4D Induced Matter From Non-Compact 5D Kaluza-Klein Gravity

by

W. N. Sajko

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Abstract

The traditional constraints associated with five-dimensional Kaluza-Klein gravity are removed. namely that the 5D metric can depend on the extra-coordinate and that this coordinate is non-compact. The assumption that the 5D theory is vacuum $\hat{R}_{AB} = 0$ is the minimal set of field equations to induce matter from 5D to 4D via a dimesional reduction. This reduction is carried out for two general types of 5D metrics : 1) the traditional Kaluza-Klein metric which unifies gravity. electromagnetism and a scalar field. and 2) conformal extra-coordinate dependent metrics which induce an effective cosmological constant and realistic neutral matter. The physical aspects such as test particle motion. the weak-field limit and gravitational waves. and the energy from a Hamiltonian perspective and conserved quantities associated with scalar-tensor theories of gravity are studied in detail. It is found that 5D relativity is a rich extension of 4D gravity that unifies geometry with 4D matter.

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Chapter 1

Introduction to 5D KKG

1.1 Historical Remarks

In 1919. just four years after the advent of general relativity and with Maxwell's theory of electromagnetism on a solid foundation. Kaluza explored the mathematical consequences of employing a higher-dimensional metric in tandem with the higher-dimensional vacuum field equations of general relativity. His idea was to deduce more than just the gravitational physics known at the time. He used one extra dimension above the four known from Einstein's general relativity to unify gravity and Maxwell's theory of electromagnetism by including the electromagnetic gauge potential as the off-diagonal components of the 5D metric tensor [1]. He went as far as to include a scalar field to measure the proper length in the extra dimension which lead to a relativistic scalar wave equation years prior to the Klein-Gordon equation in the context of relativistic quantum mechanics. At first, the inclusion of the scalar field was considered an embarrassing feature of the theory but was later resurrected by Jordon [2] and Thiry [3], and later gave rise to Brans-Dicke [4] type theories of gravity. Kaluza's investigation was an example of how the 5D Einstein vacuum field equations $\hat{R}_{AB} = 0$ in a higher-dimensional space could reproduce 4D general relativity plus induced matter, $G_{\alpha\beta} \equiv T_{\alpha\beta}$. The main observation that was derived from this exercise was that a higher-dimensional stress-energy tensor is not needed to produce 4D physics, but meaningful physics can be derived from pure geometry in five dimensions. The idea was attractive to Einstein and he presented the paper on behalf of Klein to the Prussian Academy of Sciences in 1921 (see appendix in [5] for the correspondence between Einstein and Kaluza regarding Kaluza's 5D theory). However, the experimental physics of the early 1920s did not point to any direct observational evidence for the extra dimension. Kaluza avoided this problem by reasoning that it was because the 5D metric did not have any dependence on the extra coordinate, and so experiments could not directly probe for the extra dimension. Mathematically speaking, this isometry of the 5D metric is known as Kaluza's cylinder condition

$$\frac{\partial}{\partial x^4}\hat{g}_{AB} = 0, \tag{1.1}$$

which today remains a mechanism for hiding the direct effects of the extra dimensions in modern higher-dimensional Grand Unified Theories (GUTS). He also explained the known general covariance of general relativity and the gauge invariance of classical EM by a restricted form of 5D general covariance. The restricted set of coordinate transformations involved only translations,

$$x'^{\alpha} = x^{\alpha} \qquad \text{and} \qquad x'^{4} = x^{4} + f(x^{\sigma})\,. \tag{1.2}$$

These transformations generate the gauge transformation of the off diagonal components $\hat{g}_{4\alpha} \sim A_\alpha$,

$$A'_{\alpha} = A_{\alpha} - \partial_{\alpha} f(x^{\sigma})\,. \tag{1.3}$$

Thus coordinate invariance has been translated into gauge invariance and leads us to think that gauge symmetries in nature could be explained as coordinate invariance in higher dimensions. With the advent of quantum mechanics in the mid-1920s Klein proposed a physical interpretation as to why the fifth dimension is unobservable. He reasoned that effects of the extra coordinate are not directly seen because the topology of the extra coordinate is circular and the radius of the compactified dimension is of order of the Planck length [8], [9]. Klein's explanation stemmed from quantum mechanics, and he postulated that the fields in the 5D metric were periodic in the extra coordinate. Klein then used the compactification scale as an expansion parameter in the Fourier decomposition of the fields. The motivation was to give a physical explanantion to the origin of the charge-to-mass ratio of the electron. This attempt gave the charge-to-mass ratio of the electron many orders of magnitude too large [7]. Including all the Fourier modes gave rise to the so-called mass hierchy problem with KK theories [10] in which there is an infinite tower of states which have predicted masses that are at variance with the observed values of particles. For the theory to remain finite, it is artificially cut-off at the first mode. This may seem drastic, but this type of truncation persists in modern quantum field theories such as QED and QCD which have to include a mass cut-off parameter in order to avoid divergences.

In order to circumvent the above problems associated with the compactification scale and rather than making any special assumptions about the 5D spacetime we adopt a non-compact view of the coordinates and treat them all on equal footing thus eliminating both the circular topology of the extra coordinate and the 5D cylinder condition. The theory is also postulated to be 5D generally covariant, considerably extending Kaluza's original version of covariance. This is also an extremely powerful statement when applied to 4D induced matter as we shall see thoughout the manuscript. The lifting of the two historical constraints in 5D relativity has recently produced a flurry of activity in the field of dimensionally reduced string theories ([11]-[14] and references therein). But the idea has been around for quite some time. This modern approach was advocated by Wesson and researchers [15]-[21] and goes under the name of Space-Time-Matter (STM) approach to 5D relativity [17], [18]. This theory is not limited to describe gravity, electromagetism and a scalar field but has a general enough structure to describe other forms of matter. It also has the feature that all 4D matter is a consequence of the geometry of 5D spacetime, and the type of matter one can induce into 4D to decribe physics depends on the form of the metric (ie. the symmetries) chosen in 5D. This procedure of inducing 4D matter from a 5D vacuum is guaranteed by particular embedding theorems of differential geometry due to Campbell [22] (see also [23] and [24] for general results in embeddings) and recently rediscovered in a modern context by Romero and others [25]-[27]. One theorem states that any analytic N-dimensional Riemannian manifold can be locally embedded in a $(N+1)$-dimensional Ricci-flat manifold. Thus 4D general relativity can always be locally embedded in a 5D Ricci-flat manifold. From the 5D point of view, matter in 4D is induced by a reduction from a 5D Ricci-flat manifold. Clearly the form of the 4D induced matter depends crucially on the symmmetries in the 5D space and thus on the form of the 5D metric. We will investigate two general forms for the 5D metric in the next chapter and outline the procedure for inducing 4D general relativity with source matter.

1.2 Outline

In the next chapter we derive the induced matter for two metrics: one which obeys the cylinder condition and one which has a general x^4-dependence. We also consider the effects of conformally rescaling the fields in the 5D metric and examine the consequences for the induced matter. In the case of x^4-independence, the field equations can be chosen to have the form of a scalar field minimally or non-minimally coupled to gravity depending on the parametrization of the conformal rescalings. Contact is made between Kaluza-Klein theory and other non-minimally coupled theories of gravity such as Brans-Dicke theory. We also develop a $4+1$ hypersurface foliation analogous to the $1+3$ ADM split of 4D spacetime to derive the induced matter for extra-coordinate dependent metrics. Here we foliate along the extra coordinate, and induce matter on a 4D hypersurface which is taken to be the usual 4D manifold of general relativity. It is then shown that the induced matter is a direct consequence of the geometric relations known as the Gauss-Codazzi equations, thus reinforcing our view that matter is of geometric origin. We then derive the induced matter for a special form of the 5D metric which consists of a conformal rescaling in the extra coordinate. The resulting matter contains an effective cosmological constant which depends on the scalar field and reduces to the vacuum value under certain physical limits.

We next turn to the study of particle motion in Kaluza-Klein gravity. Since we can induce matter in 4D via a 5D reduction scheme, we expect that a reduction of the 5D equations of motion for hypothetical test particle in 5D will reproduce a 4D test particle interacting with 4D matter. We will assume that the test particle motion is 5D geodesic, which agrees with our minimal input recipe for inducing 4D matter properties. This is analogous to postulating the 5D vacuum equations rather than have an ad hoc 5D stress-energy tensor T_{AB} to produce matter. We demonstrate this for the case of the traditional Kaluza-Klein metric and show that the 5D geodesic equation reduces to the 4D Lorentz force equation augmented by a scalar force term and an effective charge-to-mass ratio for charged test particles. We define the energy of test particles and we make a comparison between the 5D definition and the induced 4D definition. This assists us in identifying the effective 4D mass and charge that is consistent with the charge-to-mass ratio. For metrics which retain the extra-coordinate dependence in the fields as a conformal factor, we derive the acceleration equation and show that there exists an effective force that is of geometric origin which allows for the variation of particle rest masses. In the case of photons the force is zero as well as their mass variation. We calculate the charge to mass ratio and mass variation for some known solutions and make comments.

Chapter 4 investigates the weak-field limit of the general 5D theory as well as examining the consequences of the 5D harmonic gauge for both the weak-field form of the metric and the x^4-independent metric. We show that the 5D harmonic gauge can reproduce the 4D harmonic and Lorentz gauges. For the extra-coordinate dependent metric, the possibility of massive gravitons exists as is now being realized in non-compact versions of superstring theory [11]. We also study the propagation of null geodesics in spacetimes with a scalar field and show the deviations from the usual 4D photon propagation. We finish by giving a 5D solution which describes a combination of plane gravitational and plane electromagnetic waves.

The fifth chapter deals with the Hamiltonian derivation for Kaluza-Klein gravity and conserved quantities traditionally derived from it such as energy and angular momentum. Both forms of the 5D metric are considered. Since the Hamiltonian is derived for a general 5D metric which can depend on the extra coordinate, it is simple to make a reduction to the two special forms of the metric. In the case where the metric obeys the cylinder condition the energy of a solution reduces to previous forms considered in scalar-tensor theories of gravity as well as dimensionally reduced Kaluza-Klein gravity. We then consider conserved Komar integrals for this form of the metric and make a connection with other forms of scalar-tensor gravity via conformal rescalings to show that the energy derived from the Hamiltonian is a sum of scalar and gravitational energy for solutions of the 5D field equations.

Chapter six deals with metrics that include the extra-coordinate dependence in the 5D metric. and we show that the Hamiltonian reduces to previously-known forms in five dimensions by showing a reduction of the Hamiltonian energy to the 5D ADM energy. We then consider the special case of the conformal extra-coordinate dependent metrics and show that the 5D energy provides a valid definition for asymptotically deSitter spacetimes by making use of the properties of 5D flat backgrounds and their 4D sections. This also provides us with a unique definition of the 5D gravitational coupling constant in terms of the 4D one and the cosmological constant. We follow chapter 6 with the conclusions in chapter 7.

Two appendices are included at the end of the thesis. The first appendix derives the induced matter for two classes of metrics and is intended to give the reader more exposure to the solutions in 5D Kaluza-Klein gravity that have extra-coordinate dependence and the mechanism for inducing matter into 4D. The second appendix deals with some quantum effects of 5D Kaluza-Klein field theory. First, the thermodynamic properties of a class of spherically-symmetric solutions is discussed in the semi-classical approach to quantum gravity. Secondly, we investigate the postulate that particle properties can be derived from a massless 5D wave equation. To derive this wave equation from an action would require the input of a scalar field $\hat{\Psi}(x^\sigma, x^4)$ as a source term in the 5D Lagrangian, which would introduce a 5D stress-energy tensor T_{AB}. Although this contradicts the minimal input principle of 5D, we are removing two historical constraints (the 5D cylinder condition and the compact extra dimension), and we investigate how these two extra degrees of freedom affect the 5D wave equation. We reduce the problem to a 4D wave equation and examine the definition of particle masses for two different induced matter scenarios..

1.3 Notation

The notation used throughout the thesis is summarized here for convenient reference.

Tensors in 5D are labelled with hats and have uppercase Latin indices which run $0 \rightarrow 4$, while tensors in 4D have lowercase Greek indices which run $0 \rightarrow 3$. Lowercase Latin indices are reserved for the spatial components of the tensors and a, b, c run $1 \rightarrow 4$ while i, j, k run $1 \rightarrow 3$. We use the signature $(-, +, +, +)$ throughout and use units $\hbar = c = 8\pi G = 1$ unless explicitly stated otherwise.

Chapter 2

4D Induced Matter From 5D KKG

2.1 Introduction

It is well known that higher-dimensional theories of gravity that have symmetries associated with the extra dimensions can be reduced to four-dimensional theories with an induced energy-momentum tensor [20]. One of the most cited examples of this is classical 5D Kaluza-Klein gravity (KKG) which unifies electromagnetism and a scalar field with 4D gravity. The symmetry which allows for the reduction of the 5D theory is usually referred to as the cylinder condition, which states that the 5D metric components $\hat{g}_{AB}$ are independent of the extra coordinate $\partial_4 \hat{g}_{AB} = 0$. Mathematically stated, there exists a spacelike Killing vector $\hat{\zeta}^A = \delta^A{}_4$ associated with the extra coordinate x^4. The existence of this symmetry guarantees that the metric can be written as

$$d\hat{s}^2 = g_{\alpha\beta}(x^\sigma)dx^\alpha dx^\beta + \phi^2(x^\sigma)\left(dy + 2A_\mu(x^\sigma)dx^\mu\right)^2 . \tag{2.1}$$

Here the A_μ are interpreted as the electromagnetic (EM) vector potentials, ϕ the scalar field and we have defined $x^4 \equiv y$. This metric will be referred to as the 5D Kaluza-Klein EM metric (5D KKEM metric). This form of the metric is usually referred to as the Jordan frame metric [28] because the 5D action generates a 4D Einstein-Maxwell theory where the scalar field is non-minimally coupled to gravity. A conformal transformation is needed for the induced 4D theory to be minimally coupled, the resulting metric being in the Einstein frame. We will discuss this in more detail in the next section and discuss the dimensional reduction of the action in detail in chapter 5.

A second and more modern approach to inducing non-trivial 4D matter from a 5D vacuum is the space-time-matter (STM) theory of Wesson and co-workers (see [18], [20] and [21] for reviews). The approach is motivated by the fact that while the above metric may describe a scalar field, electromagnetism and gravity, it fails to describe any other forms of matter. Thus the cylinder condition must be removed as a constraint on the 5D theory if general forms of 4D induced matter are to arise from a 5D vacuum. With the elimination of the cylinder condition, derivatives of the metric with respect to the extra coordinate occur and we will show that non-trivial matter may be induced on hypersurfaces of $x^4 = const.$ [15], [16], [29]. This theory has had great success in deriving all the usual equations of state used in 4D cosmology [30]-[33], and derives the matter content of the 4D induced theory in terms of geometry, thus coming one step closer to unifying geometry and physics.

We now proceed to derive the induced matter for the case when the cylinder condition holds and for the more general case when the metric has extra-coordinate dependence.

2.2 x^4-Independence: The KKEM Metric

Since dimensional reduction is possible if there is a Killing vector $\hat{\zeta}^A = \delta^A{}_4$, the resulting 4D induced theory is equivalent to a 4D theory of gravity non-minimally coupled to a scalar field plus electromagnetism. Rather than doing only this case we wish to treat the problem generally, and thus conformally rescale both the scalar field and the 4D metric by

$$g_{\alpha\beta} \to \phi^{2c} g_{\alpha\beta} \qquad \text{and} \qquad \phi \to \phi^d . \tag{2.2}$$

The parameters c and d play a vital role in determining the nature of the theory and we will show how they come into play in determining the field equations. Our starting point is the 5D vacuum equations $\hat{R}_{AB} = 0$ from which we will calculate the components $\hat{R}_{\alpha\beta}$. $\hat{R}_{4\alpha}$ and $\hat{R}_{44}$. We can then form the $(\alpha\beta)$ component of the Einstein tensor and calculate $\hat{G}_{\alpha\beta} = 0$. This gives us our induced matter since this equation breaks down into the 4D Einstein tensor plus extra terms which are then atttributed to the stress-energy tensor,

$$\hat{G}_{\alpha\beta} = 0 \Rightarrow G_{\alpha\beta} \equiv T_{\alpha\beta} . \tag{2.3}$$

For the above metric, it is a routine task to calculate the tensor components and so we only quote the results :

$$\begin{aligned} \hat{G}_{\alpha\beta} = 0 \Rightarrow G_{\alpha\beta} \; &= \; \frac{(2c+d)}{\phi}\left(\nabla_\alpha \phi_\beta - g_{\alpha\beta}\Box\phi\right) + c_0 \frac{\phi_\alpha \phi_\beta}{\phi^2} \\ &\quad - c_1 \frac{\phi_\gamma \phi^\gamma}{\phi^2} g_{\alpha\beta} + 2\phi^{2(d-c)}\left(F_{\alpha\gamma}F^\gamma{}_\beta - \frac{1}{4} g_{\alpha\beta} F^{\gamma\delta}F_{\gamma\delta}\right) \end{aligned} \tag{2.4}$$

$$\hat{G}_{4\alpha} = 0 \quad \Rightarrow \quad \nabla_\alpha\left(\phi^{3d} F^{\alpha\beta}\right) = 0 \tag{2.5}$$

$$\hat{G}_{44} = 0 \quad \Rightarrow \quad \Box\phi + (2c+d-1)\frac{\phi_\alpha \phi^\alpha}{\phi^2} = \frac{1}{d}\phi^{2(d-c)+1} F^{\gamma\delta} F_{\gamma\delta} . \tag{2.6}$$

where c_0 and c_1 are the constants

$$c_0 = c(c-1) - 2cd - 2c(c+1) \qquad \text{and} \qquad c_1 = c(c-1) + cd + c(c-2)\,, \quad (2.7)$$

and $F_{\gamma\delta} \equiv \partial_\gamma A_\delta - \partial_\delta A_\gamma$. The above equations are Einstein gravity coupled to scalar and electromagnetic source terms (2.4), Maxwell's equations coupled to the scalar field (2.5) and a nonlinear scalar wave equation with an electromagnetic source (2.6). In order to obtain minimally coupled (MC) gravity we see that we must make the parameter choice $2c + d = 0$ in (2.4) to remove the terms containing second derivatives of the scalar field, while for non-minimally coupled (NMC) gravity the choice is $2c + d = 1$. The case of NMC gravity is special since it contains regular KKG, $(c, d) = (0, 1)$. It has also been shown that KKG shares a conformal correspondence with Brans-Dicke (BD) theory [34], [35]. To make the connection we must set

$$2c = 1 - p \quad \text{and} \quad d = p\,, \qquad p \equiv \sqrt{1 + 2\omega/3}\,, \quad (2.8)$$

where ω is the Brans-Dicke parameter, and so KKG can be viewed as a $\omega = 0$ BD theory. The above field equations (2.4)-(2.6) are complicated by the general (c, d) dependence and in order to make the physics more transparent we will restrict ourselves to the MC and NMC cases. They are:

<u>MC Gravity</u>: $2c + d = 0$. For this case it is advantageous to transform the scalar field by letting $\phi = e^\sigma$ so that the field equations reduce cleanly to

$$G_{\alpha\beta} = T^S_{\alpha\beta} + 2\, e^{3d\sigma}\, T^{EM}_{\alpha\beta} \quad (2.9)$$

$$\nabla_\alpha \left(e^{3d\sigma} F^{\alpha\beta} \right) = 0 \quad (2.10)$$

$$\Box\sigma = \frac{1}{d} e^{3d\sigma} F^2\,. \quad (2.11)$$

What remains is the choice for the parameter d. The conventional choice is $d = 2/\sqrt{3}$, which gives the correct coefficient for the kinetic energy of the scalar field in the dimensionally reduced action [36], as will be seen in chapter 5. This choice of d sets the parameter $c = -1/\sqrt{3}$ via the relation $2d + c = 0$, so we have $(c, d) = (-1/\sqrt{3}, 2/\sqrt{3})$ for minimally coupled gravity.

<u>NMC Gravity</u>: $2c + d = 1$. Identifying the parameters with the BD parameters we can discuss both Kaluza-Klein gravity as well as Brans-Dicke gravity for this case. The field equations reduce to:

$$G_{\alpha\beta} = T^{S}_{\alpha\beta} + \frac{\omega}{\phi^2}\left(\phi_\alpha\phi_\beta - \frac{1}{2}\phi_\gamma\phi^\gamma g_{\alpha\beta}\right) + 2\,\phi^{-\left(1-3\sqrt{1+2\omega/3}\right)}\,T^{EM}_{\alpha\beta} \tag{2.12}$$

$$\nabla_\alpha\left(\phi^{3d}F^{\alpha\beta}\right) = 0 \tag{2.13}$$

$$\Box\phi = \frac{1}{d}\phi^{3d}F^2, \quad \text{where} \quad d \equiv \sqrt{1+2\omega/3}\,. \tag{2.14}$$

As usual the stress-energy terms used in the above equations are defined as

$$T^{EM}_{\alpha\beta} \equiv F_{\alpha\gamma}F^{\gamma}{}_{\beta} - \frac{1}{4}g_{\alpha\beta}F^2 \tag{2.15}$$

$$T^{S}_{\alpha\beta} \equiv \frac{1}{\phi}\left(\nabla_\alpha\phi_\beta - g_{\alpha\beta}\,\Box\phi\right), \tag{2.16}$$

and the electromagnetic field strength is defined by $F_{\alpha\beta} \equiv \partial_\alpha A_\beta - \partial_\beta A_\alpha$.

We give here a particular class of solutions to the KKG ($\omega = 0$) field equations to which we will repeatedly return. It is a static spherically symmetric 3-parameter family discovered by Liu and Wesson [37], which is the charged version of the 2-parameter Gross-Perry-Sorkin (GPS) class of solutions [38]-[40]. The Liu-Wesson solution in the Kaluza-Klein-Jordan frame is

$$d\hat{s}^2 = -\frac{(1-k)B^a}{1-kB^{a-b}}\,dt^2 + B^{-a-b}dr^2 + r^2B^{1-a-b}d\Omega^2 + \frac{B^b - kB^a}{1-k}(dy + 2A_t\,dt)^2\,. \tag{2.17}$$

Here the function B, the timelike component of the EM vector potential, and the scalar field are given respectively by:

$$B(r) = 1 - \frac{2M(1-k)}{r} \quad A_t(r) = \frac{\sqrt{k}\,(1-B^{a-b})}{2(1-kB^{a-b})} \quad \phi^2(r) = \frac{B^b - kB^a}{1-k}\,. \tag{2.18}$$

where the parameters (a, b) satisfy

$$a^2 + b^2 + a\,b = 1\,. \tag{2.19}$$

We note that the constants (a, b) satisfy the same restriction for both the charged and neutral classes of solutions. The identification of the parameter k may be found by making a comparison of the potential A_t for large r with the standard potential for a charged point-like object $A_t = Q/r$, or by appealing to the conserved charge in Maxwell's equations. For both methods we find that the parameter k is related to the charge by

$$k = \frac{Q^2}{M^2(a-b)^2}\,, \quad a \neq b. \tag{2.20}$$

When $a = b$ we find that the vector potential A_t is zero by (2.18), which corresponds to setting the charge to zero, and hence $k = 0$ when $a = b$. Here we should note that the Liu-Wesson solutions are not the unique class of solutions to the 5D vacuum field equations, and other charged solutions exist in the literature [36], [41]-[44], as well as neutral solutions [45]-[47]. This is due to the fact that in 5D, Birkhoff's theorem does not hold when the 4D sections $y = const$ are spherically-symmetric [48]. This violation of Birkhoff's theorem also allows for spherically-symmetric time-dependent solutions [49], [50].

We briefly mention the conditions for a horizon and discuss whether the solutions can be viewed as conventional black holes for the general parameters (a, b). Horizons are usually defined as surfaces where the norm of the timelike Killing vector $\hat{\xi}^A$ vanishes, i.e. $\hat{g}_{AB}\hat{\xi}^A\hat{\xi}^B = 0$. To simplify the analysis of the solutions (2.17) we take the neutral limit $k = 0$. We find that the condition for a horizon for the neutral GPS solutions is $B^a = 0$. The existence of a horizon at $r_h = 2M$ requires that $a > 0$, which is also needed for the positivity of the gravitational mass in the large-r limit since $\hat{g}_{tt} \sim -(1 - 2M_g/r)$. The area of 2-spheres for the solution for a general distance r is $Area(r) = 4\pi r^2(1 - 2M/r)^{1-a-b}$. We observe that for the area to decrease as the radius decreases we need $1 - a - b \geq 0$. As the radial coordinate $r \to 2M$ the area converges to zero and the "horizon" is a point. Thus the solution should be referred to as a naked singularity when $1 - a - b > 0$, as noticed by several authors [45], [46], [47]. The choice $(a, b) = (1, 0)$ is the usual Schwarzschild solution, which has a horizon at $r = 2M$, and the spacetime does not have a curvature singularity there. To further set limits on the range of parameters (a, b) we turn to the definition of the mass function. In 4D, the mass function $m(r)$ is defined from the g_{rr} component of the metric. We will adopt the same definition but dimensionally extend it to 5D. The mass function is then defined by

$$\hat{g}_{rr} = \left(1 - \frac{2M}{r}\right)^{-a-b} \equiv \left(1 - \frac{2m(r)}{r}\right)^{-1}, \tag{2.21}$$

which is easily solved for $m(r)$

$$m(r) = \frac{r}{2}\left[1 - \left(1 - \frac{2M}{r}\right)^{a+b}\right]. \tag{2.22}$$

If we consider the Schwarzschild case $(a, b) = (1, 0)$ this gives

$$m(r) = M, \tag{2.23}$$

which is independent of r as expected. However, for a general (a, b) this will not be the case, and we examine the large-r limit and the limit as $r \to 2M$ which gives:

$$\text{large-}r \quad m(r) \sim M(a + b) \tag{2.24}$$

$$r \to 2M \quad \lim_{r \to 2M} m(r) = M\left[1 - \left(0^+\right)^{a+b}\right], \tag{2.25}$$

where 0^+ is symbolic for approaches zero from above. We see that for the mass function to remain positive for both limits and finite as $r \to 2M$, the parameters must satisfy $a + b \geq 0$. This constraint on (a, b) together with the constraints on the positivity of the gravitational mass, the decreasing area and the consistency relation $a^2 + ab + b^2 = 1$ force the range of values allowed for a and b to be

$$a \in \left[1, \frac{2}{\sqrt{3}}\right] \quad \text{and} \quad b \in [-1, 0] \ . \tag{2.26}$$

We will return to the possible values for these parameters when we discuss the energy derived from the Hamilonian and the Tolman mass in chapter 5. Wesson and co-workers have shown that in general the GPS solutions describe objects of stable extended distributions of scalar matter with a pressure and density related by the radiation equation of state $P = \rho/3$, and therefore should be correctly termed solitons [51], [52]. The solitons are general enough to encompass both the naked singularity cases as well as the Schwarzshild solution. For a further discussion of the physical properties of the Liu-Wesson class of solutions we refer back to [37].

2.3 x^4-Dependence: The 4+1 Induced Matter

In this section we demonstrate that a 5D vacuum Kaluza-Klein theory which has extra coordinate dependence can induce realistic matter into 4D which contains both gravity and a scalar field, as well as matter that depends on the extra coordinate. We investigate this matter for cosmological cases and show that the extra-coordinate dependence of the 4D metric is responsible for inducing an effective cosmological constant in 4D.

2.3.1 4+1 Split of the 5D Kaluza-Klein Metric

The method of inducing matter on hypersurfaces of the extra coordinate closely follows the methods used to define hypersurface evolution in time for 4D relativity.

This material is usually covered in good books on relativity [53]-[56] and we use the same mathematical method, but applied to 5D.

The 5D spacetime $(M_5, \hat{g}_{AB})$ is chosen to be foliated by 4D non-null hypersurfaces Σ_y. These surfaces are chosen to be parametrized by a global function y and the unit normal to the Σ_y is $\hat{n}_A$. Thus, $\hat{g}_{AB}$ induces a metric $\hat{h}_{AB}$ on each Σ_y by

$$\hat{h}_{AB} = \hat{g}_{AB} - \epsilon\,\hat{n}_A\hat{n}_B, \tag{2.27}$$

where $\hat{n}_A\hat{n}^A = \epsilon$ and $\hat{h}_{AB}\hat{n}^A = 0$ since the normal to Σ_y may be spacelike ($\epsilon = 1$) or timelike ($\epsilon = -1$). We choose a vector field $\hat{y}^A$ on M that satisfies

$$\hat{y}^A\hat{\nabla}_A y = 1, \tag{2.28}$$

and we decompose $\hat{y}^A$ into terms normal and parallel to Σ_y. This gives the lapse $\hat{y}_\perp = \epsilon\hat{y}^A\hat{n}_A = \hat{N}(x^\sigma, y)$ and the shift $\hat{y}_{\|}^B = \hat{y}^A\hat{h}_A^B = \hat{N}^B(x^\sigma, y)$. Here we give $\hat{y}^A$, $\hat{n}^A$, $\hat{n}_A$, and $\hat{N}^A$ in component form:

$$\begin{aligned} \hat{y}^A &= (0, 0, 0, 0, 1) \\ \hat{n}^A &= \left(-\frac{\hat{N}^\alpha}{\hat{N}}, \frac{1}{\hat{N}}\right) \\ \hat{n}_A &= \left(0, 0, 0, 0, \epsilon\hat{N}\right) \\ \hat{N}^A &= \left(\hat{N}^\alpha, 0\right). \end{aligned} \tag{2.29}$$

The 5D line element can therefore be written using (2.27) and (2.29) as

$$\begin{aligned} d\hat{s}^2 &= \hat{g}_{AB}(x^\sigma, y)dx^A dx^B \\ &= \hat{h}_{\alpha\beta}\left(dx^\alpha + \hat{N}^\alpha dy\right)\left(dx^\beta + \hat{N}^\beta dy\right) + \epsilon\hat{N}^2 dy^2. \end{aligned} \tag{2.30}$$

In matrix form the 5D metric and its inverse are

$$\hat{g}_{AB} = \begin{pmatrix} \hat{g}_{\alpha\beta} & \hat{N}_\alpha \\ \hat{N}_\beta & \epsilon/\hat{N}^2 \end{pmatrix}, \qquad \hat{g}^{AB} = \begin{pmatrix} \hat{g}^{\alpha\beta} + \epsilon\hat{N}^\alpha\hat{N}^\beta & -\epsilon\hat{N}^\alpha/\hat{N}^2 \\ -\epsilon\hat{N}^\beta/\hat{N}^2 & \epsilon/\hat{N}^2 \end{pmatrix}. \tag{2.31}$$

In order to describe the embedding of Σ_y in the 5D manifold the extrinsic curvature tensor (second fundamental form of the surfaces y = const.) must be introduced

$$\hat{K}_{AB} = -\hat{h}^{C}_{B}\hat{\nabla}_{C}\hat{n}_{A}, \tag{2.32}$$

or in matrix form

$$\hat{K}_{AB} = \begin{pmatrix} \hat{K}_{\alpha\beta} & \hat{N}^{\beta}\hat{K}_{\alpha\beta} \\ \hat{N}^{\alpha}\hat{K}_{\alpha\beta} & 0 \end{pmatrix} \tag{2.33}$$

$$\hat{K}_{\alpha\beta} = \frac{1}{2\hat{N}}\left(\nabla_{\alpha}\hat{N}_{\beta} + \nabla_{\beta}\hat{N}_{\alpha} - \frac{\partial}{\partial y}\hat{g}_{\alpha\beta}\right). \tag{2.34}$$

where ∇_α is the covariant derivative operator on Σ_y. Applying the standard projection techniques [53]-[56] and using the 5D vacuum equations $\hat{R}_{AB} = 0$ we obtain 15 equations which are broken into 10 field equations for gravity with induced matter, the four Gauss-Codazzi equations which are the covariant conservation equations on Σ_y. and one scalar wave equation for the lapse function $\hat{N}$ [29]. The equations are:

$$R_{\alpha\beta} = \frac{1}{\hat{N}}\nabla_{\alpha}\nabla_{\beta}\hat{N} + \frac{\epsilon}{\hat{N}}\left(\mathcal{L}_{\hat{N}}\hat{K}_{\alpha\beta} - \partial_y\hat{K}_{\alpha\beta}\right) + \hat{N}\left(\hat{K}\hat{K}_{\alpha\beta} - 2\hat{K}_{\alpha\gamma}\hat{K}^{\gamma}_{\beta}\right) \tag{2.35}$$

$$\nabla_{\alpha}\left(\hat{K}^{\alpha}_{\beta} - \delta^{\alpha}{}_{\beta}\hat{K}\right) = 0 \tag{2.36}$$

$$\epsilon\Box\hat{N} = \partial_y\hat{K} - \hat{N}^{\alpha}\nabla_{\alpha}\hat{K} - \hat{N}\hat{K}^{\alpha\beta}\hat{K}_{\alpha\beta}. \tag{2.37}$$

Here, $\hat{K} = \hat{g}^{\alpha\beta}\hat{K}_{\alpha\beta}$, $\Box \equiv \nabla^{\alpha}\nabla_{\alpha}$ and $\mathcal{L}_{\hat{N}}\hat{K}_{\alpha\beta}$ is the Lie derivative of $\hat{K}_{\alpha\beta}$ with respect to $\hat{N}^{\alpha}$. A large simplification of the field equations occurs when we set the shift vector to zero ($\hat{N}^{\alpha} = 0$), which in turn reduces the extrinsic curvarture to

$$\hat{K}_{\alpha\beta} = -\frac{1}{2\hat{N}}\partial_y\hat{h}_{\alpha\beta}, \quad \text{with} \quad \hat{N}_{\alpha} = 0. \tag{2.38}$$

With the above assumption, the 5D line element can be expressed in 4+1 form as

$$d\hat{s}^2 = \hat{h}_{\alpha\beta}\,dx^{\alpha}dx^{\beta} + \epsilon\hat{N}^2dy^2. \tag{2.39}$$

Since we wish to make contact with the usual 5D Kaluza-Klein theory we must compare the above metric with the standard 5D Kaluza-Klein metric [15], [16] :

$$d\hat{s}^2 = g_{\alpha\beta}(x^\sigma, y)\, dx^\alpha dx^\beta + \epsilon\phi^2(x^\sigma, y)dy^2 \,. \tag{2.40}$$

Here the scalar field has been introduced and both $g_{\alpha\beta}$ and ϕ may have y-dependence. By comparing equations (2.39) and (2.40) one can derive the relationship between the 4+1 fields and the usual 5D Kaluza-Klein fields,

$$\hat{h}_{\alpha\beta} = g_{\alpha\beta}\,(x^\sigma, y)\,, \quad \hat{N} = \phi(x^\sigma, y) \quad \text{and} \quad \hat{N}^\alpha = 0 \,. \tag{2.41}$$

Thus the 4D induced metric on Σ_y is the metric in 4D general relativity except for the extra-coordinate dependence. In order to measure the curvature of the 4D manifold as embedded in the 5D manifold we need the extrinsic curvature, which is defined as

$$\hat{K}_{\alpha\beta} = -\frac{1}{2\hat{N}}\,\partial_y\,\hat{h}_{\alpha\beta} = -\frac{1}{2\phi}\,\partial_y\,g_{\alpha\beta} \,. \tag{2.42}$$

Evaluation of the 5D vacuum field equations (2.35)-(2.37) is tedious. Forming the Einstein tensor gives the induced matter, and explicitly written out reproduce the results of Wesson and Ponce de Leon [15], [16], which with $(\dot{\ }) \equiv \frac{\partial}{\partial y}$ are:

$$T_{\alpha\beta} = \frac{1}{\phi}\nabla_\alpha\nabla_\beta\phi - \frac{\epsilon}{2\phi^2}\left\{ \begin{array}{c} \frac{\dot{\phi}}{\phi}\dot{g}_{\alpha\beta} - \ddot{g}_{\alpha\beta} + g^{\lambda\mu}\dot{g}_{\alpha\lambda}\dot{g}_{\beta\mu} \\ -\frac{1}{2}g^{\mu\nu}\dot{g}_{\mu\nu}\dot{g}_{\alpha\beta} + \frac{1}{4}g_{\alpha\beta}\left[\dot{g}^{\mu\nu}\dot{g}_{\mu\nu} + (g^{\mu\nu}\dot{g}_{\mu\nu})^2\right] \end{array} \right\} \tag{2.43}$$

$$\nabla_\alpha\left[\frac{1}{\phi}(g^{\alpha\gamma}\dot{g}_{\gamma\beta} - \delta^\alpha{}_\beta g^{\mu\nu}\dot{g}_{\mu\nu})\right] = 0 \tag{2.44}$$

$$\epsilon\phi\Box\phi = \frac{1}{2\phi}\dot{\phi}\,g^{\alpha\beta}\dot{g}_{\alpha\beta} - \frac{1}{2}\left(g^{\alpha\beta}\ddot{g}_{\alpha\beta} + \frac{1}{2}\dot{g}^{\alpha\beta}\dot{g}_{\alpha\beta}\right) . \tag{2.45}$$

Thus the induced matter crucially depends on the metric having a non-trivial dependence on the extra coordinate, and is different for each $y = constant$ hypersurface if there is y-dependence in the stress-energy $T_{\alpha\beta}$. If the *4D cylinder condition* is imposed ($\partial_y\, g_{\alpha\beta} = 0$, which is different from the *5D cylinder condition* $\partial_y\,\hat{g}_{AB} = 0$) the field equations simplify and the induced matter is only a scalar field contribution. The non-linear interactions in the wave equation disappear and only a massless scalar field remains. The stress-energy tensor in this case is traceless and so the scalar field may be thought of as having a radiation equation of state $P = \rho/3$ if the induced matter is modelled after a perfect fluid [15], [16]. There exist many solutions to the induced matter field equations for the case when the 4D cylinder condition is enforced since the theory reduces to a NMC Scalar-Tensor (ST) theory of gravity, of which Brans-Dicke theory is a well known example (see [20] and [21] for an extensive bibliography of solutions). Here, we will not restrict ourselves to this case, but will deal with a conformally rescaled form of the 4D metric in which the extra-coordinate dependence is retained.

2.3.2 Induced Matter and Conformal Rescalings

As an example of induced matter with non-trivial metric dependence on the extra coordinate we consider the following 5D metric which has recently been termed canonical in the Kaluza-Klein literature [57], but has its roots in non-compact extra-coordinate dependent 6D Kaluza-Klein theory [58] :

$$\hat{g}_{AB}(x^\sigma, l) = \begin{pmatrix} \frac{l^2}{L^2} g_{\alpha\beta}(x^\sigma, l) & 0 \\ 0 & \epsilon\phi^2(x^\sigma) \end{pmatrix} . \tag{2.46}$$

Here we have made the simplifying assumption that the scalar field is independent of the extra coordinate which we have denoted by $x^4 \equiv l$. The importance of the length scale L will shortly be seen. The shift vector in this case is zero and the induced metric on Σ_l is

$$\hat{h}_{\alpha\beta} = \frac{l^2}{L^2} g_{\alpha\beta}(x^\sigma, l) . \tag{2.47}$$

Since we have partially factored out the extra-coordinate dependence conformally. we may expect a significant algebraic reduction of the field equations. Evaluation of the 4D Ricci scalar (2.35) gives

$$R_{\alpha\beta} = \frac{\nabla_\alpha \phi_\beta}{\phi} + \frac{3\epsilon}{\phi^2 L^2}\left(1 + \frac{l}{6}(g^{\gamma\delta}\partial_l g_{\gamma\delta})\right) g_{\alpha\beta} + \frac{1}{\phi} S_{\alpha\beta} \tag{2.48}$$

where we have defined the tensor

$$S_{\alpha\beta} \equiv \frac{\epsilon l^2}{2\phi L^2}\left[\left(\frac{4}{l} + \frac{1}{2} g^{\gamma\delta}\partial_l g_{\gamma\delta}\right)\partial_l g_{\alpha\beta} - \left(g^{\gamma\delta}\partial_l g_{\alpha\gamma}\right)\partial_l g_{\beta\delta} + \partial_l^2 g_{\alpha\beta}\right], \tag{2.49}$$

which collects the derivatives with respect to the extra coordinate [59]. Each of the terms in the Ricci tensor has a physical significance. The first term is a scalar field term that will generate the scalar stress-energy tensor

$$T_{\alpha\beta}(\phi) \equiv \frac{1}{\phi}\left(\nabla_\alpha \phi_\beta - \Box\phi g_{\alpha\beta}\right) . \tag{2.50}$$

The second term in (2.48) can be used to define an *effective cosmological constant* since it is the coefficient of the metric $g_{\alpha\beta}$. Finally the last term will generate the matter stress-energy due to the metric dependence of the extra coordinate by

$$T_{\alpha\beta}(\partial_l g) \equiv S_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} S^\gamma{}_\gamma . \tag{2.51}$$

Thus the total induced energy-momentum tensor can be expressed as

$$T_{\alpha\beta} = T_{\alpha\beta}(\phi) - \Lambda_{eff} g_{\alpha\beta} + \frac{T_{\alpha\beta}(\partial_l g)}{\phi}, \tag{2.52}$$

where we have factored out the ϕ^{-1} term in the extra-coordinate stress-energy tensor in order to emphasize its role as a matter contribution and not a scalar contribution (this is analogous to what is done in Brans-Dicke theory).

In order to determine the vacuum cosmological constant we take the limit in which the 4D metric is independent of the extra coordinate so that the matter term

$T_{\alpha\beta}(\partial_l g)$ is zero. Consistency of the induced matter field equations requires $\phi = 1$. This gives for the Ricci tensor

$$R_{\alpha\beta} = \frac{3\epsilon}{L^2} g_{\alpha\beta} , \tag{2.53}$$

from which we can define the vacuum cosmological constant as

$$\Lambda_{vac} \equiv \frac{3\epsilon}{L^2} , \tag{2.54}$$

and it is here we see the effects of the signature of the extra dimension. If the extra dimension is spacelike ($\epsilon = 1$) then the vacuum cosmological constant is deSitter ($\Lambda > 0$). while if the extra dimension is timelike the cosmological constant is anti-deSitter ($\Lambda < 0$) as well as introducing a two-time metric. This creates a uniqueness problem for the physical choice of time and therefore we rule anti-deSitter spaces as unphysical, viewed as an induced 4D theory from 5D. Current astrophysical constraints deduced from observations of supernovae indicate a large contribution to the vacuum energy density of the universe $\Omega_\Lambda = \rho_\Lambda/\rho_{crit}$, as well as requiring that $\Lambda > 0$ [60]. Thus from a physical viewpoint. and a mathematical justification for a well-defined metric, we will only work with deSitter-type metrics.

The effective cosmological constant can then be defined as

$$\Lambda_{eff} \equiv \frac{\Lambda_{vac}}{\phi^2} \left(1 + \frac{l}{6} g^{\alpha\beta} \, \partial_l g_{\alpha\beta} \right) . \tag{2.55}$$

This definition can be used to express the conservation equation in terms of more physical quantities. Susbtitution of the above equation (2.55) into the conservation equation (2.44) gives

$$\nabla_\alpha \left(\phi \frac{\Lambda_{eff}}{\Lambda_{vac}} \delta^\alpha{}_\beta - \frac{l}{6\phi} g^{\alpha\gamma} \, \partial_l g_{\beta\gamma} \right) = 0 . \tag{2.56}$$

It is evident that if we assume the 4D cylinder condition $\partial_l g_{\alpha\beta} = 0$, the effective cosmological constant reduces to its vacuum value and the scalar field must be a constant which we can scale to unity without loss of generality. However, if we choose to retain the extra coordinate dependence in the 4D metric we require that $g^{-1}\partial_l g \sim l^{-1}$ in order for Λ_{vac} to be a function of the 4D spacetime coordinates only. This is satisfied for solutions that are diagonal and have power law dependence in the extra coordinate. We now present such a solution.

The example we consider is a one-parameter class of cosmological solutions found by Ponce de Leon [30]. On the induced 4D hypersurfaces they are the analogues of the $k = 0$ FRW cosmologies. The metric is

$$d\hat{s}^2 = \frac{l^2}{L^2}\left(-c^2dt^2 - \left(\frac{ct}{L}\right)^{2/\alpha}\left(\frac{l}{L}\right)^{2\alpha/(1-\alpha)} d\vec{x}\cdot d\vec{x}\right) + \left(\frac{\alpha}{1-\alpha}\right)^2\left(\frac{ct}{L}\right)^2 dl^2 \,. \quad (2.57)$$

where α is characteristic of the induced equation of state as will be shown below. The effective cosmological constant can be evaluated from (2.55) and gives

$$\Lambda_{eff} = \Lambda_{vac}\,(1-\alpha)\left(\frac{L}{\alpha ct}\right)^2 , \quad (2.58)$$

which varies as a function of time and depends on the value of α. This inverse square law for the cosmological constant is favoured in string cosmologies [61] and time-varying Λ theories from BD gravity [62] for inflation, since it gives a large Λ for early times and a negligible Λ for late times (see [63] for an extensive bibliography). The equations of state for each of the components of the induced stress-energy tensor (2.52) which has been modelled after a perfect fluid are [59]:

$$\rho_\phi c^2 = -\frac{3}{16\pi G\alpha}\left(\frac{c}{t}\right)^2 \quad P_\phi = \frac{1}{16\pi G\alpha}\left(\frac{c}{t}\right)^2 \Rightarrow P_\phi = -\frac{1}{3}\rho_\phi c^2 \quad (2.59)$$

$$\rho_\Lambda c^2 = \frac{\Lambda_{vac}(1-\alpha)}{8\pi G}\left(\frac{Lc}{\alpha t}\right)^2 \quad P_\Lambda = -\frac{\Lambda_{vac}(1-\alpha)}{8\pi G}\left(\frac{Lc}{\alpha t}\right)^2 \Rightarrow P_\Lambda = -\rho_\Lambda c^2 \quad (2.60)$$

$$\rho_m c^2 = \frac{9}{16\pi G\alpha}\left(\frac{c}{t}\right)^2 \quad P_m = -\frac{3}{16\pi G\alpha}\left(\frac{c}{t}\right)^2 \Rightarrow P_m = -\frac{1}{3}\rho_m c^2 \,. \quad (2.61)$$

where we have restored units and designated the ρ_m and P_m arising from $T_{\alpha\beta}(\partial_l g)$ as matter terms . Finally, if we treat the entire energy-momentum tensor as one fluid, we obtain from (2.52)

$$\rho c^2 = \frac{3}{8\pi G}\left(\frac{c}{\alpha t}\right)^2, \quad P = \frac{(2\alpha - 3)}{8\pi G}\left(\frac{c}{\alpha t}\right)^2, \quad P = \left(\frac{2}{3}\alpha - 1\right)\rho c^2. \qquad (2.62)$$

which is equivalent with the results found in [31]. In this case, we have a simple fluid with a linear equation of state parameter $\gamma = \frac{2}{3}\alpha$. Note that when considering $T_{\alpha\beta}(\partial_l g)$ alone as the fluid source, it is necessary to require that $\alpha < 1$ in which case the effective cosmological constant is positive. When considering both $T_{\alpha\beta}(\partial_l g)$ and $T_{\alpha\beta}(\phi)$ as the fluid source $\alpha > 0$ to ensure $\rho_\phi + \rho_m > 0$. Finally, when considering the entire energy–momentum tensor as describing a single fluid, α may have any value. In this final case we find, as in [31], there are three physical choices for α: $\alpha \in (0, 1)$ for inflation, $\alpha = 2$ for radiation, and $\alpha = 3/2$ for dust, and for the latter two values, the cosmological constant is negative ($\alpha = 0, 1$ are excluded since these choices introduce singularities into the 5D metric). Thus this metric seems to best describe inflation since it is favourable to have an effective cosmological constant which is positive.

We pause here to briefly discuss the density and pressure for the components of the induced stress-energy, and whether they satisfy the energy conditions in 4D GR. In 4D, the matter content is known from the outset and enters the field equations via the $T_{\alpha\beta}$. In the induced matter approach the 4D field equations with source $T_{\alpha\beta}$ are generated from the geometry of a 5D vacuum, and the algebraic form of the induced stress-energy tensor is directly related to symmetries of the 5D metric and its components. Matter is generated from the extra-coordinate terms $\partial_l \hat{g}_{\alpha\beta}$, the EM gauge potential $\hat{g}_{4\alpha} \sim A_\alpha$ and the scalar field $\hat{g}_{44} \sim \phi$. Since fluid stress-energy tensors are well understood in 4D relativity, we impose a fluid interpretation on the geometrically derived induced matter in order to represent 4D physical quantities such as density ρ and pressure P [64], [65].

In the context of classical 4D general relativity, the stress-energy tensor satisfies certain energy conditions which prove to be invaluable in proving singularity theorems. The constraints on the matter derived from the stress-energy include the positivity of energy density and that the energy density is larger than the pressure. There exist different versions of the energy conditions of varying strength. The energy conditions are [56] :

$$\text{weak (WEC)} \; : \; T_{\alpha\beta}v^{\alpha}v^{\beta} \geq 0 \quad \Rightarrow \quad \rho \geq 0 \quad \rho + P_i \geq 0 \tag{2.63}$$

$$\text{strong (SEC)} \; : \; R_{\alpha\beta}v^{\alpha}v^{\beta} \geq \quad 0 \Rightarrow \quad \rho + \sum_i P_i \geq 0 \quad \rho + P_i \geq 0 \tag{2.64}$$

$$\text{dominant (DEC)} \; : \; -T_{\alpha\beta}v^{\alpha}\,\text{future directed} \quad \Rightarrow \quad \rho \geq 0 \quad \rho \geq |P_i|, \tag{2.65}$$

where v^{α} is any future-directed vector field. We now check if the induced matter for the Ponce de Leon class of metrics satisfies the above energy conditions. Since this class is parametrized by the constant α we expect that the energy conditions will place restrictions on the value α can have. We will consider the value of α to range $\alpha \in (0, 1)$ and $\alpha > 1$ since these values represent physically desirable equations of state for the total induced matter (2.62). It is a simple task to check the above energy conditions for the energy densities and pressures of the different fluid components of the induced stress-energy tensor (2.59)-(2.62). We summarize the results here.

<u>WEC:</u>

The scalar energy density ($\rho_{\phi} < 0$) and pressure ($\rho_{\phi} + P_{\phi} < 0$) violate the WEC. The cosmological energy density obeys the WEC if $\alpha \in (0, 1)$ (inflation) while violates it if $\alpha > 1$ (dust, radiation). The total matter energy density and pressure are found to obey the WEC.

SEC:

The scalar energy density does not obey the SEC since $\rho_\phi < 0$ for the α range. It is also found that the cosmological term violates the SEC for the inflationary period, and the matter contribution satisfies the SEC for all α. The total stress-energy violates the SEC for inflation $\alpha \in (0, 1)$, but for dust and radiation models $(\alpha > 1)$ the SEC is satisfied.

DEC:

Here both the scalar and cosmological components violate the DEC for $\alpha > 1$, but the matter component satisfies the DEC for inflation, dust and radiation models. The total matter is found to obey the DEC for all epochs.

Although certain components of the stress-energy violate different energy conditions, the total matter violates only the SEC during inflation. We can conclude that the total induced matter is physical and preserves ideas regarding the energy conditions in 4D.

2.4 Implications of 5D Covariance

The assumption that the induced matter theory is covariant in 5D has some far-reaching implications for the 4D induced matter theory. That the theory should be 5D covariant is a natural assumption when extending a 4D theory of gravity to a 5D theory. The field equations in 5D are the vacuum equations $\hat{R}_{AB} = 0$ which are generally covariant under the coordinate transformations

$$x^A \rightarrow x'^A = x'^A(x^\sigma, x^4), \tag{2.66}$$

which include the usual 4D trasformations $x^\alpha \rightarrow x'^\alpha = x'^\alpha(x^\sigma)$ as a subset. The 4D induced matter from the 5D vacuum is covariant under these 4D coordinate transformations. but will not be covariant under 5D coordinate transformations (2.66) since the induced matter will in general depend on the 4D coordinates x^α and the extra coordinte x^4. The induced matter theory will. however. be invariant under a restricted set of 5D coordinate transformations. We have already seen an example of this with the KKEM metric (2.1). In this case the matter is only invariant under the restricted set of 5D coordinate transformations which include the general 4D ones and a translation in the extra coordinate $x^4 \to x'^4 = x^4 + f(x^\sigma)$. which generates the gauge freedom $A_\alpha \to A' = A_\alpha - \partial_\alpha f$ in the EM gauge potential. If we consider the whole set of coordinate transformations (2.66) we are not guaranteed invariance of the 4D induced matter since the 4D coordinates and the extra-coordinate may mix and thus 4D physical quantities may change their meaning. This implies that 4D physics is not invariant under 5D coordinate transformations and that we can alter the interpretation of the induced 4D matter with a 5D coordinate transformation [18].

We now give two examples which illustrate 5D covariance and its implications for the induced matter. We first consider the canonical metric (2.46). which has the form

$$d\hat{s}^2 = \frac{l^2}{L^2} g_{\alpha\beta}(x^\sigma, l)\, dx^\alpha dx^\beta + \phi^2(x^\sigma) dl^2 , \tag{2.67}$$

where we have chosen the extra dimension to be spacelike. Now let us pull out the factor of l^2/L^2 so that the 5D metric appears to be 5D conformal

$$d\hat{s}^2 = \frac{l^2}{L^2} \left(g_{\alpha\beta}(x^\sigma, l)\, dx^\alpha dx^\beta + \frac{L^2}{l^2}\phi^2(x^\sigma) dl^2 \right) . \tag{2.68}$$

We then make the coordinate transformation

$$L^2 \frac{dl^2}{l^2} = dy^2 \qquad \Rightarrow \qquad l = Le^{\pm y/L} , \tag{2.69}$$

and obtain for the 5D metric

$$d\hat{s}^2 = e^{2ay/L} \left(g_{\alpha\beta}(x^\sigma . y) dx^\alpha dx^\beta + \phi^2(x^\sigma) dy^2 \right) , \tag{2.70}$$

where L is a length scale and the parameter a satisfies $a^2 = 1$. The first case we consider is when the 4D cylinder condition is applied ($\partial_y g_{\alpha\beta} = 0 \rightarrow \phi = 1$ by the field equations) :

$$d\hat{s}^2 = e^{2ay/L}\left(g_{\alpha\beta}(x^\sigma)dx^\alpha dx^\beta + dy^2\right) . \qquad (2.71)$$

The 5D vacuum equations $\hat{R}_{AB} = 0$ for the metric (2.71) generate the Einstein field equations with a cosmological constant

$$G_{\alpha\beta} = -\Lambda g_{\alpha\beta}, \quad \Lambda \equiv \frac{3}{L^2} > 0 , \qquad (2.72)$$

while the remaining five field equations ($\hat{R}_{y\alpha} = 0$. $\hat{R}_{yy} = 0$) are identically satisfied. We can conclude that the 4D deSitter solution in 4D general relativity can be locally embedded in a 5D canonical metric (2.71) when the 4D cylinder condition is applied to the 4D metric [25] (for a discussion of embeddings of D-dimensional gravity into D+1 dimensions. see [25]-[27]). If we consider deSitter space it is simple to show that the 5D manifold is 5D Riemann flat since it is well known that 5D deSitter space is a 4D pseudosphere of constant curvature $3/L^2 \equiv \Lambda$.

The geodesic equations in the original coordinate system (2.68) have an interesting form. in which the departures from 4D geodesic motion are governed by the derivative of the 4D metric with respect to the extra coordinate [66] . This can be interpreted as a geometric force which is removed if the extra coordinate dependence is removed. As well we will show in the next chapter there is a cosmological variation in the rest masses of massive particles associated with the original coordinate system but not with the 5D conformal metric (2.70). This is an example of how a coordinate transformation may alter 4D physics.

In the second example we reinstate the extra-coordinate in the 4D metric and consider the Ponce de Leon solutions (2.57). In order to simplify the analysis we momentarily set $c = L = 1$. Thus the metric takes the form

$$d\hat{\tau}^2 = l^2 dt^2 - t^{2/\alpha} l^{2/(1-\alpha)}(dr^2 + r^2 d\Omega^2) - \frac{\alpha^2}{(1-\alpha)^2} t^2 dl^2 , \tag{2.73}$$

where we have introduced spherical coords in the 3-space. Under the coordinate transformations [18]

$$T = \frac{\alpha}{2} t^{1/\alpha} l^{1/(1-\alpha)} \left(1 + \frac{r^2}{\alpha^2}\right) - \frac{\alpha}{2(1-2\alpha)} \left(t^{-1} l^{\alpha/(1-\alpha)}\right)^{(1-2\alpha)/\alpha} \tag{2.74}$$

$$R = r t^{1/\alpha} l^{1/(1-\alpha)} \tag{2.75}$$

$$Z = \frac{\alpha}{2} t^{1/\alpha} l^{1/(1-\alpha)} \left(1 - \frac{r^2}{\alpha^2}\right) + \frac{\alpha}{2(1-2\alpha)} \left(t^{-1} l^{\alpha/(1-\alpha)}\right)^{(1-2\alpha)/\alpha} . \tag{2.76}$$

the 5D line element transforms to

$$d\hat{\tau}^2 = dT^2 - dR^2 - R^2 d\Omega^2 - dZ^2 , \tag{2.77}$$

which is just the 5D Minkowski metric, and hence is 5D Riemann flat. This is one example of how a 5D Riemann flat manifold may still have meaningful physics on a 4D hypersurface $x^4 = const$. Many other examples exist in the literature [67]-[69], which can describe cosmological models and soliton-like solutions.

To add to the above examples of induced matter from an extra-coordinate dependent metric a few more interesting solutions are presented in Appendix A. There, two types of metrics are introduced. The first is a variation of the canonical coordinate system and the second is an extra-coordinate dependent analogue of the 4D Kerr-Schild coordinate system. Both are general enough to induce a variable cosmological constant and non-trivial matter.

2.5 Final Comments

In this chapter we have given two general methods for generating 4D matter from a 5D vacuum. This matter can describe electromagnetic phenomena when the 5D space has a Killing vector associated with the extra coordinate, and can describe general forms of matter when this symmetry is removed. The most notable effect the extra-coordinate has is the generation of a variable Λ and a realistic matter stress-energy which does not violate the 4D energy conditions. For the remainder of the thesis we will investigate the physical aspects of the induced matter theory such as particle motion, the weak-field limit and gravitational waves and the energy associated with solutions of the field equations.

Chapter 3

Particle Motion

3.1 Introduction to KKG Particle Motion

This chapter deals with the 4D particle motion induced from 5D for the two metrics considered in Chapter 2. For the KKEM (2.1) metric, it is found that the 4D electric and scalar charge-to-mass ratios have a 4D spacetime dependence. This dependence of the charges on the scalar field affects the energy of test particles in 4D through the definition of an effective mass. For the canonical metric (2.46) we derive the particle trajectories from a Lagrangian and show that particles may follow four-dimensional trajectories corresponding to massive or null particles when the parameterization is chosen properly. This leads to the possibilty of a cosmological variation in the rest masses of particles in the STM approach to 5D gravity, and a consequent departure from 4D geodesic motion due to a geometric force. Some examples are considered to illustrate the weak-field limit of the acceleration of charged particles, and the variation of rest-masses in 5D cosmology.

3.2 Test Particle Motion and KKEM Metric

Let us consider the KKEM form of the 5D metric written in the Jordan frame (2.1)

$$d\hat{\tau}^2 = d\tau^2 - \phi^2(dx^4 + 2A_\alpha dx^\alpha)^2\,, \tag{3.1}$$

where $d\tau$ is the proper time and we make the assumption that particles follow 5D geodesics

$$\hat{u}^B\hat{\nabla}_B\hat{u}^A = 0\,, \tag{3.2}$$

where $\hat{u}^A \equiv dx^A/d\hat{\tau}$. From this equation the 4D acceleration equation can be extracted as well as the motion in the fifth dimension. Since we assume the 5D cylinder condition $\partial_y \hat{g}_{AB} = 0$, there exists a 5D Killing vector associated with this symmetry, namely $\hat{\zeta}^A = \delta^A{}_4$. The scalar product $\hat{\zeta}\cdot\hat{u}$ is a 5D constant of the motion by virtue of the 5D geodesic equation (3.2) and Killing's equation. The conserved quantity is

$$c \equiv \text{const.} = \hat{\zeta}\cdot\hat{u} = \hat{g}_{4A}\hat{u}^A = -\phi^2\left(\hat{u}^4 + 2A\cdot\hat{u}\right)\,. \tag{3.3}$$

The 4D part of the geodesic equation $\hat{u}^B\hat{\nabla}_B\hat{u}^\alpha$ will simplify if we use the constant of the motion and expand the 5D covariant derivative in terms of the electric and scalar parts to give

$$\hat{u}^B\hat{\nabla}_B\hat{u}^\alpha = 0 \quad\Rightarrow\quad \hat{u}^\beta\nabla_\beta\hat{u}^\alpha = cF^\alpha{}_\beta\hat{u}^\beta - c^2\frac{\phi^\alpha}{\phi^3}\,, \tag{3.4}$$

where ∇_α is the 4D covariant derivative operator associated with the metric $g_{\alpha\beta}$. This equation takes on a more familiar form after we transform the 5D velocities into 4D velocities by using the relationship between the 5D and 4D line-elements provided by equation (3.1), and the constant of the motion (3.3). Thus:

$$d\hat{\tau} = \frac{d\tau}{\sqrt{1+c^2\phi^{-2}}} \quad\Rightarrow\quad \hat{u}^\alpha \equiv \frac{du^\alpha}{d\hat{\tau}} = u^\alpha\sqrt{1+c^2\phi^{-2}}\,. \tag{3.5}$$

The 4D acceleration reduces to the induced matter anologue of the Lorentz force equation in 4D:

$$u^\beta \nabla_\beta u^\alpha = \frac{c}{\sqrt{1+c^2\phi^{-2}}} F^\alpha{}_\beta u^\beta + \frac{c^2}{\phi^3\left(1+c^2\phi^{-2}\right)} h^{\alpha\beta}\phi_\beta . \tag{3.6}$$

This is the form of the 4D equation that has been considered by workers in traditional KKG [70]-[74]. Here $h^{\alpha\beta} \equiv g^{\alpha\beta} + u^\alpha u^\beta$ is the usual 4D projection tensor and obeys the orthogonality condtion $h^{\alpha\beta}u_\alpha = 0$. When the Faraday tensor $F_{\alpha\beta}$ is non-zero the coefficient of the Lorentz term $F^\alpha{}_\beta u^\beta$ can be defined as the effective charge-to-mass ratio of the test particle:

$$\left.\frac{q}{m}\right|_{eff} = \frac{c}{\sqrt{1+c^2\phi^{-2}}} \tag{3.7}$$

This agrees with the usual Lorentz equation for a charged test particle in 4D relativity. The second term on the right hand side of (3.6) appears to be a scalar force term but a closer examination reveals that it may be written as a derivative. Thus, the Kaluza-Klein version of the Lorentz equation when $F_{\alpha\beta} \neq 0$ may be written as

$$u^\beta \nabla_\beta u^\alpha = \left.\frac{q}{m}\right|_{eff} F^\alpha{}_\beta u^\beta + h^{\alpha\beta}\partial_\beta \ln\left(\left.\frac{q}{m}\right|_{eff}\right) . \tag{3.8}$$

However, when $F_{\alpha\beta} = 0$ the Lorentz term $F^\alpha{}_\beta u^\beta$ vanishes and therefore it is not possible to define a charge-to-mass ratio for a test particle from the acceleration equation. In this case the last term on the right hand side of (3.6) may survive and represents a scalar force term which we *define* to be

$$f^\alpha(\phi) \equiv h^{\alpha\beta}\frac{\phi_\beta}{\phi} . \tag{3.9}$$

The coefficient of this scalar force term may be defined as the effective scalar charge-to-mass ratio

$$\left.\frac{\sigma}{m}\right|_{eff} = \frac{c^2}{\phi^2\left(1+c^2\phi^{-2}\right)} , \tag{3.10}$$

where σ is the scalar charge for the test particle. Although the definition for the scalar term may appear to be arbitrary, we are guided by a physical motivation provided by the gravitational field. In 4D the force terms for a test particle are a result of the Christoffel symbols which have the form (barring numerical factors) $F_g \sim g^{-1}\partial g$, and so we can plausibly extend this to define the scalar force term $F_s \sim g^{44}\partial g_{44} \sim \partial\phi/\phi$. We expect that when the test particle is at a large distance from the electro-gravitating source, the weak-field limit applies to the 4D metric and gauge fields so that the effective electric (3.7) and scalar (3.10) charge to mass ratios of the test particle tend towards constants

$$\phi \to 1 \quad \Rightarrow \quad \left.\frac{q}{m}\right|_{eff}^{\infty} = \frac{q}{m} = \frac{c}{\sqrt{1+c^2}} \quad \text{and} \quad \left.\frac{\sigma}{m}\right|_{eff}^{\infty} = \frac{\sigma}{m} = \left(\frac{q}{m}\right)^2 . \tag{3.11}$$

Since setting $c = 0$ in equation (3.11) sets $q/m = 0$ we can safely assume the test particle charge is zero, the other alternative being that the test particle mass is infinite in the asymptotic limit, which is clearly unphysical. With this assumption, the constant c may be taken to be proportional to the electric charge q. In order for the geometrized units to be consistent, the proportionality factor must have dimensions of inverse length, the simplest choice being an inverse rest mass m_o which in the asymptotic limit gives

$$\phi \to 1 \quad \Rightarrow \quad c = \frac{q}{m_0} \quad \Rightarrow \quad m = m_0\sqrt{1+\left(\frac{q}{m_0}\right)^2} . \tag{3.12}$$

These definitions for the charge and mass can formally be extended to include the spacetime dependence through the scalar field and give

$$q_{eff} = q = const \quad \text{and} \quad m_{eff} = m_0\sqrt{1+\left(\frac{q}{m_o\phi}\right)^2} . \tag{3.13}$$

Before moving on to describe the energy of the test particles we make a few comments on how the interaction between the charge of the particle and the source affect the 4D acceleration equation. As a consequence of setting $c = 0$ in (3.7) we have effectively removed the electric charge from the problem, since $q = 0$ by (3.11). This leads to neutral particles which follow geodesics since both force terms drop out of the acceleration equation (3.6). But when we consider neutral spacetimes which have $F_{\alpha\beta} = 0$, the Lorentz force is zero and so charged particles cannot be defined as stated earlier, and the scalar force remains for neutral test particles. The acceleration equation (3.6) then provides us with a possible probe for the effects of KKG using the classical tests of general relativity [75]. In addition, possible violations of the Weak Equivalence Principle could be tested by an experiment such as the Satellite Test of the Equivalence Principle (STEP) [76]. In the case of spinning test particles, the scalar interactions are expected to be small but may be tested in the future with the Gravity Probe B (GP-B) experiment [77] (see [78] for reviews of GP-B).

We will now give the definition of energy for the test particle familiar to us from classical mechanics. We can define the energy from a 5D point of view provided that the spacetime admits a timelike Killing vector $\hat{\xi}^A = \delta^A{}_t$. The 5D energy is then the conserved quantity

$$\hat{E} = \hat{\xi} \cdot \hat{u} = \hat{g}_{tA}\hat{u}^A , \tag{3.14}$$

which, when expanded in terms of 4D quantities gives,

$$\hat{E} = \xi \cdot \left(u\sqrt{1 + c^2\phi^{-2}} + c\,A \right) . \tag{3.15}$$

In the above we have used the fact that the 5D Killing vector in the Jordan frame can be expressed as

$$\hat{\xi}^A = \delta^A{}_t \quad \Rightarrow \quad \hat{\xi}^\alpha = \delta^\alpha{}_t = \xi^\alpha , \tag{3.16}$$

so the 4D component of the 5D Killing vector is just the usual timelike 4D Killing vector. When the 5D energy (3.15) is compared to the classical 4D result from general relativity

$$E = \xi \cdot \left(u + \frac{q}{m} A \right) , \tag{3.17}$$

we find that the 5D and 4D energies are related by

$$\begin{aligned} \hat{E} &= \sqrt{1 + c^2 \phi^{-2}} \left(\xi \cdot u + \frac{c}{\sqrt{1 + c^2 \phi^{-2}}} \xi \cdot A \right) \\ &= \frac{m_{eff}}{m_0} \left(\xi \cdot u + \left. \frac{q}{m} \right|_{eff} \xi \cdot A \right) \end{aligned} \tag{3.18}$$

where we have used the definitions for the effective charge to mass ratio (3.7) and the effective mass (3.13) for the test particle.

3.2.1 The Weak-Field Limit

In this section we will examine the weak-field limit of the geodesic equation. We make the usual assumptions about the weak-field limit of the metric tensor and the scalar field, namely

$$\begin{aligned} g_{\alpha\beta} &= \eta_{\alpha\beta} + h_{\alpha\beta}, \qquad |h| \ll |\eta| \\ \phi &= 1 + \chi \qquad \chi \sim -\frac{2M_S}{r} \end{aligned} \tag{3.19}$$

and the velocities,

$$u^t \approx 1 , \quad \text{and} \quad u^i \approx \frac{v^i}{c} \ll 1 . \tag{3.20}$$

By expanding the i^{th}-component of the acceleration equation (3.6) it can be shown that equation (3.6) reduces to

$$\vec{a} = -\vec{\nabla} U + \frac{q}{m} \left(\vec{E} + \vec{v} \times \vec{B} \right) + \frac{\sigma}{m} \vec{\nabla} \phi . \tag{3.21}$$

where as usual the classical gravitational potential U is defined by $h_{tt} = -2U$. Here the electric and scalar charge-to-mass ratios are their asymptotic values given by (3.11). This is the classical result apart from the scalar force term which is a Kaluza-Klein addition. As an example of what the extra terms may appear like for a solution to the induced matter field equations (2.12)-(2.14) (with $\omega = 0$), we consider the static spherically-symmetric charged solution (2.17):

$$d\hat{s}^2 = -\frac{(1-k)B^a}{1-kB^{a-b}}dt^2 + B^{-a-b}dr^2 + r^2B^{1-a-b}d\Omega^2 + \frac{B^b - kB^a}{1-k}(dy + 2A_t\,dt)^2 \,. \quad (3.22)$$

The equation for the radial acceleration of a charged test particle in the weak-field limit then can be shown to reduce to

$$\ddot{r} = -\frac{M}{r^2}(a - kb) + \frac{qQ}{mr^2} - \frac{\sigma M}{2mr^2}(b - ka)\,, \qquad k = \frac{Q^2}{M^2(a-b)^2}\,. \quad (3.23)$$

This equation motivates the definition of the metric-based gravitational and scalar masses

$$M_g = M(a - kb) \quad \text{and} \quad M_s = \frac{M}{2}(b - ka)\,. \quad (3.24)$$

which we will verify using Komar integrals in chapter 5. For charged test particles, the electrostatic term will dominate the motion. Macroscopically $Q/M \ll 1$, and we expect the parameter b to be small as well, which gives the radial accleration

$$\ddot{r} = -\frac{Ma}{r^2} + \frac{qQ}{mr^2} - \frac{\sigma Mb}{2mr^2} \quad (3.25)$$

If the source is neutral ($Q = 0$ hence $k = 0$) there remains a scalar force term which affects the motion of neutral particles;

$$\ddot{r} = -\frac{Ma}{r^2} + \frac{\sigma Mb}{2mr^2}\,. \quad (3.26)$$

This implies that the parameter b must be very small since we do not observe such a force in macroscopic physics. As well, the scalar force may be attractive or repulsive depending on the sign of σb. In fact the choice $(a, b) = (1, 0)$ in neutral spacetimes would not violate any known physics, since this is just the Schwarzschild solution trivially embedded in 5D. This choice of parameters is supported by thermodynamical arguments which we discuss in appendix B.

3.3 Test Particles and the Canonical Metric

In this section we approach particle dynamics from a 5D Lagrangian for the canonical metric and use the Euler-Lagrange equations to obtain the acceleration equation induced in 4D. When the path parameterization is chosen judiciously we then show that the components of the 5D acceleration equation reproduce the 4D geodesic equation for massive and null particles, as well as a rest-mass variation for massive particles. The rest-mass variation is a direct consequence of the STM interpretation in which the extra coordinate is related to the rest-masses of particles. We conclude by giving an example 5D solution to the field equations which is of cosmological interest and make some final comments.

3.3.1 Motion and Mass Variation

To study the dynamics in 5D Kaluza-Klein gravity with the canonical metric (2.46) we consider the action

$$\hat{I} = \int_A^B \hat{L}(x^A, \dot{x}^A)\, d\lambda = \int_A^B \sqrt{-\frac{l^2}{L^2} g_{\alpha\beta}(x^\Sigma, l) \frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda} + \phi^2(x^\Sigma, l)\frac{dl^2}{d\lambda^2}}\, d\lambda\,, \qquad (3.27)$$

where λ is an arbitrary path parameter and the velocities are coterminal at the points A, B. With these boundary conditions, extremizing the action gives the well-known Euler-Lagrange equations

$$\frac{d}{d\lambda}\left(\frac{\partial \hat{L}}{\partial u^A}\right) - \frac{\partial \hat{L}}{\partial x^A} = 0 \quad \Rightarrow \quad \frac{du^A}{d\lambda} + \hat{\Gamma}^A_{BC}\, u^B u^C = u^A \frac{d}{d\lambda}\left(\ln \hat{L}\right)\,. \qquad (3.28)$$

Here, the term on the right hand side can be set to zero if the path parameterization is chosen to be the 5D proper distance $\lambda = \hat{s}$. The 4D and l components of equation (3.28) are:

$$u^\beta \nabla_\beta u^\alpha = \frac{d}{d\lambda}\ln\left(\frac{\hat{L}}{l^2}\right) u^\alpha - g^{\alpha\beta}\left[\partial_l g_{\beta\gamma} u^\gamma - \frac{1}{2}\left(\frac{L\phi}{l}\right)^2 \partial_\beta \left(\ln \phi^2\right)\dot{l}\right]\dot{l} \qquad (3.29)$$

$$\dot{l}\left\{\Sigma\left[\ln\left(\frac{\phi^2\dot{l}}{l}\right)\right]^{\cdot}-\frac{\dot{\phi}}{\phi}\left(\frac{L\phi\dot{l}}{l}\right)^2\right\}=$$
$$\frac{1}{2}\left(\frac{l}{L\phi}\right)\left[-\Sigma+\left(\frac{L\phi\dot{l}}{l}\right)^2\right]\left[\frac{2\Sigma}{l}+\partial_l g_{\alpha\beta}u^\alpha u^\beta+\left(\frac{L\phi\dot{l}}{l}\right)^2\partial_l\ln\phi^2\right]. \quad (3.30)$$

Here we have chosen the 4D proper distance to be the path parameter $\lambda = s$ such that $u_\alpha u^\alpha \equiv \Sigma$, which is 1 for timelike paths and 0 for null paths; and $(\dot{\ })$ is shorthand for $d/d\lambda$. The extra terms on the rhs of the two above equations are a result of choosing $\lambda = s$, as opposed to the 5D proper distance $\lambda = \hat{s}$ [66]. Equation (3.30) for $\dot{l}$ is very complicated in general, but we may get a simple result by choosing

$$\left(\frac{\dot{l}}{l}\right)^2=\frac{\Sigma}{L^2\phi^2}. \quad (3.31)$$

This satisfies (3.30) identically, and causes the particle paths to be 5D null even though we have chosen the 4D proper distance $\lambda = s$ to be the path parameter. That the paths are 5D null follows from the definition of the 5D canonical line element and the assumption (3.31) for $\dot{l}$. The relation (3.31) constrains the velocity $\dot{l}$ but does not give it physical meaning. For this we turn to the STM approach to Kaluza-Klein gravity, the essential feature being that the extra coordinate can be interpreted as a geometric mass via $l = Gm/c^2$ [17]-[21]. We now look at the variation of rest mass as a function of 4D path parametrization. The solution for the rest mass of a particle is easily obtained from (3.31) and is

$$m = m_o\exp\left(\pm\sqrt{\frac{\Sigma}{L^2}}\int ds\,\phi^{-1}\right). \quad (3.32)$$

Since in 4D we have $\Sigma = 0$ for photons, this implies that the variation in a photon's rest mass is zero and its mass may consistently be set to zero. However, for 4D paths with $\Sigma = 1$, there is a variation in the rest-mass of massive particles driven by the scalar field ϕ which behaves as a Higgs-type field generating the rest-mass variation. If the 4D cylinder condition ($\partial_l g_{\alpha\beta} = 0 \rightarrow \phi = 1$) is imposed, we get a cosmological variation of the rest masses of massive particles in the deSitter vacuum ($\epsilon = 1$). We now turn our attention to the acceleration equation.

Some algebra is required to show that the 4D acceleration equation (3.29) can be reduced to the form

$$u^{\beta}\nabla_{\beta} u^{\alpha} = f^{\alpha} , \tag{3.33}$$

where f^{α} is the force per unit rest mass, given by

$$f^{\alpha} = h^{\alpha\gamma} \left(\Sigma \frac{\phi_{\gamma}}{\phi} + \partial_l g_{\gamma\beta} u^{\beta} \dot{l} \right) . \tag{3.34}$$

When the 4D cylinder condition is imposed ($\partial_l g_{\alpha\beta} = 0 \rightarrow \phi = 1$), the force term is $f^{\alpha} = 0$. This gives geodesic motion for both photons and massive particles in the pure 4D deSitter vacuum, which is the correct 4D result in general relativity. However, when the 4D cylinder condition is lifted ($\partial_l g_{\alpha\beta} \neq 0$), photons will still travel along null 4D geodesics since they obey $\Sigma = 0$ and $\dot{l} = 0$; but massive particles will experience a geometric force since $\Sigma = 1$ and $\dot{l} \neq 0$. This rest-mass variation of massive particles could in principle be used to test for the existence of the scalar field which could be inferred from particle motion in the coming Satellite Test of the Equivalence Principle (STEP) [76]. We now consider some examples to elucidate these ideas.

3.3.2 Cosmological Example

In this section we revisit the cosmological example first discussed in chapter 2. We will show that the rest masses of particles may vary in a cosmological frame which employs a comoving coordinate system, and make some comments about the observability of the geometric force for comoving particles.

Consider the one-parameter Ponce de Leon [30] class of cosmological solutions (2.57):

$$d\hat{s}^2 = \frac{l^2}{L^2}\left(-c^2dt^2 - \left(\frac{ct}{L}\right)^{2/\alpha}\left(\frac{l}{L}\right)^{2\alpha/(1-\alpha)} d\vec{x}\cdot d\vec{x}\right) + \left(\frac{\alpha}{1-\alpha}\right)^2\left(\frac{ct}{L}\right)^2 dl^2 . \quad (3.35)$$

Since the 4D metric has a non-trivial l-dependence. the cosmological constant will not be pure deSitter as seen in chapter 2. A simple analysis of (3.31) with $l \equiv Gm/c^2$ leads to rest-mass variation as follows:

$$\frac{\dot{m}}{m} = \pm\left(\frac{1-\alpha}{\alpha}\right)\frac{1}{t}. \quad (3.36)$$

For the present universe ($t \sim 15\times 10^9\, yr$) whose matter content is well approximated by dust ($\alpha = 3/2$). the variation of rest masses is approximately $2\times 10^{-11}\, yr^{-1}$ which is consistent with the classical tests of 4D general relativity [20]. [79]. [80].

The acceleration equation for the Ponce deLeon metric is simplified when evaluated in the comoving coordinate system. In general, the assumption that the spatial velocities are zero ($u^i = 0$). implies that the scalar field can only depend on time. so $\phi = \phi(t)$. Thus we can conclude that any 5D metric in the canonical form of (2.46). which has the 4D section $g_{\alpha\beta}$ written in comoving coordinates with a time-dependent scalar field. will not impart a fifth force and the motion will be 4D geodesic.

3.4 Final Comments

For the KKEM metric and the 5D geodesic equation we have been able to deduce the electric and scalar properties of 4D test particles from the induced 4D Kaluza-Klein-Lorentz equation. We found that the electric charge of test particles is constant while mass is not and has a scalar field dependence. In the weak-field limit we derive the usual Lorentz equation augmented by a scalar force term which is parametrized by the scalar mass $M_s = Mb/2$ and has to be small to agree with the classical tests of relativity.

By retaining the extra coordinate $x^4 = l$ in 5D Kaluza-Klein gravity we have seen that a 5D vacuum may induce non-trivial matter and particle dynamics on 4D hypersurfaces. We used the assumption of 5D null geodesics to induce particle motion in the 4D subspace with the added feature of variable rest mass for massive particles, once we interpreted the extra dimension as mass [17]-[20]. The acceleration of null particles remained the same as in 4D, but the motion for massive particles was modified by a geometric force. This force has a contribution from a scalar field and crucially depends on the existence of the extra dimension. This motion was investigated for the Ponce de Leon class of solutions and it was found that the geometric force is undetectable for observers in the comoving coordinate system. It seems that we should turn to l-dependent analogues of the Schwarzschild metric [68] to observe and test any deviations from the classical tests of GR due to the geometric force. Work on this is underway, and we expect to relate 5D dynamics to the upcoming Satellite Test of the Equivalence Principle [76] and Gravity Probe B (GP-B) [78].

Chapter 4

Gauges in KKG

4.1 Introduction

Five-dimensional Kaluza-Klein theory is a generalization of 4D Einstein theory and is commonly regarded as a unified theory of the gravitational. electromagnetic and scalar fields whose quantum analogues are the spin-2 graviton. spin-1 photon and the spin-0 scalar. In the present chapter we wish to give a generic account of how a gauge choice in KKG may induce novel physics in the 4D manifold. We specifically choose the 5D harmonic gauge and analyze two specific cases of this gauge. Linearized 5D gravity with an extra coordinate dependence is explored, and secondly the 5D harmonic gauge is examined for the KKEM metric.

In section 2 we take a new look at the case where the 5D metric can be written as a perturbation from 5D Minkowski space. The linearized 5D field equations can be made algebraically tractable if one chooses the 5D harmonic gauge. This gives a wave equation for the 4D components of the perturbation and a Klein-Gordon type equation for the extra part of the perturbation. These wave equations in general have sources, and can represent gravitons with finite masses. We investigate a restricted version of 5D linearized gravity in which the 5D cylinder condition holds and we choose the 5D linearized KKEM metric. In section 3, we consider the full conformally rescaled KKEM metric and again choose the 5D harmonic gauge. For the conformal parameter choice giving 4D minimally coupled gravity the 5D harmonic gauge reduces to the 4D harmonic gauge of gravity and the Lorentz gauge in electromagnetism, while for non-minimally coupled gravity these gauges are coupled to the scalar field. We then examine the geometric optics approximation to Maxwell's equations and discuss the propagation of photons in both NC and NMC gravity. We also present a plane wave solution to the field equations in the 5D harmonic gauge which is comprised of both EM plane waves and gravitational parallel plane (pp) waves.

4.2 Waves in Linearized 5D Theory

To linearize 5D gravity we assume as in the 4D problem that the metric can be written as

$$\hat{g}_{AB} = \hat{\eta}_{AB} + \hat{h}_{AB}, \qquad |\hat{h}| \ll |\hat{\eta}| \tag{4.1}$$

where $\hat{h}_{AB}$ is viewed as a small perturbation from flat 5D Minkowski space. The block diagonal form for the 5D Minkowski space is $\hat{\eta}_{AB} = (\eta_{\alpha\beta}, \epsilon)$ where ϵ can be spacelike $(+1)$ or timelike (-1). The inverse of the above metric to $O(\hat{h})$ is

$$\hat{g}^{AB} = \hat{\eta}^{AB} - \hat{h}^{AB}, \tag{4.2}$$

and the Christoffel symbols to $O(\hat{h})$ are

$$\hat{\Gamma}^{A}_{BC} = \frac{1}{2}(\partial_B \hat{h}^{A}{}_{C} + \partial_C \hat{h}^{A}{}_{B} - \partial^A \hat{h}_{BC}). \tag{4.3}$$

Here $\partial_A = \frac{\partial}{\partial x^A}$ and indices are raised and lowered using the flat-space metric ($\hat{h}^A{}_C = \hat{\eta}^{AB}\hat{h}_{BC}$). We take as our starting point the fifteen field equations representing the 5D vacuum

$$\hat{R}_{AB} = 0\,. \tag{4.4}$$

Evaluation of the 5D Ricci tensor to $O(\hat{h})$ gives

$$\hat{R}_{AB} = \frac{1}{2}(\partial_A\partial_C\hat{h}^C{}_B + \partial_B\partial_C\hat{h}^C{}_A - \hat{\Box}\hat{h}_{AB} - \partial_A\partial_B\hat{h}^C{}_C)\,, \tag{4.5}$$

where the trace of $\hat{h}_{AB}$ is defined as

$$\hat{h} \equiv \hat{h}^A{}_A = \hat{\eta}^{AB}\hat{h}_{AB}\,, \tag{4.6}$$

and the linearized 5D box operator is defined as

$$\hat{\Box} \equiv \hat{\eta}^{AB}\partial_A\partial_B\,. \tag{4.7}$$

In order to reduce the algebraic complexity of the 5D Ricci tensor it is advantageous to choose the harmonic gauge:

$$\hat{\Gamma}^C \equiv \hat{g}^{AB}\hat{\Gamma}^C_{AB} = 0\,. \tag{4.8}$$

This to first order in $\hat{h}$ gives

$$\partial_A\hat{h}^{AC} = \frac{1}{2}\partial^C\hat{h}\,, \tag{4.9}$$

which removes the first, second and fourth terms in (4.5), so that the 5D Ricci tensor reduces to

$$\hat{R}_{AB} = -\frac{1}{2}\hat{\Box}\hat{h}_{AB}\,. \tag{4.10}$$

If we now impose the 5D vacuum field equations (4.4), we obtain a wave equation for the $\hat{h}_{AB}$, namely

$$\hat{\Box}\hat{h}_{AB} = 0\,. \tag{4.11}$$

The trace of this equation gives

$$\hat{R} = -\frac{1}{2}\hat{\Box}\hat{h} = 0\,. \tag{4.12}$$

However, it is sometimes more convenient to use the tensor

$$\hat{\psi}_{AB} \equiv \hat{h}_{AB} - \frac{1}{2}\hat{\eta}_{AB}\hat{h}\,, \tag{4.13}$$

which by (4.10) and (4.12), satisfies the wave equation (a 5D Pauli-Fierz equation):

$$\hat{G^{AB}} = -\frac{1}{2}\hat{\Box}\hat{\psi}^{AB} = 0\,. \tag{4.14}$$

Now the harmonic gauge (4.9) applied to $\hat{\psi}_{AB}$ becomes

$$\partial_A\hat{\psi}^{AB} = 0\,. \tag{4.15}$$

Since this equation is a representation of the non-covariant harmonic it is not invariant under arbitrary 5D coordinate transformations. We can, however, require (as in 4D theory) that (4.15) be invariant to $O(\hat{\xi})$ under the transformation

$$x^A \to x'^A = x^A + \hat{\xi}^A\,, \tag{4.16}$$

where $\hat{\xi}^A$ is an infinitesimal 5-vector. Under this transformation $\hat{h}_{AB}, \hat{\psi}_{AB}$ and the harmonic gauge condition transform as

$$\begin{aligned} \hat{h}_{AB} \to \hat{h}'_{AB} &= \hat{h}_{AB} - \partial_B\hat{\xi}_A - \partial_A\hat{\xi}_B & (4.17)\\ \hat{\psi}_{AB} \to \hat{\psi}'_{AB} &= \hat{\psi}_{AB} - \partial_B\hat{\xi}_A - \partial_A\hat{\xi}_B + \hat{\eta}_{AB}\partial_C\hat{\xi}^C & (4.18)\\ \partial_A\psi^{AB} \to \partial'_A\hat{\psi}'^{AB} &= \partial_A\hat{\psi}^{AB} - \hat{\Box}\hat{\xi}^B\,. & (4.19) \end{aligned}$$

The invariance of (4.15) under the gauge transformation (4.16) then holds provided the following wave equation for $\hat{\xi}^B$ is satisfied:

$$\hat{\Box}\hat{\xi}^B = 0\,. \tag{4.20}$$

The transformation (4.16) represents the only gauge freedom left in the theory and is important in deducing the actual degrees of freedom of the metric $\hat{h}_{AB}$. Now $\hat{h}_{AB}$ has fifteen independent components. But we have used five coordinate degrees of freedom in (4.16) and imposed five constraints through (4.15). We therefore conclude that $\hat{h}_{AB}$ has only $15 - 5 - 5 = 5$ degrees of freedom left. These correspond to two degrees of freedom for the gravitational field, two degrees for the electromagnetic field and one degree for the scalar field. To show this we refer back to the KKEM metric (2.1)

$$d\hat{s}^2 = g_{\alpha\beta}(x^\sigma)dx^\alpha dx^\beta + \phi^2(x^\sigma)\left(dy + 2A_\mu(x^\sigma)dx^\mu\right)^2 . \tag{4.21}$$

which obeys the 5D cylinder condition. If we linearize the 4D metric by $g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}$ as well as the scalar field by $\phi = \phi_0 + \chi$, and assume that $O(h) \sim O(A) \sim O(\chi)$, then the induced matter field equations (2.12)-(2.14) (with $\omega = 0$) reduce to

$$\begin{aligned} \Box_M h_{\alpha\beta} &= -\nabla_\alpha \chi_\beta & (4.22) \\ \Box_M A^\beta - \partial^\beta \partial_\alpha A^\alpha &= 0 & (4.23) \\ \Box_M \chi &= 0 , & (4.24) \end{aligned}$$

where $\Box_M \equiv \eta^{\alpha\beta}\partial_\alpha\partial_\beta$ is the 4D Minkowski box operator. The above three equations correspond to a massless spin-2 graviton, a massless spin-1 photon and a spin-0 scalar field. The restricted set of coordinate transformations for the KKEM metric is the set of 4D transformations that preserves 4D covariance and so is easily accomodated in the linearized theory. The coordinate transformations for the extra coordinate in the KKEM metric are

$$y \to y' = y + f(x^\sigma) \quad \Rightarrow \quad A_\alpha \to A'_\alpha = A_\alpha - \partial_\alpha f . \tag{4.25}$$

The coordinate transformations for the extra coordinate in the linearized 5D theory (with the cylinder condition imposed) are

$$x^4 \to x'^4 = x^4 + \hat{\xi}^4(x^\sigma) \quad \Rightarrow \quad \hat{h}_{4\alpha} \to \hat{h}'_{4\alpha} = \hat{h}_{4\alpha} + \partial_\alpha \hat{\xi}_4 \,. \tag{4.26}$$

By making a direct comparison it is evident that

$$\hat{h}_{4\alpha} \sim A_\alpha \quad \text{and} \quad \partial_\alpha \hat{\xi}_4 \sim \partial_\alpha f \tag{4.27}$$

and the gauge condition for the infinitesimal vector component $\hat{\xi}_4$ is

$$\Box \hat{\xi}_4 = 0 \quad \Rightarrow \quad \Box f = 0 \,. \tag{4.28}$$

which is the subsidiary condition for the gauge transformations in EM under the Lorentz gauge. Thus the equivalence between the off-diagonal components in 5D linearized theory and EM vector potential has been shown. We can again infer that the $\hat{h}_{4\alpha}$ are spin-1 fields as are the A_α and the spins of $\hat{h}_{\alpha\beta}$ and $h_{\alpha\beta}$ are the same (spin-2) as well as $\hat{h}_{44}$ and χ (spin-0).

We now reduce the field equations to 4D form and interpret their meaning. We have shown that the field equations for 5D linearized theory in the harmonic gauge are

$$\hat{\Box} \hat{h}^{AB} = 0 \qquad \text{or} \qquad \hat{\Box} \hat{\psi}^{AB} = 0 \tag{4.29}$$

$$\partial_A \hat{h}^{AB} = \frac{1}{2} \partial^B \hat{h} \qquad \qquad \partial_A \hat{\psi}^{AB} = 0 \tag{4.30}$$

with general covariance being replaced by the restricted coordinate transformations

$$x^A \to x'^A = x^A + \hat{\xi}^A \,, \quad \text{and} \quad \hat{\Box} \hat{\xi}^A = 0 \,. \tag{4.31}$$

From the $(\alpha\beta)$, (4α) and (44) components of the right part of (4.29) we obtain

$$(\alpha\beta) \qquad \Box_M \hat{\psi}^{\alpha\beta} = -\epsilon\, \partial_4^2 \hat{\psi}^{\alpha\beta} \tag{4.32}$$

$$(4\alpha) \qquad \Box_M \hat{\psi}^{4\alpha} = -\epsilon\, \partial_4^2 \hat{\psi}^{4\alpha} \tag{4.33}$$

$$(44) \qquad \Box_M \hat{\psi}^{44} = -\epsilon\, \partial_4^2 \hat{\psi}^{44} \,. \tag{4.34}$$

The β and fourth components of the constraint equations (4.30) yield

$$\partial_\alpha \hat{\psi}^{\alpha\beta} = -\partial_4 \hat{\psi}^{4\beta} \quad (4.35)$$

$$\partial_\alpha \hat{\psi}^{4\alpha} = -\partial_4 \hat{\psi}^{44}, \quad (4.36)$$

and the wave equation (4.31) for $\hat{\xi}^A$ gives

$$\Box_M \hat{\xi}^\alpha = -\epsilon\, \partial_4^2 \hat{\xi}^\alpha \quad (4.37)$$

$$\Box_M \hat{\xi}^4 = -\epsilon\, \partial_4^2 \hat{\xi}^4 . \quad (4.38)$$

In the above, the 5D box operator was expanded as

$$\hat{\Box} \equiv \hat{\eta}^{AB} \partial_A \partial_B = \eta^{\alpha\beta} \partial_\alpha \partial_\beta + \epsilon\, \partial_4^2 , \quad (4.39)$$

and using the 4D Minkowski box operator $\Box_M \equiv \eta^{\alpha\beta}\partial_\alpha\partial_\beta$ equation (4.39) reduces to

$$\hat{\Box} = \Box_M + \epsilon\, \partial_4^2 . \quad (4.40)$$

The most important of the sets of equations given above is (4.32) since it describes an induced energy-momentum tensor from the definition $G^{\alpha\beta} \equiv T^{\alpha\beta}$. Its explicit form and the corresponding field equations are:

$$T^{\alpha\beta} \equiv -\epsilon\, \partial_4^2 \hat{\psi}^{\alpha\beta} , \quad (4.41)$$

$$\Box_M \hat{\psi}^{\alpha\beta} = T^{\alpha\beta} . \quad (4.42)$$

These are the equations for 4D linearized gravity, with a source term [29]. For a 5D vacuum to go to a 4D vacuum it would be necessary to have $\partial_4 \hat{g}_{AB} = 0$ (the 5D cylinder condition), $\hat{g}_{4\alpha} = 0$ and $g_{44} = const.$ (to satisfy the (4α) and (44) equations) . These conditions will not in general be satisfied by physically interesting solutions of the 5D field equations (4.4). So, in general, Kaluza-Klein theory in 5D generates an energy-momentum tensor for Einstein theory in 4D.

The question of whether the induced energy-momentum tensor is conserved ($\nabla_\alpha T^{\alpha\beta} = 0$) is now addressed. Since $T^{\alpha\beta}$ is of $O(\hat{\psi})$ we only need to verify that

$$\partial_\alpha T^{\alpha\beta} = 0\,, \tag{4.43}$$

since the product of $T^{\alpha\beta}$ and the Christoffel symbols is of $O(\hat{\psi}^2)$, which we neglect in our approximation. Taking the partial derivative of both sides of (4.42) and using the gauge conditions (4.35) we obtain

$$\partial_\alpha T^{\alpha\beta} = -\epsilon \partial_\alpha \partial_4^2 \hat{\psi}^{\alpha\beta}\,. \tag{4.44}$$

Therefore, for the induced stress-energy to be conserved in our approximation we must have $-\epsilon\partial_\alpha\partial_4^2\hat{\psi}^{\alpha\beta} = 0$. There are three cases in which this statement will hold, and they are presented here.

<u>Case1</u>: $\partial_4^2\hat{\psi}^{\alpha\beta} = 0$. This automatically sets the graviton mass to zero since this term drops out of the wave equation (4.32). The spin-1 $\hat{\psi}^{4\alpha}$ and spin-0 $\hat{\psi}^{44}$ fields also are massless by virtue of the gauge constraints and field equations.

<u>Case2</u>: $\partial_\alpha\hat{\psi}^{\alpha\beta} = 0$. For wavelike fields, this implies the transverse condition on $\hat{\psi}^{\alpha\beta}$ but may allow for a massive graviton. From the gauge conditions and the field equations we find that $\hat{\psi}^{4\alpha}$ and $\hat{\psi}^{44}$ fields are both massless.

A combination of these cases is when both of the above constraints hold, which implies that all the fields are massless.

<u>Case3</u>: The final case is when $\hat{\psi}^{4\beta} = 0$. Since $\partial_\alpha\hat{\psi}^{\alpha\beta} = -\partial_4\hat{\psi}^{4\beta}$, if $\hat{\psi}^{4\beta} = 0$ by a choice of the coordinate frame then $\partial_\alpha\hat{\psi}^{\alpha\beta} = 0$ automatically. This can be achieved by setting from the outset $\hat{h}_{4\beta} = 0$, and this defines what we will refer to as the natural frame. This condition removes the spin-1 field (up to a coordinate transformation involving the extra coordinate), and allows for either massive or massless gravitons, but constrains the scalar field to be massless.

We now look at a simple example of plane waves in the natural frame. The flat-space metric and the perturbation are

$$\hat{\eta}_{AB} = \begin{pmatrix} \eta_{\alpha\beta} & 0 \\ 0 & \epsilon \end{pmatrix} \qquad \hat{h}_{AB} = \begin{pmatrix} h_{\alpha\beta} & 0 \\ 0 & h_{44} \end{pmatrix}. \tag{4.45}$$

We can assume that $\hat{\psi}_{AB}$ has the form of a 4D gravitational wave and a scalar wave:

$$\hat{\psi}^{AB} = \begin{pmatrix} \hat{\psi}^{\alpha\beta} & 0 \\ 0 & \hat{\psi}^{44} \end{pmatrix} = \begin{pmatrix} A^{\alpha\beta} e^{i(k_\alpha x^\alpha + a x^4)} & 0 \\ 0 & A^{44} e^{i l_\alpha x^\alpha} \end{pmatrix}. \tag{4.46}$$

Here $A^{\alpha\beta}$ is the constant 4D polarization tensor. A^{44} is the amplitude of the scalar wave, and a is a constant with dimensions of inverse length which parametrizes the extra coordinate dependence. The field equations (4.32)-(4.34) simplify considerably and give

$$\Box_M \hat{\psi}^{\alpha\beta} = -\epsilon\, \partial_4^2 \hat{\psi}^{\alpha\beta} \quad \Rightarrow \quad k_\gamma k^\gamma = -\epsilon\, a^2 \tag{4.47}$$

$$\Box_M \hat{\psi}^{44} = -\epsilon\, \partial_4^2 \hat{\psi}^{44} \quad \Rightarrow \quad l_\gamma l^\gamma = 0\,. \tag{4.48}$$

We see that (4.48) can be interpreted as a massless scalar field, while (4.47) can be interpreted as a massive graviton when the parameter a is identified as

$$a = \pm\sqrt{\epsilon}\, m\,. \tag{4.49}$$

The induced stress-energy (4.41) is

$$T^{\alpha\beta} = -\epsilon\, a^2 A^{\alpha\beta}\, e^{i(k_\alpha x^\alpha + a x^4)}, \tag{4.50}$$

and its trace is given by

$$T^\alpha{}_\alpha = \epsilon\, a^2 A^\alpha{}_\alpha\, e^{i(k_\alpha x^\alpha + a x^4)}. \tag{4.51}$$

This will be zero either if the 4D polarization tensor is assumed to be trace-free or if we choose $a = 0$ (or both). This would imply a radiation-like equation of state induced on the 4D manifold representing gravitational radiation. The gauge condition (4.35) gives

$$k_\beta A^{\alpha\beta} = 0\,, \tag{4.52}$$

since the conservation of energy demands $\partial_4 \hat{\psi}^{\alpha\beta} = 0$ and so the propagation of the 4D gravitational plane wave is transverse as it is in the 4D linearized theory.

Thus far, there exists little difference in the qualitative features of 5D and 4D linearized theory, but the difference becomes apparent when the gauge freedom in $\hat{\xi}^A$ is used to obtain a transverse-traceless (TT) representation for $A^{\alpha\beta}$. The wave equations for the components of $\hat{\xi}^A$ which must be satisfied are

$$\Box_M \hat{\xi}^\alpha = -\epsilon\, \partial_4^2 \hat{\xi}^\alpha \tag{4.53}$$

$$\Box_M \hat{\xi}^4 = -\epsilon\, \partial_4^2 \hat{\xi}^4\,. \tag{4.54}$$

The choice

$$\hat{\xi}^a = (\hat{\xi}^\alpha, \hat{\xi}^4) = (-ie^\alpha e^{i(k_\gamma x^\gamma + ax^4)}, -i\epsilon e^4 e^{i(l_\gamma x^\gamma)}) \tag{4.55}$$

satisfies these wave equations, since (4.47) and (4.48) hold. In the natural frame, the transformations for the polarization components of $\hat{\psi}^{AB}$ under the gauge transformation are given by the following (see equation (4.18)) :

$$A'^{\alpha\beta} = A^{\alpha\beta} - k^\alpha e^\beta - k^\beta e^\alpha + \eta^{\alpha\beta} k_\gamma e^\gamma \tag{4.56}$$

$$A'^{4\alpha} = -a\epsilon e^\alpha e^{i(k_\gamma x^\gamma + ax^4)} - \epsilon e^4 l_\alpha e^{il_\gamma x^\gamma} \tag{4.57}$$

$$A'^{44} = A^{44} + \epsilon k_\alpha e^\alpha e^{i[(k-l)_\gamma x^\gamma + ax^4]}\,. \tag{4.58}$$

We see that off-diagonal amplitudes have been generated and that the scalar amplitude has also been changed. What is interesting about the off-diagonal components is that they are a linear superposition of plane waves with different wave vectors, and functions of all five coordinates. The choice of setting $a = e^4 = 0$ is consistent with all the equations derived above but physically limiting since is sets the off-diagonal components in the transformed natural frame to zero, and therefore removes the electromagnetic effects which are usually associated with these components. This choice also sets the graviton mass to zero, and hence the 5D theory would give a conventional TT (transverse-traceless) 4D gravitational wave and a scalar wave with an oscillating amplitude. Another simplification that takes place is if one chooses to remove the periodic behaviour from the scalar field amplitude with the choice $k_\alpha = l_\alpha$. Since l_α is null, this forces k_α to be null as well, and hence a must equal zero, which implies a massless graviton again. In this case, the off-diagonal terms survive and give a simple expression:

$$A'^{4\alpha} = -\epsilon k^\alpha e^4 e^{ik_\gamma x^\gamma} . \tag{4.59}$$

This has the form of an electromagnetic plane wave propagating with the same null wave vector as the gravitational plane wave.

Let us now consider an example of a plane gravitational wave in the natural frame propagating in the z-direction. The wave vector is

$$k_\gamma = (\omega, 0, 0, k) , \tag{4.60}$$

and obeys the conditions (4.47), (4.49) and (4.52). These give

$$\begin{aligned} \omega^2 - k^2 &= m^2 \\ \omega A^{0\alpha} &= k A^{3\alpha} . \end{aligned} \tag{4.61}$$

Performing the gauge transformations and using the above we see that the only independent components of the $\hat{\psi}^{AB}$ are

$$\begin{aligned} A'^{00} &= A^{00} - \omega e^0 - k e^3 \\ A'^{01} &= A^{01} - \omega e^1 \\ A'^{02} &= A^{02} - \omega e^2 \end{aligned}$$

$$
\begin{aligned}
A'^{11} &= A^{11} - \omega e^0 + k e^3 \\
A'^{12} &= A^{12} \\
A'^{22} &= A^{22} - \omega e^0 + k e^3 \\
A'^{44} &= A^{44} - \epsilon(\omega e^0 - k e^3).
\end{aligned} \tag{4.62}
$$

Then the choice

$$
\begin{aligned}
e^0 &= \frac{1}{2\omega}(A^{00} + \frac{1}{2}A^{11} + \frac{1}{2}A^{22}) \\
e^1 &= A^{01} \\
e^2 &= A^{02} \\
e^3 &= \frac{1}{2k}(A^{00} - \frac{1}{2}A^{11} - \frac{1}{2}A^{22})
\end{aligned} \tag{4.63}
$$

will bring the 4D polarization tensor to a transverse-traceless form. but will generate off-diagonal terms and an oscillating scalar field amplitude. The choice $e^4 = a = 0$ corresponds to setting the off-diagonal terms of $A'^{\alpha\beta}$ to zero, and hence $\hat{\psi}'^{AB}$ represents a massless TT gravitational wave and a scalar wave. If we choose $l_\alpha = k_\alpha$ (which again forces the graviton mass to zero), the 4D gravitional wave is again TT. but the off-diagonal components with the choice of $e^4 = \epsilon C/k$ (where $C \ll 1$ is a constant. since e^4 must be infinitesimal), are:

$$
\begin{aligned}
A'^{04} &= -C e^{ik_\gamma x^\gamma} \\
A'^{34} &= C e^{ik_\gamma x^\gamma}.
\end{aligned} \tag{4.64}
$$

These are simple plane waves with constant amplitude.

4.3 Waves in Conformal 5D Gravity

In this section we investigate the consequences of imposing the 5D harmonic gauge condition for the conformal KKEM metric. We then look at the propagation of electromagnetic waves and their interaction with the scalar field. To end this section we give a solution to the field equations which has physical relevance.

Let us consider the metric

$$d\hat{s}^2 = e^{2c\sigma} g_{\alpha\beta} dx^\alpha dx^\beta + e^{2d\sigma}(dx^4 + 2A_\alpha dx^\alpha)^2 \,, \tag{4.65}$$

where c and d are constants and σ is the scalar field. The requirement that this metric satisfy the 5D harmonic gauge condition (4.8) gives two equations:

$$\hat{\Gamma}^\gamma \equiv \hat{g}^{AB}\,\hat{\Gamma}^\gamma_{AB} = 0 \quad \Rightarrow \quad g^{\alpha\beta}\,\Gamma^\gamma_{\alpha\beta} + (2c+d)\,\sigma^\gamma = 0 \tag{4.66}$$

$$\hat{\Gamma}^4 \equiv \hat{g}^{AB}\,\hat{\Gamma}^4_{AB} = 0 \quad \Rightarrow \quad \nabla_\alpha A^\alpha + (2c+d)\,\sigma_\alpha A^\alpha = 0\,. \tag{4.67}$$

Here ∇_α is the 4D covariant derivative operator associated with the metric $g_{\alpha\beta}$. In order for the 5D harmonic gauge to induce the 4D harmonic gauge as well as the Lorentz gauge, we are forced to choose $2c + d = 0$ (MC gravity with $(c, d) = (-1/\sqrt{3}, 2\sqrt{3})$. With this constraint the metric takes the form

$$d\hat{s}^2 = e^{-2\sigma/\sqrt{3}} g_{\alpha\beta} dx^\alpha dx^\beta + e^{4\sigma/\sqrt{3}}(dx^4 + 2A_\alpha dx^\alpha)^2 \,, \tag{4.68}$$

which is the usual metric for Kaluza-Klein gravity in the Einstein frame [36].

We now investigate the propagation of electromagnetic waves using the Lorentz gauge (4.67) and Maxwell's equations (2.10). We postpone the use of the 4D harmonic gauge so the following equations hold covariantly in 4D for the induced matter from the Kaluza-Klein metric in the Einstein frame. The 4D harmonic gauge will however be imposed when we discuss an exact solution at the end of this section. The ansatz for the electromagnetic vector potential is

$$A_\alpha = a_\alpha e^{i\omega S} \,, \tag{4.69}$$

which represents a good approximation for large ω only. This statement is usually referred to as the geometric optics approximation [53]. Substituting this ansatz into both the Lorentz gauge condition and Maxwell's equations (2.10) and setting the coefficients of ω^2 and ω separately to zero. gives respectively

$$\nabla_\alpha A^\alpha = 0 \qquad \Rightarrow \qquad k_\alpha a^\alpha = 0 \tag{4.70}$$

$$O(\omega^2): \qquad k_\alpha k^\alpha = 0 \tag{4.71}$$

$$O(\omega): k^\alpha \nabla_\beta a^\beta - a^\alpha \nabla_\beta k^\beta - 2k^\beta \nabla_\beta a^\alpha = -2\sqrt{3}\,\sigma_\beta(k^\alpha a^\beta - k^\beta a^\alpha)\,. \tag{4.72}$$

Here $k_\alpha \equiv \partial_\alpha S$ is the gradient to the surfaces of constant phase. and as such the null condition implies that the k_α follow null geodesics:

$$k^\beta \nabla_\beta k^\alpha = 0\,. \tag{4.73}$$

Contraction of (4.72) with $\overline{a}_\alpha$ (the complex conjugate of a_α) and the orthogonality condition (4.70) allows us to obtain

$$\nabla_\alpha(a^2 k^\alpha) = -2\sqrt{3}\,\sigma_\alpha(a^2 k^\alpha)\,, \tag{4.74}$$

where $a^2 \equiv a_\alpha \overline{a}^\alpha$. If we define a photon current $j^\alpha \equiv a^2 k^\alpha$, the right-hand side of (4.74) can be interpreted as a nonconservation of photon number due to the coupling of the gradient of the scalar field to the wave vector. The equation governing the propagation of the unit polarization vector can be obtained from (4.74) by the substitution of $f^\alpha = a^\alpha/a^2$, and is

$$k^\beta \nabla_\beta f^\alpha = \sqrt{3}\,\sigma_\beta f^\beta k^\alpha + \frac{1}{2}\left(\nabla_\beta f^\beta + \frac{f^\beta \nabla_\beta a}{a}\right) k^\alpha\,. \tag{4.75}$$

This again differs from the usual 4D result due to the scalar-field coupling. However, both the nonconservation of photon number and the effect of the scalar field term on the propagation of the unit polarization vector could in principle be tested, although we expect the deviations from standard 4D results to be small in physically realistic weak-field situations.

So far we have not used the 4D harmonic gauge in deriving the above equations. We will illustrate the use of this gauge by deriving an exact, gravitational plane wave solution accompanied by a plane electromagnetic wave and a scalar wave. The choice of an electromagnetic plane wave simplifies the induced 4D field equations since $F^2 = 0$ for plane waves. This reduces the Kaluza-Klein field equations (2.12)-(2.14) along with the 4D harmonic and Lorentz gauge to

$$R^H_{\alpha\beta} = 2e^{2\sqrt{3}\sigma} F_{\alpha\gamma} F_\beta{}^\gamma + 2\sigma_\alpha \sigma_\beta \tag{4.76}$$

$$\partial_\alpha F^{\alpha\beta} + \left(\partial_\alpha \ln \sqrt{-g}\right) F^{\alpha\beta} = -2\sqrt{3}\,\sigma_\alpha F^{\alpha\beta} \tag{4.77}$$

$$\Box\sigma = g^{\alpha\beta}\partial_\alpha\partial_\beta\,\sigma = 0 \tag{4.78}$$

$$g^{\alpha\beta}\Gamma^\gamma_{\alpha\beta} = 0 \tag{4.79}$$

$$\nabla_\alpha A^\alpha = g^{\alpha\beta}\partial_\alpha A_\beta = 0. \tag{4.80}$$

Here $R^H_{\alpha\beta}$ is the Ricci tensor in the 4D harmonic gauge. We would like a 4D gravitational wave metric that would simplify these equations even further. The candidate metric should have a determinant equal to a constant, which would remove the second term on the left-hand side of (4.77), while also satisfying the 4D harmonic gauge (4.79). A simple metric which satisfies these conditions is that of an exact gravitational plane wave with parallel rays (pp-waves have a null vector field which satisfies $\nabla_\alpha k_\beta = 0$ [24]) travelling along the z-direction. Such a metric has the form :

$$ds^2 = -K(u,\xi,\bar{\xi})du^2 - 2du dv + d\xi d\bar{\xi}\,. \tag{4.81}$$

Here u and v are retarded and advanced coordinates, and $\xi = x + iy$ and $\bar{\xi} = x - iy$ are complex transverse coordinates. If we choose the electromagnetic vector potential and the scalar field to be independent of the complex transverse coordinates, we have

$$A_\alpha = A(u)\,\delta_{\alpha x} \qquad \text{and} \qquad \sigma = \sigma(u)\,. \tag{4.82}$$

The vector potential then corresponds to an electric field oscillating in the x-direction and a magnetic field oscillating in the y-direction, while the wave propagates in the z-direction. One can check that the scalar wave equation (4.78) and Maxwell's equations (4.77) are satisfied by arbitrary functions $\sigma(u)$ and $A(u)$, and that the only surviving component of the Ricci tensor is R_{uu}, which gives for (4.76) the equation

$$\partial_\xi \partial_{\bar{\xi}} K(u,\xi,\bar{\xi}) = e^{2\sqrt{3}\sigma}[\,\partial_u A(u)\,]^2 + [\,\partial_u \sigma(u)\,]^2. \qquad (4.83)$$

This equation can be integrated immediately to give

$$K(u,\xi,\bar{\xi}) = \left(e^{2\sqrt{3}\sigma}[\,\partial_u A(u)\,]^2 + [\,\partial_u \sigma(u)\,]^2\right)\xi\bar{\xi} + f(u)\xi^2 + \bar{f}(u)\bar{\xi}^2\,, \qquad (4.84)$$

where $f(u)$ is an arbitrary complex function. Since we are interested in electromagnetic and scalar waves we assume that they can be written as

$$A(u) = \mathrm{Re}\,\left(A_0 e^{i\omega u}\right) \qquad \sigma(u) = \mathrm{Im}\,\left(\sigma_0 e^{i\lambda u}\right)\,, \qquad (4.85)$$

where A_0 and σ_0 are real constants. This gives on taking the real parts only,

$$K(u,\xi,\bar{\xi}) = \left[e^{2\sqrt{3}\sigma_o \cos(\lambda u)}[\,A_0^2 \sin^2(\omega u)\,] + \sigma_0^2 \lambda^2 \sin^2(\lambda u)\right]\xi\bar{\xi}\,. \qquad (4.86)$$

This particular solution is simple and would probably repay future investigation.

4.4 Final Comments

In this chapter we have shown that 5D gravity can be linearized in the same way as 4D gravity, but that the harmonic gauge (4.8) is very restrictive; and since it is non-covariant, conservation of the induced energy-momentum tensor on the 4D subspace holds only when certain restrictions are imposed on the components of the perturbation tensor. Also, while the equations allow in principle for massive gravitons. the most natural choice of parameters which removes electromagnetic effects also reduces the graviton mass to zero. We have also shown that the implementation of the 5D harmonic gauge for the conformally rescaled KKEM metric reproduces the 4D harmonic gauge plus the Lorentz gauge provided we choose minimally coupled gravity. In the geometric optics limit the coupling of the scalar field to Maxwell's equations alters the propagation equations for photons. The most notable effect is the nonconservation of photon current. It should be noted that the modification to photon motion is not restricted to Kaluza-Klein gravity, but will occur in any theory that has a scalar field coupled to electromagnetism.

Chapter 5

A Hamiltonian Treatment of 5D KKG

5.1 Introduction

With a renewed interest in the literature concerning the energy of solutions in general relativity from a Hamiltonian point of view [81], [82], and its relation to entropy [83]-[85], we formulate a definition of energy making use of a background subtraction term for Kaluza-Klein theory that retains the extra coordinate dependence. This chapter aims to investigate the energy and angular momentum of 5D solutions. In the case where the 5D spacetimes obey the 5D cylinder condition the definition of energy from the Hamiltonian dimensionally reduces to the energy associated with 4D non-minimally coupled induced matter, and is shown to equal conserved quantities in a non-minimally coupled scalar-tensor theory. This comparison is important since it is well known that Kaluza-Klein gravity is a non-standard theory of gravity which includes at the very least a scalar field plus Jordan-frame gravity, and therefore must have conserved quantities associated with both sectors of the theory. We will show that for the examples we consider, both approaches give the same results. With the 5D cylinder condition imposed, we next study theories which can be derived from 5D KK theory via conformal transformations and dimensional reduction of the 5D Hamiltonian. These theories include the minimally-coupled (MC) and non-minimally-coupled (NMC) Einstein-Maxwell theory, and Brans-Dicke (BD) theory. The energy and angular momentum of axially symmetric stationary spacetimes are calculated for a host of examples from these different theories. We then compare the results of the previous section to how conserved quantities constructed from Killing vectors (the Komar integrals) behave under conformal transformations for the induced scalar-tensor theories, and agreement with the results from the charged GPS spacetimes of chapter 2 is shown. When appropriate, we will draw comparisons to previous definitions of energy and angular momentum in the literature [83], [87], [88].

In section 2 we will first discuss notation and set up a 1+4 ADM split of the 5D spacetime metric and derive the lapse function and shift vector. In section 3 we will derive the Hamiltonian in general, paying special attention to the concept of background fields in 5D to define a zero-point energy, and then examine certain physical limits of this definition of energy. As an example we calculate the energy for a class of 5D spherically symmetric solutions with the cylinder condition enforced, known as the GPS solutions [38]-[40]. We then extended this example to the charged version [37]. In section 4 we look at the energy associated with imposing the cylinder condition and show that it agrees with conserved quantities associated with the 4D induced scalar-tensor matter for the two GPS cases considered. It is also shown that the Hamiltonian energy or the total mass associated with these solitons is the sum of the gravitational and scalar masses. Section 5 deals with the derivation of the conserved quantities associated with the induced matter theory via Komar integrals. We show here that the total mass derived for solutions considered can be represented in a conserved scalar and gravitational Komar integrals. In section 6 we extend the Komar integrals to include other theories of gravity by considering the conformally rescaled action and its dimensional reduction. We apply the results to solutions of Brans-Dicke gravity which include rotation as well as considering the rotating version of the GPS solutions [89], [90].

5.2 A 1+4 Split and the 5D Kaluza-Klein Metric

We begin by explaining the foliation of the 5D spacetime and stating the notation that will be used. The notation used in [81] will be adopted throughout the chapter with modifications to the dimension of the spacetime. We assume that the 5D manifold M_5 with metric $\hat{g}$ can be foliated by a time function $t(x^A)$ defined over a closed interval I_t which generates a set of 4D spacelike hypersurfaces $\hat{\Sigma}_t$ labelled by t, and a vector field $\hat{t}^A$ which obeys $\hat{t}^A\hat{\nabla}_A t = 1$. The vector field $\hat{t}^A$ can be decomposed using the unit normal $\hat{n}^A$ to the sufaces $\hat{\Sigma}_t$ and the shift $\hat{N}^A$ such that $\hat{t}^A = \hat{N}\,\hat{n}^A + \hat{N}^A$. The boundary ∂M_5 of M_5 is the union of initial $\hat{\Sigma}_{t_i}$ and final $\hat{\Sigma}_{t_f}$ 4D spacelike hypersurfaces, and a timelike 4D boundary $\hat{B}$ which has a unit normal $\hat{u}^A$. The intersection of $\hat{\Sigma}_t = I_y \times \Sigma_t$ and $\hat{B}$ is a 3-surface $B_t^3 = I_y \times B_t^2$ which bounds the 4D hypersurface $\hat{\Sigma}_t$. Here I_y is a finite interval of the extra dimension. The unit normal to the surface B_t^3 is denoted by $\hat{r}^A$. The set of all B_t^3 then foliates the timelike boundary $\hat{B} = I_t \times B_t^3$, where I_t is a finite interval in time. We also assume for simplicity that the surfaces $\hat{\Sigma}_t$ and the timelike surfaces $\hat{B}$ intersect orthogonally, so that $\hat{u}_A\hat{n}^A = 0$. With the above 5D spacetime split the 5D metric can then be written in 1+4 ADM form as

$$d\hat{s}^2 = -\hat{N}^2 dt^2 + \hat{h}_{ab}\,(dx^a + \hat{N}^a dt)(dx^b + \hat{N}^b dt)\,. \tag{5.1}$$

where $\hat{h}_{ab}$ is the induced metric on $\hat{\Sigma}_t$.

Since we wish to make contact with the usual 5D Kaluza-Klein theory we must compare the above metric with the standard 5D metric written in the Jordan frame:

$$d\hat{s}^2 = \hat{g}_{AB}(x^\sigma, y)\, dx^A dx^B \tag{5.2}$$

$$= g_{\alpha\beta}\, dx^\alpha dx^\beta + \phi^2 (dy + 2A_\alpha\, dx^\alpha)^2\,. \tag{5.3}$$

To bring the metric into the Einstein frame a conformal rescaling of the 4D metric and the scalar field is necessary. A discussion of conformal transformations their role in the 5D Hamiltonian formulation is discussed in detail in [86]. Here we assume that the 4D metric can be written in the standard 1+3 ADM form [91] :

$$ds^2 = g_{\alpha\beta}(x^\sigma, y)\, dx^\alpha dx^\beta \tag{5.4}$$

$$= -N^2 dt^2 + h_{ij}\,(dx^i + N^i dt)(dx^j + N^j dt)\,. \tag{5.5}$$

By absorbing the 1+3 ADM (5.5) metric into the Kaluza-Klein metric (5.3) we can write the 5D metric in 1+3+1 form as

$$d\hat{s}^2 = -N^2 dt^2 + h_{ij}\,(dx^i + N^i dt)(dx^j + N^j dt) + \phi^2 (dy + 2A_t\, dt + 2A_i\, dx^i)^2. \tag{5.6}$$

When this metric is compared to the 1+4 form of the metric (5.1) we find that the lapse for the 1+4 metric equals the lapse associated with the 1+3+1 metric, namely

$$\hat{N} = N\,. \tag{5.7}$$

Also, the unit normal to $\hat{\Sigma}_t$ in covariant and contravariant form is

$$\hat{n}_A = (-N, 0, 0, 0, 0)\,, \qquad \hat{n}^A = \hat{g}^{AB}\hat{n}_B = \left(\frac{1}{N}, -\frac{N^i}{N}, \frac{-2A_t + N^i 2A_i}{N}\right). \tag{5.8}$$

The 4D shift vector in contravariant and covariant form is given by

$$\hat{N}^a = (N^i\,, 2A_t - 2N^i A_i)\,, \qquad \hat{N}_a = \hat{h}_{ab}\hat{N}^b = (N_i + 4\phi^2 A_t A_i\,, 2\phi^2 A_t) \tag{5.9}$$

and the induced metric on $\hat{\Sigma}_t$ is

$$\hat{h}_{ab} = \begin{pmatrix} h_{ij} + 4\phi^2 A_i A_j & 2\phi^2 A_i \\ 2\phi^2 A_j & \phi^2 \end{pmatrix}. \tag{5.10}$$

This metric comparison was first introduced by Beciu [92], but there are errors in that article concerning the shift vector that are corrected here. We note in passing that if the electromagnetic vector potential is set to zero ($A_\alpha = 0$) then the 5D metric (5.3) is block diagonal, and the 1+4 lapse function and shift vector are identical to their 1+3 counterparts.

5.3 Derivation of the Hamiltonian

The 5D action appropriate for Kaluza-Klein gravity in which the induced metric $\hat{h}_{ab}$ is held fixed is

$$\hat{I} = \frac{1}{2\hat{\kappa}} \int_{M_5} d^5x \sqrt{-\hat{g}}\, \hat{R} + \frac{1}{\hat{\kappa}} \oint_{\partial M_5} d^4x \sqrt{\hat{h}}\, \hat{K}\,, \tag{5.11}$$

where $\hat{\kappa} = 8\pi G_5$ is the five-dimensional gravitational coupling constant and $\hat{K}$ is the trace of the extrinsic curvature of the boundary. By fixing the induced metric $\hat{h}_{ab}$ on ∂M_5 one is effectively fixing the 3D metric h_{ij}, the spatial components of the electromagnetic vector potential A_i and the scalar field ϕ, all on the boundary ∂M_5. At this point it is usually assumed that the 5D spacetime has a Killing vector $\hat{\zeta} = \delta^A{}_4$. If so, one can dimensionally reduce the action to 4D, to produce the non-minimally coupled gravitational action. This dimensional reduction also introduces a length scale which is used to redefine the 5D gravitational coupling constant by

$$\frac{1}{\kappa} = \frac{1}{\hat{\kappa}} \int_{I_y} dy, \tag{5.12}$$

where the interval I_y must be finite. We do not assume this symmetry here initially and treat the extra spacelike coordinate democratically until the end where one can assume y-independence if one chooses. To begin the traditional Hamiltonian treatment, the 5D Ricci scalar must be decomposed using the 1+4 split [53]-[56] as

$$\hat{R}(\hat{g}) = R(\hat{h}) + \hat{K}_{ab}\hat{K}^{ab} - \hat{K}^2 - 2\hat{\nabla}_A\left(\hat{a}^A - \hat{n}^A\hat{K}\right). \tag{5.13}$$

Here $\hat{a}^A = \hat{n}^B\hat{\nabla}_B\hat{n}^A$ is the acceleration of the normal vector $\hat{n}^A$, and $\hat{R}(\hat{h})$ is the Ricci scalar associated with the metric $\hat{h}_{ab}$. The extrinsic curvature of the 4D hypersurface $\hat{\Sigma}_t$ and its trace are defined as

$$\hat{K}_{ab} = \frac{1}{2\hat{N}}\left(\partial_t\hat{h}_{ab} - \hat{D}_a\hat{N}_b - \hat{D}_b\hat{N}_a\right), \quad \hat{K} = \hat{h}^{ab}\hat{K}_{ab} \tag{5.14}$$

where $\hat{D}$ is the covariant derivative operator on the hypersurface Σ_t. Inserting (5.13) into the expression for the action (5.11) gives

$$\hat{I} = \frac{1}{2\hat{\kappa}}\int_{M_5} d^5x\sqrt{-\hat{g}}\left(R(\hat{h}) + \hat{K}_{ab}\hat{K}^{ab} - \hat{K}^2\right) + \frac{1}{\hat{\kappa}}\int_{\hat{B}} d^4x\sqrt{-\hat{\gamma}}\left(\hat{\Theta} - \hat{u}_A\hat{a}^A\right), \tag{5.15}$$

where we have used Gauss' theorem to remove the total divergences from (5.13), plus the fact that $\hat{n}_A\hat{a}^A = 0$, as well as the boundary condition $\hat{u}_A\hat{n}^A = 0$. The integral over the boundary $\hat{B}$ contains $\hat{\Theta}$, which is the trace of the extrinsic curvature associated with the timelike boundary $\hat{B}$, and $\hat{\gamma}$ its determinant. In order to reduce (5.15) to canonical form we need to define the momentum conjugate to the induced metric $\hat{h}_{ab}$ via

$$2\hat{\kappa}\hat{P}^{ab} \equiv \sqrt{\hat{h}}\left(\hat{K}^{ab} - \hat{h}^{ab}\hat{K}\right). \tag{5.16}$$

This can be used to simplify the kinetic term in the action. Some algebra and the use of Gauss' theorem again reveals that the action has the canonical form

$$\hat{I} = \int dt \left\{ \int_{\hat{\Sigma}_t} d^4x \, \hat{P}^{ab} \, \partial_t \hat{h}_{ab} - \hat{H} \right\} , \tag{5.17}$$

where the Hamiltonian is a sum of three terms, $\hat{H} = \hat{H}_c + \hat{H}_k + \hat{H}_m$ that are described below.

The first term $\hat{H}_c$ is the constraint term

$$\hat{H}_c = \int_{\hat{\Sigma}_t} d^4x \, \left(\hat{N}\hat{\mathcal{H}} + \hat{N}_a \hat{\mathcal{H}}^a \right) \tag{5.18}$$

$$= \int_{\hat{\Sigma}_t} d^4x \, N \left[\frac{2\hat{\kappa}}{\sqrt{\hat{h}}} \left(\hat{P}^{ab}\hat{P}_{ab} - \frac{1}{3}\hat{P}^2 \right) - \frac{\sqrt{\hat{h}}}{2\hat{\kappa}} R(\hat{h}) \right] -2 \int_{\hat{\Sigma}_t} d^4x \, \hat{N}_a \hat{D}_b \, \hat{P}^{ab} . \tag{5.19}$$

Since the lapse and the shift behave as Lagrange multipliers, a variation of the action with respect to them generates the constraints

$$\hat{\mathcal{H}} = \frac{2\hat{\kappa}}{\sqrt{\hat{h}}} \left(\hat{P}^{ab}\hat{P}_{ab} - \frac{1}{3}\hat{P}^2 \right) - \frac{\sqrt{\hat{h}}}{2\hat{\kappa}} R(\hat{h}) = 0 , \tag{5.20}$$

$$\hat{\mathcal{H}}^a = -2\hat{D}_b \hat{P}^{ab} = 0. \tag{5.21}$$

If the 5D cylinder condition is imposed, the first Hamiltonian constraint can be shown to give the usual Hamiltonian constraint for gravity and electromagnetism non-minimally coupled to the scalar field from a 1 + 3 split [86], [93]; and the second one represents Maxwell's equations coupled to the scalar field. This is to be expected since we began with a 5D metric that contained the 4D electromagnetic vector potential. If the cylinder condition is dropped these constraints would give the usual constraints in a $1+3$ split but with matter terms that would have extra-coordinate dependence.

The next term $\hat{H}_k$ in (5.17) is a boundary curvature term which will be identified with the total mass of a solution to the field equations, and is

$$\hat{H}_k = -\frac{1}{\hat{\kappa}} \int_{B_t^3} d^3x\, \hat{N}\sqrt{\hat{\sigma}}\, \hat{k}\,. \tag{5.22}$$

It can be shown [81], [82] that $\hat{k} = \hat{\Theta} - \hat{u}_a\, \hat{a}^a$ is the curvature of the boundary B_t^3 embedded in B, and $\hat{\sigma}$ is the determinant of B_t^3. The curvature term (5.22) can be reduced to a more familiar form by noting that for the $1+3+1$ form of the metric $\hat{N} = N$, $\sqrt{\hat{\sigma}} = \phi\sqrt{\sigma}$, and $B_t^3 = I_y \times B_t^2$, so that the above definition of mass reduces to

$$\hat{H}_k = -\frac{1}{\hat{\kappa}} \int_{I_y} dy \int_{B_t^2} d^2x\, N\phi\sqrt{\sigma}\, \hat{k}\,. \tag{5.23}$$

If the scalar field is a constant and the cylinder condition is imposed, we recover the usual 4D expression [81], [83].

The final term in (5.17) is the momentum term and includes the conjugate momentum defined with a different weight since the integration is over B_t^3, and is expressed as

$$\hat{H}_m = 2 \int_{B_t^3} d^3x\; \hat{r}_a \hat{N}_b\, \hat{P}_{\hat{\sigma}}^{ab} = 2 \int_{I_y} dy \int_{B_t^2} d^2x\; \hat{r}_a \hat{N}_b\, \hat{P}_{\hat{\sigma}}^{ab}, \tag{5.24}$$

where

$$\hat{P}_{\hat{\sigma}}^{ab} \equiv \frac{1}{2\hat{\kappa}} \sqrt{\hat{\sigma}}\, (\hat{K}^{ab} - \hat{h}^{ab}\hat{K}) = \frac{1}{2\hat{\kappa}}\, \phi\sqrt{\sigma}\, (\hat{K}^{ab} - \hat{h}^{ab}\hat{K}). \tag{5.25}$$

Since the constraint terms vanish for a 5D vacuum solution, we can use the curvature term (5.23) and the momentum term (5.24) to define the Hamiltonian of a 5D vacuum solution as

$$\hat{H} = -\frac{1}{\hat{\kappa}} \int_{I_y} dy \int_{B_t^2} d^2x \left(N\phi\sqrt{\sigma}\,\hat{k} - 2\,\hat{\kappa}\,\hat{r}_a \hat{N}_b \hat{P}^{ab} \right). \tag{5.26}$$

Some comments are in order about interpreting this as the total energy-momentum of a solution to the 5D field equations. Since the action in general will diverge for non-compact $\hat{\Sigma}_t$, we need to regularize the action by choosing a background metric $\bar{g}$. We require that the background metric itself be a solution, so that the physical action

$$\hat{I}_{phys} = \hat{I}(\hat{g}) - \hat{I}(\bar{g}) \tag{5.27}$$

defined for the background is zero [83]. This definition also provides a way of comparing the contribution of the gauge fields to the action and the background action since they are contained in the 5D metric. We must also demand that the topology of the extra coordinate be the same for the solution and the background so they redefine the gravitational coupling in the same manner. We can can also assume that the induced metrics on $\hat{\Sigma}_t$ agree [81], which implies that their volume elements will be the same, $\bar{N}\sqrt{\bar{\sigma}} = N\phi\sqrt{\sigma}$. The physical energy-momentum associated with a time translation is then

$$\hat{E} = -\frac{1}{\hat{\kappa}} \int_{I_y} dy \int_{B_t^2} d^2x \left[N\phi\sqrt{\sigma}\left(\hat{k} - \bar{k}\right) - 2\,\hat{\kappa}\left(\hat{r}_a \hat{N}_b \hat{P}^{ab}_{\hat{\sigma}} - \bar{r}_a \bar{N}_b \bar{P}^{ab}_{\bar{\sigma}}\right)\right]. \tag{5.28}$$

Note that the integrand may have extra-coordinate dependence, which is investigated for 5D spacetimes in the next chapter. For 5D spacetimes that are asymptotically flat the natural choice of a background is 5D flat space, and we now consider two examples of this type.

5.4 x^4-Independent Solutions

We now consider two simple examples which illustrate how to calculate the energy. Both examples are spherically-symmetric solutions, which are independent of the extra coordinate, $\partial_y \hat{g}_{AB} = 0$. In the first example we include only the timelike component of the electromagnetic gauge potential $A_t \neq 0$ in order to describe a charged solution, and in the second example we consider only the effects of a scalar field. The ubiquitous example of Kaluza-Klein theory, the magnetic monopole ($A_t = 0, A_i \neq 0$) will not be considered here since it is treated in detail in [38], [40], [43], [87], [94].

First we would like to make some general comments about these particular classes of metrics. The matching boundary $B_t^3 = I_y \times B_t^2$ will be taken to be the surface $r = R = const.$, and will be designated by $B_t^3(R) = I_y \times B_t^2(R)$. We will let $R \to \infty$ at the end of the calculation so that $B_t^2(R) \to S_t^\infty$. The background metric is chosen to be 5D flat space. Since the 1+3+1 metric (5.6) is independent of the extra coordinate and $A_i = 0$ under our initial assumptions, it can be verified that the curvature term $\hat{k}$ reduces to

$$\hat{k} = k + \frac{\phi_\alpha r^\alpha}{\phi} \qquad \text{when} \qquad A_i = 0\,, A_t \neq 0\,, \tag{5.29}$$

where k is the curvature of the B_t^2 boundary. Thus the curvature $\hat{k}$ dimensionally reduces, and includes a projection of the scalar field. This will occur for the background metric as well, but the scalar field contribution will vanish since the scalar field is constant on the matching boundary. By redefining the 5D gravitational coupling using (5.12) the expression for the energy for uncharged solutions then reduces to

$$E = -\frac{1}{\kappa} \int\limits_{B_t^2} d^2x \, N \sqrt{\sigma} \left[\phi \left(k - \overline{k} \right) + \phi_\alpha r^\alpha \right] , \tag{5.30}$$

where $\bar{k}$ denotes the background curvature term of the solution embedded in flat space. This is exactly the expression one would obtain if the starting point was the 4D action for gravity non-minimally coupled to a scalar field [95]. Since the block diagonal form of the 5D metric induces this in 4D, it is a good check of the Hamiltonian approach in 5D. In the case where a 5D solution has a scalar field which is constant (which can be normalized to one by scaling the extra coordinate) the energy reduces to the 4D definition given in [81], [83]. This again produces the correct limit since the theory is equivalent to the 4D theory embedded in 5D. We now proceed to evaluate the energy for the charged GPS class of metrics which satisfy our initial assumptions.

5.4.1 The Energy of Charged 5D Liu-Wesson Solutions

We seek to derive the energy of the charged Liu-Wesson solutions (2.17) presented in chapter 2. The metric of the 3-parameter class of solutions written in the Kaluza-Klein-Jordan frame is :

$$d\hat{s}^2 = -\frac{(1-k)B^a}{1-kB^{a-b}}\,dt^2 + B^{-a-b}dr^2 + r^2B^{1-a-b}d\Omega^2 + \frac{B^b - kB^a}{1-k}(dy + 2A_t\,dt)^2\,. \quad (5.31)$$

Here the function B, the timelike component of the EM vector potential, and the scalar field are given by (2.18):

$$B(r) = 1 - \frac{2M(1-k)}{r}\,, \quad A_t(r) = \frac{\sqrt{k}\,(1-B^{a-b})}{2(1-kB^{a-b})}\,, \quad \phi^2(r) = \frac{B^b - kB^a}{1-k}\,; \quad (5.32)$$

where (a, b) obey the consistency relation and k is defined by

$$a^2 + b^2 + a\,b = 1 \quad \text{and} \quad k = \frac{Q^2}{M^2\,(a-b)^2}\,. \quad (5.33)$$

The presence of the A_t in the above means that the shift vector is nonzero, so by (5.9)

$$\hat{N}^a = (0, 0, 0, 2\phi^2 A_t)\,, \qquad \hat{N}_a = (0, 0, 0, 2A_t) \quad (5.34)$$

and the lapse function is

$$\hat{N} = N = \sqrt{\frac{(1-k)B^a}{1-kB^{a-b}}}. \quad (5.35)$$

The non-zero shift vector would imply that the conjugate momentum term does not vanish, and hence one must use the expression (5.28) when the electromagnetic potential A_t is present. But as will be shown, the momentum contribution to the energy is zero due to the asymptotic behaviour of the potential A_t. The background metric for the charged case is the solution embedded in 5D flat space. In order for the solution and the background to agree on the surface $r = R = const.$, we will need to scale the time coordinate and translate the extra dimension so that the 5D flat space background is

$$d\bar{s}^2 = -[\,N(R)\,]^2 dt^2 + d\bar{r}^2 + \bar{r}^2 d\Omega^2 + \phi^2(R)\,[dy + 2A_t(R)dt]^2 . \quad (5.36)$$

Evaluation of the curvature term gives

$$\hat{k} = [\,B(R)\,]^{(a+b)/2-1}\left[\frac{2}{R} - \frac{2Mz}{R^2}\left(1 + a + \frac{b}{2}\right)\right] - k(a-b)\frac{Mz}{R^2}\frac{[\,B(R)\,]^{(3a-b)/2-1}}{1-k[\,B(R)\,]^{a-b}}, \quad (5.37)$$

and for the background

$$\bar{k} = \frac{2}{R}[\,B(R)\,]^{(a+b-1)/2}, \quad (5.38)$$

where $z \equiv 1 - k$. The integrand for the momentum term of the charged solution (5.31) reduces to

$$\hat{r}_a \hat{N}_b \hat{P}^{ab}_{\hat{\sigma}} = -\frac{\phi^3\sqrt{\sigma}A_t\, r_i\, h^{ij}\,\partial_j A_t}{\hat{\kappa}N} = -\frac{\phi^3\sqrt{\sigma}\,\partial_r A_t^2}{2\hat{\kappa}N\sqrt{h_{rr}}}, \quad (5.39)$$

which has a large-R behaviour of

$$\hat{r}_a \hat{N}_b \hat{P}^{ab}_{\hat{\sigma}} = \frac{Q^2}{R} + O\left(\frac{1}{R^2}\right), \quad (5.40)$$

and when integrated over the angles will not contribute to the energy in the large-R limit. Thus only the curvature term contributes to the total energy for charged solutions that are well-behaved asymptotically.

The evaluation of the curvature term gives the total energy as

$$E = -\frac{1}{\kappa}\int_{S_t^\infty} d^2\theta\, R^2 \left[\frac{2}{R}\left(1-\sqrt{B(R)}\right) - \frac{2Mz}{R^2}\left(1+a+\frac{b}{2}\right) - k(a-b)\frac{Mz}{R^2}[\,B(R)\,]^{a-b}\right]. \tag{5.41}$$

When evaluated for large-R, this gives

$$E = M\left(a+\frac{b}{2}\right) - Mk\left(\frac{a}{2}+b\right), \tag{5.42}$$

or rearranging terms,

$$E = M\left(a+\frac{b}{2}\right)(1-k) + \frac{Mk}{2}(a-b). \tag{5.43}$$

We see that when the charge is set to zero ($k = 0$) the solution reduces to the neutral GPS case and the energy reduces to a sum of gravitational (Ma) and scalar ($Mb/2$) terms.

We also see that the Hamiltonian approach gives the total energy as the sum of total gravitational and electromagnetic, but does not separate out the scalar and gravity parts. One could identify the terms containing the parameter b with the scalar field contribution since it signals the presence of the scalar field (the case $b = 0$ sets $\phi = 1$). To prove this we turn to the Komar integrals in section 5.5 .

5.4.2 The Energy of Neutral 5D GPS Solutions

We briefly give an account of how to calculate the energy of the neutral GPS solutions in order to emphasize the procedure of matching the solution and background metrics.

The Gross-Perry-Sorkin (GPS) solitons [38]-[40] are a class of solutions given by the 5D metric

$$d\hat{s}^2 = -A^a dt^2 + A^{-a-b}dr^2 + r^2 A^{1-a-b}d\Omega^2 + A^b dy^2 \tag{5.44}$$

where

$$A(r) \equiv 1 - \frac{2M}{r}. \tag{5.45}$$

This class of solutions has two independent parameters: M and only one of a or b by (5.33). This solution reduces to the Schwarzschild solution in the case $(a, b) = (1, 0)$, but in general is different [38], [52] as discussed in chapter 2. This will become important when considering the temperature and entropy of these solutions in appendix 2. We will consider the metric with (a, b) left general. The metric (5.44) has zero shift, and the lapse and scalar field are

$$N = A^{a/2}, \quad \phi = A^{b/2} \tag{5.46}$$

while the curvature of the 3-boundary is

$$\hat{k} = [\,A(R)\,]^{(a+b)/2-1}\left[\frac{2}{R} - \frac{2m}{R^2}\left(1 + a + \frac{b}{2}\right)\right]. \tag{5.47}$$

The background $\bar{g}$ is the 5D Minkowski metric in spherical coordinates with a scaled time and a scaled extra coordinate to ensure matching at the boundary. Thus:

$$d\bar{s} = -[\,A(R)\,]^{a}dt^2 + d\bar{r}^2 + \bar{r}^2 d\Omega^2 + [\,A(R)\,]^{b}dy^2, \tag{5.48}$$

where $\bar{r}^2 = r^2\, A^{1-a-b}$ to ensure the same metric on B_t^2. The background curvature on the boundary is

$$\bar{k} = \frac{2}{\bar{r}(R)} = \frac{2}{R}[\,A(R)\,]^{(a+b-1)/2}, \tag{5.49}$$

from which the energy can be calculated:

$$E = -\frac{1}{\kappa}\int_{S_t^\infty} d^2\theta\, R^2 \left[\frac{2}{R}\left(1 - \sqrt{A(R)}\right) - \frac{2M}{R^2}\left(1 + a + \frac{b}{2}\right)\right] = M\left(a + \frac{b}{2}\right). \tag{5.50}$$

Here the square root has been expanded for large R. This reduces to the energy of the Schwarzschild black hole $E = M$ when the parameters (a, b) reduce to their

Schwarzschild values $(1, 0)$. The general result depends on both a and b, and we can obviously identify the gravitational $M_g = Ma$ and scalar $M_s = Mb/2$ contribution to the total mass, which agrees with previous results [38], [87].

5.5 Conservation Laws of 4D Induced Scalar-Tensor Gravity

In this section we derive the conserved quantities associated with the induced 4D matter from a 5D vacuum. Since dimensional reduction is possible if there is a Killing vector $\hat{\zeta} = \delta^A{}_4$, the resulting 4D induced theory is equivalent to a 4D theory of gravity non-minimally coupled to a scalar field plus electromagnetism. The field equations for this induced matter from chapter 2 are :

$$G_{\alpha\beta} = T^S_{\alpha\beta} + 2\,\phi^2\,T^{EM}_{\alpha\beta} \tag{5.51}$$

$$\nabla_\alpha \left(\phi^3 F^{\alpha\beta}\right) = 0 \tag{5.52}$$

$$\Box\phi = \phi^3 F^{\alpha\beta} F_{\alpha\beta}\,. \tag{5.53}$$

We expect the definitions of conserved quantities defined in regular 4D general relativity, such as the Komar mass and the electrical charge, to be modified by the presence of the scalar field. Fortunately, conserved quantities associated with Killing vectors have already been defined for Brans-Dicke theory [96]-[99]. We briefly review the development of the conserved quantities using the notation of [96]. Consider the following vector

$$\Theta^\beta = \nabla_\alpha B^{\alpha\beta}\,, \tag{5.54}$$

where the antisymmetric tensor $B^{\alpha\beta}$ is defined as

$$B^{\alpha\beta} = -\frac{1}{\kappa}\left[\nabla^\alpha\left(\phi\,\xi^\beta\right) - \nabla^\beta\left(\phi\,\xi^\alpha\right)\right]\,, \tag{5.55}$$

ξ^α is a 4D Killing vector and ϕ is the scalar field. It follows that since $B^{\alpha\beta}$ is antisymmetric, the vector Θ^α has zero divergence:

$$\nabla_\alpha \Theta^\alpha = 0 \,. \tag{5.56}$$

Then we can define a conserved quantity

$$C = \int_{\Sigma_t} d^3\Sigma_\alpha \, \nabla_\beta B^{\alpha\beta} = \frac{1}{2} \oint_{S_t} dS_{\alpha\beta} \, B^{\alpha\beta} \,. \tag{5.57}$$

where Gauss' theorem has been used. Substituting the expression for $B^{\alpha\beta}$ into the integral, and using the anti-symmetry properties of $B^{\alpha\beta}$ and $dS_{\alpha\beta}$, one can show that the conserved quantity C is

$$C = -\frac{1}{\kappa} \oint_{S_t} dS_{\alpha\beta} \, \phi^\alpha \xi^\beta - \frac{1}{\kappa} \oint_{S_t} dS_{\alpha\beta} \, \phi \, \nabla^\alpha \xi^\beta \,. \tag{5.58}$$

When the 2-surface approaches infinity ($S_t \to S_t^\infty$), and setting $\xi^\alpha = \xi^\alpha{}_t$ as a time-like Killing vector which generates an asymptotic time translation, the conserved quantity can be identified with the total energy [56]:

$$E_T = -\frac{1}{\kappa} \oint_{S_t^\infty} dS_{\alpha\beta} \, \phi^\alpha \xi_t^\beta - \frac{1}{\kappa} \oint_{S_t^\infty} dS_{\alpha\beta} \, \phi \, \nabla^\alpha \xi_t^\beta \,. \tag{5.59}$$

Thus the total energy is a sum of the scalar energy and the modified gravitational Komar energy:

$$E_s = -\frac{1}{\kappa} \oint_{S_t^\infty} dS_{\alpha\beta} \, \phi^\alpha \xi_t^\beta \,, \tag{5.60}$$

$$E_g = -\frac{1}{\kappa} \oint_{S_t^\infty} dS_{\alpha\beta} \, \phi \, \nabla^\alpha \xi_t^\beta \,. \tag{5.61}$$

When the scalar field is set to one ($\phi = 1$) the energy of the scalar field (5.60) is zero and the gravitational energy (5.61) reduces exactly to the gravitational Komar integral. For the case of the uncharged GPS solutions considered in the last section, the evaluation of the scalar and gravitational energies is a simple task and gives as expected

$$E_s = \frac{Mb}{2} \qquad E_g = Ma\,, \tag{5.62}$$

which is the same total energy defined from the Hamiltonian,

$$E_T = E_s + E_g = M\left(a + \frac{b}{2}\right). \tag{5.63}$$

For the charged soliton, the expressions for the gravitational and scalar energies are still valid. The 4D metric components and the scalar field are modified by the presence of the parameter k. To find an expression for this parameter we use Gauss' theorem on Maxwell's equations (5.52), giving

$$Q = \frac{1}{8\pi}\oint_{S_t^\infty} dS_{\alpha\beta}\left(\phi^3 F^{\alpha\beta}\right) = M\sqrt{k}(a-b), \tag{5.64}$$

and so

$$k = \frac{Q^2}{M^2(a-b)^2} \quad a \neq b\,, \tag{5.65}$$

but when

$$a = b \Longrightarrow Q = 0, \tag{5.66}$$

which is in agreement with the value derived from the large r behaviour of A_t. The calculation of the scalar and gravitational energies for the charged GPS solutions is again straightforward and the results are

$$E_s = \frac{M}{2}(b - ak) \tag{5.67}$$

$$E_g = M(a - kb)\,. \tag{5.68}$$

Therefore the total conserved energy for the charged solution is

$$E_T = M\left(a + \frac{b}{2}\right) - Mk\left(\frac{a}{2} + b\right), \tag{5.69}$$

and this result agrees with the expression for the total energy derived from the Hamiltonian (5.43).

Here we should mention that the total energy E_T may become negative for certain choices of (a, b). Under a rescaling of the 5D metric $\hat{g}_{AB} \rightarrow \lambda^2 \hat{g}_{AB}$, where $\lambda = const$, the total energy (5.59) rescales as $E_T \rightarrow \lambda^3 E_T$. If the total energy can be negative then there exists no lower bound, which is an undesirable feature of any field theory. Thus in the case of the neutral GPS solutions we should restrict ourselves to the case when the parameters (a, b) satisfy $a + b/2 \geq 0$ for the total energy to be positive. If we demand that the total energy (5.69) of the Liu-Wesson solutions is positive, this effectively sets an upper bound on the charge to mass ratio of the source

$$\frac{Q^2}{M^2} \leq \frac{\left(a + \frac{b}{2}\right)(a - b)^2}{\left(\frac{a}{2} + b\right)}. \tag{5.70}$$

where we have used the definition (5.65) for k.

5.5.1 Tolman Mass for the GPS Solutions

Here we derive the Tolman mass for the induced matter for the simple case of the neutral GPS solutions which have the scalar field as the source of the stress-energy tensor $T_{\alpha\beta} = \nabla_\alpha \phi_\beta / \phi$. The conventional 4D definition for the Tolman mass is

$$M_{Tol} = \frac{1}{4\pi} \oint_{\Sigma_t} d^3x \sqrt{h}\, n^\alpha \xi^\beta \left(T_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} T^\gamma{}_\gamma\right), \tag{5.71}$$

where Σ_t is the spacelike surface orthogonal to the timelike Killing vector ξ^α. This integral can be shown to reduce to

$$M_{Tol} = \frac{1}{4\pi} \oint_{\Sigma_t} d^3x \sqrt{-g}\, (\rho + 3P) \tag{5.72}$$

for the case of a perfect fluid [53]. When the Tolman mass (5.71) is evaluated for the neutral GPS solutions (5.44) the result can be expressed as

$$M_{Tol} = Ma\left(1 - \frac{2M}{r}\right)^{-b/2}\Bigg|_{2M}^{R} \tag{5.73}$$

where R and $2M$ are the outer and inner limits of integration on the spacelike hypersurface Σ_t. Since we demand that the gravitational mass $M_g = Ma \geq 0$ we see that the Tolman mass will remain finite and positive only if $b \leq 0$ (2.26). This agrees with our assumptions first made in chapter 2. Although the Tolman mass depends on the scalar field through the stress-energy tensor. the asymptotic limit $R \to \infty$ is independent of the scalar field and only measures the gravitational contribution. This however will not be the case for finite radial distances R, where the scalar field will make a contribution.

5.6 Conformal Rescalings and the 5D KKEM Metric

We want to investigate 4D theories that can be obtained from a conformal rescaling and a dimensional reduction of the 5D KKEM (2.1) metric which obeys the cylinder condition. We begin with the 5D action

$$\begin{aligned}\hat{I} &= \frac{1}{2\hat{\kappa}}\int_{M_5} d^5x\,\sqrt{-\hat{g}}\,\hat{R} + \frac{\epsilon}{\hat{\kappa}}\oint_{\partial M_5} d^4x\,\sqrt{\hat{h}}\,\hat{\mathcal{K}} \\ &= \frac{1}{2\hat{\kappa}}\int_{I_y} dy\int_{M_4} d^4x\,\sqrt{-\hat{g}}\,\hat{R} + \frac{\epsilon}{\hat{\kappa}}\int_{I_y} dy\oint_{\partial M_4} d^3x\,\sqrt{\hat{h}}\,\hat{\mathcal{K}}\end{aligned} \tag{5.74}$$

where $\hat{\kappa} = 8\pi G_5$ is the five-dimensional gravitational coupling constant, ϵ is 1 or -1 according to whether the unit normal to the boundary is spacelike or timelike, and $\hat{\mathcal{K}}$ is the trace of the extrinsic curvature of the boundary. The 5D manifold and its boundary are taken to be the products

$$M_5 = I_y \times M_4 \qquad \text{and} \qquad \partial M_5 = I_y \times \partial M_4 \, . \tag{5.75}$$

At this point it is usually assumed that the 5D spacetime has a spacelike Killing vector $\hat{\zeta}^A = \delta^A{}_y$, in which case one could dimensionally reduce to 4D to produce the gravitational action with the electromagnetic and scalar terms. This dimensional reduction also introduces a length scale which is used to formally redefine the 5D gravitational coupling constant by (5.12). The 5D action $\hat{I}$, from (5.74), thus reduces to an effective 4D action

$$I = \frac{1}{2\kappa}\int_{M_4} d^4x\sqrt{-g}\left(\phi R - 2\Box\phi - \phi^3 F^2\right) + \frac{\epsilon}{\kappa}\oint_{\partial M_4} d^3x\,\sqrt{h}\left(\hat{\mathcal{K}} + \frac{\phi\cdot n}{\phi}\right), \tag{5.76}$$

where we have used the fact that the extrinsic curvature reduces as

$$\sqrt{\hat{h}}\,\hat{\mathcal{K}} = \phi\sqrt{h}\left(\frac{1}{\phi\sqrt{-g}}\nabla_\alpha(\phi\sqrt{-g}\,n^\alpha)\right) = \sqrt{h}\left(\mathcal{K} + \frac{\phi\cdot n}{\phi}\right). \tag{5.77}$$

Here $\mathcal{K}$ is the extrinsic curvature of the boundary ∂M_4 and n_α is the unit normal of this boundary. This action is recognized as the NMC Einstein-Maxwell action and we wish to investigate how the action transforms under the conformal rescaling of the metric and the dilaton as

$$g_{\alpha\beta} \to \phi^{2c} g_{\alpha\beta} \qquad \text{and} \qquad \phi \to \phi^d , \tag{5.78}$$

where (c, d) are constants. It is useful to see how each term in the action rescales under this transformation. We have for the volume and boundary terms

$$\sqrt{-g}\,\phi R \;\to\; \sqrt{-g}\,\phi^{2c+d}\left(R + 6c(1-c)\frac{\phi_\alpha\phi^\alpha}{\phi^2} - 6c\frac{\Box\phi}{\phi}\right), \tag{5.79}$$

$$\sqrt{-g}\,\Box\phi \;\to\; d\sqrt{-g}\,\phi^{2c+d}\left((2c+d-1)\frac{\phi_\alpha\phi^\alpha}{\phi^2} + \frac{\Box\phi}{\phi}\right), \tag{5.80}$$

$$\sqrt{-g}\,\phi^3 F^2 \;\to\; \sqrt{-g}\,\phi^{3d} F^2 , \tag{5.81}$$

$$\sqrt{h}\,\mathcal{K} \;\to\; \sqrt{h}\,\phi^{2c+d}\left(\mathcal{K} + 3c\frac{\phi\cdot n}{\phi}\right). \tag{5.82}$$

Adding the above terms in the 4D action and applying Gauss' theorem on the $\Box\phi$ term, it can be verified that the action transforms as

$$\begin{aligned} I \to I(c,d) &= \frac{1}{2\kappa}\int\limits_{M_4} d^4x\sqrt{-g}\,\phi^{2c+d}\left(R + 6c(c+d)\frac{\phi_\alpha\phi^\alpha}{\phi^2} - \phi^{2(d-c)}F^2\right) \\ &+\frac{\epsilon}{\kappa}\int\limits_{I_y} dy \oint\limits_{\partial M_4} d^3x\,\sqrt{h}\,\phi^{2c+d}\mathcal{K}. \end{aligned} \tag{5.83}$$

This is general, holding for any choice of the parameters c and d. However, to reduce the above action to a MC theory it is evident that $2c+d=0$, otherwise the theory is NMC. For a MC theory the conventional choices for the parameters are $c=-1/\sqrt{3}$ and $d=2/\sqrt{3}$ [36], and the 5D KKEM metric (2.1) transforms from the Jordan frame to the Einstein frame.

5.6.1 Energy and Conformal Transformations

The standard procedure for deriving the energy from the action begins with a 1+3 ADM decomposition of the 4D metric $g_{\alpha\beta}$ and any other gauge or matter fields in terms of the lapse function N and the shift vector N^α. Detailed derivations already exist in the literature [81], [93], [91], and rather than repeat the analysis here for the action (5.83) we simply mention that extra care must be taken to include the factor ϕ^{2c+d} properly, the rest of the derivation being similar. Thus the energy-momentum associated with the timelike vector field $t_\mu = Nn_\mu + N_\mu$ for rescaled action (5.83) is

$$\begin{aligned} E(c,d) &= -\frac{1}{\kappa}\int\limits_{S_t^2} d^2x\,\phi^{2c+d}N\sqrt{\sigma}\left[\left(k-\overline{k}\right) + (2c+d)\frac{\phi_\alpha r^\alpha}{\phi} + r_\alpha A_\beta\,\phi^{2(d-c)}F^{\alpha\beta}\right] \\ &+\frac{1}{\kappa}\int\limits_{S_t^2} d^2x\,\phi^{2c+d}\sqrt{\sigma}\,r_\alpha N_\beta\left(K^{\alpha\beta} - h^{\alpha\beta}K\right), \end{aligned} \tag{5.84}$$

where the subtraction of the background curvature term designated by $\bar{k}$ has been included to define a zero-point energy. Here n_μ is the unit normal to the surfaces of constant time and $K_{\alpha\beta} = \nabla_\alpha n_\beta$ is the extrinsic curvature of these surfaces. The curvature term of the 2-boundary is defined as $k = \nabla_\alpha r^\alpha + n^\alpha n^\beta \nabla_\beta r_\alpha$, where r_α is the unit normal to the 2-surface. The first integral represents the energy associated with a time translation whereas the second integral is the negative of the angular momentum when the shift is identified with an asymptotic rotation. The minus sign is included to account for the usual right-hand-rule definition of angular momentum [56]. Details of the character of the background spacetime and its contribution to the definition of energy are discussed in [81] and [83]. We now turn our attention to the calculation of the energy and angular momentum of 4D axially symmetric stationary (ASS) solutions in MC and NMC theories derived from neutral 5D Kaluza-Klein gravity. For charged solutions the reader is referred to [100].

We wish to derive the energy and angular momentum of ASS solutions in sufficient generality so as to cover both MC and NMC theories. We will adopt the following form of the 4D ASS metric and decompose it into ADM form

$$\begin{aligned} ds^2 &= -(N^2 - N_i N^i)dt^2 + h_{ij}(dx^i + N^i\, dt)(dx^j + N^j\, dt) && (5.85)\\ &= -e^{2\nu}dt^2 + e^{2\psi}(d\phi - \omega\, dt)^2 + e^{2\alpha}dr^2 + e^{2\beta}d\theta^2\,, && (5.86) \end{aligned}$$

from which the lapse function N, shift vector N^i and 3-metric h_{ij} can be obtained in the form:

$$N = e^\nu \qquad N^i = (0,0,-\omega) \qquad N_i = (0,0,-\omega e^{2\psi})\,, \tag{5.87}$$

$$h_{ij} = \begin{pmatrix} e^{2\alpha} & 0 & 0 \\ 0 & e^{2\beta} & 0 \\ 0 & 0 & e^{2\psi} \end{pmatrix}. \tag{5.88}$$

The unit normal to the spacelike hypersurface $t = const.$ and the unit normal to the surfaces $r = const.$ are

$$n_\alpha = e^\nu(-1,0,0,0) \qquad n^\alpha = e^{-\nu}(1,0,0,\omega)\,, \tag{5.89}$$

$$r_\alpha = e^\alpha(0,1,0,0) \qquad r^\alpha = e^{-\alpha}(0,1,0,0)\,. \tag{5.90}$$

Evaluation of the curvature term for the spherical 2-boundary gives

$$k = \nabla_\alpha r^\alpha + n^\alpha n^\beta \nabla_\beta r_\alpha = e^{-\alpha}\partial_r\left(\ln e^{\beta+\psi}\right)\,. \tag{5.91}$$

For a background which matches the metric and scalar field on the 2-boundary the expression for the energy associated with a time translation reduces to

$$E(c,d) = -\frac{1}{\kappa}\int_{S_t^2} d\Omega\,\phi^{2a+b}\,e^{\nu+\psi+\beta}\left[\partial_r\left(\ln e^{\psi+\beta}\right)(e^\alpha - 1) + (2c+d)\,e^{-\alpha}\partial_r\left(\ln\phi\right)\right]\,. \tag{5.92}$$

The angular momentum associated with an asymptotic rotation for the ASS metric (5.86) can also be evaluated as

$$J(c,d) = -\frac{1}{2\kappa}\int_{S_t^2} d\Omega\,\phi^{2c+d}\,e^{3\psi+\beta-\nu-\alpha}\,\partial_r\omega\,, \tag{5.93}$$

and we now proceed to evaluate $E(c,d)$ and $J(c,d)$ for some examples.

The first example we consider is the $N+1$ dimensional vacuum solutions of Myers and Perry [101]. The choice $N = 4$ is then the 5D Kaluza-Klein vacuum, and we note that the corresponding 5D solution can be viewed as a 4D induced-matter solution of BD gravity [34]. The 5D metric is

$$d\hat{s}^2 = g_{\alpha\beta}dx^\alpha dx^\beta + \phi^2 dy^2 \tag{5.94}$$

$$d\hat{s}^2 = -\left(\frac{r^2+j^2-\mu}{r^2+j^2+\frac{j^2\sin^2\theta}{\rho^2}}\right)dt^2 + A\sin^2\theta(d\phi - \frac{j\Delta}{A}\,dt)^2 + \psi dr^2 + \rho^2 d\theta^2 + (r\cos\theta)^2 dy^2\,, \tag{5.95}$$

where the metric components are given by

$$\rho^2 = r^2 + j^2\cos^2\theta \tag{5.96}$$

$$\Delta = \frac{\mu}{\rho^2} \tag{5.97}$$

$$A = r^2 + j^2(1 + \Delta\sin^2\theta) \tag{5.98}$$

$$\psi = \frac{\rho^2}{r^2 + j^2 - \mu}. \tag{5.99}$$

The BD solution obtained by conformally scaling the 4D metric and scalar field is

$$ds^2_{BD} = (r\cos\theta)^{1-p} g_{\alpha\beta}dx^\alpha dx^\beta \qquad \text{with} \qquad \phi = (r\cos\theta)^p\,. \tag{5.100}$$

Since the scalar field has this particular angular dependence, one can verify that both the energy and angular momentum of this solution are zero since the integration of the scalar field over the solid angle is zero. This contradicts the results originally given in [101], where for general N the energy and angular momentum are

$$E_{N+1} = \frac{(N-1)A_{N-1}}{16\pi G_{N+1}}\mu \qquad [L^{N-2}] \tag{5.101}$$

$$J_{N+1} = \frac{A_{N-1}}{8\pi G_{N+1}}\mu j \qquad [L^N]\,. \tag{5.102}$$

Here A_{N-1} is the area of the $N-1$ sphere and G_{N+1} is the dimensionally dependent gravitational coupling. (We have also included the dimensions of the energy and angular momentum.) Although this gives the correct results for 4D theory ($N = 3$), it gives nonzero results for 5D theory ($N = 4$) and there seems to be a contradiction with the induced-matter calculations. This is resolved by the following reasoning. In the large-r region, we have for a general dimension N, $g_{tt} = -1 + \mu/r^{N-2}$, so that for $N = 4$ there is no $O(\frac{1}{r})$ term and hence no mass. This would agree with the induced matter results. The dimensionality of μ also suggests that it may represent a charge, but without a mass present we conclude that for $N = 4$ the solution is unrealistic. Thus care must be exercised in reducing higher-dimensional solutions to 4D.

The next example we examine follows the proposed rotating solutions of Krori and Bhattacharjee [89], [90] (for other forms of Kerr-type metrics in scalar-tensor theory see [102]-[106]). They have given a Kerr-type solution for BD theory by utilizing the complex coordinate transformation of Newman and Janis [107] on the original BD static spherically-symmetric solution [4]. Instead of using the the 4D BD metric as the seed metric for the Newman-Janis algorithm (recently summarized in [108]), we choose to work in 5D and use the closely related GPS metric [38]-[40] as the seed metric,

$$d\hat{s}^2 = -A^a dt^2 + A^{-a-b} dr^2 + r^2 A^{1-a-b} d\Omega^2 + A^b dy^2 \,, \tag{5.103}$$

where $A \equiv 1 - 2M/r$. The scalar field for this metric is defined as $\phi \equiv A^{b/2}$. This class of solutions has two independent parameters: M and only one of a or b by the consistency relation (2.19). After using the Newman-Janis algorithm on the 4D part of the 5D metric, we find a new 4D Kerr-type metric which has the form of (5.86),

$$e^{2\nu} = \frac{\Delta}{\Sigma\, B^{1-a} + j^2 \sin^2\theta\,(2B^{b/2} - B^{a+b})} \tag{5.104}$$

$$e^{2\psi} = \left(\Sigma\, B^{1-a-b} + j^2 \sin^2\theta \left(2B^{-b/2} - B^{a}\right)\right) \sin^2\theta \tag{5.105}$$

$$\omega = \frac{j\left(B^{-b/2} - B^{a}\right)}{\Sigma\, B^{1-a-b} + j^2 \sin^2\theta\,(2B^{-b/2} - B^{a})} \tag{5.106}$$

$$e^{2\alpha} = \frac{\Sigma}{\Delta} B^{1-a-b} \tag{5.107}$$

$$e^{2\beta} = \Sigma\, B^{1-a-b} \,, \tag{5.108}$$

where scalar field is redefined to be

$$\phi = B^{b/2} \,. \tag{5.109}$$

The new metric components are :

$$B = 1 - \frac{2Mr}{\Sigma} \tag{5.110}$$
$$\Sigma = r^2 + j^2 \sin^2\theta \tag{5.111}$$
$$\Delta = r^2 - 2mr + j^2 . \tag{5.112}$$

It can be verified that the above solution reduces to the 4D Kerr solution embedded in 5D when the parameters take on their Schwarzschild values $(a, b) = (1, 0)$. Since this metric is asymptotically flat, the definitions of energy and angular momentum can be applied and yield

$$E_T = M\left(a + \frac{b}{2}\right) \qquad \text{and} \qquad J_T = Mj\left(a + \frac{b}{2}\right). \tag{5.113}$$

The energy agrees with previous calculations using various methods [38], [87], [94], [100], and the angular momentum has the form expected. Hence when the parameter combination $a + b/2 \neq 0$ the total energy and total angular momentum are related by

$$j = \frac{J_T}{E_T}, \tag{5.114}$$

which agrees with the physical interpretation of this parameter being the angular momentum per unit mass. If the combination $a + b/2 = 0$, the total energy and angular momentum are zero. One might expect that the solution might be 5D (or even 4D) Riemann flat when the total energy is zero (even in the case without rotation) but this is not the case because there is an exchange of scalar and gravitational energy which curves the 4D space. This 5D metric goes over to the BD solution when the conformal factors are appropriately chosen as described before.

5.6.2 Conformal Komar Integrals For S-T Gravity

In this section we look at the conserved quantities associated with the induced 4D matter from a 5D vacuum and how they behave under conformal transformations of the type (5.78). The conserved quantity is a sum of a scalar field component and a gravitational component,

$$C = C_s + C_g = -\frac{c_0}{\kappa} \oint_{S_t} dS_{\alpha\beta}\, \phi^{\alpha} \psi^{\beta} - \frac{c_0}{\kappa} \oint_{S_t} dS_{\alpha\beta}\, \phi\, \nabla^{\alpha} \psi^{\beta} . \tag{5.115}$$

Here ψ^{α} is a Killing vector. ϕ is the scalar field and c_0 is an appropriate constant that will be specified once a Killing vector is chosen. We now wish to see the effect of the conformal rescalings on the definition of the Komar integrals for the 4D induced matter. Under the conformal rescalings (5.78), the surface element and the Killing vector rescale as

$$dS_{\alpha\beta} \to \phi^{4c} dS_{\alpha\beta} \quad \text{and} \quad \psi^{\alpha} \to \psi^{\alpha} , \quad \psi_{\alpha} \to \phi^{2c} \psi_{\alpha} . \tag{5.116}$$

After a little algebra one can show that the conserved scalar C_s and gravitational C_g quantities rescale to:

$$C_s \;\to\; C_s(c, d) = -\frac{c_0 d}{\kappa} \oint_{S_t^{\infty}} dS_{\alpha\beta}\; \phi^{2c+d-1}\, \phi^{\alpha} \psi^{\beta} , \tag{5.117}$$

$$C_g \;\to\; C_g(c, d) = -\frac{c_0}{\kappa} \oint_{S_t^{\infty}} dS_{\alpha\beta}\, \phi^{2c+d-1} \left(2c\, \phi^{\alpha} \psi^{\beta} + \phi \nabla^{\alpha} \psi^{\beta}\right) . \tag{5.118}$$

Thus the scalar and gravitational quantities are not invariant under the conformal rescalings, and neither is their sum

$$C = C_s + C_g = -\frac{c_0}{\kappa} \oint_{S_t} dS_{\alpha\beta}\, \phi^{2c+d-1} \left((2c + d)\phi^{\alpha} \psi^{\beta} + \phi \nabla^{\alpha} \psi^{\beta}\right) . \tag{5.119}$$

The total energy can be defined when the Killing vector ψ^{α} is identified with ξ_t^{α} which generates asymptotic time translations and the constant c_0 is chosen as $c_0 =$

1. The angular momentum is defined when $\psi^\alpha = \xi^\alpha_\phi$, which generates asymptotic rotations and $c_0 = -1/2$. However, for this choice of Killing vector one finds upon explicit evaluation that $dS_{\alpha\beta}\phi^\alpha\xi^\beta_\phi = 0$ since $r_\alpha\xi^\alpha_\phi = 0$ and $n_\alpha\xi^\alpha_\phi = 0$. Thus the scalar contribution to the total angular momentum is zero. Hence the total energy and total angular momentum can be defined as:

$$E_T(c,d) = -\frac{1}{\kappa}\oint\limits_{S_t^\infty} dS_{\alpha\beta}\,\phi^{2c+d-1}\left((2c+d)\phi^\alpha\xi^\beta_t + \phi\nabla^\alpha\xi^\beta_t\right) , \tag{5.120}$$

and

$$J_T(c,d) = \frac{1}{2\kappa}\oint\limits_{S_t^\infty} dS_{\alpha\beta}\,\phi^{2c+d-1}\left(\phi\nabla^\alpha\xi^\beta_\phi\right) . \tag{5.121}$$

The choice $2c + d = 0$ which we know reduces to a minimally coupled theory in the Einstein frame metric reduces the energy and angular momentum to the usual 4D Komar definitions. However, as mentioned above, to make the connection to Brans-Dicke theory the parameters must satisfy $2c + d = 1$ which leaves the total energy and angular momentum invariant (compare (5.115) and (5.119)). We can therefore conclude that the Komar energy and angular momentum of the induced matter derived from the 5D vacuum field equations using the 5D metric in the Jordan frame are equivalent to the total energy and angular momentum in 4D Brans-Dicke theory. This is somewhat unexpected since this holds for all ω even though Kaluza-Klein gravity is the special case $\omega = 0$.

To finish this section, we would like to give the explicit form of the energy and angular momentum for 5D metrics which have axially-symmetric-stationary 4D parts and evaluate them for the two examples from the previous section. One can verify that expressions (5.120) and (5.121) reduce to:

$$E(c,d) = -\frac{1}{\kappa}\int\limits_{S_t^2} d\Omega\,\phi^{2c+d}\,e^{\psi+\beta-\alpha-\nu}\left[\partial_r e^{2\nu} + \frac{1}{2}e^{2\psi}\partial_r\omega^2 - (2c+d)\,e^{2\nu}\partial_r\ln\phi\right] , \tag{5.122}$$

and

$$J(c,d) = -\frac{1}{2\kappa}\int_{S_t^2} d\Omega\,\phi^{2c+d}\,e^{3\psi+\beta-\alpha-\nu}\,\partial_r\omega\,. \tag{5.123}$$

For the Myers and Perry solution (5.95) it is verified that the energy and angular momentum are zero due to the angular dependence of the scalar field. For the Kerr-type GPS solution the energy and angular momentum agree with the results of the last section. $E_T = M(a+b/2)$ and $J_T = jM(a+b/2)$.

5.7 Final Comments

We have given a Hamiltonian formulation of Kaluza-Klein gravity in the Jordan frame that is unrestricted in the extra coordinate. When the 5D metric has a Killing vector $\hat{\zeta} = \delta^A{}_4$ associated with the extra coordinate, the Hamiltonian reduces to the Hamiltonian of 4D gravity and electromagnetism non-minimally coupled to a scalar field. The total energy for the neutral GPS and charged Liu-Wesson solutions was calculated, and shown in both cases to reduce to a sum of scalar and gravitational masses. The scalar $M_s = M(b-ak)/2$ and gravitational $M_g = M(a-kb)$ masses were shown to agree with conserved Komar integrals for solutions of the 4D induced theory. Positivity of the total energy constrained the GPS parameters to obey $a + b/2 \geq 0$ and for the Liu-Wesson solution set an upper bound on the charge-to-mass ratio of the solution. These results were then extended into the conformally rescaled theory and examined for a class of rotating GPS solutions, where the energy agreed with the neutral result and the total angular momentum was found to be $J = Mj(a+b/2)$. In short, we have shown that the 5D Hamiltonian approach naturally contains the 4D one, and that quantities like the energy and angular momentum in 5D are natural extensions of those in 4D.

Chapter 6

The 5D ADM Mass

In the previous chapter we derived a general expression for the energy (5.28) for a solution of the 5D field equations from a Hamiltonian perspective. Here we wish to show the versatility of this definition by showing its equivalence to previous definitions of energy considered in Kaluza-Klein theory [87]. In the context of 4D general relativity, Hawking and Horowitz [83] have shown the equivalence between the energy derived from the Hamiltonian and the classical ADM result [91] for flat backgrounds, and to the definition considered by Abbott and Deser [88] for non-flat backgrounds. Their derivation did not depend on the dimensionality of spacetime and so it is easy to extend it to 5D. We briefly outline their proof of the equivalence between the Hamiltonian energy and the ADM energy here, and show that it reduces to the Deser-Soldate energy. A detailed derivation may also be found in [56].

The definition of the ADM energy in 5D can be dimensionally extended from the 4D definition without much effort and is

$$M_{ADM} = \frac{1}{2\hat{\kappa}} \int_S d^3x \sqrt{\tilde{\sigma}}\, \hat{r}^a \left(\hat{D}^b \hat{\gamma}_{ab} - \hat{D}_a \hat{\gamma}\right) , \tag{6.1}$$

where $\hat{r}^a$ is the unit normal to the asymptotic surface element which can be taken as the product of a 2-sphere S^2 times an interval of the real line I_y for the extra coordinate, $\hat{\gamma}_{ab} \equiv \hat{h}_{ab} - \tilde{h}_{ab}$ is the difference between the spatial metric under consideration and the background reference metric. Also $\hat{D}_a$ is the covariant derivative operator associated with the background metric $\tilde{h}_{ab}$ which will also be used to raise and lower indices. The energy defined from the Hamiltonian (5.28) is

$$M_{Ham} = -\frac{1}{\hat{\kappa}} \int_{I_y} dy \int_{S^2_\infty} d^2x \sqrt{\tilde{\sigma}} \left(\hat{k} - \tilde{k}\right) , \tag{6.2}$$

where we have made the choice of a unit time translation $\hat{N} = 1$ and $\hat{N}^a = 0$ in the definition for the energy. To show that these two definitions are equivalent we need to appeal to two coordinate systems on the asymptotic surface S. One coordinate system for $\hat{h}_{ab}$ and another for the background metric $\tilde{h}_{ab}$. We must also take care that the background spacetime redefines the gravitational coupling constant in the same manner as the original spacetime through the integral over the extra coordinate. The choice of coordinates simplifies the analysis and is permitted since the expressions for the ADM and Hamiltonian energy are coordinate invariant. In a neighbourhood of S we can construct coordinate systems such that

$$\hat{h}_{ab} dx^a dx^b = dr^2 + \hat{B}_{lm}(x^i, x^4)\, dx^l dx^m \tag{6.3}$$

$$\tilde{h}_{ab} dx^a dx^b = dr^2 + \tilde{B}_{lm}(x^i, x^4)\, dx^l dx^m . \tag{6.4}$$

Since the background is chosen such that the metrics agree on S, we have $\hat{h}_{ab} = \tilde{h}_{ab}$ on S. We find that in these coordinates the curvature of the boundary term can be expressed as

$$\hat{k} = \frac{1}{2} \hat{h}^{ab}\, \partial_r\, \hat{h}_{ab} , \tag{6.5}$$

which reduces the Hamiltonian energy (6.2) to

$$M_{Ham} = -\frac{1}{2\hat{\kappa}} \int_{I_y} dy \int_{S^2_\infty} d^2x \sqrt{\hat{\sigma}} \left(\hat{h}^{ab}\, \partial_r \hat{h}_{ab} - \bar{h}^{ab}\, \partial_r \bar{h}_{ab}\right) . \tag{6.6}$$

But since the metrics are equivalent $\hat{h}_{ab} = \bar{h}_{ab}$ on S, and using the definition $\hat{\gamma}_{ab} = \hat{h}_{ab} - \bar{h}_{ab}$, the expression for M_{Ham} reduces to

$$M_{Ham} = -\frac{1}{2\hat{\kappa}} \int_{I_y} dy \int_{S^2_\infty} d^2x \sqrt{\hat{\sigma}}\, \hat{h}_{ab}\, \partial_r \hat{\gamma}_{ab} . \tag{6.7}$$

The next step is to show that the ADM definition (6.1) reduces to the same result. From the integrand of (6.1) we have

$$\hat{r}^a\, \hat{D}^b\, \hat{\gamma}_{ab}\Big|_S = \hat{D}^b\, (\hat{r}^a\, \hat{\gamma}_{ab})\Big|_S - \left(\hat{\gamma}_{ab}\, \hat{D}^b\, \hat{r}^a\right)\Big|_S = 0 , \tag{6.8}$$

where the first term is zero since the unit normal vector is orthogonal to both metrics. and the second term is zero by definition on S. The remaining term can be simplified. and gives

$$\begin{aligned} \hat{r}^a\, \hat{D}_a\, \hat{\gamma}\Big|_S &= \hat{r}^a\, \bar{h}^{bc}\, \hat{D}_a\, \hat{\gamma}_{bc}\Big|_S & (6.9)\\ &= \bar{h}^{ab}\, \partial_r\, \hat{\gamma}_{ab}\Big|_S + 2\, \hat{r}^a\, \bar{h}^{bc}\, \hat{\Gamma}^e_{ab}\, \hat{\gamma}^{ec}\Big|_S & (6.10)\\ &= \bar{h}^{ab}\, \partial_r\, \hat{\gamma}_{ab}\Big|_S . & (6.11) \end{aligned}$$

In going from the second to third line we have used the fact that $\hat{\gamma}_{ab} = 0$ on S by definition. Thus the ADM mass reduces to

$$\begin{aligned} M_{ADM} &= \frac{1}{2\hat{\kappa}} \int_S d^3x \sqrt{\hat{\sigma}}\, \hat{r}^a \left(\hat{D}^b\, \hat{\gamma}_{ab} - \hat{D}_a\, \hat{\gamma}\right) & (6.12)\\ &= -\frac{1}{2\hat{\kappa}} \int_{I_y} dy \int_{S^2_\infty} d^2x \sqrt{\hat{\sigma}}\, \hat{h}_{ab}\, \partial_r\, \hat{\gamma}_{ab} , & (6.13) \end{aligned}$$

which is equivalent to the energy from the Hamiltonian (6.7). This was derived with the assumption that the lapse was asymptotically $\hat{N} = 1$, but can be extended to include a general lapse that is not asymptotically flat [83]. With a general lapse function, the definition of energy can accomodate both anti-deSitter spacetimes [88], and deSitter spacetimes [88], [109], [110]. Before we move on to discuss the mass of deSitter spacetimes from a 5D point of view we wish to examine the limits of the 5D ADM mass.

In the case that the 5D cylinder condition holds and the background is flat, the form of the ADM mass reduces to

$$M_{ADM} = \frac{1}{2\hat{\kappa}} \int_{S} d^3x \, \sqrt{\hat{\sigma}} \, \hat{r}_a \left(\partial_b \hat{\gamma}_{ab} - \partial_a \hat{\gamma} \right) \tag{6.14}$$

$$= -\frac{1}{2\kappa} \int_{S^2_\infty} d^2x \, \sqrt{\sigma} \, \hat{r}_i \left(\partial_j \hat{\gamma}_{ij} - \partial_i \hat{\gamma}_{jj} - \partial_i \hat{\gamma}_{44} \right) , \tag{6.15}$$

which is the Deser-Soldate mass for 5D Kaluza-Klein gravity. To show that this indeed agrees with the mass from the Hamilitonian and to illustrate the mechanics involved in performing the calculation we evaluate it for the Liu-Wesson solutions. To start, it is advantageous to work in comoving coordinates which are defined by

$$r = \rho \left(1 + \frac{Mz}{2\rho} \right) , \tag{6.16}$$

and then take the large-ρ limit which behaves like $\rho \sim r$. The only components of the 5D metric we need is the 3D spatial metric and the $\hat{g}_{44}$ part of the metric. If we make the transformation to comoving coordinates and take the large-ρ limit, we find that the 3D spatial metric can be expressed in Cartesian coordinates as

$$\hat{g}_{ij} dx^i dx^j = \left(1 + \frac{2Mz(a+b)}{r} \right) \frac{d\vec{x} \cdot d\vec{x}}{r^2} , \tag{6.17}$$

and the extra-coordinate part of the metric is given by

$$\hat{g}_{44} dy^2 = \left(1 - \frac{2Mz(b-ak)}{r} \right) dy^2 . \tag{6.18}$$

By defining the 3D spatial unit vector $\hat{r}^i = x^i/r$ we can write the $\hat{\gamma}_{ij}$ and $\hat{\gamma}_{44}$ components as

$$\hat{\gamma}_{ij} = \frac{2Mz(a+b)}{r}\hat{r}_i\hat{r}_j \quad \text{and} \quad \hat{\gamma}_{44} = -\frac{2Mz(b-ak)}{r}, \tag{6.19}$$

where we have used flat space as the background. A simple evaluation of (6.16) gives the ADM mass

$$M_{ADM} = M\left(a + \frac{b}{2}\right) - k\left(\frac{a}{2} + b\right), \tag{6.20}$$

which agrees with the result obtained from the Hamiltonian. This was a simple example in which the 5D cylinder condition was assumed. We now turn to the case when this constraint is lifted from the 5D metric.

6.1 Asymptotically deSitter Spacetimes

Whereas the previous section dealt with spacetimes which obeyed the 5D cylinder condition ($\partial_y \hat{g}_{AB} = 0$), we now turn our attention to 5D canonical spacetimes which explicitly depend on the extra coordinate. This extra coordinate dependence has the form of a 5D conformal factor and the 5D metric can be written as

$$\begin{aligned} d\hat{s}^2 &= \hat{g}_{AB}(x^\sigma, y)dx^A dx^B && (6.21)\\ &= e^{-2y/L}\left(g_{\alpha\beta}(x^\sigma)dx^\alpha dx^\beta + dy^2\right), && (6.22)\end{aligned}$$

where the range for the extra coordinate is $y \in [0, \infty)$.

Although the 4D cylinder condition of the 4D metric is still enforced ($\partial_y g_{\alpha\beta} = 0$), there exists non-trivial induced matter from this metric. The field equations for the 5D vacuum $\hat{R}_{AB} = 0$ generate the ten Einstein field equations with a positive cosmological constant (see chapter 2):

$$R_{\alpha\beta} = \Lambda g_{\alpha\beta} \qquad \text{where} \qquad \Lambda \equiv \frac{3}{L^2} \, . \tag{6.23}$$

The remaining five field equations $\hat{R}_{y\alpha} = 0$, $\hat{R}_{yy} = 0$ are satisfied identically. Thus the y-dependent conformal factor is responsible for generating a 4D deSitter vacuum from a pure 5D vacuum. As stated before, any 4D deSitter vacuum solution of Einstein gravity can be embedded in a 5D Kaluza-Klein vacuum via the canonical metric (6.22). Here we will examine the Hamiltonian for the canonical metric.

One must be careful, however, in choosing a background spacetime since it will involve a redefinition of the 5D gravitational coupling $\hat{\kappa}$. We therefore must restrict the background spacetime to have the same topology associated with the extra coordinate as the original solution, so the redefinition of $\hat{\kappa}$ will be the same for the solution as well as the background. The total physical energy for a solution can then be defined, and we recall the definition here for time translations

$$\hat{E} = -\frac{1}{\hat{\kappa}} \int_{I_y} dy \int_{B_t^2} d^2x \, \hat{N} \sqrt{\hat{\sigma}} \left(\hat{k} - \tilde{k} \right) \, . \tag{6.24}$$

For 5D spacetimes that are not asymptotically flat the background has to be chosen carefully. In the case of 5D canonical spacetimes which are asymptotically deSitter in their 4D sections, the background can be chosen as 5D flat, which can be written as a 4D deSitter spacetime trivially embedded in 5D (deSitter space can be viewed as a 4D pseudosphere embedded in 5D). Thus by extending the procedure of including a flat background subtraction term from 4D to 5D, we are able to include deSitter spaces as backgrounds automatically since they are 5D flat but have 4D curved background sections. For the 5D canonical spacetimes we expect the definition of energy (6.24) to be a valid definition of energy for 4D deSitter spacetimes, and to reduce to the proposed definition of energy in deSitter spacetime by Abbott and Deser [88] and later in refs. [109], [110]. We now proceed to demonstrate this.

Since the canonical metric has the extra coordinate as a conformal factor on the 4D metric $g_{\alpha\beta}$ we expect that there would be large simplifications to the terms in the action. This is reinforced by the fact that the induced Einstein-deSitter equations are independent of the extra coordinate, but not of the length scale L which defines the cosmological constant. We therefore expect this length parameter to play an important role in the action when reducing from 5D to 4D.

We start this decomposition by splitting the 5D canonical metric into $1+3+1$ form

$$
\begin{aligned}
d\hat{s}^2 &= -\hat{N}^2 dt^2 + \hat{h}_{ab}(dx^a + \hat{N}^a dt)(dx^b + \hat{N}^b dt) &(6.25)\\
&= e^{-2y/L}\left(-N^2 dt^2 + h_{ij}(dx^i + N^i dt)(dx^j + N^j dt) + dy^2\right). &(6.26)
\end{aligned}
$$

From this decomposition, the geometrical quantities between the 5D and 4D theory can be determined. The unit normal and shift vector to the hypersurface $\hat{\Sigma}_t$ are defined by

$$\hat{n}_A = (-\hat{N},0,0,0,0) = (-e^{-y/L}N,0,0,0,0) = e^{-y/L}(n_\alpha,0)\,. \qquad (6.27)$$

$$\hat{n}^A = \left(\frac{1}{\hat{N}},-\frac{\hat{N}^a}{\hat{N}}\right) = e^{y/L}\left(\frac{1}{N},-\frac{N^i}{N},0\right) = e^{y/L}(n^\alpha,0)\,, \qquad (6.28)$$

and the induced metric on $\hat{\Sigma}_t$ is

$$\hat{h}_{ab} = e^{-2y/L}\begin{pmatrix} h_{ij} & 0 \\ 0 & 1 \end{pmatrix}. \qquad (6.29)$$

One can show that the extrinsic curvature (5.14) has the components

$$\hat{K}_{ab} = e^{-y/L}\begin{pmatrix} K_{ij} & 0 \\ 0 & 0 \end{pmatrix}, \qquad (6.30)$$

where K_{ij} is the extrinsic curvature associated the hypersurface Σ_t. These definitions can be used to express the conjugate momenta (5.16) in a straightforward manner from their definitions. After some algebra we find that the Ricci scalar for $\hat{\Sigma}_t$ can be expressed as

$$\hat{R}(\hat{h}) = R(h) - 2\Lambda\,. \tag{6.31}$$

One can also show that the action (5.15) reduces cleanly to the formal result one expects in $1+3$ with a cosmological constant, but with an overall factor of

$$\frac{\kappa}{\hat{\kappa}}\int_0^\infty e^{-3y/L}dy \tag{6.32}$$

which is defined to be unity so that the gravitational couplings between 4D and 5D can be related. This defines the 5D gravitational coupling as

$$\hat{\kappa} \equiv \frac{L\kappa}{3} = \frac{\kappa}{\sqrt{3\Lambda}}\,. \tag{6.33}$$

Thus it is appropriate that the metric was initially referred to as canonical, since it maps the 5D Hamiltonian in canonical form to the 4D Hamiltonian in canonical form but with a cosmological constant.

The boundary terms $\hat{H}_k + \hat{H}_m$ which define the energy-momentum of 5D solutions in canonical form also reduce to their $1+3$ expressions times the redefinition of the gravitational coupling, giving

$$E = -\frac{1}{\kappa}\int_{B_t^2} d^2x\,\left[N\sqrt{\sigma}\left(k-\overline{k}\right) - 2\,\kappa\, r_a N_b\, P_\sigma^{ab}\right]\,. \tag{6.34}$$

Here $\overline{k}$ denotes the background curvature term of the 4D deSitter background. Now, Hawking and Horowitz [83] have shown the equivalence of the above definition of total energy to the ADM energy for a general lapse function N. This includes the Abbott-Deser energy for 4D anti-deSitter spacetimes [88] when N behaves asymptotically anti-deSitter, and the extension to deSitter spacetimes can be easily made, although care must be taken in defining the integration region in deSitter space due to the cosmological event horizon. This is elucidated in [88], where the authors calculate the energy of the Schwarzschild-deSitter solution and show the expected result $E = M$. We have therefore shown that (5.28) is a valid definition for the energy induced from 5D into 4D, which includes extra-coordinate dependence and gives the energy for deSitter spacetimes.

6.2 Final Comments

The 5D definition for the energy presented in detail in the last chapter was shown to agree with the ADM definition for Kaluza-Klein metrics which are independent of the extra-coordinate. The Deser-Soldate mass was evaluated for the Liu-Wesson solutions and shown to agree with previous results. For the canonical metrics which induce a cosmological constant in 4D, the definition of 5D energy was shown to reduce to the 4D result for deSitter spacetimes considered by Abbott and Deser. Here we used the fact that 4D solutions with a vacuum cosmological constant can be embedded in the 5D canonical metric, and the natural background for the calculation of energy for 4D deSitter spaces is 5D flat space.

Chapter 7

Conclusions

Five-dimensional Kaluza-Klein gravity can be regarded as the simplest higher dimensional generaliztion of classical general relativity. The extra degrees of freedom in the metric tensor and the dependence of the metric components on the extra coordinate are sufficient to induce matter in 4D from a 5D vacuum $\hat{R}_{AB} = 0$. We found no need to introduce higher-dimensional matter through a stress-energy tensor $\hat{T}_{AB}$ as is done in some unified field theories such as superstrings, or interpret what such matter means physically. The matter induced into 4D is thus a consequence of pure geometry in 5D and there is no distinction between source and field in 5D, but this distinction is made in 4D via the induced matter equations $\hat{G}_{\alpha\beta} = 0 \rightarrow G_{\alpha\beta} \equiv T_{\alpha\beta}$.

The nature of the 4D induced matter in general depends on the symmetries of the 5D metric. The induced matter was calculated for two distinct types of metrics: 1) the traditional Kaluza-Klein metric which is independent of the extra coordinate and unifies 4D gravity, electromagnetism and a scalar field, and 2) the canonical metrics which have an extra-coordinate dependent conformal factor and lead to the cosmological constant and neutral matter.

The role of the extra cooordinate in these two cases is very different. For the KKEM metrics, the Killing symmetry of extra-coordinate plays a vital role in the algebraic reduction of the 5D vacuum into a 4D spacetime with matter. This has historically been referred to as the 5D cylinder condition. Because of this symmetry, the off-diagonal components of the 5D metric can describe the EM vector potential, and the extra metric component the scalar field. We note that the induced matter is independent of any length scale associated with the extra dimension which is also a result of the Killing symmetry. In assuming the 5D geodesic equation, the 4D particle dynamics that arise from this metric point to no physical identification of the extra coordinate, but the conserved quantity $c = \hat{\zeta} \cdot \hat{u}$ is related to the charge q of associated test particles. This is a consequence of the modified 4D Kaluza-Klein-Lorentz equation, which is derived from the 5D geodesic equation. For the canonical metrics the situation is drastically different. *The matter that is induced into 4D has its origins in the metric dependence on the extra coordinate.* The length scale L associated with the extra coordinate does appear in the induced matter field equations and defines the vacuum cosmological constant $\Lambda \equiv 3/L^2$. As well, the effective cosmological constant and neutral matter depend crucially on x^4-dependence of the 4D metric $g_{\alpha\beta}(x^\sigma, x^4)$. Another difference is that the particle motion derived for the canonical metric does give the extra coordinate meaning. Since there exist three fundamental units for neutral dynamics, namely, time, length and mass, we expect that the fifth dimension can be geometrized via a combination of fundamental constants. This is the case and we set $x^4 = Gm/c^2$, with the extra coordinate interpreted as a mass. By making a judicious choice of path parameterization for the 4D motion we find that the 5D equations of motion from a Lagrangian derivation allow for the rest-mass variation of particles, and that their motion in 4D is affected by a geometric force. Photons are exempt from this force but massive particles will feel a type of 'fifth' force. This force is also related to the dependence of the 4D metric on the extra coordinate and is zero if the 4D cylinder condition is imposed. The geometric force was calculated for the astrophysically important Ponce de Leon class of metrics and shown to be zero for comoving particles. Hence massive particles must be in a non-comoving frame for the geometric force to be realized.

The physics induced into 4D is not only affected by coordinate symmetries, but also by 5D gauge choices. The linearized version of 5D gravity in which the 5D harmonic gauge was imposed, reproduced 4D linearized gravity with the possibility of a massive graviton. The extra degrees of freedom in the metric where shown to be associated with the photon and a scalar field. From a field theoretical point of view, massive gravitons are associated with a modification of the gravitational potential which effectively reduces the range of the gravitational force. Although this possibility exists for the induced matter theory, the natural choice is to set the graviton mass to zero. By imposing the 5D harmonic gauge for the conformally rescaled KKEM metric, we found that the 4D harmonic and Lorentz gauges could be induced simultaneously with the parameter choice for MC gravity, $2c + d = 0$. Although the 5D harmonic gauge is not covariant, it seems as though the natural choice for the conformally rescaled metric is the MC form which simplifies the gauge choices and the induced matter field equations. Especially important are Maxwell's equations, which are coupled to a scalar field. The scalar field in this case plays the role of a spacetime-dependent dielectric. In the geometric optics approximation it was found that the scalar field coupling to the Faraday tensor, $\phi_\alpha F^{\alpha\beta}$, was responsible for the nonconservation of the photon current and a change in the propagation of the unit polarization vector. Although we expect that these effects should be small, in order to agree with the classical tests of 4D GR, they are non-zero and could possibly be detected. It should also be noted that these deviations from the usual 4D results are not solely a Kaluza-Klein effect. In any scalar-tensor theory in which the scalar field is coupled to Maxwell's equations, such as Brans-Dicke gravity or the low-energy limit of superstring theory, the propagation of photons will be affected.

The Hamiltonian approach to 5D Kaluza-Klein gravity was derived in general and then applied to the two different metrics. Due to the generality of the expression for the energy, various expressions for the energy previously given in the literature such as the ADM energy, the Deser-Soldate energy and the Abbott-Deser energy were shown to be included in the 5D approach. For the KKEM metrics, the total energy or inertial mass was calculated from the Hamiltonian for the Liu-Wesson class of solutions, and was found to be the sum of gravitational and scalar mass, agreeing with the type of decomposition that occurs in scalar-tensor theories such as Brans-Dicke gravity. The calculations where also carried out for a neutral-rotating extension of the GPS solutions, and the angular momentum per unit inertial mass was found to be independent of any scalar-tensor parameters.

The difference between the inertial mass and the gravitational mass is a violation of the Weak Equivalence Principle (WEP). The extent of this violation is dependent on the gravitational and scalar parameters a and b in the neutral GPS solutions. Although the Schwarzschild values $(a, b) = (1, 0)$ do not violate the WEP, a general choice for (a, b) will however. The experimental accuracy for the violation of the WEP is of the order 10^{-11} for terrestrial experiments [79], but this will be greatly improved upon by the satellite experiment STEP [76] which will put very tight constraints on the parameters (a, b).

The Hamiltonian for the canonical metric was shown to accomodate the definition of energy for deSitter spacetimes, and can reduce to the Abbott-Deser energy. This is a natural consequence of the 5D approach since the definition of energy involves the reference to a background metric. The 5D Minkowski metric can be viewed as an embedded 4D psuedosphere representing deSitter space. Since any solution to the 4D Einstein field equations with a cosmological constant can be embedded in a 5D canonical metric, we used a 5D flat background as the reference spacetime. This is equivalent to using a 4D deSitter background to calculate the 4D energy for these solutions, which is the case for the Abbott-Deser energy.

Although all of the above results could in principle be derived for a general N-dimensional theory of gravity, a five-dimensional vacuum theory is preferred for its simplicity, the physical identification of the extra-coordiate as a mass and for the non-trivial matter that can be induced into 4D via the extra-coordinate dependence in the metric. The physical interpretation of the extra-coordinates in higher-dimensional theories seems to be lacking and their experimental consequences tenuous at this time, whereas this is not the case for the modern version of Kaluza-Klein gravity.

Appendix A

Induced Matter Examples

In this appendix we derive the induced matter for two x^4-dependent solutions of the 5D vacuum field equations $\hat{R}_{AB} = 0$. The first solution is a class of metrics which have been termed shell-like [68], [69] in the Kaluza-Klein literature. They induce matter on the 4D hypersurfaces $x^4 = const$ but are 5D Riemann flat. This fact shows that 5D Minkowski metrics can induce non-trivial matter in 4D, provided the 4D sections have an extra-coordinate dependence introduced by a particular choice of coordinates. The second solution is a Kerr-Schild type of metric which exploits the simplification of the field equations as a result of the null vector used in the ansatz for the 4D metric $\hat{g}_{\alpha\beta} = \eta_{\alpha\beta} - 2f(x^\sigma, l)k_\alpha k_\beta$. Again the induced matter is found and shown to give physically meaningful results.

A.1 Shell-like Solutions

The shell-like solutions to the 5D vacuum field equations due to Wesson and Liu [69] are a two-parameter class of solutions :

$$d\hat{s}^2 = \frac{l^2}{L^2}\left(-A^2c^2dt^2 + B^2dr^2 + C^2r^2d\Omega^2\right) + dl^2, \tag{A.1}$$

where the functions A, B and C are

$$A = \frac{1}{B} + \frac{k_2 L}{l}, \tag{A.2}$$

$$B = \frac{1}{\sqrt{1-\frac{r^2}{L^2}}}, \tag{A.3}$$

$$C = 1 + \frac{k_3 L^2}{rl}, \tag{A.4}$$

(note that this form can be expressed as the form given in [69] by letting $k_2 \to k_2/k_1$ and $t \to k_1 t$). A discussion of the spacetime structure is given in [69]. This metric reduces to the deSitter metric when the parameters k_2 and k_3 are both zero. Through the inclusion of the extra-coordinate, this metric may be interpreted as a generalization of the deSitter vacuum with an effective cosmological constant given by (2.55)

$$\Lambda_{eff} = \frac{C + 2AB}{L^2 ABC}, \tag{A.5}$$

which reduces to the vacuum deSitter value in the limit $k_2 = k_3 = 0$ ($A = B^{-1}, C = 1$) as does the metric (A.1). For the metric (A.1) the matter $T_{\alpha\beta}(\partial_l g)$ and total stress-energy $T_{\alpha\beta}$ tensors, defined by equations (2.51) and (2.52), give:

$$T_{\alpha\beta} = -\mathrm{diag}\left[\frac{(1+2C)}{L^2C^2}, \frac{(AB+2C)}{L^2ABC^2}, \frac{C+AB+1}{L^2ABC}, \frac{C+AB+1}{L^2ABC}\right], \tag{A.6}$$

$$T_{\alpha\beta}(\partial_l g) = \mathrm{diag}\left[\frac{(AB-C^2)}{L^2ABC^2}, \frac{(AB-C^2)+2C(1-AB)}{L^2ABC^2}, \frac{(1-AB)}{L^2ABC}, \frac{(1-AB)}{L^2ABC}\right]. \tag{A.7}$$

The total stress-energy tensor (A.6) reduces to the vacuum deSitter value $\rho = -P = 3/L^2$ when $k_2 = k_3 = 0$.

A.2 Kerr-Schild Solutions

In this section we will study the induced matter produced by two Kerr-Schild metrics which have explicit extra-coordinate dependence.

The first 5D metric we investigate is of the form

$$\hat{g}_{AB} = \begin{pmatrix} \eta_{\alpha\beta} - 2f(x^\sigma, l)\, k_\alpha k_\beta & 0 \\ 0 & \phi^2(x^\sigma, l) \end{pmatrix}, \tag{A.8}$$

where k_α is a null vector with respect to the metric $g_{\alpha\beta}$ $k_\alpha k^\alpha = 0$ and $\eta_{\alpha\beta}$ is the flat 4D Minkowski metric. Since k_α is null, the raising and lowering of indices can be done with the flat-space metric, $k^\alpha = g^{\alpha\beta} k_\beta = \eta^{\alpha\beta} k_\beta$. The extrinsic curvature and its trace can be calculated from (2.42) and have the form

$$\hat{K}_{\alpha\beta} = \frac{1}{\phi}\, \dot{f}\, k_\alpha k_\beta\,, \qquad \hat{K} = \hat{g}^{\alpha\beta} \hat{K}_{\alpha\beta} = 0\,, \tag{A.9}$$

where the trace is zero since the inner product of the null vectors is zero, and where we have defined overdot as $(\dot{\ }) \equiv \partial/\partial l$. The induced matter can easily be calculated from (2.43)-(2.45) and has a simple form due to the algebraic reduction in using the null vector k_α. The induced matter field equations for the above metric (A.8) are:

$$G_{\alpha\beta} = T_{\alpha\beta} = \frac{1}{\phi} \nabla_\alpha \phi_\beta - \partial_l \left(\frac{\dot{f}}{\phi}\right) \frac{k_\alpha k_\beta}{\phi}\,, \tag{A.10}$$

$$\nabla_\alpha \left(\frac{\dot{f}\, k^\alpha k^\beta}{\phi}\right) = \partial_\alpha \left(\frac{\dot{f}\, k^\alpha k^\beta}{\phi}\right) = 0\,, \tag{A.11}$$

$$\Box \phi = 0\,. \tag{A.12}$$

The stress-energy tensor (A.10) has two components and represents a massless scalar field and a null radiation field. It is also traceless $T^{\gamma}{}_{\gamma} = 0$ which suggests a radiation equation of state $P = \rho/3$. The conservation equation (A.11) can be viewed as a constraint equation between the null vector and the scalar field where the second equality follows from the fact that the null vector is orthogonal to the Christoffel symbols for Kerr-Schild coordinates [24]. Since we are interested in non-trivial induced matter we will assume that $\dot{f} \neq 0$.

The next metric we consider is the canonical form of the Kerr-Schild metric used above. The metric is

$$\hat{g}_{AB} = \begin{pmatrix} \frac{l^2}{L^2}\left(\eta_{\alpha\beta} - 2f(x^\sigma, l)\, k_\alpha k_\beta\right) & 0 \\ 0 & \phi^2\,(x^\sigma) \end{pmatrix}, \tag{A.13}$$

Due to the conformal factor l^2/L^2 in front of the 4D Kerr-Schild metric, the extrinsic curvature and its trace both gain an extra term:

$$\hat{K}_{\alpha\beta} = -\frac{l^2}{\phi\, L^2}\left(\frac{1}{l}\, g_{\alpha\beta} - \dot{f}\, k_\alpha k_\beta\right) \quad \text{and} \quad \hat{K} = -\frac{4}{\phi\, l}. \tag{A.14}$$

The resulting induced matter (2.52) from the 5D vacuum is:

$$T_{\alpha\beta} = \frac{1}{\phi}\,\nabla_\alpha \phi_\beta - \frac{3}{\phi^2 L^2} g_{\alpha\beta} - \frac{1}{\phi^2 L^2}\left(4l\dot{f} + l^2 \ddot{f}\right) k_\alpha k_\beta\,, \tag{A.15}$$

and the constraint equation and scalar wave equation are

$$\nabla_\alpha \left[\frac{1}{\phi}\left(3g^{\alpha\beta} + l\dot{f}\, k^\alpha k^\beta\right)\right] = 0\,, \tag{A.16}$$

$$\Box\phi = 0\,. \tag{A.17}$$

The above induced stress-energy can be viewed as a 3-component fluid having a scalar field component and null radiation as before, but the second term describes an effective cosmological constant (2.55)

$$\Lambda_{eff}(x^\sigma, l) = \frac{\Lambda_{vac}}{\phi^2}\,. \tag{A.18}$$

The exact form of the effective cosmological constant will be determined by a solution to the field equations. In particular, the generalization to the Schwarzschild-deSitter solution where the function $f(x^\sigma, l)$ has a non-trivial l-dependence would be desirable. This would also lead to variations of the rest-masses of particles in Schwarzschild-deSitter spacetime (see chapter 3). The first steps towards this have been taken in [68] and work on this problem continues.

Appendix B

Quantum Effects in KKG

Here we wish to give a fairly brief account of some semi-classical and quantum effects in Kaluza-Klein gravity. First we examine the temperature of the GPS solutions and find that the Schwarzschild solution is favoured in the class of solutions, and we discuss the surface gravity and temperature for the canonical metrics and relate 4D and 5D results. Secondly, we look at the five-dimensional Klein-Gordon equation for the KKEM and canonical metrics. We rely upon the gauge fields in the 5D KKEM metric and the extra-coordinate dependence in the canonical metric to induce properties such as charge and mass when the problem is reduced from 5D to 4D. In this way, the quantum properties of particles have their origins in a higher dimensional wavefunction and metric.

B.1 Temperature and Stability

In this section we look into the issues of thermodynamical and classical stability for the GPS solutions.

B.1.1 Entropy of the GPS Solutions

In order to investigate the temperature of the GPS solitons we appeal to the method of using Euclidean metrics. In what follows we will only consider the uncharged soliton since it simplifies the analysis. The first step in defining the Euclidean metric is to Wick rotate the time $t = -i\tau$ and remove any conical singularities by defining a period $\hat{\beta}$ for τ that may have been introduced as a result of the Wick rotation. The conical singularity for the uncharged GPS soliton is removed if we define the period to be

$$\hat{\beta} = \frac{8\pi M}{a} \lim_{r \to 2M} A^{1-a-b/2} . \tag{B.1}$$

The value of this period is governed by the parameters (a, b). Using the classical Hamiltonian expression for energy (5.50) the period can be rewritten as

$$\hat{\beta} = \frac{8\pi M}{a} \lim_{r \to 2M} A^{(M-E_T)/M} = \begin{cases} \infty & M < E_T \\ 8\pi M & M = E_T \\ 0 & M > E_T . \end{cases} \tag{B.2}$$

In this form the period depends on whether the total classical energy is greater than, less than, or equal to the Schwarzschild result. However, it is obvious that the only sensible case is when $E = M$ or $(a, b) = (1, 0)$ because otherwise the period $\hat{\beta}$ is ill-valued. This implies that the Schwarzschild solution is the only Euclidean solution with a well-defined period and thus unique among the GPS class. This is due to the fact the Schwarzschild solution is the only metric among the members of the GPS class with a conventional horizon [52]. It also follows that the entropy of this system is one-quarter of the area, as expected. Thus we can conclude that the Schwarzschild solution is thermodynamically unique among members of the GPS class of solutions since it is the only member with a well-defined temperature and entropy.

If we now turn to the canonical metrics with the 4D cylinder condition imposed, it is easy to show that the 5D and 4D temperatures are equivalent. In general, the surface gravity on the horizon is defined as

$$\hat{\kappa}_h^2 = \hat{h}^{ab}\,\partial_a \hat{N}\,\partial_b \hat{N} \tag{B.3}$$

when the metric is written in $1+4$ ADM form. The surface gravity is related to the period of the Euclidean time coordinate and the temperature by

$$\hat{\beta} = \frac{2\pi}{\hat{\kappa}_h} \quad \text{and} \quad \hat{T} = \hat{\beta}^{-1}\,. \tag{B.4}$$

One can check that this reproduces the above temperature for the GPS solutions. If we expand the terms in the 5D surface gravity (B.3) for the canonical metric we find

$$\hat{\kappa}_h^2 = {\kappa_h}^2 + \frac{\Lambda}{3}\,N^2\bigg|_{r_h}\,, \tag{B.5}$$

where κ_h is the 4D surface gravity and the last term is zero since it defines the location of the horizon(s). Thus for the canonical metric with the 4D cylinder condition obeyed, the 5D and 4D surface gravities are the same, hence the 5D and 4D temperatures are the same

$$\hat{T} = T\,. \tag{B.6}$$

We can thus conclude that when the 4D cylinder condition holds, all classical and thermodynamical properties in 4D deSitter spaces are the same in 5D canonical spaces.

B.2 5D Wave Equations

The 4D Klein-Gordon equation for a massive particle is given by

$$\hat{p}_\alpha \hat{p}^\alpha \psi(x^\sigma) = -m^2 \psi(x^\sigma) \tag{B.7}$$

$$\Box \psi(x^\sigma) = m^2 \psi(x^\sigma) \tag{B.8}$$

where the 4-momenta operators are $\hat{p}_\alpha = -i\nabla_\alpha$, and the mass is a constant. We are motivated to dimensionally extend this wave equation to a massless higher-dimensional scalar wave equation for two reasons. The first reason being that derivatives with respect to the extra coordinate acting on a higher-dimensional wave function will generate particle properties such as mass and charge. The second reason is that by including gauge fields in the higher-dimensional metric we expect to induce wave equations that depend on the gauge fields.

In 4D relativity the 4-momenta obey $p_\alpha p^\alpha = -m^2$, and so a 5D theory must be able to reproduce this in an appropriate limit. The simplest extention to higher-dimensional theories is to assume that 5-momenta $P^A = (P^\alpha, P^4)$ obey the null condition

$$P_A P^A = 0 \quad \Rightarrow \quad P_\alpha P^\alpha = -P_4 P^4 \,. \tag{B.9}$$

If we quantize this relation by the usual substitution $\hat{P}_A \rightarrow -i\hbar\hat{\nabla}_A$, and introduce a 5D wavefunction we have

$$\hat{P}_A \hat{P}^A \hat{\Psi}(x^\sigma, x^4) = \hat{g}^{AB} \hat{\nabla}_A \hat{\nabla}_B \hat{\Psi}\,(x^\sigma, x^4) = \frac{1}{\sqrt{-\hat{g}}} \partial_A \left(\sqrt{-\hat{g}}\, \hat{g}^{AB} \partial_B \hat{\Psi}\right) = 0 \,. \tag{B.10}$$

We will examine this equation for the two types of spacetimes encountered in the main body of the thesis, the KKEM and canonical spacetimes.

B.2.1 x^4-Independence : The KKEM Metric

The KKEM metric (2.1) first presented in chapter 2 has an inverse given by

$$\hat{g}^{AB}(x^\sigma) = \begin{pmatrix} \phi^{-2c} g_{\alpha\beta} & -\phi^{-2c} A^\alpha \\ -\phi^{-2c} A^\beta & \phi^{-2d} + \phi^{-2c} A^2 \end{pmatrix} \qquad \text{(B.11)}$$

where we have used the conformally rescaled metric to include both the MC and NMC cases, and we have suppressed the factor of 2 for simplicity. A simple expansion of the 5D wave equation results in

$$\begin{aligned} &\partial_\alpha \left(\sqrt{-g}\phi^{2c+d} g^{\alpha\beta}\partial_\beta \hat{\Psi}\right) - \partial_\alpha \left(\sqrt{-g}\phi^{2c+d} A^\alpha \partial_4 \hat{\Psi}\right) \\ &-\phi^{2c+d}\sqrt{-g}A^\alpha \partial_\alpha \partial_4 \hat{\Psi} + \phi^{2c+d}\sqrt{-g}\left(\phi^{2(c-d)} + A^2\right)\partial_4^2 \hat{\Psi} = 0 . \qquad \text{(B.12)} \end{aligned}$$

where we have separated the derivatives with respect to the extra-coordinate from the 4D spacetime derivatives. It is clear that to uncouple the scalar field from the wavefunction $\hat{\Psi}$ we must choose $2c + d = 0$, with $(c, d) = (-1/\sqrt{3}, 2/\sqrt{3})$, which is the choice for MC gravity (see chapter 2). The wave equation (B.12) reduces to

$$\Box\hat{\Psi} - \nabla_\alpha A^\alpha\, \partial_4 \hat{\Psi} - 2A^\alpha \partial_\alpha \partial_4 \hat{\Psi} + A^2 \partial_4^2\, \hat{\Psi} = -\phi^{-2\sqrt{3}}\, \partial_4^2 \hat{\Psi} . \qquad \text{(B.13)}$$

If we now make the assumption that the 5D wavefunction is separable in terms of a 4D wavefunction and a function of the extra coordinate

$$\hat{\Psi}(x^\sigma . y) = \psi(x^\sigma) e^{i f(x^4)} , \qquad \text{(B.14)}$$

the wave equation takes the form

$$g^{\alpha\beta}\left(\nabla_\alpha - i\partial_4 f(x^4) A_\alpha\right)\left(\nabla_\beta - i\partial_4 f(x^4) A_\beta\right)\psi = \left(\partial_4 f(x^4)\right)^2 \phi^{-2\sqrt{3}}\,\psi . \qquad \text{(B.15)}$$

This form is suggestive of the Klein-Gordon equation for a charged particle and we are compelled to make the identification $f(x^4) = qx^4$, and we can identify the induced mass of the charged particle as

$$m_{eff}^2 = q^2 \phi^{-2\sqrt{3}} . \qquad \text{(B.16)}$$

Thus given a solution to the 5D vacuum field equations $\hat{R}_{AB} = 0$ we can solve for the 4D wavefunction on a curved-electromagnetic background and identify the charge-to-mass ratio of the particles with (B.16).

If we use the expression for the scalar field from the 5D Liu-Wesson vacuum solution

$$\phi^2 = \frac{B^b - kB^a}{1-k} \qquad B \equiv 1 - \frac{2M(1-k)}{r}, \tag{B.17}$$

we find that the charge-to-mass ratio

$$\left|\frac{q}{m}\right|_{eff} = \phi^{\sqrt{3}} = \left(\frac{B^b - kB^a}{1-k}\right)^{\sqrt{3}}, \tag{B.18}$$

in general depends on the choice of (a, b). In the large-r limit this gives

$$\left|\frac{q}{m}\right|_{eff} \sim 1 - \frac{\sqrt{3}M(b - ak)}{r}. \tag{B.19}$$

This gives the extreme result $q = m$ for $r \to \infty$ which is problematic for elementary particles.

If we consider neutral spacetimes ($A_\alpha = 0$) we do not encounter the gauge potential in the Klein-Gordon equation, and thus avoid the identification of the function $f(x^4) = qx^4$. The wave equation for neutral spacetimes reduces to

$$\Box\psi = -\left(\partial_4 f(x^4)\right)^2 \phi^{-2\sqrt{3}}\psi, \tag{B.20}$$

and we can either have a massless scalar field if $f(x^4) = const$, or a massive scalar field with the choice $f(x^4) = x^4/\lambda_c$ where $\lambda_c = \hbar/mc$ is the Compton wavelength of the particle with units restored. This identification supports the geometrization of the extra-coordinate in terms atomic units [19].

B.2.2 x^4-Dependence : The Canonical Metric

Whereas the above example removed the non-compact nature of the extra coordinate we now move on to remove both traditional constraints in Kaluza-Klein theory. We will consider a 5D scalar wave equation in which the 5D metric depends on the extra-coordinate and is non-compact. Examples of this nature, but with mass terms assumed from the outset have been condsidered previously in [111], and more recently for gravitons in superstring theories [11].

We take the 5D conformal form of the canonical metric (6.22)

$$\hat{g}^{AB}(x^\sigma) = \begin{pmatrix} e^{-2y/L}\, g_{\alpha\beta} & 0 \\ 0 & e^{-2y/L} \end{pmatrix} \tag{B.21}$$

where we assumed the 4D cylinder condition $\partial_y g_{\alpha\beta} = 0$ and have left out the scalar field ϕ for simplicity. Although this form of the metric is simple, it induces the non-trivial Einstein-deSitter field equations $G_{\alpha\beta} = -\Lambda g_{\alpha\beta}$ ($\Lambda \equiv 3/L^2$) from the 5D vacuum $\hat{R}_{AB} = 0$. Expanding the 5D box operator for the 5D canonical metric yields

$$\left(\Box - \frac{3}{L}\partial_y + \partial_y^2\right)\hat{\Psi} = 0\,. \tag{B.22}$$

If we factor the 5D wavefunction as done previously we arrive at the 4D Klein-Gordon equation

$$\Box\psi + \left[i\partial_y^2 f - \frac{3i}{L}\partial_y f - (\partial_y f)^2\right]\psi = 0\,. \tag{B.23}$$

The term in the brackets is suggestive of an induced mass, and in order for the mass term to be real the fuction $f(y)$ must satisfy

$$f(y) = f_0 e^{3y/L} \quad \Rightarrow \quad m_{eff}^2 = (\partial_y f)^2 = \frac{3\Lambda\hbar^2}{c^2} e^{2\sqrt{3\Lambda}y}\,, \tag{B.24}$$

where f_0 is a dimensionless constant which we set to unity, and we have inserted units in the final answer. Since the coordinate range for the extra coordinate is

$y \in [0, \infty)$ we find that the mass of these cosmologically induced particles grows with the distance in the extra-coordinate. This result may be due to the simplicity of this model and could be alleviated by lifting the 4D cylinder condition and including the scalar field.

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