Scheduling in Large Scale MIMO Downlink Systems

by

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I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

This dissertation deals with the problem of scheduling in wireless MIMO (Multiple-Input Multiple-Output) downlink systems. The focus is on the large-scale systems when the number of subscribers is large.

In part one, the problem of user selection in MIMO Broadcast channel is studied. An efficient user selection algorithm is proposed and is shown to achieve the sum-rate capacity of the system asymptotically (in terms of the number of users), while requiring (i) low-complexity precoding scheme of zero-forcing beam-forming at the base station, (ii) low amount of feedback from the users to the base station, (iii) low complexity of search.

Part two studies the problem of MIMO broadcast channel with partial Channel State Information (CSI) at the transmitter. The necessary and sufficient conditions for the amount of CSI at the transmitter (which is provided to via feedback links from the receivers) in order to achieve the sum-rate capacity of the system are derived. The analysis is performed in various singnal to noise ratio regimes.

In part three, the problem of sum-rate maximization in a broadcast channel with large number of users, when each user has a stringent delay constraint, is studied. In this part, a new definition of fairness, called short-term fairness is introduced. A scheduling algorithm is proposed that achieves: (i) Maximum sum-rate throughput and (ii) Maximum short-term fairness of the system, simultaneously, while satisfying the delay constraint for each individual user with probability one.

In part four, the sum-rate capacity of MIMO broadcast channel, when the channels are Rician fading, is derived in various scenarios in terms of the value of the Rician factor and the distribution of the specular components of the channel.

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and

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List of Notations and Abbreviations

MIMO	Multiple-Input Multiple-Output
MISO	Multiple-Input Single-Output
SNR	Signal to Noise Ratio
SINR	Signal to Interference plus Noise Ratio
CSI	Channel State Information
ZFBF	Zero-Forcing Beam-Forming
DPC	Dirty Paper Coding
BS	Base Station
MS	Mobile Station
CDF	Cumulative Distribution Function
pdf	Probability Density Function
AWGN	Additive White Gaussian Noise
Boldface Upper-Case Letters	Matrices
Boldface Lower-Case Letters	Vectors
$\lVert\mathbf{h}\rVert$	Norm of vector $\mathbf{h}$
$\lVert\mathbf{H}\rVert$	Frobenius norm of matrix $\mathbf{H}$
$(\mathbf{H})^H$	Transpose conjugate of $\mathbf{H}$
$\lvert\mathbf{H}\rvert$	Determinant of $\mathbf{H}$
$\mathrm{Tr}(\mathbf{H})$	Trace of $\mathbf{H}$
$\mathbf{H} \succeq 0$	Matrix $\mathbf{H}$ is positive semi-definite
$\mathbf{H} \succeq \mathbf{G}$	$\mathbf{H} - \mathbf{G} \succeq 0$
$\mathbf{1}_n$	n-dimensional vector with all entries equal to one
$\mathbf{I}_n$	n-dimensional identity matrix
$\mathbf{0}$	The vector of all zeros

$\mathcal{CN}(\mathbf{0}, \mathbf{I})$	Circularly symmetric Gaussian distribution with zero mean and unit variance
$\vert\mathcal{S}\vert$	Cardinality of the set $\mathcal{S}$
$\ln(.)$	Natural logarithm
$\mathbb{E}\{.\}$	Expectation operation
$\Pr\{\mathscr{B}\}$	Probability of event $\mathscr{B}$
$f_X(.)$	pdf of the random variable X
$F_X(.)$	CDF of the random variable X
$\Re\{x\}$	The real part of x
RH(.)	Right hand side of the equations
N	Number of users
M	Number of transmit antennas
K	Number of receive antennas
P	Total transmit power
$\mathcal{C}_{\text{sum}}$	Sum-rate Capacity
$\mathcal{R}$	Achievable sum-rate
$\Gamma(.)$	Gamma function
$\lfloor z \rfloor$	Floor of z
$\lceil z \rceil$	Ceiling of z
$Q(.)$	Complementary CDF of Normal distribution
$f(n) = o(g(n))$	$\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$
$f(n) = O(g(n))$	$\lim_{n\to\infty} \frac{f(n)}{g(n)} < \infty$
$f(n) = \omega(g(n))$	$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \infty$
$f(n) = \Omega(g(n))$	$\lim_{n\to\infty} \frac{f(n)}{g(n)} > 0$
$f(n) = \Theta(g(n))$	$\lim_{n\to\infty} \frac{f(n)}{g(n)} = c$, where $0 < c < \infty$

$f(n) \sim g(n)$	$\lim_{n\to\infty} \frac{f(n)}{g(n)} = 1$
$f(n) \gtrsim g(n)$	$\lim_{n\to\infty} \frac{f(n)}{g(n)} \geq 1$
$A \approx B$	approximate equality between A and B, such that if one replaces A by B in the equations, the results still hold.
g (in chapter 3)	a large number ($g \gg 1$), but not necessarily a function of the number of users.

Chapter 1

Introduction

1.1 MIMO Systems

Recently, there has been an increasing demand for high speed wireless multimedia services. Traditionally, the achievable bit rate of a communication link is limited by the available bandwidth and power. In traditional wireless systems based on single transmit and single receive antenna, the only way to increase the bit rate is to increase either the bandwidth or the power budget. Another interesting approach to increase the bit rate in wireless systems without increasing the bandwidth or power budget is to use multiple transmit and/or multiple receive antennas. This transforms the channel into a Multiple-Input Multiple-Output (MIMO) system. MIMO systems offer two main advantages: spatial diversity (independent fading for different antennas) and multiplexing gain (creating multiple transmission channels). More precisely, using multiple antennas at the transmitter and the receiver can increase the transmission rate up to $\min(M, K)$, where M is the number of transmit antennas and K is the number of receive antennas [1, 2], or increase the reliability of transmission up to MK [3]. MIMO systems can be designed to sacrifice diversity to support high transmission rates (Bell Labs Layered Space-Time (BLAST)), or to sacrifice the rate to create diversity (space-time codes). Space-time codes, invented by Tarokh *et al.* [3], are a new family of codes for transmission of data using multiple transmit antennas over Rayleigh or Rician wireless channels using a trellis structure. More generally, space-time code is a combination of channel coding, modulation and transmit and receive diversity. BLAST [2] is based on using the independence of the fading between pairs of antennas to create multiple transmission channels. These channels overlap in time and frequency, however, these are separate in space.

Most of the research work reported in the literature on MIMO systems have addressed one of these two extreme solutions. However, in many cases, an intermediate solution providing an appropriate tradeoff between "rate" and "diversity" may be more appropriate. It is also desirable that such an intermediate solution can adjust the tradeoff point in an adaptive fashion depending on the channel condition.

MIMO systems with either transmit diversity or with spatial multiplexing has proved their ability in terms of increasing the spectral efficiency of the wireless systems. The large spectral efficiency obtained by using MIMO systems for a point-to-point wireless communication, suggests applying MIMO systems into network wireless systems.

1.2 MIMO Downlink Systems

Network information theory generalizes Shannon point-to-point (two terminals) communication to systems with more than two terminals. This general framework allows us to consider transmission of more than one source, and/or over more than one channel. For many years, theoretical studies in this subject have shown potential for realizing high gains over conventional point-to-point communication techniques. However, the lack of practical schemes has limited the application of these ideas in the past. With the advances in hardware, these subjects have received considerable attention over the last few years and are widely believed to provide a viable solution for the future wireless networks.

One of the most important aspect of wireless communications is cellular communications in which a Base Station (BS) communicates with several mobile users. In a cellular systems, the communications takes place in two scenarios; (i) *downlink scenario* in which the BS transmits data to the users, and (ii) *uplink scenario* in which the users transmit data to the BS. When the BS is equipped with multiple antennas, the downlink channel can be modeled as a MIMO Broadcast Channel (MIMO-BC) and the uplink channel can be modeled as MIMO Multiple-Access Channel (MIMO-MAC).

Unlike the MIMO-MAC, finding the capacity region of MIMO-BC is challenging and can not be performed using the conventional coding methods. This is due to the fact that MIMO-BC belongs to the category of non-degraded broadcast channels. Recently, there has been a lot of interest in characterizing the capacity region of this channel [4,5,6,7,8]. In [5]- [7], it has been shown that the sum-rate capacity of MIMO broadcast channels can be achieved by exploiting the surprising result of Costa [9] on known-interference cancellation at the transmitter, so-called Dirty Paper Coding (DPC). Briefly, Costa showed that the capacity of the standard scalar single-user additive white Gaussian noise channel is *unchanged* in the presence of an independent additive Gaussian interferer, provided that the interferer's signal in known noncausally to the transmitter. This result effectively shows that, while encoding the desired user's signal, the transmitter can perform a pre-cancellation of the interfering signal without a power or rate penalty. Several researchers have investigated practical techniques to achieve the sum-rate capacity (the maximum achievable sum of long-term average data rates transmitted to all users) promised by dirty paper coding. Nested lattices are used in [10] for the interference channel, as well as the general multiuser channel. Trellis coding for the broadcast channel is presented in [11], [12] as a practical technique for the multiuser channel.

There has been another line of work studying the capacity of MIMO broadcast channels. Randomization form of the fading channels in a wireless network for different users motivates utilizing a new technique called *multiuser diversity* [16], [17]. Traditionally, diversity can be achieved over fading channels either over space (multiple antenna in reception and/or transmission), over time (interleaving) or in frequency (using of RAKE receiver in spread spectrum systems). In a point-to-multipoint wireless network, multiuser diversity can be obtained exploiting the time varying characteristics of the users' channels. Multiuser diversity gain arises from the fact that in a system with many users whose channels vary independently, the overall throughput is maximized by allocating the channel resource to the user which, at that time, can best exploited. The fading rate and the dynamic range of the channel fluctuations are essential parameters for the exploitation of multiuser diversity, i.e., the larger the fluctuation, the larger the diversity gain. In the single-input single-output (SISO) case, it has been shown that transmitting to the user with the strongest channel in the given time slot is a strategy that can achieve sum-rate capacity [35]. However, in the case of multiple-antenna transmitter, simultaneous transmission of data streams to multiple users is required to achieve the sum-rate capacity. The challenge is to find the best sub-set of users for achieving the sum-rate capacity (user selection problem), which is addressed in the second chapter of this dissertation.

Unlike the point-to-point MIMO link, in the MIMO-BC, it is crucial for the transmitter to have the channel state information for all the users. This is obtained by providing a feedback channel from the receivers to the transmitter, which is very impractical for large-scale networks. Hence, it is interesting to study the problem of MIMO-BC with partial channel state information at the transmitter. More precisely, what is the performance degradation in the system when the transmitter knows only partial information about the users' channels (instead of the whole information), and what is the minimum amount of the channel information at the transmitter (or equivalently, the minimum amount of feedback from the users to the transmitter), in order not to have any degradation in the system performance? This problem is addressed in chapter 3.

One of the most important challenges in wireless networks is to provide the subscribers their quality of service demands, including throughput, delay, and fairness constraints. Most of the conventional scheduling schemes try to either maximize the throughput of the system (by exploiting multiuser diversity) or to maximizing the fairness (like the Round-Robin scheduling). Although lots of schedulings proposed in the literature have considered both throughput and fairness in the scheduling, it is interesting to know whether or not one can simultaneously maximize the throughput and fairness in a wireless system. Chapter 4 studies this problem in a large-scale wireless downlink system.

Most of the research done in the literature on MIMO broadcast channels have considered either a deterministic model or Rayleigh fading model for the channels. It is important to study how would the results change if a different model is used for the channels. In chapter 5, the capacity of MIMO-BC is derived assuming Rician fading model for the channels.

1.3 Summary of the Dissertation

In chapter 2, we consider a Rayleigh fading MIMO-BC with large number of users and propose an efficient sub-optimum algorithm that assigns the coordinates of transmission space to different users in order to achieve the best performance in terms of the sum-rate throughput. It is assumed that the zero-forcing beamforming is used at the base station as the precoding scheme. The algorithm starts by setting a threshold value. By applying Singular Value Decomposition (SVD) to all users' channel matrices, only the eigenvectors whose corresponding singular values are above the set threshold are considered. Then, among these candidate eigenvectors, the algorithm chooses a set of size M which are nearly orthogonal to each other. For the asymptotic case of $N \to \infty$, we give the necessary and sufficient conditions for the threshold value in order to achieve the optimum sum-rate capacity, such that the difference between the sum-rates approaches zero. Moreover, it is demonstrated that the complexity of search and the amount of feedback required at the base station is significantly reduced. Simulation results indicate that the proposed algorithm performs well for practical scenarios as well.

In chapter 3, a large-scale Rayleigh fading MIMO-BC is considered, in which the channel state information is provided from the users to the transmitter via feedback links. First, we define the amount of feedback as the average number of users who send information to the BS. In the fixed and low SNR regimes, our results show that it is not possible to achieve the maximum sum-rate with a finite amount of feedback. Moreover, in the fixed SNR regime, in order to reduce the gap between the achieved sum-rate and the optimum value to zero, the amount of feedback must be greater than $\ln \ln \ln N$. In the second part, we define the amount of feedback as the number of information bits sent to the BS. In the fixed SNR regime, our analysis shows that the minimum amount of feedback, in order to reduce the gap to the optimum sum-rate to zero, scales as $\Theta(\ln \ln \ln N)$, which can be achieved using the Random Beam-Forming scheme proposed in [26]. However, the optimality of Random Beam-Forming only holds for the region $\ln P \neq \Omega(\ln N)$. In the regime of $\ln P = \Omega(\ln N)$, we consider two cases. In the case of $K < M$, we prove that the minimum amount of feedback bits to reduce the gap between the achievable sum-rate and the maximum sum-rate to zero grows logarithmically with SNR, which is achievable by the "Generalized Random Beam-Forming" scheme, proposed in [51]. In the case of $K = M$, we show that by using the Random Beam-Forming scheme and the amount of feedback not growing with SNR the maximum sum-rate capacity is achievable.

In chapter 4, we consider a *hard* delay constraint D for each user, which is enforced by the application or physical limitations (e.g. buffer size). We define a dropping event as the event that there exists a user who does not meet the desired delay constraint. We propose a scheduling scheme for maximizing the throughput of the system, while satisfying the delay constraint for all users. The proposed scheduling algorithm works based on setting a threshold on the channel gain of the users and among the users with channel gains above the threshold, the user with the minimum *Packet Expiry Countdowns* (PEC), which is defined as the remaining time to the expiration of that users' packet, is served. By doing asymptotic analysis, it is proved that by selecting the threshold level properly, the proposed scheduling algorithm achieves the maximum throughput, maximum fairness, and minimum delay in the network, simultaneously, in the asymptotic case of $N \rightarrow \infty$. The analysis is based on characterizing the probability mass function of PEC in terms of N, D , and the threshold value, and evaluating the network dropping probability accordingly. It is also demonstrated that the Round-Robin (RR) scheduling, which focuses on maximizing the fairness and minimizing the delay in the network, and Multi-User Diversity (MUD) scheduling, which focuses on maximizing the throughput in the system, are two extreme cases of the proposed algorithm, where the former suffers from the weak performance in terms of throughput and the latter increases the network delay by a factor of $\log N$. Moreover, we have introduced a new notion of performance in the network, called "Average Throughput", which is defined as the product of the packet arrival rate and the amount of information per channel use in each packet, and proved that the proposed algorithm maximizes the *Minimum Average Throughput* in a broadcast channel. It is also established that the proposed algorithm reaches the boundaries of the capacity region and stability region of the underlying system, simultaneously, in the asymptotic case of $N \rightarrow \infty$. The proposed algorithm is also generalized to MIMO Broadcast Channels (MIMO-BC) by modifying the Random Beam-Forming scheme proposed in [26]. It is shown that the proposed algorithm is capable of achieving the maximum throughput, maximum fairness, and minimum delay, simultaneously, in the asymptotic case of $N \rightarrow \infty$, in a MIMO-BC.

In chapter 5, we consider a Rician MIMO-BC, in which a transmitter equipped with M antennas communicates with N ($N \gg 1$) single-antenna users. The channels are assumed to be perfectly known at both the transmitter and receiver sides. The asymptotic (in terms of the number of users) sum-rate capacity of the system, as well as the capacity-achieving strategies, are derived. The main results of the chapter are as follows: i) in the region of $\mathcal{K} = o(\log N)$, where $\mathcal{K}$ denotes the *Rician factor*, the sum-rate capacity scales as $M \log(1 + \frac{P}{M}\eta)$, where P denotes the SNR and $\eta \triangleq \frac{\log N}{1+\mathcal{K}}$, which is achieved by Zero-Forcing Beam-Forming (ZFBF) along with a low-complexity user selection algorithm that considers only the scattered component of the users' channels, ii) in the region $\mathcal{K} = \omega(\log N)$, in the case of co-located transmit antennas, the capacity scales as $\log(1 + MP)$, which is achieved by TDMA, iii) in the region $\mathcal{K} = \omega(\log N)$, in the case of isotropically-distributed specular components, the sum-rate capacity behaves as $M \log(1 + P)$, which is achieved by ZFBF, along with a user selection algorithm that considers only the specular component of the users' channels. Simulation results confirm the validity of analytical results.

Chapter 6 presents a summary of the thesis contributions and discusses several future research directions.

Chapter 2

User Selection in MIMO Broadcast Channels

2.1 Introduction

Multiple-input multiple-output (MIMO) systems have proved their ability to achieve high bit rates on a scattering wireless network [1]. In a MIMO broadcast channel, the base station equipped with multiple antennas communicates with several multiple-antenna users. Recently, there has been a lot of interest in characterizing the capacity region of this channel [4, 5, 6, 7, 8]. In [5]- [7], it has been shown that the sum-rate capacity of MIMO broadcast channels can be achieved by applying dirty-paper coding (DPC) [9] at the transmitter. Practical schemes for approximate implementation of DPC are proposed in [10], [11], [12], [13], [14], [15]. However, achieving the theoretical limits promised by DPC faces many challenges.

In a network with a large number of users, the base station can increase the throughput by selecting the best set of users to communicate with. This results in the so-called “multiuser diversity” gain [16], [17]. However, achieving the optimum multiuser diversity gain requires an exhaustive search over all possible combination of the users, which is not practical for large-scale networks. To overcome this problem, references [18] and [19] propose sub-optimum methods for user selection. These methods exploit the multiuser diversity gain, but are based on assuming DPC at the base station.

To avoid the complexity of DPC, the simple precoding scheme of “zero-forcing beam-forming”, which is also called “channel inversion”, is considered by some authors [20], [21], [4], [22]. In these works, it is assumed that the users are equipped with a single antenna. Using zero-forcing beam-forming, the downlink channel with M transmit antennas is decomposed into $N \leq M$ interference-free subchannels, serving N users. Unfortunately, in cases that the number of users is equal to the number of transmit antennas, this method does not offer a good performance [22]. However, the case of $N > M$ is more common in practical networks. In this case, selecting the best set of users improves the performance of this scheme significantly [8] , [23] (multiuser diversity gain). Due to the high complexity of selecting the best set, reference [24] proposes a suboptimum algorithm for user selection in order to maximize the sum-rate. This algorithm is based on using zero-forcing beam-forming at the transmitter. The complexity of this algorithm is shown to be $O(M^3N)$.

To achieve a good performance by using zero-forcing beam-forming, the selected sub-channels must have high gains and be nearly orthogonal to each other. As the number of users increases, it becomes easier to satisfy these requirements. However, the exhaustive search for selecting the best set of users is very complex. In [25], the authors propose a suboptimum algorithm for selecting such a set of users in a downlink environment with large number of single-antenna users. This algorithm is similar to the greedy algorithm proposed in [18], with the difference in using an orthogonality threshold for selecting the users in each step. As a result, the channel vectors of the selected users become nearly orthogonal to each other with considerable gains. It has been shown that using this algorithm, the optimum sum-rate throughput of the system is asymptotically achieved as $N \to \infty$. However, in their approach, the base station must have perfect Channel State Information (CSI) for all users.

To avoid the huge amount of feedback required by providing perfect CSI to the base station, reference [26] proposes a downlink transmission scheme based on random beam-forming relying on partial CSI at the transmitter. In this scheme, the base station randomly constructs M orthogonal beams and transmits data to the users with the maximum Signal to Interference plus Noise Ratio (SINR) for each beam. Therefore, only the value of maximum SINR, and the index of the beam for which the maximum SINR is achieved, are fed back to the base station for each user. This significantly reduces the amount of feedback. Reference [26] shows that when the number of users tends to infinity, the optimum sum-rate throughput can be achieved. However, for practical number of users, it does not perform well [25].

In this chapter, we consider a Rayleigh fading MIMO-BC with large number of users and propose an efficient sub-optimum algorithm that assigns the coordinates of transmission space to different users in order to achieve the best performance in terms of the sum-rate throughput. It is assumed that the zero-forcing beam-forming is used at the base station as the precoding scheme. The algorithm starts by setting a threshold value. By applying Singular Value Decomposition (SVD) to all users' channel matrices, only the eigenvectors whose corresponding singular values are above the set threshold are considered. Then, among these candidate eigenvectors, the algorithm chooses a set of size M which are nearly orthogonal to each other. For the asymptotic case of $N \rightarrow \infty$, we give the necessary and sufficient conditions for the threshold value in order to achieve the optimum sum-rate capacity, such that the difference between the sum-rates approaches zero. The proposed algorithm follows the same approach as that of [25], with a difference in the user selection strategy. The main advantage of our algorithm is that the coordinates are selected among the eigenvectors with singular values above a given threshold, and for the rest of the eigenvectors no information is sent to the base station. Therefore, the complexity of search and the amount of feedback required at the base station is significantly reduced. Moreover, we give the necessary and sufficient conditions for the threshold value in order to achieve the optimum sum-rate, such that the difference between the achievable sum-rate and the optimum value approaches zero.

This chapter is organized as follows. In section 2.2, we introduce the system model, and describe the proposed algorithm in section 2.3. Sections 2.4 and 2.5 are devoted to analyzing the performance, in terms of the sum-rate throughput, and the complexity of our proposed algorithm, respectively. Finally, section 2.6 concludes the chapter.

2.2 System Model

In this work, a MIMO-BC in which a base station equipped with M antennas communicates with N users, each equipped with K antennas, is considered. The channel between each user and the base station is modeled as a zero-mean circularly symmetric Gaussian matrix (Rayleigh fading). The received vector by user k can be written as

$$\mathbf{y}_k = \mathbf{H}_k \mathbf{x} + \mathbf{n}_k, \tag{2.1}$$

where $\mathbf{x} \in \mathbb{C}^{M\times 1}$ is the transmitted signal, $\mathbf{H}_k \in \mathbb{C}^{K\times M}$ is the channel matrix from the transmitter to the kth user, which is assumed to be perfectly known at the receiver side and provided to the BS via a noiseless feedback channel [1], and $\mathbf{n}_k \in \mathbb{C}^{K\times 1} \sim \mathcal{CN}(\mathbf{0}, \mathbf{I}_K)$ is the noise vector at this receiver. We assume that the transmitter has an average power constraint P, i.e. $\mathbb{E}\left\{\mathrm{Tr}(\mathbf{x}\mathbf{x}^{\mathrm{H}})\right\} \leq P$. The power constraint is assumed to be *per frame*. In other words, the power constraint is independent of the channel realization. The channels are assumed to be quasi-static block fading, in which each channel $\mathbf{H}_k$ is drawn randomly at the start of each transmission frame and remains constant for the whole transmission frame, and changes independently to another realization in the start of the next frame. The frame itself is assumed to be long enough to allow communication at rates close to the capacity. Defining the sum-rate capacity of the system in the channel realization $\mathcal{H} \triangleq \{\mathbf{H}_k\}_{k=1}^{N}$, when the transmitter has perfect CSI about all users' channels, as $\mathcal{C}_{\mathrm{sum}}(\mathcal{H})$, the average sum-rate capacity, denoted as $\mathcal{C}_{\mathrm{sum}}$, is defined as the average over time of $\mathcal{C}_{\mathrm{sum}}(\mathcal{H})$, which is by the ergodicity of the channel, equal to $\mathbb{E}_{\mathcal{H}}\left\{\mathscr{R}_{\mathrm{Opt}}(\mathcal{H})\right\}$.

[1] As we will show later, the BS does not need to have the perfect CSI about all the users' channels. However, the partial CSI that the BS receives through feedback is based on the perfect CSI that the receivers have.

$\mathcal{C}_{\text{sum}}$ is shown in [5] to be equal to

$$\mathcal{C}_{\text{sum}} = \mathbb{E}_{\mathcal{H}} \left\{ \max_{\substack{\mathbf{Q}_k \\ \sum \mathrm{Tr}(\mathbf{Q}_k)=\mathrm{P}}} \ln \left| \mathbf{I}_M + \sum_{k=1}^{N} \mathbf{H}_k^H \mathbf{Q}_k \mathbf{H}_k \right| \right\}, \tag{2.2}$$

where $\mathbf{Q}_k$ is the transmit covariance matrix of the kth user. The capacity achieving transmission strategy is shown to involve at least M, and at most M^2 data streams in total [27]. However, experimental results show that M data streams are adequate to achieve a significant portion of the capacity [18], [19].

As discussed earlier, the capacity achieving strategy in a downlink environment requires applying dirty-paper coding at the base station, which is not practical in many applications. For this reason, it is desirable to utilize a precoding scheme with less complexity. Among the known precoding schemes, zero-forcing beam-forming has received considerable attention, as it uses a simple structure of channel matrix inversion. This scheme results in having M interference-free sub-channels. Although this scheme does not yield a good performance for the case $M = N$ [22][2], for the case of $N > M$, which is more common in wireless networks, by selecting an appropriate set of dimensions, the corresponding performance is shown to be good [25], [24], [28]. In this work, using zero-forcing beam-forming at the base station, we propose an efficient algorithm to find M coordinates for data transmission, focusing on maximizing the sum-rate throughput.

2.3 Proposed Algorithm

As mentioned earlier, to maximize the sum-rate using zero-forcing beam-forming, the selected eigenvectors must be nearly orthogonal to each other, and their cor-

[2]The result is derived for the case of single-antenna users

responding singular values be sufficiently large. The measure of orthogonality between two $M \times 1$ vectors υ and ψ is defined as,

$$z(\upsilon, \psi) = \frac{|\upsilon^H \psi|^2}{\|\upsilon\|^2 \|\psi\|^2}. \tag{2.3}$$

It is evident that the smaller is $z(\upsilon, \psi)$, the more orthogonal will be υ and ψ.

Using Singular Value Decomposition (SVD), $\mathbf{H}_k$ can be written as

$$\mathbf{H}_k = \mathbf{U}_k \mathbf{\Lambda}_k \mathbf{V}_k^H, \tag{2.4}$$

where $\mathbf{\Lambda}_k$ is a $K \times M$ diagonal matrix containing the singular values of $\mathbf{H}_k$, $\mathbf{U}_k$ and $\mathbf{V}_k$ are $K \times K$ and $M \times M$ unitary matrices, respectively. Multiplying both sides of (2.1) by $\mathbf{U}_{k,j}^H$, where $\mathbf{U}_{k,j}$ is the jth column of $\mathbf{U}_k$, it is easy to show that

$$r_{k,j} = \mathbf{g}_{k,j} \mathbf{x} + w_{k,j}. \tag{2.5}$$

In the above equation, $r_{k,j} = \mathbf{U}_{k,j}^H \mathbf{y}_k$, $\mathbf{g}_{k,j} = \sqrt{\lambda_j(k)} \mathbf{V}_{k,j}^H$, where $\mathbf{V}_{k,j}$ is the jth column of $\mathbf{V}_k$ and $\sqrt{\lambda_j(k)}$ is the jth singular value of $\mathbf{H}_k$, corresponding to $\mathbf{V}_{k,j}$, and $w_{k,j} \sim \mathcal{CN}(0,1)$ is AWGN. This equation suggests that for selecting the dimensions with high gains, the norm of the equivalent channel introduced by (2.5), namely $\mathbf{g}_{k,j}$, which is equal to $\sqrt{\lambda_j(k)}$, can be compared with a threshold. This threshold is set by the base station at the beginning of the transmission. Using such a threshold reduces the amount of feedback and the size of search space for selecting the coordinates. To satisfy the orthogonality criterion, the base station can perform an exhaustive search for finding the "most orthogonal set"[3] among the pre-selected eigenvectors. Due to the large complexity of exhaustive search, the coordinates can be chosen one by one. In other words, in each step the eigenvector

[3]In general, the orthogonality of a set $\{\mathbf{h}_i\}_{i=1}^M$ can be measured by the orthogonality defect, defined as $\frac{\prod_{i=1}^M \|\mathbf{h}_i\|^2}{|\mathbf{H}\mathbf{H}^\mathbf{H}|}$, where $\mathbf{H} = [\mathbf{h}_1^T | \cdots | \mathbf{h}_M^T]^T$.

which is the most orthogonal to the previously selected coordinates, is selected. The first coordinate is chosen as the eigenvector with the maximum corresponding singular value. The steps of the algorithm are given in the following:

Proposed Algorithm (Algorithm 1):

1. Using SVD, each user computes the eigenvectors and singular values of its channel matrix and sends back the singular values which are larger than a predetermined threshold t, along with their corresponding "right" eigenvectors[4], to the base station. The indices of these eigenvectors form the following set:
$$\mathcal{S}_0 = \{(k,j) | \quad \lambda_j(k) > t\}. \tag{2.6}$$

2. Base station selects the index in $\mathcal{S}_0$, corresponding to the maximum eigenvalue. Let us define this index as (s_1, d_1), i.e., the d_1th eigenvector of the s_1th user.

3. Define
$$\mathcal{S}_1 \quad = \quad \mathcal{S}_0 - \{(s_1, d_1)\},$$
and
$$\gamma_{k,j}(1) \quad = \quad z(\mathbf{V}_{s_1,d_1}, \mathbf{V}_{k,j}), \quad \forall (k,j) \in \mathcal{S}_1, \tag{2.7}$$
where $z(.,.)$ is defined in (2.3). Note that as $\|\mathbf{V}_{k,j}\| = \|\mathbf{V}_{s_1,d_1}\| = 1$, $z(\mathbf{V}_{s_1,d_1}, \mathbf{V}_{k,j}) = |\mathbf{V}^H_{s_1,d_1}\mathbf{V}_{k,j}|^2$.

[4]In the SVD of $\mathbf{H}$ as $\mathbf{U\Lambda V}$, $\mathbf{U}$ is called *right eigenvector matrix* and $\mathbf{V}$ is called *left eigenvector matrix*.

4. For $2 \le m \le M$, repeat the followings:

$$\begin{aligned}(s_m, d_m) &= \arg \min_{(k,j)\in \mathcal{S}_{m-1}} \gamma_{k,j}(m-1) \\ \mathcal{S}_m &= \mathcal{S}_{m-1} - \{(s_m, d_m)\} \\ \gamma_{k,j}(m) &= z(\mathbf{V}_{s_m,d_m}, \mathbf{V}_{k,j}) + \gamma_{k,j}(m-1), \qquad \forall (k,j) \in \mathcal{S}_m. \end{aligned} \tag{2.8}$$

In the above, $\gamma_{k,j}(m-1) = \sum_{i=1}^{m-1} z(\mathbf{V}_{s_i,d_i}, \mathbf{V}_{k,j})$ is used as the measure of orthogonality between a candidate eigenvector $\mathbf{V}_{k,j}$ and the set of previously selected eigenvectors, $\{\mathbf{V}_{s_i,d_i}\}_{i=1}^{m-1}$. Since these eigenvectors are nearly orthogonal to each other by the algorithm, with a good approximation, $\gamma_{k,j}(m-1)$ can be interpreted as the square magnitude of the projection of $\mathbf{V}_{k,j}$ over the sub-space spanned by $\{\mathbf{V}_{s_i,d_i}\}_{i=1}^{m-1}$. It is obvious that the smaller is this projection, the more orthogonal will be $\mathbf{V}_{k,j}$ to this sub-space. The recursive structure of $\gamma_{k,j}(m)$ facilitates its computation at each step of the algorithm.

After selecting the dimensions, we construct the “selected coordinate matrix” as

$$\boldsymbol{\mathcal{H}} = \left[\mathbf{g}_{s_1,d_1}^T \,|\, \mathbf{g}_{s_2,d_2}^T \,|\, \cdots \,|\, \mathbf{g}_{s_M,d_M}^T\right]^T. \tag{2.9}$$

Using zero-forcing beam-forming, the transmitted vector $\mathbf{x}$ can be written as

$$\mathbf{x} = \boldsymbol{\mathcal{H}}^{-1}\mathbf{u}, \tag{2.10}$$

where $\mathbf{u} = [u_{s_1,d_1}, \cdots, u_{s_M,d_M}]^T$ is the information vector. Using (2.5) and (2.10), the received signal over the mth coordinate is equal to

$$\begin{aligned} r_{s_m,d_m} &= \mathbf{U}_{s_m,d_m}^H \mathbf{y}_{s_m} \\ &= \mathbf{g}_{s_m,d_m}\mathbf{x} + w_{s_m,d_m} \\ &= \mathbf{g}_{s_m,d_m}\boldsymbol{\mathcal{H}}^{-1}\mathbf{u} + w_{s_m,d_m} \\ &= u_{s_m,d_m} + w_{s_m,d_m}. \end{aligned} \tag{2.11}$$

It can be seen that by applying zero-forcing beam-forming, the downlink channel is decomposed to M interference-free sub-channels.

2.4 Performance Analysis

In this section, we examine the performance of our proposed algorithm in terms of the sum-rate throughput. First, we consider the asymptotic case of $N \to \infty$.

2.4.1 Asymptotic Analysis

The sum-rate capacity of MIMO-BC has been shown to scale as $M \ln \ln N$, as N tends to infinity [26]. This implies that to achieve the optimum sum-rate, the singular values corresponding to the selected dimensions must behave like $\ln N$. In other words, the threshold value should scale as $\ln N$. The following theorems indicates this fact with more details:

Theorem 2.1 *The necessary condition to achieve* $\lim_{N\to\infty} \mathcal{C}_{\text{sum}} - \mathcal{R}_{\text{Prop}} = 0$ *is having*

$$t = \ln N + (M + K - 2) \ln \ln N - \rho(N), \tag{2.12}$$

where $\rho(N)$ *satisfies*

$$\rho(N) \quad = \quad o(\ln N),$$

and

$$\rho(N) \quad = \quad \ln \ln \ln \ln N + \ln[\Gamma(K)\Gamma(M)] + \omega\left(\frac{1}{\ln \ln \ln N}\right),$$

where $\Gamma(n) \triangleq (n-1)!$*, for integer values of* n*.*

Proof - We show that by violating any of the above conditions, the optimum sum-rate can not be achieved.

The necessity of $\rho(N) \sim o(\ln N)$*:*

It is sufficient to show that

$$\lim_{N\to\infty} \frac{t}{\ln N} = 1. \tag{2.13}$$

For this purpose, we consider the following cases:

Case I; $\lim_{N\to\infty} t = \infty, \lim_{N\to\infty} \dfrac{t}{\ln N} < 1$**:** The achievable sum-rate of the proposed method, denoted by $\mathcal{R}_{\text{Prop}}$, can be upper-bounded as

$$\begin{aligned} \mathcal{R}_{\text{Prop}} &\leq \mathbb{E}\left\{ \max_{\substack{P_i \\ \sum_{i=1}^{M} P_i = P}} \sum_{i=1}^{M} \ln(1 + P_i \|\mathbf{g}_{s_i,d_i}\|^2) \right\} \\ &= \mathbb{E}\left\{ \max_{\substack{P_i \\ \sum_{i=1}^{M} P_i = P}} \sum_{i=1}^{M} \ln(1 + P_i \lambda_{d_i}(s_i)) \right\}, \end{aligned} \tag{2.14}$$

where $\mathbf{g}_{s_i,d_i}$ and $\lambda_{d_i}(s_i)$ are defined in (2.5).

Since the optimum sum-rate is shown to be $M \ln\left(\frac{P}{M} \ln N + O(\ln\ln N)\right)$ [26], we have

$$\begin{aligned}\mathcal{C}_{\text{sum}} - \mathcal{R}_{\text{Prop}} &\geq M \ln\left(\frac{P}{M}\ln N + O(\ln\ln N)\right) - \\ &\quad \mathbb{E}\left\{\max_{\substack{P_i \\ \sum_{i=1}^M P_i = P}} \sum_{i=1}^{M} \ln(1 + P_i \lambda_{d_i}(s_i))\right\}, \\ &= M \ln\left(\frac{P}{M}\ln N + O(\ln\ln N)\right) - \\ &\quad \mathbb{E}\left\{\max_{\substack{P_i \\ \sum_{i=1}^M P_i = P}} \sum_{i=1}^{M} \ln(P_i \lambda_{d_i}(s_i)) + \ln\left(1 + \frac{1}{P_i \lambda_{d_i}(s_i)}\right)\right\}.\end{aligned} \tag{2.15}$$

The right hand side of the above equation can be written as follows:

$$\begin{aligned}\text{RH}(2.15) &\overset{(a)}{\geq} \min_{\substack{P_i \\ \sum_{i=1}^M P_i = P}} \ln\left(\frac{(P/M)^M}{\prod_{i=1}^M P_i}\right) + M\ln\left(\ln N + O(\ln\ln N)\right) \\ &\quad - \sum_{i=1}^{M} \mathbb{E}\left\{\ln \lambda_{d_i}(s_i)\right\} + O\left(\frac{1}{t}\right) \\ &= M\ln\left(\ln N + O(\ln\ln N)\right) - \sum_{i=1}^{M} \mathbb{E}\left\{\ln \lambda_{d_i}(s_i)\right\} + O\left(\frac{1}{t}\right) \\ &\overset{(b)}{\geq} M\ln\left(\ln N + O(\ln\ln N)\right) - \mathbb{E}\left\{\max_{k=1,\cdots,N} \ln \lambda_{\max}(\mathbf{H}_k)\right\} \\ &\quad -(M-1)\mathbb{E}\left\{\ln \lambda / \lambda > t\right\} + O\left(\frac{1}{t}\right),\end{aligned} \tag{2.16}$$

where $\lambda_{\max}(\mathbf{A})$ is the maximum singular value of $\mathbf{A}\mathbf{A}^H$, and λ is a random variable, denoting an unordered eigenvalue of a $K \times K$ Wishart matrix. (a) comes from writing $\ln(P/M \ln N + O(\ln\ln N))$ as $\ln(P/M) + \ln(\ln N + O(\ln\ln N))$ and $\ln(1 + P_i\lambda_{d_i}(s_i))$ as $\ln P_i + \ln \lambda_{d_i}(s_i) + \ln\left(1 + \frac{1}{P_i\lambda_{d_i}(s_i)}\right)$, using the approximation $\ln(1+x) \sim O(x)$, $x \ll 1$, noting that the solution to the maximization problem (2.14) satisfies

$P_i\lambda_{s_i}(d_i) \gg 1$, $i = 1, \cdots, M$. (b) results from the fact that excluding the largest maximum singular value from the set of singular values, which are greater than t, reduces the expectation in the second line of (2.16). In writing (b), we also used the fact that the eigenvectors and their corresponding singular values of a circularly symmetric Gaussian matrix are independent. The distribution of λ, denoting as $f(\lambda)$ is derived in [1] as

$$f(\lambda) = \frac{1}{K}\sum_{i=0}^{K-1}\frac{i!}{(M-K+i)!}[L_i^{M-K}(\lambda)]^2\lambda^{M-K}\exp(-\lambda), \tag{2.17}$$

where $L_i^{M-K}(\lambda)$ is the associated Laguerre polynomial of order k [29]. Using the above equation, it is easy to show that

$$\begin{aligned}\mathbb{E}\left\{\ln\lambda/\lambda > t\right\} &= \frac{\int_t^\infty \ln\lambda f(\lambda)d\lambda}{1-F(t)}\\ &= \ln t + \frac{\int_t^\infty \frac{1-F(\lambda)}{\lambda}d\lambda}{1-F(t)}\\ &= \ln t + O\left(\frac{1}{t}\right),\end{aligned} \tag{2.18}$$

where $F(.)$ stands for the CDF of λ. Moreover, we can write

$$\mathbb{E}\left\{\max_{k=1,\cdots,N}\ln\lambda_{\max}(\mathbf{H}_k)\right\} \leq \mathbb{E}\left\{\max_{k=1,\cdots,N}\ln\|\mathbf{H}_k\|^2\right\}, \tag{2.19}$$

where $\|\mathbf{A}\|^2$ denotes the Frobenius norm of matrix $\mathbf{A}$. In [26], it has been shown that with probability one,

$$\max_{k=1,\cdots,N}\|\mathbf{H}_k\|^2 = \ln N + O(\ln\ln N).$$

Therefore,

$$\mathbb{E}\left\{\max_{k=1,\cdots,N}\ln\lambda_{\max}(\mathbf{H}_k)\right\} \leq \ln\left(\ln N + O(\ln\ln N)\right) \tag{2.20}$$

Combining (2.16), (2.18), and (2.20), we get

$$\mathcal{C}_{\text{sum}} - \mathcal{R}_{\text{Prop}} \geq (M-1)\ln\frac{\ln N}{t} + O\left(\frac{\ln\ln N}{\ln N}\right) + O\left(\frac{1}{t}\right). \tag{2.21}$$

Consequently, for $\lim_{N\to\infty} t = \infty$ and $\lim_{N\to\infty} \dfrac{t}{\ln N} < 1$, $\lim_{N\to\infty} \mathcal{C}_{\text{sum}} - \mathcal{R}_{\text{Prop}} \neq 0$.

Case II*;* $\lim_{N\to\infty} t = c$*, where* c *is a constant*: In this case, (2.16) can be written as

$$\begin{aligned}\mathcal{C}_{\text{sum}} - \mathcal{R}_{\text{Prop}} &\geq M\ln\left(\frac{P}{M}\ln N + O(\ln\ln N)\right) - \sum_{i=1}^{M}\mathbb{E}\left\{\ln(1+P\lambda_{d_i}(s_i))\right\}\\ &\geq M\ln\left(\frac{P}{M}\ln N + O(\ln\ln N)\right) - \\ &\quad \mathbb{E}\left\{\ln\left(1+P\max_{k=1,\cdots,N}\lambda_{\max}(\mathbf{H}_k)\right)\right\} - \\ &\quad (M-1)\mathbb{E}\left\{\ln(1+P\lambda)/\lambda > t\right\}.\end{aligned} \tag{2.22}$$

Similar to (2.20), it is easy to see that

$$\mathbb{E}\left\{\ln\left(1+P\max_{k=1,\cdots,N}\lambda_{\max}(\mathbf{H}_k)\right)\right\} \leq \ln P + \ln(\ln N + O(\ln\ln N)). \tag{2.23}$$

Moreover, since $\mathbb{E}\left\{\ln(1+P\lambda)\right\} < \infty$, we have $\mathbb{E}\left\{\ln(1+P\lambda)/\lambda > t\right\} = O(1)$. Hence,

$$\mathcal{C}_{\text{sum}} - \mathcal{R}_{\text{Prop}} \geq (M-1)\ln\ln N + O(1). \tag{2.24}$$

As a result, $\lim_{N\to\infty} \mathcal{C}_{\text{sum}} - \mathcal{R}_{\text{Prop}} \neq 0$. This completes the proof of

$$\lim_{N\to\infty}\frac{t}{\ln N} < 1 \Rightarrow \lim_{N\to\infty}\mathcal{C}_{\text{sum}} - \mathcal{R}_{\text{Prop}} \neq 0.$$

Case III; $\lim_{N\to\infty} \dfrac{t}{\ln N} > 1$**:** Let us define p_k as the probability that the maximum singular value of a randomly chosen user k is greater than t. In [19], it is shown that for a $K \times M$ matrix $\mathbf{A}$, whose entries are i.i.d Gaussian with zero mean and variance one, we have

$$\Pr\{\lambda_{\max}(\mathbf{A}) > t\} = \frac{t^{M+K-2}\exp(-t)}{\Gamma(M)\Gamma(K)}\left[1 + O\left(t^{-1}\right)\right]. \quad (2.25)$$

Therefore,

$$p_k = \frac{t^{M+K-2}\exp(-t)}{\Gamma(M)\Gamma(K)}\left[1 + O\left(t^{-1}\right)\right], \quad (2.26)$$

which is independent of k, and we denote it with p. We define L as the number of users whose maximum singular values are greater than t. Since L is a binomial random variable with parameter p, $\mathbb{E}\{L\} = Np$.

Using Theorem 3.2 in the next chapter, we can write

$$\mathcal{C}_{\text{sum}} - \mathcal{R}_{\text{Prop}} \geq (1-p)^N(\mathcal{R}_1 - \mathcal{R}_{\mathcal{A}}^{\text{NCSI}}), \quad (2.27)$$

where $\mathcal{R}_1 = \mathbb{E}\left\{\max_{\substack{\mathbf{Q}_n \\ \sum \text{Tr}(\mathbf{Q}_n)=\text{P}}} \ln\left|\mathbf{I}_M + \sum_{n=1}^{N}\mathbf{H}_n^H\mathbf{Q}_n\mathbf{H}_n\right|\middle|\mathcal{A}\right\}$, $\mathcal{A}$ is the event that $L = 0$, and $\mathcal{R}_{\mathcal{A}}^{\text{NCSI}}$ stands for the sum-rate of MIMO-BC when no CSI is available at the base station, conditioned on $\mathcal{A}$. In [31], it has been shown that

$$\mathcal{R}^{\text{NCSI}} = \mathbb{E}_{\mathbf{H}_k}\left\{\ln\left|\mathbf{I} + \frac{P}{M}\mathbf{H}_k\mathbf{H}_k^H\right|\right\}. \quad (2.28)$$

Since $\lim_{N\to\infty}\frac{t}{\ln N} > 1$, using (2.26), it can be easily shown that $Np \to 0$. As a result, with a similar approach as in [30], we have

$$\begin{aligned}\mathcal{R}_{\mathcal{A}}^{\text{NCSI}} &= \mathbb{E}_{\mathbf{H}_k|\mathcal{A}}\left\{\ln\left|\mathbf{I} + \frac{P}{M}\mathbf{H}_k\mathbf{H}_k^H\right|\middle|\mathcal{A}\right\} \\ &= O(1). \quad (2.29)\end{aligned}$$

Moreover, we can write

$$\begin{aligned}\mathcal{R}_1 &\geq \mathbb{E}\left\{\ln(1 + P\theta_{\max})\left|\theta_{\max} < t\right.\right\} \\ &\geq \mathbb{E}\left\{\ln(1 + P\theta_{\max})\left|\theta_{\max} < t, \theta_{\max} > \ln N\right.\right\}\Pr\{\theta_{\max} > \ln N|\theta_{\max} < t\} \\ &\geq \ln(1 + P\ln N)\vartheta, \quad (2.30)\end{aligned}$$

where $\theta_{\max} \triangleq \max_k \lambda_{\max}(\mathbf{H}_k)$, and $\vartheta \triangleq \Pr\{\theta_{\max} > \ln N | \theta_{\max} < t\}$. Using (2.26), ϑ can be written as follows:

$$\vartheta = \frac{\left(1 - \frac{t^{M+K-2}e^{-t}(1+O(t^{-1}))}{\Gamma(M)\Gamma(K)}\right)^N - \left(1 - \frac{[\ln N]^{M+K-2}(1+O([\ln N]^{-1}))}{N\Gamma(M)\Gamma(K)}\right)^N}{\left(1 - \frac{t^{M+K-2}e^{-t}(1+O(t^{-1}))}{\Gamma(M)\Gamma(K)}\right)^N}. \quad (2.31)$$

Since $\lim_{N\to\infty} \frac{t}{\ln N} > 1$, it can be shown that $\vartheta = 1 - o(\frac{1}{N})$. Substituting ϑ in (2.30), yields

$$\mathcal{R}_1 \geq \ln(1 + P\ln N)\left(1 - o\left(\frac{1}{N}\right)\right). \quad (2.32)$$

Using the above equation and (2.29), the right hand side of (2.27) can be lower-bounded as,

$$\begin{aligned} \text{RH}(2.27) &\geq (1-p)^N[\ln\ln N + O(1)] \\ &= e^{-Np(1+O(p))}[\ln\ln N + O(1)] \\ &= \ln\ln N. \end{aligned} \quad (2.33)$$

The last line in the above equation follows from $\lim_{N\to\infty} \frac{t}{\ln N} > 1$, which incurs $Np \to 0$. As a result, $\mathcal{C}_{\text{sum}} - \mathcal{R}_{\text{Prop}} \neq 0$. This completes the proof for the necessity of $\rho(N) \sim o(\ln N)$.

The necessity of $\rho(N) = \ln\ln\ln\ln N + \ln[\Gamma(K)\Gamma(M)] + \omega\left(\frac{1}{\ln\ln\ln N}\right)$:

Let $\rho(N) = \ln\ln\ln\ln N + \ln[\Gamma(M)\Gamma(K)] + \sigma(N)$. Suppose that

$$\rho(N) \neq \ln\ln\ln\ln N + \ln[\Gamma(K)\Gamma(M)] + \omega\left(\frac{1}{\ln\ln\ln N}\right), \quad (2.34)$$

which incurs $\sigma(N) = O\left(\frac{1}{\ln\ln\ln N}\right)$, or $\sigma(N) < 0$. Using (2.27), we have

$$\mathcal{C}_{\text{sum}} - \mathcal{R}_{\text{Prop}} \geq (1-p)^N[\mathcal{R}_1 - \mathcal{R}_{\mathcal{A}}^{\text{NCSI}}]. \quad (2.35)$$

Similar to (2.29) and (2.32), under the assumption of (2.34), it can be shown that

$$\begin{aligned} \mathcal{R}_1 &\geq \ln(1+P\ln N)\left(1-o\left(\frac{1}{N}\right)\right), \\ \mathcal{R}_{\mathcal{A}}^{\mathrm{NCSI}} &= O(1). \end{aligned} \tag{2.36}$$

Using the above equations and (2.26), we can write

$$\begin{aligned} \mathcal{C}_{\mathrm{sum}} - \mathcal{R}_{\mathrm{Prop}} &\geq \left(1 - \frac{t^{M+K-2}\exp(-t)}{\Gamma(M)\Gamma(K)}\left[1+O(t^{-1})\right]\right)^N [\ln\ln N + O(1)] \\ &= \left(1 - \frac{e^{\rho(N)}}{N\Gamma(M)\Gamma(K)}\left[1+O\left(\frac{\ln\ln N}{\ln N}\right)\right]\right)^N [\ln\ln N + O(1)] \\ &= \exp\left\{-\frac{e^{\rho(N)}}{\Gamma(M)\Gamma(K)}\right\}[1+o(1)]\,[\ln\ln N + O(1)] \\ &= \exp\left\{-e^{\sigma(N)}\ln\ln\ln N\right\}[\ln\ln N + O(1)]\,[1+o(1)] \\ &= M\exp\left\{[1-e^{\sigma(N)}]\ln\ln\ln N\right\}[1+o(1)]\,. \end{aligned} \tag{2.37}$$

Under the assumption of (2.34), in the case of $\sigma(N) = O\left(\frac{1}{\ln\ln\ln N}\right)$, i.e.,

$$\lim_{N\to\infty} \sigma(N)\ln\ln\ln N = c < \infty,$$

using (2.37), we have

$$\begin{aligned} \mathcal{C}_{\mathrm{sum}} - \mathcal{R}_{\mathrm{Prop}} &\geq \exp\left\{[-\sigma(N) + O(\sigma^2(N))]\ln\ln\ln N\right\}[1+o(1)] \\ &= \exp\left\{-\sigma(N)\ln\ln\ln N\right\}[1+o(1)]\,. \end{aligned} \tag{2.38}$$

Hence,

$$\begin{aligned} \lim_{N\to\infty} \mathcal{C}_{\mathrm{sum}} - \mathcal{R}_{\mathrm{Prop}} &\geq e^{-c} \\ &\neq 0. \end{aligned} \tag{2.39}$$

Also, in the case of $\sigma(N) < 0$, using (2.37), we have

$$\begin{aligned} \lim_{N\to\infty} \mathcal{C}_{\mathrm{sum}} - \mathcal{R}_{\mathrm{Prop}} &\geq 1 \\ &\neq 0. \end{aligned} \tag{2.40}$$

This completes the proof for the necessity of $\rho(N) = \ln\ln\ln\ln N + \ln[\Gamma(K)\Gamma(M)] + \omega\left(\frac{1}{\ln\ln\ln N}\right)$.

■

Theorem 2.2 *The sufficient condition to achieve* $\lim_{N\to\infty} \mathcal{C}_{\text{sum}} - \mathcal{R}_{\text{Prop}} = 0$ *is having*

$$t = \ln N + (M + K - 2)\ln\ln N - \rho(N), \tag{2.41}$$

where $\rho(N)$ *satisfies*

$$\rho(N) = o(\ln N),$$

and

$$\rho(N) = \ln\ln\ln\ln N + \omega(1).$$

Proof - First, we state and prove lemmas 2.3-2.7. In Lemma 2.3, we show that the pre-selected eigenvectors in the first step of Algorithm 1 must correspond to the maximum singular values of some users' channel matrices, with probability one. Having this, in Lemma 2.4, we obtain the number of pre-selected users in the first step of Algorithm 1 in terms of $\rho(N)$. Then, by deriving the pdf of the orthogonality measure defined in Lemma 2.5, in Lemma 2.7, we give a lower-bound on the measure of orthogonality between the selected eigenvectors in terms of $\rho(N)$. Finally, the theorem is proved by obtaining a lower-bound on the achievable rate of the proposed scheme and showing that if $\rho(N)$ satisfies the conditions in Theorem 2.2, the difference between this lower-bound and the sum-rate capacity of MIMO-BC approaches zero, as $N \to \infty$.

Lemma 2.3 *Assuming $K > 1$, define Ω_J as the probability of existing at least one user from which J eigenvectors ($J > 1$) are selected in Algorithm 1. Setting $t = \ln N + (M + K - 2)\ln\ln N - \rho(N)$, in which $\rho(N)$ satisfies the conditions of Theorem 2.2, we have*

$$\Omega_J = O\left(\frac{e^{o(\ln N)}}{N^{J-1}}\right). \tag{2.42}$$

Proof- Consider the following event [5]:

$$A_k = \{\lambda_i(k) > t, \quad i = 1, \cdots J, \quad \lambda_i(k) < t, \quad i = J+1, \cdots, K\}. \tag{2.43}$$

We have

$$\begin{aligned} \|\mathbf{H}_k\|^2 &= \mathrm{Tr}\{\mathbf{H}_\mathrm{k}\mathbf{H}_\mathrm{k}^\mathrm{H}\} \\ &= \sum_{i=1}^{K} \lambda_i(k) \\ &\geq \sum_{i=1}^{J} \lambda_i(k). \end{aligned} \tag{2.44}$$

Since $t = \ln N + o(\ln N)$, we can write

$$\Pr\{A_k\} \leq \Pr\{\|\mathbf{H}_k\|^2 \geq J\ln N + o(\ln N)\}, \tag{2.45}$$

As $\|\mathbf{H}_k\|^2$ has a chi-square distribution with $2MK$ degrees of freedom [32], the right hand side of (2.45) can be written as

$$\begin{aligned} \Pr\left\{\|\mathbf{H}_k\|^2 \geq J\ln N + o(\ln N)\right\} &= \int_{J\ln N + o(\ln N)}^{\infty} \frac{x^{MK-1}\exp(-x)}{\Gamma(MK)}dx \\ &= \sum_{m=0}^{MK-1} \frac{[J\ln N + o(\ln N)]^m}{m!} e^{-J\ln N + o(\ln N)} \\ &= \frac{\left([J\ln N]^{MK-1} + o([\ln N]^{MK-1})\right) e^{o(\ln N)}}{N^J (MK-1)!} \\ &= \Psi_J \frac{[\ln N]^{MK-1} e^{o(\ln N)}}{N^J}[1 + o(1)], \end{aligned} \tag{2.46}$$

[5]We have assumed that the singular values are in the decreasing order, i.e., $\lambda_1 > \lambda_2 > \cdots > \lambda_K$

where $\Psi_J = \frac{J^{MK-1}}{(MK-1)!}$. Using (2.45), and (2.46), we can write Ω_J as

$$\begin{aligned}
\Omega_J &= 1 - \prod_{k=1}^{N} \left(1 - \Pr\{A_k\}\right) \\
&\leq 1 - \left[1 - \Psi_J \frac{[\ln N]^{MK-1} e^{o(\ln N)}}{N^J} [1 + o(1)]\right]^N \\
&= 1 - \exp\left\{N \ln \left[1 - \Psi_J \frac{[\ln N]^{MK-1} e^{o(\ln N)}}{N^J} [1 + o(1)]\right]\right\} \\
&= 1 - \exp\left\{-\Psi_J \frac{[\ln N]^{MK-1} e^{o(\ln N)}}{N^{J-1}} [1 + o(1)]\right\} \\
&= O\left(\frac{e^{o(\ln N)}}{N^{J-1}}\right). \qquad (2.47)
\end{aligned}$$

■

As a result, $\lim_{N\to\infty} \Omega_J = 0$, for $J > 1$. This implies that as $N \to \infty$, with probability one, at most one eigenvector for each user is likely to be selected by this algorithm. This eigenvector corresponds to the maximum singular value of that user.

Lemma 2.4 *Let $t = \ln N + (M + K - 2) \ln \ln N - \rho(N)$, in which $\rho(N)$ satisfies the conditions of Theorem 2.2, and L be the number of users being selected in the first step of Algorithm 1. Then, as $N \to \infty$, with probability one*

$$L = \frac{e^{\rho(N)}}{\Gamma(M)\Gamma(K)} \left[1 + o\left(1\right)\right]. \qquad (2.48)$$

Proof- Using (2.26), the probability of a randomly chosen user k being pre-selected in the first step of Algorithm 1 can be calculated as,

$$\begin{aligned} p &= \Pr\{\lambda_{\max}(\mathbf{H}_k) > t\} \\ &= \frac{t^{M+K-2}e^{-t}}{\Gamma(M)\Gamma(K)}\left(1 + O(t^{-1})\right) \\ &= \frac{e^{\rho(N)}}{N\Gamma(M)\Gamma(K)}\left[1 + o(1)\right] \\ &= \frac{\ln\ln\ln N e^{q(N)}}{N\Gamma(M)\Gamma(K)}\left[1 + o(1)\right], \end{aligned} \tag{2.49}$$

where $q(N) = \rho(N) - \ln\ln\ln\ln N$. Consider the following probability:

$$\xi = \Pr\left\{Np(1-\epsilon) < L < Np(1+\epsilon)\right\}, \tag{2.50}$$

where $\epsilon = \sqrt{2\Gamma(M)\Gamma(K)e^{\frac{-q(N)}{4}}}$. Note that since $q(N) = \omega(1)$, we have $\lim_{N\to\infty}\epsilon = 0$. ξ can be computed as

$$\begin{aligned} \xi &= \sum_{l=\lceil Np(1-\epsilon)\rceil}^{\lfloor Np(1+\epsilon)\rfloor}\binom{N}{l}p^l(1-p)^{N-l} \\ &\approx 1 - Q\left(\frac{Np - Np(1-\epsilon)}{\sqrt{Np(1-p)}}\right) - Q\left(\frac{Np(1+\epsilon) - Np}{\sqrt{Np(1-p)}}\right) \\ &= 1 - 2Q\left(\frac{\sqrt{Np}\epsilon}{\sqrt{1-p}}\right) \\ &\approx 1 - \frac{2\sqrt{1-p}}{\sqrt{2\pi}\sqrt{Np}\epsilon}\exp\left(-\frac{Np\epsilon^2}{2(1-p)}\right). \end{aligned} \tag{2.51}$$

Substituting p from (2.49), and having $\epsilon^2 = 2\Gamma(M)\Gamma(K)e^{\frac{-q(N)}{2}}$, we have

$$\xi = 1 - O\left(\frac{e^{\frac{-q(N)}{4}}}{\sqrt{\ln\ln\ln N}}\right)\exp\left\{-\ln\ln\ln N e^{\frac{q(N)}{2}}\left[1 + o(1)\right]\right\} \tag{2.52}$$

Thus, $\lim_{N\to\infty}\xi = 1$. Finally, using (2.49) and (2.52), with probability one we have

$$\begin{aligned} L &= Np\left(1+O(\epsilon)\right) \\ &= \frac{e^{\rho(N)}}{\Gamma(M)\Gamma(K)}\left[1+o\left(1\right)\right]. \end{aligned} \tag{2.53}$$

■

Since $\rho(N) = o(\ln N)$, from Lemma 2.4, it is evident that $\lim_{N\to\infty}\frac{L}{N} = 0$. Therefore, only a small fraction of users are pre-selected. This results in reducing the amount of feedback sent to the base station.

As shown in Lemma 2.3, in the asymptotic case of $N \to \infty$, at most one eigenvector from each user is likely to be selected. This eigenvector corresponds to the maximum singular value of that user's channel matrix, and is denoted by $\mathbf{V}_{i,\max}$. Hence, for the sake of simplicity of notation, we define the measure of orthogonality between the users i and j, denoted by $\mathcal{O}(i,j)$, as the orthogonality measure between $\mathbf{V}_{i,\max}$ and $\mathbf{V}_{j,\max}$, defined in (2.3) as $z\left(\mathbf{V}_{i,\max},\mathbf{V}_{j,\max}\right)$. In other words,

$$\mathcal{O}(i,j) = |\mathbf{V}_{i,\max}^H\mathbf{V}_{j,\max}|^2. \tag{2.54}$$

Lemma 2.5 *The pdf of $\mathcal{O}(i,j)$ defined in (2.54) can be computed from*

$$f_{\mathcal{O}(i,j)}(z) = (M-1)(1-z)^{M-2}. \tag{2.55}$$

Proof- In Appendix A.

Definition 2.6 *A set $\mathcal{S} = \{\mathbf{h}_i\}_{i=1}^M$, in which $\mathbf{h}_i \in \mathbb{C}^{1\times M}$, is called ϵ-orthogonal if we have $z(\mathbf{h}_i,\mathbf{h}_j) < \epsilon$, for every $\mathbf{h}_i \neq \mathbf{h}_j \in \mathcal{S}$.*

Lemma 2.7 *Let* $t = \ln N + (M+K-2)\ln\ln N - \rho(N)$*, where* $\rho(N)$ *satisfies the conditions of Theorem 2.2. Then, as* $N \to \infty$*, the selected coordinates by Algorithm 1 construct an* $\epsilon(N)$*-orthogonal set, with probability one, where* $\epsilon(N) = e^{-\frac{q(N)}{M}}$*, and* $q(N) = \rho(N) - \ln\ln\ln\ln N$*.*

Proof- After selecting the first user, s_1, with largest maximum singular value, the user which is most orthogonal to s_1 is selected. In other words,

$$s_2 = \arg \min_{l \in \mathcal{S}_1} \mathcal{O}(l, s_1), \tag{2.56}$$

where $\mathcal{S}_1$ is defined in (2.7). First, we show that the users s_1 and s_2 are with probability one $\epsilon(N)$-orthogonal to each other, or equivalently, $\mathcal{O}(s_2, s_1) < \epsilon(N)$. To do this, consider the following probability:

$$\mu = \Pr\left\{\mathcal{O}(s_2, s_1) < \epsilon(N)\right\}. \tag{2.57}$$

Using (2.55), this probability can be written as

$$\begin{aligned}
\mu &= \Pr\left\{\min_l \mathcal{O}(l, s_1) < \epsilon(N)\right\} \\
&= 1 - \left(\Pr\left\{\mathcal{O}(l, s_1) > \epsilon(N)\right\}\right)^{L-1} \\
&= 1 - \left(\int_{\epsilon(N)}^{1} (M-1)(1-z)^{M-2} dz\right)^{L-1} \\
&= 1 - \left[1 - \epsilon(N)\right]^{(L-1)(M-1)} \\
&= 1 - \exp\left\{-(L-1)(M-1)\ln\left[1-\epsilon(N)\right]\right\} \\
&= 1 - \exp\left\{-(L-1)(M-1)\left[\epsilon(N) + O\left(\epsilon^2(N)\right)\right]\right\}.
\end{aligned} \tag{2.58}$$

Defining the event $\mathcal{D} = \{Np(1-\epsilon) < L < Np(1+\epsilon)\}$, with p and ϵ defined in (2.49) and (2.50), and using (2.52), a lower bound for μ is found as,

$$\begin{aligned}
\mu &\geq \Pr\{\mathcal{D}\}\left[1 - \exp\left\{-(Np(1-\epsilon)-1)(M-1)\left[\epsilon(N) + O\left(\epsilon^2(N)\right)\right]\right\}\right] \\
&= \left[1 - O\left(\frac{e^{\frac{-q(N)}{4}}}{\sqrt{\ln\ln\ln N}}\right)\exp\left\{-\ln\ln\ln N e^{\frac{q(N)}{2}}[1+o(1)]\right\}\right] \times \\
&\quad \left[1 - \exp\left\{-\frac{\ln\ln\ln N e^{\frac{(M-1)q(N)}{M}}}{\Gamma(M-1)\Gamma(K)}[1+o(1)]\right\}\right]. \qquad (2.59)
\end{aligned}$$

Since $q(N) \sim \omega(1)$, the above probability approaches one as $N \to \infty$. Therefore, with probability one users s_1 and s_2 are $\epsilon(N)$-orthogonal to each other.

Now, assume that m users, which construct an $\epsilon(N)$-orthogonal set $\mathcal{A}_m$, are selected up to the mth step of Algorithm 1. We show that the selected user in the $(m+1)$th step of this algorithm, s_{m+1}, is such that with probability one, $\mathcal{A}_{m+1} = \mathcal{A}_m + \{s_{m+1}\}$ is $\epsilon(N)$-orthogonal, or equivalently, s_{m+1} is $\epsilon(N)$-orthogonal to all users in $\mathcal{A}_m$. To this end, we define the following probability:

$$\nu_{k,m} = \Pr\{\mathcal{O}(s_1,k) < \alpha, \mathcal{O}(s_2,k) < \alpha, \cdots, \mathcal{O}(s_m,k) < \alpha\}, \qquad (2.60)$$

where $\alpha = \frac{\epsilon(N)}{M}$. $\nu_{k,m}$ is the probability that a randomly selected user k is α-orthogonal to all users in $\mathcal{A}_m$. This probability can be written as

$$\nu_{k,m} = \Pr\{\mathcal{O}(s_1,k) < \alpha\}\prod_{i=2}^{m}\kappa_i, \qquad (2.61)$$

where $\kappa_i = \Pr\{\mathcal{O}(s_i,k) < \alpha \mid \mathcal{O}(s_1,k) < \alpha, \cdots, \mathcal{O}(s_{i-1},k) < \alpha\}$. From (2.55), the first term in the right hand side of the above equation can be written as

$$\begin{aligned}
\Pr\{\mathcal{O}(s_1,k) < \alpha\} &= \int_0^{\alpha}(M-1)(1-z)^{M-2}dz \\
&= 1-(1-\alpha)^{M-1} \\
&= (M-1)\alpha + O(\alpha^2). \qquad (2.62)
\end{aligned}$$

In Appendix B, it has been proved that

$$\kappa_i = (M-i)\alpha + O(\alpha^{3/2}). \tag{2.63}$$

Hence, using (2.61), (2.62), and (2.63), we can write

$$\begin{aligned}
\nu_{k,m} &= \left[(M-1)\alpha + O(\alpha^2)\right] \prod_{i=2}^{m} \left[(M-i)\alpha + O(\alpha^{3/2})\right] \\
&= \frac{\Gamma(M)}{\Gamma(M-m)}\alpha^m + O\left(\alpha^{m+1/2}\right) \\
&= \frac{\Gamma(M)}{\Gamma(M-m)M^m}\left[[\epsilon(N)]^m + O\left([\epsilon(N)]^{(m+1/2)}\right)\right]
\end{aligned} \tag{2.64}$$

Now, we define ω_m as the probability of existing at least one user α-orthogonal to the users in the set $\mathcal{A}_m$. Noting that $\nu_{k,m}$ is the same for all k, we obtain,

$$\begin{aligned}
\omega_m &= 1 - \prod_{k=1}^{L-m}(1-\nu_{k,m}) \\
&= 1 - \exp\left\{(L-m)\ln(1-\nu_{k,m})\right\} \\
&= 1 - \exp\left\{(L-m)\left[-\nu_{k,m} + O(\nu_{k,m}^2)\right]\right\}.
\end{aligned} \tag{2.65}$$

Similar to (2.59), we can compute ω_m as,

$$\begin{aligned}
\omega_m = & \left[1 - O\left(\frac{e^{\frac{-q(N)}{4}}}{\sqrt{\ln\ln\ln N}}\right)\exp\left\{-\ln\ln\ln N e^{\frac{q(N)}{2}}[1+o(1)]\right\}\right] \times \\
& \left[1 - \exp\left\{-\frac{\ln\ln\ln N e^{\frac{(M-m)q(N)}{M}}}{\Gamma(M-m)M^m\Gamma(K)}[1+o(1)]\right\}\right].
\end{aligned} \tag{2.66}$$

Since $m \leq M-1$, it follows that $\lim_{N\to\infty}\omega_m = 1$. In other words, as N tends to infinity, with probability one there exists at least one user u_{m+1}, α-orthogonal to all users in $\mathcal{A}_m$.

Consider user s_{m+1} which is selected in the $(m+1)$th step of Algorithm 1. Obviously, we have

$$\begin{aligned}\sum_{j=1}^{m}\mathcal{O}(s_{m+1},s_j) &\leq \sum_{j=1}^{m}\mathcal{O}(u_{m+1},s_j)\\ &\leq m\alpha\\ &= \frac{m\epsilon(N)}{M}\\ &\leq \epsilon(N).\end{aligned}\tag{2.67}$$

Knowing the fact that $\mathcal{O}(s_{m+1},s_j)\geq 0$, for $j=1,\cdots m$, we can write

$$\mathcal{O}(s_{m+1},s_j)\leq \epsilon(N),\quad j=1,\cdots m$$

which means that with probability one, s_{m+1} is $\epsilon(N)$-orthogonal to the users in the set $\mathcal{A}_m$, and consequently, $\mathcal{A}_{m+1}$ is an $\epsilon(N)$-orthogonal set.

Let us define $\mathcal{X}_m$ as the event that the set $\mathcal{A}_m$ is $\epsilon(N)$-orthogonal. We can write

$$\Pr\{\mathcal{X}_M\}=\Pr\{\mathcal{X}_2\}\prod_{m=3}^{M}\Pr\{\mathcal{X}_m|\mathcal{X}_{m-1}\}.\tag{2.68}$$

From (2.59) and (2.66), the above probability is lower-bounded as

$$\begin{aligned}\Pr\{\mathcal{X}_M\} &\geq \mu\prod_{m=2}^{M-1}\omega_m\\ &\geq \left[1-O\left(\frac{e^{\frac{-q(N)}{4}}}{\sqrt{\ln\ln\ln N}}\right)\exp\left\{-\ln\ln\ln Ne^{\frac{q(N)}{2}}[1+o(1)]\right\}\right]^{M-1}\times\\ &\quad \left[1-\exp\left\{-\frac{\ln\ln\ln Ne^{\frac{(M-1)q(N)}{M}}}{\Gamma(M-1)\Gamma(K)}[1+o(1)]\right\}\right]\times\\ &\quad \prod_{m=2}^{M-1}\left[1-\exp\left\{-\frac{\ln\ln\ln Ne^{\frac{(M-m)q(N)}{M}}}{\Gamma(M-m)M^m\Gamma(K)}[1+o(1)]\right\}\right]\\ &= 1-\exp\left\{-\frac{\ln\ln\ln Ne^{\frac{q(N)}{M}}}{M^{M-1}\Gamma(K)}[1+o(1)]\right\}.\end{aligned}\tag{2.69}$$

Therefore, $\lim_{N\to\infty} \Pr\{\mathcal{X}_M\} = 1$. In other words, the selected coordinates by Algorithm 1, with probability one, construct an $\epsilon(N)$-orthogonal set as N tends to infinity, which completes the proof of Lemma 2.7.

■

As mentioned earlier, after selecting the coordinates, the "selected coordinate matrix", $\boldsymbol{\mathcal{H}}$, is constructed using (2.9). By applying zero-forcing beam-forming, the information vector, $\mathbf{u}$, is multiplied by $\boldsymbol{\mathcal{H}}^{-1}$ to construct the transmitted signal as (2.10). Using (2.11), we can write

$$\mathbf{r} = \mathbf{u} + \mathbf{w}, \tag{2.70}$$

where $\mathbf{r} = [r_{s_1,d_1}, \cdots, r_{s_M,d_M}]^T$, $\mathbf{u} = [u_{s_1,d_1}, \cdots, u_{s_M,d_M}]^T$, and $\mathbf{w} = [w_{s_1,d_1}, \cdots, w_{s_M,d_M}]^T$. Having the power constraint P for $\mathbf{x}$, the sum-rate capacity can be computed as [4],

$$\mathcal{R}_{\text{Prop}} = \mathbb{E}_{\boldsymbol{\mathcal{H}}} \left\{ \max_{\substack{P_m \\ \sum_{m=1}^{M} \gamma_m P_m \leq P}} \sum_{m=1}^{M} \ln(1 + P_m) \right\}, \quad \gamma_m = \left[\left(\boldsymbol{\mathcal{H}}^H \boldsymbol{\mathcal{H}} \right)^{-1} \right]_{m,m}, \tag{2.71}$$

where $[\mathbf{A}]_{i,j}$ denotes the entry of matrix $\mathbf{A}$ in the ith row and the jth column. The optimal P_m's in (2.71) can be obtained by "water-filling". Here, we assume that P_m's are all equal (uniform power allocation). Thus,

$$P_m = \frac{P}{\text{Tr}\left\{ \left[\boldsymbol{\mathcal{H}}^{\text{H}} \boldsymbol{\mathcal{H}} \right]^{-1} \right\}}. \tag{2.72}$$

Consequently,

$$\mathcal{R}_{\text{Prop}}^{\text{U}} = \mathbb{E}_{\boldsymbol{\mathcal{H}}} \left\{ M \ln \left(1 + \frac{P}{\text{Tr}\left\{ \left[\boldsymbol{\mathcal{H}}^{\text{H}} \boldsymbol{\mathcal{H}} \right]^{-1} \right\}} \right) \right\}, \tag{2.73}$$

where $\mathcal{R}_{\text{Prop}}^{\text{U}}$ stands for the sum-rate achieving by the proposed method, when the power is uniformly allocated among the coordinates.

Having defined $\mathcal{X}_M$ in (2.68) and using (2.69), the above equation can be written as follows:

$$\begin{aligned}
\mathcal{R}^{\mathrm{U}}_{\mathrm{Prop}} &= \mathbb{E}_{\boldsymbol{\mathcal{H}}}\left\{ M \ln\left(1+\frac{P}{\mathrm{Tr}\left\{\left[\boldsymbol{\mathcal{H}}^{\mathrm{H}}\boldsymbol{\mathcal{H}}\right]^{-1}\right\}}\right)\middle| \mathcal{X}_M \right\} \Pr\{\mathcal{X}_M\} + \\
&\quad \mathbb{E}_{\boldsymbol{\mathcal{H}}}\left\{ M \ln\left(1+\frac{P}{\mathrm{Tr}\left\{\left[\boldsymbol{\mathcal{H}}^{\mathrm{H}}\boldsymbol{\mathcal{H}}\right]^{-1}\right\}}\right)\middle| \mathcal{X}_M^C \right\} (1-\Pr\{\mathcal{X}_M\}) \\
&\geq \mathbb{E}_{\boldsymbol{\mathcal{H}}}\left\{ M \ln\left(1+\frac{P}{\mathrm{Tr}\left\{\left[\boldsymbol{\mathcal{H}}^{\mathrm{H}}\boldsymbol{\mathcal{H}}\right]^{-1}\right\}}\right)\middle| \mathcal{X}_M \right\} \Pr\{\mathcal{X}_M\} \\
&= \left(1-\exp\left\{-\frac{\ln\ln\ln N e^{\frac{q(N)}{M}}}{M^{M-1}\Gamma(K)}[1+o(1)]\right\}\right) \times \\
&\quad \mathbb{E}_{\boldsymbol{\mathcal{H}}}\left\{ M \ln\left(1+\frac{P}{\mathrm{Tr}\left\{\left[\boldsymbol{\mathcal{H}}^{\mathrm{H}}\boldsymbol{\mathcal{H}}\right]^{-1}\right\}}\right)\middle| \mathcal{X}_M \right\}, \qquad (2.74)
\end{aligned}$$

where $\mathcal{X}_M^C$ is the complement of $\mathcal{X}_M$.

From Algorithm 1, it is obvious that the corresponding singular values of the selected eigenvectors are greater that $t = \ln N + (M+K-2)\ln\ln N - \rho(N)$. However, the following lemma which is proved in Appendix C, states that the singular values of all selected dimensions, with probability one, can not exceed $\ln N + (M+K-1)\ln\ln N$:

Lemma 2.8 *Let* $t = \ln N + (M+K-1)\ln\ln N$*. Then,*

$$\eta = Pr\left\{\max_{k=1,\cdots,N} \lambda_{\max}(\mathbf{H}_k) > t\right\} = O\left(\frac{1}{\ln N}\right). \qquad (2.75)$$

As a result of this lemma, the singular values corresponding to the all selected dimensions can be expressed as $\ln N + o(\ln N)$.

To compute the conditional probability $\mathbb{E}_{\boldsymbol{\mathcal{H}}}\left\{M\ln\left(1+\frac{P}{\mathrm{Tr}\left\{\left[\boldsymbol{\mathcal{H}}^{\mathrm{H}}\boldsymbol{\mathcal{H}}\right]^{-1}\right\}}\right)\middle|\, \mathcal{X}_M\right\}$, we define $\boldsymbol{\mathcal{B}} = \boldsymbol{\mathcal{H}}\boldsymbol{\mathcal{H}}^H$. Conditioned on $\mathcal{X}_M$, i.e., having $\epsilon(N)$-orthogonality among the selected dimensions, using (2.9), and the results of Lemma 2.7 and Lemma 2.8, we can write

$$\boldsymbol{\mathcal{B}}_{ii} = \|\mathbf{g}_{s_i,d_i}\|^2 = \ln N + f(N), \tag{2.76}$$

and

$$\begin{aligned} |\boldsymbol{\mathcal{B}}_{ij}| &= \sqrt{\|\mathbf{g}_{s_i,d_i}\|^2\|\mathbf{g}_{s_j,d_j}\|^2 z\left(\mathbf{V}_{s_i,d_i},\mathbf{V}_{s_j,d_j}\right)} \\ &= \sqrt{O(\ln N)\times O(\ln N)\times O\left(\epsilon(N)\right)} \\ &= O(\epsilon(N)\ln N), \end{aligned} \tag{2.77}$$

where $f(N) = o(\ln N)$. In Appendix D it has been shown that any diagonal element of $\boldsymbol{\mathcal{B}}^{-1}$ can be expressed as $[\ln N]^{-1} + O\left(\frac{h(N)}{\ln N}\right)$, where

$$h(N) \triangleq \max\left(\frac{f(N)}{\ln N}, \epsilon(N)\right) = o(1). \tag{2.78}$$

Having this, and using the fact that $\mathrm{Tr}\left\{\left[\boldsymbol{\mathcal{H}}^{\mathrm{H}}\boldsymbol{\mathcal{H}}\right]^{-1}\right\} = \mathrm{Tr}\left\{\boldsymbol{\mathcal{B}}^{-1}\right\}$, we can write

$$\mathbb{E}_{\boldsymbol{\mathcal{H}}}\left\{M\ln\left(1+\frac{P}{\mathrm{Tr}\left\{\left[\boldsymbol{\mathcal{H}}^{\mathrm{H}}\boldsymbol{\mathcal{H}}\right]^{-1}\right\}}\right)\middle|\, \mathcal{X}_M\right\} = \mathbb{E}_{\boldsymbol{\mathcal{B}}}\left\{M\ln\left(1+\frac{P}{\mathrm{Tr}\left\{\boldsymbol{\mathcal{B}}^{-1}\right\}}\right)\middle|\, \mathcal{X}_M\right\}$$

$$\begin{aligned} &= M\ln\left(1+\frac{P}{M[\ln N]^{-1}+O\left(\frac{h(N)}{\ln N}\right)}\right) \\ &= M\ln\left(1+\frac{P}{M[\ln N]^{-1}\left[1+O\left(h(N)\right)\right]}\right) \\ &= M\ln\left(\frac{P}{M}\ln N + O(h(N)\ln N)\right). \end{aligned} \tag{2.79}$$

From (2.74) and (2.79), we have

$$\begin{aligned}\mathcal{R}^{\mathrm{U}}_{\mathrm{Prop}} &\geq M\ln\left(\frac{P}{M}\ln N + O(h(N)\ln N)\right)\times \\ &\left(1-\exp\left\{-\frac{\ln\ln\ln N e^{\frac{q(N)}{M}}}{M^{M-1}\Gamma(K)}[1+o(1)]\right\}\right). \qquad (2.80)\end{aligned}$$

Since adaptive power allocation (using "water-filling") results in higher sum-rate than that of uniform power allocation, we have $\mathcal{R}_{\mathrm{Prop}} \geq \mathcal{R}^{\mathrm{U}}_{\mathrm{Prop}}$. Having the fact that [26]

$$\mathcal{C}_{\mathrm{sum}} = M\ln\left(\frac{P}{M}\ln N + O(\ln\ln N)\right), \qquad (2.81)$$

and using (2.80), we have

$$\begin{aligned}\mathcal{C}_{\mathrm{sum}} - \mathcal{R}_{\mathrm{Prop}} &\leq M\ln\left(\frac{P}{M}\ln N + g_1(N)\right) - \\ &\quad M\ln\left(\frac{P}{M}\ln N + g_2(N)\right)(1-g_3(N)) \\ &= M\ln\left(1+\frac{Mg_1(N)}{P\ln N}\right) - M\ln\left(1+\frac{Mg_2(N)}{P\ln N}\right) + \\ &\quad Mg_3(N)\ln\left(\frac{P}{M}\ln N + g_2(N)\right), \qquad (2.82)\end{aligned}$$

where $g_1(N) = O(\ln\ln N)$, $g_2(N) = O\left(h(N)\ln N\right)$, and

$$g_3(N) = \exp\left\{-\frac{\ln\ln\ln N e^{\frac{q(N)}{M}}}{M^{M-1}\Gamma(K)}[1+o(1)]\right\} = o\left(\frac{1}{\ln\ln N}\right). \qquad (2.83)$$

From (2.78) and (2.83), and Using the approximation $\ln(1+x) \approx x$, for $x \ll 1$, and we can write

$$\begin{aligned}\mathcal{C}_{\mathrm{sum}} - \mathcal{R}_{\mathrm{Prop}} &\sim M\left(\frac{M[g_1(N)-g_2(N)]}{P\ln N}\right) + Mg_3(N)\ln\left(\frac{P}{M}\ln N + g_1(N)\right) \\ &= o(1). \qquad (2.84)\end{aligned}$$

Consequently,

$$\lim_{N\to\infty} \mathcal{C}_{\text{sum}} - \mathcal{R}_{\text{Prop}} = 0, \tag{2.85}$$

which completes the proof of Theorem 2.2.

■

Theorem 2.2 implies that using Algorithm 1, and applying zero-forcing beamforming at the base station, the same performance as when the optimum user selection algorithm and optimum precoding scheme is utilized, can asymptotically be achieved.

Remark 1- Although in the proof of Theorem 2.2, we showed that $\lim_{N\to\infty} \mathcal{C}_{\text{sum}} - \mathcal{R}_{\text{Prop}} = 0$, it is interesting to minimize the order of difference.
Rewriting (2.84), we get

$$\mathcal{C}_{\text{sum}} - \mathcal{R}_{\text{Prop}} = O\left(\varrho(N)\right) + \exp\left\{-\frac{\ln\ln\ln N e^{1/\epsilon(N)}}{M^{M-1}\Gamma(K)}[1+o(1)]\right\} O(\ln\ln N), \tag{2.86}$$

where $\varrho(N) = \max\left(h(N), \frac{\ln\ln N}{\ln N}\right)$, and $h(N)$ is defined in (2.78). Hence, in order to minimize the order of difference, we must have $h(N) = O\left(\frac{\ln\ln N}{\ln N}\right)$, which incurs $\epsilon(N) = O\left(\frac{\ln\ln N}{\ln N}\right)$ and $f(N) = O(\ln\ln N)$. As a result,

$$\begin{aligned} q(N) &= -M \ln \epsilon(N) \\ &= M \ln\ln N - M \ln\ln\ln N + \psi(N), \end{aligned} \tag{2.87}$$

where $\psi(N)$ is an arbitrary function with the condition $\lim_{N\to\infty} \psi(N) = c > 0$. Hence, using the definition of $q(N)$ in Lemma 2.7, we can write

$$t = \ln N + (K-2)\ln\ln N + M \ln\ln\ln N - \ln\ln\ln\ln N - \psi(N). \tag{2.88}$$

Also, to guarantee $f(N) = O(\ln \ln N)$, we must have

$$t = \ln N + O(\ln \ln N), \tag{2.89}$$

which means $\psi(N) = O(\ln \ln N)$. Having these conditions on t, we can guarantee $\mathcal{C}_{\text{sum}} - \mathcal{R}_{\text{Prop}} = O\left(\frac{\ln \ln N}{\ln N}\right)$.

Remark 2- It is important to note that satisfying $\lim_{N\to\infty} \mathcal{C}_{\text{sum}} - \mathcal{R}_{\text{Prop}} = 0$, is much more challenging than that of $\lim_{N\to\infty} \frac{\mathcal{R}_{\text{Prop}}}{\mathcal{C}_{\text{sum}}} = 1$. The following lemma, which is proved in Appendix E, clarifies this fact:

Lemma 2.9 *Suppose that in Algorithm 1, $t = \ln N$, and the coordinates are chosen randomly among the pre-selected eigenvectors. Then,*

$$\lim_{N\to\infty} \frac{\mathcal{R}_{\text{Prop}}}{\mathcal{C}_{\text{sum}}} = 1. \tag{2.90}$$

The above lemma states that to satisfy $\lim_{N\to\infty} \frac{\mathcal{R}_{\text{Prop}}}{\mathcal{C}_{\text{sum}}} = 1$, the orthogonality among the coordinates is not a necessary condition.

2.4.2 Comparison with other Downlink Strategies

In this section, we compare the performance of our proposed scheme with some other downlink strategies in terms of sum-rate capacity. To have a good measure for comparison, we give the following definition:

Definition 2.10 *For a MIMO-BC in which a base station, and average power constraint P communicating to N users, using strategy S, the* ***multiplexing gain***

is defined as [6]

$$r_S = \lim_{P\to\infty} \frac{\mathcal{R}_S(P,N)}{\ln P}, \tag{2.91}$$

and the ***multiuser diversity gain*** *is defined as*

$$d_S = \lim_{N\to\infty} \frac{\mathcal{R}_S(P,N)}{r_S \ln\ln N}, \tag{2.92}$$

where $\mathcal{R}_S(P,N)$ is the achievable sum-rate.

Lemma 2.11 *Using the proposed algorithm, and applying zero-forcing beam-forming, we can achieve $r = M$, and $d = 1$, which are the maximum achievable values in a MIMO-BC.*

Proof- Appendix F.

Time Division Multiple Access (TDMA)

In this scheme, the base station only serves one user in each time slot. Hence, to achieve the maximum sum-rate, the user which has the maximum single-user capacity should be served. Because of its simplicity, this strategy is widely used in the downlink of the cellular networks. The achievable sum-rate of this scheme can be written as

$$\mathcal{R}_{\text{TDMA}} = \mathbb{E}\left\{ \max_k \max_{\substack{\mathbf{Q}_k \\ \text{Tr}\{\mathbf{Q}_\text{k}\}=\text{P}}} \ln \left| \mathbf{I}_{K\times K} + \mathbf{H}_k \mathbf{Q}_k \mathbf{H}_k^H \right| \right\}, \tag{2.93}$$

[6] More precisely, as in [33], r is the maximum achievable multiplexing gain when diversity gain approaches zero.

where $\mathbf{Q}_k$ is obtained by "water-filling". Using (2.91) and (2.92), and the result of Lemma 2.3 in [34], the *multiplexing gain* and *multiuser diversity gain* for this scheme can be obtained as,

$$\begin{aligned} r_{\text{TDMA}} &= \lim_{P\to\infty} \frac{\mathcal{R}_{\text{TDMA}}(P,N)}{\ln P} \\ &= \lim_{P\to\infty} \frac{\mathbb{E}\left\{\max_k\left(\sum_{i=1}^{\min(M,K)} \ln\left(\frac{P\lambda_i(k)}{\min(M,K)}\right)\right)\right\}}{\ln P} \\ &= \min(M,K), \end{aligned} \tag{2.94}$$

and,

$$\begin{aligned} d_{\text{TDMA}} &= \lim_{N\to\infty} \frac{\mathcal{R}_{\text{TDMA}}(P,N)}{\min(M,K)\ln\ln N} \\ &= 1. \end{aligned} \tag{2.95}$$

Hence, this scheme achieves the full multiuser diversity gain, while achieving the full multiplexing gain only in the case of $K \geq M$.
Although this method has been shown to be optimal for single-antenna broadcast channel ($M = 1$) [35], for the case of $M > K \geq 1$, as a result of losing the multiplexing gain, this method is no longer optimum [7].

From the proof of the Lemma 2.3 in [34], it can be observed that the upper and lower bounds for $\mathcal{R}_{\text{TDMA}}$ have the same behavior asymptotically almost surely, when $N \to \infty$. In other words[8],

$$\begin{aligned} K\ln\left(1+\frac{P}{K}\max_k \lambda_{\min}(\mathbf{H}'_k{\mathbf{H}'_k}^H)\right) &\approx K\ln\left(1+\frac{P}{K^2}\max_k \text{Tr}(\mathbf{H}_{\text{k}}\mathbf{H}_{\text{k}}^{\text{H}})\right) \\ &\approx K\ln(1+\frac{P}{K^2}\ln N), \end{aligned} \tag{2.96}$$

[7]For the case of $K \geq M$, this scheme is not optimal either. This fact will be discussed in more details later.

[8] It is assumed that $K \leq M$.

where $\mathbf{H}'_k$ $(K \times K)$ is a truncated version of $\mathbf{H}_k$ by omitting the $M-K$ columns of $\mathbf{H}_k$. From (2.96), and having the fact that $\lambda_{\min}(\mathbf{H}'_k{\mathbf{H}'_k}^H) \leq \lambda_{\min}(\mathbf{H}_k{\mathbf{H}_k}^H)$, the following observations can be obtained:

Observation 1- For the user which maximizes the single-user capacity in (2.93), (l), all the eigenvalues should be of the same order. In other words,

$$\lambda j(\mathbf{H}_l\mathbf{H}_l^H) = \frac{\ln N}{K} + O(\ln\ln N), \qquad j = 1, \cdots, K. \tag{2.97}$$

As a result of this, $\mathbf{H}_l\mathbf{H}_l^H$ tends to the identity matrix.

Observation 2- The user with maximum single-user capacity has the maximum $\lambda_{\min}$, asymptotically.

For the case of $K \geq M$, similar to (2.96), the asymptotic sum-rate capacity can be computed as

$$\mathcal{R}_{\text{TDMA}} \approx M \ln\left(\frac{P}{M^2}\ln N\right). \tag{2.98}$$

In this case, it can be easily shown that $\lim_{N\to\infty}\frac{\mathcal{R}_{\text{TDMA}}}{\mathcal{C}_{\text{sum}}} = 1$. In other words, the optimum sum-rate can asymptotically be achieved. However, the selected dimensions by TDMA belong to the same user and have the asymptotic behavior of $\frac{\ln N}{M}$, while in our proposed method the selected dimensions belong to different users with the asymptotic behavior of $\ln N$. Moreover, we have

$$\begin{aligned} \mathcal{C}_{\text{sum}} - \mathcal{R}_{\text{TDMA}} &\approx M\ln\left(1 + \frac{P}{M}\ln N\right) - M\ln\left(1 + \frac{P}{M^2}\ln N\right) \\ &\sim M \ln M. \end{aligned} \tag{2.99}$$

As can be observed from figure 2.2, this gap affects the performance significantly, especially when M is large.

Random Selection

In this method, the base station randomly selects M users for transmission. This results in having fairness in the system. This strategy can also be regarded as Round-Robin scheduling algorithm, when the users are randomly divided into groups of size M, and the base station serves one group in each time slot.

In Appendix G, it is shown that using multiple dimensions for transmission results in having *multiplexing gain* equal to M. However, because of random selection of the users, this scheme does not provide multiuser diversity gain. More precisely,

$$\begin{aligned} d_{\text{RS}} &= \lim_{N\to\infty} \frac{\mathcal{R}_{\text{RS}}(P,N)}{M \ln\ln N} \\ &= \lim_{N\to\infty} \frac{\mathbb{E}_{\mathbf{H}_1,\cdots,\mathbf{H}_M}\left\{\max_{\substack{\mathbf{Q}_m \\ \sum \text{Tr}(\mathbf{Q}_\text{m})=\text{P}}} \ln\left|\mathbf{I}_M + \sum_{m=1}^{M} \mathbf{H}_m^H \mathbf{Q}_m \mathbf{H}_m\right|\right\}}{M \ln\ln N} \\ &= \lim_{N\to\infty} \frac{O(1)}{M \ln\ln N} \\ &= 0. \end{aligned} \tag{2.100}$$

As a result of lacking multiuser diversity gain, this scheme shows a weak performance especially for large number of users. (Figure 2.2)

2.4.3 Simulation Results

So far, we have shown that as N tends to infinity, our scheme achieves the optimum sum-rate which scales like $M \ln\left(\frac{P}{M}\ln N\right)$. In this section, simulation results are provided to examine the performance of our proposed scheme in practical networks with finite number of users.

Figure 2.1 shows the optimum threshold (computed by exhaustive search) as a function of the number of users for $M = 2, K = 1$, and $M = 4, K = 1$. These curves show that the optimum threshold for each N, lies between $\ln N - \ln\ln N$, and $\ln N$, which is consistent with (2.88), in which we showed that the optimum threshold behaves as $\ln N + (K-2)\ln\ln N + O(\ln\ln\ln N)$, which lies between $\ln N + (K-2)\ln\ln N$ and $\ln N + (K-1)\ln\ln N$. Note that, in general, the optimum threshold value depends on both SNR and N. However, as N increases (or SNR increases) this dependency decreases.

Figures 2.2 presents the plots of the corresponding sum-rate versus the number of users for different number of transmit and receive antennas. The Signal to Noise Ratio (SNR), which is equal to the transmitted power P, is fixed to 10 dB in all curves. For comparison, the plots of the sum-rate when using TDMA and Random Selection algorithms, as well as the optimum scheme of dirty-paper coding are also given. For Random Selection algorithm, it is assumed that the optimum precoding scheme of dirty-paper coding is used.

Figure 2.3 depicts the plots of sum-rate capacity versus SNR (P), for $M = 2, K = 1$ and $M = 4, K = 1$. The number of users is fixed to 100 in both curves. It can be observed that the sum-rate achieving by the proposed scheme shows a linear increase with $\ln P$ in high SNRs with the slope equal to M. This confirms achieving the multiplexing gain of M by the proposed scheme. The fading model we have considered in our work is Rayleigh fading. However, it is interesting to investigate the performance of our proposed algorithm for more general fading models. Figure 2.4 depicts the achievable sum-rate of the proposed algorithm, as well as the maximum sum-rate and achievable sum-rates of TDMA and Random Selection schemes, versus the number of users . The fading model is assumed to be Rician with Rician Factor [9] equal to one. It is also assumed that $M = 2$, $K = 1$

[9] *Rician Factor* is defined as the ratio of the power of Line of Sight (LOS) to the power of

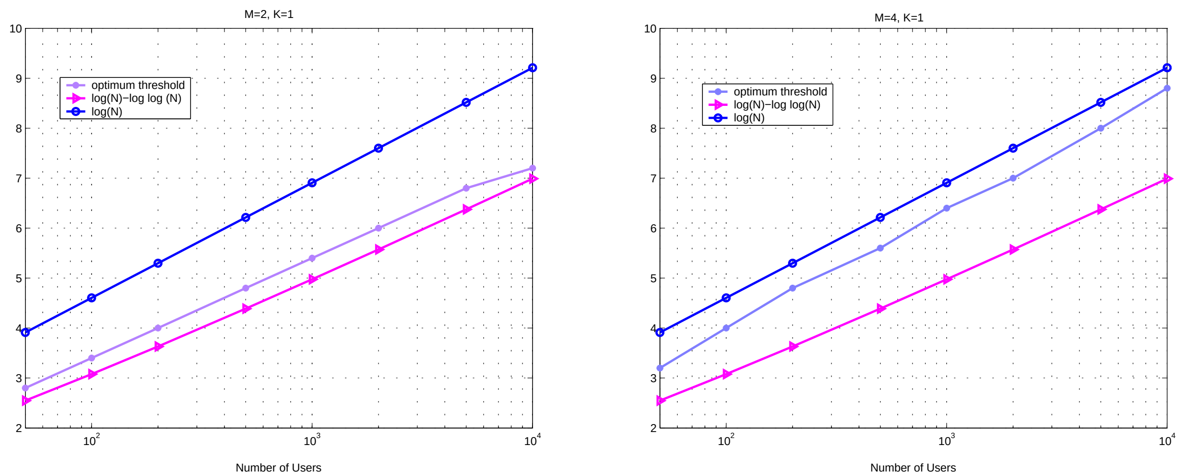

Figure 2.1: Optimum threshold versus the number of users.

and $P = 10$ dB. As can be observed, the proposed algorithm almost achieves the capacity of the system. However, the convergence rate of the sum-rates is slower than that of the Rayleigh fading case.

2.5 Complexity Analysis

2.5.1 Amount of Feedback

As can be observed in the proposed algorithm, only the eigenvectors that belong to $\mathcal{S}_0$, defined in (2.6), must be sent back to the base station, along with their corresponding singular values. For the asymptotic case of $N \to \infty$, from Lemma 2.4, we conclude that the cardinality of $\mathcal{S}_0$ scales as $\frac{e^{\rho(N)}}{\Gamma(M)\Gamma(K)}$. Assuming that for each eigenvector and its singular value $2M$ real values must be fed back, the total number of real values required at the base station is asymptotically equal to

Non-Line of Sight (NLOS) component.

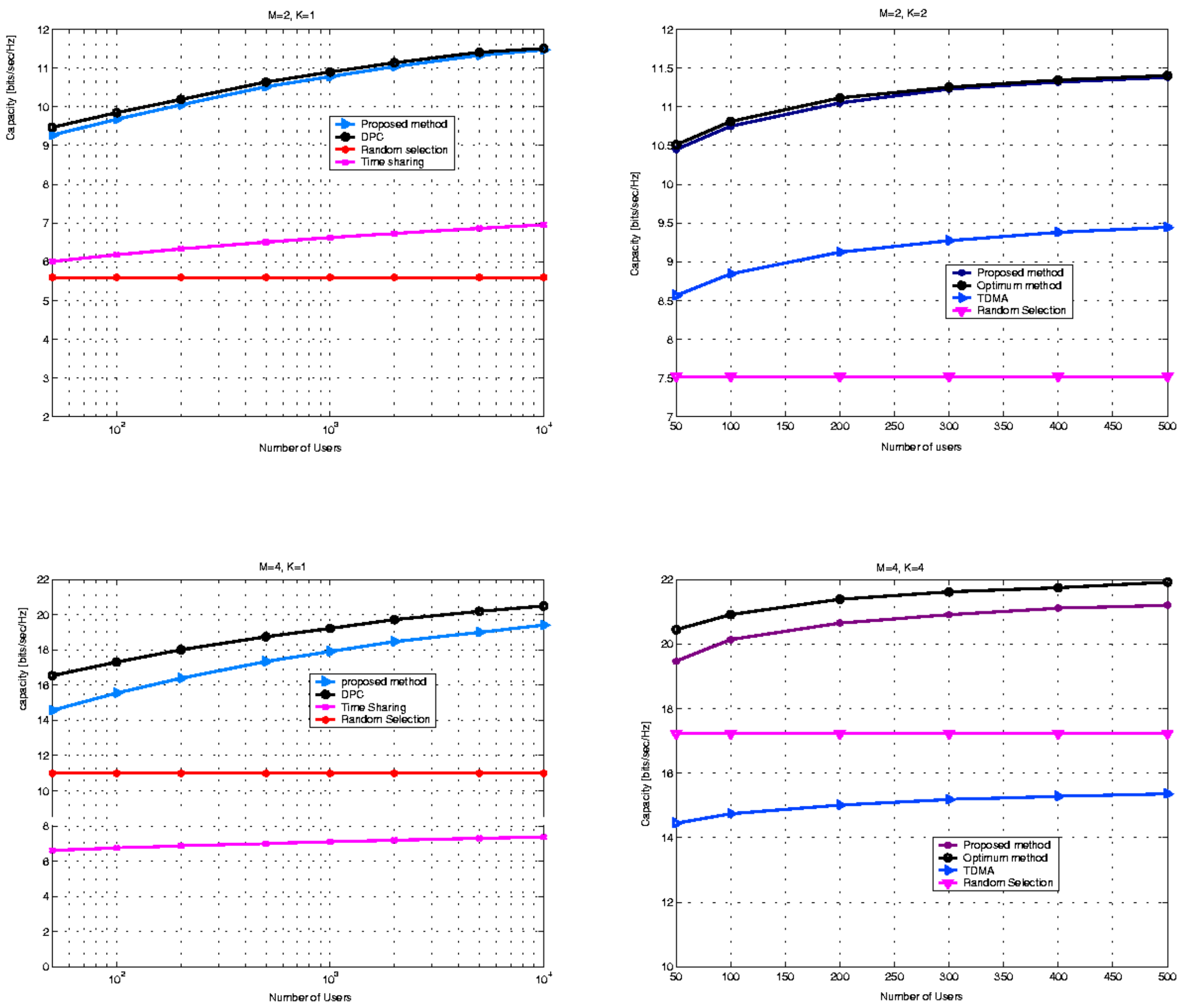

Figure 2.2: Sum-rate capacity versus the number of users, $P = 10$dB.

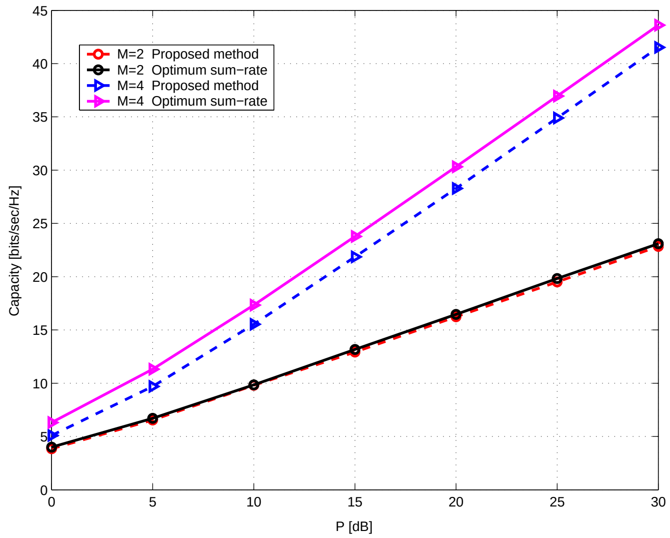

Figure 2.3: Sum-rate capacity versus transmit power, $N = 100, K = 1$.

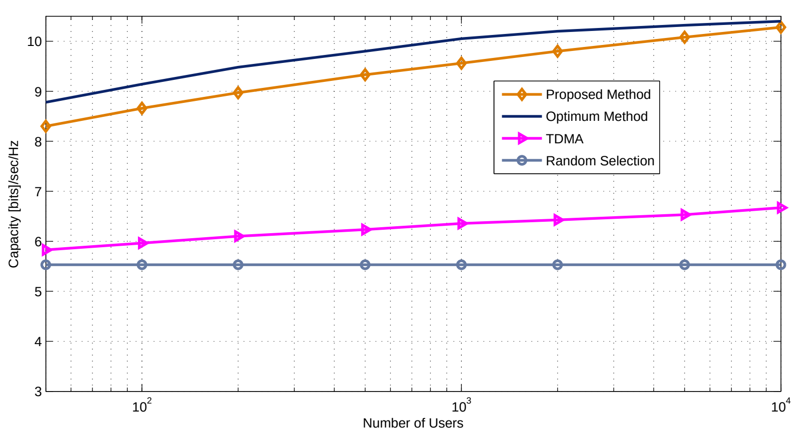

Figure 2.4: Sum-rate versus the number of users, Rician fading with Rician Factor=1.

$\frac{2Me^{\rho(N)}}{\Gamma(M)\Gamma(K)}$.

From Theorem 2.1, we observe that to achieve the optimum sum-rate, i.e., $\lim_{N\to\infty} \mathcal{C}_{\text{sum}} - \mathcal{R}_{\text{Prop}} = 0$, the following condition must be satisfied:

$$\rho(N) \quad = \quad \ln\ln\ln\ln N + \ln[\Gamma(K)\Gamma(M)] + \omega\left(\frac{1}{\ln\ln\ln N}\right). \tag{2.101}$$

As a result,

$$\mathcal{N}_{\text{Prop}} \quad = \quad 2M \ln\ln\ln N + \omega(1), \tag{2.102}$$

where $\mathcal{N}_{\text{Prop}}$ stands for the amount of feedback (in terms of the total number of real values required at the base station) in the proposed method. From the above equation, it follows that the minimum amount of feedback required to achieve the optimum performance is lower-bounded by $\ln\ln\ln N$, in the proposed algorithm. However, in [30], it has been shown that the same result holds for any other strategies.

In order to guarantee $\lim_{N\to\infty} \mathcal{C}_{\text{sum}} - \mathcal{R}_{\text{Prop}} = 0$ in the proposed scheme, using Theorem 2.2, the following condition must be satisfied:

$$\mathcal{N}_{\text{Prop}} \quad = \quad \omega(\ln\ln\ln N). \tag{2.103}$$

Note that the computation of $\gamma_{k,j}$'s in Algorithm 1 (eq. (2.8)) can be performed in the mobile sides, which reduces the amount of feedback further. This idea is described in details as the following algorithm:

Algorithm 2 (Modified version of Algorithm 1):

1. Set the thresholds t and β.

2. Define

$$\mathcal{S}_0 = \{(k,j)| \quad \lambda_j(k) > t\}.$$

   For all $(k,j) \in \mathcal{S}_0$, send $\lambda_j(k)$ to the base station.

3. Let $(s_1, d_1) = \arg \quad \max_{(k,j)\in\mathcal{S}_0} \lambda_j(k)$. Base station informs the user s_1 to feed back the eigenvector corresponding to its maximum singular value and after receiving it, sends these information to all the users in $\mathcal{S}_0 - \{(s_1, d_1)\}$.

4. Define $\gamma_{k,j}(0) = 0$ for all $(k,j) \in \mathcal{S}_0$. For $m = 1$ to $M-1$ the following steps are repeated:

   – Define $\mathcal{S}_m = \left\{(k,j)|(k,j) \in \mathcal{S}_{m-1}, |\mathbf{V}^H_{s_m,d_m}\mathbf{V}_{k,j}|^2 < \beta\right\}$ and $\gamma_{k,j}(m) = |\mathbf{V}^H_{s_m,d_m}\mathbf{V}_{k,j}|^2 + \gamma_{k,j}(m-1)$, for all $(k,j) \in \mathcal{S}_m$. All users in $\mathcal{S}_m$ feed back their corresponding $\gamma_{k,j}(m)$ to the base station.

   – Select $(s_{m+1}, d_{m+1}) = \arg \quad \min_{(k,j)\in\mathcal{S}_m} \quad \gamma_{k,j}(m)$. Base station inform the user s_m to feedback its d_mth eigenvector, and after receiving, sends it to all users in $\mathcal{S}_m - -\{(s_m, d_m)\}$.

For the asymptotic case of $N \to \infty$, having $t = \ln N + (M + K - 2)\ln\ln N -$

$\ln\ln\ln\ln N - q(N)$ and $\beta = e^{-\frac{q(N)}{M}}$, and using equations (2.53) and (2.64), we have

$$\begin{aligned}
\mathcal{N}_{\text{Prop}} &= \sum_{m=0}^{M-1} |\mathcal{S}_m| + 2M^2 \\
&= \sum_{m=0}^{M-1} L \times \Pr\{k \in \mathcal{S}_m | k \in \mathcal{S}_0\} + 2M^2 \\
&= L + L\sum_{m=1}^{M-1} O(e^{-\frac{mq(N)}{M}}) + 2M^2 \\
&= L\left[1 + O\left(e^{-\frac{q(N)}{M}}\right)\right] \\
&= \frac{e^{\rho(N)}}{\Gamma(M)\Gamma(K)}.
\end{aligned} \tag{2.104}$$

Figure 2.5 depicts the plots of the required amount of feedback versus the number of users for $M = 2, K = 1$ and $M = 4, K = 1$, when Algorithm 1 and Algorithm 2 are used. The measure for the amount of feedback is defined as the number of real components per user that should be sent to the base station. In these curves, the optimum values for the thresholds (t and β) are found by exhaustive search. Since the optimum threshold t is used in Algorithm 2, the achievable sum-rate of this algorithm is the same as that of Algorithm 1.

Although Algorithm 2 decreases the amount of feedback significantly, however, it increases the feedback delay. This can degrade the performance of the system in practical scenarios, as the CSI can become outdated.

2.5.2 Search Complexity

Since at the first step of the algorithm, only a fraction of eigenvectors are pre-selected, the size of the search space for next steps is decreased from NK to L, which is defined in Lemma 2.4. As can be observed, at the mth step of the

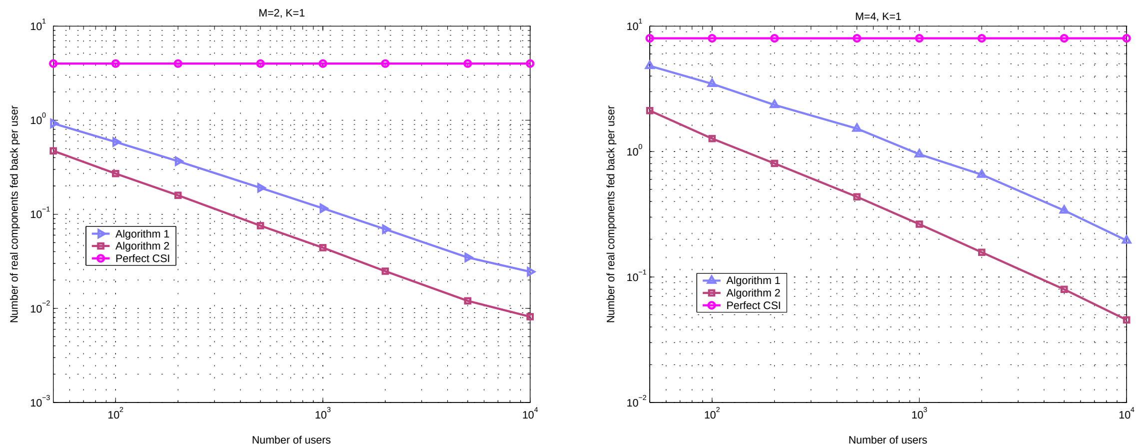

Figure 2.5: Amount of feedback

algorithm, the base station searches for the dimension with the smallest $\gamma_{k,j}(m-1)$ among $\mathcal{S}_{m-1}$, which requires $L-m+1$ searches. Therefore, the total number of searches for selecting the desired set is equal to $\sum_{m=1}^{M}(L-m+1) = ML - \frac{M(M-1)}{2}$, which is linear in L. Again, we can restrict our search space if the modified algorithm stated in the previous section is used.

As mentioned earlier, the best M eigenvectors for maximizing the sum-rate capacity can be found by exhaustive search. In this case, the size of the search space is equal to $\binom{NK}{M}$.

In the asymptotic case of $N \to \infty$, from Theorem 2.2, the total number of searches in the proposed algorithm is $\Theta(e^{\rho(N)}) = o(N)$ and can be as low as $\omega(\ln \ln \ln N)$, which is much less than that of exhaustive search ($\Theta(N^M)$), and also the algorithm in [25] ($\Theta(N)$). Therefore, using our algorithm the complexity of search at the base station is decreased significantly.

2.6 Conclusion

In this chapter, we have considered a downlink communication system, in which a base station equipped with M transmit antennas communicates with N users, each equipped with K receive antennas. We have proposed an efficient suboptimum algorithm for selecting a set of users in order to maximize the sum-rate throughput of the system, using zero-forcing beam-forming at the base station. For the asymptotic case of $N \to \infty$, we have derived the necessary and sufficient conditions to achieve the optimum sum-rate capacity, such that $\lim_{N\to\infty} \mathcal{C}_{\text{sum}} - \mathcal{R}_{\text{Prop}} = 0$. We have also investigated the complexity of our scheme in terms of the required amount of feedback from the users to the base station, as well as the number of searches needed for selecting the coordinates. The proposed algorithm is compared with some other downlink strategies like TDMA and Random Selection algorithms.

Chapter 3

Feedback in MIMO Broadcast Channels

3.1 Introduction

Multiple-Input Multiple-Output (MIMO) systems have proved their ability to achieve high bit rates in a scattering wireless network. In a point-to-point scenario, it has been shown that the capacity scales linearly with the minimum number of transmit and receive antennas, regardless of the availability of Channel State Information (CSI) at the transmitter [1] [2]. This linear increase is so-called *multiplexing gain*.

In a MIMO Broadcast Channel (MIMO-BC), a BS equipped with multiple antennas communicates with several multiple-antenna users. Recently, there has been a lot of interest in characterizing the capacity region of this channel [5], [6], [7], [8]. In these works, it has been shown that the sum-rate capacity of MIMO-BC grows linearly with the minimum number of transmit and receive antennas, provided that both transmitter and receiver sides have perfect CSI. Moreover, in a network with a large number of users, the BS can increase the throughput by selecting the best set of users to communicate with. This results in the so-called *multiuser diversity gain* [16], [17].

Unlike the point-to-point scenario, in MIMO-BC it is crucial for the transmitter to have CSI. It has been shown that MIMO-BC without CSI at the BS is degraded [31]. Moreover, for the case of single antenna users, multiplexing gain reduces to one, and multiuser diversity gain disappears [40].

Due to the weak performance of having no CSI at the BS, some authors have considered MIMO-BC with partial CSI [41, 42, 43, 26, 44, 45, 46, 47, 48, 49, 50, 51]. In [41], the authors have proposed a user selection strategy in a single-antenna broadcast channel, which exploits the maximum sum-rate capacity with only one bit feedback per user. This idea has been generalized for MIMO-BC in [42], using the idea of antenna selection. In [43], the authors propose a scalable feedback protocol, in which time slots for channel feedback correspond not to users, but to a channel value. Through asymptotic analysis, this scheme is shown to achieve the asymptotic sum-rate capacity of MIMO-BC, with the amount of feedback scaling as $\ln N$.

Reference [26] proposes a downlink transmission scheme based on random beam-forming, relying on partial CSI at the transmitter. In this scheme, the BS randomly constructs M orthogonal beams and transmits data to the users with the maximum Signal to Interference plus Noise Ratio (SINR) for each beam. Therefore, only the value of maximum SINR, and the index of the beam for which the maximum SINR is achieved, are fed back to the BS for each user. This significantly reduces the amount of feedback. Reference [26] shows that when the number of users tends to infinity, the optimum sum-rate throughput can be achieved. In [44], a variant of Random Beam-Forming is introduced and shown to achieve the maximum sum-rate capacity of MIMO-BC with only one bit feedback per user.

Reference [45] considers a downlink channel where a transmitter with M antennas communicates with M single-antenna receivers. It is assumed that receivers have perfect CSI, but the transmitter only has the quantized information regarding the channel instantiation. This reference shows that assuming Zero-Forcing Beam-Forming (ZFBF) precoding at the transmitter, the full multiplexing gain can be achieved with partial CSI, if the quality of the CSI is increased linearly with the SNR. This result is generalized in [46] to the case of multiple-antenna receivers, when the number of receive antennas is less than M, and also in [47] to the case of multiple antenna receivers, where the aggregate number of receive antennas equals the number of transmit antennas and the transmitter performs block diagonalization. In [48], the authors compare the performance of quantized (digital) channel feedback versus analog channel feedback for MIMO-BC and show that the digital feedback is potentially superior, when the feedback channel uses per channel coefficient is larger than 1. In [49], the authors consider a MIMO-BC when a transmitter with two antennas transmits data to two single-antenna receivers. They show that if the transmitter has the channel state with finite precision, the maximum achievable multiplexing gain is upper-bounded by $\frac{2}{3}$ [1]. In fact, references [45, 46, 47, 48, 49] study the performance degradation of MIMO-BC due to the imperfect CSI, at the high SNR regime. The size of the network (the number of users) is assumed to be fixed in these references.

In [50], we have considered a downlink scheme based on ZFBF and have proved

[1] It is assumed that the transmitted signal and the channel coefficients are real.

that when the number of users, N, tends to infinity, the maximum sum-rate capacity is achievable with the amount of feedback scaling as $\omega(\ln \ln \ln N)$. In [51], the authors have considered a MIMO-BC with large number of users at high SNR. They have shown that it is possible to achieve the maximum multiplexing gain with the amount of feedback per user decreasing with N. However, it is still required that the feedback load per user grows logarithmically with SNR. Two essential questions arise here: i) Is it possible to achieve the maximum sum-rate capacity with finite feedback in a large network ($N \to \infty$)? ii) If not, what is the minimum feedback rate (in terms of N and SNR) in order to achieve the sum-rate capacity of the system?

In this chapter, we aim to answer the above questions. First, we define the amount of feedback as the average number of users who send information to the BS. In the fixed and low SNR regimes, our results show that it is not possible to achieve the maximum sum-rate with a finite amount of feedback. Moreover, in the fixed SNR regime, in order to reduce the gap between the achieved sum-rate and the optimum value to zero, the amount of feedback must be greater than $\ln \ln \ln N$. In the second part, we define the amount of feedback as the number of information bits sent to the BS. In the fixed SNR regime, our analysis shows that the minimum amount of feedback, in order to reduce the gap to the optimum sum-rate to zero, scales as $\Theta(\ln \ln \ln N)$, which can be achieved using the Random Beam-Forming scheme proposed in [26]. However, the optimality of Random Beam-Forming only holds for the region $\ln P \neq \Omega(\ln N)$. In the regime of $\ln P = \Omega(\ln N)$, we consider two cases. In the case of $K < M$, we prove that the minimum amount of feedback bits to reduce the gap between the achievable sum-rate and the maximum sum-rate to zero grows logarithmically with SNR, which is achievable by the "Generalized

Random Beam-Forming" scheme, proposed in [51]. In the case of $K = M$, we show that by using the Random Beam-Forming scheme and the amount of feedback not growing with SNR the maximum sum-rate capacity is achievable.

The rest of the chapter is organized as follows: In section 3.2, we introduce the system model, while section 3.3 is devoted to the asymptotic analysis of the amount of feedback. Section 3.4 concludes the chapter.

3.2 System Model

In this work, we consider a MIMO-BC in which a BS equipped with M antennas communicates with N users, each equipped with K antennas, where we assume that $K \leq M$. The channel between each user and the BS is modeled as a zero-mean circularly symmetric Gaussian matrix (Rayleigh fading). The received vector by user k can be written as

$$\mathbf{y}_k = \mathbf{H}_k \mathbf{x} + \mathbf{n}_k, \tag{3.1}$$

where $\mathbf{x} \in \mathbb{C}^{M \times 1}$ is the transmitted signal, $\mathbf{H}_k \in \mathbb{C}^{K \times M}$ is the channel matrix from the transmitter to the kth user, which is assumed to be perfectly known at the receiver side and partially known (or completely unknown) at the transmitter side, and $\mathbf{n}_k \in \mathbb{C}^{K \times 1} \sim \mathcal{CN}(\mathbf{0}, \mathbf{I}_K)$ is the noise vector at this receiver. We assume that the transmitter has an average power constraint P, i.e. $\mathbb{E}\left\{\mathrm{Tr}(\mathbf{x}\mathbf{x}^{\mathrm{H}})\right\} \leq P$. The power constraint is assumed to be *per frame*. In other words, the power constraint is independent of the channel realization. The channels are assumed to be quasi-static block fading, in which the channel is drawn randomly at the start of each transmission frame, remains constant for the whole transmission frame, and changes independently to another realization in the start of the next frame.

The frame itself is assumed to be long enough to allow communication at rates close to the capacity. Defining the sum-rate capacity of the system in the channel realization $\mathcal{H} \triangleq \{\mathbf{H}_i\}_{i=1}^{N}$, when the transmitter has perfect CSI about all users' channels, as $\mathcal{C}_{\text{sum}}(\mathcal{H})$, the average sum-rate capacity, denoted as $\mathcal{C}_{\text{sum}}$, is defined as the average over time of $\mathcal{C}_{\text{sum}}(\mathcal{H})$, which is by the ergodicity of the channel, equal to $\mathbb{E}_{\mathcal{H}}\left\{\mathcal{C}_{\text{sum}}(\mathcal{H})\right\}$. Similarly, for any scheme $\mathcal{S}$ [2], $\mathcal{R}_{\mathcal{S}}$ is defined as $\mathbb{E}_{\mathcal{H}}\left\{\mathcal{R}_{\mathcal{S}}(\mathcal{H})\right\}$, where $\mathcal{R}_{\mathcal{S}}(\mathcal{H})$ denotes the achievable sum-rate of scheme $\mathcal{S}$. It is assumed that there is a separate error-free feedback channel from each user to the BS. The parameters of interest in this chapter are: i) $\mathcal{N}_{\mathcal{S}}$; the number of users who send feedback to the BS (or equivalently, the number of active feedback channels), and ii) $\mathcal{F}_{\mathcal{S}}$; the total amount of information conveyed through all the feedback channels.

3.3 Asymptotic Analysis

3.3.1 The average number of users sending feedback to the BS

In this section, we define the amount of feedback as the average number of users who send feedback to the BS. It is assumed that the SNR (P) is fixed. In Theorems 3.2-3.4, we provide the necessary and sufficient conditions in order to achieve $\lim_{N\to\infty} \frac{\mathcal{R}_{\mathcal{S}}}{\mathcal{C}_{\text{sum}}} = 1$ and $\lim_{N\to\infty} \mathcal{C}_{\text{sum}} - \mathcal{R}_{\mathcal{S}} = 0$. Before that, we give the definition of the user selection strategy as follows:

Definition 3.1 *The user selection strategy S is defined as the decision rule in*

[2]Here, by scheme we mean the way the transmitter selects the user to communicate with, the way it allocates the power between the users, and the way it performs precoding.

which each user i, based on its knowledge about its own channel [3], *decides whether or not to send feedback to the BS. More precisely, the user selection strategy S can be defined as a binary indicator variable $I_S(i)$, $i = 1, \cdots, N$, which is equal to 1 if the user i sends feedback to the BS and 0 otherwise. Note that the user selection strategy is assumed to be fixed during the whole transmission period and agreed in advance between the BS and the users.*

Theorem 3.2 *Consider a MIMO-BC with N users ($N \to \infty$), which utilizes a fixed user selection strategy S. Let $\mathcal{N}_S$ be the number of users who send information to the BS in this strategy. Then, the necessary and sufficient condition to achieve $\lim_{N\to\infty} \frac{\mathcal{R}_\mathcal{S}}{\mathcal{C}_{\text{sum}}} = 1$ is having*

$$\mathbb{E}\{\mathcal{N}_S\} = g, \tag{3.2}$$

where $g \gg 1$.

Proof- *Necessary Condition-* Let us denote $\mathcal{G}_S$ as the set of users who send information to the BS using strategy S. In other words, $\mathcal{G}_S$ is the set of users for which $I_S(k) = 1$. Define $p_S(k)$ as the probability that user k belongs to $\mathcal{G}_S$, i.e, $\Pr\{I_S(k) = 1\}$. Since we consider a homogeneous network, this probability is independent of k, and we denote it by p_S. Therefore, $\mathcal{N}_S = |\mathcal{G}_S|$ is a Binomial random variable with parameters (N, p_S), and we have $\mathbb{E}\{\mathcal{N}_S\} = Np_S$.

To compute $\mathcal{C}_{\text{sum}}$ and $\mathcal{R}_\mathcal{S}$, we use the basic Bayes formula. In general term, if we have a partitioning $(\mathcal{P}, \mathcal{P}^C)$ of the sample space of the channel realizations $\mathcal{H}$,

[3] Note that since the users are not aware of the other users' channels, their decisions are solely based on their own channels.

and for any function of the channel realizations $\mathcal{F}(\mathcal{H})$, we have

$$\begin{aligned}\mathbb{E}_{\mathcal{H}}\left\{\mathcal{F}(\mathcal{H})\right\} &= \mathbb{E}_{\mathcal{H}|\mathcal{P}}\left\{\mathcal{F}(\mathcal{H})|\mathcal{H}\in\mathcal{P}\right\}\Pr\{\mathcal{H}\in\mathcal{P}\}+\\ &\quad \mathbb{E}_{\mathcal{H}|\mathcal{P}^C}\left\{\mathcal{F}(\mathcal{H})|\mathcal{H}\in\mathcal{P}^C\right\}\Pr\{\mathcal{H}\in\mathcal{P}^C\}. \end{aligned} \tag{3.3}$$

Here, the strategy S is defined to partition the sample space to $\mathcal{P}=\mathcal{A}_S$ and $\mathcal{P}^C=\mathcal{A}_S^C$, where $\mathcal{A}_S$ is the set of all channel realizations for which $\mathbf{I}_S\triangleq(I_S(1),\cdots,I_S(N))=\mathbf{0}$, in other words, the set of all realizations that no users are sending feedback to the BS under user selection strategy S, which occurs with probability $(1-p_S)^N$, and $\mathcal{A}_S^C$ is the complement of $\mathcal{A}_S$ (Figure 3.1).

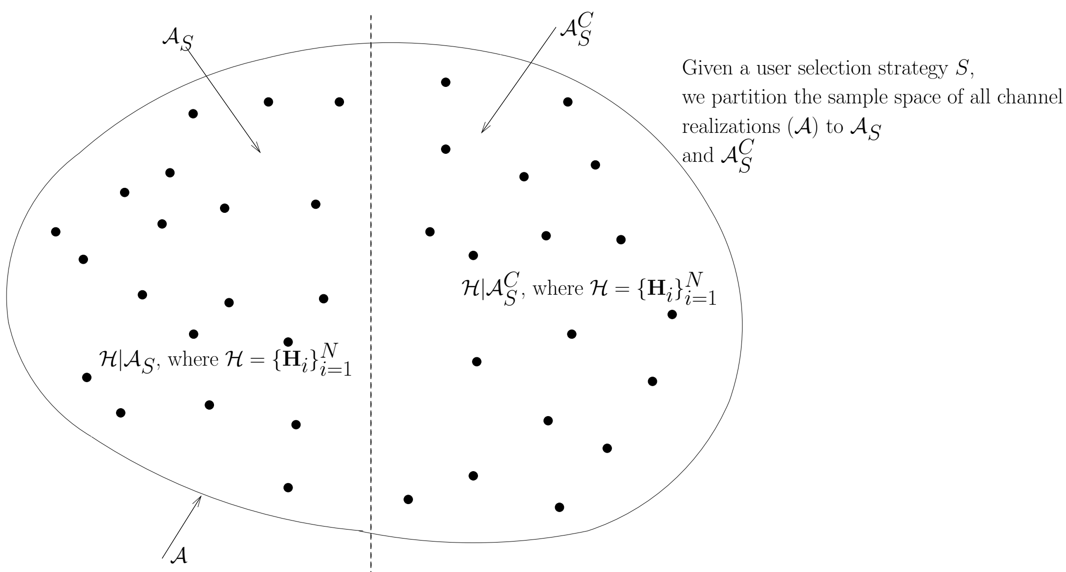

Figure 3.1: Definition of the events $\mathcal{A}_S$ and $\mathcal{A}_S^C$

Substituting $\mathcal{F}(\mathcal{H})$ by $\mathcal{R}_{\mathcal{S}}(\mathcal{H})$, the achievable rate of scheme $\mathcal{S}$ when the channel realization is $\mathcal{H}$, using the above equation, we have

$$\begin{aligned}\mathcal{R}_{\mathcal{S}} &= \mathbb{E}_{\mathcal{H}}\left\{\mathcal{R}_{\mathcal{S}}(\mathcal{H})\right\}\\ &= \mathbb{E}_{\mathcal{H}|\mathcal{A}_S}\left\{\mathcal{R}_{\mathcal{S}}(\mathcal{H})|\mathcal{A}_S\right\}\Pr\{\mathcal{A}_S\}+\mathbb{E}_{\mathcal{H}|\mathcal{A}_S^C}\left\{\mathcal{R}_{\mathcal{S}}(\mathcal{H})|\mathcal{A}_S^C\right\}\Pr\{\mathcal{A}_S^C\}. \end{aligned} \tag{3.4}$$

Note that for any realization of the channels $\mathcal{H} = \{\mathbf{H}_i\}_{i=1}^N$, the maximum achievable sum-rate equals to:

$$\mathcal{C}_{\rm sum}(\mathcal{H}) = \max_{\substack{\mathbf{Q}_n \\ \sum \mathrm{Tr}(\mathbf{Q}_n)=\mathrm{P}}} \ln\left|\mathbf{I}_M + \sum_{n=1}^{N} \mathbf{H}_n^H \mathbf{Q}_n \mathbf{H}_n\right|, \tag{3.5}$$

which is derived based on the perfect CSI assumption at the BS. Therefore, for any scheme $\mathcal{S}$, we have

$$\mathbb{E}_{\mathcal{H}|\mathcal{A}_S^C}\left\{\mathcal{R}_{\mathcal{S}}(\mathcal{H})|\mathcal{A}_S^C\right\} \le \mathcal{R}_2, \tag{3.6}$$

where

$$\mathcal{R}_2 = \mathbb{E}_{\mathcal{H}|\mathcal{A}_S^C}\left\{\max_{\substack{\mathbf{Q}_n \\ \sum \mathrm{Tr}(\mathbf{Q}_n)=\mathrm{P}}} \ln\left|\mathbf{I}_M + \sum_{n=1}^{N} \mathbf{H}_n^H \mathbf{Q}_n \mathbf{H}_n\right| \,\middle|\, \mathcal{A}_S^C\right\}.$$

Note that $\mathcal{R}_2$ is the expected value of the maximum sum-rate (assuming perfect CSI) corresponding to those channel realizations in $\mathcal{A}_S^C$. Also, since conditioned on $\mathcal{A}_S$, no users send feedback to the BS, there is no CSI at the transmitter. Hence, for any scheme $\mathcal{S}$, we have

$$\mathbb{E}_{\mathcal{H}|\mathcal{A}_S}\left\{\mathcal{R}_{\mathcal{S}}(\mathcal{H})|\mathcal{A}_S\right\} \le \mathcal{R}_{\mathcal{A}_S}^{\rm NCSI}, \tag{3.7}$$

where $\mathcal{R}_{\mathcal{A}_S}^{\rm NCSI} = \mathbb{E}_{\mathcal{H}|\mathcal{A}_S}\left\{\mathcal{C}_{\rm sum}^{\rm NCSI}(\mathcal{H})|\mathcal{A}_S\right\}$, i.e., the maximum expected sum-rate when the CSI is not available at the BS, conditioned on $\mathcal{A}_S$. Combining (3.4), (3.6) and (3.7), we obtain

$$\begin{aligned} \mathcal{R}_{\mathcal{S}} &\le \Pr\{\mathcal{A}_S\}\mathcal{R}_{\mathcal{A}_S}^{\rm NCSI} + \Pr\{\mathcal{A}_S^C\}\mathcal{R}_2 \\ &= (1-p_S)^N \mathcal{R}_{\mathcal{A}_S}^{\rm NCSI} + \left[1-(1-p_S)^N\right]\mathcal{R}_2. \end{aligned} \tag{3.8}$$

Moreover, if we substitute $\mathcal{F}(\mathcal{H})$ by $\mathcal{C}_{\rm sum}(\mathcal{H})$ in (3.3), and define

$$\mathcal{R}_1 = \mathbb{E}_{\mathcal{H}|\mathcal{A}_S}\left\{\mathcal{C}_{\rm sum}(\mathcal{H})|\mathcal{A}_S\right\},$$

noting that $\mathcal{R}_2 = \mathbb{E}_{\mathcal{H}|\mathcal{A}_S^C}\left\{\mathcal{C}_{\rm sum}(\mathcal{H})|\mathcal{A}_S^C\right\}$, we conclude

$$\mathcal{C}_{\rm sum} = \Pr\{\mathcal{A}_S\}\mathcal{R}_1 + \Pr\{\mathcal{A}_S^C\}\mathcal{R}_2. \tag{3.9}$$

Subtracting both sides of (3.8) and (3.9), we obtain

$$\mathcal{C}_{\rm sum} - \mathcal{R}_{\mathcal{S}} \;\; \geq \;\; (1-p_S)^N(\mathcal{R}_1 - \mathcal{R}_{\mathcal{A}_S}^{\rm NCSI}). \tag{3.10}$$

Since

$$\max_{\substack{\mathbf{Q}_n \\ \sum \mathrm{Tr}(\mathbf{Q}_n)=\mathrm{P}}} \ln\left|\mathbf{I}_M + \sum_{n=1}^{N}\mathbf{H}_n^H\mathbf{Q}_n\mathbf{H}_n\right| \geq \ln\left(1 + P\max_{j,k}\|\mathbf{H}_{j,k}\|^2\right),$$

we have

$$\mathcal{R}_1 \;\; \geq \;\; \mathbb{E}\left\{\ln\left(1+P\max_{j,k}\|\mathbf{H}_{j,k}\|^2\right)\middle|\,\mathcal{A}_S\right\}, \tag{3.11}$$

where $\mathbf{H}_{j,k}$ denotes the jth row of $\mathbf{H}_k$. The right hand side of (3.11) can be lower-bounded as,

$$\mathrm{RH}(3.11) \;\; \geq \;\; \mathbb{E}\left\{\ln\left(1+P\max_{j,k}\|\mathbf{H}_{j,k}\|^2\right)\middle|\,\mathcal{A}_S,\mathscr{C}_t\right\}\Pr\left\{\mathscr{C}_t|\mathcal{A}_S\right\}, \tag{3.12}$$

where $\mathscr{C}_t$ is the event that $\max_{j,k}\|\mathbf{H}_{j,k}\|^2 > t$, for some chosen t. Hence,

$$\begin{aligned}\mathrm{RH}(3.11) \;\; &\geq \;\; \ln(1+Pt)\frac{\Pr\{\mathcal{A}_S,\mathscr{C}_t\}}{\Pr\{\mathcal{A}_S\}} \\ &\geq \;\; \ln(1+Pt)\frac{1-\Pr\{\mathcal{A}_S^C\}-\Pr\{\mathscr{C}_t^C\}}{\Pr\{\mathcal{A}_S\}} \\ &= \;\; \ln(1+Pt)\left(1-\frac{\Pr\{\mathscr{C}_t^C\}}{\Pr\{\mathcal{A}_S\}}\right),\end{aligned} \tag{3.13}$$

where $\mathscr{C}_t^C$ is the complement of $\mathscr{C}_t$. $\Pr\{\mathscr{C}_t^C\}$ can be computed as

$$\begin{aligned}\Pr\{\mathscr{C}_t^C\} \;\; &= \;\; \Pr\left\{\max_{j,k}\|\mathbf{H}_{j,k}\|^2 \leq t\right\} \\ &\stackrel{(a)}{=} \;\; \left(1-\sum_{m=0}^{M-1}\frac{t^m}{m!}e^{-t}\right)^{NK},\end{aligned} \tag{3.14}$$

where (a) comes from the fact that $\|\mathbf{H}_{j,k}\|^2$ has chi-square distribution with $2M$ degrees of freedom [32]. Now, assume that

$$\mathbb{E}\{\mathcal{N}_S\} = Np_S \neq g, \tag{3.15}$$

i.e., $Np_S = O(1)$. Choosing $t = \frac{\ln N}{2}$, from (3.14), we obtain

$$\Pr\{\mathscr{C}_t^C\} = e^{-\frac{K\sqrt{N}(\ln N)^{M-1}}{2^{M-1}(M-1)!}[1+o(1)]}. \tag{3.16}$$

Moreover, noting $\Pr\{\mathcal{A}_S\} = (1-p_S)^N$ and $Np_S = O(1)$, we have

$$\Pr\{\mathcal{A}_S\} = \Theta(1). \tag{3.17}$$

Substituting (3.16) and (3.17) in (3.13) yields

$$\begin{aligned} \text{RH}(3.11) \quad &\geq \quad \ln\left(1 + \frac{P}{2}\ln N\right)\left(1 - \Theta\left(e^{-\frac{K\sqrt{N}(\ln N)^{M-1}}{2^{M-1}(M-1)!}[1+o(1)]}\right)\right) \\ &= \quad \ln\ln N + O(1). \end{aligned} \tag{3.18}$$

Moreover, using the fact that in a homogeneous MIMO-BC (when the users' channels have the same statistical behavior) with no CSI at the transmitter, the maximum sum-rate is achieved by time-sharing between the users [31], we can write

$$\begin{aligned} \mathcal{R}_{\mathcal{A}_S}^{\text{NCSI}} \quad &= \quad \mathbb{E}_{\mathbf{H}_k|\mathcal{A}_S}\left\{\ln\left|\mathbf{I} + \frac{P}{M}\mathbf{H}_k\mathbf{H}_k^H\right| \,\middle|\, \mathcal{A}_S\right\} \\ &\leq \quad K\mathbb{E}_{\mathbf{H}_k|\mathcal{A}_S}\left\{\ln\left(1 + \frac{P}{M}\|\mathbf{H}_k\|^2\right) \,\middle|\, \mathcal{A}_S\right\} \\ &\overset{(a)}{\leq} \quad K\ln\left(1 + \frac{P}{M}\mathbb{E}_{\mathbf{H}_k|\mathcal{A}_S}\left\{\|\mathbf{H}_k\|^2 \,\middle|\, \mathcal{A}_S\right\}\right) \\ &\overset{(b)}{\leq} \quad K\ln\left(1 + \frac{P}{M}\frac{\mathbb{E}_{\mathbf{H}_k}\{\|\mathbf{H}_k\|^2\}}{\Pr\{\mathcal{A}_S\}}\right) \\ &= \quad K\ln\left(1 + \frac{PK}{\Pr\{\mathcal{A}_S\}}\right) \\ &\overset{(3.17)}{=} \quad \Theta(1), \end{aligned} \tag{3.19}$$

where (a) comes from the concavity of ln function and (b) comes from the fact that $\mathbb{E}_{\mathbf{H}_k}\left\{\|\mathbf{H}_k\|^2\right\} \geq \mathbb{E}_{\mathbf{H}_k|\mathcal{A}_S}\left\{\|\mathbf{H}_k\|^2 | \mathcal{A}_S\right\} \Pr\{\mathcal{A}_S\}$. Combining (3.11), (3.18), and (3.19), and substituting in (3.10), under the assumption of (3.15), we get

$$\begin{aligned} \mathcal{C}_{\text{sum}} - \mathcal{R}_{\mathcal{S}} &\geq \left(1 - \frac{\mathbb{E}\{\mathcal{N}_S\}}{N}\right)^N [\ln\ln N + O(1)] \\ &\approx e^{-\mathbb{E}\{\mathcal{N}_S\}} \ln\ln N. \\ \Rightarrow \frac{\mathcal{R}_{\mathcal{S}}}{\mathcal{C}_{\text{sum}}} &\leq 1 - \frac{e^{-\mathbb{E}\{\mathcal{N}_S\}} \ln\ln N}{\mathcal{C}_{\text{sum}}}. \end{aligned} \tag{3.20}$$

As a result, noting that $\mathcal{C}_{\text{sum}} \sim M \ln\ln N$ [26], we obtain

$$\mathbb{E}\{\mathcal{N}_S\} \neq g \Rightarrow \lim_{N\to\infty} \frac{\mathcal{R}_{\mathcal{S}}}{\mathcal{C}_{\text{sum}}} \neq 1. \tag{3.21}$$

Sufficient Condition- Let us define the strategy S as selecting M users randomly among the following set:

$$\mathcal{G}_S = \{k | \lambda_{\max}(\mathbf{H}_k) > t\}, \tag{3.22}$$

where $\lambda_{\max}(\mathbf{H}_k)$ is the maximum singular value of $\mathbf{H}_k\mathbf{H}_k^H$, and t is a threshold value. After selecting the users, the BS performs ZFBF, where the coordinates are chosen as the eigenvectors, corresponding to the maximum singular values of the selected users. In [19], it has been shown that for a $K \times M$ matrix $\mathbf{A}$, whose elements are i.i.d Gaussian, we have

$$p_S \triangleq \Pr\{\lambda_{\max}(\mathbf{A}) > t\} = \frac{t^{M+K-2}e^{-t}(1 + O(e^{-t}t^{-1}))}{\Gamma(M)\Gamma(K)}. \tag{3.23}$$

Hence,

$$\begin{aligned} \mathbb{E}\{\mathcal{N}_S\} &= N p_S \\ &= N \frac{t^{M+K-2}e^{-t}(1 + O(e^{-t}t^{-1}))}{\Gamma(M)\Gamma(K)}. \end{aligned} \tag{3.24}$$

Having $\mathbb{E}\{\mathcal{N}_S\} = g$, yields,

$$t = \ln N + (M + K - 2) \ln \ln N - g. \tag{3.25}$$

Utilizing ZFBF at the BS, and defining

$$\mathcal{R}^* \triangleq M \mathbb{E}_{\boldsymbol{\mathcal{H}}} \left\{ \ln \left(1 + \frac{P}{\mathrm{Tr}\left\{ \left[\boldsymbol{\mathcal{H}}^{\mathrm{H}} \boldsymbol{\mathcal{H}} \right]^{-1} \right\}} \right) \middle| \, |\mathcal{G}_S| \geq M \right\},$$

we can write

$$\mathcal{R}_{\mathcal{S}} \;\; \geq \;\; \mathcal{R}^* \mathrm{Pr}\{|\mathcal{G}_S| \geq M\}, \tag{3.26}$$

where $\boldsymbol{\mathcal{H}} = \left[\mathbf{g}^T_{s_1,\max} |\, \mathbf{g}^T_{s_2,\max} | \cdots |\, \mathbf{g}^T_{s_m,\max} \right]^T$ in which $\mathbf{g}_{s_i,\max} = \sqrt{\lambda_{\max}(\mathbf{H}_{s_i})} \mathbf{V}^H_{s_i,\max}$, $i = 1, \cdots, m$, and $\mathbf{V}_{s_i,\max}$ is the eigenvector corresponding to maximum singular value of the ith selected user (s_i), and $m = \min(M, |\mathcal{G}_S|)$.

$\eta_S \triangleq \mathrm{Pr}\{|\mathcal{G}_S| \geq M\}$ can be computed as follows:

$$\begin{aligned} \eta_S &= 1 - \mathrm{Pr}\{|\mathcal{G}_S| < M\} \\ &= 1 - \sum_{m=0}^{M-1} \binom{N}{m} p_S^m (1 - p_S)^{N-m} \\ &\overset{(a)}{\geq} 1 - \sum_{m=0}^{M-1} \frac{(N p_S)^m}{m!} e^{-(N-m) p_S}, \end{aligned} \tag{3.27}$$

where (a) results from the facts that $\binom{N}{m} \leq \frac{N^m}{m!}$ and $(1 - p_S)^{N-m} \leq e^{-(N-m)p_S}$. Since $N p_S = g$, we have $\eta_S \approx 1$.

Moreover, we can lower-bound $\mathcal{R}^*$ as

$$\mathcal{R}^* \;\; \geq \;\; M \ln P - M \mathbb{E}_{\boldsymbol{\mathcal{H}}} \left\{ X(\boldsymbol{\mathcal{H}}) |\, |\mathcal{G}_S| \geq M \right\}, \tag{3.28}$$

where $X(\boldsymbol{\mathcal{H}}) \triangleq \ln \left(\mathrm{Tr} \left\{ \left[\boldsymbol{\mathcal{H}}^{\mathrm{H}} \boldsymbol{\mathcal{H}} \right]^{-1} \right\} \right)$. In Appendix E, it has been shown that

$$\mathbb{E}_{\boldsymbol{\mathcal{H}}} \left\{ X(\boldsymbol{\mathcal{H}}) |\; |\mathcal{G}_S| \geq M \right\} \;\; \leq \;\; \ln \frac{M}{t} + (M - 1) \ln(2M^2). \tag{3.29}$$

Using the above equation and (3.28) and selecting $t > \ln N$, yields,

$$\mathcal{R}^* \geq M \ln\left(\frac{P \ln N}{M}\right) - M(M-1)\ln(2M^2). \tag{3.30}$$

Substituting $\mathcal{R}^*$ and η_S in (3.26), and having the fact that $\mathcal{C}_{\text{sum}} \sim M \ln \ln N$ [26], yields

$$\lim_{N\to\infty} \frac{\mathcal{R}_{\mathcal{S}}}{\mathcal{C}_{\text{sum}}} = 1. \tag{3.31}$$

■

Remark - Although in Theorem 3.2 it is established that for having $\lim_{N\to\infty} \frac{\mathcal{R}_{\mathcal{S}}}{\mathcal{C}_{\text{sum}}} = 1$, it is required that $\mathbb{E}\{\mathcal{N}_S\} \to \infty$, as shown in the proof of the sufficient condition, $\mathbb{E}\{\mathcal{N}_S\}$ does not need to scale with N.

Theorem 3.3 *For any user selection strategy S, the necessary condition to achieve $\lim_{N\to\infty} \mathcal{C}_{\text{sum}} - \mathcal{R}_{\mathcal{S}} = 0$ is having*

$$\mathbb{E}\{\mathcal{N}_S\} = \ln \ln \ln N + g. \tag{3.32}$$

Proof - Assume that

$$\mathbb{E}\{\mathcal{N}_S\} \neq \ln \ln \ln N + g. \tag{3.33}$$

Similar to (3.10), we can write

$$\mathcal{C}_{\text{sum}} - \mathcal{R}_{\mathcal{S}} \quad \geq \quad (1-p_S)^N [\mathcal{R}_1 - \mathcal{R}_{\mathcal{A}_S}^{\text{NCSI}}]. \tag{3.34}$$

Following the same approach as in Theorem 3.2, under the assumption of (3.33), we can show that $\mathcal{R}_1 \geq \ln \ln N + O(1)$, and $\mathcal{R}_{\mathcal{A}_S}^{\text{NCSI}} = O(\ln \ln \ln N)$. Hence,

$$\begin{aligned} \mathcal{C}_{\text{sum}} - \mathcal{R}_{\mathcal{S}} \quad &\geq \quad (1-p_S)^N \left[\ln \ln N + O(\ln \ln \ln N)\right] \\ &\stackrel{(a)}{=} \quad e^{-\mathbb{E}\{\mathcal{N}_S\}[1+O(p_S)]} \left[\ln \ln N + O(\ln \ln \ln N)\right] \\ &\stackrel{(b)}{=} \quad e^{-(\mathbb{E}\{\mathcal{N}_S\} - \ln \ln \ln N)} \left[1 + o(1)\right]. \end{aligned} \tag{3.35}$$

(a) comes from the facts that $\mathbb{E}\{\mathcal{N}_S\} = Np_S$ and $\ln(1-p_S) = p_S + O(p_S^2)$, and (b) results from writing $\ln\ln N$ as $e^{\ln\ln\ln N}$, noting that $e^{\mathbb{E}\{\mathcal{N}_S\}O(p_S)} = 1+o(1)$. Defining $d \triangleq \mathbb{E}\{\mathcal{N}_S\} - \ln\ln\ln N$, it follows that

$$\mathcal{C}_{\text{sum}} - \mathcal{R}_{\mathcal{S}} \gtrsim e^{-d}, \tag{3.36}$$

meaning that in order to have $\lim_{N\to\infty} \mathcal{C}_{\text{sum}} - \mathcal{R}_{\mathcal{S}} = 0$, we must have $d \to \infty$. Note that d does not need to scale with N. In fact, as shown in (3.36), $-d$ gives a lower bound on $\ln(\mathcal{C}_{\text{sum}} - \mathcal{R}_{\mathcal{S}})$, which must approach $-\infty$, if we want to have $\mathcal{C}_{\text{sum}} - \mathcal{R}_{\mathcal{S}} \to 0$. As a result,

$$\mathbb{E}\{\mathcal{N}_S\} \neq \ln\ln\ln N + g \Rightarrow \lim_{N\to\infty} \mathcal{C}_{\text{sum}} - \mathcal{R}_{\mathcal{S}} \neq 0. \tag{3.37}$$

■

The above theorem simply implies that if $\mathbb{E}\{\mathcal{N}_S\}$ does not have an infinite difference to $\ln\ln\ln N$, it is not possible to achieve $\lim_{N\to\infty} \mathcal{C}_{\text{sum}} - \mathcal{R}_{\mathcal{S}} = 0$.

Theorem 3.4 *A sufficient condition to achieve* $\lim_{N\to\infty} \mathcal{R}_{\text{opt}} - \mathcal{R}_{\mathcal{S}} = 0$ *is having*

$$\mathbb{E}\{\mathcal{N}_S\} = M\ln\ln\ln N + g. \tag{3.38}$$

Proof - Consider the Random Beam-Forming strategy, introduced in [26]. In this strategy, the BS randomly constructs M orthogonal beams and transmits data to the users with the maximum SINR for each beam. Assuming each user's antenna as a separate user, we define the following set:

$$\mathcal{G}_{\text{RBF}}^{(m)} = \{k | \exists i, \quad \text{SINR}_{k,i}^{(m)} > t\}, \quad m = 1, \cdots, M, \tag{3.39}$$

where $\text{SINR}_{k,i}^{(m)}$ is the received SINR over the ith antenna of the kth user, for the mth transmitted beam. $\mathcal{G}_{\text{RBF}} = \bigcup_{m=1}^{M} \mathcal{G}_{\text{RBF}}^{(m)}$ is the set of users who send feedback to the BS. The achievable sum-rate by this scheme, denoted by $\mathcal{R}_{\mathrm{RBF}}$, is lower-bounded as

$$\begin{aligned}\mathcal{R}_{\mathrm{RBF}} &\geq M\ln(1+t)\Pr\left\{\bigcap_{m=1}^{M}\mathscr{D}_m\right\}\\ &\geq M\ln(1+t)\left(1-\sum_{m=1}^{M}\Pr\{\mathscr{D}_m^C\}\right), \qquad (3.40)\end{aligned}$$

where $\mathscr{D}_m$ is the event that $|\mathcal{G}_{\mathrm{RBF}}^{(m)}| \geq 1$, and $\mathscr{D}_m^C$ is the complement of $\mathscr{D}_m$.

For a randomly chosen user k, we define

$$\begin{aligned}p_k^{(m)} &\triangleq \Pr\{k \in \mathcal{G}_{\mathrm{RBF}}^{(m)}\}\\ &= \Pr\left\{\bigcup_{i=1}^{K}\mathscr{B}_{k,i}^{(m)}\right\}\\ &\leq \sum_{i=1}^{K}\eta_{k,i}^{(m)}, \qquad (3.41)\end{aligned}$$

where $\mathscr{B}_{k,i}^{(m)}$ is the event that $\mathrm{SINR}_{k,i}^{(m)} > t$ and $\eta_{k,i}^{(m)} \triangleq \Pr\{\mathscr{B}_{k,i}^{(m)}\}$, which is independent of k, i, m, and we denote it by η. Moreover, $p_k^{(m)}$ is independent of k, m, and is denoted by p. Hence, $p \leq K\eta$.

To evaluate the right hand side of (3.40), first we compute $\Pr\{\mathscr{D}_m^C\}$ as follows:

$$\begin{aligned}\Pr\{\mathscr{D}_m^C\} &= (1-\eta)^{KN}\\ &\leq \left(1-\frac{p}{K}\right)^{KN}. \qquad (3.42)\end{aligned}$$

Therefore,

$$\begin{aligned}\mathrm{RH}(3.40) &\geq M\ln(1+t)\left[1-M\left(1-\frac{p}{K}\right)^{KN}\right]\\ &\geq M\ln(1+t)[1-Me^{-Np}]. \qquad (3.43)\end{aligned}$$

Under the condition of (3.38), which implies that $\mathbb{E}\{\mathcal{N}_{\mathrm{RBF}}\} = M\ln\ln\ln N + g$, and having the facts that $\mathbb{E}\{\mathcal{N}_{\mathrm{RBF}}\} \approx MNp$ and $\eta = \frac{e^{-Mt/P}}{(1+t)^{M-1}}$ [26] and writing p

as $p = T\eta$, where T tends to a constant in the interval $[1, K]$ as $N \to \infty$ [4], we can write

$$\begin{aligned} NT\frac{e^{-Mt/P}}{(1+t)^{M-1}} &\approx \ln\ln\ln N + g. \\ \Rightarrow t &\approx \frac{P}{M}\Big[\ln N - (M-1)\ln\Big(\frac{P}{M}\ln N\Big) - \\ & \ln\big(\ln\ln\ln N + g\big) + \ln T\Big]. \end{aligned} \tag{3.44}$$

Substituting t in (3.43) yields

$$\begin{aligned} \mathcal{R}_{\text{RBF}} &\geq M\ln\left(1 + \frac{P}{M}\ln N + O(\ln\ln N)\right) \times \\ & \left(1 - Me^{-Np}\right). \end{aligned} \tag{3.45}$$

Using the above equation and having the facts that $\mathcal{C}_{\text{sum}} = M\ln\left(1 + \frac{P}{M}\ln N + O(\ln\ln N)\right)$ [26], and $\mathbb{E}\{\mathcal{N}_{\text{RBF}}\} \approx MNp$, we have

$$\begin{aligned} \mathcal{C}_{\text{sum}} - \mathcal{R}_{\text{RBF}} &\leq O\left(\frac{\ln\ln N}{\ln N}\right) + M^2 e^{-(\frac{\mathbb{E}\{\mathcal{N}_{\text{RBF}}\}}{M} - \ln\ln\ln N)}[1 + o(1)] \\ &= o(1), \end{aligned} \tag{3.46}$$

where (a) follows from the fact that $\mathbb{E}\{\mathcal{N}_{\text{RBF}}\} = M\ln\ln\ln N + g$. Consequently, $\lim_{N\to\infty}\mathcal{C}_{\text{sum}} - \mathcal{R}_{\text{RBF}} = 0$.

■

[4]This results from the fact that for any sets $\{\mathcal{A}_i\}_{i=1}^K$, we have $\Pr\{\mathcal{A}_i\} \leq \Pr\{\bigcup_{i=1}^K \mathcal{A}_i\} \leq \sum_{i=1}^K \Pr\{\mathcal{A}_i\}$, which incurs that $\eta \leq p \leq K\eta$. Defining $T \triangleq \frac{p}{\eta}$, it follows that $1 \leq T \leq K$. Note that, T can be any arbitrary function of N. However, when $N \to \infty$, T converges to a constant number between 1 and K.

3.3.2 Amount of bits fed back to the BS

In this section, we study the minimum amount of feedback required at the BS, in terms of the number of bits [5], in order to achieve the maximum sum-rate capacity. It is assumed that the SNR (P) is fixed and the number of bits fed back by each user is an integer.

Theorem 3.5 *The necessary and sufficient condition to achieve* $\lim_{N\to\infty} \frac{\mathcal{R}_S}{\mathcal{C}_{\text{sum}}} = 1$ *for any user selection strategy* S *is having*

$$\mathbb{E}\{\mathcal{F}_S\} = g, \tag{3.47}$$

where $\mathcal{F}_S$ *is the total number of bits fed back to the BS.*

Proof- *Necessary condition-* The proof of the necessary condition easily follows from Theorem 3.2, and the fact that the number of bits fed back by each user is an integer.

Sufficient Condition- Consider the Random Beam-Forming scheme. Given any function $f(N) \triangleq \mathbb{E}\{\mathcal{N}_S\} = g$, we set the threshold t as the solution to the following equation:

$$\frac{e^{-Mt/P}}{(1+t)^{M-1}} = \frac{f(N)}{MNT}, \tag{3.48}$$

where T is a constant between 1 and K. By selecting t as the above equation, using the same approach as in the proof of Theorem 3.4, it can be shown that $\lim_{N\to\infty} \frac{\mathcal{R}_S}{\mathcal{C}_{\text{sum}}} = 1$. Since the users in $\mathcal{G}_{\text{RBF}}^{(m)}$ only need to send the index m to the BS,

[5] In fact, it is more precise to express the amount of feedback in terms of *binits*, as it is assumed that the users who do not send any information to the BS do not contribute to the total amount of feedback.

the total amount of feedback bits is equal to $\lceil \log_2(M) \rceil f(N) = g$. Consequently, it is possible to achieve $\lim_{N\to\infty} \frac{\mathcal{R}_{\mathcal{S}}}{\mathcal{C}_{\text{sum}}} = 1$, with *any* infinitely large average number of feedback bits (but not necessarily scaling with N).

■

Theorem 3.6 *The necessary and sufficient condition to achieve* $\lim_{N\to\infty} \mathcal{C}_{\text{sum}} - \mathcal{R}_{\mathcal{S}} = 0$ *is having*

$$\mathbb{E}\{\mathcal{F}_S\} = \Theta(\ln \ln \ln N) + g. \tag{3.49}$$

Proof- The proof follows from Theorems 3.3 and 3.4, with the same approach as that of Theorem 3.5.

■

Remark 1- From the above theorems, it follows that the Random Beam-forming scheme is optimum in the fixed SNR regime, in the sense of achieving the maximum sum-rate with the minimum order of the required amount of feedback.

Remark 2- Using the conventional ZFBF (with the user selection algorithm as in the proof of the sufficient condition in Theorem 3.2), assuming that the selected users quantize the eigenvectors corresponding to their maximum singular values and feed back the quantization indices to the BS, from [45], it can be shown that

$$\mathcal{R}_{\text{ZF}} - \mathcal{R}_{\text{ZF}}^{\text{Q}} \leq M \ln \left(1 + P\gamma(\ln N) 2^{-\frac{B}{M-1}}\right), \tag{3.50}$$

where $\mathcal{R}_{\text{ZF}}$ denotes the achievable sum-rate of ZFBF when the BS has perfect CSI from all the selected users, $\mathcal{R}_{\text{ZF}}^{\text{Q}}$ is the achievable sum-rate when the BS only has the quantization indices of the selected users' channels, B is the number of quantized bits for each selected user, and γ is a constant depending on the quantization method, which is shown to be lower-bounded by $\frac{M-1}{M}$ [45]. From the above equation, it follows that in order to achieve $\lim_{N\to\infty} \frac{\mathcal{R}_{\mathrm{ZF}}^{Q}}{\mathcal{C}_{\mathrm{sum}}} = 1$, we must have $B \geq \frac{M-1}{\ln 2} \ln\ln N + o(\ln\ln N)$, and in order to achieve $\lim_{N\to\infty} \mathcal{C}_{\mathrm{sum}} - \mathcal{R}_{\mathrm{ZF}}^{Q} = 0$, the condition $B \geq \frac{M-1}{\ln 2} \ln\ln N + g$ must be satisfied. In other words, the minimum required amount of feedback to achieve the maximum sum-rate must scale at least as $\ln\ln N$. This implies that although the proposed user selection algorithm in Theorem 3.2, along with utilizing ZFBF, is shown to be optimal in terms of the average number of users who send feedback to the BS, in terms of the average number of feedback bits, it is not optimal.

3.3.3 Variable SNR Scenario

In the previous section, the SNR (P) is assumed to be fixed. In this section, we study the scaling law of the minimum amount of feedback in order to achieve the maximum sum-rate, when the SNR itself is a function of N. To this end, we consider two special regimes of *low SNR* and *high SNR*. Since achieving the optimum sum-rate requires the square magnitudes of the selected coordinates to behave as $\ln N$, the effective SNR of the selected links scales as $P \ln N$. Hence, *low SNR* and *high SNR* regimes are defined by the regions of $P \ln N = o(1)$ and $P \ln N = \omega(1)$, respectively.

Low SNR Regime

In this regime, it can be shown that [39]

$$\mathcal{C}_{\mathrm{sum}} \sim P\mathbb{E}\{\eta_{\max}\}, \tag{3.51}$$

where $\eta_{\max} \triangleq \max_k \lambda_{\max}(\mathbf{H}_k)$. In other words, the optimum strategy requires the BS to perform beam-forming on the eigenvector corresponding to the maximum of the largest eigenvalues among the users. Having the fact that $\mathbb{E}\{\eta_{\max}\} \sim \ln N$ [19], it follows that in the low SNR regime, as $\mathcal{R}_{\text{opt}} \sim P \ln N = o(1)$, the achievability of the optimum sum-rate for a given strategy S is defined by $\lim_{N\to\infty} \frac{\mathcal{R}_{\mathcal{S}}}{\mathcal{R}_{\text{opt}}} = 1$.

Theorem 3.7 *The necessary and sufficient condition in order to achieve the optimum sum-rate throughput in the low SNR regime is:*

$$\mathbb{E}\{\mathcal{N}_S\} = g,$$

and

$$\mathbb{E}\{\mathcal{F}_S\} = g.$$

Proof - Following the approach of Theorem 3.2 and using the equations (3.10), (3.13), (3.14), and (3.19), we have

$$\mathcal{C}_{\text{sum}} - \mathcal{R}_{\mathcal{S}} \geq (1-p_S)^N \left(\mathcal{R}_1 - \mathcal{R}_{\mathcal{A}_S}^{\text{NCSI}}\right), \tag{3.52}$$

$$\begin{aligned} \mathcal{R}_1 &\geq \ln(1+Pt)\left(1 - \frac{\left(1-\sum_{m=0}^{M-1}\frac{t^m}{m!}e^{-t}\right)^{NK}}{(1-p_S)^N}\right) \\ &\overset{(a)}{\approx} Pt\left(1 - \frac{\left(1-\sum_{m=0}^{M-1}\frac{t^m}{m!}e^{-t}\right)^{NK}}{(1-p_S)^N}\right), \end{aligned} \tag{3.53}$$

and

$$\mathcal{R}_{\mathcal{A}_S}^{\text{NCSI}} \leq K \ln\left(1 + \frac{PK}{(1-p_S)^N}\right). \tag{3.54}$$

(a) comes from the low-SNR assumption and the fact that for $x \ll 1$, $\ln(1+x) \approx x$. Under the assumption of $\mathbb{E}\{\mathcal{N}_S\} = Np_S \neq g$ and choosing $t = \frac{\ln N}{2}$, we have

$\mathcal{R}_1 \sim \frac{P \ln N}{2}$ and $\mathcal{R}_{\mathcal{A}_S}^{\text{NCSI}} = \Theta(P)$. Noting that $\mathcal{C}_{\text{sum}} \sim P \ln N$, we can write

$$\begin{aligned}\frac{\mathcal{R}_{\mathcal{S}}}{\mathcal{C}_{\text{sum}}} &\leq 1 - (1 - p_S)^N \frac{\mathcal{R}_1 - \mathcal{R}_{\mathcal{A}_S}^{\text{NCSI}}}{\mathcal{C}_{\text{sum}}} \\ &\approx 1 - \frac{e^{-\mathbb{E}\{\mathcal{N}_S\}}}{2} \\ &< 1. \end{aligned} \tag{3.55}$$

As a result,

$$\mathbb{E}\{\mathcal{N}_S\} \neq g \Rightarrow \lim_{N \to \infty} \frac{\mathcal{R}_{\mathcal{S}}}{\mathcal{C}_{\text{sum}}} < 1. \tag{3.56}$$

The necessity of $\mathbb{E}\{\mathcal{F}_S\} = g$ directly follows from the above equation.

Sufficient condition - In this part, we prove that for any given $g \gg 1$, one can achieve the maximum sum-rate such that $\mathbb{E}\{\mathcal{N}_S\} \leq g$ and $\mathbb{E}\{\mathcal{F}_S\} \leq g$. Assume that the users in the following set:

$$\mathcal{G}_S \triangleq \{k | \lambda_{\max}(\mathbf{H}_k) > t\}, \tag{3.57}$$

where

$$t \triangleq \max\left(\ln N + (M + K - 2) \ln \ln N - \frac{1}{2} \ln(g), \ln N\right), \tag{3.58}$$

quantize the eigenvector corresponding to their maximum singular value, using a quantization code book $\mathcal{W}$, which consists of $L = 2^{\frac{\sqrt{g}}{2}}$ randomly selected unit vectors in the M-dimensional space (Random Vector Quantization (RVQ)). The BS selects one of the users in $\mathcal{G}_S$ at random and serves this user, performing beamforming on the direction of its quantized eigenvector. The achievable sum-rate of this scheme can be lower-bounded as

$$\begin{aligned}\mathcal{R}_{\mathcal{S}} &\geq \mathbb{E}\left\{\ln\left(1 + Pt|\mathbf{\Phi}^H \widehat{\mathbf{\Phi}}|^2\right)\right\} \left[1 - (1 - p_S)^N\right] \\ &\approx Pt\,\mathbb{E}\left\{|\mathbf{\Phi}^H \widehat{\mathbf{\Phi}}|^2\right\} \left[1 - (1 - p_S)^N\right] \\ &\overset{(a)}{\geq} Pt\,\mathbb{E}\left\{|\mathbf{\Phi}^H \widehat{\mathbf{\Phi}}|^2\right\} \left[1 - e^{-Np_S}\right], \end{aligned} \tag{3.59}$$

where $p_S \triangleq \Pr\{k \in \mathcal{G}_S\}$ for a randomly chosen k, $\mathbf{\Phi}$ denotes the eigenvector corresponding to the maximum singular value of the selected user, and $\widehat{\mathbf{\Phi}}$ denotes the quantized version of $\mathbf{\Phi}$. (a) comes from the fact that $(1-p_S)^N \leq e^{-Np_S}$. Using (3.23), we can write

$$\begin{aligned} p_S &= \frac{t^{M+K-2}e^{-t}}{\Gamma(M)\Gamma(K)}\left[1+O(e^{-t}t^{-1})\right] \\ &\stackrel{(a)}{\approx} \frac{1}{\Gamma(M)\Gamma(K)}\min\left(\frac{\sqrt{g}}{N},\frac{(\ln N)^{M+K-2}}{N}\right) \\ \Rightarrow e^{-Np_S} &\approx e^{-\Theta\left(\min\left(\sqrt{g},(\ln N)^{M+K-2}\right)\right)}, \end{aligned} \tag{3.60}$$

where (a) comes from (3.58). We have

$$\begin{aligned} \theta &\triangleq |\mathbf{\Phi}^H\widehat{\mathbf{\Phi}}|^2 \\ &= \max_{\substack{\mathbf{c}_l \\ \mathbf{c}_l\in\mathcal{W}}} |\mathbf{\Phi}^H\mathbf{c}_l|^2. \end{aligned} \tag{3.61}$$

From Appendix A, it follows that the pdf of $\theta_l \triangleq |\mathbf{\Phi}^H\mathbf{c}_l|^2$ is obtained from

$$f_{\theta_l}(\theta_l) = (M-1)(1-\theta_l)^{M-2}, \quad 0 \leq \theta_l \leq 1. \tag{3.62}$$

Hence,

$$\begin{aligned} F_\theta(\theta) &= \left[F_{\theta_l}(\theta)\right]^L \\ &= \left[1-(1-\theta)^{M-1}\right]^L. \end{aligned} \tag{3.63}$$

From the above equation, $\mathbb{E}\{\theta\}$ can be lower-bounded as

$$
\begin{aligned}
\mathbb{E}\{\theta\} &= \int_0^1 \theta f_\theta(\theta) \mathrm{d}\theta \\
&= \int_0^1 (1 - F_\theta(\theta)) \mathrm{d}\theta \\
&= \int_0^1 \left(1 - \left[1 - (1-\theta)^{M-1}\right]^L\right) \mathrm{d}\theta \\
&= \int_0^1 \left(1 - \left[1 - \mu^{M-1}\right]^L\right) \mathrm{d}\mu \\
&\overset{(a)}{\geq} 1 - \int_0^1 e^{-L\mu^{M-1}} \mathrm{d}\mu \\
&\overset{(b)}{=} 1 - \frac{L^{-\frac{1}{M-1}}}{M-1} \int_0^L u^{\frac{2-M}{M-1}} e^{-u} \mathrm{d}u \\
&\overset{(c)}{\geq} 1 - \frac{L^{-\frac{1}{M-1}}}{M-1} \left[\int_0^1 u^{\frac{2-M}{M-1}} \mathrm{d}u + \int_1^\infty e^{-u} \mathrm{d}u\right] \\
&= 1 - L^{-\frac{1}{M-1}} \left(1 + \frac{e^{-1}}{M-1}\right) \\
&\overset{(d)}{=} 1 - 2^{-\frac{\sqrt{g}}{2(M-1)}} \left(1 + \frac{e^{-1}}{M-1}\right).
\end{aligned}
\tag{3.64}
$$

In the above equation, (a) comes from the fact that $\left[1 - \mu^{M-1}\right]^L \leq e^{-L\mu^{M-1}}$, (b) results from the change of variable $u = L\mu^{M-1}$. (c) comes from the fact that as $M \geq 2$, $\frac{2-M}{M-1} \leq 0$, and as a result, for $u \geq 1$, $u^{\frac{2-M}{M-1}} \leq 1$. (d) follows from the definition of L as $2^{\frac{\sqrt{g}}{2}}$. Combining (3.51), (3.58), (3.59), (3.60), and (3.64), and the fact that $\mathbb{E}\{\eta_{\max}\} = \ln N + O(\ln \ln N)$ [19], yields,

$$
\begin{aligned}
\lim_{N\to\infty} \frac{\mathcal{R}_{\mathcal{S}}}{\mathcal{C}_{\text{sum}}} &= \lim_{N\to\infty} \frac{Pt\left[1 - 2^{-\frac{\sqrt{g}}{2(M-1)}} \left(1 + \frac{e^{-1}}{M-1}\right)\right]\left[1 - e^{-\Theta\left(\min\left(\sqrt{g}, (\ln N)^{M+K-2}\right)\right)}\right]}{P\mathbb{E}\{\eta_{\max}\}} \\
&= 1.
\end{aligned}
\tag{3.65}
$$

Moreover, we have

$$\begin{aligned}\mathbb{E}\{\mathcal{N}_S\} &= Np_S \\ &\overset{(3.60)}{\leq} \sqrt{g} \\ &< g, \end{aligned} \tag{3.66}$$

and

$$\begin{aligned}\mathbb{E}\{\mathcal{F}_S\} &= \mathbb{E}\{\mathcal{N}_S\}\log_2(L) \\ &\leq \sqrt{g}\,\frac{\sqrt{g}}{2} \\ &< g, \end{aligned} \tag{3.67}$$

which completes the proof of Theorem 3.7.

■

High SNR Regime

The sum-rate capacity in this regime can be written as [26],

$$\mathcal{C}_{\text{sum}} = M \ln\left(\frac{P}{M}\ln N + O(P\ln\ln N)\right). \tag{3.68}$$

Theorem 3.8 *i) The necessary condition to achieve* $\lim_{N\to\infty}\frac{\mathcal{R}_S}{\mathcal{C}_{\text{sum}}} = 1$ *in the case of* $K < M$*, and also* $K = M$ *and SNR regime of* $\ln P = O(\ln\ln N)$*, is having* $\mathbb{E}\{\mathcal{N}_S\} = g$*. ii) in the case of* $K = M$*, and the regime of* $\ln P = \omega(\ln\ln N)$*, it is possible to achieve* $\lim_{N\to\infty}\frac{\mathcal{R}_S}{\mathcal{C}_{\text{sum}}} = 1$ *without any CSI at the BS.*

Proof - *Proof of i)*: Similar to the proof of Theorem 3.2, we can write

$$\mathcal{C}_{\text{sum}} - \mathcal{R}_{\text{S}} \;\geq\; (1-p_S)^N(\mathcal{R}_1 - \mathcal{R}_{\mathcal{A}_S}^{\text{NCSI}}). \tag{3.69}$$

From [19], $\mathcal{R}_1$ can be lower bounded as

$$\mathcal{R}_1 \geq \mathbb{E}\left\{ \sum_{j=1}^{M} \ln\left(1 + \frac{P}{M}\sigma_j^2\right) \middle| \mathcal{A}_S \right\}, \tag{3.70}$$

where

$$\begin{aligned} \sigma_j^2 = \max_k \max_{\mathbf{x}} \ & \mathbf{x}^H \mathbf{H}_k^H \mathbf{H}_k \mathbf{x} \\ s.t. \quad & \mathbf{x}^H \mathbf{x} = 1 \\ & \mathbf{\Xi}_j^H \mathbf{x} = 0, \end{aligned} \tag{3.71}$$

and $\mathbf{\Xi}_j \triangleq [\mathbf{v}_1|\cdots|\mathbf{v}_{j-1}]$, in which $\mathbf{v}_i$, $i = 1, \cdots, j-1$, is the optimizing parameter $\mathbf{x}$, in the maximization of σ_i^2. In other words, the maximizing parameter $\mathbf{x}$ is found in the null space of the previously selected coordinates. Defining $\mathscr{C}_t \triangleq \left\{\bigcap_{j=1}^{M} \left(\sigma_j^2 > t\right)\right\}$, similar to (3.13), we can write

$$\begin{aligned} \mathcal{R}_1 &\geq M \ln\left(1 + \frac{P}{M}t\right)\left(1 - \frac{\Pr\{\mathscr{C}_t^C\}}{\Pr\{\mathcal{A}_S\}}\right) \\ &\overset{(a)}{\geq} M \ln\left(1 + \frac{P}{M}t\right)\left(1 - \frac{\sum_{j=1}^{M} \Pr\{\sigma_j^2 \leq t\}}{\Pr\{\mathcal{A}_S\}}\right), \end{aligned} \tag{3.72}$$

where (a) comes from the union bound for the probability. From [19], Lemma 3, we have

$$\Pr\{\sigma_j^2 \leq t\} \leq \sum_{i=N-j+1}^{N} \binom{N}{i} G_{K,M-j+1}(t)^i \left[1 - G_{K,M-j+1}(t)\right]^{N-i}, \tag{3.73}$$

where $G_{n,m}(t)$ is defined in [19], Lemma 1.

Setting $t = \frac{\ln N}{2}$, and using the result of [19], Appendix IV, on the asymptotic behavior of $G_{n,m}(t)$ for large t, we have

$$\begin{aligned}\Pr\left\{\sigma_j^2 \leq \frac{\ln N}{2}\right\} &\leq \sum_{i=N-j+1}^{N} \binom{N}{i} \left[1 - \Theta\left(\frac{(\ln N)^{M+K-j-1}}{\sqrt{N}}\right)\right]^i \times \\ &\quad \left[\Theta\left(\frac{(\ln N)^{M+K-j-1}}{\sqrt{N}}\right)\right]^{N-i} \\ &\leq N^{j-1} e^{-\Theta\left(\sqrt{N}(\ln N)^{M+K-j-1}\right)} \\ &= o\left(N^{j-1} e^{-\sqrt{N}}\right). \end{aligned} \tag{3.74}$$

Substituting in (3.72), we obtain

$$\mathcal{R}_1 \geq M \ln\left(1 + \frac{P \ln N}{2M}\right)\left(1 - \frac{o(N^{M-1}e^{-\sqrt{N}})}{\Pr\{\mathcal{A}_S\}}\right). \tag{3.75}$$

Assuming $Np_S \neq g$, noting that $\Pr\{\mathcal{A}_S\} = (1-p_S)^N$, incurs $\Pr\{\mathcal{A}_S\} = \Theta(1)$, which yields

$$\mathcal{R}_1 \geq M \ln\left(1 + \frac{P \ln N}{2M}\right)\left(1 - o(N^{M-1}e^{-\sqrt{N}})\right). \tag{3.76}$$

Moreover, using (3.19), under the condition of $Np_S \neq g$, we have

$$\mathcal{R}_{\mathcal{A}_S}^{\text{NCSI}} \leq K \ln\left(\frac{P}{M}\right) + \Theta(1). \tag{3.77}$$

Substituting in (3.69), yields

$$\mathcal{C}_{\text{sum}} - \mathcal{R}_{\text{S}} \geq (1-p_S)^N \left[(M-K)\ln\left(\frac{P}{M}\ln N\right) + K \ln\ln N + \Theta(1)\right]. \tag{3.78}$$

In the case of $K < M$, from the above equation and noting $\mathcal{C}_{\text{sum}} \sim M \ln\left(\frac{P}{M}\ln N\right)$, it follows that

$$\begin{aligned}\frac{\mathcal{R}_{\text{S}}}{\mathcal{C}_{\text{sum}}} &\lesssim 1 - \frac{(1-p_S)^N(M-K)}{M} \\ &\approx 1 - \frac{e^{-Np_S}(M-K)}{M}. \end{aligned} \tag{3.79}$$

Hence, having $Np_S \neq g$ results in

$$\lim_{N,P\to\infty} \frac{\mathcal{R}_{\rm S}}{\mathcal{C}_{\rm sum}} < 1. \tag{3.80}$$

Moreover, in the case of $K = M$, similar to (3.79), we can write

$$\frac{\mathcal{R}_{\rm S}}{\mathcal{C}_{\rm sum}} \lesssim 1 - \frac{(1-p_S)^N \ln\ln N}{\ln P + \ln\ln N}. \tag{3.81}$$

Therefore, for the regime of $\ln P = O(\ln\ln N)$, having $Np_S \neq g$ incurs $\lim_{N\to\infty} \frac{\mathcal{R}_{\rm S}}{\mathcal{C}_{\rm sum}} \neq 1$.

Proof of ii): In the case of $K = M$ and $\ln P = \omega(\ln\ln N)$, assume that no CSI is available at the BS. In this case, the best strategy, as mentioned earlier, is time-sharing between the users. The achievable sum-rate in this case can be written as

$$\begin{aligned}
\mathcal{R}_{\rm S} &= \mathbb{E}\left\{\ln\left|\mathbf{I} + \frac{P}{M}\mathbf{H}_k\mathbf{H}_k^H\right|\right\} \\
&\approx M\ln(\frac{P}{M}) + \mathbb{E}\left\{\ln\left|\mathbf{H}_k\mathbf{H}_k^H\right|\right\} \\
&= M\ln P + \Theta(1).
\end{aligned} \tag{3.82}$$

As a result,

$$\begin{aligned}
\lim_{N\to\infty} \frac{\mathcal{R}_{\rm S}}{\mathcal{C}_{\rm sum}} &= \lim_{N\to\infty} \frac{M\ln P}{M\ln P + M\ln\ln N} \\
&= 1.
\end{aligned} \tag{3.83}$$

■

Theorem 3.9 *The necessary condition to achieve* $\lim_{N\to\infty} \mathcal{C}_{\rm sum} - \mathcal{R}_{\mathcal{S}} = 0$ *in the case of* $K = M$ *is having*

$$\mathbb{E}\{\mathcal{N}_S\} = \ln\ln\ln N + g, \tag{3.84}$$

and in the case of $K < M$ *is having*

$$\mathbb{E}\{\mathcal{N}_S\} = \ln\ln(P\ln N) + g, \tag{3.85}$$

for the values of P *satisfying* $\ln\ln(P\ln N) = o(N)$.

Proof - The proof easily follows from (3.78) and the approach used in the proof of Theorem 3.3.

■

Theorem 3.9 implies that in the case of $K = M$, the average number of users sending feedback to the BS does not need to grow with the SNR [6]. In the case of $K < M$, writing $\ln\ln(P\ln N)$ as $\ln\ln\ln N + \ln\left(1 + \frac{\ln P}{\ln\ln N}\right)$, it turns out that for the values of P such that $\ln P = O(\ln\ln N)$, the condition $\mathbb{E}\{\mathcal{N}_S\} = \ln\ln(P\ln N) + g$ is equivalent to $\mathbb{E}\{\mathcal{N}_S\} = \ln\ln\ln N + g$, which implies that $\mathbb{E}\{\mathcal{N}_S\}$ does not need to grow with SNR. Moreover, for the values of P satisfying $\ln P = \omega(\ln\ln N)$, the condition (3.85) reduces to $\mathbb{E}\{\mathcal{N}_S\} = \ln\ln P + g$, which incurs that the average number of users sending feedback to the BS must grow at least double logarithmic with SNR.

In the previous section, we have observed that the Random Beam-forming scheme introduced in [26] is asymptotically optimal in the sense of achieving the maximum sum-rate with the minimum order of the required amount of feedback, in the fixed SNR regime. The question here is for what ranges of SNR this optimality holds. The following theorem answers this question:

Theorem 3.10 *The necessary and sufficient condition to achieve* $\lim_{N\to\infty} \mathcal{C}_{\text{sum}} -$

[6]This statement will be made rigorous in the proof of Theorem 3.16.

$\mathcal{R}_{\text{RBF}} = 0$ *is having* [7]

$$\ln P \neq \Omega(\ln N) \text{ (or equivalently, } \ln P = |o(\ln N)|\text{)}. \tag{3.86}$$

Proof - *Necessary condition* - The sum-rate throughput of Random Beam-forming scheme can be upper-bounded as

$$\begin{aligned} \mathcal{R}_{\text{RBF}} &= \mathbb{E}\left\{ \sum_{m=1}^{M} \ln\left(1 + \text{SINR}_{\max}^{(m)}\right) \right\} \\ &\leq M \ln\left(1 + \mathbb{E}\{\text{SINR}_{\max}^{(m)}\}\right), \end{aligned} \tag{3.87}$$

where $\text{SINR}_{\max}^{(m)}$ denotes the maximum received SINR over the mth transmitted beam. Defining $X_{\max} \triangleq \text{SINR}_{\max}^{(m)}$, for all values of t, we can write

$$\begin{aligned} \mathbb{E}\{X_{\max}\} &= \int_0^\infty x f_{X_{\max}}(x) \mathrm{d}x \\ &= \int_0^\infty \left[1 - F_{X_{\max}}(x)\right] \mathrm{d}x \\ &\leq t + \int_t^\infty \left[1 - F_{X_{\max}}(x)\right] \mathrm{d}x, \quad t \geq 0. \end{aligned} \tag{3.88}$$

Having the fact that $F_X(x) = 1 - \frac{e^{-\frac{Mx}{P}}}{(1+x)^{M-1}}$ [26], where $X \triangleq \text{SINR}_{i,k}^{(m)}$, we can write

$$\mathbb{E}\{X_{\max}\} \leq t + \int_t^\infty \left[1 - \left(1 - \frac{e^{-\frac{Mx}{P}}}{(1+x)^{M-1}}\right)^{NK}\right] \mathrm{d}x. \tag{3.89}$$

Assuming that $\ln P = \Omega(\ln N)$, i.e., $\lim_{N\to\infty} \frac{\ln P}{\ln N} = c$, where $c > 0$, we define

$$t \triangleq \begin{cases} \frac{P}{M}[\ln N - \frac{1}{2}\ln P], & c < 1; \\ \frac{P}{2M}\ln N, & c \geq 1. \end{cases} \tag{3.90}$$

[7]It is assumed that each received antenna is treated as a separate user.

Substituting t in (3.89) yields,

$$\begin{aligned}
\mathbb{E}\{X_{\max}\} &\leq t + \int_t^\infty \left(1 - \exp\left\{-\frac{NKe^{-\frac{Mx}{P}}}{(1+x)^{M-1}}\left[1 + O\left(\frac{e^{-\frac{Mx}{P}}}{(1+x)^{M-1}}\right)\right]\right\}\right)\mathrm{d}x \\
&\overset{(a)}{\leq} t + \int_t^\infty \frac{NKe^{-\frac{Mx}{P}}}{(1+x)^{M-1}}\left[1 + O\left(\frac{e^{-\frac{Mx}{P}}}{(1+x)^{M-1}}\right)\right]\mathrm{d}x \\
&\leq t + \int_t^\infty \frac{NKe^{-\frac{Mx}{P}}}{(1+x)^{M-1}}\mathrm{d}x\left[1 + O\left(\frac{e^{-\frac{Mt}{P}}}{(1+t)^{M-1}}\right)\right] \\
&\overset{(b)}{=} t + \int_t^\infty \frac{NKe^{-\frac{Mx}{P}}}{(1+x)^{M-1}}\mathrm{d}x\left[1 + O\left(\frac{1}{\sqrt{N}}\right)\right] \\
&\overset{(c)}{\leq} t + \int_t^\infty \frac{NKe^{-\frac{Mx}{P}}}{(\frac{P}{M})^{M-1}}\mathrm{d}x\left[1 + O\left(\frac{1}{\sqrt{N}}\right)\right] \\
&\leq t + \left(\frac{P}{M}\right)^{2-M} NKe^{-\frac{Mt}{P}}\left[1 + O\left(\frac{1}{\sqrt{N}}\right)\right] \\
&\overset{(d)}{=} t + NKe^{-\frac{Mt}{P}}\left[1 + O\left(\frac{1}{\sqrt{N}}\right)\right] \\
&\leq \begin{cases} \frac{P}{M}[\ln N - \frac{1}{2}\ln P]\left[1 + O\left(\frac{1}{\sqrt{P}}\right)\right], & c < 1; \\ \frac{P}{2M}\ln N\left[1 + O\left(\frac{\sqrt{N}}{P}\right)\right], & c \geq 1, \end{cases}
\end{aligned} \tag{3.91}$$

where (a) comes from the fact that $1 - e^{-x} \leq x, \forall x$, (b) comes from the fact that $t \geq \frac{P}{2M}\ln N$ (from (3.90)), which incurs $\frac{e^{-\frac{Mt}{P}}}{(1+t)^{M-1}} \leq \frac{1}{\sqrt{N}}$, (c) comes from the fact that since $t \geq \frac{P}{2M}\ln N$, for $x > t$, we have $1 + x > \frac{P}{M}$, and (d) comes from the fact that $M \geq 2$ and as a result $\left(\frac{P}{M}\right)^{2-M} \leq 1$. Noting that $\mathcal{C}_{\text{sum}} = M\ln\left(\frac{P\ln N}{M}\right) + o(1)$, and using (3.87), (3.89), (3.90), and (3.91), we can write

$$\mathcal{C}_{\text{sum}} - \mathcal{R}_{\text{RBF}} \geq \begin{cases} -\ln\left(1 - \frac{\ln P}{2\ln N}\right) + O\left(\frac{1}{\sqrt{P}}\right), & c < 1; \\ \ln(2) - \ln\left[1 + O\left(\frac{\sqrt{N}}{P}\right)\right], & c \geq 1. \end{cases} \tag{3.92}$$

Noting that in the case of $c \geq 1$, $\frac{\sqrt{N}}{P} = o(1)$, it follows from the above equation that

$$\ln P = \Omega(\ln N) \Rightarrow \lim_{N\to\infty} \mathcal{C}_{\text{sum}} - \mathcal{R}_{\text{RBF}} \neq 0. \tag{3.93}$$

Sufficient condition - Assume that $\ln P \neq \Omega(\ln N)$. $\mathcal{R}_{\text{RBF}}$ can be lower-bounded as

$$\begin{aligned} \mathcal{R}_{\text{RBF}} &\geq M\ln(1+t)\Pr\left\{\text{SINR}_{\max}^{(1)} > t, \cdots, \text{SINR}_{\max}^{(M)} > t\right\} \\ &\geq M\ln(1+t)\left[1 - \sum_{m=1}^{M}\Pr\left\{\text{SINR}_{\max}^{(m)} \leq t\right\}\right] \\ &= M\ln(1+t)\left[1 - M(1-\eta)^{NK}\right] \\ &\geq M\ln(1+t)\left[1 - Me^{-NK\eta}\right], \end{aligned} \tag{3.94}$$

where $\eta \triangleq \Pr\{\text{SINR}_{i,k}^{(m)} \leq t\} = \frac{e^{-\frac{Mt}{P}}}{(1+t)^{M-1}}$ [26]. Setting

$$t = \frac{P}{M}\left[\ln N - (M-1)\ln\frac{P}{M} - M\ln\ln N\right],$$

it is easy to show that $\eta \geq \frac{\ln N}{N}$ and hence,

$$\mathcal{R}_{\text{RBF}} \geq M\ln\left(1 + \frac{P}{M}\left[\ln N - (M-1)\ln\frac{P}{M} - M\ln\ln N\right]\right)\left(1 - \frac{M}{N^K}\right). \tag{3.95}$$

Since $\ln P \neq \Omega(\ln N)$, it follows from the above equation that $\lim_{N\to\infty} \mathcal{C}_{\text{sum}} - \mathcal{R}_{\text{RBF}} = 0$.

■

Theorem 3.10 implies that the Random Beam-forming scheme is not capable of achieving the maximum sum-rate when $\ln P = \Omega(\ln N)$. In other words, the Random Beam-forming scheme is not efficient in the high SNR regime. In fact, it is easy to show that the multiplexing gain of this scheme is zero. In the region of $\ln P = |o(\ln N)|$, following the approach of Theorem 3.4, it can be shown that with the number of feedback bits scaling as $M\lceil\log_2 M\rceil \ln\ln(P\ln N) + g$, the maximum sum-rate capacity can be achieved.

The weak performance of Random Beam-Forming in the high SNR regime is due to the fact that the interference from the other users dominates the noise term. It can be shown that in order to achieve the maximum sum-rate, we must have $\lim_{P\to\infty} I(P) = 0$, where I denotes the interference term. In other words, the interference term must be negligible compared to the noise. The Random Beam-Forming scheme can be considered as the quantization of the users' channel vectors by M orthogonal code words. Since the number of code words is fixed, the quantization error, which is translated to the interference, grows with the SNR. This suggests that at high SNRs the channel of the users must be known at the BS with higher precision. This can be performed by increasing the size of the quantization code book and more efficient methods of channel quantization. Some efficient algorithms for channel quantization have been proposed in [52, 53, 54, 55].

Theorem 3.11 *Consider a MIMO-BC with N users ($N \to \infty$), each equipped with K receive antennas, in which the base station communicates with M of them with the total power constraint P ($P \to \infty$). Assume that each user quantizes its channel matrix and sends the quantization index to the transmitter. Then, for any quantization method chosen by the users, any user selection strategy and any known precoding scheme chosen by the transmitter, the necessary condition to* *achieve* $\lim_{N\to\infty} \mathcal{C}_{\text{sum}} - \mathcal{C}_{\text{sum}}^{\text{Q}} = 0$*, in the case of* $K < M$*, is having*

$$\mathbb{E}\{\mathcal{F}_Q\} \quad \geq \quad \ln\ln(P\ln N) + g' + \frac{1}{\ln 2}\sum_{i=1}^{M-K}\left[(M-i)\ln(P\ln N) - \ln N + g''\right]^+, \tag{3.96}$$

and in the case of $K = M$ *is having*

$$\mathbb{E}\{\mathcal{F}_Q\} \quad \geq \quad \ln\ln\ln N + g, \tag{3.97}$$

for some $g' \gg 1$*,* $g'' \gg 1$ *and* $g \gg 1$*, where* $\mathcal{F}_{\text{Q}}$ *and* $\mathcal{C}_{\text{sum}}^{\text{Q}}$ *are the total number of bits fed back to the BS, and the maximum achievable sum-rate, when the BS only has the quantized CSI, respectively, and* $a^+ \triangleq \max(0, a)$*.*

Proof - In order to prove the theorem, we assume that the BS selects M users, and transmits $\mathbf{x}_1, \cdots, \mathbf{x}_M$, with covariance matrices $\mathbf{Q}_1, \cdots, \mathbf{Q}_M$, respectively. Since for a fixed set of transmit covariance matrices, Dirty-Paper Coding is proved to achieve the Marton's region [7] (which is proved to be the highest known achievable region in BC), we consider this coding scheme for the proof of this theorem. In Lemmas 3.12-3.14, we state the necessary conditions for the transmit covariance matrices and the selected users, in order to achieve the maximum sum-rate capacity. Then, in Lemma 3.15, we associate those conditions with the size of quantization codebooks, utilized for the quantization of the selected users' channel matrices. Combining the results of the lemmas, the theorem is proved.

Lemma 3.12 *The transmit covariance matrices of the selected users, maximizing the sum-rate capacity in a MIMO-BC with* $N \to \infty$ *users, are rank one, with probability one.*

Proof - Assume that the selected users are indexed by 1 to M. Then, the sum-rate capacity can be written as [5]

$$\mathcal{C}_{\text{sum}} = \mathbb{E}\left\{ \max_{\substack{\mathbf{Q}_i,\pi \\ \sum \text{Tr}\{\mathbf{Q}_i\}\leq P}} \sum_{i=1}^{M} \ln \left| \mathbf{I} + \mathbf{H}_{\pi(i)}\mathbf{Q}_{\pi(i)}\mathbf{H}_{\pi(i)}^H \left(\mathbf{I} + \mathbf{H}_{\pi(i)} \left(\sum_{j>i} \mathbf{Q}_{\pi(i)} \right) \mathbf{H}_{\pi(i)}^H \right)^{-1} \right| \right\}, \tag{3.98}$$

where the expectation is taken over the channel matrices $\mathbf{H}_1, \cdots, \mathbf{H}_M$. Using the duality between the MIMO-BC and MIMO Multiple Access Channel (MIMO-MAC), expressed in [5], the sum-rate capacity can be written as follows:

$$\mathcal{C}_{\text{sum}} = \mathbb{E}_{\mathbf{H}_1,\cdots,\mathbf{H}_M} \max_{\substack{\mathbf{P}_i \\ \sum \text{Tr}\{\mathbf{P}_i\}\leq P}} \ln \left| \mathbf{I} + \sum_{i=1}^{M} \mathbf{H}_i^H \mathbf{P}_i \mathbf{H}_i \right|, \tag{3.99}$$

where $\mathbf{P}_i$'s are the corresponding covariance matrices in the dual MIMO-MAC. We first prove that to achieve the maximum sum-rate capacity, $\mathbf{P}_i$'s must be rank one, with probability one.

Since $\mathbf{P}_i$'s are positive semi-definite, we can write them as $\mathbf{U}_i^H \mathbf{\Lambda}_i \mathbf{U}_i$, for some unitary matrix $\mathbf{U}_i$ and diagonal matrix $\mathbf{\Lambda}_i$. Defining $\mathbf{Z}_i \triangleq \mathbf{U}_i \mathbf{H}_i$ and writing $\mathbf{\Lambda}_i = \text{Diag}(\rho_{i1}, \cdots, \rho_{iK})$, we have

$$\begin{aligned} \ln \left| \mathbf{I} + \sum_{i=1}^{M} \mathbf{H}_i^H \mathbf{P}_i \mathbf{H}_i \right| &= \ln \left| \mathbf{I} + \sum_{i=1}^{M} \mathbf{Z}_i^H \mathbf{\Lambda}_i \mathbf{Z}_i \right| \\ &= \ln \left| \mathbf{I} + \sum_{i=1}^{M} \sum_{l=1}^{K} \rho_{il} \mathbf{Z}_i(l)^H \mathbf{Z}_i(l) \right|, \end{aligned} \tag{3.100}$$

where $\mathbf{Z}_i(l)$ denotes the lth row of $\mathbf{Z}_i$. Having the fact that $|\mathbf{A}| \leq \left(\frac{\text{Tr}(\mathbf{A})}{M}\right)^M$ for any positive semi-definite matrix $\mathbf{A}$, the right hand side of the above equation can be upper-bounded as

$$\ln \left| \mathbf{I} + \sum_{i=1}^{M} \sum_{l=1}^{K} \rho_{il} \mathbf{Z}_i(l)^H \mathbf{Z}_i(l) \right| \leq M \ln \left(1 + \frac{\sum_{i=1}^{M} \sum_{l=1}^{K} \rho_{il} \|\mathbf{Z}_i(l)\|^2}{M} \right). \tag{3.101}$$

Now, assume that there exists a user k, such that $\rho_{kl} = \Theta(P)$ and $\rho_{kj} = \Theta(P)$, for some $1 \leq l, j \leq K$. In other words, this matrix is asymptotically of rank at least 2. We have

$$\begin{aligned} \|\mathbf{Z}_k(l)\|^2 + \|\mathbf{Z}_k(j)\|^2 &\leq \|\mathbf{Z}_k\|^2 \\ &= \|\mathbf{H}_k\|^2. \end{aligned} \tag{3.102}$$

In [26], it has been shown that $\|\mathbf{H}_k\|^2_{\max} < \ln N + MK \ln\ln N$, with probability one. This incurs that at least one of $\|\mathbf{Z}_k(l)\|^2$ and $\|\mathbf{Z}_k(j)\|^2$ must be less than $\frac{\ln N + MK \ln\ln N}{2}$. Without loss of generality, assume that $\|\mathbf{Z}_k(j)\|^2 < \frac{\ln N + MK \ln\ln N}{2}$. Having ρ_{kj} allocated to the coordinate (k, j) and using (3.101), yields

$$\begin{aligned} \mu &\leq \max_{\substack{\rho_{il} \\ (i,l)\neq(k,j) \\ \sum \rho_{il} = P - \rho_{kj}}} M \ln\left(1 + \frac{\sum_{i=1}^{M}\sum_{l=1}^{K} \rho_{il}\|\mathbf{Z}_i(l)\|^2}{M}\right) \\ &= M \ln\left(1 + \frac{\max_{\substack{\rho_{il} \\ (i,l)\neq(k,j) \\ \sum \rho_{il} = P - \rho_{kj}}} \sum_{i=1}^{M}\sum_{l=1}^{K} \rho_{il}\|\mathbf{Z}_i(l)\|^2}{M}\right) \\ &\overset{(a)}{\leq} M \ln\left(1 + \frac{\rho_{kj}}{2M}\ln N + O(\ln\ln N) + \frac{P - \rho_{kj}}{M}\|\mathbf{Z}\|^2_{\max}\right), \end{aligned} \tag{3.103}$$

where

$$\mu \triangleq \max_{\substack{\rho_{il} \\ (i,l)\neq(k,j) \\ \sum \rho_{il} = P - \rho_{kj}}} \ln\left|\mathbf{I} + \sum_{i=1}^{M}\sum_{l=1}^{K} \rho_{il}\mathbf{Z}_i(l)^H\mathbf{Z}_i(l)\right|,$$

and $\|\mathbf{Z}\|^2_{\max} \triangleq \max_{i,l} \|\mathbf{Z}_i(l)\|^2$. In the above equation, (a) comes from the fact that the solution to the maximization problem in the second line is to allocate the rest of the available power $(P - \rho_{kj})$ to the coordinate with the highest norm. By a similar argument as before, we can show that $\|\mathbf{Z}\|^2_{\max} < \ln N + MK \ln\ln N$, with probability one. Hence, using the above equation,

$$\text{RH (3.103)} \leq M \ln\left(1 + \frac{P - \frac{\rho_{kj}}{2}}{M}[\ln N + O(\ln\ln N)]\right). \tag{3.104}$$

Having the fact that $\mathcal{C}_{\text{sum}} = M \ln\left(\frac{P}{M}\ln N + O(\ln\ln N)\right)$, and using the above equation, we have

$$\mathcal{C}_{\text{sum}} - \text{RH (3.103)} \quad \geq \quad M \ln\left(1 - \frac{\rho_{kj}}{2P}\right) + O\left(\frac{\ln\ln N}{\ln N}\right). \tag{3.105}$$

Hence, having $\rho_{kj} = \Theta(P)$, incurs $\lim_{N\to\infty} \mathcal{C}_{\text{sum}} - \text{RH (3.103)} > 0$. In other words, in order to have $\lim_{N\to\infty} \mathcal{C}_{\text{sum}} - \text{RH (3.103)} = 0$, for each user k, there must be at most one ρ_{km} scaling as $\Theta(P)$, and the rest must scale as $o(P)$. In the following, we will show that with probability one, for each user exactly one ρ_{km} is non-zero, and the rest are zero.

Using (3.101) and having the fact that $\sum_{i=1}^{K} \|\mathbf{Z}_k(i)\|^2 < \ln N + MK \ln\ln N$ with probability one, it follows that the right hand side of (3.101) is upper-bounded by $M \ln\left(\frac{P}{M}\ln N\right)$, which is proved to be the maximum achievable sum-rate throughput in MIMO-MAC. Hence, in order to achieve the maximum sum-rate, the inequality in (3.101) must be turned into the equality, which means that $\sum_{i=1}^{M}\sum_{l=1}^{K} \rho_{il}\mathbf{Z}_i(l)^H\mathbf{Z}_i(l)$ must behave like $\frac{P}{M}\ln N(\mathbf{I}+o(\mathbf{I}))$[8]. Moreover, since from each user at most one singular value can scale as fast as $\ln N$ [50], it follows that the maximum singular values of the selected users must scale as $\ln N$, and their corresponding powers must scale as $\frac{P}{M} + o(P)$.

Now, assume that there exists i, l such that $\lim_{N\to\infty} \frac{\|\mathbf{Z}_i(l)\|^2}{\ln N} < 1$, but $\rho_{il} \neq 0$. In the above, we have seen that $\rho_{il} = o(P)$. The sum-rate can be upper-bounded

[8] $\mathbf{A} = o(\mathbf{I})$ means that all the singular values of $\mathbf{A}$ are $o(1)$.

as

$$\begin{aligned}
\mathcal{R} &\leq \mathcal{C}_{\text{sum}}(P-\rho_{il}) + \\
&\quad \ln\left|\mathbf{I} + \rho_{il}\|\mathbf{Z}_i(l)\|^2\phi_i(l)^H\phi_i(l)\left(\mathbf{I} + \sum_{\substack{(j,m)\\(j,m)\neq(i,l)}}\rho_{jm}\mathbf{Z}_j(m)^H\mathbf{Z}_j(m)\right)^{-1}\right| \\
&\overset{(a)}{=} \mathcal{C}_{\text{sum}}(P-\rho_{il}) + \ln\left|\mathbf{I} + \frac{\rho_{il}\|\mathbf{Z}_i(l)\|^2}{\frac{P-\rho_{il}}{M}\ln N(1+o(1))}\phi_i(l)^H\phi_i(l)\right| \\
&\overset{(b)}{=} M\ln\left(\frac{P-\rho_{il}}{M}\ln N(1+o(1))\right) + \ln\left(1 + \frac{\rho_{il}\|\mathbf{Z}_i(l)\|^2}{\frac{P-\rho_{il}}{M}\ln N(1+o(1))}\right) \\
&\overset{(c)}{=} M\ln\left(\frac{P}{M}\ln N(1+o(1))\right) - \frac{M\rho_{il}}{P}\left(1 - \frac{\|\mathbf{Z}_i(l)\|^2}{\ln N}\right) + o(\frac{\rho_{il}}{P}), \qquad (3.106)
\end{aligned}$$

where $\phi_i(l) \triangleq \frac{\mathbf{Z}_i(l)}{\|\mathbf{Z}_i(l)\|}$, and $\mathcal{C}_{\text{sum}}(P-\rho_{il})$ denotes the maximum sum-rate when the power constraint is $P-\rho_{il}$. (a) comes from the fact that achieving the maximum throughput of $\mathcal{C}_{\text{sum}}(P-\rho_{il})$ requires that

$$\mathbf{I} + \sum_{\substack{(j,m)\\(j,m)\neq(i,l)}}\rho_{jm}\mathbf{Z}_j(m)^H\mathbf{Z}_j(m) = \frac{P-\rho_{il}}{M}\ln N\left(\mathbf{I} + o(\mathbf{I})\right).$$

(b) comes from the fact that $\mathcal{C}_{\text{sum}}(P-\rho_{il}) = M\ln\left(\frac{P-\rho_{il}}{M}\ln N(1+o(1))\right)$, and finally (c) results from the fact that $\rho_{il} = o(P)$, and using the approximation $\ln(1+x) \approx x$, for $x \ll 1$. Suppose that instead of allocating ρ_{il} to the coordinate (i,l), it is allocated to the coordinate corresponding to the maximum eigenvalue of any of the selected users. Let us denote the achievable sum-rate of the system in this case by $\mathcal{R}^*$. Since the maximum singular values of the selected users scale as $\ln N$, the second term in the last line of the above equation scales as $o(\frac{\rho_{il}}{P})$ and we have

$$\mathcal{R}^* - \mathcal{R} = \frac{M\rho_{il}}{P}\left(1 - \frac{\|\mathbf{Z}_i(l)\|^2}{\ln N}\right) + o(\frac{\rho_{il}}{P}). \qquad (3.107)$$

As a result, if $\rho_{il} > 0$, $\mathcal{R}^* > \mathcal{R}$, which incurs that in order to achieve the maximum sum-rate ρ_{il} must be zero with probability one. Having this and the fact that from each user at most one coordinate has the gain scaling as fast as $\ln N$ with probability one [50], it follows that to achieve the maximum sum-rate in the dual MIMO-MAC, the transmit covariance matrices must be rank one with probability one. Using the result of [5], the following equation holds between the covariance matrix of the user with the encoding order j in the MIMO-BC, denoted by $\mathbf{Q}_{\pi(j)}$, and the covariance matrix of the user with the reverse decoding order j in the dual MIMO-MAC, denoted by $\mathbf{P}_{\pi(j)}$:

$$\mathbf{Q}_{\pi(j)} = \mathbf{M}_{\pi(j)}\mathbf{P}_{\pi(j)}\mathbf{M}_{\pi(j)}^H, \tag{3.108}$$

where $\mathbf{M}_{\pi(j)}$ is an $M \times K$ matrix. Since $\mathbf{P}_{\pi(j)}$ is proved to be a rank one matrix with probability one, it follows from the above equation that $\mathbf{Q}_{\pi(j)}$ is also rank one with probability one, which completes the proof of Lemma 3.12.

■

Lemma 3.12 implies that the transmit covariance matrix for the jth user can be written as

$$\mathbf{Q}_j = \rho_j \mathbf{\Phi}_j \mathbf{\Phi}_j^H, \tag{3.109}$$

where $\mathbf{\Phi}_j$ is a unit vector and ρ_j is the allocated power to the jth user.

Lemma 3.13 *The necessary condition for achieving the maximum sum-rate is that* $\{\mathbf{\Phi}_j\}_{j=1}^M$*, defined in the above equation, form a semi-orthogonal basis for* $\mathbb{C}^M$*, i.e,* $|\mathbf{\Phi}_j^H \mathbf{\Phi}_i| = o(1)$*,* $i \neq j$*, with probability one.*

Proof - The sum-rate can be upper-bounded as

$$\begin{aligned}
\mathcal{R} &\overset{(a)}{\leq} \mathbb{E}\left\{\sum_{i=1}^{M} \ln\left|\mathbf{I}+\mathbf{H}_i\mathbf{Q}_i\mathbf{H}_i^H\right|\right\} \\
&\overset{(3.109)}{=} \mathbb{E}\left\{\sum_{i=1}^{M} \ln\left|\mathbf{I}+\rho_i\mathbf{H}_i\mathbf{\Phi}_i\mathbf{\Phi}_i^H\mathbf{H}_i^H\right|\right\} \\
&= \mathbb{E}\left\{\sum_{i=1}^{M} \ln\left(1+\rho_i\|\mathbf{H}_i\mathbf{\Phi}_i\|^2\right)\right\} \\
&= \mathbb{E}\left\{\sum_{i=1}^{M} \ln\left(1+\rho_i\sum_{l=1}^{K}\lambda_l(i)\left|\mathbf{v}_l^H(i)\mathbf{\Phi}_i\right|^2\right)\right\} \\
&= \mathbb{E}\left\{\sum_{i=1}^{M} \ln\left(1+\rho_i\left[\lambda_1(i)\left|\mathbf{v}_1^H(i)\mathbf{\Phi}_i\right|^2+\sum_{l=2}^{K}\lambda_l(i)\left|\mathbf{v}_l^H(i)\mathbf{\Phi}_i\right|^2\right]\right)\right\},
\end{aligned} \tag{3.110}$$

where (a) comes from ignoring the interference terms, $\lambda_l(i)$ denotes the lth ordered singular value of $\mathbf{H}_i\mathbf{H}_i^H$, and $\mathbf{v}_l(i)$ denotes its corresponding eigenvector. Having the facts that $\lambda_1(i) = \ln N + o(\ln N)$, which has been proved to be the necessary condition to achieve the maximum sum-rate (in Lemma 3.12), and $\|\mathbf{H}_i\|^2 = \sum_l \lambda_l(i) = \ln N + o(\ln N)$, with probability one [26], it follows that $\sum_{l=2}^{K}\lambda_l(i)\left|\mathbf{v}_l^H(i)\mathbf{\Phi}_i\right|^2 = o(\ln N)$. Having this and $\mathcal{C}_{\text{sum}} = M\ln\left(\frac{P}{M}\ln N + o(\ln N)\right)$ [26], it follows that to achieve the maximum sum-rate we must have $\lambda_1(i)\left|\mathbf{v}_1^H(i)\mathbf{\Phi}_i\right|^2 = \ln N[1+o(1)]$, $\forall i, 1 \leq i \leq M$. Noting $\lambda_1(i) = \ln N + O(\ln\ln N)$, we conclude $\left|\mathbf{v}_1^H(i)\mathbf{\Phi}_i\right|^2 = 1+o(1)$, $\forall 1 \leq i \leq M$. In other words, the coordinate of the transmit covariance matrix for each user is almost in the direction of the eigenvector corresponding to the maximum singular value of that user.

The rate of the ith encoded user can be upper-bounded as

$$\begin{aligned}\mathcal{R}_{\pi(i)} &= \mathbb{E}\left\{\ln\left|\mathbf{I}+\mathbf{H}_{\pi(i)}\mathbf{Q}_{\pi(i)}\mathbf{H}_{\pi(i)}^H\left(\mathbf{I}+\mathbf{H}_{\pi(i)}\left(\sum_{j>i}\mathbf{Q}_{\pi(j)}\right)\mathbf{H}_{\pi(i)}^H\right)^{-1}\right|\right\} \\ &\le \frac{1}{M-i}\sum_{j=i+1}^{M}\mathbb{E}\left\{\ln\left|\mathbf{I}+\mathbf{H}_{\pi(i)}\mathbf{Q}_{\pi(i)}\mathbf{H}_{\pi(i)}^H\left(\mathbf{I}+\mathbf{H}_{\pi(i)}\mathbf{Q}_{\pi(j)}\mathbf{H}_{\pi(i)}^H\right)^{-1}\right|\right\}.\end{aligned} \tag{3.111}$$

Substituting $\mathbf{Q}_{\pi(i)}$ and $\mathbf{Q}_{\pi(j)}$ from (3.109) yields

$$\begin{aligned}\mathcal{R}_{\pi(i)} &\le \frac{1}{M-i}\sum_{j=i+1}^{M}\mathbb{E}\left\{\ln\left|\mathbf{I}+\rho_{\pi(i)}\eta_{\pi(i)}\mathbf{\Psi}_{\pi(i)}\mathbf{\Psi}_{\pi(i)}^H\left(\mathbf{I}+\rho_{\pi(j)}I_{\pi(j)}^{\pi(i)}\mathbf{\Omega}_{\pi(j)}\mathbf{\Omega}_{\pi(j)}^H\right)^{-1}\right|\right\} \\ &\overset{(a)}{=} \frac{1}{M-i}\sum_{j=i+1}^{M}\mathbb{E}\left\{\ln\left(1+\rho_{\pi(i)}\eta_{\pi(i)}\mathbf{\Psi}_{\pi(i)}^H\left[\mathbf{I}-\frac{\rho_{\pi(j)}I_{\pi(j)}^{\pi(i)}}{1+\rho_{\pi(j)}I_{\pi(j)}^{\pi(i)}}\mathbf{\Omega}_{\pi(j)}\mathbf{\Omega}_{\pi(j)}^H\right]\mathbf{\Psi}_{\pi(i)}\right)\right\} \\ &= \frac{1}{M-i}\sum_{j=i+1}^{M}\mathbb{E}\left\{\ln\left(1+\rho_{\pi(i)}\eta_{\pi(i)}\right)\right\}+ \\ &\quad \mathbb{E}\left\{\ln\left(1-\frac{\rho_{\pi(i)}\eta_{\pi(i)}}{1+\rho_{\pi(i)}\eta_{\pi(i)}}\frac{\rho_{\pi(j)}I_{\pi(j)}^{\pi(i)}}{1+\rho_{\pi(j)}I_{\pi(j)}^{\pi(i)}}|\mathbf{\Psi}_{\pi(i)}^H\mathbf{\Omega}_{\pi(j)}|^2\right)\right\} \\ &\overset{(b)}{\le} \frac{1}{M-i}\sum_{j=i+1}^{M}\mathbb{E}\left\{\ln\left(1+\rho_{\pi(i)}\eta_{\pi(i)}\right)\right\}+ \\ &\quad \ln\left(1-\mathbb{E}\left\{\frac{\rho_{\pi(i)}\eta_{\pi(i)}}{1+\rho_{\pi(i)}\eta_{\pi(i)}}\frac{\rho_{\pi(j)}I_{\pi(j)}^{\pi(i)}}{1+\rho_{\pi(j)}I_{\pi(j)}^{\pi(i)}}|\mathbf{\Psi}_{\pi(i)}^H\mathbf{\Omega}_{\pi(j)}|^2\right\}\right),\end{aligned} \tag{3.112}$$

where $\eta_{\pi(i)} \triangleq \|\mathbf{H}_{\pi(i)}\mathbf{\Phi}_{\pi(i)}\|^2$, $I_{\pi(j)}^{\pi(i)} \triangleq \|\mathbf{H}_{\pi(i)}\mathbf{\Phi}_{\pi(j)}\|^2$, $\mathbf{\Psi}_{\pi(i)} \triangleq \frac{\mathbf{H}_{\pi(i)}\mathbf{\Phi}_{\pi(i)}}{\|\mathbf{H}_{\pi(i)}\mathbf{\Phi}_{\pi(i)}\|}$, $\mathbf{\Omega}_{\pi(j)} \triangleq \frac{\mathbf{H}_{\pi(i)}\mathbf{\Phi}_{\pi(j)}}{\|\mathbf{H}_{\pi(i)}\mathbf{\Phi}_{\pi(j)}\|}$. (a) comes from the facts $|\mathbf{I}+\mathbf{AB}| = |\mathbf{I}+\mathbf{BA}|$ and

$$\left(\mathbf{I}+\rho_{\pi(j)}I_{\pi(j)}^{\pi(i)}\mathbf{\Omega}_{\pi(j)}\mathbf{\Omega}_{\pi(j)}^H\right)^{-1} = \mathbf{I}-\frac{\rho_{\pi(j)}I_{\pi(j)}^{\pi(i)}}{1+\rho_{\pi(j)}I_{\pi(j)}^{\pi(i)}}\mathbf{\Omega}_{\pi(j)}\mathbf{\Omega}_{\pi(j)}^H,$$

and (b) comes from the concavity of ln function. From the above equation, and noting the facts that $\mathbb{E}\left\{\ln\left(1+\rho_{\pi(i)}\eta_{\pi(i)}\right)\right\} \le \ln\left(\frac{P}{M}\ln N + o(\ln N)\right)$ and $\mathcal{C}_{\text{sum}} =$

$M \ln \left(\frac{P}{M} \ln N + o(\ln N)\right)$, it follows that in order to achieve the maximum sum-rate, the term

$$\ln\left(1 - \mathbb{E}\left\{\frac{\rho_{\pi(i)}\eta_{\pi(i)}}{1+\rho_{\pi(i)}\eta_{\pi(i)}}\frac{\rho_{\pi(j)}I_{\pi(j)}^{\pi(i)}}{1+\rho_{\pi(j)}I_{\pi(j)}^{\pi(i)}}|\mathbf{\Psi}_{\pi(i)}^{H}\mathbf{\Omega}_{\pi(j)}|^2\right\}\right)$$

must approach zero for all i and $j > i$, which incurs that

$$\mathbb{E}\left\{\frac{\rho_{\pi(i)}\eta_{\pi(i)}}{1+\rho_{\pi(i)}\eta_{\pi(i)}}\frac{\rho_{\pi(j)}I_{\pi(j)}^{\pi(i)}}{1+\rho_{\pi(j)}I_{\pi(j)}^{\pi(i)}}|\mathbf{\Psi}_{\pi(i)}^{H}\mathbf{\Omega}_{\pi(j)}|^2\right\} = o(1) \quad \forall i, j > i.$$

Since $\rho_{\pi(i)} \to \infty$ (as $P \to \infty$), and $\eta_{\pi(i)} \sim \ln N$, the term $\frac{\rho_{\pi(i)}\eta_{\pi(i)}}{1+\rho_{\pi(i)}\eta_{\pi(i)}} \approx 1$, with probability one. Writing $\mathbf{v}_1(\pi(i))$ as $\alpha_{\pi(i)}\mathbf{\Phi}_{\pi(i)} + \mathbf{v}_1(\pi(i))^{\perp}$ and $\mathbf{\Phi}_{\pi(i)}$ as $\gamma_{\pi(i)}\mathbf{v}_1(\pi(i)) + \mathbf{\Phi}_{\pi(i))}^{\perp}$, where $\alpha_{\pi(i)} \triangleq \mathbf{\Phi}_{\pi(i)}^{H}\mathbf{v}_1(\pi(i))$, $\gamma_{\pi(i)} \triangleq \mathbf{v}_1(\pi(i))^{H}\mathbf{\Phi}_{\pi(i)}$, $\mathbf{v}_1(\pi(i))^{\perp}$ denotes the projection of $\mathbf{v}_1(\pi(i))$ over the null space of $\mathbf{\Phi}_{\pi(i)}$ and $\mathbf{\Phi}_{\pi(i)}^{\perp}$ denotes the projection of $\mathbf{\Phi}_{\pi(i)}$ over the null space of $\mathbf{v}_1(\pi(i))$, $\chi \triangleq \mathbb{E}\left\{\frac{\rho_{\pi(j)}I_{\pi(j)}^{\pi(i)}}{1+\rho_{\pi(j)}I_{\pi(j)}^{\pi(i)}}|\mathbf{\Psi}_{\pi(i)}^{H}\mathbf{\Omega}_{\pi(j)}|^2\right\}$

can be written as

$$
\begin{aligned}
\chi &= \mathbb{E}\left\{ \frac{\rho_{\pi(j)} I_{\pi(j)}^{\pi(i)}}{1+\rho_{\pi(j)} I_{\pi(j)}^{\pi(i)}} \frac{\left| \mathbf{\Phi}_{\pi(i)}^H \mathbf{H}_{\pi(i)}^H \mathbf{H}_{\pi(i)} \mathbf{\Phi}_{\pi(j)} \right|^2}{\eta_{\pi(i)} I_{\pi(j)}^{\pi(i)}} \right\} \\
&= \mathbb{E}\left\{ \frac{\rho_{\pi(j)}}{1+\rho_{\pi(j)} I_{\pi(j)}^{\pi(i)}} \frac{\left| \left[\gamma_{\pi(i)} \mathbf{v}_1(\pi(i)) + \mathbf{\Phi}_{\pi(i))}^{\perp} \right]^H \mathbf{H}_{\pi(i)}^H \mathbf{H}_{\pi(i)} \mathbf{\Phi}_{\pi(j)} \right|^2}{\eta_{\pi(i)}} \right\} \\
&\overset{(a)}{\geq} \mathbb{E}\left\{ \frac{\rho_{\pi(j)}}{1+\rho_{\pi(j)} I_{\pi(j)}^{\pi(i)}} \frac{\left(\left|\gamma_{\pi(i)}\right| \left| \mathbf{v}_1(\pi(i))^H \mathbf{H}_{\pi(i)}^H \mathbf{H}_{\pi(i)} \mathbf{\Phi}_{\pi(j)} \right| - \left| \left(\mathbf{\Phi}_{\pi(i)}^{\perp} \right)^H \mathbf{H}_{\pi(i)}^H \mathbf{H}_{\pi(i)} \mathbf{\Phi}_{\pi(j)} \right| \right)^2}{\eta_{\pi(i)}} \right\} \\
&\overset{(b)}{\geq} \mathbb{E}\left\{ \frac{\rho_{\pi(j)}}{1+\rho_{\pi(j)} I_{\pi(j)}^{\pi(i)}} \frac{\left(\left|\gamma_{\pi(i)}\right| \lambda_{\max}(\pi(i)) \left| \mathbf{v}_1(\pi(i))^H \mathbf{\Phi}_{\pi(j)} \right| - \lambda_{\max}(\pi(i)) \| \mathbf{\Phi}_{\pi(i)}^{\perp} \| \right)^2}{\eta_{\pi(i)}} \right\} \\
&\overset{(c)}{\geq} \mathbb{E}\left\{ \frac{\rho_{\pi(j)} \lambda_{\max}(\pi(i))}{1+\rho_{\pi(j)} I_{\pi(j)}^{\pi(i)}} \left(\left|\gamma_{\pi(i)}\right| \left| \left[\alpha_{\pi(i)} \mathbf{\Phi}_{\pi(i)} + \mathbf{v}_1(\pi(i))^{\perp} \right]^H \mathbf{\Phi}_{\pi(j)} \right| - \left\| \mathbf{\Phi}_{\pi(i)}^{\perp} \right\| \right)^2 \right\} \\
&\overset{(d)}{\geq} \mathbb{E}\left\{ \left(\left|\gamma_{\pi(i)}\right| \left|\alpha_{\pi(i)}\right| \left| \mathbf{\Phi}_{\pi(i)}^H \mathbf{\Phi}_{\pi(j)} \right| - \| \mathbf{v}_1(\pi(i))^{\perp} \| - \left\| \mathbf{\Phi}_{\pi(i)}^{\perp} \right\| \right)^2 \right\},
\end{aligned} \tag{3.113}
$$

where $\lambda_{\max}(\pi(i))$ denotes the maximum singular value of $\mathbf{H}_{\pi(i)}^H \mathbf{H}_{\pi(i)}$. (a) comes from the fact that $|a+b|^2 \geq (|a|-|b|)^2$. (b) results from the facts that $\mathbf{v}_1(\pi(i))$ is the eigenvector corresponding to the maximum singular value of $\mathbf{H}_{\pi(i)}$, and hence, $\mathbf{v}_1(\pi(i))^H \mathbf{H}_{\pi(i)}^H \mathbf{H}_{\pi(i)} = \lambda_{\max}(\pi(i)) \mathbf{v}_1(\pi(i))^H$, and also $\left| \left(\mathbf{\Phi}_{\pi(i)}^{\perp} \right)^H \mathbf{H}_{\pi(i)}^H \mathbf{H}_{\pi(i)} \mathbf{\Phi}_{\pi(j)} \right|^2 \leq \left\| \mathbf{\Phi}_{\pi(i)}^{\perp} \right\|^2 \lambda_{\max}(\pi(i))$. (c) comes from the fact that $\eta_{\pi(i)} = \left\| \mathbf{H}_{\pi(i)} \mathbf{\Phi}_{\pi(i)} \right\|^2 \leq \lambda_{\max}(\pi(i))$, and finally (d) results from the facts that $I_{\pi(j)}^{\pi(i)} = \left\| \mathbf{H}_{\pi(i)} \mathbf{\Phi}_{\pi(j)} \right\|^2 \leq \lambda_{\max}(\pi(i))$,

$$
\left| \left[\alpha_{\pi(i)} \mathbf{\Phi}_{\pi(i)} + \mathbf{v}_1(\pi(i))^{\perp} \right]^H \mathbf{\Phi}_{\pi(j)} \right| \geq \left|\alpha_{\pi(i)}\right| \left| \mathbf{\Phi}_{\pi(i)}^H \mathbf{\Phi}_{\pi(j)} \right| - \| \mathbf{v}_1(\pi(i))^{\perp} \|,
$$

and $|\gamma_{\pi(i)}| < 1$. Since $\left| \mathbf{v}_1^H(\pi(i)) \mathbf{\Phi}_{\pi(i)} \right| = 1 + o(1)$, it follows that $|\alpha_{\pi(i)}| = |\gamma_{\pi(i)}| = 1 + o(1)$ and $\| \mathbf{v}_1(\pi(i))^{\perp} \| = \left\| \mathbf{\Phi}_{\pi(i)}^{\perp} \right\| = o(1)$. Hence, the necessary condition to achieve the maximum sum-rate is having $\left|\mathbf{\Phi}_{\pi(i)}^H \mathbf{\Phi}_{\pi(j)}\right|^2 = o(1), \forall i, j > i$, with probability one. In other words, $\mathbf{\Phi}_{\pi(i)}$ and $\mathbf{\Phi}_{\pi(j)}$ must be semi-orthogonal to each other with probability one, which completes the proof of Lemma 3.12.

Remark - It is worth to note that the right hand side of (3.110) achieves the maximum sum-rate of $M \ln\left(1 + \frac{P}{M} \ln N[1 + o(1)]\right)$ if the power is uniformly allocated to the coordinates, almost surely. In other words, $\rho_i = \frac{P}{M}[1 + o(1)]$.

Lemma 3.14 *Defining* $\epsilon_i \triangleq \mathbf{v}_1^H(\pi(i))\mathbf{\Upsilon}_i$, *where* $\mathbf{\Upsilon}_i \triangleq \left[\mathbf{\Phi}_{\pi(i+1)} | \cdots | \mathbf{\Phi}_{\pi(M)}\right]$, *and* $\mathbf{v}_1(\pi(i))$ *denotes the eigenvector corresponding to the maximum eigenvalue of the* i*th encoded user, assuming Dirty-paper Coding, the necessary condition to have* $\lim_{N\to\infty} \mathcal{C}_{\text{sum}} - \mathcal{R} \to 0$, *in the case* $K < M - i + 1$ *is* $\|\epsilon_i\|^2 = o\left(\frac{1}{P \ln N}\right)$ *and in the case* $K \geq M - i + 1$ *is* $\|\epsilon_i\|^2 = o(1)$, *with probability one.*

Proof - Consider the user with the encoding order i. The rate of this user can be upper-bounded as

$$\begin{aligned}
\mathcal{R}_{\pi(i)} &\leq \mathbb{E}\left\{\ln\left|\mathbf{I} + \mathbf{H}_{\pi(i)}\mathbf{Q}_{\pi(i)}\mathbf{H}_{\pi(i)}^H\left(\mathbf{I} + \mathbf{H}_{\pi(i)}\left[\sum_{j=i+1}^{M}\mathbf{Q}_{\pi(j)}\right]\mathbf{H}_{\pi(i)}^H\right)^{-1}\right|\right\} \\
&= \mathbb{E}\left\{\ln\left|\mathbf{I} + \rho_{\pi(i)}\mathbf{H}_{\pi(i)}\mathbf{\Phi}_{\pi(i)}\mathbf{\Phi}_{\pi(i)}^H\mathbf{H}_{\pi(i)}^H\left(\mathbf{I} + \mathbf{H}_{\pi(i)}\left[\sum_{j=i+1}^{M}\rho_{\pi(j)}\mathbf{\Phi}_{\pi(j)}\mathbf{\Phi}_{\pi(j)}^H\right]\mathbf{H}_{\pi(i)}^H\right)^{-1}\right|\right\}.
\end{aligned} \tag{3.114}$$

Writing the SVD of $\mathbf{H}_{\pi(i)}$ as $\mathbf{U}_{\pi(i)}\mathbf{\Lambda}_{\pi(i)}\mathbf{V}_{\pi(i)}^H$, we have

$$\mathcal{R}_{\pi(i)} \leq \mathbb{E}\left\{\ln\left|\mathbf{I} + \rho_{\pi(i)}\lambda_1(\pi(i))\mathbf{\Psi}_{\pi(i)}\mathbf{\Psi}_{\pi(i)}^H\mathbf{W}\right|\right\}, \tag{3.115}$$

where $\mathbf{W} \triangleq (\mathbf{I} + \mathbf{G})^{-1}$, in which $\mathbf{G} \triangleq \mathbf{\Lambda}_{\pi(i)}\mathbf{V}_{\pi(i)}^H\left[\sum_{j=i+1}^{M}\rho_{\pi(j)}\mathbf{\Phi}_{\pi(j)}\mathbf{\Phi}_{\pi(j)}^H\right]\mathbf{V}_{\pi(i)}\mathbf{\Lambda}_{\pi(i)}^T$, and $\mathbf{\Psi}_{\pi(i)} \triangleq \frac{\mathbf{\Lambda}_{\pi(i)}\mathbf{V}_{\pi(i)}^H\mathbf{\Phi}_{\pi(i)}}{\sqrt{\lambda_1(\pi(i))}}$. Having the facts that $\mathbf{v}_1^H(\pi(i))\mathbf{\Phi}_{\pi(i)} = 1 + o(1)$,

$\mathbf{v}_j^H(\pi(i))\mathbf{\Phi}_{\pi(i)} = o(1)$, $j \neq 1$ (Lemma 3.13), $\lambda_1(\pi(i)) \sim \ln N$, and $\lambda_j(\pi(i)) = o(\ln N)$, $j \neq 1$ (Lemma 3.12), we have $\mathbf{\Psi}_{\pi(i)} = [1 + o(1), o(1), \cdots, o(1)]^T$. In other words, as $N \to \infty$, $\mathbf{\Psi}_{\pi(i)}$ approaches to the vector $[1, 0, \cdots, 0]^T$. Using $|\mathbf{I} + \mathbf{AB}| = |\mathbf{I} + \mathbf{BA}|$, we can write

$$\begin{aligned} \mathcal{R}_{\pi(i)} &\leq \mathbb{E}\left\{\ln\left(1 + \rho_{\pi(i)}\lambda_1(\pi(i))\mathbf{\Psi}_{\pi(i)}^H \mathbf{W} \mathbf{\Psi}_{\pi(i)}\right)\right\} \\ &\approx \mathbb{E}\left\{\ln\left(1 + \rho_{\pi(i)}\lambda_1(\pi(i))\mathbf{W}_{11}\left[1 + o(1)\right]\right)\right\}, \end{aligned} \tag{3.116}$$

where $\mathbf{A}_{ij}$ denotes the (i, j)th entry of matrix $\mathbf{A}$. Using the concavity of ln function, and having the facts that $\lambda_1(\pi(i)) = \ln N + o(\ln N)$ with probability one, we have

$$\mathcal{R}_{\pi(i)} \leq \ln\left(1 + \rho_{\pi(i)}(\ln N)\mathbb{E}\left\{\mathbf{W}_{11}\right\}\left[1 + o(1)\right]\right). \tag{3.117}$$

Since the necessary condition to achieve the maximum sum-rate is having $\mathcal{R}_{\pi(i)} = \ln(\frac{P}{M}\ln N) + o(1)$, $\forall i$, the above equation implies that the necessary condition to have $\lim_{N\to\infty} \mathcal{C}_{\text{sum}} - \mathcal{R} = 0$ is having $\mathbb{E}\left\{\mathbf{W}_{11}\right\} = 1 + o(1)$, which incurs that $\mathbf{W}_{11}$ must scale as $1 + o(1)$, with probability one. In the following, we calculate $\mathbf{W}_{11}$.

$\mathbf{G} = \mathbf{\Lambda}_{\pi(i)}\mathbf{V}_{\pi(i)}^H \left[\sum_{j=i+1}^M \rho_{\pi(j)}\mathbf{\Phi}_{\pi(j)}\mathbf{\Phi}_{\pi(j)}^H\right] \mathbf{V}_{\pi(i)}\mathbf{\Lambda}_{\pi(i)}^T$ can be written as

$$\mathbf{G} = \mathbf{Z}\mathbf{\Theta}\mathbf{\Theta}^H\mathbf{Z}^H, \tag{3.118}$$

where $\mathbf{Z} \triangleq \left[\sqrt{\lambda_1(\pi(i))}\mathbf{v}_1(\pi(i))| \cdots |\sqrt{\lambda_K(\pi(i))}\mathbf{v}_K(\pi(i))\right]^H$, and

$$\mathbf{\Theta} \triangleq \left[\sqrt{\rho_{\pi(i+1)}}\mathbf{\Phi}_{\pi(i+1)}| \cdots |\sqrt{\rho_{\pi(M)}}\mathbf{\Phi}_{\pi(M)}\right].$$

$\mathbf{Z}\mathbf{\Theta}$ can be written as $\left[\mathbf{\Xi}^T|\mathbf{\Omega}^T\right]^T$, where $\mathbf{\Xi} \triangleq \sqrt{\lambda_1(\pi(i))}\mathbf{v}_1^H(\pi(i))\mathbf{\Theta}$ and $\mathbf{\Omega} \triangleq \mathbf{Z}_r\mathbf{\Theta}$, and

$$\mathbf{Z}_r \triangleq \left[\sqrt{\lambda_2(\pi(i))}\mathbf{v}_2(\pi(i))\,|\cdots|\,\sqrt{\lambda_K(\pi(i))}\mathbf{v}_K(\pi(i))\right]^H.$$

Substituting in the above equation yields

$$\mathbf{G} = \begin{bmatrix} \|\boldsymbol{\Xi}\|^2 & \boldsymbol{\Xi}\boldsymbol{\Omega}^H \\ \boldsymbol{\Omega}\boldsymbol{\Xi}^H & \boldsymbol{\Omega}\boldsymbol{\Omega}^H \end{bmatrix}. \tag{3.119}$$

As a result, $\mathbf{W}_{11}$ can be written as

$$\begin{aligned} \mathbf{W}_{11} &= \frac{|\mathbf{I}+\boldsymbol{\Omega}\boldsymbol{\Omega}^H|}{|\mathbf{I}+\mathbf{G}|} \\ &= \frac{|\mathbf{I}+\boldsymbol{\Omega}\boldsymbol{\Omega}^H|}{(1+\|\boldsymbol{\Xi}\|^2)\,|\mathbf{I}+\boldsymbol{\Omega}\boldsymbol{\Omega}^H| + \sum_{j=2}^{K}(-1)^{j+1}\mathbf{G}_{1j}|\Delta(\mathbf{C}_{1j})|}, \end{aligned} \tag{3.120}$$

where $\Delta(\mathbf{C}_{1j})$ denotes the minor of $\mathbf{C}_{1j}$ and $\mathbf{C} \triangleq \mathbf{G}+\mathbf{I}$. $|\Delta(\mathbf{C}_{1j})|$ can be computed as

$$|\Delta(\mathbf{C}_{1j})| = \sum_{\substack{i \\ 1,j\notin\mathcal{A}_i}} |\Delta_{\mathcal{A}_i}(\mathbf{G}_{1j})|\,, \tag{3.121}$$

where $\Delta_{\mathcal{A}_i}(\mathbf{G}_{1j})$ denotes a sub-matrix of $\Delta(\mathbf{G}_{1j})$, resulted from deleting the rows and columns corresponding to the elements in $\mathcal{A}_i$, and $\mathcal{A}_i$ is an arbitrary subset of $\{1, 2, \cdots, K\}$. Note that $\Delta_{\emptyset}(\mathbf{G}_{1j}) = \Delta(\mathbf{G}_{1j})$, where $\emptyset$ denotes the null set. Similarly, we can write

$$|\mathbf{I}+\boldsymbol{\Omega}\boldsymbol{\Omega}^H| = \sum_{\substack{i \\ 1\notin\mathcal{A}_i}} |\Delta_{\mathcal{A}_i}(\mathbf{G}_{11})|\,. \tag{3.122}$$

Substituting (3.121) and (3.122) in (3.120), after some manipulations, we obtain

$$\mathbf{W}_{11} = \frac{|\mathbf{I}+\boldsymbol{\Omega}\boldsymbol{\Omega}^H|}{|\mathbf{I}+\boldsymbol{\Omega}\boldsymbol{\Omega}^H| + |\mathbf{G}| + \|\boldsymbol{\Xi}\|^2\delta_1 + \sum_{j=2}^{K}(-1)^{j+1}\mathbf{G}_{1j}\delta_j}, \tag{3.123}$$

where $\delta_1 \triangleq \sum_{\substack{i \\ 1\notin\mathcal{A}_i \\ \mathcal{A}_i\neq\emptyset}} |\Delta_{\mathcal{A}_i}(\mathbf{G}_{11})|$ and $\delta_j \triangleq \sum_{\substack{i \\ 1,j\notin\mathcal{A}_i \\ \mathcal{A}_i\neq\emptyset}} |\Delta_{\mathcal{A}_i}(\mathbf{G}_{1j})|$. Two situations can occur here:

- *Case I;* $K \geq M - i + 1$: In this case, since $\mathbf{G}$ is of rank at most $M - i$, $|\mathbf{G}| = 0$ in the above equation. We have observed that in order to achieve the maximum sum-rate $\rho_{\pi(j)} = \frac{P}{M}[1 + o(1)]$, which incurs $|\mathbf{G}_{lk}| = \Theta\left(Pf^{(1)}(\boldsymbol{\lambda})\right)$, $k, l \neq 1$, where $\boldsymbol{\lambda} \triangleq [\lambda_2(\pi(i)), \cdots, \lambda_K(\pi(i))]$, and $f^{(m)}(\boldsymbol{\lambda})$ denotes a function of $\boldsymbol{\lambda}$, with order m [9]. Having this, it can be easily proved that

$$\|\boldsymbol{\Xi}\|^2\delta_1 + \sum_{j=2}^{K}(-1)^{j+1}\mathbf{G}_{1j}\delta_j = \Theta\left(P^{K-2}\|\boldsymbol{\Xi}\|^2 f^{(K-2)}(\boldsymbol{\lambda})\right),$$

  and

$$|\mathbf{I} + \boldsymbol{\Omega}\boldsymbol{\Omega}^H| = \Theta\left(P^{K-1}f^{(K-2)}(\boldsymbol{\lambda})g^{(1)}(\boldsymbol{\lambda})\right). \quad (3.124)$$

  Using this and (3.123), it follows that the necessary condition to satisfy $\mathbf{W}_{11} = 1 + o(1)$ is having $\|\boldsymbol{\Xi}\|^2 = o\left(Pg^{(1)}(\boldsymbol{\lambda})\right)$. Since $g^{(1)}(\boldsymbol{\lambda}) = o(\ln N)$, this condition can be written as $\|\boldsymbol{\Xi}\|^2 = o\left(P \ln N\right)$.

- *Case II;* $K < M - i + 1$: In this case, $\mathbf{G}$ is full-rank with probability one and with a similar argument as in the previous part, we can show that

$$|\mathbf{G}| = \Theta\left(\|\boldsymbol{\Xi}\|^2 P^{K-1} f^{(K-2)}(\boldsymbol{\lambda})g^{(1)}(\boldsymbol{\lambda})\right).$$

  Hence, using (3.123) and (3.124), the necessary condition to satisfy $\mathbf{W}_{11} = 1 + o(1)$ is having $\|\boldsymbol{\Xi}\|^2 = o\left(1\right)$.

Having the facts that $\rho_{\pi(j)} \sim \frac{P}{M}$ and $\lambda_1(\pi(i)) \sim \ln N$, we have $\|\epsilon_i\|^2 \sim \frac{\|\boldsymbol{\Xi}_i\|^2}{P \ln N}$. Therefore, the conditions of $\|\boldsymbol{\Xi}\|^2 = o\left(P \ln N\right)$ and $\|\boldsymbol{\Xi}\|^2 = o\left(1\right)$ are translated into $\|\epsilon_i\|^2 = o\left(1\right)$ and $\|\epsilon_i\|^2 = o\left(\frac{1}{P \ln N}\right)$, respectively, which completes the proof of

[9] A function $f(x_1, \cdots, x_n)$ is said to be of order m, if it can be written as $\sum_j c_j \prod_{l=1}^{n} x_l^{\alpha_l(j)}$, where $\sum_{l=1}^{n} \alpha_l(j) = m$, $\forall j$.

Lemma 3.14.

■

Remark - Note that since

$$\|\epsilon_i\|^2 = \sum_{j=i+1}^{M} |\mathbf{v}_1^H(\pi(i))\mathbf{\Phi}_{\pi(j)}|^2,$$

it follows that for case 1,

$$|\mathbf{v}_1^H(\pi(i))\mathbf{\Phi}_{\pi(j)}|^2 = o\,(1)\,, \quad i+1 \le j \le M,$$

and for case 2,

$$|\mathbf{v}_1^H(\pi(i))\mathbf{\Phi}_{\pi(j)}|^2 = o\left(\frac{1}{P\ln N}\right), \quad i+1 \le j \le M.$$

In other words, achieving the maximum sum-rate imposes an orthogonality constraint between the eigenvector corresponding to the maximum singular value of each user and the coordinates of the transmitted signal for users with higher encoding orders. This orthogonality constraint is much more restrictive in the second case.

In Lemmas 3.12-3.14, we have proved that, for any user selection strategy and any known precoding scheme, in order to achieve the maximum sum-rate capacity, the following constraints must be satisfied with probability one:

- The maximum singular values of selected users must behave as $\ln N$.
- The transmit covariance matrices must be rank one.
- The transmit coordinates must be almost orthogonal to each other. Moreover, they must be almost in the direction of the eigenvectors corresponding to the maximum singular values of the selected users.

- The transmit power must be allocated almost uniformly among the selected users.

Having the above constraints satisfied, depending on the number of receive antennas, an orthogonality constraint must be satisfied between the eigenvector corresponding to the maximum singular value of each user and the transmit coordinates of the users with higher encoding orders, with probability one. Now, the question is that, taking the effect of quantization into account, how accurate should the BS know the channels of the selected users such that the above constraints are satisfied. For this purpose, we focus on the last constraint and associate $\|\epsilon_i\|^2$ with the size of the quantization cookbook for the ith encoded user in the following lemma:

Lemma 3.15 *Let L_i be the size of the codebook used for the quantization of $\mathbf{H}_{\pi(i)}$. Then, for any quantization method and any value of θ, we have*

$$Pr\{\|\epsilon_i\|^2 > \theta\} \geq \left[\max\left(0, 1 - L_i \binom{M-1}{i-1} \theta^{M-i}\right)\right]^N. \tag{3.125}$$

Proof - Since the transmitter only knows the quantized information about the channel matrices, we can write $\mathbf{v}_1(\pi(i))$ as $\widehat{\mathbf{v}}_1(\pi(i)) + \Delta\mathbf{v}_1(\pi(i))$, where $\widehat{\mathbf{v}}_1(\pi(i))$ is perfectly known by the transmitter and can be considered as a deterministic vector, and $\Delta\mathbf{v}_1(\pi(i))$ is unknown to the transmitter. Hence, we have

$$\begin{aligned} \epsilon_i &= \left[\widehat{\mathbf{v}}_1(\pi(i)) + \Delta\mathbf{v}_1(\pi(i))\right]^H \boldsymbol{\Upsilon}_i \\ &= \mathbf{b}_{\pi(i)} + \Delta\mathbf{x}_{\pi(i)}, \end{aligned} \tag{3.126}$$

where $\mathbf{b}_{\pi(i)} \triangleq \widehat{\mathbf{v}}_1^H(\pi(i))\boldsymbol{\Upsilon}_i$ is a $1 \times (M-i)$ vector, known to the transmitter, while $\Delta\mathbf{x}_{\pi(i)} \triangleq \Delta\mathbf{v}_1^H(\pi(i))\boldsymbol{\Upsilon}_i$ is an unknown $1 \times (M-i)$ vector. We can write

$$\|\epsilon_i\|^2 \geq \min_n \|\mathbf{b}_n + \Delta\mathbf{v}_1^H(n)\boldsymbol{\Upsilon}_i\|^2, \tag{3.127}$$

where $\mathbf{v}_1(n)$ denotes the eigenvector corresponding to the maximum singular value of the nth user, $\Delta\mathbf{v}_1(n)$ denotes the error in $\mathbf{v}_1(n)$ due to the quantization of $\mathbf{H}_n$, and $\mathbf{b}_n \triangleq \widehat{\mathbf{v}}_1^H(n)\boldsymbol{\Upsilon}_i$. In fact, in the above equation, it is assumed that all users quantize their channel matrices, and $\|\epsilon_i\|^2$ is lower-bounded by the minimum error. Since $\Delta\mathbf{v}_1(n)$ are i.i.d random variables, it follows that $\mu_n \triangleq \|\mathbf{b}_n + \Delta\mathbf{x}_n\|^2$, where $\Delta\mathbf{x}_n \triangleq \Delta\mathbf{v}_1^H(n)\boldsymbol{\Upsilon}_i$, are independent from each other. Hence,

$$\Pr\{\|\epsilon_i\|^2 > \theta\} \quad \geq \quad \prod_{n=1}^{N} \xi_n, \tag{3.128}$$

where $\xi_n \triangleq \Pr\{\mu_n > \theta\}$. ξ_n can be lower-bounded as follows:

$$\begin{aligned} \xi_n &\overset{(a)}{\geq} 1 - \Pr\left\{\bigcup_{l=1}^{L_i} \|\mathbf{x}_n - \mathbf{d}_l\|^2 \leq \theta\right\} \\ &\overset{(b)}{\geq} \max\left(0, 1 - \sum_{l=1}^{L_i} \Pr\left\{\|\mathbf{x}_n - \mathbf{d}_l\|^2 \leq \theta\right\}\right), \end{aligned} \tag{3.129}$$

where $\mathbf{c}_l$, $l = 1, \cdots, L_i$, are the corresponding quantization code words for the quantization of $\mathbf{x}_n \triangleq \mathbf{v}_1^H(n)\boldsymbol{\Upsilon}_i$, and $\mathbf{d}_l \triangleq \mathbf{c}_l - \mathbf{b}_n$. (a) comes from the fact that all the quantization bits are not necessarily utilized for the quantization of $\mathbf{x}_n$ [10], and (b) results from the union bound for the probability.

Since the columns of $\boldsymbol{\Upsilon}_i$, namely $\{\boldsymbol{\Phi}_{\pi(j)}\}_{j=i+1}^{M}$, are semi-orthogonal to each other, $\mathbf{x}_n \triangleq \mathbf{v}_1^H(n)\boldsymbol{\Upsilon}_i$ can be approximated by $\mathbf{y}_n$, which denotes the projection of $\mathbf{v}_1(n)$ over the $(M-i)$-dimensional sub-space spanned by $\{\boldsymbol{\Phi}_{\pi(j)}\}_{j=i+1}^{M}$. More precisely,

$$\mathbf{x}_n = \mathbf{y}_n \left[\mathbf{I} + o(\mathbf{I})\right]. \tag{3.130}$$

[10] In fact, if we denote the original quantization code words, utilized for the quantization of $\mathbf{H}_n$, by $\{\mathbf{e}_l\}_{l=1}^{L_i}$, we can write $\mathbf{c}_l = f(\mathbf{e}_l)$, $1 \leq l \leq L_i$, where $f(.)$ is a mapping which depends on the quantization method. Since the mapping $f(.)$ is not necessarily one-to-one, it follows that the number of distinct elements in the set $\{\mathbf{c}_l\}_{l=1}^{L_i}$ is at most L_i.

As $\mathbf{v}_1(n)$ is an isotropically distributed unit vector in $\mathbb{C}^{1\times M}$, the pdf of $\mathbf{y}_n$ can be computed from [36] as

$$p(\mathbf{y}_n) = \frac{(M-1)!}{\pi^{M-i}(i-1)!}\left(1-\|\mathbf{y}_n\|^2\right)^{i-1}, \quad \|\mathbf{y}_n\| \leq 1. \tag{3.131}$$

Combining (3.130) and (3.131), $\Pr\{\|\mathbf{x}_n - \mathbf{d}_l\|^2 \leq \theta\}$ can be computed as

$$\begin{aligned}
\Pr\left\{\|\mathbf{x}_n - \mathbf{d}_l\|^2 \leq \theta\right\} &= \int_{C_{M-i}(\mathbf{d}_l,\sqrt{\theta})} p(\mathbf{x}_n)\mathrm{d}\mathbf{x}_n \\
&\overset{(3.130)}{\approx} \int_{C_{M-i}(\mathbf{d}_l,\sqrt{\theta})} p(\mathbf{y}_n)\mathrm{d}\mathbf{y}_n \\
&\overset{(a)}{\leq} \frac{(M-1)!}{\pi^{M-i}(i-1)!}\int_{C_{M-i}(\mathbf{d}_l,\sqrt{\theta})} \mathrm{d}\mathbf{y}_n \\
&= \frac{(M-1)!}{\pi^{M-i}(i-1)!}\mathrm{vol}\left(C_{M-i}(\mathbf{d}_l,\sqrt{\theta})\right) \\
&\overset{(b)}{=} \binom{M-1}{i-1}\theta^{M-i},
\end{aligned} \tag{3.132}$$

where $C_m(\mathbf{t}, r)$ denotes the m-dimensional sphere (in the complex space) centered at $\mathbf{t}$ with radius r, and $\mathrm{vol}(\mathbf{v})$ denotes the volume of the region $\mathbf{v}$. (a) comes from the fact that that from (3.131), $p(\mathbf{y}_n) \leq \frac{(M-1)!}{\pi^{M-i}(i-1)!}$, and (b) results from the fact that the volume of a sphere with radios d in the m-dimensional complex space is equal to $\frac{\pi^m}{m!}d^{2m}$. Substituting (3.132) in (3.129), we have

$$\xi_n \geq \max\left(0, 1 - L_i\binom{M-1}{i-1}\theta^{M-i}\right). \tag{3.133}$$

Substituting in (3.128), Lemma 3.15 easily follows.

∎

In Lemma 3.14, we have shown that in order to achieve the maximum sum-rate, in the case $K < M - i + 1$, we must have $\|\epsilon_i\|^2 = o\left(\frac{1}{P\ln N}\right)$ and in the case $K \geq M - i + 1$, we must have $\|\epsilon_i\|^2 = o(1)$, with probability one. In other words, in the first case,

$$\Pr\left\{\|\epsilon_i\|^2 > \frac{1}{P\ln N}\right\} = o(1), \tag{3.134}$$

and in the second case,

$$\Pr\left\{\|\epsilon_i\|^2 > 1\right\} = o(1). \tag{3.135}$$

Combining the above equations with (3.125), it follows that for the user with the encoding order i, such that $i \leq M - K$, we must have

$$\left(1 - L_i\binom{M-1}{i-1}\left[\frac{1}{P\ln N}\right]^{M-i}\right)^N = o(1) \Rightarrow L_i = \omega\left(\frac{[P\ln N]^{M-i}}{N}\right), \tag{3.136}$$

and for the users with the encoding order greater than $M - K$,

$$L_i = \omega\left(\frac{1}{N}\right). \tag{3.137}$$

Therefore, in the case of $K < M$, the total amount of feedback can be written as

$$\begin{aligned}\mathbb{E}\{\mathcal{F}_Q\} &\overset{(a)}{\geq} \mathbb{E}\{\mathcal{N}_Q\} + \sum_{i=1}^{M-K}\left[\log_2(L_i)\right]^+ \\ &\overset{(b)}{=} \ln\ln(P\ln N) + g' + \frac{1}{\ln 2}\sum_{i=1}^{M-K}\left[(M-i)\ln(P\ln N) - \ln N + g''\right]^+,\end{aligned} \tag{3.138}$$

for some $g' \gg 1$ and $g'' \gg 1$, where $\mathcal{N}_Q$ denotes the number of users who send feedback to the BS. (a) comes from the fact that at least $\mathcal{N}_Q$ users send one bit and $(M - K)$ users each send $[\log_2(L_i)]^+$ bits to the BS, where L_i is computed from (3.136). (b) results from (3.85) and (3.136).

In the case of $K = M$, (3.137) does not impose any constraints on L_i. Hence, the total amount of feedback can be lowered-bounded as

$$\begin{aligned}\mathbb{E}\{\mathcal{F}_Q\} &\geq \mathbb{E}\{\mathcal{N}_Q\} \\ &= \ln\ln\ln N + g,\end{aligned} \tag{3.139}$$

which completes the proof of Theorem 3.11.

■

Although the above theorem gives us the necessary conditions for the amount of feedback to achieve the maximum sum-rate, the achievability of those conditions is not clear. A subsequent theorem gives the sufficient condition for achieving the maximum sum-rate.

From the above theorem the following observations can be made:

i) In the case of $K < M$, for the asymptotic scenario of $P \to \infty$, the minimum amount of feedback per user in order to achieve the maximum sum-rate grow logarithmically with SNR. More precisely, in the region $\ln P = \omega(\ln N)$, the total amount of feedback must be at least $\frac{(M-K)(M+K-1)}{2\ln 2} \ln P$, which means that the minimum amount of feedback per user must be $\frac{(M-K)(M+K-1)}{2\ln 2} \frac{\ln P}{N}$. This logarithmic growth is also shown for the fixed-size networks in [45], when the BS performs ZFBF. Moreover, for the fixed SNR scenario, this theorem implies that the minimum amount of feedback bits per user does not need to grow with N, which agrees with the result of Theorem 3.6, where we showed that the maximum sum-rate is achievable by a fixed amount of feedback per user.

ii) The more interesting observation is that, in the case of $K = M$, the above theorem does not impose any constraints on the minimum amount of feedback bits per user, even for the asymptotic scenario of $P \to \infty$. One may argue that this is not surprising as in this case, the transmitter can select the user which maximizes the single-user capacity (with a fixed amount of feedback per user, regardless of SNR), and communicates with that user, without knowing its channel. In [50], we have shown that this argument is not valid, as $\lim_{N\to\infty} \mathcal{C}_{\text{sum}} - \mathcal{R}_{\text{TDMA}} = M \ln M$. In other words, there is a constant gap between the achieving sum-rate and the maximum sum-rate. In fact, the reason that this case differs form the case $K < M$ is the "interference hiding". Since each user has M coordinates and the number of interfering coordinates is $M-1$, the transmitter can wisely hide the interference coordinates in the null-space of the signal coordinate, and thus the receiver does not see any interference. In other words, the transmitter does not need to "mitigate" the interference, which requires much more precise information about the channels. As a result, unlike the case $K < M$, the total amount of feedback does not need to grow with SNR.

Theorem 3.16 *The sufficient condition for achieving the maximum sum-rate, such that* $\lim_{N,P\to\infty} \mathcal{C}_{\text{sum}} - \mathcal{R} = 0$*, in the case of* $K < M$ *is*

$$\mathbb{E}\{\mathcal{F}_Q\} = \frac{1}{\ln 2}\left[M(M-1)\ln P - M(K-1)\ln\ln N - o(\ln N)\right]^+ + \omega(\ln\ln(P\ln N)), \tag{3.140}$$

and in the case of $K = M$ *is*

$$\mathbb{E}\{\mathcal{F}_Q\} = M\ln\ln\ln N + g. \tag{3.141}$$

Proof - The proof is based on the two algorithms given in the following, in the cases $K < M$ and $K = M$. We show that by using these algorithms one can achieve the maximum sum-rate throughput of the system in each case, while the total amount of feedback satisfies (3.140) and (3.141), respectively.

Case $K < M$:

Consider the following algorithm:

1. Set the thresholds t, β, and ϵ.

2. Define

$$\mathcal{S}_0 = \{k| \quad \lambda_{\max}(k) > t\},$$

where $\lambda_{\max}(k)$ is the the maximum singular value of the kth user.

3. All users in $\mathcal{S}_0$ quantize the eigenvector corresponding to the maximum singular value of their channel matrix, denoted by $\mathbf{v}_k$, using the quantization code book $\mathcal{C} = \{\mathbf{c}_1, \cdots, \mathbf{c}_{2^B}\}$, where $\{\mathbf{c}_l\}_{l=1}^{2^B}$ are i.i.d. unit vectors with uniform distribution (RVQ). The quantized vector of $\mathbf{v}_k$, denoted by $\widehat{\mathbf{v}}_k$ is selected as

$$\widehat{\mathbf{v}}_k = \arg \max_{\mathbf{c}_l \in \mathcal{C}} |\mathbf{v}_k^H \mathbf{c}_l|.$$

4. All the users in the set

$$\mathcal{S}_1 = \left\{ k \in \mathcal{S}_0 \middle| \quad |\mathbf{v}_k^H \widehat{\mathbf{v}}_k|^2 > 1 - \epsilon \right\}$$

send one bit to the BS. The BS selects one user in $\mathcal{S}_1$ at random and inform this user (s_1) to feed back its eigenvector. User s_1 feeds back the quantization index corresponding to its eigenvector to the BS. The BS sends this index to all the users in the set $\mathcal{S}_1 - \{s_1\}$.

5. For $m = 2$ to M the following steps are repeated:

   - Define $\mathcal{S}_m = \left\{ k \in \mathcal{S}_{m-1} \middle| \quad |\mathbf{v}_k^H \widehat{\mathbf{v}}_{s_{m-1}}|^2 < \beta \right\}$. All users in $\mathcal{S}_m$ send one bit to the BS.
   - The BS selects one user in $\mathcal{S}_m$ at random and informs this user (s_m) to feed back its corresponding eigenvector.
   - User s_m feeds back the quantization index corresponding to its eigenvector to the BS. The BS sends this index to all the users in the set $\mathcal{S}_m - \{s_m\}$.

6. After selecting the users and receiving their quantized eigenvectors, the BS forms the beams $\{\mathbf{\Phi}_{s_m}\}_{m=1}^{M}$, such that $\mathbf{\Phi}_{s_m}$ is in the null-space of $\widehat{\mathbf{v}}_{s_j}$, $j \neq m$ (Zero-Forcing Beam-Forming). In other words, $\mathbf{\Phi}_{s_m}^H \widehat{\mathbf{v}}_{s_j} = 0$, $\forall j \neq m$.

7. The BS forms the transmitted signal as

$$\mathbf{x} = \sum_{j=1}^{M} \mathbf{\Phi}_{s_j} x_{s_j}, \tag{3.142}$$

where $x_{s_j} \sim \mathcal{CN}(0, \frac{P}{M})$ is the intended signal for the user s_j.

8. At the receiver s_m, the received vector $\mathbf{y}_{s_m}$ is multiplied by $\mathbf{u}_{s_m}^H$, where $\mathbf{u}_{s_m}$ denotes the left eigenvector corresponding to the maximum eigenvalue of the user s_m, to form $r_{s_m} = \mathbf{u}_{s_m}^H \mathbf{y}_{s_m}$. Then, the decoding is performed.

Defining the event $\mathcal{Q} \triangleq \bigcap_{m=1}^{M} \{|\mathcal{S}_m| \neq 0\}$, the sum-rate can be upper-bounded as

$$\begin{aligned} \mathcal{R} &= \Pr\{\mathcal{Q}\} \mathcal{R}_{\mathcal{Q}} + \Pr\left\{\mathcal{Q}^C\right\} \mathcal{R}_{\mathcal{Q}^C} \\ &\geq \Pr\{\mathcal{Q}\} \mathcal{R}_{\mathcal{Q}} \\ &\overset{(a)}{\geq} \left[1 - \sum_{m=1}^{M} \Pr\{|\mathcal{S}_m| = 0\}\right] \mathcal{R}_{\mathcal{Q}}, \end{aligned} \tag{3.143}$$

where $\mathcal{R}_{\mathcal{Q}}$ denotes the average sum-rate conditioned on $\mathcal{Q}$ and (a) comes from the union bound for the probability. To compute $\mathcal{R}_{\mathcal{Q}}$, we calculate the rate of each user conditioned on $\mathcal{Q}$. For this purpose, the received signal by the s_mth user is simplified as follows:

$$\begin{aligned} r_{s_m} &= \mathbf{u}_{s_m}^H \mathbf{y}_{s_m} \\ &= \mathbf{u}_{s_m}^H \left[\mathbf{H}_{s_m} \mathbf{x} + \mathbf{n}_{s_m} \right] \\ &\overset{(a)}{=} \sqrt{\lambda_{\max}(s_m)} \mathbf{v}_{s_m}^H \mathbf{x} + z_{s_m} \\ &= \sqrt{\lambda_{\max}(s_m)} \mathbf{v}_{s_m}^H \mathbf{\Phi}_{s_m} x_{s_m} + \sum_{j \neq m} \sqrt{\lambda_{\max}(s_m)} \mathbf{v}_{s_m}^H \mathbf{\Phi}_{s_j} x_{s_j} + z_{s_m}, \end{aligned} \tag{3.144}$$

where $z_{s_m} \sim \mathcal{CN}(0,1)$ is AWGN and (a) comes from writing SVD for $\mathbf{H}_{s_m}$. In the above equation, the first term contains the desired signal and the rest are the interference and noise terms. Hence, the rate of this user can be written as

$$\mathcal{R}_{s_m} = \mathbb{E}\left\{ \ln \left(1 + \frac{\frac{P}{M} \lambda_{\max}(s_m) \left| \mathbf{v}_{s_m}^H \mathbf{\Phi}_{s_m} \right|^2}{\sum_{j \neq m} \frac{P}{M} \lambda_{\max}(s_m) \left| \mathbf{v}_{s_m}^H \mathbf{\Phi}_{s_j} \right|^2 + 1} \right) \right\}. \tag{3.145}$$

We can write

$$\mathbf{v}_{s_m} = \alpha_{s_m}^{\parallel} \widehat{\mathbf{v}}_{s_m} + \widehat{\mathbf{v}}_{s_m}^{\perp}, \tag{3.146}$$

where $\alpha_{s_m}^{\parallel} \triangleq \widehat{\mathbf{v}}_{s_m}^H \mathbf{v}_{s_m}$ and $\widehat{\mathbf{v}}_{s_m}^{\perp}$ is the projection of $\mathbf{v}_{s_m}$ over the sub-space perpendicular to $\widehat{\mathbf{v}}_{s_m}$. Using the above equation, $\left| \mathbf{v}_{s_m}^H \mathbf{\Phi}_{s_j} \right|^2$ can be written as

$$\begin{aligned} \left| \mathbf{v}_{s_m}^H \mathbf{\Phi}_{s_j} \right|^2 &= \left| \left(\alpha_{s_m}^{\parallel} \widehat{\mathbf{v}}_{s_m} + \widehat{\mathbf{v}}_{s_m}^{\perp} \right)^H \mathbf{\Phi}_{s_j} \right|^2 \\ &\overset{(a)}{=} \left| \left(\widehat{\mathbf{v}}_{s_m}^{\perp} \right)^H \mathbf{\Phi}_{s_j} \right|^2 \\ &\leq \left\| \widehat{\mathbf{v}}_{s_m}^{\perp} \right\|^2 \\ &= 1 - \left| \widehat{\mathbf{v}}_{s_m}^H \mathbf{v}_{s_m} \right|^2, \end{aligned} \tag{3.147}$$

where (a) comes from the fact that $\widehat{\mathbf{v}}_{s_m}^H \mathbf{\Phi}_{s_j} = 0$, $j \neq m$, by the algorithm. Conditioned on $\mathcal{Q}$, we have $\lambda_{\max}(s_m) > t$ and $\left| \widehat{\mathbf{v}}_{s_m}^H \mathbf{v}_{s_m} \right|^2 > 1 - \epsilon$. Therefore, the rate of the s_mth user, conditioned on $\mathcal{Q}$, can be lower-bounded as

$$\mathcal{R}_{s_m|\mathcal{Q}} \geq \ln \left(1 + \frac{\frac{Pt}{M} \left| \mathbf{v}_{s_m}^H \mathbf{\Phi}_{s_m} \right|^2}{1 + \frac{Pt\epsilon(M-1)}{M}} \right). \tag{3.148}$$

In Appendix H, we have shown that having $\beta = o(1)$ and $\epsilon = o(1)$ guarantees $\left|\mathbf{v}_{s_m}^H \mathbf{\Phi}_{s_m}\right|^2 = 1 + o(1)$. Having this, it follows that choosing $t = \ln N + o(\ln N)$ and $\epsilon = o\left(\frac{1}{P \ln N}\right)$ incurs $\mathcal{R}_{s_m|\mathcal{Q}} = \ln\left(1 + \frac{P}{M} \ln N + o(\ln N)\right)$. Similarly, we can show that the same rate is achievable for the other selected users. Hence, $\mathcal{R}_{\mathcal{Q}} = M \ln\left(1 + \frac{P}{M} \ln N + o(\ln N)\right)$ and as a result, $\lim_{P,N\to\infty} \mathcal{C}_{\text{sum}} - \mathcal{R}_{\mathcal{Q}} = 0$. Using this fact and (3.143), it follows that the sufficient condition to achieve $\lim_{P,N\to\infty} \mathcal{C}_{\text{sum}} - \mathcal{R} = 0$ is $\left[\sum_{m=1}^{M} \Pr\{|\mathcal{S}_m| = 0\}\right] \mathcal{R}_{\mathcal{Q}} = o(1)$, which incurs $\Pr\{|\mathcal{S}_m| = 0\} = o\left(\frac{1}{\ln(P \ln N)}\right)$. Since $\mathcal{S}_M \subseteq \mathcal{S}_{M-1} \subseteq \cdots \subseteq \mathcal{S}_1$, it suffices to consider only $\mathcal{S}_M$. Defining $q_k \triangleq \Pr\{k \in \mathcal{S}_M\}$ for a randomly chosen user k, we have

$$q_k = \Pr\left\{\lambda_{\max}(k) > t, |\mathbf{v}_k^H \widehat{\mathbf{v}}_{s_m}|^2 < \beta, m = 1, \cdots, M-1, |\mathbf{v}_k^H \widehat{\mathbf{v}}_k|^2 > 1 - \epsilon\right\} \tag{3.149}$$

Since the events $\mathcal{A}_1 \triangleq \{\lambda_{\max}(k) > t\}$, $\mathcal{A}_2 \triangleq \left\{|\mathbf{v}_k^H \widehat{\mathbf{v}}_{s_m}|^2 < \beta, m = 1, \cdots, M-1\right\}$ and $\mathcal{A}_3 \triangleq \{|\mathbf{v}_k^H \widehat{\mathbf{v}}_k|^2 > 1 - \epsilon\}$ are independent of each other, q_k can be written as $\prod_{i=1}^{3} q_{ki}$, where $q_{ki} \triangleq \Pr\{\mathcal{A}_i\}$. We have

$$\begin{aligned} q_{k1} &\overset{(a)}{=} \Theta\left(e^{-t} t^{M+K-2}\right), \\ q_{k2} &\overset{(b)}{=} \Theta(\beta^{M-1}), \end{aligned} \tag{3.150}$$

where (a) comes from [19], and (b) comes from [50]. Furthermore,

$$\begin{aligned} q_{k3} &= 1 - \Pr\left\{|\mathbf{v}_k^H \widehat{\mathbf{v}}_k|^2 < 1 - \epsilon\right\} \\ &= 1 - \prod_{l=1}^{L} \Pr\left\{|\mathbf{v}_k^H \mathbf{c}_l|^2 < 1 - \epsilon\right\} \\ &\overset{(a)}{=} 1 - \left(1 - \epsilon^{M-1}\right)^L \\ &\approx 1 - e^{-L\epsilon^{M-1}} \\ &\leq L\epsilon^{M-1}, \end{aligned} \tag{3.151}$$

where $L \triangleq 2^B$ and (a) results from Appendix A. Combining (3.150) and (3.151), we can write

$$\begin{aligned} \Pr\{|\mathcal{S}_M| = 0\} &\approx (1 - q_k)^N \\ &= (1 - q_{k1}q_{k2}q_{k3})^N \\ &\geq \left[1 - \Theta\left(e^{-t}t^{M+K-2}\beta^{M-1}L\epsilon^{M-1}\right)\right]^N \\ &\approx \exp\left\{-\Theta\left(Ne^{-t}t^{M+K-2}\beta^{M-1}L\epsilon^{M-1}\right)\right\}. \end{aligned} \tag{3.152}$$

Hence, in order to have $\Pr\{|\mathcal{S}_M| = 0\} = o\left(\frac{1}{\ln(P\ln N)}\right)$, it suffices to have

$$L = \Theta\Big(\left(\ln\ln(P\ln N) + g\right)(\beta\epsilon)^{-(M-1)}N^{-1}e^t t^{-(M+K-2)}\Big). \tag{3.153}$$

Choosing $\beta = o(1)$, $t = (1-\alpha)\ln N$, and $\epsilon = \frac{\delta}{P\ln N}$, where $\alpha, \delta = o(1)$, and substituting in the above equation, we obtain

$$\begin{aligned} L &= \Theta\Big(\left(\ln\ln(P\ln N) + g\right)[P\ln N]^{M-1}(\beta\delta)^{-(M-1)}N^{-\alpha}[\ln N]^{-(M+K-2)}\Big) \\ &= \Theta\Big(\left(\ln\ln(P\ln N) + g\right)P^{M-1}[\ln N]^{-(K-1)}(\beta\delta)^{-(M-1)}N^{-\alpha}\Big). \end{aligned} \tag{3.154}$$

Having $B = [\log_2(L)]^+$, yields

$$B = \frac{1}{\ln 2}\left[(M-1)\ln P - (K-1)\ln\ln N + \ln\ln\ln(P\ln N) + g - o(\ln N)\right]^+ \tag{3.155}$$

Using the above equation, the total amount of feedback can be written as

$$\begin{aligned} \mathbb{E}\{\mathcal{F}_Q\} &= MB + \sum_{m=1}^{M}\mathbb{E}\{|\mathcal{S}_m|\} \\ &= MB + \sum_{m=1}^{M}(N-m+1)\Pr\{k \in \mathcal{S}_m\} \\ &\overset{(a)}{=} MB + \omega(\ln\ln(P\ln N)) \\ &= [M(M-1)\ln P - M(K-1)\ln\ln N - o(\ln N)]^+ + \omega(\ln\ln(P\ln N)), \end{aligned} \tag{3.156}$$

where (a) comes from the fact that selecting L as in (2.53), results in $N\Pr\{k \in \mathcal{S}_M\} = \ln\ln(P\ln N) + g$, and hence, $N\Pr\{k \in \mathcal{S}_m\} \sim N\Pr\{k \in \mathcal{S}_M\}\beta^{m-M} = \omega(\ln\ln(P\ln N))$.

Case $K = M$:

Consider the following algorithm:

1. Set the thresholds t and ϵ.

2. Define

$$\mathcal{S}_0 = \{k| \quad \lambda_{\max}(k) > t\},$$

   where $\lambda_{\max}(k)$ is the the maximum singular value of the kth user.

3. The BS selects a unit vector $\mathbf{\Phi}_{s_1}$ at random and sends this vector to all users in $\mathcal{S}_0$.

4. All the users in the set

$$\mathcal{S}_1 = \left\{k \in \mathcal{S}_0 \middle| \quad |\mathbf{v}_k^H \mathbf{\Phi}_{s_1}|^2 > 1 - \epsilon\right\},$$

   where $\mathbf{v}_k$ denotes the eigenvector corresponding to the maximum eigenvalue of user k, send one bit to the BS. The BS selects one user in $\mathcal{S}_1$ at random indexed by s_1.

5. For $m = 2$ to M the following steps are repeated:

   – The BS selects a unit vector $\mathbf{\Phi}_{s_m}$ such that it is orthogonal to the previously chosen vectors $\{\mathbf{\Phi}_{s_j}\}_{j=1}^{m-1}$, and sends it to the users in $\mathcal{S}_0$.

   – Define $\mathcal{S}_m = \left\{k \in \mathcal{S}_0 \middle| \quad |\mathbf{v}_k^H \mathbf{\Phi}_{s_m}|^2 > 1 - \epsilon\right\}$. All users in $\mathcal{S}_m$ send one bit to the BS.

– The BS selects one user in $\mathcal{S}_m$ at random indexed by s_m.

6. The BS forms the transmitted signal as

$$\mathbf{x} = \sum_{m=1}^{M} \mathbf{\Phi}_{s_m} x_{s_m}, \tag{3.157}$$

where $x_{s_m} \sim \mathcal{CN}(0, \frac{P}{M})$ is the intended signal for the user s_m.

7. At the receiver s_m, the received vector is multiplied by $\mathbf{R}_{s_m}^{-1/2}$, where

$$\mathbf{R}_{s_m} \triangleq \mathbf{I} + \sum_{j \neq m} \frac{P}{M} \mathbf{H}_{s_m} \mathbf{\Phi}_{s_j} \mathbf{\Phi}_{s_j}^H \mathbf{H}_{s_m}^H,$$

to form $\mathbf{r}_{s_m} = \mathbf{R}_{s_m}^{-1/2} \mathbf{y}_{s_m}$. Then, the decoding is performed.

As can be observed, this algorithm is very similar to the previous algorithm, with the difference in the quantization code book and decoding. In this algorithm, the quantization code book contains only one code word at each step, which is variable and decided by the BS, while in the previous algorithm the quantization code book is fixed and the number of code words grow with SNR. Moreover, the receiver uses all coordinates for decoding the signal, while in the previous algorithm the decoding is only performed in one coordinate. In fact, in the case of $K < M$, using all the coordinates does not provide any gain, while in the case of $K = M$, it does. In the case of $K = M$, if any of the sets $\mathcal{S}_m$, $m = 1, \cdots, M$, is empty, the BS selects any user at random and communicates with that user, setting the transmit covariance matrix equal to $\frac{P}{M}\mathbf{I}$. This provides a rate scaling as $M \ln P$, without requiring any amount of feedback.

Defining the event $\mathcal{Q} \triangleq \bigcap_{m=1}^{M} \{|\mathcal{S}_m| \neq 0\}$, similar to (3.143), we can write

$$\begin{aligned}\mathcal{R} &= \Pr\{\mathcal{Q}\}\mathcal{R}_{\mathcal{Q}} + [1 - \Pr\{\mathcal{Q}\}]\,\mathcal{R}_{\text{RS}}^{\mathcal{Q}^C} \\ &= \mathcal{R}_{\mathcal{Q}} - [1 - \Pr\{\mathcal{Q}\}]\left[\mathcal{R}_{\mathcal{Q}} - \mathcal{R}_{\text{RS}}^{\mathcal{Q}^C}\right] \\ &\geq \mathcal{R}_{\mathcal{Q}} - \left(\sum_{m=1}^{M} \Pr\{|\mathcal{S}_m| = 0\}\right)\left[\mathcal{R}_{\mathcal{Q}} - \mathcal{R}_{\text{RS}}^{\mathcal{Q}^C}\right], \qquad (3.158)\end{aligned}$$

where $\mathcal{R}_{\text{RS}}^{\mathcal{Q}^C}$ denotes the achievable rate, when the BS selects one user at random and communicates with that user, conditioned on $\mathcal{Q}^C$. It is easy to show that $\mathcal{R}_{\text{RS}}^{\mathcal{Q}^C} = M \ln P + \Theta(1)$.

The rate of the user s_m, conditioned on $\mathcal{Q}$, can be computed as

$$\mathcal{R}_{s_m|\mathcal{Q}} = \mathbb{E}\left\{\ln\left|\mathbf{I} + \frac{P}{M}\mathbf{H}_{s_m}\mathbf{\Phi}_{s_m}\mathbf{\Phi}_{s_m}^H\mathbf{H}_{s_m}^H\mathbf{R}_{s_m}^{-1}\right| \,\Big|\, \mathcal{Q}\right\}. \qquad (3.159)$$

For $\epsilon = o(1)$ and $t \sim \ln N$, and using the equations (3.116) and (3.123), it follows that

$$\begin{aligned}\mathcal{R}_{s_m|\mathcal{Q}} &\geq \mathbb{E}\left\{\ln\left(1 + \frac{P}{M}t(1-\epsilon)\mathbf{W}_{11}\right)\right\} \\ &= \ln\left(1 + \frac{P}{M}\ln N[1 + o(1)]\right), \qquad (3.160)\end{aligned}$$

where $\mathbf{W} = \mathbf{R}_{s_m}^{-1}$. Hence,

$$\mathcal{R}_{\mathcal{Q}} = M \ln\left(1 + \frac{P}{M}\ln N[1 + o(1)]\right), \qquad (3.161)$$

and as a result, $\mathcal{C}_{\text{sum}} - \mathcal{R}_{\mathcal{Q}} = o(1)$. Therefore, having the fact that $\mathcal{R}_{\mathcal{Q}} - \mathcal{R}_{\text{RS}}^{\mathcal{Q}^C} \sim M \ln\ln N$, we can show that $\eta_m \triangleq \Pr\{|\mathcal{S}_m| \neq 0\} = o\left(\frac{1}{\ln\ln N}\right)$, $\forall m$, guarantees $\mathcal{C}_{\text{sum}} - \mathcal{R} = o(1)$. η_m can be written as $(1 - q_m)^N$, where $q_m \triangleq \Pr\{k \in \mathcal{S}_m\}$, for a randomly chosen user k. q_m can be computed as

$$\begin{aligned}q_m &= \Pr\{\lambda_{\max}(k) > t\}\Pr\{|\mathbf{v}_k^H\mathbf{\Phi}_{s_m}|^2 > 1 - \epsilon\} \\ &\overset{(a)}{\approx} \frac{e^{-t}t^{M+K-2}}{\Gamma(M)\Gamma(K)}\epsilon^{M-1}, \qquad (3.162)\end{aligned}$$

where (a) comes from [19] and [50]. Consequently,

$$\begin{aligned}\eta_m &\approx \left[1 - \frac{e^{-t}t^{M+K-2}}{\Gamma(M)\Gamma(K)}\epsilon^{M-1}\right]^N \\ &\approx e^{-N\frac{e^{-t}t^{M+K-2}}{\Gamma(M)\Gamma(K)}\epsilon^{M-1}}. \end{aligned} \tag{3.163}$$

Choosing $\epsilon = \frac{1}{\ln N}$ and $t = \ln N + (K-1)\ln\ln N - \ln\ln\ln\ln N - \ln\Gamma(M)\Gamma(K) - \omega\left(\frac{1}{\ln\ln\ln N}\right)$ results in $\eta_m = o\left(\frac{1}{\ln\ln N}\right)$ and hence, having $\lim_{N,P\to\infty} \mathcal{C}_{\text{sum}} - \mathcal{R} = 0$. The amount of feedback can be computed from

$$\begin{aligned}\mathbb{E}\{\mathcal{F}_Q\} &= \mathbb{E}\left\{\sum_{m=1}^{M}|\mathcal{S}_m|\right\} \\ &= N\sum_{m=1}^{M} q_m \\ &\overset{(a)}{\approx} \sum_{m=1}^{M}\ln(\eta_m^{-1}) \\ &= M\ln\ln\ln N + g, \end{aligned} \tag{3.164}$$

where (a) comes from the fact that $\eta_m = (1-q_m)^N \approx e^{-Nq_m}$.

■

Remark 1- Comparing the necessary and sufficient conditions on the minimum amount of feedback for achieving the maximum sum-rate, it turns out that the proposed algorithm in the case of $K < M$ is asymptotically optimal by a constant multiplicative factor, in terms of the required amount of feedback, in the region $\ln P = \omega(\ln N)$. Moreover, in the case $K = M$, the proposed algorithm is optimal by a constant multiplicative factor, in terms of the required amount of feedback, for all ranges of SNR.

Remark 2- Comparing the two cases $K < M$ and $K = M$, it follows that the minimum amount of feedback in the first case grows logarithmically with SNR while in the second case it does not grow with SNR.

Remark 3- In the case of $K < M$, when $\ln P \neq \Omega(\ln N)$, it is possible to achieve the maximum sum-rate by using a finite-size quantization code book for all the users (Random Beam-Forming). However, in the case of $\ln P = \omega(\ln N)$, the size of the quantization code book must grow polynomially with SNR. In the case of $K = M$, it is possible to achieve the maximum sum-rate with finite rate quantization for all ranges of SNR. In other words, Random Beam-Forming is always optimal in this case. Note that, however, the decoding must be performed in all the coordinates.

Remark 4- The first algorithm can be considered as the generalization of Random Beam-Forming, when the number of beams vary with SNR. This algorithm is very similar to the algorithm proposed in [51], with the difference in limiting the number of candidate users and thus reducing the amount of feedback furthermore.

3.4 Conclusion

In this chapter, the minimum required amount of feedback in order to achieve the maximum sum-rate capacity in a MIMO-BC with large number of users and different ranges of SNR is studied. In the fixed SNR and low SNR regimes, we have proved that to achieve the maximum sum-rate the total amount of feedback from the users to the BS must be infinity. However, it does not need to scale with N. Moreover, in the fixed SNR regime, in order to reduce the gap to the sum-rate capacity to zero, the amount of feedback must scale at least as $\ln \ln \ln N$, which is achievable by the Random Beam-Forming scheme introduced in [26]. Moreover, it is shown that the optimality of Random Beam-Forming scheme only holds for the region $\ln P \neq \Omega(\ln N)$. In the regime of $\ln P = \Omega(N)$, we consider two cases.

In the case of $K < M$, we prove that the minimum amount of feedback in order to reduce the gap between the achievable sum-rate and the maximum sum-rate to zero grows logarithmically with SNR, which is achievable by the "Generalized Random Beam-Forming" scheme proposed in [51]. In the case of $K = M$, we show that by using the Random Beam-Forming scheme with the total amount of feedback not growing with SNR, the maximum sum-rate capacity is achieved, provided that the decoding is performed in all the received coordinates.

Chapter 4

Fairness in the Scheduling

4.1 Introduction

With the development of personal communication services, one of the major concerns in supporting data applications is providing quality of service (QoS) for all subscribers. In most real-time applications, high data rates and small transmission delays are desired. Most data-scheduling schemes proposed for current systems have concentrated on the system throughput by exploiting multiuser diversity [17, 56, 57, 58, 59]. In cellular networks, by applying multiuser diversity, the time-varying nature of the fading channel is exploited to increase the spectral efficiency of the system. It is shown that transmitting to the user with the highest signal to noise ratio (SNR) provides the system with maximum sum-rate throughput [60]. The opportunistic transmission is proposed in Qualcomm's High Data Rate (HDR) system [56].

Although applying multiuser diversity through the scheme in [60] achieves the maximum system throughput, QoS demands, including fairness and delay constraints, provoke designing more appropriate scheduling schemes. The schemes that consider delay constraints have been studied extensively in [17, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 75, 72, 73, 74]. In [61], the authors propose an algorithm which maintains a balance between the throughput maximization, delay, and outage probability in a multiple access fading channel. The tradeoff between the average delay and the average transmit power in fading environments is analyzed in [62]. In [63, 64], authors propose scheduling metrics that combine multiuser diversity gain with the delay constraints. In [65], the scheduling scheme is designed based on maximizing the effective capacity [76] which is characterized by data rate, delay bound, and delay-bound violation probability triplet. The throughput-delay tradeoff of the multicast channel is analyzed for different schemes in a single cell system [66]. This trade-off has been obtained for more general network topologies in [67]. In the static random network with n nodes, the results of [67] show that the optimal tradeoff between throughput T_n and delay D_n is given by $D_n = \Theta(nT_n)$. They also show that the same result is achieved in random mobile networks, when $T_n = O(1/\sqrt{n \ln n})$. The first studies on achieving a high throughput and low delay in ad-hoc wireless networks are framed in [58], [68], and [69]. This line of work is further expanded in [67, 70, 71] by using different mobility models such as the random walk and the Brownian mobility models. Neely and Modiano [71] consider the delay-throughput tradeoff only for mobile ad-hoc networks. They investigate the delay characteristics by using the redundant packets transmission through multiple paths. In [72], the authors have proposed and compared different scheduling achemes based on the users' channel qualities and their remaining job times, in the downlink of a MIMO wireless cellular packet data system in fast and slow channel variation scenarios. In [73], the authors have analytically characterized the scheduling gain achieved by opportunistic schedulers with both single-user and multi-user multiplexing, and showed that the average delay grow double-exponentially with the overall throughput, with any opportunistic (single-user time-sharing or multi-user multiplexing) scheduling. In [74], the authors consider a wireless downlink communication system, where the channels are characterized by frequency-selective fading, modeled as a set of M parallel block-fading channels, and a frequency-flat distance-dependent path loss. They compare delay-limited systems (which impose hard fairness) with variable-rate systems (which impose proportional fairness), in terms of the achieved system spectral efficiency C (bit/s/Hz) versus E_b/N_0, and find simple iterative resource allocation algorithms that converge to the optimal delay-limited throughput for orthogonal (FDMA/TDMA) and optimal (superposition/interference cancellation) signaling. In the limit of large number of users and finite M, the authors find closed-form expressions for C as a function of E_b/N_0 and show that in this limit, the optimal allocation policy consists of letting each user transmit on its best subchannel only.

In [75], the delay is defined as the minimum number of channel uses that guarantees all n users successfully receive m packets. Reference [75] studies the statistical properties of the underlaying delay function. However, the delay constraint is assumed to be *soft*, meaning that this scheme aims to minimize the total *average* network delay and there is not any delay constraints for the individual users.

In this chapter, we consider a *hard* delay constraint D for each user, which is enforced by the application or physical limitations (e.g. buffer size). We define a dropping event as the event that there exists a user who does not meet the desired delay constraint. We propose a scheduling scheme for maximizing the throughput of the system, while satisfying the delay constraint for all users. The proposed scheduling algorithm works based on setting a threshold on the channel gain of the users and among the users with channel gains above the threshold, the user with the minimum *Packet Expiry Countdown*s (PEC), which is defined as the remaining time to the expiration of that users' packet, is served. By doing asymptotic analysis, it is proved that by selecting the threshold level properly, the proposed scheduling algorithm achieves the maximum throughput, maximum fairness, and minimum delay in the network, simultaneously, in the asymptotic case of $N \to \infty$. The analysis is based on characterizing the probability mass function of PEC in terms of N, D , and the threshold value, and evaluating the network dropping probability accordingly. It is also demonstrated that the Round-Robin (RR) scheduling, which focuses on maximizing the fairness and minimizing the delay in the network, and Multi-User Diversity (MUD) scheduling, which focuses on maximizing the throughput in the system, are two extreme cases of the proposed algorithm, where the former suffers from the weak performance in terms of throughput and the latter increases the network delay by a factor of $\ln N$. Moreover, we have introduced a new notion of performance in the network, called "Average Throughput", which is defined as the product of the packet arrival rate and the amount of information per channel use in each packet, and proved that the proposed algorithm maximizes the *Minimum Average Throughput* in a broadcast channel. It is demonstrated that the proposed scheduling outperforms the conventional multiuser diversity scheduling and Round-Robin scheduling in terms of the *Minimum Average Throughput*, by factors $\ln N$ and $\ln \ln N$, respectively. It is also established that the proposed algorithm reaches the boundaries of the capacity region and stability region of the underlying system, simultaneously. The proposed algorithm is also generalized to MIMO Broadcast Channels (MIMO-BC) by modifying the Random Beam-Forming scheme proposed in [26]. It is shown that the proposed algorithm is capable of achieving the maximum throughput, maximum fairness, and minimum delay, simultaneously, in the asymptotic case of $N \to \infty$. Moreover, it maximizes the *Minimum Average Throughput* in a MIMO-BC.

The rest of the chapter is organized as follows. In section 4.2, the system model is introduced and the proposed algorithm is described. Section 4.3 is devoted to the asymptotic analysis of the proposed algorithm. Section 4.4 describes the generalization of the proposed algorithm for MIMO-BC, and finally, section 4.5 concludes the chapter.

4.2 System Model and Proposed Algorithm

4.2.1 System Model, Assumptions, and Definitions

In this chapter, a downlink environment in which a single-antenna Base Station (BS) communicates with a large number (N) single-antenna users, is considered. We assume a homogeneous network, where the channel between each user and the BS is modelled as a zero-mean complex Gaussian random variable (Rayleigh fading). The received signal at the kth terminal can be written as

$$y_k = h_k x + n_k, \tag{4.1}$$

where x denotes the transmitted signal by the BS, which is assumed to be Gaussian with the power constraint P, i.e., $\mathbb{E}\{|x|^2\} \leq P$ [1], $h_k \sim \mathcal{CN}(0, 1)$ denotes the channel

[1]Note that the power constraint here is *per frame*, i.e, is independent of the channel realizations.

coefficient between the BS and the kth terminal, and $n_k \sim \mathcal{CN}(0,1)$ is AWGN. We assume that block coding for error free transmission is performed over frames, where the information content of a frame is called packet. In addition, we assume that the frame length is constant (unit of time), while the information content of a frame can potentially vary depending on the capacity of the corresponding channel realization. As we will see later, the proposed method results in almost equal information content (packet length in bits) for all the frames. It is also assumed that *only one user* is served during each frame. The channel coefficients are assumed to be constant for the duration of a frame, and change independently at the start of the next frame (block fading model). The frame itself is assumed to be long enough to allow communication at rates close to the capacity. This model is also used in [75] and [26].

It is assumed that the users have delay constraint D. In other words, the delay between two consecutive received packets should not be greater than the duration of D frames. Otherwise, the transmitted packet will be dropped. The *network dropping event*, denoted by $\mathscr{B}$, is defined as the event that dropping occurs for any user in the network. We define a parameter ν for each user, which denotes the *expiry countdown* of that user's packet, i.e., the remaining time to the expiration of the packet. ν is expressed in terms of an integer multiple of the frame length. At the end of each frame, the *expiry countdown* of each user is decremented by one, except for the user which is served during that frame. For this user, the *expiry countdown* is set to D at the start of the next frame. Therefore, for all users $\nu \leq D$ (Fig. 4.1). Since the channel model is independent block fading, and the network topology and the proposed scheduling algorithm are symmetric with respect to the users, it can be easily shown that there exists a steady state for the system

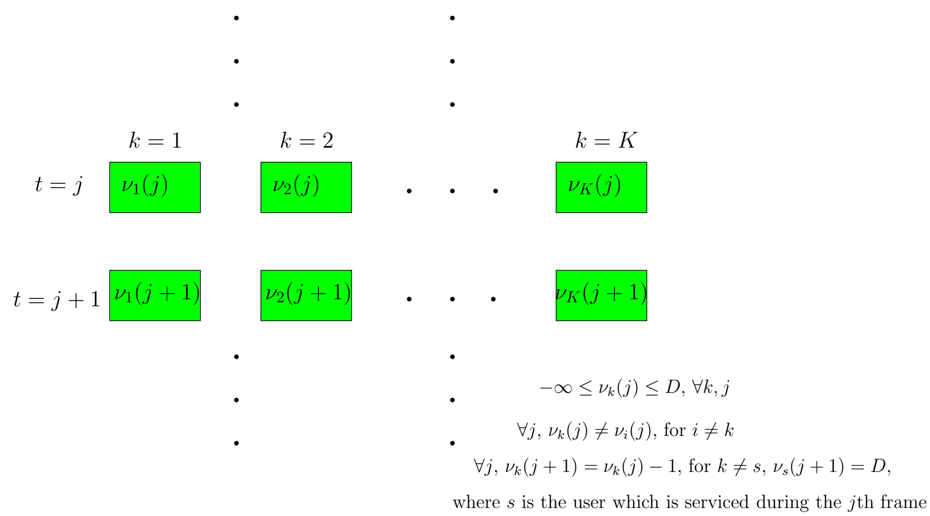

Figure 4.1: A Schematic figure for the *expiry countdown*.

(no matter what the initial state is), in which the statistical behavior of the users' expiry countdowns is independent of the time index. In the steady state, since in each frame only one user is served by the transmitter, the expiry countdown of the users are distinct in each time. All the results derived in this chapter are based on the assumption that the system is in the steady state.

In this chapter, we are interested in maximizing the *throughput* and *fairness* in the network. First, we give the definitions of *throughput* and *fairness*:

Definition 4.1 *The **throughput** is defined as the average sum-rate of the system, when the average is computed over all the channel realizations.*

Definition 4.2 *Consider a scheduling $\mathscr{S}$. Then, the **Fairness Factor** (FF) for this scheduling is defines as*

$$FF(\mathscr{S}) \triangleq \frac{D_{\min}(\mathscr{S})}{N}, \tag{4.2}$$

where $D_{\min}(\mathscr{S})$ *denotes the minimum value of* D *such that* $Pr\{\mathscr{B}\} \to 0$*, using scheduling* $\mathscr{S}$*.*

Definition 4.3 *A scheduling* $\mathscr{S}$ *is said to achieve the maximum fairness, if* $FF(\mathscr{S}) = 1$ [2].

4.2.2 Proposed Scheduling Algorithm

The proposed scheduling algorithm is described as follows:

Algorithm 1:

1) The BS chooses a threshold Θ, and sends it to all users.

2) Let us define

$$\mathcal{S} \triangleq \{k|\ |h_k|^2 \geq \Theta\}. \tag{4.3}$$

   All users in $\mathcal{S}$ send a confirmation message to the BS.

3) Among the users in $\mathcal{S}$, the BS serves the one with the minimum ν (*expiry countdown*).

In the proposed algorithm, the threshold Θ is set to trade-off the throughput vs. the fairness in the system. If Θ is chosen to be very large, then the scheduling tends to maximize the throughput. If Θ is chosen to be very small, the algorithm tends to maximize the fairness in the network.

[2] This definition is motivated by the fact that for Round-Robin scheduling (which is known to be the most fair scheduling), $D_{\min} = N$.

4.3 Asymptotic Analysis

In this section, we analyze the network dropping probability, denoted as $\Pr\{\mathscr{B}\}$, in terms of the number of users N, and the delay constraint D, for the proposed scheduling. We consider the asymptotic case of $N \to \infty$ and derive the condition for D such that $\Pr\{\mathscr{B}\} \to 0$. To this end, the probability mass function (pmf) of ν, denoted as $f_\nu(\nu)$, is characterized in terms of D, N, and Θ. First, we consider two special cases of the proposed algorithm:

4.3.1 Special Case I; $\Theta = 0$:

In this case, the user with the minimum ν is served. In other words, the quality of channel does not play any role in the scheduling. The set $\mathcal{S}$ which is defined in (4.3) is simply the set of all users.

Theorem 4.4 *For $\Theta = 0$, $f_\nu(\nu)$ can be obtained as follows:*

$$f_\nu(\nu) = \begin{cases} \frac{1}{N} & D - N + 1 \leq \nu \leq D \\ 0 & \nu \leq D - N \end{cases} . \tag{4.4}$$

Proof - Let us define $\nu_{\min}(t) \triangleq \min_{k \in \mathcal{S}} \nu_k(t)$, where $\nu_k(t)$ denotes the *expiry countdown* for the kth user at time t. We have

$$\begin{aligned} \Pr\{\nu_{\min}(t) = l\} &\overset{(a)}{=} \sum_{k=1}^{N} \Pr\{\nu_k(t) = l, \nu_i(t) > l, i \neq k\} \\ &\overset{(b)}{=} N\Pr\{\nu_1(t) = l, \nu_2(t) > l, \cdots, \nu_N(t) > l\} \\ &= N\Pr\{\nu_1(t) = l\}\Pr\{\nu_2(t) > l, \cdots, \nu_N(t) > l |\, \nu_1(t) = l\}, \end{aligned} \tag{4.5}$$

where (a) follows from the fact that as in each channel use only one user is served, the random variables $\nu_i(t)$'s are distinct in each time slot t, and (b) results from the symmetry between the users. We have

$$\Pr\{\nu_2(t) > l, \cdots, \nu_N(t) > l |\, \nu_1(t) = l\} = 0, \text{ for } \quad l > D - N + 1, \tag{4.6}$$

which results from the fact that for $l > D-N+1$, there are at most $N-2$ possible choices for each of $\nu_i(t)$, $i = 2, \cdots, N$, and since $\nu_i(t)$ are distinct, the assignment problem has no solution. Moreover, we can write,

$$\Pr\{\nu_k(t) = l - 1\} \quad = \quad \Pr\{\nu_k(t-1) = l, \mathscr{X}_k^C(t-1)\}, \tag{4.7}$$

where $\mathscr{X}_k(t-1)$ represents the event that user k is served during the $(t-1)$th frame, and $\mathscr{X}_k^C(t-1)$ denotes the complement of $\mathscr{X}_k(t-1)$. Since we are studying the behavior of the system in its steady state condition, it follows that the statistical properties of $\nu_k(t)$ and $\mathscr{X}_k(t-1)$ are independent of the time index. Hence, we can drop the time index in the above equation and write

$$\begin{aligned} \Pr\{\nu_k = l - 1\} &= \Pr\{\nu_k = l, \mathscr{X}_k^C\} \\ &= \Pr\{\nu_k = l\}\left(1 - \Pr\{\mathscr{X}_k | \nu_k = l\}\right) \\ &= \Pr\{\nu_k = l\}\left(1 - \Pr\{\nu_{\min} = l | \nu_k = l\}\right). \end{aligned} \tag{4.8}$$

Combining (4.5) and (4.8), and noting that $\Pr\{\nu_k = l\} = f_\nu(l)$ and $\Pr\{\nu_{\min} = l | \nu_k = l\} = \Pr\{\nu_2 > l, \cdots, \nu_N > l | \nu_1 = l\}$ (by the symmetry), we have

$$f_\nu(l-1) = f_\nu(l) - f_\nu(l)\Pr\{\nu_2 > l, \cdots, \nu_N > l | \nu_1 = l\}. \tag{4.9}$$

Substituting (4.6) in (4.9), we get

$$f_\nu(l) = f_\nu(l-1), \text{ for } \qquad D - N + 2 \leq l \leq D. \tag{4.10}$$

Since during each frame, exactly one user is served, there is always one user with *expiry countdown* equal to D in the system. In other words,

$$\Pr\left\{\bigcup_{k=1}^{N} (\nu_k = D)\right\} = 1. \tag{4.11}$$

Since the events $\nu_k = D$, $k = 1, \cdots, N$, are mutually exclusive, it follows that

$$\begin{aligned}\sum_{k=1}^{N} \Pr\{\nu_k = D\} &= 1 \\ \Rightarrow f_\nu(D) &\overset{(a)}{=} \frac{1}{N},\end{aligned} \tag{4.12}$$

where (a) comes from the fact that $\Pr\{\nu_k = D\}$ is the same for all k, and is equal to $f_\nu(D)$. Combining (4.10) and (4.12), we have

$$f_\nu(l) = \frac{1}{N}, \qquad D - N + 1 \leq l \leq D. \tag{4.13}$$

Since $\sum_{l=-\infty}^{D} f_\nu(l) = 1$, from the above equation it follows that

$$f_\nu(l) = 0, \qquad l \leq D - N, \tag{4.14}$$

which completes the proof of Theorem 4.4.

■

The above theorem implies that the pmf of ν is a step function which is only non-zero in the interval $[D - N + 1, D]$. Since the probability of dropping for any given user can be expressed as $\sum_{l=-\infty}^{0} f_\nu(l)$, it follows from the above equation that for $D \geq N$, the dropping probability for each user is zero and as a result, the network dropping probability is zero.

This scheduling is exactly the Round-Robin scheduling, when the users are served based on a pre-determined order. One can observe that this scheduling is the most fair scheduling ($FF = 1$), as all the users have the same opportunity for being served, regardless of their channel quality. However, due to disregarding the effect of channel quality in the scheduling, the achievable throughput is not good. More precisely, it can be easily shown that the achievable throughput of this scheduling scales as $\Theta(1)$.

4.3.2 Special case II; $\Theta = \max_k |h_k|^2$:

In this scheduling, $|\mathcal{S}| = 1$. In other words, the user with the best channel quality is served during each frame. This results in the conventional scheduling to exploit the multiuser diversity and achieves the maximum sum-rate throughput in the system [77].

Theorem 4.5 *For the Special Case II, $f_\nu(\nu)$ is equal to*

$$f_\nu(\nu) = \frac{1}{N}\left(1 - \frac{1}{N}\right)^{D-\nu} u(D-\nu), \tag{4.15}$$

where $u(.)$ denotes the unit step function.

Proof - Similar to (4.8), we can write

$$\begin{aligned} f_\nu(l-1) &= f_\nu(l)\left(1 - \Pr\{\mathscr{X}_k | \nu_k = l\}\right) \\ &\overset{(a)}{=} f_\nu(l)\left(1 - \Pr\{\mathscr{X}_k\}\right) \\ &\overset{(b)}{=} f_\nu(l)\left(1 - \frac{1}{N}\right), \end{aligned} \tag{4.16}$$

where (a) comes from the fact that the selection of users is performed regardless of the value of their *expiry countdown*. (b) results from the fact that the fading process is block-wise independent, and as a result, the probability that the channel norm of any user is the highest during a frame is $\frac{1}{N}$. From the above equation, the pmf of ν can be written as

$$f_\nu(l) = f_\nu(D)\left(1 - \frac{1}{N}\right)^{D-l}, \quad l \leq D. \tag{4.17}$$

From (4.16) and noting that $\sum_{l=-\infty}^{D} f_\nu(l) = 1$, we have $f_\nu(D) = \frac{1}{N}$. Hence,

$$f_\nu(l) = \frac{1}{N}\left(1 - \frac{1}{N}\right)^{D-l} u(D-l), \tag{4.18}$$

where $u(.)$ denotes the unit step function. Hence, the pmf of ν follows the exponential distribution with the parameter $1 - \frac{1}{N}$.

Theorem 4.6 *For $N \to \infty$, the necessary and sufficient condition to have $Pr\{\mathscr{B}\} \to 0$ for the special case II is*

$$D = N \ln N + \omega(N). \tag{4.19}$$

Proof - *Sufficient Condition:* Using (4.18), the dropping probability for a user k, denoted as $\Pr\{\mathscr{B}_k\}$, can be written as

$$\begin{aligned}
\Pr\{\mathscr{B}_k\} &= \sum_{l=-\infty}^{0} f_\nu(l) \\
&= \sum_{l=-\infty}^{0} \frac{1}{N}\left(1 - \frac{1}{N}\right)^{D-l} \\
&= \left(1 - \frac{1}{N}\right)^{D} \\
&\sim e^{-\frac{D}{N}}.
\end{aligned} \tag{4.20}$$

The network dropping probability ($\Pr\{\mathscr{B}\}$) can be written as $\Pr\{\bigcup_{k=1}^{N} \mathscr{B}_k\}$. Using the union bound for the probability, we have

$$\begin{aligned}
\Pr\{\mathscr{B}\} &\leq \sum_{k=1}^{N} \Pr\{\mathscr{B}_k\} \\
&\overset{(4.20)}{\sim} N e^{-\frac{D}{N}} \\
&= e^{-\frac{D - N \ln N}{N}}.
\end{aligned} \tag{4.21}$$

Hence, having $D = N \ln N + \omega(N)$ guarantees $\Pr\{\mathscr{B}\} \to 0$.

Necessary Condition: We can write

$$\Pr\{\mathscr{B}\} = 1 - \Pr\left\{\bigcap_{k=1}^{N} \mathscr{B}_k^C\right\}. \tag{4.22}$$

The dropping event for the kth user, $\mathscr{B}_k$, is equivalent to $\nu_k \leq 0$. Hence, the above equation can be written as

$$\begin{aligned}
\Pr\{\mathscr{B}\} &= 1 - \Pr\{\nu_1 > 0, \cdots, \nu_N > 0\} \\
&= 1 - \Pr\{\nu_1 > 0\} \prod_{k=2}^{N} \Pr\{\nu_k > 0 | \nu_1 > 0, \nu_2 > 0, \cdots, \nu_{k-1} > 0\} \\
&= 1 - \Pr\{\nu_1 > 0\} \prod_{k=2}^{N} \left(\frac{\sum_{\substack{(a_1,\cdots,a_{k-1}) \\ 1 \leq a_i \leq D}} f_{\nu_1,\cdots,\nu_{k-1}}(a_1,\cdots,a_{k-1})}{\Pr\{\nu_1 > 0, \cdots, \nu_{k-1} > 0\}} \times \right. \\
&\quad \left. \Pr\{\nu_k > 0 | \nu_1 = a_1, \nu_2 = a_2, \cdots, \nu_{k-1} = a_{k-1}\} \right) \qquad (4.23) \\
&\stackrel{(a)}{=} 1 - \Pr\{\nu_1 > 0\} \prod_{k=2}^{N} \left(\frac{\sum_{\substack{(a_1,\cdots,a_{k-1}) \\ 1 \leq a_i \leq D}} f_{\nu_1,\cdots,\nu_{k-1}}(a_1,\cdots,a_{k-1})}{\Pr\{\nu_1 > 0, \cdots, \nu_{k-1} > 0\}} \times \right. \\
&\quad \left. \Pr\{\nu_k > 0 | \nu_k \notin \{a_1, a_2, \cdots, a_{k-1}\}\} \right) \\
&= 1 - \Pr\{\nu_1 > 0\} \prod_{k=2}^{N} \left(\frac{\sum_{\substack{(a_1,\cdots,a_{k-1}) \\ 1 \leq a_i \leq D}} f_{\nu_1,\cdots,\nu_{k-1}}(a_1,\cdots,a_{k-1})}{\Pr\{\nu_1 > 0, \cdots, \nu_{k-1} > 0\}} \times \right. \\
&\quad \left. \frac{\Pr\{\nu_k > 0\} - \sum_{i=1}^{k-1} f_{\nu_k}(a_i)}{1 - \sum_{i=1}^{k-1} f_{\nu_k}(a_i)} \right) \qquad (4.24) \\
&\stackrel{(b)}{\geq} 1 - \prod_{k=1}^{N} \Pr\{\nu_k > 0\} \\
&\stackrel{(4.20)}{=} 1 - \left[1 - \left(1 - \frac{1}{N} \right)^D \right]^N \\
&\stackrel{(c)}{\geq} 1 - e^{-N\left(1 - \frac{1}{N}\right)^D}, \qquad (4.25)
\end{aligned}$$

where (a) follows from the fact that the only dependency among ν_k's is that they are distinct random variables, (b) results from the fact that $\frac{\Pr\{\nu_k > 0\} - \sum_{i=1}^{k-1} f_{\nu_k}(a_i)}{1 - \sum_{i=1}^{k-1} f_{\nu_k}(a_i)} \leq \Pr\{\nu_k > 0\}$, and (c) results from the fact that $(1-x)^n \leq e^{-nx}$, $\forall n > 0, x < 1$. It follows from the above equation that in order to have $\Pr\{\mathscr{B}\} \to 0$, we must have $e^{-N\left(1-\frac{1}{N}\right)^D} \to 1$, which incurs $N\left(1-\frac{1}{N}\right)^D \to 0$. Since $N \to \infty$, we can write

$$\begin{aligned} N\left(1-\frac{1}{N}\right)^D &= Ne^{D\ln(1-\frac{1}{N})} \\ &= Ne^{-\frac{D}{N}(1+O(1/N))} \\ &= e^{-\frac{D-N\ln N}{N}(1+O(1/N))}. \end{aligned} \tag{4.26}$$

Hence, $N\left(1-\frac{1}{N}\right)^D \to 0$ is equivalent to $\frac{D-N\ln N}{N} \to \infty$, which incurs $D = N\ln N + \omega(N)$. This completes the proof of Theorem 4.6.

■

The above theorem states that the minimum delay constraint in order to have small dropping probability in the network must scale as fast as $N\ln N$. Compared to the Round-Robin scheduling (Case I), we have a factor of $\ln N$ increase in the Fairness Factor (or equivalently, a factor of $\ln N$ increase in the network delay), which is due to ignoring ν in the scheduling [3]

4.3.3 Proposed Algorithm; The general case:

In the previous sections, we have studied our proposed scheduling algorithm in two extreme cases, where one extreme focuses on achieving the maximum fairness, and the other extreme on achieving the maximum sum-rate throughput. In general, it is possible to have a trade-off between the fairness and throughput, by adjusting the threshold value. Now, the question is, whether or not, it is possible to simultaneously achieve the maximum throughput and the maximum fairness of

[3] It should be noted that this scheduling is *long-term fair*, i.e., all the users are equally served over a long period of time. However, with our definition of fairness (which can be called *short-term* fairness), this scheduling is away from the maximum fairness by a factor of $\ln N$.

the system. The following theorem shows this is indeed possible in the asymptotic case of $N \to \infty$.

Theorem 4.7 *Consider the proposed algorithm in the asymptotic case of $N \to \infty$. Then, for the values of Θ satisfying*

$$\ln N - 2 \ln \ln N < \Theta < \ln N - 1.5 \ln \ln N, \tag{4.27}$$

one can simultaneously achieve:
I- Maximum Throughput:

$$\lim_{N \to \infty} \mathcal{C}_{\text{sum}} - \mathcal{R} = 0, \tag{4.28}$$

in which $\mathcal{C}_{\text{sum}}$ denotes the maximum achievable sum-rate in the broadcast channel and $\mathcal{R}$ denotes the achievable sum-rate of the proposed algorithm, and
II- Maximum Fairness:

$$\lim_{N \to \infty} \frac{D}{N} = 1, \quad \textit{while } Pr\{\mathscr{B}\} \to 0 \text{ (}\textit{or equivalently, } \lim_{N \to \infty} FF = 1\text{)}. \tag{4.29}$$

Proof - The steps of the proof are as follows: in Lemma 4.8, we study the behavior of $f_\nu(l)$ and derive a difference equation satisfied by $f_\nu(l)$. In Lemma 4.9, we derive an explicit solution for this difference equation. Based on this solution, in Lemma 4.10, we present a sufficient condition such that the conditions $\lim_{N \to \infty} \frac{D}{N} \to 1$ and $\Pr\{\mathscr{B}\} \to 0$ are satisfied simultaneously. Finally, the theorem is proved by deriving a lower-bound on the achievable sum-rate, based on the threshold level given in (4.27).

Lemma 4.8 *Defining $D_0 = D - \sqrt{N} n_0(n_0 - 1)$, where $n_0 = 3(\ln N)^2$, for $D_0 \leq l \leq D$, we have $f_\nu(l) = \frac{1}{N}[1 - o(1/N)]$, and for $l < D_0$, $f_\nu(l)$ satisfies the following* *difference equation:*

$$f_\nu(l) - f_\nu(l-1) \;\; = \;\; \eta f_\nu(l) \left[1 - pF_\nu(l)\right]^{N-1} \left[1 + O(1/\sqrt{N})\right], \qquad (4.30)$$

where $p = e^{-\Theta}$, $\eta \triangleq \frac{p}{1-p}$, *and* $F_\nu(.)$ *denotes the CDF of* ν*.*

Proof - Similar to (4.8), we have

$$f_\nu(l-1) = f_\nu(l) \left(1 - \Pr\{\nu_{\min} = l, k \in \mathcal{S} | \nu_k = l\}\right), \qquad (4.31)$$

where $\nu_{\min} = \min_k\{\nu_k | k \in \mathcal{S}\}$. Having the fact that

$$p \triangleq \Pr\{k \in \mathcal{S}\} = e^{-\Theta}, \qquad (4.32)$$

which is resulted from the exponential distribution for $|h_k|^2$ (as a result of the Complex Gaussian distribution for h_k), and the independence between the users' channels, it follows that $|\mathcal{S}|$ is a Binomial random variable with parameters (N, p).

As a result, we have

$$
\begin{aligned}
\Pr\{\nu_{\min} = l, k \in \mathcal{S} | \nu_k = l\} &= \sum_{n=1}^{N} \Pr\{\nu_{\min} = l, k \in \mathcal{S}, |\mathcal{S}| = n | \nu_k = l\} \\
&= \sum_{n=1}^{N} \Pr\{|\mathcal{S}| = n, k \in \mathcal{S} \ |\nu_k = l\} \times \\
&\quad \Pr\{\nu_{\min} = l | \ \nu_k = l, |\mathcal{S}| = n, k \in \mathcal{S}\} \\
&\stackrel{(a)}{=} \sum_{n=1}^{N} \Pr\{|\mathcal{S}| = n, k \in \mathcal{S}\} \times \\
&\quad \Pr\{\nu_{\min} = l | \ \nu_k = l, |\mathcal{S}| = n, k \in \mathcal{S}\} \\
&= \sum_{n=1}^{N} \binom{N-1}{n-1} p^n (1-p)^{N-n} \times \\
&\quad \Pr\{\nu_{\min} = l | \ \nu_k = l, |\mathcal{S}| = n, k \in \mathcal{S}\} \\
&= \sum_{n=1}^{N} \binom{N-1}{n-1} p^n (1-p)^{N-n} \times \\
&\quad \Pr\{\nu_i > l, i \in \mathcal{S}, i \neq k | \ \nu_k = l, |\mathcal{S}| = n, k \in \mathcal{S}\},
\end{aligned}
\tag{4.33}
$$

where (a) comes from the fact that the events $|\mathcal{S}| = n$ and $k \in \mathcal{S}$ are independent of the event $\nu_k(t) = l$. In fact, the event $\nu_k(t) = l$ is a function of $\{h_k(j)\}_{k=1}^{N}$, $j < t$, while the events $|\mathcal{S}(t)| = n$ and $k \in \mathcal{S}(t)$ are functions of $\{h_k(t)\}_{k=1}^{N}$, and because of the independent block fading assumption, are independent of $\{h_k(j)\}_{k=1}^{N}$, $j < t$, and consequently independent of $\nu_k(t) = l$.

To evaluate the right hand side of the above equation, we need to find the following probability:

$$
\Pr\{\nu_i > l, i \in \mathcal{S}, i \neq k | \ \nu_k = l, |\mathcal{S}| = n, k \in \mathcal{S}\}, \tag{4.34}
$$

which is, by symmetry, equal to

$$
\Pr\{\nu_1 > l, \cdots, \nu_{n-1} > l | \ \nu_n = l\}, \tag{4.35}
$$

noting that $\nu_k(t)$ and $h_k(t)$ are independent random variables. An upper-bound on this probability can be given as bellow:

$$\begin{aligned}
\Pr\{\nu_1 > l, \cdots, \nu_{n-1} > l |\, \nu_n = l\} &= \Pr\{\nu_1 > l |\, \nu_n = l\} \times \\
&\quad \prod_{i=2}^{n-1} \Pr\{\nu_i > l |\, \nu_1 > l, \cdots, \nu_{i-1} > l, \nu_n = l\} \qquad (4.36) \\
&\overset{(a)}{\leq} \left[\Pr\{\nu_i > l |\, \nu_n = l\}\right]^{n-1} \\
&\overset{(b)}{=} \left[\frac{G_\nu(l)}{1 - f_\nu(l)}\right]^{n-1}, \qquad (4.37)
\end{aligned}$$

where (a) follows from (4.24), in which we have shown that $\Pr\{\nu_i > l |\, \nu_1 > l, \cdots, \nu_{i-1} > l\} \leq \Pr\{\nu_i > l\}$, and by following the same approach we can show $\Pr\{\nu_i > l |\, \nu_1 > l, \cdots, \nu_{i-1} > l, \nu_n = l\} \leq \Pr\{\nu_i > l |\, \nu_n = l\}$, and (b) results from the fact that the only dependency between ν_i and ν_n is that they are distinct, and hence $(\nu_i > l |\, \nu_n = l)$ is equivalent to $(\nu_i > l |\, \nu_i \neq l)$, with the probability of $\frac{G_\nu(l)}{1 - f_\nu(l)}$, where $G_\nu(l) \triangleq 1 - F_\nu(l)$.

In order to lower-bound $\Pr\{\nu_i > l |\, \nu_1 > l, \cdots, \nu_{i-1} > l, \nu_n = l\}$, we need to derive an upper-bound on $f_\nu(l)$. Since $f_\nu(l)$ is an increasing function of l (from (4.31)), it follows that

$$f_\nu(l) \leq f_\nu(D), \quad \forall l. \qquad (4.38)$$

However, unlike the previous cases, $f_\nu(D) \neq \frac{1}{N}$. This results from the fact that using the proposed algorithm in the general case, it is probable that no user is served. Defining the event $\mathscr{X}(t) \triangleq \bigcup_{k=1}^{N} \mathscr{X}_k(t)$ as the event of serving at least one user in frame t, we have

$$\begin{aligned}\Pr\{\mathscr{X}(t)\} &= \Pr\{|\mathcal{S}(t)| > 0\} \\ &= 1 - \prod_{k=1}^{N} \Pr\{|h_k|^2 < \Theta\} \\ &= 1 - \left(1 - e^{-\Theta}\right)^N. \end{aligned} \tag{4.39}$$

Noting that $\ln N - 2\ln\ln N < \Theta < \ln N - 1.5\ln\ln N$, we have $\frac{(\ln N)^{1.5}}{N} < e^{-\Theta} < \frac{(\ln N)^2}{N}$, and hence, $\left(1 - e^{-\Theta}\right)^N \lesssim e^{-(\ln N)^{1.5}}$. Moreover, $\Pr\{\mathscr{X}(t)\}$ in terms of $f_\nu(D)$ can be written as

$$\begin{aligned}\Pr\{\mathscr{X}(t)\} &= \sum_{k=1}^{N} \Pr\{\nu_k(t+1) = D\} \\ &= N f_\nu(D), \end{aligned} \tag{4.40}$$

where the first line comes from the distinction of ν_k's and the second line follows from the symmetry between the users and dropping the time index. Combining (4.39) and (4.40) yields,

$$f_\nu(D) = \frac{1}{N}\left[1 - \left|O\left(e^{-(\ln N)^{1.5}}\right)\right|\right], \tag{4.41}$$

which is less than $\frac{1}{N}$. Combining (4.38) with the above equation yields

$$f_\nu(l) \leq \frac{1}{N}, \quad \forall l. \tag{4.42}$$

Similar to (4.23) and (4.24), we can lower-bound $q_2 \triangleq \Pr\{\nu_i > l | \, \nu_1 > l, \cdots, \nu_{i-1} > l, \nu_n = l\}$ as follows:

$$\begin{aligned}
q_2 &= \frac{\sum_{\substack{(a_1,\cdots,a_{i-1})\\ l<a_k\le D}} f_{\nu_1,\cdots,\nu_{i-1},\nu_n}(a_1,\cdots,a_{i-1},l)}{\Pr\{\nu_1 > l,\cdots,\nu_{i-1} > l, \nu_n = l\}} \times \\
&\quad \Pr\{\nu_i > l | \, \nu_1 = a_1,\cdots,\nu_{i-1} = a_{i-1}, \nu_n = l\} \\
&= \frac{\sum_{\substack{(a_1,\cdots,a_{i-1})\\ l<a_k\le D}} f_{\nu_1,\cdots,\nu_{i-1},\nu_n}(a_1,\cdots,a_{i-1},l)}{\Pr\{\nu_1 > l,\cdots,\nu_{i-1} > l, \nu_n = l\}} \times \\
&\quad \Pr\left\{\nu_i > l | \, \nu_i \notin \{a_1,\cdots,a_{i-1},l\}\right\} \\
&= \frac{\sum_{\substack{(a_1,\cdots,a_{i-1})\\ l<a_k\le D}} f_{\nu_1,\cdots,\nu_{i-1},\nu_n}(a_1,\cdots,a_{i-1},l)}{\Pr\{\nu_1 > l,\cdots,\nu_{i-1} > l, \nu_n = l\}} \times \\
&\quad \frac{\Pr\{\nu_i > l\} - \sum_{k=1}^{i-1} f_{\nu_i}(a_k)}{1 - \sum_{k=1}^{i-1} f_{\nu_i}(a_k) - f_{\nu_i}(l)} \\
&\ge \frac{\sum_{\substack{(a_1,\cdots,a_{i-1})\\ l<a_k\le D}} f_{\nu_1,\cdots,\nu_{i-1},\nu_n}(a_1,\cdots,a_{i-1},l)}{\Pr\{\nu_1 > l,\cdots,\nu_{i-1} > l, \nu_n = l\}} \times \\
&\quad \left(\Pr\{\nu_i > l\} - \sum_{k=1}^{i-1} f_{\nu_i}(a_k)\right) \\
&\overset{(a)}{\ge} G_\nu(l) - \frac{i-1}{N},
\end{aligned} \tag{4.43}$$

where (a) follows from the fact that $f_{\nu_i}(a_k) \le \frac{1}{N}, \forall a_k$ (equation (4.42)). From the above equation and (4.36), $\Pr\{\nu_1 > l,\cdots,\nu_{n-1} > l | \, \nu_n = l\}$ can be lower-bounded as

$$\Pr\{\nu_1 > l,\cdots,\nu_{n-1} > l | \, \nu_n = l\} \ge \prod_{i=0}^{n-2} \left(G_\nu(l) - \frac{i}{N}\right). \tag{4.44}$$

Using the above equation, and defining $n_0 \triangleq 3(\ln N)^2$ and $D_0 \triangleq D - \sqrt{N} n_0(n_0 - 1)$, a lower-bound on $\Pr\{\nu_1 > l, \cdots, \nu_{n-1} > l | \, \nu_n = l\}$ is given as,

$$\Pr\{\nu_1 > l, \cdots, \nu_{n-1} > l | \, \nu_n = l\} \geq g(n, l), \tag{4.45}$$

where

$$g(n,l) \triangleq \begin{cases} \prod_{i=0}^{n-2} \left(G_\nu(l) - \frac{i}{N} \right) & l < D_0 \text{ and } n \leq n_0 \\ 0 & \text{Otherwise.} \end{cases} \tag{4.46}$$

As we will see later, the form in (4.46) is more convenient to carry out our subsequent derivations.

From (4.33), (4.35), (4.36), and (4.37), an upper-bound on $\Pr\{\nu_{\min} = l, k \in \mathcal{S} | \nu_k = l\}$ can be obtained as follows:

$$\begin{aligned}
\Pr\{\nu_{\min} = l, k \in \mathcal{S} | \nu_k = l\} &\leq \sum_{n=1}^{N} \binom{N-1}{n-1} p^n (1-p)^{N-n} \left(\frac{G_\nu(l)}{1 - f_\nu(l)} \right)^{n-1} \\
&= \eta \sum_{n=1}^{N} \binom{N-1}{n-1} p^{n-1} (1-p)^{N-n+1} \left(\frac{G_\nu(l)}{1 - f_\nu(l)} \right)^{n-1} \\
&= \eta \sum_{n=0}^{N-1} \binom{N-1}{n} \left(\frac{p\, G_\nu(l)}{1 - f_\nu(l)} \right)^{n} (1-p)^{N-n} \\
&= \eta \left(\frac{p\, G_\nu(l)}{1 - f_\nu(l)} + 1 - p \right)^{N-1} \\
&\overset{(a)}{<} \eta \left(p\, G_\nu(l) \Big(1 + \frac{2}{N} \Big) + 1 - p \right)^{N-1} \\
&= \eta \, (1 - p\, F_\nu(l))^{N-1} \left(1 + \frac{2p\, G_\nu(l)}{N(1 - pF_\nu(l))} \right)^{N-1} \\
&\overset{(b)}{\leq} \eta \, (1 - p\, F_\nu(l))^{N-1} \left(1 + \frac{2p\, G_\nu(l)}{N(1 - p)} \right)^{N-1} \\
&\sim \eta \, (1 - p\, F_\nu(l))^{N-1} \, e^{\frac{2p\, G_\nu(l)}{1-p}} \\
&\overset{(c)}{=} \eta \, (1 - p\, F_\nu(l))^{N-1} \, [1 + O(p)],
\end{aligned} \tag{4.47}$$

where $\eta \triangleq \frac{p}{1-p}$. (a) comes from the facts that $\forall l,\ f_\nu(l) \leq \frac{1}{N}$ (equation (4.42)), and for x sufficiently small, $\frac{1}{1-x} < 1 + 2x$, (b) results from $F_\nu(l) \leq 1$, and (c) follows from the fact that since $\ln N - 2\ln\ln N < \Theta < \ln N - 1.5\ln\ln N$, we have $\frac{(\ln N)^{1.5}}{N} < p = e^{-\Theta} < \frac{(\ln N)^2}{N}$, which implies that $p = o(1)$.

Moreover, from (4.33), (4.35), and (4.45), a lower-bound on $\Pr\{\nu_{\min} = l, k \in \mathcal{S} | \nu_k = l\}$, for $l < D_0$, is given as follows:

$$\begin{aligned}
\Pr\{\nu_{\min} = l, k \in \mathcal{S} | \nu_k = l\} &\geq \sum_{n=1}^{N} \binom{N-1}{n-1} p^n (1-p)^{N-n} g(n,l) \\
&= \sum_{n=1}^{n_0} \binom{N-1}{n-1} p^n (1-p)^{N-n} \prod_{i=0}^{n-2} \left(G_\nu(l) - \frac{i}{N} \right) \\
&= \sum_{n=1}^{n_0} \binom{N-1}{n-1} p^n (1-p)^{N-n} G_\nu(l)^{n-1} \prod_{i=0}^{n-2} \left(1 - \frac{i}{N G_\nu(l)} \right).
\end{aligned} \tag{4.48}$$

By repeated application of (4.31) and using (4.47) to upper-bound $\Pr\{\nu_{\min} = l, k \in \mathcal{S} | \nu_k = l\}$, we obtain

$$\begin{aligned}
f_\nu(D) - f_\nu(D_0) &\leq \sum_{l=D_0}^{D} \eta f_\nu(l) \left(1 - pF_\nu(l)\right)^{N-1} [1 + O(p)] \\
&\overset{(a)}{\leq} \frac{\eta(D - D_0 + 1)}{N} \left(1 - p + p\frac{D - D_0}{N} \right)^{N-1} [1 + O(p)] \\
&\leq \frac{\eta(D - D_0 + 1)}{N} e^{-(N-1)p\left(1 - \frac{D-D_0}{N}\right)} [1 + O(p)],
\end{aligned} \tag{4.49}$$

where (a) comes from the fact that $f_\nu(l) \leq \frac{1}{N}$ and as a result $F_\nu(l) \geq 1 - \frac{D-l}{N}$, which implies that $F_\nu(l) \geq 1 - \frac{D-D_0}{N}$ for $l \geq D_0$. Having the facts that $D - D_0 \sim 9\sqrt{N}(\ln N)^4$ and $\ln N - 2\ln\ln N < \Theta < \ln N - 1.5\ln\ln N$, which results in $\frac{(\ln N)^{1.5}}{N} < p < \frac{(\ln N)^2}{N}$, and $\eta = \frac{p}{1-p} \sim p$, the right hand side of the above equation can be upper-bounded as

$$\text{RH}(4.49) \lesssim \frac{9(\ln N)^6}{N^{3/2}} e^{-(\ln N)^{1.5}}. \tag{4.50}$$

Substituting in (4.49) and using (4.41), noting that $e^{-(\ln N)^{1.5}} = o(1/N)$, we obtain

$$f_\nu(D_0) = \frac{1}{N}\left[1 + o(1/N)\right]. \tag{4.51}$$

Since $f_\nu(l)$ is an increasing function of l, it follows from the above equation that

$$f_\nu(l) = \frac{1}{N}\left[1 + o(1/N)\right], \quad \forall l, D_0 \le l \le D. \tag{4.52}$$

The above equation incurs that for $l < D_0$, $G_\nu(l) \gtrsim \frac{D-D_0}{N} = \frac{n_0(n_0-1)}{\sqrt{N}}$. As a result, $\prod_{i=0}^{n-2}\left(1 - \frac{i}{NG_\nu(l)}\right)$ in (4.48) can be lower-bounded as

$$\begin{aligned}
\prod_{i=0}^{n-2}\left(1 - \frac{i}{NG_\nu(l)}\right) &\overset{(a)}{\gtrsim} \prod_{i=0}^{n_0-2}\left(1 - \frac{i}{\sqrt{N}n_0(n_0-1)}\right) \\
&\overset{(b)}{\approx} \prod_{i=0}^{n_0-2} e^{-\frac{i}{\sqrt{N}n_0(n_0-1)}} \\
&= e^{-\frac{(n_0-1)(n_0-2)}{2\sqrt{N}n_0(n_0-1)}} \\
&= 1 + O\Big(1/\sqrt{N}\Big),
\end{aligned} \tag{4.53}$$

where (a) follows from the fact that $n \le n_0$, and (b) results from the fact that as $i < n_0$, $\frac{i}{\sqrt{N}n_0(n_0-1)} \ll 1$, which implies that $1 - \frac{i}{\sqrt{N}n_0(n_0-1)} \approx e^{-\frac{i}{\sqrt{N}n_0(n_0-1)}}$. Moreover, similar to (4.47), we can write $\Psi \triangleq \sum_{n=1}^{n_0}\binom{N-1}{n-1}p^n(1-p)^{N-n}G_\nu(l)^{n-1}$ as

$$\begin{aligned}
\Psi &= \eta\left[(1 - pF_\nu(l))^{N-1} - \sum_{n=n_0}^{N-1}\binom{N-1}{n}p^n(1-p)^{N-n}G_\nu(l)^n\right] \\
&\ge \eta\left[(1 - pF_\nu(l))^{N-1} - \sum_{n=n_0}^{N-1}\binom{N-1}{n}p^n(1-p)^{N-n}\right] \\
&\overset{(a)}{\approx} \eta\left[(1 - pF_\nu(l))^{N-1} - Q\left(\frac{n_0 - (N-1)p}{\sqrt{(N-1)p(1-p)}}\right)\right],
\end{aligned} \tag{4.54}$$

where (a) results from the Gaussian approximation for a Binomial distribution with parameters (n, p), when $np \to \infty$. Noting $n_0 = 3[\ln N]^2$ and $p < \frac{[\ln N]^2}{N}$, it follows that $n_0 \geq 3(N-1)p$. Substituting in the above equation, and having the fact that $Q(x) \approx \frac{1}{\sqrt{2\pi}x} e^{-x^2/2}$ for large enough x, the right hand side of the above equation can be lower-bounded as

$$\text{RH (4.54)} \quad \geq \quad \eta \left[(1 - pF_\nu(l))^{N-1} - e^{-2(N-1)p} \right]. \tag{4.55}$$

Having the facts that $(1 - pF_\nu(l))^{N-1} \sim e^{-(N-1)pF_\nu(l)} \geq e^{-(N-1)p}$, RH (4.55) can be lower-bounded as

$$\begin{aligned} \text{RH (4.55)} \quad &\geq \quad \eta \left(1 - pF_\nu(l)\right)^{N-1} \left[1 - e^{-(N-1)p}\right] \\ &\overset{(a)}{=} \quad \eta \left(1 - pF_\nu(l)\right)^{N-1} \left[1 + O(1/N)\right], \end{aligned} \tag{4.56}$$

where (a) follows from the fact that as $p > \frac{(\ln N)^{1.5}}{N}$, we have $e^{-(N-1)p} = O(1/N)$. Combining (4.48), (4.53), (4.54), (4.55), and (4.56), we have

$$\Pr\{\nu_{\min} = l, k \in \mathcal{S} | \nu_k = l\} \quad \gtrsim= \quad \eta \left(1 - pF_\nu(l)\right)^{N-1} \left[1 + O\left(1/\sqrt{N}\right)\right], \tag{4.57}$$

for $l < D_0$. Combining (4.47) and (4.57), noting that $p = o(1/\sqrt{N})$, yields

$$\Pr\{\nu_{\min} = l, k \in \mathcal{S} | \nu_k = l\} \quad = \quad \eta \left(1 - pF_\nu(l)\right)^{N-1} \left[1 + O\left(1/\sqrt{N}\right)\right], \tag{4.58}$$

for $l < D_0$. Substituting in (4.31), we have

$$f_\nu(l) - f_\nu(l-1) = \eta f_\nu(l) \left(1 - pF_\nu(l)\right)^{N-1} \left[1 + O\left(1/\sqrt{N}\right)\right], \quad l < D_0. \tag{4.59}$$

Moreover, for $D_0 \leq l \leq D$, from (4.52), we have $f_\nu(l) = \frac{1}{N}\left[1 + o(1/N)\right]$, which completes the proof of Lemma 4.8.

■

Lemma 4.9 *The solution to the difference equation (4.30), in the asymptotic case of $N \to \infty$, is*

$$f_\nu(l) \sim \frac{\frac{\varphi}{(N-1)p} e^{(N-1)p} e^{\varphi(l-D_0)}}{1 + e^{(N-1)p} e^{\varphi(l-D_0)}} \quad l < D_0, \tag{4.60}$$

for some $\varphi = \eta \left[1 + O\left(\frac{1}{\sqrt{N}}\right)\right]$.

Proof - Rewriting (4.30), we have

$$\begin{aligned} f_\nu(l) - f_\nu(l-1) &= \eta f_\nu(l)\left(1 - pF_\nu(l)\right)^{N-1}\left[1 + O\left(1/\sqrt{N}\right)\right] \\ &\overset{(a)}{=} \eta f_\nu(l) e^{-(N-1)pF_\nu(l)[1+O(p)]}\left[1 + O\left(1/\sqrt{N}\right)\right] \\ &= \eta f_\nu(l) e^{-(N-1)pF_\nu(l)}\left[1 + O(Np^2)\right]\left[1 + O\left(1/\sqrt{N}\right)\right] \\ &\overset{(b)}{=} \eta f_\nu(l) e^{-(N-1)pF_\nu(l)}\left[1 + O\left(1/\sqrt{N}\right)\right] \quad l < D_0, \end{aligned} \tag{4.61}$$

where (a) comes from the fact that $(1+x)^n = e^{xn[1+O(x)]}$ for $x = o(1)$, and (b) results from the fact that $p < \frac{[\ln N]^2}{N}$ and as a result, $Np^2 = o\left(1/\sqrt{N}\right)$.

Now, consider the following differential equation:

$$x'(u) = \varphi x(u) e^{-(N-1)pX(u)} \quad u < D_0, \tag{4.62}$$

with the boundary conditions: $x(-\infty) = X(-\infty) = 0$, and $X(D_0) = 1 - \frac{D-D_0}{N}$, in which u is a continuous variable, and $X(u) = \int_{-\infty}^{u} x(t)dt$, and $\varphi = \eta \left[1 + O\left(\frac{1}{\sqrt{N}}\right)\right]$. Writing the Taylor series for $x(u-1)$ about u, we have

$$x(u) - x(u-1) = x'(u) + \sum_{n=2}^{\infty} \frac{(-1)^{n+1} x^{(n)}(u)}{n!}. \tag{4.63}$$

For the second derivative of (4.62), we have

$$\begin{aligned} x''(u) &= \varphi x'(u) e^{-(N-1)pX(u)} - \varphi(N-1)px(u)^2 e^{-(N-1)pX(u)} \\ &= \varphi x'(u) e^{-(N-1)pX(u)} - (N-1)px'(u)x(u). \end{aligned} \tag{4.64}$$

From the above equation, noting that with the given boundary conditions for the differential equation in (4.62), we have $e^{-(N-1)pX(u)} \leq 1$ (which follows from the facts that $x'(u) \geq 0$ and $x(u) \geq 0$, which incurs $X(u) \geq 0$), and $x(u) \leq \frac{1}{N}$ (which follows from solving (4.62) with the boundary condition $X(D_0) = 1 - \frac{D-D_0}{N}$), it is easy to see that $|x''(u)| < \varphi|x'(u)|$. Similarly, we can show that $|x^{(n)}(u)| < 2^{n-1}\varphi^n|x'(u)|$. Substituting in (4.63), noting that $\varphi \sim \eta \sim p < \frac{[\ln N]^2}{N}$, yields

$$\begin{aligned} x(u) - x(u-1) &= x'(u)[1 + O(\varphi)] \\ &\overset{(a)}{=} \varphi x(u) e^{-(N-1)pX(u)} \left[1 + O\left(\varphi\right)\right] \\ &\overset{(b)}{=} \eta x(u) e^{-(N-1)pX(u)} \left[1 + O\left(1/\sqrt{N}\right)\right] \quad u < D_0, \end{aligned} \tag{4.65}$$

where (a) comes from (4.62) and (b) follows from the facts that $\varphi = \eta\left[1 + O\left(\frac{1}{\sqrt{N}}\right)\right]$ and $\varphi = O(1/\sqrt{N})$. We also have

$$\begin{aligned} X(u) &\overset{(a)}{=} \sum_{v=-\infty}^{u} [X(v) - X(v-1)] \\ &\overset{(b)}{=} \sum_{v=-\infty}^{u} \left[x(v) + \sum_{n=1}^{\infty} \frac{(-1)^n x^{(n)}(v)}{(n+1)!}\right] \\ &\overset{(c)}{=} \sum_{v=-\infty}^{u} x(v)\left[1 + O(\varphi)\right], \end{aligned} \tag{4.66}$$

where (a) results from the fact that $X(-\infty) = 0$, (b) follows from writing the Tailor series for $X(v-1)$ about v, and (c) comes from the the fact that $|x'(v)| \leq \varphi x(v)$, $\forall v$ (4.62), and also $|x^{(n)}(v)| < 2^{n-1}\varphi^n|x'(v)|$, demonstrated earlier. Defining

$Z(u) \triangleq \sum_{v=-\infty}^{u} x(v)$ and using the above equation and (4.65), we have

$$\begin{aligned} x(u) - x(u-1) &= \eta x(u) e^{-(N-1)pX(u)} \left[1 + O\left(1/\sqrt{N}\right)\right] \\ &= \eta x(u) e^{-(N-1)pZ(u)[1+O(\varphi)]} \left[1 + O\left(1/\sqrt{N}\right)\right] \\ &\stackrel{(a)}{=} \eta x(u) e^{-(N-1)pZ(u)} \left[1 + O(Np^2)\right] \left[1 + O\left(1/\sqrt{N}\right)\right] \\ &\stackrel{(b)}{=} \eta x(u) e^{-(N-1)pZ(u)} \left[1 + O\left(1/\sqrt{N}\right)\right], \end{aligned} \tag{4.67}$$

where (a) results from the fact that $\varphi \sim p$, and (b) follows from the fact that $p < \frac{[\ln N]^2}{N}$ and as a result, $Np^2 \sim o\left(1/\sqrt{N}\right)$ (similar to (b) in (4.61)). The above equation incurs that the solution of (4.62) also satisfies (4.61). More precisely, for any value of l, $l < D_0$, there exists a φ such that $\varphi = \eta\left[1 + O\left(\frac{1}{\sqrt{N}}\right)\right]$, and $f_\nu(l) \sim x(l)$, where $f_\nu(l)$ is the solution of (4.61) and $x(l)$ is the solution of (4.62) at $u = l$. This suggests us to solve the differential equation (4.62), instead of the difference equation (4.61), assuming the same boundary conditions. The boundary conditions are $x(-\infty) = f_\nu(-\infty) = 0$ and $X(D_0) = F_\nu(D_0) = 1 - \frac{D-D_0}{N}$. The second condition comes from the fact that $f_\nu(l) \approx \frac{1}{N}$, for $l \geq D_0$.

By taking the integral from both sides of (4.62), we obtain

$$x(u) = -\frac{\varphi}{(N-1)p} e^{-(N-1)pX(u)} + c. \tag{4.68}$$

Noting that $X(-\infty) = x(-\infty) = 0$, $c = \frac{\varphi}{(N-1)p}$. Substituting $e^{-(N-1)pX(u)}$ by $\frac{x'(u)}{\varphi x(u)}$ from (4.62), we come up with the following differential equation:

$$\frac{x'(u)}{\varphi x(u)\left[1 - \frac{(N-1)p}{\varphi} x(u)\right]} = 1, \tag{4.69}$$

which can be solved as follows:

$$\begin{aligned}\frac{x'(u)}{x(u)} + \frac{\frac{(N-1)p}{\varphi}x'(u)}{1-\frac{(N-1)p}{\varphi}x(u)} &= \varphi \\ \Rightarrow \ln\frac{x(u)}{1-\frac{(N-1)p}{\varphi}x(u)} &= \varphi u + b, \end{aligned} \tag{4.70}$$

where b is the constant of the integration, to be determined by the other boundary condition. Solving the above equation, $x(u)$ can be written as

$$x(u) = \frac{Ae^{\varphi u}}{1+\frac{A(N-1)p}{\varphi}e^{\varphi u}}, \tag{4.71}$$

where $A = e^b$. Using (4.68) and (4.71), we have

$$X(u) = \frac{1}{(N-1)p}\ln\left(1+\frac{A(N-1)p}{\varphi}e^{\varphi u}\right). \tag{4.72}$$

Applying the condition $X(D_0) = 1 - \frac{D-D_0}{N}$ yields

$$\begin{aligned} A &= \frac{\varphi}{(N-1)p}\left[e^{(N-1)p(1-\frac{D-D_0}{N})}-1\right]e^{-\varphi D_0} \\ &\approx \frac{\varphi}{(N-1)p}e^{(N-1)p-\varphi D_0}, \end{aligned} \tag{4.73}$$

where the second line comes from the facts that $(N-1)p \gg 1$ (since $p > \frac{(\ln N)^{1.5}}{N}$) and $p(D-D_0) \ll 1$ (since $p < \frac{(\ln N)^2}{N}$ and $D - D_0 \sim 9\sqrt{N}(\ln N)^4$). Substituting A in (4.71), we have

$$x(u) \sim \frac{\frac{\varphi}{(N-1)p}e^{(N-1)p}e^{\varphi(u-D_0)}}{1+e^{(N-1)p}e^{\varphi(u-D_0)}}. \tag{4.74}$$

One can easily check that $x(D_0) \sim \frac{1}{N}$, which is consistent with (4.51). Combining (4.74) with the fact that $f_\nu(l) \sim x(l)$, Lemma 4.9 easily follows.

■

Although the derived analytical pmf in (4.74) is valid in the asymptotic regime of

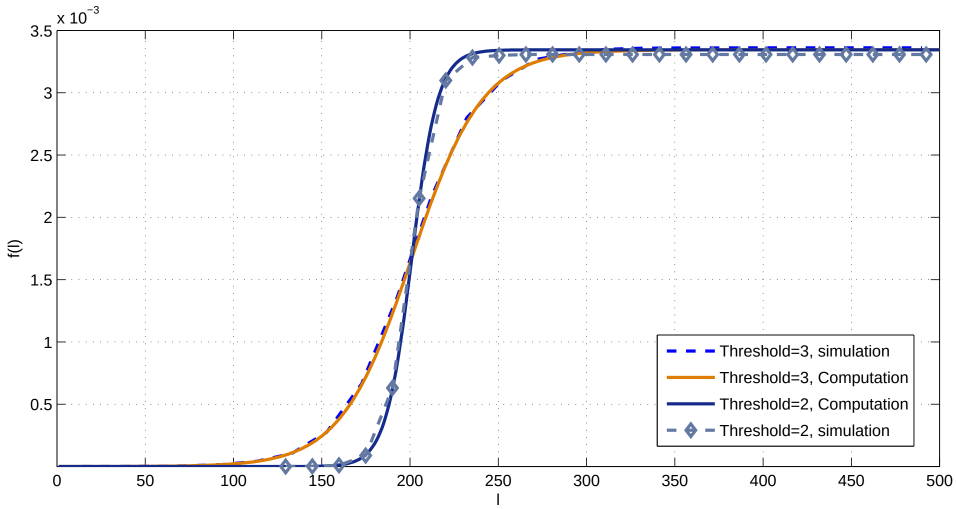

Figure 4.2: $f_\nu(l)$; comparison between simulation and computation.

$N \to \infty$, figure 4.2 shows that the analytical expression in (4.74) indeed works for finite number of users. In this figure, $f_\nu(l)$ is depicted for the proposed scheduling algorithm with the threshold values of 2 and 3, assuming $N = 300$ and $D = 500$. As can be observed, the curves derived by simulation almost follow the curves derived by computation of $f_\nu(l)$ from (4.74).

Figure 4.3 shows the plots of $f_\nu(l)$ for different values of threshold Θ. The plots of $f_\nu(l)$ for the Round-Robing scheduling and the maximum-throughput scheduling are also given for comparison. It is observed that as the value of threshold decreases, $f_\nu(l)$ merges to that of Round-Robin scheduling, while by increasing the threshold value, it merges to that of the maximum-throughput scheduling.

Lemma 4.10 *Setting* $D_0 = \frac{p}{\varphi}(N-1) + \frac{\ln N}{\varphi}$, *for some* φ *such that* $\varphi = \eta \left[1 + O\left(\frac{1}{\sqrt{N}}\right)\right]$, *yields* $Pr\{\mathscr{B}\} \to 0$, *while satisfying* $\lim_{N\to\infty} \frac{D}{N} = 1$.

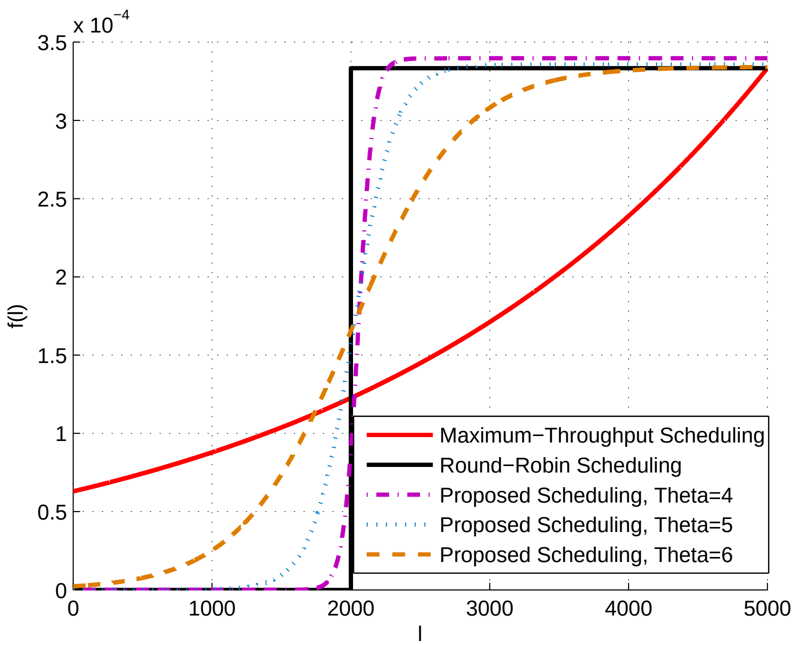

Figure 4.3: Comparison of $f_{\nu}(l)$ for different schedulings; $D = 5000, N = 3000$.

Proof - We have seen earlier that the dropping probability for each user is equal to $F_\nu(0)$. Using the union bound for the probability, it follows that having $F_\nu(0) \sim o(\frac{1}{N})$ guarantees $\Pr\{\mathscr{B}\} \to 0$. Using (4.72) and (4.73), we have

$$F_\nu(0) \sim X(0) \quad = \quad \frac{1}{(N-1)p} \ln\left(1 + e^{(N-1)p - \varphi D_0}\right), \tag{4.75}$$

for some $\varphi = \eta\left[1 + O\left(\frac{1}{\sqrt{N}}\right)\right]$. From the above equation, the condition $F_\nu(0) = o\left(\frac{1}{N}\right)$ can be equivalently written as

$$e^{(N-1)p - \varphi D_0} = o(p).$$

It can be easily verified that having $D_0 = \frac{p}{\varphi}(N-1) + \frac{\ln N}{\varphi}$, results in $e^{(N-1)p}e^{-\varphi D_0} = \frac{1}{N}$, which satisfies the above condition (since $\frac{1}{N} = o(p)$). Furthermore, since $\Theta < \ln N - 1.5 \ln\ln N$, it follows that $\varphi \sim \eta \sim p > \frac{[\ln N]^{1.5}}{N}$, which incurs that $\frac{\ln N}{\varphi} \lesssim \frac{N}{\sqrt{\ln N}}$. Combining this with the facts that $\lim_{N\to\infty} \frac{p}{\varphi} = 1$ and $D = D_0 + 9\sqrt{N}[\ln N]^4$ (which follows from the definition of D_0), we have $\lim_{N\to\infty} \frac{D}{N} = 1$. This completes the proof of Lemma 4.10.

■

The achievable sum-rate of the proposed algorithm can be lowered-bounded as follows:

$$\begin{aligned} \mathcal{R} \quad &= \quad \mathcal{R}_{\mathscr{X}} \Pr\{\mathscr{X}\} + \mathcal{R}_{\mathscr{X}^C} \Pr\{\mathscr{X}^C\} \\ &\geq \quad \mathcal{R}_{\mathscr{X}} \Pr\{\mathscr{X}\} \\ &\overset{(a)}{\geq} \quad \ln(1 + P\Theta) \Pr\{\mathscr{X}\} \\ &\overset{(4.39)}{\geq} \quad \ln(1 + P\Theta)\left[1 - \left|O\left(e^{-(\ln N)^{1.5}}\right)\right|\right]. \end{aligned} \tag{4.76}$$

where $\mathcal{R}_{\mathscr{X}}$ and $\mathcal{R}_{\mathscr{X}^C}$ denote the achievable sum-rate conditioned on $\mathscr{X}$ and $\mathscr{X}^C$, respectively, and $\mathscr{X}^C$ (complement of $\mathscr{X}$) is defined as the event that $|\mathcal{S}| = 0$. In the above equation, (a) follows from the fact that conditioned on $\mathscr{X}$, the channel gain of the selected user is greater than Θ, and hence, the achievable sum-rate is lower-bounded by $\ln(1 + P\Theta)$.

From the above equation and noting the facts that $\mathcal{C}_{\text{sum}} \sim \ln(1 + P \ln N + O(\ln \ln N))$ [26], and $\Theta > \ln N - 2 \ln \ln N$, we have

$$\begin{aligned} \mathcal{C}_{\text{sum}} - \mathcal{R} &= O\left(\frac{\ln \ln N}{\ln N}\right) \\ \Rightarrow \lim_{N\to\infty} \mathcal{C}_{\text{sum}} - \mathcal{R} &= 0. \end{aligned} \tag{4.77}$$

Combining the above equation with Lemma 4.10 completes the proof of Theorem 4.7.

■

Remark 1- Since $D = N$ is the smallest delay constraint in order not to have any dropping in the network, the above theorem simply implies that the proposed scheduling algorithm is capable of achieving the maximum throughput and minimum network delay, simultaneously.

Remark 2- Assume that the information data delivered to the users are put in *packets*, which are stored in the transmitter buffer and each packet is mapped to a coded frame, consisting of n channel uses, and transmitted over the channel (Fig. 4.4). Assume that the Packet Arrival Rate (PAR) for user k to be fixed and equal to r_k (measured as the number of arrived packets per unit time, i.e., one frame duration) and the amount of information in each packet of that user to be $n\mathcal{R}_k$. In order to have arbitrary small outage probability, $\mathcal{R}_k$, $k = 1, \cdots, N$, must be inside the capacity region of the underlying broadcast channel, which implies that $\mathcal{R}_k \leq \mathcal{C}_{\text{sum}}$, $\forall k$. Moreover, in order to have arbitrarily small dropping probability in the network, the vector consisting of the PAR of the users, denoted by $\mathbf{r} = (r_1, \cdots, r_N)$, must be inside the *stability region* of the network [78]. More

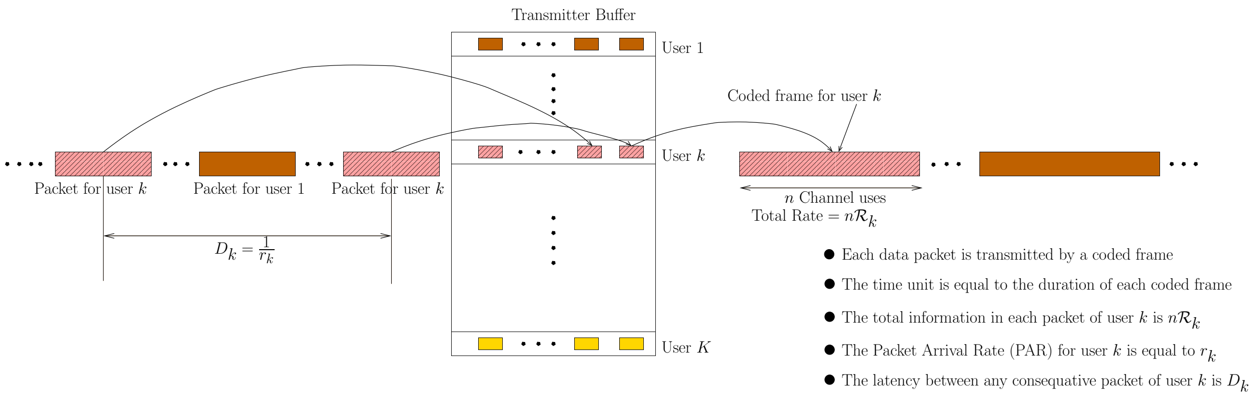

Figure 4.4: Transmission of data packets over the broadcast channel

specifically, for $r_1 = r_2 = \cdots = r_N = r$, this condition reduces to $r \leq \frac{1}{N}$ [4]. From this discussion, it follows that the maximum r and $\mathcal{R}_k$, $k = 1, \cdots, N$, in order not to have any dropping or outage in the network scale as $\frac{1}{N}$ and $\mathcal{C}_{\text{sum}}$, respectively. The above theorem states that the proposed scheduling is capable of achieving the maximum values of r and $\mathcal{R}_k$, $k = 1, \cdots, N$, simultaneously. In other words, the proposed algorithm reaches the boundary of the *capacity region* and *stability region* of the network, simultaneously. The following corollary illustrates this fact from a different perspective:

Corollary 1 *Consider a Broadcast system illustrated in Fig. 4.4, where the transmitter has the buffer size of one packet for each user and the Packet Arrival Rate (PAR) for the kth user is r_k and the amount of information in each packet for user k is $n\mathcal{R}_k$. Let us define the "average throughput" of user k (normalized per*

[4] Note that this is based on the assumption that at each frame, only one user is served.

channel use) as [5]

$$\mathscr{T}_k \triangleq r_k \mathcal{R}_k. \tag{4.78}$$

Then, for any scheduling scheme, any rate vector $\mathbf{R} = (\mathcal{R}_1, \cdots, \mathcal{R}_N)$ *supported by the channel (decoding error approaches zero), and for any PAR vector* $\mathbf{r} = (r_1, \cdots, r_N)$*, the necessary condition for Pr*$\{\mathscr{B}\} \to 0$ *is having*

$$\mathscr{T}_{\min} \triangleq \min_k \mathscr{T}_k \lesssim \frac{\ln \ln N}{N}, \tag{4.79}$$

which is achievable by the proposed algorithm.

Proof - *Necessary Condition* - Consider a long interval of time T. Defining $\mathcal{A}_k(t)$ as the indicator variable taking one when the user k is served during the frame t, and taking zero otherwise, we have

$$\sum_{k=1}^{N} \mathcal{A}_k(t) \mathcal{R}_k \leq \mathcal{C}_{\text{sum}}, \quad \forall t, 1 \leq t \leq T. \tag{4.80}$$

The above equation comes from the fact that the rates $(\mathcal{R}_1, \cdots, \mathcal{R}_N)$ must be supported by the channel. Taking the summation with respect to t, we can write

$$\sum_{t=1}^{T} \sum_{k=1}^{N} \mathcal{A}_k(t) \mathcal{R}_k \leq \mathcal{C}_{\text{sum}} T. \tag{4.81}$$

Since $\Pr\{\mathscr{B}\} \to 0$, the arrival rate of the packets must be less than or equal to their service rate, over a long period of time, almost surely. In other words, $\sum_{t=1}^{T} \mathcal{A}_k(t) \gtrsim T r_k$, $\forall k, 1 \leq k \leq N$, with probability one. Substituting in the above

[5]This definition is motivated by the fact that there is a time delay of $\frac{1}{r_k}$ between two consecutive packets of user k, and as a result, the average amount of information per channel use delivered to user k is equal to $r_k \mathcal{R}_k$.

equation yields

$$\sum_{k=1}^{N} \mathscr{T}_k = \sum_{k=1}^{N} r_k \mathcal{R}_k \lesssim \mathcal{C}_{\text{sum}} \overset{(a)}{\sim} \ln(P \ln N), \tag{4.82}$$

where (a) comes from [26]. Combining (4.78) and (4.82), yields

$$\begin{aligned} \mathscr{T}_{\min} &\leq \frac{\sum_{k=1}^{N} \mathscr{T}_k}{N} \\ &\lesssim \frac{\ln \ln N}{N} + \frac{\ln P}{N} \\ &\sim \frac{\ln \ln N}{N}. \end{aligned} \tag{4.83}$$

Sufficient Condition - Consider the proposed algorithm, with the condition of Theorem 4.7, i.e., $\ln N - 2 \ln \ln N < \Theta < \ln N - 1.5 \ln \ln N$. It is realized from Lemma 4.10 that selecting $r_k = \frac{1}{D}$ for all users, where D is obtained as follows:

$$D = \frac{p}{\varphi}(N-1) + \frac{\ln N}{\varphi} + 9\sqrt{N}[\ln N]^4,$$

guarantees $\Pr\{\mathscr{B}\} \to 0$. Furthermore, the channel can support the rate

$$\mathcal{R}_k = \ln\left[1 + P(\ln N - 2 \ln \ln N)\right],$$

with probability $\Pr\{\mathscr{X}\}$ (which is almost equal to 1 from (4.39)), for all users. Hence,

$$\begin{aligned} \mathscr{T}_{\min} &\geq \frac{\ln\left[1 + P(\ln N - 2 \ln \ln N)\right]}{D} \\ &\sim \frac{\ln \ln N}{N}. \end{aligned} \tag{4.84}$$

■

In the above corollary, the *minimum average throughput*, denoted by $\mathscr{T}_{\min}$, is defined as the measure of performance. The average throughput itself can be interpreted as the average amount of information (per channel use) delivered to a user over a long period of time. This measure is suitable for the real-time applications, where the packets have certain amount of information and certain arrival rates. Note that in the above corollary, we have assumed that the users have the buffer size of one, which is a very restrictive assumption in wireless networks. For the realistic scenarios, this constraint is more relaxed. However, since we have shown the optimality of our proposed scheduling for this assumption, it easily follows that this optimality holds for more relaxed assumptions, as well.

Computing $\mathscr{T}_{\min}$ for the two special cases of the proposed algorithm, i.e., maximum-throughput scheduling ($\mathscr{T}_{\min}^{MT}$) and Round-Robin scheduling ($\mathscr{T}_{\min}^{RR}$), yields,

$$\begin{aligned} \mathscr{T}_{\min}^{MT} &\sim \frac{\ln\ln N}{N \ln N}, \\ \mathscr{T}_{\min}^{RR} &\sim \frac{1}{N}. \end{aligned} \tag{4.85}$$

Therefore, the proposed algorithm outperforms these conventional scheduling algorithms by a factor of $\ln N$ and $\ln\ln N$, respectively.

The above corollary states that the proposed scheduling scheme maximizes the *minimum average throughput* of the system while making the network dropping probability arbitrarily small in the asymptotic regime of $N \to \infty$, for all the threshold values in the interval $[\ln N - 2\ln\ln N, \ln N - 1.5\ln\ln N]$. However, for finite number of users, it is not possible to simultaneously maximize the *minimum average throughput* and make the network dropping probability zero. In fact, for a given constraint on the dropping probability, the *minimum average throughput* will

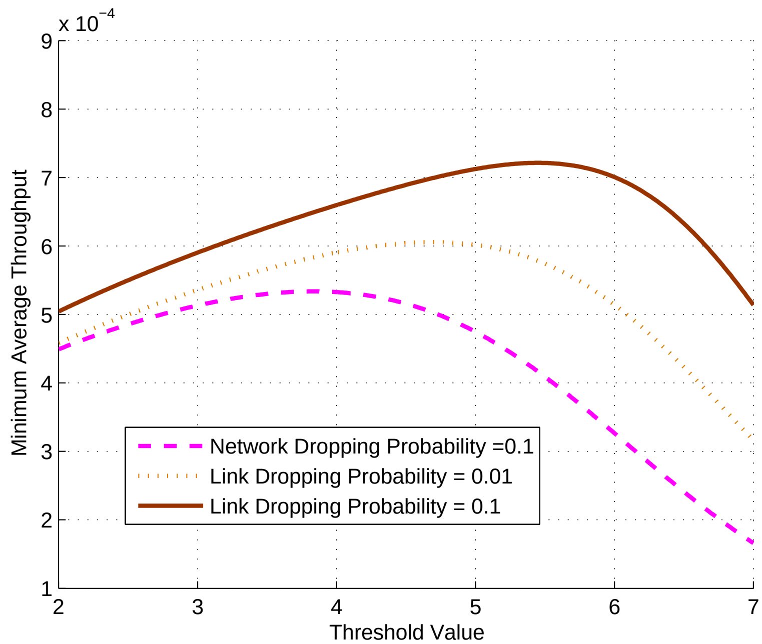

Figure 4.5: *Minimum average throughput* vs. the threshold value.

be a function of the threshold value, which is desired to be maximized. Figure 4.5 shows the plots of the *minimum average throughput* versus the threshold value, for different assumptions on the link and network dropping probabilities. The number of users N is set to 3000 and the SNR value P is set to $0\,dB$. As can be observed, for each plot, there is an optimum threshold value for which the *minimum average throughput* is maximized. Moreover, by making the constraint on the dropping probability more restrictive, the optimum threshold value decreases.

4.4 Extension to the MIMO-BC

So far, we have assumed that the transmitter and the receivers are all equipped with single antennas. In this section, we assume that the transmitter has M antennas, while the receivers have single antennas. The main difference between this case and the previous case is that for SISO-BC, serving one user at each time (TDMA) is optimal in terms of achieving the maximum throughput of the system [77], while in the MIMO-BC, this is not the case. Therefore, we must apply some modifications to our proposed algorithm, to make it suitable for MIMO-BC.

4.4.1 System Model and Proposed Algorithm

The channel model for the kth user is assumed to be

$$y_k = \mathbf{h}_k \mathbf{x} + n_k, \tag{4.86}$$

where $\mathbf{x} \in \mathbb{C}^{M\times 1}$ is the transmitted signal with the power constraint $\mathbb{E}\{\mathbf{x}^H \mathbf{x}\} \leq P$, $\mathbf{h}_k \in \mathbb{C}^{1\times M} \sim \mathcal{CN}(\mathbf{0}, \mathbf{I})$ is the channel vector, $n_k \sim \mathcal{CN}(0,1)$ is AWGN, and y_k is the received signal by the kth user.
Algorithm 2:

1) Set the threshold Υ.

2) The BS selects M orthogonal unit vectors, denoted by $\mathbf{\Phi}_1, \cdots, \mathbf{\Phi}_M$, randomly, and sends it to all users.

3) Among each of the following sets:

$$\mathcal{S}_m = \{k| \quad \mathrm{SINR}_k^{(m)} > \Upsilon\}, \quad m = 1, \cdots, M, \tag{4.87}$$

the BS serves the user with the minimum *expiry countdown*. In the above equation, $\text{SINR}_k^{(m)} \triangleq \frac{\frac{P}{M}|\mathbf{h}_k \mathbf{\Phi}_m^H|^2}{1+\sum_{j\neq m}\frac{P}{M}|\mathbf{h}_k \mathbf{\Phi}_j^H|^2}$ is the received Signal to Interference plus Noise Ratio (SINR) on the mth transmitted beam, by the kth user.

As can be observed, this algorithm is a variant of Random-Beam-Forming scheme proposed in [26], where the *expiry countdown* is considered in the scheduling.

4.4.2 Asymptotic Analysis

In this section, we analyze the performance of the proposed algorithm in the asymptotic case of $N \to \infty$. Similar to the SISO case, it is interesting to investigate the possibility of achieving the maximum throughput and fairness of the system, simultaneously, which is performed in the following theorem:

Theorem 4.11 *Using Algorithm 2, for the values of* Υ *satisfying*

$$\frac{P}{M}\left[\ln N - (M+1)\ln\ln N\right] < \Upsilon < \frac{P}{M}\left[\ln N - (M+0.5)\ln\ln N\right], \qquad (4.88)$$

we have $\lim_{N\to\infty} \mathcal{C}_{\text{sum}} - \mathcal{R} = 0$*, and* $\lim_{N\to\infty} \frac{MD}{N} = 1$*, while satisfying* $Pr\{\mathscr{B}\} \to 0$.

Proof - Using the same approach as in the proof of Theorem 4.7, we first derive $f_\nu(\nu)$ in terms of N, D, and Υ. Consider the following sets:

$$\mathcal{S}'_m \triangleq \left\{k \,\middle|\, k \in \mathcal{A}_m,\ \text{SINR}_k^{(m)} > \Upsilon\right\}, \quad m = 1, \cdots, M, \qquad (4.89)$$

where $\mathcal{A}_m \triangleq \{k |\ |\mathbf{h}_k \mathbf{\Phi}_m^H|^2 > |\mathbf{h}_k \mathbf{\Phi}_j^H|^2, \forall j \neq m\}$. For simplicity of analysis, we assume that the step 3 of Algorithm 2 works based on $\mathcal{S}'_m$ instead of $\mathcal{S}_m$. It is obvious that $\mathcal{S}'_m \subset \mathcal{S}_m$. However, since $\sum_{m=1}^M |\mathbf{h}_k \mathbf{\Phi}_m^H|^2 = \|\mathbf{h}_k\|^2 < \ln N + O(\ln\ln N)$, with probability one [26], it follows that having $\text{SINR}_k^{(m)} > \Upsilon$, where

$\Upsilon \sim \beta\frac{P}{M}\ln N$ and $\beta > \frac{1}{2}$, yields $k \in \mathcal{A}_m$. This implies that for the values of Υ satisfying (4.88), we have $\mathcal{S}'_m = \mathcal{S}_m$, with probability one. Rewriting (4.8), we have

$$f_\nu(l-1) = f_\nu(l)\left(1 - \Pr\{\mathscr{X}_k|\nu_k = l\}\right). \tag{4.90}$$

$\Pr\{\mathscr{X}_k|\nu_k = l\}$ can be written as follows:

$$\begin{aligned}
\Pr\{\mathscr{X}_k|\nu_k = l\} &\overset{(a)}{=} \Pr\{\mathscr{X}_k, k \in \mathcal{S}'|\nu_k = l\} \\
&= \sum_{m=1}^{M} \Pr\{\mathscr{X}_k, k \in \mathcal{S}'|\nu_k = l, \mathscr{F}_m\}\Pr\{\mathscr{F}_m|\nu_k = l\} \\
&\overset{(b)}{=} \sum_{m=1}^{M} \Pr\{\mathscr{X}_k, k \in \mathcal{S}'|\nu_k = l, \mathscr{F}_m\}\Pr\{\mathscr{F}_m\} \\
&\overset{(c)}{=} \frac{1}{M}\sum_{m=1}^{M} \Pr\{\mathscr{X}_k, k \in \mathcal{S}'_m|\nu_k = l, \mathscr{F}_m\} \\
&\overset{(d)}{=} \Pr\{\mathscr{X}_k, k \in \mathcal{S}'_m|\nu_k = l, \mathscr{F}_m\},
\end{aligned} \tag{4.91}$$

where $\mathcal{S}' \triangleq \bigcup_{m=1}^{M} \mathcal{S}'_m$, and $\mathscr{F}_m \triangleq \{k \in \mathcal{A}_m\}$. In the above equation, (a) results from the fact that $\mathscr{X}_k \subseteq (k \in \mathcal{S}')$, in order words, the necessary condition for user k to be served is being in any of the sets $\mathcal{S}'_m$, $s = 1, \cdots, M$. (b) results from the independence of the events $\nu_k = l$ and $\mathscr{F}_m$ [6]. (c) follows from the fact that conditioned on $\mathscr{F}_m$, i.e. $k \in \mathcal{A}_m$, $k \in \mathcal{S}'$ incurs $k \in \mathcal{S}'_m$, and also the fact that $\Pr\{\mathscr{F}_m\} = \frac{1}{M}$. (d) follows from the symmetry between the terms $\Pr\{\mathscr{X}_k, k \in \mathcal{S}'_m|\nu_k = l, \mathscr{F}_m\}$, $m = 1, \cdots, M$.

[6] In fact, $\mathscr{F}_m(t)$ is a function of $\{\mathbf{h}_k(t)\}_{k=1}^N$, while the event $\nu_k(t) = l$ is a function of $\{\mathbf{h}_k(j)\}_{k=1}^N$, $j < t$. Since the channel model is assumed to be independent block fading, the independence of $\nu_k = l$ and $\mathscr{F}_m$ easily follows.

We have

$$\begin{aligned}
\Pr\{\mathscr{X}_k, k \in \mathcal{S}'_m | \nu_k = l, \mathscr{F}_m\} &\overset{(a)}{=} \sum_{n=1}^{N}\sum_{s=n}^{N} \Pr\{\mathscr{X}_k, k \in \mathcal{S}'_m, |\mathcal{S}'_m| = n, |\mathcal{A}_m| = s \mid \nu_k = l, \mathscr{F}_m\} \\
&\overset{(b)}{=} \sum_{n=1}^{N}\sum_{s=n}^{N} \Pr\{|\mathcal{A}_m| = s|\mathscr{F}_m\} \times \\
&\quad \Pr\left\{k \in \mathcal{S}'_m, |\mathcal{S}'_m| = n \mid |\mathcal{A}_m| = s, \mathscr{F}_m\right\} \\
&\times \quad \Pr\{\mathscr{X}_k | \nu_k = l, |\mathcal{S}'_m| = n, |\mathcal{A}_m| = s, k \in \mathcal{S}'_m\}. \qquad (4.92)
\end{aligned}$$

In the above equation, (a) follows from the fact that $\mathcal{S}'_m \subset \mathcal{A}_m$, and hence $s \geq n$. (b) results from the facts that the events $|\mathcal{A}_m| = s$ and $k \in \mathcal{S}'_m$ are independent of $\nu_k = l$ (As explained in the footnote), and $k \in \mathcal{S}'_m$ is a subset of $\mathscr{F}_m$.

$\Pr\{|\mathcal{A}_m| = s|\mathscr{F}_m\}$ can be computed as

$$\begin{aligned}
\Pr\{|\mathcal{A}_m| = s|\mathscr{F}_m\} &= \frac{\Pr\{|\mathcal{A}_m| = s, k \in \mathcal{A}_m\}}{\Pr\{k \in \mathcal{A}_m\}} \\
&\overset{(a)}{=} M\binom{N-1}{s-1}\left(\frac{1}{M}\right)^{s}\left(\frac{M-1}{M}\right)^{N-s}, \qquad (4.93)
\end{aligned}$$

where (a) follows from the facts that $\Pr\{k \in \mathcal{A}_m\} = \frac{1}{M}$, and $|\mathcal{A}_m|$ is a Binomial random variable with parameters $(N, \frac{1}{M})$. In order to compute

$$\Pr\left\{k \in \mathcal{S}'_m, |\mathcal{S}'_m| = n \big| |\mathcal{A}_m| = s, \mathscr{F}_m\right\},$$

we first compute $q \triangleq \Pr\{k \in \mathcal{S}'_m | \mathscr{F}_m\}$ as follows:

$$\begin{aligned}
q &= \frac{\Pr\{k \in \mathcal{S}'_m, k \in \mathcal{A}_m\}}{\Pr\{k \in \mathcal{A}_m\}} \\
&\overset{(a)}{=} \frac{\Pr\{k \in \mathcal{S}_m, k \in \mathcal{A}_m\}}{\Pr\{k \in \mathcal{A}_m\}} \\
&= Mp\Pr\{k \in \mathcal{A}_m | k \in \mathcal{S}_m\}, \qquad (4.94)
\end{aligned}$$

where $p \triangleq \Pr\{k \in \mathcal{S}_m\} = \frac{e^{-\frac{M\Upsilon}{P}}}{(1+\Upsilon)^{M-1}}$ [26]. In the above equation, (a) results from the fact that $(k \in \mathcal{S}'_m) = (k \in \mathcal{S}_m) \bigcap (k \in \mathcal{A}_m)$. Note that as $\Pr\{k \in \mathcal{A}_m | k \in \mathcal{S}_m\} \approx 1$, it follows that $q \approx Mp$. Having q from the above equation, we can write

$$\Pr\left\{k \in \mathcal{S}'_m, |\mathcal{S}'_m| = n \Big| |\mathcal{A}_m| = s, \mathscr{F}_m\right\} = \binom{s-1}{n-1} q^n (1-q)^{s-n}. \tag{4.95}$$

Substituting $\Pr\{|\mathcal{A}_m| = s|\mathscr{F}_m\}$ and $\Pr\left\{k \in \mathcal{S}'_m, |\mathcal{S}'_m| = n \Big| |\mathcal{A}_m| = s, \mathscr{F}_m\right\}$ from (4.94) and (4.95), and noting that conditioned on $|\mathcal{S}'_m| = n$, $\mathscr{X}_k$ is independent of $|\mathcal{A}_m| = s$, RH (4.92) can be written as

$$\begin{aligned}
\text{RH}(4.92) &= \sum_{n=1}^{N}\sum_{s=n}^{N} M\binom{N-1}{s-1}\left(\frac{1}{M}\right)^{s}\left(\frac{M-1}{M}\right)^{N-s}\binom{s-1}{n-1} q^n (1-q)^{s-n} \times \\
&\quad \Pr\{\mathscr{X}_k|\nu_k = l, |\mathcal{S}'_m| = n, k \in \mathcal{S}'_m\} \\
&= M\left(\frac{M-1}{M}\right)^{N}\sum_{n=1}^{N}\binom{N-1}{n-1}\left(\frac{q}{1-q}\right)^{n} \times \\
&\quad \Pr\{\mathscr{X}_k|\nu_k = l, |\mathcal{S}'_m| = n, k \in \mathcal{S}'_m\} \times \\
&\quad \sum_{s=n}^{N}\binom{N-n}{s-n}\left(\frac{1-q}{M-1}\right)^{s} \\
&= M\left(\frac{M-1}{M}\right)^{N}\sum_{n=1}^{N}\binom{N-1}{n-1}\left(\frac{q}{1-q}\right)^{n} \times \\
&\quad \Pr\{\mathscr{X}_k|\nu_k = l, |\mathcal{S}'_m| = n, k \in \mathcal{S}'_m\} \times \\
&\quad \left(\frac{1-q}{M-1}\right)^{n}\left[1 + \frac{1-q}{M-1}\right]^{N-n} \\
&= M\sum_{n=1}^{N}\binom{N-1}{n-1}\left(\frac{q}{M}\right)^{n}\left(1 - \frac{q}{M}\right)^{N-n} \times \\
&\quad \Pr\{\mathscr{X}_k|\nu_k = l, |\mathcal{S}'_m| = n, k \in \mathcal{S}'_m\}.
\end{aligned} \tag{4.96}$$

As can be observed, the above equation is very similar to (4.33), and by a similar argument we can show that

$$\begin{aligned}
\Pr\{\nu_i > l, i = 1, \cdots, n, i \neq k|\nu_k = l, |\mathcal{S}'_m| = n, k \in \mathcal{S}'_m\} \leq \\
\Pr\{\mathscr{X}_k \mid \nu_k = l, |\mathcal{S}'_m| = n, k \in \mathcal{S}'_m\} \leq \\
\Pr\{\nu_i \geq l, i = 1, \cdots, n, i \neq k \mid \nu_k = l, |\mathcal{S}'_m| = n, k \in \mathcal{S}'_m\}.
\end{aligned} \tag{4.97}$$

In the above equation, the first inequality results from the fact that having $\nu_i > l$, $i \neq k$, implies that the kth user has the minimum *expiry countdown* among $\mathcal{S}'_m$, and hence, will be selected. The second inequality follows from the fact that the kth user must have the minimum expiry countdown in $\mathcal{S}'_m$ in order to be selected, i.e., no user in $\mathcal{S}'_m$ should have a smaller *expiry countdown*. Noting the symmetry of the problem with respect to the users and the fact that the events $\nu_i > l$ (or $\nu_i \geq l$) are independent of $|\mathcal{S}'_m| = n$ and $k \in \mathcal{S}'_m$, the upper bound can be written as $\Pr\{\nu_1 \geq l, \cdots, \nu_{n-1} \geq l | \nu_n = l\}$, which is by the chain rule equal to

$$\Pr\{\nu_1 \geq l, \cdots, \nu_{n-1} \geq l | \nu_n = l\} = \Pr\{\nu_1 \geq l | \nu_n = l\} \times \prod_{i=2}^{n-1} \Pr\{\nu_i \geq l | \nu_1 \geq l, \cdots, \nu_{i-1} \geq l, \nu_n = l\}. \tag{4.98}$$

Consider the following probability:

$$\Pr\{\nu_i = l_1 | \nu_j = l_2\}, \quad i \neq j. \tag{4.99}$$

For $l_1 = l_2$, the above probability can be upper-bounded as

$$\Pr\{\nu_i = l_1 | \nu_j = l_1\} \leq f_\nu(l_1). \tag{4.100}$$

The above inequality comes from the fact that $\Pr\{\nu_i = l_1, \nu_j = l_1\} \leq \Pr^2\{\nu_i = l_1\} = f_\nu^2(l)$, which is shown in Appendix I. A brief explanation of this would be, there are $M(M-1)$ possibilities for the users i and j to be selected in the same frame (since there are M possibilities for assigning each of them to any of the beams and they can not be assigned to the same beam), while in the term $\Pr^2\{\nu_i = l_1\}$ all the M^2 possibilities are encountered.

Also, for $l_1 \neq l_2$, we have

$$f_\nu(l_1) \leq \Pr\{\nu_i = l_1 | \nu_j = l_2\} \leq \frac{f_\nu(l_1)}{1 - f_\nu(l_2)}. \tag{4.101}$$

To prove the above equation, first we note that the ratio $\frac{\Pr\{\nu_i=l_1|\nu_j=l_2\}}{f_\nu(l_1)}$ is the same for all $l_1 \neq l_2$. In other words, the condition $\nu_j = l_2$ scales the probabilities of the outcomes $\nu_i = l_1$ by the same value for $l_1 \neq l_2$ in the conditional sample space. To establish (4.101), let us denote $x \triangleq \frac{\Pr\{\nu_i=l_1|\nu_j=l_2\}}{f_\nu(l_1)}, l_1 \neq l_2$. We have

$$\begin{aligned} \sum_{u \neq l_2} \Pr\{\nu_i = u|\nu_j = l_2\} + \Pr\{\nu_i = l_2|\nu_j = l_2\} &= 1. \\ \Rightarrow \sum_{u \neq l_2} f_\nu(u)x + \Pr\{\nu_i = l_2|\nu_j = l_2\} &= 1 \\ \Rightarrow x = \frac{1 - \Pr\{\nu_i = l_2|\nu_j = l_2\}}{1 - f_\nu(l_2)}. \end{aligned} \tag{4.102}$$

Therefore,

$$\begin{aligned} \Pr\{\nu_i = l_1|\nu_j = l_2\} &= f_\nu(l_1)x \\ &= \frac{f_\nu(l_1)\left[1 - \Pr\{\nu_i = l_2|\nu_j = l_2\}\right]}{1 - f_\nu(l_2)}. \end{aligned} \tag{4.103}$$

Using (4.100) and the fact that $\Pr\{\nu_i = l_2|\nu_j = l_2\} \geq 0$, (4.101) easily follows.

Using (4.100) and (4.101), the upper-bound in (4.98), denoted by t_1, can be further upper-bounded as

$$\begin{aligned}
t_1 &= \Pr\{\nu_i \geq l | \nu_1 \geq l, \cdots, \nu_{i-1} \geq l, \nu_n = l\} \\
&= \sum_{\substack{(a_1,\cdots,a_{i-1}) \\ l\leq a_1 \leq D,\cdots,l\leq a_{i-1}\leq D}} \frac{f_{\nu_1,\cdots,\nu_{i-1}}(a_1,\cdots,a_{i-1}|\nu_n = l)}{\Pr\{\nu_1 \geq l,\cdots,\nu_{i-1}\geq l|\nu_n = l\}} \times \\
&\quad \Pr\{\nu_i \geq l \,|\, \nu_1 = a_1,\cdots,\nu_{i-1} = a_{i-1}, \nu_n = l\} \\
&= \sum_{\substack{(a_1,\cdots,a_{i-1}) \\ l\leq a_1 \leq D,\cdots,l\leq a_{i-1}\leq D}} \frac{f_{\nu_1,\cdots,\nu_{i-1}}(a_1,\cdots,a_{i-1}|\nu_n = l)}{\Pr\{\nu_1 \geq l,\cdots,\nu_{i-1}\geq l|\nu_n = l\}} \times \\
&\quad \left[\Pr\{\nu_i \geq l, \mathscr{Y} \,|\, \mathscr{Q}\} + \Pr\{\nu_i \geq l, \mathscr{Y}^C \,|\, \mathscr{Q}\}\right] \\
&= \sum_{\substack{(a_1,\cdots,a_{i-1}) \\ l\leq a_1 \leq D,\cdots,l\leq a_{i-1}\leq D}} \frac{f_{\nu_1,\cdots,\nu_{i-1}}(a_1,\cdots,a_{i-1}|\nu_n = l)}{\Pr\{\nu_1 \geq l,\cdots,\nu_{i-1}\geq l|\nu_n = l\}} \times \\
&\quad \Big[\Pr\{\mathscr{Y} \,|\, \mathscr{Q}\}\Pr\{\nu_i \geq l \,|\, \mathscr{Y}, \mathscr{Q}\} + \\
&\quad \Pr\{\mathscr{Y}^C \,|\, \mathscr{Q}\}\Pr\{\nu_i \geq l \,|\, \mathscr{Y}^C, \mathscr{Q}\}\Big] \\
&\leq \sum_{\substack{(a_1,\cdots,a_{i-1}) \\ l\leq a_1 \leq D,\cdots,l\leq a_{i-1}\leq D}} \frac{f_{\nu_1,\cdots,\nu_{i-1}}(a_1,\cdots,a_{i-1}|\nu_n = l)}{\Pr\{\nu_1 \geq l,\cdots,\nu_{i-1}\geq l|\nu_n = l\}} \times \\
&\quad \Big[\Pr\{\mathscr{Y} \,|\, \mathscr{Q}\} + \Pr\{\nu_i \geq l \,|\, \mathscr{Y}^C, \mathscr{Q}\}\Big] \\
&\overset{(a)}{\leq} \sum_{\substack{(a_1,\cdots,a_{i-1}) \\ l\leq a_1 \leq D,\cdots,l\leq a_{i-1}\leq D}} \frac{f_{\nu_1,\cdots,\nu_{i-1}}(a_1,\cdots,a_{i-1}|\nu_n = l)}{\Pr\{\nu_1 \geq l,\cdots,\nu_{i-1}\geq l|\nu_n = l\}} \times \\
&\quad \left[\sum_{k=1}^{i-1}\Pr\{\nu_i = a_k\} + \Pr\{\nu_i = l\} + \Pr\{\nu_i \geq l \,|\, \mathscr{Y}^C, \mathscr{Q}\}\right] \\
&\overset{(b)}{\leq} \sum_{\substack{(a_1,\cdots,a_{i-1}) \\ l\leq a_1 \leq D,\cdots,l\leq a_{i-1}\leq D}} \frac{f_{\nu_1,\cdots,\nu_{i-1}}(a_1,\cdots,a_{i-1}|\nu_n = l)}{\Pr\{\nu_1 \geq l,\cdots,\nu_{i-1}\geq l|\nu_n = l\}} \times \\
&\quad \left[\sum_{k=1}^{i-1} f_\nu(a_k) + f_\nu(l) + \frac{\Pr\{\nu_i \geq l\} - \sum_{k=1}^{i-1} f_\nu(a_k) - f_\nu(l)}{1 - \sum_{k=1}^{i-1} f_\nu(a_k) - f_\nu(l)}\right] \\
&\overset{(c)}{\leq} \frac{Mi}{N} + G_\nu(l-1).
\end{aligned} \tag{4.104}$$

where $\mathscr{Y} \triangleq \bigcup_{k=1}^{i-1}\{\nu_i = \nu_k\} \bigcup \{\nu_i = \nu_n\}$ and $\mathscr{Q} \triangleq \{\nu_1 = a_1, \cdots, \nu_{i-1} = a_{i-1}, \nu_n = l\}$. In the above equation, (a) results from (4.100), which incurs that $\Pr\{\mathscr{Y}|\mathscr{Q}\} \leq \sum_{k=1}^{i-1} \Pr\{\nu_i = a_k\} + \Pr\{\nu_i = l\} = \sum_{k=1}^{i-1} f_\nu(a_k) + f_\nu(l)$, (b) results from (4.101), noting that conditioned on $\mathscr{Y}^C, \mathscr{Q}$, the points $a_1, \cdots, a_{i-1}, l$ are excluded from the sample space. (c) results from: (i) upper-bounding $f_\nu(a_k)$, $k = 1, \cdots, i-1$, and $f_\nu(l)$ by $\frac{M}{N}$, which is due to the facts that $f_\nu(l) \leq f_\nu(D)$ and $f_\nu(D) = \Pr\{\mathscr{X}_k\} \leq \frac{M}{N}$, where $\Pr\{\mathscr{X}_k\}$ is the probability that user k is being selected in a frame [7], and (ii) upper-bounding $\frac{\Pr\{\nu_i \geq l\} - \sum_{k=1}^{i-1} f_\nu(a_k) - f_\nu(l)}{1 - \sum_{k=1}^{i-1} f_\nu(a_k) - f_\nu(l)}$ by $\Pr\{\nu_i \geq l\} = G_\nu(l-1)$.

Using the above equation and (4.98), the upper bound in (4.97) can be upper-bounded as

$$\Pr\{\nu_1 \geq l, \cdots, \nu_{n-1} \geq l | \nu_n = l\} \leq \prod_{i=1}^{n-1} \left(G_\nu(l-1) + \frac{Mi}{N} \right). \tag{4.105}$$

[7]In fact, $\Pr\{\mathscr{X}_k\} \leq \frac{M}{N}$ follows from the union bound on the probability. More precisely, denoting $\mathscr{X}_k^{(m)}$ as the event that user k is assigned to beam m, using the same argument as in the SISO case, one can show that $\Pr\{\mathscr{X}_k^{(m)}\} \leq \frac{1}{N}$, and hence, $\Pr\{\mathscr{X}_k\} = \Pr\{\bigcup_{m=1}^{M} \mathscr{X}_k^{(m)}\} \leq \frac{M}{N}$.

Moreover, to lower-bound the lower bound in (4.97), we first lower-bound $t_2 \triangleq \Pr\{\nu_i > l | \nu_1 > l, \cdots, \nu_{i-1} > l, \nu_n = l\}$ as follows:

$$\begin{aligned}
t_2 &\geq \sum_{\substack{(a_1,\cdots,a_{i-1}) \\ l<a_1\leq D,\cdots,l<a_{i-1}\leq D}} \frac{f_{\nu_1,\cdots,\nu_{i-1}}(a_1,\cdots,a_{i-1}|\nu_n = l)}{\Pr\{\nu_1 > l,\cdots,\nu_{i-1} > l|\nu_n = l\}} \times \\
&\quad \Pr\{\nu_i > l \,|\, \nu_1 = a_1,\cdots,\nu_{i-1} = a_{i-1}, \nu_n = l\} \\
&= \sum_{\substack{(a_1,\cdots,a_{i-1}) \\ l<a_1\leq D,\cdots,l<a_{i-1}\leq D}} \frac{f_{\nu_1,\cdots,\nu_{i-1}}(a_1,\cdots,a_{i-1}|\nu_n = l)}{\Pr\{\nu_1 > l,\cdots,\nu_{i-1} > l|\nu_n = l\}} \times \\
&\quad \Big[\Pr\{\mathscr{Y} \,|\, \mathscr{Q}\}\Pr\{\nu_i > l \,|\, \mathscr{Y}, \mathscr{Q}\} + \\
&\quad \Pr\{\mathscr{Y}^C \,|\, \mathscr{Q}\}\Pr\{\nu_i > l \,|\, \mathscr{Y}^C, \mathscr{Q}\}\Big] \\
&\geq \sum_{\substack{(a_1,\cdots,a_{i-1}) \\ l<a_1\leq D,\cdots,l<a_{i-1}\leq D}} \frac{f_{\nu_1,\cdots,\nu_{i-1}}(a_1,\cdots,a_{i-1}|\nu_n = l)}{\Pr\{\nu_1 > l,\cdots,\nu_{i-1} > l|\nu_n = l\}} \times \\
&\quad \Pr\{\mathscr{Y}^C \,|\, \mathscr{Q}\}\Pr\{\nu_i > l \,|\, \mathscr{Y}^C, \mathscr{Q}\} \\
&\overset{(a)}{\geq} \sum_{\substack{(a_1,\cdots,a_{i-1}) \\ l<a_1\leq D,\cdots,l<a_{i-1}\leq D}} \frac{f_{\nu_1,\cdots,\nu_{i-1}}(a_1,\cdots,a_{i-1}|\nu_n = l)}{\Pr\{\nu_1 > l,\cdots,\nu_{i-1} > l|\nu_n = l\}} \times \\
&\quad \Pr\{\mathscr{Y}^C \,|\, \mathscr{Q}\}\Pr\{\nu_i > l\} \\
&= \sum_{\substack{(a_1,\cdots,a_{i-1}) \\ l<a_1\leq D,\cdots,l<a_{i-1}\leq D}} \frac{f_{\nu_1,\cdots,\nu_{i-1}}(a_1,\cdots,a_{i-1}|\nu_n = l)}{\Pr\{\nu_1 > l,\cdots,\nu_{i-1} > l|\nu_n = l\}} \times \\
&\quad (1 - \Pr\{\mathscr{Y} \,|\, \mathscr{Q}\})\Pr\{\nu_i > l\} \\
&\overset{(b)}{\geq} \sum_{\substack{(a_1,\cdots,a_{i-1}) \\ l<a_1\leq D,\cdots,l<a_{i-1}\leq D}} \frac{f_{\nu_1,\cdots,\nu_{i-1}}(a_1,\cdots,a_{i-1}|\nu_n = l)}{\Pr\{\nu_1 > l,\cdots,\nu_{i-1} > l|\nu_n = l\}} \times \\
&\quad \left(1 - \sum_{k=1}^{i-1} f_\nu(a_k) - f_\nu(l)\right)\Pr\{\nu_i > l\} \\
&\overset{(c)}{\geq} G_\nu(l) - \frac{Mi}{N},
\end{aligned} \tag{4.106}$$

where (a) results from (4.101) which implies that $\Pr\{\nu_i > l \,|\, \mathscr{Y}^C, \mathscr{Q}\} \geq \Pr\{\nu_i > l\}$, (b) follows from (4.100), which incurs that $\Pr\{\mathscr{Y}|\mathscr{Q}\} \leq \sum_{k=1}^{i-1} f_\nu(a_k) + f_\nu(l)$. Finally, (c) results from the fact that $f_\nu(\nu) \leq \frac{M}{N}$, and writing $\Pr\{\nu_i > l\}$ as $G_\nu(l)$.

Using the above equation, the lower-bound in (4.97) can be lower-bounded as

$$\Pr\{\nu_1 > l, \cdots, \nu_{n-1} > l|\nu_n = l\} \geq \prod_{i=1}^{n-1} \left(G_\nu(l) - \frac{Mi}{N} \right). \tag{4.107}$$

Similar to the approach used in the SISO case, by defining $n_0 = 3(\ln N)^2$ and $D_0 = D - \sqrt{N} n_0(n_0 - 1)$, first we show that for $D_0 \leq l \leq D$, we have $f_\nu(l) \sim \frac{M}{N}$. For this purpose, by repeated application of (4.90), and using (4.91), (4.92), (4.96), (4.97), and (4.105), we have

$$f_\nu(D) - f_\nu(D_0) \leq \sum_{l=D_0+1}^{D} \mathcal{W}_l, \tag{4.108}$$

where $\mathcal{W}_l \triangleq M \sum_{n=1}^{N} \binom{N-1}{n-1} (\frac{q}{M})^n \left(1 - \frac{q}{M}\right)^{N-n} \prod_{i=1}^{n-1} \left(G_\nu(l-1) + \frac{Mi}{N}\right)$. In Appendix J, it has been shown that $\mathcal{W}_l$ is upper-bounded as $M \frac{(\ln N)^2}{N} e^{-(\ln N)^{1.5}}$, which implies that

$$\begin{aligned} f_\nu(D) - f_\nu(D_0) &\leq (D - D_0) M \frac{(\ln N)^2}{N} e^{-(\ln N)^{1.5}} \\ &\sim 9M \frac{(\ln N)^6}{\sqrt{N}} e^{-(\ln N)^{1.5}} \\ &= o\left(e^{-(\ln N)^{1.5}}\right). \end{aligned} \tag{4.109}$$

Moreover, $f_\nu(D)$ can be written as $\Pr\{\mathscr{X}_k\}$ [8], which denotes the probability that user k is selected in a frame. This probability can be expressed as $\Pr\{\bigcup_{m=1}^{M} \mathscr{X}_k^{(m)}\}$, where $\mathscr{X}_k^{(m)}$ denotes the event that the kth user is assigned to the mth beam.

[8]More precisely, $f_{\nu_k(t)}(D) = \Pr\{\mathscr{X}_k(t-1)\}$, where the the time index are removed due to the steady state condition.

Defining $\mathscr{X}^{(m)} \triangleq \bigcup_{k=1}^{N} \mathscr{X}_k^{(m)}$, which is the probability that the mth beam is assigned to some user, we have

$$\begin{aligned}
\Pr\{\mathscr{X}^{(m)}\} &= 1 - \Pr\{|\mathcal{S}'_m| = 0\} \\
&= 1 - (1 - \Pr\{k \in \mathcal{S}'_m\})^N \\
&\overset{(a)}{=} 1 - \left(1 - \frac{q}{M}\right)^N \\
&= 1 - e^{-Nq/M} \\
&\overset{(b)}{\geq} 1 - e^{-(\ln N)^{1.5}},
\end{aligned} \tag{4.110}$$

where (a) follows from the definition of q in (4.94), and (b) results from the fact that $\frac{q}{M} \sim p > \frac{(\ln N)^{1.5}}{N}$, Having the fact that the events $\mathscr{X}_k^{(m)}$, $k = 1, \cdots, N$ are mutually exclusive, i.e., beams can not be assigned to multiple users simultaneously, we have

$$\begin{aligned}
\Pr\{\mathscr{X}^{(m)}\} = \sum_{k=1}^{N} \Pr\{\mathscr{X}_k^{(m)}\} &\geq 1 - e^{-(\ln N)^{1.5}} \\
\Rightarrow \Pr\{\mathscr{X}_k^{(m)}\} &\geq \frac{1}{N}\left(1 - e^{-(\ln N)^{1.5}}\right),
\end{aligned} \tag{4.111}$$

where the second line results from the symmetry between the users. Moreover, since the sets $\mathcal{S}'_m$, $m = 1, \cdots, M$ are disjoint, it follows that the events $\mathscr{X}_k^{(m)}$, $m = 1, \cdots, M$ are mutually exclusive. Therefore, using the above equation,

$$\Pr\{\mathscr{X}_k\} = \sum_{m=1}^{M} \Pr\{\mathscr{X}_k^{(m)}\} \geq \frac{M}{N}\left(1 - e^{-(\ln N)^{1.5}}\right). \tag{4.112}$$

Combining the above equation with (4.109), it follows that

$$f_\nu(l) = \frac{M}{N}\left[1 + o\left(Ne^{-(\ln N)^{1.5}}\right)\right], \quad D_0 \leq l \leq D. \tag{4.113}$$

In other words, in the interval $[D_0, D]$, $f_\nu(l)$ is almost constant.

In the region $l < D_0$, by defining the following functions:

$$g_u(n,l) = \begin{cases} \prod_{i=1}^{n-1}\left(G_\nu(l-1) + \frac{iM}{N}\right), & n \le n_0 \\ 1 & n > n_0 \end{cases}, \tag{4.114}$$

and

$$g_l(n,l) = \begin{cases} \prod_{i=1}^{n-1}\left(G_\nu(l) - \frac{iM}{N}\right), & n \le n_0 \\ 0 & n > n_0 \end{cases}, \tag{4.115}$$

where $n_0 = 3(\ln N)^2$, using the equations (4.97), (4.105), and (4.107), it follows that

$$g_l(n,l) \le \Pr\{\mathscr{X}_k|\nu_k = l, |\mathcal{S}'_m| = n, k \in \mathcal{S}'_m\} \le g_u(n,l), \tag{4.116}$$

where $\Pr\{\mathscr{X}_k|\nu_k = l, |\mathcal{S}'_m| = n, k \in \mathcal{S}'_m\}$ is the probability we need to find in order to compute $\Pr\{\mathscr{X}_k|\nu_k = l\}$ in (4.96). From the above equation, $\Pr\{\mathscr{X}_k|\nu_k = l\}$ can be upper-bounded as follows:

$$\begin{aligned} \Pr\{\mathscr{X}_k|\nu_k = l\} &\le M\sum_{n=1}^{N}\binom{N-1}{n-1}\left(\frac{q}{M}\right)^n\left(1-\frac{q}{M}\right)^{N-n} g_u(n,l) \\ &\overset{(a)}{=} \eta\sum_{n=0}^{N-1}\binom{N-1}{n}\left(\frac{q}{M}\right)^n\left(1-\frac{q}{M}\right)^{N-n} g_u(n+1,l) \\ &= \eta\sum_{n=0}^{n_0}\binom{N-1}{n}\left(\frac{q}{M}\right)^n\left(1-\frac{q}{M}\right)^{N-n}\prod_{i=1}^{n}\left(G_\nu(l-1)+\frac{iM}{N}\right) + \\ &\quad \eta\sum_{n_0+1}^{N-1}\binom{N-1}{n}\left(\frac{q}{M}\right)^n\left(1-\frac{q}{M}\right)^{N-n} \\ &= \eta\sum_{n=0}^{n_0}\binom{N-1}{n}\left(\frac{q}{M}\right)^n\left(1-\frac{q}{M}\right)^{N-n} G_\nu(l-1)^n \times \\ &\quad \prod_{i=1}^{n}\left(1+\frac{iM}{NG_\nu(l-1)}\right) + \\ &\quad \eta\sum_{n_0+1}^{N-1}\binom{N-1}{n}\left(\frac{q}{M}\right)^n\left(1-\frac{q}{M}\right)^{N-n}, \end{aligned} \tag{4.117}$$

where $\eta = \frac{q}{1-\frac{q}{M}}$. In the above equation, (a) results from taking the terms $\frac{\frac{q}{M}}{1-\frac{q}{M}}$ outside the summation and make a change of variable $n-1$ to n. Since $f_\nu(l) \sim \frac{M}{N}$ for $D_0 \leq l \leq D$, it follows that $G_\nu(D_0) \sim \frac{M(D-D_0)}{N} = \frac{Mn_0(n_0-1)}{\sqrt{N}}$, which implies that $G_\nu(l-1) \geq \frac{Mn_0(n_0-1)}{\sqrt{N}}$, for $D_0 \leq l \leq D$. Therefore, the term $\prod_{i=1}^{n}\left(1+\frac{iM}{NG_\nu(l-1)}\right)$ can be written as

$$\begin{aligned}
\prod_{i=1}^{n}\left(1+\frac{iM}{NG_\nu(l-1)}\right) &\leq \prod_{i=1}^{n}\left(1+\frac{i}{\sqrt{N}n_0(n_0-1)}\right) \\
&\overset{(a)}{\approx} 1+\sum_{i=1}^{n}\frac{i}{\sqrt{N}n_0(n_0-1)} \\
&= 1+\frac{n(n+1)}{2\sqrt{N}n_0(n_0-1)} \\
&\overset{(b)}{=} 1+O\left(\frac{1}{\sqrt{N}}\right), \qquad (4.118)
\end{aligned}$$

where (a) results from the fact that as $i \leq n_0$, $\frac{i}{\sqrt{N} n_0(n_0-1)} \ll 1$, and (b) follows from $n \leq n_0$. Having the above equation, RH (4.117) can be written as

$$
\begin{aligned}
\text{RH (4.117)} &= \eta \sum_{n=0}^{n_0} \binom{N-1}{n} \left(\frac{q}{M}\right)^n \left(1 - \frac{q}{M}\right)^{N-n} G_\nu(l-1)^n \left[1 + O\left(\frac{1}{\sqrt{N}}\right)\right] + \\
&\quad \eta \sum_{n_0+1}^{N-1} \binom{N-1}{n} \left(\frac{q}{M}\right)^n \left(1 - \frac{q}{M}\right)^{N-n} \\
&\leq \eta \sum_{n=0}^{N-1} \binom{N-1}{n} \left(\frac{q}{M}\right)^n \left(1 - \frac{q}{M}\right)^{N-n} G_\nu(l-1)^n \left[1 + O\left(\frac{1}{\sqrt{N}}\right)\right] + \\
&\quad \eta \sum_{n_0+1}^{N-1} \binom{N-1}{n} \left(\frac{q}{M}\right)^n \left(1 - \frac{q}{M}\right)^{N-n} \\
&\overset{(a)}{=} \eta \left[1 - \frac{q}{M} F_\nu(l-1)\right]^{N-1} \left[1 + O\left(\frac{1}{\sqrt{N}}\right)\right] + \\
&\quad \eta Q\left(\frac{n_0 - (N-1)\frac{q}{M}}{\sqrt{(N-1)\frac{q}{M}\left(1 - \frac{q}{M}\right)}}\right) \\
&\overset{(b)}{\leq} \eta e^{-(N-1)\frac{q}{M}[F_\nu(l) - f_\nu(l)]} \left[1 + O\left(\frac{1}{\sqrt{N}}\right)\right] + \eta e^{-2(N-1)\frac{q}{M}} \\
&\overset{(c)}{=} \eta e^{-(N-1)\frac{q}{M} F_\nu(l)} \left[1 + O\left(\frac{1}{\sqrt{N}}\right) + e^{-(N-1)\frac{q}{M}}\right] \\
&\overset{(d)}{=} \eta e^{-(N-1)\frac{q}{M} F_\nu(l)} \left[1 + O\left(\frac{1}{\sqrt{N}}\right)\right]. \qquad (4.119)
\end{aligned}
$$

In the above equation, (a) follows from approximating the tale of the Binomial random variable with the Gaussian $Q(.)$ function. In deriving (b), we first approximate $\left[1 - \frac{q}{M} F_\nu(l-1)\right]^{N-1}$ by $e^{-(N-1)\frac{q}{M} F_\nu(l-1)} = e^{-(N-1)\frac{q}{M}[F_\nu(l) - f_\nu(l)]}$, which follows from $q \ll 1$. Using the fact that as $\frac{P}{M}[\ln N - (M+1)\ln\ln N] < \Upsilon < \frac{P}{M}[\ln N - (M+0.5)\ln\ln N]$, we have $\frac{q}{M} < \frac{(\ln N)^2}{N}$, which implies that $n_0 > 3(N-1)\frac{q}{M}$, and also the fact that for $x \gg 1$, $Q(x) < e^{-x^2/2}$, $Q\left(\frac{n_0 - (N-1)\frac{q}{M}}{\sqrt{(N-1)\frac{q}{M}\left(1 - \frac{q}{M}\right)}}\right)$ is upper-bounded as $e^{-2(N-1)\frac{q}{M}}$. (c) results from the facts that: (i) as $f_\nu(l) \leq$

$\frac{M}{N}$, we have $e^{(N-1)\frac{q}{M}f_\nu(l)} = 1 + O(q) = 1 + O\left(\frac{1}{\sqrt{N}}\right)$, and ii) since $F_\nu(l) \leq 1$, $e^{-(N-1)\frac{q}{M}F_\nu(l)} \geq e^{-(N-1)\frac{q}{M}}$, and as a result, $e^{-2(N-1)\frac{q}{M}} \leq e^{-(N-1)\frac{q}{M}}e^{-2(N-1)\frac{q}{M}F_\nu(l)}$. Finally, (d) follows from the fact that $e^{-(N-1)\frac{q}{M}} = o\left(\frac{1}{\sqrt{N}}\right)$, which is due to the fact that $q > M\frac{(\ln N)^{1.5}}{N}$.

Similar to (4.117) and (4.119), a lower-bound for $\Pr\{\mathscr{X}_k|\nu_k = l\}$ can be given as follows:

$$\begin{aligned}
\Pr\{\mathscr{X}_k|\nu_k = l\} &\geq M\sum_{n=1}^{N}\binom{N-1}{n-1}\left(\frac{q}{M}\right)^n\left(1-\frac{q}{M}\right)^{N-n}g_u(n,l) \\
&= \eta\sum_{n=0}^{n_0}\binom{N-1}{n}\left(\frac{q}{M}\right)^n\left(1-\frac{q}{M}\right)^{N-n}\prod_{i=1}^{n}\left(G_\nu(l)-\frac{iM}{N}\right) \\
&= \eta e^{-(N-1)\frac{q}{M}F_\nu(l)}\left[1+O\left(\frac{1}{\sqrt{N}}\right)\right]. \qquad (4.120)
\end{aligned}$$

Comparing (4.119) and (4.120), it follows that

$$\Pr\{\mathscr{X}_k|\nu_k = l\} = \eta e^{-(N-1)\frac{q}{M}F_\nu(l)}\left[1+O\left(\frac{1}{\sqrt{N}}\right)\right]. \qquad (4.121)$$

Substituting in (4.91), we reach the following difference equation in the region $l < D_0$:

$$f_\nu(l) - f_\nu(l-1) = \eta f_\nu(l)e^{-(N-1)\frac{q}{M}F_\nu(l)}\left[1+O\left(\frac{1}{\sqrt{N}}\right)\right]. \qquad (4.122)$$

Comparing the above equation with (4.30), it is realized that the above difference equation is the same as the difference equation obtained in the SISO case, with the difference in replacing N by $\frac{N}{M}$, and p by q. Therefore, all the results stated in Lemmas 4.8-4.10 are valid for the MIMO case, by substituting N by $\frac{N}{M}$, which completes the proof of Theorem 4.11.

■

In fact, algorithm 2 basically separates the MIMO-BC into M "virtual" SISO-BCs by assigning the users to the beam for which the maximum SINR is attained.

Therefore, the analysis of $f_\nu(l)$ is similar to the case of SISO-BC, discussed in the previous section. However, there are two main differences: i) In SISO-BC, all the users are always served by the same transmitter, while in MIMO-BC the users are switched independently between the virtual transmitters, from frame to frame. This causes $\nu_1, \cdots, \nu_N$ (The packet expiry countdown of the users) not to be necessarily distinct. However, we have shown in the proof of Theorem 4.11 that this does not affect the analysis. ii) The sizes of the virtual SISO-BCs ($\mathcal{A}_m$) are not fixed. In fact, $|\mathcal{A}_m|$, $m = 1, \cdots, M$, are Binomial random variables with parameters $(N, \frac{1}{M})$. Using Gaussian approximation for the Binomial distribution, we can write

$$\Pr\left\{\frac{N}{M}(1-\epsilon) < |\mathcal{A}_m| < \frac{N}{M}(1+\epsilon)\right\} \approx 1 - 2Q\left(\frac{\frac{N}{M}\epsilon}{\sqrt{\frac{N}{M}(1-\frac{1}{M})}}\right). \qquad (4.123)$$

Setting $\epsilon \triangleq \sqrt{\frac{2(M-1)\ln N}{N}}$, and using the approximation $Q(x) \approx \frac{1}{\sqrt{2\pi}x}e^{-\frac{x^2}{2}}$ for $x \gg 1$, the above equation can be written as

$$\Pr\left\{\frac{N}{M}(1-\epsilon) < |\mathcal{A}_m| < \frac{N}{M}(1+\epsilon)\right\} = 1 - o\left(\frac{1}{N}\right). \qquad (4.124)$$

Therefore, with probability one, the size of the sets $\mathcal{A}_m$ scales as $\frac{N}{M}\left[1 - O\left(\sqrt{\frac{\ln N}{N}}\right)\right]$. Following the above discussions, MIMO-BC can be considered as M parallel SISO-BCs, each serving approximately $\frac{N}{M}$ users. The network dropping event ($\mathscr{B}$) can be considered as the union of the dropping events for the SISO sub-channels, denoted by $\mathscr{B}_m$, $m = 1, \cdots, M$. From the union bound for the probability, we have

$$\begin{aligned}\Pr\{\mathscr{B}\} &\leq \sum_{m=1}^{M} \Pr\{\mathscr{B}_m\} \\ &= M\Pr\{\mathscr{B}_m\}, \qquad (4.125)\end{aligned}$$

where the second line comes from the symmetry between the events $\mathscr{B}_m$. Following the steps of proof for Theorem 4.7, and setting $\frac{[\ln N]^{1.5}}{N} < p < \frac{[\ln N]^2}{N}$ and $D = \frac{p}{\varphi}\frac{N}{M} + \frac{\ln N}{\varphi} + 9\sqrt{N}[\ln N]^4$, guarantees $\Pr\{\mathscr{B}_m\} \to 0$, and hence, $\Pr\{\mathscr{B}\} \to 0$. Note that as $p \sim \frac{e^{-\frac{M\Upsilon}{P}}}{(1+\Upsilon)^{M-1}}$ [26], the condition $\frac{[\ln N]^{1.5}}{N} < p < \frac{[\ln N]^2}{N}$ incurs that

$$\frac{P}{M}\left[\ln N - (M+1)\ln\ln N\right] < \Upsilon < \frac{P}{M}\left[\ln N - (M+0.5)\ln\ln N\right]. \qquad (4.126)$$

Noting that $\mathcal{C}_{\text{sum}} \sim M\ln(1+\frac{P}{M}\ln N + O(\ln\ln N))$ [26], it follows that $\lim_{N\to\infty} \mathcal{C}_{\text{sum}} - \mathcal{R} = 0$.

■

Theorem 4.11 implies that the proposed scheduling algorithm is capable of achieving the maximum sum-rate throughput, while guaranteeing $\lim_{N\to\infty} \frac{MD_{\min}}{N} = 1$, where $D_{\min}$ is the minimum value of D such that $\Pr\{\mathscr{B}\} \to 0$. Noting that $\lceil \frac{N}{M} \rceil$ is the minimum value of D in MIMO-BC to have $\Pr\{\mathscr{B}\} \to 0$, (using Round-Robin scheduling, assuming that M users are served during each frame), it follows that the proposed scheme achieves the maximum sum-rate and maximum fairness in the network, simultaneously.

Defining the *minimum average throughput* as in (4.79), it is straightforward to show that for the proposed algorithm,

$$\mathscr{T}_{\min} \sim \frac{M\ln\ln N}{N}, \qquad (4.127)$$

which is asymptotically the maximum achievable value in MIMO-BC.

4.5 Conclusion

In this chapter, a single-antenna broadcast channel with large (N) number of users is considered. It has been assumed that all users have hard delay constraint D. We have proposed a scheduling algorithm for maximizing the throughput of the system, while satisfying the delay constraint for all users. By characterizing the network dropping probability, in terms of N, D, and the threshold value in the algorithm, it has been shown that by using the proposed algorithm, it is possible to achieve the maximum throughput and maximum fairness in the network, simultaneously, in the asymptotic case of $N \to \infty$. Moreover, we have introduced a performance measure in the network, called "Minimum Average Throughput", and proved that the proposed algorithm maximizes the maximum *minimum average throughput* in a broadcast channel. Finally, the proposed algorithm is generalized for (MIMO-BC), and shown to be optimum in the sense of achieving the maximum throughput and maximum fairness in the network, simultaneously, in the asymptotic case of $N \to \infty$.

Chapter 5

Capacity of Rician MIMO Broadcast Channels

5.1 Introduction

Multiple-input multiple-output (MIMO) systems have proved their ability to achieve high bit rates on a scattering wireless network [1, 2]. In a MIMO broadcast channel, the base station equipped with multiple antennas communicates with several users. Recently, there has been a lot of interest in characterizing the capacity region of this channel [5, 6, 7, 8]. In these works, it has been demonstrated that the sum-rate capacity of MIMO broadcast channels can be achieved by applying dirty-paper coding (DPC) [9] at the transmitter.

Despite the fact that the sum-rate capacity of Gaussian MIMO-BC is known, it is still interesting to study the behavior of sum-rate capacity in various scenarios. [79] compares the achievable sum-rate of MIMO-BC for DPC to that achieved by using linear precoding schemes, and characterizes the gap between the achievable sum-rates in the high SNR regime. [39] compares the achievable sum-rate of DPC to that of Time Division Multiple Access (TDMA) for a Gaussian MIMO-BC. [26] considers a MIMO-BC with a large number of users and shows that i) the sum-rate capacity of the system scales as $M \ln \ln N$, when N is the number of users in the network, and ii) a simple scheme of "Random Beam-Forming" asymptotically achieves the sum-rate capacity as $N \to \infty$. References [25, 50, 80] consider the same network set-up and prove that one can achieve the sum-rate capacity of the system by utilizing Zero-Forcing Beam-Forming at the transmitter, provided that the user selection is performed wisely. In [34] the scaling laws of the sum-rate for fading MIMO Gaussian broadcast channels using time-sharing to the strongest user, DPC and beamforming, is derived for the asymptotic case of $N \to \infty$. In all the mentioned papers ([79]- [34]), the channel model is assumed to be Rayleigh fading. Therefore, it is of interest to investigate the sum-rate capacity of MIMO-BC, assuming more general channel models.

One of the most widely-used models for the wireless channels is Rician fading. This model is suitable for wireless links when there is a line of sight (LOS) link between the transmitter and receiver. Several papers in the literature consider Rician fading in the context of point-to-point MIMO communications. In [81], the authors derive the exact capacity of MIMO Rician channel, when perfect Channel State Information (CSI) is available at the receiver, but the transmitter has neither instantaneous nor statistical CSI. Reference [82] studies the capacity of MIMO Rician channel in the high and low SNR regimes, for both coherent and non-coherenet communications. it is shown in [82] that in the low SNR regime, the specular component of the channel completely determines the form of the optimum signal whereas in the high SNR regime it has no effect on the optimum signal structure. In [83], the authors consider the min-capacity of a MIMO Rician channel with an unknown deterministic specular component. [84] analyzes the capacity of a MIMO Rician channel with isotropically random rank-one specular component, when the channel is unknown at both the transmitter and receiver sides.

In this chapter, we consider a Rician MIMO-BC, in which a transmitter equipped with M antennas communicates with N ($N \gg 1$) single-antenna users. The channels are assumed to be perfectly known at both the transmitter and receiver sides. The asymptotic (in terms of the number of users) sum-rate capacity of the system, as well as the capacity-achieving strategies, are derived. The main results of the chapter are as follows: i) in the region of $\mathcal{K} = o(\ln N)$, where $\mathcal{K}$ denotes the *Rician factor*, the sum-rate capacity scales as $M \ln(1 + \frac{P}{M}\eta)$, where P denotes the SNR and $\eta \triangleq \frac{\ln N}{1+\mathcal{K}}$, which is achieved by Zero-Forcing Beam-Forming (ZFBF) along with a low-complexity user selection algorithm that considers only the scattered component of the users' channels, ii) in the region $\mathcal{K} = \omega(\ln N)$, in the case of co-located transmit antennas, the capacity scales as $\ln(1 + MP)$, which is achieved by TDMA, iii) in the region $\mathcal{K} = \omega(\ln N)$, in the case of isotropically-distributed specular components, the sum-rate capacity behaves as $M \ln(1 + P)$, which is achieved by ZFBF, along with a user selection algorithm that considers only the specular component of the users' channels. Simulation results confirm the validity of analytical results.

The rest of the chapter is organized as follows. In 5.2, we introduce the system model. Section 5.3 is devoted to analyzing the asymptotic sum-rate capacity of the underlying system. Some simulation results are presented in section 5.4, and finally, section 5.5 concludes the chapter.

5.2 System Model

In this work, a MIMO-BC in which a base station equipped with M antennas communicates with N users, each equipped with single antennas, is considered. The received signal by user k can be written as

$$y_k = \mathbf{H}_k \mathbf{x} + n_k, \tag{5.1}$$

where $\mathbf{x} \in \mathbb{C}^{M \times 1}$ is the transmitted signal, $\mathbf{H}_k \in \mathbb{C}^{1 \times M}$ is the channel vector from the transmitter to the kth user, which is assumed to be perfectly known at the receiver side and provided to the BS via a noiseless feedback channel [1], and $n_k \sim \mathcal{CN}(0, 1)$ is the AWGN at this receiver.

Under Rician channel model, $\mathbf{H}_k$ can be written as

$$\mathbf{H}_k = \sqrt{1 - r_k}\mathbf{G}_k + \sqrt{r_k M}\mathbf{b}_k, \tag{5.2}$$

where $\mathbf{G}_k$ is a circularly symmetric zero mean unit variance Gaussian vector, reflecting the scattered component and $\mathbf{b}_k$ is a unit-norm vector representing the specular component of the channel, and r_k is a constant related to the Rician factor $\mathcal{K}_k$ [2] via $r_k = \frac{\mathcal{K}_k}{\mathcal{K}_k + 1}$. We consider two scenarios for $\mathbf{b}_k$: (i) The entries of $\mathbf{H}_k$ are i.i.d Gaussian with mean b_k and variance $1 - |b_k|^2$, where b_k is a complex number satisfying $|b_k|^2 = r_k$. In this case, it is easy to observe that $\mathbf{b}_k = \frac{e^{j\theta_k}}{\sqrt{M}}\mathbf{1}$, where $\mathbf{1}$ is the vector of all ones. This model corresponds to the case that the transmit

[1] In fact, the BS does not need to have the perfect CSI about all the users' channels. However, the partial CSI that the BS receives through feedback is based on the perfect CSI that the receivers have.

[2] *Rician factor* is defined as the ratio of the power of the specular component to the power of the scattered component.

antennas are co-located, and consequently, the specular components from all transmit antennas to each of the users are equal [3]. ii) The vector $\mathbf{b}_k$ is isotropically distributed in the unit sphere. This model has been used in [84]. It is assumed that r_k is fixed for all the users during the whole transmission period and is equal to a constant r, i.e., $r_1 = r_2 = \cdots = r_N = r$.

We assume that the transmitter has an average power constraint P, i.e.,

$$\mathbb{E}\left\{\mathrm{Tr}(\mathbf{x}\mathbf{x}^*)\right\} \leq P.$$

The power constraint is assumed to be *per frame*. In other words, the power constraint is independent of the channel realization. The channels are assumed to be quasi-static block fading, in which each channel $\mathbf{H}_k$ is drawn randomly at the start of each transmission frame and remains constant for the whole transmission frame, and changes independently to another realization in the start of the next frame. The frame itself is assumed to be long enough to allow communication at rates close to the capacity. Defining the sum-rate capacity of the system in the channel realization $\mathcal{H} \triangleq \{\mathbf{H}_k\}_{k=1}^N$, when the transmitter has perfect CSI about all users' channels, as $\mathcal{C}_{\text{sum}}(\mathcal{H})$, the average sum-rate capacity, denoted as $\mathcal{C}_{\text{sum}}$, is defined as the average over time of $\mathcal{C}_{\text{sum}}(\mathcal{H})$, which is by the ergodicity of the channel, equal to $\mathbb{E}_{\mathcal{H}}\left\{\mathcal{C}_{\text{sum}}(\mathcal{H})\right\}$. $\mathcal{C}_{\text{sum}}$ is shown in [5] to be equal to

$$\mathcal{C}_{\text{sum}} = \mathbb{E}_{\mathcal{H}}\left\{\max_{\substack{P_k \\ \sum P_k = P}} \ln\left|\mathbf{I}_M + \sum_{k=1}^{N}\mathbf{H}_k^* P_k \mathbf{H}_k\right|\right\}, \tag{5.3}$$

where P_k is the transmit power allocated to the kth user.

[3] Note that however, the specular components from each transmit antenna to different users are not necessarily equal.

5.3 Asymptotic Analysis; Capacity Computation

In this section, we compute the capacity of MIMO-BC under Rician fading, in the asymptotic scenario of $N \to \infty$. To this end, we consider two cases; (i) $\mathcal{K} = o(\ln N)$ and (ii) $\mathcal{K} = \omega(\ln N)$. For each case, we provide a lower-bound and upper-bound for the capacity and prove that as $N \to \infty$, these bounds converge to each other.

5.3.1 $\mathcal{K} = o(\ln N)$

Theorem 5.1 *The capacity of the underlying MIMO-BC in the case of* $\mathcal{K} = o(\ln N)$ *equals*

$$\mathcal{C}_{\text{sum}} = M \ln \left(1 + \frac{P}{M} \frac{\ln N}{1 + \mathcal{K}}\right) + o(1), \tag{5.4}$$

which is asymptotically achievable by ZFBF.

Proof - The proof is based on the upper-bound and lower-bound given as follows:

Upper-bound

Using (5.2), the upper-bound for the sum-rate capacity can be derived as [26]

$$\begin{aligned}
\mathcal{C}_{\text{sum}} &\leq M\mathbb{E}\left\{\ln\left(1+\frac{P}{M}\|\mathbf{H}\|^2_{\max}\right)\right\} \\
&\leq M\mathbb{E}\left\{\ln\left(1+\frac{P}{M}\left[\sqrt{1-r}\|\mathbf{G}_k\|+\sqrt{rM}\|\mathbf{b}_k\|\right]^2_{\max}\right)\right\} \\
&\overset{(a)}{=} M\mathbb{E}\left\{\ln\left(1+\frac{P}{M}\max_k\left[\sqrt{\frac{1}{1+\mathcal{K}}}\|\mathbf{G}_k\|+\sqrt{\frac{\mathcal{K}M}{1+\mathcal{K}}}\right]^2\right)\right\} \\
&= M\mathbb{E}\left\{\ln\left(1+\frac{P}{M}\frac{1}{1+\mathcal{K}}\|\mathbf{G}\|^2_{\max}\right)+\right. \\
&\qquad \left.\ln\left(1+\frac{\frac{P}{M}\left(\frac{2\sqrt{\mathcal{K}M}}{1+\mathcal{K}}\|\mathbf{G}\|_{\max}+\frac{M\mathcal{K}}{1+\mathcal{K}}\right)}{1+\frac{P}{M}\frac{1}{1+\mathcal{K}}\|\mathbf{G}\|^2_{\max}}\right)\right\} \\
&\overset{(b)}{\leq} M\mathbb{E}\left\{\ln\left(1+\frac{P}{M}\frac{1}{1+\mathcal{K}}\|\mathbf{G}\|^2_{\max}\right)+\right. \\
&\qquad \left.\frac{2\sqrt{\mathcal{K}M}\|\mathbf{G}\|_{\max}}{\frac{M}{P}(1+\mathcal{K})+\|\mathbf{G}\|^2_{\max}}+\frac{\mathcal{K}M}{\frac{M}{P}(1+\mathcal{K})+\|\mathbf{G}\|^2_{\max}}\right\} \\
&\overset{(c)}{\leq} M\ln\left(1+\frac{P}{M}\frac{1}{1+\mathcal{K}}\mathbb{E}\left\{\|\mathbf{G}\|^2_{\max}\right\}\right)+M\mathbb{E}\left\{\frac{2\sqrt{\mathcal{K}M}\|\mathbf{G}\|_{\max}}{\frac{M}{P}(1+\mathcal{K})+\|\mathbf{G}\|^2_{\max}}\right\}+ \\
&\qquad M\mathbb{E}\left\{\frac{\mathcal{K}M}{\frac{M}{P}(1+\mathcal{K})+\|\mathbf{G}\|^2_{\max}}\right\},
\end{aligned} \tag{5.5}$$

where (a) follows from the facts that $r=\frac{\mathcal{K}}{1+\mathcal{K}}$ and $\|\mathbf{b}_k\|=1$, (b) results from upper-bounding $\ln(1+x)$ by x, and (c) follows from the concavity of $\ln(.)$ function which incurs that $\mathbb{E}\left\{\ln\left(1+\frac{P}{M}\frac{1}{1+\mathcal{K}}\|\mathbf{G}\|^2_{\max}\right)\right\}\leq\ln\left(1+\frac{P}{M}\frac{1}{1+\mathcal{K}}\mathbb{E}\left\{\|\mathbf{G}\|^2_{\max}\right\}\right)$. Defining $A\triangleq\frac{M}{P}(1+\mathcal{K})$, $t\triangleq\ln N+(M-3)\ln\ln N$, and $\mathfrak{A}$ as the event that $\|\mathbf{G}\|^2_{\max}\leq t$,

we have

$$\begin{aligned}\mathbb{E}\left\{\frac{2\sqrt{\mathcal{K}M}\|\mathbf{G}\|_{\max}}{A+\|\mathbf{G}\|^2_{\max}}\right\} &= \mathbb{E}\left\{\frac{2\sqrt{\mathcal{K}M}\|\mathbf{G}\|_{\max}}{A+\|\mathbf{G}\|^2_{\max}}\middle|\mathfrak{A}\right\}\Pr\{\mathfrak{A}\}+\\ &\quad\ \mathbb{E}\left\{\frac{2\sqrt{\mathcal{K}M}\|\mathbf{G}\|_{\max}}{A+\|\mathbf{G}\|^2_{\max}}\middle|\mathfrak{A}^C\right\}\Pr\{\mathfrak{A}^C\}\\ &\le \sqrt{\frac{\mathcal{K}M}{A}}\Pr\{\mathfrak{A}\}+\frac{2\sqrt{\mathcal{K}Mt}}{A+t},\end{aligned}\tag{5.6}$$

where the second line results from the fact that $\frac{2\|\mathbf{G}\|_{\max}}{A+\|\mathbf{G}\|^2_{\max}} \le \frac{1}{\sqrt{A}}$ and also the function $\frac{2\|\mathbf{G}\|_{\max}}{A+\|\mathbf{G}\|^2_{\max}}$ is decreasing for $\|\mathbf{G}\|^2_{\max} \ge A$, noting that as $A = o(\ln N)$ (since $\mathcal{K} = o(\ln N)$), we have $t > A$. By a similar approach, the third term in RH(3.110) can be upper-bounded as

$$\mathbb{E}\left\{\frac{\mathcal{K}M}{\frac{M}{P}(1+\mathcal{K})+\|\mathbf{G}\|^2_{\max}}\right\} \le \frac{\mathcal{K}M}{A}\Pr\{\mathfrak{A}\}+\frac{\mathcal{K}M}{A+t}.\tag{5.7}$$

In Appendix K, it has been shown that $\Pr\{\mathfrak{A}\} = o\left(\frac{1}{N}\right)$. Noting that $\frac{\mathcal{K}M}{A} = O(1)$ and $\mathcal{K} = o(\ln N)$, which incurs that $\mathcal{K} = o(t)$, we have

$$\mathbb{E}\left\{\frac{2\sqrt{\mathcal{K}M}\|\mathbf{G}\|_{\max}}{A+\|\mathbf{G}\|^2_{\max}}\right\} = o(1),\tag{5.8}$$

and

$$\mathbb{E}\left\{\frac{\mathcal{K}M}{\frac{M}{P}(1+\mathcal{K})+\|\mathbf{G}\|^2_{\max}}\right\} = o(1).\tag{5.9}$$

Substituting in (3.110), the upper-bound on the sum-rate capacity can be written as

$$\begin{aligned}\mathcal{C}_{\text{sum}} &\le M\ln\left(1+\frac{P}{M}\frac{1}{1+\mathcal{K}}\mathbb{E}\left\{\|\mathbf{G}\|^2_{\max}\right\}\right)+o(1)\\ &= M\ln\left(1+\frac{P\ln N}{M(1+\mathcal{K})}\right)+o(1),\end{aligned}\tag{5.10}$$

where the second line follows from the fact that $\mathbb{E}\left\{\|\mathbf{G}\|^2_{\max}\right\} = \ln N + O(\ln\ln N)$ [26].

Achievability: Scheduling based the scattered component

Consider the following algorithm:

Algorithm 1

- Set the threshold $t = \ln N + (M-3)\ln\ln N$
- Among the users in the following set:

$$\mathcal{S} \triangleq \{k| \ \|\mathbf{G}_k\|^2 > t\}, \tag{5.11}$$

  select one user at random. Call this user s_1, and define $\mathcal{S}_1 \triangleq \mathcal{S} - \{s_1\}$.

- For $m = 2$ to M, repeat the following:
  - Denote the set of selected users up to the $(m-1)$th step as $\mathcal{A}_m \triangleq \{s_1, \cdots, s_{m-1}\}$. Define $\mathcal{S}_m \triangleq \mathcal{S} - \mathcal{A}_m$.
  - Define $\boldsymbol{\mathcal{P}}_m$ as the sub-space spanned by the scattered channel components of the users selected in the previous steps, i.e., $\{\mathbf{v}_{s_j}\}_{j=1}^{m-1}$, where $\mathbf{v}_k \triangleq \frac{\mathbf{G}_k}{\|\mathbf{G}_k\|}$, $k = 1, \cdots, N$.
  - Let $\{\mathbf{\Phi}_j\}_{j=1}^{m-1}$ be $m-1$ orthonormal bases for $\boldsymbol{\mathcal{P}}_m$. Then,

$$s_m = \arg \min_{k \in \mathcal{S}_m} \sum_{j=1}^{m-1} \left| \mathbf{v}_k \mathbf{\Phi}_j^H \right|. \tag{5.12}$$

In the above algorithm, the user selection is solely performed based on the scattered component of the channel. First, the users with scattered channel gains above the threshold t are candidated. After that, the algorithm tries to find a set of semi-orthogonal channel vectors out of the candidate users. To this end, at each step of the algorithm, the user whose scattered channel vector is the most orthogonal to the sub-space spanned by the previously selected users' scattered channel vectors is selected. After selecting the users, the BS performs zero-forcing beam-forming on the (whole) channel vectors of the selected users. Defining $\mathbb{H} \triangleq \left[\mathbf{H}_{s_1}^T | \cdots | \mathbf{H}_{s_M}^T\right]^T$ and $\mathbf{u} = [u_1, \cdots, u_M]^T$ as the information vector for the selected users, we have

$$\mathbf{x} = \mathbb{H}^{-1}\mathbf{u}. \tag{5.13}$$

Therefore, the achievable sum-rate of this scheme can be written as

$$\mathcal{R} = M\mathbb{E}_{\mathbb{H}}\left\{\ln\left(1 + \frac{P}{\mathrm{Tr}\left\{[\mathbb{H}^H\mathbb{H}]^{-1}\right\}}\right)\right\}. \tag{5.14}$$

Defining $\mathfrak{B}$ as the event that $L \triangleq |\mathcal{S}| > \ln N$, $\mathfrak{C}$ as $\delta(\mathbb{G}^H) > 1 + 2M(\ln N)^{-\frac{1}{2(M-1)}}$, and $\mathfrak{D}$ as the event that $\|\mathbf{G}\|_{\max}^2 \leq t^+$, where $\delta(\mathbf{A})$ denotes the orthogonality defect [37] of $\mathbf{A}$, $\|\mathbf{G}\|_{\max}^2 \triangleq \max_k \|\mathbf{G}_k\|^2$, and $t^+ \triangleq \ln N + M\ln\ln N$, we have

$$\begin{aligned}
\mathcal{R} &= M\mathbb{E}_{\mathbb{H}|\mathfrak{B},\mathfrak{C},\mathfrak{D}}\left\{\ln\left(1 + \frac{P}{\mathrm{Tr}\left\{[\mathbb{H}^H\mathbb{H}]^{-1}\right\}}\right)\middle|\,\mathfrak{B},\mathfrak{C},\mathfrak{D}\right\}\Pr\{\mathfrak{B},\mathfrak{C},\mathfrak{D}\} + \\
&\quad M\mathbb{E}_{\mathbb{H}|\mathfrak{B}^C\cup\mathfrak{C}^C\cup\mathfrak{D}^C}\left\{\ln\left(1 + \frac{P}{\mathrm{Tr}\left\{[\mathbb{H}^H\mathbb{H}]^{-1}\right\}}\right)\middle|\,\mathfrak{B}^C\cup\mathfrak{C}^C\cup\mathfrak{D}^C\right\} \times \\
&\quad \Pr\{\mathfrak{B}^C\cup\mathfrak{C}^C\cup\mathfrak{D}^C\} \\
&\geq M\mathbb{E}_{\mathbb{H}|\mathfrak{B},\mathfrak{C},\mathfrak{D}}\left\{\ln\left(1 + \frac{P}{\mathrm{Tr}\left\{[\mathbb{H}^H\mathbb{H}]^{-1}\right\}}\right)\middle|\,\mathfrak{B},\mathfrak{C},\mathfrak{D}\right\}\Pr\{\mathfrak{B},\mathfrak{C},\mathfrak{D}\} \\
&\geq \left(M\ln P - M\mathbb{E}_{\mathbb{H}|\mathfrak{B},\mathfrak{C}}\left\{\ln\left(\mathrm{Tr}\left\{\left[\mathbb{H}^H\mathbb{H}\right]^{-1}\right\}\right)\middle|\,\mathfrak{B},\mathfrak{C},\mathfrak{D}\right\}\right)\Pr\{\mathfrak{B}\}\Pr\{\mathfrak{C}|\mathfrak{B}\} \times \\
&\quad \Pr\{\mathfrak{D}|\mathfrak{B},\mathfrak{C}\}.
\end{aligned} \tag{5.15}$$

In Appendix L, it has been shown that $\Pr\{\mathfrak{B}\} = 1 + o\left(\frac{1}{N}\right)$ and $\Pr\{\mathfrak{C}|\mathfrak{B}\} = 1 + o\left(\frac{1}{\ln N}\right)$, and $\Pr\{\mathfrak{D}|\mathfrak{B},\mathfrak{C}\} = 1 + O\left(\frac{1}{\ln^2 N}\right)$.

Defining $\mathbb{G} \triangleq \left[\mathbf{G}_{s_1}^T | \cdots | \mathbf{G}_{s_M}^T\right]^T$, and $\mathbb{B} \triangleq \left[\mathbf{b}_{s_1}^T | \cdots | \mathbf{b}_{s_M}^T\right]^T$, the term $\mathrm{Tr}\left\{\left[\mathbb{H}^H\mathbb{H}\right]^{-1}\right\}$ can be written as

$$\begin{aligned}
\mathrm{Tr}\left\{\left[\mathbb{H}^H\mathbb{H}\right]^{-1}\right\} &= \mathrm{Tr}\left\{\left[\left(\sqrt{\frac{1}{\mathcal{K}+1}}\mathbb{G}+\sqrt{\frac{M\mathcal{K}}{1+\mathcal{K}}}\mathbb{B}\right)^H\left(\sqrt{\frac{1}{\mathcal{K}+1}}\mathbb{G}+\sqrt{\frac{M\mathcal{K}}{1+\mathcal{K}}}\mathbb{B}\right)\right]^{-1}\right\} \\
&= (\mathcal{K}+1)\mathrm{Tr}\left\{\left[\mathbb{G}^H\mathbb{G}+\sqrt{M\mathcal{K}}\left(\mathbb{G}^H\mathbb{B}+\mathbb{B}^H\mathbb{G}\right)+M\mathcal{K}\mathbb{B}^H\mathbb{B}\right]^{-1}\right\} \\
&\overset{(a)}{\leq} (\mathcal{K}+1)\mathrm{Tr}\left\{\left[\mathbb{G}^H\mathbb{G}-2M\sqrt{\mathcal{K}\mathrm{Tr}\{\mathbb{G}^H\mathbb{G}\}}\mathbf{I}\right]^{-1}\right\} \\
&\overset{(b)}{\leq} (\mathcal{K}+1)\mathrm{Tr}\left\{\left[\lambda_{\min}\left\{\mathbb{G}^H\mathbb{G}\right\}\mathbf{I}-2M\sqrt{\mathcal{K}\mathrm{Tr}\{\mathbb{G}^H\mathbb{G}\}}\mathbf{I}\right]^{-1}\right\} \\
&= M(\mathcal{K}+1)\left(\lambda_{\min}\left\{\mathbb{G}^H\mathbb{G}\right\}-2M\sqrt{\mathcal{K}\mathrm{Tr}\{\mathbb{G}^H\mathbb{G}\}}\right)^{-1}. \qquad (5.16)
\end{aligned}$$

In the above equation, (a) follows from the facts that for any two positive definite matrices $\mathbf{A}$ and $\mathbf{B}$: i) if $\mathbf{A} \preceq \mathbf{B}$, then $\mathrm{Tr}\{\mathbf{A}\} \leq \mathrm{Tr}\{\mathbf{B}\}$, ii) if $\mathbf{A} \preceq \mathbf{B}$, then $\mathbf{B}^{-1} \preceq \mathbf{A}^{-1}$, iii) $\mathbb{B}^H\mathbb{B} \succeq \mathbf{0}$, and iv) $\mathbb{G}^H\mathbb{B}+\mathbb{B}^H\mathbb{G} \succeq -2\sqrt{M\mathrm{Tr}\{\mathbb{G}^H\mathbb{G}\}}\mathbf{I}$. The latter results from the fact that for any $M \times M$ matrices $\mathbf{A}$ and $\mathbf{B}$, and any $M \times 1$ unit norm vector $\mathbf{x}$, we have

$$\begin{aligned}
\mathbf{x}^H\left(\mathbf{A}^H\mathbf{B}+\mathbf{B}^H\mathbf{A}\right)\mathbf{x} &= 2\Re\{\mathbf{x}^H\mathbf{A}^H\mathbf{B}\mathbf{x}\} \\
&\geq -2\left|\mathbf{A}\mathbf{x}\right|\left|\mathbf{B}\mathbf{x}\right| \\
&\geq -2\sqrt{\lambda_{\max}(\mathbf{A})\lambda_{\max}(\mathbf{B})} \\
&\geq -2\sqrt{\mathrm{Tr}\{\mathbf{A}^H\mathbf{A}\}\mathrm{Tr}\{\mathbf{B}^H\mathbf{B}\}}, \qquad (5.17)
\end{aligned}$$

where $\lambda_{\max}(\mathbf{A})$ denotes the maximum eigenvalue of $\mathbf{A}^H\mathbf{A}$. This implies that

$$|\lambda_i(\mathbf{C})| \leq 2\sqrt{\mathrm{Tr}\{\mathbf{A}^H\mathbf{A}\}\mathrm{Tr}\{\mathbf{B}^H\mathbf{B}\}},$$

$i = 1, \cdots, M$, where $\lambda_i(\mathbf{C})$ denotes the ith singular value of $\mathbf{C} = \mathbf{A}^H\mathbf{B} + \mathbf{B}^H\mathbf{A}$. Hence,

$$\mathbf{A}^H\mathbf{B} + \mathbf{B}^H\mathbf{A} \succeq -2\sqrt{\mathrm{Tr}\{\mathbf{A}^H\mathbf{A}\}\mathrm{Tr}\{\mathbf{B}^H\mathbf{B}\}}\mathbf{I}. \tag{5.18}$$

Substituting $\mathbf{A}$ by $\mathbb{G}$ and $\mathbf{B}$ by $\mathbb{B}$, noting that $\mathrm{Tr}\{\mathbf{B}^H\mathbf{B}\} = M$, (a) follows. Also, (b) results from the fact that $\mathbb{G}^H\mathbb{G} \succeq \lambda_{\min}\left(\mathbb{G}^H\mathbb{G}\right)\mathbf{I}$.

Conditioned on $\mathfrak{B}$ and $\mathfrak{D}$, $\mathrm{Tr}\{\mathbb{G}^H\mathbb{G}\}$ is upper-bounded by Mt^+. Defining $\varepsilon \triangleq 2M(\ln N)^{-\frac{1}{2(M-1)}}$, conditioned on $\mathfrak{C}$, we have

$$\begin{aligned}
\delta\left(\mathbb{G}^H\right) = \frac{\prod_{i=1}^M \|\mathbf{G}_i\|^2}{|\mathbb{G}^H\mathbb{G}|} &< 1+\varepsilon \\
\stackrel{(a)}{\Longrightarrow} \frac{t^M}{\prod_{i=1}^M \lambda_i\left(\mathbb{G}^H\mathbb{G}\right)} &< 1+\varepsilon \\
\stackrel{(b)}{\Longrightarrow} \frac{t^M}{\lambda_{\min}\left(\mathbb{G}^H\mathbb{G}\right)\left[\frac{\mathrm{Tr}(\mathbb{G}^H\mathbb{G})-\lambda_{\min}(\mathbb{G}^H\mathbb{G})}{M-1}\right]^{M-1}} &< 1+\varepsilon \\
\stackrel{(c)}{\Longrightarrow} \frac{t^M}{\lambda_{\min}\left(\mathbb{G}^H\mathbb{G}\right)\left[\frac{Mt^+-\lambda_{\min}(\mathbb{G}^H\mathbb{G})}{M-1}\right]^{M-1}} &< 1+\varepsilon,
\end{aligned} \tag{5.19}$$

where (a) follows from the fact that conditioned on $\mathfrak{B}$, we have $\|\mathbf{G}_i\|^2 \geq t$, (b) results from the fact that knowing $\lambda_{\min}\left(\mathbb{G}^H\mathbb{G}\right)$, the product of the rest of the eigenvalues is maximized when they are equal, i.e.,

$$\prod_{i=1}^M \lambda_i\left(\mathbb{G}^H\mathbb{G}\right) \leq \lambda_{\min}\left(\mathbb{G}^H\mathbb{G}\right)\left[\frac{Mt^+ - \lambda_{\min}\left(\mathbb{G}^H\mathbb{G}\right)}{M-1}\right]^{M-1},$$

and (c) follows from the fact that conditioned on $\mathfrak{D}$, $\mathrm{Tr}\left(\mathbb{G}^H\mathbb{G}\right) < Mt^+$.

Defining $\gamma \triangleq \frac{\lambda_{\min}\left(\mathbb{G}^H\mathbb{G}\right)}{t^+}$, from the above equation, we can write

$$\frac{\gamma(M-\gamma)^{M-1}}{(M-1)^{M-1}} > \frac{(t/t^+)^M}{1+\varepsilon}. \tag{5.20}$$

Since $t = \ln N + (M-3)\ln\ln N$ and $t^+ = \ln N + M\ln\ln N$, it follows that $\frac{t}{t^+} > 1 - \frac{3\ln\ln N}{\ln N}$. Hence, using the inequality $(1-x)^n \geq 1 - nx$, for $0 \leq x \leq 1$, we have $\left(\frac{t}{t^+}\right)^M > 1 - \frac{3M\ln\ln N}{\ln N}$. Moreover, using the fact that $\frac{1}{1+\varepsilon} > 1 - \varepsilon$, the above equation can be rewritten as

$$\frac{\gamma(M-\gamma)^{M-1}}{(M-1)^{M-1}} > 1 - \psi, \tag{5.21}$$

where $\psi \triangleq \frac{3M\ln\ln N}{\ln N} + \varepsilon$. Since the function $f(\gamma) = \frac{\gamma(M-\gamma)^{M-1}}{(M-1)^{M-1}}$ is an increasing function of γ over the interval $[0, 1]$, writing the Tailor series of $f(\gamma)$ about 1, noting that $f(1) = 1$, $f'(1) = 0$, and $f''(1) = -\frac{M}{M-1}$, we have

$$\begin{aligned} \frac{\gamma(M-\gamma)^{M-1}}{(M-1)^{M-1}} &> 1 - \psi \\ \Longrightarrow \frac{M(1-\gamma)^2}{2(M-1)} &< \psi \\ \Longrightarrow \gamma > 1 - \sqrt{\frac{2(M-1)\psi}{M}}&. \end{aligned} \tag{5.22}$$

Having the fact that $\psi = O\left((\ln N)^{-\frac{1}{2(M-1)}}\right)$, the above equation yields that conditioned on $\mathfrak{B}$, $\mathfrak{C}$ and $\mathfrak{D}$,

$$\begin{aligned} \lambda_{\min}\left(\mathbb{G}^H\mathbb{G}\right) &= t^+\left[1 + O\left((\ln N)^{-\frac{1}{4(M-1)}}\right)\right] \\ &= \ln N\left[1 + O\left((\ln N)^{-\frac{1}{4(M-1)}}\right)\right], \end{aligned} \tag{5.23}$$

where the second line follows from the fact that $t^+ = \ln N + M\ln\ln N = \ln N\left[1 + O\left(\frac{\ln\ln N}{\ln N}\right)\right] = \ln N\left[1 + O\left((\ln N)^{-\frac{1}{4(M-1)}}\right)\right]$. Substituting in (5.16) yields that conditioned on $\mathfrak{B}$,

$\mathfrak{C}$, and $\mathfrak{D}$,

$$\begin{aligned}
\mathrm{Tr}\left\{\left[\mathbb{H}^H\mathbb{H}\right]^{-1}\right\} &\leq M(\mathcal{K}+1)\left(t^+\left[1+O\left((\ln N)^{-\frac{1}{4(M-1)}}\right)\right]-2M\sqrt{\mathcal{K}Mt^+}\right)^{-1} \\
&= \frac{M(\mathcal{K}+1)}{t^+}\left[1+O\left((\ln N)^{-\frac{1}{4(M-1)}}\right)+O\left(\frac{1}{\sqrt{t^+}}\right)\right] \\
&\overset{(a)}{=} \frac{M(\mathcal{K}+1)}{t^+}\left[1+O\left((\ln N)^{-\frac{1}{4(M-1)}}\right)\right] \\
&= \frac{M(\mathcal{K}+1)}{\ln N}\left[1+O\left((\ln N)^{-\frac{1}{4(M-1)}}\right)\right], \qquad (5.24)
\end{aligned}$$

where (a) follows from the fact that $\frac{1}{\sqrt{t^+}} = O\left(\frac{1}{\sqrt{\ln N}}\right) = o\left((\ln N)^{-\frac{1}{4(M-1)}}\right)$. Substituting in (5.15) yields

$$\begin{aligned}
\mathcal{R} &\geq M\ln\left(\frac{P\ln N}{M(1+\mathcal{K})}\left[1+O\left((\ln N)^{-\frac{1}{4(M-1)}}\right)\right]\right)\Pr\{\mathfrak{B},\mathfrak{C},\mathfrak{D}\} \\
&\overset{(b)}{=} \left[M\ln\left(\frac{P\ln N}{M(1+\mathcal{K})}\right)+O\left((\ln N)^{-\frac{1}{4(M-1)}}\right)\right]\left[1+O\left(\frac{1}{\ln N}\right)\right] \\
&= M\ln\left(\frac{P\ln N}{M(1+\mathcal{K})}\right)+O\left((\ln N)^{-\frac{1}{4(M-1)}}\right). \qquad (5.25)
\end{aligned}$$

Since $\mathcal{K} = o(\ln K)$, it follows that $\ln\left(\frac{P\ln N}{M(1+\mathcal{K})}\right) = \ln\left(1+\frac{P\ln N}{M(1+\mathcal{K})}\right)+o(1)$. Noting this fact and comparing the above lower-bound with the upper-bound derived in (5.10) completes the proof of Theorem 5.1.

■

5.3.2 $\mathcal{K} = \omega(\ln N)$

Co-located transmit antennas

In this scenario, the specular components from all transmit antennas to each receiver are equal. In other words, $\mathbf{b}_k = \frac{e^{i\theta_k}}{\sqrt{M}}\mathbf{1}_M$, where $\mathbf{1}_M$ the all-one vector with size M. However, the scattered component of all users' channels follow the circularly symmetric complex Gaussian distribution. The following theorem gives the capacity of MIMO-BC in this scenario:

Theorem 5.2 *The capacity of MIMO-BC in the case of* $\mathcal{K} = \omega(\ln N)$ *and co-located transmit antennas scales as*

$$\mathcal{C}_{\text{sum}} = \ln(1 + MP) + o(1), \tag{5.26}$$

which is achievable by TDMA.

Proof - Like the proof of Theorem 5.1, we first give an upper-bound on the sum-rate capacity and then, by giving an achievable rate which is asymptotically equal to the upper-bound the theorem is proved.

Upper-bound: Writing the sum-rate capacity of MIMO-BC from (5.3), we have

$$\begin{aligned}
\mathcal{C}_{\text{sum}} &= \mathbb{E}_{\mathcal{H}}\left\{\max_{\substack{P_k\\ \sum P_k=P}} \ln\left|\mathbf{I}_M+\sum_{k=1}^{N}\mathbf{H}_k^H P_k\mathbf{H}_k\right|\right\}\\
&= \mathbb{E}_{\mathfrak{G}}\left\{\max_{\substack{P_k\\ \sum P_k=P}} \ln\left|\mathbf{I}_M+\sum_{k=1}^{N}\left[\sqrt{1-r_k}\mathbf{G}_k+\sqrt{r_kM}\mathbf{b}_k\right]^H P_k\left[\sqrt{1-r_k}\mathbf{G}_k+\sqrt{r_kM}\mathbf{b}_k\right]\right|\right\}\\
&= \mathbb{E}_{\mathfrak{G}}\Bigg\{\max_{\substack{P_k\\ \sum P_k=P}} \ln\left|\mathbf{I}_M+rM\sum_{k=1}^{N}\mathbf{b}_k^H P_k\mathbf{b}_k\right|+\\
&\qquad \ln\left|\mathbf{I}_M+\left(\sqrt{r(1-r)M}\sum_{k=1}^{N}\left[\mathbf{G}_k^H P_k\mathbf{b}_k+\mathbf{b}_k^H P_k\mathbf{G}_k\right]+(1-r)\sum_{k=1}^{N}\mathbf{b}_k^H P_k\mathbf{b}_k\right|\right)\mathbf{P}\Bigg\}\\
&= \mathbb{E}_{\mathfrak{G}}\Bigg\{\max_{\substack{P_k\\ \sum P_k=P}} \ln\left|\mathbf{I}_M+r\mathbf{1}_M^H\left(\sum_{k=1}^{N}P_k\right)\mathbf{1}_M\right|+\\
&\qquad \ln\left|\mathbf{I}_M+\left(\sqrt{r(1-r)M}\sum_{k=1}^{N}\left[\mathbf{G}_k^H P_k\mathbf{b}_k+\mathbf{b}_k^H P_k\mathbf{G}_k\right]+(1-r)\sum_{k=1}^{N}\mathbf{b}_k^H P_k\mathbf{b}_k\right)\mathbf{P}\right|\Bigg\}\\
&\overset{(a)}{\le} \ln\left(1+rMP\right)+\\
&\qquad \mathbb{E}_{\mathfrak{G}}\left\{\max_{\substack{P_k\\ \sum P_k=P}} \ln\left|\mathbf{I}_M+\sqrt{r(1-r)M}\sum_{k=1}^{N}\left[\mathbf{G}_k^H P_k\mathbf{b}_k+\mathbf{b}_k^H P_k\mathbf{G}_k\right]+(1-r)\sum_{k=1}^{N}\mathbf{G}_k^H P_k\mathbf{G}_k\right|\right\}\\
&\overset{(b)}{\le} \ln\left(1+rP\right)+\\
&\qquad M\mathbb{E}_{\mathfrak{G}}\left\{\max_{\substack{P_k\\ \sum P_k=P}} \ln\left(1+\frac{\sum_{k=1}^{N}2P_k\left(\sqrt{r(1-r)M}\,\mathrm{Tr}\left\{\mathbf{G}_k^H\mathbf{b}_k\right\}+(1-r)\mathrm{Tr}\left\{\mathbf{G}_k^H\mathbf{G}_k\right\}\right)}{M}\right)\right\},
\end{aligned} \tag{5.27}$$

where $\mathfrak{G} \triangleq \{\mathbf{G}_k\}_{k=1}^{N}$ and $\mathbf{P} \triangleq \left(\mathbf{I}_M + rP\mathbf{1}_M^H\mathbf{1}_M\right)^{-1}$. In the above equation, (a) follows from i) $|\mathbf{I}+\mathbf{AB}| = |\mathbf{I}+\mathbf{BA}|$, and hence, $\left|\mathbf{I}_M + rP\mathbf{1}_M^H\mathbf{1}_M\right| = 1 + rP\mathbf{1}_M\mathbf{1}_M^H$, noting that $\mathbf{1}_M\mathbf{1}_M^H = M$, and ii) as $\mathbf{P} \preceq \mathbf{I}$, we have

$$\ln\left|\mathbf{I}_M + \left(\sqrt{r(1-r)M}\sum_{k=1}^{N}\left[\mathbf{G}_k^H P_k \mathbf{b}_k + \mathbf{b}_k^H P_k \mathbf{G}_k\right] + (1-r)\sum_{k=1}^{N}\mathbf{b}_k^H P_k \mathbf{b}_k\right)\mathbf{P}\right| \leq$$

$$\ln\left|\mathbf{I}_M + \sqrt{r(1-r)M}\sum_{k=1}^{N}\left[\mathbf{G}_k^H P_k \mathbf{b}_k + \mathbf{b}_k^H P_k \mathbf{G}_k\right] + (1-r)\sum_{k=1}^{N}\mathbf{b}_k^H P_k \mathbf{b}_k\right|.$$

Moreover, (b) results from the fact that for any $\mathbf{A} \succeq 0$, $|\mathbf{A}| \leq \left(\frac{\mathrm{Tr}\{\mathbf{A}\}}{M}\right)^M$. Noting that $\mathrm{Tr}\{\mathbf{G}_k^H\mathbf{b}_k\} = \mathrm{Tr}\{\mathbf{b}_k^H\mathbf{G}_k\} \leq \|\mathbf{G}_k\|\|\mathbf{b}_k\| = \frac{1}{\sqrt{M}}\|\mathbf{G}_k\|\|$, and $\mathrm{Tr}\{\mathbf{G}_k^H\mathbf{G}_k\} = \|\mathbf{G}_k\|^2$, the second term in the right hand side of the above equation, denoted by R_2, can be further upper-bounded as follows:

$$\begin{aligned}
R_2 &\leq M\mathbb{E}_{\mathfrak{G}}\left\{\max_{\substack{P_k \\ \sum P_k = P}} \ln\left(1 + \frac{\sum_{k=1}^N 2P_k\left(\sqrt{r(1-r)}\|\mathbf{G}_k\| + (1-r)\|\mathbf{G}_k\|^2\right)}{M}\right)\right\} \\
&\overset{(a)}{=} M\mathbb{E}\left\{\ln\left(1 + \frac{P\left(\sqrt{r(1-r)}\|\mathbf{G}\|_{\max} + (1-r)\|\mathbf{G}\|_{\max}^2\right)}{M}\right)\right\} \\
&\overset{(b)}{\leq} M\ln\left(1 + \frac{P\left(\frac{\mathbb{E}\{\|\mathbf{G}\|_{\max}\}}{\sqrt{1+\mathcal{K}}} + \frac{\mathbb{E}\{\|\mathbf{G}\|_{\max}^2\}}{1+\mathcal{K}}\right)}{M}\right) \\
&\overset{(c)}{=} M\ln\left(1 + \frac{P\left(\frac{O(\sqrt{\ln N})}{\sqrt{1+\mathcal{K}}} + \frac{O(\ln N)}{1+\mathcal{K}}\right)}{M}\right) \\
&\overset{(d)}{=} o(1),
\end{aligned} \tag{5.28}$$

where $\|\mathbf{G}\|_{\max} = \max_k \|\mathbf{G}_k\|$. In the above equation, (a) results from the fact that the solution to the optimization problem in (5.28) is to allocate all the transmit power to the user with the highest scattered gain. (b) follows from i) the concavity of ln function along with the Jensen's inequality which enables us to move the expectation inside the ln, and ii) the fact that $r = \frac{\mathcal{K}}{1+\mathcal{K}}$, which incurs that $r \leq 1$, and $1 - r = \frac{1}{1+\mathcal{K}}$. (c) results from [26], in which it is shown that $\|\mathbf{G}\|^2_{\max} = \ln N + O(\ln \ln N)$ with probability one, which incurs that $\mathbb{E}\{\|\mathbf{G}\|^2_{\max}\} = O(\ln N)$ and $\mathbb{E}\{\|\mathbf{G}\|_{\max}\} = O(\sqrt{\ln N})$, and finally, (d) follows from the assumption of $\mathcal{K} = \omega(\ln N)$ and the fact that $\ln(1 + o(1)) = o(1)$. Substituting R_2 in (5.27) yields

$$
\begin{aligned}
\mathcal{C}_{\text{sum}} &\leq \ln(1 + rPM) + o(1) \\
&\leq \ln(1 + PM) + o(1), \qquad (5.29)
\end{aligned}
$$

where the last line comes from the fact that $r \leq 1$.

Achievability - In order to show that the sum-rate given in (5.26) is achievable, we propose a random selection scheme, in which the transmitter selects a user at random and communicates with that user. Therefore, the maximum achievable rate is equal to the capacity of a MISO Rician channel, expressed as bellow:

$$
\begin{aligned}
\mathcal{R} &= \mathbb{E}_{\mathbf{H}_k}\left\{ \max_{\substack{\mathbf{Q} \\ \text{Tr}\{\mathbf{Q}\} \leq P}} \ln\left(1 + \mathbf{H}_k \mathbf{Q} \mathbf{H}_k^H\right) \right\} \\
&= \mathbb{E}\left\{\ln\left(1 + P\|\mathbf{H}_k\|^2\right)\right\} \\
&\geq \mathbb{E}\left\{\ln\left(1 + P\left|\sqrt{rM} - \sqrt{1-r}\|\mathbf{G}_k\|\right|^2\right)\right\}. \qquad (5.30)
\end{aligned}
$$

Let us define $\mathfrak{E}$ as the event that $\|\mathbf{G}_k\|^2 < \ln N$. $\mathcal{R}$ can be lower-bounded as

$$
\begin{aligned}
\mathcal{R} &\geq \mathbb{E}\left\{\ln\left(1 + P\left|\sqrt{rM} - \sqrt{1-r}\|\mathbf{G}_k\|\right|^2\right) \middle| \mathfrak{E}\right\} \Pr\{\mathfrak{E}\} \\
&\stackrel{(a)}{=} \ln\left(1 + PrM + o(1)\right) \Pr\{\mathfrak{E}\} \\
&\stackrel{(b)}{=} \ln(1 + PM) + o(1). \qquad (5.31)
\end{aligned}
$$

In the above equation, (a) follows from the assumption of $\mathcal{K} = \omega(\ln N)$, which implies that conditioned on $\mathfrak{E}$, $\sqrt{1-r}\|\mathbf{G}_k\| = \frac{\|\mathbf{G}_k\|}{\sqrt{\mathcal{K}+1}} = o(1)$. (b) follows from i) as $\|\mathbf{G}_k\|^2$ has Chi-Square distribution with $2M$ degrees of freedom, $\Pr\{\mathfrak{E}\} \sim$

$\frac{\ln^M N}{(M-1)!N} = o(1)$ and ii) as $r = \frac{\mathcal{K}}{\mathcal{K}+1}$ and $\mathcal{K} = \omega(\ln N)$, we have $r = 1 + o\left(\frac{1}{\ln N}\right)$. This completes the proof of achievability and hence, the proof of Theorem 5.2.

■

The case of isotropic specular components

In this case, it is assumed that the specular component of all users' channels, i.e., $\mathbf{b}_k$, $k = 1, \cdots, N$, are isotropically distributed in the unit sphere. The difference between this case and the previous case is that in the case of co-located transmit antennas, there is only one available coordinate in the system (the coordinate of $\mathbf{1}_M$) for transmission, and as a result, we don't have the M-fold capacity increase, as we expect in Gaussian MIMO-BC. However, in this case, by wisely selecting the users one can achieve the M-fold capacity increase. The following theorem gives the capacity in this case:

Theorem 5.3 *The capacity of Rician MIMO-BC in the case of* $\mathcal{K} = \omega(\ln N)$ *and isotropic specular components is equal to*

$$\mathcal{C}_{\text{sum}} = M \ln(1 + P) + o(1). \tag{5.32}$$

Proof - *Upper-bound:* In [26], Appendix B, an upper-bound on the capacity of MIMO-BC is given as

$$\begin{aligned}
\mathcal{C}_{\text{sum}} &\leq M\mathbb{E}\left\{\ln\left(1+\frac{P}{M}\|\mathbf{H}\|_{\max}^2\right)\right\} \\
&\leq M\mathbb{E}\left\{\ln\left(1+\frac{P}{M}\left|\sqrt{rM}+\sqrt{1-r}\|\mathbf{G}\|_{\max}\right|^2\right)\right\} \\
&\overset{(a)}{\leq} M\ln\left(1+\frac{P}{M}\mathbb{E}\left\{\left|\sqrt{rM}+\sqrt{1-r}\|\mathbf{G}\|_{\max}\right|^2\right\}\right) \\
&\overset{(b)}{=} M\ln\left(1+\frac{P}{M}\left|\sqrt{rM}+o(1)\right|^2\right) \\
&= M\ln(1+rP)+o(1) \\
&\overset{(c)}{=} M\ln(1+P)+o(1). \qquad (5.33)
\end{aligned}$$

In the above equation, (a) follows from the concavity of ln function along with the Jensen's inequality, (b) results from the fact that $\|\mathbf{G}\|_{\max} = O(\ln N)$ and since $1-r = \frac{1}{1+\mathcal{K}} = o\left(\frac{1}{\ln N}\right)$, we have $\sqrt{1-r}\|\mathbf{G}\|_{\max} = o(1)$, and (c) results from $r = 1+o(1)$.

Achievability; Scheduling based on specular component Consider the following algorithm:

Algorithm 2

- select one user at random. Call this user s_1, and define $\mathcal{S}_1 \triangleq \mathcal{S} - \{s_1\}$.
- For $m = 2$ to M, repeat the following:
  - Denote the set of selected users up to the $(m-1)$th step as $\mathcal{A}_m \triangleq \{s_1, \cdots, s_{m-1}\}$. Define $\mathcal{S}_m \triangleq \mathcal{S} - \mathcal{A}_m$.
  - Define $\boldsymbol{\mathcal{P}}_m$ as the sub-space spanned by the specular channel components of the users selected in the previous steps, i.e., $\{\mathbf{b}_{s_j}\}_{j=1}^{m-1}$.

– Let $\{\mathbf{\Phi}_j\}_{j=1}^{m-1}$ be $m-1$ orthonormal bases for $\boldsymbol{\mathcal{P}}_m$. Then,

$$s_m = \arg\min_{k\in\mathcal{S}_m} \sum_{j=1}^{m-1} \left|\mathbf{b}_k \mathbf{\Phi}_j^H\right|. \tag{5.34}$$

- After selecting the users, the BS performs zero-forcing beam-forming on the (whole) channel vectors of the selected users. Defining $\mathbb{H} \triangleq \left[\mathbf{H}_{s_1}^T|\cdots|\mathbf{H}_{s_M}^T\right]^T$ and $\mathbf{u} = [u_1, \cdots, u_M]^T$ as the information vector for the selected users, we have

$$\mathbf{x} = \mathbb{H}^{-1}\mathbf{u}. \tag{5.35}$$

Defining the event $\mathfrak{F} \triangleq \{\delta(\mathbb{B}^H) < 1+\epsilon\}$ and $\mathfrak{Q} \triangleq \left\{\mathrm{Tr}\{\mathbb{G}^H\mathbb{G}\} < \ln N\right\}$, where $\mathbb{B} = \left[\mathbf{b}_{s_1}^T|\cdots|\mathbf{b}_{s_M}\right]^T$, $\mathbb{G} = \left[\mathbf{G}_{s_1}^T|\cdots|\mathbf{G}_{s_M}\right]^T$, and $\epsilon \triangleq 2MN^{-\frac{1}{2(M-1)}}$, similar to (5.15), we have

$$\mathcal{R} \geq M\mathbb{E}_{\mathbb{H}|\mathfrak{F},\mathfrak{Q}} \left\{ \ln\left(1 + \frac{P}{\mathrm{Tr}\left\{[\mathbb{H}^H\mathbb{H}]^{-1}\right\}}\right) \middle| \mathfrak{F}, \mathfrak{Q} \right\} \Pr\{\mathfrak{F}, \mathfrak{Q}\}. \tag{5.36}$$

Since $\mathbf{b}_k$'s are isotropic unit vectors, $\Pr\{\mathfrak{F}\}$ can be computed similar to $\Pr\{\mathfrak{C}|\mathfrak{B}\}$, which is performed in Appendix L, and shown to be $1 + o(\frac{1}{N})$ [4]. Moreover, since the scattered component is not considered in the selection, it follows that $\mathbb{G}$ can be considered as an $M \times M$ circularly symmetric complex Gaussian matrix, and as a result, $\mathrm{Tr}\{\mathbb{G}^H\mathbb{G}\}$ has Chi-Square distribution with $2M^2$ degrees of freedom which implies that $\Pr\left\{\mathrm{Tr}\{\mathbb{G}^H\mathbb{G}\} > \ln N\right\} = \frac{[\ln N]^{M^2-1}e^{-\ln N}}{(M-1)!}[1+o(1)] = O\left(\frac{[\ln N]^{M^2-1}}{N}\right)$. Therefore, $\Pr\{\mathfrak{Q}\} = 1 + O\left(\frac{[\ln N]^{M^2-1}}{N}\right) = o\left(\frac{1}{\sqrt{N}}\right)$. Having computed $\Pr\{\mathfrak{F}\}$ and $\Pr\{\mathfrak{Q}\}$, noting that as the specular and scattered components of the channels are independent, $\mathfrak{F}$ and and $\mathfrak{Q}$ are also independent, we have

$$\Pr\{\mathfrak{F}, \mathfrak{Q}\} = 1 + o\left(\frac{1}{\sqrt{N}}\right). \tag{5.37}$$

[4] To this end, it is sufficient to substitute $\ln N$ by N in the steps of proof.

Similar to (5.16), $\text{Tr}\left\{\left[\mathbb{H}^H\mathbb{H}\right]^{-1}\right\}$ can be upper-bounded as

$$\begin{aligned}\text{Tr}\left\{\left[\mathbb{H}^H\mathbb{H}\right]^{-1}\right\} &\leq (\mathcal{K}+1)\text{Tr}\left\{\left(M\mathcal{K}\mathbb{B}^H\mathbb{B} - 2M\sqrt{\mathcal{K}\text{Tr}\{\mathbb{G}^H\mathbb{G}\}}\mathbf{I}\right)^{-1}\right\} \\ &\leq (\mathcal{K}+1)\text{Tr}\left\{\left(M\mathcal{K}\lambda_{\min}\left(\mathbb{B}^H\mathbb{B}\right)\mathbf{I} - 2M\sqrt{\mathcal{K}\text{Tr}\{\mathbb{G}^H\mathbb{G}\}}\mathbf{I}\right)^{-1}\right\} \\ &= \left(1+\frac{1}{\mathcal{K}}\right)\left(\lambda_{\min}\left(\mathbb{B}^H\mathbb{B}\right) - 2\sqrt{\frac{\text{Tr}\{\mathbb{G}^H\mathbb{G}\}}{\mathcal{K}}}\right)^{-1}. \end{aligned} \tag{5.38}$$

Conditioned on $\mathfrak{Q}$, we have $\text{Tr}\{\mathbb{G}^H\mathbb{G}\} < \ln N$, and since $\mathcal{K} = \omega(\ln N)$, it follows that $2\sqrt{\frac{\text{Tr}\{\mathbb{G}^H\mathbb{G}\}}{\mathcal{K}}} = o(1)$. Moreover, conditioned on $\mathfrak{F}$, i.e., $\delta(\mathbb{B}^H) < 1+\epsilon$, and following the equations (5.19)-(5.23) with $t = t^+ = 1$, we have

$$\lambda_{\min}\left(\mathbb{B}^H\mathbb{B}\right) = 1 + O\left(N^{-\frac{1}{4(M-1)}}\right). \tag{5.39}$$

Combining the above equation with (5.38) yields

$$\text{Tr}\left\{\left[\mathbb{H}^H\mathbb{H}\right]^{-1}\right\} \leq 1 + o(1). \tag{5.40}$$

Substituting in (5.36), noting (5.37), we have

$$\mathcal{R} \geq M\ln(1+P) + o(1), \tag{5.41}$$

which completes the proof of Theorem 5.3.

■

Remark - Comparing the sum-rate capacity of the system in the two cases of co-located transmit antennas and isotropic specular components when $\mathcal{K} = \omega(\ln N)$, it follows that in the first case, the capacity grows logarithmically with M, while in the second case it scales linearly with M. Moreover, since $(1+x)^M > 1 + Mx$, $\forall x, M$, it follows that

$$\mathcal{C}_{\text{sum}}^{isotropic} \geq \mathcal{C}_{\text{sum}}^{co-located}. \tag{5.42}$$

5.3.3 $\mathcal{K} = \Theta(\ln N)$, Isotropic specular components

The following theorem gives the asymptotic sum-rate in this case:

Theorem 5.4 *The sum-rate capacity of the system in the case of* $\mathcal{K} = \Theta(\ln N)$ *and isotropic specular components can be obtained as*

$$\mathcal{C}_{\text{sum}} = M \ln \left(1 + P \left[1 + \sqrt{\frac{\eta}{M}} \right]^2 \right) + o(1), \tag{5.43}$$

where $\eta \triangleq \lim_{N\to\infty} \frac{\ln N}{\mathcal{K}}$.

Proof - *Upper-bound:* Similar to (5.33), we can write

$$\begin{aligned}
\mathcal{C}_{\text{sum}} &\leq M \ln \left(1 + \frac{P}{M} \mathbb{E} \left\{ \left| \sqrt{rM} + \sqrt{1-r} \|\mathbf{G}\|_{\max} \right|^2 \right\} \right) \\
&= M \ln \left(1 + \frac{P}{M} \mathbb{E} \left\{ \left| \sqrt{rM} + \sqrt{\frac{\|\mathbf{G}\|^2_{\max}}{1+\mathcal{K}}} \right|^2 \right\} \right) \\
&\overset{(a)}{=} M \ln \left(1 + \frac{P}{M} \left[\sqrt{rM} + \sqrt{\eta[1+o(1)]} \right]^2 \right) \\
&\overset{(b)}{=} M \ln \left(1 + P \left[1 + \sqrt{\frac{\eta}{M}} \right]^2 \right) + o(1),
\end{aligned} \tag{5.44}$$

where (a) follows from the facts that i) $\|\mathbf{G}\|^2_{\max} = \ln N + o(\ln N)$, with probability one, and ii) $\eta \sim \frac{\ln N}{\mathcal{K}+1}$, and (b) results from the fact that as $\mathcal{K} = \Theta(\ln N)$, we have $r \sim 1$.

Achievability; Scheduling based on both specular and scattered components: Consider the following algorithm:

Algorithm 3:

- Select the thresholds $t = \ln N - 2.5 \ln \ln N$ and $\gamma = \frac{2}{\ln N}$.

- Construct the following set:

$$\mathcal{S}_0 \triangleq \{k \mid \Re(\mathbf{v}_k, \mathbf{b}_k) \geq 1 - \gamma\}, \tag{5.45}$$

  where $\Re(x)$ denotes the real part of x, and $\mathbf{v}_k \triangleq \frac{\mathbf{G}_k}{\|\mathbf{G}_k\|}$, $k = 1, \cdots, N$.

- Among the users in the following set:

$$\mathcal{S} \triangleq \{k \in \mathcal{S}_0 | \ \|\mathbf{G}_k\|^2 > t\}, \tag{5.46}$$

  select one user at random. Call this user s_1, and define $\mathcal{S}_1 \triangleq \mathcal{S} - \{s_1\}$.

- For $m = 2$ to M, repeat the following:

  – Denote the set of selected users up to the $(m-1)$th step as $\mathcal{A}_m \triangleq \{s_1, \cdots, s_{m-1}\}$. Define $\mathcal{S}_m \triangleq \mathcal{S} - \mathcal{A}_m$.

  – Define $\boldsymbol{\mathcal{P}}_m$ as the sub-space spanned by the scattered channel components of the users selected in the previous steps, i.e., $\{\mathbf{v}_{s_j}\}_{j=1}^{m-1}$.

  – Let $\{\mathbf{\Phi}_j\}_{j=1}^{m-1}$ be $m-1$ orthonormal bases for $\boldsymbol{\mathcal{P}}_m$. Then,

$$s_m = \arg \min_{k \in \mathcal{S}_m} \sum_{j=1}^{m-1} \left| \mathbf{v}_k \mathbf{\Phi}_j^H \right|. \tag{5.47}$$

  – After selecting the users, the BS performs zero-forcing beam-forming on the (whole) channel vectors of the selected users, i.e.,

$$\mathbf{x} = \mathbb{H}^{-1}\mathbf{u}. \tag{5.48}$$

As can be observed, the above algorithm is very similar to Algorithm 1, with the difference in putting an extra constraint for the user selection, which is, the scattered and specular components of the selected users must be almost in the same direction.

Defining the events $\mathfrak{B}$, $\mathfrak{C}$, and $\mathfrak{D}$ as in the proof of the achievability part of Theorem 5.1, similar to (5.15), we have

$$\begin{aligned}\mathcal{R} \quad \geq \quad & M\mathbb{E}_{\mathbb{H}|\mathfrak{B},\mathfrak{C},\mathfrak{D}}\left\{\ln\left(1+\frac{P}{\mathrm{Tr}\left\{[\mathbb{H}^H\mathbb{H}\}]^{-1}\right\}}\right)\middle|\mathfrak{B},\mathfrak{C},\mathfrak{D}\right\}\times \\ & \Pr\{\mathfrak{B}\}\Pr\{\mathfrak{C}|\mathfrak{B}\}\Pr\{\mathfrak{D}|\mathfrak{B},\mathfrak{C}\}. \end{aligned} \tag{5.49}$$

$\Pr\{\mathfrak{C}|\mathfrak{B}\}$ and $\Pr\{\mathfrak{D}|\mathfrak{B},\mathfrak{C}\}$ can be computed from Appendix L as $1+o\left(\frac{1}{\ln N}\right)$ and $1+O\left(\frac{1}{\ln N}\right)$, respectively. For computing $\Pr\{\mathfrak{B}\}$, we first compute $\xi \triangleq \Pr\{k\in\mathcal{S}\}$ as follows:

$$\begin{aligned}\xi \quad &= \quad \Pr\{\Re(\mathbf{v}_k,\mathbf{b}_k)>1-\gamma, \|\mathbf{G}_k\|^2>t\} \\ &\overset{(a)}{=} \quad \Pr\{\Re(\mathbf{v}_k,\mathbf{b}_k)>1-\gamma\}\Pr\{\|\mathbf{G}_k\|^2>t\} \\ &\overset{(b)}{\geq} \quad \Pr\{z(\mathbf{v}_k,\mathbf{b}_k)>1-0.5\gamma\}\Pr\left\{\cos\left[\Theta(\mathbf{v}_k\mathbf{b}_k^H)\right]>1-0.5\gamma\right\}\Pr\{\|\mathbf{G}_k\|^2>t\} \\ &\overset{(c)}{=} \quad (0.5\gamma)^{M-1}\frac{\sqrt{\gamma}}{\pi}\frac{t^{M-1}e^{-t}}{(M-1)!}\left[1+O\left(1/t\right)\right] && (5.50) \\ &\overset{(d)}{=} \quad \frac{\sqrt{2}\ln^2 N}{\pi(M-1)!N}\left[1+O\left(\frac{\ln\ln N}{\ln N}\right)\right], && (5.51)\end{aligned}$$

where $\Theta(x)$ denotes the phase of a complex number x, and for any $1\times M$ vectors $\mathbf{u}$ and $\mathbf{v}$, $z(\mathbf{u},\mathbf{v})$ is defined as $\frac{|\mathbf{u}\mathbf{v}^H|^2}{\|\mathbf{u}\|^2\|\mathbf{v}\|^2}$. In the above equation, (a) follows from the facts that i) $z(\mathbf{v}_k,\mathbf{b}_k)$ is a function of only the direction of $\mathbf{G}_k$ and for Gaussian vectors, norm and direction are independent, and ii) $\mathbf{b}_k$ and $\mathbf{v}_k$ are independent of each other. (b) comes from the fact that since $\Re(\mathbf{v}_k,\mathbf{b}_k)=z(\mathbf{v}_k,\mathbf{b}_k)\cos\left[\Theta(\mathbf{v}_k\mathbf{b}_k^H)\right]$, having the events $z(\mathbf{v}_k,\mathbf{b}_k)>1-0.5\gamma$ and $\cos\left[\Theta(\mathbf{v}_k\mathbf{b}_k^H)\right]>1-0.5\gamma$ yields $\Re(\mathbf{v}_k,\mathbf{b}_k)>1-\gamma$, and also the fact that the norm and phase of $\mathbf{v}_k\mathbf{b}_k^H$ are independent of each other. (c) results from i) as $\mathbf{b}_k$ and $\mathbf{v}_k$ are two independent isotropic unit vectors, the pdf of $Z \triangleq z(\mathbf{v}_k, \mathbf{b}_k)$ is computed in Lemma 2.5, as

$$p_Z(z) = (M-1)(1-z)^{M-2} \quad \Longrightarrow \quad \Pr\{Z > 1-\gamma\} = \gamma^{M-1}, \tag{5.52}$$

ii) for small enough x, $\cos(x) \approx 1 - \frac{x^2}{2}$, and hence, the event $\cos\left[\Theta(\mathbf{v}_k \mathbf{b}_k^H)\right] > 1 - 0.5\gamma$ is equivalent to $|\Theta(\mathbf{v}_k, \mathbf{b}_k)| < \sqrt{\gamma}$, and since $\Theta(\mathbf{v}_k, \mathbf{b}_k)$ is uniformly distributed between 0 and 2π, we have $\Pr\left\{\cos\left[\Theta(\mathbf{G}_k \mathbf{b}_k^H)\right] > 1 - 0.5\gamma\right\} \approx \frac{\sqrt{\gamma}}{\pi}$, and iii) Since $\|\mathbf{G}_k\|^2$ has Chi-square distribution with $2M$ degrees of freedom [32], it can be shown that

$$\Pr\{\|\mathbf{G}_k\|^2 > t\} = \frac{t^{M-1}e^{-t}}{(M-1)!}\left[1 + O\left(1/t\right)\right]. \tag{5.53}$$

Finally, (d) follows from substitution of $t = \ln N - 2.5 \ln\ln N$ and $\gamma = \frac{2}{\ln N}$ in (5.50).

Similar to (5.16), $\mathrm{Tr}\left\{\left[\mathbb{H}^H\mathbb{H}\right]^{-1}\right\}$ can be written as

$$\mathrm{Tr}\left\{\left[\mathbb{H}^H\mathbb{H}\right]^{-1}\right\} \;=\; \mathrm{Tr}\left\{\left[\frac{1}{\mathcal{K}+1}\mathbb{G}^H\mathbb{G} + \frac{\sqrt{M\mathcal{K}}}{\mathcal{K}+1}\left(\mathbb{G}^H\mathbb{B} + \mathbb{B}^H\mathbb{G}\right) + \frac{M\mathcal{K}}{\mathcal{K}+1}\mathbb{B}^H\mathbb{B}\right]^{-1}\right\}. \tag{5.54}$$

In Appendix M, it has been shown that $z(\mathbf{G}_{s_i}, \mathbf{G}_{s_j}) \leq \epsilon$, for $i \neq j$, where $\epsilon = 1 - \frac{1}{1+2M(\ln N)^{-\frac{1}{2(M-1)}}} \approx 2M(\ln N)^{-\frac{1}{2(M-1)}}$. As a result, conditioned on $\mathfrak{B}$, $\mathfrak{C}$, and $\mathfrak{D}$, we have

$$\left|\mathbf{G}_{s_i}\mathbf{G}_{s_j}^H\right|^2 \leq t^{+2}\epsilon. \tag{5.55}$$

Moreover, since conditioned on $\mathfrak{B}$, $\|\mathbf{G}_{s_i}\|^2 > t$, we have

$$\mathbb{G}^H\mathbb{G} \succeq \mathbb{D}, \tag{5.56}$$

where $\mathbb{D}$ is an $M \times M$ matrix with $\mathbb{D}_{ii} = t$ and $\mathbb{D}_{ij} = \mathbf{G}_{s_i}\mathbf{G}_{s_j}^H$. Since $\frac{t^+}{t} = 1 + O\left(\frac{\ln\ln N}{\ln N}\right)$, from (5.55) it follows that $\frac{\mathbb{D}}{t} = \mathbf{I} + \epsilon O(\mathbf{I})$, where $O(\mathbf{I})$ denotes a matrix whose eigenvalues are $O(1)$. Moreover, since $\ln N \sim \eta\mathcal{K}$ and $t = \ln N - 2.5 \ln\ln N$, we have

$$\frac{1}{\mathcal{K}+1}\mathbb{G}^H\mathbb{G} \succeq \eta\mathbf{I} + o(\mathbf{I}). \tag{5.57}$$

For computing $\frac{\sqrt{M\mathcal{K}}}{\mathcal{K}+1}\mathbb{F}$, where $\mathbb{F} \triangleq \mathbb{G}^H\mathbb{B} + \mathbb{B}^H\mathbb{G}$, we need to find $\mathbb{F}_{ij} = \mathbf{G}_{s_i}\mathbf{b}_{s_j}^H + \mathbf{b}_{s_i}\mathbf{G}_{s_j}^H = 2\Re\left(\mathbf{G}_{s_i}\mathbf{b}_{s_j}^H\right) = 2\|\mathbf{G}_{s_i}\|\Re\left(\mathbf{v}_{s_i}\mathbf{b}_{s_j}^H\right)$, $\forall i, j$. For $i = j$, due to the algorithm, we have $\mathbb{F}_{ii} \geq 2\sqrt{t}(1-\gamma)$. Also, for $i \neq j$, $\mathbb{F}_{ij}$ can be upper-bounded as $2\left|\mathbf{G}_{s_i}\mathbf{b}_{s_j}^H\right|$. Writing $\mathbf{b}_{s_j}$ as $\alpha_j^{\|}\mathbf{v}_{s_j} + \alpha_j^{\perp}\mathbf{v}_{s_j}^{\perp}$, where $\mathbf{v}_{s_j}^{\perp}$ is perpendicular to $\mathbf{v}_{s_j}$, $\alpha_j^{\|} = \mathbf{b}_{s_j}\mathbf{v}_{s_j}^H$, and $\|\mathbf{v}_{s_j}^{\perp}\| = 1$. Hence,

$$\begin{aligned}
\mathbb{F}_{ij} &\leq 2\left|\mathbf{G}_{s_i}\mathbf{b}_{s_j}^H\right| \\
&= 2\left|\mathbf{G}_{s_i}\left(\alpha_j^{\|}\mathbf{v}_{s_j} + \alpha_j^{\perp}\mathbf{v}_{s_j}^{\perp}\right)^H\right| \\
&\overset{(a)}{\leq} 2\left|\mathbf{G}_{s_i}\mathbf{v}_{s_j}^H\right| + \left|\alpha_j^{\perp}\right| \|\mathbf{G}_{s_i}\| \\
&\overset{(b)}{\leq} 2\left[\sqrt{t^+\epsilon} + \sqrt{2\gamma t^+}\right] \\
&= \sqrt{t^+}O(\sqrt{\epsilon}).
\end{aligned} \tag{5.58}$$

where (a) follows from i) $|a+b| \leq |a| + |b|$, ii) $|\alpha_j^{\|}| \leq 1$, and iii) $\left|\mathbf{G}_{s_i}\mathbf{v}_{s_j}^{\perp\,H}\right| \leq \|\mathbf{G}_{s_i}\|$, and (b) results from i) $\left|\mathbf{G}_{s_i}\mathbf{v}_{s_j}^H\right| = \|\mathbf{G}_{s_i}\|\sqrt{z(\mathbf{G}_{s_i}, \mathbf{G}_{s_j})}$, which is conditioned on $\mathfrak{C}$ and $\mathfrak{D}$ upper-bounded by $\sqrt{t^+\epsilon}$, and ii) $\left|\alpha_j^{\perp}\right| = \sqrt{1 - \left|\alpha_j^{\|}\right|^2} = \sqrt{1 - \left|\mathbf{b}_{s_j}\mathbf{v}_{s_j}^H\right|^2} \leq \sqrt{1 - \left[\Re\left(\mathbf{b}_{s_j}\mathbf{v}_{s_j}^H\right)\right]^2} \leq \sqrt{1-(1-\gamma)^2} \leq \sqrt{2\gamma}$. This implies that $\mathbb{F} = 2\sqrt{t}\left[\mathbf{I} + \sqrt{\epsilon}O(\mathbf{I})\right]$. Consequently, noting that $\ln N \sim \eta\mathcal{K}$ and $t = \ln N - 2.5\ln\ln N$, we have

$$\frac{\sqrt{M\mathcal{K}}}{\mathcal{K}+1}\mathbb{F} = 2\sqrt{\eta M}\left[\mathbf{I} + o(\mathbf{I})\right]. \tag{5.59}$$

Finally, having the facts that $\left[\mathbb{B}^H\mathbb{B}\right]_{ii} = \|\mathbf{b}_i\|^2 = 1$, and for $i \neq j$,

$$\begin{aligned}
\left|\left[\mathbb{B}^H\mathbb{B}\right]_{ij}\right| &= \left|\mathbf{b}_{s_i}\mathbf{b}_{s_j}^H\right| \\
&= \left|\left(\alpha_i^{\|}\mathbf{v}_{s_i} + \alpha_i^{\perp}\mathbf{v}_{s_i}^{\perp}\right)\left(\alpha_j^{\|}\mathbf{v}_{s_j} + \alpha_j^{\perp}\mathbf{v}_{s_j}^{\perp}\right)^H\right| \\
&\overset{(a)}{\leq} \left|\mathbf{v}_{s_i}\mathbf{v}_{s_j}^H\right| + |\alpha_i^{\perp}| + |\alpha_j^{\perp}| + |\alpha_i^{\perp}||\alpha_j^{\perp}| \\
&\overset{(b)}{\leq} \sqrt{\epsilon} + 2\sqrt{2\gamma} + 4\gamma, \\
&= O(\sqrt{\epsilon}),
\end{aligned} \tag{5.60}$$

in which (a) follows from the facts that i) $|\alpha_i^{\|}| \leq 1, |\alpha_j^{\|}| \leq 1, |\alpha_i^{\perp}| \leq 1, |\alpha_j^{\perp}| \leq 1$, and ii) $\left|\mathbf{v}_{s_i}\mathbf{v}_{s_j}^H\right| \leq 1, \left|\mathbf{v}_{s_i}^{\perp}\mathbf{v}_{s_j}^H\right| \leq 1, \left|\mathbf{v}_{s_i}^{\perp}\mathbf{v}_{s_j}^{\perp H}\right| \leq 1$, and (b) results from the facts that i) $|\alpha_i^{\perp}| \leq \sqrt{2\gamma}$ and conditioned on $\mathfrak{C}$, $\left|\mathbf{v}_{s_i}\mathbf{v}_{s_j}^H\right| \leq \sqrt{\epsilon}$, we have $\mathbb{B}^H\mathbb{B} = \mathbf{I} + \sqrt{\epsilon}O(\mathbf{I})$, and consequently,

$$\frac{M\mathcal{K}}{\mathcal{K}+1}\mathbb{B}^H\mathbb{B} = M\mathbf{I} + \sqrt{\epsilon}O(\mathbf{I}). \tag{5.61}$$

Combining (5.54), (5.57), (5.59), and (5.61) yields

$$\begin{aligned}
\mathrm{Tr}\left\{\left[\mathbb{H}^H\mathbb{H}\right]^{-1}\right\} &= \mathrm{Tr}\left\{\left[\eta\mathbf{I} + o(\mathbf{I}) + 2\sqrt{\eta M}\mathbf{I} + o(\mathbf{I}) + M\mathbf{I} + \sqrt{\epsilon}O(\mathbf{I})\right]^{-1}\right\} \\
&= \mathrm{Tr}\left\{\left[(\sqrt{\eta} + \sqrt{M})^2\mathbf{I} + o(\mathbf{I})\right]^{-1}\right\} \\
&\overset{((a)}{=} \frac{M}{(\sqrt{\eta} + \sqrt{M})^2} + o(1),
\end{aligned} \tag{5.62}$$

where (a) follows from the fact that $[\mathbf{I} + o(\mathbf{I})]^{-1} = \mathbf{I} + o(\mathbf{I})$. Substituting in (5.49) yields

$$\begin{aligned} \mathcal{R} &\geq M \ln \left(1 + P \left[1 + \sqrt{\frac{\eta}{M}} \right]^2 + o(1) \right) \Pr\{\mathfrak{B}\} \Pr\{\mathfrak{C}|\mathfrak{B}\} \Pr\{\mathfrak{D}|\mathfrak{B}, \mathfrak{C}\} \\ &= \left[M \ln \left(1 + P \left[1 + \sqrt{\frac{\eta}{M}} \right]^2 \right) + o(1) \right] \left[1 + O\left(\frac{1}{\ln N} \right) \right] \\ &= M \ln \left(1 + P \left[1 + \sqrt{\frac{\eta}{M}} \right]^2 \right) + o(1). \end{aligned} \tag{5.63}$$

This completes the proof of Theorem 5.4.

■

5.4 Simulation Results

In this section, we examine the analytical results in the previous section by simulation. Figures 5.1-5.3 present the plots of the sum-rate capacity versus the number of users, for different values of Rician factor $\mathcal{K} = 1$, $\mathcal{K} = 10$, and $\mathcal{K} = 100$, respectively. The SNR (P) is assumed to be 10 dB in these figures and the number of transmit antennas M is set to 2. Also, the plots of the achievable sum-rate for ZFBF and TDMA are given for comparison. The user selection algorithm used for ZFBF is the same as Algorithm 1 in chapter 2. As can be observed in th figures the following observations can be made: i) The sum-rate capacity of the system in the case of isotropic specular components is larger than the sum-rate capacity in the case of co-located transmit antennas. ii) In the case of isotropic specular components, ZFBF performs well for all values of $\mathcal{K}$, while in the case of co-located transmit antennas the performance of ZFBF is degraded significantly by increasing $\mathcal{K}$. iii) in the case of co-located transmit antennas and $K = 100$, the sum-rate

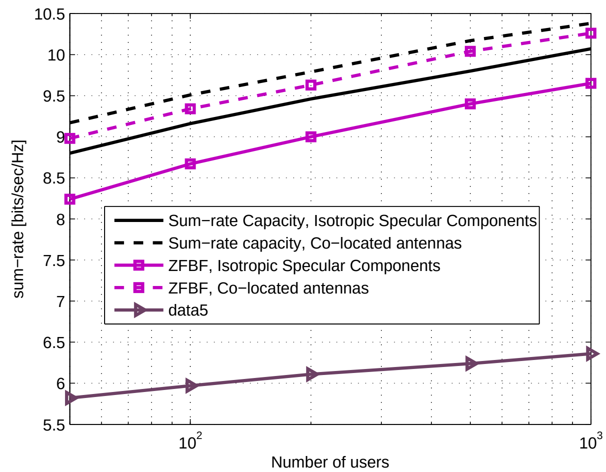

Figure 5.1: Sum-rate capacity versus the number of users; $\mathcal{K} = 1$.

of TDMA is almost close to the sum-rate capacity, which is compatible with the result of Theorem 5.2.

Figure 5.4 presents the plots of sum-rate capacity versus SNR for different values of Rician factor and two cases of isotropic specular components and co-located transmit antennas. It is assumed that $N = 100$ and $M = 2$ in this figure. As can be observed, by increasing the value of the Rician factor, the difference between the sum-rate capacity of the system in the two cases of isotropic specular components and co-located transmit antennas increases. Moreover, the slope of the curves in the case of isotropic specular components is equal to 2, regardless of the value of $\mathcal{K}$, while the slope of the curves in the case of co-located transmit antennas decreases with $\mathcal{K}$, but increases with SNR. However, for high values of SNR, the slope of all curves approaches 2, implying that the multiplexing gain of the system is 2, regardless of the distribution of the specular components and the value of the Rician factor.

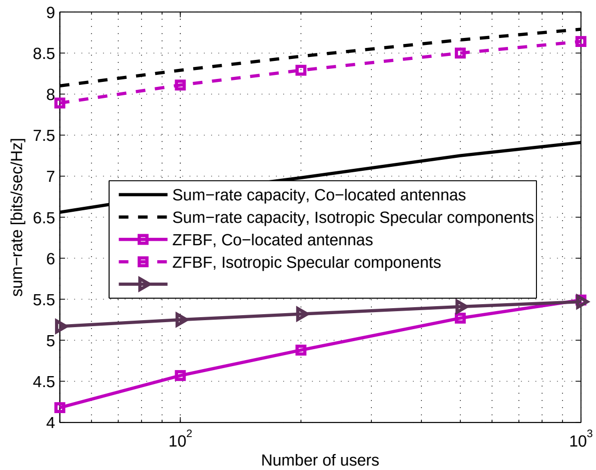

Figure 5.2: Sum-rate capacity versus the number of users; $\mathcal{K} = 10$.

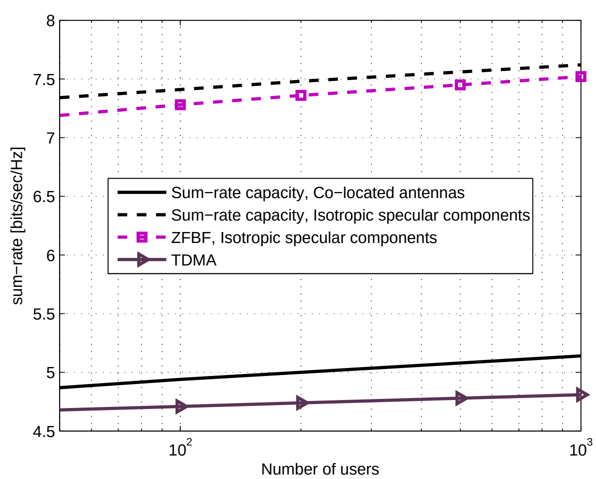

Figure 5.3: Sum-rate capacity versus the number of users; $\mathcal{K} = 100$.

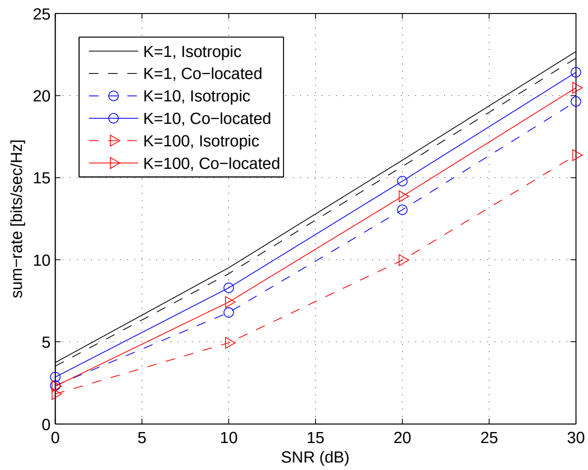

Figure 5.4: Sum-rate capacity versus SNR; $N = 100$.

5.5 Conclusion

In this chapter, we have derived the asymptotic sum-rate capacity of MIMO-BC with large number of users in a Rician fading environment. It is observed that in the region $\mathcal{K} = o(\ln N)$, the capacity achieving strategy is exactly the same as the Rayleigh fading case. In the region $\mathcal{K} = \omega(\ln N)$, the sum-rate capacity depends on the distribution of the specular component; in the case of co-located transmit antennas, it is demonstrated that TDMA achieves the sum-rate capacity and the capacity grows logarithmically with the number of transmit antennas. In the case of isotropically distributed specular components, ZFBF along with a user selection strategy which selects M users with semi-orthogonal specular components is shown to be optimum. Moreover, the sum-rate capacity grows linearly with the number of transmit antennas.

Chapter 6

Conclusion and Future Research

This dissertation focuses on scheduling in large-scale MIMO downlink systems.

In chapter 2, we consider a Rayleigh fading MIMO-BC with large number of users and propose an efficient sub-optimum algorithm that assigns the coordinates of transmission space to different users in order to achieve the best performance in terms of the sum-rate throughput. It is assumed that the zero-forcing beamforming is used at the base station as the precoding scheme. The algorithm starts by setting a threshold value. By applying Singular Value Decomposition (SVD) to all users' channel matrices, only the eigenvectors whose corresponding singular values are above the set threshold are considered. Then, among these candidate eigenvectors, the algorithm chooses a set of size M which are nearly orthogonal to each other. For the asymptotic case of $N \to \infty$, we give the necessary and sufficient conditions for the threshold value in order to achieve the optimum sum-rate capacity, such that the difference between the sum-rates approaches zero. Moreover, it is demonstrated that the complexity of search and the amount of feedback required at the base station is significantly reduced. Simulation results indicate that the proposed algorithm performs well for practical scenarios as well.

In chapter 3, a large-scale Rayleigh fading MIMO-BC is considered, in which the channel state information is provided from the users to the transmitter via feedback links. First, we define the amount of feedback as the average number of users who send information to the BS. In the fixed and low SNR regimes, our results show that it is not possible to achieve the maximum sum-rate with a finite amount of feedback. Moreover, in the fixed SNR regime, in order to reduce the gap between the achieved sum-rate and the optimum value to zero, the amount of feedback must be greater than $\ln \ln \ln N$. In the second part, we define the amount of feedback as the number of information bits sent to the BS. In the fixed SNR regime, our analysis shows that the minimum amount of feedback, in order to reduce the gap to the optimum sum-rate to zero, scales as $\Theta(\ln \ln \ln N)$, which can be achieved using the Random Beam-Forming scheme proposed in [26]. However, the optimality of Random Beam-Forming only holds for the region $\ln P \neq \Omega(\ln N)$. In the regime of $\ln P = \Omega(\ln N)$, we consider two cases. In the case of $K < M$, we prove that the minimum amount of feedback bits to reduce the gap between the achievable sum-rate and the maximum sum-rate to zero grows logarithmically with SNR, which is achievable by the "Generalized Random Beam-Forming" scheme, proposed in [51]. In the case of $K = M$, we show that by using the Random Beam-Forming scheme and the amount of feedback not growing with SNR the maximum sum-rate capacity is achievable.

In chapter 4, we consider a *hard* delay constraint D for each user, which is enforced by the application or physical limitations (e.g. buffer size). We define a dropping event as the event that there exists a user who does not meet the desired delay constraint. We propose a scheduling scheme for maximizing the throughput of the system, while satisfying the delay constraint for all users. The proposed scheduling algorithm works based on setting a threshold on the channel gain of the users and among the users with channel gains above the threshold, the user with the minimum *Packet Expiry Countdown*s (PEC), which is defined as the remaining time to the expiration of that users' packet, is served. By doing asymptotic analysis, it is proved that by selecting the threshold level properly, the proposed scheduling algorithm achieves the maximum throughput, maximum fairness, and minimum delay in the network, simultaneously, in the asymptotic case of $N \to \infty$. The analysis is based on characterizing the probability mass function of PEC in terms of N, D , and the threshold value, and evaluating the network dropping probability accordingly. It is also demonstrated that the Round-Robin (RR) scheduling, which focuses on maximizing the fairness and minimizing the delay in the network, and Multi-User Diversity (MUD) scheduling, which focuses on maximizing the throughput in the system, are two extreme cases of the proposed algorithm, where the former suffers from the weak performance in terms of throughput and the latter increases the network delay by a factor of $\log N$. Moreover, we have introduced a new notion of performance in the network, called "Average Throughput", which is defined as the product of the packet arrival rate and the amount of information per channel use in each packet, and proved that the proposed algorithm maximizes the *Minimum Average Throughput* in a broadcast channel. It is also established that the proposed algorithm reaches the boundaries of the capacity region and stability region of the underlying system, simultaneously, in the asymptotic case of $N \to \infty$. The proposed algorithm is also generalized to MIMO Broadcast Channels (MIMO-BC) by modifying the Random Beam-Forming scheme proposed in [26]. It is shown that the proposed algorithm is capable of achieving the maximum throughput, maximum fairness, and minimum delay, simultaneously, in the asymptotic case of $N \to \infty$, in a MIMO-BC.

In chapter 5, we consider a Rician MIMO-BC, in which a transmitter equipped with M antennas communicates with N ($N \gg 1$) single-antenna users. The channels are assumed to be perfectly known at both the transmitter and receiver sides. The asymptotic (in terms of the number of users) sum-rate capacity of the system, as well as the capacity-achieving strategies, are derived. The main results of the chapter are as follows: i) in the region of $\mathcal{K} = o(\log N)$, where $\mathcal{K}$ denotes the *Rician factor*, the sum-rate capacity scales as $M \log(1 + \frac{P}{M}\eta)$, where P denotes the SNR and $\eta \triangleq \frac{\log N}{1+\mathcal{K}}$, which is achieved by Zero-Forcing Beam-Forming (ZFBF) along with a low-complexity user selection algorithm that considers only the scattered component of the users' channels, ii) in the region $\mathcal{K} = \omega(\log N)$, in the case of co-located transmit antennas, the capacity scales as $\log(1 + MP)$, which is achieved by TDMA, iii) in the region $\mathcal{K} = \omega(\log N)$, in the case of isotropically-distributed specular components, the sum-rate capacity behaves as $M \log(1 + P)$, which is achieved by ZFBF, along with a user selection algorithm that considers only the specular component of the users' channels. Simulation results confirm the validity of analytical results.

6.1 Future Research Directions

The dissertation can be continued in several directions as briefly explained in what follows.

The results of the chapters 2 and 3 is based on the assumption that the feedback links between the transmitter and the receivers are noise-less. A natural extension to these results can be considering noise in the feedback links and study the effect of the feedback channel noise on the results and also efficient ways of channel quantization and transmission through the feedback links.

In chapters 2-4, it is assumed that the channels are block fading, i.e., there is no correlation between the channel gains in the consecutive blocks, and also, there is no correlation between the transmit antennas or any of the receivers' antennas. It is interesting to investigate the effect of the temporal or spatial correlation on the results of these chapters. Moreover, in chapter 4, the arrival process of the packets is assumed to be deterministic with a constant rate. An extension to the results of this chapter is to consider other arrival processes (like Poisson) and study the possibility of simultaneously achieving the maximum throughput and maximum short-term fairness in this scenario.

In chapter 5, the asymptotic sum-rate capacity of the systems is derived in terms of the number of users and the Rician factor. However, the SNR is assumed to be fixed throughout this chapter. Hence, it is interesting to consider a variable SNR scenario (like in chapter 3), and investigate the behavior of the sum-rate capacity with SNR.

Appendix A

Proof of Lemma 2.5

In this appendix, we derive the probability density function of $\mathcal{O}(i,j) = |\mathbf{V}_{i,\max}^H \mathbf{V}_{j,\max}|^2$. For simplicity of notation, $\mathbf{V}_{i,\max}$ is denoted by ϕ_i, and $\mathbf{V}_{j,\max}$ is denoted by ϕ_j. Since ϕ_i and ϕ_j are the eigenvectors of two independent matrices whose entries are independent $\mathcal{CN}(0,1)$, it follows from [36] that ϕ_i and ϕ_j are independent isotropically distributed unit vectors in $\mathbb{C}^M$, with the following probability density function:

$$f_{\phi_i}(\phi) = f_{\phi_j}(\phi) = \frac{\Gamma(M)}{\pi^M}\delta(\phi^H \phi - 1). \tag{A.1}$$

Moreover, this probability density function does not change by multiplying any $M \times M$ unitary matrix $\mathbf{\Theta}$, i.e.,

$$f_{\mathbf{\Theta}\phi_i}(\phi) = f_{\phi_i}(\phi). \tag{A.2}$$

Now, define $u = \phi_i^H \phi_j$, and let $\mathbf{\Theta}$ be a unitary matrix whose first row is equal to ϕ_i. We can write

$$\begin{aligned} u &= \phi_i^H \mathbf{\Theta}^H \mathbf{\Theta} \phi_j \\ &= [\mathbf{\Theta}\phi_i]^H \, \mathbf{\Theta}\phi_j \\ &= [1\,0\,\cdots\,0]\,\phi_j^{'} \\ &= \phi_j^{'}(1), \end{aligned} \tag{A.3}$$

where $\phi_j^{'} = \mathbf{\Theta}\phi_j$, and $\phi_j^{'}(1)$ is the first element of $\phi_j^{'}$. Since $\mathbf{\Theta}$ is unitary, ϕ_j and $\phi_j^{'}$ have the same pdf. Hence, the probability density function of $\phi_j^{'}(1)$ is the same as that of $\phi_j(1)$, and can be computed as [36]

$$f_u(u) = f_{\phi_j(1)}(u) = \frac{M-1}{\pi}\left(1-|u|^2\right)^{M-2}. \tag{A.4}$$

Using the above equation, the probability density function of $\mathcal{O}(i,j) = |u|^2$ will be equal to

$$\begin{aligned} f_{\mathcal{O}(i,j)}(z) &= f_{|u|^2}(z) \\ &= \frac{f_{|u|}(\sqrt{z})}{2\sqrt{z}} \\ &= \frac{2\pi\sqrt{z} f_u(\sqrt{z})}{2\sqrt{z}} \\ &= (M-1)(1-z)^{M-2}. \end{aligned} \tag{A.5}$$

Appendix B

Proof of (2.63)

Since the selected vectors $\left\{\mathbf{V}_{s_j,\max}\right\}_{j=1}^{i-1}$ are nearly orthogonal to each other, they form a basis for the sub-space spanned by them. We call this sub-space $\mathcal{P}_{i-1}$. In the following, we denote $\mathbf{V}_{k,\max}$, the eigenvector corresponding to the maximum singular value of user k, by ϕ_k for the simplicity of notation.

Any vector $\mathbf{v} \in \mathbb{C}^M$ can be represented as

$$\mathbf{v} = \mathbf{v}^{\perp} + \sum_{j=1}^{i-1} \left\langle \phi_{s_j}, \mathbf{v} \right\rangle \phi_{s_j}, \tag{B.1}$$

where $\mathbf{v}^{\perp}$ is the project of $\mathbf{v}$ on the null space of $\mathcal{P}_{i-1}$, denoted by $\mathcal{P}_{i-1}^{\perp}$, and $\left\langle \phi_{s_j}, \mathbf{v} \right\rangle = \phi_{s_j}^H \mathbf{v}$.

Defining the event $\mathcal{C}_i = \{\mathcal{O}(s_1, k) < \alpha, \cdots, \mathcal{O}(s_{i-1}, k) < \alpha\}$ [1], the conditional probability in (2.63) can be written as

$$\kappa_i = \text{Prob}\left\{\mathcal{O}(s_i, k) < \alpha | \quad \mathcal{C}_i\right\}. \tag{B.2}$$

[1]Recall the definition of α which is $\frac{\epsilon(N)}{M}$.

Using (2.54), we can write $\mathcal{C}_i$ by

$$\mathcal{C}_i = \left\{ |\phi_{s_1}^H \phi_k|^2 < \alpha, \cdots, |\phi_{s_{i-1}}^H \phi_k|^2 < \alpha \right\}. \tag{B.3}$$

Hence, (B.2) can be expressed as

$$\kappa_i = \text{Prob}\left\{ |\phi_{s_i}^H \phi_k|^2 < \alpha \;\middle|\; |\phi_{s_1}^H \phi_k|^2 < \alpha, \cdots, |\phi_{s_{i-1}}^H \phi_k|^2 < \alpha \right\}. \tag{B.4}$$

Using (B.1), we can write ϕ_k as

$$\phi_k = \phi_k^\perp + \sum_{j=1}^{i-1} \left\langle \phi_{s_j}, \phi_k \right\rangle \phi_{s_j}, \tag{B.5}$$

and ϕ_{s_i} as

$$\phi_{s_i} = \phi_{s_i}^\perp + \sum_{j=1}^{i-1} \left\langle \phi_{s_j}, \phi_{s_i} \right\rangle \phi_{s_j}. \tag{B.6}$$

Hence, $|\phi_{s_i}^H \phi_k|^2$ can be computed as,

$$|\phi_{s_i}^H \phi_k|^2 = \Bigg| \left\langle \phi_{s_i}^\perp, \phi_k^\perp \right\rangle + \sum_{j=1}^{i-1} \left\langle \phi_{s_i}, \phi_{s_j} \right\rangle \left\langle \phi_{s_j}, \phi_k \right\rangle + \sum_{j=1}^{i-1} \sum_{\substack{l=1 \\ l \neq j}}^{i-1} \left\langle \phi_{s_i}, \phi_{s_j} \right\rangle \left\langle \phi_{s_l}, \phi_k \right\rangle \left\langle \phi_{s_j}, \phi_{s_l} \right\rangle \Bigg|^2. \tag{B.7}$$

Defining

$$\begin{aligned} u_1 &= \left\langle \phi_{s_i}^\perp, \phi_k^\perp \right\rangle, \\ u_2 &= \sum_{j=1}^{i-1} \left\langle \phi_{s_i}, \phi_{s_j} \right\rangle \left\langle \phi_{s_j}, \phi_k \right\rangle, \\ u_3 &= \sum_{j=1}^{i-1} \sum_{\substack{l=1 \\ l \neq j}}^{i-1} \left\langle \phi_{s_i}, \phi_{s_j} \right\rangle \left\langle \phi_{s_l}, \phi_k \right\rangle \left\langle \phi_{s_j}, \phi_{s_l} \right\rangle, \end{aligned} \tag{B.8}$$

we have

$$|\phi_{s_i}^H \phi_k|^2 = |u_1|^2 + |u_2|^2 + |u_3|^2 + 2\Re\{u_1 u_2^H\} + 2\Re\{u_2 u_3^H\} + 2\Re\{u_1 u_3^H\}, \quad \text{(B.9)}$$

where $\Re\{x\}$ denotes the real part of x. An upper bound for $|\phi_{s_i}^H \phi_k|^2$ is given by

$$|\phi_{s_i}^H \phi_k|^2 < |u_1|^2 + |u_2|^2 + |u_3|^2 + 2|u_1|(|u_2| + |u_3|) + 2|u_2||u_3|. \quad \text{(B.10)}$$

Having the facts that $\|\phi_k^\perp\|^2 < \|\phi_k\|^2 = 1$, and $\|\phi_{s_i}^\perp\|^2 < \|\phi_{s_i}\|^2 = 1$, we can write

$$\begin{aligned} |\phi_{s_i}^H \phi_k|^2 &< \frac{|u_1|^2}{\|\phi_k^\perp\|^2 \|\phi_{s_i}^\perp\|^2} + 2\frac{|u_1|}{\|\phi_k^\perp\|\|\phi_{s_i}^\perp\|}(|u_2| + |u_3|) + (|u_2| + |u_3|)^2 \\ &= \mathcal{O}\left(\phi_k^\perp, \phi_{s_i}^\perp\right) + 2\sqrt{\mathcal{O}\left(\phi_k^\perp, \phi_{s_i}^\perp\right)}(|u_2| + |u_3|) + (|u_2| + |u_3|)^2 \\ &= \left(\sqrt{\mathcal{O}\left(\phi_k^\perp, \phi_{s_i}^\perp\right)} + |u_2| + |u_3|\right)^2. \end{aligned} \quad \text{(B.11)}$$

Also, a lower bound for $|\phi_{s_i}^H \phi_k|^2$ can be given as

$$\begin{aligned} |\phi_{s_i}^H \phi_k|^2 &> |u_1|^2 - 2|u_1|(|u_2| + |u_3|) - 2|u_2||u_3| \\ &> \mathcal{O}\left(\phi_k^\perp, \phi_{s_i}^\perp\right) \|\phi_k^\perp\|^2 \|\phi_{s_i}^\perp\|^2 - 2\sqrt{\mathcal{O}\left(\phi_k^\perp, \phi_{s_i}^\perp\right)}(|u_2| + |u_3|)\|\phi_k^\perp\|\|\phi_{s_i}^\perp\| - 2|u_2||u_3| \\ &> \mathcal{O}\left(\phi_k^\perp, \phi_{s_i}^\perp\right) \|\phi_k^\perp\|^2 \|\phi_{s_i}^\perp\|^2 - 2\sqrt{\mathcal{O}\left(\phi_k^\perp, \phi_{s_i}^\perp\right)}(|u_2| + |u_3|) - 2|u_2||u_3|. \end{aligned} \quad \text{(B.12)}$$

Using (B.5) and (B.6), we have

$$\|\phi_k^\perp\|^2 = 1 - \sum_{j=1}^{i-1} |\phi_{s_j}^H \phi_k|^2 + \sum_{j=1}^{i-1} \sum_{\substack{l=1 \\ l \neq j}}^{i-1} \langle \phi_k, \phi_{s_j} \rangle \langle \phi_{s_j}, \phi_{s_l} \rangle \langle \phi_{s_l}, \phi_k \rangle, \quad \text{(B.13)}$$

and

$$\|\phi_{s_i}^\perp\|^2 = 1 - \sum_{j=1}^{i-1} |\phi_{s_j}^H \phi_{s_i}|^2 + \sum_{j=1}^{i-1} \sum_{\substack{l=1 \\ l \neq j}}^{i-1} \langle \phi_{s_i}, \phi_{s_j} \rangle \langle \phi_{s_j}, \phi_{s_l} \rangle \langle \phi_{s_l}, \phi_{s_i} \rangle. \quad \text{(B.14)}$$

Conditioned on $\mathcal{C}_i$, and knowing that the set $\{\phi_{s_j}\}_{j=1}^{i}$ is $\epsilon(N)$-orthogonal (or equivalently, $M\alpha$-orthogonal, i.e., $|\phi_{s_j}^H \phi_{s_l}|^2 < M\alpha, j, l = 1, \cdots, i$), from (B.8) we conclude the followings:

$$\begin{aligned}
|u_2| &< (i-1)\sqrt{M}\alpha, \\
|u_3| &< (i-1)(i-2)M\alpha^{3/2}, \\
\|\phi_k^{\perp}\|^2 &> 1-(i-1)\alpha-(i-1)(i-2)\sqrt{M}\alpha^{3/2}, \\
\|\phi_{s_i}^{\perp}\|^2 &> 1-(i-1)M\alpha-(i-1)(i-2)M^{3/2}\alpha^{3/2}. \qquad \text{(B.15)}
\end{aligned}$$

Therefore, using (B.11), (B.12), and (B.15) the upper bound and lower bound for $|\phi_{s_i}^H \phi_k|^2$ can be rewritten as

$$|\phi_{s_i}^H \phi_k|^2 < \left(\sqrt{\mathcal{O}\left(\phi_k^{\perp}, \phi_{s_i}^{\perp}\right)} + (i-1)\sqrt{M}\alpha + (i-1)(i-2)M\alpha^{3/2}\right)^2, \qquad \text{(B.16)}$$

and

$$|\phi_{s_i}^H \phi_k|^2 > A.\mathcal{O}\left(\phi_k^{\perp}, \phi_{s_i}^{\perp}\right) - 2B\sqrt{\mathcal{O}\left(\phi_k^{\perp}, \phi_{s_i}^{\perp}\right)} - C, \qquad \text{(B.17)}$$

where

$$\begin{aligned}
A = &\left(1-(i-1)\alpha-(i-1)(i-2)\sqrt{M}\alpha^{3/2}\right) \times \\
&\left(1-(i-1)M\alpha-(i-1)(i-2)M\sqrt{M}\alpha^{3/2}\right),
\end{aligned}$$

$B = (i-1)\sqrt{M}\alpha + (i-1)(i-2)M\alpha^{3/2}$, and $C = 2(i-1)^2(i-2)M^{3/2}\alpha^{5/2}$.

Using (B.2), (B.16), and (B.17) we have

$$\begin{aligned}
\kappa_i &> \text{Prob}\left\{\left[\sqrt{\mathcal{O}\left(\phi_k^{\perp}, \phi_{s_i}^{\perp}\right)} + (i-1)\sqrt{M}\alpha + (i-1)(i-2)M\alpha^{3/2}\right]^2 < \alpha\right\} \\
&= \text{Prob}\left\{\mathcal{O}\left(\phi_k^{\perp}, \phi_{s_i}^{\perp}\right) < \left[\sqrt{\alpha} - (i-1)\sqrt{M}\alpha + (i-1)(i-2)M\alpha^{3/2}\right]^2\right\} \\
&= \text{Prob}\left\{\mathcal{O}\left(\phi_k^{\perp}, \phi_{s_i}^{\perp}\right) < \alpha - 2(i-1)\sqrt{M}\alpha^{3/2} + O(\alpha^2)\right\}, \qquad \text{(B.18)}
\end{aligned}$$

and

$$\begin{aligned}
\kappa_i &< \text{Prob}\left\{A.\mathcal{O}\left(\phi_k^\perp, \phi_{s_i}^\perp\right) - 2B\sqrt{\mathcal{O}\left(\phi_k^\perp, \phi_{s_i}^\perp\right)} - C < \alpha\right\} \\
&= \text{Prob}\left\{\sqrt{\mathcal{O}\left(\phi_k^\perp, \phi_{s_i}^\perp\right)} < \frac{B + \sqrt{B^2 + A(C+\alpha)}}{A}\right\} \\
&= \text{Prob}\left\{\sqrt{\mathcal{O}\left(\phi_k^\perp, \phi_{s_i}^\perp\right)} < \sqrt{\alpha} + (i-1)\sqrt{M}\alpha + O(\alpha^{3/2})\right\} \\
&= \text{Prob}\left\{\mathcal{O}\left(\phi_k^\perp, \phi_{s_i}^\perp\right) < \alpha + 2(i-1)\sqrt{M}\alpha^{3/2} + O(\alpha^2)\right\}. \qquad \text{(B.19)}
\end{aligned}$$

Since $\phi_k^\perp$ and $\phi_{s_i}^\perp$ are the projections of ϕ_k and ϕ_{s_i} over $\mathcal{P}_{i-1}^\perp$, a $(M-i+1)$-dimensional subspace of $\mathbb{C}^{M\times 1}$, $\frac{\phi_k^\perp}{\|\phi_k^\perp\|}$, and $\frac{\phi_{s_i}^\perp}{\|\phi_{s_i}^\perp\|}$, can be considered as uniformly distributed unit vectors in $\mathcal{P}_{i-1}^\perp$. Therefore, using Lemma 2.5, the probability density function for $\mathcal{O}\left(\phi_k^\perp, \phi_{s_i}^\perp\right)$ can be given as

$$f_{\mathcal{O}\left(\phi_k^\perp, \phi_{s_i}^\perp\right)}(z) = (M-i)(1-z)^{M-i-1}. \qquad \text{(B.20)}$$

Having (B.20), and using (B.18) and (B.19) we can write

$$\begin{aligned}
\kappa_i &< \int_0^{\alpha+2(i-1)\sqrt{M}\alpha^{3/2}+O(\alpha^2)} (M-i)(1-z)^{M-i-1}dz \\
&= 1 - \left[1 - \alpha - 2(i-1)\sqrt{M}\alpha^{3/2} + O(\alpha^2)\right]^{M-i} \\
&= (M-i)\alpha + 2(M-i)(i-1)\sqrt{M}\alpha^{3/2} + O(\alpha^2), \qquad \text{(B.21)}
\end{aligned}$$

and

$$\begin{aligned}
\kappa_i &> \int_0^{\alpha-2(i-1)\sqrt{M}\alpha^{3/2}+O(\alpha^2)} (M-i)(1-z)^{M-i-1}dz \\
&= 1 - \left[1 - \alpha + 2(i-1)\sqrt{M}\alpha^{3/2} + O(\alpha^2)\right]^{M-i} \\
&= (M-i)\alpha - 2(M-i)(i-1)\sqrt{M}\alpha^{3/2} + O(\alpha^2). \qquad \text{(B.22)}
\end{aligned}$$

From (B.21) and (B.22) we conclude

$$\kappa_i = (M-i)\alpha + O(\alpha^{3/2}). \qquad \text{(B.23)}$$

Appendix C

Proof of Lemma 2.8

Let us define

$$p \quad = \quad \mathrm{Prob}\left\{\lambda_{\max}(\mathbf{H}_k) > t\right\}, \tag{C.1}$$

where $t = \ln N + (M+K-1)\ln\ln N$. Using (2.25), the above probability probability can be written as

$$\begin{aligned} p &= \frac{t^{M+K-2}\exp(-t)}{\Gamma(M)\Gamma(K)}\left[1+O(t^{-1})\right] \\ &= \frac{\left[\ln N + (M+K-1)\ln\ln N\right]^{M+K-2} + O\left([\ln N]^{M+K-3}\right)}{\Gamma(M)\Gamma(K)e^{\ln N+(M+K-1)\ln\ln N}} \\ &= \frac{1}{N\ln N\Gamma(M)\Gamma(K)} + O\left(\frac{\ln\ln N}{N[\ln N]^2}\right). \end{aligned} \tag{C.2}$$

Using the above equation, the probability in (2.75) can be computed as,

$$\begin{aligned} \eta &= 1-(1-p)^N \\ &= 1-\exp\left(-Np+O(Np^2)\right) \\ &= 1-\exp\left[-\frac{1}{\Gamma(M)\Gamma(K)\ln N}+O\left(\frac{\ln\ln N}{[\ln N]^2}\right)\right] \\ &= 1-\left[1-\frac{1}{\Gamma(M)\Gamma(K)\ln N}+O\left(\frac{\ln\ln N}{[\ln N]^2}\right)\right] \\ &= O\left(\frac{1}{\ln N}\right). \end{aligned} \tag{C.3}$$

Appendix D

We have observed that $\boldsymbol{\mathcal{B}} = \boldsymbol{\mathcal{H}}\boldsymbol{\mathcal{H}}^H$ is an $M \times M$ matrix whose diagonal elements behave like $\ln N + f(N)$, where $f(N) = o(\ln N)$, and its non-diagonal elements scale as $O(\epsilon(N) \ln N)$. For simplicity of notation, we define $\theta(N) = \ln N + f(N)$ and $\varphi(N) = O(\epsilon(N) \ln N)$.

Let us define $\mathcal{A}_m$ as a $m \times m$ matrix whose diagonal elements scale like $\theta(N)$, and, its non-diagonal elements scale like $\varphi(N)$. Hence, all diagonal elements of $\boldsymbol{\mathcal{B}}^{-1}$ can be written as $\dfrac{|\mathcal{A}_{M-1}|}{|\mathcal{A}_M|}$.

It can be easily shown that

$$
\begin{aligned}
|\mathcal{A}_m| &= [\theta(N)]^m + O([\theta(N)]^{m-2}[\varphi(N)]^2) \\
&= [\ln N]^m + O\left([\ln N]^m h(N)\right), \quad m = 2, \cdots, M.
\end{aligned} \tag{D.1}
$$

where $h(N) = \max\left(\frac{f(N)}{\ln N}, \epsilon(N)\right) = o(1)$. Consequently, we can write any diagonal element of $\boldsymbol{\mathcal{B}}^{-1}$ as

$$
\begin{aligned}
[\boldsymbol{\mathcal{B}}^{-1}]_{ii} &= \frac{[\ln N]^{M-1} + O\left([\ln N]^{M-1} h(N)\right)}{[\ln N]^M + O\left([\ln N]^M h(N)\right)} \\
&= [\ln N]^{-1} + O\left(h(N)[\ln N]^{-1}\right).
\end{aligned} \tag{D.2}
$$

Appendix E

Proof of Lemma 2.9

For the proposed method, we have seen that the achievable sum-rate can be lower-bounded as

$$\begin{aligned} \mathcal{R}_{\text{Prop}} &\geq \mathbb{E}_{\boldsymbol{\mathcal{H}}}\left\{M \ln\left(1 + \frac{P}{\text{Tr}\left\{\left[\boldsymbol{\mathcal{H}}^{\text{H}}\boldsymbol{\mathcal{H}}\right]^{-1}\right\}}\right)\right\} \\ &\geq M \ln P - M\mathbb{E}_{\boldsymbol{\mathcal{H}}}\left\{\ln\left(\text{Tr}\left\{\left[\boldsymbol{\mathcal{H}}^{\text{H}}\boldsymbol{\mathcal{H}}\right]^{-1}\right\}\right)\right\}. \end{aligned} \tag{E.1}$$

where $\boldsymbol{\mathcal{H}}$ is the " selection coordinate matrix", defined in (2.9).

In [37], it has been shown that

$$\|\mathbf{b}_i\|^2\|\mathbf{a}_i\|^2 \leq \delta(\mathbf{B}), \qquad i = 1, \cdots, M, \tag{E.2}$$

where $\mathbf{b}_i$, $i = 1, \cdots, M$, are the columns of $\mathbf{B}$, a $M \times M$ matrix with the orthogonality defect $\delta(\mathbf{B})$, and $\mathbf{a}_i$, $i = 1, \cdots, M$, are the columns of $\mathbf{A} = (\mathbf{B}^{-1})^H$. Similarly, we can write

$$\|\mathbf{b}_i\|^2\|\mathbf{a}_i\|^2 \leq \delta(\mathbf{A}), \qquad i = 1, \cdots, M. \tag{E.3}$$

Defining $\mathbf{B} = \boldsymbol{\mathcal{H}}^{-1}$, and using the above equation, we can write

$$\begin{aligned}\mathrm{Tr}\left(\left[\boldsymbol{\mathcal{H}}\boldsymbol{\mathcal{H}}^{\mathrm{H}}\right]^{-1}\right) &= \sum_{i=1}^{M} \|\mathbf{b}_i\|^2 \\ &\leq \sum_{i=1}^{M} \frac{\delta(\boldsymbol{\mathcal{H}}^{\mathbf{H}})}{\|\mathbf{a}_i\|^2},\end{aligned} \tag{E.4}$$

where $\mathbf{a}_i$, is the ith column of $\boldsymbol{\mathcal{H}}^{\mathbf{H}}$, which is equal to $\mathbf{g}_{s_i}^H$. Having the fact that $\|\mathbf{g}_{s_i}\|^2 \geq t$ (by the algorithm), we can rewrite (E.4) as

$$\mathrm{Tr}\left(\left[\boldsymbol{\mathcal{H}}\boldsymbol{\mathcal{H}}^{\mathrm{H}}\right]^{-1}\right) \leq \frac{M\delta\left(\boldsymbol{\mathcal{H}}^{\mathbf{H}}\right)}{t}. \tag{E.5}$$

Defining $X(\boldsymbol{\mathcal{H}}) = \ln \mathrm{Tr}\left(\left[\boldsymbol{\mathcal{H}}\boldsymbol{\mathcal{H}}^{\mathrm{H}}\right]^{-1}\right)$, $Y(\boldsymbol{\mathcal{H}}) = \ln \dfrac{M\delta\left(\boldsymbol{\mathcal{H}}^{\mathbf{H}}\right)}{t}$, $Z(\boldsymbol{\mathcal{H}}) = \ln \delta(\boldsymbol{\mathcal{H}}^{\mathbf{H}})$, and $F_W(.)$ as the CDF of the random variable W, we have

$$\begin{aligned}\mathbb{E}\{X(\boldsymbol{\mathcal{H}})\} &\leq \mathbb{E}\{Y(\boldsymbol{\mathcal{H}})\} \\ &= \ln \frac{M}{t} + \mathbb{E}\{Z(\boldsymbol{\mathcal{H}})\} \\ &= \ln \frac{M}{t} + \int_0^{\infty} z f_{Z(\boldsymbol{\mathcal{H}})}(z) dz \\ &= \ln \frac{M}{t} + \int_0^{\infty} \left[1 - F_{Z(\boldsymbol{\mathcal{H}})}(z)\right] dz \\ &= \ln \frac{M}{t} + \int_1^{\infty} \left[1 - F_{\delta(\boldsymbol{\mathcal{H}}^{\mathbf{H}})}(e^z)\right] dz.\end{aligned} \tag{E.6}$$

It can be easily shown that $\delta(\boldsymbol{\mathcal{H}}^{\mathbf{H}}) = \delta(\boldsymbol{\Psi})$, where $\boldsymbol{\Psi} = [\boldsymbol{\Psi}_1 | \cdots | \boldsymbol{\Psi}_M]$ is the matrix consisting of the normalized columns of $\boldsymbol{\mathcal{H}}^{\mathbf{H}}$, i.e., $\boldsymbol{\Psi}_i = \dfrac{\boldsymbol{\mathcal{H}}^{\mathbf{H}}{}_i}{\|\boldsymbol{\mathcal{H}}^{\mathbf{H}}{}_i\|}$, $i = 1, \cdots, M$. Since the rows of $\boldsymbol{\mathcal{H}}$ are chosen randomly among the pre-selected eigenvectors, and due to the fact that the eigenvalues of a zero-mean circularly symmetric Gaussian matrix are independent of their corresponding eigenvectors, $\boldsymbol{\Psi}$ can be considered as a $M \times M$ matrix whose column are M randomly selected unit vectors. We have

$$\begin{aligned} \delta(\mathbf{\Psi}) &= \frac{1}{|\mathbf{\Psi}|^2} \\ &= \frac{1}{\prod_{i=1}^{M-1}(1-\gamma_i)}, \end{aligned} \tag{E.7}$$

where γ_i is the square norm of the project of $\mathbf{\Psi}_{i+1}$ over the sub-space spanned by $\{\mathbf{\Psi}_j\}_{j=1}^{i}$, $\mathcal{P}_i$. Now, consider $\mathbf{\Phi}_1, \cdots, \mathbf{\Phi}_M$, to be an orthonormal basis for the M-dimensional space, where $\{\mathbf{\Phi}_j\}_{j=1}^{i}$ are a basis for $\mathcal{P}_i$. Therefore, $\mathbf{\Psi}_{i+1}$ can be represented as $(\psi_{1,i+1}, \cdots, \psi_{i,i+1}, 0, \cdots, 0)$, where $\psi_{j,i+1}$ is the project of $\mathbf{\Psi}_{i+1}$ over $\mathbf{\Phi}_j$. In [36], the joint probability density function of $\mathbf{\Psi}_{i+1}^{(i)} = (\psi_{1,i+1}, \cdots, \psi_{i,i+1})$ is given as,

$$f_{\mathbf{\Psi}_{i+1}^{(i)}}(\psi) = \frac{\Gamma(M)}{\pi^i \Gamma(M-i)} \left(1 - \|\psi\|^2\right)^{M-i-1}. \tag{E.8}$$

Using the above equation, the probability density function of $\gamma_i = \|\mathbf{\Psi}_{i+1}^{(i)}\|^2$ can be written as

$$f_{\gamma_i}(z) = \frac{\Gamma(M)}{\Gamma(i)\Gamma(M-i)} z^{i-1}(1-z)^{M-i-1}, \tag{E.9}$$

which corresponds to the Beta distribution with parameters $(i, M-i)$.

Using (E.7), (E.9), and independence of γ_i's [4], we have

$$\begin{aligned} \text{Prob}\left\{\delta(\mathbf{\Psi}) > r\right\} &\leq \text{Prob}\left\{\max_i \gamma_i > 1 - r^{-\frac{1}{M-1}}\right\} \\ &= 1 - \prod_{i=1}^{M-1} I_{i,M-i}\left(1 - r^{-\frac{1}{M-1}}\right), \qquad r \geq 1, \end{aligned} \tag{E.10}$$

where $I_{r,s}(.)$ denotes the *Incomplete Beta Function*, with parameters (r, s). In [38], it has been shown that

$$I_{r,s}(x) = \frac{\Gamma(r+s)x^r(1-x)^{s-1}}{\Gamma(r+1)\Gamma(s)} + I_{r+1,s-1}(x), \quad \forall r, s \in \mathbb{Z}^+, \tag{E.11}$$

which incurs that

$$I_{r,s}(x) \geq I_{r+1,s-1}(x), \quad \forall x \in [0,1]. \tag{E.12}$$

Consequently,

$$\begin{aligned} I_{i,M-i}(x) &\geq I_{M-1,1}(x) \\ &= x^{M-1}, \qquad i = 1, \cdots, M-1. \end{aligned} \tag{E.13}$$

Using (E.13) and (E.10), we can write,

$$\text{Prob}\left\{\delta(\boldsymbol{\Psi}) > r\right\} \leq 1 - \left(1 - \sqrt[M-1]{1/r}\right)^{(M-1)^2}. \tag{E.14}$$

Combining (E.6) and (E.14), we have

$$\begin{aligned} \mathbb{E}\{X(\boldsymbol{\mathcal{H}})\} &\leq \ln\frac{M}{t} + \int_1^\infty \left[1 - \left(1 - e^{\frac{-r}{M-1}}\right)^{(M-1)^2}\right] dr \\ &= \ln\frac{M}{t} + \sum_{m=1}^{(M-1)^2} \binom{(M-1)^2}{m} (-1)^{m+1} \int_1^\infty e^{-\frac{mr}{M-1}} dr \\ &= \ln\frac{M}{t} + \sum_{m=1}^{(M-1)^2} \binom{(M-1)^2}{m} (-1)^{m+1} \frac{M-1}{m} e^{\frac{-m}{M-1}} \\ &= \ln\frac{M}{t} + (M-1) \sum_{m=1}^{(M-1)^2} \frac{1 - \left(1 - e^{-\frac{1}{M-1}}\right)^m}{m} \\ &\leq \ln\frac{M}{t} + (M-1) \sum_{m=1}^{(M-1)^2} \frac{1}{m} \\ &\leq \ln\frac{M}{t} + (M-1)[2\ln(M-1) + 1]. \end{aligned} \tag{E.15}$$

Substituting (E.15) into (E.1) and having $t = \ln N$, we get

$$\mathcal{R}_{\text{Prop}} \geq M \ln\left(\frac{P}{M}\ln N\right) - M(M-1)[2\ln(M-1)+1]. \tag{E.16}$$

As a result,

$$\lim_{N\to\infty} \frac{\mathcal{R}_{\text{Prop}}}{\mathcal{C}_{\text{sum}}} = 1. \tag{E.17}$$

Appendix F

Proof of Lemma 2.11

Achievability of the maximum multiplexing gain

Using (E.1), the multiplexing gain achieved by the proposed method, denoted by r_{Prop}, can be lower-bounded as

$$\begin{aligned} r_{\text{Prop}} &\geq \lim_{P\to\infty} \frac{M \ln P - M \mathbb{E}_{\boldsymbol{\mathcal{H}}} \left\{ \ln \left(\operatorname{Tr} \left\{ \left[\boldsymbol{\mathcal{H}}^{\mathrm{H}} \boldsymbol{\mathcal{H}} \right]^{-1} \right\} \right) \right\}}{\ln P} \\ &= M - M \lim_{P\to\infty} \frac{\mathbb{E}_{\boldsymbol{\mathcal{H}}} \left\{ \ln \operatorname{Tr} \left\{ \left[\boldsymbol{\mathcal{H}} \boldsymbol{\mathcal{H}}^{\mathbf{H}} \right]^{-1} \right\} \right\}}{\ln P}. \end{aligned} \tag{F.1}$$

Following the proof of Lemma 2.9 in Appendix E, and using equations (E.6), and (E.15), and the union bound for the probability, we have

$$\begin{aligned} \mathbb{E}_{\boldsymbol{\mathcal{H}}} \left\{ \ln \operatorname{Tr} \left\{ \left[\boldsymbol{\mathcal{H}} \boldsymbol{\mathcal{H}}^{\mathbf{H}} \right]^{-1} \right\} \right\} &\leq \ln \frac{M}{t} + \binom{L}{M} \int_1^\infty \left[1 - \left(1 - e^{\frac{-r}{M-1}} \right)^{(M-1)^2} \right] dr \\ &\leq \ln \frac{M}{t} + (M-1)[2 \ln(M-1) + 1] \binom{L}{M}, \end{aligned} \tag{F.2}$$

where L is the number of preselected eigenvectors in the first step of Algorithm 1. Since $L \leq NK$, we have $\mathbb{E}_{\boldsymbol{\mathcal{H}}} \left\{ \ln \operatorname{Tr} \left\{ \left[\boldsymbol{\mathcal{H}} \boldsymbol{\mathcal{H}}^{\mathbf{H}} \right]^{-1} \right\} \right\} < \infty$, the second term in (F.1) approaches zero, and as a result $r_{\text{Prop}} \geq M$.

For the optimum strategy, the sum-rate can be upper-bounded as [39],

$$\mathcal{C}_{\text{sum}} \leq M\mathbb{E}_{\|\mathbf{H}\|_{\max}} \left\{ \ln \left(1 + \frac{P}{M} \|\mathbf{H}\|_{\max}^2 \right) \right\}, \tag{F.3}$$

where $\|\mathbf{H}\|_{\max}^2$ is the maximum Frobenius norm of all channel matrices. This random variable can be considered as the maximum of N $\chi^2(2MK)$ random variables which has the pdf of the form

$$f_{\|\mathbf{H}\|_{\max}^2}(x) = N \frac{x^{MK-1} \exp(-x)}{\Gamma(MK)} \gamma(x, MK)^{N-1}, \tag{F.4}$$

where $\gamma(x, MK) = \int_x^\infty \frac{u^{MK} \exp(-u)}{\Gamma(MK)} du$. So, using (F.3) and (F.4), we can write the upper bound for the sum-rate as

$$\mathcal{C}_{\text{sum}} \leq M \int_0^\infty \ln(1 + \frac{P}{M} x) N \frac{x^{MK-1} \exp(-x)}{\Gamma(MK)} \gamma(x, MK)^{N-1} dx. \tag{F.5}$$

Thus, using the above equation, we have

$$\begin{aligned} r_{\text{Opt}} &= \lim_{P \to \infty} \frac{\mathcal{C}_{\text{sum}}}{\ln P} \\ &\leq \frac{M \ln P + \int_0^\infty M \ln(\frac{x}{M}) N \frac{x^{MK-1} \exp(-x)}{\Gamma(MK)} \gamma(x, MK)^{N-1} dx}{\ln P} \\ &= M. \end{aligned} \tag{F.6}$$

Since for any values of P and N, $\mathcal{C}_{\text{sum}}(P, N)$ is the maximum achievable sum-rate , r_{Opt} will be the maximum achievable multiplexing gain in MIMO-BC. Hence, using the above equation and having the fact that $r_{\text{Prop}} \geq M$, we conclude $r_{\text{Opt}} = r_{\text{Prop}} = M$ Therefore, the proposed method achieves the maximum *multiplexing gain* in MIMO-BC.

Achievability of the optimum multiuser diversity gain

In the proof of theorem 2, we observed that the sum-rate achieved by the proposed strategy, as well as the optimum one, scales like $M \ln \left(\frac{P}{M} \ln N\right)$. Hence, using (F.8) the *multiuser diversity gain* for the optimal scheme, denoted by d_{Opt} is equal to

$$\begin{aligned} d_{\text{Opt}} &= \lim_{N\to\infty} \frac{\mathcal{C}_{\text{sum}}}{r_{\text{Opt}} \ln \ln N} \\ &= \lim_{N\to\infty} \frac{M \ln \left(\dfrac{P}{M} \ln N\right)}{M \ln \ln N} \\ &= 1. \end{aligned} \tag{F.7}$$

and for the proposed method,

$$\begin{aligned} d_{\text{Prop}} &= \lim_{N\to\infty} \frac{\mathcal{R}_{\text{Prop}}}{r_{\text{Prop}} \ln \ln N} \\ &= \lim_{N\to\infty} \frac{M \ln \left(\dfrac{P}{M} \ln N\right)}{M \ln \ln N} \\ &= 1. \end{aligned} \tag{F.8}$$

Therefore, the proposed method achieves the maximum *multiuser diversity gain* in MIMO-BC. This, completes the proof of Lemma 2.11.

Appendix G

Multiplexing Gain in Random Selection Method

In this appendix, we prove that the Random selection strategy achieves the maximum *multiplexing gain*, i.e., $r_{\text{RS}} = M$. For this purpose, we consider the the precoding scheme of zero-forcing beam-forming. We assume that the coordinates are chosen randomly among the eigenvectors corresponding to the maximum singular value of each user's channel matrix. Therefore, similar to (F.1), we have

$$r_{\text{RS}}^{\text{ZFBF}} \geq M - M \lim_{P\to\infty} \frac{\mathbb{E}_{\boldsymbol{\mathcal{H}}}\left\{\ln \text{Tr}\left\{\left[\boldsymbol{\mathcal{H}}^{\text{H}}\boldsymbol{\mathcal{H}}\right]^{-1}\right\}\right\}}{\ln P}, \tag{G.1}$$

where $\boldsymbol{\mathcal{H}} = \left[\mathbf{g}_{s_1,\max}^T \,\middle|\, \mathbf{g}_{s_2,\max}^T \,\middle|\, \cdots \,\middle|\, \mathbf{g}_{s_M,\max}^T\right]^T$, and the users $s_1, \cdots, s_M$ are selected randomly. Defining $\mathbf{B} = \boldsymbol{\mathcal{H}}^{-1}$, similar to (E.4), we can write

$$\text{Tr}\left\{\left[\mathcal{H}\mathcal{H}^{\mathcal{H}}\right]^{-1}\right\} \leq \sum_{i=1}^{M} \frac{\delta(\boldsymbol{\mathcal{H}}^{\mathbf{H}})}{\|\mathbf{a}_i\|^2}, \tag{G.2}$$

where $\mathbf{a}_i$ is the ith column of $\boldsymbol{\mathcal{H}}^{\mathbf{H}}$, which is equal to $\mathbf{g}_{s_i}$. Noting that $\|\mathbf{g}_{s_i}\|^2 = \lambda_{\max}(\mathbf{H}_{s_i})$, we have

$$\begin{aligned} \mathrm{Tr}\left\{\left[\mathcal{H}\mathcal{H}^{\mathcal{H}}\right]^{-1}\right\} &\leq \sum_{i=1}^{M} \frac{\delta(\boldsymbol{\mathcal{H}}^{\mathbf{H}})}{\lambda_{\max}(\mathbf{H}_{s_i})} \\ &\leq \sum_{i=1}^{M} \frac{M\delta(\boldsymbol{\mathcal{H}}^{\mathbf{H}})}{\|\mathbf{H}_{s_i}\|^2}. \end{aligned} \tag{G.3}$$

Using (E.6), (E.15), and (G.3) we can write

$$\begin{aligned} \mathbb{E}_{\boldsymbol{\mathcal{H}}}\left\{\ln \mathrm{Tr}\left\{\left[\boldsymbol{\mathcal{H}}^{\mathrm{H}}\boldsymbol{\mathcal{H}}\right]^{-1}\right\}\right\} &\leq \mathbb{E}\left\{\ln\left(\sum_{i=1}^{M}\frac{M\delta(\boldsymbol{\mathcal{H}}^{\mathbf{H}})}{\|\mathbf{H}_{s_i}\|^2}\right)\right\} \\ &= \ln M + \mathbb{E}\left\{\ln\delta(\boldsymbol{\mathcal{H}}^{\mathbf{H}})\right\} + \mathbb{E}\left\{\ln\left(\sum_{i=1}^{M}\frac{1}{\|\mathbf{H}_{s_i}\|^2}\right)\right\} \\ &\leq \ln M + (M-1)[2\ln(M-1)+1] + \\ &\quad \ln\left[M\mathbb{E}\left\{\frac{1}{\|\mathbf{H}_{s_i}\|^2}\right\}\right] \\ &\leq M[2\ln(M-1)+1] + \\ &\quad \ln\left[\int_0^{\infty} x^{-1}.\frac{x^{MK-1}\exp(-x)}{\Gamma(MK)}dx\right] \\ &= M[2\ln(M-1)+1] - \ln(MK-1). \end{aligned} \tag{G.4}$$

Using (G.1) and (G.4), and noting that $r_{\mathrm{RS}}^{\mathrm{ZFBF}} \leq r_{\mathrm{RS}} \leq M$, we conclude $r_{\mathrm{RS}} = M$.

Appendix H

To evaluate $\mathbf{v}_{s_m}^H \mathbf{\Phi}_{s_m}$, we define $\boldsymbol{\mathcal{P}}_m$ as the sub-space defined by the vectors $\{\widehat{\mathbf{v}}_{s_i}\}_{i \neq m}$. We can write

$$\mathbf{v}_{s_m} = \mathbf{v}_{s_m}^{\parallel} + \mathbf{v}_{s_m}^{\perp}, \tag{H.1}$$

where $\mathbf{v}_{s_m}^{\parallel}$ is the projection of $\mathbf{v}_{s_m}$ over $\boldsymbol{\mathcal{P}}_m$, and $\mathbf{v}_{s_m}^{\perp}$ is the projection of $\mathbf{v}_{s_m}$ over $\boldsymbol{\mathcal{P}}_m^{\perp}$, and $\boldsymbol{\mathcal{P}}_m^{\perp}$ denotes the sub-space perpendicular to $\boldsymbol{\mathcal{P}}_m$. Since $\mathbf{\Phi}_m$ is perpendicular to all the vectors in the set $\{\widehat{\mathbf{v}}_{s_i}\}_{i\neq m}$, it belongs to $\boldsymbol{\mathcal{P}}_m^{\perp}$, and we have

$$
\begin{aligned}
\left|\mathbf{v}_{s_m}^H \boldsymbol{\Phi}_{s_m}\right|^2 &= \left|\left(\mathbf{v}_{s_m}^{\|} + \mathbf{v}_{s_m}^{\perp}\right)^H \boldsymbol{\Phi}_{s_m}\right|^2 \\
&= \left|\boldsymbol{\Phi}_{s_m}^H \mathbf{v}_{s_m}^{\perp}\right|^2 \\
&= \|\mathbf{v}_{s_m}^{\perp}\|^2 \\
&= 1 - \|\mathbf{v}_{s_m}^{\|}\|^2 \\
&\overset{(a)}{\approx} 1 - \sum_{i\neq m} \left|\mathbf{v}_{s_m}^H \widehat{\mathbf{v}}_{s_i}\right|^2 \\
&\overset{(b)}{\geq} 1 - \sum_{i=1}^{m-1} \beta - \sum_{i=m+1}^{M} \left|\mathbf{v}_{s_m}^H \widehat{\mathbf{v}}_{s_i}\right|^2 \\
&\overset{(c)}{=} 1 - (m-1)\beta - \sum_{i=m+1}^{M} \left|\left(\alpha_m^{\|}\widehat{\mathbf{v}}_{s_m} + \widehat{\mathbf{v}}_{s_m}^{\perp}\right)^H \left(\gamma_i^{\|}\mathbf{v}_{s_i} + \mathbf{v}_{s_i}^{\perp}\right)\right|^2 \\
&\overset{(d)}{\geq} 1 - (m-1)\beta - \sum_{i=m+1}^{M} \left(\left|\widehat{\mathbf{v}}_{s_m}^H \mathbf{v}_{s_i}\right| + \|\widehat{\mathbf{v}}_{s_m}^{\perp}\| + \|\mathbf{v}_{s_i}^{\perp}\|\right)^2 \\
&\overset{(e)}{\geq} 1 - (m-1)\beta - \sum_{i=m+1}^{M} \left(\sqrt{\beta} + \sqrt{\mu_m} + \sqrt{\mu_i}\right)^2 \\
&\overset{(f)}{\geq} 1 - (m-1)\beta - 3\sum_{i=m+1}^{M} (\beta + \mu_m + \mu_i) \\
&\overset{(g)}{\geq} 1 - (3M - 2m - 1)\beta - 6(M-m)\epsilon. \qquad \text{(H.2)}
\end{aligned}
$$

In the above equation, (a) follows from the fact that $\{\widehat{\mathbf{v}}_{s_i}\}_{i\neq m}$ form an semi-orthogonal basis for $\boldsymbol{\mathcal{P}}_i$. To see this, we evaluate $\left|\widehat{\mathbf{v}}_{s_i}^H \widehat{\mathbf{v}}_{s_j}\right|^2$, $i, j \neq m$, for $i > j$. For this purpose, we write $\widehat{\mathbf{v}}_{s_i}$ as $\gamma_i^{\|}\mathbf{v}_{s_i} + \mathbf{v}_{s_i}^{\perp}$, in which $\mathbf{v}_{s_i}^{\perp}$ denotes the projection of $\widehat{\mathbf{v}}_{s_i}$

over the subspace perpendicular to $\mathbf{v}_{s_i}$, and $\gamma_i^{\|} \triangleq \mathbf{v}_{s_i}^H \widehat{\mathbf{v}}_{s_i}$. Then, we have

$$\begin{aligned}
\left|\widehat{\mathbf{v}}_{s_i}^H \widehat{\mathbf{v}}_{s_j}\right|^2 &= \left|\left(\gamma_i^{\|}\mathbf{v}_{s_i} + \mathbf{v}_{s_i}^{\perp}\right)^H \widehat{\mathbf{v}}_{s_j}\right|^2 \\
&\le \left(\left|\gamma_i^{\|}\right| \left|\mathbf{v}_{s_i}^H \widehat{\mathbf{v}}_{s_j}\right| + \left|\left[\mathbf{v}_{s_i}^{\perp}\right]^H \widehat{\mathbf{v}}_{s_j}\right|\right)^2 \\
&\le \left(\sqrt{\beta} + \|\mathbf{v}_{s_i}^{\perp}\|\right)^2 \\
&\le \left(\sqrt{\beta} + \sqrt{\epsilon}\right)^2 \\
&= o(1), \qquad \text{(H.3)}
\end{aligned}$$

where the first inequality results from the fact that $|a+b|^2 \le (|a|+|b|)^2$, $\forall a, b$, the second inequality follows from the facts that $\left|\gamma_i^{\|}\right| \le 1$, $\left|\mathbf{v}_{s_i}^H \widehat{\mathbf{v}}_{s_j}\right| < \sqrt{\beta}$ (by the algorithm), and $\left|\left[\mathbf{v}_{s_i}^{\perp}\right]^H \widehat{\mathbf{v}}_{s_j}\right| \le \|\mathbf{v}_{s_i}^{\perp}\|$, the third inequality results from the fact that $\|\mathbf{v}_{s_i}^{\perp}\|^2 = 1 - \left|\mathbf{v}_{s_i}^H \widehat{\mathbf{v}}_{s_i}\right|^2$, which is by the algorithm upper-bounded by ϵ, and finally, the last line follows from the assumptions of $\epsilon = o(1)$ and $\beta = o(1)$.

The inequality (b) in (H.2) comes from the fact that $\left|\mathbf{v}_{s_m}^H \widehat{\mathbf{v}}_{s_i}\right|^2 < \beta$ for $i < m$ by the algorithm. The equality (c) results from writing $\mathbf{v}_{s_m}$ as $\alpha_m^{\|}\widehat{\mathbf{v}}_{s_m} + \widehat{\mathbf{v}}_{s_m}^{\perp}$ and $\widehat{\mathbf{v}}_{s_i}$ as $\gamma_i^{\|}\mathbf{v}_{s_i} + \mathbf{v}_{s_i}^{\perp}$ with the assumption of $\widehat{\mathbf{v}}_{s_m}^H \widehat{\mathbf{v}}_{s_m}^{\perp} = 0$, and $\mathbf{v}_{s_i}^H \mathbf{v}_{s_i}^{\perp} = 0$. Hence, it follows that $\alpha_m^{\|} = \widehat{\mathbf{v}}_{s_m}^H \mathbf{v}_{s_m}$, $\gamma_i^{\|} = \mathbf{v}_{s_i}^H \widehat{\mathbf{v}}_{s_i}$, $\|\widehat{\mathbf{v}}_{s_m}^{\perp}\|^2 = 1 - \left|\alpha_m^{\|}\right|^2$, and $\|\mathbf{v}_{s_i}^{\perp}\|^2 = 1 - \left|\gamma_i^{\|}\right|^2$. Inequality (d) follows from the fact that $\left|\gamma_i^{\|}\right| < 1$, $\left|\alpha_m^{\|}\right| < 1$, $\left|\widehat{\mathbf{v}}_{s_m}^H \mathbf{v}_{s_i}^{\perp}\right| < \left\|\mathbf{v}_{s_i}^{\perp}\right\|$ and $\left|\mathbf{v}_{s_i}^H \widehat{\mathbf{v}}_{s_m}^{\perp}\right| < \left\|\widehat{\mathbf{v}}_{s_m}^{\perp}\right\|$. Inequality (e) comes from the fact that $\left|\widehat{\mathbf{v}}_{s_m}^H \mathbf{v}_{s_i}\right|^2 < \beta$ for $i > m$ by the algorithm, and defining $\mu_m \triangleq \left\|\widehat{\mathbf{v}}_{s_m}^{\perp}\right\|^2 = 1 - \left|\mathbf{v}_{s_m}^H \widehat{\mathbf{v}}_{s_m}\right|^2$ and $\mu_i \triangleq \left\|\mathbf{v}_{s_i}^{\perp}\right\|^2 = 1 - \left|\mathbf{v}_{s_i}^H \widehat{\mathbf{v}}_{s_i}\right|^2$. Inequality (f) comes from the fact that $\forall a, b, c$, $(a+b+c)^2 \le 3(a^2+b^2+c^2)$, and finally, (g) results from the fact that $\left|\mathbf{v}_{s_m}^H \widehat{\mathbf{v}}_{s_m}\right|^2 > 1 - \epsilon$ for all $1 \le m \le M$. From the above equation, it can be observed that having $\beta = o(1)$ and $\epsilon = o(1)$ yields $\left|\mathbf{v}_{s_m}^H \mathbf{\Phi}_{s_m}\right|^2 = 1 + o(1)$.

Appendix I

Proof of (4.100)

From the definition of $\nu_i(t)$, we have

$$\begin{aligned}
\Pr\{\nu_i(t)=l_1,\nu_j(t)=l_1\} &= \Pr\left\{\nu_i(\psi)=D,\nu_j(\psi)=D,\bigcap_{l=\psi}^{t}\mathscr{X}_i^C(l),\bigcap_{l=\psi}^{t}\mathscr{X}_j^C(l)\right\} \\
&\overset{(a)}{=} \Pr\left\{\mathscr{X}_i(\psi-1),\mathscr{X}_j(\psi-1),\bigcap_{l=\psi}^{t}\mathscr{X}_i^C(l),\bigcap_{l=\psi}^{t}\mathscr{X}_j^C(l)\right\} \\
&= \Pr\left\{\mathscr{X}_i(\psi-1)\right\}\Pr\left\{\mathscr{X}_j(\psi-1)\,|\,\mathscr{X}_i(\psi-1)\right\}\times \\
&\quad \Pr\left\{\bigcap_{l=\psi}^{t}\mathscr{X}_i^C(l)\,\middle|\,\mathscr{X}_i(\psi-1),\mathscr{X}_j(\psi-1)\right\}\times \\
&\quad \Pr\left\{\bigcap_{l=\psi}^{t}\mathscr{X}_j^C(l)\,\middle|\,\mathscr{X}_i(\psi-1),\mathscr{X}_j(\psi-1),\bigcap_{l=\psi}^{t}\mathscr{X}_i^C(l)\right\} \\
&\overset{(b)}{=} \Pr\left\{\mathscr{X}_i(\psi-1)\right\}\Pr\left\{\mathscr{X}_j(\psi-1)\,|\,\mathscr{X}_i(\psi-1)\right\}\times \\
&\quad \Pr\left\{\bigcap_{l=\psi}^{t}\mathscr{X}_i^C(l)\,\middle|\,\mathscr{X}_i(\psi-1)\right\}\times \\
&\quad \Pr\left\{\bigcap_{l=\psi}^{t}\mathscr{X}_j^C(l)\,\middle|\,\mathscr{X}_i(\psi-1),\mathscr{X}_j(\psi-1),\bigcap_{l=\psi}^{t}\mathscr{X}_i^C(l)\right\} \\
&\overset{(c)}{=} \Pr\{\nu_i(t)=l_1\}\Pr\left\{\mathscr{X}_j(\psi-1)\,|\,\mathscr{X}_i(\psi-1)\right\}\times \\
&\quad \Pr\left\{\bigcap_{l=\psi}^{t}\mathscr{X}_j^C(l)\,\middle|\,\mathscr{X}_i(\psi-1),\mathscr{X}_j(\psi-1),\bigcap_{l=\psi}^{t}\mathscr{X}_i^C(l)\right\},
\end{aligned} \tag{I.1}$$

where $\psi \triangleq t-D+l_1$. In the above equation, (a) comes from the fact that the events $\nu_i(\psi)=D$ and $\mathscr{X}_i(\psi-1)$ are equivalent [1]. (b) results from the fact that conditioned on $\mathscr{X}_i(\psi-1)$, $\bigcap_{l=\psi}^{t}\mathscr{X}_i^C(l)$ is independent of $\mathscr{X}_j(\psi-1)$ [2]. Finally, (c) follows from writing $\Pr\{\mathscr{X}_i(\psi-1)\}\Pr\left\{\bigcap_{l=\psi}^{t}\mathscr{X}_i^C(l)\,\middle|\,\mathscr{X}_i(\psi-1)\right\}$ as $\Pr\{\nu_i(t)=l_1\}$. For computing $\sigma \triangleq \Pr\left\{\bigcap_{l=\psi}^{t}\mathscr{X}_j^C(l)\,\middle|\,\mathscr{X}_i(\psi-1),\mathscr{X}_j(\psi-1),\bigcap_{l=\psi}^{t}\mathscr{X}_i^C(l)\right\}$, we have

$$\Pr\left\{\bigcap_{l=\psi}^{t}\mathscr{X}_j^C(l)\,\middle|\,\mathscr{X}_j(\psi-1),\bigcap_{l=\psi}^{t}\mathscr{X}_i^C(l)\right\}=\sigma\mu+\sigma^*(1-\mu), \tag{I.2}$$

where $\sigma^* \triangleq \Pr\left\{\bigcap_{l=\psi}^{t}\mathscr{X}_j^C(l)\,\middle|\,\mathscr{X}_j(\psi-1),\bigcap_{l=\psi-1}^{t}\mathscr{X}_i^C(l)\right\}$ and

$$\mu \triangleq \Pr\left\{\mathscr{X}_i(\psi-1)\,\middle|\,\mathscr{X}_j(\psi-1),\bigcap_{l=\psi}^{t}\mathscr{X}_i^C(l)\right\}.$$

From the above equation, σ can be written as

$$\begin{aligned}
\sigma &= \frac{\Pr\left\{\bigcap_{l=\psi}^{t}\mathscr{X}_j^C(l)\,\middle|\,\mathscr{X}_j(\psi-1),\bigcap_{l=\psi}^{t}\mathscr{X}_i^C(l)\right\}-(1-\mu)\sigma^*}{\mu}\\
&\overset{(a)}{\leq} \frac{\Pr\left\{\bigcap_{l=\psi}^{t}\mathscr{X}_j^C(l)\,\middle|\,\mathscr{X}_j(\psi-1)\right\}-(1-\mu)\sigma^*}{\mu}\\
&\overset{(b)}{\leq} \frac{\Pr\left\{\bigcap_{l=\psi}^{t}\mathscr{X}_j^C(l)\,\middle|\,\mathscr{X}_j(\psi-1)\right\}-(1-\mu)\Pr\left\{\bigcap_{l=\psi}^{t}\mathscr{X}_j^C(l)\,\middle|\,\mathscr{X}_j(\psi-1),\mathscr{Z}_j\right\}}{\mu}\\
&\overset{(c)}{\approx} \Pr\left\{\bigcap_{l=\psi}^{t}\mathscr{X}_j^C(l)\,\middle|\,\mathscr{X}_j(\psi-1)\right\},
\end{aligned} \tag{I.3}$$

[1] In fact, if we have $\mathscr{X}_i(\psi-1)$, i.e., the user i is served in the $(\psi-1)$th frame, in the next frame its expiry countdown will be set to D. In other words, $\mathscr{X}_i(\psi-1)$ results in $\nu_i(\psi)=D$. By a similar argument one can conclude that $\nu_i(\psi)=D$ results in $\mathscr{X}_i(\psi-1)$. Therefore, this two events are equivalent.

[2] In fact, since in each frame M users are served with probability one, conditioned on $\mathscr{X}_i(\psi-1)$, there are $M-1$ other users which are served in the same frame. Since the rest of users are all the same for the ith user (because of the homogeneity of the network), it follows that the condition $\mathscr{X}_j(\psi-1)$ does not change the conditional probability $\Pr\left\{\bigcap_{l=\psi}^{t}\mathscr{X}_i^C(l)\,\middle|\,\mathscr{X}_i(\psi-1)\right\}$.

where $\mathscr{Z}_j$ denotes the event that user j is excluded from the network, and hence is never served. (a) comes from the fact that the event $\bigcap_{l=\psi}^{t} \mathscr{X}_i^C(l)$ reduces $\Pr\left\{\bigcap_{l=\psi}^{t} \mathscr{X}_j^C(l) \,\middle|\, \mathscr{X}_j(\psi-1)\right\}$. (b) results from the fact that

$$\sigma^* \geq \Pr\left\{\bigcap_{l=\psi}^{t} \mathscr{X}_j^C(l) \,\middle|\, \mathscr{X}_j(\psi-1), \mathscr{Z}_j\right\},$$

which is due to the fact that excluding the jth user from the network, increases the chance of user i to be served during each frame and as a result, reduces the conditional probability $\Pr\left\{\bigcap_{l=\psi}^{t} \mathscr{X}_j^C(l) \,\middle|\, \mathscr{X}_j(\psi-1), \bigcap_{l=\psi-1}^{t} \mathscr{X}_i^C(l)\right\}$. (c) follows from the fact that as $N \to \infty$, the effect of excluding the user j from the network on the conditional probability $\Pr\left\{\bigcap_{l=\psi}^{t} \mathscr{X}_j^C(l) \,\middle|\, \mathscr{X}_j(\psi-1)\right\}$ is negligible. In other words,

$$\Pr\left\{\bigcap_{l=\psi}^{t} \mathscr{X}_j^C(l) \,\middle|\, \mathscr{X}_j(\psi-1), \mathscr{Z}_j\right\} \approx \Pr\left\{\bigcap_{l=\psi}^{t} \mathscr{X}_j^C(l) \,\middle|\, \mathscr{X}_j(\psi-1)\right\}.$$

Substituting σ from the above equation in the right hand side of (I.1) yields

$$\begin{aligned}
\Pr\{\nu_i(t)=l_1, \nu_j(t)=l_1\} &\leq \Pr\{\nu_i(t)=l_1\}\Pr\left\{\mathscr{X}_j(\psi-1) \,|\, \mathscr{X}_i(\psi-1)\right\} \times \\
&\quad \Pr\left\{\bigcap_{l=\psi}^{t} \mathscr{X}_j^C(l) \,\middle|\, \mathscr{X}_j(\psi-1)\right\} \\
&\overset{(a)}{=} \Pr\{\nu_i(t)=l_1\}\Pr\{\nu_j(t)=l_1\}\frac{\Pr\left\{\mathscr{X}_j(\psi-1) \,|\, \mathscr{X}_i(\psi-1)\right\}}{\Pr\{\mathscr{X}_j(\psi-1)\}} \\
&\overset{(b)}{\approx} \Pr\{\nu_i(t)=l_1\}\Pr\{\nu_j(t)=l_1\}\frac{M-1}{M}, \qquad \text{(I.4)}
\end{aligned}$$

where (a) follows from the fact that $\Pr\left\{\mathscr{X}_j(\psi-1)\right\}\Pr\left\{\bigcap_{l=\psi}^{t} \mathscr{X}_i^C(l) \,\middle|\, \mathscr{X}_i(\psi-1)\right\} = \Pr\{\nu_i(t)=l_1\}$, and (b) results from the fact that $\Pr\{\mathscr{X}_j(\psi-1)\} \sim \frac{M}{N}$ (which we have shown earlier in the paper in (4.112)) and also $\Pr\left\{\mathscr{X}_j(\psi-1) \,|\, \mathscr{X}_i(\psi-1)\right\} \sim \frac{M-1}{N}$. The latter is due to the fact that conditioned on $\mathscr{X}_i(\psi-1)$, the network can be considered as a $(N-1)$-user broadcast channel, in which $(M-1)$ beams are to be assigned to $(M-1)$ users. Hence, the probability of assigning a beam to a randomly selected user is $\frac{M-1}{N-1} \approx \frac{M-1}{N}$. From (I.4), (4.100) easily follows.

Appendix J

For upper-bounding the right hand side of (4.108), we use the fact that

$$G_\nu(l-1) \leq \frac{M(D-l+1)}{N}, \tag{J.1}$$

which follows from the fact that $f_\nu(l) \leq \frac{M}{N}$, $\forall l$, and consequently, $G_\nu(l-1) = \sum_{\nu=l}^{D} f_\nu(\nu) \leq \frac{M(D-l+1)}{N}$. Having the above equation, RH (4.108) can be upper-bounded as follows:

$$
\begin{aligned}
\text{RH (4.108)} \quad &\leq M\sum_{n=1}^{N}\binom{N-1}{n-1}\left(\frac{q}{M}\right)^n\left(1-\frac{q}{M}\right)^{N-n}\prod_{i=1}^{n-1}\left(\frac{M(D-l+1)}{N}+\frac{Mi}{N}\right)\\
&= \eta\sum_{n=0}^{N-1}\binom{N-1}{n}\left(\frac{q}{M}\right)^n\left(1-\frac{q}{M}\right)^{N-n}\prod_{i=1}^{n}\left(\frac{M(D-l+1)}{N}+\frac{Mi}{N}\right)\\
&= \eta\sum_{n=0}^{N-1}\binom{N-1}{n}\left(\frac{q}{M}\right)^n\left(1-\frac{q}{M}\right)^{N-n}\left(\frac{M(D-l+1)}{N}\right)^n\times\\
&\quad \prod_{i=1}^{n}\left(1+\frac{1}{D-l+1}i\right)\\
&\overset{(a)}{\leq} \eta\sum_{n=0}^{N-1}\binom{N-1}{n}\left(\frac{q}{M}\right)^n\left(1-\frac{q}{M}\right)^{N-n}\left(\frac{M(D-l+1)}{N}\right)^n\prod_{i=1}^{n}(1+i)\\
&= \eta\sum_{n=0}^{N-1}\binom{N-1}{n}\left(\frac{q}{M}\right)^n\left(1-\frac{q}{M}\right)^{N-n}\left(\frac{M(D-l+1)}{N}\right)^n(n+1)!\\
&\overset{(b)}{=} \eta\left(1-\frac{q}{M}\right)^N\sum_{n=0}^{N-1}\frac{(N-1)!}{N^n(N-n-1)!}(n+1)\left[(D-l+1)\eta\right]^n\\
&\overset{(c)}{\leq} \eta\left(1-\frac{q}{M}\right)^N\sum_{n=0}^{N-1}(n+1)\left[(D-l+1)\eta\right]^n\\
&\overset{(d)}{\leq} \eta\left(1-\frac{q}{M}\right)^N\frac{1}{\left[1-(D-l+1)\eta\right]^2}\\
&\sim \eta\left(1-\frac{q}{M}\right)^N\\
&\overset{(e)}{\leq} \frac{M(\ln N)^2}{N}e^{-(\ln N)^{1.5}},
\end{aligned}
\tag{J.2}
$$

where $\eta = \frac{q}{1-\frac{q}{M}}$. In the above equation, (a) follows from the fact that $D-l+1 \geq 1$ (since $l \leq D$). (b) follows from writing $\binom{N-1}{n}$ as $\frac{(N-1)!}{n!(N-n-1)!}$ and canceling out $n!$ by $(n+1)!$, which leaves the term $n+1$ in the numerator. (c) results from the fact that $\frac{(N-1)!}{(N-n-1)!} = (N-1)(N-2)\cdots(N-n) \leq N^n$, which leads to having $\frac{(N-1)!}{N^n(N-n-1)!} \leq 1$. (d) follows from upper-bounding the sum $\sum_{n=0}^{N-1}(n+1)\left[(D-l+1)\eta\right]^n$ by an infinite sum $\sum_{n=0}^{\infty}(n+1)\left[(D-l+1)\eta\right]^n$ which equals to $\frac{1}{[1-(D-l+1)\eta]^2}$, noting that since $D-l \leq D-D_0 \leq 9\sqrt{N}(\ln N)^4$ and $\eta \sim q \sim Mp \leq \frac{(\ln N)^2}{N}$ [1], we have $(D-l+1)\eta \ll 1$. Finally, (e) results from upper-bounding $\eta \sim Mp$ by $\frac{M(\ln N)^2}{N}$, which is explained in the footnote, and also approximating $\left(1-\frac{q}{M}\right)^N$ by $e^{-\frac{Nq}{M}} \sim e^{-Np}$ which is upper-bounded by $e^{-(\ln N)^{1.5}}$, which is due to the fact that as $\Upsilon < \frac{P}{M}\left(\ln N-(M+0.5)\ln\ln N\right)$, $p = \frac{e^{-M\Upsilon/P}}{\left(1+\frac{M\Upsilon}{P}\right)^{M-1}} > \frac{(\ln N)^{1.5}}{N}$.

[1] As it is shown in the paper, since $\frac{P}{M}\left(\ln N-(M+1)\ln\ln N\right) < \Upsilon < \frac{P}{M}\left(\ln N-(M+0.5)\ln\ln N\right)$, we have $p = \frac{e^{-M\Upsilon/P}}{\left(1+\frac{M\Upsilon}{P}\right)^{M-1}} < \frac{(\ln N)^2}{N}$.

Appendix K

We have

$$\begin{aligned}\Pr\{\mathfrak{A}\} &= \Pr\{\|\mathbf{G}\|^2_{\max} \leq t\} \\ &= \left(\Pr\{\|\mathbf{G}\|^2_k \leq t\}\right)^N. \end{aligned} \tag{K.1}$$

Using the fact that $\|\mathbf{G}_k\|^2$ has Chi-square distribution with $2M$ degrees of freedom [32], we have

$$\Pr\{\|\mathbf{G}\|^2_k \leq t\} = 1 - \sum_{m=0}^{M-1} \frac{t^m}{m!} e^{-t}. \tag{K.2}$$

Substituting $t = \ln N + (M-3)\ln\ln N$, the above equation can be rewritten as

$$\Pr\{\|\mathbf{G}\|^2_k \leq t\} = 1 - \frac{\ln^2 N}{N}\left[1 + O\left(\frac{\ln\ln N}{\ln N}\right)\right]. \tag{K.3}$$

Substituting in (K.1), it is concluded that

$$\begin{aligned}\Pr\{\mathfrak{A}\} &= \left(1 - \frac{\ln^2 N}{N}\left[1 + O\left(\frac{\ln\ln N}{\ln N}\right)\right]\right)^N \\ &= o\left(\frac{1}{N}\right). \end{aligned} \tag{K.4}$$

Appendix L

Calculation of $\Pr\{\mathfrak{B}\}$:

Consider a randomly selected user k. Using (K.2) and (K.3), we have

$$\Pr\{k \in \mathcal{S}\} = \frac{\ln^2 N}{N} \left[1 + O\left(\frac{\ln \ln N}{\ln N} \right) \right]. \tag{L.1}$$

Therefore, $L = |\mathcal{S}|$ is a Binomial random variable with parameters (N, p), where $p \triangleq \Pr\{k \in \mathcal{S}\}$. Using the Gaussian approximation for Binomial distribution, we have

$$\begin{aligned}
\Pr\{\mathfrak{B}\} &= \sum_{l=\lfloor \ln N \rfloor + 1}^{N} \binom{N}{l} p^l (1-p)^{N-l} \\
&\approx Q\left(\frac{\ln N - Np}{\sqrt{Np(1-p)}} \right) \\
&= 1 - Q\left(\frac{Np - \ln N}{\sqrt{Np(1-p)}} \right) \\
&\stackrel{(a)}{\geq} 1 - e^{-\frac{(Np - \ln N)^2}{2Np(1-p)}} \\
&\geq 1 - Ne^{-\frac{Np}{2}} \\
&\stackrel{(b)}{=} 1 - o\left(1/N\right),
\end{aligned} \tag{L.2}$$

where (a) results from the fact that $Q(x) \leq e^{-x^2/2}$ and (b) follows from the fact that $Np \sim \frac{\ln^2 N}{(M-1)!}$, which incurs that $e^{-Np/2} = o(N^{-2})$.

Calculation of $\Pr\{\mathfrak{C}|\mathfrak{B}\}$:

Using equation (E.7) in Appendix E, we have

$$\delta(\mathbb{G}^H) = \delta(\mathbf{\Psi}) = \frac{1}{\prod_{i=1}^{M-1} \beta_i}, \tag{L.3}$$

where $\mathbf{\Psi} \triangleq \left[\mathbf{v}_{s_1}^H | \cdots | \mathbf{v}_{s_M}^H\right]$, and β_i denotes the projection of $\mathbf{v}_{s_{i+1}}$ over $\boldsymbol{\mathcal{P}}_{i+1}^{\perp}$, which denotes the null space of $\boldsymbol{\mathcal{P}}_{i+1}$, the subspace spanned by $\{\mathbf{v}_{s_j}\}_{j=1}^{i}$. Defining $\epsilon \triangleq 1 - \frac{1}{1+2M(\ln N)^{-\frac{1}{2(M-1)}}}$, from (L.3), $\Pr\{\mathfrak{C}^C|\mathfrak{B}\}$ can be written as

$$\begin{aligned} \Pr\{\mathfrak{C}|\mathfrak{B}\} &= \Pr\left\{\prod_{i=1}^{M-1} \beta_i < 1-\epsilon \,\middle|\, \mathfrak{B}\right\} \\ &\overset{(a)}{\leq} \sum_{i=1}^{M-1} \Pr\left\{\beta_i < 1 - \frac{\epsilon}{M} \,\middle|\, \mathfrak{B}\right\} \\ &\overset{(b)}{\leq} \sum_{i=1}^{M-1} \Pr\left\{\beta_i < 1 - (\ln N)^{-\frac{1}{2(M-1)}} \,\middle|\, \mathfrak{B}\right\}, \end{aligned} \tag{L.4}$$

where (a) follows from the fact that the event $\prod_{i=1}^{M-1} \beta_i < 1-\epsilon$ is a subset of the event $\bigcup_{i=1}^{M-1} \left\{\beta_i < 1 - \frac{\epsilon}{M}\right\}$. To show this, we observe that if none of the events $\left\{\beta_i < 1 - \frac{\epsilon}{M}\right\}_{i=1}^{M-1}$ occur, it means that $\prod_{i=1}^{M-1} \beta_i > \left(1 - \frac{\epsilon}{M}\right)^M > 1-\epsilon$. Also, (b) results from the fact that as $N \to \infty$, $2M(\ln N)^{-\frac{1}{2(M-1)}} < 1$.

From the algorithm, β_i can be written as $1 - \min_{k \in \mathcal{S}_{i+1}} z_{k,i}$, where $z_{k,i}$ denotes the projection of $\mathbf{v}_k$ over $\boldsymbol{\mathcal{P}}_{i+1}$. The probability density function (pdf) of $z_{k,i}$ is given in the equation (E.9) in Appendix E as [1]

$$p_{z_{k,i}}(z) = \frac{\Gamma(M)}{\Gamma(M-i)\Gamma(i)} z^{i-1}(1-z)^{M-i-1}. \quad \text{(L.5)}$$

Since $\mathbf{V}_k$'s are i.i.d. random variables (since the channel vector of users are independent of each other), it follows that $z_{k,i}$'s are also i.i.d.. Hence, defining $\theta \triangleq 1 - (\ln N)^{-\frac{1}{2(M-1)}}$, we have

$$\begin{aligned} \Pr\left\{\beta_i < \theta \middle| \mathfrak{B}\right\} &\overset{(a)}{=} \left(\Pr\left\{z_{k,i} > 1-\theta\right\}\right)^{L-i} \\ &= \left(1 - I_{i,M-i}(1-\theta)\right)^{L-i} \\ &\overset{(b)}{\leq} \left(1-(1-\theta)^{M-1}\right)^{L-i} \\ &= \left(1 - \frac{1}{\sqrt{\ln N}}\right)^{L-i} \\ &\overset{(c)}{\leq} \left(1 - \frac{1}{\sqrt{\ln N}}\right)^{\ln N - i} \\ &\sim e^{-\sqrt{\ln N}} \\ &= o\left(\frac{1}{\ln N}\right). \end{aligned} \quad \text{(L.6)}$$

In the above equation, (a) results from the fact that $|\mathcal{S}_{i+1}| = L - i$, and (b) follows from the the fact that $I_{i,M-i}(\theta) \geq I_{M-1,1}(\theta) = \theta^{M-1}$. (c) comes from the fact that conditioned on $\mathfrak{B}$, $L > \ln N$. Combining the avove equation with (L.4) yields $\Pr\{\mathfrak{C}^C|\mathfrak{B}\} = o\left(\frac{1}{\ln N}\right)$, which implies that $\Pr\{\mathfrak{C}|\mathfrak{B}\} = 1 + o\left(\frac{1}{\ln N}\right)$.

Computation of $\Pr\{\mathfrak{D}|\mathfrak{B},\mathfrak{C}\}$

To compute $\Pr\{\mathfrak{D}|\mathfrak{B},\mathfrak{C}\}$, we first note that since the norm and direction of circularly symmetric complex Gaussian vectors are independent of each other and hav-

[1] Note that since the norm and direction of circularly symmetric complex Gaussian vectors are independent of each other, the distribution of $z_{k,i}$ is independent of the condition $\mathfrak{B}$.

ing the facts that $\mathfrak{B}$ and $\mathfrak{D}$ depend solely on the norm of $\{\mathbf{G}_k\}_{k=1}^N$ and $\mathfrak{C}$ depends only on the direction of these vectors, it follows that $\mathfrak{B}$ and $\mathfrak{D}$ are independent of $\mathfrak{C}$. Therefore, we can ignore $\mathfrak{C}$ in the condition and write

$$\begin{aligned}
\Pr\{\mathfrak{D}|\mathfrak{B},\mathfrak{C}\} &= \Pr\{\mathfrak{D}|\mathfrak{B}\} \\
&= \frac{\Pr\{\mathfrak{D},\mathfrak{B}\}}{\Pr\{\mathfrak{B}\}} \\
&\overset{(a)}{\geq} \frac{\Pr\{\mathfrak{D}\} - \Pr\{\mathfrak{B}^C\}}{\Pr\{\mathfrak{B}\}}.
\end{aligned} \tag{L.7}$$

Since we have already computed $\Pr\{\mathfrak{B}\}$ in this appendix, it suffices to compute $\Pr\{\mathfrak{D}\}$. $\Pr\{\mathfrak{D}^C\}$ is computed in Appendix A, and shown to scale as $O\left(\frac{1}{\ln N}\right)$. Hence,

$$\begin{aligned}
\Pr\{\mathfrak{D}|\mathfrak{B},\mathfrak{C}\} &= \frac{1 + O\left(\frac{1}{\ln N}\right) + o\left(\frac{1}{N}\right)}{1 + o\left(\frac{1}{N}\right)} \\
&= 1 + O\left(\frac{1}{\ln N}\right).
\end{aligned} \tag{L.8}$$

Appendix M

Conditioned on $\mathfrak{C}$, we have $\prod_{i=1}^{M-1} \beta_i > \frac{1}{1+2M(\ln N)^{-\frac{1}{2(M-1)}}} = 1-\epsilon$. Since $\beta_i \leq 1$, this incurs that $\beta_i \geq 1-\epsilon$, $\forall i$. In other words, $\gamma_i \leq \epsilon$, where γ_i denotes the projection of $\mathbf{v}_{s_{i+1}}$ over $\boldsymbol{\mathcal{P}}_{i+1}$. Now, consider $\{\boldsymbol{\Phi}_j\}_{j=1}^{i}$ as j orthonormal bases for $\boldsymbol{\mathcal{P}}_{i+1}$. Since $\mathbf{v}_{s_j} \in \boldsymbol{\mathcal{P}}_{i+1}$, $\forall j \leq i$, we can write

$$\mathbf{v}_{s_j} = \sum_{l=1}^{i} a_l \boldsymbol{\Phi}_l, \tag{M.1}$$

where $\sum_{l=1}^{i} |a_l|^2 = 1$. Therefore, for all $i, j,\ j \leq i$, we have

$$\begin{aligned} z(\mathbf{G}_{s_{i+1}}, \mathbf{G}_{s_j}) &= z(\mathbf{v}_{s_{i+1}}, \mathbf{v}_{s_j}) \\ &= \left|\mathbf{v}_{s_{i+1}} \mathbf{v}_{s_j}^H\right|^2 \\ &= \left|\sum_{l=1}^{i} a_l^H \left(\mathbf{v}_{s_{i+1}} \boldsymbol{\Phi}_l^H\right)\right|^2 \\ &\overset{(a)}{\leq} \sum_{l=1}^{i} \left|\mathbf{v}_{s_{i+1}} \boldsymbol{\Phi}_l^H\right|^2 \\ &= \gamma_i \\ &\leq \epsilon, \end{aligned} \tag{M.2}$$

where (a) follows from the fact that $\left|\sum_{l=1}^{i} a_l b_l\right|^2 \leq \left(\sum_{l=1}^{i} |a_l|^2\right)\left(\sum_{l=1}^{i} |b_l|^2\right)$, noting that $\sum_{l=1}^{i} |a_l|^2 = 1$.

Bibliography

[1] E. Telatar, "Capacity of multi-antenna Gaussian channels," *European Trans. on Telecommunications*, vol. 10, no. 6, pp. 585– 595, November 1999.

[2] G. J. Foschini and M.J. Gans, "On limits of wireless communications in a fading environment when using multiple antennas," *Wireless Personal Commun.*, vol. 6, pp. 311–335, 1998.

[3] V. Tarokh, N. Seshadri, and A.R. Calderbank, "Space-time codes for high data rate wireless communications: Performance analysis and code construction," *IEEE Trans. on Inform. Theory*, vol. 44, No. 2, pp. 744–765, March 1998.

[4] G. Caire and S. Shamai, "On the achievable throughput of a multiantenna Gaussian broadcast channel," *IEEE Trans. on Inform. Theory*, vol. 49, pp. 1691 1706,, July 2003.

[5] S. Vishwanath, N. Jindal, and A. Goldsmith, "Duality, achievable rates, and sum-rate capacity of Gaussian MIMO broadcast channels," *IEEE Trans. on Inform. Theory*, vol. 49, no. 10, pp. 2658–2668, October 2003.

[6] P. Viswanath and D. N. C. Tse, "Sum capacity of the vector Gaussian broadcast channel and uplink-downlink duality," *IEEE Trans. on Inform. Theory*, vol. 49, pp. 1912–1921, Aug. 2003.

[7] Wei Yu and John Cioffi, "Sum capacity of vector Gaussian broadcast channels," *IEEE Trans. on Inform. Theory*, vol. 50, pp. 1875–1892, Sep. 2004.

[8] H. Weingarten, Y. Steinberg, and S. Shamai, "The capacity region of the Gaussian MIMO broadcast channel," in *Conference on Information Sciences and Systems*, 2004.

[9] M. Costa, "Writing on dirty paper," *IEEE Trans. on Inform. Theory*, vol. 29, pp. 439–441, May 1983.

[10] R. Zamir, S. Shamai, and U. Erez, "Nested linear/lattice codes for structured multiterminal binning," *IEEE Trans. Inform. Theory*, vol. 48, pp. 12501276, June 2002.

[11] W. Yu and J. Cioffi, "Trellis precoding for the broadcast channel," in *Proceedings of 2001 IEEE GlobeCom*, Nov. 2001, pp. 1344–1348.

[12] J. Kusuma and K. Ramchandran, "Communicating by cosets and applications to broadcast," in *Proc. Conference on Information Sciences and Systems*, Princeton University, 2002.

[13] U. Erez, S. Shamai, , and R. Zamir, "Capacity and lattice-strategies for cancelling known interference," *IEEE Trans. on Inform. Theory*, vol. 51, pp. 3820–3833, Nov. 2005.

[14] U. Erez and Stephan ten Brink, "Approaching the dirty paper limit for canceling known interference," in *41st Annual Allerton Conference on Communication, Control, and Computing*, Oct. 2003.

[15] H. Farmanbar and Amir K. Khandani, "On precoding for channels with known interference at the transmitter," in *Proc. Conference on Information Sciences and Systems*, The Johns Hopkins University, Baltimore, MD, 2005.

[16] R. Knopp and P. Humblet, "Information capacity and power control in single-cell multiuser communications," in *IEEE Int. Conf. Commun.*, June 1995, vol. 1, pp. 331–335.

[17] P. Viswanath, D. Tse, and R. Laroia, "Opportunistic beamforming using dumb antennas," *IEEE Trans. on Inform. theory*, vol. 48, pp. 1277–1294, June 2002.

[18] Zhenyu Tu and R. S. Blum, "Multiuser diversity for a dirty paper approach," *IEEE Communications Letters*, vol. 7, no. 8, pp. 370–372, Aug. 2003.

[19] Mohammad A. Maddah-Ali, Mehdi Ansari, and Amir K. Khandani, "An efficient algorithm for user selection and signalling over MIMO multiuser systems," Technical Report UW-ECE 2005-09, 2004.

[20] Derek Gerlach and A. Paulraj, "Adaptive transmitting antenna arrays with feedback," *IEEE Signal Processing Letters*, vol. 1, no. 10, pp. 150–152, October 1994.

[21] H. Hochwald and S. Vishwanath, "Space-time multiple access: Linear growth in the sum rate," in *40th Annual Allerton Conf. Communications, Control and Computing,*, Allerton, IL, Oct. 2002.

[22] C. B. Peel, B. M. Hochwald, , and A. L. Swindlehurst, “A vector-perturbation technique for near-capacity multiple-antenna multi-user communications-part i: Channel inversion and regularization,” *IEEE Trans. on Communications*, Vol 53, pp. 195-202, Jan. 2005.

[23] H. Viswanathan, S. Venkatesan, and H. Huang, “capacity evaluation of cellular networks with known-interference cancellation,” *IEEE J. Select. Areas Commun.*, vol. 21, pp. 802–811, June 2003.

[24] Goran Dimic and Nicholas D. Sidiropoulos, “Low-complexity downlink beamforming for maximum sum capacity,” in *Proceeding of IEEE International Conference on Acoustics, Speech, and Signal Processing (ICASSP 2004)*, May 2004, vol. 4, pp. iv–701 – iv–704.

[25] Taesang Yoo and Andrea Goldsmith, “On the optimality of Multi-Antenna Broadcast Scheduling Using Zero-Forcing Beamforming,” *IEEE JSAC Special Issue on 4G Wireless Systems*, Vol. 24, No. 3, March 2006.

[26] M. Sharif and B. Hassibi, “On the capacity of MIMO broadcast channel with partial side infonnation,” *IEEE Trans. on Inform. Theory*, vol. 51, pp. 506–522, Feb. 2005.

[27] Wei Yu and Wonjong Rhee, “Degrees of Freedom in Wireless Multiuser Spatial Multiplex Systems with Multiple Antennas,” *IEEE Trans. on Communications*, vol. 54, no. 10, pp. 1747-1753, October 2006.

[28] Manish Airy, Robert W. Heath Jr, and Sanjay Shakkottai, “Multi-user diversity for the multiple antenna broadcast channel with linear receivers: Asymp-

totic analysis," in *Proceeding of 38th IEEE Asimolar Conference on Signals, systems, and Computers*, Pacific Grove, CA, November 2004.

[29] I. S. Gradshteyn and I. M. Ryzhik, *Tables of Integrals, Series, and Products*, New York: Academic Press, 1980, Section 8.97.

[30] Alireza Bayesteh and Amir K. Khandani, "How much feedback is required in MIMO broadcast channels?," *submitted to IEEE Trans. Inform. Theory*, UW Technical Report UW-ECE 2007-10 (available online at http://cst.uwaterloo.ca/), March 2007.

[31] A. Amraoui, G. Kramer, and S. Shamai, "Coding for the MIMO broadcast channel," in *Proc. of International Symposium on Information Theory*, Yokohama, Japan, June 29 - July 4 2003, p. 296.

[32] Sheldon Ross, *A First Course in Probability*, Prentice-Hall Inc., sixth edition, 2002.

[33] Lizhong Zheng and D. N. C. Tse, "Diversity and multiplexing: A fundamental tradeoff in multiple-antenna channels," *IEEE Trans. Inform. Theory*, vol. 47, no. 6, pp. 1073–1096, May 2003.

[34] M. Sharif and B. Hassibi, "A comparison of time-sharing, DPC, and beamforming for MIMO broadcast channels with many users," *IEEE Trans. on Communications*, Vol. 55, pp. 11-15, Jan. 2007.

[35] D. Tse, "Optimal power allocation over parallel Gaussian broadcast channels," Unpublished.

[36] T.L. Marzetta and B.M Hochwald, "Capacity of a mobile multiple-antenna communication link in Rayleigh flat fading," vol. 45, pp. 139–157, Jan. 1999.

[37] Mahmoud Taherzadeh, Amin Mobasher, and Amir Keyvan Khandani, "Communication over MIMO broadcast channels using lattice-basis reduction," To Appear in *IEEE Transactions on Information Theory*, Dec. 2007.

[38] Ingram Olkin, Leon J. Gleser, and Cyrus Derman, *Probability Models and Applications*, McMillian College Publishing Company Inc., second edition, 1994, Page 460.

[39] N. Jindal and A. Goldsmith, "Dirty paper coding vs. TDMA for MIMO broadcast channels," in *IEEE Int. Conf. Commun.*, June 2004, vol. 2, pp. 682–686.

[40] S. Jafar and A. Goldsmith, "Isotropic fading vector broadcast channels: The scalar upperbound and loss in degrees of freedom," *IEEE Trans. on Inform. Theory*, vol. 51, pp. 848 – 857, March 2005.

[41] Shahab Sanayei and Aria Nosratinia, "Opportunistic Downlink Transmission With Limited Feedback," *IEEE Trans. on Inform. Theory*, Vol. 53, No. 11, Nov. 2007, pp. 4363-4372.

[42] Shahab Sanayei and Aria Nosratinia, "Opportunistic beamforming with limited feedback," *Proc. of the thirty-ninth Asilomar Conference on Signals, Systems and Computers*, Nov. 2005, pp. 648-652.

[43] Rajiv Agarwal, Chan-Soo Hwang, and John Cioffi, "Scalable Feedback Protocol for Achieving Sum-Capacity of the MIMO BC with Finite Feedback," *Submitted to ICC2007*, Stanford Technical Report, Sep 2006.

[44] J. Diaz, O. Simeone, and Y. Bar-Ness, "Asymptotic Analysis of Reduced-Feedback for MIMO Gaussian Broadcast Channels," *submitted to IEEE Trans. On Inform. Theory (IT607)*, April 5, 2006, accepted pending revision, September 2006.

[45] Nihar Jindal, "MIMO broadcast channels with finite rate feedback," *IEEE Trans. on Inform. Theory*, Vol. 52, No. 11, Nov. 2006, pp. 5045-5060.

[46] Nihar Jindal, "A feedback reduction technique for MIMO broadcast channels," in *Proc. of International Symposium on Information Theory*, Seattle, WA, July 2006, pp. 2699-2703.

[47] N. Ravindran and N. Jindal, "Limited Feedback-based Block Diagonalization for the MIMO Broadcast Channel," *Submitted to IEEE Journal Sel. Areas in Communications,* Nov. 2007.

[48] G. Caire, N. Jindal, M. Kobayashi, and N. Ravindran, " Multiuser MIMO Downlink Made Practical: Achievable Rates with Simple Channel State Estimation and Feedback Schemes," *Submitted to IEEE Trans. on Inform. Theory*, Nov. 2007.

[49] A. Lapidoth, S. Shamai, and M. Wigger, "On the capacity of a MIMO fading broadcast channel with imperfect transmitter side-information," in *Proceedings of Allerton Conf. on Commun., Control, and Comput.*, Sept. 2005.

[50] Alireza Bayesteh and Amir K. Khandani, "On the user selection for MIMO broadcast channels," *To Appear in IEEE Trans. Inform. Theory*, March 2008, Available online at http://cst.uwaterloo.ca/, Technical Report UW-ECE 2005-16.

[51] Taesang Yoo and Nihar Jindal and Andrea Goldsmith , "Multi-antenna broadcast channels with limited feedback and user selection," *IEEE Journal Sel. Areas in Communications*, Vol. 25, No. 7, pp. 1478-1491, Sept. 2007.

[52] D. J. Love, R. W. Heath Jr., W. Santipach, and M. L. Honig , "What is the value of limited feedback for MIMO channels?," *IEEE Commun. Mag.*, vol. 42, no. 10, pp. 54–59, Oct. 2004.

[53] Mehdi Ansari Sadrabadi, Amir K. Khandani and Farshad Lahouti, "Channel feedback quantization for high data rate MIMO systems," *IEEE Trans. on Wireless Communications*, vol. 5, no. 12, pp. 3335–3338, Dec. 2006.

[54] J.C. Roh and B. D. Rao, "Multiple antenna channels with partial channel state information at the transmitter," *IEEE Trans. Wireless Commun.*, vol. 3, no. 2, pp. 677–688, March 2004.

[55] V. Lau, Y. Liu, and T.-A. Chen, "On the design of MIMO block-fading channels with feedback-link capacity constraint," *IEEE Trans. Commun.*, vol. 52, no. 1, pp. 62–70, Jan. 2004.

[56] P. Bender, P. Black, M. Grob, R. Padovani, N. Sindhushayana, and A. Viterbi, "CDMA/HDR: A bandwidth efficient high-speed wireless data service for nomadic users," *IEEE Communications Magazine*, pp. 70–77, July 2000.

[57] A. Jalali, R. Padovani, and R. Pankaj , "Data throughput of CDMA/HDR: A high efficiency, high data rate personal wireless system," in *Proc. IEEE Vehicular Tech. Conference*, vol. 3, pp. 1854–1858, May 2000.

[58] X. Liu, E. N. Chong, and N. B. Shroff, "Opportunistic transmission scheduling with resource-sharing constraints in wireless networks," *IEEE JSAC*, vol. 19, pp. 2053–2064, Oct. 2001.

[59] S. Borst, "User-level performance of channel-aware scheduling algorithms in wireless data networks," *IEEE/ACM Trans. Networking*, vol. 13, pp. 636–647, June 2005.

[60] R. Nnopp and P. A. Humblet, "Information capacity and power control in single-cell multiuser communications," *IEEE ICC'95*, vol. 1, pp. 331–335, June 1995.

[61] I. Bettesh and S. Shamai, "A low delay algorithm for the multiple access channel with Rayleigh fading," *in Proc. IEEE Personal, Indoor and Mobile Radio Commun.*, vol. 3, pp. 1367–1372, Sept. 1998.

[62] R. A. Berry and R. G. Gallager, "Communication over fading channels with delay constraints," *IEEE Trans. Inform. Theory*, vol. 48, pp. 1135–1149, May 2002.

[63] M. Andrews, N. Numaran, N. Ramanan, A.L. Stoylar, R. Vijayakumar and P. Whiting, "Providing quality of service over a shared wireless link," *IEEE Commun. Mag.*, vol. 39, pp. 150–154, Feb. 2001.

[64] R. Srinivasan and J. S. Baras, "Understanding the trade-off between multiuser diversity gain and delay - an analytical approach," *IEEE Vehicular Technology Conference*, vol. 5, pp. 2543–2547, May 2004.

[65] D. Wu and R. Negi , "Utilizing multiuser diversity for efficient support of quality of service over a fading channel," *IEEE Trans. Vehicular Techn.*, vol. 54, pp. 1198–1206, May 2005.

[66] P.N. Gopala and H. El Gamal, "On the throughput-delay tradeoff in cellular multicast," *International Conference on Wireless Networks, Communications and Mobile Computing*, vol. 2, pp. 1401–1406, June 2005.

[67] A. El Gamal, J. Mammen, B. Prabhakar, and D. Shah, "Optimal throughput-delay scaling in wireless networks - part I: The fluid model," *IEEE Trans. on Information Theory*, vol. 52, no. 6, pp. 2568–2592, June 2006.

[68] N. Bansal and Z. Liu, "Capacity, delay and mobility in wireless ad-hoc networks," *in Proc. IEEE INFOCOM*, April 2003, pp. 1553–1563.

[69] S. Toumpis and A. J. Goldsmith, "Large wireless networks under fading, mobility, and delay constraints," *in Proc. IEEE INFOCOM*, March 2004, pp. 609–619.

[70] G. Sharma L. Xiaojun, R. R. Mazumdar, and N. B. Shroff, "Degenerate delay-capacity tradeoffs in ad-hoc networks with brownian mobility," *IEEE Trans. on Information Theory*, vol. 52, no. 6, pp. 2777–2784, June 2006.

[71] M. J. Neely and E. Modiano, "Capacity and delay tradeoffs for ad hoc mobile networks," *IEEE Trans. on Information Theory*, vol. 51, no. 6, pp. 1917–1937, June 2005.

[72] M. Airy, S. Shakkottai and R. Heath, "Limiting queuing models for scheduling in multi-user MIMO systems," *IASTED Conference on Communications, Internet and Information Technology*, Scottsdale, AZ, November 17–19, 2003.

[73] Manish Airy, Sanjay Shakkottai and Robert W. Heath Jr, "Scheduling for the MIMO Broadcast Channel: Delay-Capacity Tradeoff," Preprint.

[74] Giuseppe Caire, Ralf R. Mller and Raymond Nnopp, "Hard Fairness Versus Proportional Fairness in Wireless Communications: The Single-Cell Case," *IEEE Trans. on Information Theory*, vol. 53, no. 4, pp. 1366–1385, April 2007.

[75] Masoud sharif and Babak Hassibi , "A delay analysis for opportunistic transmission in fading broadcast channels," in *Proc. IEEE, INFOCOM*, vol. 4, pp. 2720–2730, March 2005.

[76] D. Wu and R. Negi, "Effective capacity: a wireless link model for support of quality of service," *IEEE Trans. Wireless Commun.*, vol. 2, pp. 630–643, July 2003.

[77] D. Tse, "Optimal power allocation over parallel Gaussian broadcast channels." Unpublished.

[78] D. Bertsekas and R. Gallagher, *Data Networks.* Prentice Hall, Englewood Cliffs, NJ, 2nd edition, 1991.

[79] J. Lee and N. Jindal, "High SNR Analysis for MIMO Broadcast Channels: Dirty Paper Coding vs. Linear Precoding," *IEEE Trans. on Inform. Theory*, Vol. 53, No. 12, pp. 4787-4792, Dec. 2007.

[80] C. Swannack, E. Uysal-Biyikoglu, and G. W. Wornell, "Low Complexity Multiuser Scheduling for Maximizing Throughput in the MIMO Broadcast Channel," *in Proc. Allerton Conf. Commun., Contr., and Computing*, Oct. 2004.

[81] Ming Kang and Mohamed-Slim Alouini, "Capacity of MIMO Rician Channels," *Trans. on Wireless Communi.*, Vol. 50, pp. 112-122, Jan. 2006.

[82] M. Godavarti and A. O. Hero III, "Multiple antenna capacity in a deterministic Rician fading channel," *Submitted to IEEE Trans. on Information Theory*

[83] Mahesh Godavarti, Alfred O. Hero, III, and Thomas L. Marzetta, "Min-Capacity of a Multiple-Antenna Wireless Channel in a Static Rician Fading Environment," *IEEE Trans. on Communi.*, Vol. 4, pp. 1715-1723, July 2005.

[84] Mahesh Godavarti, Thomas L. Marzetta, and Shlomo Shamai (Shitz), "Capacity of a Mobile Multiple-Antenna Wireless Link With Isotropically Random Rician Fading," *IEEE Transactions on Information Theory*, Vol. 49, pp. 3330-3334, Dec. 2003.