Packing Unit Disks

by

Benjamin J. Lafreniere

A thesis
presented to the University of Waterloo
in fulfillment of the
thesis requirement for the degree of
Master of Mathematics
in
Computer Science

Waterloo, Ontario, Canada, 2008

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Benjamin J. Lafreniere

Abstract

Given a set of unit disks in the plane with union area A, what fraction of A can be covered by selecting a pairwise disjoint subset of the disks? Richard Rado conjectured 1/4 and proved 1/4.41. In this thesis, we consider a variant of this problem where the disjointness constraint is relaxed: selected disks must be k-colourable with disks of the same colour pairwise-disjoint. Rado's problem is then the case where $k = 1$, and we focus our investigations on what can be proven for $k > 1$.

Motivated by the problem of channel-assignment for Wi-Fi wireless access points, in which the use of 3 or fewer channels is a standard practice, we show that for $k = 3$ we can cover at least 1/2.09 and for $k = 2$ we can cover at least 1/2.82. We present a randomized algorithm to select and colour a subset of n disks to achieve these bounds in $O(n)$ expected time. To achieve the weaker bounds of 1/2.77 for $k = 3$ and 1/3.37 for $k = 2$ we present a deterministic $O(n^2)$ time algorithm.

We also look at what bounds can be proven for arbitrary k, presenting two different methods of deriving bounds for any given k and comparing their performance. One of our methods is an extension of the method used to prove bounds for $k = 2$ and $k = 3$ above, while the other method takes a novel approach.

Rado's proof is constructive, and uses a regular lattice positioned over the given set of disks to guide disk selection. Our proofs are also constructive and extend this idea: we use a k-coloured regular lattice to guide both disk selection and colouring. The complexity of implementing many of the constructions used in our proofs is dominated by a lattice positioning step. As such, we discuss the algorithmic issues involved in positioning lattices as required by each of our proofs. In particular, we show that a required lattice positioning step used in the deterministic $O(n^2)$ algorithm mentioned above is 3SUM-hard, providing evidence that this algorithm is optimal among algorithms employing such a lattice positioning approach. We also present evidence that a similar lattice positioning step used in the constructions for our better bounds for $k = 2$ and $k = 3$ may not have an efficient exact implementation.

Acknowledgements

I would like to thank my supervisor, Anna Lubiw, for her ideas and guidance, and for helping me to decide which path to pursue next in life. Anna, along with Peter Brass and Ferran Hurtado were my coauthors for publications based on this research, and were a pleasure to work with. For their contributions and input, they deserve my most sincere thanks.

I would like to thank my readers, Timothy Chan and Craig Kaplan, for their valuable comments and critical reviews. Their input has certainly improved the quality of this thesis. Thanks also to Stephane Durocher for suggesting that I look into the Weber point problem, and to Ross Willard and Stephen New, who both provided valuable advice on a number of analysis problems that arose during the course of this work.

The problem that I discuss in this thesis was introduced to me by Anna Lubiw, and suggested to Anna by Srinivasan Keshav. Work on the problem began at the 5th McGill-INRIA Workshop on Computational Geometry in Computer Graphics at McGill's Bellairs Research Institute in 2006. Thanks to S. Keshav and the participants of the McGill-INRIA workshop for their contributions.

For their financial support, I would like to gratefully acknowledge the National Sciences and Engineering Research Council of Canada, the Ontario Ministry of Training, Colleges, and Universities, and the University of Waterloo.

Finally, on a personal note, I am ever grateful to my friends and family for being themselves.

Contents

List of Tables **vii**

List of Figures **viii**

1 Introduction **1**

1.1 The Basic Problem . . . 1

1.2 The k-Colour Variant . . . 2

1.3 Outline . . . 3

2 Background **5**

2.1 Wireless Network Deployment . . . 5

2.2 Related Work . . . 6

2.3 Rado's Proof . . . 9

3 Basic Bounds **12**

3.1 A Bound of $c_3 \gtrapprox 1/2.77$ . . . 12

3.2 Minimum Disk-Hexagon Intersection . . . 14

3.3 A Bound of $c_2 \gtrapprox 1/3.37$ . . . 17

3.4 Algorithms . . . 20

4 Weighted Bounds **23**

4.1 A Bound of $c_3 \gtrapprox 1/2.09$ . . . 24

4.2 A Bound of $c_2 \gtrapprox 1/2.82$ . . . 28

4.3 Algorithmic Issues . . . 31

5 Extension to k Colours **34**

5.1 k-Colour Bounds . . . 34

5.2 What about a square lattice? . . . 40

5.3 Algorithms . . . 42

6 Conclusion **44**

References **45**

Appendix **48**

A Lemma 3.2 Formulas **48**

A.1 Derived Formula for $0 \leq \theta \leq \arccos(\frac{2}{3}) - \frac{\pi}{6}$ 48

A.2 Derived Formula for $\arccos(\frac{2}{3}) - \frac{\pi}{6} \leq \theta \leq \frac{\pi}{6}$ 49

List of Tables

5.1 Bounds on c_k derived using Theorem 5.2 37
5.2 Bounds on c_k derived using Theorem 5.4 39
5.3 Bounds on c_k derived using Theorems 5.2 and 5.4 around $k = 1477$ 40

List of Figures

1.1 An arrangement of disks demonstrating that $c_1 \leq 1/4$ 1
1.2 Arrangements of disks demonstrating that $c_2 \leq 1/2$ and $c_3 \lessapprox 1/1.41$ 3

2.1 A triangular lattice ensures that selected disks are pairwise disjoint. 10
2.2 Fundamental cells of regular triangular and square lattices 10
2.3 Translating portions of $\cup D$ to the fundamental cell 11

3.1 A finer 3-coloured lattice 13
3.2 The Voronoi tessellation of a regular triangular lattice 14
3.3 Sliding disk d outward from the center of h 15
3.4 Calculating the intersection area as a function of angle θ 15
3.5 Plot showing the intersection area between disk d and a hexagon h 16
3.6 Arrangement in which the lower bound is realized for all disks 17
3.7 A 2-coloured lattice with side length $2\sqrt{2}$ 18
3.8 Sliding d to the right results in a smaller intersection between d and S . . . 19
3.9 Plot showing the area of intersection between d and S 20

4.1 Selecting the maximum number of disks is not necessarily best 24
4.2 Lower bounding $w(p)$ using the largest inscribed disk 25
4.3 The union of a set of disks before and after a radius scaling 27
4.4 Exploring the intersection between d and S 29

5.1 Distance $\sqrt{i^2 + ij + j^2}$ occurs in the unit triangular lattice. 35
5.2 Each Voronoi cell is coloured to match its associated lattice point. 38
5.3 Scaling used by Theorem 5.4 38
5.4 Plot of Theorem 5.2 vs. Theorem 5.4 for $1 \leq k \leq 2000$ 39
5.5 Plot of Theorem 5.2 vs. Theorem 5.6 for $k \leq 200$ 41
5.6 Plot of Theorem 5.2 vs. Theorem 5.6 for $k \leq 2000$ 42
5.7 Plot of Theorem 5.4 vs. Theorem 5.7 for $k \leq 2000$ 42

5.8 Plot of Theorem 5.4 vs. Theorem 5.7 for $2000 \leq k \leq 50,000$ 43

Chapter 1

Introduction

1.1 The Basic Problem

Richard Rado [30] studied the following problem: Given any arrangement of open unit disks D in the plane, what is the largest c such that we can select a pairwise disjoint subset of disks that cover at least a fraction c of the area of the union of D? Clearly $c \leq \frac{1}{4}$, corresponding to the case shown in Figure 1.1 where a large number of unit disks share a very small intersection—the common intersection prevents us from selecting more than a single disk, which has area π, while the union area of all disks approaches 4π. Rado conjectured that this upper bound is tight—i.e., $c = \frac{1}{4}$—and proved a lower bound of $c \geq \frac{\pi}{8\sqrt{3}} \approx \frac{1}{4.41}$ [30].

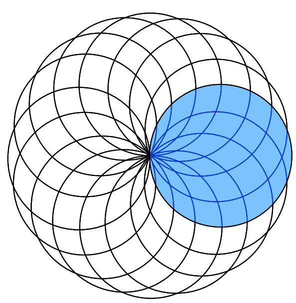

Figure 1.1: Given a set of unit disks arranged in a circle with a very small mutual intersection, the largest fraction of the area we can cover by selecting disjoint disks is $\frac{\pi}{4\pi} = \frac{1}{4}$.

If we imagine that the unit disks in D each represent the coverage area of a wireless access point placed at that disk's center, then Rado's problem models the following wireless deployment problem: How to choose a subset of access points to cover the largest area while avoiding regions of interference. Interference occurs at points that are overlapped by the coverage of two access points; this causes a dead spot where devices can communicate with neither.

In reality, wireless devices are more sophisticated than described above and use multiple channels to prevent interference. Each access point is assigned a specific channel and interference only occurs in regions that are covered by two access points operating on the same channel. With this as motivation, we consider a k*-coloured* variant of Rado's problem which relaxes the disjointness constraint to allow multiple "channels". Selected disks are each assigned one of k available colours and only disks which have been assigned the same colour are required to be pairwise disjoint. The goal remains to cover as much area as possible under these constraints. In the next section, we state this problem more formally and present a detailed outline of our results.

1.2 The k-Colour Variant

Rado's problem can be viewed as the 1-colour case of a more general unit disk packing problem: Given an arrangement of open unit disks D in the plane and a fixed number of colours k, we want to find the largest c_k such that we can always select and k-colour a subset of disks C such that same-coloured disks are pairwise disjoint and the union area of C covers at least a fraction c_k of the union area of D. It should be noted that the term "disk packing" is more commonly used for the geometric problem of positioning congruent copies of a disk such that they cover a given region as densely as possible without overlapping [36]. Instead, here the goal is to select a subset of the set D of candidate unit disks (whose positions are fixed) in order to cover as much of the union area of D as possible.

Motivated by the problem of channel-assignment for IEEE 802.11 wireless networks, in which the use of 3 or fewer channels is a standard practice, we begin by proving bounds for $k = 2$ and $k = 3$. We show that, for any given arrangement of unit disks, $c_2 \gtrapprox \frac{1}{2.82}$ and $c_3 \gtrapprox \frac{1}{2.09}$. We also present a randomized algorithm to select and colour a subset of n disks to achieve these bounds in $O(n)$ expected time. To achieve weaker bounds of $c_3 \gtrapprox \frac{1}{2.77}$ and $c_2 \gtrapprox \frac{1}{3.37}$ we present a deterministic $O(n^2)$ time algorithm.

In the spirit of Rado's conjecture that $c_1 = 1/4$, we make conjectures for c_2 and c_3. Specifically, we conjecture that the arrangement of disks shown in Figure 1.1 also allows the smallest fraction of area to be covered for $k = 2$ and $k = 3$, and thus $c_2 = 1/2$ and $c_3 \approx 1/1.41$ (see Figure 1.2).

We also look at what bounds can be proven for arbitrary k, presenting two different methods for deriving bounds for any given k and comparing their performance. One method is a generalization of the method used to prove our bounds on c_2 and c_3 while the other takes a novel approach.

Rado's proof for $k = 1$ is constructive, and uses a regular lattice positioned over the given set of disks to guide disk selection. Our proofs are also constructive and extend this idea: we use a k-coloured regular lattice to guide both disk selection and colouring. The

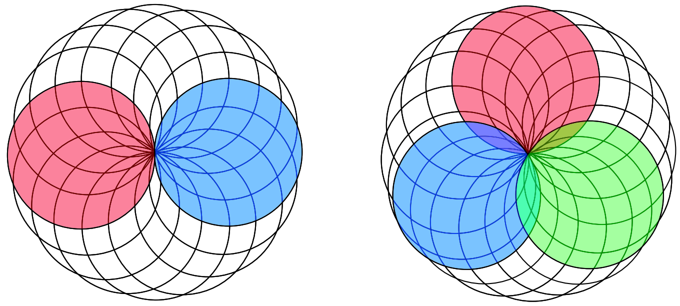

Figure 1.2: Given a set of disks arranged in a circle with a very small mutual intersection, the largest fraction of area we can cover is $\frac{2\pi}{4\pi} = \frac{1}{2}$ with two colours (left) and approximately $\frac{1}{1.41}$ with three colours (right).

complexity of implementing many of the constructions used in our proofs are dominated by a lattice positioning step. As such, we investigate the algorithmic issues involved in positioning lattices as required by each of our proofs. In particular, we show that a required lattice positioning step used in our $O(n^2)$ deterministic algorithm mentioned above is 3SUM-hard, providing evidence that our algorithm is optimal among algorithms using such a lattice positioning approach. We also present evidence that a similar lattice positioning step used in the constructions for our better bounds on c_2 and c_3 is a difficult problem.

1.3 Outline

The rest of the thesis is organized as follows. In Chapter 2 we review the background and motivation for our variant of Rado's problem. In particular, we detail the wireless network motivation for our variant, examine literature on related problems, and review the details of Rado's proof.

In Chapter 3 we prove bounds of $c_2 \gtrapprox \frac{1}{3.37}$ and $c_3 \gtrapprox \frac{1}{2.77}$ and present an $O(n^2)$ time deterministic algorithm and an $O(n)$ expected time randomized algorithm for selecting and colouring disks to satisfy these bounds. We also show that a lattice positioning step used in our deterministic algorithm is 3SUM-hard.

In Chapter 4 we build on the approach used in Chapter 3 and prove bounds of $c_2 \gtrapprox \frac{1}{2.82}$ and $c_3 \gtrapprox \frac{1}{2.09}$. We also discuss the algorithmic issues involved in implementing the constructions used in these proofs, and present an $O(n)$ expected time randomized algorithm for selecting and colouring disks to achieve these bounds.

In Chapter 5 we look at what can be proven for $k > 3$, presenting two methods for deriving bounds for arbitrary k. One method is a generalization of the approach used to prove bounds in Chapter 4, while the other follows a novel approach. We compare these two methods, identifying for which k each method derives better bounds. We also discuss algorithmic issues for these methods. In particular, we present an efficient algorithm for selecting and colouring disks to achieve the bounds derived using our novel method.

Finally, Chapter 6 concludes the thesis with a discussion of interesting open problems and directions in which the work presented here could be extended.

Chapter 2

Background

In this chapter we present the motivation for our variant of Rado's problem, examine literature on related problems, and review the details of Rado's proof. Section 2.1 motivates our k-colour variant of Rado's problem, describing in detail the wireless networking deployment problem that it models. This section is not strictly essential to understand the following chapters. Section 2.2 reviews literature on Rado's problem and a number of other problems related to our own. Finally, in Section 2.3 we present Rado's proof and, in particular, review the proof of an important lemma proven by Rado [30]. We will refer to this material frequently in the chapters that follow.

2.1 Wireless Network Deployment

Wi-Fi (IEEE 802.11) wireless networks are becoming a ubiquitous feature in modern businesses, universities, and cities. In a typical Wi-Fi deployment scenario, a set of candidate locations are determined for wireless access points (APs) [5]. A subset of the candidate locations must be chosen along with a channel assignment for each installed AP in order to maximize the area covered by the wireless network while minimizing interference. Interference occurs when two APs using the same channel are within range of one another. This has the effect that users within range of both APs are prevented from communicating with either. This type of interference, caused by clients and APs within the wireless network, is called internal interference. This is to contrast with external interference, caused by devices outside of the wireless network (e.g., microwaves, cordless phones, or APs from other wireless networks). Apart from interference, AP coverage areas may be irregular (i.e., not uniquely determined by the Euclidean distance between a client and the AP) and may change over time due to change in the environment in which the AP is deployed. For a more detailed discussion of these and other complications faced by wireless network deployments, see the introduction to [5].

To avoid interference in a deployment, the number and location of APs can be varied, as can the channel and the power level of each deployed AP. While these could all be considered as free variables, the realities of wireless network deployment present constraints of varying severity on each. For example, the set of potential locations for APs is strongly constrained by where it is physically possible and aesthetically acceptable to place AP hardware. In contrast, the number of APs is typically only bounded by the ever-lowering cost of wireless networking hardware. Finally, available power levels and channel assignments are tightly restricted by the wireless networking protocol being used, government regulations on radio spectrum, and the capabilities of the specific AP hardware. As it is relevant to our results, we will discuss in greater detail the constraints placed on channel selection.

The IEEE 802.11 standard establishes a number of requirements on the radio frequency characteristics of 802.11 hardware. The 2.4-GHz band used by 802.11 devices is broken into 11 channels for the North American domain and 13 channels for the European domain. Unfortunately, the number of effective channels is much lower due to interference between channels with center frequencies close to one another. A study by wireless hardware maker Cisco Systems recommends that wireless network deployments only use three channels (1, 6, and 11 for the North American domain). They find that even a four-channel scheme can cause unacceptable degradation of service in systems with a high volume of users [15].

Motivated by the discussion above, we consider the set of candidate locations for APs to be fixed and place no restriction on the number of APs to be installed. Each installed AP is assigned one of three available channels and to simplify matters we consider all installed APs as having the same power level. This simplification is not entirely unjustified, as the problem of power control for IEEE 802.11 network devices is a complex research area in its own right [4]. Under these restrictions, the wireless network deployment problem is analogous to the 3-colour version of Rado's problem discussed in Chapter 1. Each disk corresponds to the coverage area of a potential AP placed at the disk's center, and a 3-coloured disk packing represents a deployment of APs with channel assignments.

2.2 Related Work

General Problem Context

Rado's problem and our generalization to k-colours are related to a number of well-known problems in mathematics and computer science. From set theory, Rado's problem is similar to the *set packing* problem. A set packing is a set of subsets of a given set that are pairwise disjoint. In the set packing problem, we are given a set of subsets of a finite set S as well as a number k and asked to determine if a set packing of the given subsets exists with cardinality k [34]. The associated optimization problem, maximum set packing, asks for a set packing of largest cardinality and has been shown to be NP-hard [22]. Rado's problem can be viewed as set packing though with an infinite set S, namely the whole plane. Each subset is a unit disk in the plane. The other difference is that Rado wasn't interested in finding an optimum set cover, but rather in proving worst case bounds on the cardinality of an optimum cover as a fraction of $1/\pi$ times the area of the union of all disks. This makes the problem a geometric one.

From graph theory, Rado's problem is closely related to the *maximum independent set* (MIS) problem on *unit-disk graphs*. An *independent set* in a graph is a subset of vertices that are pairwise non-adjacent. A unit-disk graph is an intersection graph for a set of congruent disks in the plane—vertices represent disks and edges connect pairs of vertices whose associated disks have a non-empty intersection. The MIS problem is to find the largest cardinality independent set in a graph, and is known to be NP-complete for general graphs [22] as well as for unit-disk graphs [16]. Rado's problem is then to prove worst case bounds on the cardinality of an optimal independent set of a unit-disk graph as a fraction of $1/\pi$ times the area of the union of all disks.

The *unit-disk k-colourability problem* (UD k-CP) can be viewed as a generalization of the MIS problem to multiple colours, and shares some similarities with our problem. A graph is called k-colourable if we can find a partitioning of its vertices into k independent sets, or equivalently to colour the vertices of the graph using at most k colours such that no two same-coloured vertices are adjacent. The unit-disk k-colourability problem is to determine if a given unit-disk graph is k-colourable. This problem has been shown to be NP-complete for any fixed $k \geq 3$ [23]. Our problem is similar to the UD k-CP problem, but we are looking for a k-colourable subset of disks in the unit-disk graph instead of determining if the entire graph is k-colourable. That is, our goal is to prove quantitative bounds on the maximum union area of a k-colourable subset of disks in a unit-disk graph, relative to the union area of all disks in the graph.

The History of Richard Rado's Problem

Richard Rado's problem was preceded by a similar problem for arbitrary (i.e., not necessarily congruent) axis-parallel squares. Tibor Radó investigated the problem of proving bounds on the fraction c of the union area of a set of such squares coverable by a pairwise disjoint subset. He conjectured that $c = 1/4$ and proved $c \geq 1/9$ using a greedy algorithm [33]. Interestingly, 45 years later Ajtai demonstrated a construction involving several hundred squares that disproved T. Radó's conjecture [2]. R. Rado improved T. Radó's bound for arbitrary axis-parallel squares to 1/8.75 [30], a bound later improved by Zalgaller to 1/8.60 [37], and Bereg et al. to 1/8.4797 [10].

As well as improving T. Radó's bound for arbitrary axis-parallel squares, R. Rado considered a number of variants of this problem, including the variant for congruent disks (equivalently unit disks) discussed in Chapter 1. In a series of publications titled "On Covering Theorems" [30, 31, 32] he investigated these types of problems for classes of convex geometric objects including disks, centrally symmetric convex geometric objects, and arbitrary convex geometric objects. He also considered the problem in more general settings such as higher dimensions. R. Rado proved that $c \geq \frac{\pi}{8\sqrt{3}} \approx 1/4.41$ for sets of congruent disks [30]. Though his bound for congruent disks has so far stood the test of time, recent work by Bereg et al. has improved a number of R. Rado's other bounds, including those for arbitrary radii disks, centrally symmetric convex sets, and arbitrary convex sets [10].

Related Problems

We believe this is the first work to study the k-colour generalization of Rado's problem. However, there are a number of similar and related problems that have received attention in the literature.

In the *1-covered* variant of Rado's problem, selected disks are allowed to overlap but we only count area where there is no "interference"—i.e., the area of the set of points covered by only one selected disk. This variant can also be generalized to k-colours, leading to a *k-colour 1-covered* variant where the problem is colour a subset of the given disks using at most k colours to maximize the area of $\{p \in \mathbb{R}^2 \mid \text{for some colour, point } p \text{ is in exactly one disk of that colour}\}$, called the *1-covered area*. Note that, unlike usual colouring, disks assigned the same colour are allowed to overlap here.

Asano et al. [7] proved that it is always possible to achieve at least approximately 1/4.37 1-covered area relative to the union area of all disks using only one colour. This problem has also been considered with respect to two other optimization problems. For the problem of selecting disks to maximize the 1-covered area using one colour, work has focused on approximation algorithms (though no proof yet exists, it is suspected that this problem is NP-hard). Asano et al. [7] present a 5.83-approximation algorithm with polynomial runtime. Chen et al. [14] show that the problem admits a polynomial time approximation scheme when the ratio of the radius of the largest disk over the radius of the smallest disk is a constant.

The other related and well-explored optimization problem is *conflict-free colouring*; here the goal is to minimize the number of colours needed to 1-cover the whole area, i.e. the union of the given disks. Even et al. [19] proved that $O(\log n)$ colours are always sufficient and sometimes necessary for any given disks of general radii. They also proved that, when the ratio between the radius of the largest and smallest disk in the input set is a constant, the necessary number of colours is bounded by the log of the maximum number of disks residing in a square with unit diagonal length [19]. Alon et al. [3] have shown that, if each disk intersects at most k others, then $O(\log^3 k)$ colours are sufficient for a conflict-free colouring, which improves on the above $O(\log n)$ bound when k is much smaller than n. There is also work on online algorithms for conflict-free colouring [20], and on conflict-free colouring of regions other than disks [26].

Recent research has also looked at the problem of efficiently selecting subsets of congruent disks (equivalently unit disks) to cover a fraction of their union area. In [9], Bereg et al. present an $O(n \log n)$ algorithm for selecting a subset of a set of congruent disks to satisfy a $c \geq 1/(5 + 4/\pi) > 1/6.2733$ bound, and a linear-time approximation scheme that approximately achieves Rado's 1/4.41 bound, running in $O(n/\epsilon^2)$ and selecting a set of disks to satisfy a $c \geq 1/(8\sqrt{3}/\pi + \epsilon)$ bound for any $\epsilon > 0$. For the equivalent problem on sets of arbitrary radii disks, they present an $O(n^2)$ algorithm to satisfy a $c > 1/8.4898$ bound.

We will also look at the algorithmic issues involved in selecting subsets of unit disks to cover a fraction of their union area, but in the context of our k-colour generalization of Rado's problem. That is, the research above looks at the problem for $k = 1$, while our work looks at the problem for $k > 1$.

2.3 Rado's Proof

As was mentioned in Chapter 1, Rado proved that $c_1 \geq \frac{\pi}{8\sqrt{3}} \approx \frac{1}{4.41}$ and conjectured the lower bound $c_1 \geq \frac{1}{4}$ [30]. According to Rado this result was first discovered by Besicovitch but never published, and was later rediscovered and published by himself. In this section we will review Rado's proof in detail. The purpose of this is to give a background on the problem, and also because we build on ideas used in Rado's proof in the chapters that follow.

We start by defining some notation. Let D be a set of unit disks with union area A, and let $\cup D$ denote the union of all disks in D.

Rado's proof uses a regular triangular lattice with side length 4 to guide disk selection. By positioning such a lattice over D and selecting one arbitrary disk containing each lattice point that intersects $\cup D$, the set of selected disks must be pairwise disjoint (see Figure 2.1). If we commit to using such a lattice to guide disk selection, we can prove a lower bound on c_1 by proving a lower bound on how many points of $\cup D$ we can position our lattice to intersect. That is, supposing that we prove that it is always possible to position our lattice to intersect $\cup D$ in at least m points, then we can always select a subset of disks with area πm (m pairwise disjoint disks each with area π) and $c_1 \geq \frac{\pi m}{A}$.

To get a value for m we apply Lemma 2.1, proven below, which gives a lower bound on the number of points in a region of the plane that we can intersect with a given regular lattice. Lemma 2.1 uses the concept of the *fundamental cell* of a lattice. For a regular triangular lattice the fundamental cell F consists of a pair of adjacent triangles (see Figure 2.2). Lemma 2.1 states that we can position the lattice to contain at least $\frac{A}{\alpha}$ points in $\cup D$, where α is the area of the fundamental cell. Since $\alpha = 2\left(4\sqrt{3}\right) = 8\sqrt{3}$ for our lattice, by Lemma 2.1 we can position the lattice to contain at least $m = \frac{A}{8\sqrt{3}}$ points in $\cup D$ and therefore $c_1 \geq \frac{\pi}{8\sqrt{3}}$.

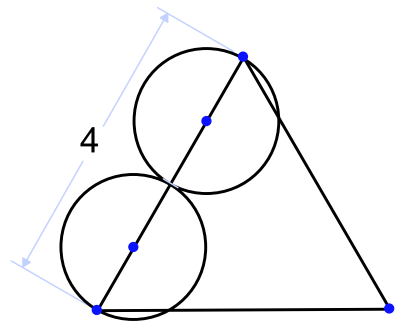

Figure 2.1: A triangular lattice ensures that selected disks are pairwise disjoint.

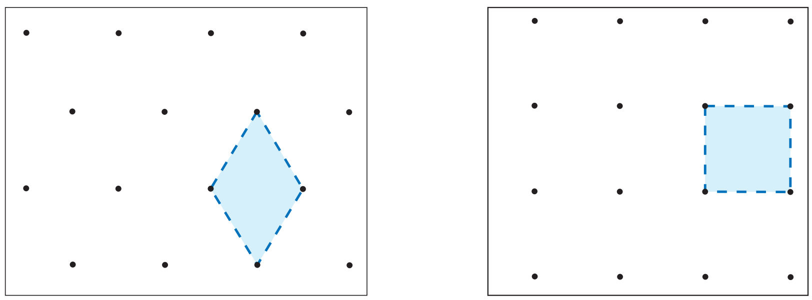

Figure 2.2: The respective fundamental cells of a regular triangular lattice (left) and a regular square lattice (right).

Lemma 2.1 [30]. *Given a region of the plane G with area A, and a regular lattice with a fundamental cell of area α, the lattice can always be positioned such that it contains $\frac{A}{\alpha}$ points in G.*

Proof. Given an arbitrary placement of the lattice, each translate of the fundamental cell F of the lattice can be translated to F along with whatever parts of G they contain (see Figure 2.3). The translated parts of G may "overlap" in F—there may be points on the fundamental cell intersecting multiple translated portions of G.

Supposing a point p in F intersects k translated portions of G, then repositioning the lattice such that p is a lattice point ensures that k lattice points intersect G. In this case, we refer to k as the *depth* of point p in the fundamental cell. Since the area of G is A, the total area of all portions of G translated to F is clearly also A. Therefore, we have portions of G with total area A translated to a region of area α and a point of depth at least $\frac{A}{\alpha}$ must exist. □

In the next chapter we will use Lemma 2.1 and a method very similar to that used above to prove preliminary bounds on c_2 and c_3.

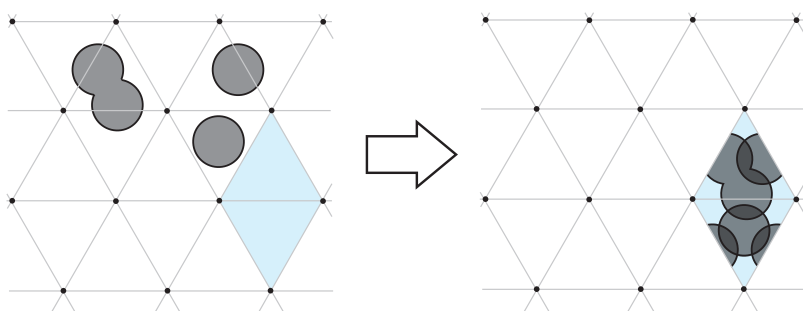

Figure 2.3: Translating portions of $\cup D$ to the fundamental cell (highlighted in blue).

Chapter 3

Basic Bounds

In this chapter we prove bounds of $c_2 \gtrapprox 1/3.37$ and $c_3 \gtrapprox 1/2.77$ and present algorithms to select and colour subsets of disks to achieve these bounds. The proofs in this chapter also serve as a stepping stone to the improved bounds that we will prove in Chapter 4.

The proofs presented in this chapter build on ideas from Rado's proof that $c_1 \gtrapprox 1/4.41$ [30]. In particular, our approach is similar in that we will use a regular lattice to guide disk selection, and we will make use of Lemma 2.1. Because of this, we will assume familiarity with Section 2.3.

The rest of this chapter is organized as follows. We begin by proving that $c_3 \gtrapprox 1/2.77$ in Section 3.1, since this proof shares the most similarity with Rado's proof. Section 3.2 proves a lemma required by our proof from Section 3.1. In Section 3.3 we prove that $c_2 \gtrapprox 1/3.37$. Finally, in Section 3.4 we present an $O(n^2)$ time deterministic algorithm and an $O(n)$ expected time algorithm to achieve the bounds on c_2 and c_3 presented in this chapter. We also prove that a lattice positioning step used in our deterministic algorithm is 3SUM-hard, providing evidence that this algorithm is optimal among algorithms that use such a lattice positioning approach.

3.1 A Bound of $c_3 \gtrapprox 1/2.77$

As mentioned above, our overall method of deriving bounds is similar to that of Rado. We position a regular lattice over the set of disks and, for each lattice point that falls in $\cup D$, we select a single disk containing that point. The side length of the lattice guarantees that our selection meets the required disjointness constraints, and we can bound the number of disks selected using Lemma 2.1. The key observation that allows us to derive a better bound for $k = 3$ than $k = 1$ is that the relaxed disjointness constraint allows us to use a finer lattice, in turn allowing us to select more disks. We will also use the lattice as a convenient way to assign colours to selected disks.

Theorem 3.1. *Let D be a collection of unit disks in the plane with union area A. For C a 3-coloured subset of D with same-coloured disks pairwise disjoint, let A_C denote C's union area. There exists a C such that $\frac{A_C}{A} \gtrapprox \frac{1}{2.77}$.*

Proof. We use a triangular lattice with side length $\frac{4\sqrt{3}}{3}$. The points of the lattice are 3-coloured such that no two lattice points of the same colour are adjacent. Now, for any placement of the lattice, we select a subset C of D as follows: for each lattice point p in the union of D, select a disk containing p and assign the disk the colour of p. The side length of the lattice ensures that no disk contains two lattice points so the selection and colouring are well-defined. It also ensures that disks assigned the same colour are pairwise disjoint (see Figure 3.1).

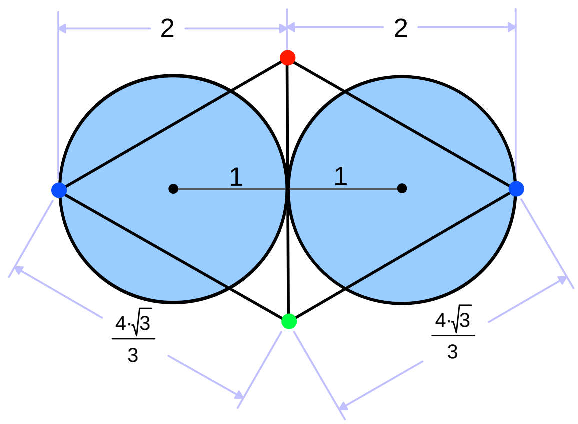

Figure 3.1: A finer 3-coloured lattice ensures that same-coloured selected disks are pairwise disjoint.

Also observe that, by Lemma 2.1, we can position the lattice to intersect the union area of D in at least $\frac{A\sqrt{3}}{8}$ points, so $|C| \geq \frac{A\sqrt{3}}{8}$. While same-coloured disks in C are pairwise disjoint, differently coloured disks may not be, so $|C|\pi$ is only an upper bound on A_C.

To derive a lower bound we will partition the union of C using the triangular lattice's Voronoi tessellation, which has regular hexagonal cells of side length $\frac{4}{3}$ and vertices at the barycenters of the triangular lattice (see Figure 3.2). Suppose disk $d \in C$ contains lattice point p which lies in hexagonal cell h. If we count only the area of $d \cap h$, and sum over all d, this gives a lower bound on A_C. Thus if we establish a lower bound Δ on the minimum possible area of $d \cap h$ then $A_C \geq |C|\Delta \geq \frac{A\sqrt{3}}{8}\Delta$. In Lemma 3.2, which we will prove in the next section, we show that $\Delta \approx 1.6645$. From the lower bound on A_C we reach our desired lower bound on c_3 of

$$c_3 \geq \frac{A_C}{A} = \frac{\sqrt{3}}{8}\Delta \approx 1/2.77$$

□

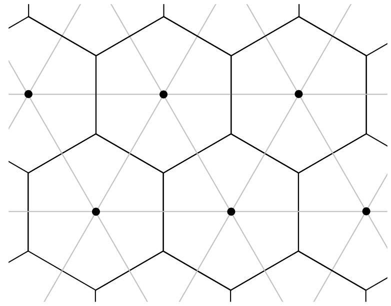

Figure 3.2: The Voronoi tessellation of a regular triangular lattice forms a grid of regular hexagonal cells.

3.2 Minimum Disk-Hexagon Intersection

Lemma 3.2. *Given a regular hexagon h with center point X and side length $\frac{4}{3}$, and any unit disk d containing point X, the minimum area of intersection Δ between h and d is approximately* 1.6645*, or more precisely*

$$\Delta \;=\; \frac{\sqrt{3}}{36}+\frac{\sqrt{11}}{12}+\frac{\pi}{2}-\frac{1}{2}\arctan\left(\frac{5\sqrt{3}-\sqrt{11}}{5+\sqrt{11}\sqrt{3}}\right)$$

Proof. Our first claim is that the minimum area of intersection is achieved by a disk d positioned such that X, the center point of hexagon h, lies on its boundary. Suppose this is not the case. By symmetry, it suffices to consider possible placements of Y, the center point of d, within the intersection of wedge BXK and the unit disk centered at X in Figure 3.3. For any position of Y, moving Y to the right along a line parallel to AB decreases the area of intersection, since the portion of $d-h$ lying above the supporting line of AB stays the same and the portion of $d-h$ below the supporting line of AB strictly increases (by containment). Thus we can move Y to the right until it lies either on XK or the boundary of the disk centered at X. For Y on XK, moving Y toward K decreases $d \cap h$ because when the diameter of d parallel to BC lies strictly inside h, the area of $d-h$ increases (by containment), and when the diameter is not strictly contained in h, the area of $d \cap h$ decreases (by containment).

Thus we can restrict our attention to the minimum area of intersection with h among disks whose boundary contains point X. To find this minimum, we assume that $X=(0,0)$ and express the area of intersection $f(\theta)$ in terms of angle θ between the center point B of a disk d, the center X of h, and the x-axis. There are two general cases to consider, illustrated in Figure 3.4. Case 1 occurs when d contains two vertices of h. Case 2 occurs when d only contains a single vertex of h. Note that in either case we can express the area of intersection as the sum of the area of a polygon and a circle sector. For instance, in

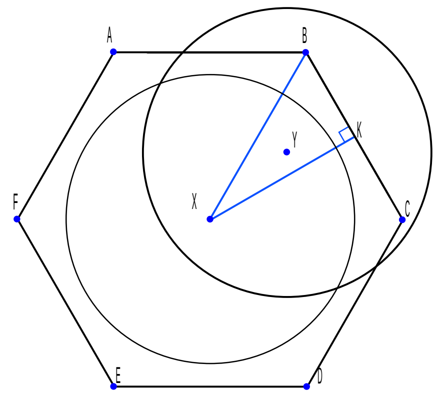

Figure 3.3: Sliding disk d outward from the center of h results in a smaller intersection between d and h.

Figure 3.4 (left) the area of intersection is the sum of the area of polygon $ABCED$ and the area of the sector of d interior to angle ABC.

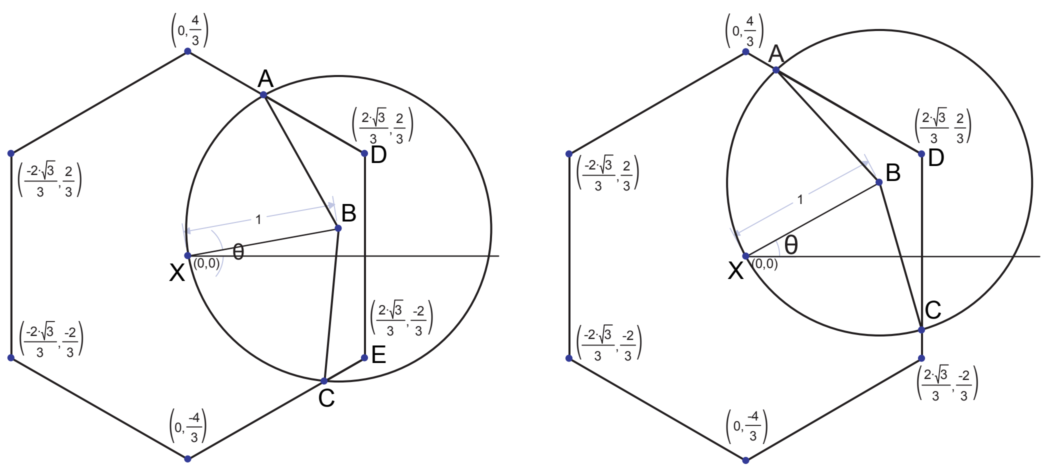

Figure 3.4: Calculating the intersection area as a function of angle θ. Case 1 (left) and Case 2 (right).

By symmetry, we need only consider the area of intersection for $0 \leq \theta \leq \frac{\pi}{6}$. As a result, we can use the two cases shown in Figure 3.4 to derive a formula for $f(\theta)$. Specifically, we use the symbolic geometry package Geometry Expressions to derive formulas relating θ and the intersection points between the boundaries of d and h (points A and C in Figure 3.4). See Appendix A for the derived formulas for these points, along with formulas for the other points in the two cases shown in Figure 3.4.

From these formulas we express the area of intersection in terms of θ using standard formulas for the area of polygons and circle sectors. This gives us a formula $f_1(\theta)$ for the area of intersection for $0 \leq \theta \leq \arccos(\frac{2}{3}) - \frac{\pi}{6}$ (that is, the region covered by Case 1)

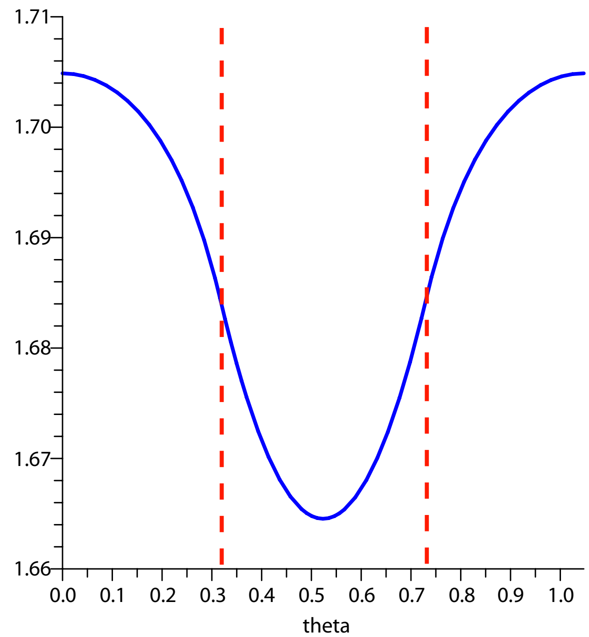

Figure 3.5: Plot showing the area of intersection for $0 \leq \theta \leq \frac{\pi}{3}$.

and a formula $f_2(\theta)$ for the area of intersection for $\arccos(\frac{2}{3}) - \frac{\pi}{6} \leq \theta \leq \frac{\pi}{6}$ (Case 2), see Appendix A Sections A.1 and A.2. By symmetry, we can use $f_1(\theta)$ and $f_2(\theta)$ to build a formula $f(\theta)$ for the area of intersection on the interval $0 \leq \theta \leq \frac{\pi}{3}$:

$$f(\theta) = \begin{cases} f_1(\theta) & \text{if } 0 \leq \theta < \arccos(\frac{2}{3}) - \frac{\pi}{6} \\ f_2(\theta) & \text{if } \arccos(\frac{2}{3}) - \frac{\pi}{6} \leq \theta < \frac{\pi}{6} \\ f_2(\frac{\pi}{3} - \theta) & \text{if } \frac{\pi}{6} \leq \theta < \arccos(\frac{2}{3}) \\ f_1(\frac{\pi}{3} - \theta) & \text{if } \arccos(\frac{2}{3}) \leq \theta \leq \frac{\pi}{3} \end{cases}$$

A plot showing $f(\theta)$ for the interval $0 \leq \theta \leq \frac{\pi}{3}$ is given in Figure 3.5. Using Maple we find that $f'(\theta)$ (the first derivative of $f(\theta)$) is 0 for $\theta = \frac{\pi}{6}$. Thus, by symmetry, the minimum intersection occurs when X, the center point of d and a vertex of of h are collinear. Computing the value of $f(\theta)$ at any one of these points gives our value for Δ, specifically

$$\begin{aligned} \Delta &= \frac{\sqrt{3}}{36} + \frac{\sqrt{11}}{12} + \frac{\pi}{2} - \frac{1}{2} \arctan\left(\frac{5\sqrt{3} - \sqrt{11}}{5 + \sqrt{11}\sqrt{3}}\right) \\ &\approx 1.6645 \end{aligned}$$

□

Our proof of Lemma 3.2 also shows that the lower bound $A_C \geq |C|\Delta$ from the proof of Theorem 3.1 is tight, as can be seen in the example in Figure 3.6 where the union of C is exactly partitioned by the hexagons and each disk intersects its hexagon in the minimum area Δ. Note, however, that this does not mean that our bound on c_3 is tight. Indeed, for the example shown in Figure 3.6 we could capture the whole area by 3-colouring the disks, and in Chapter 4 we prove a better bound on c_3.

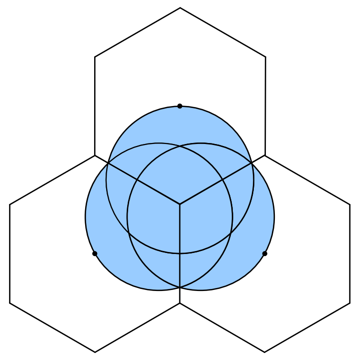

Figure 3.6: In this arrangement of selected disks, the lower bound on the contribution of each disk is realized.

3.3 A Bound of $c_2 \gtrapprox 1/3.37$

A straightforward corollary of Theorem 3.1 gives us a bound on c_2. Specifically, we can follow the construction used to prove Theorem 3.1, but then discard all disks of one colour. By discarding the colour with the least number of selected disks, we can guarantee that at least 2/3 of the selected disks remain, leading to a bound of $c_2 \gtrapprox \frac{2}{3} \cdot \frac{1}{2.77} \gtrapprox 1/4.16$. In this section we prove a better bound directly, but we will revisit this idea of proving a bound for a larger number of colours and then discarding some in Chapter 5, when we present methods to derive bounds for arbitrary k.

The proof of Theorem 3.3 is essentially the same as that of Theorem 3.1, but instead of using a 3-coloured regular triangular lattice, we use a 2-coloured square lattice.

Theorem 3.3. *Let D be a collection of unit disks in the plane with union area A. For C a 2-coloured subset of D with same-coloured disks pairwise disjoint, let A_C denote C's union area. There exists a C such that $\frac{A_C}{A} \gtrapprox \frac{1}{3.37}$.*

Proof. We use a regular square lattice with side length $2\sqrt{2}$. The points of the lattice are 2-coloured such that no two same-coloured lattice points are adjacent. For any placement of the lattice, we select and colour a subset C of D as in the proof of Theorem 3.1. The size of the lattice ensures that the selection and colouring are well-defined and disks assigned the same colour are pairwise disjoint (see Figure 3.7).

The fundamental cell of this regular square lattice is simply a $2\sqrt{2} \times 2\sqrt{2}$ square in the lattice, so by Lemma 2.1 we can position the lattice to intersect the union area of D in at

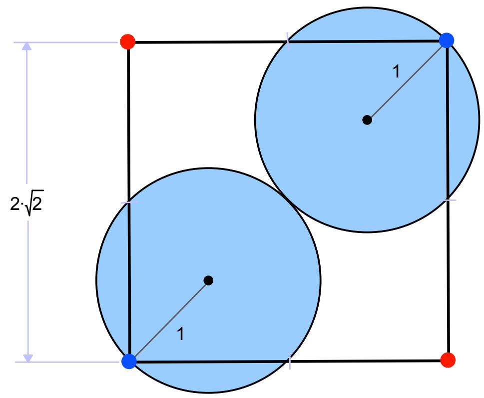

Figure 3.7: A 2-coloured lattice with side length $2\sqrt{2}$ ensures that same-coloured selected disks are pairwise disjoint.

least $\frac{A}{8}$ points and $|C| \geq \frac{A}{8}$. As in the proof of Theorem 3.1, differently coloured selected disks may not be pairwise disjoint and so we derive a lower bound on A_C by counting, for each selected disk, the minimum intersection Δ between a selected disk and the Voronoi cell of its selecting lattice point. By Lemma 3.4, proven below, $\Delta \approx 2.3749$ and we reach our desired lower bound on c_2 of

$$c_2 \geq \frac{A_C}{A} = \frac{1}{8}\Delta \approx 1/3.37$$

□

Lemma 3.4. *Given a square S with side length $2\sqrt{2}$ and center point X, and any unit disk d containing point X, the minimum area of intersection Δ between S and d is approximately 2.3749, or more precisely*

$$\Delta \quad = \pi - \left(\arccos\left(\sqrt{2}-1\right) - \left(\sqrt{2}-1\right)\sqrt{4-2\sqrt{2}-\left(2-\sqrt{2}\right)^2}\right)$$

Proof. Starting with the top left corner and proceeding in clockwise order, let the corner points of S be points A, B, D, and C. Let K be the midpoint of segment BD and let Y denote the center point of d (see Figure 3.8).

Our first claim is that the minimum area of intersection is achieved by a disk d positioned such that X, the center point of square S, lies on d's boundary. Suppose this is not the case. By symmetry, it suffices to consider possible placements of Y within the intersection of wedge BXK and a unit disk centered at X (see Figure 3.8). For any position of Y in this region, moving Y to the right along a line parallel to AB decreases the area of intersection, since the portion of $d - S$ lying above the supporting line of AB stays the same, and the portion of $d - S$ to the right of the supporting line of AB strictly increases (by containment). Thus we can move Y to the right until it lies on the boundary of the disk centered at X.

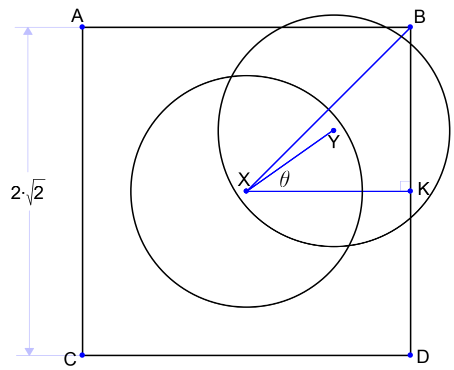

Figure 3.8: Sliding the disk d (centered at Y) to the right results in a smaller intersection between d and S.

Therefore we can restrict our attention to the minimum area of intersection with S among disks whose boundary contains point X. To find this minimum, we consider the area of intersection in terms of the interior angle θ between line segments YX and XK. Our claim is that the minimum intersection occurs when $\theta = 0$. Increasing θ on the interval $[0, \arcsin(\sqrt{2} - 1)]$ increases the intersection between d and S by decreasing the height of the circle segment to the right of the supporting line of BD. At $\theta = \arcsin(\sqrt{2} - 1)$ the circle segment above the supporting line of AB appears and as θ increases on the interval $[\arcsin(\sqrt{2}-1), \pi/4]$ the height of this circle segment increases while the height of the circle segment to the right of the supporting line of BD continues to decrease. Since the distance between X and B is 2, the circle segment above the supporting line of AB will always be disjoint with the circle segment to the right of the supporting line of BD and the area of intersection in terms of θ on this interval is given by

$$f(\theta) = \pi - s(1 - \sqrt{2} + \cos x) - s(1 - \sqrt{2} + \sin x)$$

where $s(h)$ is the function for the area of a circle segment with height h on a unit disk stated in Fact 3.5 below.

Fact 3.5. *The area $s(h)$ of a circle segment with height h on a unit-radius disk is given by*

$$s(h) = \arccos(1 - h) - (1 - h)\sqrt{2h - h^2}$$

Figure 3.9 shows a plot of $f(\theta)$ for $\arcsin(\sqrt{2}-1) \leq \theta \leq \pi/4$. Using Maple we find that $f'(\theta)$—the first derivative of $f(\theta)$—is negative on this interval except at $\pi/4$ where it is zero, demonstrating that the area of intersection continues to increase as θ increases on this interval. Therefore, the minimum area of intersection occurs when $\theta = 0$ and we get the desired value for Δ of

$$\begin{aligned}\Delta &= \pi - s(2-\sqrt{2}) \\ &= \pi - \left(\arccos\left(\sqrt{2}-1\right) - \left(\sqrt{2}-1\right)\sqrt{4-2\sqrt{2}-\left(2-\sqrt{2}\right)^2}\right)\end{aligned}$$

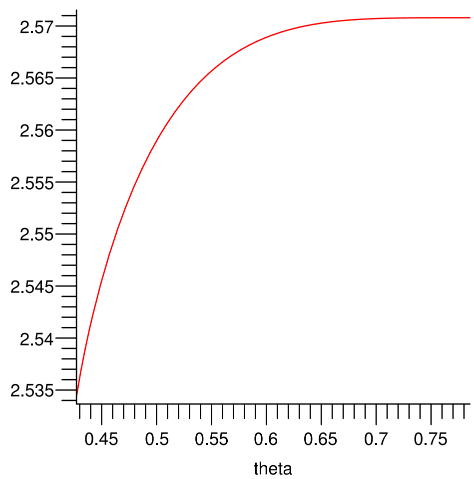

Figure 3.9: Plot showing the area of intersection between d and S for $\arcsin(\sqrt{2}-1) \leq \theta \leq \pi/4$.

□

3.4 Algorithms

In this section we discuss algorithms to select and colour subsets of a set of unit disks D to realize the bounds on c_2 and c_3 proven earlier in the chapter. Recall that both of our bound proofs were constructive and essentially the same, except for the type of lattice used to select disks. Our algorithms follow the constructive method used by these proofs. The constructions are straightforward to implement efficiently, with the exception of the step where we position the lattice such that the number of lattice points lying in $\cup D$ satisfies the bound prescribed by Lemma 2.1 (i.e., at least $\frac{A}{\alpha}$ points where α is the area of the fundamental cell for the lattice). We present several algorithms for solving this lattice positioning problem. We first give an $O(n^2)$ time exact algorithm which finds a translation of a given lattice which intersects $\cup D$ in the maximum number of points. Next, we prove that it is 3SUM-hard to determine whether a lattice positioning exists which intersects $\cup D$ in a given number of points, providing evidence that our exact algorithm is optimal. Finally, we present a randomized $O(n)$ expected time algorithm which finds a lattice positioning which intersects $\cup D$ in at least $\frac{A}{\alpha}$ points.

Theorem 3.6. *Given a regular lattice with a fundamental cell of area α, we can find a lattice positioning that intersects $\cup D$ in at least $\frac{A}{\alpha}$ points in $O(n^2)$ time.*

Proof. The proof of Lemma 2.1 is also constructive, and we follow its approach to design our lattice positioning algorithm. In the proof of Lemma 2.1 we initially position the lattice arbitrarily and then translate all cells in the lattice along with portions of $\cup D$ to a fundamental cell F, in order to find a point of maximum depth among the translated portions of $\cup D$. Thus the problem is reduced to capturing the translated portions of $\cup D$ so that we can compute a point of maximum depth. Our basic idea involves translating all of the disks and then computing and traversing their arrangement. For the lattices used to prove bounds in Theorems 3.1 and 3.3, no disk d intersects more than 4 translates of F. Thus we make up to 4 translated copies of d, and record which translate of F they came from. Note that the lattices used to prove Theorems 3.1 and 3.3 do not allow copies of the same disk to overlap in F. Computing this set of translated disks, call it D', takes $O(n)$ time. Computing $\mathcal{A}(D')$, the arrangement of D', takes $O(n^2)$ time using the incremental insertion algorithm of Chazelle and Lee [13].

It is now easy to traverse $\mathcal{A}(D')$ to compute maximum depth in D'—the depth increases when we enter a disk and decreases when we exit. However, this is not quite what we want; we want depth with respect to portions of $\cup D$ translated to F. This is different from depth in D' because of disks that originally overlapped in D. Our solution is to traverse $\mathcal{A}(D')$ while maintaining the current depth c_i in each translate i of F. Note that there are $O(n)$ translates of F that intersect $\cup D$ so the number of variables used to maintain depth will be linear in n. We also maintain a count c of the number of non-zero c_is. The face in the arrangement that attains the maximum value of c over all faces of $\mathcal{A}(D')$ is the desired cell, and translating the lattice such that a lattice point lies in this face gives us the desired lattice positioning. Since $\mathcal{A}(D')$ will have $O(n^2)$ cells and the extra overhead to keep track of depth with regard to each translate of F is negligible, the runtime of this algorithm is dominated by the time taken to traverse $\mathcal{A}(D')$ and is therefore in $O(n^2)$. □

We now prove that the lattice positioning problem discussed above is 3SUM-hard. A problem is *3SUM-hard* if it is harder than the problem of determining whether a set S of n integers contains three elements $a, b, c \in S$ such that $a + b + c = 0$. The best known algorithms for this problem take $O(n^2)$ and it is an open problem to do better [21].

Theorem 3.7. *The following problem is 3SUM-hard: Given an integer k and a set D of n unit disks in the plane, determine whether a given regular lattice can be positioned such that it intersects the union area of D in at least k points.*

Proof. We show that our problem is harder than the known 3SUM-hard problem of determining whether there is a point of depth k in a set of unit radius disks in the plane. The more general problem for variable radius disks was proven 3SUM-hard in [6] and the reduction is easily modified to produce unit radius disks.

Our reduction is as follows. Given a set D of unit radius disks in the plane, place a figure T whose shape matches that of the fundamental cell of the lattice large enough to contain all of D. Expand T to a lattice, and translate each disk of D to a different cell in the lattice that has the same orientation. Let the translated set of disks be D'. Then there is a point of depth k in D if and only if the lattice can be translated to intersect D' in k points. The reduction takes linear time. □

Theorem 3.7 provides evidence that the algorithm used to prove Theorem 3.6 is optimal among exact deterministic algorithms. However, if we are willing to sacrifice determinism or accept an approximate solution, we can devise faster algorithms. In Theorem 3.8 below we present a randomized algorithm with $O(n)$ expected runtime.

Theorem 3.8. *Given a regular lattice with a fundamental cell of area α, we can find a lattice positioning that intersects $\cup D$ in at least $\frac{A}{\alpha}$ points in $O(n)$ expected time.*

Proof. As in Theorem 3.6, we start by positioning our lattice arbitrarily over the set of disks and translating all portions of $\cup D$ to the fundamental cell. We then randomly select points on the fundamental cell and test their depth until we find a point which satisfies the bound prescribed by Lemma 2.1 (i.e., $\frac{A}{\alpha}$ points where α is the area of the fundamental cell for the lattice).

From the proof of Lemma 2.1 it is clear that the average depth of a point on the fundamental cell will be $\frac{A}{\alpha}$. Therefore, with high probability we will only need to test a constant number of points before finding one equal or greater to this depth. It takes $O(n)$ time to translate all disks to the fundamental cell, and $O(n)$ time to test the depth of a given point by iterating over all disks. Therefore, the expected runtime is in $O(n)$. □

Another option is to find a point of approximately maximum depth on the fundamental cell. For this problem, Afshani et al. [1] give a $(1 + \epsilon)$-factor approximation algorithm for finding a point of maximum depth in an arrangement of n disks with runtime $O(n \log n)$ in n and polynomial in $1/\epsilon$.

In the next chapter, we use a weighted version of the proof method used in this chapter to demonstrate improved bounds for c_2 and c_3. We also discuss the algorithmic issues involved in implementing this weighted approach.

Chapter 4

Weighted Bounds

In this chapter we build on the approach used in Chapter 3 and demonstrate improved bounds of $c_2 \gtrapprox \frac{1}{2.82}$ and $c_3 \gtrapprox \frac{1}{2.09}$. We also discuss the algorithmic issues involved in implementing the constructions used in these proofs, and present an $O(n)$ expected time randomized algorithm for selecting and colouring disks to achieve these bounds.

In proving bounds on c_2 and c_3 in Chapter 3 we only counted the intersection of a disk with its selecting Voronoi cell, and we took the minimum possible value for that intersection. If we commit to the first idea but try and improve on the second, then we should try to maximize the intersection of a disk with its selecting Voronoi cell. Clearly the maximum intersection occurs when the selecting lattice point is at the center of the selected disk. This suggests that we can use a weighting function that prefers placing a lattice point in the center of a disk to do a more intricate analysis of the contribution of each disk.

Looking at this from another direction, in our proofs from Chapter 3 we optimized the number of disks selected rather than the area of the intersection between selected disks and their selecting Voronoi cells. This approach can be improved because, among all subsets of disks that can be selected using a lattice, the largest subset of disks does not necessarily cover the largest area. For example, in Figure 4.1 selecting the three intersecting disks using the lattice positioning shown gives a subset with a union area of approximately 4.99 (specifically 3Δ where $\Delta \approx 1.6645$ as proven in Lemma 3.2), while an alternate lattice positioning selecting two disjoint disks gives a subset with union area $2\pi \approx 6.28$.

The above points suggest that we can improve our bounds from Chapter 3 by using a more sophisticated criterion for lattice positioning based on the area contributed by selected disks rather than the number of disks selected. In the following sections, we will use this approach to prove better bounds for c_2 and c_3, and in Chapter 5 we extend this approach to give bounds on c_k for any given k. We begin by proving our improved bound on c_3 in the next section.

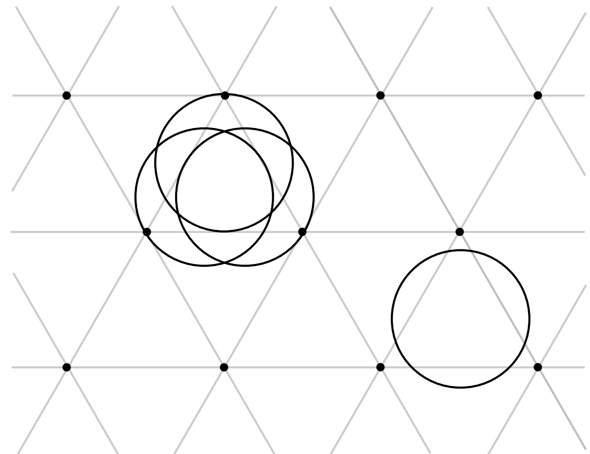

Figure 4.1: The lattice positioning shown selects the maximum number of disks, however an alternate lattice positioning selects fewer disks while covering more area.

4.1 A Bound of $c_3 \gtrapprox 1/2.09$

As mentioned above, we will stick with the idea of using a lattice to select subsets of disks that satisfy the required disjointness constraints. We also keep the accounting scheme of counting the intersection of a disk with its selecting Voronoi cell. The major difference from our approach in Chapter 3 is that we will derive a bound on the lattice positioning based on how much area is selected in this accounting scheme, instead of the number of selected disks. To implement this more sophisticated criterion for lattice positioning, we define a weight function $w : \mathbb{R}^2 \to \mathbb{R}_{\geq 0}$ defined over all points on the plane and use the following, weighted version of Lemma 2.1 to give a lower bound on the maximum value of the weight of a lattice placement $W(L)$.

Lemma 4.1. *Given a weight function $w : \mathbb{R}^2 \to \mathbb{R}_{\geq 0}$, let G be a bounded region of the plane with a total weight $B = \int_{x \in G} w(x)$. A regular lattice with a fundamental cell of area α can always be positioned such that the sum of the weights at intersection points between the lattice and G is at least $\frac{B}{\alpha}$.*

Proof. As in the proof of Lemma 2.1 we consider translating portions of G onto a fundamental cell. After translation, each point p in the fundamental cell is assigned the sum of the weights of p's intersection with each translated portion of G. Supposing that a point p in the fundamental cell has weight m, then positioning L such that p is a lattice point ensures that $W(L) = m$. Thus, if we can prove that the total weight of the portions of G translated to the fundamental cell is B, then a point in the fundamental cell (and therefore a lattice positioning) with weight at least $\frac{B}{\alpha}$ must exist. □

We are now ready to present our improved bound on c_3.

Theorem 4.2. *Let D be a collection of unit disks in the plane with union area A. For C a 3-coloured subset of D with same-coloured disks pairwise disjoint, let A_C denote C's union area. There exists a C such that $\frac{A_C}{A} \gtrapprox \frac{1}{2.09}$ and thus $c_3 \gtrapprox \frac{1}{2.09}$.*

Proof. We'll start by defining our weight function. For point p, let $H(p)$ be a regular hexagon of side length $\frac{4}{3}$ centered at p such that p and a vertex of $H(p)$ lie on a line parallel to the y-axis. Thus $H(p)$ is the Voronoi cell of p if our triangular lattice is translated to include point p. For $p \in \cup D$, let $d(p)$ be the disk in D containing p whose intersection with $H(p)$ has maximum area. Now let $w(p) = area(H(p) \cap d(p))$ for $p \in \cup D$, $w(p) = 0$ otherwise. Then $w(p)$ measures the area contributed by including p in the lattice (given our method of choosing disks using the lattice, and our accounting scheme of counting only the area of the disk in the Voronoi cell).

We want to choose a lattice L to maximize $W(L) = \sum_{p \in L} w(p)$. Establishing a lower bound on the maximum value of $W(L)$ for lattices of the type used in our proof of Theorem 3.1 will give us a lower bound on c_3. We can apply Lemma 4.1 to get such a lower bound, provided we can find B, the total weight of $\cup D$.

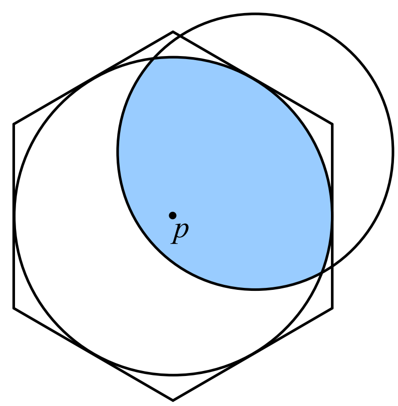

Figure 4.2: Considering the intersection between a selected disk and the largest disk that can be contained in the disk's Voronoi cell provides a simple lower bound for $w(p)$ in terms of distance from p to the nearest disk center.

The value of B will be the integral of $w(p)$ over $p \in \cup D$. For our chosen $w(p)$ this is difficult to compute exactly, but we can obtain a lower bound. Our first step is to replace the true weight function $w(p)$ with a lower bound $w_l(p)$ in which we replace the hexagon by its inscribed circle (see Figure 4.2). Specifically, $w_l(p)$ is the area of the intersection of two disks: a disk of radius $2/\sqrt{3}$ centered at p, and the unit disk in D whose center is closest to p. Note that $w_l(p)$ depends only on the distance, r, from p to the nearest disk center in D. We will overload the notation and define $w_l(r)$, for $r \in [0, 1]$, to be the area of the intersection of a unit disk and a disk of radius $2/\sqrt{3}$ whose centers are distance r apart. We can then write the following explicit formula for $w_l(r)$.

$$w_l(r) = \begin{cases} \pi & \text{if } 0 \le r \le \frac{2}{\sqrt{3}} - 1 \\ \beta(r) & \text{if } \frac{2}{\sqrt{3}} - 1 < r \le 1 \end{cases}$$

where $\beta(r)$ is given by

$$\beta(r) = \arccos\left(\frac{1}{2}\,\frac{r^2 - \frac{1}{3}}{r}\right) + \frac{4}{3}\,\arccos\left(\frac{1}{4}\,\frac{\left(r^2 + \frac{1}{3}\right)\sqrt{3}}{r}\right) -$$
$$\frac{1}{2}\sqrt{\left(-r + 1 + \frac{2}{\sqrt{3}}\right)\left(r + 1 - \frac{2}{\sqrt{3}}\right)\left(r - 1 + \frac{2}{\sqrt{3}}\right)\left(r + 1 + \frac{2}{\sqrt{3}}\right)}$$

We will express B as an integral in terms of variable r. Note that the points on the boundary of $\cup D$ are precisely the points at distance 1 from the closest disk center. More generally, we can capture the points that are distance r from the closest disk center as follows. For each unit disk $d \in D$, let d_r be a disk of radius r at the same center. Let D_r be $\{d_r : d \in D\}$. Then the points that are distance r from the closest disk center are precisely the points on the boundary of $\cup D_r$. Let $p(r)$ be the length of the boundary of $\cup D_r$. As $w(r) \ge w_l(r) \ge 0$, we obtain:

$$B = \int_0^1 p(r)w(r)\,dr \ge \int_0^1 p(r)w_l(r)\,dr$$

We prove a lower bound on the latter integral that eliminates $p(r)$.

Lemma 4.3. *For $\gamma(r)$ a continuous non-increasing function with a piecewise continuous differential $\gamma'(r)$,*

$$\int_0^1 p(r)\gamma(r)\,dr \ge 2A \int_0^1 r\gamma(r)\,dr$$

Since $w_l(r)$ is clearly a continuous non-increasing function with a piecewise continuous derivative, we can plug $w_l(r)$ into Lemma 4.3 to get $B \ge \int_0^1 p(r)w_l(r)\,dr \ge 2A \int_0^1 rw_l(r)\,dr$. Substituting the expression for $w_l(r)$ and evaluating the integral using Maple we obtain a lower bound on B of $2.207A$. Therefore, by Lemma 4.1 a lattice positioning with weight at least $\frac{\sqrt{3}}{8}B \gtrapprox \frac{A}{2.09}$ must exist, and we arrive at our desired lower bound of $c_3 \gtrapprox \frac{1}{2.09}$. $\square$

It remains to prove Lemma 4.3.

Proof of Lemma 4.3. We want to prove $\int_0^1 p(r)\gamma(r)\,dr \geq 2A\int_0^1 r\gamma(r)\,dr$. Equivalently, we want to prove that

$$\int_0^1 [p(r) - 2rA]\gamma(r)\,dr \geq 0$$

Let $A(r)$ be a function for the area of $\cup D_r$, which is related to the perimeter by the fact that $p(r) = A'(r)$. Define $f(r) = A(r) - r^2 A$. Recall that $A = A(1)$ is the area of $\cup D$. Note that $f(0) = f(1) = 0$. Now $f'(r) = A'(r) - 2rA = p(r) - 2rA$, and the inequality we want to prove is transformed into $\int_0^1 f'(r)\gamma(r)\,dr \geq 0$.

We apply integration by parts, noting that f' is continuous, and that γ' is piecewise continuous.

$$\int_0^1 f'(r)\gamma(r)\,dr \quad = \quad f(r)\gamma(r)\Big|_0^1 - \int_0^1 f(r)\gamma'(r)\,dr \quad = \quad -\int_0^1 f(r)\gamma'(r)\,dr$$

We will prove below that $A(r) \geq r^2 A$. Thus $f(r) \geq 0$ for all $r \in [0,1]$. Recall that $\gamma(r)$ is a non-increasing function, so $\gamma'(r) \leq 0$ for all $r \in [0,1]$. The integral of a negative function is negative, and this completes the proof. $\square$

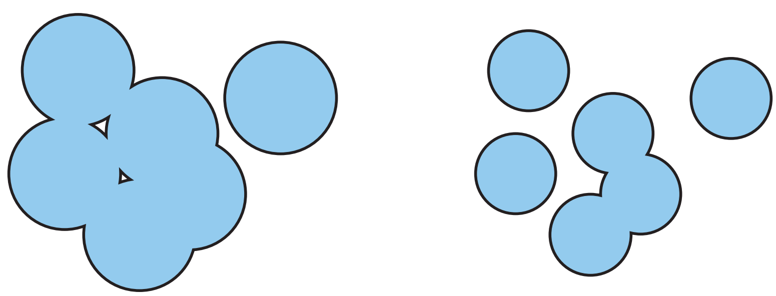

Figure 4.3: The union of a set of disks before (left) and after a radius scaling has been applied (right).

Claim 4.1. *Given a collection of unit disks with union area A, if we scale the radius of each disk by $r \in [0,1]$ then the union area of the scaled disks will be at least $r^2 A$.*

Proof. We want to scale the radius of each disk by r. We accomplish this in two steps. First we scale the whole plane by a factor of r. This reduces the area to $r^2 A$. It also decreases the distance between the centers of any two disks by a factor of r. The second step is to translate each scaled disk back to its original position. During the translation the distance between the centers of any two disks increases continuously. By a result of Bollobás [11], the union area of a set of congruent disks moving apart from one another continuously cannot decrease, and therefore the area of the final arrangement of scaled disks is at least $r^2 A$. □

4.2 A Bound of $c_2 \gtrapprox 1/2.82$

As Theorem 4.2 was a weighted version of Theorem 3.1, Theorem 4.4 presented in this section is a weighted version of Theorem 3.3. As in Theorem 3.3 we use a 2-coloured regular square lattice with side length $2\sqrt{2}$, and derive a bound on the union area of selected disks based on a weight function defined on $\cup D$.

Theorem 4.4. *Let D be a collection of unit disks in the plane with union area A. For C a 2-coloured subset of D with same-coloured disks pairwise disjoint, let A_C denote C's union area. There exists a C such that $\frac{A_C}{A} \gtrapprox \frac{1}{2.82}$.*

Proof. As in the proof of Theorem 4.2, we start by defining our weight function. For point p, let $S(p)$ be the $2\sqrt{2} \times 2\sqrt{2}$ square centered at p. Thus $S(p)$ is the Voronoi cell of p if our square lattice is translated to include point p. For $p \in \cup D$, let $d(p)$ be the disk containing p whose intersection with $S(p)$ has maximum area. Let $w(p) = area(S(p) \cap d(p))$ for $p \in \cup D$, $w(p) = 0$ otherwise. Then $w(p)$ measures the area contributed by including p in the lattice (given our method of choosing disks using the lattice, and our accounting scheme of counting only the area of a disk contained in its selecting Voronoi cell).

We want to compute a lower bound on B—the integral of $w(p)$ over $p \in \cup D$—and then apply Lemma 4.1. Recall that our approach in Theorem 4.2 was to lower bound $w(p)$ with the function $w_l(p)$ where we replaced the Voronoi cell with its largest inscribed circle. Since the value of $w_l(p)$ depended only on the distance from p to the nearest disk center in D, we overloaded the notation to $w_l(r)$ for $r \in [0, 1]$. Then $w_l(r)$ gave a lower bound on the weight of any point $p \in \cup D$ with distance r to the nearest disk center and we applied Lemma 4.1 with $w_l(r)$ to compute a lower bound on B. Following the same approach with a 2-coloured square lattice leads to a bound of $c_2 \gtrapprox 1/2.95$. We derive a better bound here by following a similar approach but using a different method to lower bound $w(p)$.

As mentioned above, $w(p)$ measures the intersection area of a $2\sqrt{2} \times 2\sqrt{2}$ square centered at p with some unit disk $d(p)$ containing p. Since point p is specified by (a) the distance r between p and the center point of $d(p)$ and (b) the interior angle θ between the x-axis and the segment joining these two points, we can replace $w(p)$ with $w(r, \theta)$. We then lower bound $w(r, \theta)$ with a new function $w_s(r)$ by minimizing over θ (i.e., $w_s(r) = \min_\theta w(r, \theta)$). Thus $w_s(r)$ gives a lower bound on the weight of any point $p \in \cup D$ with distance r to the nearest disk center.

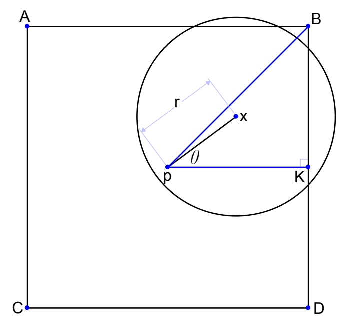

Figure 4.4: Exploring the intersection between a unit disk centered at x, and a $2\sqrt{2} \times 2\sqrt{2}$ square centered at p.

Lemma 4.5. *Using the above notation, for any $r \in [0, 1]$, the minimum value of $w(r, \theta)$ occurs at $\theta = 0$.*

Proof. Let S be a $2\sqrt{2} \times 2\sqrt{2}$ square with center point p. Starting with the top left corner and proceeding in clockwise order, let the corner points of S be denoted A, B, D, and C. Let d be a unit disk whose center point x is distance r from p. Let K be the midpoint of segment BD. Then θ as discussed above will be the interior angle between segments xp and pK (see Figure 4.4).

By symmetry we need only consider $0 \leq \theta \leq \pi/4$. We will refer to the circle segment of the disk lying above the supporting line of AB as the *top circle segment* and the circle segment of the disk lying to the right of the supporting line of BD as the *right circle segment*. Line segment pB has length 2, so for all $r \in [0, 1]$ and $\theta \in [0, \pi/4]$ the top and right circle segments will always be disjoint and will be the only portions of d lying outside of S. Let $s_{top}(r, \theta)$ and $s_{right}(r, \theta)$ respectively denote the area of the top and right circle segments for a given $r \in [0, 1]$ and $\theta \in [0, \pi/4]$. Using the function $s(h)$ for the area of a circle segment with height h on a unit disk as stated in Fact 3.5, we can write out the formulas for s_{top} and s_{right} as follows

$$s_{top}(r, \theta) = \begin{cases} s(1 + r\sin\theta - \sqrt{2}) & \text{if } 1 + r\sin\theta - \sqrt{2} \geq 0 \\ 0 & \text{if } 1 + r\sin\theta - \sqrt{2} < 0 \end{cases}$$

$$s_{right}(r, \theta) = \begin{cases} s(1 + r\cos\theta - \sqrt{2}) & \text{if } 1 + r\cos\theta - \sqrt{2} \geq 0 \\ 0 & \text{if } 1 + r\cos\theta - \sqrt{2} < 0 \end{cases}$$

Using $s_{top}(r, \theta)$ and $s_{right}(r, \theta)$ we can write the formula for $w(r, \theta)$, giving the area of intersection between d and S for a given $r \in [0, 1]$ and $\theta \in [0, \pi/4]$.

$$w(r, \theta) = \pi - s_{top}(r, \theta) - s_{right}(r, \theta)$$

We will prove that, for all $r \in [0, 1]$, $w(r, 0) \leq w(r, \theta)$ for all $\theta \in [0, \pi/4]$. We do so by considering a number of cases corresponding to possible values of r.

- $r \in [0, \sqrt{2} - 1)$: In this case d will be completely contained within S for all θ so it is trivially true that $w(r, \theta)$ attains a minimum at $\theta = 0$.
- $r \in [\sqrt{2} - 1, 2 - \sqrt{2})$: In this range $s_{top}(r, \theta)$ will always be 0, since $1 + r \sin \theta - \sqrt{2}$ (the height of the top circle segment) is less than 0 for $r < 2 - \sqrt{2}$ and all $\theta \in [0, \pi/4]$. Moreover, as θ increases on the interval $0 \leq \theta \leq \pi/4$ the height (and thus area) of the right circle segment will decrease, increasing the area of intersection between d and S. Thus $w(r, \theta)$ attains a minimum at $\theta = 0$ for this case as well.
- $r \in [2 - \sqrt{2}, 1]$: For $0 \leq \theta < \arcsin\left(\frac{\sqrt{2}-1}{r}\right)$ we have a similar situation to the previous case, where there is no top circle segment and $s_{right}(r, \theta)$ decreases as θ increases. At $\theta = \arcsin\left(\frac{\sqrt{2}-1}{r}\right)$ the top circle segment appears and its height increases with θ on the interval $\arcsin\left(\frac{\sqrt{2}-1}{r}\right) \leq \theta \leq \pi/4$. We will show that $s_{right}(r, \theta)$ decreases faster than $s_{top}(r, \theta)$ increases on this interval and thus the area of intersection between d and S will continue to increase with θ on this interval.

  Let $\lambda_{top}(r, \theta)$ and $\lambda_{right}(r, \theta)$ be the respective partial derivatives of $s_{top}(r, \theta)$ and $s_{right}(r, \theta)$ with respect to θ. Then $\lambda_{top}(r, \theta)$ and $\lambda_{top}(r, \theta)$ are given by the equations below

$$\lambda_{top}(r, \theta) = \frac{\partial s_{top}(r, \theta)}{\partial \theta} = 2r \cos \theta \sqrt{2r\sqrt{2} \sin \theta - r^2 \sin^2 \theta - 1}$$

$$\lambda_{right}(r, \theta) = \frac{\partial s_{right}(r, \theta)}{\partial \theta} = -2r \sin \theta \sqrt{2r\sqrt{2} \cos \theta - r^2 \cos^2 \theta - 1}$$

  We want to prove that $\lambda_{top}(r, \theta) \leq -\lambda_{right}(r, \theta)$, for all $r \in [2 - \sqrt{2}, 1]$ and $\theta \in [\arcsin(\frac{\sqrt{2}-1}{r}), \pi/4]$. Simplifying, we get the following inequality:

$$2r\sqrt{2} \left(\sin \theta \cos^2 \theta - \cos \theta \sin^2 \theta\right) + \sin^2 \theta - \cos^2 \theta \leq 0$$

  Since $\sin^2 \theta - \cos^2 \theta \leq 0$ for $0 \leq \theta \leq \pi/4$, this will be true for all $r \in [2 - \sqrt{2}, 1]$ if it is true when $r = 1$. Substituting in $r = 1$ and simplifying to remove all occurrences of $\sin \theta$ yields the following inequality:

$$16\cos^6\theta - 8\sqrt{2}\cos^5\theta - 20\cos^4\theta + 12\sqrt{2}\cos^3\theta + 4\cos^2\theta - 4\sqrt{2}\cos\theta + 1 \geq 0$$

Using Maple to find zeros for the left hand side of the above inequality, we find that there is a zero at $\theta = \pi/4$ and no other zeroes on the interval $0 \leq \theta \leq \pi/4$. Further, the left hand side is greater than or equal to zero for any $\theta \in [0, \pi/4)$ and thus the inequality holds on this region. Therefore, the area of intersection between d and S continues to increase on the interval $\arcsin\left(\frac{\sqrt{2}-1}{r}\right) \leq \theta \leq \pi/4$ and $w(r,\theta)$ attains a minimum at $\theta = 0$ in this case as well.

The cases above show that, for all $r \in [0,1]$, a minimum area intersection between d and S occurs at $\theta = 0$ and this completes our proof.

□

By Lemma 4.5, for all $r \in [0,1]$, $w(r,0) \leq w(r,\theta)$ and we can let $w_s(r) = w(r,0)$. We observe that $w_s(r)$ is a continuous non-increasing function of r with a piecewise continuous derivative, and that we can write the following explicit formula for it

$$w_s(r) = \begin{cases} \pi & \text{if } 0 \leq r \leq \sqrt{2}-1 \\ \\ \pi - s(r+1-\sqrt{2}) & \text{if } \sqrt{2}-1 < r \leq 1 \end{cases}$$

where $s(h)$ is the function for the area of a circle segment with height h on a unit disk as stated in Fact 3.5.

Applying Lemma 4.3 with $w_s(r)$ and evaluating the integral using Maple, we obtain a lower bound on B of approximately $2.834A$. Then by Lemma 4.1 a lattice positioning must exist with weight at least $\frac{1}{8}B \gtrapprox A/2.82$ and $c_2 \gtrapprox \frac{1}{2.82}$. □

4.3 Algorithmic Issues

The proofs of Theorems 4.2 and 4.4 are constructive, so an obvious question is whether they can be implemented efficiently, as with the constructions of Theorems 3.1 and 3.3 discussed in Section 3.4. As with the constructions of Theorems 3.1 and 3.3, the main algorithmic issue is the lattice positioning step. For the problem of finding a lattice positioning of maximum weight, we present evidence that a deterministic exact algorithm is unlikely to exist. However, we also present a simple randomized algorithm with $O(n)$ expected runtime that finds a lattice positioning with weight at least that prescribed by Lemma 4.1.

Finding a maximum weight lattice positioning appears to be difficult

The weighted lattice positioning problem can be broken into two parts: finding the weight of each point in $\cup D$ and finding a lattice positioning that maximizes the sum of weights at all lattice points. For the first part, recall that ideally we want the weight of a candidate lattice point to reflect exactly how much area the best disk d selectable by that point will contribute to the union area of all selected disks. This is difficult to determine since differently coloured selected disks may intersect, and so we settled on a scheme of counting the intersection area of d with the lattice Voronoi cell of its selecting point. We further simplified this in the proofs of Theorems 4.2 and 4.4 to a lower bound function $\gamma(\cdot)$ on the intersection area of d with the lattice Voronoi cell of the candidate point. In both Theorems 4.2 and 4.4 $\gamma(\cdot)$ depends only on the distance between the candidate lattice point and the center point of a selected disk and is a monotone non-increasing function. This makes it easy to decide which disk to select for a given candidate lattice point: we always select the disk whose center point is closest among disks containing the candidate lattice point. In this way, for our lower bound on the weight function, it is easy to compute the weight of any single candidate lattice point p in $\cup D$: the weight is simply $\gamma(r)$ where r is the distance between p and the center of the nearest disk containing p.

The second part of the lattice positioning problem is more difficult. Here we need to find a lattice positioning that maximizes the total weight of all lattice points in $\cup D$. Equivalently, we need to find a point of maximum weight in the fundamental cell after translating all portions of $\cup D$ there. This appears to be a very challenging problem. Consider a situation where all disks in D are disjoint but all overlap when translated to the fundamental cell. In this case, we need to find a point x in the fundamental cell to solve a problem of the form: maximize $\sum_{p \in P} \gamma(dist(x, p))$ where P is the set of center points of disks in D and $\gamma(\cdot)$ is a monotone non-increasing function. This is similar in many ways to the Weber point problem in two dimensions, which is not encouraging.

In the Weber point problem, given a set of points P we are asked to find a point $M(P)$ (sometimes called a Weber point) such that $\sum_{p \in P} -dist(M(P), p)$ is maximized. This problem has been studied extensively, and has proven to be quite challenging (for background on the problem see [17]). In general, no polynomial algorithm has been discovered, nor has the problem been shown to be NP-hard [25]. Moreover, for the Weber point problem in two or more dimensions, it has been proven that there is no exact algorithm for finding the location of $M(P)$ under models of computation where the root of an algebraic equation is obtained using arithmetic operations and the extraction of kth roots [8].

In some ways the Weber point problem in two dimensions is simpler than our problem of finding a point of maximum weight on the fundamental cell. In both cases we are seeking a point of maximum weight where the weight of any point p in the plane is a function of the distance between p and each of a set of points in P (respectively, the center points of the disks in D). Whereas the Weber point problem uses the simplest such weight function—simply the sum of distances to the input points themselves—our weight function is more complicated, involving the intersecting area of disks. It might be argued that our weight function falls to 0 fairly quickly, simplifying our problem. However, translating disks to the respective fundamental cells of the regular lattices used in our proofs of Theorems 3.1 and 3.3 forces center points of translated disks to be at most distance 4 apart, mitigating the advantage this might give us.

We feel that the above arguments provide strong evidence that finding a maximum-weight lattice positioning is difficult, and efficient exact algorithms are unlikely to exist for all but simple arrangements of disks. On a more encouraging note, approximation algorithms have been successful for solving the Weber point problem. Recent work by Bose et al. [12] has given a deterministic ϵ-approximation algorithm with runtime $O(n \log n)$ in terms of n and polynomial in $1/\epsilon$. As well, Bose et al. [12] and Indyk [29] have presented randomized ϵ-approximation algorithms with running times linear in n and polynomial in $1/\epsilon$.

In the next section, we give a simple randomized algorithm with $O(n)$ expected runtime that finds a lattice positioning with weight at least that prescribed by Lemma 4.1.

Finding a lattice positioning to satisfy Lemma 4.1 in $O(n)$ expected time

If we are not concerned with finding a lattice positioning of maximum weight, we can find a lattice positioning of at least the weight prescribed by Lemma 4.1 quickly using a simple randomized approach.

Theorem 4.6. *Given a regular lattice with a fundamental cell of area α, and a set of unit disks D with total weight $B = \int_{p \in \cup D} w(p)$, we can find a lattice positioning in $O(n)$ expected time such that the sum of weights at intersection points between the lattice and $\cup D$ is at least B/α.*

Proof. The proof is essentially the same as that of Theorem 3.8. We first position our lattice arbitrarily over the set of disks and translate portions of $\cup D$ to the fundamental cell. We then test the weight of randomly selected points on the fundamental cell until we find a point with weight at least $\frac{B}{\alpha}$.

From the proof of Lemma 4.1 it is clear that the average weight of a point on the fundamental cell will be $\frac{B}{\alpha}$. Therefore, with high probability we will only need to test a constant number of points before finding one equal or greater than this weight. We can compute the weight of any given point in $O(n)$ time by iterating over all disks, and therefore the expected runtime is in $O(n)$. □

Chapter 5

Extension to k Colours

In this chapter we build on the concepts introduced for $k \leq 3$ and present two methods of demonstrating general bounds for any fixed k. Up to this point we have focused on cases where $k \leq 3$ colours are available. As mentioned in Section 2.1, this is relevant for channel assignment in Wi-Fi networks where the use of three or fewer channels is a standard practice. However, other wireless communication systems such as GSM (Global System for Mobile communication) operate on a larger frequency range than Wi-Fi and have more available channels [27]. As well, disregarding the wireless networking motivation, the question of what bounds can be proven for arbitrary k is interesting in its own right.

5.1 k-Colour Bounds

The first method that we will present is a generalization of the proof method used for $k = 3$ in Chapter 4. The second method follows a different approach from what we have seen so far.

Both methods presented in this section use a coloured regular triangular lattice to guide disk selection and colouring, though the lattice is used differently in each. As in the proofs presented in previous chapters, a key requirement is a regular lattice whose points are coloured to guarantee some minimum distance between those assigned the same colour. We start by identifying a subset of values for k for which lattices with this desirable property exist.

Lemma 5.1. *For all $k \in \{i^2 + ij + j^2 \mid i, j \in \mathbb{N}\}$, and any given distance m we can k-colour a regular triangular lattice of side length $\frac{m}{\sqrt{k}}$ such that same-coloured lattice points are at least distance m apart.*

Proof. Consider the unit triangular lattice L_{unit} and suppose we designate one point in the lattice as the origin O. Standing at the origin and walking i units along the lattice in one direction, we will arrive at another lattice point Y. If we then turn $\pi/3$ radians and walk

j units along the lattice in this new direction we will arrive at another lattice point X. Clearly the distance from X back to the origin will be $\sqrt{i^2+ij+j^2}$ (see Figure 5.1) and the supporting line of segment OX will intersect lattice points at regular $\sqrt{i^2+ij+j^2}$ intervals. Therefore for all $k \in \{i^2+ij+j^2 \mid i,j \in \mathbb{N}\}$, distance $\sqrt{k}$ occurs in the unit triangular lattice and by $\frac{\pi}{3}$ rotational symmetry, an entire sublattice with side length $\sqrt{k}$ exists. Each triangle in the sublattice will have area $k\frac{\sqrt{3}}{4}$ and each triangle in the unit lattice will have area $\frac{\sqrt{3}}{4}$ so we can partition L_{unit} into k triangular sublattices of side length $\sqrt{k}$ and assign each a unique colour. Finally, scaling L_{unit} by a factor of $\frac{m}{\sqrt{k}}$ gives us the desired lattice. □

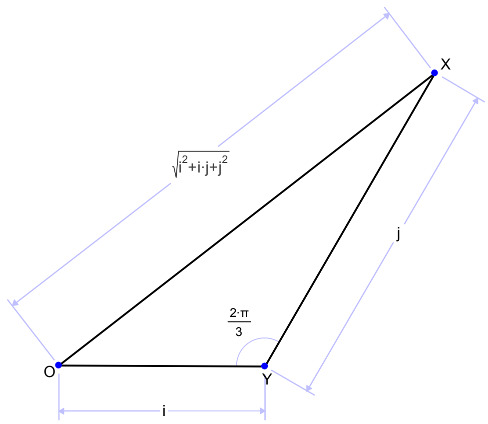

Figure 5.1: Distance $\sqrt{i^2+ij+j^2}$ occurs in the unit triangular lattice.

Given a distance m and a number of colours k, Lemma 5.1 tells us that a k-coloured regular triangular lattice exists containing k sublattices of side length m so long as $k \in \{i^2+ij+j^2 \mid i,j \in \mathbb{N}\}$. Using this, Theorem 5.2 generalizes the proof method used in Theorem 4.2 to demonstrate a bound for any k such that $k \in \{i^2+ij+j^2 \mid i,j \in \mathbb{N}\}$.

Theorem 5.2. *Given k colours, where $k \in \{i^2+ij+j^2 \mid i,j \in \mathbb{N}\}$ there exists a k-coloured subset C of disks such that same-coloured disks are disjoint. Let A_C denote C's union area, then*

$$c_k \geq \frac{A_C}{A} \geq \frac{k}{4\sqrt{3}} \int_0^1 rg\left(r, 1, \frac{2}{\sqrt{k}}\right) dr$$

where $g(d_0, r_1, r_2)$ is a function for the area of intersection between two disks of respective radii r_1 and r_2 and distance d_0 between their center points (stated in Fact 5.3 below).

Proof. As mentioned above, our proof is a straightforward generalization of the proof of Theorem 4.2 from Section 4.1. By Lemma 5.1, we know that a k-coloured regular triangular lattice exists with distance m between same-coloured lattice points and side length $\frac{m}{\sqrt{k}}$.

By choosing $m = 4$ (twice the diameter of a unit disk) we ensure that disks selected and coloured using this lattice will satisfy the required disjointness constraints. Using this lattice, we proceed as in the proof of Theorem 4.2 and define a weight function. For point p, let $H(p)$ be the regular hexagon of side length $\frac{4}{\sqrt{3k}}$ centered at p. $H(p)$ is the Voronoi cell of p if our lattice was translated to include point p. For $p \in \cup D$ let $d(p)$ be the disk containing p whose intersection with $H(p)$ has maximum area. Let $w(p) = area(H(p) \cap d(p))$ for $p \in \cup D$, $w(p) = 0$ otherwise. Then $w(p)$ measures the area contributed by including p in the lattice (given our method of choosing disks based on the lattice, and our accounting scheme of counting only the area of the disk in the Voronoi cell).

We want to compute a lower bound on B—the integral of $w(p)$ over $p \in \cup D$—and then apply Lemma 4.1. As in the proof of Theorem 4.2 we will lower bound $w(p)$ with a function $w_l(p)$ which replaces the hexagonal Voronoi cell $H(p)$ with its largest inscribed circle. Since the value of $w_l(p)$ only depends on the distance from p to the nearest disk center in D, we overload the notation to $w_l(r)$ for $r \in [0, 1]$. The inscribed circle within $H(p)$ will have radius $\frac{2}{\sqrt{k}}$, and so $w_l(r) = g\left(r, 1, \frac{2}{\sqrt{k}}\right)$ where $g(d_0, r_1, r_2)$ is a function for the area of intersection between two disks of respective radii r_1 and r_2 and distance d_0 between their center points as stated in Fact 5.3 below.

Fact 5.3. *The area of intersection $g(d_0, r_1, r_2)$ between two disks with respective radii r_1 and r_2 and distance d_0 between their center points is given by the formula below. We use a standard formula $\beta(d_0, r_1, r_2)$ for the area of intersection between two disks whose boundaries intersect, and then take into account the cases where one disk is completely contained in the other, or the two disks are disjoint.*

$$g(d_0, r_1, r_2) = \begin{cases} \min\{\pi {r_1}^2, \pi {r_2}^2\} & \textit{if } 0 \le d < \max\{r_1, r_2\} - \min\{r_1, r_2\} \\ \beta(d_0, r_1, r_2) & \textit{if } \max\{r_1, r_2\} - \min\{r_1, r_2\} \le d < r_1 + r_2 \\ 0 & \textit{if } d \ge r_1 + r_2 \end{cases}$$

where $\beta(d_0, r_1, r_2)$ is given by the standard formula

$$\begin{aligned} \beta(d_0, r_1, r_2) = & {r_1}^2 \arccos\left(\frac{d^2 + {r_1}^2 - {r_2}^2}{2dr_1}\right) + {r_2}^2 \arccos\left(\frac{d^2 + {r_2}^2 - {r_1}^2}{2dr_2}\right) \\ & - \frac{1}{2}\sqrt{(-d + r_1 + r_2)(d + r_1 - r_2)(d - r_1 + r_2)(d + r_1 + r_2)} \end{aligned}$$

Since $w_l(r)$ gives the intersection area of two disks with center points distance r apart, it is clearly a continuous non-increasing function of r and its derivative will be piecewise continuous. Therefore, we can apply Lemma 4.3 with $w_l(r)$ to get the following lower bound on B:

$$B \geq \int_0^1 p(r) w_l(r)\, dr \geq 2A \int_0^1 rg\left(r, 1, \frac{2}{\sqrt{k}}\right) dr$$

Finally, since the fundamental cell of the lattice has area $\frac{8\sqrt{3}}{k}$, by Lemma 4.1 a lattice positioning must exist with weight at least

$$\left(\frac{k}{8\sqrt{3}}\right) B = A \frac{k}{4\sqrt{3}} \int_0^1 rg\left(r, 1, \frac{2}{\sqrt{k}}\right) dr$$

and we arrive at our desired bound of

$$c_k \geq \frac{k}{4\sqrt{3}} \int_0^1 rg\left(r, 1, \frac{2}{\sqrt{k}}\right) dr$$

□

Using Maple to evaluate the resulting function from Theorem 5.2 for the values of $k \leq 16$ which satisfy $k \in \{i^2 + ij + j^2 \mid i, j \in \mathbb{N}\}$ produces the bounds shown in Table 5.1.

k	1	3	4	7	9	12	13	16
$c_k \gtrapprox$	$\frac{1}{4.41}$	$\frac{1}{2.09}$	$\frac{1}{1.88}$	$\frac{1}{1.61}$	$\frac{1}{1.53}$	$\frac{1}{1.46}$	$\frac{1}{1.44}$	$\frac{1}{1.40}$

Table 5.1: Bounds on c_k derived using Theorem 5.2

Our second method of deriving bounds demonstrates weaker bounds than those of Theorem 5.2 for small k, but surpasses Theorem 5.2 as k grows large (greater than 1483 colours). The proof is also simpler and more self-contained than that of Theorem 5.2, using only Lemma 4.3, and Claim 4.1 from Section 4.1.

Theorem 5.4 below states the result of our second method. The proof of Theorem 5.4 also uses a k-coloured triangular lattice to select and colour sets of disks, but in a different manner than the proofs we've presented so far. We start with a k-coloured regular triangular lattice and colour each lattice Voronoi cell to match its associated lattice point (see Figure 5.2). Then, for each Voronoi cell containing the center point of one or more disks, we select one such disk and colour it to match the Voronoi cell. Observe that we can ensure that disks selected and coloured in this manner are pairwise disjoint by scaling the lattice such that the enclosing circle around same-coloured Voronoi cells are distance 2 apart (see Figure 5.3).

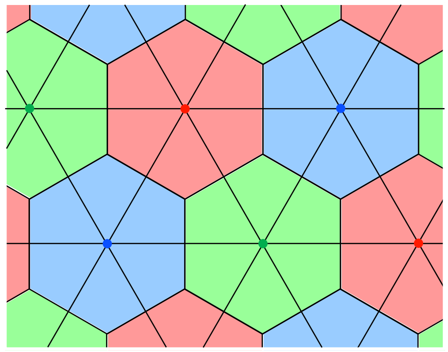

Figure 5.2: Each Voronoi cell is coloured to match its associated lattice point.

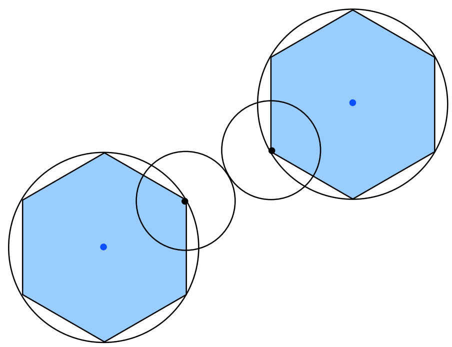

Figure 5.3: Scaling such that distance 2 separates the enclosing circles of same-coloured Voronoi cells guarantees that same-coloured selected disks will be pairwise disjoint.

Theorem 5.4. *Given k colours, where $k \in \{i^2 + ij + j^2 \mid i, j \in \mathbb{N}\}$ there exists a k-coloured subset C of disks such that same-coloured disks are disjoint. Let A_C denote C's union area, then $c_k \geq \frac{A_C}{A} \geq \frac{1}{(1+\delta_k)^2}$ where $\delta_k = \frac{2}{\sqrt{3}}\left(\frac{2}{\sqrt{k}-\frac{2}{\sqrt{3}}}\right)$.*

Proof. By Lemma 5.1 a k-coloured regular triangular lattice L exists with side length 1 and distance at least $\sqrt{k}$ between same-coloured lattice points. We can scale L such that distance 2 separates the smallest enclosing disks of Voronoi cells of same-coloured lattice points by applying a scaling factor of $\alpha_k = \frac{2}{\sqrt{k}-\frac{2}{\sqrt{3}}}$.

Now, let each disk in D be associated with the lattice Voronoi cell containing that disk's center. For each Voronoi cell with one or more associated disks we select one associated disk arbitrarily and colour it to match the Voronoi cell's lattice point. Note that by our choice of lattice size, same-coloured selected disks cannot intersect.

If a point p is in $\cup D$ but is not in any selected disk, then the disk covering p intersects another disk with center in the same Voronoi cell, and the distance between their center points is less than the diameter of the Voronoi cell $\delta_k = \frac{2}{\sqrt{3}}\alpha_k$. Therefore, if all selected disks were blown up by a factor of $1 + \delta_k$, p would be covered by some blown-up selected disk. That is, the union area of all blown-up selected disks would be at least A and thus the union area of all selected disks must be at least $A\frac{1}{(1+\delta_k)^2}$ by Claim 4.1 from Section 4.1, and we have shown that $c_k \geq \frac{1}{(1+\delta_k)^2}$.

□

k	1	3	4	7	9	12	13	16
$c_k \gtrapprox$	—	$\frac{1}{25.00}$	$\frac{1}{13.93}$	$\frac{1}{6.50}$	$\frac{1}{5.07}$	$\frac{1}{4.00}$	$\frac{1}{3.77}$	$\frac{1}{3.28}$

Table 5.2: Bounds on c_k derived using Theorem 5.4

Table 5.2 shows bounds derived using Theorem 5.4 for $k \leq 16$. There is no bound for $k = 1$ since the lattice scaling step would need to scale the lattice by a negative scalar. Figure 5.4 shows a plot of bounds derived using Theorems 5.2 and 5.4 up to $k = 2000$, from which it appears that the bounds proven by Theorem 5.4 eventually surpass those of Theorem 5.2. Table 5.3 shows in detail the the bounds produced by Theorems 5.4 and 5.2 around $k = 1483$, where we can observe the bounds produced by Theorem 5.4 overcome those of Theorem 5.2.

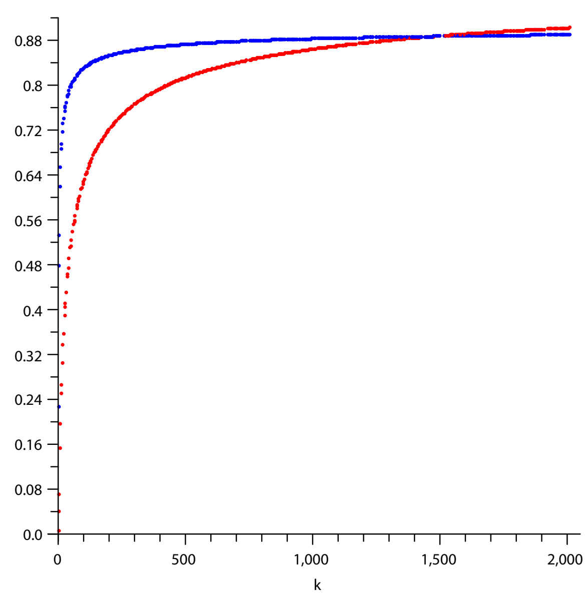

Figure 5.4: Plot comparing the bounds derived using Theorem 5.2 (blue) and Theorem 5.4 (red) for $1 \leq k \leq 2000$

Strictly speaking, Theorems 5.2 and 5.4 don't let us derive bounds for any exact number of colours k, only $k \in \{i^2 + ij + j^2 \mid i, j \in \mathbb{N}\}$. Moreover, the number of such k less than a given $x \in \mathbb{N}$ is given by $\Theta(\frac{x}{\sqrt{\log x}})$, so the set of such k is thin (density 0). However, for values of k which are not in this set, we can still derive a bound by letting k_l be the largest element of the set $\{i^2 + ij + j^2 \mid i, j \in \mathbb{N}\}$ less than k and let k_h be the smallest element

	k	1468	1471	1477	1483	1488	1489
Theorem 5.2	$c_k \gtrapprox$	$\frac{1}{1.12764}$	$\frac{1}{1.12761}$	$\frac{1}{1.12756}$	$\frac{1}{1.12751}$	$\frac{1}{1.12747}$	$\frac{1}{1.12746}$
Theorem 5.4	$c_k \gtrapprox$	$\frac{1}{1.12816}$	$\frac{1}{1.12802}$	$\frac{1}{1.12774}$	$\frac{1}{1.12747}$	$\frac{1}{1.12724}$	$\frac{1}{1.12720}$

Table 5.3: Bounds on c_k derived using Theorems 5.2 and 5.4 around $k = 1477$

of $\{i^2 + ij + j^2 \mid i, j \in \mathbb{N}\}$ greater than k. Then $c_k \geq \max\left(c_{k_l}, \frac{k}{k_h} c_{k_h}\right)$. That is, we can either choose not to use $k - k_l$ colours and use the bound c_{k_l}, or we can derive a bound c_{k_h} using k_h colours and then throw away disks of the $k_h - k$ colours whose disks contribute the least to the area of selected disks.

5.2 What about a square lattice?

In this section we consider equivalent theorems to Theorems 5.2 and 5.4 using a square lattice. While different numbers of colours can be used to colour a square lattice, it appears that this doesn't allow us to derive any improved bounds.

In both Chapters 3 and 4 we presented a bound for $k = 3$ proven using a triangular lattice and a bound for $k = 2$ proven using a square lattice. Our proofs for $k = 2$ used a square lattice because it can be easily 2-coloured and scaled to guarantee distance 4 between same-coloured lattice points. Generalizing this to arbitrary k, we can prove the following square-lattice counterpart of Lemma 5.1, defining a set of values for k for which we can k-colour a square lattice while guaranteeing some distance between same-coloured points.

Lemma 5.5. *For all $k \in \{i^2 + j^2 \mid i, j \in \mathbb{N}\}$, and any given distance m we can k-colour a regular square lattice of side length $\frac{m}{\sqrt{k}}$ such that same-coloured lattice points are at least distance m apart.*

Proof. The proof is the same as that of Lemma 5.1 but modified for a square lattice (distance $\sqrt{k}$ occurs in the unit square lattice for $k \in \{i^2 + j^2 \mid i, j \in \mathbb{N}\}$ and the lattice has $\frac{\pi}{4}$ rotational symmetry). □

Using Lemma 5.5 we can easily modify the proofs of Theorems 5.2 and 5.4 to yield the following square-lattice equivalents.

Theorem 5.6. *Given k colours, where $k \in \{i^2 + j^2 \mid i, j \in \mathbb{N}\}$ we can select and colour a k-coloured subset C of disks such that same-coloured disks are disjoint, and letting A_C denote C's union area,*

$$c_k \geq \frac{A_C}{A} \geq \frac{k}{8} \int_0^1 r g\left(r, 1, \frac{2}{\sqrt{k}}\right) dr$$

where $g(d_0, r_1, r_2)$ is a function for the area of intersection between two disks of respective radii r_1 and r_2 and distance d_0 between their center points (stated in Fact 5.3).

Theorem 5.7. *Given k colours, where $k \in \{i^2 + j^2 \mid i, j \in \mathbb{N}\}$ we can select and colour a k-coloured subset C of disks such that same-coloured disks are disjoint, and letting A_C denote C's union area, $c_k \geq \frac{A_C}{A} \geq \frac{1}{(1+\delta_k)^2}$ where $\delta_k = \frac{2\sqrt{2}}{\sqrt{k}-\sqrt{2}}$.*

Unfortunately, this doesn't appear to allow us to derive any better bounds for arbitrary k. Observe that for values of k which fall in both $\{i^2 + j^2 \mid i, j \in \mathbb{N}\}$ and $\{i^2 + ij + j^2 \mid i, j \in \mathbb{N}\}$, Theorem 5.2 will always give a better bound than Theorem 5.6 since the integral is the same and the scalar outside the integral is smaller for Theorem 5.6. This doesn't say anything about values of k for which Theorem 5.6 is valid but Theorem 5.2 is not, or vice versa. Likewise, it doesn't say anything about bounds derived for values of k for which neither Theorem 5.2 or 5.6 hold, using the technique discussed at the end of the last section. However, exploring the data up to $k = 2000$ indicates that the bounds from Theorem 5.6 are still weaker in these cases than those of Theorem 5.2. This can be seen in Figure 5.5 showing bounds derived using Theorem 5.2 and Theorem 5.6 for $k \leq 200$, and Figure 5.6 for $k \leq 2000$.

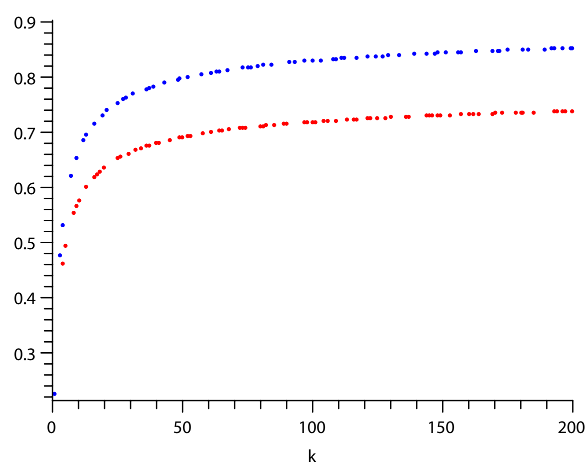

Figure 5.5: Plot comparing the bounds derived using Theorem 5.2 (blue) and Theorem 5.6 (red) for $k \leq 200$

Similarly, for values of k in both $\{i^2 + j^2 \mid i, j \in \mathbb{N}\}$ and $\{i^2 + ij + j^2 \mid i, j \in \mathbb{N}\}$, Theorem 5.4 will clearly give a better bound than Theorem 5.7, and Figures 5.7 and 5.8 show that Theorem 5.4 derives better bounds than Theorem 5.7 up to at least $k = 50,000$.

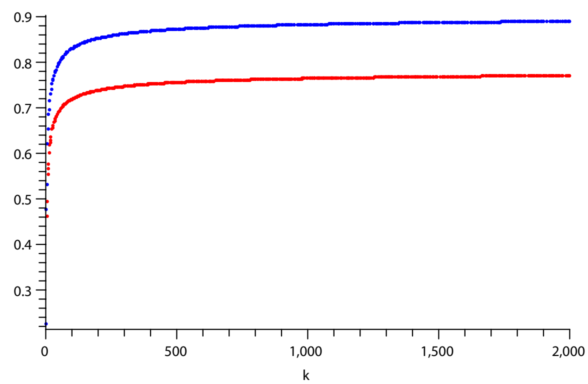

Figure 5.6: Plot comparing the bounds derived using Theorem 5.2 (blue) and Theorem 5.6 (red) for $k \leq 2000$

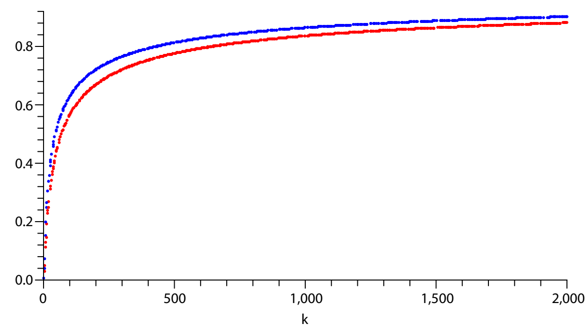

Figure 5.7: Plot comparing the bounds derived using Theorem 5.4 (blue) and Theorem 5.7 (red) for $k \leq 2000$

5.3 Algorithms

In this section we look at the algorithmic issues involved in selecting and colouring sets of disks to satisfy the bounds proven in this chapter. Since the method used in Theorem 5.2 is a generalization of the construction used to prove Theorems 4.2 and 4.4, the algorithmic results discussed in Section 4.3 apply to it as well. As such, we focus on the construction used to prove Theorem 5.4.

The construction used by Theorem 5.4 is the simplest construction we have seen so far, not requiring a complex lattice positioning step. Recall that the construction involves first arbitrarily positioning a k-coloured lattice over the set of points. Then, for each lattice point whose Voronoi cell contains the center point of at least one disk, we select one of these associated disks arbitrarily and colour it to match the selecting point.

This can be implemented in $O(n)$ time, where n is the size of the set of input disks D. We simply iterate through each disk $d \in D$, checking if d's associated lattice point p has been used to select and colour a disk yet. If it has not, we select d and colour it to match

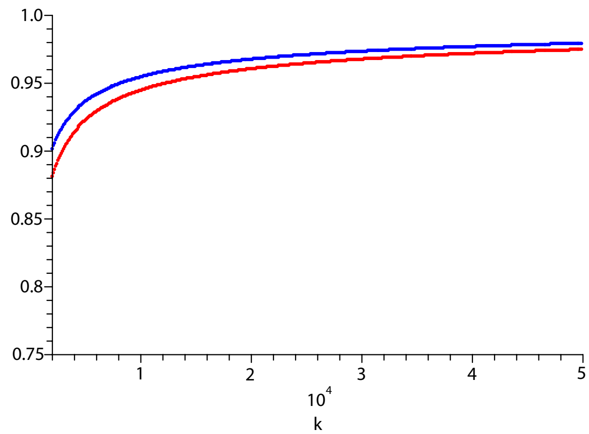

Figure 5.8: Plot comparing the bounds derived using Theorem 5.4 (blue) and Theorem 5.7 (red) for $2000 \leq k \leq 50,000$

p, and record that p has been used to select and colour a disk. There will be at most n lattice points with associated disks, since each disk has exactly one associated lattice point (assuming a convention for resolving disks with centers equidistant to two or more lattice points). We can use a hash table to record lattice points that have already selected disks and so the runtime of the algorithm is dominated by the time to iterate over the n input disks, and is in $O(n)$.

Chapter 6

Conclusion

In this thesis, we have introduced a k-coloured generalization of a problem studied by Rado in which the goal is to select a subset of a set of unit disks to cover as large a fraction of their union area as possible. We have demonstrated a number of bounds for $k = 2, 3$ as well as two methods of deriving bounds for arbitrary k. All of our bound proofs are constructive and in some way use a coloured lattice as a guide for disk selection and colouring. We have examined the algorithmic complexity of implementing each of these constructions and have presented efficient algorithms for selecting and colouring sets of disks to satisfy a number of our bounds.

There are a number of directions in which our work can be extended. Foremost, the bounds on c_k proven here leave room for improvement. For instance, for $k = 2$ the arrangement of disks in Figure 1.2 show that $c_2 \leq 1/2$ while the best known lower bound is $c_2 \gtrapprox 1/2.82$ (Theorem 4.4). Similarly, for $k = 3$, Figure 1.2 shows that $c_3 \lessapprox 1/1.41$ while the best known lower bound is $c_3 \gtrapprox 1/2.09$ (Theorem 4.2). Even for $k = 1$, a problem posed more than 60 years ago, there exists a gap between the best known upper bound of $c_1 \leq 1/4$ by the arrangement of disks shown in Figure 1.1 and Rado's result of $c_1 \geq 1/4.41$ [30]. Moreover, the arrangements shown in Figures 1.1 and 1.2 are only conjectured to be the worst possible arrangements, suggesting that it may be possible to close the gap from above as well.

Here we have assumed that k is a small constant in relation to the number of input disks n, but there are open questions if this is not the case. To start, what bounds can be proven on c_k when k is related to n? Clearly $c_k = 1$ for any $k \geq n$ since we can simply select all disks and assign each a unique colour. For $k < n$, the arrangement of disks shown in Figure 1.1 demonstrates that $c_k < 1$. It would be interesting to see what bounds could be proven when we have, for instance, $O(\log n)$ available colours, or n/m available colours for some fixed constant $m > 1$.

Other open problems include generalizing the algorithms and bounds presented here to higher dimensions, arbitrary radii disks, or classes of geometric objects other than disks.

References

[1] P. Afshani and T. M. Chan. On Approximate Range Counting and Depth. *Proceedings of the 23rd ACM Symposium on Computational Geometry (SoCG 2007)*, pp.337–343, 2007.

[2] M. Ajtai. The solution of a problem of T. Rado. *Bulletin de l'Académie Polonaise des Sciences, Série des Sciences Mathématiques, Astronomiques et Physiques*, v.21, pp.61–63, 1973.

[3] N. Alon and S. Smorodinsky. Conflict-Free Colorings of Shallow Discs. *Proceedings of the 22nd Annual ACM Symposium on Computational Geometry (SoCG 2006)*, pp.41–43, 2006.

[4] F. B. Abdesslem, L. Iannone, M. D. de Amorim, K. Kabassanov, and S. Fdida. On the feasibility of power control in current IEEE 802.11 devices. *Proceedings of the 4th IEEE International Conference on Pervasive Computing and Communications Workshops (PERCOMW '06)*, pp.468-473, 2006.

[5] N. Ahmed and S. Keshav. A Successive Refinement Approach to Wireless Infrastructure Network Deployment. *IEEE Wireless Communications and Networking Conference (WCNC 2006)*, v.1, pp.511–519, 2006.

[6] B. Aronov and S. Har-Peled. On Approximating the Depth and Related Problems. *Proceedings of the 16th ACM-SIAM Symposium on Discrete Algorithms (SODA 2005)*, pp.886–894, 2005.

[7] T. Asano, P. Brass, and S. Sasahara. Disc Covering Problem with Application to Digital Halftoning. *Proceedings of the International Conference on Computer Science and Applications (ICCSA 2004)*, LNCS 3045, pp.11–21, 2004.

[8] C. Bajaj. The algebraic degree of geometric optimization problems. *Discrete and Computational Geometry*, v.3, pp.177–191, 1988.

[9] S. Bereg, A. Dumitrescu, and M. Jiang. Maximum Area Independent Sets in Disk Intersection Graphs. *International Journal of Computational Geometry and Applications (IJCGA)*, forthcoming 2008.

[10] S. Bereg, A. Dumitrescu, and M. Jiang. On Covering Problems of Rado. Proceedings of the 12th Scandinavian Workshop on Algorithm Theory (SWAT 2008), LNCS 5124, pp.294–305, 2008.

[11] B. Bollobás. Area of the union of disks. *Elemente der Mathematik*, v.23, pp.60–61, 1968.

[12] P. Bose, A. Maheshwari, and P. Morin. Fast Approximations for Sums of Distances, Clustering and the Fermat-Weber Problem. *Computational Geometry: Theory and Applications*, v.24(3), pp.135–146, 2003.

[13] B. M. Chazelle and D. T. Lee. On a Circle Placement Problem. *Computing*, v.36(1–2), pp.1–16, 1986.

[14] Z. Chen, B. Fu, Y. Tang, and B. Zhu. A PTAS for a Disc Covering Problem Using Width-Bounded Separators. *Journal of Combinatorial Optimization*, v.11(2), pp.203–217, 2006.

[15] Cisco Systems, Inc. Channel Deployment Issues for 2.4-Ghz 802.11 WLANs. *http://www.cisco.com/univercd/cc/td/doc/product/wireless/airo1200/accsspts/techref/channel.pdf*, Retrieved April 15, 2007.

[16] B. N. Clark, C. J. Colbourn, and D. S. Johnson. Unit disk graphs. *Discrete Mathematics*, v.86, pp.165–177, 1990.

[17] Z. Drezner, K. Klamroth, A. Schöbel, and G. O. Wesolowsky. The Weber problem. In *Facility Location: Applications and Theory*, Springer, eds. Z. Drezner and H. W. Hamacher, pp.1–36, 2002.

[18] S. Durocher and D. Kirkpatrick. The Projection Median of a Set of Points. *Computational Geometry: Theory and Applications*, to appear 2008, 21 pages.

[19] G. Even, Z. Lotker, D. Ron, and S. Smorodinsky. Conflict-Free Colorings of Simple Geometric Regions with Applications to Frequency Assignment in Cellular Networks. *Proceedings of the 43rd Annual IEEE Symposium on Foundations of Computer Science (FOCS 2002)*, pp.691–700, 2002.

[20] A. Fiat, H. Kaplan, M. Levy, J. Matoušek, E. Mossel, J. Pach, M. Sharir, S. Smorodinsky, U. Wagner, and E. Welzl. Online Conflict-Free Coloring for Intervals. *Proceedings of the 16th Annual ACM-SIAM Symposium on Discrete Algorithms (SODA 2005)*, pp.545–554, 2005.

[21] A. Gajentaan and M. H. Overmars. On a class of $O(n^2)$ problems in computational geometry. *Computational Geometry: Theory and Applications*, v.5, pp.165–185, 1995.

[22] M. R. Garey and D. S. Johnson. *Computers and Intractability: A Guide to the Theory of NP-Completeness*. W. H. Freeman, New York, USA, 1979.

[23] A. Gräf, M. Stumpf, and G. Weißenfels. On Coloring Unit Disk Graphs. *Algorithmica*, v.20(3), pp.277–293, 1998.

[24] D. A. Gusev. Anti-Correlation Digital Halftoning. *Technical Report*, TR No. 513, Computer Science Department, Indiana University, Bloomington, 1998.

[25] S. L. Hakimi. Location Theory. In *Handbook of Discrete and Combinatorial Mathematics*, CRC Press, ed. K. H. Rosen, pp.986–995, 2000.

[26] S. Har-Peled and S. Smorodinsky. On Conflict-Free Coloring of Points and Simple Regions in the Plane. *Proceedings of the 19th Annual ACM Symposium on Computational Geometry (SoCG 2003)*, pp.114–123, 2003.

[27] L. Harte, R. Levine, and G. Livingston. *GSM Superphones: Technologies and Services*. McGraw-Hill Professional, New York, USA, 1998.

[28] H. Z. Hel-Or, X. M. Zhang, and B. A. Wandell. Adaptive cluster dot dithering. *Journal of Electronic Imaging*, v.8(2), pp.133–144, 1999.

[29] P. Indyk. Sublinear Time Algorithms for Metric Space Problems. *Proceedings of the Symposium on the Theory of Computing*, v.31, pp.428–434, 1999.

[30] R. Rado. Some Covering Theorems (I). *Proceedings of the London Mathematical Society*, s2-51(4), pp.241–265, 1947.

[31] R. Rado. Some Covering Theorems (II). *Proceedings of the London Mathematical Society*, s2-53(4), pp.243–267, 1951.

[32] R. Rado. Some Covering Theorems (III). *Journal of the London Mathematical Society*, s1-43(1), pp.127–130, 1968.

[33] T. Radó. Sur un problème relatif à un théorème de Vitali. *Fundamenta Mathematicae*, v.11, pp.228–229, 1928.

[34] S. S. Skiena. *The Algorithm Design Manual*. Springer-Verlag, New York, USA, 1998.

[35] S. Smorodinsky. *Combinatorial Problems in Computational Geometry* Ph.D Dissertation, School of Computer Science, Tel-Aviv University, 2003.

[36] G. F. Tóth. Packing and Covering. In *Handbook of Discrete and Computational Geometry*, CRC Press, eds. K. E. Goodman and J. O'Rourke, v.2, pp.19–41, 1997.

[37] V. A. Zalgaller. Remarks on a problem of Rado. *Matem. Prosv.*, v.5, pp.141–148, 1960.

Appendix A

Lemma 3.2 Formulas

A.1 Derived Formula for $0 \leq \theta \leq \arccos(\frac{2}{3}) - \frac{\pi}{6}$ (see Fig. 3.4 (left))

$$A_x(\theta) = \frac{1}{3}\sqrt{3} - \frac{1}{2}\sqrt{1 - \left(-\frac{2}{3}\sqrt{3} + \frac{1}{2}\sin(\theta)\sqrt{3} + \frac{1}{2}\cos(\theta)\right)^2}\sqrt{3} - \frac{1}{4}\sin(\theta)\sqrt{3} + \frac{3}{4}\cos(\theta)$$

$$A_y(\theta) = 1 + \frac{1}{2}\sqrt{1 - \left(-\frac{2}{3}\sqrt{3} + \frac{1}{2}\sin(\theta)\sqrt{3} + \frac{1}{2}\cos(\theta)\right)^2} + \frac{1}{4}\sin(\theta) - \frac{1}{4}\cos(\theta)\sqrt{3}$$

$$B_x(\theta) = \cos(\theta)$$

$$B_y(\theta) = \sin(\theta)$$

$$C_x(\theta) = \frac{1}{3}\sqrt{3} - \frac{1}{2}\sqrt{1 - \left(\frac{2}{3}\sqrt{3} + \frac{1}{2}\sin(\theta)\sqrt{3} - \frac{1}{2}\cos(\theta)\right)^2}\sqrt{3} + \frac{1}{4}\sin(\theta)\sqrt{3} + \frac{3}{4}\cos(\theta)$$

$$C_y(\theta) = -1 - \frac{1}{2}\sqrt{1 - \left(\frac{2}{3}\sqrt{3} + \frac{1}{2}\sin(\theta)\sqrt{3} - \frac{1}{2}\cos(\theta)\right)^2} + \frac{1}{4}\sin(\theta) + \frac{1}{4}\cos(\theta)\sqrt{3}$$

$$D_x(\theta) = \frac{2}{3}\sqrt{3}$$

$$D_y(\theta) = \frac{2}{3}$$

$$E_x(\theta) = \frac{2}{3}\sqrt{3}$$

$$E_y(\theta) = -\frac{2}{3}$$

$$f_1(\theta) = \frac{1}{2}\left(\left|\begin{array}{cc} A_x(\theta) & B_x(\theta) \\ A_y(\theta) & B_y(\theta) \end{array}\right| + \left|\begin{array}{cc} B_x(\theta) & C_x(\theta) \\ B_y(\theta) & C_y(\theta) \end{array}\right| + \left|\begin{array}{cc} C_x(\theta) & E_x(\theta) \\ C_y(\theta) & E_y(\theta) \end{array}\right| + \left|\begin{array}{cc} E_x(\theta) & D_x(\theta) \\ E_y(\theta) & D_y(\theta) \end{array}\right| + \left|\begin{array}{cc} D_x(\theta) & A_x(\theta) \\ D_y(\theta) & A_y(\theta) \end{array}\right|\right)$$
$$+ \frac{1}{2}\left(\pi + \arctan\left(\frac{-(-B_x(\theta) + C_x(\theta))(-A_y(\theta) + B_y(\theta)) + (B_y(\theta) - C_y(\theta))(A_x(\theta) - B_x(\theta))}{(-B_x(\theta) + C_x(\theta))(A_x(\theta) - B_x(\theta)) + (B_y(\theta) - C_y(\theta))(-A_y(\theta) + B_y(\theta))}\right)\right)$$

A.2 Derived Formula for $\arccos(\frac{2}{3}) - \frac{\pi}{6} \leq \theta \leq \frac{\pi}{6}$ (see Fig. 3.4 (right))

$$A_x(\theta) = \frac{1}{3}\sqrt{3} - \frac{1}{2}\sqrt{1 - \left(-\frac{2}{3}\sqrt{3} + \frac{1}{2}\sin(\theta)\sqrt{3} + \frac{1}{2}\cos(\theta)\right)^2}\sqrt{3} - \frac{1}{4}\sin(\theta)\sqrt{3} + \frac{3}{4}\cos(\theta)$$

$$A_y(\theta) = 1 + \frac{1}{2}\sqrt{1 - \left(-\frac{2}{3}\sqrt{3} + \frac{1}{2}\sin(\theta)\sqrt{3} + \frac{1}{2}\cos(\theta)\right)^2} + \frac{1}{4}\sin(\theta) - \frac{1}{4}\cos(\theta)\sqrt{3}$$

$$B_x(\theta) = \cos(\theta)$$

$$B_y(\theta) = \sin(\theta)$$

$$C_x(\theta) = \frac{2}{3}\sqrt{3}$$

$$C_y(\theta) = -\frac{1}{3}\sqrt{-3 + 12\cos(\theta)\sqrt{3} - 9\left(\cos(\theta)\right)^2} + \sin(\theta)$$

$$D_x(\theta) = \frac{2}{3}\sqrt{3}$$

$$D_y(\theta) = \frac{2}{3}$$

$$f_2(\theta) = \frac{1}{2}\left(\begin{vmatrix} A_x(\theta) & B_x(\theta) \\ A_y(\theta) & B_y(\theta) \end{vmatrix} + \begin{vmatrix} B_x(\theta) & C_x(\theta) \\ B_y(\theta) & C_y(\theta) \end{vmatrix} + \begin{vmatrix} C_x(\theta) & D_x(\theta) \\ C_y(\theta) & D_y(\theta) \end{vmatrix} + \begin{vmatrix} D_x(\theta) & A_x(\theta) \\ D_y(\theta) & A_y(\theta) \end{vmatrix}\right)$$
$$+ \frac{1}{2}\left(\pi + \arctan\left(\frac{-(-B_x(\theta) + C_x(\theta))(-A_y(\theta) + B_y(\theta)) + (B_y(\theta) - C_y(\theta))(A_x(\theta) - B_x(\theta))}{(-B_x(\theta) + C_x(\theta))(A_x(\theta) - B_x(\theta)) + (B_y(\theta) - C_y(\theta))(-A_y(\theta) + B_y(\theta))}\right)\right)$$