Branched Covering Constructions and the Symplectic Geography Problem

by

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I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

We apply branched covering techniques to construct minimal simply-connected symplectic 4-manifolds with small χ_h values. We also use these constructions to provide an alternate proof that for each $s \geq 0$, there exists a positive integer $\lambda(s)$ such that each pair $(j, 8j+s)$ with $j \geq \lambda(s)$ is realized as $(\chi_h(M), c_1^2(M))$ for some minimal simply-connected symplectic M. The smallest values of $\lambda(s)$ currently known to the author are also explicitly computed for $0 \leq s \leq 99$. Our computations in these cases populate 19 952 points in the (χ, c)-plane not previously realized in the existing literature.

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Chapter 1

Introduction

The two main integer topological invariants used in the study of 4-dimensional manifolds are the signature σ and the well-known Euler characteristic e. For closed 4-manifolds which admit either complex or symplectic structures, we can also define the integer valued holomorphic Euler characteristic χ_h and the square of the first Chern class c_1^2 by

$$\chi_h = \frac{e+\sigma}{4} \qquad c_1^2 = 2e + 3\sigma.$$

We can ask which combinations of these invariants are realizable by 4-manifolds of a given type. These questions are typically referred to as *geography problems*. For example, the *complex geography problem* from algebraic geometry has been well studied, and asks which integer pairs are realizable as $(\chi_h(M), c_1^2(M))$ for a minimal complex surface (a real 4-dimensional manifold) M of general type. In recent years, the question has been again posed, with the restriction of M admitting a complex structure relaxed to only requiring that M admit a symplectic structure. The *symplectic geography problem* asks which integer pairs are realizable as $(\chi_h(M), c_1^2(M))$ for some minimal simply-connected symplectic M. The symplectic geography problem has been completely resolved in the case of integer pairs (χ, c) with $0 \leq c < 8\chi$; all such points have been shown to correspond to (χ_h, c_1^2) for minimal simply-connected symplectic 4-manifolds. The solution of the problem in the case of points with $c \geq 8\chi$ has progressed at a slower pace however. While infinite families of minimal simply-connected symplectic manifolds with $c_1^2 \geq 8\chi_h$ have been constructed, examples of such manifolds are comparatively rare, and typically have large χ_h values. Until recently, examples of such manifolds with $c_1^2 \geq 8\chi_h$ and $\chi_h < 61$ have been absent from the literature.

Recent work by A. Akhmedov and B.D. Park generalized constructions by A. Stipsicz to present examples of minimal simply-connected symplectic 4-manifolds with $c_1^2 \geq 8\chi_h$ and χ_h values as small as 24. These examples can then be used as "base points" to construct examples realizing large wedge-shaped regions of the (χ, c)-plane. These techniques realize a large number of integer pairs as $(\chi_h(M), c_1^2(M))$ (for minimal simply-connected symplectic M) that were unrealized by previous constructions.

The purpose of this thesis is to construct infinite families, following [3], of those examples introduced by Akhmedov and B.D. Park. It is now well-known (see [2, 27, 5]) that for any $s \geq 0$, there exists an integer $\lambda(s)$ such that that $(j, 8j + s)$ is realized as $(\chi_h(M), c_1^2(M))$ by a minimal simply-connected symplectic manifold M for every $j \geq \lambda(s)$. In this thesis we use Akhmedov and B.D. Park's constructions to provide an alternate proof of this result, as well as to explicitly formulate the smallest possible $\lambda(s)$ values presently known to the author for $0 \leq s \leq 99$ (see Table 7.1 and Theorem 7.2.3). In terms of the symplectic geography problem, we populate 7082 new points in (χ, c)-plane with $8\chi \leq c \leq 8\chi + 99$.

1.1 Outline

This thesis will be presented as follows: Chapter 2 will contain a brief review of many of the definitions and results needed later on. No attempt is made to develop the material in generality beyond what will be needed in later chapters. The reader is recommended to consult the listed references for additional details or a more in-depth development of the subject matter. Chapter 2 begins with Section 2.1 by stating briefly the definitions of some important 4-manifold invariants, as well as stating some basic results regarding them. This includes a brief discussion (see Section 2.1.2) on the classification of simply-connected 4-manifolds by their intersection forms. Sections 2.2 introduces symplectic 4-manifolds and other important related items, while Section 2.3 outlines certain topological properties of symplectic 4-manifolds. This includes reviewing Gompf's symplectic sum operation, as well as briefly discussing the relation between symplectic manifolds and Lefschetz fibrations and pencils. The symplectic geography problem is discussed in Chapter 3, along with the current status of attempts at achieving a solution to this problem. In Chapter 4 we construct two infinite families of symplectic 4-manifolds following Akhmedov and B.D. Park's constructions in [5]. Chapter 5 begins with a discussion in Section 5.1.2 about a particular surgery along Lagrangian tori embedded in symplectic 4-manifolds which yields new symplectic 4-manifolds. This surgery operation is used in Section 5.2 to construct manifolds from [4], which are then used in Chapter 6 as building blocks to construct infinite families of irreducible simply-connected symplectic 4-manifolds with positive signature (see Sections 6.2 and 6.3). Explicit examples from these families are computed in Chapter 7, allowing us to prove our main result, Theorem 7.2.3.

Throughout the discussion that follows, all manifolds will be assumed to be closed, connected, orientable, and compact unless otherwise stated.

Chapter 2

Background Material

2.1 The topology of 4-dimensional manifolds

2.1.1 4-manifold invariants

We begin by recalling the definitions of some important 4-manifold invariants. The following invariants will be defined for manifolds with empty boundary, though they naturally generalize to manifolds M with $\partial M \neq \emptyset$. We begin with the well-known *Euler characteristic*, which is defined for an m-dimensional manifold M as

$$e(M) = \sum_{i=0}^{m} (-1)^i b_i(M),$$

where $b_i(M)$ denotes the i^{th} betti number of M, i.e. the rank of the i^{th} homology group $H_i(M;\mathbb{Z})$.

Now let M denote a compact, oriented topological 4-manifold. Given such a manifold M, we can define a map

$$Q_M : H^2(M;\mathbb{Z}) \times H^2(M;\mathbb{Z}) \to \mathbb{Z}$$

by

$$Q_M(\alpha, \beta) = \langle \alpha \cup \beta, [M] \rangle,$$

where $\alpha, \beta \in H^2(M;\mathbb{Z})$, and $[M] \in H_4(M;\mathbb{Z}) \cong \mathbb{Z}$ is the fundamental class of M associated with its given orientation. The form Q_M is called the *intersection form* of M, and the value $Q_M(\alpha, \beta)$ is called the *intersection product* of α and β. Q_M is both symmetric and bilinear. Note that since Q_M is bilinear, it vanishes on all torsion elements of $H^2(M;\mathbb{Z})$; hence we can think of Q_M as being defined on the free part of $H^2(M;\mathbb{Z})$. Furthermore, since by Poincaré duality $H^2(M;\mathbb{Z}) \cong H_2(M;\mathbb{Z})$, Q_M can also be thought of as a pairing on $H_2(M;\mathbb{Z})$ modulo its torsion subgroup. Note that if we take the 4-manifold M with the opposite orientation (which we denote by $\overline{M}$), then $Q_{\overline{M}} = -Q_M$.

Although we have chosen to define the intersection form Q_M of a 4-manifold algebraically using the cup and Kronecker products, there is an equivalent, more intuitive geometric definition which will be of use to us later. Let $\alpha, \beta \in H^2(M;\mathbb{Z})$ be two cohomology classes with Poincaré duals $a = PD(\alpha)$ and $b = PD(\beta)$. Let Σ_a and Σ_b be oriented 2-dimensional submanifolds of M which represent a and b respectively. In other words, Σ_a is an oriented surface with fundamental class $[\Sigma_a]$ and embedding $i_a : \Sigma_a \hookrightarrow M$ such that $(i_a)_*[\Sigma_a] = a \in H_2(M;\mathbb{Z})$. Likewise $(i_b)_*[\Sigma_b] = b$, where $i_b : \Sigma_b \hookrightarrow M$ is the embedding of Σ_b in M (note that every homology class of a smooth 4-manifold can be represented by some embedded submanifold). Σ_a and Σ_b can be chosen generically so that they intersect transversely in a finite number of points. At each of these intersection points p, we choose positively oriented bases for $T_p\Sigma_a$ and $T_p\Sigma_b$. Concatenating these two bases yields either a positively or negatively oriented basis for T_pM. Assign the point p a value of $+1$ if the orientation is positive, and -1 if it is negative. Summing these values over all $p \in \Sigma_a \cap \Sigma_b$ yields an integer value which we denote by $a \cdot b$.

Proposition 2.1.1. *For a smooth 4-manifold M with $\alpha, \beta \in H^2(M;\mathbb{Z})$, $a = PD(\alpha)$ and $b = PD(\beta)$, $Q_M(\alpha, \beta) = a \cdot b$.*

There are thus two equivalent ways to think about and compute intersection products in smooth 4-manifold M; one algebraic and one geometric. Both will be of use to us later on.

2.1.2 Integral unimodular symmetric bilinear forms and 4-manifolds

We find it useful to make a few important definitions regarding integral bilinear forms in general. Let Q be a symmetric bilinear form over a finitely generated free abelian group G. Choosing a basis of G, we can write Q as a matrix Q'. If $\det(Q') = \pm 1$, then Q is said to be *unimodular*. The *rank* of Q, denoted by $\text{rank}(Q)$, is defined as the rank of the group G. The *signature* of Q, $\sigma(Q)$, is defined as the number of positive eigenvalues of Q' minus the number of negative eigenvalues of Q' (where Q' is extended and diagonalized over $\mathbb{R}$). Q is said to be *positive definite* if $\text{rank}(Q) = \sigma(Q)$, *negative definite* if $\text{rank}(Q) = -\sigma(Q)$, and *indefinite* otherwise. Finally, Q is said to be *even* if $Q(\alpha, \alpha) \equiv 0 \pmod 2$ for all $\alpha \in G$, and is *odd* otherwise.

Our primary concern will be with integral symmetric bilinear forms which arise as Q_M for some 4-manifold M. We define the *signature* $\sigma(M)$ of 4-manifold M to be the signature of Q_M. As a result of Poincaré duality, we have the following lemma:

Lemma 2.1.2. *The intersection form of a closed 4-manifold is unimodular.*

Because of Lemma 2.1.2 the study of integral unimodular forms plays an important role in the study of 4-manifolds. We might also ask to what extent the converse of Lemma 2.1.2 holds. In other words, which symmetric integral bilinear unimodular forms are realized as the intersection forms of 4-manifolds? The following theorem, due to M. Freedman, provides an answer to this question:

Theorem 2.1.3 (Freedman's Classification Theorem)**.** *For any integral symmetric unimodular form Q, there is a closed simply-connected topological 4-manifold M such that $Q \cong Q_M$. If Q is* *even, then this manifold is unique up to homeomorphism. If Q is odd, then there are exactly two distinct homeomorphism types with intersection form Q, at least one of which will not admit any smooth structure.*

It follows from this classification that simply-connected *smooth* 4-dimensional manifolds are uniquely determined by their intersection form (up to homeomorphism). We can take take this classification even further, by applying J.-P. Serre's classification of indefinite forms.

Theorem 2.1.4 (Serre's Classification Theorem)**.** *Let Q_1 and Q_2 be two indefinite symmetric bilinear integral unimodular forms. Then Q_1 and Q_2 are isomorphic if and only if they have the same rank, signature and parity.*

By Freedman's classification, this implies that smooth simply-connected manifolds with indefinite intersection forms are uniquely determined by the rank, signature and parity of their intersection forms. Followings Serre's theorem, we might hope to find a similar classification of definite unimodular forms. Unfortunately no such classification exists. While the number of definite forms of a given rank is finite, it grows unwieldily large very quickly. For instance, for even definite forms of rank $8, 16$ and 24, we have only $1, 2$, and 24 forms respectively of each rank, while we have over 10^7 and 10^{51} even definite forms of rank 32 and 40 respectively (note that the rank of even unimodular forms is always divisible by 8).

Abandoning attempts to classify all definite unimodular forms, we might at least hope for some classification of which such forms correspond to manifolds admitting smooth structures. Donaldson proved the following theorem answering this question:

Theorem 2.1.5 (Donaldson's Theorem)**.** *Suppose that the intersection form Q_M of a simply-connected smooth 4-manifold M is positive definite, and $\text{rank}(Q_M) = m$. Then*

$$Q_M \cong \bigoplus m\,[+1]\,.$$

If Q_M is negative definite, then

$$Q_M \cong \bigoplus m\,[-1]\,.$$

These are the only two definite forms realizable as the intersection forms of simply-connected smooth 4-manifolds.

Theorem 2.1.5, along with the fact that $Q_{\overline{M}} = -Q_M$, implies that simply-connected smooth 4-manifolds with definite intersection form are uniquely determined (up to homeomorphism) by the rank of their intersection forms. Combining both Donaldson's theorem and Serre's classification of indefinite forms with Freedman's theorem, we have that the homeomorphism classes of simply-connected smooth 4-manifolds are uniquely determined by the signature, rank and parity of their intersection forms.

Not all combinations of these invariants occur in smooth 4-manifolds however. For example, we have the following restriction on the signature of even intersection forms corresponding to smooth 4-manifolds:

Theorem 2.1.6 (Rohlin [29]). *If M is an oriented, closed, simply-connected smooth 4-manifold with even intersection form, then $\sigma(M) \equiv 0 \pmod{16}$.*

Note that no such restrictions arise in the case of manifolds with odd intersection forms. This can be seen by noting that for $a, b \geq 0$, the manifolds $a\mathbb{CP}^2 \# b\overline{\mathbb{CP}}^2$ are simply-connected smooth 4-manifolds with odd intersection forms $Q_{a,b} \simeq a[+1] \oplus b[-1]$. It follows that $\text{rank}(Q_{a,b}) = a + b$ and $\sigma(Q_{a,b}) = a - b$, and for appropriate choices of a and b we can realize all integer pairs (r, s) with $r \equiv s \pmod 2$, $r \geq s$, and $r \geq 1$, by $r = \text{rank}(Q_{a,b})$ and $s = \sigma(Q_{a,b})$.

We make one final definition before ending this section:

Definition 2.1.7 (Irreducibility). A 4-manifold M is said to be *irreducible* if for every smooth decomposition $M = M_1 \# M_2$ (i.e. M is diffeomorphic to $M_1 \# M_2$), either M_1 or M_2 is homeomorphic (but not necessarily diffeomorphic) to S^4.

It is important that the definition of irreducibility require that the decomposition of M into the connect sum be smooth. By Freedman's theorem, any smooth manifold M with odd intersection form is homeomorphic to $a\mathbb{CP}^2 \# b\overline{\mathbb{CP}}^2$, for some nonnegative integers a and b. If M were only required to be homeomorphic to $M_1 \# M_2$, it would imply that the only irreducible smooth manifolds with odd intersection forms would be $\mathbb{CP}^2$ and $\overline{\mathbb{CP}}^2$. Similarly, we would not have any irreducible smooth manifolds with even intersection form. The only homeomorphism types of irreducible smooth 4-manifolds would thus be $\mathbb{CP}^2$, $S^2 \times S^2$ and S^4; not a very interesting situation.

2.1.3 Blow-ups and blow-downs

We begin this section by recalling the definitions of the *blow-up* and *blow-down* operations on smooth oriented 4-manifolds. These definitions are contained in the following lemma:

Lemma 2.1.8. *Let M be a smooth 4-manifold and let $p \in M \backslash \partial M$ be any point. Then there exists a smooth 4-manifold M' (called the* blow-up *of M at p), and a smooth map $\pi : M' \to M$, such that $\pi^{-1}(p) = E \subset M'$ is an embedded sphere with $[E]^2 = -1$, and the restriction $\pi|_{M' \backslash E} : M' \backslash E \to M \backslash \{p\}$ is a diffeomorphism.*

Conversely, suppose that M' is a smooth manifold with an embedded sphere E satisfying $[E]^2 = -1$. Then there exists a smooth manifold M (called the blow-down *of M') and map $\pi : M' \to M$ which contracts E to a point and, as above, restricts to a diffeomorphism between the complement $M' \backslash E$ and $M \backslash \{p\}$.*

The embedded -1-sphere $E \subset M'$ is called the *exceptional sphere*. Topologically, the blow up M' of a 4-manifold M is just the connect sum $M' = M \# \overline{\mathbb{CP}}^2$, where the exceptional sphere E corresponds to a copy of $\overline{\mathbb{CP}}^1$ in the $\overline{\mathbb{CP}}^2$ summand. Suppose that $\Sigma \subset M$ is a surface with $p \in \Sigma$. Let M' be the blow up of M at p with map $\pi : M' \to M$. Then the preimage $\pi^{-1}(\Sigma)$ will consist of the union of the exceptional sphere E and a surface Σ' which is diffeomorphic to Σ and which intersects E transversely in a single point. This subset of M' fails to be a manifold, and is referred to as the *total transform* of Σ. Taking the preimage of $\Sigma \backslash \{p\}$ under π yields a punctured surface disjoint from E. The topological closure of this preimage $\overline{\pi^{-1}(\Sigma \backslash \{p\})}$ is a 2-dimensional submanifold $\widetilde{\Sigma}$ of M', which is diffeomorphic to the original surface $\Sigma \subset M$ and which intersects E transversely in a single point. The surface $\widetilde{\Sigma}$ is defined to be the *proper transform* of Σ.

The importance of the blow-up and blow-down operations lies in the fact that they allow us to resolve positive intersection points between transversely intersecting 2-dimensional submanifolds. The blow-up operation on M at p can be thought of as replacing p with the exceptional sphere $E = \overline{\mathbb{CP}}^1$, which can itself be interpreted as the set of all complex lines passing through the origin of $\mathbb{C}^2$. All of complex lines that were passing through p in M are now passing through their corresponding direction in $\overline{\mathbb{CP}}^1$, and are thus disjoint in M'. If Σ_1 and Σ_2 are two transversely intersecting surfaces with positive intersection at p (and only at p), then their proper transforms $\widetilde{\Sigma}_1$ and $\widetilde{\Sigma}_2$ in M' will be disjoint.

Since by Lemma 2.1.8 any manifold containing a -1-sphere can be blown down, we make the following definition:

Definition 2.1.9 (Minimality)**.** A 4-manifold M is called *minimal* if it does not contain an embedded 2-sphere E with $[E]^2 = -1$.

Note that any manifold with such a sphere E can be blown-down to a minimal manifold. This minimal manifold is not always unique however. For example, blowing up $\mathbb{CP}^2$ twice and $\mathbb{CP}^1 \times \mathbb{CP}^1$ once can produce the same complex surface (cf. pg. 46 of [16]), while both $\mathbb{CP}^2$ and $\mathbb{CP}^1 \times \mathbb{CP}^1$ are minimal.

2.2 Symplectic manifolds

2.2.1 Definitions and examples

Let V be an m-dimensional vector space. A *skew-symmetric bilinear form* on V is a map

$$\omega : V \times V \to \mathbb{R}$$

which is linear in both arguments and which satisfies $\omega(v, w) = -\omega(w, v)$ for all $v, w \in V$. Note that this later condition implies that $\omega(v, v) = 0$ for all $v \in V$. We say that ω is *nondegenerate* if for each vector $v \in V$, there exists a vector $w \in V$ such that $\omega(v, w) \neq 0$. If the vector space V admits a nondegenerate skew-symmetric bilinear form ω, then an easy argument shows that $m = \dim(V)$ must be even. In this case we call the pair (V, ω) a *symplectic vector space*, while ω is called a *linear symplectic structure* on V. We will often stray from this notation when referring to (V, ω) by omitting the symplectic structure ω and referring to the symplectic vector space simply as V. Given such a vector space, it is always possible to find a basis $\{e_1, \ldots, e_n, f_1, \ldots f_n\}$ (where $m = 2n$) which satisfies

$$\omega(e_i, e_j) = \omega(f_i, f_j) = 0$$

and

$$\omega(e_i, f_j) = \delta_i^j$$

for all $1 \leq i, j \leq n$. Such a basis is called a *symplectic basis*.

Example 2.2.1. Given any basis $\mathcal{B} = \{a_1, \ldots a_n, b_1, \ldots b_n\}$ of a $2n$-dimensional vector space V, the form

$$\omega = \sum_{i=1}^{n} a_i^* \wedge b_i^*$$

(where $\{a_1^*, \ldots a_n^*, b_1^*, \ldots, b_n^*\}$ is the dual basis to $\mathcal{B}$) is a linear symplectic structure, making (V, ω) into a symplectic vector space with symplectic basis $\mathcal{B}$.

We wish to extend the notion of symplectic structures smoothly to the tangent bundle of a differentiable manifold M. To this end, we recall that a *differential p-form* on M is a smooth section of the p^{th} exterior power of the cotangent bundle T^*M. In particular, a differential 2-form ω is a map which assigns to each point $x \in M$ a skew-symmetric bilinear form $\omega_x : T_xM \times T_xM \to \mathbb{R}$ which varies smoothly with x. We say that ω is *nondegenerate* if ω_x is nondegenerate for each $x \in M$. Furthermore, ω is said to be *closed* if the exterior derivative of ω is zero, i.e. $d\omega \equiv 0$.

Definition 2.2.2. A differential 2-form ω on a a smooth manifold M is called a *symplectic structure* if it is both closed and nondegenerate. In this case the pair (M, ω) is called a *symplectic manifold*.

Note that the nondegeneracy condition implies that $\dim(T_xM)$ is even for each $x \in M$, and hence that $m = \dim(M)$ is also even. Furthermore, if $m = 2n$, then ω is nondegenerate on M if and only if the n^{th} wedge power $\omega^n = \omega \wedge \cdots \wedge \omega$ is nowhere vanishing on M. Since in this case ω^n is a nowhere vanishing m-form on an m-dimensional manifold, it follows that M is orientable. In fact, a symplectic structure defines a choice of orientation on a symplectic manifold. Under this orientation, a choice of basis $\{X_1, \ldots, X_{2n}\}$ of T_pM is positively oriented if $\omega^n(X_1, \ldots, X_{2n}) > 0$. Since ω^n is nonvanishing, this defines a consistent orientation on M.

Example 2.2.3. Our first example of a symplectic manifold is the Euclidean space $\mathbb{R}^{2n}$ with symplectic structure

$$\omega_o = \sum_{i=1}^{n} dx_i \wedge dx_{n+i}$$

where $\{x_1, \ldots, x_{2n}\}$ are the standard coordinates on $\mathbb{R}^{2n}$. For each $p \in M$

$$\left\{ \left(\frac{\partial}{\partial x_1}\right)_p, \ldots, \left(\frac{\partial}{\partial x_{2n}}\right)_p \right\}$$

is a symplectic basis of T_pM under the symplectic structure ω_o.

As we will discuss later, all symplectic manifolds are locally modeled after $(\mathbb{R}^{2n}, \omega_o)$. Our next examples are likewise straightforward but will be included as they will be needed later on:

Example 2.2.4. Given any smooth manifold M, the cotangent bundle can be seen to be a symplectic manifold, with canonical symplectic structure. Let $(x_1, \ldots, x_n)$ be a local coordinate chart on some subset $U \subset M$. Recall that at any $p \in U$, the set $\{(dx_1)_p, \ldots, (dx_n)_p\}$ forms a basis for T_p^*M. Let $(\zeta_1, \ldots, \zeta_n)$ be the corresponding coordinate functions for this basis. In other words, the ζ_i are maps $\zeta_i : T_p^*U \to \mathbb{R}$ such that given any $\zeta \in T_p^*U$, we can write $\zeta = \sum_{i=1}^n \zeta_i (dx_i)_p$. Define the 2-form ω on T^*U as

$$\omega = \sum_{i=1}^n dx_i \wedge d\zeta_i.$$

It can be shown that thus defined, ω is independent of the coordinate chart used, and hence is intrinsic to the manifold T^*M itself. It is likewise easy to see that ω is a symplectic form, thus making (T^*M, ω) a symplectic manifold for any smooth M.

Example 2.2.5. A volume form on a closed surface (real 2-dimensional manifold) is a symplectic form, and hence all oriented closed surfaces admit symplectic forms. Let (Σ, ω) and (Σ', ω') be two such surfaces with symplectic forms ω and ω'. Let $\pi_1 : \Sigma \times \Sigma' \to \Sigma$ and $\pi_2 : \Sigma \times \Sigma' \to \Sigma'$ be the natural projections onto the first and second factors. Then $(\pi_1)^*\omega + (\pi_2)^*\omega'$ is a symplectic structure on $\Sigma \times \Sigma'$. This symplectic structure is called the *product symplectic structure* on $\Sigma \times \Sigma'$.

Now that we have given a few examples of symplectic manifolds, we proceed to make a few more definitions which will be needed later on:

Definition 2.2.6. A *symplectomorphism* is a diffeomorphism $\phi : M_1 \to M_2$ between two symplectic manifolds (M_1, ω_1) and (M_2, ω_2) such that $\phi^*\omega_2 = \omega_1$. In this case, M_1 and M_2 are said to be *symplectomorphic.*

An important result by Darboux shows that locally, every $2n$-dimensional symplectic manifold is symplectomorphic to $(\mathbb{R}^{2n}, \omega_o)$:

Theorem 2.2.7 (Darboux)**.** *If (M, ω) is a symplectic manifold with $p \in M$, then there is an open set $U \subset M$ with $p \in U$ and coordinates $(x_1, \ldots x_n, y_1, \ldots, y_n)$ centered at p such that*

$$\omega = \sum_{i=1}^n dx_i \wedge dy_i$$

on U.

This implies that unlike the local structure of a Riemannian manifold, the symplectic form provides no interesting local invariants. Everything looks locally like $(\mathbb{R}^{2n}, \omega_o)$.

We conclude this section by discussing briefly two classes of submanifolds of a symplectic manifold (M, ω). Let N be such a submanifold, and let $i : N \hookrightarrow M$ be its embedding.

Definition 2.2.8. N is said to be a *symplectic submanifold* of M if $(N, i^*\omega)$ is itself a symplectic manifold. N is said to be a *Lagrangian submanifold* of M if $\dim(N) = \frac{1}{2}\dim(M)$ and $i^*\omega \equiv 0$.

Example 2.2.9. The surfaces $\Sigma \times \{\text{pt}\}$ and $\{\text{pt}\} \times \Sigma'$ are both symplectic submanifolds of $(\Sigma\times\Sigma', (\pi_1)^*\omega+(\pi_2)^*\omega')$, while for simple closed curves $\alpha \subset \Sigma$ and $\alpha' \subset \Sigma'$ the torus $\alpha\times\alpha' \subset \Sigma\times\Sigma'$ is a Lagrangian submanifold.

The following theorem of Gompf allows us change a given symplectic structure on 4-manifold an arbitrarily small amount so that a given Lagrangian submanifold becomes symplectic. In other words, we can perturb the symplectic structure on a 4-manifold in such a way that a given Lagrangian submanifold becomes symplectic under the new symplectic form.

Theorem 2.2.10 (Gompf [14])**.** *Let (M, ω) be a closed, symplectic 4-manifold with $\Sigma \subset M$ a closed, connected, oriented, Lagrangian submanifold with $[\Sigma] \neq 0 \in H_2(M;\mathbb{Z})$. Then there is an arbitrarily small perturbation ω' of ω such that (M, ω') is symplectic and Σ is a symplectic submanifold of M.*

2.3 The topology of symplectic 4-manifolds

2.3.1 Symplectic blow-ups and minimality

Let (M, ω) be a symplectic 4-manifold, and let $p \in M$ be any point. As discussed in Section 2.1.3, M can be blown-up at p to obtain the blow-up $M' = M\#\overline{\mathbb{CP}}^2$. Let $\pi : M' \to M$ be the corresponding blow-down map. It can be shown then that M' admits a symplectic form which, away from E, is equal to the pull-back $\pi^*\omega$. Furthermore, the form ω' can be chosen so that the exceptional sphere E is a symplectic submanifold of M'. Conversely, any symplectic 4-manifold M' with a symplectic sphere E satisfying $[E]^2 = -1$ can be blown-down to obtain a symplectic manifold M. We can therefore use the blow-up and blow-down operations in the symplectic setting as in the smooth case. We thus restate the following definition for symplectic manifolds:

Definition 2.3.1. A symplectic manifold (M, ω) is said to be *minimal* if it does not contain a symplectic sphere E of self-intersection -1.

It can be shown that the homology class of any smoothly embedded sphere $E \subset M$ with $[E]^2 = -1$ can be represented by a symplectic -1-sphere. Definition 2.3.1 is thus equivalent to Definition 2.1.9 in the symplectic category.

As in the smooth case, any symplectic manifold can be blown down to obtain a minimal symplectic manifold. Again however, this minimal manifold may not be unique. Note that for a smooth manifold M' the condition of being minimal implies that M' cannot be decomposed as $M' = M\#\overline{\mathbb{CP}}^2$ for some smooth manifold M. In the symplectic setting, minimality implies more:

Theorem 2.3.2 (Taubes [19, 34])**.** *A simply-connected, minimal symplectic 4-manifold X is irreducible.*

Thus not only can a minimal symplectic 4-manifold M' not be expressed as $M' = M\#\overline{\mathbb{CP}}^2$ for any M, it cannot be written as $M' = M_1\#M_2$ for any M_1, M_2 not homeomorphic to S^4. Theorem 2.3.2 implies that in the symplectic setting, minimality and irreducibility are equivalent.

2.3.2 Almost-complex structures and Chern characteristic classes

Definition 2.3.3 (Almost-Complex Structure)**.** Let M be a $2n$-manifold, and let $\pi : TM \to M$ be the natural projection. A smooth, fiberwise linear map $J : TM \to TM$ satisfying $\pi \circ J = \pi$ and $J^2 = -\mathrm{id}_{TM}$ is called an *almost-complex structure* on M.

Note that the existence of an almost-complex structure on a manifold organizes the tangent bundle TM as a complex bundle, where the map J corresponds to multiplication by the imaginary unit i. We restrict our attention to almost-complex structures whose induced orientation on each tangent space is the same as the orientation from M.

Definition 2.3.4. Given a symplectic manifold (M, ω) and an almost-complex structure J on M, we say that J and ω are *compatible* if $\omega(Jx, Jy) = \omega(x, y)$ and $\omega(x, Jx) > 0$ for all nonzero $x, y \in TM$.

It turns out that for symplectic manifolds, we can always find such a compatible almost-complex structure. In fact, even more can be said:

Proposition 2.3.5. *Any symplectic manifold (M, ω) admits an almost-complex structure compatible with ω. Furthermore, the space of all such compatible structures is contractible.*

Thus for any symplectic manifold M, we can organize TM as a complex vector bundle by choosing an almost-complex structure J compatible with the symplectic form on M. Using the complex bundle structure induced by J, we can define *Chern classes* $c_i(M, J) \in H^{2i}(M; \mathbb{Z})$ for the tangent bundle TM of M. Recall that if $E \to X$ is a complex vector bundle of rank n, then there exists a unique $c(E) = c_0(E) + c_1(E) + \cdots + c_n(E) \in H^*(X; \mathbb{Z})$, where $c_i(E) \in H^{2i}(X; \mathbb{Z})$ for each i, which satisfies the following axioms:

AI. $c_0(E) = 1$ is the unity of $H^*(X; \mathbb{Z})$.

AII. $f^*(c_i(E)) = c_i(f^*E)$ for each i and all continuous maps $f : Y \to X$.

AIII. $c(E_1 \oplus E_2) = c(E_1) \cup c(E_2)$ for any two bundles $E_1 \to X$ and $E_2 \to X$.

AIV. If $\tau \to \mathbb{CP}^n$ is the tautological line bundle and $h \in H_{2n-2}(\mathbb{CP}^n; \mathbb{Z})$ is the class representing the hypersurface $H = \{[x_1, \ldots, x_{n+1}] \mid x_{n+1} = 0\}$ with its natural orientation, then $c(\tau) = 1 + PD(h)$.

The class $c_i(E) \in H^{2i}(X; \mathbb{Z})$ is called the i^{th} *Chern class* of the bundle $E \to X$, while $c(E)$ is called the *total Chern class*. See [18] for a proof of the existence and uniqueness of these classes. The class $c_i(E)$ can be thought of as describing the obstruction to finding a (complex) codimension $i - 1$ trivial subbundle over the restriction of E to the $2i$ skeleton of X (see Section 5.6 of [16]).

In the case that our bundle E is the tangent bundle $TM \to M$ for some manifold M with complex vector space structure induced by J, we denote these classes by $c_i(M, J) \in H^{2i}(M; \mathbb{Z})$. In general, the Chern classes of the tangent bundle of a manifold will depend on our choice of J; for a symplectic manifold (M, ω) however, by choosing an almost-complex structure compatible with ω, we get well-defined Chern classes. This follows from the fact that the space of almost-complex structures compatible with ω is contractible. We may thus omit the almost-complex structure from the notation, and instead denote the Chern classes of a symplectic manifold (M, ω) by $c_i(M, \omega)$, or when the symplectic form is understood, simply by $c_i(M)$. Having well-defined Chern classes on symplectic manifolds, we can adapt many familiar results to the symplectic setting. In particular the following result, known as the *adjunction formula*, which relates the self-intersection and Euler characteristic of a symplectic submanifold, will be used frequently in the following discussion:

Theorem 2.3.6 (Adjunction Formula). *Let (M, ω) be a symplectic 4-manifold, with symplectic submanifold $\Sigma \subset M$. Then*

$$\langle c_1(M, \omega), [\Sigma] \rangle = [\Sigma]^2 + e(\Sigma).$$

2.3.3 Symplectic fiber sums

We will often be interested in taking two symplectic 4-manifolds M_1 and M_2 and gluing them together in such a way that the resulting manifold admits a symplectic structure uniquely determined by the symplectic structures on M_1 and M_2. Gompf described such a construction in [14]. Let M_1 and M_2 be symplectic 4-manifolds with smooth, closed, connected 2-dimensional symplectic (or Lagrangian) submanifolds $F_i \subset M_i$ of the same genus. Suppose further that $[F_1]^2 = -[F_2]^2$. Then the normal bundles νF_i of $F_i \subset M_i$ are isomorphic under a fiber orientation-reversing diffeomorphism $\Psi : \nu F_1 \to \nu F_2$. Restricting Ψ to $\partial(\nu F_1)$ gives an orientation-reversing diffeomorphism $\widetilde{\Psi} : \partial(\nu F_1) \to \partial(\nu F_2)$.

Definition 2.3.7. In the above situation, the *symplectic fiber sum* $M_1 \#_{\widetilde{\Psi}} M_2$ is the quotient space of $M_1 \backslash \nu F_1$ and $M_2 \backslash \nu F_2$ formed by identifying $\partial(\nu F_1)$ and $\partial(\nu F_2)$ under the gluing map $\widetilde{\Psi}$.

The usefulness of this construction is demonstrated by the following theorems due to Gompf and Usher:

Theorem 2.3.8 (Gompf [14]). *Given the above situation, the symplectic fiber sum $M_1 \#_{\widetilde{\Psi}} M_2$ admits a symplectic structure which can be chosen so that the symplectic (Lagrangian) submanifolds of $M_i \backslash \nu F_i$ are symplectic (respectively Lagrangian) in $M_1 \#_{\widetilde{\Psi}} M_2$.*

Theorem 2.3.9 (Usher [35]). *Let $M = X_1 \#_{\widetilde{\Psi}} X_2$ be the symplectic fiber sum of two 4-manifolds along symplectic surfaces $F_i \subset X_i$ of positive genus g. Then*

(i) *if either $X_1 \backslash F_1$ or $X_2 \backslash F_2$ contains an embedded symplectic -1-sphere, then M is not minimal;*

(ii) *if one of the summands* X_i *(say* X_1*) admits the structure of an* S^2*-bundle over a surface of genus* g *such that* F_1 *is a section of this fiber bundle, then* M *is minimal if and only if* X_2 *is minimal;*

(iii) *in all other cases,* M *is minimal.*

Theorem 2.3.8 allows us to choose the symplectic structure on the fiber sum of two manifolds so that it agrees with the symplectic structures of the summands in terms of their symplectic and Lagrangian submanifolds. Theorem 2.3.9 gives us, among other things, that the symplectic fiber sum of two minimal symplectic manifolds is minimal.

2.3.4 Lefschetz fibrations and symplectic manifolds

We want to better understand how the existence of a symplectic structure affects the topology of a 4-manifold. In particular, we would like to find a topological characterization of manifolds which admit symplectic structures. Such a characterization was given by Donaldson. He showed that any 4-manifold which admits a symplectic structure also admits, after the removal of a finite number of singular points, a particular fibration structure. Furthermore, the singular points of this fibration are limited to a certain type. These structures, called *Lefschetz pencils*, were first considered by Solomon Lefschetz in relation to his study of algebraic varieties in [20].

Definition 2.3.10. Let M be a closed, connected, oriented, smooth 4-manifold. A *Lefschetz pencil* on M is a nonempty finite set $B \subset M$ (called the *base locus*), and a map $f : M \backslash B \to \mathbb{CP}^1$ such that

1. for each $b \in B$ there are orientation-preserving local complex coordinates centered at b under which f is given by $f(u, v) = [u, v]$ (projectivization of $\mathbb{C}^2 - \{0\} \to \mathbb{CP}^1$), and
2. for each critical point p of f there are orientation-preserving local complex coordinates around p, under which f is given by $f(u, v) = u^2 + v^2$ (for suitable smooth coordinates on $\mathbb{CP}^1$).

For each $t \in \mathbb{CP}^1$, we call the set $F_t = f^{-1}(t) \cup B \subset M$ the *fiber* over t.

Definition 2.3.11. Let M be a compact, connected, oriented, smooth 4-manifold, and let Σ be a compact, connected, oriented surface. A *Lefschetz fibration* on M is a map $f : M \to \Sigma$ with $f^{-1}(\partial\Sigma) = \partial M$, which is modeled around each critical point in orientation-preserving local complex coordinates by $f(u, v) = u^2 + v^2$ (again relative to some smooth local coordinates on Σ).

We make a few brief observations. Blowing up a Lefschetz pencil at each point of the base locus yields a Lefschetz fibration over $\mathbb{CP}^1$. The generic fibers of a Lefschetz fibration will all be smooth, closed surfaces of the same diffeomorphism type. Furthermore, we can perturb the fibration f so that is is injective on its set of critical points. In other words, each singular fiber will have a unique singularity. Around each critical point, by writing f in complex coordinates as $f(u,v) = u^2 + v^2$, we have that $f^{-1}(0)$ is given by the planes $\{(u,v) \mid u = iv\}$ and $\{(u,v) \mid u = -iv\}$, which intersect transversely at $(u,v) = (0,0)$. Thus each singular fiber is a smoothly immersed surface, where the critical point corresponds to a transverse self-intersection. In this coordinate chart nearby fibers F_ε will be given locally by $u^2 + v^2 = \varepsilon$ for small $\varepsilon \in \mathbb{C}$. Considering some such fiber F_ε near the singular fiber F_0, we obtain F_0 from F_ε by shrinking a loop in F_ε to a single point, thus creating the transverse self-intersection in F_0. The cycle we collapse to obtain the transverse self-intersection is called the *vanishing cycle* of the singular fiber F_0. More explicitly, we can take a nearby fiber F_t corresponding to some $t \in \mathbb{R}^+ \subset \mathbb{C}$ (any fiber can be made to correspond to such a t by multiplying f by a unit complex number). Intersecting F_t with $\mathbb{R}^2 \subset \mathbb{C}^2$ yields a circle $(\mathcal{R}(u))^2 + (\mathcal{R}(v))^2 = t$ in F_t (where $\mathcal{R}(z)$ denoted the real part of z). This circle is the vanishing cycle of the critical point. Taking t to zero along the positive real axis will correspond to the shrinking of the vanishing cycle in fibers progressively closer to F_0. We can in fact obtain a tubular neighborhood νF_0 of the singular fiber F_0 by taking a tubular neighborhood νF_t of a nearby fiber and attaching a 2-handle to $\partial(\nu F_t)$. This 2-handle is attached along the vanishing cycle in a fiber $F_s \in \partial(\nu F_t)$ with $0 < s < t$. It thus follows that the vanishing cycles are nullhomotopic in M (see Section 8.2 of [16]).

Finally, it is easy to see that if M admits a Lefschetz fibration $f : M \to \Sigma$ and we blow-up a point $p \in M$, the resulting manifold $M \# \overline{\mathbb{CP}}^2$ will also admit a Lefschetz fibration over Σ. The fibration is obtained by composing the corresponding blow-down map $M \# \overline{\mathbb{CP}}^2 \to M$ with f. The fibers in the new fibration will remain unchanged except for the fiber containing p, which will have the point p replaced with the exceptional sphere. We call a Lefschetz fibration *relatively minimal* if no fiber contains a sphere of self-intersection -1. It is well-known that any Lefschetz fibration can be blown down to obtain a relatively minimal one. Stipsicz proved the following result, showing the equivalence of minimality and relative minimality for most Lefschetz fibrations:

Theorem 2.3.12 (Stipsicz [33])**.** *A genus g Lefschetz fibration $f : M \to \Sigma$ with $g > 0$ is minimal if and only if it is relatively minimal.*

As mentioned above, our interest in Lefschetz fibrations lies in their connection to symplectic manifolds. The following theorem due to Gompf establishes half of this important relationship:

Theorem 2.3.13 (Gompf [15])**.** *Let M be a closed 4-manifold which admits a Lefschetz fibration $f : M \to \Sigma$ with fiber F. Then M admits a symplectic structure ω with symplectic fibers if and only if the homology class $[F]$ of F is nonzero in $H_2(M;\mathbb{R})$. Furthermore if $s_1, \ldots, s_n$ is a finite set of sections of f, then ω can be chosen so that each of the s_i are symplectic.*

Note that by Remark 10.2.22(a) and Exercise 8.4.15(b) in [16] the requirement of $[F] \neq 0 \in H_2(M;\mathbb{R})$ is satisfied automatically for any fibration of genus greater than 1. In other words, a 4-manifold M which admits a genus g Lefschetz fibration with $g \geq 2$ will always admit a symplectic structure with symplectic fiber, and can be chosen so that a given set of sections as above will also be symplectic.

Corollary 2.3.14. *Any* 4*-manifold which admits a Lefschetz pencil admits a symplectic structure.*

Proof. Let M admit a Lefschetz pencil with base locus B. Recall that by Definition 2.3.10 $B \neq \emptyset$. Blowing up at each point of B gives us a Lefschetz fibration $f : M \# k\overline{\mathbb{CP}}^2 \to \mathbb{CP}^1$, where $k = |B|$. Note that each of the k exceptional spheres $E_1, \ldots, E_k$ will intersect each of the fibers of f transversely in a single point, and are thus the images of sections of f. Thus $[F] \cdot [E_i] = \pm 1$, where $[F]$ is the class in $H_2(M \# k\overline{\mathbb{CP}}^2; \mathbb{Z})$ represented by a fiber, whence $[F] \neq 0 \in H_2(M \# k\overline{\mathbb{CP}}^2; \mathbb{Z})$. By Theorem 2.3.13, we can equip $M \# k\overline{\mathbb{CP}}^2$ with a symplectic structure such that each of the exceptional spheres is a symplectic manifold. Since the blow-up and blow-down processes can be carried out symplectically, we can blow-down $M \# k\overline{\mathbb{CP}}^2$ to recover the manifold M with symplectic structure inherited from the structure on $M \# k\overline{\mathbb{CP}}^2$. □

The converse of this result, namely that all symplectic 4-manifolds admit Lefschetz pencils, was proved by Donaldson (see [11]). Combining these results yields:

Theorem 2.3.15. *A 4-manifold M admits a symplectic structure if and only if it admits a Lefschetz pencil.*

We thus have a completely topological characterization of the manifolds which admit symplectic structures as desired. This relationship will prove vital in the discussion which follows.

We state briefly one final result which will be needed in later chapters as we compute the fundamental groups of manifolds which admit Lefschetz fibrations.

Proposition 2.3.16. *Let $f : M \to \Sigma$ be a Lefschetz fibration with connected fiber F. Then the maps $F \hookrightarrow M \to \Sigma$ induce an exact sequence $\pi_1(F) \to \pi_1(M) \to \pi_1(\Sigma) \to 1$.*

Chapter 3

The Symplectic Geography Problem

3.1 Geography problems

Recall that for a symplectic 4-manifold (M, ω) the Chern classes $c_1(M, \omega)$ and $c_2(M, \omega)$ are well-defined elements of $H^2(M; \mathbb{Z})$ and $H^4(M; \mathbb{Z})$ respectively. From these two classes we can construct integer topological invariants of M:

$$c_2(M) = \langle c_2(M, \omega), [M] \rangle$$
$$c_1^2(M) = PD(c_1(M, \omega)) \cdot PD(c_1(M, \omega)) = \langle c_1(M, \omega) \cup c_1(M, \omega), [M] \rangle.$$

Note that despite the similarity in notation between the Chern cohomology classes and the integer-valued invariants defined above, little confusion arises in practice. Although the Chern classes of (M, ω) depend upon our choice of ω (as they depend on our choice of almost-complex structure), the integers defined above do not. They depend only on the homeomorphism type of M. Furthermore, since the top Chern class of a complex vector bundle is equal to its Euler class, we have that $c_2(M)$ is in fact equal to $e(M)$, the well-known Euler characteristic of M.

Recall that for a complex surface S, the *holomorphic Euler characteristic* $\chi_h(S)$ is defined as $\chi_h(S) = \frac{1}{12}(c_1^2(S) + c_2(S)) = \frac{1}{4}(e(S) + \sigma(S))$. Since all closed symplectic 4-manifolds satisfy $1 - b_1(M) + b_2^+(M) \equiv 0 \pmod 2$ (see Corollary 10.1.10 of [16]), we have that

$$\begin{aligned} e(M) + \sigma(M) &= (2 - b_1(M) + b_2(M) - b_3(M)) + (b_2^+(M) - b_2^-(M)) \\ &= 2 - 2b_1(M) + 2b_2^+(M) \end{aligned}$$

is divisible by 4 (where we used that fact that $b_1(N) = b_3(N)$ for all 4-manifolds N). Thus for a symplectic 4-manifold M, the expression $\frac{1}{4}(e(M) + \sigma(M))$ always takes values in $\mathbb{Z}$, and we can extend the definition of the holomorphic Euler characteristic to the symplectic category. Finally, by the Hirzebruch signature theorem (see Theorem 1.4.12 of [16]), we have that the signature $\sigma(M)$ of M can be expressed as

$$\sigma(M) = \frac{1}{3}(c_1^2(M) - 2c_2(M)) = \frac{1}{3}(c_1^2(M) - 2e(M)),$$

whence $c_1^2(M) = 3\sigma(M) + 2e(M)$.

For any symplectic 4-manifold M we thus have four integer-valued topological invariants $e(M)$, $\sigma(M)$, $c_1^2(M)$ and $\chi_h(M)$, any two of which determine the values of the other two. For example, given $c_1^2(M)$ and $\chi_h(M)$ we can compute

$$\sigma(M) = c_1^2(M) - 8\chi_h(M) \quad \text{and} \quad e(M) = 12\chi_h(M) - c_1^2(M).$$

It is a very natural question to ask which combinations of these invariants are actually realized by symplectic 4-manifolds. For instance, we can ask for what ordered pairs $(\chi, c) \in \mathbb{Z} \times \mathbb{Z}$ does there exist a symplectic manifold M such that $\chi_h(M) = \chi$ and $c_1^2(M) = c$? Questions of this nature are commonly referred to as *geography problems*, as they concern which lattice points in the plane are realized as given characteristic numbers for some manifold M with desired properties.

For example, the classical *complex geography problem*, first posed by Persson in [28], asks which ordered pairs of positive integers (χ, c) are realized as the pair $(\chi_h(S), c_1^2(S))$ for some minimal complex surface S of general type. In this paper, we will be concerned with an analogous question in the symplectic category. In [14] Gompf proved that every finitely presentable group is realized as the fundamental groups of an infinite family of symplectic 4-manifolds. Given this lack of restriction on the possible fundamental groups of symplectic 4-manifolds, we restrict our search to ordered pairs realized by *simply-connected* symplectic 4-manifolds. We will also further strengthen the requirements by asking that M not only be simply-connected and symplectic, but also that it isn't the blow-up of some simpler manifold (in other words, is minimal). Our problem can thus be summarized as determining which pairs $(\chi, c) \in \mathbb{Z} \times \mathbb{Z}$ are realized by $(\chi_h(M), c_1^2(M))$ for some minimal simply-connected symplectic 4-manifold M. Note that although the problem can be asked in terms of $e(M)$ and $\sigma(M)$, it has historically been studied using the invariants χ_h and c_1^2, a practice we will continue in this thesis.

We begin by making a few observations. First, by the classification theorems in Section 2.1, smooth simply-connected 4-manifolds are uniquely determined up to homeomorphism by the rank, signature and parity of their intersection forms. For a simply-connected symplectic manifold M, both $\sigma(M)$ and $\text{rank}(Q_M) = b_2(M)$ are in turn determined by $\chi_h(M)$ and $c_1^2(M)$ by the above formula for $\sigma(M)$ and

$$b_2(M) = e(M) - 2 = 12\chi_h - c_1^2(M) - 2.$$

Thus for each ordered pair there can be at most two homeomorphism types of irreducible, simply connected symplectic 4-manifolds, one for each parity of intersection form. Thus the geography question can be further specialized to require that the desired manifolds have either even or odd intersection form. Note that by Rohlin's theorem (Theorem 2.1.6), if M is smooth and has even intersection form, then $\sigma(M) \equiv 0 \pmod{16}$. Hence the case of even intersection forms is significantly more restricted than case of odd intersection forms. We will not impose any conditions on the parity of the intersection forms in our discussion.

We can also observe that for any simply-connected symplectic manifold M we have $\chi_h(M) = \frac{1}{4}(e(M) + \sigma(M)) = \frac{1}{4}(2 + 2b_2^+(M))$, and hence $\chi_h(M) \geq 1$. Moreover, from Taubes ([19, 34]), we have that $c_1^2(M) \geq 0$ for minimal symplectic manifolds. We thus focus our attention on ordered pairs $(a, b) \in \mathbb{N}^+ \times \mathbb{N}$ in the (χ, c)-plane.

3.2 Current results

The case of lattice points in the (χ, c)-plane corresponding to manifolds with negative signature (equivalently points with $c < 8\chi$) has been well understood for quite some time. In [2] the authors constructed simply-connected minimal symplectic 4-manifolds which realized all but a finite number of pairs (χ, c) with $c \leq 8\chi - 2$ (corresponding to signatures less than -1). The authors also constructed simply-connected minimal symplectic 4-manifolds which realized points of the form $(\chi, c) = (k, 8k)$ and $(\chi, c) = (j, 8j-1)$ for all integers $k \geq 46$ and $j \geq 49$, corresponding to signatures of 0 and -1 respectively.

All but four of the pairs with $c \leq 8\chi - 2$ that were not realized by the constructions in [2] were realized in the previous literature. In [4] Akhmedov and B.D. Park constructed irreducible simply-connected symplectic 4-manifolds which realized these remaining 4 points, as well as all points of the form $(\chi, c) = (k, 8k-1)$ for $k \geq 1$. This completes the sympletic geography problem for the negative signature case. In other words, for every pair $(\chi, c) \in \mathbb{Z} \times \mathbb{Z}$ with $\chi \geq 1$ and $0 \leq c \leq 8\chi - 1$, there exists a simply-connected minimal symplectic 4-manifold M with $\chi_h(M) = \chi$ and $c_1^2(M) = c$.

The case of points with $c \geq 8\chi$ is far from complete however. Examples of minimal simply-connected symplectic manifolds realizing points in this region of the plane have been scarce relative to the negative signature case. As in the negative signature case, where the possible $c_1^2(M)$ values are bounded below by zero, we might hope to find an upper bound for the $c_1^2(M)$ values realizable by minimal simply-connected symplectic M. To date no such upper bound has been proven. As all simply-connected complex surfaces admit symplectic structures (Theorem 10.1.4 of [16], see also [8]), it is natural to turn to the classical complex geography problem for possible insights into the geography of minimal simply-connected symplectic manifolds. For example, it is well-known that the *Bogomolov-Miyoaka-Yau inequality* (B-M-Y) $c_1^2(S) \leq 9\chi_h(S)$ holds for all minimal complex surfaces of general type. One might then ask if this relation in fact holds for all minimal simply-connected symplectic 4-manifolds as well. Fintushel and Stern conjectured that it does indeed hold in this case, though to this date neither it nor its negation have been proven. Even though aside from $\mathbb{CP}^2$ ($c_1^2(\mathbb{CP}^2) = 9$, $\chi_h(\mathbb{CP}^2) = 1$) no such simply-connected examples have been found with $c_1^2 = 9\chi_h$, there have been families of minimal simply-connected symplectic manifolds constructed which converge to this line. In [32] Stipsicz constructed a family $\{C_n \mid n \geq 1\}$ with $c_1^2(C_n) = 900n^2 + 376n + 4$ and $\chi_h(C_n) = 100n^2 + 62n + 3$, which thus realize a sequence of points in the (χ, c)-plane converging to $c = 9\chi$. Niepel constructed a similar family of minimal simply-connected symplectic K_n in [25] converging to the line $c = 9\chi$, which satisfy $c_1^2(K_n) = 3n^7 + 20n^5 - 24n^4 + 6n^3 + 2$ and $\chi_h(K_n) = \frac{1}{3}(n^7 + 8n^5) - 3n^4 + n^3 + 2$. Note that even the smallest examples in these families have large characteristic values. For example, the smallest two examples with nonnegative signature constructed by Stipsicz in [32] have $\chi_h(C_2) = 527$ and $\chi_h(C_3) = 1089$, while the smallest nonnegative signature examples in the family constructed in [25] have $\chi_h(K_3) = 1163$ and $\chi_h(K_4) = 7490$. It seems to be a common difficulty when constructing minimal simply-connected symplectic manifolds above the $c = 8\chi$ line to find ones with smaller characteristic values.

Recent work has not only focused on finding families converging to the B-M-Y line, but also on finding families which fill in large areas of the (χ, c)-plane. In [26] for example, J. Park was able to show that all but a finite number of *allowable* lattice points (for manifolds with even intersection forms) satisfying $0 \leq c \leq 8.76\chi$ correspond to minimal simply-connected symplectic 4-manifolds (he actually showed there is a line $c = f(\chi)$ with slope greater than 8.76 such that all allowable lattice points with $c \leq f(\chi)$ in the first quadrant correspond to such manifolds). It is important to note that the lattice points he populates are restricted to allowable points for symplectic manifolds with *even* intersection form. Recall that such manifolds satisfy $\sigma \equiv 0 \pmod{16}$, (equivalently $c_1^2 \equiv 8\chi_h \pmod{16}$), and hence the allowable points do not include the vast majority of integer points in the (χ, c)-plane.

In [27], J. Park succeeded in populating large regions of the plane using manifolds with odd intersection form, by starting with non-simply connected complex surfaces $H(n^2)$ lying on the B-M-Y line (constructed by Stipsicz in [32]), and performing repeated fiber sum operations with other building block manifolds to kill the fundamental group. For each odd integer $n \geq 1$ and each $10 \leq k \leq 18$, he constructed a simply connected irreducible symplectic 4-manifold $Z_{n,k}$ with positive signature satisfying $c_1^2(Z_{n,k}) = 225n^2 + 248n + 35 - k$ and $\chi_h(Z_{n,k}) = 25n^2 + 31n + 5$. Note that these manifolds also approach the B-M-Y line as $n \to \infty$. These manifolds were then pieced together with various combinations of other building block manifolds, using the symplectic fiber sum, to populate large regions of the plane above the line $c = 8\chi$ with simply-connected irreducible symplectic manifolds. In particular, he proved the following:

Theorem 3.2.1 ([27])**.** *There is an increasing sequence $\{m_i\}$ with $m_i \to 9$ such that every integer lattice point (χ, c) satisfying $0 \leq c \leq m_i\chi$ and $\chi \geq D_i$ (for some constants D_i), is realized by a simply connectd irreducible symplectic 4-manifold with odd intersection form, which admits infinitely many (both symplectic and non-symplectic) exotic smooth structures.*

The sequence m_i is the quotient $c_1^2(M_i)/\chi_h(M_i)$ for some simply connected irreducible symplectic M_i (with $i \geq 1$ odd) constructed in [27], and is given explicitly by

$$m_i = \frac{225i^2 + 1148i + 1413}{25i^2 + 143.5i + 178.5},$$

while the D_i are given by $D_i = 25i^2 + 143.5i + 178.5$. Note that although the statement of Theorem 3.2.1 only covers lattice points with $\chi \geq D_1 = 347$, [27] contains the construction of minimal simply-connected symplectic 4-manifolds with smaller χ_h values. For each integer $j \geq 1$, using the manifolds $Z_{n,k}$ and extending the constructions in the proof of Theorem 3.2.1, we can find minimal simply-connected symplectic 4-manifolds which realize all lattice points in the region

$$R_j = \{(\chi, c) \mid \chi \geq 100j^2 - 38j - 1 \text{ and } 900j^2 - 404j - 6 \leq c \leq 8\chi + 100j^2 - 100j + 10\}. \quad (3.1)$$

For example, the methods in [27] can be used to construct minimal simply-connected symplectic 4-manifolds corresponding to lattice points in the regions

$$R_1 = \{(\chi, c) \mid \chi \geq 61 \text{ and } 490 \leq c \leq 8\chi + 10\}$$

and

$$R_2 = \{(\chi, c) \mid \chi \geq 323 \text{ and } 2786 \leq c \leq 8\chi + 210\},$$

even though they contain 99 802 points not covered by Theorem 3.2.1.

Before moving on to the main constructions of this thesis, we must briefly mention one recent result which will be useful in populating large regions of the plane by minimal simply-connected symplectic 4-manifolds. The result is due to Akhmedov and B.D. Park, and allows us to construct an infinite number of minimal simply-connected symplectic 4-manifolds in a wedge like region of the (χ, c)-plane once we construct a certain base example. To be more precise:

Theorem 3.2.2 ([5]). *Let M be a closed symplectic 4-manifold that contains a symplectic torus T of self-intersection zero. Suppose further that the inclusion induced homomorphism $\pi_1(\partial(\nu T)) \to \pi_1(M \backslash \nu T)$ is trivial, where νT is a tubular neighborhood of T with boundary $\partial(\nu T)$. Then for any pair $(a, b) \in \mathbb{Z} \times \mathbb{Z}$ satisfying*

$$a \geq 1 \quad \textit{and} \quad 0 \leq b \leq 8a,$$

there exists a symplectic 4-manifold N with $\pi_1(N) = \pi_1(M)$,

$$\chi_h(N) = \chi_h(M) + a \quad \textit{and} \quad c_1^2(N) = c_1^2(M) + b.$$

If M is minimal then N is minimal as well. Furthermore, if $b < 8a$, or if $b = 8a$ and M has an odd intersection form, then N has an odd indefinite intersection form.

Thus, if we have some minimal simply-connected symplectic 4-manifold M with sympectic torus as above, then Theorem 3.2.2 allows us to construct such manifolds which realize all lattice points in a wedge-like region of the (χ, c)-plane given by $\chi \geq \chi_h$ and $c_1^2(M) \leq c \leq 8\chi - 8\chi_h(M) + c_1^2(M)$.

If we only require $0 \leq b \leq 8a - 1$, then the fundamental group condition in Theorem 3.2.2 can be relaxed to only requiring that the inclusion induced homomorphism $\pi_1(T) \to \pi_1(M)$ be trivial (see Theorem 2 of [4]). In this case, N is constructed by taking the symplectic fiber sum of M with certain symplectic manifolds constructed in [2] and [4]. These symplectic manifolds are constructed following similar recipes as the minimal simply-connected symplectic manifolds used to realize the lattice points (χ, c) with $0 \leq c \leq 8\chi - 1$, though they need not be simply-connected. In [5], Akhmedov and B.D. Park strengthened Theorem 2 of [4] to allow pairs with $b = 8a$ (under the slightly stronger fundamental group conditions). In this case, the desired manifold N is obtained from again using the symplectic fiber sum operation, where this time the second summand is constructed by omitting a single Luttinger surgery in the construction of $Y_n(1)$ (see Section 5.2 below).

3.3 Overview of constructions

The primary object of this thesis is to construct minimal simply-connected symplectic 4-manifolds with positive signature which realize points in the (χ, c)-plane not realized in [25, 27, 32]. The examples we construct will have the smallest χ_h values of all such manifolds currently known to the author. To do this, we follow the constructions in [3]. In Sections 4.2 and 4.3 we construct certain branched coverings from [5] over the product spaces $\Sigma_g \times \Sigma_g$ (for closed genus g Riemann surfaces Σ_g), giving rise to Lefschetz fibrations and hence symplectic manifolds which lie close to the B-M-Y line but have nontrivial fundamental groups. In Section 5.2 we construct a family of pairwise nondiffeomorphic 4-manifolds $\{Y_n(m) \mid m = 1, 2, 3, \ldots\}$ for each integer $n \geq 2$, as constructed in [4]. When $m = 1$, these manifolds are symplectic. We then present general fiber sum constructions (from [3]) in Sections 6.2.1 and 6.3.1, using the $Y_n(m)$ and relatively minimal Lefschetz fibrations (such as those constructed in Sections 4.2 and 4.3). It is shown that by choosing the gluing diffeomorphism of these fiber sums carefully, the result is simply-connected, and when $m = 1$, minimal and symplectic. These constructions populate previously unrealized points in the (χ, c)-plane, including points with particularly small χ_h values. Using Theorem 3.2.2, we are then able to use these newly constructed manifolds as the starting point to populate large wedge shaped regions of the plane. Using these constructions, we are able to realize 19 952 new lattice points not realized in the previous literature.

Chapter 4

Branched Covering Constructions of Symplectic 4-Manifolds

4.1 Branched covering construction

The purpose of this section is to give a brief outline of a construction due to Hirzebruch which yields branched coverings of 4-dimensional manifolds. To begin, we recall that the set of isomorphism classes of $U(1)$-bundles over a smooth 4-manifold M along with the tensor product of bundles forms a group $\mathcal{L}_M$, which is canonically isomorphic to $H^2(M;\mathbb{Z})$, the second integral cohomology group of M. This isomorphism $\Upsilon : \mathcal{L}_M \to H^2(M;\mathbb{Z})$ sends the line bundle $L \to M$ to its first Chern class $c_1(L)$. (This fact serves as the starting point of an equivalent alternate definition of the Chern classes $c(L) \in H^*(M;\mathbb{Z})$.)

Suppose that M is a smooth 4-manifold. Let B be a closed 2-dimensional submanifold of M, and suppose that $[B] \in H_2(M;\mathbb{Z})$ is divisible by d. In other words, there exists some $[A] \in H_2(M;\mathbb{Z})$ such that $d[A] = [B]$. Let $PD([B])$ denote the Poincaré dual of $[B]$ in $H^2(M;\mathbb{Z})$. By our above comments, $\Upsilon^{-1}(PD([B]))$ is a complex line bundle, which we will denote by $L_B \to M$, with $c_1(L_B) = PD([B])$. Since $c_1(L_B)$ is the top Chern class of $L_B \to M$, it is equal to its Euler class $e(L_B)$. By definition of the Euler class, this implies that $PD(e(L_B)) = [B]$ is the homology class of the zero set of a generic section of $L_B \to M$. We can thus fix a generic section $s_B : M \to L_B$, with $s_B^{-1}(0) = B$. We likewise have a complex line bundle $\Upsilon^{-1}(PD([A]))$, which we denote by $L_A \to M$, with generic section $s_A : M \to L_A$ whose zero set is A. Furthermore, we let $\pi : L_A \to M$ denote its bundle projection.

Since $PD([B]) = d \cdot PD([A])$, we have that $L_B = L_A \otimes \cdots \otimes L_A = L_A^{\otimes d}$. We define a subset of the total space L_A by

$$N = \{x_p \in L_A \mid x_p^{\otimes d} = s_B(p)\}.$$

Consider the restriction $\pi|_N : N \to M$ of the bundle projection $\pi : L_A \to M$. This defines a d-fold cyclic branched covering of M which is branched along B. For a d-fold branched covering $\pi : N \to M$ constructed in this way we have the following:

Lemma 4.1.1 (See [16]). *If $\pi : N \to M$ is the d-fold branched covering branched along $B \subset M$ constructed above, we have*

$$c_1(N) = f^* \left(c_1(M) - \frac{d-1}{d} PD([B]) \right)$$

and

$$c_1^2(N) = d \left(PD(c_1(M)) - \frac{d-1}{d} [B] \right)^2 .$$

4.2 First family of symplectic 4-manifolds with positive signature

Following Section 2 of [5], we construct our first family of symplectic manifolds with positive signature by starting with an oriented genus $g > 0$ Riemann surface Σ_g. Σ_g can be constructed by taking two concentric spheres $S^2(1)$ and $S^2(2)$ of radius 1 and 2 in $\mathbb{R}^3$, and connecting them with $g+1$ tubes (see Figure 4.2). If the tubes are arranged to be centered around rays in the xy-plane beginning at the origin, spaced at $\frac{2\pi}{g+1}$ radians apart, then we can define a map $\gamma : \Sigma_g \to \Sigma_g$ which is rotation of Σ_g around the z-axis by an angle of $\frac{2\pi}{g+1}$. Clearly γ has exactly 4 fixed points (the points where the z-axis intersects $S^2(1)$ and $S^2(2)$), and has order $g+1$.

For each integer $1 \le i \le g+1$, let $\Gamma_i = \text{graph}(\gamma^i) = \{(x, \gamma^i(x)) \mid x \in \Sigma_g\} \subset \Sigma_g \times \Sigma_g$. Thus $\Gamma_1, \ldots, \Gamma_{g+1}$ are all genus g surfaces in $\Sigma_g \times \Sigma_g$, which are disjoint except for 4 points $\{p_1, p_2, p_3, p_4\}$, which they all share in common. Note that for $i < j$, the graph Γ_j is the image of Γ_i under the orientation preserving diffeomorphism $\text{id}_{\Sigma_g} \times \gamma^{j-i} : \Sigma_g \times \Sigma_g \to \Sigma_g \times \Sigma_g$. Thus $[\Gamma_i]^2 = [\Gamma_j]^2$ for all i and j. Since $\Gamma_{g+1} \simeq \Sigma_g$ is the diagonal in $\Sigma_g \times \Sigma_g$, the tangent bundle $T\Gamma_{g+1} \simeq T\Sigma_g$ is isomorphic to the normal bundle $\nu\Gamma_{g+1}$ of Γ_{g+1} in $\Sigma_g \times \Sigma_g$. Since $\langle e(\nu S), [S] \rangle = [S]^2$ for any closed, orientable surface embedded in an oriented 4-manifold (where we tacitly denote the Euler class of the bundle νS here by $e(\nu S)$), we have that

$$[\Gamma_{g+1}]^2 = \langle e(\nu\Gamma_{g+1}), [\Gamma_{g+1}] \rangle = \langle e(T\Sigma_g), [\Gamma_{g+1}] \rangle = \langle e(T\Gamma_{g+1}), [\Gamma_{g+1}] \rangle,$$

which is just equal to the Euler characteristic $e(\Gamma_{g+1}) = 2 - 2g$ of Γ_{g+1}. Thus $[\Gamma_i]^2 = 2 - 2g$ for $1 \le i \le g+1$.

Let $B = \Gamma_1 \cup \cdots \cup \Gamma_{g+1}$. We compute the homology class $[B]$ of B in $H_2(\Sigma_g \times \Sigma_g; \mathbb{Z})$ in the following lemma:

Lemma 4.2.1. *For B as above, the homology class of B in $H_2(\Sigma_g \times \Sigma_g; \mathbb{Z})$ is given by $[B] = (g+1)([\Sigma_g \times \{\text{pt}\}] + [\{\text{pt}\} \times \Sigma_g])$.*

Proof. Although B is not a smooth surface (it fails to be a manifold at the points $(p_i, p_i) \in \Sigma_g \times \Sigma_g$), it does define a homology class in $H_2(\Sigma_g \times \Sigma_g; \mathbb{Z})$ which will be equal to $[\Gamma_1] + \cdots + [\Gamma_{g+1}]$. By the Künneth theorem, $H_2(\Sigma_g \times \Sigma_g; \mathbb{Z}) \cong \bigoplus (4g^2 + 2) \mathbb{Z}$, and is generated by the classes

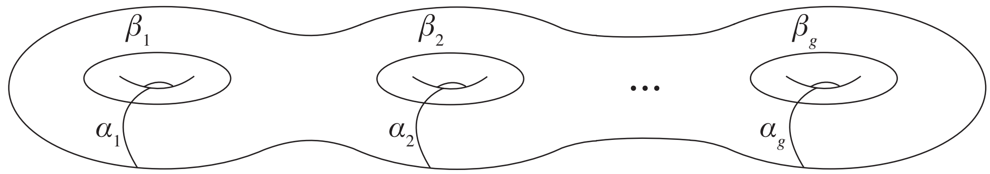

Figure 4.1: Generators of $H_1(\Sigma_g;\mathbb{Z})$.

$[\Sigma_g \times \{\text{pt}\}]$, $[\{\text{pt}\} \times \Sigma_g]$, and $4g^2$ tori classes $[\mu_i \times \mu_j]$, where μ_i and μ_j are any two of the standard generators $\{\alpha_1, \dots \alpha_g, \beta_1, \dots, \beta_g\}$ of $H_1(\Sigma_g;\mathbb{Z})$ as in Figure 4.1. We thus write

$$\begin{aligned}[B] &= a\,[\Sigma_g \times \{\text{pt}\}] + b\,[\{\text{pt}\} \times \Sigma_g] + \sum_{i,j=1}^{g} c_{ij}\,[\alpha_i \times \alpha_j] + \sum_{i,j=1}^{g} d_{ij}\,[\alpha_i \times \beta_j] \\ &\quad + \sum_{i,j=1}^{g} e_{ij}\,[\beta_i \times \alpha_j] + \sum_{i,j=1}^{g} f_{ij}\,[\beta_i \times \beta_j]\end{aligned}$$

for some integers a, b, c_{ij}, d_{ij}, e_{ij} and f_{ij}.

Let $\text{pr}_1 : \Sigma_g \times \Sigma_g \to \Sigma_g$ be the projection map onto the first factor. Since $(\text{pr}_1)_*[\Sigma_g \times \{\text{pt}\}] = [\Sigma_g]$, while $(\text{pr}_1)_*[\{\text{pt}\} \times \Sigma_g] = (\text{pr}_1)_*[\mu_i \times \mu_j] = 0$, it follows that

$$(\text{pr}_1)_*[B] = a[\Sigma_g].$$

As each of the Γ_i is a graph of a self-diffeomorphism of Σ_g, we also have that $(\text{pr}_1)_*[\Gamma_i] = [\Sigma_g]$ for each i. Thus

$$(\text{pr}_1)_*[B] = (\text{pr}_1)_*([\Gamma_1] + \cdots + [\Gamma_{g+1}]) = (g+1)\,[\Sigma_g]\,,$$

and hence $a = g+1$. Similarly, $b = g+1$.

Using the intersection pairing on $H_2(\Sigma_g \times \Sigma_g;\mathbb{Z})$, we will proceed to show that each of the c_{ij} is zero. Consider the class $[\beta_i \times \beta_j]$. Note that $[\beta_i \times \beta_j] \cdot [\alpha_i \times \alpha_j] = \pm 1$, while $[\beta_i \times \beta_j] \cdot [P] = 0$ for all of the other generators $[P]$ of $H_2(\Sigma_g \times \Sigma_g;\mathbb{Z})$. Thus

$$[B] \cdot [\beta_i \times \beta_j] = \pm c_{ij}.$$

Recall that the map $\gamma : \Sigma_g \to \Sigma_g$ is defined by rotating the surface Σ_g in Figure 4.2 by $\frac{2\pi}{g+1}$ radians. Note that γ_* cyclically permutes the classes $[\beta_1], \dots, [\beta_g], -([\beta_1] + \cdots + [\beta_g])$. If $i < j$, then $(\gamma^{j-i})_*[\beta_i] = [\beta_j]$, while for $i \geq j$, $(\gamma^{g-i+j+1})_*[\beta_i] = [\beta_j]$. In either case, we can isotope β_j so that it is parallel and disjoint from the image $\gamma^k(\beta_i)$ (for the appropriate k). Thus, we have that $\beta_i \times \beta_j$ is disjoint from $\Gamma_k = \{(x, \gamma^k(x)) \mid x \in \Sigma_g\}$. Similarly, since we can isotope all of the β_t so that they are disjoint, we have that $\beta_i \times \beta_j$ is disjoint from each of the Γ_t, and hence that $[B] \cdot [\beta_i \times \beta_j] = \pm c_{ij} = 0$. Similarly, using the fact that $[B] \cdot [\alpha_i \times \alpha_j] = 0$ we can show that each f_{ij} is also zero.

To compute the d_{ij} values, we again note that $[B] \cdot [\beta_i \times \alpha_j] = \pm d_{ij}$. As noted above, for appropriate choice of k, we have that $(\gamma^k)_*[\beta_i] = [\beta_j]$. If we isotope α_j and β_j so that $\beta_j = \gamma^k(\beta_i)$,

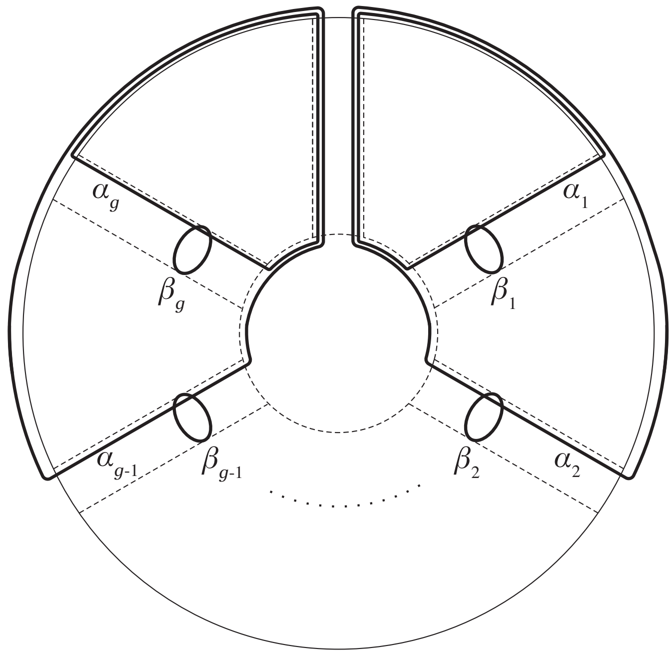

Figure 4.2: Alternate view of Σ_g with $H_1(\Sigma_g;\mathbb{Z})$ generators (thickened lines).

and so that α_j and β_j intersect in exactly one point, then $\beta_i \times \alpha_j$ will intersect $(\beta_i \times \beta_j) \cap \Gamma_k$ in exactly one point. It is easy to see that this will be the only intersection point of $\beta_i \times \alpha_j$ with Γ_k. We can see however, from Figure 4.2, that we can arrange that $\beta_i \times \alpha_j$ will also intersect Γ_{g-i+1}, in precisely one point, whose sign will be opposite that of the intersection of $\beta_i \times \alpha_j$ and Γ_k. As we can isotope these two intersection points so that they are away from the fixed point set of γ (and hence distinct points), and since $(\beta_i \times \alpha_j) \cap \Gamma_t = \emptyset$ for all other $1 \leq t \leq g+1$, we can see that $[B] \cdot [\beta_i \times \alpha_j] = \pm d_{ij} = 0$.

To compute the values of the e_{ij} we first note that the action of γ_* on the subgroup generated by $\{[\alpha_1], \ldots, [\alpha_g]\}$ is given by the matrix

$$\begin{bmatrix} -1 & -1 & \cdots & -1 & -1 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}.$$

Applying similar arguments as in the above cases, we can show that $e_{ij} = f_{ij} = 0$ for all i and j. □

Since $\Sigma_g \times \Sigma_g$ is a symplectic manifold manifold with the product symplectic structure, it has well defined Chern classes $c_i \in H^*(\Sigma_g \times \Sigma_g;\mathbb{Z})$. From the adjunction formula, we have that $\langle c_1(\Sigma_g \times \Sigma_g), [\Sigma_g \times \{\text{pt}\}]\rangle = \langle c_1(\Sigma_g \times \Sigma_g), [\{\text{pt}\} \times \Sigma_g]\rangle = 2-2g$, and $\langle c_1(\Sigma_g \times \Sigma_g), [\mu_i \times \mu_j]\rangle = 0$

(where μ_i and μ_j are generators of $H_1(\Sigma_g \times \Sigma_g)$ as above). Thus

$$PD\left(c_1\left(\Sigma_g \times \Sigma_g\right)\right) = 2\left(1-g\right)\left(\left[\Sigma_g \times \{\text{pt}\}\right] + \left[\{\text{pt}\} \times \Sigma_g\right]\right).$$

Blowing up $\Sigma_g \times \Sigma_g$ at the fixed points p_1, p_2, p_3, and p_4, we obtain $g+1$ disjoint surfaces $\widetilde{\Gamma}_i$ in $\left(\Sigma_g \times \Sigma_g\right) \# 4\overline{\mathbb{CP}}^2$, which are the proper transforms of the Γ_i. Let $\widetilde{B} = \widetilde{\Gamma}_1 \cup \cdots \cup \widetilde{\Gamma}_{g+1}$ be the proper transform of $B = \Gamma_1 \cup \cdots \cup \Gamma_{g+1}$. We have that $[\widetilde{B}]$ is given by

$$[\widetilde{B}] = (g+1)\left(\left[\Sigma_g \times \{\text{pt}\}\right] + \left[\{\text{pt}\} \times \Sigma_g\right] - \sum_{i=1}^{4} [e_i]\right)$$

where e_i is the exceptional sphere of the i^{th} blow-up. Furthermore

$$PD(c_1((\Sigma_g \times \Sigma_g)\#4\overline{\mathbb{CP}}^2)) = 2\left(1-g\right)\left(\left[\Sigma_g \times \{\text{pt}\}\right] + \left[\{\text{pt}\} \times \Sigma_g\right]\right) - \sum_{i=1}^{4} [e_i].$$

Since $[\widetilde{B}]$ is divisible by $g+1$, we can construct a $(g+1)$-fold cyclic branched cover of $\left(\Sigma_g \times \Sigma_g\right) \# 4\overline{\mathbb{CP}}^2$, branched along $\widetilde{B}$ (see Section 4.1). Furthermore, since

$$\begin{aligned} H_2((\Sigma_g \times \Sigma_g)\#4\overline{\mathbb{CP}}^2;\mathbb{Z}) &\cong H_2(\Sigma_g \times \Sigma_g;\mathbb{Z}) \oplus 4H_2(\overline{\mathbb{CP}}^2;\mathbb{Z}) \\ &\cong \bigoplus(4g^2+6)\mathbb{Z} \end{aligned}$$

is torsion-free, this cyclic branched covering will be uniquely constructed. Denote this cyclic $(g+1)$-fold branched covering by $\phi : X_g(1) \to (\Sigma_g \times \Sigma_g)\#4\overline{\mathbb{CP}}^2$.

Consider the following composition of maps

$$X_g(1) \longrightarrow \left(\Sigma_g \times \Sigma_g\right) \# 4\overline{\mathbb{CP}}^2 \longrightarrow \Sigma_g \times \Sigma_g \xrightarrow{\text{pr}_1} \Sigma_g.$$

This can be seen to be a Lefschetz fibration over Σ_g, which we will denote by $\varphi_{g,1} : X_g(1) \to \Sigma_g$. The preimage of a generic point $z \in \Sigma_g$ in $(\Sigma_g \times \Sigma_g)\#4\overline{\mathbb{CP}}^2$ will be a surface of genus g which intersects the branch locus $\widetilde{B} \subset (\Sigma_g \times \Sigma_g)\#4\overline{\mathbb{CP}}^2$ in $g+1$ points. Hence a regular fiber of $\varphi_{g,1}$ is a $(g+1)$-fold covering of Σ_g, branched at $g+1$ points. By the Riemann-Hurwitz formula, this implies that for a regular fiber F

$$\begin{aligned} e\left(F\right) &= (g+1)\,e\left(\Sigma_g\right) - g\left(g+1\right) \\ &= 2 - 3g^2 - g, \end{aligned}$$

and hence F is a genus $\frac{1}{2}g\left(3g+1\right)$ surface.

Lemma 4.2.2. *For $1 \leq i \leq g+1$ there exist sections $s_i : \Sigma_g \to X_g(1)$ of the fibration $\varphi_{g,1}$ whose images $s_i(\Sigma_g)$ in $X_g(1)$ are genus g surfaces of self-intersection -2.*

Proof. Since ϕ is branched along the union of the proper transforms $\widetilde{\Gamma}_i$, ϕ maps injectively onto these surfaces. Thus for each i we can lift $\widetilde{\Gamma}_i$ uniquely to a surface $\overline{\Gamma}_i$ of genus g in $X_g(1)$. Note that $\overline{\Gamma}_i$ will intersect each fiber of the fibration $\varphi_{g,1} : X_g(1) \to \Sigma_g$ in exactly one point; hence it will be the image of a section of $\varphi_{g,1}$. In other words, since the restriction of the map $\varphi_{g,1}$ to $\overline{\Gamma}_i$ can be seen to be both injective and surjective, it is a diffeomorphism of $\overline{\Gamma}_i$ to Σ_g. The inverse of these diffeomorphism are the desired sections s_i of $\varphi_{g,1}$ with images $\overline{\Gamma}_i$.

Denote the union $\overline{\Gamma}_1 \cup \cdots \cup \overline{\Gamma}_{g+1}$ by $\overline{B}$ (i.e. the lift of $\widetilde{B}$ in $X_g(1)$). By the adjunction formula, we have that

$$\begin{aligned}\langle c_1(X_g(1)), [\overline{B}]\rangle &= [\overline{B}]^2 + \chi(\overline{B}) \\ &= ([\overline{\Gamma}_1] + \cdots + [\overline{\Gamma}_{g+1}])^2 + \chi(\overline{\Gamma}_1 \cup \cdots \cup \overline{\Gamma}_{g+1}) \\ &= (g+1)([\overline{\Gamma}_i]^2 + \chi(\overline{\Gamma}_i)) \\ &= (g+1)([\overline{\Gamma}_i]^2 + 2 - 2g)\end{aligned}$$

since the $\overline{\Gamma}_i$ are disjoint surfaces of genus g. Thus

$$[\overline{\Gamma}_i]^2 = \frac{1}{g+1}\langle c_1(X_g(1)), [\overline{B}]\rangle - 2 + 2g.$$

Recall that $X_g(1)$ is constructed by choosing a complex line bundle $L_{\widetilde{B}} \to (\Sigma_g \times \Sigma_g)\#4\overline{\mathbb{CP}}^2$ with a generic section whose zero set is equal to $\widetilde{B}$. Since

$$c_1(L_{\widetilde{B}}) = e(L_{\widetilde{B}}) = PD[\widetilde{B}] = (g+1)PD\left([\Sigma_g \times \{\text{pt}\}] + [\{\text{pt}\} \times \Sigma_g] - \sum_{i=1}^{4}[e_i]\right)$$

is divisible by $(g+1)$, we can use the class

$$[A] = \left([\Sigma_g \times \{\text{pt}\}] + [\{\text{pt}\} \times \Sigma_g] - \sum_{i=1}^{4}[e_i]\right) \in H_2((\Sigma_g \times \Sigma_g)\#4\overline{\mathbb{CP}}^2; \mathbb{Z})$$

as above to specify the $(g+1)$-fold branched covering $\phi : X_g(1) \to (\Sigma_g \times \Sigma_g)\#4\overline{\mathbb{CP}}^2$ branched along $\widetilde{B}$.

From Lemma 4.1.1 we have that

$$\begin{aligned}c_1(X_g(1)) &= \phi^*(c_1((\Sigma_g \times \Sigma_g)\#4\overline{\mathbb{CP}}^2) - gPD[A]) \\ &= \phi^* PD((2-3g)([\Sigma_g \times \{\text{pt}\}] + [\{\text{pt}\} \times \Sigma_g]) + (g-1)\sum_{i=1}^{4}[e_i]).\end{aligned}$$

Thus

$$\begin{aligned}\langle c_1(X_g(1)), [\overline{B}]\rangle &= \langle \phi^*(c_1((\Sigma_g \times \Sigma_g)\#4\overline{\mathbb{CP}}^2) - gPD[A]), [\overline{B}]\rangle \\ &= \langle c_1((\Sigma_g \times \Sigma_g)\#4\overline{\mathbb{CP}}^2) - gPD[A], \phi_*[\overline{B}]\rangle \\ &= \langle c_1((\Sigma_g \times \Sigma_g)\#4\overline{\mathbb{CP}}^2) - gPD[A], [\widetilde{B}]\rangle \\ &= \left\langle PD\left((2-3g)([\Sigma_g \times \{\text{pt}\}] + [\{\text{pt}\} \times \Sigma_g]) + (g-1)\sum_{i=1}^{4}[e_i]\right),\right.\end{aligned}$$

$$
\begin{aligned}
& \quad (g+1)\left([\Sigma_g \times \{\mathrm{pt}\}] + [\{\mathrm{pt}\} \times \Sigma_g] - \sum_{i=1}^{4}[e_i]\right)\Bigg\rangle \\
= & \quad (2-3g)\left([\Sigma_g \times \{\mathrm{pt}\}] + [\{\mathrm{pt}\} \times \Sigma_g]) + (g-1)\sum_{i=1}^{4}[e_i]\right) \\
& \quad \cdot (g+1)\left([\Sigma_g \times \{\mathrm{pt}\}] + [\{\mathrm{pt}\} \times \Sigma_g] - \sum_{i=1}^{4}[e_i]\right) \\
= & \quad -2g(g+1),
\end{aligned}
$$

implying that $[\overline{\Gamma}_i]^2 = -2$. □

Using the Riemann-Hurwitz formula, we compute the Euler characteristic of $X_g(1)$ as follows:

$$
\begin{aligned}
e\left(X_g(1)\right) &= (g+1)\, e((\Sigma_g \times \Sigma_g)\#4\overline{\mathbb{CP}}^2) - ge(\widetilde{B}) \\
&= (g+1)\, e((\Sigma_g \times \Sigma_g)\#4\overline{\mathbb{CP}}^2) - g\sum_{i=1}^{g+1} e(\widetilde{\Gamma}_i) \\
&= (g+1)\left(e(\Sigma_g)^2 + 4e(\overline{\mathbb{CP}}^2) - 8e(D^4)\right) - g\,(g+1)\, e\left(\Sigma_g\right) \\
&= 2\,(g+1)\left(3g^2 - 5g + 4\right),
\end{aligned}
$$

(where we used the facts that $e(\Sigma_g) = 2-2g$, $e(\overline{\mathbb{CP}}^2) = 3$ and $e(D^4) = 1$). Furthermore, from Lemma 4.1.1 we have that

$$
\begin{aligned}
c_1^2(X_g(1)) &= (g+1)\left(PD(c_1((\Sigma_g \times \Sigma_g)\#4\overline{\mathbb{CP}}^2)) - \frac{g}{g+1}[\widetilde{B}]\right)^2 \\
&= (g+1)\left((2-3g)\left([\Sigma_g \times \{\mathrm{pt}\}] + [\{\mathrm{pt}\} \times \Sigma_g]\right) + (g-1)\sum_{i=1}^{4}[e_i]\right)^2 \\
&= 2\,(g+1)\left(7g^2 - 8g + 2\right),
\end{aligned}
$$

from which we obtain

$$
\begin{aligned}
\sigma(X_g(1)) &= \frac{c_1^2\left(X_g(1)\right) - 2e\left(X_g(1)\right)}{3} \\
&= \frac{2}{3}(g+1)(g^2+2g-6)
\end{aligned}
$$

and

$$
\begin{aligned}
\chi_h(X_g(1)) &= \frac{e\left(X_g(1)\right) + c_1^2\left(X_g(1)\right)}{12} \\
&= \frac{1}{6}(g+1)(10g^2-13g+6).
\end{aligned}
$$

For each integer $u \geq 2$, let $\Theta_u : \Sigma_k \to \Sigma_g$ be a u-fold unbranched covering of Σ_g, where $k = u(g-1)+1$. Let $\varphi_{g,u} : X_g(u) \to \Sigma_k$ denote the pull-back fibration of $\varphi_{g,1}$ by Θ_u. It follows that $\varphi_{g,u}$ is again a Lefschetz fibration with regular fiber a genus $\frac{1}{2}g\,(3g+1)$ surface. Each section

s_i of $\varphi_{g,1}$ will pull-back to a section of $\varphi_{g,u}$. A parallel copy of $\overline{\Gamma}_i$ in $X_g(1)$ will pull-back to a parallel copy of the image of the pull-back section, with each intersection point in $X_g(1)$ yielding u intersection points of the same sign in $X_g(u)$. Hence $\varphi_{g,u}$ admits sections whose images are genus k surfaces with self-intersection $-2u$. Since $X_g(u)$ can be seen to be a u-fold unbranched cover of $X_g(1)$, we have that $e(X_g(u)) = u \cdot e(X_g(1))$, $\sigma(X_g(u)) = u \cdot \sigma(X_g(1))$, $\chi_h(X_g(u)) = u \cdot \chi_h(X_g(1))$, and $c_1^2(X_g(u)) = u \cdot c_1^2(X_g(1))$ (from Section 7.1 of [16], using $[B] = 0$).

Consider now the fibration

$$X_g(1) \longrightarrow (\Sigma_g \times \Sigma_g)\#4\overline{\mathbb{CP}}^2 \longrightarrow \Sigma_g \times \Sigma_g,$$

which we can pull back by $\Theta_u \times \Theta_u : \Sigma_k \times \Sigma_k \to \Sigma_g \times \Sigma_g$. The total space of this fibration is a $(g+1)$-fold branched cover of $\Sigma_k \times \Sigma_k$, and will be denoted by $\widetilde{X}_g(u^2)$. The composition

$$\widetilde{X}_g(u^2) \longrightarrow \Sigma_k \times \Sigma_k \xrightarrow{\text{pr}_1} \Sigma_k$$

will be denoted by $\widetilde{\varphi}_{g,u}$, and can be seen to be a Lefschetz fibration of $\widetilde{X}_g(u^2)$ over Σ_k. The preimage of a generic point $z \in \Sigma_k$ in $\widetilde{X}_g(u^2)$ will be a $(g+1)$-fold covering of Σ_k branched at $u(g+1)$ points. Thus for a generic fiber F of $\widetilde{\varphi}_{g,u}$

$$\begin{aligned} e(F) &= (g+1)e(\Sigma_k) - ug(g+1) \\ &= u(g+1)(2-3g), \end{aligned}$$

and hence is a genus $\frac{1}{2}u(g+1)(3g-2)+1$ surface. The $\overline{\Gamma}_i$ will again give rise to sections of $\widetilde{\varphi}_{g,u}$ whose images will be genus k surfaces of self-intersection $-2u$. Since $\widetilde{X}_g(u^2)$ is a u^2-fold unbranched cover of $X_g(1)$ it follows that $e(\widetilde{X}_g(u^2)) = u^2 \cdot e(X_g(1))$, $\sigma(\widetilde{X}_g(u^2)) = u^2 \cdot \sigma(X_g(1))$, $\chi_h(\widetilde{X}_g(u^2)) = u^2 \cdot \chi_h(X_g(1))$, and $c_1^2(\widetilde{X}_g(u^2)) = u^2 \cdot c_1^2(X_g(1))$.

4.3 Second family of symplectic 4-manifolds with positive signature

A second way of constructing a genus $g > 0$ Riemann surface Σ_g is by identifying diametrically opposite edges of a $4g$-gon, so that the word given by its boundary is

$$a_1 a_2 \cdots a_{2g} a_1^{-1} a_2^{-1} \cdots a_{2g}^{-1}$$

(see Exercise IV.5.6 of [23]). This $4g$-gon can be divided into two $(2g+1)$-gons along a diagonal d so that the boundaries of the two $(2g+1)$-gons are

$$a_1 a_2 \cdots a_{2g} d \quad \text{and} \quad a_1^{-1} a_2^{-1} \cdots a_{2g}^{-1} d^{-1}.$$

Viewed as regular polygons, these $(2g+1)$-gons can be rotated by $\frac{2\pi}{2g+1}$ and then reglued, yielding a self-diffeomorphism $\delta : \Sigma_g \to \Sigma_g$ of order $2g+1$ (see Figure 4.3). This self-diffeomorphism will

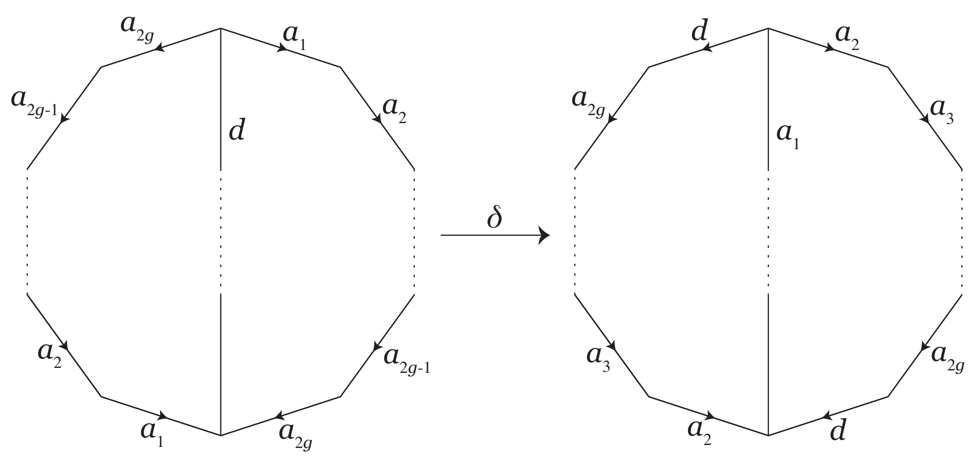

Figure 4.3: Construction of self-diffeomorphism $\delta : \Sigma_g \to \Sigma_g$.

have three fixed points: the 2 centers of rotation of the $(2g+1)$-gons, and the single vertex (after identification of the edges).

As in section 4.2, we consider the graphs $\Delta_i = \text{graph}(\delta^i) = \{(x, \delta^i(x) \mid x \in \Sigma_g\} \subset \Sigma_g \times \Sigma_g$. These Δ_i will be disjoint except for three points $\{q_1, q_2, q_3\}$, where they all intersect transversely. Let $D = \Delta_1 \cup \cdots \cup \Delta_{2g+1}$. The homology class $[D] \in H_2(\Sigma_g \times \Sigma_g; \mathbb{Z})$ is given by the following lemma:

Lemma 4.3.1. *For D as above, the homology class of D in $H_2(\Sigma_g \times \Sigma_g; \mathbb{Z})$ is given by $[D] = (2g+1)([\Sigma_g \times \{\text{pt}\}] + [\{\text{pt}\} \times \Sigma_g])$.*

The proof of Lemma 4.3.1 consists of a computation similar to the computation of $[B]$ in Section 4.2.

Blowing up $\Sigma_g \times \Sigma_g$ three times at the points q_1, q_2, and q_3, we obtain $2g+1$ disjoint surfaces $\widetilde{\Delta}_1, \ldots, \widetilde{\Delta}_{2g+1}$, which are the proper transforms of the Δ_i. Denoting the union of these proper transforms by $\widetilde{D}$, we have that

$$[\widetilde{D}] = (2g+1)\left([\Sigma_g \times \{\text{pt}\}] + [\{\text{pt}\} \times \Sigma_g] - \sum_{i=1}^{3} [e_i]\right)$$

where again e_i is the exceptional sphere of the i^{th} blow-up, and that

$$PD(c_1((\Sigma_g \times \Sigma_g)\#3\overline{\mathbb{CP}}^2)) = 2(1-g)([\Sigma_g \times \{\text{pt}\}] + [\{\text{pt}\} \times \Sigma_g]) - \sum_{i=1}^{3} [e_i].$$

Since $[\widetilde{D}]$ is divisible by $2g+1$ we can take the $(2g+1)$-fold covering $Z_g(1)$ of $(\Sigma_g \times \Sigma_g)\#4\overline{\mathbb{CP}}^2$ branched along $\widetilde{D}$. As in Section 4.2, since

$$H_2((\Sigma_g \times \Sigma_g)\#4\overline{\mathbb{CP}}^2; \mathbb{Z}) \cong \bigoplus (4g^2+5)\mathbb{Z}$$

is torsion-free, $Z_g(1)$ constructed in this way will be unique. Furthermore, as in Section 4.2, the composition

$$Z_g(1) \longrightarrow (\Sigma_g \times \Sigma_g)\,\#3\overline{\mathbb{CP}}^2 \longrightarrow \Sigma_g \times \Sigma_g \xrightarrow{\mathrm{pr}_1} \Sigma_g$$

will yield a Lefschetz fibration of $Z_g(1)$ over Σ_g, which will be denoted by $\psi_{g,1} : Z_g(1) \to \Sigma_g$. A regular fiber F of $\psi_{g,1}$ is a $(2g+1)$-fold covering of Σ_g, branched at $2g+1$ points. Hence

$$\begin{aligned} e(F) &= (2g+1)e(\Sigma_g) - 2g(2g+1) \\ &= -8g^2 + 2 \end{aligned}$$

and thus F is a genus $4g^2$ surface.

Lemma 4.3.2. *For $1 \leq i \leq 2g+1$ there exist sections $s_i' : \Sigma_g \to Z_g(1)$ of the fibration $\psi_{g,1}$ whose images $s_i'(\Sigma_g)$ in $Z_g(1)$ are genus g surfaces of self-intersection -1.*

Proof. As in the proof of Lemma 4.2.2, we can lift the $\widetilde{\Delta}_i$ to genus g surfaces $\overline{\Delta}_i$ in $Z_g(1)$. The section s_i' will again be given by the inverse of the restriction of $\psi_{g,1}$ to $\overline{\Delta}_i$. Let $\overline{D}$ be the union of the $\overline{\Delta}_i$. Then

$$[\overline{\Delta}_i]^2 = \frac{1}{2g+1}\langle c_1(Z_g(1)), [\overline{D}]\rangle - 2 + 2g.$$

If $\phi' : Z_g(1) \to (\Sigma_g \times \Sigma_g)\#3\overline{\mathbb{CP}}^2$ is the covering map constructed above, and

$$[A'] = [\Sigma_g \times \{\mathrm{pt}\}] + [\{\mathrm{pt}\} \times \Sigma_g] - \sum_{i=1}^{3}[e_i],$$

then

$$\begin{aligned} \langle c_1(Z_g(1)), [\overline{D}]\rangle &= \langle (\phi')^*(c_1((\Sigma_g \times \Sigma_g)\#3\overline{\mathbb{CP}}^2) - 2gPD[A']), [\overline{D}]\rangle \\ &= \left\langle PD\left((2-4g)([\Sigma_g \times \{\mathrm{pt}\}] + [\{\mathrm{pt}\} \times \Sigma_g]) + (2g-1)\sum_{i=1}^{3}[e_i]\right),\right. \\ &\qquad \left.(2g+1)\left([\Sigma_g \times \{\mathrm{pt}\}] + [\{\mathrm{pt}\} \times \Sigma_g] - \sum_{i=1}^{3}[e_i]\right)\right\rangle \\ &= (2g+1)(1-2g), \end{aligned}$$

whence $[\overline{\Delta}_i]^2 = -1$. □

Computing the Euler characterstic of $Z_g(1)$ yields

$$\begin{aligned} e\left(Z_g(1)\right) &= (2g+1)\,e((\Sigma_g \times \Sigma_g)\,\#3\overline{\mathbb{CP}}^2) - 2ge(\widetilde{D}) \\ &= (2g+1)\,e((\Sigma_g \times \Sigma_g)\,\#3\overline{\mathbb{CP}}^2) - 2g\sum_{i=1}^{2g+1} e(\widetilde{\Delta}_i) \\ &= (2g+1)\left(e\left(\Sigma_g\right)^2 + 3e(\overline{\mathbb{CP}}^2) - 6e(D^4)\right)) - 2g\left(2g+1\right)e\left(\Sigma_g\right) \\ &= (2g+1)(8g^2 - 12g + 7), \end{aligned}$$

while

$$\begin{aligned}
c_1^2(Z_g(1)) &= (2g+1)\left(PD(c_1((\Sigma_g\times\Sigma_g)\#3\overline{\mathbb{CP}}^2)) - \frac{2g}{2g+1}[\widetilde{D}]\right)^2\\
&= (2g+1)\left((2-4g)\left([\Sigma_g\times\{\mathrm{pt}\}]+[\{\mathrm{pt}\}\times\Sigma_g]\right)+(2g-1)\sum_{i=1}^{3}[e_i]\right)^2\\
&= 5(2g+1)(4g^2-4g+1).
\end{aligned}$$

Hence

$$\begin{aligned}
\sigma(Z_g(1)) &= \frac{c_1^2\left(Z_g(1)\right)-2e\left(Z_g(1)\right)}{3}\\
&= \frac{1}{3}(2g+1)(4g^2+4g-9)
\end{aligned}$$

and

$$\begin{aligned}
\chi_h(Z_g(1)) &= \frac{e\left(Z_g(1)\right)+c_1^2\left(Z_g(1)\right)}{12}\\
&= \frac{1}{3}(2g+1)(7g^2-8g+3).
\end{aligned}$$

Note that $Z_2(1)$ here is the 4-manifold $H(1)$ constructed by A. Stipsicz in [31].

Following the procedure in Section 4.2, we can pull back $\psi_{g,1} : Z_g(1) \to \Sigma_g$ by a u-fold unbranched cover $\Theta_u : \Sigma_k \to \Sigma_g$ (where again, $k = u(g-1)+1$). We will denote the resulting pull-back fibration by $\psi_{g,u} : Z_g(u) \to \Sigma_k$. This defines another Lefschetz fibration, whose regular fiber is a genus $4g^2$ surface. Furthermore, the sections s_i' will pull back to sections of $\psi_{g,u}$ with images genus k surfaces with self-intersection $-u$. We also have that $e(Z_g(u)) = u\cdot e(Z_g(1))$, $\sigma(Z_g(u)) = u\cdot\sigma(Z_g(1))$, $\chi_h(Z_g(u)) = u\cdot\chi_h(Z_g(1))$, and $c_1^2(Z_g(u)) = u\cdot c_1^2(Z_g(1))$.

We can also pull back the fibration

$$Z_g(1) \longrightarrow (\Sigma_g\times\Sigma_g)\#3\overline{\mathbb{CP}}^2 \longrightarrow \Sigma_g\times\Sigma_g$$

by $\Theta_u\times\Theta_u$, to yield a fibration whose total space will be denoted by $\widetilde{Z}_g(u^2)$. We will denote by $\widetilde{\psi}_{g,u}$ composition

$$\widetilde{Z}_g(u^2) \longrightarrow \Sigma_k\times\Sigma_k \xrightarrow{\mathrm{pr}_1} \Sigma_k,$$

which is a genus $u(4g^2-1)+1$ Lefschetz fibration, having sections whose images are genus k surfaces with self-intersection $-u$. Again, since $\widetilde{Z}_g(u^2)$ is a u^2-fold unbranched cover of $Z_g(1)$, it follows that $e(\widetilde{Z}_g(u^2)) = u^2\cdot e(Z_g(1))$, $\sigma(\widetilde{Z}_g(u^2)) = u^2\cdot\sigma(Z_g(1))$, $\chi_h(\widetilde{Z}_g(u^2)) = u^2\cdot\chi_h(Z_g(1))$, and $c_1^2(\widetilde{Z}_g(u^2)) = u^2\cdot c_1^2(Z_g(1))$.

Chapter 5

Surgery Along Lagrangian Tori and Construction of $Y_n(m)$

5.1 Torus surgery in symplectic 4-manifolds

Before constructing our final building block manifolds, we first define a surgery method along embedded Lagrangian tori in symplectic 4-manifolds. This surgery, known as *Luttinger surgery*, was applied to embedded Lagrangian tori in $\mathbb{R}^4$ by Luttinger ([21]), and later to embedded Lagrangian tori in general symplectic manifolds by Auroux, Donaldson and Katzarkov ([6]). It is a special case of a more general class of torus surgeries. We recall first briefly the definition of this general class of surgery, and discuss how these surgeries affect the fundamental groups of the manifolds to which they are applied, before giving the more detailed description from [6] of the special case of Luttinger surgery. The importance of Luttinger surgery lies in the fact that it yields symplectic manifolds, whose symplectic form is unchanged away from a neighborhood of the surgery.

5.1.1 p/q-torus surgery

We restrict our definition of torus surgery to surgery along a Lagrangian torus T in a symplectic 4-manifold (M, ω). Let $\mathcal{U}$ be a tubular neighborhood of T in M. Since T is Lagrangian, we can choose $\mathcal{U}$ so that it has a unique framing (i.e. an identification of $\mathcal{U}$ with $T \times D^2$) defined by the property that $T \times \{d\}$ is Lagrangian for each $d \in D^2$. This framing is called the *Lagrangian framing.* Since under the above identification $T \times \{d\}$ is isotopic to $T \times \{d'\}$ for any choices of $d, d' \in D^2$, we will refer to $T \times \{d\}$ simply as *the Lagrangian pushoff* of T for any d. Let γ be an oriented, essential, simple closed loop in T. Under this framing we can uniquely (up to isotopy) push off γ to a curve γ' in $\partial\mathcal{U} = T^3$ by setting $\gamma' = \gamma \times \{d_o\}$ for some $d_o \in \partial D^2$. Let $\mu = \{\text{pt}\} \times \partial D^2$ be a meridian of T in $\partial\mathcal{U}$. Given relatively prime integers p and q, *p/q-surgery* on T with respect to γ is the process in which $\mathcal{U}$ in M is replaced by $T^2 \times D^2$, resulting in the manifold

$$M(T, \gamma, p/q) = (M \backslash \mathcal{U}) \cup_{\lambda} (T^2 \times D^2),$$

where the gluing map $\lambda : T^2 \times \partial D^2 \to \partial(M\backslash\mathcal{U})$ satisfies $\lambda_*([\{\text{pt}\} \times \partial D^2]) = q[\gamma'] + p\,[\mu]$ in $H_1(\partial(M\backslash\mathcal{U}); \mathbb{Z})$. We will follow the notation in [12] by denoting the p/q-surgery operation on a torus T with respect to γ by $(T, \gamma, p/q)$ when the manifold M is understood.

We will be interested in computing the effect that the above surgery has on $\pi_1(M)$. We will make use of the following lemmas:

Lemma 5.1.1. *Let M, T, μ, γ, and γ' be as above. Then*

$$\pi_1(M(T, \gamma, p/q)) \cong \pi_1(M\backslash\mathcal{U})/\langle \mu^p \cdot (\gamma')^q \rangle.$$

Proof. Let $i : \partial(M\backslash\mathcal{U}) \hookrightarrow M\backslash\mathcal{U}$ and $j : \partial(T^2 \times D^2) \hookrightarrow T^2 \times D^2$ be the inclusion maps. Fix a basepoint $x \in \partial(M\backslash\mathcal{U})$ to be used as the basepoint for all the fundamental groups in the following discussion (though we will suppress it from our notation). By the Seifert-van Kampen theorem,

$$\pi_1(M(T, \gamma, p/q)) \cong (\pi_1(M\backslash\mathcal{U}) * \pi_1(T^2 \times D^2))/N,$$

where $*$ denotes the free product of groups, and N is the normal subgroup generated by elements of the form $(i \circ \lambda)_*(z) \cdot j_*(z)^{-1}$ for $z \in \pi_1(\partial(T^2 \times D^2))$ (note that the basepoints of $\pi_1(T^2 \times D^2)$ and $\pi_1(\partial(T^2 \times D^2))$ are taken to be $\lambda^{-1}(x)$). By our choice of gluing diffeomorphism λ, we have that $\lambda_*(\{\text{pt}\} \times \partial D^2) = \mu^p \cdot (\gamma')^q \in \pi_1(\partial(M\backslash\mathcal{U}))$, whence $(i \circ \lambda)_*(\{\text{pt}\} \times \partial D^2) = \mu^p \cdot (\gamma')^q \in \pi_1(M\backslash\mathcal{U})$. Furthermore, $j_*(\{\text{pt}\} \times \partial D^2) = 1 \in \pi_1(T^2 \times D^2)$. Thus $(i \circ \lambda)_*(\{\text{pt}\} \times \partial D^2) \cdot j_*(\{\text{pt}\} \times \partial D^2)^{-1} = \mu^p \cdot (\gamma')^q$ is a generator of N.

Note that $\pi_1(T^2 \times D^2) \cong \pi_1(T^2) \times \pi_1(D^2) \cong \pi_1(T^2)$ is entirely carried by $\partial(T^2 \times D^2) = T^2 \times \partial D^2 = T^2 \times S^1$. In other words, the inclusion induced homomorphism $j_* : \pi_1(\partial(T^2 \times D^2)) \to \pi_1(T^2 \times D^2)$ is surjective. Thus for any $y \in (\pi_1(M\backslash\mathcal{U}) * \pi_1(T^2 \times D^2))/N$, any terms from $\pi_1(T^2 \times D^2)$ in the expression of y can be written as $j_*(z_i)$ for some $z_i \in \pi_1(\partial(T^2 \times D^2))$. Using the relations introduced by N, we can rewrite each such term $j_*(z_i)$ as $(i \circ \lambda)_*(z_i)$. Thus every element of $(\pi_1(M\backslash\mathcal{U}) * \pi_1(T^2 \times D^2))/N$ can be written as an element of the subgroup $\pi_1(M\backslash\mathcal{U})/(N \cap \pi_1(M\backslash\mathcal{U}))$. After discarding the generators of N not in $\pi_1(M\backslash\mathcal{U})$, the lemma follows. □

As a result of Lemma 5.1.1, we will need to know the effect that removing a tubular neighborhood of a Lagrangian torus has on the fundamental group of its ambient manifold. We thus consider the following lemma, which limits the nontrivial elements introduced to the fundamental group of a 4-manifold when any closed surface Σ with trivial normal bundle is removed. Note that this lemma covers the cases when Σ is a Lagrangian torus or a genus g surfaces of self-intersection zero, as both imply that the normal bundle of Σ is trivial. Though the result is rather intuitive, we will state and prove it formally owing to its frequent usage in what follows.

Lemma 5.1.2. *Let M be a 4-manifold, and let $\Sigma \subset M$ be a submanifold which is a closed genus g surface of self-intersection zero. Let $i : \pi_1(M\backslash\Sigma) \to \pi_1(M)$ be the inclusion induced homomorphism. Then the kernel of i is the normal subgroup generated by a meridian of Σ.*

Proof. We begin by fixing a basepoint in $\nu\Sigma\backslash\Sigma$ to be used for all of the fundamental groups in the following discussion. Since the normal bundle $\nu\Sigma$ is assumed to be trivial, we can fix a trivialization $\nu\Sigma \cong \Sigma \times D^2$. Therefore $\pi_1(\nu\Sigma) \cong \pi_1(\Sigma) \times \pi_1(D^2) \cong \pi_1(\Sigma)$, and choosing a standard presentation of $\pi_1(\Sigma)$ yields a presentation

$$\pi_1(\nu\Sigma) = \langle c_1, d_1, \ldots c_g, d_g \mid \prod_{j=1}^{g}[c_j, d_j]\rangle.$$

Consider now the open subset $(M\backslash\Sigma)\cap\nu\Sigma = \nu\Sigma\backslash\Sigma$, which can be thought of as the complement of the zero section in the normal bundle. Note that by the above trivialization, this is homeomorphic to $\Sigma \times \widehat{D}^2$, where $\widehat{D}^2$ is the punctured disk, and hence is homotopically equivalent to $\Sigma \times S^1$. Thus $\pi_1(\nu\Sigma\backslash\Sigma) \cong \pi_1(\Sigma \times S^1) \cong \pi_1(\Sigma) \times \pi_1(S^1) \cong \pi_1(\Sigma) \times \mathbb{Z}$. We thus obtain the presentation

$$\pi_1(\nu\Sigma\backslash\Sigma) = \langle \hat{c}_1, \hat{d}_1, \ldots \hat{c}_g, \hat{d}_g, \mu \mid \prod_{j=1}^{g}[\hat{c}_j, \hat{d}_j], [\hat{c}_1, \mu], [\hat{d}_1, \mu], \ldots, [\hat{c}_g, \mu], [\hat{d}_g, \mu]\rangle.$$

Choose each of the $\hat{c}_i$ and $\hat{d}_i$ so that $h(\hat{c}_i) = c_i$ and $h(\hat{d}_i) = d_i$, where $h : \pi_1(\nu\Sigma\backslash\Sigma) \to \pi_1(\nu\Sigma)$ is the inclusion induced homomorphism. Note that μ represents a meridian of Σ, and that $h(\mu) = 1$. Furthermore, fix a presentation

$$\pi_1(M\backslash\Sigma) = \langle S \mid R\rangle,$$

where S and R are respectively the sets of generators and relators of $\pi_1(M\backslash\Sigma)$.

Let $l : \pi_1(\nu\Sigma\backslash\Sigma) \to \pi_1(M\backslash\Sigma)$ be the inclusion induced homomorphism. By the Seifert-van Kampen theorem we can write

$$\begin{aligned}\pi_1(M) &= \langle c_1, d_1, \ldots, c_g, d_g, S \mid R, \prod_{j=1}^{g}[c_j, d_j], h(\mu)l(\mu)^{-1}, h(\hat{c}_1)l(\hat{c}_1)^{-1}, \ldots, h(\hat{d}_g)l(\hat{d}_g)^{-1}\rangle \\ &= \langle c_1, d_1, \ldots, c_g, d_g, S \mid R, \prod_{j=1}^{g}[c_j, d_j], l(\mu), c_1 l(\hat{c}_1)^{-1}, \ldots, d_g l(\hat{d}_g)^{-1}\rangle,\end{aligned}$$

since $h(\hat{c}_i) = c_i, h(\hat{d}_j) = d_j$, and $h(\mu) = 1$. We can thus express each of the $c_1, d_1, \ldots, c_g, d_g$ in $\pi_1(M)$ as words in S, and we obtain a simplified presentation

$$\pi_1(M) = \langle S \mid R, \prod_{j=1}^{g}[l(\hat{c}_j), l(\hat{d}_j)], l(\mu)\rangle.$$

Clearly, as $\prod_{j=1}^{g}[l(\hat{c}_j), l(\hat{d}_j)] = l(\prod_{j=1}^{g}[\hat{c}_j, \hat{d}_j]) = 1$ in $\pi_1(M\backslash\Sigma)$ (since $\prod_{j=1}^{g}[\hat{c}_j, \hat{d}_j] = 1$ in $\pi_1(\nu\Sigma\backslash\Sigma)$), it follows that $\prod_{j=1}^{g}[l(\hat{c}_j), l(\hat{d}_j)]$ is in the normal subgroup generated by the relators in R, and

$$\pi_1(M) = \langle S \mid R, l(\mu)\rangle.$$

The inclusion induced homomorphism $i : \pi_1(M\backslash\Sigma) \to \pi_1(M)$ is the identity on the generating set S, hence $\ker(i) = \langle l(\mu)\rangle$, where $l(\mu)$ is a meridian of Σ in $\pi_1(M\backslash\Sigma)$. □

Remark 5.1.3. Lemma 5.1.2 tells us that by removing a closed 2 dimensional submanifold with trivial normal bundle (or a tubular neighborhood of such), the only non-trivial loops we might create arise from the meridian of the surface. Combining Lemmas 5.1.1 and 5.1.2, we see that although p/q-torus surgery $(T, \gamma, p/q)$ introduces a relation $\mu^p \cdot (\gamma')^q$, it may create a nontrivial loop $\mu \in \pi_1(M(T, \gamma, p/q))$.

Lemma 5.1.4. *Suppose that Σ is a closed embedded surface of self-intersection zero in a 4-manifold M whose meridian is nullhomotopic in $M\backslash\Sigma$. Then $\pi_1(M\backslash\Sigma) \cong \pi_1(M)$.*

Proof. By Lemma 5.1.2 it remains only to show that the inclusion induced homomorphism $\pi_1(M\backslash\Sigma) \to \pi_1(M)$ is surjective. For any loop in M, we can homotope it so that it is transverse to, and thus disjoint from Σ. It is thus homotopic to a loop in $M\backslash\Sigma$, and the lemma follows. □

Lemmas 5.1.2 and 5.1.4 imply that the sequence

$$1 \longrightarrow \langle\mu\rangle \longrightarrow \pi_1(M\backslash\nu\Sigma) \longrightarrow \pi_1(M) \longrightarrow 1$$

(where the middle two homomorphisms are the natural inclusions) is exact.

5.1.2 Luttinger sugery

Although the manifold $M(T, \gamma, p/q)$ will not in general be symplectic, when $p = 1$ the above surgery procedure does yield manifolds with a unique symplectic structure, determined by the symplectic structure on the original manifold M. More specifically, by Theorem 9.3 of [10], there is a tubular neighborhood $\mathcal{U}$ of T in M which is symplectomorphic to a neighborhood $\mathcal{U}_o$ of the zero section of the cotangent bundle $T^*T \cong T \times \mathbb{R}^2$, with its canonical symplectic structure (see Example 2.2.4). Symplectically identify two such neighborhoods. Identify T itself with $\mathbb{R}^2/\mathbb{Z}^2$ so that $\gamma \subset T$ is identified with the first coordinate axis, and so that the positive direction of the axis coincides with the orientation of γ. Letting (x_1, x_2) denote the corresponding coordinates on T, with (y_1, y_2) being the dual coordinates on the cotangent fibers, the symplectic form is given by $\omega = dx_1 \wedge dy_1 + dx_2 \wedge dy_2$. Clearly in these coordinates, for small enough ε and ε', $T' = \{(x_1, x_2, \varepsilon, \varepsilon') \mid 0 \leq x_1, x_2 < 1\}$ is the Lagrangian pushoff of T. Likewise, fixing any $0 \leq u, v < 1$, a meridian of T is given by $\mu(t) = (u, v, w\cos(t), w\sin(t))$, for $0 \leq t \leq 2\pi$ and small enough w.

Choose $r > 0$ so that the set $U_r = T \times [-r, r] \times [-r, r]$ is contained in $\mathcal{U}_o$. Let $s : [-r, r] \to [0, 1]$ be a smooth function such that $s(t) = 0$ for $t \leq -\frac{r}{3}$, $s(t) = 1$ for $t \geq \frac{r}{3}$, and $\int_{-r}^{r} ts'(t)dt = 0$. For each $k \in \mathbb{Z}$, define the map $\Lambda_k : U_r\backslash U_{r/2} \to U_r\backslash U_{r/2}$ by

$$\Lambda_k(x_1, x_2, y_1, y_2) = \begin{cases} (x_1, x_2, y_1, y_2) & \text{when } y_2 < \frac{r}{2}, \\ (x_1 + k \cdot s(y_1), x_2, y_1, y_2) & \text{when } y_2 \geq \frac{r}{2}. \end{cases}$$

It is easy to see that Λ_k is a diffeomorphism for each $k \in \mathbb{Z}$, and that it satisfies the desired properties of having trivial action on $H_1(T;\mathbb{Z})$ and having $(\Lambda_k)_*[\mu] = [\mu]+k[\gamma']$ in $H_1(U_r\backslash U_{r/2};\mathbb{Z})$. Likewise, a simple computation verifies that Λ_k preserves the symplectic form (i.e. $(\Lambda_k)^*(\omega) = \omega$). We thus construct a new symplectic 4-manifold by removing the neighborhood $U_{r/2}$ of T from M, and gluing U_r back by identifying the neighborhoods $U_r\backslash U_{r/2}$ in $M\backslash U_{r/2}$ and U_r by the symplectomorphism Λ_k. More formally,

$$M(T,\gamma,1/k) = (M\backslash U_{r/2}) \cup_{\Lambda_k} U_r.$$

The surgery procedure described above on the symplectic 4-manifold M is called *Luttinger surgery* and will still be denoted by $(T,\gamma,1/k)$. The isotopy class of the gluing map Λ_k is actually uniquely determined by the action on the meridian μ, and hence uniquely determined by γ and k given T. Thus $M(T,\gamma,1/k)$ has a symplectic structure uniquely determined up to isotopy, which is independent of the choices made in the above construction (Proposition 2.2 in [6]). Away from T the symplectic structures on M and $M(T,\gamma,1/k)$ will be the same.

5.2 Construction of $Y_n(m)$

Having all the surgery tools in place, we proceed to construct a family of irreducible pairwise nondiffeomorphic 4-manifolds $\{Y_n(m) \mid m = 1,2,3,\ldots\}$ for each integer $n \geq 2$. These families are constructed in Section 2 of [4] by performing Luttinger and torus surgeries on $\Sigma_2 \times \Sigma_n$. Let a_i and b_i $(i = 1,2)$ be the generators of $\pi_1(\Sigma_2)$, and let c_i and d_i $(i = 1,\ldots,n)$ be the generators of $\pi_1(\Sigma_n)$ as shown in Figure 5.1. In order to choose disjoint Lagrangian tori on which to perform our surgeries, we choose loops a_i' and a_i'' (for $i = 1,2$) which are parallel and disjoint from the generators a_i of $\pi_1(\Sigma_2)$ as in Figure 5.1. We likewise do the same for the other generators b_i, c_j and d_j (for $i = 1,2$ and $j = 1,\ldots,n$). As mentioned in Example 2.2.9, the cartesian products formed from these loops will be Lagrangian tori in $\Sigma_2 \times \Sigma_n$ under the product symplectic structure.

Recall that $\pi_1(\Sigma_2 \times \Sigma_n)$ is generated by $a_i \times \{y\}$, $b_i \times \{y\}$, $c_j \times \{x\}$ and $d_j \times \{x\}$ (for $i = 1,2$ and $j = 1,\ldots,n$), which loops we also respectively denote by a_i, b_i, c_j and d_j. Furthermore, the following relations hold in $\pi_1(\Sigma_2 \times \Sigma_n)$:

$$[a_1,b_1][a_2,b_2] = 1, \quad \prod_{j=1}^{n}[c_j,d_j] = 1,$$
$$[a_i,c_j] = 1, \quad [a_i,d_j] = 1, \quad [b_i,c_j] = 1, \quad \text{and} \quad [b_i,d_j] = 1.$$

We also have that $e(\Sigma_2 \times \Sigma_n) = e(\Sigma_2)e(\Sigma_n) = 4n-4$, and we can explicitly compute the intersection form of $\Sigma_2 \times \Sigma_n$ to show that $\sigma(\Sigma_2 \times \Sigma_n) = 0$.

We will also consider based loops (with basepoints the respective vertices x and y) arising from the loops in Figure 5.1. For example, we can create a based loop from a_i' by starting at the vertex x located at the initial point of a_i and the terminal point of b_i, traversing backwards along b_i to the initial point of a_i', then following a_i' until reaching b_i again, finally returning in

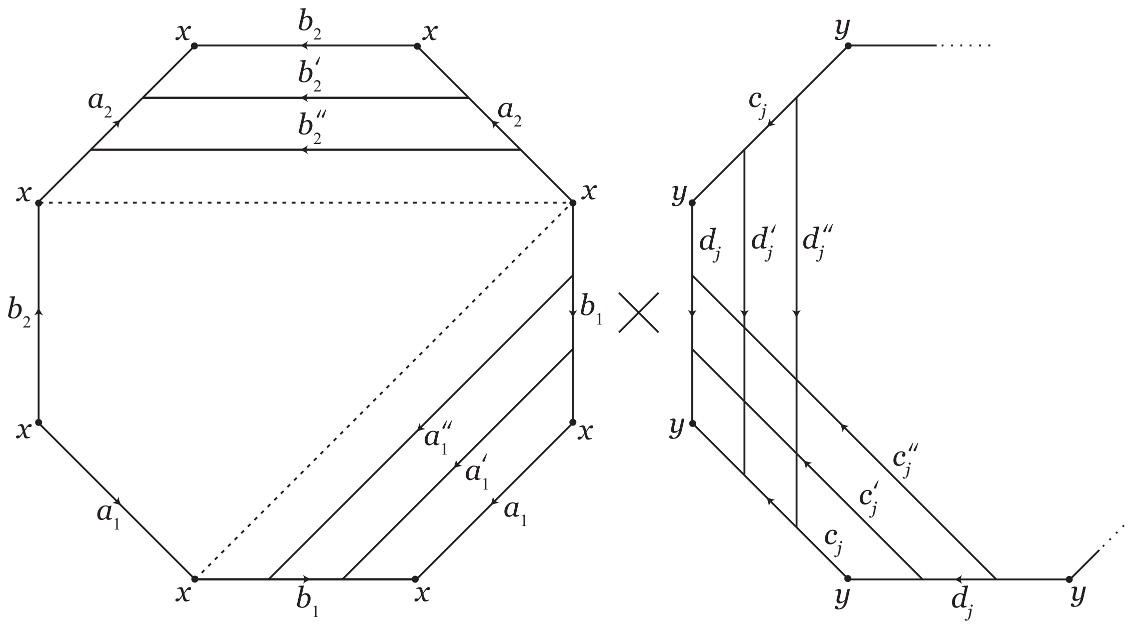

Figure 5.1: $\Sigma_2 \times \Sigma_n$ with generators of Lagrangian tori.

the forwards direction of b_i back to x. Likewise, a based loop is create from a_i'' by beginning at the initial point of b_i, following it in the forward direction until reaching the initial point of a_i'', traversing a_i'' in the forward direction, and returning to x by following b_i again but in the opposite direction. We likewise construct based loops from $b_i', b_i'', c_j', c_j'', d_j'$ and d_j''. We will not distinguish in notation between the based loops and the unbased loops they arise from. Note that the based loops a_i, a_i' and a_i'' are all disjoint except for the base point x, and similarly for the other families of loops.

For each $n \geq 2$ and $m \geq 1$ we define $Y_n(m)$ to be the manifold obtained from $\Sigma_2 \times \Sigma_n$ by performing the $2n+4$ torus surgeries

$$\begin{gathered}(a_1' \times c_1', a_1', -1), \quad (b_1' \times c_1'', b_1', -1), \quad (a_2' \times c_2', a_2', -1), \quad (b_2' \times c_2'', b_2', -1), \\ (a_2' \times c_1', c_1', +1), \quad (a_2'' \times d_1', d_1', +1), \quad (a_1' \times c_2', c_2', +1), \quad (a_1'' \times d_2', d_2', +m), \\ (b_1' \times c_3', c_3', -1), \quad (b_2' \times d_3', d_3', -1), \ \ldots, \ (b_1' \times c_n', c_n', -1), \quad (b_2' \times d_n', d_n', -1).\end{gathered} \tag{5.1}$$

Note that all of the surgered tori are disjoint. Furthermore, since $\Sigma_2 \times \Sigma_n$ carries a symplectic structure, if $m = 1$ the surgeries are all Luttinger, implying that $Y_n(1)$ is a symplectic manifold. We now proceed to determine the effect that the torus surgeries have on the fundamental group of $\Sigma_2 \times \Sigma_n$:

Lemma 5.2.1 ([4, 12]). *$\pi_1(Y_n(m), (x, y))$ is generated by the loops a_i, b_i, c_j and d_j (for $i = 1, 2$ and $j = 1, \ldots, n$), and the following relations hold:*

$$[b_1^{-1}, d_1^{-1}] = a_1, \quad [a_1^{-1}, d_1] = b_1, \quad [b_2^{-1}, d_2^{-1}] = a_2, \quad [a_2^{-1}, d_2] = b_2,$$

$$
\begin{gathered}
[d_1^{-1}, b_2^{-1}] = c_1, \quad [c_1^{-1}, b_2] = d_1, \quad [d_2^{-1}, b_1^{-1}] = c_2, \quad [c_2^{-1}, b_1]^m = d_2, \\
[a_1^{-1}, d_3^{-1}] = c_3, \quad [a_2^{-1}, c_3^{-1}] = d_3, \ \ldots, \ [a_1^{-1}, d_n^{-1}] = c_n, \quad [a_2^{-1}, c_n^{-1}] = d_n, \\
[a_1, c_1] = 1, \quad [a_1, c_2] = 1, \quad [a_1, d_2] = 1, \quad [b_1, c_1] = 1, \\
[a_2, c_1] = 1, \quad [a_2, c_2] = 1, \quad [a_2, d_1] = 1, \quad [b_2, c_2] = 1, \\
[b_1, c_3] = 1, \quad [b_2, d_3] = 1, \ \ldots, \ [b_1, c_n] = 1, \quad [b_2, d_n] = 1, \\
[a_1, b_1][a_2, b_2] = 1, \quad \prod_{j=1}^{n} [c_j, d_j] = 1.
\end{gathered} \tag{5.2}
$$

Proof. Let $i \in \{1, 2\}$ and $j \in \{1, \ldots, n\}$. To fix some notation for the following discussion, let $g, \overline{g} \in \{a_i, b_i\}$ and $h, \overline{h} \in \{c_j, d_j\}$ with $g \neq \overline{g}$ and $h \neq \overline{h}$. We will be interested in performing surgery along the Lagrangian tori of the form $g' \times h', g' \times h''$, and $g'' \times h'$ (from (5.1)).

Consider first the torus $g' \times h'$. We construct the Lagrangian pushoff of $g' \times h'$ by constructing individual isotopies of g' and h'. Let $G : S^1 \times [0,1] \to \Sigma_2$ be an isotopy from g' to g. In other words, G is a homotopy with $G(\cdot, 0) = g'$, $G(\cdot, 1) = g$, and such that $G(\cdot, t)$ is an embedding of S^1 into Σ_2 for any $t \in [0,1]$. Such an isotopy is easy to construct (cf. [12]), and can be chosen so that the path of the isotopy is entirely contained within the trapezoid in Figure 5.1 whose top edge is g' and whose bottom edge is g. We likewise let $H : S^1 \times [0,1] \to \Sigma_n$ be a similar isotopy from h' to h. Clearly the product $G \times H$ gives an isotopy of $g' \times h'$ to $g \times h$, with the property that the image of the embeddings $G(\cdot, t) \times H(\cdot, t) : S^1 \times S^1 \to \Sigma_2 \times \Sigma_n$ is a Lagrangian torus for each $t \in [0,1]$. Thus $g \times h$ is the Lagrangian pushoff of $g' \times h'$. Note that the pushoff $g \times h$ contains the basepoint (x, y). Furthermore the pushoff of $g' \times \{y\}$ in the Lagrangian framing is the based loop $g \times \{y\}$, while the pushoff of $\{x\} \times h'$ is $\{x\} \times h$ (also a based loop).

Similarly, we can isotope g'' and h'' to parallel copies of g'' and h'' respectively, shown by dashed lines in Figure 5.1 (the dashed lines have been omitted from the diagram of Σ_n for clarity). These parallel copies are based loops and are given in $\pi_1(\Sigma_2)$ and $\pi_1(\Sigma_n)$ as conjugates $\overline{g}g(\overline{g})^{-1}$ and $\overline{h}h(\overline{h})^{-1}$ respectively. For example, the parallel copy of a_1'' is represented by $b_1 a_1 b_1^{-1}$ in $\pi_1(\Sigma_2)$, while the parallel copy of c_2'' is given by $d_2 c_2 d_2^{-1}$ in $\pi_1(\Sigma_n)$. As above, we can construct Lagrangian isotopies from whence it follows that the Lagrangian pushoff of $g'' \times h'$ is homotopic to the product $(\overline{g}g(\overline{g})^{-1}) \times h$ while the pushoff of $g' \times h''$ is homotopic to $g \times (\overline{h}h(\overline{h})^{-1})$. For example, the Lagrangian pushoff of $a_2'' \times d_1'$ is $(b_2 a_2 b_2^{-1}) \times d_1$. Note that as mentioned in [12], although the homotopy between $g'' \times h'$ (respectively $g' \times h''$) and its Lagrangian pushoff needs to be Lagrangian, the homotopy between $(\overline{g}g(\overline{g})^{-1}) \times h$ (respectively $g \times (\overline{h}h(\overline{h})^{-1})$) and the Lagrangian pushoff does not need to be. Thus we can push $\overline{g}g(\overline{g})^{-1}$ and $\overline{h}h(\overline{h})^{-1}$ off to ensure that the paths of their homotopies are disjoint from the Lagrangian tori we will be performing surgery on, and hence won't be affected by any of the surgeries. Furthermore the pushoffs of $g'' \times \{y\}$ and $\{x\} \times h''$ under the Lagrangian framings are homotopic to $(\overline{g}g(\overline{g})^{-1}) \times \{y\}$ and $\{x\} \times (\overline{h}h(\overline{h})^{-1})$ respectively. For example, the pushoff of $a_1'' \times \{y\}$ is homotopic to $(b_1 a_1 b_1^{-1}) \times \{y\}$, while the pushoff of $\{x\} \times c_2''$ is homotopic to $\{x\} \times (d_2 c_2 d_2^{-1})$. Note that the paths of all the Lagrangian push-offs we've chosen are disjoint from the other tori we will be performing surgery on; hence they will not be affected by these surgeries.

Finally, we find meridians of the Lagrangian tori. Since each of these tori have trivial normal bundle, for a given torus T, any two meridians of T will be conjugate. Consider the torus $g' \times h'$ and the orthogonal torus $\overline{g} \times \overline{h}$. These two tori intersect transversely in a single point. The torus $\overline{g} \times \overline{h}$ contains the basepoint (x, y) and is Lagrangian; hence the commutator $[\overline{g}, \overline{h}]$ of its generators bounds a normal disk to $g' \times h'$ in the Lagrangian framing. Upon removing a tubular neighborhood of $g' \times h'$, we are left with a punctured $\overline{g} \times \overline{h}$ in the complement $(\Sigma_2 \times \Sigma_n) \backslash \nu(g' \times h')$. Consider the boundary component of this punctured torus as a based loop (i.e. connect the boundary component of the punctured torus to the basepoint (x, y) by a curve c). If the curve c is chosen so that it lies entirely in the punctured torus (which is possible since the punctured torus contains the basepoint), then we have that the based boundary component of the punctured torus is homotopic to the commutator of the generators of $\overline{g} \times \overline{h}$ with some choice of orientation. Again, this homotopy can be chosen so that its path lies completely in $\overline{g} \times \overline{h}$ and thus is disjoint from the tubular neighborhoods of the Lagrangian surgery tori. This implies that a meridian to $g' \times h'$ will be homotopic in the complement of the tubular neighborhoods of the Lagrangian surgery tori to the commutator $[\overline{g}, \overline{h}]$ (possibly with either $(\overline{g})^{-1}$ or $(\overline{h})^{-1}$ in place of $\overline{g}$ or $\overline{h}$ respectively). By our above comments, any meridian of $g' \times h'$ will be conjugate to such a meridian.

For example, consider the surgery $(a_1' \times c_1', a_1', -1)$. The Lagrangian pushoff of $a_1' \times c_1'$ is $a_1 \times c_1$, while the pushoff of the curve $a_1' \times \{y\}$ is the based curve $a_1 \times \{y\}$. A meridian of $a_1' \times c_1'$ is homotopic in the complement to $[b_1^{-1}, d_1^{-1}]$. By Lemmas 5.1.1 and 5.1.2, this surgery makes the element $[b_1^{-1}, d_1^{-1}]a_1^{-1}$ trivial in the fundamental group of the newly surgered manifold. However, the meridian $[b_1^{-1}, d_1^{-1}]$ may no longer be trivial; in particular, the elements b_1 and d_1 may not commute in $\pi_1(Y_n(m))$.

As another example, consider the surgery $(b_1' \times c_1'', b_1', -1)$. The Lagrangian pushoff of the torus $b_1' \times c_1''$ is homotopic (in the complement of the tubular neighborhoods of the surgery tori) to $b_1 \times (d_1 c_1 d_1^{-1})$, while the pushoff of the curve b_1' is b_1. A meridian to $b_1' \times c_1''$ is homotopic in the complement of the tubular neighborhoods of the surgery tori to $[a_1^{-1}, d_1]$. Arguing as above, we see that this implies that $[a_1^{-1}, d_1] = b_1$ in $\pi_1(Y_n(m))$, but that $[a_1, d_1] = 1$ may no longer hold. Each of the first $2n + 4$ relations in (5.2) are obtained in this way.

Note that since the new relations created through performing the torus surgeries arise from gluing a disk $\{\text{pt}\} \times D^2 \subset T^2 \times D^2$ into the hole created by the removal of νT for some torus T, this relation will not be affected by any future surgeries provided they are performed on tori with tubular neighborhoods disjoint from νT. Since all of our surgery tori were chosen to be disjoint, the relations formed by any surgery procedures will not be affected by later surgeries (see also Lemma 5.1.2).

The final $2n+6$ relations in (5.2) are inherited from relations in $\pi_1(\Sigma_2 \times \Sigma_n)$. By Lemma 5.1.2, these relations will still hold after the surgeries. Geometrically, we can see this as follows: the commutator relations from $\pi_1(\Sigma_2 \times \Sigma_n)$ will still hold in $\pi_1(Y_n(m))$ as the tori generated by the elements in the commutators are disjoint from the surgered tori in (5.1), and thus are not affected by the surgeries. The final two relations $[a_1, b_1][a_2, b_2] = 1$ and $\prod_{j=1}^{n}[c_j, d_j] = 1$ also still hold in $\pi_1(Y_n(m))$, as we can always find copies of both Σ_2 and Σ_n which are disjoint from the tubular neighborhoods of the surgery tori. $\square$

Since the torus surgeries neither affect the signature (by Novikov additivity) nor the Euler characteristic, we have that $\sigma(Y_n(m)) = 0$ and $e(Y_n(m)) = 4n - 4$. We can choose surfaces $\Sigma_2 \times \{p\}$ and $\{q\} \times \Sigma_n$ in $\Sigma_2 \times \Sigma_n$ which are disjoint from the tori in (5.1). These two symplectic submanifolds intersect orthogonally with respect to the symplectic product structure on $\Sigma_2 \times \Sigma_n$, and thus satisfy $[\Sigma_2 \times \{p\}] \cdot [\{q\} \times \Sigma_n] = 1$. Furthermore, these surfaces have $[\Sigma_2 \times \{p\}]^2 = [\{q\} \times \Sigma_n]^2 = 0$, and will descend to surfaces of genus 2 and n in $Y_n(m)$, which we will denote by Σ_2 and Σ_n respectively. Recall that the Luttinger surgeries performed do not affect the symplectic form of $\Sigma_2 \times \Sigma_n$ away from the surgered tori. Since $\Sigma_2 \times \{p\}$ and $\{q\} \times \Sigma_n$ are both symplectic surfaces in $\Sigma_2 \times \Sigma_n$ and were chosen away from the tori in (5.1), it follows that Σ_2 and Σ_n are both symplectic submanifolds of $Y_n(1)$ with the symplectic structure induced from the product structure on $\Sigma_2 \times \Sigma_n$. Furthermore, the surgeries will not affect the intersection of these surfaces, whence we have $[\Sigma_2] \cdot [\Sigma_n] = 1$ and $[\Sigma_2]^2 = [\Sigma_n]^2 = 0$. Since $2n - 4$ of the geometrically dual Lagrangian tori pairs (i.e. tori pairs of the form $(g \times h, \overline{g} \times \overline{h})$) will be unaffected by the $2n + 4$ surgeries, for $n \geq 3$, $Y_n(1)$ contains $4n - 8$ Lagrangian tori, each of which can be chosen to be disjoint from Σ_n.

Chapter 6

Construction of Irreducible Simply-Connected Symplectic 4-Manifolds

We now proceed with our main construction, by taking relatively minimal Lefschetz fibrations as constructed in Sections 4.2 and 4.3, and performing fiber sum operations with the manifolds $Y_n(m)$ constructed in Section 5.2.

6.1 General constructions

Let Σ_a and Σ_b denote closed Riemann surfaces of genus $a > 1$ and $b > 0$ respectively. Let $f : M \to \Sigma_b$ be a relatively minimal Lefschetz fibration of M over Σ_b, with generic fiber F diffeomorphic to Σ_a. Also assume that f has a section whose image S has self-intersection d in M. Since M admits a Lefschetz fibration of genus $a > 1$, M also admits a symplectic structure ω which can be chosen so both the section S and the fibers F are symplectic. Furthermore, since M is assumed to be relatively minimal, by Theorem 1.4 of [33], M is a minimal symplectic manifold.

For an integer $t > 0$, any t distinct fibers will be pairwise disjoint, and each will intersect S transversely in exactly one point. Choose such a t with the added property that $t \geq -d/2$. Taking the section S and t copies of F, we can symplectically resolve the t intersection points to yield a symplectic surface of genus $ta + b$ (see Section 2.1 of [16]). Denote this surface by Σ. Note that Σ represents the homology class $t\,[F] + [S] \in H_2(M;\mathbb{R})$ (i.e. $[\Sigma] = t\,[F] + [S]$). Since $[F]^2 = 0$, and $[S]^2 = d$ by assumption, the adjunction formula gives us that $\langle c_1(M,\omega), [F]\rangle = 2 - 2a$ and $\langle c_1(M,\omega), [S]\rangle = d + 2 - 2b$, whence

$$\begin{aligned} [\Sigma]^2 &= c_1(M,\omega)\,[\Sigma] - e(\Sigma) \\ &= tc_1(M,\omega)\,[F] + c_1(M,\omega)\,[S] - e(\Sigma) \\ &= 2t + d. \end{aligned}$$

Recall that our choice of t gives us that $2t + d \geq 0$. By symplectically blowing up at $2t + d$ points of Σ, we obtain a symplectic manifold $\widetilde{M} = M\#(2t+d)\overline{\mathbb{CP}}^2$, with symplectic submanifold $\widetilde{\Sigma}$ arising as the proper transform of Σ. $\widetilde{\Sigma}$ is a genus $ta + b$ surface, with self-intersection zero. Furthermore

$$\begin{aligned} e(\widetilde{M}) &= e(M) + 2t + d \\ \sigma(\widetilde{M}) &= \sigma(M) - 2t - d. \end{aligned}$$

Lemma 6.1.1. *Let* $\tilde{\iota} : \widetilde{\Sigma}^{\|} \hookrightarrow \widetilde{M} \backslash \nu\widetilde{\Sigma}$ *be the inclusion map of a parallel copy of* $\widetilde{\Sigma}$ *into the complement of a tubular neighborhood of* $\widetilde{\Sigma}$ *in* $\widetilde{M}$. *Then*

$$\frac{\pi_1(\widetilde{M}\backslash\nu\widetilde{\Sigma})}{\langle \tilde{\iota}_*(\pi_1(\widetilde{\Sigma}^{\|}))\rangle} = 1,$$

where $\langle \tilde{\iota}_*(\pi_1(\widetilde{\Sigma}^{\|}))\rangle$ *is the normal subgroup of* $\pi_1(\widetilde{M}\backslash\nu\widetilde{\Sigma})$ *generated by the image of* $\pi_1(\widetilde{\Sigma}^{\|})$ *under the induced homomorphism* $\tilde{\iota}_* : \pi_1(\widetilde{\Sigma}^{\|}) \to \pi_1(\widetilde{M}\backslash\nu\widetilde{\Sigma})$.

Proof. As Σ was constructed by resolving the intersection points between $t > 0$ copies of the fiber F and one copy of the section S, $\pi_1(\Sigma)$ will carry all of the generators of both $\pi_1(F)$ and $\pi_1(S)$. Viewing S as a parallel copy of the base Σ_b, we can see that $\pi_1(\Sigma)$ will carry the generators of $\pi_1(\Sigma_b)$ as well. If $\Sigma^{\|}$ is a parallel copy of Σ it will also carry all of the generators of both $\pi_1(F)$ and $\pi_1(\Sigma_b)$. From Proposition 2.3.16, we have the following exact sequence

$$\pi_1(F) \xrightarrow{j} \pi_1(M) \xrightarrow{f_*} \pi_1(\Sigma_b) \to 1,$$

where j is induced by the inclusion. Thus we have an isomorphism $\pi_1(M)/j(\pi_1(F)) \to \pi_1(\Sigma_b)$ sending the equivalence class of $\alpha \in \pi_1(M)$ to the projection $f_*(\alpha)$. Since $\Sigma^{\|}$ carries all of the generators of both $\pi_1(F)$ and $\pi_1(\Sigma_b)$, it follows that

$$\frac{\pi_1(M)}{\langle \iota_*(\pi_1(\Sigma^{\|}))\rangle} = 1,$$

where $\iota : \Sigma^{\|} \to M$ is the inclusion. Blowing up along points of Σ will not alter $\pi_1(M)$, and the proper transform $\widetilde{\Sigma}^{\|}$ of $\Sigma^{\|}$ will be a parallel copy of the proper transform $\widetilde{\Sigma}$ (which we can isotope to be disjoint). Thus

$$\frac{\pi_1(\widetilde{M})}{\langle \tilde{\iota}_*(\pi_1(\widetilde{\Sigma}^{\|}))\rangle} = 1.$$

Note that as the fiber F intersects the base space Σ_b transversely, F also carries a meridian of Σ_b. By Lemma 5.1.2, this completes the proof. □

6.2 Fiber sum construction of $P_n(m, f)$

Before stating the main theorem of this section, we recall that for integers $n \geq 2$ and $m \geq 1$ the 4-manifold $Y_n(m)$ constructed in Section 5.2 by Luttinger and torus surgeries on $\Sigma_2 \times \Sigma_n$ has a genus n submanifold Σ_n with self intersection zero. By choosing $n = ta+b \geq 2$, since $\Sigma_n \subset Y_n(m)$ and $\widetilde{\Sigma} \subset \widetilde{M}$ are both submanifolds of the same genus with self-intersection zero, we can construct the fiber sum

$$Y_n(m)\#_\Psi \widetilde{M}$$

for some diffeomorphism $\Psi : \partial(\nu\widetilde{\Sigma}) \to \partial(\nu\Sigma_n)$. When $m = 1$, $Y_n(1)$ is a minimal symplectic manifold with symplectic submanifold Σ_n. The fiber sum operation can be done symplectically so $Y_n(1)\#_\Psi \widetilde{M}$ is a symplectic manifold and so that the symplectic and Lagrangian submanifolds of $\widetilde{M}\backslash\nu\widetilde{\Sigma}$ and $Y_n(1)\backslash\nu\Sigma_n$ are preserved in $Y_n(1)\#_\Psi\widetilde{M}$ (cf. [14]). For a suitable choice of gluing diffeomorphism Ψ, we can ensure that $Y_n(m)\#_\Psi\widetilde{M}$ is also simply-connected and irreducible.

Theorem 6.2.1 ([3]). *Let $f : M \to \Sigma_b$ be a relatively minimal Lefschetz fibration as above, and suppose that f has at least one non-separating vanishing cycle. Let $n = ta+b \geq 2$. For a suitable diffeomorphism $\Psi : \partial(\nu\widetilde{\Sigma}) \to \partial(\nu\Sigma_n)$, the fiber sum*

$$P_n(m, f) = Y_n(m)\#_\Psi\widetilde{M}$$

along the surfaces $\widetilde{\Sigma}$ and Σ_n is simply-connected, and satisfies

$$\begin{aligned}
e(P_n(m,f)) &= e(M) + d + (8a+2)t + 8b - 8,\\
\sigma(P_n(m,f)) &= \sigma(M) - 2t - d,\\
\chi_h(P_n(m,f)) &= \chi_h(M) + 2at + 2b - 2,\\
c_1^2(P_n(m,f)) &= c_1^2(M) - d + (16a-2)t + 16b - 16.
\end{aligned}$$

$P_n(1,f)$ is irreducible, minimal, symplectic, and if $n = ta+b \geq 3$, contains a symplectic torus T of self-intersection 0 satisfying $\pi_1(P_n(1,f)\backslash T) = 1$.

Proof. We begin by computing the Euler characteristic and signature of $P_n(m,f)$:

$$\begin{aligned}
e(P_n(m,f)) &= e(\widetilde{M}) + e(Y_n(m)) - 2e(\Sigma_n)\\
&= e(M) + 2t + d + 4n - 4 - 2(2-2n)\\
&= e(M) + 8n + 2t + d - 8\\
&= e(M) + 8(ta+b) + 2t + d - 8,\\
\sigma(P_n(m,f)) &= \sigma(\widetilde{M}) + \sigma(Y_n(m))\\
&= \sigma(M) - 2t - d,
\end{aligned}$$

while the formulae $c_1^2(N) = 2e(N) + 3\sigma(N)$ and $\chi_h(N) = \frac{1}{4}(e(N) + \sigma(N))$ yield the other characteristic numbers.

Recall that $\pi_1(Y_n(m))$ is generated by the based loops a_i, b_i, c_j and d_j for $i = 1, 2$ and $j = 1, \ldots, n$, and that the relations in (5.2) hold. As above let $\Sigma_n^{\|} \subset \partial(Y_n(m)\backslash\nu\Sigma_n)$ and $\widetilde{\Sigma}^{\|} \subset \partial(\widetilde{M}\backslash\nu\widetilde{\Sigma})$ denote parallel copies of Σ_n and $\widetilde{\Sigma}$ which lie in the boundary of the complements of the tubular neighborhoods that we will be removing. Such parallel copies can be chosen since both surfaces have self-intersection zero. Fixing both a basepoint of the form (p, y) in the surface $\Sigma_n^{\|}$ (see Figure 5.1) and a curve connecting (p, y) to the basepoint (x, y) of $Y_n(m)$, we can fix a presentation

$$\pi_1(\Sigma_n^{\|}) = \langle \hat{c}_1, \hat{d}_1, \ldots, \hat{c}_n, \hat{d}_n \mid \prod_{j=1}^{n} [\hat{c}_j, \hat{d}_j] \rangle,$$

with the property that the inclusion induced homomorphism $i_* : \pi_1(\Sigma_n^{\|}) \to \pi_1(Y_n(m))$ satisfies $i_*(\hat{c}_j) = c_j$ and $i_*(\hat{d}_j) = d_j$ for $j = 1, \ldots, n$.

When forming the fiber sum $P_n(m, f)$, we choose the gluing diffeomorphism Ψ so that the induced homomorphism Ψ_* sends the element of $\pi_1(\widetilde{\Sigma}^{\|})$ which is represented by a non-separating vanishing cycle of the Lefschetz fibration $f : M \to \Sigma_b$ to any one of $\hat{c}_1, \hat{d}_1, \hat{c}_2$ or $\hat{d}_2$ in $\pi_1(\Sigma_n^{\|})$. For definiteness, choose $\hat{c}_1$. Although this non-separating vanishing cycle is non-trivial in $\pi_1(\widetilde{\Sigma}^{\|})$, it is nullhomotopic in $\widetilde{M}$ (see Section 2.3.4). Thus $\hat{c}_1$ is trivial in $\pi_1(P_n(m, f))$, from which it follows that c_1 is also trivial.

Recall that $[\Sigma_n]^2 = 0$. Note that since $[a_1, b_1][a_2, b_2]$ is homotopic in $Y_n(m)$ to a meridian of Σ_n, by Lemma 5.1.2 the kernel of the inclusion induced homomorphism $i_* : \pi_1(Y_n(m)\backslash\nu\Sigma_n) \to \pi_1(Y_n(m))$ is normally generated by $[a_1, b_1][a_2, b_2]$. Thus all of the relations which hold in $\pi_1(Y_n(m))$ (see (5.2)), except for $[a_1, b_1][a_2, b_2] = 1$, will also hold in $\pi_1(Y_n(m)\backslash\nu\Sigma_n)$. In other words, the following relations hold in $\pi_1(Y_n(m)\backslash\nu\Sigma_n)$:

$$\begin{gathered}
[b_1^{-1}, d_1^{-1}] = a_1, \quad [a_1^{-1}, d_1] = b_1, \quad [b_2^{-1}, d_2^{-1}] = a_2, \quad [a_2^{-1}, d_2] = b_2, \\
[d_1^{-1}, b_2^{-1}] = c_1, \quad [c_1^{-1}, b_2] = d_1, \quad [d_2^{-1}, b_1^{-1}] = c_2, \quad [c_2^{-1}, b_1]^m = d_2, \\
[a_1^{-1}, d_3^{-1}] = c_3, \quad [a_2^{-1}, c_3^{-1}] = d_3, \ \ldots, \ [a_1^{-1}, d_n^{-1}] = c_n, \quad [a_2^{-1}, c_n^{-1}] = d_n, \\
[a_1, c_1] = 1, \quad [a_1, c_2] = 1, \quad [a_1, d_2] = 1, \quad [b_1, c_1] = 1, \\
[a_2, c_1] = 1, \quad [a_2, c_2] = 1, \quad [a_2, d_1] = 1, \quad [b_2, c_2] = 1, \\
[b_1, c_3] = 1, \quad [b_2, d_3] = 1, \ \ldots, \ [b_1, c_n] = 1, \quad [b_2, d_n] = 1, \\
\prod_{j=1}^{n} [c_j, d_j] = 1.
\end{gathered} \tag{6.1}$$

Note that by setting $c_1 = 1$, the above relations imply that all of the a_i, b_i, c_j and d_j are trivial. Thus $\pi_1(Y_n(m)\backslash\nu\Sigma_n)/\langle c_1 \rangle = 1$. Since the image of c_1 in $\pi_1(P_n(m, f))$ is trivial, it follows that the inclusion induced homomorphism

$$\pi_1(Y_n(m)\backslash\nu\Sigma_n) \longrightarrow \pi_1(P_n(m, f)), \tag{6.2}$$

is also trivial.

Note that by identifying $\widetilde{\Sigma}^{\|}$ with $\Sigma_n^{\|}$ using Ψ, we can see that the inclusion induced homomorphism $\pi_1(\widetilde{\Sigma}^{\|}) \to \pi_1(P_n(m, f))$ will factor through (6.2) and will thus be trivial. Combining this fact with the result from Lemma 6.1.1 that $\pi_1(\widetilde{M}\backslash\nu\widetilde{\Sigma})/\langle \tilde{\iota}_*(\pi_1(\widetilde{\Sigma}^{\|})) \rangle = 1$, it follows that the inclusion induced homomorphism

$$\pi_1(\widetilde{M}\backslash\nu\widetilde{\Sigma}) \longrightarrow \pi_1(P_n(m, f))$$

is also trivial. It then follows that $\pi_1(P_n(m,f)) = 1$ by the Seifert-van Kampen theorem.

Since $f : M \to \Sigma_b$ is a relatively minimal Lefschetz fibration of genus $a > 1$ by assumption, it follows from Theorem 2.3.12 that M is a minimal symplectic manifold. Thus by Corollary 3 of [22], the exceptional spheres of the $2t+d$ blowups are the only -1-spheres in $\widetilde{M}$. Since each of these spheres intersect $\widetilde{\Sigma}$, $\widetilde{M}\backslash\nu\widetilde{\Sigma}$ does not contain an embedded symplectic sphere of self-intersection -1. Moreover, since $Y_n(1)$ is minimal, $Y_n(1)\backslash\nu\Sigma_n$ does not contain any such sphere either. $P_n(1,f)$ is thus minimal by Usher's theorem (Theorem 2.3.9). By Taubes' theorem (2.3.2), since $P_n(1,f)$ is a minimal, simply-connected, symplectic 4-manifold, it is also irreducible.

If $n \geq 3$, recall that $Y_n(1)$ contains $4n-8$ Lagrangian tori of self-intersection zero which are disjoint from Σ_n. Since the symplectic form on $P_n(1,f)$ can be chosen to preserve the symplectic and Lagrangian submanifolds of $Y_n(1)$, these tori will be Lagrangian in $P_n(1,f)$. Letting T be one such torus, by Theorem 2.2.10, we can perturb the symplectic form on $P_n(1,f)$ making T into a symplectic submanifold.

To compute $\pi_1(P_n(1,f)\backslash T)$, we set $T = a_1' \times c_3''$ as in Figure 5.1 for convenience. From Lemma 5.1.2, $\pi_1(P_n(1,f)\backslash T)$ is normally generated by the meridian $[b_1^{-1}, d_3]$ of T. In other words, all of the relations in $\pi_1(P_n(1,f))$, (including those in (6.1)), will still hold in $\pi_1(P_n(1,f)\backslash T)$ except possibly for the relation $[b_1^{-1}, d_3] = 1$. The relations in (6.1) however, along with the fact that $c_1 = 1$ in $\pi_1(P_n(1,f)\backslash T)$ imply that b_1 and d_3 are both trivial, whence $\pi_1(P_n(1,f)\backslash T) = 1$. □

Remark 6.2.2. Applying gauge-theoretic arguments, we can show that the Seiberg-Witten invariants of the $P_n(m,f)$ grow arbitrarily large as m goes to infinity. This implies that the set $\{P_n(m,f) \mid m \geq 1\}$ contains infinitely many pairwise non-diffeomorphic manifolds. It also implies that infinitely many of the $P_n(m,f)$ are non-symplectic. Furthermore, for $m \geq 2$, as $P_n(m,f)$ can be obtained from $P_n(1,f)$ by performing a $\frac{1}{m-1}$-surgery on a nullhomologous torus in $Y_n(1)\backslash\nu\Sigma_n$, it can be shown that $P_n(m,f)$ must also be irreducible.

6.3 Fiber sum construction of $Q_n(m,f)$

We can alter our above construction slightly, using $Y_{n-2}(m)\#2\overline{\mathbb{CP}}^2$ when $n \geq 4$ instead of $Y_n(m)$ in our fiber sum. In particular, we can resolve the intersection between the surfaces Σ_2 and Σ_{n-2} in $Y_{n-2}(m)$. This yields a genus n surface represented by the class $[\Sigma_2]+[\Sigma_{n-2}] \in H_2(Y_{n-2}(m);\mathbb{Z})$. Since

$$([\Sigma_2]+[\Sigma_{n-2}])^2 = [\Sigma_2]^2 + 2[\Sigma_2]\cdot[\Sigma_{n-2}] + [\Sigma_{n-2}]^2 = 2,$$

this surface has will have self-intersection 2. Blowing up at two points on this surface yields a genus n surface Σ_n' of self-intersection 0 in $Y_{n-2}(m)\#2\overline{\mathbb{CP}}^2$. When $m = 1$, the intersection resolution and blow-ups can both be carried out symplectically, and hence Σ_n' is a symplectic submanifold of $Y_{n-2}(m)\#2\overline{\mathbb{CP}}^2$ with the symplectic structure inherited from $Y_{n-2}(m)$.

Theorem 6.3.1. *Let $f : M \to \Sigma_b$ be a relatively minimal Lefschetz fibration as above, and suppose that f has at least one non-separating vanishing cycle. Let $n = ta + b \geq 4$. For a suitable diffeomorphism $\Psi' : \partial(\nu\widetilde{\Sigma}) \to \partial(\nu\Sigma_n)$, the fiber sum*

$$Q_n(m, f) = (Y_{n-2}(m)\#2\overline{\mathbb{CP}}^2)\#_{\Psi'}\widetilde{M}$$

along the surfaces $\widetilde{\Sigma}$ and Σ'_n is simply-connected, and satisfies

$$\begin{aligned} e(Q_n(m,f)) &= e(M) + d + (8a+2)t + 8b - 14, \\ \sigma(Q_n(m,f)) &= \sigma(M) - 2t - d - 2, \\ \chi_h(Q_n(m,f)) &= \chi_h(M) + 2at + 2b - 4, \\ c_1^2(Q_n(m,f)) &= c_1^2(M) - d + (16a-2)t + 16b - 34. \end{aligned}$$

$Q_n(1, f)$ is irreducible, minimal, symplectic, and if $n = ta + b \geq 5$, contains a symplectic torus T' of self-intersection 0 satisfying $\pi_1(P_n(1, f)\backslash T) = 1$.

Proof. We begin by computing

$$\begin{aligned} e(Q_n(m,f)) &= e(\widetilde{M}) + e(Y_{n-2}(m)\#2\overline{\mathbb{CP}}^2) - 2e(\Sigma'_n) \\ &= e(\widetilde{M}) + e(Y_{n-2}(m)) + 2e(\overline{\mathbb{CP}}^2) - 4e(D^4) - 2e(\Sigma'_n) \\ &= e(M) + 2t + d + 4(n-2) - 4 + 6 - 4 - 2(2-2n) \\ &= e(M) + 8n + 2t + d - 14 \\ &= e(M) + 8(ta+b) + 2t + d - 14, \\ \sigma(Q_n(m,f)) &= \sigma(\widetilde{M}) + \sigma(Y_{n-2}(m)\#2\overline{\mathbb{CP}}^2) \\ &= \sigma(\widetilde{M}) + \sigma(Y_{n-2}(m)) + 2\sigma(\overline{\mathbb{CP}}^2) \\ &= \sigma(M) - 2t - d - 2, \end{aligned}$$

while the other characteristic numbers follow as above.

Note that the exceptional sphere of each blow-up will intersect Σ'_n transversely in a single point. Let E be one of the two exceptional spheres. Then the intersection of E with $\nu\Sigma'_n$ is a disk, and $\mu = E \cap \partial(\nu\Sigma'_n)$ is a meridian of Σ'_n. Note that μ also bounds the intersection of E with $(Y_{n-2}(m)\#2\overline{\mathbb{CP}}^2)\backslash\nu\Sigma'_n$ however, which is a disk in the complement of $\nu\Sigma'_n$. Thus $[\mu] = 1 \in \pi_1((Y_{n-2}(m)\#2\overline{\mathbb{CP}}^2)\backslash\nu\Sigma'_n)$. From Lemma 5.1.4 we conclude that

$$\pi_1((Y_{n-2}(m)\#2\overline{\mathbb{CP}}^2)\backslash\nu\Sigma'_n) \cong \pi_1(Y_{n-2}(m)\#2\overline{\mathbb{CP}}^2) \cong \pi_1(Y_{n-2}(m)),$$

as the blow-ups do not affect the fundamental group.

As in the proof of Theorem 6.2.1, we can fix a presentation

$$\pi_1(\Sigma_{n-2}) = \langle \overline{c}_1, \overline{d}_1, \ldots, \overline{c}_{n-2}, \overline{d}_{n-2} \mid \prod_{i=1}^{n-2}[\overline{c}_i, \overline{d}_i]\rangle \tag{6.3}$$

and a path connecting the basepoint of Σ_{n-2} to the basepoint of $Y_{n-2}(m)\#2\overline{\mathbb{CP}}^2$ so that the inclusion induced homomorphism into $\pi_1(Y_{n-2}(m)\#2\overline{\mathbb{CP}}^2)$ maps each $\bar{c}_i$ and $\bar{d}_j$ to c_i and d_j respectively. Letting the basepoint of Σ_{n-2} also be the basepoint of Σ'_n, we can consider each generator as a loop in $\pi_1(\Sigma'_n)$. Isotope Σ'_n to a parallel copy $\Sigma'^{\|}_n$ in $\partial(\nu\Sigma'_n)$. The image of the basepoint of Σ'_n at each $t \in [0,1]$ of this homotopy will give a path connecting the basepoint of Σ'_n to the basepoint of $\Sigma'^{\|}_n$. Joining this with the path connecting the basepoint of Σ'_n with the basepoint of $Y_{n-2}(m)\#2\overline{\mathbb{CP}}^2$ yields a path connecting the basepoints of $Y_{n-2}(m)\#2\overline{\mathbb{CP}}^2$ and $\Sigma'^{\|}_n$. Using this basepoint connecting path, the image of $\bar{c}_1$ in $\pi_1(\Sigma'^{\|}_n)$ is mapped under the inclusion induced homomorphism to $c_1 \in \pi_1(Y_{n-2}(m)\#2\overline{\mathbb{CP}}^2)$.

Let $\widetilde{\Sigma}^{\|}$ denote a parallel copy of $\widetilde{\Sigma}$ in $\partial(\nu\widetilde{\Sigma})$. Choose the gluing diffeomorphism Ψ' so that Ψ'_* maps the element of $\pi_1(\widetilde{\Sigma}^{\|})$ represented by a non-separating vanishing cycle of the fibration $f : M \to \Sigma_b$ to $\bar{c}_1 \in \pi_1(\Sigma'^{\|}_n)$. With such a choice of Ψ', the image of c_1 is trivial in $\pi_1(Q_n(m,f))$. Using the relations (5.2) and the fact that $\pi_1((Y_{n-2}(m)\#2\overline{\mathbb{CP}}^2)\backslash\nu\Sigma'_n) \cong \pi_1(Y_{n-2}(m))$, we have

$$\pi_1((Y_{n-2}(m)\#2\overline{\mathbb{CP}}^2)\backslash\nu\Sigma'_n)/\langle c_1\rangle = 1,$$

and hence the inclusion induced homomorphism

$$\pi_1((Y_{n-2}(m)\#2\overline{\mathbb{CP}}^2)\backslash\nu\Sigma'_n) \longrightarrow \pi_1(Q_n(m,f)) \tag{6.4}$$

is trivial. By identifying $\widetilde{\Sigma}^{\|}$ with $\Sigma'^{\|}_n$ using Ψ', the inclusion induced homomorphism $\pi_1(\widetilde{\Sigma}^{\|}) \to \pi_1(Q_n(m,f))$ will factor through (6.4), and will thus be trivial. Since by Lemma 6.1.1 we have $\pi_1(\widetilde{M}\backslash\nu\widetilde{\Sigma})/\langle\widetilde{\iota}_*(\pi_1(\widetilde{\Sigma}^{\|}))\rangle = 1$, it follows that the inclusion induced homomorphism

$$\pi_1(\widetilde{M}\backslash\nu\widetilde{\Sigma}) \longrightarrow \pi_1(Q_n(m,f))$$

is also trivial. By the Seifert-van Kampen theorem, it once again follows that $\pi_1(Q_n(m,f)) = 1$.

As $Y_{n-2}(1)\#2\overline{\mathbb{CP}}^2$ and $\widetilde{M}$ are both symplectic manifolds with symplectic submanifolds Σ'_n and $\widetilde{\Sigma}$ respectively, the fiber sum $Q_n(1,f)$ is also symplectic.

As shown in the proof of Theorem 6.2.1, $\widetilde{M}\backslash\nu\widetilde{\Sigma}$ does not contain any spheres of self-intersection -1. Likewise, since $Y_{n-2}(1)\#2\overline{\mathbb{CP}}^2$ is the blow up of a minimal symplectic manifold, the exceptional spheres are the only spheres of self-intersection -1 (by Corollary 3 of [22]), and hence there will be no such spheres in $(Y_{n-2}(1)\#2\overline{\mathbb{CP}}^2)\backslash\nu\Sigma'_n$. As in the proof of Theorem 6.2.1, it thus follows that $Q_n(1,f)$ is minimal and irreducible.

Recall that $Y_n(m)$ contains $4n-16$ Lagrangian tori, each of which can be chosen disjoint from both Σ_{n-2} and Σ_2. The images of these tori in $Y_{n-2}(1)\#\overline{\mathbb{CP}}^2$ are disjoint from Σ'_n, and as in the proof of Theorem 6.2.1, will descend to Lagrangian tori in the fiber sum $Q_n(1,f)$. Let T' be any one of these tori, say $a'_1 \times c''_3$ for definiteness. As above, we can perturb the symplectic form on $Q_n(1,f)$, so that T' becomes a symplectic submanifold of $Q_n(1,f)$. By Lemma 5.1.2, $\pi_1(Q_n(1,f)\backslash T')$ is normally generated by the meridian $[b_1^{-1}, d_3]$ of T'. As before however, since $c_1 = 1$ implies that $b_1 = d_3 = 1$, it follows that $\pi_1(Q_n(1,f)\backslash T') = 1$. □

Remark 6.3.2. As with Theorem 6.2.1, employing gauge theoretic arguments we can show that the set $\{Q_n(m,f) \mid m \geq 1\}$ contains infinitely many pairwise non-diffeomorphic manifolds, and infinitely many non-symplectic manifolds.

Chapter 7

Geography Results

7.1 Characteristic numbers computations

Having finished all of the needed constructions, we now apply them to the symplectic geography problem to find which lattice points in the (χ, c)-plane are now realizable as $(\chi_h(M), c_1^2(M))$ for minimal simply-connected symplectic M. Recall that in Section 4.2 we defined the Lefschetz fibrations

$$\varphi_{g,u} : X_g(u) \to \Sigma_k \quad \text{and} \quad \widetilde{\varphi}_{g,u} : \widetilde{X}_g(u^2) \to \Sigma_k$$

for integers $g, u \geq 2$, where $k = u(g-1)+1$. Applying Theorem 6.2.1 to these Lefschetz fibrations, we obtain families

$$\{P_n(m, \varphi_{g,u}) \mid m, g, u \geq 1, n \geq 2\} \quad \text{and} \quad \{P_n(m, \widetilde{\varphi}_{g,u}) \mid m, g, u \geq 1, n \geq 2\} \tag{7.1}$$

of simply-connected 4-manifolds. Recall that infinitely many of the manifolds in the above families are pairwise non-diffeomorphic and non-symplectic, and that when $m = 1$ they are minimal and symplectic. Combining the formulae in Section 4.2 and Theorem 6.2.1 yields

$$\begin{aligned}\chi_h(P_n(m, \varphi_{g,u})) &= \frac{5u}{3}g^3 + \left(3t - \frac{u}{2}\right)g^2 + \left(t + \frac{5u}{6}\right)g - u \\ c_1^2(P_n(m, \varphi_{g,u})) &= 14ug^3 + 2\,(12t - u)\,g^2 + 4\,(2t + u)\,g - 2(t + 5u)\end{aligned}$$

and

$$\begin{aligned}\chi_h(P_n(m, \widetilde{\varphi}_{g,u})) &= \frac{5u^2}{3}g^3 + u\left(3t - \frac{u}{2}\right)g^2 + u\left(t - \frac{7u}{6} + 2\right)g + u(u - 2t - 2) + 2t \\ c_1^2(P_n(m, \widetilde{\varphi}_{g,u})) &= 14u^2g^3 + 2u\,(12t - u)\,g^2 + 4u\,(2t - 3u + 4)\,g + 2u(2u - 8t - 7) + 14t.\end{aligned}$$

Applying Theorem 6.3.1 to these constructions yields families

$$\{Q_n(m, \varphi_{g,u}) \mid m, g, u \geq 1, n \geq 2\} \quad \text{and} \quad \{Q_n(m, \widetilde{\varphi}_{g,u}) \mid m, g, u \geq 1, n \geq 2\}, \tag{7.2}$$

of simply-connected manifolds as above, with infinitely many pairwise non-diffeomorphic and infinitely many non-symplectic manifolds. Again, those manifolds with $m = 1$ are minimal and symplectic, and the families satisfy

$$\begin{aligned}
\chi_h(Q_n(m,\varphi_{g,u})) &= \frac{5u}{3}g^3 + \left(3t - \frac{u}{2}\right)g^2 + \left(t + \frac{5u}{6}\right)g - (u+2)\\
c_1^2(Q_n(m,\varphi_{g,u})) &= 14ug^3 + 2(12t-u)g^2 + 4(2t+u)g - 2(t+5u+9)
\end{aligned}$$

and

$$\begin{aligned}
\chi_h(Q_n(m,\widetilde{\varphi}_{g,u})) &= \frac{5u^2}{3}g^3 + u\left(3t - \frac{u}{2}\right)g^2 + u\left(t - \frac{7u}{6} + 2\right)g + u(u-2t-2) + 2t - 2\\
c_1^2(Q_n(m,\widetilde{\varphi}_{g,u})) &= 14u^2g^3 + 2u\,(12t-u)\,g^2 + 4u(2t-3u+4)g + 2u(2u-8t-7) + 14t - 18.
\end{aligned}$$

We similarly apply Theorems 6.2.1 and 6.3.1 to the Lefschetz fibrations

$$\psi_{g,u} : Z_g(u) \to \Sigma_k \quad \text{and} \quad \widetilde{\psi}_{g,u} : \widetilde{Z}_g(u^2) \to \Sigma_k$$

to obtain the families

$$\begin{gathered}
\{P_n(m,\psi_{g,u}) \mid m,g,u \geq 1, n \geq 2\}, \quad \{P_n(m,\widetilde{\psi}_{g,u}) \mid m,g,u \geq 1, n \geq 2\},\\
\{Q_n(m,\psi_{g,u}) \mid m,g,u \geq 1, n \geq 2\}, \quad \text{and} \quad \{Q_n(m,\widetilde{\psi}_{g,u}) \mid m,g,u \geq 1, n \geq 2\}.
\end{gathered} \tag{7.3}$$

These families all satisfy the same properties as mentioned above in relation to (7.1) and (7.2). Their characteristic numbers are likewise given by

$$\begin{aligned}
\chi_h(P_n(m,\psi_{g,u})) &= \frac{14u}{3}g^3 + (8t-3u)\,g^2 + \frac{4u}{3}g - u\\
c_1^2(P_n(m,\psi_{g,u})) &= 40ug^3 + 4\,(16t-5u)\,g^2 + 6ug - 2(t+5u)\\
\chi_h(P_n(m,\widetilde{\psi}_{g,u})) &= \frac{14u^2}{3}g^3 + u(8t-3u)g^2 + 2u\left(1 - \frac{u}{3}\right)g + u(u-2t-2) + 2t\\
c_1^2(P_n(m,\widetilde{\psi}_{g,u})) &= 40u^2g^3 + 4u(16t-5u)g^2 + 2u(8-5u)g + 5u(u-3) + 2t(7-8u)\\
\chi_h(Q_n(m,\psi_{g,u})) &= \frac{14u}{3}g^3 + (8t-3u)g^2 + \frac{4u}{3}g - (u+2)\\
c_1^2(Q_n(m,\psi_{g,u})) &= 40ug^3 + 4(16t-5u)g^2 + 6ug - 2(t+5u+9)\\
\chi_h(Q_n(m,\widetilde{\psi}_{g,u})) &= \frac{14u^2}{3}g^3 + u(8t-3u)g^2 + 2u\left(1 - \frac{u}{3}\right)g + u(u-2t-2) + 2(t-1)\\
c_1^2(Q_n(m,\widetilde{\psi}_{g,u})) &= 40u^2g^3 + 4u(16t-5u)g^2 + 2u(8-5u)g + 5u(u-3) + 2t(7-8u) - 18.
\end{aligned}$$

7.2 Main results

We can use any of the families of minimal simply-connected symplectic 4-manifolds in (7.1), (7.2), and (7.3) to provide an alternate proof of the following theorem:

Theorem 7.2.1. *Let $s \geq 0$ be an integer. Then there exists an integer $\lambda(s)$ such that each integer pair $(j, 8j + s)$ with $j \geq \lambda(s)$ is realized as $(\chi_h(M), c_1^2(M))$ for some minimal simply-connected symplectic M.*

Proof. Suppose first that $s \equiv 0 \pmod 2$. Choose some $u \geq 1$ and $g \geq 2$ so that

$$\frac{2}{3}u^2(g+1)(g^2+2g-6) - 2 \geq s.$$

Set

$$t = -\frac{s}{2} + u - 1 + \frac{1}{3}u^2(g+1)(g^2+2g-6).$$

These choices of parameters satisfy the conditions of Theorem 6.3.1, and hence the manifold $Q_n(1, \widetilde{\varphi}_{g,u})$ (where $n = ta + b = tu(g+1)(3g-2) + u(g-1) + t + 1 \geq 5$) will be minimal, irreducible, simply-connected, and will contain a symplectic torus T of self-intersection zero with $\pi_1(Q_n(1, \widetilde{\varphi}_{g,u}) \backslash T) = 1$. Futhermore, $\sigma(Q_n(1, \widetilde{\varphi}_{g,u})) = s$. Set $\lambda(s) = \chi_h(Q_n(1, \widetilde{\varphi}_{g,u}))$. We have that $Q_n(1, \widetilde{\varphi}_{g,u})$ realizes the point $(\chi, c) = (\lambda(s), 8\lambda(s) + s)$. Applying Theorem 3.2.2 with $a = j - \lambda(s)$ and $b = 8j - 8\lambda(s)$, where $j > \lambda(s)$, will realize all of the points $(\chi, c) = (j, 8j + s)$ for $j > \lambda(s)$.

Suppose now that $s \equiv 1 \pmod 2$. Complete the above construction for $s + 1$, finding g, u, and t so that $Q_n(1, \widetilde{\varphi}_{g,u})$ is as above with $\sigma(Q_n(1, \widetilde{\varphi}_{g,u})) = s + 1$. Set $\lambda(s) = \chi_h(Q_n(1, \widetilde{\varphi}_{g,u})) + 1$. Then all points of the form $(j, 8j + s)$ with $j \geq \lambda(s)$ will be realized by applying Theorem 3.2.2 to $Q_n(1, \widetilde{\varphi}_{g,u})$ with $a = j - \lambda(s) + 1$ and $b = 7 + 8j - 8\lambda(s)$. □

Since the families $\{P_n(m, f) \mid m \geq 1\}$ and $\{Q_n(m, f) \mid m \geq 1\}$ contain infinitely many pairwise non-diffeomorphic smooth 4-manifolds (see Remarks 6.2.2 and 6.3.2), the proof of Theorem 7.2.1 implies the following:

Corollary 7.2.2. *For each integer $s \geq 0$, there exists some $\lambda(s) \in \mathbb{Z}$ such that each integer pair $(j, 8j + s)$ with $j \geq \lambda(s)$ is realized as $(\chi_h(M_\alpha), c_1^2(M_\alpha))$ by infinitely many pairwise non-diffeomorphic smooth 4-manifolds M_α (infinitely many of which admit no symplectic structure).*

Clearly the values defined in Theorem 7.2.1 for $\lambda(s)$ are not in general the smallest values possible using these constructions. More care is needed to find these $\lambda(s)$ values. Unfortunately, there is neither a clear nor straightforward method to finding these minimal values. To aid our discussion, we define $\mathcal{M}$ to be the union of the families (7.1), (7.2), and (7.3). We also define $\mathcal{M}_s = \{M \in \mathcal{M} \mid \sigma(M) = s\}$ for convenience. Recall that in constructing the above families, we began with a genus a Lefschetz fibration $f : M \to \Sigma_b$ with a section whose image has self-intersection d. We then chose $t > 0$ so that $2t + d \geq 0$ and so that $n = ta + b \geq \mu$, where $\mu \in \{2, 3, 4, 5\}$ depends on the construction used and whether we require the resulting manifold to contain a symplectic torus of self-intersection zero whose complement is simply-connected. These restrictions on the parameters make it difficult to find minimal χ_h values in $\mathcal{M}_s$, as choosing smaller u values require us to choose larger g values to satisfy the constraints and vice versa.

Furthermore, even if we are able to find some $M \in \mathcal{M}_s$ with minimal $\chi_h(M)$ for a given s, by applying Theorem 3.2.2 to a some $N' \in \mathcal{M}_{s'}$, with $s' > s$ and sufficiently small $\chi_h(N')$, it is possible to construct $N \notin \mathcal{M}$ with $\sigma(N) = s$ and $\chi_h(N) < \chi_h(M)$. Thus for a given signature s, finding manifolds with minimal χ_h using the above constructions and Theorem 3.2.2 depends not only minimizing χ_h over M_s, but also over $\mathcal{M}_{s'}$ for $s' \geq s$.

We thus rely on a computer search to find manifolds with small χ_h values. Our results for manifolds with signature $0 \leq \sigma \leq 99$ are summarized in Table 7.1 and Theorem 7.2.3:

s	$\lambda(s)$	s	$\lambda(s)$	s	$\lambda(s)$	s	$\lambda(s)$	s	$\lambda(s)$
0	25	20	58	40	85	60	145	80	167
1	25	21	58	41	85	61	145	81	167
2	24	22	57	42	85	62	145	82	166
3	27	23	60	43	85	63	145	83	166
4	26	24	59	44	85	64	144	84	166
5	51	25	87	45	85	65	144	85	166
6	50	26	87	46	85	66	144	86	166
7	53	27	87	47	85	67	144	87	166
8	52	28	87	48	84	68	144	88	166
9	59	29	87	49	87	69	144	89	166
10	59	30	87	50	86	70	144	90	165
11	59	31	87	51	146	71	144	91	165
12	59	32	86	52	146	72	143	92	165
13	59	33	86	53	146	73	146	93	165
14	58	34	86	54	146	74	145	94	165
15	58	35	86	55	146	75	167	95	165
16	58	36	86	56	145	76	167	96	165
17	58	37	86	57	145	77	167	97	165
18	58	38	86	58	145	78	167	98	164
19	58	39	86	59	145	79	167	99	167

Table 7.1: Small $\lambda(s)$ values for $0 \leq s \leq 99$.

Theorem 7.2.3. *For each pair of values $(s, \lambda(s))$ in Table 7.1, every point of the form $(\chi, c) = (j, 8j + s)$ with $j \geq \lambda(s)$ is realized by a minimal simply-connected symplectic manifold. In other words there exists a minimal simply-connected symplectic 4-manifold M with $\chi_h(M) = j$ and $c_1^2(M) = 8j + s$ for each integer $j \geq \lambda(s)$.*

Proof. Consider the twelve minimal simply-connected symplectic 4-manifolds in Table 7.2 and their characteristic values:

M	$\chi_h(M)$	$c_1^2(M)$	$\sigma(M)$	$e(M)$
$Q_9(1, \varphi_{2,1})$	24	194	2	94
$P_9(1, \varphi_{2,1})$	26	212	4	100
$Q_{17}(1, \varphi_{2,2})$	50	406	6	194
$P_{17}(1, \varphi_{2,2})$	52	424	8	200
$Q_{18}(1, \psi_{2,1})$	57	478	22	206
$P_{18}(1, \psi_{2,1})$	59	496	24	212
$Q_{19}(1, \psi_{2,2})$	84	720	48	288
$P_{19}(1, \psi_{2,2})$	86	738	50	294
$Q_{36}(1, \psi_{2,3})$	143	1216	72	500
$P_{36}(1, \psi_{2,3})$	145	1234	74	506
$Q_{34}(1, \widetilde{\psi}_{2,2})$	164	1410	98	558
$P_{34}(1, \widetilde{\psi}_{2,2})$	166	1428	100	564

Table 7.2: Minimal simply-connected symplectic 4-manifolds with small χ_h values.

By Theorems 6.2.1 and 6.3.1, each of them contains a symplectic torus of self-intersection zero whose complement is simply-connected. Each of the pairs of values $(s, \lambda(s))$ in Table 7.1 is realized either as $(\sigma(M), \chi_h(M))$ for some M in Table 7.2, or as $(\sigma(N), \chi_h(N))$ for some N with $\chi_h(N) = \chi_h(M) + a$ and $c_1^2(N) = c_1^2(M) + b$, where M is again from Table 7.2 and the integers a and b satisfy $0 \leq b \leq 8a$ (Theorem 3.2.2). For example, choosing $M = Q_{19}(1, \psi_{2,2})$, $a = 2$ and $b = 6$, Theorem 3.2.2 implies that there exists a minimal simply-connected symplectic 4-manifold N with

$$\chi_h(N) = \chi_h(M) + a = 86$$

and

$$c_1^2(N) = c_1^2(M) + b = 726,$$

whence $\sigma(N) = c_1^2(N) - 8\chi_h(N) = 38$. The point $(\chi, c) = (86, 726)$ is thus realized by some minimal simply-connected symplectic manifold N. This point corresponds to the pair $(s, \lambda(s)) = (38, 86)$ in Table 7.1. All of the other pairs $(s, \lambda(s))$ in Table 7.1 are likewise realized, either by applying Theorem 3.2.2 as shown in the case of $(s, \lambda(s)) = (38, 86)$, or directly using manifolds from Table 7.2. These manifolds realize all of the points in the geography plane of the form $(\chi, c) = (\lambda(s), 8\lambda(s) + s)$ for each pair $(s, \lambda(s))$ in Table 7.1.

Points of the form $(\chi, c) = (j, 8j + s)$ for $j > \lambda(s)$ are also realized using Theorem 3.2.2. If $(\lambda(s), 8\lambda(s) + s)$ is realized directly by some M in Table 7.2, simply apply Theorem 3.2.2 to M with $a = j - \lambda(s)$ and $b = 8(j - \lambda(s))$ to obtain a minimal simply-connected symplectic manifold N with $(\chi_h(N), c_1^2(N)) = (j, 8j + s)$ for $j > \lambda(s)$. If the point $(\lambda(s), 8\lambda(s) + s)$ is realized by Theorem 3.2.2 as $(\chi_h(M) + a_0, c_1^2(M) + b_0)$ for some M in Table 7.2, then points of the form $(j, 8j + s)$ for $j > \lambda(s)$ will be realized by applying Theorem 3.2.2 to M using $a = a_0 + j - \lambda(s)$ and $b = b_0 + 8(j - \lambda(s))$. □

Comparing these results with the regions R_j populated in [27] (see (3.1)), Theorem 7.2.3 populates 19 952 new lattice points in the (χ, c)-plane with minimal simply-connected symplectic 4-manifolds. Using similar methods, we can also find small $\lambda(s)$ for $s \geq 100$.

Corollary 7.2.4. *For each pair $(s, \lambda(s))$ in Table 7.1 (with the possible exceptions of those with $s \equiv 0 \pmod{16}$), there exists a minimal simply-connected symplectic 4-manifold with signature s that is homeomorphic to $(2j-1)\mathbb{CP}^2\#(2j-s-1)\overline{\mathbb{CP}}^2$ for each $j \geq \lambda(s)$.*

Proof. As all smooth 4-manifolds with even intersection form satisfy $\sigma \equiv 0 \pmod{16}$ (Theorem 2.1.6), all smooth manifolds with signature not divisible by 16 must have odd intersection form. By Serre's and Freedman's classification theorems (Theorems 2.1.3 and 2.1.4), any such simply-connected manifold M is homeomorphic to

$$(2\chi_h(M) - 1)\mathbb{CP}^2\#(2\chi_h(M) - \sigma(M) - 1)\overline{\mathbb{CP}}^2.$$ □

Corollary 7.2.5. *For each pair $(s, \lambda(s))$ in Table 7.1 (with the possible exceptions of those with $s \equiv 0 \pmod{16}$) and each $j \geq \lambda(s)$, there exist infinitely many pairwise non-diffeomorphic smooth 4-manifolds homeomorphic to $(2j-1)\mathbb{CP}^2\#(2j-s-1)\overline{\mathbb{CP}}^2$ (infinitely many of which admit no symplectic structure).*

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