Algebraic Methods for Reducibility in Nowhere-Zero Flows

by

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I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Abstract

We study reducibility for nowhere-zero flows. A reducibility proof typically consists of showing that some induced subgraphs cannot appear in a minimum counter-example to some conjecture. We derive algebraic proofs of reducibility.

We define variables which in some sense count the number of nowhere-zero flows of certain type in a graph and then deduce equalities and inequalities that must hold for all graphs. We then show how to use these algebraic expressions to prove reducibility. In our case, these inequalities and equalities are linear. We can thus use the well developed theory of linear programming to obtain certificates of these proof.

We make publicly available computer programs we wrote to generate the algebraic expressions and obtain the certificates.

Acknowledgements

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Contents

List of Figures

Chapter 1

Introduction

All graphs in this thesis are multigraphs unless stated otherwise.

1.1 Nowhere-zero flows

Definition 1.1.1. An *orientation* of a graph G is a directed graph (digraph) D such that the underlying undirected graph of D is G.

We use the notation $\vec{G}$ to mean some orientation of G.

Definition 1.1.2. Let $\vec{G}$ be a digraph and Γ an Abelian group. A Γ-*flow on* $\vec{G}$ is a function $f : E(G) \to \Gamma$ such that

$$\forall v \in V(G) \qquad \sum_{e \in \delta^+_{\vec{G}}(v)} f(e) = \sum_{e \in \delta^-_{\vec{G}}(v)} f(e) \tag{1.1}$$

Here we use the notation $+$ for the operation of the Abelian group Γ, $\delta^+_{\vec{G}}(v)$ for the set of edges with v as its tail and $\delta^-_{\vec{G}}(v)$ for the set of edges with v as its head.

When clear from the context, we will write $\delta^+(v)$ for $\delta^+_{\vec{G}}(v)$ and $\delta^-(v)$ for $\delta^-_{\vec{G}}(v)$.

From now on, we will use the notation $f(S)$ for a subset S of the edges of G to mean $\sum_{e\in S} f(e)$. We write $\delta(S)$ for the set of edges between S and $V - S$. We write $\delta^+(S)$ to mean the arcs from S to $V - S$ and $\delta^-(S)$ to mean the arcs from $V - S$ to S.

Definition 1.1.3. A Γ-flow on $\vec{G}$ is a *nowhere-zero* Γ*-flow* if

$$\forall e \in E(G) \quad f(e) \neq 0$$

We are interested in knowing when a digraph $\vec{G}$ has a nowhere-zero Γ-flow. First we prove the following theorem which states that the existence of a nowhere-zero flow does not depend on the orientation.

Proposition 1.1.4. *Let G be an undirected graph and Γ an Abelian group. Let D_1 and D_2 be two orientations of G. Suppose there is a nowhere-zero Γ-flow f on D_1. Then there is a nowhere-zero Γ-flow f' on D_2.*

Proof. In D_2, some of the edges of G in D_1 are reversed. We simply reverse the flow on the reversed edges and keep the flow the same otherwise. Let

$$\begin{aligned} f'(e) &= f(e) && \text{if } D_1 \text{ and } D_2 \text{ orient } e \text{ the same way} \\ f'(e) &= -f(e) && \text{if } D_1 \text{ and } D_2 \text{ orient } e \text{ differently} \end{aligned}$$

Then f' is clearly nowhere-zero as $-0 = 0$ for all groups. Also, f' is a flow since whenever a term changes sides in Equation (1.1), we change the sign of that term. $\square$

This justifies our notation $\vec{G}$ since in most cases we will only need an arbitrary but consistent orientation of G.

Also note that the "flow condition" of Equation (1.1) must hold for every edge cut of G. This can be deduced from Equation (1.1) by summing over all vertices on one side of the edge cut. Thus we obtain the following statement.

Proposition 1.1.5. *If f is a Γ-flow on $\vec{G}$ and $\delta(S)$ is an edge cut of G then*

$$f(\delta^+(S)) = f(\delta^-(S)) \tag{1.2}$$

This proposition says that the flow into S must be equal to the flow out of S. We will refer to Equation (1.2) as the *cut-condition*.

Therefore, if G has an edge cut consisting of a single edge e (called a *bridge*), Equation (1.2) becomes $f(e) = 0$. Then $\vec{G}$ cannot have a nowhere-zero Γ-flow (for any Γ). Therefore, from now on, we will only study graphs without an edge cut (called *bridgeless graphs*).

An important result by Tutte in the theory of nowhere-zero flows states that the existence of a nowhere-zero Γ-flow only depends on the order of Γ rather than the group itself.

Theorem 1.1.6. *[18] Let Γ_1 and Γ_2 be two Abelian groups with $|\Gamma_1| = |\Gamma_2|$ then $\vec{G}$ has nowhere-zero Γ_1-flow if and only if $\vec{G}$ has a nowhere-zero Γ_2-flow*

We will not prove this theorem but note that the proof does not give a bijection between Γ_1-flows and Γ_2-flows of G.

We can define a notion similar to nowhere-zero flows for undirected graphs.

Definition 1.1.7. An undirected graph G is said to be *k-flowable* if there exists a group Γ of order k and an orientation $\vec{G}$ of G such that $\vec{G}$ has a nowhere-zero Γ-flow.

In light of Proposition 1.1.4 and Theorem 1.1.6, we may replace "there exists" in the above definition by "for all" (i.e., if such a nowhere-zero flow exists then it exists for all groups of order k and all orientations of G). Thus, k-flowability is a property of the undirected graph G.

1.2 Nowhere-zero flow conjectures

We can now state the most important conjectures in the area of nowhere-zero flows, Tutte's nowhere-zero flow conjectures.

Conjecture 1.2.1 (5-flow conjecture)**.** *[18] Every bridgeless graph is 5-flowable.*

Conjecture 1.2.2 (4-flow conjecture)**.** *[20] Every bridgeless graph without a minor isomorphic to the Petersen graph is 4-flowable.*

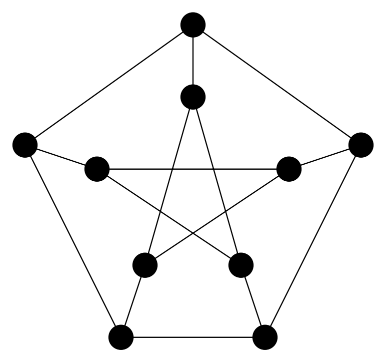


Figure 1.1: The Petersen graph

Conjecture 1.2.3 (3-flow conjecture)**.** *[17] Every 4-edge-connected graph is 3-flowable.*

Theorems have been proven by weakening the conditions in some of the conjectures.

Theorem 1.2.4 (6 flow theorem)**.** *[16] Every bridgeless graph is 6-flowable.*

Theorem 1.2.5. *[5, 6] Every 4-edge connected graph is 4-flowable.*

1.3 Nowhere-zero flow and colouring duality

As the following theorem shows, nowhere-zero flows can be viewed as an extension of colouring for non-planar graphs.

Theorem 1.3.1. *[18] Let G be a plane graph. If G is k-flowable then G^*, the dual of G, is k-vertex-colourable.*

Proof. Choose an arbitrary orientation $\vec{G}$ of G and let f be a nowhere-zero $\mathbb{Z}_k$-flow for that orientation.

We construct the dual $\vec{G^*}$ of $\vec{G}$ by rotating each (oriented) edge of $\vec{G}$ clockwise until it is aligned with the (currently) unoriented dual of the edge in G^*.

We may now think of f as a function assigning values to edges of $\vec{G^*}$. f is not a flow on $\vec{G^*}$. However, the dual of an edge cut is a cycle. Therefore, by Equation (1.2), if we walk around an undirected cycle in G^*, the sum of the f values on forward edges (edges oriented in the same direction as our walk) is equal to the sum of the f values on backward edges (edges oriented in the opposite direction). So for any (undirected) cycle C in G^*, f (on $\vec{G^*}$) satisfies

$$\sum_{e \in C, e \text{ is a forward edge}} f(e) = \sum_{e \in C, e \text{ is a backward edge}} f(e) \tag{1.3}$$

We construct a colouring c of the vertices of G^* by colouring them with elements of $\mathbb{Z}_k$.

Start by colouring an arbitrary initial vertex v by letting $c(v) = 0$. We now repeat the following step.

Suppose u is uncoloured and w is coloured. If there is an edge from u to w, set $c(u) = c(w) + f(uw)$. If there is an edge from w to u, set $c(u) = c(w) - f(wu)$.

We need to show that two adjacent vertices of G^* are coloured differently.

To do so, we show that if there is an edge from u to w then $c(u) = c(w) + f(uw)$. Since $f(uw) \neq 0$ for all arcs uw, adjacent vertices receive different colours.

Suppose $c(u) \neq c(w) + f(uw)$ for some adjacent vertices u and w in G^*. Note that the procedure for assigning colours chooses an edge at every step (with one coloured end and one uncoloured end). Furthermore, the chosen edges form a (spanning) tree T in G^* (since all vertices are coloured and we never choose an edge with two coloured ends).

Note that all edges in T satisfy $c(u) = c(w) + f(uw)$.

Let $P = \{u, w_1, ..., w_k, w\}$ be the (unique) path from u to w in T. This path is not just $\{u, w\}$ (otherwise $c(u) = c(w) + f(uw)$). This path together with uw forms a cycle. We can traverse this cycle by starting at u moving along the path and then traverse the edge uw backwards. By (1.3), this sum is 0. Since all edges of the path except uw is a tree edge, the sum is also $(c(u) - c(w_1)) + (c(w_1) - c(w_2)) + (c(w_2) - c(w_3)) + \ldots + (c(w_k) - c(w)) - f(uw)$. This telescopes to $(c(u) - c(v)) - f(uw) = 0$ so $c(u) = c(v) + f(uw)$. Contradiction.

□

As a corollary of the previous theorem, we see that the 4-flow conjecture is a strengthening of the Four Colour Theorem.

Corollary 1.3.2. *Suppose the 4-flow conjecture is true. Then every planar graph is 4-colourable.*

Proof. We show that an arbitrary plane graph G is 4-colourable. The dual G^* is planar and thus does not contain a Petersen minor (as the Petersen graph is non-planar). So G^* is 4-flowable. By Theorem 1.3.1, G is 4-colourable. □

1.4 Reducibility

In this section, we will describe what we mean by a "reducibility proof". The remainder of this thesis studies different types of reducibility proofs.

We essentially want to build a "smaller" counter-example (to some Nowhere-Zero Flow Conjecture) from another counter-example. However, we want this construction to be, in some sense, "local" so that we only need to look at and change a small part of the counter-example.

We start by showing how we can "break" a flow and a digraph into two "sides" and how we can form a flow from a flow on each of the "sides". We then proceed to "replace" one of these "sides" (with a smaller piece).

From now on, we also omit Γ and simply say "flow" to mean a Γ-flow when clear from the context. This is since we think of Γ to be some fixed Abelian group.

1.4.1 Combining flows

We start by introducing the notion of "partial flows".

Definition 1.4.1. Let $\vec{H}$ be a digraph, and E' a subset of the edges of H. A *partial* Γ*-flow on* E' is a function $f : E' \to \Gamma$ such that for all edge cuts $\delta(S) \subseteq E'$,

$$f(\delta^+_{\vec{G}}(S)) = f(\delta^-_{\vec{G}}(S)).$$

A *partial nowhere-zero* Γ*-flow on* E' is a *partial* Γ*-flow on* E' with $f(e) \neq 0\ \forall e \in E'$.

A partial flow on E' simply assigns group values to a subset E' of the edges of H and requires that the cut-condition (Proposition 1.1.5) holds for all cuts contained in E'. Note that a partial flow on $E(H)$ is just a flow (of $\vec{H}$).

We use the notation $f|_{E'}$ to denote the restriction of the function f to the domain E' (i.e., $f|_{E'} = f$ on E' and $f|_{E'}$ is undefined elsewhere).

By Proposition 1.1.5, if f is a flow then, for any subset E' of the edges, $f|_{E'}$ is a partial flow on E'. More generally, if $E'' \subseteq E'$ and f is a partial flow on E' then $f|_{E''}$ is a partial flow on E''.

Now we want to know when we can perform the reverse operation. That is, start with a partial flow on E'' and obtain a partial flow on $E' \supseteq E''$ while keeping the function the same on E''.

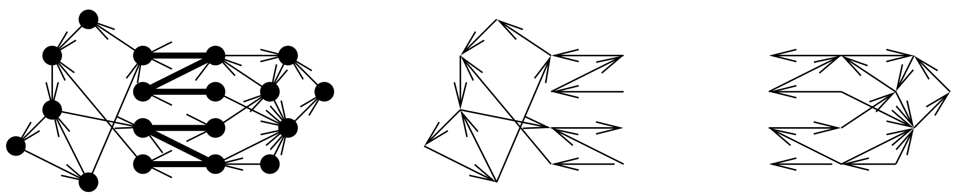
Definition 1.4.2. 1. If $E'' \subseteq E'$ then a partial flow $g : E' \to \Gamma$ on E' *extends* $f : E'' \to \Gamma$ if $g|_{E''} = f$.

2. A partial flow f on E'' *extends to* E' if there is a partial flow on E' which extends f.

The same terms for nowhere-zero flows are defined analogously.

Definition 1.4.3. Suppose $\vec{H}$ is a digraph and $R = \delta^+(S) \cup \delta^-(S)$ for some $S \subseteq V(H)$ (i.e., R is all arcs in the undirected cut $\delta(S)$). Then R splits $\vec{H}$ into the two *sides* R_1 and R_2 of the cut R:

1. The side R_1 consists of the arcs with both ends in S together with R; and
2. The side R_2 consists of the arcs with both ends in $V(H) - S$ together with R.



(a) (b) (c)

Figure 1.2: Sides of an edge cut: The original graph $\vec{H}$ in (a) with the edge cut highlighted. The edge sets in (b) and (c) are the two sides of the edge cut

With these definitions in hand, we can now state the main proposition of this section.

Proposition 1.4.4. *Let $\vec{H}$ be a digraph and R an edge cut in $\vec{H}$ with sides R_1 and R_2. Then $\vec{H}$ has a nowhere-zero flow if and only if there is a partial nowhere-zero flow on R which extends to both R_1 and R_2.*

Proof. Suppose $\vec{H}$ has a nowhere-zero flow f. Then $f|_R$ is a partial nowhere-zero flow on R. Obviously, $f|_{R_1}$ extends $f|_R$ and is a nowhere-zero flow on R_1 and $f|_{R_2}$ extends $f|_R$ and is a nowhere-zero flow on R_2. So $f|_R$ is a partial nowhere-zero flow on R which extends to both sides.

Now suppose there is a partial nowhere-zero flow f on R which extends to both sides. Let f_1 be the partial nowhere-zero flow on R_1 which extends f and f_2 be the partial nowhere-zero flow on R_2 which extends f. Since f_1 and f_2 agree on R, we can define a function g which is f_1 on R_1 and f_2 on R_2 (i.e., $g(e) = f_1(e)$ if $e \in R_1$ and $g(e) = f_2(e)$ if $e \in R_2$). The domain of g is $R_1 \cup R_2 = E(H)$ and satisfies the cut-condition for any cut which is the neighbourhood of a vertex. Therefore g is a nowhere-zero flow on $\vec{H}$. $\square$

In later sections, we will use the above proposition in the following form.

Corollary 1.4.5. *Let $\vec{H}$ be a digraph and R an edge cut in $\vec{H}$ with sides R_1 and R_2. Then $\vec{H}$ has no nowhere-zero flow if and only if every partial nowhere-zero flow on R which extends to R_1 does not extend to R_2.*

1.4.2 Replacement

In this section, we describe the operation of "replacement". We would like to be able to replace one side of a cut R in some digraph $\vec{H}$ to obtain a new digraph.

The idea is simple. If we had drawn $\vec{H}$ with R in the middle then we replace one side by simply erasing everything (arcs and vertices) on one side but leaving the "half-arcs" of R and draw something else on that side. We now define this formally. We start by introducing the notion of a "side graph".

Definition 1.4.6. A *side graph* $(\vec{G}, z, R)$ is a digraph $\vec{G}$ with a *marked vertex* $z \in V(G)$ and $\delta(z)$ is labelled by the set R.

We can obtain side graphs from a digraph and a cut R in the following way.

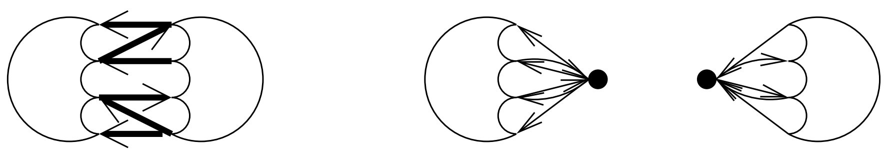
Definition 1.4.7. Let $\vec{H}$ be a digraph and $R = \delta^+(S) \cup \delta^-(S)$ be a cut in $\vec{H}$.

Let $\vec{H}_1$ be the graph obtained from $\vec{H}$ by identifying all vertices in S to a marked vertex z_1 and deleting all loops created by the identification. The edges of $\delta(z_1)$ are labelled by R using the identity map.

Let $\vec{H}_2$ be the graph obtained from $\vec{H}$ by identifying all vertices in $V(G) - S$ to a marked vertex z_2 and deleting all loops created by the identification. The edges of $\delta(z_2)$ are labelled by R using the identity map.

Then $(\vec{H}_1, z_1, R)$ and $(\vec{H}_2, z_2, R)$ are the *side graphs induced by R (in $\vec{G}$).*

We will also refer to $(\vec{H}_1, z_1, R)$ and $(\vec{H}_2, z_2, R)$ as the side graphs of R.



(a) (b) (c)

Figure 1.3: Side graphs of an edge cut: The original graph $\vec{H}$ in (a) with the edge cut highlighted. The graphs in (b) and (c) are the two side graphs of the edge cut

Note that a side of R is a set of edges whereas a side graph of R is a digraph. The edges of the side graphs of R correspond to the sides of R. Furthermore, the two notions relate to each other in the following manner.

Remark 1.4.8. *Let $\vec{H}$ be a digraph and R a cut of H. Let $\vec{H}_1$, $\vec{H}_2$ be the side graphs of R.*

Then f is a flow of $\vec{H}_1$ if and only if f is a partial flow of $\vec{H}$ on $E(H_1)$.

The same statement can be made about partial nowhere-zero flows.

The above remark is simply a restatement of Proposition 1.1.5 for partial flows.

We now want to reverse operation of splitting a digraph into two side graphs. We call this "gluing".

Definition 1.4.9. Let $(\vec{H}_1, z_1, R)$ and $(\vec{H}_2, z_2, R)$ be 2 side graphs. Suppose that the arcs incident to z_1 and the arcs incident to z_2 have opposing orientations.

Then we define $\vec{H}$, the *graph obtained by gluing* $\vec{H}_1$ and $\vec{H}_2$ as follows. $\vec{H}$ is the disjoint union of $\vec{H}_1 - \{z_1\}$ and $\vec{H}_2 - \{z_2\}$ together with an arc v_1v_2 for every arc v_1z_1 in $\vec{H}_1$ and arc z_2v_2 in $\vec{H}_1$ labelled by the same element in R.

We can now define replacement.

Definition 1.4.10. Let $\vec{H}$ be a digraph and R a cut in H which induces side graphs $\vec{H}_1$ and $\vec{H}_2$. Then the *digraph obtained by replacing* $\vec{H}_1$ *by* $\vec{H}_1'$ is the digraph obtained by gluing $\vec{H}_1'$ and $\vec{H}_2$.

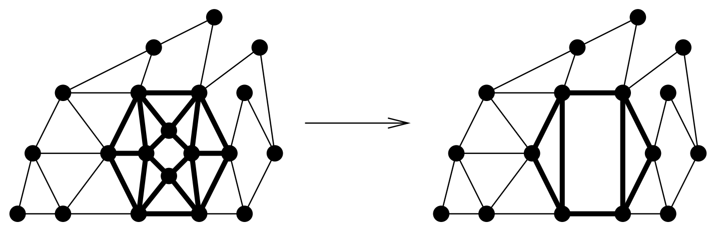


Figure 1.4: Replacement: The highlighted graph on the left is replaced with a graph with 4 fewer vertices to obtain the graph on the right

From now on, to simply notation, we will refer to $\vec{G}$ as a side graph and use $z(\vec{G})$ and $R(\vec{G})$ for the marked vertex and the labelling of edges incident to $z(\vec{G})$ respectively.

1.4.3 Minimum counter-example

In an effort to prove some of the Nowhere-Zero Flow Conjectures, properties of the minimum counter-examples were studied [3, 10, 11, 9]. For example

Theorem 1.4.11. *[3] A minimum counter-example to the 5-flow conjecture is 3-regular.*

Theorem 1.4.12. *A minimum counter-example to the 3-flow conjecture is 5-regular.*

In all these cases, by "minimum", we mean that the conjecture is true for all graphs with fewer vertices. We will also use the term *smaller graph* to mean a graph with fewer vertices. More formally, we have the following definition.

Definition 1.4.13. For a class of graphs $\mathcal{H}$ and an integer k, a *minimum* $(\mathcal{H}, k)$-*counter-example* is a graph in $\mathcal{H}$ that is not k-flowable and has the least number of vertices.

This thesis focuses on proving that some digraphs cannot appear as induced subgraphs of the minimum counter-example. This is called *reducibility* and the corresponding graphs will be deemed reducible.

Informally, a graph G is "reducible" if G can be replaced in any counter-example with a smaller graph G' and the resulting graph is still a counter-example.

Definition 1.4.14. Let $\mathcal{H}$ be a class of graphs and $\vec{\mathcal{H}}$ the orientations of graphs in $\mathcal{H}$. A graph G $(\mathcal{H}, k)$-reduces to G' if whenever $\vec{G}$ appears as a side graph of a cut R in some graph $\vec{H} \in \vec{\mathcal{H}}$, the graph $\vec{H}'$ obtained from $\vec{H}$ by replacing $\vec{G}$ with $\vec{G}'$ is a non-k-flowable graph in $\vec{\mathcal{H}}$.

We say G is $(\mathcal{H}, k)$-*reducible* if it is $(\mathcal{H}, k)$-reducible to some graph G' smaller than G.

Note that replacing $\vec{G}$ by $\vec{G}'$ can be done by replacing G by G' in H and keeping the orientation of $\vec{G}$ and $\vec{G}'$.

Now suppose that we wish to find a minimum counter-example (a smallest graph in $\mathcal{H}$ which is not k-flowable). If an orientation of such a counter-example contained $\vec{G}$ as a side graph of some cut where G is reducible then we could replace G with G' and get a smaller counter-example, which is impossible. Thus, we have the following.

Remark 1.4.15. *If G is $(\mathcal{H}, k)$-reducible then it does not appear as a subgraph in a minimum $(\mathcal{H}, k)$-counter-example.*

We will usually omit the parameters $(\mathcal{H}, k)$ when clear from the context (or the statement holds for all fixed $(\mathcal{H}, k)$).

1.4.4 Flow reducibility

In the previous section, we defined reducibility for graphs. We now wish to define reducibility for a set of flows. We will choose this definition so that if the set of flows extending to a side graph $\vec{H}$ reduces all flows extending to $\vec{H'}$ then $\vec{H}$ reduces to $\vec{H'}$.

Definition 1.4.16. Let $\mathcal{H}$ be a class of digraphs. Let $\mathcal{S}_R$ be the set of all side graphs (G, z, R) for some graph in $\mathcal{H}$ (the edges incident to z are labelled by the same set R for all graphs).

We say that $\mathcal{S}_R$ is the *set of side graphs* (of $\mathcal{H}$ containing R). An element of $\mathcal{S}_R$ is referred to as a *configuration*.

Definition 1.4.17. Let $\mathcal{S}_R$ be the side graphs of $\mathcal{H}$ containing R.

Let f be a partial flow on R. Let $\mathcal{C}$ be a set of partial flows on R.

We say that $\mathcal{C}$ *reduces* f if, for every $\vec{G} \in \mathcal{S}_R$ such that f extends to $\vec{G}$, there exists some partial flow in $\mathcal{C}$ that extends to $\vec{G}$.

In general, we will refer to the concept of having a set of flows reduce another flow as *flow reducibility.* We will refer to the reducibility of digraphs as *graph reducibility.* The following proposition shows that our definition relates flow reducibility and graph reducibility in the manner we wanted.

Proposition 1.4.18. *Let $\mathcal{S}_R$ be the side graphs of $\mathcal{H}$ containing R.*

Let $\vec{G}, \vec{G'} \in \mathcal{S}_R$. Let $\mathcal{C}$ be the set of partial flows extending to $\vec{G}$ and $\mathcal{C}'$ the set of partial flows extending to $\vec{G'}$.

If $\forall f \in \mathcal{C}'$, $\mathcal{C}$ reduces f then $\vec{G}$ reduces to $\vec{G'}$.

Proof. Suppose $\vec{G}$ appears as a side graph of R in some minimum counter-example $\vec{H} \in \mathcal{H}$. Assume that $\forall f \in \mathcal{C}'$, $\mathcal{C}$ reduces f. Let $\vec{F}$ be the other side graph of R in $\vec{H}$. Let $\vec{H'}$ be obtained by replacing $\vec{G}$ by $\vec{G'}$.

If H' is k-flowable then let f be a nowhere-zero flow on $\vec{H}'$. Then $f|_{G'}$ is a partial flow on $\vec{G}'$. Thus $f|_R \in \mathcal{C}'$ and $\mathcal{C}$ reduces $f|_R$. $f|_F$ is a flow on $\vec{F}$ so $f|_R$ extends to $\vec{F}$. By Definition 1.4.17, there exists $f' \in \mathcal{C}$ which extends to $\vec{F}$. But by definition of $\mathcal{C}$, f' extends to $\vec{G}$. Thus, by Proposition 1.4.4, f' extends to $\vec{H}$. Contradiction.

Therefore, H' is not k-flowable and $\vec{G}$ reduces to $\vec{G}'$ as required.

$\square$

1.5 Algebraic methods

In this section, we illustrate the technique in the previous section which shows that a graph is reducible. We will use an algebraic approach to prove reducibility.

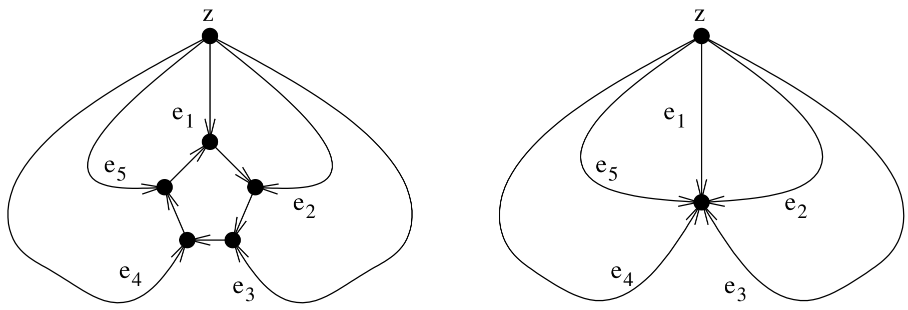
Definition 1.5.1. Fix a class $\mathcal{H}$ of graphs for which we wish to prove graphs in $\mathcal{H}$ are k-flowable. For a side graph $\vec{G}$ and partial nowhere-zero flow f on the set of edges incident to $z(\vec{G})$, we let $K_{\vec{G},f}$ denote the number of nowhere-zero flows of $\vec{G}$ that extend f.

In this thesis, we will deduce equalities and inequalities in terms of some variables we define that must hold for all digraphs and then use them for flow reducibility.

To give an idea of the methods that will be used, we now give two examples. In the first one, we use an equality for flow reducibility. In the second, we use an inequality for flow reducibility. We will not show how these equalities and inequalities are obtained as even a simple example would require quite a bit more theory.

1.5.1 Equalities examples

In this section, we show that the graph G obtained from the 5-cycle by adding a marked vertex (Fig 1.5(a)) is reducible to the graph $\vec{G}'$ obtained from the 5-star by adding a marked vertex (Fig 1.5(b)) (for bridgeless graphs and 5-flowability).



(a) (b)

Figure 1.5: We show that the side graph in (a) is reducible to the side graph in (b). We have drawn an arbitrary orientation of the edges of both graphs (with the same orientation on the neighbour of z). The edges incident with the marked vertex z are labelled $e_1, \ldots, e_5$

Let $\mathcal{H}$ be the class of bridgeless graphs and $k = 5$. There is only one Abelian group of order 5, $\mathbb{Z}_5$. Let $\mathcal{S}_R$ be the side graphs of $\mathcal{H}$ (for R). Let $\vec{G}$ be the 5-cycle (side graph in Fig 1.5(a)). Note that given a (nowhere-zero) flow f on $\vec{G}$, we can obtain another (nowhere-zero) flow g by letting $g(e) = af(e)$ for some $a \in \mathbb{Z}_5^* = \{1, 2, 3, 4\}$.

Let $R = \{e_1, e_2, e_3, e_4, e_5\}$ be the edges incident to the marked vertex in $\vec{G}$. We write a partial flow on R as a vector $\mathbf{a} = (a_1, a_2, a_3, a_4, a_5)$.

Note that if $(a_1, a_2, a_3, a_4, a_5)$ extends to $\vec{G}$ then so does $(a_5, a_1, a_2, a_3, a_4)$ and all cyclic permutations of the entries of the vector $\mathbf{a}$.

Let $\mathcal{C}$ be the set of partial nowhere-zero flows on R which extend to $\vec{G}$. Let $\mathcal{C}'$ be the set of partial nowhere-zero flows on R which extend to the 5-star.

Note that $\mathcal{C}'$ is the set of all partial nowhere-zero flows on R.

We claim that, up to multiplication by an element of $\mathbb{Z}_5^*$ and cyclic permutation of the entries, the flows that do not extend to $\vec{G}$ are those shown in Fig 1.6.

If e_i has flow a_i and the other edge with the same head as e_1 has flow x, then,

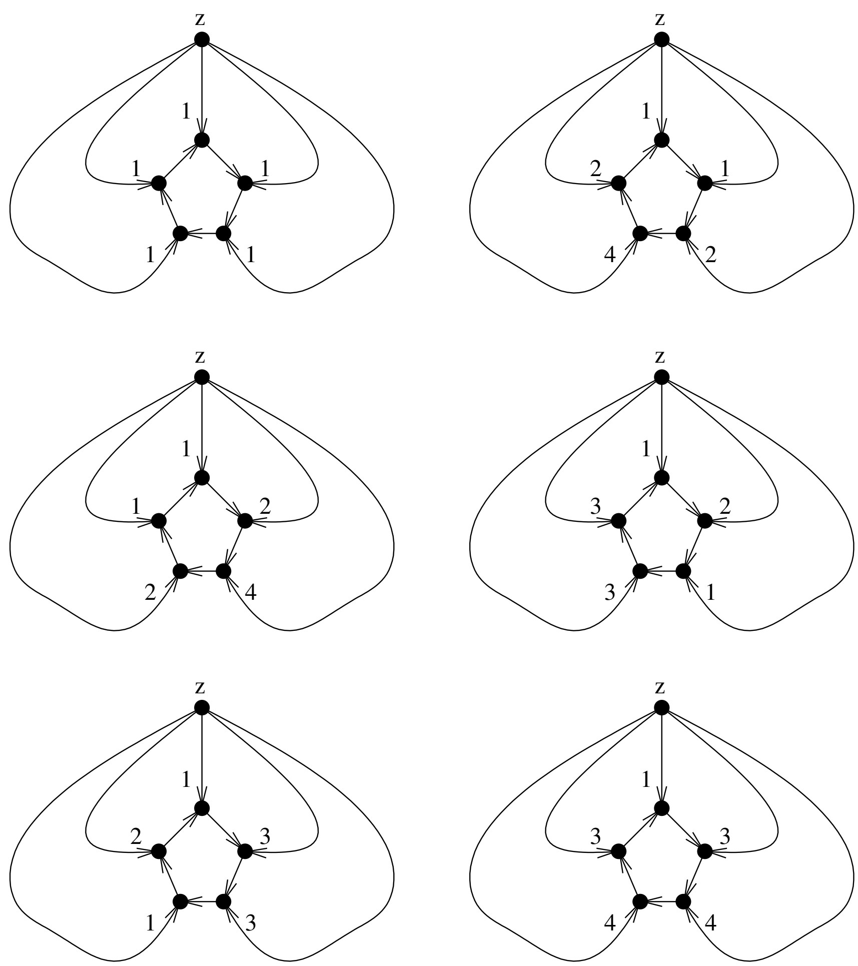


Figure 1.6: Flows that do not extend to the oriented 5-cycle with an added marked vertex $\vec{G}$ (i.e., the complement of $\mathcal{C}$ or $\mathcal{C}' - \mathcal{C}$)

cyclically around the C_5, the flows are x, $x + a_1$, $x + a_1 + a_2$, $x + a_1 + a_2 + a_3$, and $x + a_1 + a_2 + a_3 + a_4$ which we require to be non-zero. We require (of course) $a_1 + a_2 + a_3 + a_4 + a_5 = 0$. From this, the reader can check that, up to multiplication by an element of $\mathbb{Z}_5$, the only flows that do not extend to $\vec{G}$ (which are the flows that do not have a x satisfying all requirements) are those in Figure 1.6. A general discussion along these lines is given in Section 5.1.

Suppose the following equation is true for all side graphs $\vec{F} \in \mathcal{S}_R$ (we omit the index $\vec{F}$ everywhere).

$$\begin{aligned}
&4K_{(1,3,1,3,2)} + 4K_{(1,3,3,4,4)} + 4K_{(1,2,4,1,2)} + 2K_{(1,3,2,1,3)} + 2K_{(1,2,3,3,1)} \\
&\qquad - 2K_{(1,1,3,1,4)} - 2K_{(1,2,2,4,1)} - 4K_{(1,3,2,3,1)} - 2K_{(1,3,4,3,4)} \\
&= K_{(1,1,1,1,1)} + K_{(1,1,2,4,2)} + K_{(1,2,4,2,1)} + K_{(1,2,1,3,3)} + K_{(1,3,3,1,2)} + K_{(1,3,4,4,3)} \quad (1.4)
\end{aligned}$$

Suppose that $\vec{G}$ is a side graph of R in $\vec{H}$ for some $H \in \mathcal{H}$ and H is not 5-flowable. Let $\vec{F}$ be the other side graph of R in $\vec{H}$. By Corollary 1.4.5, $\vec{F}$ does not extend any flow except possibly those in $\mathcal{C}' - \mathcal{C}$. However, all variables in the left-hand side in Equation (1.4) are indexed by elements of $\mathcal{C}$. Therefore, for $\vec{F}$, the left-hand side is 0. Therefore the right-hand side is also 0. Since $K_{\vec{F},f}$ is non-negative for any $\vec{F}$ and f and all coefficients on the right-hand side are non-negative, each term on the right-hand side is non-negative. Therefore all terms on the right-hand side are equal to 0.

So, by definition of $K_{\vec{F},f}$, $\vec{F}$ does not extend any flow indexed by a variable on the right-hand side. But all flows in $\mathcal{C}' - \mathcal{C}$ appear as an index on the right-hand side. Therefore, we have deduced that $\vec{F}$ does not extend any flow in $\mathcal{C}' - \mathcal{C}$.

So $\mathcal{C}$ reduces every flow in $\mathcal{C}' - \mathcal{C}$ (by satisfying the contrapositive of Definition 1.4.17). $\mathcal{C}$ trivially reduces every flow in $\mathcal{C}$ by definition.

Thus, if H is not 5-flowable, $\vec{F}$ does not extend any nowhere-zero flow on R. Replacing $\vec{G}$ by the 5-star with a marked vertex does not create any new cuts so the resulting graph is bridgeless. Therefore, $\vec{G}$ is reducible to the 5-star.

1.5.2 Inequalities example

In this section, we show that the graph G obtained by adding a marked vertex to the dual of the Birkhoff diamond (Fig 1.7(a)) is reducible to the graph G' obtained by adding a marked vertex to the 6-star (Fig 1.7(b)) for bridgeless planar graphs and 4-flowability.

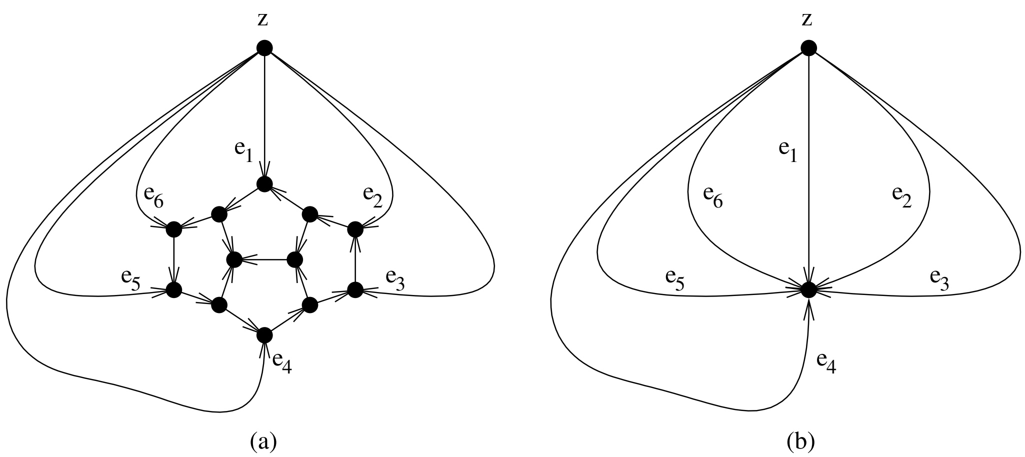


Figure 1.7: We show that the dual of the Birkhoff diamond (with an added marked vertex) in (a) is reducible to the 6-star (with an added marked vertex) in (b). We have drawn an arbitrary orientation of the edges of both graphs (with the same orientation on the neighbour of z). The neighbours of the marked vertex z are labelled $e_1, \ldots, e_6$.

Let $\vec{G}$ an orientation of G. Let $R = \{e_1, \ldots, e_6\}$ be the set of edges incident to the marked vertex.

Let $\mathcal{H}$ be the class of planar bridgeless graphs and $k = 4$. By Theorem 1.1.6, we may choose any Abelian group of order 4 so we choose it to be $\mathbb{Z}_2^2$. Let the 3 non-zero elements of $\mathbb{Z}_2^2$ be a,b and c.

Let $\mathcal{S}_R$ be the side graphs of $\mathcal{H}$ (for R). We write a partial flow on R as a vector $\mathbf{a} = (a_1, a_2, a_3, a_4, a_5, a_6)$.

Let $\mathcal{C}$ be the set of partial nowhere-zero flows on R which extend to $\vec{G}$. Let $\mathcal{C}'$ be the set of partial nowhere-zero flows on R which extend to 6-star.

Note that $\mathcal{C}'$ is the set of all partial nowhere-zero flows on R.

Note that we may permute the non-zero elements a, b, c to obtain a nowhere-zero flow from an existing nowhere-zero flow. The reader may check that up to this permutation, the only nowhere-zero flows on R which do not extend to $\vec{G}$ are those shown in Fig 1.8.

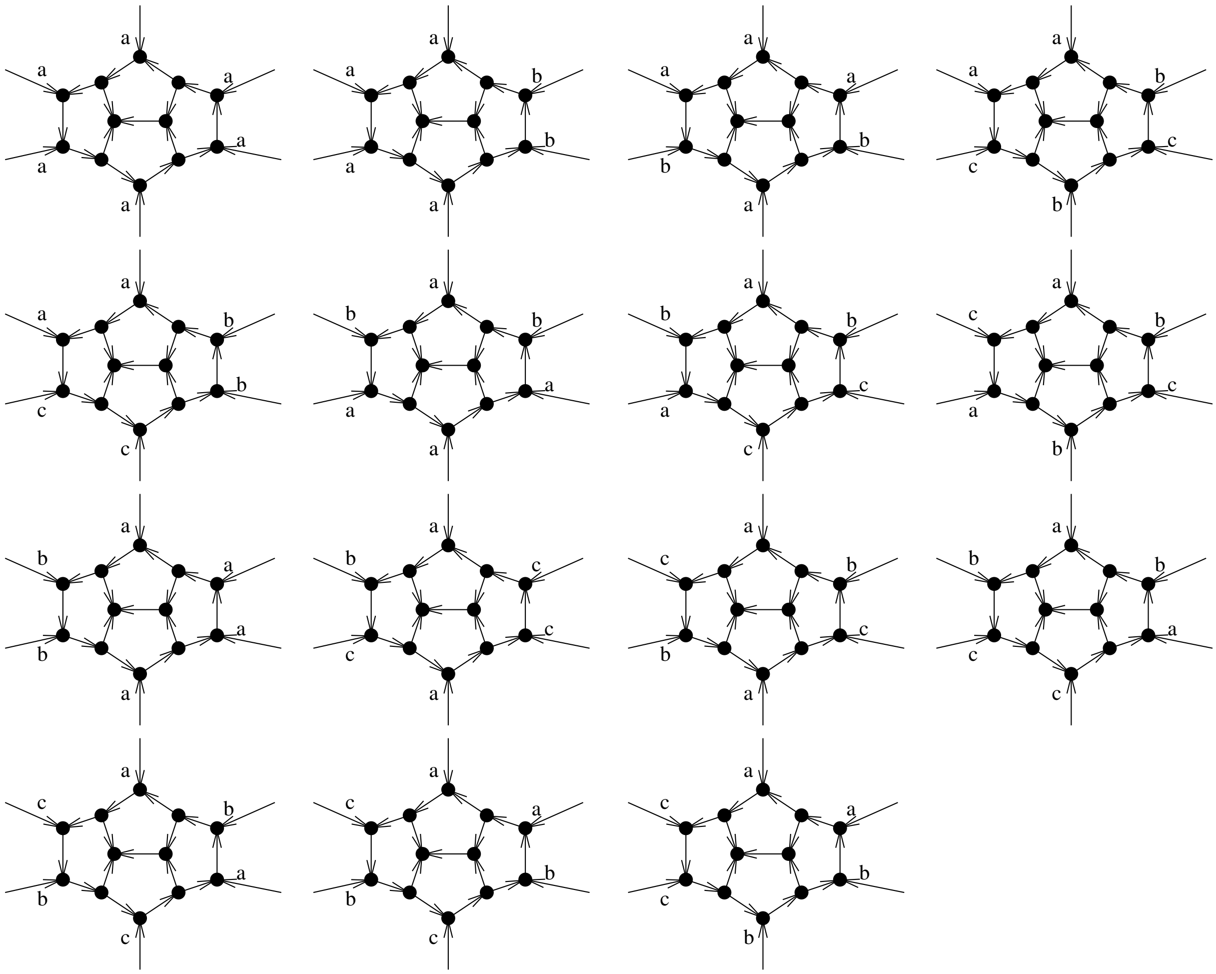


Figure 1.8: Flows that do not extend to the Birkhoff diamond dual $\vec{G}$ (i.e., the complement of $\mathcal{C}$ or $\mathcal{C}' - \mathcal{C}$). We do not draw the marked vertex.

Suppose the following inequality is true for all side graphs $\vec{F} \in \mathcal{S}_R$ (we omit the index $\vec{F}$ everywhere).

$$\begin{aligned}
&K_{(a,a,a,a,a,a)} + K_{(a,b,b,a,a,a)} + K_{(a,a,b,a,b,a)} + K_{(a,b,c,b,c,a)} + K_{(a,b,b,c,c,a)} \\
&\qquad + K_{(a,b,a,a,a,b)} + K_{(a,b,c,c,a,b)} + K_{(a,b,c,b,a,c)} + K_{(a,a,a,a,b,b)} + K_{(a,b,c,a,c,b)} \\
&\qquad + K_{(a,b,c,a,b,c)} + K_{(a,b,a,c,c,b)} + K_{(a,b,a,c,b,c)} + K_{(a,a,b,c,b,c)} + K_{(a,a,b,b,c,c)} \\
&\quad \leq 22K_{(a,a,b,b,a,a)} + 9K_{(a,b,a,a,b,a)} + 10K_{(a,a,a,b,b,a)} + 7K_{(a,b,b,b,b,a)} + 6K_{(a,b,c,c,b,a)} \\
&\quad + 9K_{(a,b,b,c,a,c)} + 29K_{(a,b,b,a,b,b)} + 8K_{(a,b,a,b,c,c)} + 17K_{(a,a,b,b,b,b)} + 7K_{(a,a,b,c,c,b)} \qquad (1.5)
\end{aligned}$$

Suppose that $\vec{G}$ is a side graph of R in $\vec{H}$ for some $H \in \mathcal{H}$ and H is not 4-flowable. Let $\vec{F}$ be the other side graph of R in $\vec{H}$. By Corollary 1.4.5, $\vec{F}$ does not extend any flow except possibly those in $\mathcal{C}' - \mathcal{C}$. However, all variables in the right-hand side in Inequality (1.5) are indexed by elements of $\mathcal{C}$. Therefore, for $\vec{F}$, the right-hand side is 0. Therefore the left-hand side is ≤ 0. Since $K_{\vec{F},f}$ is non-negative for any $\vec{F}$ and f and all coefficients on the left-hand side are non-negative, each term on the left-hand side is non-negative. Therefore all terms on the left-hand side are equal to 0.

So, by definition of $K_{\vec{F},f}$, $\vec{F}$ does not extend any flow indexed by a variable on the left-hand side. But all flows in $\mathcal{C}' - \mathcal{C}$ appear as an index on the left-hand side. Therefore, we have deduced that $\vec{F}$ does not extend any flow in $\mathcal{C}' - \mathcal{C}$.

So $\mathcal{C}$ reduces every flow in $\mathcal{C}' - \mathcal{C}$ (by satisfying the contrapositive of Definition 1.4.17). $\mathcal{C}$ trivially reduces every flow in $\mathcal{C}$ by definition.

Thus, if H is not 4-flowable, $\vec{F}$ does not extend any nowhere-zero flow on R. Replacing $\vec{G}$ by the 6-star with a marked vertex does not create any new cuts so the resulting graph is bridgeless. Therefore, $\vec{G}$ is reducible to the 6-star.

1.6 Previous work

While many attempts have been made to resolve the Nowhere-Zero Flow conjectures, only few have used algebraic methods. Kochol [10, 11] used such a method to prove that the girth of a minimum counter-example is at least 9. However, algebraic approaches have been used to attack the Four Colour Theorem. One such method, which has received much attention, uses the Birkhoff-Lewis equations [1, 21, 12, 2]. In Chapter 4, we analyse our method in order to compare it to some previous work on the topic of Nowhere-Zero Flows.

1.7 Outline

This thesis is organized as follows.

In Chapter 2, we describe how to generate and use equalities similar to Equation (1.4). In Chapter 3, we describe how to generate and use inequalities similar to the Inequality (1.5). At the end of each of these chapters, we discuss how we approach the problem computationally in practice. In Chapter 4, we analyse the arguments presented in Chapters 2 and 3 in order to compare their strength relative to one another. In Chapter 5, we describe an attempt to use a theoretical, non-computational, approach to prove the 5-flow conjecture using the argument presented in Chapter 2. In Chapter 6, we document the program used for computations. These programs apply the theory of Chapters 2 and 3.

Chapter 2

Homogeneous linear equalities

2.1 Overview

This section focuses on how we can use homogeneous linear equalities to obtain flow reducibility proofs.

Let $\mathcal{S}_R$ be the side graphs of a class of graphs $\mathcal{H}$ with cut R. Let $\mathcal{F}$ be the set of all partial flows on R. Suppose we wish to prove that all digraphs in $\mathcal{H}$ have a nowhere-zero k-flow. As in the previous section, we will omit the parameter Γ in general and assume that it is a group of order k.

As seen in the example in Section 1.5.1, we will be using equalities relating the number of flows which are true for all graphs in $\mathcal{S}_R$. We remind the reader of the definition of the variables.

Definition 2.1.1. Let $\vec{G} \in \mathcal{S}_R, f \in \mathcal{F}$. Then $K_{\vec{G},f}$ is the number of flows of $\vec{G}$ which extend f.

We now define the equalities we will study.

Definition 2.1.2. For each $f \in \mathcal{F}$, let $y_f \in \mathbb{R}$. Then

$$\sum_{f \in \mathcal{F}} y_f K_{\vec{G},f} = 0$$

is *an* $\mathcal{S}_R$*-universal linear equation* if it holds for all $\vec{G} \in \mathcal{S}_R$. Here y_f is a constant for all $f \in \mathcal{F}$.

In general, $\mathcal{S}_R$ is clear from the context so we refer to such equations as *universal equations.*

We will omit the index $\vec{G}$ in a universal equation since it holds for all graphs.

This section is organized as follows. In Section 2.2, we explain how to derive universal equations. In Section 2.3, we explain how these universal equations can be used for flow reducibility. In Section 2.4, we provide some details about computations made using the theory of universal equations.

For the remainder of this chapter, we will assume that R, $\mathcal{S}_R$, $\mathcal{H}$, $\mathcal{F}$ and k are defined as above and will not redefine them for each statement.

2.2 Generating universal equations

2.2.1 Contraction/deletion

For a fixed digraph $\vec{G}$, we may write the number $K_{G,f}$ of flows extending $f \in \mathcal{F}$ (for each $f \in \mathcal{F}$) as a vector $\mathbf{K}_{\vec{G}}$. This vector is indexed by the flows $f \in \mathcal{F}$ and the fth entry of $\mathbf{K}_{\vec{G}}$ is $K_{\vec{G},f}$.

We use the notation $\vec{G}/e$ for the graph $\vec{G}$ with the edge e contracted and $G - e$ for $\vec{G}$ with the edge e deleted.

Theorem 2.2.1 (A variation of Tutte's *contraction/deletion formula*)**.** *[18, 19] Suppose e is an edge not in R that is not a loop. Then*

$$\forall f \in \mathcal{F} \quad K_{\vec{G},f} = K_{\vec{G}/e,f} - K_{\vec{G}-e,f}.$$

Proof. Let A set of flows of $\vec{G}$ that extend f on R and are non-zero everywhere except possibly on e. Let B be the set of flows of $\vec{G}$ that extend f on R, are zero on e and non-zero everywhere else. Then $B \subseteq A$ and $A \setminus B$ is the set of nowhere-zero flows of $\vec{G}$ that extend f. Since $|A| = K_{\vec{G}/e,f}$ and $|B| = K_{\vec{G}-e,f}$, the result follows. $\square$

Similarly, in the case e is a loop, we have the following.

Theorem 2.2.2. *Suppose e is a loop not in R then*

$$\forall f \in \mathcal{F} \quad K_{\vec{G},f} = (k-1)K_{\vec{G}-e,f}.$$

Therefore, we have the following vector equation.

$$\mathbf{K}_{\vec{G}} = \begin{cases} \mathbf{K}_{\vec{G}/e} - \mathbf{K}_{\vec{G}-e} & \text{, if } e \text{ is not a loop} \\ (k-1)\mathbf{K}_{\vec{G}-e} & \text{, if } e \text{ is a loop} \end{cases} \tag{2.1}$$

As long as $\vec{G}$ has an edge not in R, we can apply the formula. So we can iterate and apply the formula to $\vec{G}/e$ and $\vec{G} - e$ until the right-hand side of the equation contains only graphs whose edges are edges of R.

Note that as we repeatedly expand the right-hand side, we may obtain vectors indexed by digraphs not in $\mathcal{S}_R$. This is not a problem since the vector Equation (2.1) hold for all digraphs so we can continue expanding.

This implies that the vector $\mathbf{K}_{\vec{G}}$ is a linear combination of vectors $\mathbf{K}_{\vec{B}}$ where each $\vec{B}$ is a digraph with only edges of R. Note that removing an isolated vertex (vertex with no neighbours) from a digraph does not change its vector of flows. Therefore, we may assume no digraph which appears as an index has an isolated vertex. Note that all vertices of R are incident to the marked vertex so the resulting graph $\vec{B}$ is connected. In fact, such a graph can be seen as a partition of the edges of R where edges are in the same part precisely if they have the same endpoints (since there are no other edges and all edges are incident to the universal vertex). Since R is finite, there is only a finite set $\mathcal{B}$ of these graphs (which can labelled by partitions). Therefore, we have shown the following.

$$\mathbf{K}_{\vec{G}} = \sum_{\vec{B} \in \mathcal{B}_R} a_{\vec{G},\vec{B}} \mathbf{K}_{\vec{B}} \tag{2.2}$$

This is true for all digraphs $\vec{G} \in \mathcal{S}_R$.

We call the digraphs in $\mathcal{B}_R$ the *basic graphs*.

2.2.2 Kochol's matrix

Let us build the matrix M_R of all vectors $\mathbf{K}_{\vec{B}}$, $B \in \mathcal{B}_R$. We claim that this is a 0-1 matrix. A partial flow f on R extends to $\vec{B}$ precisely if it is a flow on $\vec{B}$. Such an extension is unique since $\vec{B}$ has no other edges. Therefore, an entry $(M_R)_{f,\vec{B}}$ is 1 when f is a flow on $\vec{B}$ and 0 otherwise.

However, $\mathbf{K}_{\vec{G}}$ is a linear combination of the rows of M_R if and only if adding $\mathbf{K}_{\vec{G}}$ as a row to M_R does not increase its rank. But the rank is also the dimension of the column space of M_R. In other words, if the columns of M_R satisfy some linear equation then so do the entries of $\mathbf{K}_{\vec{G}}$.

But $\vec{G}$ was an arbitrary side graph of a cut R so all side graphs with $R(\vec{G}) = R$ satisfy these equations. Thus, we have shown the following.

Theorem 2.2.3. *If $\vec{G} \in \mathcal{S}$ and $\mathbf{y} \in \ker(M_R)$, then*

$$\sum_{f \in \mathcal{F}} y_f K_{\vec{G},f} = 0$$

is a universal equation.

For example, in the case of 4-flow, R is a set of 4 edges and $\mathcal{H}$ is the class of planar graphs, we have the matrix in Fig 2.1. The group chosen here is $\mathbb{Z}_2^2$, where the 3 non-zero elements are a,b and c.

Note that we may permute the non-zero elements a, b, c to obtain a nowhere-zero flow from an existing nowhere-zero flow. Therefore, we may consider nowhere-zero

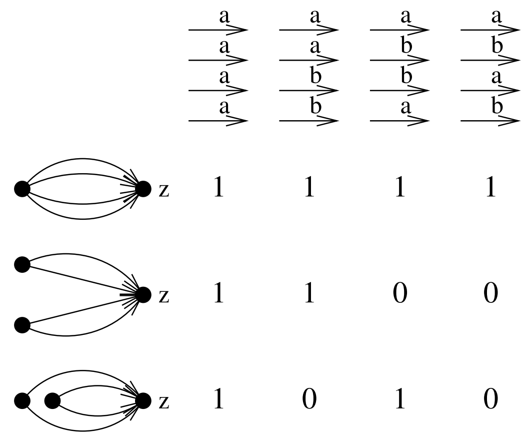


Figure 2.1: The matrix M_R where R is a set of 4 edges

flows only up to this permutation (and the columns of the matrix in Fig 2.1 shows only partial flows on R up to this permutation).

The matrix in Fig 2.1 has rank 3 so there is exactly one universal equality we may obtain from it. It is

$$K_1 + K_4 = K_2 + K_3$$

where K_i denotes the variable indexed by the flow indexing the ith column from the left. Using just this equality, we can show that the 4-cycle (with an added marked vertex) shown in Fig 2.2 is reducible (when $\mathcal{H}$ is the class of planar graphs).

Let $\vec{G}$ be an orientation of the 4-cycle with an added marked vertex. We show that $\vec{G}$ extends all flows but the one indexed by the last column by simply exhibiting a flow in each case (see Fig 2.3).

Now suppose that $\vec{G}$ is a side graph of $\vec{H}$ for a cut R, for some $H \in \mathcal{H}$. Let $\vec{F}$ be the other side graph (of $\vec{H}$ for R). Suppose that H is not 4-flowable. Then by Corollary 1.4.5, $\vec{F}$ does not extend any nowhere-zero flow on R. So we can replace $\vec{G}$ with the 4-star with an added marked vertex (Fig 2.4) and the resulting digraph still has no nowhere-zero 4 flow.

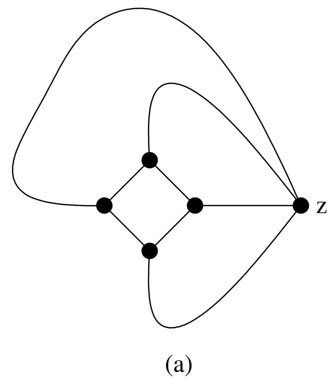


Figure 2.2: The 4-cycle with an added marked vertex G shown in (a) and an orientation $\vec{G}$ of it in (b).

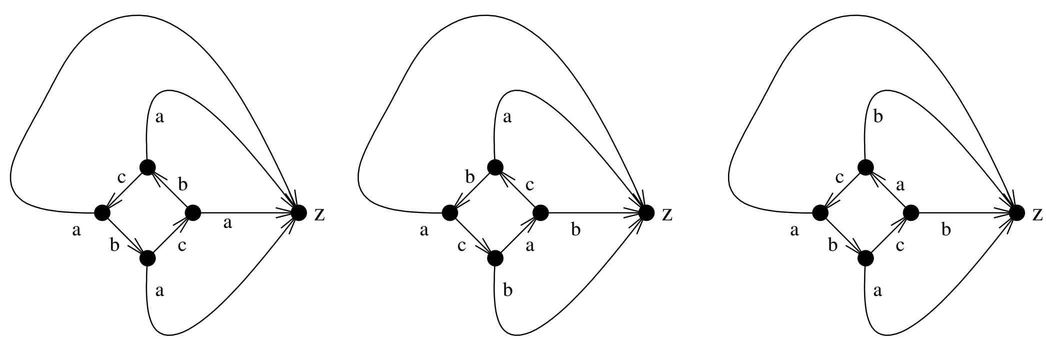


Figure 2.3: Some nowhere-zero flows on the 4-cycle (with an added marked vertex).

Clearly, the digraph obtained from the replacement is planar since it can be obtained by contracting the edges of the cycle (edges in G not in R) and deleting loops created this way.

Therefore, by definition, the 4-cycle with an added marked vertex is reducible to the 4-star with an added marked vertex.

2.3 Using universal equations for reducibility

Suppose f is a flow on R and $\mathcal{C}$ is a set of flows on R such that $K_c = 0 \ \forall c \in \mathcal{C}$. Suppose that we know of a set of equalities is universal.

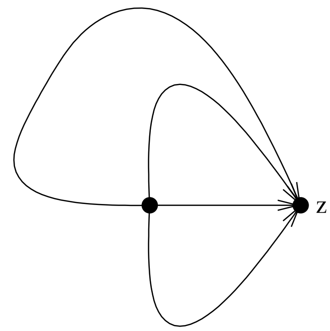


Figure 2.4: The 4-star (with an added marked vertex) which replaces the 4-cycle

Suppose that, as in the example shown in Section 1.5.1, we can deduce a universal equality of the following form.

$$\sum_{c\in\mathcal{C}} y_c K_c = \sum_{c\notin\mathcal{C}} y_c K_c \tag{2.3}$$

where $y_c \geq 0$ for all $c \notin \mathcal{C}$. Then the left-hand side sums to 0. Since K_f is non-negative for all flows f, all terms on the right-hand side are 0 as otherwise, they would not sum to 0. Thus we have proven that $K_c = 0$ for every $c \notin \mathcal{C}$, $y_c > 0$. So $\mathcal{C}$ reduces all flows c such that $y_c > 0$. To prove that $K_f = 0$, we need to find an equation of the form (2.3) with $y_f > 0$.

We may state this more formally.

Lemma 2.3.1. *Let f be a flow on R and $\mathcal{C}$ a set of flows on R.*

Suppose there is an universal equation of the form

$$\sum_{c\in\mathcal{C}} y_c K_c = \sum_{c\notin\mathcal{C}} y_c K_c \tag{2.4}$$

with the non-negative coefficients y_c on the right-hand sideand the coeffcient y_f of K_f is positive then $\mathcal{C}$ reduces f. with $y_c \geq 0$ $\forall c \notin \mathcal{C}$ and $y_f > 0$. Then $\mathcal{C}$ reduces f.

We will now show that the converse is also true. We may write a set of universal equalities in matrix form as $A\mathbf{k} = \mathbf{0}$. Note that any linear combination of the rows of A also gives a valid universal equality. Then, we know that we can prove $\mathcal{C}$ reduces f if and only if there is no solution to the following system.

$$\begin{aligned} A\mathbf{K} &= \mathbf{0} \\ K_c &= 0 \quad \forall c \in \mathcal{C} \\ K_f &> 0 \\ \mathbf{K} &\geq \mathbf{0} \end{aligned}$$

Note that any positive scalar multiple of a solution to the above system is still solution. So we may normalize and instead require

$$\begin{aligned} A\mathbf{K} &= \mathbf{0} \\ K_c &= 0 \quad \forall c \in \mathcal{C} \\ K_f &= 1 \\ \mathbf{K} &\geq \mathbf{0} \end{aligned}$$

to have no solution. We can rewrite the system in matrix form as

$$\begin{bmatrix} A \\ I_{\mathcal{C}} \\ e_f \end{bmatrix} \mathbf{K} = \mathbf{0},$$

$$\mathbf{K} \geq \mathbf{0}$$

where e_f is the vector which is 1 for the entry f and 0 everywhere else, and $I_{\mathcal{C}}$ is the set of row vectors $e_c, c \in \mathcal{C}$.

By Farkas' Lemma [4], the above has no solution if and only if there is a solution to

$$\left[A^T|I_{\mathcal{C}}^T|e_f^T\right]\mathbf{d} \geq \mathbf{0},$$

$$d_f < 0$$

where d_f is the coefficient for the equation $K_f = 1$ and $[A^T|I_{\mathcal{C}}^T|e_f^T]$ denotes the matrix formed by putting A^T, $I_{\mathcal{C}}^T$ and e_f^T side-by-side. We can rewrite the above equation as

$$A^T\mathbf{d_1} + I_{\mathcal{C}}^T\mathbf{d_2} + e_f^T\mathbf{d_3} \geq \mathbf{0}$$

The first term in sum is just a linear combination of the row vectors in A. Recall that such vectors represent universal equalities. Let $\mathbf{y} = A^T\mathbf{d_1}$.

We can now translate the second and third term into requirements on $\mathbf{y}$. There is a solution $\mathbf{d}$ with $d_f < 0$ if and only if there is a $\mathbf{y}$ such that $y_c \geq 0\ \forall c \notin \mathcal{C}$ and $y_f > 0$. This is since given such a y, we can choose entries of $\mathbf{d_2}$ large enough to make the sum non-negative for all $c \in \mathcal{C}$ and we can choose $\mathbf{d_3}$ small enough so that the non-zero entry of $e_f^T\mathbf{d_3}$ is smaller than y_f in absolute value.

Therefore, we have proven the following.

Theorem 2.3.2. *$\mathcal{C}$ reduces f (using universal linear equalities) if and only if there is $\mathbf{y}$ in the row space of A such that $y_c \geq 0\ \forall c \notin \mathcal{C}$, $y_f > 0$.*

Equivalently,

Theorem 2.3.3. *$\mathcal{C}$ reduces f (using universal linear equalities) if and only if there is a universal equality of the form*

$$\sum_{c\in\mathcal{C}} y_c K_c = \sum_{c\notin\mathcal{C}} y_c K_c \tag{2.5}$$

with $y_c \geq 0\ \forall c \notin \mathcal{C}$ and $y_f > 0$

We will refer to the above theorem as the *Farkas Criterion.*

As an easy corollary, we get the following condition.

Corollary 2.3.4. *$\mathcal{C}$ reduces f (using universal linear equalities) if there is a universal equality of the form*

$$\sum_{c\in\mathcal{C}} y_c K_c = y_f K_f \tag{2.6}$$

with $y_f > 0$.

2.3.1 Kochol's argument

To prove that the minimum counter-example to the 5-flow conjecture has girth at least 9, Kochol [10, 11] built the matrix M_R presented in Section 2.2 and then proved and used the following.

Theorem 2.3.5. *[10, 11] $\mathcal{C}$ reduces f if the column indexed by f in M_R is in the span of the columns indexed by elements of $\mathcal{C}$ in M_R.*

This theorem is equivalent to Corollary 2.3.4 since the coefficients for the linear combination (for showing the column is in the span of the others) is the same as the coefficients y_c required for the universal equality.

We refer to Theorem 2.3.5 as the *Span Criterion.*

By comparing Theorem 2.3.3 and Corollary 2.3.4, we see that the Farkas Criterion is at least as strong as the Span Criterion.

The following example shows that the Farkas Criterion is sometimes stronger than the Span Criterion when used for flow reducibility. However, we do not know of an example where the Farkas Criterion is stronger than the Span Criterion when used for graph reducibility.

Let R be the set pendant edges in the digraph in Fig 2.5 and let $\mathcal{C}$ be the set of partial flows on R which extends to the digraph in Fig 2.5.

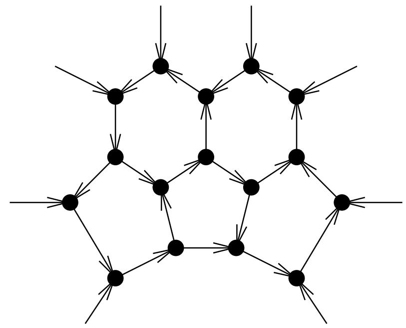


Figure 2.5: A digraph. Drawn with pendant edges (without the marked vertex). We use it to show that the Farkas Criterion is strictly stronger than the Span Criterion.

Let f be the partial flow on R with values $(1,1,1,1,1,3,1,3)$ (enumerating the edges starting at the upper left edge and going clockwise). Then, using a computer, we can see that $\mathcal{C}$ reduces f using the Farkas Criterion but not with the Span Criterion.

2.4 Computations

In this section, we describe how we used the theory from the previous two sections to make computations in order to prove reducibility.

We let A be a matrix whose rows form a basis for the kernel of M_R. From Section 2.2, we see that $A\mathbf{K} = 0$ are universal equalities.

Let f be a flow on R and $\mathcal{C}$ a set of flows on R. From Theorem 2.3.3, $\mathcal{C}$ reduces f if and only if there is a vector $\mathbf{y} \in \ker(M)$ such that $y_f > 0$ and $y_c \geq 0 \ \forall c \notin \mathcal{C}$.

We can verify the existence of such a vector by solving the following LP.

$$
\begin{aligned}
&\max y_f \\
&M\mathbf{y} = 0 \\
&y_c \geq 0 \quad \forall c \notin \mathcal{C} \\
&y_f \leq 1
\end{aligned}
\qquad (LP1)
$$

We added the constraint $y_f \leq 1$ to prevent the LP from being unbounded (as remarked in Section 2.3, positive scalar multiples of a solution are also solutions).

Since setting all y_c to be 0 gives a feasible solution, the optimal solution is at least 0. If the optimal solution is 0, then $\mathcal{C}$ does not reduce f, while otherwise the optimal solution is 1 and $\mathcal{C}$ does reduce f.

We can obtain an optimal solution $\mathbf{y}(\mathbf{f})$ for each flow $f \notin \mathcal{C}$ ($\mathcal{C}$ clearly reduces any flow in $\mathcal{C}$).

Let $\tilde{\mathbf{y}} = \sum_{f \notin \mathcal{C}} \mathbf{y}(\mathbf{f})$. The condition $y_c \geq 0 \; \forall c \notin \mathcal{C}$ forces $\tilde{\mathbf{y}}_c$ to be positive if one of the $\mathbf{y}(\mathbf{f})_c$ is positive.

Therefore, the flows f indexing the positive entries of $\tilde{\mathbf{y}}$ are exactly the flows $f \notin \mathcal{C}$ that $\mathcal{C}$ reduces. In fact, $\tilde{\mathbf{y}}$ is a certificate that those flows are reducible. Given such a $\tilde{\mathbf{y}}$, we only need to check that $M\tilde{\mathbf{y}} = 0$, $\tilde{\mathbf{y}}_c \geq 0 \; \forall c \notin \mathcal{C}$ and $\tilde{\mathbf{y}}_f > 0$ for any flow f that we claim $\mathcal{C}$ reduces.

Chapter 3

Homogeneous linear inequalities

In this section, much like the equalities found in the previous section, we will derive and use inequalities which the number of flows for any side graph (with the same labels on the edges incident to the marked vertex) must satisfy.

Let $\mathcal{S}_R$ be the side graphs of a class of graphs $\mathcal{H}$ with cut R. Let $\mathcal{F}$ be the set of all partial flows on R. Suppose we wish to prove that all digraphs in $\mathcal{H}$ have a nowhere-zero k-flow. As in the previous section, we will omit the parameter Γ in general and assume that it is a group of order k. We remind the reader of the definition of the variables.

Definition 3.0.1. Let $\vec{G} \in \mathcal{S}_R, f \in \mathcal{F}$. Then $K_{\vec{G},f}$ is the number of flows of $\vec{G}$ which extend f.

We now define the inequalities we will study.

Definition 3.0.2. For each $f \in \mathcal{F}$, let $y_f \in \mathbb{R}$. Then

$$\sum_{f \in \mathcal{F}} y_f K_{\vec{G},f} \leq 0$$

is *an $\mathcal{S}_R$-universal linear inequality* if it holds for all $\vec{G} \in \mathcal{S}_R$. Here y_f depends on $f \in \mathcal{F}$ and not on $\vec{G}$.

In general, $\mathcal{S}_R$ is clear from the context so we refer to such inequalities as *universal inequalities*.

We will omit the index $\vec{G}$ in an universal inequality since it holds for all graphs.

This chapter is organized as follows. In Section 3.1, we explain how to derive universal inequalities. In Section 3.2, we explain how these universal inequalities can be used for flow reducibility. In Section 3.3, we provide some details about computations made using the theory of universal inequalities.

For the remainder of this section, we will assume that $\mathcal{S}$, $\mathcal{H}$, $\mathcal{F}$, R and k are defined as above and will not redefine them for each statement.

3.1 Generating universal inequalities

We will only generate universal inequalities in the case where the order of the group k is 4. At the end, in Section 3.1.7, we briefly explain possible ways of extending this method for other values of k.

We may choose the group by Theorem 1.1.6, so we choose it to be $\mathbb{Z}_2^2$. Let a,b,c denote the 3 non-zero elements of $\mathbb{Z}_2^2$.

First note that, in the case of $\mathbb{Z}_2^2$, if f is a flow of $\vec{H}$, then the same function f is a flow on any orientation of H. We see this by looking at the construction in the proof of Proposition 1.1.4 and noting that every element of $\mathbb{Z}_2^2$ it its own inverse. The same argument applies for nowhere-zero flows.

Therefore, we may think of a nowhere-zero flow of $\vec{H}$ as a (non-proper) colouring of the edges of H (with elements of $\mathbb{Z}_2$). In fact, we now prove that these edges form a "postman set".

Definition 3.1.1. A *postman set* S of a graph H is a subset of the edges of H such that odd degree vertices of H are incident to an odd number of edges in S and even degree vertices of H are incident to an even number of edges in S.

Theorem 3.1.2. *[3] H is 4-flowable if and only if $E(H)$ can be partitioned into 3 postman sets.*

Proof. Choose the group to be $\mathbb{Z}_2^2$.

Given 3 postman sets P_a, P_b, P_c which partition $E(H)$, we define f as $f(e) = a$ if $e \in P_a$, $f(e) = b$ if $e \in P_b$ and $f(e) = c$ if $e \in P_c$. f is clearly nowhere-zero. Note that the sum of an even number of some fixed element of $\mathbb{Z}_2^2$ is 0. Thus the sum of the flow f around a vertex of even degree is 0 (the sum of an even number of each of the non-zero element of $\mathbb{Z}_2^2$). The sum of the flow f around a vertex of odd degree is the sum of the 3 non-zero elements (since there is an odd number of each of element in the sum and twice any element is 0). But the sum of the 3 non-zero elements of $\mathbb{Z}_2^2$ is 0. Therefore, f is a nowhere-zero flow.

Given a flow f of $\vec{H}$, let $P_i = \{e \in E(H) | f(e) = i\}$. Clearly, P_a, P_b, P_c is a partition of $E(H)$. We claim that each P_i is a postman set. Suppose not. Without loss of generality, let v be a vertex incident to the incorrect parity of edges from P_a.

If v is even, then v is incident to an odd number of edges of f. So v is incident to an odd number of $P_b \cup P_c$ edges. Without loss of generality, v is incident to an odd number of P_b edges and an even number of P_c edges. But then the sum of the flows incident to v is $a + b = c \neq 0$. Contradiction.

If v is odd, remove some edge in P_a incident to v. The sum of the flows of the remaining edges can be computed as in the previous paragraphs and so it is either b or c. But $c + a = b \neq 0$ and $b + a = c \neq 0$. Contradiction. □

However, in the case of cubic graphs, we do get a proper 3-edge-colouring.

Corollary 3.1.3. *Let H be a cubic graph.*

H is 4-flowable if and only if H is 3-edge-colourable.

Proof. By Theorem 3.1.2, we only need to show that a partition into 3 postman sets is a 3-edge-colouring. Since each vertex of H must be incident to an odd number of edges in each postman set, each vertex is incident to exactly one edge from each postman set. □

Note that when $\mathcal{H}$ is the family of bridgeless cubic graphs, a flow on a side graph is a 3-edge-colouring which is proper except around the marked vertex. We can think of this as a 3-edge-colouring by not drawing the marked vertex and letting the edges incident to the marked vertex be “pendant” (and have an open end).

We may now state the following weakening of the 4-flow conjecture (restricted to cubic graphs).

Conjecture 3.1.4. *[20] Every bridgeless cubic graph without a Petersen minor is 3-edge-colourable.*

The proof of this conjecture has been announced by N. Robertson, D. P. Sanders, P. D. Seymour and R. Thomas with a first part proven in [15].

We present arguments in the next three sections for reducibility with respect to this conjecture. This allows us to speak of 3-edge-colourings rather than flows. We will use the terminology developed for flows (such as “sides”, “side graphs” and “extensions”) for 3-edge-colourings. In those cases, we may simply think of the 3-edge-colouring as a flow on an arbitrary orientation of the edges.

We will present the arguments for general graphs in Section 3.1.5 and see that the result is essentially the same as for cubic graphs.

3.1.1 Kempe chains

Kempe chains were introduced in one of the first incorrect proofs of the Four Colour Theorem [8]. We now present this method of proof in the context of nowhere-zero $\mathbb{Z}_2^2$-flows. This is the exactly the argument used in [14] but in the dual graph.

Consider a 3-edge-colouring of some side graph F of an edge cut R of a cubic graph. We may assume that the edges are coloured with the non-zero elements of

$\mathbb{Z}_2^2$. We think of the edges of F incident to the marked vertex as "pendant edges" by removing the marked vertex but not the edges incident to it (so they only have one end). Suppose we remove all edges of some colour, say a, in F (including edges of R). Then the remaining graph, F_a, is a union of disjoint cycles (since all vertices in are incident to 2 edges in F_a) and paths ending in edges of R (because of the pendant edges).

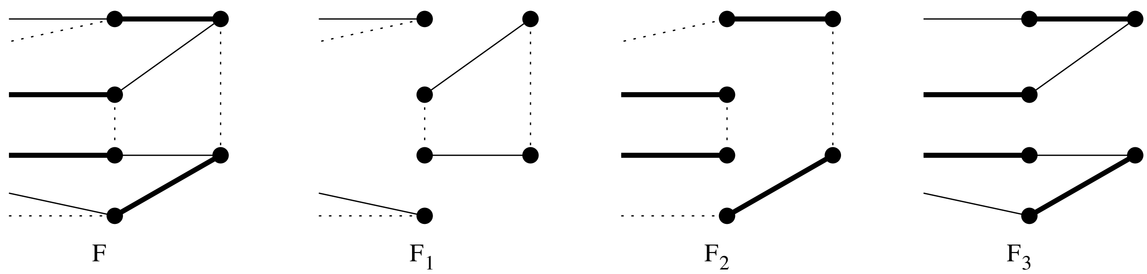


Figure 3.1: A cubic side graph F (drawn with pendant edges) and for each colour, the graph obtained by deleting all edges of that colour.

Suppose there is at least one path. We can exchange the colouring of the edges on a path in F_a (by switching colours b and c). If we exchanged the colours of the corresponding path in F, we obtain a new 3-edge-colouring of F. This new colouring differs from the original one in exactly 2 edges of R (those at the ends of the path).

Note that some edges of R may not appear in F_a (namely those coloured a). However, the remaining edges of R are joined by paths. In fact, given any 3-edge-colouring of F, we can delete all edges of colour a to obtain F_a and define a matching M on the remaining edges of R in F_a.

We can exchange the colours on each path independently of the other paths. When we make these inversions, the length of the path restricts the colours at the ends of a path. If the path has an even number of edges, the end edges are coloured the same. If the path has an even number of edges, the end edges are coloured differently. In fact, we can mark the edges of the matching with the parity of the length of the paths to produce what we call a *signed matching*. Given a 3-

edge-colouring we can build such a signed matching and call it the *signed matching induced by this 3-edge-colouring (and colour a).*

Notice that when we exchange the edges of a path in a 3-edge-colouring, the resulting 3-edge-colouring and the original 3-edge-colouring induce the same signed matching.

Definition 3.1.5. We say that a colouring of the edges of R θ*-fits* a signed matching M if:

1. edges of R that do not appear in M (as vertices) are coloured θ; all other edges of R are not coloured θ;

2. edges of R joined by odd edges of M are coloured the same colour; and

3. edges of R joined by even edges of M are coloured differently.

In the current discussion, we have assumed that $\theta = a$. In general, unless we need to consider multiple colours at the same time, we may assume $\theta = a$.

Since we can independently choose whether to make a colour inversion on each path, the following holds.

Remark 3.1.6. *A 3-edge-colourings of R that θ-fits a matching M extend to a 3-edge-colouring of F if there is a 3-edge-colouring of F that induces M.*

We may use the previous remark as follows. Suppose that we are told some colouring f of edges of R extends to F. Consider the signed matching M induced by some 3-edge-colouring of F that extends f and colour a. By the remark, each 3-edge-colouring of R that a-fits M extend to F (so we have deduced that many 3-edge-colourings of R extend to F from one colouring of R, f).

However, the power of this argument lies in the fact that we may deduce M to be one of several signed matching without knowing what F is (and therefore, without knowing what the 3-edge-colouring which extends f is). We know that M

is a perfect matching on the edges of R coloured b and c by f. Given a perfect matching of these edges, we also know what the sign should be (depending on whether the endpoints are coloured the same or differently by f).

We see an example of this type of argument used in the next section.

This is, in essence, the Kempe chain argument that we will need: the existence of a 3-edge-colouring induces some signed matching which implies the existence of other 3-edge-colourings.

3.1.2 An example

We show an example of the argument presented in the previous section. We will prove that when $\mathcal{H}$ is the set of bridgeless cubic graphs, the 4-cycle with an added marked vertex (Fig 3.2(a)) is reducible. Let G be the 4-cycle with an added marked vertex.

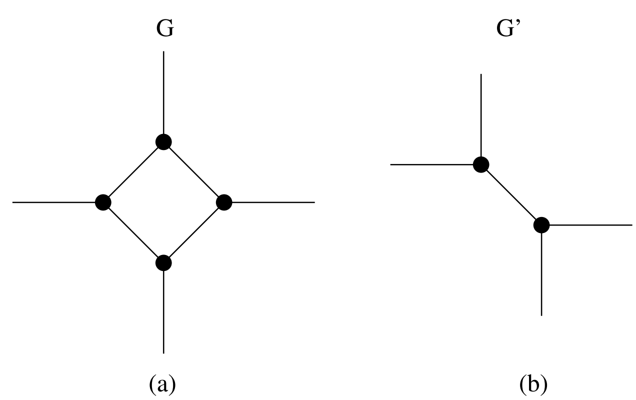


Figure 3.2: G, the 4-cycle with an added marked vertex is show in (a). G', the graph we will reduce G to is shown in (b). Both are drawn with pendant edges (without its marked vertex).

Suppose $H \in \mathcal{H}$ and H is not 3-edge-colourable. Suppose G appears as a side graph of some cut R in a graph H and F is the other side graph of R. There is only one 3-edge-colouring f that G does not extend (see Fig 3.3).

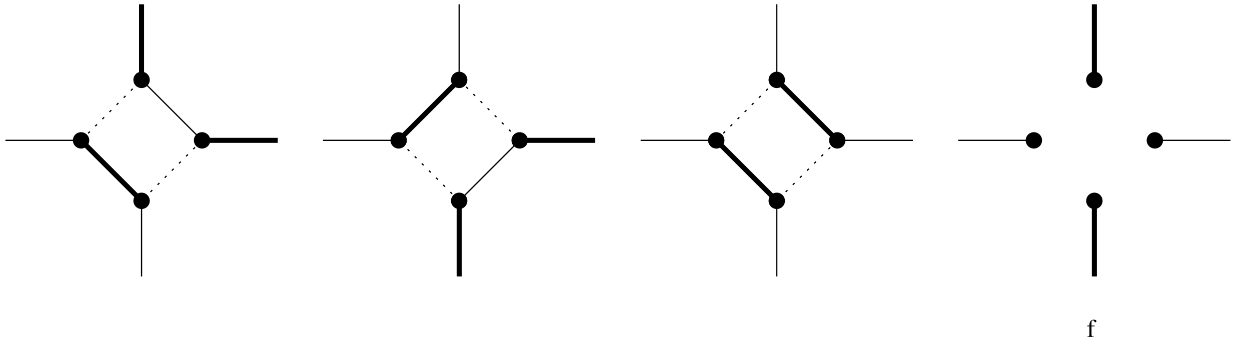


Figure 3.3: The 4-cycle extends all 3-edge-colourings except the last one.

Therefore, F extends only f (or no 3-edge-colouring at all). If F extends f then, by the previous section, we can find one of the set of paths shown in Fig 3.4.

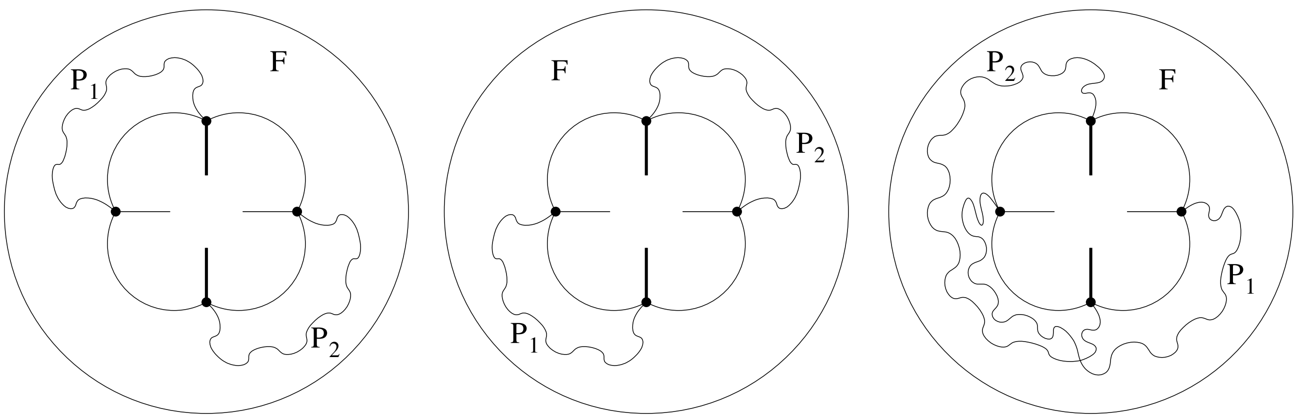


Figure 3.4: The possible paths for F, the graph on the other side.

Therefore, we can exchange the colours on the path P_2 in each case to obtain one of the 3-edge-colourings in Fig 3.3 on the cut. So F cannot extend f either (otherwise, it would extend one of the other edge colourings that it is forbidden to extend). Therefore, F extends no 3-edge-colourings and we can reduce G to anything. We may choose the graph G' in Fig 3.2.

3.1.3 Consistent sets

Although the previous example suggests that we would need an ad hoc argument every time, it is actually possible to formalize the definition of a set of 3-edge-colourings which a Kempe chain argument can reduce.

Suppose we know that some set $\mathcal{C}$ of 3-edge-colourings on R does not extend to F. Suppose that some 3-edge-colouring f on R does extend to F. Then, as stated in Section 3.1.1, for each colour θ, we can obtain a signed matching M from an extension of f to F. And thus, all edge colourings of R which θ-fit M should extend to F so all these edge colourings should not be in $\mathcal{C}$.

In a proof that $\mathcal{C}$ reduces f, we could repeatedly apply the argument in the previous paragraph to obtain a signed matching from a colouring and a colour and a set of colourings from the signed matching.

Let us think of a sufficient condition for $\mathcal{C}$ not to reduce f using a Kempe chain argument. At the last "step" in the proof, a matching that f θ-fits was obtained and then used to reduce f. However if such all such matching were "blocked", meaning that they cannot be induced by some f' throughout the proof then $\mathcal{C}$ does not reduce f. A simple condition for these matchings to be "block" is to have all f' which could induce such a matching be themselves "blocked". But, for colourings, "blocked" simply means that $\mathcal{C}$ does not reduce f'.

Then we can repeat the requirements on all such f' (i.e.: all matchings that. In doing so, we may obtain a large list of colourings that $\mathcal{C}$ must not reduce. But if none of these colourings is in $\mathcal{C}$ then $\mathcal{C}$ does not reduce any colourings in this list.

However, it is possible that all colourings extending f (and colour θ) induce the same signed matching M_θ. In fact, the argument used cannot determine whether a graph where all colourings extending f and θ induce M exists. Therefore, we only need one blocked matching per colour θ in order to have $\mathcal{C}$ not reduce f (rather than have all matchings that f θ-fit blocked). Therefore, we have shown the following.

Proposition 3.1.7. *Suppose there a set $\mathcal{D}$ of 3-edge-colourings such that for every*

$f \in \mathcal{D}$ and every θ, there is a signed matching M for which every f' that θ-fits M is also in $\mathcal{D}$, then $\mathcal{C}$ cannot reduce any edge colouring in $\mathcal{D}$.

We will call such a set $\mathcal{D}$ a "consistent set" which we now define more formally.

Definition 3.1.8. A set of edge colourings $\mathcal{D}$ is *consistent* if there is a set of signed matchings $\mathcal{M}_\theta$ for each colour θ such that:

1. for each $f \in \mathcal{D}$ and each θ, f θ-fits some matching $M_\theta \in \mathcal{M}_\theta$;
2. if an edge colouring f θ-fits some $M_\theta \in \mathcal{M}_\theta$ (for some θ) then $f \in \mathcal{D}$.

Note that the union of two consistent sets is consistent, since the union of their sets of signed matching show the union is consistent. Therefore, we can speak of a *maximum consistent set* $\mathcal{D}^*(\mathcal{C})$ disjoint from $\mathcal{C}$.

We can also show that if some f is not in any consistent set disjoint from $\mathcal{C}$ (equivalently, not in $\mathcal{D}^*(\mathcal{C})$) then $\mathcal{C}$ reduces f. This is because, if $\mathcal{C}$ does not reduce f then there must be some side graph which extends f but not $\mathcal{C}$. From such a side graph, we can obtain the set of edge colourings that it extends and the set of signed matchings from each of the edge colourings. These sets form a consistent set. Therefore, f is in this consistent set. Contradiction.

Therefore, we have proven the following theorem throughout this section.

Theorem 3.1.9. *[14] $\mathcal{C}$ reduces f (using Kempe chains) if and only if $f \notin \mathcal{D}^*(\mathcal{C})$*

3.1.4 Inequalities

We may strengthen the argument in Section 3.1.1 by actually counting the number of 3-edge-colourings of G which extend a colouring f of the edges of R. We can also count the number of 3-edge-colourings of G which induce a signed matching as well.

Let F be a side of an edge cut R of some cubic graph.

Definition 3.1.10. Let $S_{F,f}$ denote the set of 3-edge-colourings of F which extends a colouring f of the edges of R. Let $K_{F,f} = |S_{F,f}|$.

Let $T_{F,M,\theta,f}$ denote the set of 3-edge-colourings of F which extends a colouring f of the edges of R and induce the signed matching M with colour θ. Let $m_{F,M,\theta,f} = |T_{F,M,\theta,f}|$.

To simplify notation, we will no longer write the parameter F for the graph (and understand that it is always F). Theorems shall be proven for every graph F.

Note that the definition implies $m_{M,\theta,f}$ is only non-zero when f θ-fits M.

We now prove the "numerical equivalent" of statements in Section 3.1.1.

Theorem 3.1.11.

$$\forall f, \theta \quad \sum_{M} m_{M,\theta,f} \geq K_f$$

Proof. We will simply show the inclusion $S_f \subseteq \bigcup_M T_{M,\theta,f}$ for every f and every θ.

Let θ and f be fixed. Suppose $c \in S_f$. Then c extends f. Using c, we can build the signed matching M with colour θ as described in Section 3.1.1. Therefore, by definition, $c \in T_{M,\theta,f}$. □

Theorem 3.1.12. *Suppose f θ-fits M and f' is any colouring of the edges of R. Then $K_f \geq m_{M,\theta,f'}$.*

Proof. The case where f' does not θ-fit M is trivial as $T_{M,\theta,f'}$ is empty. So suppose f' is consistent with M.

Thus, by exchanging the colouring on the appropriate paths corresponding to edges of M, we can obtain a colouring in $T_{M,\theta,f'}$ from a colouring in $T_{M,\theta,f}$ and vice versa. In fact, this gives a bijection between $T_{M,\theta,f'}$ and $T_{M,\theta,f}$. But clearly $T_{M,\theta,f} \subseteq S_f$.

□

Let $m_{M,\theta} = \max_{f'} m_{M,\theta,f'}$.

Corollary 3.1.13.

$$\forall f, \theta \qquad \sum_M m_{M,\theta} \geq K_f \tag{3.1}$$

$$\forall f, \theta, M \text{ s.t. } f\ \theta\text{-fits } M \qquad K_f \geq m_{M,\theta} \tag{3.2}$$

Proof. (3.1) follows immediately from the fact that $m_{M,\theta} \geq m_{M,\theta,f}$

Fixing f,θ and M in Theorem 3.1.12 and taking the maximum over all f' gives (3.2) □

3.1.5 Non-cubic graphs

In this section, we show that the arguments presented in the previous sections can also be used non-cubic graphs as well. We will see that the same universal inequalities can be derived.

We will show that removing all edges of a postman set in a flow (partition into 3 postman sets) still induces a matching amongst the remain edges. We will still find paths on which we exchange edge colours and these inversions will still be independent of each other.

We still view a flow on (side) graphs as an edge colouring but it is now a non-proper 3-edge-colouring for which each colour class is a postman set. We will call them flows here to avoid repeating this property every time.

Suppose we have such a colouring of a side graph F of some cut R. Then, since the edges of colour a forms a postman set, we can delete all edges of colour a to obtain F_a. Then the remaining graph has only vertices of even degree (since we removed an odd number of edges incident to odd degree vertices and an even number of edges incident to even degree vertices).

We now prove that we can find paths as we did in Section 3.1.1.

Lemma 3.1.14. *Let F_a be an graph with only even degree vertices. Let z be a vertex in F_a. Then there exist edge-disjoint closed walks in F_a such that every edge* *incident to z is contained in exactly one walk and every walk contains exactly two edges incident to z.*

This is essentially saying that we can find edge-disjoint paths matching the pendant edges. We will see how to use these paths in the next lemma.

Proof. We proceed by induction on the degree of z. If z had degree 0 then we choose an empty set of walks.

Suppose the lemma is true for degree d and z has degree $d + 2$. Start at z and walk along the graph until we return to z (this can be done as all vertices we enter through the walk have even degree at least 2). This is a closed walk W which contains exactly 2 edges incident to z (or we would have stopped earlier). Remove the edges of W from F_a and note that the resulting graph has only vertices of even degree. By induction, we may find a set of closed walks as in the lemma. Add W to this set of walks (it is clearly disjoint) and we obtain a set of closed walks with the required properties. $\square$

The closed walks found by the previous lemma consists only of edges coloured b and c. Although they do not alternate in edges of colours b and c, we show that we can still exchange the colours on them.

Lemma 3.1.15. *Let f be a flow on a side graph F. Let W be a closed walk consisting only of edges of flow value b and c. If f' is obtained from f by exchanging the flow values on W, then f' is a flow.*

Proof. Since W is a closed walk, if $v \in V(F)$ is incident to an even number of edges with flow b (in f), then v is incident to an even number of flow c. Similarly, if $v \in V(F)$ is incident to an odd number of edges with edges with flow b, then v is incident to an odd number of edges with flow c.

Since we are exchanging, this is also true in f'. Therefore f' is a flow as the parity is conserved. $\square$

Note that, as before, when we make this inversion, if the edges incident to z were coloured the same, then they are coloured the same after the inversion and if the edges incident to z were coloured differently, then they are coloured differently after the inversion.

Therefore, we can define a *signed matching induced by a flow f (and colour a)* to be the matching obtained from Lemma 3.1.14 where two edges are matched together if they are in the same walk and the sign is determined by whether or not two matched edges are coloured the same in f.

Note that we only needed one signed matching for each flow in the previous sections. We could obtain more matchings for non-cubic graphs (because there may be more than one possible choice of walks) but we always obtain at least one signed matching from each flow.

From this definition we can then use the theory in the previous sections without any changes.

3.1.6 Planarity

We note that in the case we wish to prove the Four Colour Theorem, we may further require that the edges of the signed matching do not cross (since a side graph of a planar graph is planar).

3.1.7 Other k-flows

In this section, we explore possible ways of extending the method of generating universal inequalities using Kempe chains to other groups.

The main problem which arises with other groups is the inability to find a matching of the edges of the cut R or even a path on which we can modify the flow.

For example, if we could find a path with both ends in R such that we could add x to all forward arcs and subtract x to all backwards arcs then making this modification would yield a new flow with the flow changed on two edges of R. In fact, exchanging colours b and c on a path (for $\mathbb{Z}_2^2$-flows) corresponds to adding a on every edge of the path.

When the group is $\mathbb{Z}_2^2$, we could view the flows as edge colourings and only consider the underlying undirected graph G of $\vec{G}$. For $\mathbb{Z}_k$ flows, we may consider $d(G)$ the digraph where each edge of G is replaced with a digon. Then we can define f' on $d(G)$ from a flow f on $\vec{G}$: if uv is an arc of $\vec{G}$, then $f'(uv) = f(e)$; otherwise, $f'(uv) = -f(e)$. i.e., define f' to be the same as f for all arc in $\vec{G}$ and define f' to be the inverse for all "reverse arcs".

Then we are simply looking for a directed cycle (or path if we remove the marked vertex z and consider edges in R as pendant) which contains two arcs in R (from different edges) and does not contain an arc uv with $f'(uv) = -x$.

Unfortunately, for $\mathbb{Z}_k$ flows in general, there may be directed cuts where all flows on the cut edges have value x. This prevents the existence of the path we are looking for.

3.2 Using universal inequalities for reducibility

As in Section 2.3, we will now see how to use universal inequalities to obtain reducibility arguments.

Suppose f is a flow on R and $\mathcal{C}$ is a set of flows on R such that $K_c = 0\ \forall c \in \mathcal{C}$. Suppose that we know of a set of inequalities is universal.

Suppose that, as in the example shown in Section 1.5.2, we can deduce an universal equality of the form

$$\sum_{c \notin \mathcal{C}} y_c K_c \leq \sum_{c \in \mathcal{C}} y_c K_c, \tag{3.3}$$

where $y_c \geq 0$ for all $c \notin \mathcal{C}$. Then the right-hand side sums to 0. Since K_f is non-negative for all flows f, all terms on the left-hand side are 0 as otherwise, their sum would be greater than 0. Thus we have proven that $K_c = 0$ for every $c \notin \mathcal{C}$, $y_c > 0$. So $\mathcal{C}$ reduces all flows c such that $y_c > 0$. To prove that $K_f = 0$, we need to find an inequality of the form (3.3) with $y_f > 0$.

We may state this more formally.

Lemma 3.2.1. *Let f be a flow on R and $\mathcal{C}$ a set of flows on R.*

If there is an universal inequality of the form

$$\sum_{c \notin \mathcal{C}} y_c K_c \leq \sum_{c \in \mathcal{C}} y_c K_c \tag{3.4}$$

with the non-negative coefficients on the left-hand sideand the coeffcient y_f of K_f is positive then $\mathcal{C}$ reduces f.

We now use the argument in Section 2.3 to show that the converse is also true. We may write a set of universal inequalities in matrix form as $A\mathbf{k} \leq \mathbf{0}$. Note that any non-negative linear combination of the rows of A also gives a valid universal inequality. Then, we know that we can prove $\mathcal{C}$ reduces f if and only if there is no solution to the following system.

$$\begin{aligned} A\mathbf{K} &\leq \mathbf{0} \\ K_c &= 0 \quad \forall c \in \mathcal{C} \\ \mathbf{K} &\geq \mathbf{0} \\ K_f &> 0 \end{aligned}$$

We can rewrite this as

$$\begin{aligned} A\mathbf{K} &\leq \mathbf{0} \\ -\mathbf{K} &\leq \mathbf{0} \\ K_c &\leq 0 \quad \forall c \in \mathcal{C} \\ K_f &> 0 \end{aligned}$$

We can again rewrite the system in matrix form as

$$\begin{bmatrix} A \\ -I \\ I_{\mathcal{C}} \end{bmatrix} \mathbf{K} \leq \mathbf{0}$$

$$e_f \mathbf{K} > 0$$

where e_f is the vector which is 1 for the entry f and 0 everywhere else and $I_{\mathcal{C}}$ is the set of row vectors $e_c, c \in \mathcal{C}$.

By Farkas' Lemma [4] (instead of $A^T y \geq 0$, $b^T y < 0$, we have $A^T y \leq 0$, $b^T y > 0$), the above has no solution if and only if there is a solution to

$$[A^T| - I|I_{\mathcal{C}}]\mathbf{d} = e_f$$

where $\mathbf{d} \geq \mathbf{0}$. Again we can write this as the sum of 3 vectors and let $\mathbf{y} = A^T \mathbf{d}$.

$$\mathbf{y} - I\mathbf{d_2} + I_{\mathcal{C}}\mathbf{d_3} = e_f$$

The second vector $-I\mathbf{d_2}$ turns the equation into an inequality (since we can increase the value of $\mathbf{d_2}$ if any coordinate is too high.

$$\mathbf{y} + I_{\mathcal{C}}\mathbf{d_3} \geq e_f$$

The $I_{\mathcal{C}}\mathbf{d_3}$ indicates that we have need to choose $\mathbf{y}$ so that the inequality is satisfied for rows not indexed by elements of $\mathcal{C}$.

Thus, we have translated the second and third term into requirement on $\mathbf{y}$. There is a solution $\mathbf{d} \geq \mathbf{0}$ if and only if there is a $\mathbf{y}$ such that $y_c \geq 0\ \forall c \notin \mathcal{C}$ and $y_f \geq 1$. We can normalize this so that we only need $y_f > 0$.

Therefore, we have proven the following.

Theorem 3.2.2. *$\mathcal{C}$ reduces f (using universal linear inequalities) if and only if there is $\mathbf{y}$ which is a non-negative linear combination of columns of A such that $y_c \geq 0\ \forall c \notin \mathcal{C}$, $y_f > 0$.*

Equivalently,

Theorem 3.2.3. *$\mathcal{C}$ reduces f (using universal linear equalities) if and only if there is a universal inequality of the form*

$$\sum_{c \notin \mathcal{C}} y_c K_c \leq \sum_{c \in \mathcal{C}} y_c K_c \tag{3.5}$$

with $y_c \geq 0\ \forall c \notin \mathcal{C}$ and $y_f > 0$.

We will refer to the above theorem as the *Farkas Criterion* (for inequalities).

3.3 Computations

In this section, we describe how we used the theory from the previous two sections to make computations in order to prove reducibility.

When we derived the universal inequalities using Kempe chains in Section 3.1, we had introduced some auxiliary variables $m_{M,\theta}$.

In practice, we will solve an LP with the auxiliary variables in it but will only look at the value of the K variables in the solution. In theory, we could also take the projection and obtain inequalities without the auxiliary variables.

Let $C\mathbf{K}+B\mathbf{m} \leq 0$ be the set of universal inequalities obtained in Section 3.1 (we put all variables on one side of the inequality). Let $A = [C|B]$. So the inequalities become $A\binom{\mathbf{K}}{\mathbf{m}} \leq 0$

Let f be a flow on R and $\mathcal{C}$ a set of flows on R. From Theorem 3.2.3, $\mathcal{C}$ reduces f if and only if there is a vector $\mathbf{y}$ that is a non-negative linear combination of the columns of A such that $y_f > 0$ and $y_c \geq 0\ \forall c \notin \mathcal{C}$. Note that this second non-negativity condition must also hold for indices corresponding to the m variables (although they are not indexed by a flow c). So $\mathcal{C}$ simply refers to the columns of C (which are columns of A) indexed by elements of $\mathcal{C}$ and $c \notin \mathcal{C}$ refers to all other columns of A.

We can verify the existence of such a vector by solving the following LP.

$$
\begin{aligned}
&\max\, y_f \\
&\quad \mathbf{y} = A\mathbf{z} \\
&\quad \mathbf{z} \geq \mathbf{0} \qquad\qquad\qquad (LP2) \\
&\quad y_c \geq 0 \qquad \forall c \notin \mathcal{C} \\
&\quad y_f \leq 1
\end{aligned}
$$

We added the constraint $y_f \leq 1$ to prevent the LP from being unbounded (as remarked in Section 2.3, positive scalar multiples of a feasible solution are also solutions).

Since setting all y_c to be 0 and $\mathbf{z}$ to be $\mathbf{0}$ gives a feasible solution, the optimal solution is at least 0. If the optimal solution is 0, then $\mathcal{C}$ does not reduce f, while otherwise the optimal solution is 1 and $\mathcal{C}$ does reduce f.

We can obtain an optimal solution $\mathbf{y}(\mathbf{f})$ for each flow $f \notin \mathcal{C}$ ($\mathcal{C}$ clearly reduces any flow in $\mathcal{C}$).

Let $\tilde{\mathbf{y}} = \sum_{f \notin \mathcal{C}} \mathbf{y}(\mathbf{f})$. The condition $y_c \geq 0\ \forall c \notin \mathcal{C}$ forces $\tilde{\mathbf{y}}_c$ to be positive if one of the $\mathbf{y}(\mathbf{f})_c$ is positive.

We can also keep track of $\mathbf{z}$ and let $\mathbf{z}(\mathbf{f})$ be the value of the z variables in an optimal solution when maximizing y_f. Let $\tilde{\mathbf{z}} = \sum_{f \notin \mathcal{C}} \mathbf{z}(\mathbf{f})$. Note that $\tilde{\mathbf{y}} = A\tilde{\mathbf{z}}$ since each individual term satisfies the equality.

Therefore, the flows f indexing the positive entries of $\tilde{\mathbf{y}}$ are exactly the flows $f \notin \mathcal{C}$ that $\mathcal{C}$ reduces. In fact, $(\tilde{\mathbf{y}}, \tilde{\mathbf{z}})$ is a certificate that those flows are reducible. Given $(\tilde{\mathbf{y}}, \tilde{\mathbf{z}})$, we only need to check that $\tilde{\mathbf{y}} = A\tilde{\mathbf{z}}$, $\tilde{\mathbf{z}} \geq \mathbf{0}$, $\tilde{\mathbf{y}}_c \geq 0\ \forall c \notin \mathcal{C}$ and $\tilde{\mathbf{y}}_f > 0$ for any flow f that we claim $\mathcal{C}$ reduces.

Chapter 4

Argument strength

We presented two different arguments in the last two sections: one with universal equalities and one with universal inequalities. We can see from the computations sections that we can simply combine the two sets of restrictions to obtain a stronger argument. We first derive the "Farkas Criterion" in that case. However, it is not clear the we obtain a strictly stronger argument. Using the definition of graph reducibility and flow reducibility, we may quantify the strength of an argument.

Definition 4.0.1. We says that an argument A is *stronger* than an argument B if whenever $\mathcal{C}$ reduces f using argument B (where R is a cut, f is a partial flow R and $\mathcal{C}$ is a set of partial flows on R), $\mathcal{C}$ reduces f using argument A.

We says that an argument A is *strictly stronger* than an argument B if A is stronger than B and there is a $\mathcal{C}$ and f (where R is a cut, f is a partial flow R and $\mathcal{C}$ is a set of partial flows on R) such that $\mathcal{C}$ reduces f using argument A but $\mathcal{C}$ does not reduce f using argument B.

Define stronger and strictly stronger for graph reducibility similarly.

4.1 Combining equalities and inequalities

In this section, we define an argument which is the combination of the universal equalities argument and universal inequalities argument.

Suppose we have a set of universal equalities $A\mathbf{k} = 0$ and a set of universal inequalities $A'\mathbf{K} \leq 0$. Not too surprisingly, we will show the following *Farkas Criterion*.

Theorem 4.1.1. *Let $A\mathbf{K} = 0$ be a set of universal equalities and $A'\mathbf{K} \leq 0$ a set of universal inequalities.*

$\mathcal{C}$ reduces f (using combined equalities and inequalities) if and only if there is a universal inequality of the form

$$\sum_{c \in \mathcal{C}} y_c K_c \geq \sum_{c \notin \mathcal{C}} y_c K_c \tag{4.1}$$

with $y_c \geq 0$ $\forall c \notin \mathcal{C}$ and $y_f > 0$ which can be obtained from the sum of a linear combination of the rows of A and a non-negative linear combination of the rows of A'.

Proof. We can rewrite the equalities as inequalities $A\mathbf{K} \leq 0$ and $-A\mathbf{K} \leq 0$. The theorem now follows by the inequalities version of the theorem (Theorem 3.2.3). Since we have both $A'\mathbf{K} \leq 0$ and $-A'\mathbf{K} \leq 0$, the requirement for a non-negative combination is removed. $\square$

Computationally, we see that the K variables in (LP1) and (LP2) are the same (with the same interpretation), we may simply combine these constraints to formulate a new LP.

$$
\begin{aligned}
\max\ & y_f \\
\mathbf{y} &= \mathbf{y_1} + \mathbf{y_2} \\
M^T\mathbf{y_1} &= 0 \\
\mathbf{y_2} &= A'\mathbf{z} \\
\mathbf{z} &\geq \mathbf{0} \qquad\qquad\qquad\qquad (LP3) \\
(y_1)_c &\geq 0 \qquad \forall c \notin \mathcal{C} \\
(y_2)_c &\geq 0 \qquad \forall c \notin \mathcal{C} \\
y_f &\leq 1
\end{aligned}
$$

4.2 Consistent sets and inequalities

In this section, we show that the argument using inequalities derived in Section 3.1.4 has the same strength as the Kempe chain argument (using consistent sets) developed in Section 3.1.3. For the remainder of this section, we think of a flow f on R and a set of flows $\mathcal{C}$ on R as fixed but arbitrary.

The argument in Section 3.1.4 shows that $\mathcal{C}$ reduces f if and only if the following LP has an optimum of 0.

$$
\begin{aligned}
\max\ & K_f \\
\sum_M m_{M,\theta} &\geq K_f \qquad \forall f, \theta \\
K_f &\geq m_{M,\theta} \quad \forall f, \theta, M \text{ s.t. } f\ \theta\text{-fits } M \qquad (LP4) \\
K_c &= 0 \qquad\quad \forall c \in \mathcal{C}
\end{aligned}
$$

We now prove the following theorem which shows that both arguments have the same strength.

Theorem 4.2.1. *$f \notin \mathcal{D}^*(\mathcal{C})$ if and only if the optimum to (LP4) is 0 for f.*

Proof. $\Leftarrow$: We prove the contrapositive.

Suppose that $f \in \mathcal{D}^*(\mathcal{C})$. Let $\mathcal{M}_\theta^*$ be the set of signed matchings in the definition of a consistent set.

For each $c \in \mathcal{D}^*(\mathcal{C})$, let $K_c = 1$. For each $M_\theta \in \mathcal{M}_\theta$, let $m_{M_\theta,\theta} = 1$. We will show that this is a feasible solution to the LP.

By condition (1) in the definition of a consistent set, inequalities of the form (3.1) are satisfied since whenever the right-hand side is 1, the matching which θ-fit the corresponding edge colouring is set to 1 on the left-hand side.

By condition (2) in the definition of a consistent set, inequalities of the form (3.2) are satisfied.

This solution has objective value 1. But this contradicts the assumption the maximum is 0.

$\Rightarrow$: We prove the contrapositive.

Suppose the optimum to (LP4) is unbounded. Then there is a feasible solution where $k_f > 0$. We build a consistent set by looking at this feasible solution.

- Let $\mathcal{D} = \{f : k_f > 0\}$.
- For each θ, let $\mathcal{M}_\theta = \{M : m_{M,\theta} > 0\}$.

Since equations of the form (3.1) are satisfied, condition (1) of Definition 3.1.8 is met.

Since equations of the form (3.2) are satisfied, condition (2) of Definition 3.1.8 is met.

Therefore, f is in the consistent set $\mathcal{D}$. $\square$

4.3 General universal equalities

We have derived some equalities in Section 2.2 and some inequalities in Section 3.1. However, there may be more universal linear equalities and inequalities. In the case there are more, we may want to know the strength of an argument when we use all such equalities or all such inequalities. We will refer to these as the *general equalities argument* and *general inequalities argument*. We do not know how to analyze the strength of these arguments in general.

In this section, we show that there are no other equalities than those found in Section 2.2.

Theorem 4.3.1. *Let M_R be Kochol's matrix. Let*

$$\sum_c y_c K_c = 0 \tag{4.2}$$

be a universal equation. Then $\mathbf{y} \in \ker(M_R)$.

Proof. Since (4.2) is true for all digraphs $\vec{G}$, it is true for all basic graphs $\vec{B} \in \mathcal{B}_R$. Therefore, by definition, $\mathbf{y} \in \ker(M_R)$. □

This is not too surprising. We had shown using the contraction/deletion formula that all equations satisfied by the basic graphs is satisfied by all graphs. Here we simply noted that an equation satisfied by all graphs is satisfied by all basic graphs.

4.4 Replacing equalities with inequalities

From the proof of Theorem 4.1.1, we see that we can easily replace an equality with two inequalities so that the constraint on the variables is the same. However, in this section, we see that, for the purpose of flow reducibility, we may replace the equalities with even weaker inequalities (so that now the solution space changes).

Given a universal equation, we can move all terms with negative coefficient to the other side of the equation so that it is of the form.

$$\sum_{c \in P} a_c K_c = \sum_{c \in N} b_c K_c \tag{4.3}$$

where $a_c > 0 \ \forall c \in P$ and $b_c > 0 \ \forall c \in N$. P and N simply denote the coefficients which were positive and negative. We claim that we may replace such equalities with the set of inequalities

$$\begin{aligned} \forall f \in N \quad \sum_{c \in P} a_c K_c &\geq b_f K_f \\ \forall f \in P \quad a_f K_f &\leq \sum_{c \in N} b_c K_c \end{aligned}$$

without losing any strength for flow reducibility. However, whereas we only need to replace $A\mathbf{K} = 0$ by $A\mathbf{K} \leq 0$ and $A\mathbf{K} \geq 0$, now we must make the replacement for every equation that is a linear combination of the rows of A.

By Theorem 2.3.3, $\mathcal{C}$ reduces f if and only if there is an equation of the form

$$\sum_{c \in \mathcal{C}} y_c K_c = \sum_{c \notin \mathcal{C}} y_c K_c$$

with $y_c \geq 0 \ \forall c \notin \mathcal{C}$ and $y_f > 0$. But then the inequality

$$\sum_{c \in \mathcal{C}} y_c K_c \geq y_f K_f$$

also suffices to prove $\mathcal{C}$ reduces f. Therefore, we can make the replacement we have suggested without losing any strength in reducibility.

We may deduce the following theorems

Theorem 4.4.1. *In the case of 4-flows, $\mathcal{H}$ is the class of all bridgeless planar graphs and the cut R has fewer than 5 edges, the Kempe chain argument is stronger than the equalities argument.*

Proof. We choose the group to be $\mathbb{Z}_2^2$ and interpret a flow as a (non-proper) 3-edge-colouring as in Section 3.1.

We will show that all the inequalities of the form (4.3) can be obtained using a Kempe chain argument.

For $|R| = 2$ and $|R| = 3$, there is only one partial flow on the edges of R so there are no non-trivial equalities or inequalities.

For the case $|R| = 4$, we have seen that the matrix M_R is the one shown in Figure 2.1. There is only one equality since the matrix has rank 3. It is

$$K_1 + K_4 = K_2 + K_3$$

Thus we would replace it by

$$\begin{aligned} K_1 + K_4 &\geq K_2 \\ K_1 + K_4 &\geq K_3 \\ K_1 &\leq K_2 + K_3 \\ K_4 &\leq K_2 + K_3 \end{aligned}$$

But they are exactly the (non-trivial) inequalities we obtain from Corollary 3.1.13. $\square$

Proposition 4.4.2. *In the case of 4-flows, $\mathcal{H}$ is the class of all bridgeless graphs and the cut R has fewer than 6 edges, the Kempe chain argument is stronger than the equalities argument.*

Proof. We choose the group to be $\mathbb{Z}_2^2$ and interpret a flow as a (non-proper) 3-edge-colouring as in Section 3.1.

We will show that all the inequalities of the form (4.3) can be obtained using a Kempe chain argument.

For $|R| = 2$ and $|R| = 3$, there is only one partial flow on the edges of R so there are no non-trivial equalities or inequalities.

In the case $|R| = 4$, there are only 4 partial flows up to permutation of the colour classes. We compute the rank of the matrix M_R to be 4. Therefore, only trivial equalities (which permute the colour classes) exist.

It remains to prove the case where $|R| = 5$. Note that a partial flow on R must have exactly 3 edges of one colour and 1 edge of each of the other colours (for the sum to be zero). So there are only 10 partial flows (up to permutation of the colour classes). We can compute the matrix M_R and see that it has rank 10. Therefore, only trivial equalities (which permute the colour classes) exist.

□

Thus the following is immediate from the proof of the theorem and the fact that there is at least one inequality when $|R| = 5$.

Corollary 4.4.3. *In the case of 4-flows where the cut R has 5 edges, the Kempe chain argument is strictly stronger than the equalities argument.*

Chapter 5

5 flow

We have attempted to resolve a generalized version of a conjecture posed by Kochol [10]. Namely, we would like to know, in the case of 5-flows and all bridgeless graphs, if all cycles are reducible. Because of Theorem 1.4.11, we may reduce only cubic graphs. We start by characterizing the set of flows which extend to a cycle. From this and Farkas Criterion, we will be able to determine the form of a vector $\mathbf{y}$ that we are looking for in the kernel of Kochol's matrix M_R. We then show how we could obtain (inductively) some elements of the kernel of M_R for arbitrary R.

Kochol[10] had previous shown using the Span Criterion that all cycles of length at most 8 are reducible. Using programs in Chapter 5, we were able to verify this and prove that the cycle of length 9 is also reducible. Kochol conjectured that the Span Condition could be used to prove reducibility of all cycles and thus prove the 5-Flow Conjecture.

However, we note that this conjecture can be answered in the negative.

Theorem 5.0.4. *[13] Suppose $|R| \geq 22$ and $\mathcal{C}$ is the set of partial flows on R which extends to the cycle of length $|R|$. Then the dimension of the span of the columns indexed by elements of $\mathcal{C}$ in M_R is (strictly) smaller than the rank of M_R.*

It is therefore immediate that there is at least one column of M_R that the columns indexed by $\mathcal{C}$ do not span. So the Span Criterion cannot prove that $\mathcal{C}$

reduces the partial flow indexing that column.

Corollary 5.0.5. *The Span Criterion is not sufficient to prove the 5-flow conjecture.*

We do not know if Span Criterion can be used for cycles of length between 10 and 21.

However, the above corollary does not imply that the Farkas Criterion is also insufficient to prove reducibility of all cycles.

5.1 Cycles

Let G be a cycle of length n. We may orient the edges of a cycle as in Figure 5.1.

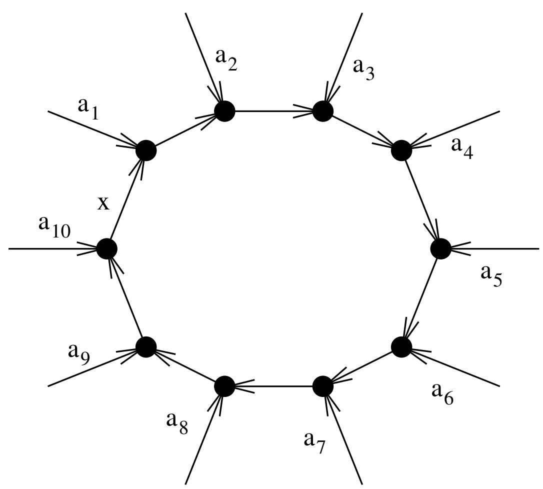


Figure 5.1: An orientation $\vec{G}$ of the edges of a cycle drawn without the marked vertex.

Let f be a partial flow on R. We may write the values of f as a vector $(a_1, \ldots, a_n)$ as shown in Figure 5.1. Now consider a flow f' extending f to $\vec{G}$. f' has some flow value x on the edge marked x in Figure 5.1. But the vertex with flow a_1 and x in must have the same flow out. Since there is only one edge out, it must have value $x + a_1$.

In fact, we may continue this process until we obtain all the values of f' (in terms of x). They are $x + \sum_{i=1}^{j} a_i$ (where the sum may be empty). Let $b_j = \sum_{i=1}^{j} a_i$ (so $b_0 = 0$). If f' is a nowhere-zero flow, x must take some value in $\mathbb{Z}_5$. This happens if there is a solution to

$$\begin{aligned} x + b_0 &\neq 0 \\ x + b_1 &\neq 0 \\ x + b_2 &\neq 0 \\ &\vdots \\ x + b_{n-1} &\neq 0 \end{aligned}$$

But this system has a solution precisely if the set $\{b_j\}_{j=0}^{n-1}$ is missing a number in $\mathbb{Z}_5$. This is since if y is missing, we may let $x = -y$ and similarly, if f' is a nowhere-zero flow with some value x on that edge, then $y = -x$ is missing from that list of numbers.

So $\mathcal{C}$, the set of partial flows that extend to G, are the flows with $\{b_j\}_{j=0}^{n-1}$ missing at least one number.

For example, the flow $(a_1, \ldots, a_6) = (1, 2, 3, 3, 2, 4)$ has $(b_0, \ldots, b_5) = (0, 1, 3, 1, 4, 1)$ which is missing the number 2. Thus this flow extends to the cycle of length 6 (show in Fig 5.2). However, the flow $(a_1, \ldots, a_6) = (1, 2, 3, 3, 3, 3)$ has $(b_0, \ldots, b_5) = (0, 1, 3, 1, 4, 2)$ does not extend (since all 5 numbers appear).

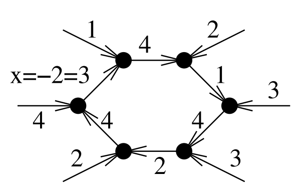


Figure 5.2: A flow extending to the cycle of length 6.

5.2 Cycle with a contracted edge

Suppose we now contract an edge of G in the previous section to obtain G'. If we have contracted the edge with weight $x + b_j$ then we have effectively removed the requirement that $x + b_j$ is non-zero. By rotation, we may assume that $j = n - 1$.

Therefore, the partial flows that extend to G' are those missing at least one number in $\{b_i\}_{i=0}^{n-2}$.

The flows $\mathcal{C}'$ that extend to G' but not to G are those where all numbers appear in $\{b_i\}_{i=0}^{n-1}$ but the number b_{n-1} appears exactly once (as the $(n-1)$th element).

Therefore, if a universal equation of the following form exists for all R, then we have proven the 5-flow conjecture.

$$\sum_{c \in \mathcal{C}} y_c K_c = \sum_{c \notin \mathcal{C}} y_c K_c$$

with $y_c \geq 0 \ \forall c \notin \mathcal{C}$ and $y_c > 0 \ \forall c \in \mathcal{C}'$ where C is the set of flows with $\{b_i\}_{i=0}^{n-1}$ missing at least one number and $\mathcal{C}'$ is the set of flows with all numbers appearing in $\{b_i\}_{i=0}^{n-1}$ but b_{n-1} appears only once.

5.3 Generating kernel vectors

In this section, we describe how to obtain "formulae" (as a function of $|R|$) for vectors which appear in the kernel of Kochol's matrix M_R.

We will always orient the edges of R away from the marked vertex in this section. Suppose $|R| = n$.

Again, we write a partial flow on R as a vector $(a_1, \ldots, a_n)$ of values on the edges of R.

From a vector in the kernel of M_R, we can obtain a vector in the kernel of $M_{R'}$ where R' is R with 2 added edges. This can be done as follows.

Theorem 5.3.1. *[13] Let R be an edge cut and R' is R with 2 added edges.*

Let $\mathbf{y} \in \ker(M_R)$. i.e., $\sum_f y_f K_f = 0$ is a universal equation.

For each column vector indexed by a partial flow $f = (a_1, \ldots, a_n)$ on R with coefficient y_f, we replace it with the term $y_f \sum_{s=1}^{4} K_{(a_1,\ldots,a_n,s,5-s)}$. ie

$$\sum_f y_f \sum_{s=1}^{4} K_{(a_1,\ldots,a_n,s,5-s)} = 0$$

is a universal equation.

We may choose any 2 coordinate to extend on rather than the least two coordinates.

This allows us to start with a vector in the kernel of M_R when $|R| = 4$ and obtain a vector for each $M_{R'}$ where $|R'|$ is even.

Unfortunately, we are not able to prove the existence of a kernel vector of the form described in the previous section.

Chapter 6

Programs

We have written a set of programs in order to obtain some computational results. These programs are freely available at `www.math.uwaterloo.ca/~z14li/nzf/`.

6.1 Required programs

- Python 2.4.3 (free, see `www.python.org`). May also work with earlier versions.
- GNU Bash 3.1.17 (free, see `ftp://ftp.gnu.org/pub/bash/`). May also work with earlier versions.
- GCC 4.0.3 (free, see `gcc.gnu.org`). May also work with earlier versions or other C compilers.
- One of either:
  - lp_solve 5.5 (free, see `lpsolve.sourceforge.net/5.5/`). Earlier versions may also work but it needs to understand free variables.
  - CPLEX 10

6.2 Sample run

In this section, we show how we obtain the equality in Section 1.5.1 and the inequality in Section 1.5.2.

6.2.1 Inequalities example

We choose the group to be $\mathbb{Z}_2^2$ and view all flows as (non-proper) 3-edge-colourings where each colour class is a postman set (see Theorem 3.1.2).

We assume that the file `graphbd` contains the following

```
18
15
12
1 4 7,0 2 9,1 3 11,2 4 12,0 3 5,4 6 13,5 7 14,0 6 8,7 9 15,
1 8 10,9 11 16,2 10 17
12 3,13 5,14 6,15 8,16 10,17 11
```

This simply encodes the Birkhoff diamond dual with the vertices and edges labelled as in Fig 6.1. Note that the other end of a pendant edge is labelled but it is not a vertex.

We start by determining all flows which extend to the Birkhoff diamond dual. Running `python flowextnz22.py graphbd` gives the output

```
[[0, 2, 4, 5], [1, 3], []] ,
[[0, 1, 4, 5], [2, 3], []] ,
[[0, 2, 3, 5], [1, 4], []] ,
[[0, 1, 2, 5], [3, 4], []] ,
[[0, 5], [1, 2, 3, 4], []] ,
```

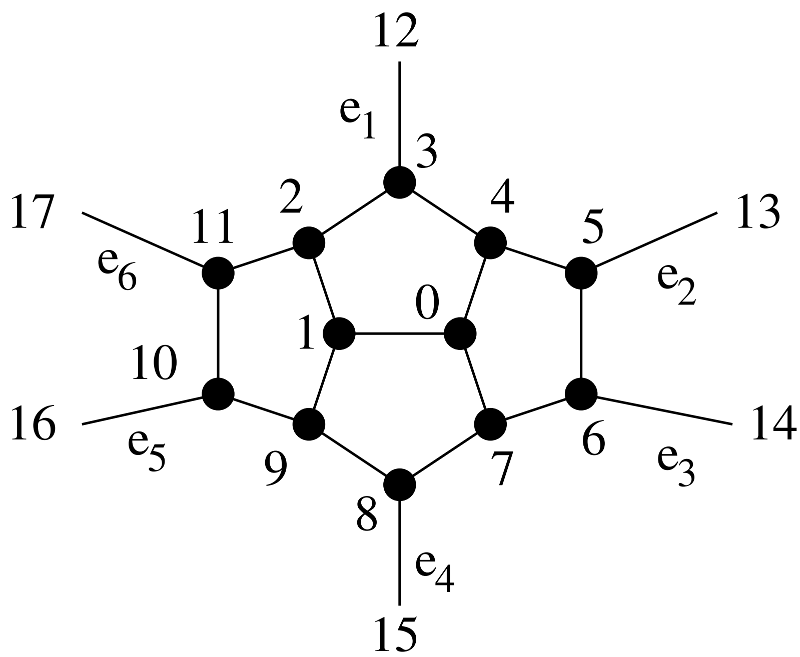


Figure 6.1: The Birkhoff diamond dual encoded by the file `graphbd`.

```
[[0, 5], [1, 4], [2, 3]] ,
[[0, 1, 3, 4], [2, 5], []] ,
[[0, 1, 2, 4], [3, 5], []] ,
[[0, 4], [1, 2, 3, 5], []] ,
[[0, 4], [1, 2], [3, 5]] ,
[[0, 3], [1, 2, 4, 5], []] ,
[[0, 3], [1, 2], [4, 5]] ,
[[0, 2], [1, 3, 4, 5], []] ,
[[0, 2], [1, 3], [4, 5]] ,
[[0, 1], [2, 3, 4, 5], []] ,
[[0, 1], [2, 5], [3, 4]] ,
```

This is the set of flows $\mathcal{C}$.

Flows are given as partitions of the edges of R into 3 sets: the ith set represents the edges of the ith colour. For example, the first line tells us that the flow where edges e_1, e_3, e_5, e_6 are coloured 1 and edges e_2, e_4 are coloured 2 extends to the Birkhoff diamond dual.

These flows are now also stored in the file `listcol` in python's "pickle" format.

We now generate the LP for $\mathcal{C}$ by running `python matching.py -p -s=6 -nm -c`. The flags tell the program to generate the LP in CPLEX format for the planar case where $|R| = 6$ and to use inequalities only (use `-l` instead of `-c` here to generate the LP in lp format (for `lp_solve`)).

The inequalities in Corollary 3.1.13 can be seen in the file `primallp`. The first few lines are

```
k0-m0<0
k0-m1-m2-m3-m4-m5<0
k0-m1-m2-m3-m4-m5<0
m0-k0<0
m1-k0<0
m2-k0<0
m3-k0<0
m4-k0<0
m5-k0<0
m1-k0<0
m2-k0<0
m3-k0<0
m4-k0<0
m5-k0<0
```

All inequalities are non-strict (although the `<` symbol appears instead of $\leq$).

Note that the variables are indexed by number. To retrieve the corresponding index in terms of signed matchings, colourings and colours, we may consult the file `varlist`. For example, it contains the lines

```
KVARLIST
0 [[0, 1, 2, 3, 4, 5], [], []]
```

and

```
MVARLIST
0 [][]
```

which tells us that the first inequality in `primallp` is `k[[0, 1, 2, 3, 4, 5], [], []]<m[][]` (the K variable indexed by the colouring where all edges are coloured 1 is bounded by the m variable indexed by the empty matching (and any non-1 colour)).

What we really want is to solve the dual to find an inequality needed to prove reducibility. So we run `./cplexsolvedual` (we would run `./lpssolvedual` with `lp_solve` here). This creates the file `reducible` which contains

```
cy0
cy1
cy5
cy9
cy10
cy11
cy15
cy16
cy18
cy20
cy21
cy24
cy25
```

```
cy29
cy30
```

This tells us that all the 3-edge-colourings listed above are reducible by $\mathcal{C}$. This happens to be all the colourings in the complement (see the file `ccomplement`). We may obtain a certificate by adding the above constraints to the dual LP. We do this by running `./cplexcert` (this would be `./lpscert` in the case we are using `lp_solve`).

```
cy0          1.000000
cy1          1.000000
cy2        -18.000000
cy3        -15.000000
cy4         -3.000000
cy5          1.000000
cy6        -11.000000
cy7         -9.000000
cy8         -5.000000
cy9          1.000000
cy10         1.000000
cy11         1.000000
cy12        -1.000000
cy13        -6.000000
cy14        -7.000000
cy15         1.000000
cy16         1.000000
cy17        -7.000000
cy18         1.000000
cy19       -15.000000
cy20         1.000000
```

```
cy21                    1.000000
cy22                   -1.000000
cy24                    1.000000
cy25                    1.000000
cy26                   -7.000000
cy28                   -5.000000
cy29                    1.000000
cy30                    1.000000
```

is the certificate we obtain. This is exactly Equation (1.5). The LP which yield this solution is now stored in the file `duallpcert`.

6.2.2 Equalities example

There is only one group of order 5, it is $\mathbb{Z}_5$.

We assume that the file `graph5cycle` contains the following

```
10
5
5
1 4 5,0 2 6,1 3 7,2 4 8,0 3 9
5 0,6 1,7 2,8 3,9 4
```

This simply encodes the 5 with the vertices and edges labelled as in Fig 6.2. Note that the other end of a pendant edge is labelled but it is not a vertex. This is simply to indicate the direction of the flow (the direction of the other edges need not be predetermined).

We run the meta-script `./metameta 5ring 5 5 1`

The matrix is written to the file `lookup`. The first few lines are

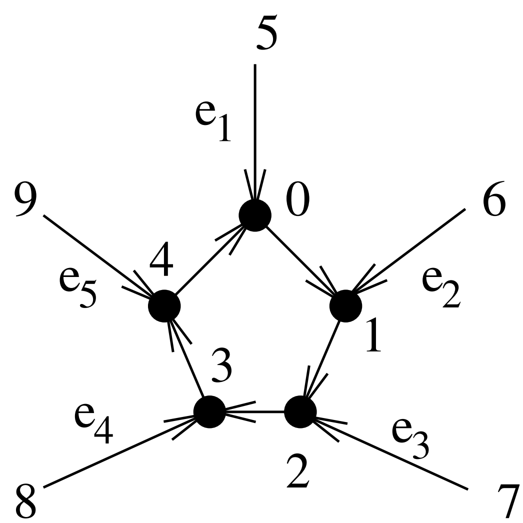


Figure 6.2: The 5-cycle encoded by the file `graph5cycle`.

```
11213 1 0 0 0 1 1 0 0 1 0 0
21113 1 0 0 0 0 1 1 0 1 0 0
32313 1 0 0 1 0 0 0 1 1 0 0
12113 1 0 0 0 1 0 1 0 1 0 0
23313 1 0 1 1 0 0 0 0 1 0 0
33213 1 0 1 0 0 0 0 1 1 0 0
11222 1 0 0 1 0 0 1 0 1 0 0
21122 1 0 0 0 1 0 0 1 1 0 0
32322 1 0 1 0 0 1 0 0 1 0 0
12122 1 0 1 0 0 1 0 0 1 0 0
```

This is the transpose of the matrix seen in Section 2.2.2. The first column is the flow which indexes that particular row. The remain columns are the columns of the matrix.

The flows extending to the 5 ring are stored in the file `gvect`. The first few lines are

```
11134
```

```
11143
11224
11233
11314
11323
11332
11341
11413
11422
```

Each line contains one flow on the pendant edges. It is the flow value on $e_1, \ldots, e_5$. This is the set $\mathcal{C}$.

The file `gbs5ringfe` contains

```
Good: [0]
Good: [1]
Good: [2]
Good: [3]
Good: [4]
```

which indicates that the 5 ring reduces to any graph that is the 5 ring with one edge deleted. Looking at `badvect`, we see that it is empty which indicates that $\mathcal{C}$ reduces all flows and thus the 5-cycle can be reduced to any graph (such as the 5-star we chose in Section 1.5.1).

We can obtain a certificate for reducibility by running `./getcert 5ring`. The following certificate is obtained.

```
x1                                8.000000
```

y2	16.000000
x41	8.000000
x65	1.000000
x66	1.000000
x67	1.000000
x68	1.000000
x80	16.000000
y83	16.000000
x87	1.000000
x91	1.000000
y92	16.000000
x95	1.000000
x99	1.000000
x141	1.000000
x145	1.000000
x149	1.000000
x153	1.000000
x158	1.000000
x162	1.000000
y163	8.000000
x166	1.000000
y168	8.000000
x170	1.000000
x175	1.000000
x180	1.000000
x183	1.000000
x188	1.000000
x191	1.000000
x194	8.000000
x196	1.000000

```
x199                          1.000000
x204                          1.000000
```

Here the coefficient of K_i is $x_i - y_i$. We may look up the indices in `lookupnum`. After combining terms belonging to the same equivalence class (with respect to multiplication by elements in $\mathbb{Z}_5^*$), we obtain Equation (1.4).

6.3 Quick Start

6.3.1 $\mathbb{Z}_2^2$-flows

Run

```
python flowextnz22.py graphbd
python matching.py -c
./cplexsolvedual
./cplexcert
```

6.3.2 $\mathbb{Z}_n$-flows

Run

```
./metameta 113 6 5 4
./getcert 113
```

6.4 Matching.py

`matching.py` - Used to generate the primal and dual LP and other related data for nowhere-zero $\mathbb{Z}_2^2$-flows.

6.4.1 Synopsis

```
python matching.py [OPTION]
```

6.4.2 Description

- **OPTIONS**

  -np, –non-planar Allow all graphs and Kempe chains including non-planar ones. Used for the nowhere-zero 4-flow conjecture.

  -p, –planar Default. Allow only planar graph and Kempe chains (so they cannot cross). Used for the Four Colour Theorem.

  -f, –tofiles Default. Write output to various files (see FILES section).

  -o, –tostdout Default. Write all output to screen (stdout).

  -k, –usechains Default. Use Kempe chain inequalities.

  -nk, –nochains Do not use Kempe chain inequalities.

  -m, –usematrix Default. Use matrix kernel equalities.

  -nm, –nomatrix Do not use matrix kernel equalities.

  -r, –readlistcol Default. Read list of colours $\mathcal{C}$ from file **listcol** and set corresponding variables in the primal and dual LP.

  -h, –hardcodedcol Do not read list of colours $\mathcal{C}$ from file. Read the list from the one hardcoded into the program (currently the empty list). Set the corresponding variables in the primal and dual LP.

  -c, –cplexformat Writes the output LPs in CPLEX (lp) format.

  -l, –lpformat Default. Writes the output LPs in LP format. Can be used with `lp_solve` for example.

  -s=SIZE, –size==SIZE Default is 6. Sets the number of edges in the cut (incident to the marked vertex) to SIZE.

- **FILES**

  If the program is writing to files (default behaviour) then the following files are (over)written.

  primallp Basic set of constraints for the primal LP.

  duallp Basic set of constraints for the dual LP.

  primallpsetcols Set k variables with index in $\mathcal{C}$ to 0.

  duallpsetcols Set cy variables with index in $\mathcal{C}$ to be free.

  matrices Matrix M_i where i is the ring size. RREF form of the matrix and row operations performed.

  matrixkernel Primal constraints for the vector $\mathbf{k}$ to be in span(M_i).

  varlist Look-up table for lists of variables (to convert from number to index).

  planarparts List of all planar partitions (basic graphs) for this ring size.

  ccomplement List of indices not in $\mathcal{C}$.

- **VARIABLES INDICES**

  Ring edges are labelled from 0 to $i-1$.

  Edge colours are labelled 0, 1 and 2 for the 3 non-zero elements of $\mathbb{Z}_2^2$ (i.e., 0 does not mean the element 0 of $\mathbb{Z}_2^2$)

  Edge colouring (up to permutation of the colour classes) of the ring are a list of 3 lists. The p list is the list of edges coloured p. Eg: [[0,1],[2,5],[3,6]] means that edges 0 and 1 are coloured 0, edges 2 and 5 are coloured 1 and edges 3 and 6 are coloured 2.

  0 is always in the first list. The smallest number not appearing in the first list is always in the second list.

  Signed matchings are represented by 2 lists: a list of edges (denoted by a list of 2 vertices) and a list of signs. The pth element of the first list has sign corresponding to the pth element of the second list. Eg:

[[0,3],[1,2]][-1,1] mean the signed matching where 0 is matched to 3 with sign -1 and 1 is matched to 2 with sign 1.

To avoid enumerating the same matching twice, the smallest number is always in the first list, the remaining smallest number is always in the second list, etc.

- **VARIABLES**

  Variables in the various LP files are labelled as follows.

  k Indexed by an edge colouring (up to permutation of the colour classes) of the ring.

  m Indexed by a signed matching and a colour.

  d Coefficients for inequalities of the form (3.1)

  f Coefficients for inequalities of the form (3.2)

6.5 flowextnz22.py

`flowextnz22.py` - Used to generate list of $\mathbb{Z}_2^2$-flows which extend to a cubic graph.

6.5.1 Synopsis

```
python flowextnz22.py [FILE]
```

6.5.2 Description

`flowextnz22.py` reads the graphfile (see Section 6.8 for file specifications) FILE and outputs all $\mathbb{Z}_2^2$-flow which extend to this graph.

Writes the result to standard output as well as the file **listcol** which is overwritten. **listcol** stores a list in Python's "pickle" format.

6.6 flowextn.py

`flowextn.py` - Used to generate list of $\mathbb{Z}_n$-flows which extend to a graph.

6.6.1 Synopsis

```
python flowextn.py [FILE] [FLOWFILE] [MODE]
```

6.6.2 Description

`flowextn.py` reads the graphfile FILE and the list of flows from FLOWFILE (see Section 6.8 for file specifications).

If MODE is 0, it outputs all $\mathbb{Z}_n$-flow in FLOWFILE which extend to this graph.

If MODE is greater than 0, it tries to delete all combinations of MODE edges from the graph and outputs the graphs which extend all flows in FLOWFILE.

6.7 metameta

`metameta` - High level script for $\mathbb{Z}_n$-flows.

6.7.1 Synopsis

```
./metameta [FILE] [SIZE] [FLOW] [DELETE EDGES]
```

6.7.2 Description

`metameta` reads from the graphfile graphFILE (eg: `./metameta bd` reads from the file `graphbd`) which has a cut of size SIZE (number of edges incident to the marked vertex) and computes the $\mathbb{Z}_{\text{FLOW}}$ flows which extend to it and tries to reduce the graph to a smaller graph by deleting edges. It also tries to see if adding planarity helps.

6.8 Graph file

The graphs that the programs take as input are stored in "graph files" which are formatted as follows.

The file consist of 5 lines.

Line 1 contains a single number, the number of vertices in the graph. Here, the other side of a pendant edge counts as a vertex.

Line 2 contains a single number, the number of non-pendant edges in the graph (number edges excluding those incident to the marked vertex).

Line 3 contains a single number, the number of vertices in the graph. Here, the other side of a pendant edge does not counts as a vertex.

Line 4 contains a comma separated list of lists. Each of the list (in the list) is an adjacency list. Elements (neighbours) in an adjacency list is separated by spaces. The ith list corresponds to the adjacency list of the ith vertex.

Line 5 contains the pendant edges. The list of edges is comma separated. Each edge consist of 2 elements (the ends of the edge) which are separated by a space. The edge is directed from the first element to the second element.

Chapter 7

Conclusion

We have developed some algebraic methods for proving reducibility in nowhere-zero flows, analyzed the strength of some of these methods and shown how other reducibility proofs, namely Kochol's matrix and Kempe chains, fit into this scheme. We were able to compare some of these methods with respect to flow reducibility.

In the case where the equalities and inequalities are linear, we haven shown that these methods can provide a certificate which we can check.

However, many questions remain open.

7.1 Open questions and future work

1. Throughout our analysis, we were not able to obtain a complete proof of k-flowability (for all bridgeless graphs) for any k. It would we interesting to know if such a proof exists using these methods. If this was not possible, we would like to know if it is possible to obtain a "reducibility" type proof that all bridgeless graphs are k-flowable. We believe that this second task is possible and that it is only a matter of formulating the 6-Flow Theorem [16] or the 8-Flow Theorem[7] into an inductive proof and turning that proof by induction into a reducibility proof.

2. In Section 3.1.7, we noted some problems which arise when we attempt to use a Kempe chain argument to generate inequalities for groups of order $k \neq 4$. Still, we do not know if such problems actually prevent us from deriving any inequality in the case $k \neq 4$. More generally, is it possibly to obtain inequalities in the case $k \neq 4$ that cannot be obtained from universal equalities?

3. In Section 4.3, we have shown that we obtained all equalities using the argument in Section 2.2. However, we do not have a similar result in the case of inequalities case. Can we obtain all inequalities from equalities and Kempe chain arguments?

4. We know that there are cases where the inequalities argument is strictly stronger than the equalities argument (for example for the case $|R| = 5$, $k = 4$ where the Kernel of the M_R matrix is empty. Can we also show that the equalities argument can prove flow reducibility in some cases where the inequalities argument would not suffice?

   This is equivalent to asking whether the combined inequalities and equalities argument derived in Section 4.1 is strictly stronger than the inequalities argument alone.

5. Assuming that we could obtain all universal inequalities, is the space of intersection of all these half-planes polyhedral?

6. One possible application of these algebraic methods is toward finding a shorter proof of the Four Colour Theorem. However, to do so, we need that the combined inequalities and equalities be strictly stronger than inequalities alone (see Question 3). Section 4.4 suggests that this will not be possible. If we could prove that inequalities were stronger for cuts of size at most 13, we would know that the equalities would not be useful for proving the Four Colour Theorem. This is since the current proof only reduces graphs with these cut sizes.

7. In this thesis, we have obtained homogeneous linear equalities and inequalities which are universal. Are there other types of equalities and inequalities which are also universal? We would not expect expressions which are non-homogeneous since it seems easy to multiply the number of flows of a graph by adding a disjoint component to the graph.

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