Forays into Mathematical Physics

by

Jonathan Hackett

A thesis
presented to the University of Waterloo
in fulfillment of the
thesis requirement for the degree of
Master of Science
in
Physics

Waterloo, Ontario, Canada, 2007

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Abstract

Two different works in mathematical physics are presented:

A construction of conformal infinity in null and spatial directions is constructed for the Rainbow-flat space-time corresponding to doubly special relativity. From this construction a definition of asymptotic DSRness is put forward which is compatible with the correspondence principle of Rainbow gravity. Furthermore a result equating asymptotically flat space-times with asymptotically DSR spacetimes is presented.

An overview of microlocality in braided ribbon networks is presented. Following this, a series of definitions are presented to explore the concept of microlocality and the topology of ribbon networks. Isolated substructure of ribbon networks are introduced, and a theorem is proven that allows them to be relocated. This is followed by a demonstration of microlocal translations. Additionally, an investigation into macrolocality and the implications of invariants in braided ribbon networks are presented.

Acknowledgements

I would like to thank Lee Smolin, Fotini Markopoulou-Kalamara, Sundance Bilson-Thompson and Louis Kauffman for their guidance and discussions. I would also like to thank both the University of Waterloo and the Perimeter Institute for Theoretical Physics for their support of this work.

Dedication

This is dedicated to my fiancee Sonia and to my family.

Contents

List of Figures

> It seems likely that this process of increasing abstraction will continue in the future and that advance in physics is to be associated with a continual modification and generalisation of the axioms at the base of the mathematics rather than with a logical development of any one mathematical scheme on a fixed foundation.
>
> The theoretical worker in the future will therefore have to proceed in a more indirect way. The most powerful method of advance that can be suggested at present is to employ all the resources of pure mathematics in attempts to perfect and generalise the mathematical formalism that forms the existing basis of theoretical physics, and after each success in this direction, to try to interpret the new mathematical features in terms of physical entities...

- Paul A.M. Dirac

Chapter 1

Introduction

Presented here are two very different forays into mathematical physics. Both of the papers were developed within a different mathematical framework, and both of the frameworks represents a different attempt at making progress in theoretical physics. As these frameworks fall into rather different fields and so to prevent a comprehensive unified introduction to both of them would be impossible. Instead, an overview of the principles of mathematical physics are presented to allow the reader to see the underlying principles that led to the research.

The two quotes presented before the introduction are from [3] and present a scheme under which the work presented within falls. Though Dirac's scheme is by no means accepted as the foundation of modern mathematical physics, it is an exceptionally insightful description of the dilemma faced by modern mathematical physicists. There is a temptation to use the strength of modern mathematics to push the already developed physical theories to new limits. The works presented here are not examples of such vehicles, they are examples more in tune with Dirac's vision of the field.

The first paper presents work done on Rainbow Gravity, an extension of Doubly Special Relativity to a general theory. Though Rainbow Gravity does not immediately present itself as the latest frontiers of mathematics, it is based upon work in Doubly Special relativity and provides an important comparison point for the work done in the larger field. Meanwhile, Doubly Special Relativity and its related works are using recent advances in mathematics - such as quantum group theory and non-commutative geometry - to form the foundations of their theories.

The other paper presents work done in Braided Ribbon Quantum Gravity. This paper follows Dirac's description of mathematical physics: it introduces new mathematics, and then proceeds to relate this mathematical advancement to a physical phenomenon, that of translations of particles.

Chapter 2

Asymptotic Flatness in Rainbow Gravity[1]

2.1 Introduction

The idea of a fundamental length scale has emerged in multiple approaches to quantum gravity. This length typically identified as the Planck length l_p, is expected to be the scale at which quantum gravitational corrections to our present theories would be required. However, the idea that when probing below the Planck length we will require new physics to describe the resultant phenomena is in direct contradiction with special relativity. How can we reconcile the idea of a fundamental length scale when special relativity allows lengths to contract? This apparent paradox of quantum gravity is what gave the impetus behind the original work in doubly special relativity (DSR) [4, 5, 6, 7]. Doubly special relativity fixes the planck length and attempts to occupy the position of a flat space-time limit of quantum gravity.

Recent work on Doubly special relativity has been spurred on by current experiments that could provide a fertile testing ground for its results. Experiments such as GLAST (The Gamma Ray Large Area Telescope) and AUGER (a cosmic ray detector array) [8, 9] provide the opportunity to test the GZK cutoff and the constancy of the speed of light, both of which are subjects which DSR is capable of making predictions for. The other reason for the increased excitement in DSR is the possibility of it providing increased insight into quantum gravity. The ingredients that lead to DSR are not dependent upon any particular attempt towards a quantum theory of gravity and in fact are based solely upon attempting to combine the ideas of special relativity and a fundamental length scale. Due to the simplicity of its construction, it is possible that DSR could provide not only hints into the structure of space-time in a complete theory of quantum gravity, but that it could place restrictions upon one as well.

The picture of space-time resulting from DSR is in some ways still an open question. Some approaches to DSR (particularly those that involve Hopf algebras) have resulted in a picture of space-time which is non-commutative, these approaches though interesting have yet to present any physical predictions, primarily due to the difficulty in construction large-scale behaviour from non-commutative spacetimes. Fortunately there is an alternative to non-commutative geometry as the arena in which the space-time of DSR is understood, by attempting to extend DSR to general relativity Magueijo and Smolin put forward Rainbow gravity. In Rainbow gravity, the need for non-commutative geometry is avoided by introducing a metric which 'runs' with respect to the energy at which it is probed. Not only does this approach avoid the need for non-commutative geometry to describe the space-time of DSR, it allows DSR to be extended to a theory of gravity much like general relativity.

The intent of this paper is to explore the implications of Rainbow gravity further. Of particular interest is the asymptotic behaviour of the theory as this is an area where other approaches to DSR have not yet produced results. To this end we will investigate the ideas of conformal infinity and asymptotic DSRness (in analogy with asymptotically flat space-times in General relativity) in Rainbow gravity.

2.2 Rainbow Gravity

Rainbow gravity [10] is an attempt at construction an extension of DSR into a general relativity framework which has at its foundation the proposal that the geometry of a space-time 'runs' with the energy scale at which the geometry is being probed. The implication of this is that the metric of a space-time becomes energy dependant. Fortunately a natural proposal for the manner in which the metric could vary with respect to energy emerged from previous work on DSR [11], which is the form that we shall use. In this sense we shall refer to as a 'rainbow metric' $g_{ab}(E)$ the family of fundamental forms parameterized by the energy of the particle probing it.

Rainbow gravity is governed by two principles, the correspondence principle and the modified equivalence principle. These two principles are stated as follows [10]

Correspondence principle
In the limit of low energies relative to the Planck Energy, standard general relativity is recovered. That is for any rainbow metric $g_{ab}(E)$ corresponding to a standard metric g_{ab} the following limit holds true.

$$\lim_{\frac{E}{E_{pl}} \to 0} g_{ab}(E) = g_{ab} \tag{2.1}$$

Modified Equivalance principle
Given a region of space-time with a radius of curvature R such that

$$R >> E_{pl}^{-1} \tag{2.2}$$

then freely falling observers measure particles and fields with energies E observe the laws of physics to be the same as modified special relativity to first order in $\frac{1}{R}$ so long as:

$$\frac{1}{R} << E << E_{pl} \tag{2.3}$$

where E_{pl} is the Planck energy. Thus they can consider themselves to be inertial observers in a rainbow flat space-time (to first order in $\frac{1}{R}$) and use a family of energy dependent orthonormal frames locally given by

$$e_0 = f^{-1}(\frac{E}{E_{pl}})\tilde{e}_0 \tag{2.4}$$

$$e_i = g^{-1}(\frac{E}{E_{pl}})\tilde{e}_i \tag{2.5}$$

with a metric

$$g(E) = \eta^{ab} e_a \otimes e_b \tag{2.6}$$

We shall assume the existence of these two functions $f(E)$ and $g(E)$ which are strictly greater than zero for small values of E; the small range of restriction is to allow for the possibility that at significantly greater energies the geometry of space-time could take on a significantly different character. The implications of this assumption will be explored further in the context of asymptotic flatness.

One way to satisfy these principles is to require that the rainbow metric for any space-time actually be a family of metrics given by energy-dependent orthonormal frame fields - as presented above - which must satisfy a 'Rainbow Einstein equation'

$$G_{\mu\nu}(E) = 8\pi G(E) T_{\mu\nu}(E) + g_{\mu\nu}\Lambda(E) \tag{2.7}$$

where Newton's constant and the cosmological constant are now allowed to vary with the energy so long as they obey the correspondence principle.

This is the form of Rainbow gravity which will be used to study the idea of asymptotic DSRness in the following sections. It should be noted that conformal mappings of rainbow gravity space-times pointwise with respect to the energy (at specific energies instead of treating the energy as a dimension) are possible due to the similarity between the Rainbow Einstein Equations and the original Einstein equations. All Rainbow metrics are actually solutions to Einstein's equations in a mathematical sense (treating the functions solely as mathematical concepts, instead of allowing them to correspond to physical quantities) with the caveat that for energies where $G(E)$ varies from Newton's constant that the equation is slightly modified, but not in a manner which would impact the behaviour of solutions under conformal mappings, nor their compactifications.

2.3 Conformal infinity in Rainbow Minkowski space-times

In order to understand the asymptotic behaviour of space-times in Rainbow gravity it is useful to have a consistent manner in which to evaluate quantities 'infinitely far away' in null and spatial directions. Additionally in order to be able to ascribe the title of 'asymptotically' DSR to a space-time we need to be able to identify the behaviour of the flat DSR space-time at these asymptotic locations. To do this we shall extend the ideas of $\mathcal{I}$ and $\imath_0$ (the concepts of null and spatial infinity) from asymptotic flatness to DSR, proceeding in a similar manner to previous demonstrations of these completions of minkowski space-time. [12]

We begin by considering the components of the metric of the flat DSR spacetime (we shall call this the deformed Minkowski space-time or Rainbow space-time) g_{ab}

$$ds^2 = \frac{-dt^2}{f^2(E)} + \frac{dr^2}{g^2(E)} + \frac{r^2}{g^2(E)} d\Omega^2 \tag{2.8}$$

Where $d\Omega^2$ is the angular component of the spatial directions. By setting this equal to zero (and likewise setting the angular component to zero) we are able to identify the speed of light as a function of the energy:

$$c = \frac{dr}{dt} = \frac{g(E)}{f(E)} \tag{2.9}$$

It is interesting to note that this will allow c to be non-constant. This allows us to construct (energy dependent) null co-ordinates given by

$$v = t + \frac{f(E)}{g(E)} r \tag{2.10}$$

$$u = t - \frac{f(E)}{g(E)} r \tag{2.11}$$

and change the metric accordingly to:

$$ds^2 = \frac{1}{f^2(E)} \left(-dv du + \frac{1}{4} (v - u)^2 d\Omega^2 \right) \tag{2.12}$$

At this point we see that the only difference between this metric and the minkowski space-time metric in null co-ordinates is a factor of $\frac{1}{f^2(E)}$. This means that we can perform a conformal mapping of this space-time into a restriction of the Einstein static universe by using a conformal factor of

$$\Theta^2 = \frac{4}{f^2(E)\,(1 + v(E)^2)\,(1 + u(E)^2)} \tag{2.13}$$

such that the new metric $\bar{g}_{ab}$ is related to the old by

$$\hat{g}_{ab} = \Theta^2 g_{ab} \tag{2.14}$$

The only restrictions being that

$$\frac{1}{f^2(E)} \neq 0 \tag{2.15}$$

and that

$$g(E) \neq 0 \tag{2.16}$$

for all E. This mapping is made clear by choosing new co-ordinates of:

$$T(E) = \tan^{-1}(v(E)) + \tan^{-1}(u(E)) \tag{2.17}$$
$$R(E) = \tan^{-1}(v(E)) - \tan^{-1}(u(E)) \tag{2.18}$$

The ranges of the new co-ordinates being

$$-\pi < T(E) + R(E) < \pi \tag{2.19}$$
$$-\pi < T(E) - R(E) < \pi \tag{2.20}$$
$$0 <= R \tag{2.21}$$

and the components of the new metric being

$$\hat{ds}^2 = -dT^2 + dR^2 + \sin^2(R)d\Omega^2 \tag{2.22}$$

From here we are able to extend the original space-time to the boundary of the larger space-time to yield an identification of the 'infinity' of the deformed minkowski space-time as follows:

Future Null infinity($\mathcal{I}^+$) is identified with $T(E) = \pi - R(E)$ for $0 < R < \pi$
Past Null infinity($\mathcal{I}^-$) is identified with $T(E) = -\pi + R(E)$ for $0 < R < \pi$
Spatial infinity($\imath^0$) is identified with $R(E) = \pi$, $T(E) = 0$.

We therefore now have an identification of conformal infinity of the deformed minkowski space-time with two reasonably physical restrictions given by equations 2.15 and 2.16. This allows us to examine asymptotic properties, and additionally examine the concept of asymptotic DSRness in curved space-times.

2.4 Asymptotically DSR space-times

We now wish to build on the identification of $\mathcal{I}^+$, $\mathcal{I}^-$, and $\imath^0$ for deformed Minkowski space-time by using them to define an 'asymptotically DSR space-time'. To do this we shall rely on the definitions of asymptotically flat space-times [12, 13, 14] and expand them to incorporate the running metric of Rainbow gravity.

There is a concern however that as Rainbow gravity requires two as of yet unknown functions - $f(E)$ and $g(E)$ - that we cannot assume that our definitions of $\mathcal{I}^+$, $\mathcal{I}^-$, and $\imath^0$ hold for all values of E in the deformed minkowski space.

We therefore shall define an interval χ by

χ is the interval $(0, \varphi)$, where φ is the smallest value greater than zero such that when $\frac{E}{E_{pl}} = \varphi$, $f(E)$ or $g(E)$ fails to satisfy the restrictions given by equations 2.15 and 2.16.

We will only address the concept of asymptotic DSRness where $\frac{E}{E_{pl}} \;\epsilon\; \chi$ as our concepts of conformal infinity are ill defined outside of this interval. This corresponds to the fact that the functions $f(E)$ or $g(E)$ tend towards zero as the upper energy bound of DSR is reached. Such a restriction on the domain in which we can define asymptotic structures is natural given the intent of Rainbow Gravity.

Within the interval χ, we shall require that for a space-time to be considered asymptotically DSR at spatial infinity it must satisfy the following requirements (analogous to those of asymptotic flatness conditions [13, 14])

Definition 1 A 'rainbow spacetime' (M,$g_{ab}(E)$) is considered asymptotically DSR at spatial infinity if there exists a set of space-times defined by the parameter E ($\hat{M}(E)$,$\hat{g}_{ab}(E)$) where each space-time is smooth everywhere except at a point $\imath^0(E)$ where $\hat{M}$ is C^1 and $\hat{g}_{ab}$ is C^0, and that there exists an imbedding of $M(E)$ into its respective $\hat{M}$ satisfying:

(req. 1) The union of the closures of the causal future and causal past of $\imath^0(E)$ is equal to the complement of $M(E)$ in $\hat{M}(E)$, i.e.

$$\bar{J}^{+}(\imath^0(E)) \cup \bar{J}^{-}(\imath^0(E)) = \hat{M} - M \tag{2.23}$$

(req. 2) There exists a function Θ on $\hat{M}$ that is C^2 at $\imath^0(E)$ and smooth everywhere else satisfying:

$$\hat{g}_{ab} = \Theta^2 g_{ab} \tag{2.24}$$
$$\Theta(\imath^0) = 0 \tag{2.25}$$
$$\hat{\nabla}_a \Theta(\imath^0) = 0 \tag{2.26}$$
$$\hat{\nabla}_a \hat{\nabla}_b \Theta(\imath^0) = 0 \tag{2.27}$$

for all values of $\frac{E}{E_{pl}} \;\epsilon\; \chi$.

The motivation behind this definition is the desire for rainbow gravity to be consistent at each value of E. Given any single value of E within χ all standard rules of general relativity should apply and therefore the requirement for a spacetime to be asymptotically DSR should be that it be able to be mapped to the deformed minkowski space in a manner corresponding to the manner in which asymptotically flat spacetimes are mapped to minkowski space through the Einstein static universe. For a rainbow spacetime to be asymptotically DSR however, it must satisfy this requirement at all energies within the interval χ however as we desire a definition which is dependent upon the spacetime, not upon the specific energy at which it is being probed. It should be noted that this requires that only spacetimes which are asymptotically flat in the low energy limit can be asymptotically DSR.

Expanding this approach, we are able to expand the standard definition of an Asymptotically flat and Empty spacetime as follows:

Definition 2 A 'rainbow spacetime' (M,$g_{ab}(E)$) is considered asymptotically empty and DSR at null and spatial infinity if it is asymptotically DSR at spatial infinity (as defined above) and satisfies

> ***(req. 1)*** On the union of boundaries of the causal future and causal past of $\imath^0(E)$, $\Theta = 0$ and excepting $\imath^0(E)$, $\hat{\nabla}_a\Theta \neq 0$ on the same.
> ***(req. 2)*** There exists a neighbourhood $N(E)$ of the union of boundaries of the causal future and causal past of $\imath^0(E)$ in $\hat{M}(E)$ such that $(N(E), \hat{g}_{ab}(E))$ is strongly causal and time orientable, and in the intersection of $N(E)$ and the image of $M(E)$ in $\hat{M}$, $R_{ab}(E) = 0$ (where $R_{ab}(E)$ corresponds to the original physical metric $g_{ab}(E)$)
> ***(req. 3)*** The map of null directions at $\imath^0(E)$ into the space of inegral curves of $n^a = \hat{g}^{ab}\hat{\nabla}_a\Theta$ on $\mathcal{I}^+(\mathcal{E})$ and $\mathcal{I}^-(\mathcal{E})$ is a diffeomorphism.
> ***(req. 4)*** For a smooth function, ω, on the complement of $\imath^0(E)$ in $M\hat{(E)}$ with $\omega > 0$ on the union of the image of $M(E)$ in $M\hat{(E)}$ with $\mathcal{I}^+(\mathcal{E})$ and $\mathcal{I}^-(\mathcal{E})$ which satisfies $\hat{\nabla}_a(\omega^4 n^a) = 0$ on the union of $\mathcal{I}^+(\mathcal{E})$ and $\mathcal{I}^-(\mathcal{E})$ the vector field $\omega^{-1}n^a$ is complete on the union of $\mathcal{I}^+(\mathcal{E})$ and $\mathcal{I}^-(\mathcal{E})$.

for all values of $\frac{E}{E_{pl}}$ ϵ χ.

These two definitions provide a means of identifying whether those concepts that are defined through asymptotic behaviour (such as the energy of an isolated system) carry over to Rainbow gravity in a consistent manner.

2.5 Correspondence between asymptotic flatness and dsr

It is a natural question to inquire into whether an asymptotically flat space-time will - when extended to rainbow gravity - correspond to an asymptotically dsr space-time. This would seem to be likely, however it requires a proof in order to justify this inclination.

Theorem 1 Given any Rainbow gravity metric of the form

$$ds^2 = -\frac{A^2(t,r,\Omega)}{f^2(E)}dt^2 + \frac{B^2(t,r,\Omega)}{g^2(E)}dr^2 + \frac{r^2C^2(t,r,\Omega)}{g^2(E)}d\Omega^2 \qquad (2.28)$$

where Ω is shorthand for the solid angle of the two angular co-ordinates. If the metric is asymptotically flat in the limit as $\frac{E}{E_{pl}}$ goes to zero, the rainbow gravity metric is asymptotically flat within χ.

Proof As the metric is in a form with a radial spatial co-ordinate we can therefore find the speed of light by setting the distance and the change in angular components to zero. We therefore get

$$\frac{dr}{dt} = \frac{g(E)A(t,r,\Omega)}{f(E)B(t,r,\Omega)} \tag{2.29}$$

as the speed of light, and can therefore choose Energy dependent null co-ordinates $u(E)$ and $v(E)$ in the form of:

$$v = t + \frac{f(E)B(t,r,\Omega)}{g(E)A(t,r,\Omega)}r \tag{2.30}$$

$$u = t - \frac{f(E)B(t,r,\Omega)}{g(E)A(t,r,\Omega)}r \tag{2.31}$$

This changes the metric to the form:

$$ds^2 = \frac{-A^2(t,r,\Omega)}{f^2(E)}dvdu + \frac{C^2(t,r,\Omega)}{4B^2(t,r,\Omega)f^2(E)}\left(v-u\right)^2 d\Omega^2 \tag{2.32}$$

This is however just $\frac{1}{f^2(E)}$ times the null metric for the original asymptotically flat space-time but with energy dependant null co-ordinates. The fact that the co-ordinates are energy dependent is of no concern as the definition of asymptotic DSRness is pointwise with respect to energy and for each energy these co-ordinates are fixed. Likewise the functions $A(t,r,\Omega)$,$B(t,r,\Omega)$, and $C(t,r,\Omega)$ are still constant with respect to energy as they were constant with respect to the original co-ordinates. The construction of the conformal factor is identical to that used in the derivation of conformal infinity in DSR in section 2.3. We therefore find that we can construct a conformal factor $\Theta^2(E,t,r,\Omega)$ from the asymptotically flat space-time's factor (denoted by $\Theta^2(t,r,\Omega)$) by

$$\Theta^2(E,t,r,\Omega) = \frac{1}{f^2(E)}\Theta^2(t,r,\Omega) \tag{2.33}$$

which satisfies the requirements for the space to be asymptotically DSR.

2.6 Conclusion

The process of conformally mapping DSR into the Einstein static universe allowed for a natural recreation of the definitions of the conformal infinities and allowed the definitions for asymptotic DSR behaviour to follow from natural requirements. These definitions, though highly stringent in their point-wise nature are put forward as minimal (at least given the current knowledge of the functions $f(E)$ and $g(E)$) should Rainbow gravity undergo further refinement it could be possible that these restrictions could be relaxed based upon the running of the metric being 'well behaved'.

Though it is suspected that in general all asymptotically flat metrics will correspond to asymptotically DSR rainbow metrics, only a small result in that direction is presented herein with further work on extending the class of metrics for which this is true still in progress.

Chapter 3

Locality and Translations in Braided Ribbon Networks[2]

3.1 Introduction

In the last century, there have been repeated discoveries of underlying structure. Moving from macroscopic objects, to atoms, to components of the nuclei, to quarks, it has been demonstrated repeatedly that the differences between supposedly fundamental particles are, in fact, merely consequences of the composite structure of underlying reality. It only seems a natural progression that such an approach of looking for underlying structure be used to explain the particles of the standard model. Attempts towards this end, dubbed preon models,[16, 17, 18, 19, 20] met with many obstacles, but still there was something deeper that presented itself as a difficulty. The difficulty is that, as such a process does not have an end, we can continue to suppose that below the currently understood structure is another set of *more* fundamental particles. This idea quickly becomes unappealing at a philosophical level, or even a practical level, as the question then becomes "What could make it end?". The idea that the preons would be as fundamental as possible, such as those in [22], provides a way of achieving the desired end. One way to achieve this end is to suggest that the preons be composed of structure within space-time. This suggestion gains further appeal by its convergence with recent approaches to quantum gravity.

Such a preon model was recently proposed in [22] and then extended to the idea of quantum gravity in [23]. The idea of having a composite model of particle physics that is based upon topology in quantum gravity is appealing. The most obvious basis for its appeal is that such a theory may be viewed as progress towards a grand unified theory.

I shall investigate some features of this model and the topology of the structures that it introduces. Based on this I will discuss the evolution algebra of this theory, and demonstrate that translations of the large scale structures are a feature of the theory.

3.2 Braided Ribbon Networks

The theory of braided ribbon networks [23] is concerned with two-dimensional surfaces in a compact 3-manifold. These surfaces are composed of the unions of 'trinions' - intersections of three 'ribbons' - and are scored to divide the surface into clearly demarcated trinions (fig.3.1).

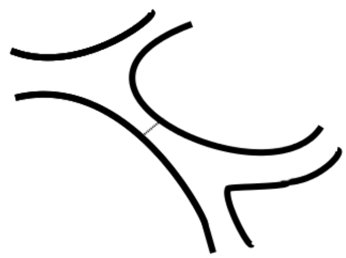


Figure 3.1: Two trinions with a scored ribbon

We allow the ribbons to be braided through the punctures in the surface, and we also allow the ribbons to be twisted by multiples of 2π (fig.3.2). This network evolves under $\mathcal{A}_{evol}$, the algebra generated by the elements A_1, A_2 and A_3 (fig.3.3). By viewing the trinions as nodes, we can consider the surfaces to be graphs. The theory is then similar to loop quantum gravity in its structure, though with some additional allowances for the labelings of the graph. We also note that a graph can be changed to a ribbon graph by 'framing' the edges of the graph: turning the one-dimensional edges into two-dimensional surfaces.

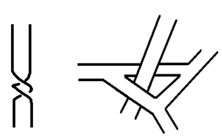


Figure 3.2: Twists and Braidings

The reduced link of a graph is taken by treating each edge of the ribbons to be a strand and then excluding unlinked unknotted strands. A subsystem is then defined as a section of the graph where its reduced link does not intersect the rest of the graph.

The first generation of the standard model is then proposed to be generated by placing 2π twists on the strands of the two crossing capped braid of three ribbons (fig.3.4), subject to the restrictions that all the twistings on a braid must be in the same direction.

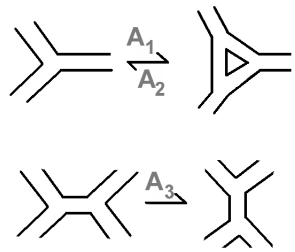


Figure 3.3: Generators of $\mathcal{A}_{evol}$

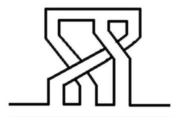


Figure 3.4: Proposed form of the first generation

3.3 Topology

In order to properly discuss the idea of a translation we must first discuss the topology with respect to which the translations shall occur. Ribbon networks present difficulties in this regard as there are several distinct classes of topologies. We begin by considering the topology inherent in the idea of neighbours from a graph theory perspective.

The Microlocal Metric Space

Consider a ribbon graph Γ consisting of N nodes and M ribbons having some braiding and twisting content. We construct a new metrical space $\tilde{\Gamma}$ as follows: let X be the set of trinions within the ribbon graph. We shall take each trinion x within X as a node in a pseudograph, and construct edges for this pseudograph in the natural way: by making an edge between two nodes if their respective trinions share a scored ribbon. This is the reverse of the framing process that can be used to construct a ribbon network.

The Microlocal Distance Function
Considering the set of all possible paths between two nodes on the pseudograph, the distance between the nodes is the minimum number of edges in any such path. This satisfies the four requirements for a distance function: that it is non-negative for any choice of two nodes, that it is strictly positive

for any two non-identical nodes, that it is reflexive and that it satisfies the triangle inequality. This metric is equivalent to the standard metric of graph theory.

Thus, the set X of nodes, along with the microlocal distance function, create a metric space and, therefore, have a standard topology $\mathcal{T}_1$ defined by the open balls given by the microlocal distance function on X. $\mathcal{T}_1$ is thus the induced graph topology of the graph Γ.

The microlocal topology surprisingly contains very little information about the structure of the ribbon network. We should therefore consider topologies that contain information about the braidings and twists of the ribbons.

Ribbon Topology
The *Ribbon Topology* is defined to be the topology corresponding to taking the ribbon network as a bounded two dimensional surface with a Euclidean metric. The Euclidean metric, together with the bounded space of the ribbons, then becomes a metric space. Again, the open balls generate the topology $\mathcal{T}_2$. In contrast to $\mathcal{T}_1$, $\mathcal{T}_2$ is able to differentiate between graphs like those in figure 3.5.

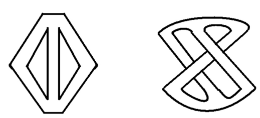


Figure 3.5: Networks that can be differentiated by $\mathcal{T}_2$

Braided Ribbon Topology
The *Braided Ribbon Topology* is defined to be any topology of the ribbon network that includes the braiding and the twisting of the ribbons. We shall call this topology $\mathcal{T}_3$. As the twisting is 'invisible' to anything living directly on the ribbons, this topology has to appeal to the higher space that the structure is embedded within. $\mathcal{T}_3$ would then be able to differentiate between graphs like those in figure 3.6. In the same way that we have referred to the 'microlocal' character of objects, we shall refer to the 'braided-local' character of objects.

If we consider $\mathcal{T}_1$, $\mathcal{T}_2$ and $\mathcal{T}_3$ to be topologies on the original network Γ, we can see that each successive topology is finer than the last, with $\mathcal{T}_3$ being the finest.

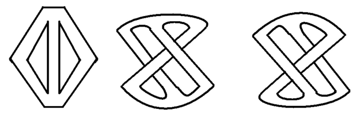


Figure 3.6: Networks that can be differentiated by $\mathcal{T}_3$

3.4 $\mathcal{A}_{evol}$ and Microlocal Translations

We shall now demonstrate that there are indeed translations of braided-local structures with respect to the microlocal distance function. Also, we shall demonstrate that even when using a more general notion of microlocal distance we can nonetheless demonstrate situations where braided-local structures have undergone a translation. These translations are generated by $\mathcal{A}_{evol}$.

We shall first introduce a series of definitions and then prove a result using them.

Ribbon Connected
Two nodes a and b are *Ribbon Connected* if there exists a sequence of $N+1$ nodes x_n such that $x_1 = a$, $x_{N+1} = b$ and for each n the trinion with node x_n and the trinion with node x_{n+1} share a scored ribbon. This is equivalent to the nodes being connected in the graph $\tilde{\Gamma}$.

Connected Ribbon Network
A *Connected Ribbon Network* is a set of nodes X, such that all nodes in X are ribbon connected to all other nodes in X.

Edge Segments
Consider the edges of a ribbon graph as a metric space E onto itself. This space is essentially a collection of 1-d spaces which can be mapped to the unit circle with the distance between two points being the minimum angle between the points on the unit circle. An *Edge Segment* is then any connected subset of E with a non-empty interior. We consider only sets of non-empty interior to avoid singleton sets that can produce difficulties in later considerations.

Edge Connected
Two edge segments a and b are *Edge Connected* if they are connected in the metric space E.

3.4.1 Isolated Substructures

In order to demonstrate translations within ribbon networks we must first define a special class of elements within ribbon networks.

> ***Isolated Substructure***
> An *Isolated Substructure* is a ribbon connected set of nodes where a closed surface can be placed around it with exactly one ribbon intersecting the surface. We call this ribbon the Isolated Substructure's "tether".

It should be understood that isolated substructures are not the same as 'subsystems' as defined by [23]. This is readily apparent by considering the form of of the reduced link of an isolated substructure.

It is interesting to note that, though the definition of an isolated substructure appears to be restrictive at first glance, there are a significant number of structures that can be 'packed up' into the form of an isolated substructure. For instance, all of the example definitions of particles from [23] can be changed into isolated substructures through the use of exchange moves from $\mathcal{A}_{evol}$ as shown in figure 3.7.

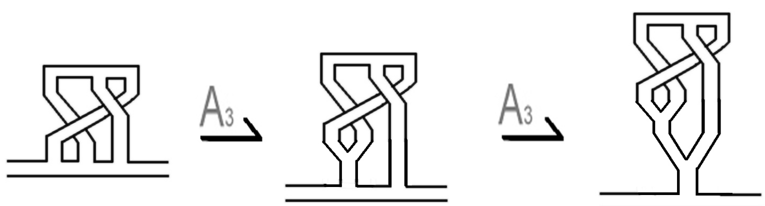


Figure 3.7: Transforming a capped braid into an isolated substructure

> ***Replaceable Edge Segments***
> An edge segment is *Replaceable* if it can be unambiguously replaced by an isolated substructure's tether. Specifically, a *Replaceable Edge Segment* cannot be the edge of a node as this would cause four valent nodes - such nodes are prohibited by our construction.

These definitions together allow us to consider the dynamics of isolated substructures under the generators of $\mathcal{A}_{evol}$. We can consider a graph Γ to be composed of a set of isolated substructures attached to replaceable edge segments of a second graph Λ. As we do not require that all such isolated substructures be so removed, this procedure can be done without ambiguity.

Theorem
Given a finite closed network Γ with two edge connected replaceable edge segments a and b, there exists a sequence of generators of $\mathcal{A}_{evol}$ such that a graph Γ_a - composed of Γ with an isolated substructure A tethered to a - evolves to Γ_b, where Γ_b is composed of the same graph Γ but with A now tethered to b.

Proof
We shall proceed by induction on the number of nodes between a and b, say N. As a and b are edge connected, the node created by A being tethered to a is ribbon connected to the two nodes that are at either side of b. We shall label these nodes x_0 (for the node created by A at a) through x_{N+1} in such a way that each x_j shares a single ribbon with x_{j+1}. The nodes on either side of b are then labeled x_N and x_{N+1}.

Before we perform this induction, we need to show the ability to move an isolated substructure through intermediate topological structures that are not composed of nodes. These are comprised of three categories: knots, twists and braidings. Examples of each of these is shown in figure 3.8. As isolated substructures only have a single connection to the outside network, we can move it past this 'terrain' through the following procedures.

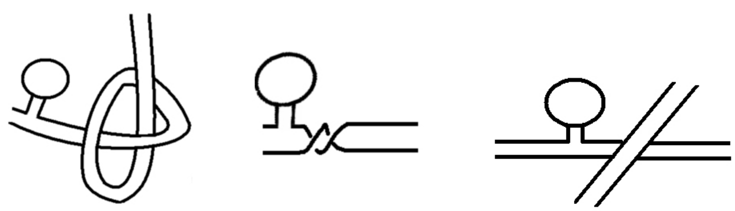


Figure 3.8: Examples of Terrain

For each knot the isolated substructure is pulled through the knot by stretching out the knot until the substructure can pass through it. As the substructure is unconnected except through its tether, this leaves the network unchanged other than the reversal of the position of the knot and the substructure.

For each twist (consisting of a rotation by π), the isolated substructure can run along the edge of the twist. Alternatively, one can view the *procedure as deforming the network itself by twisting the segment with the isolated substructure in a manner that undoes the twist on the one side and create a twist on the other.*

For each braiding, we deform the ribbon of the braid by sliding it over the isolated substructure to the other side. This is reminiscent of the Reidemeister move of the second kind.

Our lemma is that we can move an isolated substructure tethered to a node R so that it is tethered with no intermediate 'terrain' between it and some edge segment t (which is not a component of a piece of 'terrain') that connects the node R to its nearest neighbours and is edge connected to the edge which the isolated substructure is tethered to. This is proven by induction on the number of elements of 'terrain' between R and t and the use of the above prescriptions. A consequence of this is that the same method can be used for a node S which has microlocal distance 1 to R. This ability shall be used heavily in our proof.

Now, returning to the proof, we shall first prove the case of $N = 1$. We apply the above lemma to move the isolated substructure through any intermediate terrain between x_0 and x_1, giving us A tethered to a new node x_0' (we shall use primes to denote nodes that have undergone some change) that is immediately adjacent to x_1 with no intermediate terrain. We then perform an exchange move from $\mathcal{A}_{evol}$ on the node x_0' and x_1 to move x_0' onto the edge on the other side of x_1'. We can then again use the above lemma (in its more general case) to move x_0' to its final resting place at b.

Now we shall assume that the case of $N - 1$ nodes is correct and prove the case of N nodes. The prescription for this is analogous to the $N = 1$ case. Given that there is a method for moving past $N - 1$ nodes (by inductive hypothesis), we shall use that method to change the situation to a single intermediate node, and then invoke the method of the $N = 1$ case to bypass the final node.

The preceding gives the inductive argument and completes the proof.

To demonstrate translations we will need a further tool. We therefore consider also the following lemma:

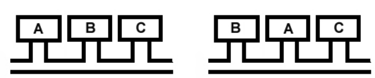
Lemma *Translations Through an Isolated Substructures*
Given an isolated substructure A that has been moved to the edge of the tether of another substructure B, it is possible to translate A to the opposite edge of the tether of B.

Proof
Due to the above theorem, it only remains to show that the two edges of

the tether of an isolated substructure are edge connected. Proceeding by contradiction, we assume that they are not edge connected. As we see that an edge of the network enters the isolated substructure and does not exit, there must be some terminus of the edge within the isolated substructure. However, such a situation is impossible, as the edges of a ribbon network must form closed links or terminate at some boundary (which we have not introduced into the theory of ribbon networks). We therefore have a contradiction. Thusly we see that if an isolated substructure can be moved to the edge of a tether, by the above theorem, it can be moved to the other edge.

3.4.2 Microlocal Translations

The application of the theorem is straightforward and results in the ability to demonstrate translations under the microlocal distance function. For instance, it is possible to construct a sequence of moves of $\mathcal{A}_{evol}$ such that figure 3.9a evolves to figure 3.9b. Under the microlocal distance function, the isolated substructure A is now less distant from the isolated substructure C (measuring the distance between substructures from the node at which they are tethered).



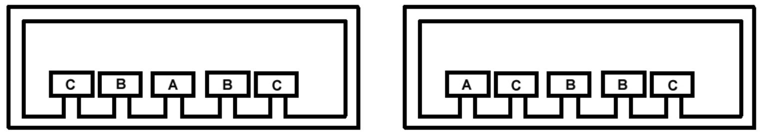
(a) (b)

Figure 3.9: Microlocal Translations

Even applying a more restrictive definition of distances, we can demonstrate translations in some form. Consider the following definition of closeness:

α-closer
An isolated substructure A is said to be *α-closer* to an isolated substructure B than it is to another substructure C, if for all paths along the ribbons of the network, leaving the node at which A is tethered and intersecting the node at which C is tethered the path intersects the node at which B is tethered.

By expanding our definitions slightly to allow us to consider isolated substructures with identical structure to be treated equally, we can show that it is possible to evoke a translation. Specifically, it is possible to take a situation where a substructure A is α-closer to substructures of type B than to those of type C and to apply a series of moves of $\mathcal{A}_{evol}$ such that the reverse is true afterwards. For instance, consider figure 3.10a and figure 3.10b. Thus we see that we have translations even under stringent requirements, thereby concluding our result.



(a) (b)

Figure 3.10: α-closer Translations

3.5 Macro, Micro and Braided Locality

The braided-local structure of the braid network is characterized by the reduced link of the structure. The reduced link of a braid network can be shown to be invariant under the generators of $\mathcal{A}_{evol}$, by applying the definition of the reduced link to the graphical representations of the generators. As a result, it is clear that the braided-local content of a braid network is invariant under the generators of $\mathcal{A}_{evol}$. This invariance is a double edged sword. On one hand, it allows us to assign some meaning to these invariant structures, as was done by the authors of [23]. On the other hand, it means that there is no way that these microlocal moves can provide any form of dynamics in this content. As a result, I suggest that, to construct a theory of quantum gravity containing particle physics from a ribbon network, it is necessary to consider the existence of a second evolution algebra, which I shall call $\mathcal{A}_{braid}$.

In [24] and [25], the concept of macrolocality in networks is put forward as the locality derived from the classical metric that would arise for a network with a space-time as its classical limit. In [24], the authors then remind us that there is no need for macrolocality to be coincidental to microlocality. It seems to be a consequence of the ideas of [23] to suggest that, though microlocality and macrolocality are not necessarily coincidental, the braided-local content - the invariants that we associate with particles - should be part of the bridge of the gap between the two. I therefore suggest that $\mathcal{A}_{braid}$ could be the bridge between microlocality and macrolocality.

For future consideration, I outline some general possibilities of $\mathcal{A}_{braid}$. Regardless, it should be noted that any such algebra that could provide macrolocal dynamics, particle interactions included, would need to alter the reduced link of the network if the identifications in [23] are to be considered seriously.

Nearly Microlocal Algebra
A candidate based upon the assumption that any move within the second evolution algebra should be as close to being microlocal as possible is called a *Nearly Microlocal Algebra*. This could be completed by introducing moves involving next to nearest neighbor nodes. This suggestion corresponds to the idea that there is a degree of coincidence between

microlocality and macrolocality (again, we should remember that such a coincidence is not needed).[24]

Braid Algebra
An algebra based upon moves that alter the braiding content of the network in ways that are roughly equivalent to elements of the standard braid group is referred to as a *Braid Algebra*. Also, it can contain moves that allow the composition of multiple braided isolated substructures.

Anti-Microlocal Algebra
An algebra premised upon the idea that microlocality should be dual or completely unrelated to macrolocality is called an *Anti-Microlocal Algebra*. Such an algebra can be constructed from a set of moves that act upon the reduced links of a graph. Such moves could be realized through the following algorithm:

> Take the reduced link of the graph Γ and apply a move that composes or interacts parts of the reduced link (whether through cutting and repairing links, or through allowing links that correspond in some manner to annihilate each other). Then take the new reduced link and equate it with a superposition of all graphs $\Gamma_{x'}$. That any such graph $\Gamma_{x'}$ should exist should be provable by a generalization of the theorem that allows the construction of a closed braid that corresponds to any link. [26]

It is possible that a stronger candidate would draw upon multiple such programs.

3.6 Conclusion

The above results give rise to several key points. First, the results are restricted to isolated substructures, without which it is impossible to bypass the terrain within the network. Second, the definitions in the previous section may not necessarily apply if labels are introduced to the network. Despite these restrictions, the result remains promising and integral to attempts to attempts to develop Ribbon networks into a theory of quantum gravity with matter. The primary candidates for the fundamental particles within such a theory are all examples of systems that can be made into isolated substructures. Indeed the form of the fundamental particles was the motivation for demonstrating translations.

The demonstration of these translations provides great promise in further developing this model into a theory that involves particle dynamics. However several key obstacles remain. As discussed in section 3.5, without adding more structure, in the form of a a second evolution algebra (or at the least, expanding the original evolution algebra), it is impossible to have any particle interactions. Indeed, even the case that one might expect to be easiest to demonstrate - that of particle and anti-particle annihilating one another - is impossible without some modification. Developing candidates for $\mathcal{A}_{braid}$ remains the subject of ongoing work.

Bibliography

[1] J. Hackett, Class. Quant. Grav. **23**, 3833 (2006) [arXiv:gr-qc/0509103].

[2] J. Hackett, arXiv:hep-th/0702198.

[3] P. Dirac *Quantised Singularities in the Electromagnetic Field*, Proceedings of the Royal Society of London. Series A, Vol. 133, No. 821. (Sept. 1, 1931), pp.60-72.

[4] J. Kowalski-Glikman, *Introduction to Doubly Special Relativity*, hep-th/0405273, (2004) submitted to Lecture Notes in Physics.

[5] G. Amelino-Camelia *Status of Relativity with observer-independent length and velocity scales* in *New Developements in Fundamental Interaction Theories: 37th Karpacz Winterschool of Theoretical Physics* ed J. Lukierski, (2001) (American Institute of Physics) gr-qc/0106004.

[6] G. Amelino-Camelia *Testable scenario for Relativity with minimum-length* Phys.Lett. B**510** (2001) 255-263, hep-th/0012238.

[7] G. Amelino-Camelia *Relativity in Space-times with short-distance structure governed by an observer-independent (Planckian) length scale* Int.J.Mod.Phys. **D11** (2002) 35-60, gr-qc/0012051.

[8] A. de Angelis, *Glast, the Gamma-ray Large Area Space Telescope*, in *New worlds in Astroparticle physics, Proceedings of the Third International Workshop* ed. A. Mouro et al, 2001 (World Scientific), astro-ph/0009271.

[9] N. Hayashida, K. Honda, N. Inoue et. al. *Updated Agasa event list above* 4×10^9 eV Astrophys. J. **522** (1999) 255, astro-ph/0008102.

[10] L. Smolin and J. Magueijo, *Gravity's Rainbow* Class.Quant.Grav. **21** (2004) 1725-1736, gr-qc/0305055.

[11] J. Magueijo and L. Smolin, *Lorentz invariance with an invariant energy scale*, Phys.Rev.Lett. **88** (2002), hep-th/0112090.

[12] R. Wald *General Relativity*, 1984. (Chicago, The university of Chicago Press)

[13] A. Ashtekar and R.O. Hansen *A unified Treatment of Null and Spatial Infinity in General Relatvity. I. Universal Structure, Asymptotic Symmetries, and Conserved Quantities at Spatial Infinity* J. Math. Phys. **19** 1542-1566, 1978

[14] A. Ashtekar *Asymptotic Structure of the Gravitational Field at Spatial Infinity* in *General Relativity and Gravitation* vol. **2** ed. A. Held, 1980. (New York: Plenum).

[15] L. Smolin and J. Magueijo, *Generalized Lorentz invariance with an invariant energy scale* Phys.Rev. D **67** (2003) 044017. hep-th/0207085

[16] J. C. Pati and A. Salam, "Lepton Number As The Fourth Color," Phys. Rev. D **10**, 275 (1974).

[17] H. Terazawa and K. Akama, "Dynamical Subquark Model Of Pregauge And Pregeometric Interactions," Phys. Lett. B **96**, 276 (1980).

[18] H. Harari, "A Schematic Model Of Quarks And Leptons," Phys. Lett. B **86**, 83 (1979).

[19] M. A. Shupe, "A Composite Model Of Leptons And Quarks," Phys. Lett. B **86**, 87 (1979).

[20] H. Harari and N. Seiberg, "A Dynamical Theory For The Rishon Model," Phys. Lett. B **98**, 269 (1981).

[21] S. O. Bilson-Thompson, "A topological model of composite preons," arXiv:hep-ph/0503213.

[22] S. O. Bilson-Thompson, "A topological model of composite preons" arXiv:hep-ph/0503213.

[23] S. O. Bilson-Thompson, F. Markopoulou and L. Smolin, "Quantum gravity and the standard model" arXiv:hep-th/0603022.

[24] F. Markopoulou and L. Smolin, "Disordered locality in loop quantum gravity states," arXiv:gr-qc/0702044.

[25] F. Markopoulou, "Towards gravity from the quantum," arXiv:hep-th/0604120. Expanded version of contribution to book 'Towards Quantum Gravity'. Edited by D. Oriti. Cambridge University Press, 2006.

[26] J.W. Alexander "A lemma on systems o knotted curves" Proc. Nat. Acad. Science USA 9 (1923), 93-95.