mathbf{e}_u \neq \mathbf{0}\}.$$

The eccentricity of a vertex bounds the size of the eigenvalue support, a result that can be found, for example, in Section 5.2 of Coutinho and Godsil [42]. This result can also be found in Fiol, Garriga, and Yebra [74], though the set-up and proof are different.

**4.1.1 Lemma.** *Let $u$ be a vertex in graph $G$ with eccentricity $e$. Then*

$$|\Phi_u| \geq e + 1.$$

*Proof.* Let $U$ be the cyclic $A$-module generated by $\mathbf{e}_u$, and let

$$S := \{E_r \mathbf{e}_u : \theta_r \in \Phi_u\}.$$

Note that $S$ is a set of eigenvectors corresponding to distinct eigenvalues in the support of $u$.

Recall that for any eigenvalue $\theta_r$, there exists a polynomial $f_r$ such that $f_r(A) = E_r$, and therefore $S$ is contained in $U$. For $i \geq 0$, we use the spectral decomposition to see that

$$A^i \mathbf{e}_u = \sum_{\theta_r \in \Phi_u} E_r \mathbf{e}_u,$$

and the elements of $S$ span $U$. Since the elements of $S$ are linearly independent, they form a basis for $U$.

Let $0 \leq r \leq e$, and let $v$ be at distance $r$ from $u$. Then

$$\mathbf{e}_v A^r \mathbf{e}_u \neq 0,$$

but for all $s < r$, we have

$$\mathbf{e}_v A^s \mathbf{e}_u = 0.$$

Thus $A^r \mathbf{e}_u$ cannot be expressed as a linear combination of $A^0 \mathbf{e}_u, \ldots, A^{r-1} \mathbf{e}_u$. It follows that $\mathbf{e}_u, A\mathbf{e}_u, \ldots, A^e \mathbf{e}_u$ are linearly independent, so $U$ has dimension at least $e + 1$. □

The global version of this result is even better known and can be found, for instance, in Section 2. 5 of Godsil [80].

**4.1.2 Corollary.** *A graph with diameter $d$ has at least $d + 1$ distinct eigenvalues.*

Let $u$ be a vertex with eccentricity $e$. The vertex $u$ is *spectrally extremal* if $|\Phi_u| = e + 1$. Equivalently, $u$ is spectrally extremal if there exists a polynomial $p_{e+1}$ of degree $e + 1$ such that

$$p_{e+1}(A)\, \mathbf{e}_u = \sum_{\theta_r \in \Phi_u} p_{e+1}(\theta_r)\, E_r \mathbf{e}_u = \mathbf{0}.$$

This allows us to rewrite our definition of locally distance-regular vertices by replacing the condition that $F^u_{e+1}$ exists with the condition that $|\Phi_u| = e + 1$. In other words, $u$ is locally distance-regular if it is spectrally extremal and there exists a sequence of orthogonal polynomials $F^u_0, \ldots, F^u_e$ such that for all $0 \leq i \leq e$, the polynomial $F^u_i(x)$ satisfies

$$F^u_i(A)\, \mathbf{e}_u = A_i \mathbf{e}_u.$$

## 4.2 Pseudo-Distance-Regular

One way to prove Theorem 2.4.1, sketched by Godsil [81], relies on the following result.

**4.2.1 Lemma.** *Let $G$ be a graph and let $u$ be a locally distance-regular vertex with valency $k_u$. If $v$ is adjacent to $u$, then for $i \geq 0$ we have*

$$\langle \mathbf{e}_u, A^i \mathbf{e}_v \rangle = \frac{1}{k_u} \langle \mathbf{e}_u, A^{i+1} \mathbf{e}_u \rangle.$$

*Proof.* Let $U$ be the cyclic $A$-module generated by $\mathbf{e}_u$. Let $F_0, \ldots, F_e$ be the sequence of local distance polynomials associated to $u$. For $0 \leq i \leq e$, let $\mathbf{z}_i = F_i(A)\,\mathbf{e}_u$. Because $U$ has dimension $|\Phi_u|$, the linearly independent vectors $\mathbf{z}_0, \ldots, \mathbf{z}_e$ form a basis for $U$.

If $v$ is adjacent to $u$, then the projection of $v$ on $U$ is

$$\sum_{i=0}^{e} \frac{\langle \mathbf{e}_v, \mathbf{z}_i \rangle}{\langle \mathbf{z}_i, \mathbf{z}_i \rangle} \mathbf{z}_i = \frac{1}{\langle \mathbf{z}_1, \mathbf{z}_1 \rangle} \mathbf{z}_1 = \frac{1}{k_u} A \mathbf{e}_u.$$

Therefore, for all $i \geq 0$, we have

$$\langle \mathbf{e}_v - \frac{1}{k_u} A \mathbf{e}_u, A^i \mathbf{e}_u \rangle = 0,$$

and so

$$\langle \mathbf{e}_v, A^r \mathbf{e}_u \rangle = \frac{1}{k_u} \langle \mathbf{e}_u, A^{i+1} \mathbf{e}_u \rangle.$$

□

In the proof, we used the fact that $u$ was spectrally extremal to obtain the basis $\mathbf{z}_0, \ldots, \mathbf{z}_e$. We also used the fact that $u$ was locally distance-regular when we concluded that $\langle \mathbf{e}_u, \mathbf{z}_i \rangle = 0$ unless $i = 1$. This conclusion doesn't require that $\mathbf{z}_i$ be a 01-vector.

A vertex $u$ of eccentricity $e$ is *pseudo-distance-regular* if $|\Phi_u| = e + 1$ and there exists a sequence of polynomials $f_0^u, \ldots, f_e^u$ such that for all $0 \leq i \leq e$, the polynomial $f_i^u(x)$ has degree $i$, and the nonzero entries of $f_i^u(A)\,\mathbf{e}_u$ are precisely the entries indexed by vertices at distance $i$ from $u$. Note that this sequence of pseudo distance polynomials is orthogonal under the $u$-local inner product. Pseudo-distance-regular vertices were introduced by Fiol, Garriga, and Yebra [74] with a different definition.

In truth, the nonzero entries are not as arbitrary as our definition makes them sound. Let $u$ be a pseudo-distance-regular vertex of eccentricity $e$, and for $0 \leq i \leq e$ let

$$\mathbf{z}_i = f_i^u(A)\,\mathbf{e}_u$$

be nonzero precisely on the entries indexed by vertices at distance $i$ from $u$. Let $U$ be the cyclic $A$-module generated by $\mathbf{e}_u$. Clearly $\mathbf{z}_0, \ldots, \mathbf{z}_e$ forms a basis for $U$, but so too do a set of eigenvectors with distinct eigenvalues. In particular, we can write the Perron vector $\mathbf{p}$ as

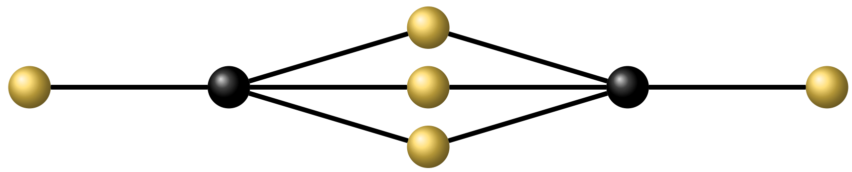

Figure 4.1: Pseudo-distance-regular

a linear combination of $\mathbf{z}_0, \ldots, \mathbf{z}_e$. Since $\mathbf{p}$ has all positive entries, and the nonzero entries of $\mathbf{z}_0, \ldots, \mathbf{z}_e$ are disjoint, it follows that for all $0 \leq i \leq e$, the nonzero entries of $\mathbf{z}_i$ must be some scalar multiple of $\mathbf{p}$.

Fiol, Garriga, and Yebra [74] defined pseudo-distance-regular vertices by fixing this scalar and giving a combinatorial interpretation of the intersection coefficients. Later in the same paper, they proved that a vertex was pseudo-distance-regular if and only if a sequence of pseudo-distance-polynomials with a fixed normalization exists. Although we don't fix the normalization in our definition, a vertex is pseudo-distance-regular in the sense defined here if and only if it is pseudo-distance-regular as defined by Fiol, Garriga, and Yebra [74].

Every locally distance-regular vertex is pseudo-distance-regular, and we will show that if every vertex in a graph is pseudo-distance-regular, they are all locally distance-regular. However, there are vertices which are pseudo-distance-regular and not locally distance-regular.

*4.2.2 Example.* Consider the graph obtained by taking the path on five vertices and cloning the centre vertex $n$ times. Figure 4.1 shows the result of cloning the centre vertex twice.

Let $u$ be one of the cloned centre vertices. We define

$$f_1^u(x) = x,$$

$$f_2^u(x) = x^2 - 2,$$

and

$$f_3^u(x) = x f_2^u(x) - (2n+1)\, f_1^u(x)\,.$$

For a vertex $v$, we have

$$\mathbf{e}_v^T f_2^u(A)\, \mathbf{e}_u = \begin{cases} 1 & v \text{ end vertex} \\ 2 & v \text{ centre vertex} \\ 0 & \text{otherwise} \end{cases}.$$

Further, since

$$f_3^u(A)\, \mathbf{e}_u = \mathbf{0},$$

we see that $v$ is pseudo-distance-regular.

## 4.3 Globally Pseudo-Distance-Regular

We wish to give a new proof that if every vertex in a graph is pseudo-distance-regular, then the graph is distance-regular or distance-biregular. Key to the proof is a variation of Lemma 4.2.1 and the notion of cospectrality.

Two graphs are *cospectral* if they share the same spectrum. Two vertices $u$ and $v$ of a graph $G$ are *cospectral* if $G \setminus u$ is cospectral to $G \setminus v$. There is a long list of equivalent characterizations of cospectral vertices that can be found in Godsil and Smith [86], but we are primarily interested in the following equivalence.

**4.3.1 Theorem** (Godsil and Smith [86])**.** *Let $G$ be a graph with vertices $u, v$. The following are equivalent:*

*(a) Vertices $a$ and $b$ are cospectral;*

*(b) For all $i \geq 0$, we have*

$$\mathbf{e}_u^T A^i \mathbf{e}_u = \mathbf{e}_v^T A^i \mathbf{e}_v;$$

*and*

*(c) For every spectral idempotent $E_r$, we have*

$$\mathbf{e}_u^T E_r \mathbf{e}_u = \mathbf{e}_v^T E_r \mathbf{e}_v.$$

By condition (c), we know that cospectral vertices $u$ and $v$ have the same eigenvalue support. Further, for any polynomials $f, g$ we have

$$\langle f, g \rangle_u = \sum_{\theta_r \in \Phi} \mathbf{e}_u^T E_r \mathbf{e}_u f(\theta_r)\, g(\theta_r) = \sum_{\theta_r \in \Phi} \mathbf{e}_v^T E_r \mathbf{e}_v f(\theta_r)\, g(\theta_r) = \langle f, g \rangle_v.$$

Thus, cospectral vertices have the same inner product.

The proof of Lemma 4.2.1 goes through with pseudo-distance-regular vertices.

**4.3.2 Lemma.** *Let $G$ be a graph and let $u$ be a pseudo-distance-regular vertex. Let $k_u$ be the valency of $u$. If $v$ is adjacent to $u$, then for $i \geq 0$ we have*

$$\langle \mathbf{e}_u, A^i \mathbf{e}_v \rangle = \frac{1}{k_u} \langle \mathbf{e}_u, A^{i+1} \mathbf{e}_u \rangle. \qquad \square$$

This gives us the following corollaries.

**4.3.3 Corollary.** *Let $G$ be a graph that is pseudo-distance-regular at vertices $u$ and $v$, with $v$ adjacent to $u$. If the valency of $u$ is the same as the valency of $v$, then $u$ and $v$ are cospectral.*

*Proof.* For all $i \geq 0$, we have

$$\frac{1}{k_u}\langle \mathbf{e}_u, A^{i+1}\mathbf{e}_u\rangle = \langle \mathbf{e}_v, A^i\mathbf{e}_u\rangle = \langle \mathbf{e}_u, A^i\mathbf{e}_v\rangle = \frac{1}{k_v}\langle \mathbf{e}_v, A^{i+1}\mathbf{e}_v\rangle.$$

Since $k_u = k_v$, by Theorem 4.3.1, vertices $u$ and $v$ are cospectral.

**4.3.4 Corollary.** *Let $G$ be a graph that is pseudo-distance-regular at vertices $u, v$ and $w$, with $u$ adjacent to $v$ and $w$. If the valency of $v$ is the same as the valency of $w$, then $v$ and $w$ are cospectral.*

*Proof.* Let $k$ be the valency of $u$, and let $\ell$ be the valency of vertices $v$ and $w$. Since $u$ is pseudo-distance-regular, for $i \geq 0$, we have

$$\langle \mathbf{e}_u, A^i\mathbf{e}_v\rangle = \frac{1}{k}\langle \mathbf{e}_u, A^{i+1}\mathbf{e}_u\rangle = \langle \mathbf{e}_u, A^i\mathbf{e}_w\rangle.$$

On the other hand, since $v$ is pseudo-distance-regular, we have

$$\langle \mathbf{e}_u, A^i\mathbf{e}_v\rangle = \langle \mathbf{e}_v, A^i\mathbf{e}_u\rangle = \frac{1}{\ell}\langle \mathbf{e}_v, A^{i+1}\mathbf{e}_v\rangle.$$

Since $w$ is also pseudo-distance-regular, we have

$$\frac{1}{\ell}\langle \mathbf{e}_v, A^{i+1}\mathbf{e}_v\rangle = \langle \mathbf{e}_u, A^i\mathbf{e}_w\rangle = \frac{1}{\ell}\langle \mathbf{e}_w, A^{i+1}\mathbf{e}_w\rangle,$$

and by Theorem 4.3.1, $v$ and $w$ are cospectral. □

We are now ready to prove the main result of this section.

**4.3.5 Theorem.** *Let $G$ be a graph that is pseudo-distance-regular at every vertex. Then $G$ is distance-regular or distance-biregular.*

*Proof.* Let $u$ be a vertex of eccentricity $e$. We can assume without loss of generality that if $u$ has pseudo-distance polynomials $f_0^u, \dots, f_e^u$ and $v$ is at distance $i$ from $u$, then

$$\mathbf{e}_v^T f_i^u(A)\, \mathbf{e}_u = \mathbf{e}_v^T \mathbf{p}.$$

Then every vertex adjacent to $u$ has the same entry of the Perron vector. Since our choice of $u$ was arbitrary, either the Perron vector is constant, or there is a partition of the vertices into two independent sets such that the Perron vector is constant on the cells of the partition. For a vertex $v$, we let $k_v$ denote the valency of $v$.

If the Perron vector is constant, we have

$$\rho \mathbf{e}_v^T \mathbf{p} = \mathbf{e}_v^T A\mathbf{p} = \sum_{w\sim v} \mathbf{e}_w^T \mathbf{p} = k_v \mathbf{e}_v^T \mathbf{p},$$

so the graph is $\rho$-regular. By Corollary 4.3.3, every vertex of $G$ is cospectral and since every vertex is spectrally extremal, every vertex has the same eccentricity $d$. Further, the graph inner product is the same as the vertex inner product for any vertex.

Consider the sequence of polynomials $F_0, \dots, F_d$ defined by

$$F_i(x) = \sqrt{|V(G)|} f_i^u(x)$$

for $0 \leq i \leq d$. This is, up to normalization, the unique sequence of orthogonal polynomials with respect to the $G$-inner product, so it is also a sequence of pseudo-distance polynomials for every vertex. The normalization for $u$ tells us that for any $0 \leq i \leq d$, we have

$$\|F_i\|_u^2 = |\{v \in V(G) : d(u,v) = i\}| = \mathbf{e}_u^T A_i \mathbf{1} \mathbf{e}_u = F_i(\rho)$$

since $G$ is regular and thus $\mathbf{1}$ is a polynomial of $A$. Since the normalization does not depend on the choice of vertex, we must have that $F_0, \dots, F_d$ are the distance-polynomials for $G$. Thus $G$ is distance-regular.

Now suppose that $G$ is bipartite and the Perron vector is constant on the cells of the partition. For $u \in \beta$ we have

$$\rho \mathbf{e}_u^T \mathbf{p} = \mathbf{e}_u^T A \mathbf{p} = k_u \mathbf{e}_v^T \mathbf{p},$$

so letting $p_\beta$ be the value on the vertices in $\beta$, and $p_\gamma$ be the value of $\mathbf{p}$ on the vertices in $\gamma$, we have

$$\frac{\rho}{k_u} = \frac{p_\beta}{p_\gamma},$$

and in particular the valency of vertices in $\beta$ are constant. The same argument applies for $\gamma$, so $G$ is semiregular.

By Corollary 4.3.4, every vertex in the same cell of the partition is cospectral. Let $d$ be the covering radius of $\beta$, let $u \in \beta$, and let $F_0^\beta, \dots, F_d^\beta$ be the sequence of pseudo-distance polynomials for $u$ normalized such that for all $0 \leq i \leq d$, we have

$$\|F_{2i}^\beta\|_u^2 = \frac{1}{2\,|\beta|} F_{2i}^\beta(\rho)$$

and

$$\|F_{2i+1}^\beta\|_u^2 = \frac{\sqrt{\ell}}{2\,|\beta|\sqrt{k}} F_{2i+1}^\beta(\rho)\,.$$

We can similarly let $d'$ be the covering radius of $\gamma$, let $v \in \gamma$, and define the sequence of pseudo-distance polynomials for $v$ normalized such that

$$\|F_{2i+1}^\gamma\|_v^2 = \frac{\sqrt{k}}{2\,|\gamma|\sqrt{\ell}} F_{2i+1}^\gamma(\rho)$$

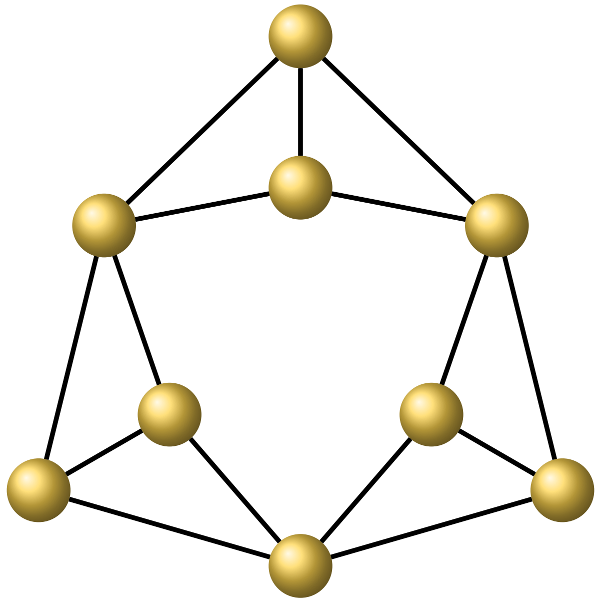

Figure 4.2: Maximal eccentricity, minimal eigenvalue support

and

$$\|F_{2i}^{\gamma}\|_u^2 = \frac{1}{2\,|\gamma|} F_{2i}^{\gamma}(\rho)\,.$$

These are the distance polynomials for the cells of the partition, so $G$ is distance-biregular. □

It is not difficult to distinguish between when a graph is distance-regular and when it is distance-biregular. In particular, if every vertex is pseudo-distance-regular and the graph is regular, then it is distance-regular. If it is bipartite, then it is distance-biregular. If it is regular and bipartite, it is both distance-regular and distance-biregular.

The condition that every vertex is spectrally extremal is not strong enough to force the graph to be distance-regular or distance-biregular.

*4.3.6 Example.* Consider the graph $G$ in Figure 4.2. Its spectrum is

$$\left\{\left(\frac{1}{2}+\frac{\sqrt{33}}{2}\right)^{(1)},2^{(2)},(-1)^{(5)},\left(\frac{1}{2}-\frac{\sqrt{33}}{2}\right)^{(1)}\right\},$$

and every vertex has eccentricity three even though the graph is not distance-regular or distance-biregular.

## 4.4 Diametral Characterization

Characterizations of pseudo-distance-regular vertices extend to distance-regular and distance-biregular graphs. Our characterization for distance-regular graphs improves the characterization of Fiol, Garriga, and Yebra [74], and our characterization of distance-biregular graphs is new. The following characterization of pseudo-distance-regular vertices was proved by Fiol, Garriga, and Yebra [74], though our formulation and proof are new.

**4.4.1 Theorem.** *Let $G$ be a connected graph with vertex $u$ of eccentricity $e$. Then $u$ is pseudo-distance-regular and only if $|\Phi_u| = e+1$ and there exists a polynomial $p$ of degree $e$ such that $p(A)\,\mathbf{e}_u$ is nonzero precisely on the vertices at distance $e$ from $u$.*

*Proof.* By definition, if $G$ is pseudo-distance-regular at $u$ then the pseudo-distance polynomial $p_e(A)\,\mathbf{e}_u$ has the desired property. Thus we may let $u$ be a spectrally extremal vertex with eccentricity $e$ and suppose there exists a polynomial $p$ of degree $e$ such that $p(A)\,\mathbf{e}_u$ is nonzero precisely on the vertices at distance $e$ from $u$.

Let $p_0, \ldots, p_{e-1}$ be a sequence of orthogonal polynomials with respect to the $u$-inner product. If $0 \leq i < e$ then for any vertex $v$ at distance $e$ from $u$, we have $\mathbf{e}_v^T A^i \mathbf{e}_u = 0$, and therefore

$$\langle p, p_i \rangle = \mathbf{e}_u^T p(A)\, p_i(A)\,\mathbf{e}_u = 0,$$

Since $p$ is orthogonal to $p_0, \ldots, p_{e-1}$, we may append it to our sequence of polynomials as $p_e$. We also append

$$p_{e+1}(x) = \prod_{\theta_r \in \Phi_u} (x - \theta_r)\,.$$

to our sequence.

It is true by our definitions that the only nonzero entries of $p_{e+1}(A)\,\mathbf{e}_u$ and $p_e(A)\,\mathbf{e}_u$ are indexed by vertices at distance, respectively, $e+1$ and $e$ from $u$. Proceeding inductively, we may assume that there exists some $i \leq e$ such that for all $j \geq i$, the only nonzero entries of $p_j(A)\,\mathbf{e}_u$ are at distance $j$ from $u$.

Then by the three-term recurrence for orthogonal polynomials, there exist coefficients $b_{i-1}, a_i, c_{i+1}$ such that

$$Ap_i(A)\,\mathbf{e}_u = b_{i-1}p_{i-1}(A)\,\mathbf{e}_u + a_i p_i(A)\,\mathbf{e}_u + c_{i+1}p_{i+1}(A)\,\mathbf{e}_u.$$

On the other hand,

$$\mathbf{e}_v^T A p_i(A)\,\mathbf{e}_u = \sum_{w \sim v} \mathbf{e}_w^T p_i(A)\,\mathbf{e}_u.$$

By the inductive hypothesis, the nonzero entries of $p_{i+1}(A)\,\mathbf{e}_u$ and $p_i(A)\,\mathbf{e}_u$, are indexed by vertices at distance, respectively, $i+1$ and $i$ from $u$. Thus, we must have that the nonzero entries of $p_{i-1}(A)\,\mathbf{e}_u$ can only be at distance $i-1$ from $u$. Since $p_{i-1}$ is a polynomial of degree $i-1$, we conclude $p_{i-1}(A)\,\mathbf{e}_u$ is nonzero precisely on the vertices at distance $i-1$ from $u$. $\square$

Fiol, Garriga, and Yebra [74] applied this globally to prove that a graph with diameter $d$ is distance-regular if and only if every vertex has eccentricity $d$ and eigenvalue support of size $d+1$, and there exists a polynomial $p$ of degree $d$ such that $p(A) = A_d$. We improve this characterization.

**4.4.2 Theorem.** *Let $G$ be a connected graph with diameter $d$. Then $G$ is distance-regular if and only if $G$ has $d+1$ distinct eigenvalues and there exists a polynomial $f$ of degree $d$ such that $f(A) = A_d$.*

*Proof.* A distance-regular graph clearly has the desired properties, so assume that $G$ has $d+1$ distinct eigenvalues and a polynomial $f$ of degree $d$ exists such that $f(A) = A_d$.

We first prove that every vertex in $G$ has eccentricity $d$. Suppose otherwise, and let $u$ be a vertex of eccentricity $d-1$. Then we have

$$\mathbf{0} = A_d \mathbf{e}_u = f(A)\, \mathbf{e}_u = \sum_{\theta_r \in \Phi u} f(\theta_r)\, E_r \mathbf{e}_u,$$

and since $E_r \mathbf{e}_u$ are eigenvectors for distinct eigenvalues, they must be linearly independent. Thus every eigenvalue in $\Phi_u$ is a root of $f$. By Lemma 4.1.1 we know that $|\Phi_u| \geq d$, and since $f(A)$ is nonzero, there must be precisely one eigenvalue $\theta_s$ of $G$ which is not a root of $f$. Thus we have

$$f(A) = f(\theta_s)\, E_s.$$

Let $v$ be a vertex of eccentricity $d$. By Lemma 4.1.1, all $d+1$ eigenvalues are in $\Phi_v$. On the other hand, we have

$$f(\theta_s)\, \mathbf{e}_v^T E_s \mathbf{e}_v = \mathbf{e}_v^T f(A)\, \mathbf{e}_v = \mathbf{e}_v^T A_d \mathbf{e}_v = 0.$$

Since $f(\theta_s) \neq 0$, this implies that $\mathbf{e}_v^T E_s \mathbf{e}_v = 0$, and since the spectral idempotents are positive semidefinite, this means that $\theta_s \notin \Phi_v$, which is a contradiction. Thus every vertex in $G$ has eccentricity $d$.

Let $u$ be an arbitrary vertex. Since there are only $d+1$ distinct eigenvalues, $u$ must be spectrally extremal, and by assumption we have

$$f(A)\, \mathbf{e}_u = A_d \mathbf{e}_u,$$

so by Theorem 4.4.1, we see $u$ is pseudo-distance-regular. Thus every vertex is pseudo-distance-regular. In fact, if we apply the proof of Theorem 4.4.1 using the graph inner product instead of the vertex inner product, we see that not only is every vertex pseudo-distance-regular, but it has the same sequence of pseudo-distance polynomials. Then by Theorem 4.3.5, the graph $G$ is distance-regular. □

Fiol [70] gave several characterizations of distance-regular graphs, including the characterization of Theorem 4.4.2, and asked for distance-biregular analogues. We give such an analogue with the following theorem.

**4.4.3 Theorem.** *Let $G = (\beta \cup \gamma, E)$ be a connected bipartite graph with diameter $d$. Then $G$ is distance-biregular if and only if $G$ has $d+1$ distinct eigenvalues and there exist polynomials $f^{\beta}$ and $f^{\gamma}$ of degree $d$ such that for $\pi \in \{\beta, \gamma\}$, we have*

$$f^{\pi}(A)\,\mathbf{E}_{\pi} = A_d \mathbf{E}_{\pi}.$$

*Proof.* These conditions clearly hold if $G$ is distance-biregular, so we may assume that $G$ is a graph with $d+1$ distinct eigenvalues and polynomials $f^{\beta}, f^{\gamma}$. Assume without loss of generality that $d$ is the covering radius of $\beta$, and let $d'$ be the covering radius of $\gamma$.

We begin by showing that every vertex in $\beta$ has eccentricity $d$ and every vertex in $\gamma$ has eccentricity $d'$. If $d' = d$, then the same argument as in the proof of Theorem 4.4.2 applies, and so every vertex has eccentricity $d$. Thus, we can assume $d' = d-1$ and $d$ is even.

Suppose there exists a vertex $u \in \beta$ with eccentricity $d-2$. Then since

$$\mathbf{0} = A_d \mathbf{e}_u = f^{\beta}(A)\,\mathbf{e}_u,$$

we again see that every eigenvalue in $\Phi_u$ is a root of $f^{\beta}$. Since $f^{\beta}(A)$ is nonzero, there must be at most two eigenvalues $\theta_r, \theta_s$ of $G$ which are not roots of $f^{\beta}$. Then

$$f^{\beta}(A) = f^{\beta}(\theta_r)\,E_r + f^{\beta}(\theta_s)\,E_s.$$

Since $d$ is even, the off-diagonal blocks of $f^{\beta}(A)$ are zero and Lemma 2.6.5 tells us that $\theta_r = -\theta_s$.

Let $v \in \beta$ have eccentricity $d$. Then all $d+1$ eigenvalues of $G$ must be in $\Phi_v$. On the other hand, we have

$$0 = \mathbf{e}_v^T f^{\beta}(A)\,\mathbf{e}_v = 2f^{\beta}(\theta_r)\,\mathbf{e}_v^T R_r \mathbf{e}_v \neq 0.$$

Therefore every vertex in $\beta$ has eccentricity $d$, from which it immediately follows that every vertex in $\gamma$ has eccentricity $d-1$.

Since there are only $d+1$ distinct eigenvalues, every vertex in $\beta$ is spectrally extremal, and for any vertex $u \in \beta$, we know

$$f^{\beta}(A)\,\mathbf{e}_u = A_d \mathbf{e}_u,$$

so by Theorem 4.4.1, every vertex in $\beta$ is pseudo-distance-regular.

If $d' = d$, an analogous argument shows $\gamma$ is pseudo-distance-regular. Otherwise, $d' = d-1$. Because

$$f^{\gamma}(A)\,\mathbf{E}_{\gamma} = \mathbf{0},$$

we see that for any vertex $v \in \gamma$, we must have $|\Phi_v| \leq d = d'+1$, so vertices in $\gamma$ are spectrally extremal. Further, since we have already shown that the set $\beta$ is pseudo-distance-regular, we know that there exists a polynomial $p^{\beta}_{d-1}$ of degree $d-1$ such that $p^{\beta}_{d-1}(A)\,\mathbf{E}_{\beta}$ is nonzero precisely on the vertices at distance $d-1$ from vertices in $\beta$. But $p^{\beta}_{d-1}$ is symmetric, so $p^{\beta}_{d-1}(A)\,\mathbf{E}_{\gamma}$ must also be nonzero precisely on the vertices at distance $d-1$ from vertices in $\gamma$. Then by Theorem 4.4.1 every vertex in $\gamma$ is pseudo-distance-regular, so by Theorem 4.3.5, the graph $G$ is distance-biregular. □

## 4.5 Distance-Regular Halved Graphs

In Theorem 3.2.3, we proved that a distance-biregular graph has distance-regular halved graphs. A natural question is to ask if the converse of Theorem 3.2.3 holds. That is, if a bipartite graph has distance-regular halved graphs, is it distance-biregular? The short answer is no, but there is a longer answer that uses the characterization of Theorem 4.4.3.

The path on four vertices has distance-regular halved graphs, but is not distance-biregular. More generally, given a distance-biregular graph, we can remove an edge in a four-cycle. The halved graphs are the same, but the resulting graph is not distance-biregular.

Let $G = (\beta \cup \gamma, E)$ be a distance-biregular graph with biadjacency matrix $N$ and halved graphs $H_\beta, H_\gamma$ respectively. The adjacency matrix of the halved graphs are related to the original graph by

$$NN^T = c_2^\beta A(H_\beta) + k \tag{4.5.1}$$

and

$$N^T N = c_2^\gamma A(H_\gamma) + \ell. \tag{4.5.2}$$

The earlier construction of breaking four-cycles no longer preserves this relationship, leading to a rephrasing of our original question.

**4.5.1 Question.** *If a bipartite graph has distance-regular halved graphs satisfying Equations 4.5.1 and 4.5.2, is it distance-biregular?*

The short answer is still no.

*4.5.2 Example.* Consider the graph in Figure 4.3. It has spectrum

$$\left\{3^{(1)}, \sqrt{3}^{(4)}, 1^{(3)}, (-1)^{(3)}, \left(-\sqrt{3}\right)^{(4)}, (-3)^{(1)}\right\}.$$

Since it has diameter 4, but 6 distinct eigenvalues, it cannot be distance-regular.

The halved graphs are isomorphic, so we speak of them with the singular $H$. Let $N$ be the biadjacency matrix. We have

$$NN^T = 3I + A(H)\,,$$

so the halved graph satisfies Equation 4.5.1.

Further, $H$ has spectrum

$$\left\{6^{(1)}, 0^{(4)}, (-2)^{(3)}\right\},$$

so since it is a connected, regular graph with diameter two and three distinct eigenvalues, it is strongly regular.

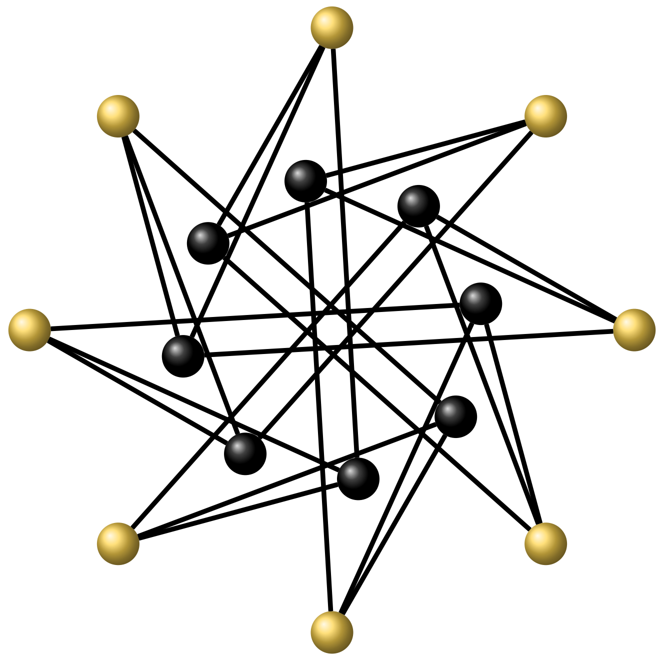

Figure 4.3: $\overline{4K_2}$ Halved Graphs

*4.5.3 Example.* Consider the graph in Figure 4.4, where the vertices on the opposite sides of the grid with the same label are identified. This graph was first described by Delorme [55].

The graph has diameter five, and spectrum

$$\left\{3^{(1)}, \sqrt{5}^{(6)}, 1^{(9)}, (-1)^{(9)}, \left(-\sqrt{5}\right)^{(6)}, (-3)^{(1)}\right\},$$

so the halved graph has diameter two and spectrum

$$\left\{6^{(1)}, 2^{(6)}, (-2)^{(9)}\right\},$$

and therefore the halved graph is strongly regular.

However, there are two paths of length 3 from the black vertex $(2,2)$ to the gold vertex $(2,1)$, but only one walk of length 3 from black $(2,2)$ to gold $(2,1)$. Therefore $G$ is not distance-biregular.

*4.5.4 Remark.* The graph in Figure 4.4 is notable for another reason. Fiol [70] asked whether a graph of diameter $d$ and $d+1$ distinct eigenvalues is distance-regular if there exists an orthogonal sequence of polynomials $F_0, \ldots, F_d$ such that $F_{d-1}(A) = A_{d-1}$. The answer is no, since the graph of Figure 4.4 has an orthogonal sequence of polynomials with $F_4$ defined

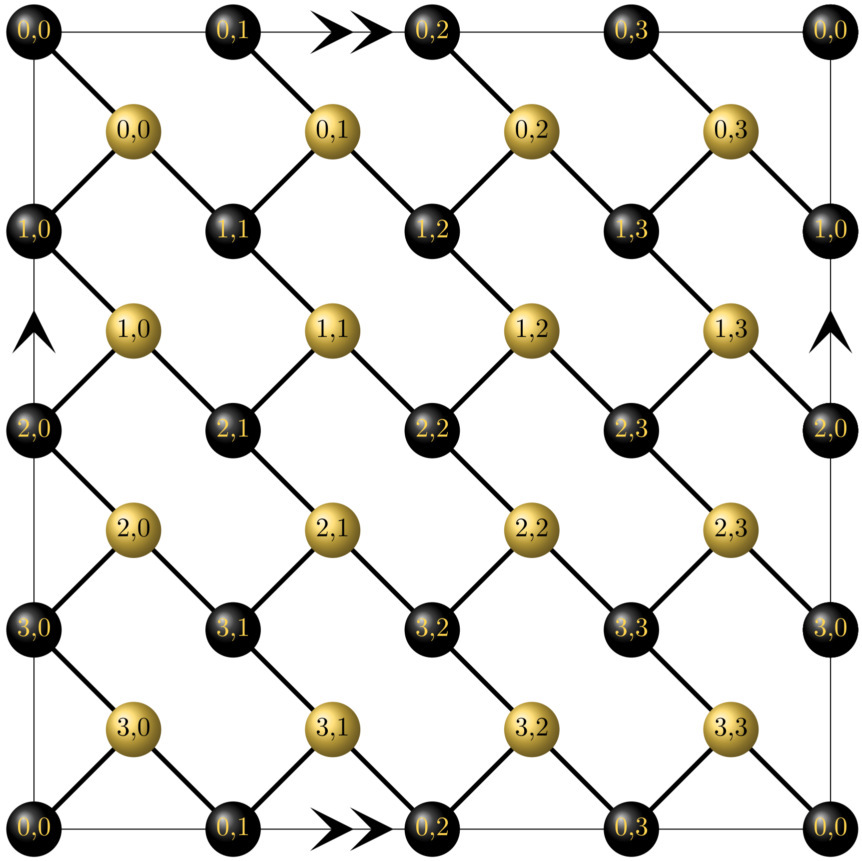


Figure 4.4: Shrikhande Halved Graphs

by

$$F_4(x) := \frac{1}{2}x^4 - 4x^2 + \frac{9}{2}$$

satisfying $F_4(A) = A_4$, but is not distance-regular.

Both the graph of Figure 4.3 and Figure 4.4 are regular. This turns out to be necessary to counterexamples to Question 4.5.1.

**4.5.5 Theorem.** *Let $G = (\beta \cup \gamma, E)$ be a $(k, \ell)$ semiregular bipartite graph with $k < \ell$ and biadjacency matrix $N$. If the halved graphs $H_\beta$ and $H_\gamma$ are distance-regular and there exist $r, s$ satisfying*

$$NN^T = rA(H_\beta) + k \tag{4.5.3}$$

*and*

$$N^T N = sA(H_\gamma) + \ell,$$

*then $G$ is distance-biregular.*

*Proof.* Let $\theta_0 > \cdots > \theta_{t-1}$ be the positive eigenvalues of $G$. Note that $G$ has $2t+1$ eigenvalues. Then the $t+1$ distinct eigenvalues of $A(H_\beta)$ are

$$\left\{\frac{1}{r}\big(\theta_i^2 - k\big) : 0 \le i \le t\right\} \bigcup \left\{-\frac{k}{r}\right\}$$

since $|\beta| > |\gamma|$. Similarly, the distinct eigenvalues of $A(H_\gamma)$ are

$$\left\{\frac{1}{s}\big(\theta_i^2 - \ell\big) : 0 \le i \le t-1\right\}$$

and possibly $-\frac{\ell}{s}$ depending on whether 0 is an eigenvalue of $N^T N$ or not.

Since $H_\beta$ is distance-regular, we know that it has diameter $t$. Similarly, $H_\gamma$ has diameter $t$ or $t-1$ depending on whether or not 0 is an eigenvalue of $N^T N$. Therefore, $G$ has diameter $2t$ or $2t+1$. Since $G$ only has $2t+1$ distinct eigenvalues, it must have diameter $2t$.

Now, since $H_\beta$ is distance-regular, we know there exists a polynomial $F_t^\beta$ of degree $t$ such that

$$F_t^\beta(A(H_\beta)) = A_t(H_\beta)\,.$$

Then using Equation 4.5.3, we have a polynomial $f^\beta$ of degree $2t$ satisfying

$$f^\beta(A(G))\,\mathbf{E}_\beta = A_{2t}(G)\,\mathbf{E}_\beta.$$

We can similarly obtain $f^\gamma$ satisfying

$$f^\gamma(A(G)) = A_{2t}(G)\,\mathbf{E}_\gamma$$

from the $t$-th distance polynomial for $H_\gamma$. Then by Theorem 4.4.3, we know $G$ is distance-biregular. □

*4.5.6 Remark.* The assumption that $k < \ell$ let us conclude that 0 is an eigenvalue of $NN^T$. Otherwise, it is possible that $G$ has $2t+2$ distinct eigenvalues. If $G$ has diameter $2t$ and $2t+2$ distinct eigenvalues, then we cannot apply Theorem 4.4.3, because the vertices are not spectrally extremal. This is what happens with the graph in Figure 4.3. If $G$ has diameter $2t+1$ and $2t+2$ distinct eigenvalues, then we cannot use the even distance polynomials to compute the diametral distance adjacency matrix, so we cannot apply Theorem 4.4.3, as is the case with the graph in Figure 4.4.

## 4.6 Spectral Excess

Let $G$ be a connected graph with vertex $u$ of eccentricity $e$. Let $\rho$ be the largest eigenvalue with corresponding Perron vector $\mathbf{p}$. For $0 \leq i \leq e$, we define the vector $\mathbf{p}_i$ by

$$\mathbf{e}_v^T \mathbf{p}_i = \begin{cases} \mathbf{e}_v^T \mathbf{p} & d(u,v) \leq i \\ 0 & d(u,v) > i \end{cases}.$$

We say that a polynomial $g_i^u(x)$ of degree $i$ is the *$u$-local $i$-excess polynomial* if it satisfies

$$g_i^u(A)\,\mathbf{e}_u = \mathbf{p}_i.$$

*4.6.1 Remark.* The *$i$-excess* of a vertex $u$ is the number of vertices that are at distance at least $i+1$ from $u$. Fiol and Garriga [73] gave a bound on the excess of a vertex in terms of the largest eigenvalue and the Perron vector. This motivates the *spectral excess theorem*, a characterization of distance-regular graphs when the excess of a vertex matches a spectral value. Although the connection between the spectrum and the excess is less obvious in our treatment, we ultimately derive spectral excess theorems for distance-regular and distance-biregular graphs.

If $u$ is spectrally extremal and has a sequence of excess polynomials $g_0, \ldots, g_e$, then the sequence $g_0, g_1 - g_0, g_2 - g_1, \ldots, g_e - g_{e-1}$ gives us a sequence of pseudo-distance polynomials. In fact, by Theorem 4.4.1, vertex $u$ is pseudo-distance-regular if and only if is spectrally extremal and the $(e-1)$-excess polynomial $g_{e-1}$ exists. Fiol and Garriga [73] used the notion of excess polynomials to give a spectral excess theorem for distance-regular graphs. Our set-up is different, and we expand their results to a characterization of distance-biregular graphs.

Let $N_i(u)$ denote the set of vertices at distance at most $i$ from $u$. We combine and reformulate several results from Fiol and Garriga [73] to obtain the following result.

**4.6.2 Proposition.** *Let $u$ be a vertex of eccentricity $e$. Let $0 \leq i \leq e$, and let $G_i$ be a polynomial of degree $i$ with $\|G_i\|_u = 1$. Then*

$$G_i(\rho) \leq \frac{1}{\mathbf{e}_u^T \mathbf{p}} \sqrt{\sum_{v \in N_i(u)} (\mathbf{e}_u^T \mathbf{p})^2} \tag{4.6.1}$$

*with equality if and only if the $i$-excess polynomial exists.*

*Proof.* Consider

$$\langle G_i(A)\,\mathbf{e}_u, \mathbf{p} \rangle.$$

By the spectral decomposition, we see this equals $G_i(\rho)\,\mathbf{e}_u^T \mathbf{p}$.

On the other hand, we can view $G_i(A)\,\mathbf{e}_u$ as a vector indexed by vertices. Since $G_i$ is a polynomial of degree $i$, we know the entries of the vector indexed by vertices at distance at least $i+1$ from $u$ must be zero. By Cauchy-Schwarz, we see

$$\langle G_i(A)\,\mathbf{e}_u, \mathbf{p}\rangle \leq \|G_i\|_u \|\mathbf{p}_i\| = \sqrt{\sum_{v\in N_i(u)} (\mathbf{e}_v^T \mathbf{p})^2}$$

If equality holds, then $G_i(A)\,\mathbf{e}_u$ is a scalar multiple of $\mathbf{p}_i$. In particular, to have the correct norm, we must have

$$G_i(A)\,\mathbf{e}_u = \frac{1}{\sqrt{\sum_{v\in N_i(u)} (\mathbf{e}_v^T \mathbf{p})^2}} \mathbf{p}_i.$$

so

$$\mathbf{e}_u^T \mathbf{p} G_i(\rho)\, G_i(A)\,\mathbf{e}_u = \mathbf{p}_i$$

and the $i$-excess polynomial exists.

Conversely, if the $i$-excess polynomial $g_i$ exists, then by the spectral decomposition

$$g_i(\rho)\,\mathbf{e}_u^T \mathbf{p} = \mathbf{p}^T g_i(\rho)\,\mathbf{p}\mathbf{p}^T \mathbf{e}_u = \mathbf{p}^T g_i(A)\,\mathbf{e}_u = \mathbf{p}^T \mathbf{p}_i = \sum_{v\in N_i(u)} \left(\mathbf{e}_v^T \mathbf{p}\right)^2,$$

so $\frac{g_i(x)}{\|g_i\|_u}$ is a polynomial of $u$-norm one achieving the bound of Equation 4.6.1. □

Applied globally, this gives us a characterization of distance-regular graphs, as obtained by Fiol and Garriga [73]. We reprove it here using a different formulation, because it gives us a preview of the main ideas used to prove a distance-biregular analogue with less involved casework.

**4.6.3 Theorem** (Fiol and Garriga [73]). *Let $G$ be a regular connected graph with diameter $d$ and largest eigenvalue $\rho$. It is distance-regular if and only if it has $d+1$ distinct eigenvalues and there exists a polynomial $f$ of degree $d$ such that for all vertices $u$, we have*

$$\|f\|_G^2 = f(\rho) = |\{v \in V(G) : d(u,v) = d\}|\,. \tag{4.6.2}$$

*Proof.* If $G$ is distance-regular, then the distance polynomial $F_d$ has the desired properties. Otherwise, suppose $f$ is a polynomial satisfying Equation 4.6.2. Let $n$ be the number of vertices of $G$ and let $t = f(\rho)$ be the number of vertices at distance $d$ from any vertex in the graph. Let $f_d(x) = f(x)$ and $f_{d+1}(x)$ be the minimal polynomial of $G$. We extend this to an orthogonal sequence $f_{d+1}, f_d, \dots, f_0$.

Define

$$q(x) = n \prod_{\theta_r \neq \rho} \frac{x - \theta_r}{\rho - \theta_r}.$$

We have

$$\|q\|_G^2 = \frac{1}{n} \sum_{u \in V(G)} \mathbf{e}_u^T q(A)^2 \, \mathbf{e}_u = \frac{1}{n} \sum_{r=0}^{d} m_r q(\theta_r)^2 = n$$

and

$$\langle q, f \rangle_G = \frac{1}{n} \sum_{r=0}^{d} m_r q(\theta_r) \, f(\theta_r) = t.$$

The projection of $q$ onto $f$ is

$$\left( \frac{\langle q, f \rangle_G}{\|f\|_G^2} \right) f = f,$$

and so we must be able to write $q - f$ as a linear combination of $f_0, \ldots, f_{d-1}$. Thus

$$Q(x) := \frac{1}{\sqrt{n-t}} (q(x) - f(x))$$

is a polynomial of degree at most $d-1$. Note that

$$\|Q\|_G^2 = \frac{1}{n-t} \Big( \|q\|_G^2 - 2\langle q, f \rangle_G + \|f\|_G^2 \Big) = 1.$$

Since $G$ is regular, the Perron vector is $\frac{1}{\sqrt{n}}\mathbf{1}$, thus for any vertex $u$, we have

$$Q(\rho) = \frac{1}{\sqrt{n-t}} (n-t) = \sqrt{n-t} = \frac{1}{\mathbf{e}_u^T \mathbf{p}} \sqrt{\sum_{v \in N_{d-1}(u)} (\mathbf{e}_v^T \mathbf{p})^2}.$$

By Proposition 4.6.2 we know that $\|Q\|_u \geq 1$. Therefore

$$n \leq \sum_{u \in V(G)} \|Q\|_u^2 = \sum_{u \in V(G)} \sum_{r=0}^{d} \mathbf{e}_u^T E_r \mathbf{e}_u Q(\theta_r)^2 = \sum_{r=0}^{d} m_r Q(\theta_r)^2 = n\|Q\|_G^2 = n.$$

and since equality holds, it must be the case that $\|Q\|_u = 1$ for all vertices $u$. Then by Proposition 4.6.2, we know the $(d-1)$-excess polynomial exists, so by Theorem 4.4.1, every vertex is pseudo-distance-regular. Therefore by Theorem 4.3.5, $G$ is distance-regular. □

We can use similar techniques to give a new characterization of distance-biregular graphs.

**4.6.4 Theorem.** *Let $G$ be a semiregular bipartite connected graph with diameter $d$ and largest eigenvalue $\rho$. It is distance-biregular if and only if it has $d+1$ distinct eigenvalues and there exist polynomials $f^\beta, f^\gamma$ of degree $d$ such that for $\pi \in \{\beta, \gamma\}$ and all vertices $u \in \pi$, we have*

$$\|f^\pi\|_\pi^2 = f^\pi(\rho) = \left|\{v \in V(G) : d(u,v) = d\}\right|. \tag{4.6.3}$$

*Proof.* If $G$ is distance-biregular, then the distance polynomials $F_d^\beta, F_d^\gamma$ have the desired property. If $d$ is odd and $G$ has $d+1$ distinct eigenvalues, then by Lemma 2.6.3 the graph $G$ is regular, and so the inner products determined by $\beta$ and $\gamma$ are the same. Thus the polynomials $f^\beta$ and $f^\gamma$ must be the same, so we use Theorem 4.6.3 to conclude $G$ is distance-regular.

Now, assume without loss of generality that $\beta$ has covering radius $d$ with $d$ even and $f^\beta$ satisfies Equation 4.6.3. As before, we let $f_d^\beta = f^\beta$ and expand this to an orthogonal sequence of polynomials $f_{d+1}^\beta, f_d^\beta, \ldots, f_0^\beta$. Let $t = f^\beta(\rho)$ be the number of vertices at distance $d$ from any vertex in $\beta$.

Let

$$q^\beta(x) = |\beta| \prod_{\theta_r \neq \rho} \frac{x-\theta_r}{\rho-\theta_r}.$$

Note $\|q^\beta\|_\beta = |\beta|$ and $\langle q^\beta, f^\beta \rangle = t$, so

$$q^\beta - f^\beta = q^\beta - \frac{\langle f^\beta, q^\beta \rangle_\beta}{\|f^\beta\|_\beta} f^\beta$$

is orthogonal to $f_d^\beta$. We define

$$Q^\beta(x) := \frac{1}{\sqrt{|\beta| - t}} \Big( q^\beta(x) - f^\beta(x) \Big)$$

and note that $Q^\beta$ is a polynomial of degree at most $d-1$ satisfying

$$\|Q^\beta\|_\beta^2 = \frac{1}{|\beta| - t} \Big( \|q^\beta\|_\beta^2 - 2\langle q^\beta, f^\beta \rangle_\beta + \|f^\beta\|_\beta^2 \Big) = 1.$$

Since $G$ is bipartite semiregular, for any vertex $u \in \beta$, we have

$$\mathbf{e}_u^T \mathbf{p} = \frac{1}{\sqrt{2\,|\beta|}},$$

so since $d$ is even,

$$Q^\beta(\rho) = \frac{1}{\sqrt{|\beta| - t}} (|\beta| - t) = \frac{1}{\mathbf{e}_u^T \mathbf{p}} \sqrt{\sum_{v \in N_i(u)} (\mathbf{e}_v^T \mathbf{p})^2}.$$

By Proposition 4.6.2, we know that $\|Q\|_u \geq 1$, and so

$$|\beta| \leq \sum_{u \in \beta} \|Q\|_u^2 = \sum_{r=0}^{d} m_r^\beta Q(\theta_r)^2 = |\beta| \, \|Q^\beta\|_\beta^2 = |\beta| \, ,$$

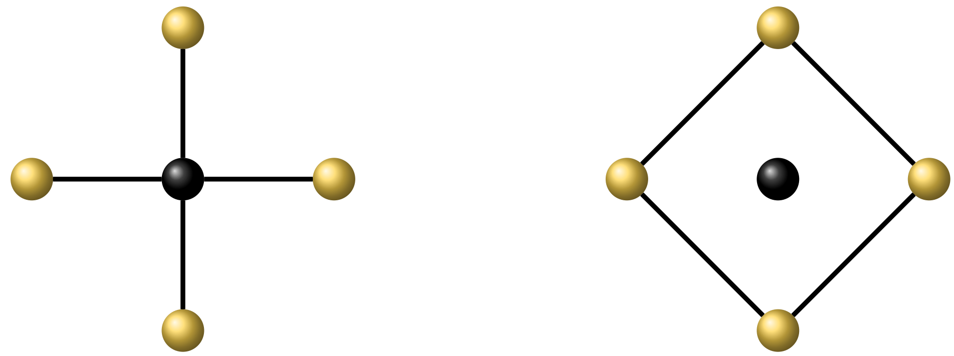

Figure 4.5: Cospectral graphs, not both semiregular

and since equality holds it must be the case that $\|Q\|_u = 1$ for every vertex $u \in \beta$. Then by Proposition 4.6.2 and Theorem 4.4.1, every vertex in $\beta$ is pseudo-distance-regular.

If $\gamma$ also has covering radius $d$, the same argument tells us every vertex in $\gamma$ is pseudo-distance-regular. Otherwise we know that for every vertex $u \in \gamma$, we have

$$\sum_{\theta_r \in \Phi_u} m_r^\gamma f^\gamma(\theta_r)^2 = 0,$$

and so every vertex in $\gamma$ is spectrally extremal. Further, since every vertex in $\beta$ is pseudo-distance-regular, we know that the polynomial $f_{d-1}^\beta$ from our orthogonal sequence is a polynomial of degree $d-1$ such that $f_{d-1}^\beta(A)\,\mathbf{E}_\beta$ is nonzero precisely on the vertices at distance $d-1$ from vertices in $\beta$. Since $f_{d-1}^\beta(A)$ is symmetric, it follows that $f_{d-1}^\beta(A)\,\mathbf{E}_\gamma$ is nonzero precisely on the vertices at distance $d-1$ from vertices in $\gamma$. Thus by Theorem 4.4.1, every vertex in $\gamma$ is pseudo-distance-regular. Then since every vertex in the bipartite graph $G$ is pseudo-distance-regular, Theorem 4.3.5 tells us that $G$ is distance-biregular. □

Fiol [72] previously proved a spectral excess theorem for distance-biregular graphs with three cases and four different polynomials. Theorem 4.6.4 is much cleaner, and a closer analogue to the spectral excess theorem of Fiol and Garriga [73].

## 4.7 Spectrum of a Distance-Biregular Graph

Recall that the inner product used in Theorem 4.6.3 is determined by the spectrum. Thus, the characterization of Theorem 4.6.3 is spectral in nature, since it shows that a graph is distance-regular if and only the number of vertices at maximal distance matches a particular spectral quantity. A well-known result which can be found, for instance, in Cvetković, Doob, and Sachs [44], says that any graph with the spectrum of a connected regular graph is regular with the same valency. However, the same is not true for connected semiregular graphs, as evident by Figure 4.5.

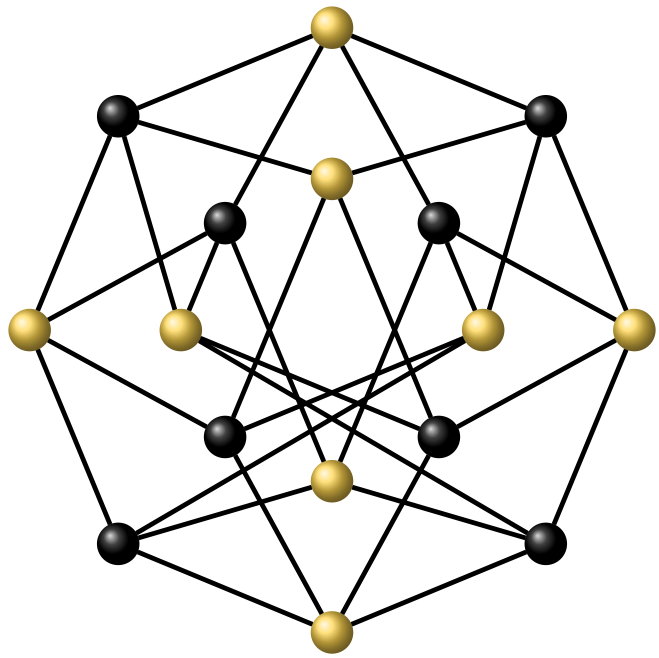

Figure 4.6: Graph cospectral to $Q_4$

Van Dam and Haemers [45] proved that for a distance-regular graph, the spectrum determined the intersection array, and we extended that to distance-biregular graphs in Theorem 2.7.2. However, our extension used the additional information of the valencies of the cells of the partition. It is unclear if knowing the valencies is necessary to determine a feasible array from the spectrum.

**4.7.1 Problem.** *If two distance-biregular graphs are cospectral, do they have the same intersection array?*

Even in the better-studied case where intersection array of a distance-regular graph is determined by the spectrum, it is not true that any graph cospectral to a distance-regular is itself distance-regular.

*4.7.2 Example.* Consider the graph in Figure 4.6. Hoffman [97] constructed this graph, and showed that it is cospectral to the bipartite distance-regular hypercube of Figure 2.1, even though the graph in Figure 4.6 is not distance-regular. Thus neither distance-regularity nor distance-biregularity are determined by the spectrum.

Recall that a connected bipartite regular graph with diameter three and four distinct eigenvalues is distance-regular. The graph of Figure 4.6 has diameter four, so in general there is no additional spectral information that allows us to completely characterize which graphs of diameter four are distance-biregular.

The example of Hoffman [97] and characterization of Cvetković, Doob, and Sachs [44] led Brouwer, Cohen, and Neumaier [28] to ask whether a graph cospectral to a distance-regular graph of diameter three is distance-regular. Haemers [88] answered that question in the negative, but provided certain spectral or structural conditions that force distance-regularity.

Theorem 4.6.3 gives us one condition, but we have seen a more tangible condition in Proposition 2.1.5 with Moore graphs. If $G$ is a $k$-regular graph, then the girth of $G$ is the smallest positive integer $i$ such that

$$\operatorname{tr}\Big(F_i^k(A)\Big) = \sum_r m_r F_i^k(\theta_r) \neq 0,$$

and as such is determined by the spectrum. Thus we know that any graph with the spectrum of a Moore graph is distance-regular.

Brouwer and Haemers [29] extended this by proving that any graph cospectral with a distance-regular with diameter $d$ and girth $2d-1$ is distance-regular. Subsequently, Van Dam and Haemers [46] gave an alternate proof that used Theorem 4.6.3. This was improved on more recently by Abiad, Van Dam, and Fiol [3] who proved that any graph with diameter $d$, $d+1$ distinct eigenvalues, and girth $2d-1$ is distance-regular.

More generally, the question of when a graph with the spectrum of a distance-regular graph is distance-regular has also been considered by Van Dam and Haemers [45] and Haemers and Spence [91]. However, the corresponding question for distance-biregular graphs has barely been considered.

**4.7.3 Problem.** *When is a graph with the spectrum of a distance-biregular graph distance-biregular?*

Example 2.2.7 showed us that any semiregular bipartite graph with diameter $d$ and girth $2d$ is a distance-biregular. For regular graphs, Van Dam and Haemers [45] proved that any graph cospectral to a bipartite distance-regular graph of diameter $d$ and girth $2d-2$ is distance-regular. Abiad, Van Dam, and Fiol [3] loosened the conditions and considered a bipartite graph $G$ with diameter $d$ and $d+1$ distinct eigenvalues. They proved that if $G$ has girth at least $2d-2$, then $G$ is distance-regular. We can extend this to distance-biregular graphs.

**4.7.4 Theorem.** *Let $G$ be a connected semiregular bipartite graph with diameter $d$ and $d+1$ distinct eigenvalues. If $G$ has girth $g \geq 2d-2$, then $G$ is distance-biregular.*

*Proof.* Suppose without loss of generality that $\beta$ has covering radius $d$. Let $u \in \beta$.

Since $g \geq 2d-2$, for all $0 \leq i \leq d-2$ we know that there is a unique non-backtracking walk of length at most $i$ from $u$ to a vertex at distance $i$ from $u$. Then by Lemma 2.2.4, we have

$$F_i^{k,\ell}(A)\,\mathbf{e}_u = A_i\mathbf{e}_u$$

for all $0 \leq i \leq d-2$. Similarly, for $v \in \gamma$ we have

$$F_i^{\ell,k}(A)\,\mathbf{e}_v = A_i\mathbf{e}_v.$$

If $d$ is even, we define

$$f_d^{\beta}(x) := |\beta| \prod_{\substack{\theta_r>0\\ \theta_r\neq\sqrt{k\ell}}} \frac{x^2-\theta_r^2}{k\ell-\theta_r^2} - \sum_{i=0}^{\frac{d}{2}-1} F_{2i}^{k,\ell}(x)\,.$$

Note that $f_d^{\beta}$ is a polynomial of degree $d$ such that

$$f_d^{\beta}(A)\,\mathbf{E}_B = \begin{pmatrix}\mathbf{1}_{|\beta|}\\ 0\end{pmatrix} - \sum_{i=0}^{\frac{d}{2}-1} A_{2i}\mathbf{E}_\beta = A_{2d}\mathbf{E}_\beta.$$

The same argument applies to

$$f_d^{\gamma}(x) := |\gamma| \prod_{\substack{\theta_r>0\\ \theta_r\neq\sqrt{k\ell}}} \frac{x^2-\theta_r^2}{k\ell-\theta_r^2} - \sum_{i=0}^{\frac{d}{2}-1} F_{2i}^{\ell,k}(x)\,,$$

so by Theorem 4.4.3, $G$ is distance-biregular.

If $d$ is odd, then by Lemma 2.6.3 we know $k=\ell$, so we can define

$$f_d(x) := |\beta|\,x \prod_{\substack{\theta_r>0\\ \theta_r\neq\sqrt{k\ell}}} \frac{x^2-\theta_r^2}{k\ell-\theta_r^2} - \sum_{i=1}^{\frac{d-1}{2}-1} F_{2i+1}^{k,\ell}(x)\,,$$

and we have

$$f_d^{\beta}(A) = \begin{pmatrix}\mathbf{0} & \mathbf{1}_{|\beta|,|\gamma|}\\ \mathbf{1}_{|\gamma|,|\beta|} & \mathbf{0}\end{pmatrix} - \sum_{i=1}^{\frac{d-1}{2}-1} A_{2i+1} = A_d,$$

so by Theorem 4.4.2, $G$ is distance-regular. □

# Chapter 5

# Spectral Moore Bound

Besides their nice combinatorial and algebraic properties, one of the reasons distance-regular graphs are so well studied is that they arise as extremal examples of graphs with particular structural or spectral conditions. The canonical text on distance-regular graphs by Brouwer, Cohen and Neumaier [28] spends multiple sections in the first chapter setting up families of distance-regular graphs as extremal examples. In this chapter, we extend this extremal perspective to distance-biregular graphs, with an emphasis on variants of the Moore problem.

Cioabă, Koolen, Nozaki, and Vermette [37] obtained an upper bound on the number of vertices a graph with fixed valency and second-largest eigenvalue could have, and proved that when the bound is tight, the graph is distance-regular. This problem can be seen as a spectral version of the Moore problem, as described by Cioabă [34]. Cioabă, Koolen, and Nozaki [36] strengthened this spectral Moore bound for bipartite regular graphs, and Cioabă, Koolen, Mimura, Nozaki, and Okuda [35] proved a hypergraph version of the bound. The motivating question for this chapter is to extend the spectral Moore bound, and the characterization of when the bound is tight, to bipartite semi-regular graphs.

**5.0.1 Question.** *What is the maximum number of vertices that a semiregular bipartite graph with fixed valencies and second-largest eigenvalue can have? What graphs meet this bound?*

We will answer Question 5.0.1 by deriving a bound on the size of the cells of a bipartite semiregular graph, and prove that when the bound is tight, the graph is distance-biregular. The eigenvalues of the halved graphs are determined by the eigenvalues of a distance-biregular graph, and the halved graphs are regular, so we can compare this bound to the bound obtained by Cioabă, Koolen, Nozaki, and Vermette [37] for regular graphs. Further, a hypergraph can be represented as a bipartite incidence graph, where one cell of the partition is the vertices and the other cell is the hyperedges. Thus, as noted by Cioabă, Koolen, Mimura, Nozaki, and Okuda [35], the hypergraph bound can also be interpreted as an answer to Question 5.0.1. However, in Section 5.7 we describe infinite families of distance-biregular graphs which are tight for the bound we derive in this chapter, but are not tight when considering the halved graphs or the associated hypergraphs.

Many of the results in this chapter can be found in the author's paper [110].

## 5.1 Moore Graphs Revisited

Let $G$ be a $k$-regular graph of diameter $d$. If we fix a vertex $u$, there are $k$ vertices at distance one from $u$, and at most $k(k-1)^{i-1}$ vertices at distance $i$ from $u$. Summing over all possible distances gives us an upper bound on the number of vertices,

$$|V(G)| \leq 1 + \sum_{i=1}^{d} k(k-1)^{i-1}. \tag{5.1.1}$$

This bound was obtained by Moore, who asked for a characterization of graphs that meet the bound [98]. Trivial examples come from the complete graphs and the odd cycles. Although the Moore bound is structural, the extremal examples are characterized by algebraic graph theory. Hoffman and Singleton [98] observed that Petersen graph is a *Moore graph*, and also constructed the eponymous Hoffman-Singleton graph. By computing the multiplicities of the eigenvalues, they proved that the only other possible Moore graphs of diameter two would be 57-regular graphs on 3,250 vertices. The existence of such graphs is still open.

Hoffman and Singleton [98] also proved that there are no nontrivial Moore graphs of diameter three. Damerell [49] and independently Bannai and Ito [17] proved the more general result that the only Moore graphs of diameter at least three are the odd cycles. The idea behind both proofs is to define a notion of feasibility similar to Definition 3.4.1, then prove that no Moore graph with diameter at least three is feasible.

Variations of the Moore bound and Moore graphs have been considered to describe graphs with close to a maximal number of vertices for the valency and diameter, or to consider the Moore bound in a restricted setting. Miller and Širáň [116] gave an overview of these variations. One variation of particular interest to our work is a bipartite version of the Moore bound.

If $G$ is bipartite, then there are many edges from a vertex at distance $d$ to vertices at distance $d-1$, so the bound of Equation 5.1.1 significantly overcounts the number of vertices. We can improve this by bounding one cell of the partition at a time. Specifically, we fix a vertex $u$ and bound the size of the cell of the partition which does not contain vertices at distance $d$ from $u$.

Let $G = (\beta \cup \gamma, E)$ be a $(k,\ell)$-semiregular graph where $\beta$ has covering radius $d$. If $d$ is odd, we bound $\beta$ by choosing a vertex $u \in \beta$ and bounding the vertices at an even distance from $u$. This gives us

$$|\beta| \leq 1 + \sum_{i=0}^{\frac{d-3}{2}} k(\ell-1)^{i+1}(k-1)^i \,. \tag{5.1.2}$$

If $d$ is even, we bound $\beta$ by choosing a vertex $u \in \gamma$ and counting the vertices at an odd distance from $u$ to get

$$|\beta| \leq \ell \sum_{i=0}^{\frac{d-2}{2}} (k-1)^i(\ell-1)^i \,. \tag{5.1.3}$$

When these bounds are tight for both cells of the partition, $G$ is a generalized polygon. If Equation 5.1.3 holds for both cells of the partition, then $k = \ell$, leading Yebra, Fiol, and Fábrega [157] to propose a semiregular version of the Moore bound for odd diameter. More recently, Araujo-Pardo, Dalfó, Fiol, and López [8] improved these bounds, and, for certain choices of $k$ and $\ell$, constructed families of infinite $(k,\ell)$-semiregular graphs meeting these bounds.

One way to improve on the Moore bound uses the eigenvalues and the sequence of polynomials associated to $k$-regular trees described in Example 2.1.1. The following result was observed by several authors, see, for instance, Dinitz, Schapira, and Shahaf [62] or Section 3. 2 of Miller and Širáň [116].

**5.1.1 Theorem.** *[Dinitz, Schapira, and Shahaf [62]]Let $G$ be a connected, $k$-regular graph of diameter $d$ with distinct eigenvalues $k = \theta_0 > \theta_1 > \cdots > \theta_t$. Let $\lambda = \max\{|\theta_1|, |\theta_t|\}$. Then*

$$|V(G)| \leq \sum_{i=0}^{d} F_i^k(k) - \sum_{i=0}^{d} F_i^k(\lambda)\,.$$

As discussed by Cioabă [34], when $\lambda$ is large, $\sum_{i=0}^{d} F_i^k(\lambda)$ is positive so Theorem 5.1.1 provides an improvement over the Moore bound, though when $\lambda$ is small, the Moore bound is better, even if it is rarely tight. Thus a spectral version of the Moore bound asks for an upper bound on the number of vertices where $\lambda$ is small. Towards that direction, we have the following question.

**5.1.2 Question.** *What is the maximum number of vertices that a regular graph with given valency and second-largest eigenvalue can have?*

The answer to Question 5.1.2 has more than motivational connections to Moore graphs. As we shall see, the bound itself and the extremal examples have strong similarities to the Moore bound and Moore graphs.

## 5.2 Spectral Moore Bounds

Cioabă, Koolen, Nozaki, and Vermette [37] proved a spectral Moore bound to answer Question 5.1.2 for regular graphs, and characterized when the bound is tight. Subsequently, Cioabă, Koolen, and Nozaki [36] improved the bound for bipartite regular graphs, and Cioabă, Koolen, Mimura, Nozaki, and Okuda [35] extended the bound to hypergraphs. In this section, we sketch the major ideas in the proof of the spectral Moore bound to set up for our own extension to semiregular bipartite graphs.

One of the major ideas is the linear programming bound of Nozaki [125]. It was introduced to study *extremal expanders*, graphs that minimize the second-largest eigenvalue over all regular graphs on a fixed number of vertices. Using the linear programming bound, he proved that any regular graph with girth at least twice the diameter is an extremal expander. By the work of Abiad, Van Dam, and Fiol [3], any such graph is also distance-regular. The linear programming bound is also a key tool in the spectral Moore bound of Cioabă, Koolen, Nozaki, and Vermette [37].

**5.2.1 Theorem.** *[Linear Programming Bound [125]]Let $G$ be a connected $k$-regular graph with distinct eigenvalues $k = \theta_0 > \theta_1 > \cdots > \theta_t$. If there exists a polynomial*

$$f(x) := \sum_{i=0}^{s} h_i F_i^k(x)$$

*such that*

- *For eigenvalues $\theta_r \neq \theta_0$, we have $f(\theta_r) \leq 0$;*
- *The coefficient $h_0$ is positive; and*
- *For $1 \leq i \leq t$, the coefficient $h_i$ is non-negative,*

*then*

$$|V(G)| \leq \frac{f(k)}{h_0}.$$

*Proof.* If there exists such a polynomial, we have

$$\operatorname{tr}(f(A)) = \sum_{i=0}^{t} h_i \operatorname{tr}\Big(F_i^k(A)\Big) \geq h_0 \operatorname{tr}\Big(F_0^k(A)\Big) = h_0 \left|V(G)\right|.$$

Using the spectral decomposition, we also have

$$\operatorname{tr}(f(A)) = \sum_{r=0}^{t} m_r f(\theta_r) \leq f(k)\,.$$

Combining this gives us the desired result. □

When the bound of Theorem 5.2.1 is tight, we have that $f(\theta_r) = 0$ for all $r \neq 0$, and for all $1 \leq i \leq s$, we have $h_i \operatorname{tr}\big(F_i^k(A)\big) = 0$. In particular, if for some $1 \leq i \leq s$ we have $h_i > 0$, then $\operatorname{tr}\big(F_i^k(A)\big) = 0$. By Lemma 2.1.4, we know $\operatorname{tr}\big(F_i^k(A)\big)$ counts the closed non-backtracking walks on length $i$. Thus if $h_i > 0$ for all $1 \leq i \leq s$, the graph $G$ must have girth at least $s$.

We can now outline the steps involved in the proof of the spectral Moore bound.

**5.2.2 Theorem** (Spectral Moore Bound [37])**.** *Let $G$ be a $k$-regular graph with second-largest eigenvalue $\theta_1 < 2\sqrt{k-1}$. Let $t, c > 0$ be constants determined by the choice of $k$ and $\lambda$. Then*

$$|V(G)| \leq 1 + \sum_{i=0}^{t-3} k(k-1)^i + \frac{k(k-1)^t - 2}{c}.$$

*If equality holds, $G$ is distance-regular.*

*Proof.* For a positive integer $t$ and some positive number $c$, let $T(k,t,c)$ be the $t \times t$ tridiagonal matrix

$$\begin{pmatrix} 0 & k & & & \\ 1 & 0 & k-1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & 0 & k-1 \\ & & & c & k-c \end{pmatrix}.$$

Using properties of the orthogonal polynomials $\left(F_i^k\right)_{i\geq 0}$, Cioabă, Koolen, Nozaki, and Vermette [37] proved that there exists some $t, c$ such that $T(k,t,c)$ has second-largest eigenvalue $\theta_1$.

Now, let $k = \lambda_0 > \lambda_1 > \cdots > \lambda_{t-1}$ be the distinct eigenvalues of $T(k,t,c)$. We claim that

$$f(x) := \frac{1}{c}(x-\lambda_1)\prod_{r=2}^{t-1}(x-\lambda_r)^2$$

satisfies the conditions of Theorem 5.2.1. The non-trivial step is showing that the $h_i$ are non-negative. In fact, Cioabă, Koolen, Nozaki, and Vermette [37] proved that the $h_i$ were all positive.

The third step is to show that Theorem 5.2.1 gives the desired bound.

The fourth step is to characterize what happens when the bound is tight. Since all the coefficients $h_i$ are positive, we know that $G$ has girth at least $2t-3$. Then by showing that $G$ has at most $t$ distinct eigenvalues, we can apply the result of Abiad, Van Dam, and Fiol [3] to conclude that $G$ is distance-regular. □

This outline also applies to prove other spectral Moore bounds, with a different sequence of orthogonal polynomials. Cioabă, Koolen, and Nozaki [36] described the even and odd distance polynomials for a $k$-regular tree and used that to improve the linear programming bound and spectral Moore bound when the graph is bipartite. Cioabă, Koolen, Mimura, Nozaki, and Okuda [35] used a family of polynomials that counted the non-backtracking walks on a uniform regular hypergraph to develop a version of the linear programming bound and spectral Moore bound for hypergraphs. In the remainder of this chapter, we develop a linear programming bound and spectral Moore bound for semiregular bipartite graphs.

As noted by Cioabă, Koolen, and Nozaki [37], one consequence of the spectral Moore bound is the following famous result.

**5.2.3 Corollary** (Alon-Boppana [6])**.** *If $k \geq 3$ and $\theta < 2\sqrt{k-1}$, there are only finitely many $k$-regular graphs with second-largest eigenvalue at most $\theta$.*

The result of Alon and Boppana [6] is a major asymptotic result in the study of graph expanders. Alternative proofs have been given by, for instance, Friedman [77] and Nilli [122, 123], and strengthenings and generalizations have been given by, for instance, Serre [131], Cioabă [33], and Mohar [117].

## 5.3 Linear Programming Bound for Semiregular Graphs

We begin by proving a version of Nozaki's [125] linear programming bound, given in Theorem 5.2.1, for semiregular bipartite graphs. To do so, we consider the sequence of even distance polynomials associated to the semiregular tree described in Section 3.2.

**5.3.1 Theorem.** *Let $G = (\beta \cup \gamma, E)$ be a semiregular graph with valencies $k, \ell$. Let $\sqrt{k\ell} = \theta_0 > \theta_1 > \cdots > \theta_t$ be the set of distinct, non-negative eigenvalues in the eigenvalue support of vertices of $\beta$. Suppose there exists a polynomial*

$$f(x) = \sum_{i=0}^{s} h_i P_i^{k,\ell}(x)$$

*such that*

- *For eigenvalues $\theta_r \neq \theta_0$, we have $f\big(\theta_r^2\big) \leq 0$;*
- *The coefficient $h_0$ is positive;*
- *For $1 \leq i \leq s$, the coefficient $h_i$ is non-negative.*

*Then*

$$|\beta| \leq \frac{f(k\ell)}{h_0}.$$

*Proof.* Using the block decomposition of a bipartite graph, we have

$$F_{2i}^{k,\ell}(A)\,\mathbf{E}_\beta = P_i^{k,\ell}\big(A^2\big)\,\mathbf{E}_\beta = \begin{pmatrix} P_i^{k,\ell}\big(NN^T\big) \\ \mathbf{0} \end{pmatrix}.$$

Then by Lemma 2.2.5, $P_i^{k,\ell}\big(NN^T\big)$ counts the number of non-backtracking walks of length $2i$ beginning in $\beta$, so in particular it must have non-negative entries. Then we have

$$f(k\ell) \geq \sum_{r=0}^{t} m_r^\beta f\big(\theta_r^2\big) = \operatorname{tr}\big(f\big(NN^T\big)\big) = \sum_{i=0}^{t} h_i \operatorname{tr}\Big(P_i^{k,\ell}\big(NN^T\big)\Big) \geq h_0\,|\beta|\,,$$

or

$$|\beta| \leq \frac{f(k\ell)}{h_0}. \qquad \square$$

Similar to the regular case, if equality holds and the coefficients $h_i$ are all positive, then the girth of $G$ must be at least $2s+2$. When $k = \ell$, this is the linear programming bound of

Cioabă, Koolen, and Nozaki [36]. Because the sequence of polynomials used is different, the bound is different than the linear programming bound for hypergraphs of Cioabă, Koolen, Mimura, Nozaki, and Okuda [35].

In Theorem 5.2.1, we can replace the condition that $h_0 > 0$ with the condition that $h_0 = 1$. Then as observed by Nozaki [125], we use the bound to get the linear program.

$$\begin{aligned}
&\text{Minimize } F_1^k(k)\,h_1 + F_2^k(k)\,h_2 + \cdots + F_s^k(k)\,h_s \\
&\text{Subject to} - \sum_{i=1}^{s} F_i^k(\theta_r)\,h_i \geq 1 && 1 \leq r \leq t \\
&\qquad\qquad h_i \geq 0 && 1 \leq i \leq s.
\end{aligned}$$

Taking the dual gives us the linear program

$$\begin{aligned}
&\text{Maximize } m_1 + \cdots + m_t \\
&\text{Subject to} - \sum_{r=1}^{d} F_i(\theta_r)\,m_j \leq F_i^k(k) && 1 \leq i \leq s \\
&\qquad\qquad m_r \geq 0 && 1 \leq r \leq d.
\end{aligned}$$

Previously, Delsarte, Goethals, and Seidel [58] used different sequences of orthogonal polynomials coming from coding theory and spherical designs to derive this dual bound.

We could similarly set up Theorem 5.3.1 as a linear program, and consider the bound that comes from taking the dual linear program. The hope is that, as with regular graphs, this bound could also be interpreted in terms of designs.

**5.3.2 Problem.** *In the context of designs, how can we interpret the dual linear programming bounds for semiregular bipartite graphs of Theorem 5.3.1? What about the dual of hypergraph linear programming bound [35]?*

Nozaki [125] gave a thorough overview of the linear programming bound for regular graphs, the bound on spherical designs from Delsarte, Goethals, and Seidel [58], and related results in design theory and coding theory, with an emphasis on the algebraic structure linking the two. The more recent work on spectral Moore bounds has been focused on the graphs, but it would be worth investigating some of the deeper connections back to coding theory.

The linear programming bound of Nozaki [125] was originally developed to study extremal expanders. We extend the notion of extremal expanders to semiregular bipartite graphs in Section 5.5, for which we need to develop some further theory.

## 5.4 Further Properties of Orthogonal Polynomials

Many of the same ideas used by Nozaki [125] to study extremal expanders were used in the proof of the spectral Moore bound [37]. In this section, we set up some of these common ideas.

One well-known tool in the theory of both graph spectra and orthogonal polynomials is interlacing. A polynomial $f$ *interlaces* the polynomial $g$ if between any two zeros of $g$, there is a zero of $f$. It is well known that if $(p_i)_{i\geq 0}$ is a sequence of orthogonal polynomials, then for any positive integer $n$, the polynomial $p_n$ interlaces $p_{n+1}$. A close modification of the proof, such as the one that can be found in Beals and Wong [18], gives the following result.

**5.4.1 Proposition.** *Let $(p_i)_{i\geq 0}$ be a sequence of orthogonal polynomials. Then for any $\alpha \in \mathbb{R}$, the polynomial $p_{n-1}$ interlaces $p_n + \alpha p_{n-1}$.*

*Proof.* Let $n \geq 1$ and let $r_1 < r_2$ be two consecutive zeros of $p_n$. By Corollary 2.3.3, we have

$$p_{n-1}(x)\, p_n'(x) - p_{n-1}'(x)\, p_n(x) > 0,$$

so

$$p_{n-1}(x)\big(p_n'(x) + \alpha p_{n-1}'(x)\big) - p_{n-1}'(x)(p_n(x) + \alpha p_{n-1}(x)) > 0.$$

For $r_1, r_2$ this simplifies as

$$(p_{n-1}(p_n + \alpha p_{n-1}))'(r_1) > 0$$

and

$$(p_{n-1}(p_n + \alpha p_{n-1}))'(r_2) > 0$$

Since $r_1$ and $r_2$ are consecutive zeros, $(p_n(r_1) + \alpha p_{n-1}(r_1))'$ and $(p_n(r_2) + \alpha p_{n-1}(r_2))'$ have different signs. Therefore, $p_{n-1}(r_1)$ and $p_{n-1}(r_2)$ have different signs, and so by the intermediate value theorem, $p_{n-1}$ has a zero in the interval $(r_1, r_2)$. □

This leads to the following result, which was proven in a more general case by Cohn and Kumar [38].

**5.4.2 Theorem.** *Let $p_0, \dots, p_n$ be a sequence of orthogonal polynomial such that $p_n$ has a positive leading coefficient. Let $\alpha \in \mathbb{R}$, and let $r_1 > \cdots > r_n$ be the roots of $p_n(x) + \alpha p_{n-1}(x)$. Then*

$$\frac{p_n(x) + \alpha p_{n-1}(x)}{x - r_1}$$

*has positive coefficients in terms of $p_0, \dots, p_{n-1}$.*

*Proof.* Since $r_1$ is a root of $p_n + \alpha p_{n-1}$ we have $\alpha = \frac{-p_n(r_1)}{p_{n-1}(r_1)}$. By Lemma 5.4.1 we know $r_1$ is not a root of $p_{n-1}$. Then we have

$$\frac{p_n(x) + \alpha p_{n-1}(x)}{x - r_1} = \frac{p_{n-1}(r_1)(p_n(x) + \alpha p_{n-1}(x))}{p_{n-1}(r_1)(x - r_1)} = \frac{p_{n-1}(r_1)\, p_n(x) - p_n(r_1)\, p_{n-1}(x)}{p_{n-1}(r_1)(x - r_1)}.$$

By Lemma 2.3.2

$$\frac{p_{n-1}(r_1)\, p_n(x) - p_n(r_1)\, p_{n-1}(x)}{x - r_1} = \frac{1}{c_n} \sum_{i=0}^{n-1} \hat{b}_{i,n-1} p_i(r_1)\, p_i(x)\,.$$

and thus

$$\frac{p_n(x) + \alpha p_{n-1}(x)}{x - r_1} = \frac{1}{c_n} \sum_{i=0}^{n-1} \hat{b}_{i,n-1} \frac{p_i(r_1)}{p_{n-1}(r_1)} p_i(r_1)\, p_i(x)\,. \tag{5.4.1}$$

By the condition on our three-term recurrence, $\hat{b}_{i,n-1}$ is positive for all $0 \le i \le n-1$. Since $p_n$ has a positive leading coefficient, $c_n$ is also positive. By Lemma 5.4.1, $p_{n-1}$ interlaces $p_n + \alpha p_{n-1}$, and, since 0 is a real number, $p_{i-1}$ interlaces $p_i$ for all $1 \le i \le n-1$. In particular, $r_1$, the largest root of $p_n + \alpha p_{n-1}$, must be greater than the largest root of $p_i$, so $p_i(r_1) > 0$. Thus the coefficient of $p_i(x)$ in Equation 5.4.1 is positive for all $0 \le i \le n-1$. □

There are also a couple more properties that will be useful for the families of polynomials associated to semiregular trees.

Note that

$$P_1^{k,\ell}(x) = x - k,$$
$$P_2^{k,\ell}(x) = x^2 - (2k + \ell - 2)\, x + k(k-1)\,,$$

and, for all $i \ge 2$, the sequence satisfies the three-term recurrence

$$P_{i+1}^{k,\ell}(x) = (x - (k + \ell - 2))\, P_i(x) - (k-1)(\ell - 1)\, P_{i-1}(x)\,.$$

Similarly,

$$I_1^{k,\ell}(x) = x - (k + \ell - 1)$$

and, for $i \ge 1$, we have

$$I_{i+1}^{k,\ell}(x) = (x - (k + \ell - 2))\, I_i^{k,\ell}(x) - (k-1)(\ell - 1)\, I_{i-1}^{k,\ell}(x)\,. \tag{5.4.2}$$

Let $G = (\beta \cup \gamma, E)$ be a distance-biregular graph and let $\pi \in \{\beta, \gamma\}$. Let $u \in \pi$ and let $v$ be at distance $h$ from $u$. The *intersection number* $p^\pi(i, j; h)$ is the number of vertices at distance $i$ from $u$ and distance $j$ from $v$. Note that this is equal to

$$\mathbf{e}_u^T A_i A_j \mathbf{e}_v = \mathbf{e}_u^T F_i^\pi(A)\, F_j^{\pi(v)} \mathbf{e}_v,$$

so it is independent of the specific choice of $u$.

In particular, since semiregular trees are distance-biregular, we get the following result.

**5.4.3 Lemma.** *For $i, j \geq 0$, there exist positive coefficients $p^{k,\ell}(i, j; h)$ such that:*

$$P_i^{k,\ell}\big(x^2\big)\, P_j^{k,\ell}\big(x^2\big) = \sum_{h=|i-j|}^{i+j} p^{k,\ell}(2i, 2j; 2h)\, P_h^{k,\ell}\big(x^2\big)\,,$$

$$P_i^{k,\ell}\big(x^2\big)\, I_j^{k,\ell}\big(x^2\big)\, x = \sum_{h=|i-j|}^{i+j} p^{k,\ell}(2i, 2j+1; 2h+1)\, I_h^{k,\ell}\big(x^2\big)\, x,$$

$$I_i^{k,\ell}\big(x^2\big)\, x P_j^{k,\ell}\big(x^2\big) = \sum_{h=|i-j|}^{i+j} p^{k,\ell}(2i+1, 2j; 2h+1)\, I_h^{k,\ell}\big(x^2\big)\, x,$$

*and*

$$I_i^{k,\ell}\big(x^2\big)\, x P_j^{k,\ell}\big(x^2\big)\, x = \sum_{h=|i-j|}^{i+j+1} p^{k,\ell}(2i+1, 2j+1; 2h)\, P_h^{k,\ell}\big(x^2\big)\,. \qquad \square$$

We wish to introduce two more families of polynomials. For $i \geq 0$, we define

$$Q_i^{k,\ell}(x) = \sum_{j=0}^{i} P_j^{k,\ell}(x)\,.$$

Note that

$$Q_1^{k,\ell}(x) = x - (k-1)$$

and, for $i \geq 1$, we have

$$Q_{i+1}^{k,\ell}(x) = (x - (k+\ell-2))\, Q_i(x) - (k-1)(\ell-1)\, Q_{i-1}(x)\,.$$

Similarly

$$J_i^{k,\ell}(x) = \sum_{j=0}^{i} I_j^{k,\ell}(x)\,,$$

satisfies

$$J_1^{k,\ell}(x) = x - (k+\ell-2)$$

and

$$I_{i+1}^{k,\ell}(x) = (x - (k+\ell-2))\, P_i(x) - (k-1)(\ell-1)\, P_{i-1}(x)$$

for $i \geq 1$.

## 5.5 Semiregular Extremal Expanders

Let $G = (\beta \cup \gamma, E)$ be a $(k, \ell)$ semiregular bipartite graph. If the second-largest eigenvalue of $G$ is minimal over all $(k, \ell)$-semiregular bipartite graphs on cells of size $|\beta|$ and $|\gamma|$, then $G$ is an *semiregular extremal expander*. Nozaki [125] proved that a regular graph with girth twice the diameter is an extremal expander. We extend that result for semiregular bipartite graphs.

**5.5.1 Theorem.** *Let $G = (\beta \cup \gamma, E)$ be a bipartite semiregular graph with diameter $d$ and $d+1$ distinct eigenvalues. If the girth of $G$ is at least $2d-2$, then $G$ is an extremal expander.*

*Proof.* Suppose the covering radius $d_\beta$ of $\beta$ is odd and let $d^* = \frac{d_\beta - 1}{2}$. Let $\theta_0 > \theta_1 > \cdots > \theta_{d^*}$ be the distinct positive eigenvalues in the eigenvalue support of vertices in $\beta$. In other words, $\theta_0, \ldots, \theta_{d^*}$ are the eigenvalues for which $E_r \mathbf{E}_\beta$ is nonzero.

We define

$$f^\beta(x) = \left(x - \theta_1^2\right) \prod_{r=2}^{d^*} \left(x - \theta_r^2\right)^2.$$

We wish to compute the coefficients of $f^\beta(x)$ when it is written in the basis of $P_0^{k,\ell}, \ldots, P_{2d^*-1}^{k,\ell}$.

Recall that $X_i$ is the $i$ distance matrix of the halved graph of $G$ induced by $\beta$. Since $G$ has girth at least $2d-2$ for $0 \leq i \leq d^* - 1$, we have

$$P_i^{k,\ell}\left(NN^T\right) = X_i.$$

Further, there must be some constant $c$ such that

$$P_{d^*}^{k,\ell}\left(NN^T\right) = cX_{d^*}.$$

Let

$$H(x) = 2\,|\beta| \prod_{r=1}^{d^*} \frac{x - \theta_r^2}{k\ell - \theta_r^2}.$$

Note that

$$H\left(NN^T\right) = \mathbf{1}_{|\beta|} = \sum_{i=0}^{d^*-1} P_i^{k,\ell}\left(NN^T\right) + \frac{1}{c} P_{d^*}^{k,\ell}\left(NN^T\right).$$

Then there exists a constant

$$e := 2\,|\beta| \prod_{r=1}^{d^*} \frac{1}{k\ell - \theta_r^2} > 0$$

such that

$$f^\beta(x) = \frac{1}{e^2} \frac{H(x)^2}{x - \theta_1^2} = \frac{1}{ce^2} \frac{Q_{d^*}^{k,\ell}(x) - (c-1)\, Q_{d^*-1}^{k,\ell}(x)}{x - \theta_1^2} \left( \sum_{i=0}^{d^*-1} P_i^{k,\ell}(x) + \frac{1}{c} P_{d^*}^{k,\ell}(x) \right)$$

By Theorem 5.4.2, we know that

$$\frac{Q^{k,\ell}_{d^*}(x) - (c-1)\, Q^{k,\ell}_{d^*-1}(x)}{x - \theta_1^2}$$

has positive coefficients in terms of $Q^{k,\ell}_0, \ldots, Q^{k,\ell}_{d^*-1}$, and therefore it has positive coefficients in terms of $P^{k,\ell}_0, \ldots, P^{k,\ell}_{d^*-1}$. Then by Lemma 5.4.3, we know that $f(x)$ has positive coefficients in terms of $P^{k,\ell}_0, \ldots, P^{k,\ell}_{d-2}$. Thus we may apply Theorem 5.3.1. Since $f\left(\theta_r^2\right) = 0$ for all $\theta_r \neq \pm\sqrt{k\ell}$, and since $G$ has girth at least $2d-2$, we see that the bound of Theorem 5.3.1 is tight.

If $d_\beta$ is even, we let $d^* = \frac{d_\beta}{2}$ and let $\theta_0 > \theta_1 > \cdots > \theta_{d^*-1}$ be the distinct positive of in the eigenvalue support of vertices in $\beta$. Let

$$f^\beta(x) = \left(x - \theta_1^2\right) x^2 \prod_{r=2}^{d^*-1} \left(x - \theta_r^2\right)^2 .$$

We once again have constants $c, e > 0$ such that

$$\sum_{i=0}^{d^*-2} I_i(x)\sqrt{x} + \frac{1}{c} I_{d^*-1}(x)\sqrt{x} = e\sqrt{x} \prod_{r=1}^{d^*-1} \left(x - \theta_r^2\right).$$

Thus

$$f^\beta(x) = \frac{1}{ce^2} \frac{J_{d^*-1}(x)\sqrt{x} - (c-1)\, J_{d^*-2}(x)\sqrt{x}}{x - \theta_1^2} \left( \sqrt{x} \sum_{i=0}^{d^*-2} I^{k,\ell}_i(x) + \frac{\sqrt{x}}{c} I^{k,\ell}_{d^*-1}(x)\, x \right),$$

so by Theorem 5.4.2 and Lemma 5.4.3, we know that $f^\beta$ has positive coefficients in terms of $P_0, \ldots, P_{d-2}$. Thus we can apply Theorem 5.3.1, and the bound is tight.

Suppose that $G$ is not an extremal expander. Then let $G' = (\beta', \gamma')$ be a $(k, \ell)$ semiregular bipartite graph with $|\beta'| = |\beta|$ and $|\gamma'| = |\gamma|$, but second-largest eigenvalue $\theta_1' < \theta_1$. Then $f^\beta$ still satisfies the conditions of Theorem 5.3.1, so it must attain the bound. In particular, $G'$ must have girth at least $2d-2$, and every nontrivial eigenvalue of $G'$ must be the square root of a root of $f^\beta$. By construction, this means that the eigenvalues of $G'$ are a subset of the eigenvalues of $G$, and since $\theta_1' < \theta_1$, they must be a proper subset. This implies that $G'$ has at most $d-1$ distinct eigenvalues, so it has diameter at most $d-2$. This contradicts the fact that $G'$ has girth at least $2d-2$, and therefore $G$ must be an extremal expander. □

Nozaki [125] used the linear programming bound to prove that any graph with girth at least twice the diameter is an extremal expander. Damerell and Georgiacodis [50] showed that graphs with this property must have diameter at most six. For bipartite regular graphs, Cioabă, Koolen, and Nozaki [36] proved that if $g \geq 2d-2$, then $d \neq 11$ and $d \leq 14$. Similar restrictions on the diameter presumably exist for semiregular graphs.

**5.5.2 Problem.** *Let $G$ be a distance-biregular graph with minimum valency at least three, diameter $d$, and girth $g \geq 2d-2$. What are the possible values of $d$?*

## 5.6 Spectral Moore Bound for Semiregular Graphs

Let $b(k,\ell,\theta)$ be the maximum number of vertices of valency $k$ in a $(k,\ell)$-semiregular bipartite graph with second-largest eigenvalue at most $\theta$. We wish to use the linear programming bound to upper bound $b(k,\ell,\theta)$ . The idea is to use a similar function as in Theorem 5.5.1, though the numerical evaluations matter more.

Let $B(k,\ell,2t+1,c)$ be the $(2t+1)\times(2t+1)$ tridiagonal matrix with lower diagonal $(1,\dots,1,c,\ell)$, zero along the main diagonal, and row sum alternating between $k$ and $\ell$. Then $B(k,\ell,2t+1,c)$ has the form

$$\begin{pmatrix} 0 & k & & & & & \\ 1 & 0 & \ell-1 & & & & \\ & 1 & 0 & k-1 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & 1 & 0 & \ell-1 & \\ & & & & c & 0 & k-c \\ & & & & & \ell & 0 \end{pmatrix}.$$

For $i \le 2t-1$, it follows from induction that the determinant of the principal $i\times i$ matrix formed by the first $i$ rows and $i$ columns is $F_i^{k,\ell}(x)$ . Expanding along the bottom two rows and right column of $xI - B(k,\ell,2t+1,c)$, the determinant is

$$x^2 F_{2t-1}^{k,\ell}(x) - c(k-1)\,xF_{2t-2}^{k,\ell}(x) - k(\ell-c)\,F_{2t-1}^{k,\ell}(x)\,.$$

This is equivalent to

$$x\Big(x^2 I_{t-1}^{k,\ell}(x^2) - c(k-1)\Big(I_{t-1}^{k,\ell}(x^2) - (\ell-1)\,I_{t-2}^{k,\ell}(x^2)\Big) - k(\ell-c)\,I_{t-1}^{k,\ell}(x^2)\Big)\,,$$

which simplifies to

$$x\Big(\big(x^2-k\ell\big)\,I_{t-1}^{k,\ell}(x^2) + c\Big(I_{t-1}^{k,\ell}(x^2) - (k-1)(\ell-1)\,I_{t-2}^{k,\ell}(x^2)\Big)\Big)\,. \tag{5.6.1}$$

Using Equation 5.4.2, we compute that

$$I_{t-1}^{k,\ell}(x^2) - (k-1)(\ell-1)\,I_{t-2}^{k,\ell}(x^2) = \big(x^2-k\ell+1\big)\,I_{t-2}^{k,\ell}(x^2) - (k-1)(\ell-1)\,I_{t-3}^{k,\ell}(x^2)\,.$$

We can inductively rewrite Equation 5.6.1 as

$$x\big(x^2-k\ell\big)\left(I_{t-1}^{k,\ell}(x^2) + c\sum_{i=0}^{t-2} I_i^{k,\ell}(x^2)\right),$$

which simplifies to

$$x\big(x^2-k\ell\big)\big((c-1)\,J_{t-2}(x^2) + J_{t-1}(x^2)\big)\,.$$

The eigenvalues of $B(k, \ell, 2t+1, c)$ are the roots of $\det(xI - B(k, \ell, 2t+1, c))$. Therefore, the nontrivial, nonzero eigenvalues of $B(k, \ell, 2t+1, c)$ are the square roots of the roots of the expression

$$(c-1)\, J_{t-2}(z) + J_{t-1}(z)\,.$$

We can similarly define $B(k, \ell, 2t, c)$ to be the $2t \times 2t$ tridiagonal matrix with lower diagonal $(1, \ldots, 1, c, k)$, zero along the main diagonal, and row sum alternating between $k$ and $\ell$. We wish to use the eigenvalues of $B(k, \ell, t, c)$ to define a function and apply the semiregular linear programming bound. The following result relates the second-largest eigenvalue of the graph to the second-largest eigenvalue of $B(k, \ell, t, c)$.

**5.6.1 Proposition.** *Let $k, \ell \geq 3$ and let $0 < \theta < \sqrt{k-1} + \sqrt{\ell - 1}$. Then there exists a matrix $B(k, \ell, t, c)$ with second-largest eigenvalue $\theta$.*

*Proof.* Similar to McKay's approach [115] to Example 2.3.7, Godsil and Mohar [83] derived the weight function for semiregular trees. Letting $p = \sqrt{(k-1)(\ell-1)}$, their results gives us a weight function

$$\alpha(x) = \frac{k\ell\sqrt{-\left(x^2 - k\ell + (p-1)^2\right)\left(x^2 - k\ell + (p+1)^2\right)}}{\pi(k+\ell)(k\ell - x^2)\,|x|}$$

on the interval $\left[-\sqrt{k-1} - \sqrt{\ell-1}, \sqrt{k-1} + \sqrt{\ell-1}\right]$.

As we saw in the proof of Theorem 2.3.1, for $i \geq 0$ the polynomial $F_i^{k,\ell}$ defines a discrete measure and an interval containing the roots of $F_i^{k,\ell}$. These measures converge to the measure obtained by Godsil on Mohar on the interval $\left[-\sqrt{k-1} - \sqrt{\ell-1}, \sqrt{k-1} + \sqrt{\ell-1}\right]$. Thus there exists some $s$ such that $F_s(\theta) < 0$ but $F_i(\theta) \geq 0$ for $i \leq s-1$.

Suppose $s$ is odd, and let $t = \frac{s-1}{2}$. Then $I_t^{k,\ell}(\theta^2) < 0$ but $I_i^{k,\ell}(\theta^2) \geq 0$ for $i \leq t-1$.

Let

$$c = \frac{-I_t^{k,\ell}(\theta^2)}{J_{t-1}(\theta^2)}.$$

Note that by construction, $c > 0$.

We have that

$$\begin{aligned} J_t(\theta^2) + (c-1)\, J_{t-1}(\theta^2) &= J_t(\theta^2) - J_{t-1}(\theta^2) + cJ_{t-1}(\theta^2) \\ &= I_t^{k,\ell}(\theta^2) - \frac{I_t^{k,\ell}(\theta^2)}{J_{t-1}(\theta^2)} J_{t-1}(\theta^2) \\ &= 0, \end{aligned}$$

so $\theta$ is a root of $J_t(z^2) + (c-1)\, J_{t-1}(z^2)$, and thus an eigenvalue of $B(k, \ell, 2t+3, c)$.

It remains to show that $\theta$ is in fact the second-largest eigenvalue. We have

$$I_2^{k,\ell}(x) - (k-1)(\ell-1)\, I_1^{k,\ell}(x) = (x-k)\, J_1(x)\,,$$

so we may inductively assume that, for $i \geq 2$, we have

$$I_i^{k,\ell}(x) - (k-1)(\ell-1)\, I_{i-1}^{k,\ell} = (x - k\ell)\, J_{i-1}(x)\,.$$

Expanding out using Equation 5.4.2, we see that

$$\begin{aligned} I_{i+1}^{k,\ell}(x) - (k-1)(\ell-1)\, I_i^{k,\ell}(x) &= (x-k\ell)\, I_i^{k,\ell}(x) + I_i^{k,\ell}(x) - (k-1)(\ell-1)\, I_{i-1}^{k,\ell}(x) \\ &= (x-k\ell)\, I_i^{k,\ell}(x) + (x-k\ell)\, J_{i-1}(x) \\ &= (x-k\ell)\, J_i(x)\,. \end{aligned}$$

By Lemma 5.4.1, the zeros of $I_{t-1}$ interlace $I_t^{k,\ell} - (k-1)(\ell-1)\, I_{t-1}^{k,\ell}$, and the nontrivial eigenvalues of $I_t^{k,\ell} - (k-1)(\ell-1)\, I_{t-1}^{k,\ell}$ interlace the zeros of $J_{t-1}$. In turn, the zeros of $J_{t-1}$ interlace the zeros of $J_{t-1} + (c-1)\, J_t$. In particular, if $\theta$ is not the second-largest eigenvalue of $B(k,\ell,2t+3,c)$, then it must be strictly less than the second-largest zero of $I_{t-1}^{k,\ell}(x^2)$, which contradicts our choice of $t$. Thus we have found a matrix $B(k,\ell,2t+3,c)$ with second-largest eigenvalue $\theta$.

The proof if $s$ is even is identical, only using the families of polynomials $P^{k,\ell}$ and $Q^{k,\ell}$ instead of $I^{k,\ell}$ and $J^{k,\ell}$. □

We can now prove a spectral Moore bound for semiregular bipartite graphs.

**5.6.2 Theorem.** *If $\theta$ is the second-largest eigenvalue of $B(k,\ell,2t+1,c)$, then*

$$b(k,\ell,\theta) \leq \ell \sum_{i=0}^{t-2} (\ell-1)^i (k-1)^i + \frac{\ell(\ell-1)^{t-1}(k-1)^{t-1}}{c}.$$

*The graph obtaining this bound is distance-biregular.*

*Proof.* Consider the matrix $xI - B(k,\ell,2t+1,c)$. The determinant is

$$x\big(x^2 - k\ell\big)\Big((c-1)\, J_{t-2}\big(x^2\big) + J_{t-1}\big(x^2\big)\Big)\,.$$

Let $\sqrt{k\ell} = \lambda_0 > \lambda_1 > \cdots > \lambda_{t-1}$ be the positive eigenvalues of $B(k,\ell,2t+1,c)$. Then

$$\prod_{r=1}^{t-1} \big(x - \lambda_r^2\big) = (c-1)\, J_{t-2}(x) + J_{t-1}(x)\,.$$

Therefore, we can define

$$\begin{aligned} f(x) &= x\big(x-\theta^2\big) \prod_{r=2}^{t-1} \big(x-\lambda_r^2\big)^2 \\ &= \frac{(c-1)\, J_{t-2}(x) + J_{t-1}(x)}{x-\theta^2} \left( c\sqrt{x} \sum_{i=0}^{t-2} I_i^{k,\ell}(x) + \sqrt{x} I_{t-1}^{k,\ell}(x) \right). \end{aligned}$$

For all eigenvalues $\theta_r$ of $G$, we have $f\left(\theta_r^2\right) \leq 0$.

By Proposition 5.4.2 and the definition of $J_i$, we know there exists positive coefficients $j_0, \ldots, j_{t-2}$ such that

$$\frac{(c-1)\sqrt{x} J_{t-2}(x) + \sqrt{x} J_{t-1}(x)}{x - \theta^2} = \sqrt{x} \sum_{i=0}^{t-2} j_i I_i^{k,\ell}(x)\,. \tag{5.6.2}$$

Thus we may write

$$f(x) = \left(\sqrt{x} \sum_{i=0}^{t-2} j_i I_i^{k,\ell}(x)\right)\left(c\sqrt{x} \sum_{i=0}^{t-2} I_i^{k,\ell}(x) + \sqrt{x} I_{t-1}^{k,\ell}(x)\right)$$

for $j_i > 0$.

By Lemma 5.4.3, we get that $h_i > 0$ for all $i = 0, \ldots, 2t-2$. We further note that

$$p^{k,\ell}(2i+1, 2j+1; 0) = k(\ell-1)^i (k-1)^i\, \delta_{i,j} = k I_i^{k,\ell}(k\ell)\, \delta_{i,j},$$

so by Equation 5.6.2 we have

$$h_0 = ck \sum_{i=0}^{t-2} j_i I_i^{k,\ell}(k\ell) = ck \frac{(c-1)\, J_{t-2}(k\ell) + J_{t-1}(k\ell)}{k\ell - \theta^2}$$

Applying Theorem 5.3.1, we then get

$$\begin{aligned} |B| &\leq \frac{f(k\ell)}{h_0} \\ &= \ell \sum_{i=0}^{t-2} I_i^{k,\ell}(k\ell) + \ell \frac{I_{t-1}^{k,\ell}(k\ell)}{c} \\ &= \ell \sum_{i=0}^{t-2} (\ell-1)^i (k-1)^i + \frac{\ell(\ell-1)^{t-1}(k-1)^{t-1}}{c}. \end{aligned}$$

If equality holds, it must hold in Theorem 5.3.1. This implies every nonzero, nontrivial eigenvalue of $G$ is a square root of a zero of a $t-1$ degree polynomial. It follows that $G$ must have at most $2t+1$ distinct eigenvalues, and thus $G$ has diameter $d \leq 2t$. Further, $G$ has girth at least $4t-2$, which means $d \geq 2t-1$. If $d = 2t-1$, then $G$ is a generalized polygon, which is distance-biregular. Otherwise we can use Theorem 4.7.4 to conclude $G$ is distance-biregular. □

A close variation of this proof gives us an analogue for even matrices.

**5.6.3 Theorem.** *If $\theta$ is the second-largest eigenvalue of $B(k, \ell, 2t, c)$, then*

$$b(k, \ell, \theta) \leq 1 + k \sum_{i=1}^{t-2} (\ell-1)^i (k-1)^{i-1} + \frac{k(\ell-1)^{t-1}(k-1)^{t-2}}{c}.$$

*Proof.* Apply Theorem 5.3.1 with

$$f(x) = \frac{\Big((c-1)\, Q^{k,\ell}_{t-2}(x) + Q^{k,\ell}_{t-1}(x)\Big)^2}{x - \theta^2}. \qquad \square$$

The same way that the spectral Moore bound for regular graphs gave a new proof of the Alon-Boppana theorem, our semiregular spectral Moore bound leads to a new proof of the following result of Feng and Li [67] in bipartite expanders.

**5.6.4 Theorem** (Feng-Li [67])**.** *If $k, \ell \geq 3$ and $\theta < \sqrt{k-1} + \sqrt{\ell - 1}$, then there are only finitely many bipartite $(k,\ell)$-semiregular graphs that have second-largest eigenvalue at most $\theta$.*

## 5.7 Comparison to Previous Bounds

The semiregular spectral Moore bound is tight for partial geometries.

*5.7.1 Example.* Let $1 < \alpha < \ell$ be an integer such that a partial geometry pg $(\ell-1, k-1, \alpha)$ exists with $k, \ell \geq 2$. Then $pg(\ell-1,k-1,\alpha)$ has second-largest eigenvalue $\sqrt{k+\ell-\alpha-1}$, which is also the second-largest eigenvalue of

$$\begin{pmatrix} 0 & k & & & \\ 1 & 0 & \ell-1 & & \\ & 1 & 0 & k-1 & \\ & & \alpha & 0 & \ell-\alpha \\ & & & k & 0 \end{pmatrix}.$$

Applying the bound of Theorem 5.6.2, we get that

$$b\Big(k,\ell,\sqrt{k+\ell-\alpha-1}\Big) \leq \ell + \frac{\ell(\ell-1)(k-1)}{\alpha} = \frac{\ell((\ell-1)(k-1)+\alpha)}{\alpha},$$

which is precisely the number of points in a partial geometry, so this bound is tight.

However, the regular spectral Moore bound of Cioabă, Koolen, Nozaki, and Vermette [37] is not tight for the halved graphs of proper partial geometries.

*5.7.2 Example.* Let $1 < \alpha < \ell$ be an integer such that a partial geometry pg $(\ell-1, k-1, \alpha)$ exists with $k, \ell \geq 2$. The second-largest eigenvalue of the point graph is $\ell - 1 - \alpha$. Using the bound in [37], we see the number of points is at most

$$b(k(\ell-1)\,, \ell-1-\alpha) \leq 1 + k(\ell-1) + \frac{k(\ell-1)(k(\ell-1)-1)(\ell-\alpha)}{k(\ell-1)-1-(\ell-1-\alpha)^2}.$$

Note that, since $2 \leq \alpha \leq \ell - 1$, and $2 \leq k$, we have

$$k(\alpha-1)(k(\ell-1)-1) + (k(\ell-1)-1) + (k-1)(\ell-1-\alpha)^2 > 0.$$

Therefore,

$$\alpha k(k(\ell-1)-1) > (k-1)\Big(k(\ell-1)-1-(\ell-1-\alpha)^2\Big),$$

which we can rewrite as

$$\frac{k(k(\ell-1)-1)}{k(\ell-1)-1-(\ell-1-\alpha)^2} > \frac{k-1}{\alpha}.$$

Multiplying both sides by $(\ell-1)(\ell-\alpha)$ gives us

$$\frac{k(\ell-1)(k(\ell-1)-1)(\ell-\alpha)}{k(\ell-1)-1-(\ell-1-\alpha)^2} > \frac{(k-1)(\ell-1)(\ell-\alpha)}{\alpha}.$$

We compute that

$$1+k(\ell-1)+\frac{(k-1)(\ell-1)(\ell-\alpha)}{\alpha} = \frac{\ell((k-1)(\ell-1)+\alpha)}{\alpha}.$$

This in turn implies that

$$1+k(\ell-1)+\frac{k(\ell-1)(k(\ell-1)-1)(\ell-\alpha)}{k(\ell-1)-1-(\ell-1-\alpha)^2} > \frac{\ell(\ell-1)(k-1)+\alpha}{\alpha},$$

and therefore, the bound given in [37] by considering the point graph is not tight.

We can also view partial geometries as hypergraphs, but the hypergraph bound of Cioabă, Koolen, Mimura, Nozaki, and Okuda [35] is not tight for partial geometries.

*5.7.3 Example.* Let $1 < \alpha < \ell$ be an integer such that a partial geometry pg $(\ell-1, k-1, \alpha)$ exists with $k, \ell \geq 2$. The point graph is the same as before, with second eigenvalue $\ell-1-\alpha$. Using the bound from [35], we have that the number of points is bounded above by

$$1+k(\ell-1)+\frac{k(k-1)(\ell-1)^2(\ell-\alpha)}{k(\ell-1)+(\alpha-1)(\ell-1-\alpha)}.$$

Since $\alpha, k, \ell \geq 2$, we have that

$$(\alpha-1)(k-1)(\ell-1)+\alpha(\alpha-1) > 0.$$

Therefore,

$$\alpha k(\ell-1) > k(\ell-1)+(\alpha-1)(\ell-1-\alpha),$$

which we can rewrite as

$$\frac{k(\ell-1)}{k(\ell-1)+(\alpha-1)(\ell-1-\alpha)} > \frac{1}{\alpha}.$$

Multiplying both sides by $(k-1)(\ell-1)(\ell-\alpha)$ gives us

$$\frac{k(k-1)(\ell-1)^2(\ell-\alpha)}{k(\ell-1)+(\alpha-1)(\ell-1-\alpha)} > \frac{(k-1)(\ell-1)(\ell-\alpha)}{\alpha}.$$

We compute that

$$1+k(\ell-1)+\frac{(k-1)(\ell-1)(\ell-\alpha)}{\alpha} = \frac{\ell((k-1)(\ell-1)+\alpha)}{\alpha}.$$

It follows that

$$1+k(\ell-1)+\frac{k(k-1)(\ell-1)^2(\ell-\alpha)}{k(\ell-1)+(\alpha-1)(\ell-1-\alpha)}$$

and therefore, the bound given in [35] is also not tight for partial geometries.

Thus, using constructions of infinite families such as those by De Clerck, Dye, and Thas [53] or Mathon [113], we have infinite families of distance-biregular graphs where the bound in Theorem 5.6.2 is tight, but the bounds by Cioabă et al [35, 37] are not.

# Chapter 6

# Future Work

Throughout this thesis, we have made use of the bipartite structure for both the distance adjacency matrices and the spectral idempotents. In Section 2.6, we saw that we could write the distance adjacency matrices in terms of the spectral idempotents, and in the proof of Theorem 4.7.4 we wrote the spectral idempotent for the largest eigenvalue in terms of the distance adjacency matrices. In fact, it is generally true that we can write the spectral idempotents in terms of the distance adjacency matrices, giving us two bases for the algebra

$$\langle \begin{pmatrix} \mathbf{0} & N \\ \mathbf{0} & \mathbf{0} \end{pmatrix}, \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ N^T & \mathbf{0} \end{pmatrix} \rangle.$$

For distance-regular graphs, the equivalent algebra generalizes to an association scheme. Association schemes are commutative and contain the identity, so the algebra of distance-biregular graphs is not an association schemes. However, there is a similarly nice algebraic structure of a coherent configuration associated to distance-biregular graphs, proposed by Delorme [55, 56], which we expand on in this chapter. We conclude with some problems for further research, including the problems that have been posed throughout the thesis, as well as additional problems suggested by the coherent configuration.

## 6.1 Coherent Configurations

Let $G$ be a distance-regular graph of diameter $d$, and let $A_0, \ldots, A_d$ be the distance adjacency matrices. We have

(a) $A_0 = I$;

(b) $\sum_{i=0}^{d} A_i = J$;

(c) $A_i^T = A_i$ for $0 \leq i \leq d$; and

(d) $A_i A_j \in \operatorname{span}\{A_0, \ldots, A_d\}$ for $0 \leq i, j \leq d$.

Thus, the distance adjacency matrices of a distance-regular graph satisfy the necessary for properties of an *association scheme*.

Association schemes are a useful tool in algebraic combinatorics. They were introduced by Bose and Shinamoto [27] in the study of designs. The use of association schemes in the context of coding theory was developed by Delsarte [57], and the connections between distance-regular graphs and association schemes has been given considerable attention in the books of Brouwer, Cohen, and Neumaier [28] and Bannai and Ito [15].

Unless the graph is regular, the distance adjacency matrices of a distance-biregular graph will not form an association scheme. However, distance-biregular graphs still have a strong algebraic structure.

Let $\mathcal{A}$ be a set of 01-matrices. They form a *coherent configuration* if they satisfy the following:

(a) There exists a subset $S \subseteq \mathcal{A}$ such that

$$\sum_{M \in S} M = I;$$

(b) $\sum_{M \in \mathcal{A}} M = J$;

(c) If $M \in \mathcal{A}$, then $M^T \in \mathcal{A}$; and

(d) For any $M_1, M_2 \in \mathcal{A}$, the matrix $M_1 M_2$ is a linear combination of matrices in $\mathcal{A}$.

Recall from Section 3.2 that $X_0, X_1, \ldots,$ and $Y_0, Y_1, \ldots$ are the distance adjacency matrices for the halved graph. From Section 3.5, we further have that $N_1, N_3, \ldots$ are the distance-biadjacency matrices. We claim that, for a distance-biregular graph, the set of matrices

$$\mathcal{A} = \left\{ \begin{pmatrix} X_i & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & Y_i \end{pmatrix} : 0 \le i \le \frac{d}{2} \right\} \bigcup \left\{ \begin{pmatrix} 0 & N_{2i+1} \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ N_{2i+1}^T & 0 \end{pmatrix} : 0 \le i \le \frac{d-1}{2} \right\}$$

is a coherent configuration. It is clear that

$$\begin{pmatrix} X_0 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & Y_0 \end{pmatrix} = I,$$

and

$$\sum_{M \in \mathcal{A}} M = \sum_{i=0}^{\lfloor \frac{d}{2} \rfloor} \begin{pmatrix} X_i & 0 \\ 0 & Y_i \end{pmatrix} + \sum_{i=0}^{\lfloor \frac{d-1}{2} \rfloor} \begin{pmatrix} 0 & N_{2i+1} \\ N_{2i+1}^T & 0 \end{pmatrix} = \sum_{i=0}^{d} A_i = J,$$

establishing (a) and (b). The set $\mathcal{A}$ is partitioned into the distance adjacency matrices of the halved graph induced by $\beta$ and $\gamma$, which are symmetric, and the distance biadjacency matrices, where the transpose is explicitly included. Thus (c) holds.

The proof of (d) is morally similar to the distance-regular case, but it breaks into more involved casework. We need to prove that for any $i, j$ we have

$$\begin{pmatrix} X_i & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} X_j & 0 \\ 0 & 0 \end{pmatrix} \in \operatorname{span}(\mathcal{A}), \tag{6.1.1}$$

$$\begin{pmatrix} X_i & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & N_{2j+1} \\ 0 & 0 \end{pmatrix} \in \operatorname{span}(\mathcal{A}), \tag{6.1.2}$$

$$\begin{pmatrix} 0 & N_{2i+1} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ N_{2j+1}^T & 0 \end{pmatrix} \in \operatorname{span}(\mathcal{A}), \tag{6.1.3}$$

and the remaining cases follow from symmetry or by switching $\beta$ and $\gamma$.

Equation 6.1.1 follows from Theorem 3.2.3, since the halved graphs form an association scheme.

Let $\hat{d} = \lfloor \frac{d-1}{2} \rfloor$. Suppose by induction there exists some $k \geq 0$ such that

$$\left(NN^T\right)^k I_j\left(NN^T\right) N \in \operatorname{span}\left\{N_1, \ldots, N_{2\hat{d}+1}\right\}$$

for all $0 \leq j \leq \hat{d}$. Now consider

$$\left(NN^T\right)^{k+1} I_j\left(NN^T\right) N = \left(NN^T\right)^k \left(NN^T I_j\left(NN^T\right)\right) N.$$

Since $I_0\left(NN^T\right), \ldots, I_{\hat{d}}\left(NN^T\right)$ form a sequence of orthogonal polynomials, they satisfy a three-term recurrence and we can write $NN^T I_j\left(NN^T\right)$ as a linear combination of $I_{j-1}\left(NN^T\right)$, $I_j\left(NN^T\right)$, and $I_{j+1}\left(NN^T\right)$. Then the inductive hypothesis proves that

$$\left(NN^T\right)^{k+1} I_j\left(NN^T\right) N \in \operatorname{span}\left\{N_1, \ldots, N_{2\hat{d}+1}\right\}$$

Since $P_i\left(NN^T\right)$ is a polynomial in $NN^T$, this establishes Equation 6.1.2.

We could also prove Equation 6.1.1 directly using this same method by replacing $I_j$ with $P_j$.

Equation 6.1.3 is equivalent to the statement that, for $0 \leq i, j \leq \hat{d}$ we have

$$I_i\left(NN^T\right) NN^T I_j\left(NN^T\right) \in \operatorname{span}\left\{X_0, \ldots, X_{\lfloor \frac{d}{2} \rfloor}\right\}.$$

Note that for any $0 \leq j \leq \hat{d}$, we can write $I_j$ as a linear combination of $P_0, \ldots, P_j$. Thus we may suppose by induction that for any $0 \leq j \leq \hat{d}$, there exists $k \geq 0$ such that

$$\left(NN^T\right)^{k+1} P_j\left(NN^T\right) \in \operatorname{span}\left\{X_0, \ldots, X_{\lfloor \frac{d}{2} \rfloor}\right\},$$

and the same argument used above establishes 6.1.3.

## 6.2 Adjacency Algebra

Consider the set of spectral idempotents, split into block matrices as described in Section 2.6. This gives us the set

$$\mathcal{A} = \left\{ \begin{pmatrix} 2R_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}, \begin{pmatrix} \mathbf{0} & 2D_r \\ \mathbf{0} & \mathbf{0} \end{pmatrix}, \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ 2D_r^T & \mathbf{0} \end{pmatrix}, \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 2L_r \end{pmatrix} : \theta_r \geq 0 \right\}.$$

Let $0 \leq 2i \leq d$. Recall that $F_{2i}^{\beta}, F_{2i}^{\gamma}$ are even functions, so we have

$$\begin{pmatrix} X_i \\ \mathbf{0} \end{pmatrix} = F_{2i}^{\beta}(A)\, \mathbf{E}_\beta = \sum_{\theta_r} F_{2i}^{\beta}(\theta_r)\, E_r \mathbf{E}_\beta = \sum_{\theta_r \geq 0} F_{2i}^{\beta}(\theta_r) \begin{pmatrix} 2R_r \\ \mathbf{0} \end{pmatrix} \tag{6.2.1}$$

and similarly

$$\begin{pmatrix} \mathbf{0} \\ Y_i \end{pmatrix} = F^{\gamma}_{2i}(A)\, \mathbf{E}_{\gamma} = \sum_{\theta_r \geq 0} F^{\gamma}_{2i}(\theta_r) \begin{pmatrix} \mathbf{0} \\ 2L_r \end{pmatrix}. \tag{6.2.2}$$

Now let $0 \leq 2i+1 \leq d$. Since $F_{2i+1}$ is an odd function, we have

$$\begin{pmatrix} \mathbf{0} \\ N_{2i+1} \end{pmatrix} = F_{2i+1}(A)\, \mathbf{E}_{\beta} = \sum_{\theta_r} F_{2i+1}(\theta_r)\, E_r \mathbf{E}_{\beta} = \sum_{\theta_r \geq 0} F_{2i+1}(\theta_r) \begin{pmatrix} \mathbf{0} \\ 2D_r \end{pmatrix}, \tag{6.2.3}$$

and by symmetry

$$\begin{pmatrix} N^T_{2i+1} \\ \mathbf{0} \end{pmatrix} = \sum_{\theta_r \geq 0} F_{2i+1}(\theta_r) \begin{pmatrix} 2D^T_r \\ \mathbf{0} \end{pmatrix}. \tag{6.2.4}$$

Thus we have

$$\mathcal{A} \subseteq \operatorname{span}(\mathcal{B}).$$

The scalars $F^{\beta}_i(\theta_r)\,, F^{\gamma}_i(\theta_r)$ are the *eigenvalues* of the configuration.

Now, let $\theta_r \geq 0$ be an eigenvalue. Recall that there exists a polynomial of degree $d$

$$f(x) := \prod_{s \neq r} \frac{x - \theta_s}{\theta_r - \theta_s}$$

such that

$$f(A) = E_r = \begin{pmatrix} R_r & D_r \\ D^T_r & L_r \end{pmatrix}.$$

Since $f(x)$ is a polynomial of degree $d$, we can write it in the basis $F^{\beta}_0, \dots, F^{\beta}_d$. This tells us there exist coefficients $q^{\beta}_r(0)\,, \dots, q^{\beta}_r(d)$ such that

$$\begin{pmatrix} 2R_r \\ \mathbf{0} \end{pmatrix} = \frac{1}{|\beta|} \sum_{2i=0}^{d} q^{\beta}_r(2i) \begin{pmatrix} X_i \\ \mathbf{0} \end{pmatrix}, \tag{6.2.5}$$

and

$$\begin{pmatrix} \mathbf{0} \\ 2D_r \end{pmatrix} = \frac{\sqrt{\ell}}{\sqrt{k}\,|\beta|} \sum_{2i+1=0}^{d} q^{\beta}_r(2i+1) \begin{pmatrix} \mathbf{0} \\ N_{2i+1} \end{pmatrix}. \tag{6.2.6}$$

We can similarly write $f(x)$ in the basis of $F^{\gamma}_0, \dots, F^{\gamma}_d$ to conclude that

$$\begin{pmatrix} 2D^T_r \\ \mathbf{0} \end{pmatrix} = \frac{\sqrt{k}}{\sqrt{\ell}\,|\gamma|} \sum_{2i+1=0}^{d} q^{\gamma}_r(2i+1) \begin{pmatrix} N^T_{2i+1} \\ \mathbf{0} \end{pmatrix}, \tag{6.2.7}$$

and

$$\begin{pmatrix} \mathbf{0} \\ 2L_r \end{pmatrix} = \frac{1}{|\gamma|} \sum_{2i=0}^{d} q^{\gamma}_r(2i) \begin{pmatrix} \mathbf{0} \\ Y_i \end{pmatrix}. \tag{6.2.8}$$

The coefficients $q_r^\beta(i)\,, q_r^\gamma(i)$ are called the *dual eigenvalues*. Note that as with the eigenvalues, $q_r^\beta(2i+1) = q_r^\gamma(2i+1)\,.$

Pulling everything together, we see that

$$\operatorname{span}(\mathcal{A}) = \operatorname{span}(\mathcal{B})\,,$$

and we have two distinguished sets of bases for this algebra.

We have

$$\begin{pmatrix} 2R_r & \mathbf{0} \\ \mathbf{0} & 2L_r \end{pmatrix} = E_r + E_{-r} = (E_r - E_{-r})^2 = \begin{pmatrix} \mathbf{0} & 2D_r \\ 2D_r^T & \mathbf{0} \end{pmatrix}^2 = \begin{pmatrix} 4D_rD_r^T & \mathbf{0} \\ \mathbf{0} & 4D_r^TD_r \end{pmatrix}$$

For $\theta_r \neq \pm\theta_s$, then since $E_rE_s = \mathbf{0}$, we must have

$$R_rR_s + D_rD_s^T = \mathbf{0}.$$

Since $2R_r, 2R_s$ are distinct spectral idempotents of $NN^T$, they are orthogonal. Thus $R_rR_s = \mathbf{0}$, from which it follows $D_rD_s^T = \mathbf{0}$.

The *Scur product* of two matrices $A$ and $B$ is the entry-wise product defined by $(A \circ B)_{i,j} = A_{i,j}B_{i,j}$. Note that the distance matrices of our coherent configuration are idempotent under Schur multiplication, and therefore our coherent configuration is closed under Schur multiplication. Switching from the distance basis and normal matrix multiplication to the spectral basis and the Schur product gives us dual version of definitions and results.

One useful result is that the dual eigenvalues can be computed from the eigenvalues. To see this, we let $0 \leq 2i, r \leq d$. Using the spectral basis, we have

$$\operatorname{tr}(X_i 2R_r) = \operatorname{tr}\Big(F_{2i}^\beta(\theta_r)\,2R_r\Big) = m_r^\beta F_{2i}^\beta(\theta_r)\,,$$

whereas under the basis of adjacency matrices we have

$$\operatorname{tr}(X_i 2R_r) = \operatorname{sum}(X_i \circ 2R_r) = \frac{1}{|\beta|}\operatorname{sum}\Big(q_r^\beta(2i)\,X_i\Big) = k_{2i}^\beta q_r^\beta(2i)\,.$$

An analogous argument gives us

$$m_r^\gamma F_{2i}^\gamma(\theta_r) = k_{2i}^\gamma q_r^\gamma(2i)\,.$$

For $1 \leq 2i+1 \leq d$ and $0 \leq r \leq d$, we have

$$\operatorname{tr}\big(N_{2i+1}2D_r^T\big) = \operatorname{tr}\Big(F_{2i+1}^\beta(\theta_r)\,2R_r^T\Big) = m_r^\beta F_{2i+1}^\beta(\theta_r)$$

and

$$\begin{aligned} \operatorname{tr}\big(N_{2i+1}2D_r^T\big) &= \operatorname{sum}\big(N_{2i+1}^T \circ 2D_r^T\big) \\ &= \frac{\sqrt{k}}{\sqrt{\ell}\,|\gamma|}\operatorname{sum}\big(q_r^\gamma(2i+1)\,N_{2i+1}^T\big) \\ &= \frac{\sqrt{k}}{\sqrt{\ell}}k_{2i+1}^\gamma q_r^\gamma(2i+1)\,. \end{aligned}$$

## 6.3 Intersection Numbers

The intersection numbers defined in Section 5.4 also arise in the coherent configuration. If

$$\begin{pmatrix} X_i & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}, \begin{pmatrix} X_j & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \in \mathcal{A},$$

then we have

$$\begin{pmatrix} X_i & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \begin{pmatrix} X_j & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} = \sum_{2h=0}^{d} p^\beta(2i, 2j; 2h) \begin{pmatrix} X_h & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix},$$

and similarly for $p^\beta(2i, 2j+1; 2h+1)\,, p^\beta(2i+1, 2j; 2h+1)\,, p^\beta(2i+1, 2j+1; 2h)$. We can also use $\gamma$ instead of $\beta$ to get the corresponding intersection numbers for the other cell of the partition.

We can write the intersection numbers recursively to express them in terms of the parameters of the distance-biregular graph. Let $\pi \in \{\beta, \gamma\}$. Note that $p^\pi(0, i; h) = \delta_{ih}$ and $p^\pi(0, j; h) = \delta_{jh}$. Further,

$$b_i^\pi = p^\pi(i, 1; i+1)$$

and

$$c_i^\pi = p^\pi(i, 1, i-1)\,.$$

**6.3.1 Lemma.** *Let $G = (\beta \cup \gamma, E)$ be a distance-biregular graph with diameter $d$. For $i \geq 1$ and $j \geq 2$, we have*

$$p^\beta(i, j+1; 2h) = \frac{1}{c_{j+1}^\beta}\Big(c_{i+1}^\beta p^\beta(i+1, j; 2h) + b_{i-1}^\beta p^\beta(i-1, j; 2h) - b_{j-1}^\beta p^\beta(i, j-1; 2h)\Big)$$

*and*

$$\begin{aligned} p^\beta(i, j+1; 2h+1) = &\frac{1}{c_{j+1}^\gamma}\Big(c_{i+1}^\beta p^\beta(i+1, j; 2h+1) + b_{i-1}^\beta p^\beta(i-1, j; 2h+1)\Big) \\ &- \frac{1}{c_{j+1}^\gamma} b_{j-1}^\gamma p^\beta(i, j-1; 2h+1)\,. \end{aligned}$$

*Proof.* Let $u \in \beta$ and let $v$ be at distance $2h$ from $u$. Then we have

$$\begin{aligned}
p^\beta(i, j+1; 2h) &= \mathbf{e}_u^T F_i^\beta(A)\, F_{j+1}^\beta(A)\, \mathbf{e}_v \\
&= \frac{1}{c_{j+1}^\beta} \mathbf{e}_u^T \Big( F_i^\beta(A) \Big( A F_j^\beta(A) - b_{j-1}^\beta F_{j-1}^\beta(A) \Big) \Big)\, \mathbf{e}_v \\
&= \frac{1}{c_{j+1}^\beta} \mathbf{e}_u^T \Big( A F_i^\beta(A)\, F_j^\beta(A) - b_{j-1}^\beta F_{2i}^\beta(A)\, F_{j-1}^\beta(A) \Big)\, \mathbf{e}_v \\
&= \frac{1}{c_{j+1}^\beta} \mathbf{e}_u^T \Big( c_{i+1}^\beta F_{i+1}^\beta(A)\, F_j^\beta(A) + b_{i-1}^\beta F_{i-1}^\beta(A)\, F_j^\beta(A) - b_{j-1}^\beta F_i^\beta(A)\, F_{j-1}^\beta(A) \Big)\, \mathbf{e}_v \\
&= \frac{1}{c_{j+1}^\beta} \Big( c_{i+1}^\beta p^\beta(i+1, j; 2h) + b_{i-1}^\beta p^\beta(i-1, j; 2h) - b_{j-1}^\beta p^\beta(i, j-1; 2h) \Big)\,.
\end{aligned}$$

Similarly, if $v$ is at distance $2h+1$ from $u$ then

$$\begin{aligned}
p^\beta(i, j+1; 2h+1) &= \mathbf{e}_u^T F_i^\beta(A)\, F_{j+1}^\gamma(A)\, \mathbf{e}_v \\
&= \frac{1}{c_{j+1}^\gamma} \mathbf{e}_u^T \Big( A F_i^\beta(A)\, F_j^\beta(A) - b_{j-1}^\beta F_i^\beta(A)\, F_{j-1}^\gamma(A) \Big)\, \mathbf{e}_v \\
&= \frac{1}{c_{j+1}^\gamma} \Big( c_{i+1}^\beta p^\beta(i+1, j; 2h+1) + b_{i-1}^\beta p^\beta(i-1, j; 2h+1) \Big) \\
&\quad - \frac{1}{c_{j+1}^\gamma} b_{j-1}^\gamma p^\beta(i, j-1; 2h+1)\,.
\end{aligned}$$

□

Since the intersection numbers count vertices in a graph, they must be non-negative integers.

## 6.4 Krein Parameters

We can define duals to the intersection numbers and, even though they do not have a combinatorial interpretation, they still give us a feasibility condition for distance-biregular graphs.

Let $\theta_r, \theta_s$ be eigenvalues. Then since the coherent configuration is closed under Schur multiplication, there exist coefficients $\rho_{rs}(t)\,, \lambda_{rs}(t)\,, \Delta_{rs}(t)$ called *Krein parameters* such that

$$\begin{pmatrix} 2R_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \circ \begin{pmatrix} 2R_s & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} = \frac{1}{|\beta|} \sum_{t=0}^{d} \rho_{rs}(t) \begin{pmatrix} 2R_t & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}; \tag{6.4.1}$$

$$\begin{pmatrix} \mathbf{0} & 2D_r \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \circ \begin{pmatrix} \mathbf{0} & 2D_s \\ \mathbf{0} & \mathbf{0} \end{pmatrix} = \frac{\sqrt{\ell}}{\sqrt{k}\,|\beta|} \sum_{t=0}^{d} \Delta_{rs}(t) \begin{pmatrix} \mathbf{0} & 2D_t \\ \mathbf{0} & \mathbf{0} \end{pmatrix}; \tag{6.4.2}$$

$$\begin{pmatrix} \mathbf{0} & \mathbf{0} \\ 2D_r^T & \mathbf{0} \end{pmatrix} \circ \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ 2D_s^T & \mathbf{0} \end{pmatrix} = \frac{\sqrt{k}}{\sqrt{\ell}\,|\gamma|} \sum_{t=0}^{d} \Delta_{rs}(t) \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ 2D_t^T & \mathbf{0} \end{pmatrix}; \tag{6.4.3}$$

and

$$\begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 2L_r \end{pmatrix} \circ \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 2L_s \end{pmatrix} = \frac{1}{|\gamma|} \sum_{t=0}^{d} \lambda_{rs}(t) \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 2L_t \end{pmatrix}. \tag{6.4.4}$$

Let $\theta_t$ be a non-negative eigenvalue in the support of $\beta$. We have

$$\rho_{rs}(t) \begin{pmatrix} 2R_t & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} = |\beta| \begin{pmatrix} 2R_t & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \left( \begin{pmatrix} 2R_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \circ \begin{pmatrix} 2R_s & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \right).$$

Taking the trace of both sides gives us

$$\rho_{rs}(t) = \frac{|\beta|}{m_t^\beta} \operatorname{tr}(2R_t(2R_r \circ 2R_s)) = \frac{|\beta|}{m_t^\beta} \operatorname{sum}(2R_t \circ 2R_r \circ 2R_s).$$

Using Equation 6.2.5, we have

$$2R_t \circ 2R_r \circ 2R_s = \frac{1}{|\beta|^3} \sum_{2i=0}^{d} q_t^\beta(2i)\, q_r^\beta(2i)\, q_s^\beta(2i)\, X_i,$$

so

$$\rho_{rs}(t) = \frac{1}{|\beta|\, m_t^\beta} \sum_{2i=0}^{d} q_r^\beta(2i)\, q_s^\beta(2i)\, q_t^\beta(2i)\, k_{2i}^\beta.$$

An analogous argument gives

$$\lambda_{rs}(t) = \frac{1}{|\gamma|\, m_t^\gamma} \sum_{2i=0}^{d} q_r^\gamma(2i)\, q_s^\gamma(2i)\, q_t^\gamma(2i)\, k_{2i}^\gamma.$$

Letting $\theta_t$ be a positive eigenvalue, we have

$$\begin{pmatrix} \mathbf{0} & \mathbf{0} \\ 2D_t^T & \mathbf{0} \end{pmatrix} \left( \begin{pmatrix} \mathbf{0} & 2D_r \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \circ \begin{pmatrix} \mathbf{0} & 2D_s \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \right) = \frac{\sqrt{\ell}}{\sqrt{k}\,|\beta|} \Delta_{rs}(t) \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & L_t \end{pmatrix}.$$

Taking the trace of both sides gives us

$$\Delta_{rs}(t) = \frac{\sqrt{k}\,|\beta|}{\sqrt{\ell} m_t^\gamma} \operatorname{tr}\big(2D_t^T(2D_r \circ 2D_s)\big) = \frac{\sqrt{k}\,|\beta|}{\sqrt{\ell} m_t^\gamma} \operatorname{sum}(2D_r \circ 2D_s \circ 2D_t).$$

Using Equation 6.2.7, we have

$$2D_r \circ 2D_s \circ 2D_t = \left( \frac{\sqrt{\ell}}{\sqrt{k}\,|\beta|} \right)^3 \sum_{2i+1=1}^{d} q_r^\beta(2i+1)\, q_s^\beta(2i+1)\, q_t^\beta(2i+1) \begin{pmatrix} \mathbf{0} & N_{2i+1} \\ \mathbf{0} & \mathbf{0} \end{pmatrix},$$

so

$$\Delta_{rs}(t) = \frac{\ell}{k\,|\beta|} \sum_{2i+1=1}^{d} q_r^\beta(2i+1)\, q_s^\beta(2i+1)\, q_t^\beta(2i+1)\, k_{2i+1}^\beta.$$

We are now ready to compute the Krein inequalities for distance-biregular graphs. This was previously done by Delorme [55, 56], and the proof is similar to the standard proofs for distance-regular graphs.

**6.4.1 Proposition.** *Let $G$ be a distance-biregular graph of diameter $d$. For all $0 \le r, s, t \le d$, the Krein parameters satisfy*

$$\rho_{rs}(t) \ge 0,$$
$$\lambda_{rs}(t) \ge 0,$$

*and*

$$\rho_{rs}(t)\,\lambda_{rs}(t) \ge \Delta_{rs}(t)^2\,.$$

*Proof.* Let $0 \le r, s \le d$. Since $E_r, E_s$ are positive semidefinite, so too is $E_r \circ E_s$. Because $R_r \circ R_s$ and $L_r \circ L_s$ are principal submatrices, they are also positive semidefinite. Since

$$R_r \circ R_s = \sum_t \rho_{rs}(t)\, R_t,$$

we see that $\rho_{rs}(t)$ are the eigenvalues of $R_r \circ R_s$, and thus they must be non-negative. A similar argument applies for $\lambda_{rs}(t)$.

We also have

$$E_r \circ E_s = \sum_t \left( \rho_{rs}(t) \begin{pmatrix} 2R_t & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} + \lambda_{rs}(t) \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 2L_t \end{pmatrix} + \Delta_{rs}(t) \begin{pmatrix} \mathbf{0} & 2D_t \\ 2D_t^T & \mathbf{0} \end{pmatrix} \right),$$

from which we get the positive-semidefinite quadratic form

$$\rho_{rs}(t)\, x_1^2 + \lambda_{rs}(t)\, x_2^2 + 2\Delta_{rs}(t)\, x_1 x_2,$$

which in turn gives us the inequality

$$\rho_{rs}(t)\,\lambda_{rs}(t) \ge \Delta_{rs}^2.$$ □

## 6.5 Feasibility and Examples

In Chapter 3, we proposed a set of feasible criteria for distance-biregular graphs, which we used to prepare the tables in Appendix A and Appendix B. We have annotated those tables with the constructions of graphs from Chapter 3, citations to other constructions, and notes of which graphs fail to meet the conditions on the intersection numbers or Krein parameters. It would be nice to have more constructions and necessary conditions to add to the table.

We highlight one intersection array of particular interest.

**6.5.1 Problem.** *Does a distance-biregular graph with intersection array*

$$\begin{vmatrix} 8; & 1, & 2, & 21, & 8 \\ 36; & 1, & 6, & 7, & 36 \end{vmatrix}$$

*exist?*

The halved intersection arrays are $(140, 45; 1, 84)$ and $(42, 5; 1, 42)$. The complement of eight disjoint copies of $K_6$ has intersection array $(42, 5; 1, 42)$, but the existence of a strongly regular graph with intersection array $(140, 45; 1, 84)$ is unknown. Thus finding a distance-biregular graph with the intersection array in Problem 6.5.1 would lead to the existence of a new strongly regular graph.

There is another reason why the tables in Appendix B are of particular interest. Let $G = (\beta \cup \gamma, E)$ be a bipartite graph such that $\beta$ has covering radius $d$. Let $u \in \beta$ and let $v$ be at distance two from $u$. The graph $G$ is *2-$\beta$-homogeneous* if for all $w$ at distance $i$ from both $u$ and $v$, the number of vertices which are neighbours of $u$ and $v$ and at distance $i-1$ from $w$ is independent of the choice of $u, v$, and $w$. Fernández and Penjić [69] studied distance-biregular graphs which were 2-$\beta$-homogeneous, and characterized the intersection arrays. As a consequence, they proved that the only 2-$\beta$-homogeneous graphs with $c_2^\beta = 1$ are the generalized polygons. They also characterized the 2-$\beta$-homogeneous distance-biregular graphs with $d = 3$, so the next smallest case to look at is the graphs with diameter four and $c_2^\beta \neq 1$.

Several of the problems that we've posed throughout the thesis can also fall under the rough category of proving or disproving the existence of distance-biregular graphs. For the sake of completeness, we restate the problems here.

**3.3.1 Problem.** *Given a distance-regular graph $H$, is $H$ the halved graph of a distance-biregular graph?*

In chapter 6 of his thesis, Shawe-Taylor [134] proved that several infinite families, including the Hamming graphs and dual polar space graphs, do not arise as halved graphs of distance-biregular graphs. Similar techniques of looking at the subgraphs of distance-biregular graphs were considered by Suzuki [144] and Hiraki [96, 95], and could likely be applied to other families of distance-biregular graphs.

**3.6.3 Problem.** *Characterize the distance-biregular graphs where one cell of the partition has valency three.*

Biggs, Boshier, and Shawe-Taylor [22] characterized distance-regular graphs of valency three. To do so, they used the fact that $b_0^* \geq b_1^* \geq \cdots \geq b_{d-1}^*$ and $c_1^* \leq c_2^* \leq \cdots \leq c_d^*$ to break the structure of cubic distance-regular graphs into several cases and systematically describe the graphs in each case. Note that there are only 13 cubic distance-regular graphs, but the Steiner triple systems give an infinite family of distance-biregular graphs where one cell of the partition has valency three. Thus a similar approach as the one used by Biggs, Boshier, and Shawe-Taylor [22] to Problem 3.6.3 would be more involved, since it would require dealing with infinite families rather than concrete graphs. Some cases have already been considered by Suzuki [143] and Yamazaki [155]. Easier subcases of Problem 3.6.3 would ask for characterizations of the distance-biregular graphs where one cell of the partition has valency three and the other cell has some other fixed valency.

**4.7.1 Problem.** *If two distance-biregular graphs are cospectral, do they have the same intersection array?*

An intersection array being feasible imposes strong numerical conditions on the intersection coefficients. The additional criteria that two feasible intersection arrays have the same spectrum imposes even stronger numerical conditions, but it's not clear if these conditions are strong enough to force the intersection arrays to be the same. Even for distance-regular graphs, where the intersection array is determined by the spectrum, there are still cospectral pairs of non-isomorphic distance-regular graphs with the same intersection array.

**6.5.2 Problem.** *How many non-isomorphic distance-biregular graphs have the intersection arrays listed in Appendix A and Appendix B?*

This problem has been considered for partial geometries and bipartite distance-regular graphs, but not the distance-biregular graphs in Appendix B. One hopeful family is the Hadamard semi-symmetric designs. For higher orders, there are many non-equivalent Hadamard matrices, and it is likely that they would lead to non-isomorphic distance-biregular graphs.

## 6.6 Extensions of Distance-Regular Results

Any result on distance-regular graphs that does not assume the existence of odd cycles can be turned into a problem for distance-biregular graphs. Such questions led us to Theorem 4.4.3 and Theorem 4.6.4, which in turn led to to the following problem:

**4.7.3 Problem.** *When is a graph with the spectrum of a distance-biregular graph distance-biregular?*

Other results on distance-regular graphs were adapted to distance-biregular graphs by Delorme [55, 56] and Shawe-Taylor [134]. In particular, they both adapted the proof of Terwilliger [146] to bound the diameter of a distance-biregular graph of valency $k$ with given girth.

In 1984, Bannai and Ito [15] conjectured that for $k \geq 3$, there are only finitely many distance-regular graphs of valency $k$. The proof of Terwilliger [146] can be seen as a step towards that conjecture, since it proves that for $k, g \geq 3$, there are only finitely many distance-regular graphs of valency $k$ and girth $g$. Working in the same time span, it was logical that Delorme [56] and Shawe-Taylor [134] would extend this to distance-biregular graphs. Since then, the Bannai-Ito conjecture has been settled by Bannai and Ito for bipartite graphs [16] and more recently for all graphs by Bang, Dubickas, Koolen, and Moulton [14]. The bipartite case is much simpler than the general case, motivating the following question.

**6.6.1 Problem.** *Prove that for all $k, \ell \geq 3$, there are only finitely many distance-biregular graphs with valencies $k$ and $\ell$.*

If we fix $k$ and $\ell$, then there are only finitely many valencies that the halved graphs of a $(k, \ell)$-distance-biregular graph can have, so by the result of Bang, Dubickas, Koolen, and Moulton [14], we know there are only finitely many halved graphs. It follows that there are only finitely many distance-biregular graphs with valencies $(k, \ell)$ , and so we have solved Problem 6.6.1. However, we are looking for a more direct proof. For the regular case, the additional structure of bipartite graphs make for a simpler proof, and it is likely that similar structure would be useful for distance-biregular graphs.

We could also take a more algebraic perspective and, instead of asking for extensions of distance-regular graphs to distance-biregular analogues, ask for extensions of results from association schemes to the coherent configuration described in Section 6.1.

The distance adjacency matrices of a distance-regular graph have a stronger property than just forming an association scheme. For all $0 \leq i \leq d$, the matrix $A_i$ is a polynomial in $A_1$ of degree $i$, so a distance-regular graph forms a *$P$-polynomial association scheme*. Since association schemes have two products, and two bases idempotent under one of the products, we can define a dual notion to $P$-polynomial.

Let $M$ be a matrix. For $i \geq 0$, we define the *Schur power* $M^{\circ i}$ as the Schur product of $M$ with itself $i$ times. Then if we have a polynomial

$$q(x) = \sum_{i=0}^{t} h_i x^i,$$

we define

$$q \circ M := \sum_{i=0}^{t} h_i M^{\circ i}.$$

An association scheme is *$Q$-polynomial* if there exists an ordering of the spectral idempotents $E_0, \ldots, E_d$ such that, for $0 \leq i \leq d$, there exists a polynomial $q_i$ such that

$$q_i \circ E_1 = E_i.$$

The definition we have been using for distance-biregular graphs gives us a notion of $P$-polynomial for a coherent configuration with the bipartite structure set up in Section 6.1. It is thus natural to ask for a dual version.

**6.6.2 Problem.** *Investigate $Q$-polynomial coherent configurations with a bipartite structure.*

Notions of $Q$-polynomial have been proposed for general coherent configurations [142], and for graphs that are not distance-regular [147], but it would be nice to have a definition for distance-biregular graphs that unifies these two perspectives, as well as more examples. Part of Problem 6.6.2 would be studying when distance-biregular graphs are and are not $Q$-polynomial, but it would also be interesting to have examples of bipartite coherent configurations that are not $P$-polynomial. This brings us back to one of our earlier problems.

**5.3.2 Problem.** *How can we interpret the dual linear programming bounds for semiregular bipartite graphs of Theorem 5.3.1?*

Recall that the linear programming bound of Nozaki [125] has a dual bound for spherical codes. Delsarte, Goethals, and Seidel [58] proved the dual bound. They also proved that certain spherical codes have the structure of a $Q$-polynomial association scheme, a result that Nozaki [125] called dual to the result of Abiad, Van Dam, and Fiol [3] that a graph with diameter $d$, $d+1$ distinct eigenvalues, and girth at least $2d-1$ is distance-regular. Thus an interpretation for Problem 5.3.2 would give us a plausible place to look for $Q$-polynomial coherent configurations which are not necessarily distance-biregular.

## 6.7 Extremal Examples

Given a bound, a natural question is what happens when the bound is tight. Often, the extremal examples have particularly strong algebraic or combinatorial structure. In the case of regular graphs, this leads to examples of distance-regular graphs as extremal examples. In the context of bipartite graphs, similar bounds lead to distance-biregular graphs as extremal examples.

We have seen some examples of this already. Bipartite graphs with large girth are distance-biregular, and they arose as extremal examples both with respect to the spectral Moore bound and as extremal expanders. We investigated these connections in Chapter 5, and our study leaves us with the following problem.

**5.5.2 Problem.** *Let $G$ be a distance-biregular graph with minimum valency at least three, diameter $d$, and girth $g \geq 2d-2$. What are the possible values of $d$?*

Cioabă, Koolen, and Nozaki [36] answered this question for regular graphs. Their proof followed a similar strategy to the one employed by Fuglister [79] using orthogonal polynomials. A similar approach, adapted to use the orthogonal polynomials for a semiregular tree, would likely let us solve Problem 5.5.2.

One idea for studying extremal distance-biregular graphs would be to take extremal problems for distance-regular graphs and adapt them to the distance-biregular case. An example comes from a dual version of Problem 6.6.1. For distance-regular graphs, we have the following result.

**6.7.1 Theorem** (Godsil [82]). *Let $G$ be a connected distance-regular graph of valency $k$ and diameter $d$ that is not complete bipartite. Let $\theta$ be a nontrivial eigenvalue with multiplicity $m > 2$. Then $d \leq 3m - 4$ and $k \leq \frac{1}{2}(m-1)(m-2)$.*

Godsil [82] further proved that the only graph with $d = 3m - 4$ is the dodecahedron. We would like an extension to distance-biregular graphs.

**6.7.2 Problem.** *Let $G$ be a connected distance-biregular graph of valency $k$ and diameter $d > 2$. Let $\theta$ be a nontrivial eigenvalue with multiplicity $m > 2$. What are the bounds on the diameter and the valencies in terms of $m$? What graphs meet these bounds?*

Another way to study distance-biregular graphs as extremal graphs is to look at extremal problems where the known examples are distance-biregular, or closely related to families of distance-biregular graphs. One such possibility comes from a problem originally posed by Zarankiewicz [158] on the principal submatrix of a 01-matrix. Viewing the matrix as the biadjacency matrix of a graph gives the following common formulation.

**6.7.3 Problem** (Zarankiewicz [158]). *Let $m, n, r, s$ be positive integers with $m \geq r$ and $n \geq s$. Given a bipartite graph $G = (\beta \cup \gamma, E)$ with $|\beta| = m$ and $|\gamma| = n$, what is the maximum number of edges that $G$ can have without containing $K_{r,s}$ as a subgraph?*

An overview of the major bounds for the Zarankiewicz problem can be found in the first half of Section VI. 2 of Bollobas [24]. Many of the known extremal examples of graphs for the Zarankiewicz problem come from modifications of incidence structures such as generalized polygons [48] or designs [87] with a highly regular structure. Distance-biregular graphs provide a unifying structure behind some of these constructions that might lead to new examples, particularly in the unbalanced case where $r \neq s$.

# Appendix A

# Feasible Parameters for Diameter Four

| $\|\beta\|$ | $\|\gamma\|$ | Intersection Array | Notes |
|---|---|---|---|
| 15.0 | 15.0 | $\begin{vmatrix} 3; & 1, & 1, & 1, & 3 \\ 3; & 1, & 1, & 1, & 3 \end{vmatrix}$ | $GQ(2,2)$ Exists and is unique [129] |
| 9.0 | 9.0 | $\begin{vmatrix} 3; & 1, & 1, & 2, & 3 \\ 3; & 1, & 1, & 2, & 3 \end{vmatrix}$ | Pappus Graph [28] |
| 16.0 | 12.0 | $\begin{vmatrix} 3; & 1, & 1, & 2, & 3 \\ 4; & 1, & 1, & 2, & 4 \end{vmatrix}$ | $OA(3,4)$ Exists [1] |
| 12.0 | 9.0 | $\begin{vmatrix} 3; & 1, & 1, & 3, & 3 \\ 4; & 1, & 1, & 3 & \end{vmatrix}$ | $S(2,3,9)$ Exists [107] |
| 45.0 | 27.0 | $\begin{vmatrix} 3; & 1, & 1, & 1, & 3 \\ 5; & 1, & 1, & 1, & 5 \end{vmatrix}$ | $GQ(2,4)$ Exists and is unique [148] |
| 25.0 | 15.0 | $\begin{vmatrix} 3; & 1, & 1, & 2, & 3 \\ 5; & 1, & 1, & 2, & 5 \end{vmatrix}$ | $OA(3,5)$ Exists [1] |
| 36.0 | 18.0 | $\begin{vmatrix} 3; & 1, & 1, & 2, & 3 \\ 6; & 1, & 1, & 2, & 6 \end{vmatrix}$ | $OA(3,6)$ Exists [1] |
| 26.0 | 13.0 | $\begin{vmatrix} 3; & 1, & 1, & 3, & 3 \\ 6; & 1, & 1, & 3 & \end{vmatrix}$ | $S(2,3,13)$ Exists [107] |
| 49.0 | 21.0 | $\begin{vmatrix} 3; & 1, & 1, & 2, & 3 \\ 7; & 1, & 1, & 2, & 7 \end{vmatrix}$ | $OA(3,7)$ Exists [1] |
| 35.0 | 15.0 | $\begin{vmatrix} 3; & 1, & 1, & 3, & 3 \\ 7; & 1, & 1, & 3 & \end{vmatrix}$ | $S(2,3,15)$ Exists [107] |
| 64.0 | 24.0 | $\begin{vmatrix} 3; & 1, & 1, & 2, & 3 \\ 8; & 1, & 1, & 2, & 8 \end{vmatrix}$ | $OA(3,8)$ Exists [1] |
| 81.0 | 27.0 | $\begin{vmatrix} 3; & 1, & 1, & 2, & 3 \\ 9; & 1, & 1, & 2, & 9 \end{vmatrix}$ | $OA(3,9)$ Exists [1] |
| 57.0 | 19.0 | $\begin{vmatrix} 3; & 1, & 1, & 3, & 3 \\ 9; & 1, & 1, & 3 & \end{vmatrix}$ | $S(2,3,19)$ Exists [107] |
| 100.0 | 30.0 | $\begin{vmatrix} 3; & 1, & 1, & 2, & 3 \\ 10; & 1, & 1, & 2, & 10 \end{vmatrix}$ | $OA(3,10)$ Exists [1] |
| 70.0 | 21.0 | $\begin{vmatrix} 3; & 1, & 1, & 3, & 3 \\ 10; & 1, & 1, & 3 & \end{vmatrix}$ | $S(2,3,21)$ Exists [107] |
| 231.0 | 63.0 | $\begin{vmatrix} 3; & 1, & 1, & 1, & 3 \\ 11; & 1, & 1, & 1, & 11 \end{vmatrix}$ | Fails Krein Inequality: Proposition 6.4.1 |
| 121.0 | 33.0 | $\begin{vmatrix} 3; & 1, & 1, & 2, & 3 \\ 11; & 1, & 1, & 2, & 11 \end{vmatrix}$ | $OA(3,11)$ Exists [1] |
| 144.0 | 36.0 | $\begin{vmatrix} 3; & 1, & 1, & 2, & 3 \\ 12; & 1, & 1, & 2, & 12 \end{vmatrix}$ | $OA(3,12)$ Exists [1] |
| 100.0 | 25.0 | $\begin{vmatrix} 3; & 1, & 1, & 3, & 3 \\ 12; & 1, & 1, & 3 & \end{vmatrix}$ | $S(2,3,25)$ Exists [107] |

| | | | | |
|---|---|---|---|---|
| 40.0 | 40.0 | | 4; 1, 1, 1, 4<br>4; 1, 1, 1, 4 | $GQ(3,3)$ Exactly two exist [63] |
| 16.0 | 16.0 | | 4; 1, 1, 3, 4<br>4; 1, 1, 3, 4 | $OA(4,4)$ Exists [1] |
| 10.0 | 10.0 | | 4; 1, 2, 2, 4<br>4; 1, 2, 2, 4 | Fails Krein Inequality: Proposition 6.4.1 |
| 8.0 | 8.0 | | 4; 1, 2, 3, 4<br>4; 1, 2, 3, 4 | Hypercube Graph Figure 2.1 |
| 35.0 | 28.0 | | 4; 1, 1, 2, 4<br>5; 1, 1, 2, 5 | $PG(3,4,2)$ Does not exist [30] |
| 25.0 | 20.0 | | 4; 1, 1, 3, 4<br>5; 1, 1, 3, 5 | $OA(4,5)$ Exists [1] |
| 20.0 | 16.0 | | 4; 1, 1, 4, 4<br>5; 1, 1, 4 | $S(2,4,16)$ Exists [93] |
| 96.0 | 64.0 | | 4; 1, 1, 1, 4<br>6; 1, 1, 1, 6 | $GQ(3,5)$ Exists and is unique [129] |
| 36.0 | 24.0 | | 4; 1, 1, 3, 4<br>6; 1, 1, 3, 6 | $OA(4,6)$ Does not exist [1] |
| 133.0 | 76.0 | | 4; 1, 1, 1, 4<br>7; 1, 1, 1, 7 | $GQ(3,6)$ Does not exist [63] |
| 49.0 | 28.0 | | 4; 1, 1, 3, 4<br>7; 1, 1, 3, 7 | $OA(4,7)$ Exists [1] |
| 14.0 | 8.0 | | 4; 1, 2, 6, 4<br>7; 1, 3, 4 | Points/Hyperplanes of AG(2,3): Example 3.7.1<br>Hadamard design of order 8: Example 3.1.5 |
| 64.0 | 32.0 | | 4; 1, 1, 3, 4<br>8; 1, 1, 3, 8 | $OA(4,8)$ Exists [1] |
| 50.0 | 25.0 | | 4; 1, 1, 4, 4<br>8; 1, 1, 4 | $S(2,4,25)$ Exists [93] |
| 81.0 | 36.0 | | 4; 1, 1, 3, 4<br>9; 1, 1, 3, 9 | $OA(4,9)$ Exists [1] |
| 63.0 | 28.0 | | 4; 1, 1, 4, 4<br>9; 1, 1, 4 | $S(2,4,28)$ Exists [93] |
| 280.0 | 112.0 | | 4; 1, 1, 1, 4<br>10; 1, 1, 1, 10 | $GQ(3,9)$ Exists and is unique [63] |
| 100.0 | 40.0 | | 4; 1, 1, 3, 4<br>10; 1, 1, 3, 10 | $OA(4,10)$ Exists [7] |
| 176.0 | 64.0 | | 4; 1, 1, 2, 4<br>11; 1, 1, 2, 11 | $PG(3,10,2)$ |

| | | | |
|---|---|---|---|
| 121.0 | 44.0 | $\left\{\begin{array}{lllll} 4; & 1, & 1, & 3, & 4 \\ 11; & 1, & 1, & 3, & 11 \end{array}\right\}$ | $OA(4,11)$ Exists [1] |
| 144.0 | 48.0 | $\left\{\begin{array}{lllll} 4; & 1, & 1, & 3, & 4 \\ 12; & 1, & 1, & 3, & 12 \end{array}\right\}$ | $OA(4,12)$ Exists [104] |
| 111.0 | 37.0 | $\left\{\begin{array}{lllll} 4; & 1, & 1, & 4, & 4 \\ 12; & 1, & 1, & 4 & \end{array}\right\}$ | $S(2,4,37)$ Exists [93] |
| 85.0 | 85.0 | $\left\{\begin{array}{lllll} 5; & 1, & 1, & 1, & 5 \\ 5; & 1, & 1, & 1, & 5 \end{array}\right\}$ | $GQ(4,4)$ Exists and is unique [127, 128] |
| 25.0 | 25.0 | $\left\{\begin{array}{lllll} 5; & 1, & 1, & 4, & 5 \\ 5; & 1, & 1, & 4, & 5 \end{array}\right\}$ | $OA(5,5)$ Exists [1] |
| 36.0 | 30.0 | $\left\{\begin{array}{lllll} 5; & 1, & 1, & 4, & 5 \\ 6; & 1, & 1, & 4, & 6 \end{array}\right\}$ | $OA(3,4)$ Does not exist [1] |
| 30.0 | 25.0 | $\left\{\begin{array}{lllll} 5; & 1, & 1, & 5, & 5 \\ 6; & 1, & 1, & 5 & \end{array}\right\}$ | $S(2,5,25)$ Exists [93] |
| 175.0 | 125.0 | $\left\{\begin{array}{lllll} 5; & 1, & 1, & 1, & 5 \\ 7; & 1, & 1, & 1, & 7 \end{array}\right\}$ | $GQ(4,6)$ Exists [129] |
| 63.0 | 45.0 | $\left\{\begin{array}{lllll} 5; & 1, & 1, & 3, & 5 \\ 7; & 1, & 1, & 3, & 7 \end{array}\right\}$ | $PG(4,6,3)$ Exactly two exist [114] |
| 49.0 | 35.0 | $\left\{\begin{array}{lllll} 5; & 1, & 1, & 4, & 5 \\ 7; & 1, & 1, & 4, & 7 \end{array}\right\}$ | $OA(5,7)$ Exists [1] |
| 120.0 | 75.0 | $\left\{\begin{array}{lllll} 5; & 1, & 1, & 2, & 5 \\ 8; & 1, & 1, & 2, & 8 \end{array}\right\}$ | $PG(4,7,2)$ |
| 64.0 | 40.0 | $\left\{\begin{array}{lllll} 5; & 1, & 1, & 4, & 5 \\ 8; & 1, & 1, & 4, & 8 \end{array}\right\}$ | $OA(5,8)$ Exists [1] |
| 297.0 | 165.0 | $\left\{\begin{array}{lllll} 5; & 1, & 1, & 1, & 5 \\ 9; & 1, & 1, & 1, & 9 \end{array}\right\}$ | $GQ(4,8)$ Exists [129] |
| 81.0 | 45.0 | $\left\{\begin{array}{lllll} 5; & 1, & 1, & 4, & 5 \\ 9; & 1, & 1, & 4, & 9 \end{array}\right\}$ | $OA(5,9)$ Exists [1] |
| 190.0 | 95.0 | $\left\{\begin{array}{lllll} 5; & 1, & 1, & 2, & 5 \\ 10; & 1, & 1, & 2, & 10 \end{array}\right\}$ | $PG(4,9,2)$ |
| 100.0 | 50.0 | $\left\{\begin{array}{lllll} 5; & 1, & 1, & 4, & 5 \\ 10; & 1, & 1, & 4, & 10 \end{array}\right\}$ | $OA(5,10)$ |
| 82.0 | 41.0 | $\left\{\begin{array}{lllll} 5; & 1, & 1, & 5, & 5 \\ 10; & 1, & 1, & 5 & \end{array}\right\}$ | $S(2,5,41)$ Exists [93] |
| 121.0 | 55.0 | $\left\{\begin{array}{lllll} 5; & 1, & 1, & 4, & 5 \\ 11; & 1, & 1, & 4, & 11 \end{array}\right\}$ | $OA(5,11)$ Exists [1] |
| 99.0 | 45.0 | $\left\{\begin{array}{lllll} 5; & 1, & 1, & 5, & 5 \\ 11; & 1, & 1, & 5 & \end{array}\right\}$ | $S(2,5,45)$ Exists [93] |

| | | | | |
|---|---|---|---|---|
| 540.0 | 225.0 | | 5; 1, 1, 1, 5<br>12; 1, 1, 1, 12 | $GQ(4,11)$ |
| 144.0 | 60.0 | | 5; 1, 1, 4, 5<br>12; 1, 1, 4, 12 | $OA(5,12)$ Exists [104] |
| 156.0 | 156.0 | | 6; 1, 1, 1, 6<br>6; 1, 1, 1, 6 | $GQ(5,5)$ Exists [59] |
| 81.0 | 81.0 | | 6; 1, 1, 2, 6<br>6; 1, 1, 2, 6 | Exists [111] |
| 36.0 | 36.0 | | 6; 1, 1, 5, 6<br>6; 1, 1, 5, 6 | $OA(6,6)$ Does not exist [1] |
| 36.0 | 36.0 | | 6; 1, 2, 2, 6<br>6; 1, 2, 2, 6 | Fails Krein Inequality: Proposition 6.4.1 |
| 26.0 | 26.0 | | 6; 1, 2, 3, 6<br>6; 1, 2, 3, 6 | Fails Krein Inequality: Proposition 6.4.1 |
| 18.0 | 18.0 | | 6; 1, 2, 5, 6<br>6; 1, 2, 5, 6 | Hexacode Graph [28] |
| 16.0 | 16.0 | | 6; 1, 3, 3, 6<br>6; 1, 3, 3, 6 | Fails Krein Inequality: Proposition 6.4.1 |
| 12.0 | 12.0 | | 6; 1, 3, 5, 6<br>6; 1, 3, 5, 6 | Does not exist [55, 56] |
| 112.0 | 96.0 | | 6; 1, 1, 2, 6<br>7; 1, 1, 2, 7 | $PG(5,6,2)$ |
| 49.0 | 42.0 | | 6; 1, 1, 5, 6<br>7; 1, 1, 5, 7 | $OA(6,7)$ Exists [1] |
| 42.0 | 36.0 | | 6; 1, 1, 6, 6<br>7; 1, 1, 6 | $S(2,6,36)$ Does not exist [100] |
| 288.0 | 216.0 | | 6; 1, 1, 1, 6<br>8; 1, 1, 1, 8 | $GQ(5,7)$ |
| 64.0 | 48.0 | | 6; 1, 1, 5, 6<br>8; 1, 1, 5, 8 | $OA(6,8)$ Exists [1] |
| 189.0 | 126.0 | | 6; 1, 1, 2, 6<br>9; 1, 1, 2, 9 | $PG(5,8,2)$ |
| 99.0 | 66.0 | | 6; 1, 1, 4, 6<br>9; 1, 1, 4, 9 | $PG(5,8,4)$ Does not exist [30] |
| 81.0 | 54.0 | | 6; 1, 1, 5, 6<br>9; 1, 1, 5, 9 | $OA(6,9)$ Exists [1] |
| 69.0 | 46.0 | | 6; 1, 1, 6, 6<br>9; 1, 1, 6 | $S(2,6,46)$ Does not exist [100] |

| | | | |
|---|---|---|---|
| 160.0 | 96.0 | $\left\{\begin{matrix} 6; & 1, & 1, & 3, & 6 \\ 10; & 1, & 1, & 3, & 10 \end{matrix}\right\}$ | $PG(5,9,3)$ |
| 100.0 | 60.0 | $\left\{\begin{matrix} 6; & 1, & 1, & 5, & 6 \\ 10; & 1, & 1, & 5, & 10 \end{matrix}\right\}$ | $OA(6,10)$ |
| 85.0 | 51.0 | $\left\{\begin{matrix} 6; & 1, & 1, & 6, & 6 \\ 10; & 1, & 1, & 6 & \end{matrix}\right\}$ | $S(2,6,51)$ |
| 561.0 | 306.0 | $\left\{\begin{matrix} 6; & 1, & 1, & 1, & 6 \\ 11; & 1, & 1, & 1, & 11 \end{matrix}\right\}$ | $GQ(5,10)$ |
| 121.0 | 66.0 | $\left\{\begin{matrix} 6; & 1, & 1, & 5, & 6 \\ 11; & 1, & 1, & 5, & 11 \end{matrix}\right\}$ | $OA(6,11)$ Exists [1] |
| 22.0 | 12.0 | $\left\{\begin{matrix} 6; & 1, & 3, & 10, & 6 \\ 11; & 1, & 5, & 6 & \end{matrix}\right\}$ | Hadamard design of order 12: Example 3.1.5 |
| 342.0 | 171.0 | $\left\{\begin{matrix} 6; & 1, & 1, & 2, & 6 \\ 12; & 1, & 1, & 2, & 12 \end{matrix}\right\}$ | $PG(5,11,2)$ |
| 144.0 | 72.0 | $\left\{\begin{matrix} 6; & 1, & 1, & 5, & 6 \\ 12; & 1, & 1, & 5, & 12 \end{matrix}\right\}$ | $OA(6,12)$ Exists [104] |
| 122.0 | 61.0 | $\left\{\begin{matrix} 6; & 1, & 1, & 6, & 6 \\ 12; & 1, & 1, & 6 & \end{matrix}\right\}$ | $S(2,6,61)$ |
| 259.0 | 259.0 | $\left\{\begin{matrix} 7; & 1, & 1, & 1, & 7 \\ 7; & 1, & 1, & 1, & 7 \end{matrix}\right\}$ | $GQ(6,6)$ |
| 70.0 | 70.0 | $\left\{\begin{matrix} 7; & 1, & 1, & 4, & 7 \\ 7; & 1, & 1, & 4, & 7 \end{matrix}\right\}$ | $PG(6,6,4)$ |
| 49.0 | 49.0 | $\left\{\begin{matrix} 7; & 1, & 1, & 6, & 7 \\ 7; & 1, & 1, & 6, & 7 \end{matrix}\right\}$ | $OA(7,7)$ Exists [1] |
| 176.0 | 154.0 | $\left\{\begin{matrix} 7; & 1, & 1, & 2, & 7 \\ 8; & 1, & 1, & 2, & 8 \end{matrix}\right\}$ | $PG(6,7,2)$ |
| 64.0 | 56.0 | $\left\{\begin{matrix} 7; & 1, & 1, & 6, & 7 \\ 8; & 1, & 1, & 6, & 8 \end{matrix}\right\}$ | $OA(7,8)$ Exists [1] |
| 56.0 | 49.0 | $\left\{\begin{matrix} 7; & 1, & 1, & 7, & 7 \\ 8; & 1, & 1, & 7 & \end{matrix}\right\}$ | Points and hyperplanes of $AG(2,7)$ Example 3.7.1 |
| 441.0 | 343.0 | $\left\{\begin{matrix} 7; & 1, & 1, & 1, & 7 \\ 9; & 1, & 1, & 1, & 9 \end{matrix}\right\}$ | $GQ(6,8)$ Exists [5] |
| 153.0 | 119.0 | $\left\{\begin{matrix} 7; & 1, & 1, & 3, & 7 \\ 9; & 1, & 1, & 3, & 9 \end{matrix}\right\}$ | $PG(6,8,3)$ |
| 81.0 | 63.0 | $\left\{\begin{matrix} 7; & 1, & 1, & 6, & 7 \\ 9; & 1, & 1, & 6, & 9 \end{matrix}\right\}$ | $OA(7,9)$ Exists [1] |
| 550.0 | 385.0 | $\left\{\begin{matrix} 7; & 1, & 1, & 1, & 7 \\ 10; & 1, & 1, & 1, & 10 \end{matrix}\right\}$ | $GQ(6,9)$ |

| | | | | | |
|---|---|---|---|---|---|
| 280.0 | 196.0 | | $\begin{array}{ccccc} 7; & 1, & 1, & 2, & 7 \\ 10; & 1, & 1, & 2, & 10 \end{array}$ | | $PG(6, 9, 2)$ |
| 100.0 | 70.0 | | $\begin{array}{ccccc} 7; & 1, & 1, & 6, & 7 \\ 10; & 1, & 1, & 6, & 10 \end{array}$ | | $OA(7, 10)$ |
| 341.0 | 217.0 | | $\begin{array}{ccccc} 7; & 1, & 1, & 2, & 7 \\ 11; & 1, & 1, & 2, & 11 \end{array}$ | | $PG(6, 10, 2)$ |
| 231.0 | 147.0 | | $\begin{array}{ccccc} 7; & 1, & 1, & 3, & 7 \\ 11; & 1, & 1, & 3, & 11 \end{array}$ | | $PG(6, 10, 3)$ |
| 143.0 | 91.0 | | $\begin{array}{ccccc} 7; & 1, & 1, & 5, & 7 \\ 11; & 1, & 1, & 5, & 11 \end{array}$ | | $PG(6, 10, 5)$ |
| 121.0 | 77.0 | | $\begin{array}{ccccc} 7; & 1, & 1, & 6, & 7 \\ 11; & 1, & 1, & 6, & 11 \end{array}$ | | $OA(7, 11)$ Exists [1] |
| 144.0 | 84.0 | | $\begin{array}{ccccc} 7; & 1, & 1, & 6, & 7 \\ 12; & 1, & 1, & 6, & 12 \end{array}$ | | $OA(7, 12)$ Exists [104] |
| 400.0 | 400.0 | | $\begin{array}{ccccc} 8; & 1, & 1, & 1, & 8 \\ 8; & 1, & 1, & 1, & 8 \end{array}$ | | $GQ(7, 7)$ Exists [59] |
| 64.0 | 64.0 | | $\begin{array}{ccccc} 8; & 1, & 1, & 7, & 8 \\ 8; & 1, & 1, & 7, & 8 \end{array}$ | | $OA(8, 8)$ Exists [1] |
| 64.0 | 64.0 | | $\begin{array}{ccccc} 8; & 1, & 2, & 3, & 8 \\ 8; & 1, & 2, & 3, & 8 \end{array}$ | | Folded 8-cube [28] |
| 50.0 | 50.0 | | $\begin{array}{ccccc} 8; & 1, & 2, & 4, & 8 \\ 8; & 1, & 2, & 4, & 8 \end{array}$ | | Distance-regular $(8, 7, 6, 4; 1, 2, 4, 8)$ |
| 32.0 | 32.0 | | $\begin{array}{ccccc} 8; & 1, & 2, & 7, & 8 \\ 8; & 1, & 2, & 7, & 8 \end{array}$ | | Incidence Graph of $STD_2\,[8; 4]$ [9] |
| 16.0 | 16.0 | | $\begin{array}{ccccc} 8; & 1, & 4, & 7, & 8 \\ 8; & 1, & 4, & 7, & 8 \end{array}$ | | Hadamard Graph on 32 vertices [28] |
| 261.0 | 232.0 | | $\begin{array}{ccccc} 8; & 1, & 1, & 2, & 8 \\ 9; & 1, & 1, & 2, & 9 \end{array}$ | | $PG(7, 8, 2)$ |
| 135.0 | 120.0 | | $\begin{array}{ccccc} 8; & 1, & 1, & 4, & 8 \\ 9; & 1, & 1, & 4, & 9 \end{array}$ | | $PG(7, 8, 4)$ |
| 81.0 | 72.0 | | $\begin{array}{ccccc} 8; & 1, & 1, & 7, & 8 \\ 9; & 1, & 1, & 7, & 9 \end{array}$ | | $OA(8, 9)$ Exists [1] |
| 72.0 | 64.0 | | $\begin{array}{ccccc} 8; & 1, & 1, & 8, & 8 \\ 9; & 1, & 1, & 8 & \end{array}$ | | Points and hyperplanes of $AG(2, 8)$ Example 3.7.1 |
| 640.0 | 512.0 | | $\begin{array}{ccccc} 8; & 1, & 1, & 1, & 8 \\ 10; & 1, & 1, & 1, & 10 \end{array}$ | | $GQ(7, 9)$ Exists [5, 92] |
| 325.0 | 260.0 | | $\begin{array}{ccccc} 8; & 1, & 1, & 2, & 8 \\ 10; & 1, & 1, & 2, & 10 \end{array}$ | | $PG(7, 9, 2)$ |

| | | | | | |
|---|---|---|---|---|---|
| 220.0 | 176.0 | | $\begin{array}{lllll} 8; & 1, & 1, & 3, & 8 \\ 10; & 1, & 1, & 3, & 10 \end{array}$ | | $PG(7,9,3)$ |
| 100.0 | 80.0 | | $\begin{array}{lllll} 8; & 1, & 1, & 7, & 8 \\ 10; & 1, & 1, & 7, & 10 \end{array}$ | | $OA(8,10)$ |
| 121.0 | 88.0 | | $\begin{array}{lllll} 8; & 1, & 1, & 7, & 8 \\ 11; & 1, & 1, & 7, & 11 \end{array}$ | | $OA(8,11)$ Exists [1] |
| 144.0 | 96.0 | | $\begin{array}{lllll} 8; & 1, & 1, & 7, & 8 \\ 12; & 1, & 1, & 7, & 12 \end{array}$ | | $OA(8,12)$ |
| 585.0 | 585.0 | | $\begin{array}{lllll} 9; & 1, & 1, & 1, & 9 \\ 9; & 1, & 1, & 1, & 9 \end{array}$ | | $GQ(8,8)$ Exists [59] |
| 81.0 | 81.0 | | $\begin{array}{lllll} 9; & 1, & 1, & 8, & 9 \\ 9; & 1, & 1, & 8, & 9 \end{array}$ | | $OA(9,9)$ Exists [1] |
| 57.0 | 57.0 | | $\begin{array}{lllll} 9; & 1, & 3, & 3, & 9 \\ 9; & 1, & 3, & 3, & 9 \end{array}$ | | Fails Krein Inequality: Proposition 6.4.1 |
| 27.0 | 27.0 | | $\begin{array}{lllll} 9; & 1, & 3, & 8, & 9 \\ 9; & 1, & 3, & 8, & 9 \end{array}$ | | Incidence Graph of $STD_3[9;4]$ [9] |
| 250.0 | 225.0 | | $\begin{array}{lllll} 9; & 1, & 1, & 3, & 9 \\ 10; & 1, & 1, & 3, & 10 \end{array}$ | | $PG(8,9,3)$ |
| 130.0 | 117.0 | | $\begin{array}{lllll} 9; & 1, & 1, & 6, & 9 \\ 10; & 1, & 1, & 6, & 10 \end{array}$ | | $PG(8,9,6)$ |
| 100.0 | 90.0 | | $\begin{array}{lllll} 9; & 1, & 1, & 8, & 9 \\ 10; & 1, & 1, & 8, & 10 \end{array}$ | | $OA(9,10)$ |
| 90.0 | 81.0 | | $\begin{array}{lllll} 9; & 1, & 1, & 9, & 9 \\ 10; & 1, & 1, & 9 & \end{array}$ | | Points and hyperplanes of $AG(2,9)$ Example 3.7.1 |
| 891.0 | 729.0 | | $\begin{array}{lllll} 9; & 1, & 1, & 1, & 9 \\ 11; & 1, & 1, & 1, & 11 \end{array}$ | | $GQ(8,10)$ Exists [5] |
| 231.0 | 189.0 | | $\begin{array}{lllll} 9; & 1, & 1, & 4, & 9 \\ 11; & 1, & 1, & 4, & 11 \end{array}$ | | $PG(8,10,4)$ |
| 121.0 | 99.0 | | $\begin{array}{lllll} 9; & 1, & 1, & 8, & 9 \\ 11; & 1, & 1, & 8, & 11 \end{array}$ | | $OA(9,11)$ Exists [1] |
| 540.0 | 405.0 | | $\begin{array}{lllll} 9; & 1, & 1, & 2, & 9 \\ 12; & 1, & 1, & 2, & 12 \end{array}$ | | $PG(8,11,2)$ |
| 144.0 | 108.0 | | $\begin{array}{lllll} 9; & 1, & 1, & 8, & 9 \\ 12; & 1, & 1, & 8, & 12 \end{array}$ | | $OA(9,12)$ |
| 820.0 | 820.0 | | $\begin{array}{lllll} 10; & 1, & 1, & 1, & 10 \\ 10; & 1, & 1, & 1, & 10 \end{array}$ | | $GQ(9,9)$ Exists [59] |
| 100.0 | 100.0 | | $\begin{array}{lllll} 10; & 1, & 1, & 9, & 10 \\ 10; & 1, & 1, & 9, & 10 \end{array}$ | | $OA(3,4)$ Does not exist [1] |

| | | | |
|---|---|---|---|
| 190.0 | 190.0 | $\begin{vmatrix} 10; & 1, & 2, & 2, & 10 \\ 10; & 1, & 2, & 2, & 10 \end{vmatrix}$ | Distance-regular $(10, 9, 8, 8; 1, 2, 2, 10)$ |
| 100.0 | 100.0 | $\begin{vmatrix} 10; & 1, & 2, & 4, & 10 \\ 10; & 1, & 2, & 4, & 10 \end{vmatrix}$ | Distance-regular $(10, 9, 8, 6; 1, 2, 4, 10)$ |
| 82.0 | 82.0 | $\begin{vmatrix} 10; & 1, & 2, & 5, & 10 \\ 10; & 1, & 2, & 5, & 10 \end{vmatrix}$ | Distance-regular $(10, 9, 8, 5; 1, 2, 5, 10)$ |
| 50.0 | 50.0 | $\begin{vmatrix} 10; & 1, & 2, & 9, & 10 \\ 10; & 1, & 2, & 9, & 10 \end{vmatrix}$ | Distance-regular $(10, 9, 8, 1; 1, 2, 9, 10)$ |
| 28.0 | 28.0 | $\begin{vmatrix} 10; & 1, & 5, & 5, & 10 \\ 10; & 1, & 5, & 5, & 10 \end{vmatrix}$ | Fails Krein Inequality: Proposition 6.4.1 |
| 25.0 | 25.0 | $\begin{vmatrix} 10; & 1, & 5, & 6, & 10 \\ 10; & 1, & 5, & 6, & 10 \end{vmatrix}$ | Non-integral intersection numbers<br>Fails Krein Inequality: Proposition 6.4.1 |
| 20.0 | 20.0 | $\begin{vmatrix} 10; & 1, & 5, & 9, & 10 \\ 10; & 1, & 5, & 9, & 10 \end{vmatrix}$ | Distance-regular $(10, 9, 5, 1; 1, 5, 9, 10)$ |
| 506.0 | 460.0 | $\begin{vmatrix} 10; & 1, & 1, & 2, & 10 \\ 11; & 1, & 1, & 2, & 11 \end{vmatrix}$ | $PG(9, 10, 2)$ |
| 209.0 | 190.0 | $\begin{vmatrix} 10; & 1, & 1, & 5, & 10 \\ 11; & 1, & 1, & 5, & 11 \end{vmatrix}$ | $PG(9, 10, 5)$ |
| 121.0 | 110.0 | $\begin{vmatrix} 10; & 1, & 1, & 9, & 10 \\ 11; & 1, & 1, & 9, & 11 \end{vmatrix}$ | $OA(10, 11)$ Exists [1] |
| 110.0 | 100.0 | $\begin{vmatrix} 10; & 1, & 1, & 10, & 10 \\ 11; & 1, & 1, & 10 & \end{vmatrix}$ | $S(10, 100)$ |
| 1200.0 | 1000.0 | $\begin{vmatrix} 10; & 1, & 1, & 1, & 10 \\ 12; & 1, & 1, & 1, & 12 \end{vmatrix}$ | $GQ(9, 11)$ |
| 408.0 | 340.0 | $\begin{vmatrix} 10; & 1, & 1, & 3, & 10 \\ 12; & 1, & 1, & 3, & 12 \end{vmatrix}$ | $PG(9, 11, 3)$ |
| 210.0 | 175.0 | $\begin{vmatrix} 10; & 1, & 1, & 6, & 10 \\ 12; & 1, & 1, & 6, & 12 \end{vmatrix}$ | $PG(9, 11, 6)$ |
| 144.0 | 120.0 | $\begin{vmatrix} 10; & 1, & 1, & 9, & 10 \\ 12; & 1, & 1, & 9, & 12 \end{vmatrix}$ | $OA(10, 12)$ |
| 1111.0 | 1111.0 | $\begin{vmatrix} 11; & 1, & 1, & 1, & 11 \\ 11; & 1, & 1, & 1, & 11 \end{vmatrix}$ | $GQ(10, 10)$ |
| 121.0 | 121.0 | $\begin{vmatrix} 11; & 1, & 1, & 10, & 11 \\ 11; & 1, & 1, & 10, & 11 \end{vmatrix}$ | $OA(11, 11)$ Exists [1] |
| 672.0 | 616.0 | $\begin{vmatrix} 11; & 1, & 1, & 2, & 11 \\ 12; & 1, & 1, & 2, & 12 \end{vmatrix}$ | $PG(10, 11, 2)$ |
| 144.0 | 132.0 | $\begin{vmatrix} 11; & 1, & 1, & 10, & 11 \\ 12; & 1, & 1, & 10, & 12 \end{vmatrix}$ | $OA(11, 12)$ |

| | | | |
|---|---|---|---|
| 132.0 | 121.0 | $\begin{vmatrix} 11; & 1, & 1, & 11, & 11 \\ 12; & 1, & 1, & 11 & \end{vmatrix}$ | Points and hyperplanes of $AG(2,11)$ Example 3.7.1 |
| 1464.0 | 1464.0 | $\begin{vmatrix} 12; & 1, & 1, & 1, & 12 \\ 12; & 1, & 1, & 1, & 12 \end{vmatrix}$ | $GQ(11,11)$ Exists [59] |
| 144.0 | 144.0 | $\begin{vmatrix} 12; & 1, & 1, & 11, & 12 \\ 12; & 1, & 1, & 11, & 12 \end{vmatrix}$ | $OA(12,12)$ |
| 342.0 | 342.0 | $\begin{vmatrix} 12; & 1, & 2, & 2, & 12 \\ 12; & 1, & 2, & 2, & 12 \end{vmatrix}$ | Distance-regular $(12,11,10,10;1,2,2,12)$ |
| 144.0 | 144.0 | $\begin{vmatrix} 12; & 1, & 2, & 5, & 12 \\ 12; & 1, & 2, & 5, & 12 \end{vmatrix}$ | Leonard Graph [28] |
| 122.0 | 122.0 | $\begin{vmatrix} 12; & 1, & 2, & 6, & 12 \\ 12; & 1, & 2, & 6, & 12 \end{vmatrix}$ | Distance-regular $(12,11,10,6;1,2,6,12)$ |
| 72.0 | 72.0 | $\begin{vmatrix} 12; & 1, & 2, & 11, & 12 \\ 12; & 1, & 2, & 11, & 12 \end{vmatrix}$ | Distance-regular $(12,11,10,1;1,2,11,12)$ |
| 144.0 | 144.0 | $\begin{vmatrix} 12; & 1, & 3, & 3, & 12 \\ 12; & 1, & 3, & 3, & 12 \end{vmatrix}$ | Fails Krein Inequality: Proposition 6.4.1 |
| 111.0 | 111.0 | $\begin{vmatrix} 12; & 1, & 3, & 4, & 12 \\ 12; & 1, & 3, & 4, & 12 \end{vmatrix}$ | Fails Krein Inequality: Proposition 6.4.1 |
| 48.0 | 48.0 | $\begin{vmatrix} 12; & 1, & 3, & 11, & 12 \\ 12; & 1, & 3, & 11, & 12 \end{vmatrix}$ | Distance-regular $(12,11,9,1;1,3,11,12)$ |
| 36.0 | 36.0 | $\begin{vmatrix} 12; & 1, & 4, & 11, & 12 \\ 12; & 1, & 4, & 11, & 12 \end{vmatrix}$ | Suetake Graph [9] |
| 24.0 | 24.0 | $\begin{vmatrix} 12; & 1, & 6, & 11, & 12 \\ 12; & 1, & 6, & 11, & 12 \end{vmatrix}$ | Hadmard Graph on 48 vertices [28] |

$GQ(s,t)$: Generalized quadrangle. See Example 2.2.7.
$S(2,k,v)$: Steiner system. See Example 3.1.4.
$PG(s,t,\alpha)$: Partial Geoemtry. See Section 3.3.
$OA(k,n)$: Orthogonal array of degree $k$ and order $n$. This is equivalent to $PG(k-1,n-1,k-1)$ and a set of $k-2$ mutually orthogonal Latin squares. See [1].

# Appendix B

# Feasible Parameters for Diameter Four with $c_2 \neq 1$

| $\lvert\beta\rvert$ | $\lvert\gamma\rvert$ | Intersection Array | Notes |
|---|---|---|---|
| 14.0 | 8.0 | $\begin{vmatrix} 4; & 1, & 2, & 6, & 4 \\ 7; & 1, & 3, & 4 & \end{vmatrix}$ | Points/Hyperplanes of AG(2,3): Example 3.7.1<br>Hadamard matrix of order 8: Example 3.1.5 |
| 36.0 | 9.0 | $\begin{vmatrix} 4; & 1, & 2, & 12, & 4 \\ 16; & 1, & 6, & 4 & \end{vmatrix}$ | Fails Krein Inequality: Proposition 6.4.1 |
| 22.0 | 12.0 | $\begin{vmatrix} 6; & 1, & 3, & 10, & 6 \\ 11; & 1, & 5, & 6 & \end{vmatrix}$ | Hadamard matrix of order 12: Example 3.1.5 |
| 96.0 | 36.0 | $\begin{vmatrix} 6; & 1, & 2, & 6, & 6 \\ 16; & 1, & 4, & 3, & 16 \end{vmatrix}$ | Fails Krein Inequality: Proposition 6.4.1 |
| 64.0 | 24.0 | $\begin{vmatrix} 6; & 1, & 2, & 10, & 6 \\ 16; & 1, & 4, & 5, & 16 \end{vmatrix}$ | Arc of degree 2 in $PG(2,4)$: Example 3.7.2 |
| 56.0 | 21.0 | $\begin{vmatrix} 6; & 1, & 2, & 12, & 6 \\ 16; & 1, & 4, & 6 & \end{vmatrix}$ | Example 1. 2. 10 in Shawe-Taylor [134] |
| 77.0 | 22.0 | $\begin{vmatrix} 6; & 1, & 2, & 15, & 6 \\ 21; & 1, & 5, & 6 & \end{vmatrix}$ | |
| 78.0 | 13.0 | $\begin{vmatrix} 6; & 1, & 3, & 30, & 6 \\ 36; & 1, & 15, & 6 & \end{vmatrix}$ | Fails Krein Inequality: Proposition 6.4.1 |
| 225.0 | 120.0 | $\begin{vmatrix} 8; & 1, & 2, & 3, & 8 \\ 15; & 1, & 3, & 2, & 15 \end{vmatrix}$ | Fails Krein Inequality: Proposition 6.4.1 |
| 120.0 | 64.0 | $\begin{vmatrix} 8; & 1, & 2, & 6, & 8 \\ 15; & 1, & 3, & 4, & 15 \end{vmatrix}$ | Example in Delorme [55, 56] |
| 30.0 | 16.0 | $\begin{vmatrix} 8; & 1, & 4, & 14, & 8 \\ 15; & 1, & 7, & 8 & \end{vmatrix}$ | Points/Hyperplanes of AG(2,4): Example 3.7.1<br>Hadamard matrix of order 16: Example 3.1.5 |
| 288.0 | 64.0 | $\begin{vmatrix} 8; & 1, & 2, & 15, & 8 \\ 36; & 1, & 6, & 5, & 36 \end{vmatrix}$ | Fails Krein Inequality: Proposition 6.4.1 |
| 216.0 | 48.0 | $\begin{vmatrix} 8; & 1, & 2, & 21, & 8 \\ 36; & 1, & 6, & 7, & 36 \end{vmatrix}$ | |
| 39.0 | 27.0 | $\begin{vmatrix} 9; & 1, & 3, & 12, & 9 \\ 13; & 1, & 4, & 9 & \end{vmatrix}$ | Points/Hyperplanes of AG(3,3): Example 3.7.1 |
| 38.0 | 20.0 | $\begin{vmatrix} 10; & 1, & 5, & 18, & 10 \\ 19; & 1, & 9, & 10 & \end{vmatrix}$ | Hadamard matrix of order 20: Example 3.1.5 |
| 784.0 | 280.0 | $\begin{vmatrix} 10; & 1, & 2, & 4, & 10 \\ 28; & 1, & 4, & 2, & 28 \end{vmatrix}$ | Fails Krein Inequality: Proposition 6.4.1 |
| 280.0 | 100.0 | $\begin{vmatrix} 10; & 1, & 2, & 12, & 10 \\ 28; & 1, & 4, & 6, & 28 \end{vmatrix}$ | |

| | | | |
|---|---|---|---|
| 196.0 | 70.0 | 10; 1, 2, 18, 10<br>28; 1, 4, 9, 28 | |
| 46.0 | 24.0 | 12; 1, 6, 22, 12<br>23; 1, 11, 12 | Hadamard matrix of order 24: Example 3.1.5 |
| 729.0 | 378.0 | 14; 1, 2, 6, 14<br>27; 1, 3, 4, 27 | |
| 378.0 | 196.0 | 14; 1, 2, 12, 14<br>27; 1, 3, 8, 27 | |
| 54.0 | 28.0 | 14; 1, 7, 26, 14<br>27; 1, 13, 14 | Hadamard matrix of order 28: Example 3.1.5 |
| 66.0 | 45.0 | 15; 1, 5, 21, 15<br>22; 1, 7, 15 | |
| 540.0 | 225.0 | 15; 1, 3, 10, 15<br>36; 1, 6, 5, 36 | Fails Krein Inequality: Proposition 6.4.1 |
| 456.0 | 190.0 | 15; 1, 3, 12, 15<br>36; 1, 6, 6, 36 | Fails Krein Inequality: Proposition 6.4.1 |
| 288.0 | 120.0 | 15; 1, 3, 20, 15<br>36; 1, 6, 10, 36 | |
| 216.0 | 90.0 | 15; 1, 3, 28, 15<br>36; 1, 6, 14, 36 | |
| 204.0 | 85.0 | 15; 1, 3, 30, 15<br>36; 1, 6, 15 | |
| 84.0 | 64.0 | 16; 1, 4, 20, 16<br>21; 1, 5, 16 | Points/Hyperplanes of AG(4,3): Example 3.7.1 |
| 62.0 | 32.0 | 16; 1, 8, 30, 16<br>31; 1, 15, 16 | Points/Hyperplanes of AG(2,5): Example 3.7.1<br>Hadamard matrix of order 32: Example 3.1.5 |
| 1225.0 | 630.0 | 18; 1, 3, 5, 18<br>35; 1, 5, 3, 35 | Fails Krein Inequality: Proposition 6.4.1 |
| 630.0 | 324.0 | 18; 1, 3, 10, 18<br>35; 1, 5, 6, 35 | Fails Krein Inequality: Proposition 6.4.1 |
| 70.0 | 36.0 | 18; 1, 9, 34, 18<br>35; 1, 17, 18 | |
| 93.0 | 63.0 | 21; 1, 7, 30, 21<br>31; 1, 10, 21 | |
| 155.0 | 125.0 | 25; 1, 5, 30, 25<br>31; 1, 6, 25 | |

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