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TRIANGULAR OPERATOR ALGEBRAS AND SIMULTANEOUS TRIANGULARISATION

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ABSTRACT. We consider the question of whether every collection of compact operators that is contained in a triangular operator algebra (in the sense of Kadison and Singer) must be simultaneously triangularisable. The answer is shown to be affirmative if the collection consists of finite-rank operators or is a norm-closed algebra of compact operators.

1. Introduction

One of the most basic results concerning linear operators acting on a finite-dimensional, complex Hilbert space $\mathfrak{H} \simeq \mathbb{C}^n$ is that each operator T can be *triangularised*; that is, there exists an orthonormal basis $\{e_k\}_{k=1}^n$ for $\mathfrak{H}$ relative to which the matrix $[t_{ij}]$ associated to T satisfies $t_{ij} = 0$ if $i > j$. In other words, the matrix associated with T is upper triangular. More generally, a collection $\mathcal{S}$ of linear operators on a finite-dimensional space is said to be **simultaneously triangularisable** if there is an orthonormal basis such that the matrix of every operator in the collection is upper triangular with respect to that basis. In the finite-dimensional setting, simultaneous triangularisation of the collection $\mathcal{S}$ is equivalent to each of the following:

(I) there exists a maximal chain of subspaces
$$\{\mathcal{M}_k := \operatorname{span}\{e_1, e_2, \ldots, e_k\} : 0 \leq k \leq n\}$$
of $\mathfrak{H}$, each of which is left invariant by every operator in the collection; i.e. $T\mathcal{M}_k \subseteq \mathcal{M}_k$ for each k and $T \in \mathcal{S}$, and

(II) there exists an algebra $\mathfrak{A} \subseteq \mathbb{M}_n(\mathbb{C})$ such that $\mathfrak{A} \cap \mathfrak{A}^*$ is a maximal abelian, selfadjoint subalgebra (i.e., a *masa*) and $\mathfrak{A}$ contains all of the operators in the collection.

As we shall see, (I) and (II) are not equivalent on infinite-dimensional spaces. In this paper, we consider the relationship between these properties for collections of compact operators.

We denote by $\mathcal{B}(\mathfrak{H})$ the algebra of all bounded linear operators acting on the complex Hilbert space $\mathfrak{H}$, while $\mathcal{K}(\mathfrak{H})$ denotes the closed, two-sided ideal of compact operators in $\mathcal{B}(\mathfrak{H})$. A non-empty subset $\mathcal{S} \subseteq \mathcal{B}(\mathfrak{H})$ is said to be **transitive** ([11, Chapter 6]) if for each $0 \neq x \in \mathfrak{H}$, $\mathcal{S}x := \{Sx : S \in \mathcal{S}\}$ is dense in $\mathfrak{H}$. The

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terminology **irreducible** triangular algebra is often used to mean a "transitive triangular operator algebra". For any collection $\mathcal{S}$ of operators, we denote by Lat $\mathcal{S}$ the lattice of closed subspaces of $\mathfrak{H}$ that are invariant for all members of $\mathcal{S}$.

In their seminal paper on non-selfadjoint operator algebras [6], Kadison and Singer defined a **triangular operator algebra** as any algebra $\mathfrak{T}$ of bounded linear operators acting on a Hilbert space $\mathfrak{H}$ whose **diagonal** $\mathfrak{D}_{\mathfrak{T}} := \mathfrak{T} \cap \mathfrak{T}^*$ is a maximal abelian von Neumann algebra in $\mathcal{B}(\mathfrak{H})$. Five years after the publication of the Kadison-Singer paper, Ringrose [14] introduced the notion of a *nest algebra*.

Recall that a **nest** $\mathfrak{N}$ on a Hilbert space $\mathfrak{H}$ is a complete, totally ordered collection of closed subspaces of $\mathfrak{H}$ which includes $\{0\}$ and $\mathfrak{H}$ itself. To say that a nest is **complete** means that it is closed under the lattice operations of (arbitrary) intersection ($\wedge$) and closed linear spans ($\vee$). If $\mathfrak{N}$ is a nest on $\mathfrak{H}$ and $N \in \mathfrak{N}$, we define

$$N_- := \vee\{M \in \mathfrak{N} : M \subsetneq N\}.$$

It is clear that N_- is either the immediate predecessor of N in $\mathfrak{N}$ (if $N_- \neq N$), or $N_- = N$. A non-zero subspace of the form $E_N := N \ominus N_-$ is referred to as an **atom** of $\mathfrak{N}$. We say that the nest $\mathfrak{N}$ is **atomic** if $\mathfrak{H}$ is (densely) spanned by the atoms of $\mathfrak{N}$, and that $\mathfrak{N}$ is **continuous** if it has no atoms.

The nest $\mathfrak{N}$ is said to be **maximal** if it is not properly contained in any other nest, and it is known [13] that $\mathfrak{N}$ is maximal if and only if every atom of $\mathfrak{N}$ has dimension 1. Following Erdos [2], we say that the nest $\mathfrak{N}$ is **quasi-maximal** if every atom of $\mathfrak{N}$ has dimension 1 or ∞.

Given a nest $\mathfrak{N}$, Ringrose defined the corresponding **nest algebra** to be the weak-operator topology closed algebra

$$\mathcal{T}(\mathfrak{N}) := \{T \in \mathcal{B}(\mathfrak{H}) : TN \subseteq N \text{ for all } N \in \mathfrak{N}\}.$$

A collection $\mathcal{S}$ of bounded linear operators on a Hilbert space is **(simultaneously) triangularisable** if there is a maximal nest of closed subspaces of $\mathfrak{H}$ that are simultaneously invariant under all operators in the collection ([3, 7, 8, 10]), or in other words, there exists a maximal nest $\mathfrak{N}$ for which $\mathcal{S} \subseteq \mathcal{T}(\mathfrak{N})$. There are many necessary and many sufficient conditions that a collection of operators be triangularisable – see [11] for a treatment of many of the known results.

Kadison and Singer [6] asked if every operator lies in some triangular operator algebra; as they pointed out, this would represent some kind of "triangular form" even though it would not even imply that every operator has a non-trivial invariant subspace (since there are irreducible triangular operator algebras [6] – see below). This question remains open.

We consider the question: what are necessary and/or sufficient conditions that a collection of operators be contained in a triangular operator algebra?

Definition 1.1. A collection $\mathcal{S}$ of bounded linear operators acting on a complex Hilbert space $\mathfrak{H}$ is said to be **Kadison-Singer triangularisable** – abbreviated K.S.T. – if it is contained in a triangular operator algebra.

Notation. A collection $\mathcal{S}$ of bounded linear operators acting on a complex Hilbert space $\mathfrak{H}$ is S.T. if it is simultaneously triangularisable.

A very natural question is: what relations exist between K.S.T. and S.T.?

As was shown in [6], every triangular operator algebra $\mathfrak{T}$ is contained in a maximal (with respect to inclusion) triangular operator algebra $\mathfrak{T}_{\max}$. Thus $\mathcal{S} \subseteq \mathcal{B}(\mathfrak{H})$ is contained in a triangular operator algebra if and only if it is contained in a maximal triangular operator algebra. The following result of Erdos will prove useful.

Theorem 1.2 ([2, Theorem 1])**.** *Let $\mathfrak{T}$ be a maximal triangular operator algebra acting on an infinite-dimensional, separable Hilbert space $\mathfrak{H}$. Then* $\operatorname{Lat}\mathfrak{T}$ *is a quasi-maximal nest.*

Recall that a triangular operator algebra $\mathfrak{T}$ acting on an infinite-dimensional Hilbert space $\mathfrak{H}$ is said to be **irreducible** if $\operatorname{Lat}\mathfrak{T} = \{\{0\}, \mathfrak{H}\}$. Of course, if $\mathfrak{T}$ is an irreducible triangular operator algebra and $\mathfrak{T}_{\max}$ is a maximal triangular operator algebra which contains $\mathfrak{T}$, then $\mathfrak{T}_{\max}$ is again irreducible. The existence of irreducible triangular operator algebras was demonstrated in [6]. The fascinating and particularly inconvenient (for our purposes) result that exhibits the existence of maximal irreducible triangular operator algebras which are *not norm-closed* is due to Orr [9].

If $\mathfrak{T}$ is an irreducible triangular operator algebra, then $\mathfrak{T}$ is K.S.T. but not S.T.. On the other hand, if $\mathfrak{N}$ is a maximal nest such that the projections onto the subspaces in $\mathfrak{N}$ do not generate a maximal abelian selfadjoint algebra (e.g. if they generate an abelian self-adjoint algebra of uniform multiplicity 2), then the corresponding nest algebra $\mathcal{T}(\mathfrak{N})$ is S.T. but not K.S.T., since $\mathfrak{D}_{\mathcal{T}(\mathfrak{N})} = \mathcal{T}(\mathfrak{N}) \cap (\mathfrak{T}(\mathfrak{N}))^*$ is not abelian. Thus neither of K.S.T. and S.T. implies the other, in general. There are, however, implications under additional hypotheses.

Most of the known results about simultaneous triangularisability concern algebras or semigroups of compact operators [11, Chapters 7 and 8]. Thus there are natural questions concerning the relationship between K.S.T. and S.T. for more general collections of compact operators. A theorem of Erdos [2, Theorem 3] may be used to establish one implication.

Theorem 1.3. *If $\mathcal{S} \subseteq \mathcal{K}(\mathfrak{H})$ is* S.T.*, then $\mathcal{S}$ is* K.S.T..

Proof. Let $\mathfrak{N}$ be a maximal nest, and let $\mathfrak{L} \subseteq \mathcal{T}(\mathfrak{N})$ be a collection of compact operators leaving the subspaces in $\mathfrak{N}$ invariant. Let $\mathfrak{M}$ be any maximal abelian von Neumann algebra containing all of the projections onto the members of $\mathfrak{N}$. Erdos' result [2, Theorem 3] states that the algebra generated by $\mathfrak{M}$ and $\mathfrak{L}$ is triangular. □

We have been unable to determine if the converse of this theorem holds; i.e., whether K.S.T. implies S.T. for collections of compact operators. We shall show, however, that K.S.T. implies S.T. for collections of *finite-rank operators* and for *norm-closed algebras* of compact operators.

2. A reduction to irreducible triangular operator algebras

If $\mathfrak{A} \subseteq \mathcal{B}(\mathfrak{H})$ is a norm-closed algebra and $N_1, N_2 \in \operatorname{Lat}\mathfrak{A}$ satisfy $N_1 \subseteq N_2$, we refer to $P_{N_2 \ominus N_1}\mathfrak{A}|_{N_2 \ominus N_1}$ as the **compression** of $\mathfrak{A}$ to the space $N_2 \ominus N_1$. Observe that the map $A \mapsto P_{N_2 \ominus N_1} A|_{N_2 \ominus N_1}$ is a contractive homomorphism.

Let $\mathfrak{T} \subseteq \mathcal{B}(\mathfrak{H})$ be a maximal triangular operator algebra, so that $\mathfrak{N} := \operatorname{Lat}\mathfrak{T}$ would be a quasi-maximal nest (by Theorem 1.2). Then, for each $N \in \mathfrak{N}$, the space $P_{N_-}\mathcal{B}(\mathfrak{H})P_N \subseteq \mathfrak{T}$; for otherwise, $\mathfrak{T} + P_{N_-}\mathcal{B}(\mathfrak{H})P_N$ is a triangular operator algebra which properly contains $\mathfrak{T}$.

As such, the difference between $\mathfrak{T}$ and $\mathcal{T}(\mathfrak{N})$ lies in the compression of each of these algebras to infinite-dimensional atoms. Let us denote by Ω the collection of infinite-dimensional atoms of $\mathfrak{N}$; that is,

$$\Omega := \{E_N := N \ominus N_- : N \in \mathfrak{N},\ \dim E_N = \infty\}.$$

In the case of $\mathcal{T}(\mathfrak{N})$, we have that $P_{E_N}(\mathcal{T}(\mathfrak{N}))P_{E_N} = \mathcal{B}(E_N)$ for each $E_N \in \Omega$, whereas for $\mathfrak{T}$, we have that $P_{E_N}\mathfrak{T}P_{E_N}$ is an irreducible triangular operator algebra.

Proposition 2.1. *Let $\mathfrak{T} \subseteq \mathcal{B}(\mathfrak{H})$ be a maximal triangular operator algebra of operators acting on a separable Hilbert space and let $\mathfrak{L} \subseteq \mathfrak{T}$. Suppose that the compression of $\mathfrak{L}$ to each of the infinite-dimensional atoms of* Lat $\mathfrak{T}$ *is* S.T.. *Then $\mathfrak{L}$ is* S.T..

Proof. Let $\mathfrak{N} = \operatorname{Lat}\mathfrak{T}$. By Erdos' Theorem 1.2 above, $\mathfrak{N}$ is a quasi-maximal nest. Once again, let us denote by Ω the collection of infinite-dimensional atoms of $\mathfrak{N}$; that is,

$$\Omega := \{E_N := N \ominus N_- : N \in \mathfrak{N},\ \dim E_N = \infty\}.$$

Suppose that the compression of $\mathfrak{L}$ to each infinite-dimensional atom in $\operatorname{Lat}\mathfrak{T}$ is S.T.. Given $E_N \in \Omega$, there exists a maximal nest $\mathfrak{M}_{E_N}$ in E_N such that $P_{E_N}\mathfrak{L}|_{E_N} \subseteq \mathcal{T}(\mathfrak{M}_{E_N})$. It follows that if

$$\mathfrak{M} := \mathfrak{N} \cup \left(\cup_{E_N \in \Omega}\{N_- \oplus M : M \in \mathfrak{M}_{E_N}\}\right),$$

then $\mathfrak{M}$ is a maximal nest in $\mathfrak{H}$ and $\mathfrak{L} \subseteq \mathcal{T}(\mathfrak{M})$. In other words, $\mathfrak{L}$ is S.T.. □

When the family $\mathfrak{L}$ consists of compact operators, we obtain a converse result. First we recall the following result from [11] (cited as Theorem 7.3.9 there):

Theorem 2.2. *If $\mathfrak{L}$ is a simultaneously triangularisable family of compact operators acting on a separable Hilbert space, then every chain of invariant subspaces of $\mathfrak{L}$ is contained in a triangularising chain – i.e. a maximal nest.*

Theorem 2.3. *Let $\mathfrak{T}$ be a maximal triangular operator algebra acting on a separable Hilbert space and suppose that $\mathfrak{L} \subseteq \mathfrak{T}$ is a collection of compact operators. Then $\mathfrak{L}$ is* S.T. *if and only if the compression of $\mathfrak{L}$ to each of the infinite-dimensional atoms of* Lat $\mathfrak{T}$ *is* S.T..

Proof. If every compression of $\mathfrak{L}$ to each of the infinite-dimensional atoms of $\operatorname{Lat}\mathfrak{T}$ is S.T., then it follows immediately from Proposition 2.1 that $\mathfrak{L}$ is S.T..

Conversely, suppose that $\mathfrak{L}$ is S.T.. Then $\operatorname{Lat}\mathfrak{T} \subseteq \operatorname{Lat}\mathfrak{L}$ is a quasi-maximal chain of invariant subspaces for $\mathfrak{L}$, and as such extends (by Theorem 2.2) to a maximal nest $\mathfrak{M}$ of subspaces for $\mathfrak{L}$. If $E_N = N \ominus N_- \in \Omega$ is an infinite-dimensional atom of $\operatorname{Lat}\mathfrak{T}$, then $\{M \cap E_N : M \in \mathfrak{M}\}$ is a maximal nest in E_N which triangularises $P_{E_N}\mathfrak{L}|_{E_N}$. □

Of course, if $\mathfrak{T} \subseteq \mathcal{B}(\mathfrak{H})$ is any triangular operator algebra and $\mathfrak{L} \subseteq \mathfrak{T}$ is a collection of compact operators, then we may extend $\mathfrak{T}$ to a maximal triangular operator algebra $\mathfrak{T}_{\max}$, and then $\mathfrak{L}$ will be S.T. provided that the compression of $\mathfrak{L}$ to each infinite-dimensional atom of $\operatorname{Lat}\mathfrak{T}_{\max}$ is S.T.. There is no reason to believe, however that the extension of $\mathfrak{T}$ to $\mathfrak{T}_{\max}$ should be unique.

Corollary 2.4. *Let $\mathfrak{T}$ be a maximal triangular operator algebra acting on a separable Hilbert space . The following are equivalent.*

(a) $\mathfrak{T} \cap \mathcal{K}(\mathfrak{H})$ *is* S.T..
(b) *For each $E_N \in \Omega$, the compression of $\mathfrak{T}$ to E_N has no non-zero compact operators.*

Proof. That (b) implies (a) is an immediate consequence of Theorem 2.3, with $\mathfrak{L} := \mathfrak{T} \cap \mathcal{K}(\mathfrak{H})$.

Conversely, if there exists $E_N \in \Omega$ such that $P_{E_N} K|_{E_N} \neq 0$ for some $K \in \mathfrak{T} \cap \mathcal{K}(\mathfrak{H})$, then the fact that $\mathfrak{J} := P_{E_N}(\mathfrak{T} \cap \mathcal{K}(\mathfrak{H}))|_{E_N}$ is a non-zero ideal in the irreducible algebra $P_{E_N}\mathfrak{T}|_{E_N}$ implies that that $\mathfrak{J}$ is itself transitive in $\mathcal{B}(E_N)$ ([11, Theorem 7.4.6]), and thus $\mathfrak{T} \cap \mathcal{K}(\mathfrak{H})$ is not S.T.. □

We note that the above proof shows that if $\mathfrak{T}$ is a maximal triangular operator algebra and $0 \neq K \in \mathfrak{T} \cap \mathcal{K}(\mathfrak{H})$, then the (not necessarily closed) ideal of $\mathfrak{T}$ generated by K is S.T. if and only if the compression of K to each infinite-dimensional atom of Lat $\mathfrak{T}$ is zero.

3. Collections of finite-rank operators

In this section, we show that K.S.T. implies S.T. for collections of finite-rank operators.

We begin with a lemma about rank-one operators in triangular operator algebras whose diagonal is the most familiar maximal abelian von Neumann algebra.

Lemma 3.1. *Let $\mathfrak{M}$ be the maximal abelian von Neumann algebra consisting of multiplication operators by essentially bounded functions acting on the Hilbert space $\mathfrak{H} = L^2([0,1], dx)$. Then every operator of rank one in any triangular operator algebra containing $\mathfrak{M}$ is nilpotent.*

Proof. Each such rank-one operator can be written in the form

$$(h \otimes g^*)(f) := \langle f, g\rangle\, h \text{ for all } f \in L^2([0,1], dx),$$

where h and g are fixed elements of $L^2([0,1], dx)$. We show that $\langle h, g\rangle = 0$, so that $(h \otimes g^*)^2 = 0$. Since $\langle h, g\rangle = \int h\overline{g}dx$, it suffices to prove that $hg = 0$ a.e..

For each n, let M_n denote the operator of multiplication by the characteristic function χ_{E_n} of $E_n := \{t \in [0,1] : \max(|h(t)|, |g(t)|) \leq n\}$. (Of course, E_n is defined up to a set of measure zero.) It then suffices to prove that $(M_n h)(M_n g) = 0$ a.e. for all $n \geq 1$.

If there is an $N \geq 1$ for which that does not hold, let $h_0 := M_N h$ and $g_0 := M_N g$ for that N. Since $g_0, h_0 \in L^\infty([0,1], dm)$, we see that $M_{g_0}, M_{h_0} \in \mathfrak{M}$. Also, $h_0 \otimes g_0^* = M_N(h \otimes g^*)M_N$ lies in every algebra containing $\mathfrak{M}$ and $h \otimes g^*$.

Now

$$M_{g_0}(h_0 \otimes g_0^*)M_{h_0} = (g_0 h_0) \otimes (h_0 g_0)^*$$

is a multiple of a self-adjoint projection of rank at most one that is in any triangular operator algebra containing $h \otimes g^*$ and $\mathfrak{M}$, and must therefore be in $\mathfrak{M}$. Since there are no finite-rank operators other than 0 in $\mathfrak{M}$, this operator is 0. Thus $h_0 g_0 = 0$ a.e. □

Lemma 3.2. *If $\mathfrak{A}$ is a transitive algebra acting on a space of dimension greater than one, and if $\mathfrak{A}$ contains an operator of finite rank other than 0, then $\mathfrak{A}$ contains a non-nilpotent operator of rank 1.*

Proof. Since the finite-rank operators in $\mathfrak{A}$ form an ideal, there is an operator $F \in \mathfrak{A}$ of finite rank greater than or equal to 1 in $\mathfrak{A}$. Then

$$\{FT|_{F\mathfrak{H}} : T \in \mathfrak{A}\}$$

is a transitive algebra of operators acting on the finite-dimensional space $F\mathfrak{H}$, so – by Burnside's Theorem (see, e.g., [11, Theorem 1.2.2]), it consists of all operators acting on the space $F\mathfrak{H}$. In particular, it contains an operator FT of rank 1; then FTF is a rank 1 operator in $\mathfrak{A}$, so that $FTF = h \otimes g^*$ for some h and g. Since $\mathfrak{A}$ is transitive, we may choose an $A \in \mathfrak{A}$ such that Ah is not orthogonal to g. Then $A(h \otimes g^*) \in \mathfrak{A}$ is a rank-one operator that is not nilpotent. □

A result of Dong and Lu [1, Theorem 2.4] shows that if $\mathfrak{T}$ is a *norm-closed* irreducible triangular operator algebra acting on a separable Hilbert space $\mathfrak{H}$, then $\mathfrak{T}$ does not contain any finite-rank operators other than zero. Theorem 3.3 removes the condition that $\mathfrak{T}$ need be closed.

Theorem 3.3. *Let $\mathfrak{T}$ be an irreducible triangular operator algebra acting on an infinite-dimensional, separable Hilbert space $\mathfrak{H}$. Then $\mathfrak{T}$ does not contain any non-zero finite-rank operators.*

Proof. Let $\mathfrak{T}$ be an irreducible triangular operator algebra. Suppose first that the diagonal $\mathfrak{D}_{\mathfrak{T}} := \mathfrak{T} \cap \mathfrak{T}^*$ of $\mathfrak{T}$ contains at least two independent minimal projections, say P_1 and P_2. In this case, P_1 and P_2 have orthogonal ranges and each has rank equal to one, so that $P := P_1 + P_2$ has rank equal to 2. Then $P\mathfrak{T}|_{P\mathfrak{H}}$ is a transitive algebra acting on the 2-dimensional space $P\mathfrak{H}$. By Burnside's Theorem, $P\mathfrak{T}|_{P\mathfrak{H}} = \mathcal{B}(P\mathfrak{H}) \simeq \mathbb{M}_2(\mathbb{C})$. In particular, there are self-adjoint operators of the form PT_1P and PT_2P with $T_i \in \mathfrak{T}$ such that PT_1P does not commute with PT_2P, contradicting the fact that $\mathfrak{D}_{\mathfrak{T}} := \mathfrak{T} \cap \mathfrak{T}^*$ is abelian.

Thus we can assume that $\mathfrak{D}_{\mathfrak{T}}$ has at most one minimal projection, say R. Let $Q := I - R$. Then $Q\mathfrak{D}_{\mathfrak{T}}Q$ is a completely non-atomic masa on $Q\mathfrak{H}$.

A non-zero ideal of a transitive algebra is transitive ([12], Lemma 7.4.6), so the set of finite-rank operators in $\mathfrak{T}$ is transitive. Thus $Q\mathfrak{T}Q$ contains finite-rank operators other than 0.

Since the diagonal $\mathfrak{D}_{Q\mathfrak{T}Q}$ of $Q\mathfrak{T}Q$ is a completely non-atomic masa on a separable Hilbert space, it is unitarily equivalent to the algebra

$$\mathfrak{M} := \{M_\varphi : \varphi \in L^\infty([0,1], dx)\},$$

acting on $L^2([0,1], dx)$ [5, Theorem 9.4.1]. Thus, by [4], $Q\mathfrak{T}|_{Q\mathfrak{H}}$ is unitarily equivalent to a triangular operator algebra $\mathfrak{T}_0$ such that $\mathfrak{D}_{\mathfrak{T}_0} = \mathfrak{M}$ and $\mathfrak{T}_0$ contains a non-zero operator of finite rank.

However, no such $\mathfrak{T}_0$ exists, by Lemmas 3.1 and 3.2. This completes the proof. □

Corollary 3.4. *If $\mathfrak{L} \subseteq \mathcal{B}(\mathfrak{H})$ is a non-empty collection of finite-rank operators acting on an infinite-dimensional, separable Hilbert space $\mathfrak{H}$ and $\mathfrak{L}$ is* K.S.T.*, then $\mathfrak{L}$ is* S.T..

Proof. As we have seen, we may assume without loss of generality that $\mathfrak{L}$ is contained in a maximal triangular operator algebra $\mathfrak{T}$. If $F \in \mathfrak{L}$, then F is of finite rank and thus the compression of F to any infinite-dimensional atom of $\mathfrak{T}$ is again of finite rank. But the compression of $\mathfrak{T}$ to an infinite-dimensional atom is an irreducible triangular operator algebra, and so by Theorem 3.3, the compression of F to that atom is 0.

By Corollary 2.4, $\mathfrak{L} \subseteq \mathfrak{T} \cap \mathcal{K}(\mathfrak{H})$ is S.T.. □

Remark 3.5. The presence of finite-rank operators in a triangular algebra $\mathfrak{T}$ does not ensure that $\mathfrak{T}$ is S.T.. For example, one need only consider $\mathfrak{T} = \mathfrak{T}_0 \oplus \mathcal{T}_2(\mathbb{C})$, where $\mathfrak{T}_0$ is an irreducible triangular algebra and $\mathcal{T}_2(\mathbb{C})$ denotes the space of 2×2 upper-triangular complex matrices.

4. Collections of compact operators

The problem of determining whether K.S.T. implies S.T. for collections of compact operators is two-fold. First, there is the previously mentioned pathology concerning irreducible triangular operator algebras, namely: the fact that they need not be norm-closed. Because of this, at the moment the presence of a compact operator with a non-zero eigenvalue in an irreducible triangular operator algebra may not necessarily imply the presence of the corresponding Riesz projection in that algebra. Second, the compression of a norm-closed algebra to a semi-invariant subspace (i.e. a subspace of the form $N_1 \ominus N_2$, where $N_1 < N_2 \in \mathrm{Lat}\ \mathfrak{T}$) need not necessarily be closed. Because of this, in the proof of the next result, we may not *a priori* assume that our given triangular operator algebra is irreducible.

Theorem 4.1. *A norm-closed algebra* $\mathfrak{A}$ *of compact operators acting on a separable Hilbert space* $\mathfrak{H}$ *is* S.T. *if and only if it is* K.S.T..

Proof. The forward (only if) implication follows immediately from Theorem 1.3.

Let $\mathfrak{A}$ be a norm-closed algebra of compact operators, and suppose that $\mathfrak{T}$ is a triangular algebra such that $\mathfrak{A} \subseteq \mathfrak{T}$. Without loss of generality, we may assume that $\mathfrak{T}$ is maximal, in which case $\mathfrak{N} := \mathrm{Lat}\ \mathfrak{T}$ is a quasi-maximal nest by Theorem 1.2.

Suppose that $E_N = N \ominus N_-$ is an infinite-dimensional atom of $\mathfrak{N}$, and let P_{E_N} denote the orthogonal projection of $\mathfrak{H}$ onto E_N. Then the compression $\mathfrak{T}_{E_N} := P_{E_N}\mathfrak{T}|_{E_N}$ of $\mathfrak{T}$ to E_N is an irreducible triangular operator algebra. Suppose that $K \in \mathfrak{A}$ and that $K_{E_N} := P_{E_N}K|_{E_N} \neq 0$. Since $\mathfrak{T}_{E_N}$ is transitive, we may apply Lomonosov's Lemma [11, Lemma 7.3.1] to obtain an element B_0 in $\mathfrak{A}_{E_N} := P_{E_N}\mathfrak{A}|_{E_N}$ such that $1 \in \sigma(B_0 K_{E_N})$. If $B \in \mathfrak{A}$ is chosen such that $P_{E_N}B|_{E_N} = B_0$, then it is not hard to see that $1 \in \sigma(BK)$ as well.

But $BK \in \mathfrak{A}$ and $\mathfrak{A}$ is a norm-closed algebra, which implies that the (finite-rank) Riesz projection F_1 for BK corresponding to $\{1\}$ lies in $\mathfrak{A}$. Since $1 \in \sigma(B_0 K_{E_N})$, we see that the compression of F_1 to E_N is non-zero, which contradicts our earlier result showing that $\mathfrak{T}_{E_N}$ does not admit any finite-rank operators.

Thus the compression of each element of $\mathfrak{A}$ to E_N must be zero. By Corollary 2.4, $\mathfrak{A}$ is S.T.. □

Corollary 4.2. *If* $\mathfrak{T}$ *is a norm-closed irreducible triangular operator algebra acting on a separable Hilbert space* $\mathfrak{H}$*, then*

$$\mathfrak{T} \cap \mathcal{K}(\mathfrak{H}) = \{0\}.$$

We finish with the main question we have been unable to resolve.

Question. If $\mathfrak{H}$ is infinite-dimensional and separable, and $\mathfrak{T} \subseteq \mathcal{B}(\mathfrak{H})$ is triangular, irreducible, but not norm-closed, can $\mathfrak{T}$ contain a non-zero compact operator?

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