Application of Block Sieve Bootstrap to Change-Point detection in time series

by

Saad Zaman

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Author’s Declaration

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Saad Zaman

Abstract

Since the introduction of CUSUM statistic by E.S. Page (1951), detection of change or a structural break in time series has gained significant interest as its applications span across various disciplines including economics, industrial applications, and environmental data sets. However, many of the early suggested statistics, such as CUSUM or MOSUM, lose their effectiveness when applied to time series data. Either the size or power of the test statistic gets distorted, especially for higher order autoregressive moving average processes. We use the test statistic from Gombay and Serban (2009) for detecting change in the mean of an autoregressive process and show how the application of Sieve Bootstrap to the time series data can improve the performance of our test to detect change. The effectiveness of the proposed method is illustrated by applying it to econometric data sets.

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Chapter 1

Introduction

"*We change, whether we like it or not.*" - Ralph Waldo

Perhaps change occurs consistently and we are unable to identify it most of the time. Failure to recognize a change can have dire consequences. A router can fail in a network with heavy traffic causing a delay in productive capacity of the attached terminals. A conveyor belt could stop on an assembly line costing the plant considerable downtime. Even a bank could fail in an economy as observed during the 2008 financial crisis, prior to which the banks were perceived to be too big to fail.

Even though most analysis and control schemes assume a stochastic stationary process, any change in the series (mean, variance, covariance, and/or distribution) invalidates such an assumption. The study of change-point detection in time series (which is referred to as *structural break* in econometrics literature) is crucial for model building and forecasting process. Assuming homogeneity over the entire series under consideration can lead to inaccurate models and large forecasting errors in the presence of a change.

The purpose of change point analysis is to *determine if and when a change occurred* in order to correct and change the control parameters of the process. Usually identifying change starts off with testing the null hypotheses (H_0) *claiming that the process is stationary.* Against the alternative hypothesis (H_A) *which claims that the process is non-stationary* and the stationarity was violated in a specific way.

In this chapter we briefly review the history of Change-Point analysis and then discuss our main contributions and outline of our thesis.

1.1 History of Change-Point Analysis

Typically, the analysis of time series data assumes homogeneity in the parameters and the properties of the underlying data. The application of the tools and methods developed to detect change in properties of the data initiated in industrial settings to detect a break down in a process. In the beginning, Shewart control charts were used [37] which had a disadvantage of giving false alarms. After Page's introduction of the cumulative summation statistic [30], the study of change-point gained significant interest across various disciplines. Other significant contributions preceding this were made by [34] where he suggested Exponentially Weighted Moving Average (EWMA) procedure and by Shiryaev's optimum change detection scheme [38] for reducing the probability of false alarms.

Extensive studies have been conducted in the change-point literature where the above mentioned procedures were applied to various change-point problems. With the advent of computers, more properties and efficiency of these procedures were tested which revealed that while effective for independent data, these procedures have their shortcomings for the dependent data. In addition, their delay in time of detection was also spotted in various studies such as [40] and [24]. The work of [24] was particularly noticeable as it proposed some generalizations in change-point literature for detecting change using generalized likelihood ratio (GLR) schemes.

As cheap computing became easily available, the advantage of using bootstrap procedures, which was initially proposed in [15], became apparent. As discussed in [19], bootstrap methods provide asymptotic refinements. This development was significant in the literature because bootstrap critical values are more robust than the asymptotic critical values [22]. Furthermore, bootstrapping allows us to construct confidence intervals using an empirical distribution of resamples which are smaller than the asymptotic intervals and gives estimates closer to their target values as shown by various studies in literature such as [5], [21], and [22].

1.2 Main Contributions

The thesis illustrates the effectiveness of using sieve bootstrap to get critical values to detect a change in mean of an autoregressive moving average processes. The change point statistic that we primarily use is from [18]. Our main contributions are the following:

- illustration of distortion in size of test statistic from [18] when errors follow a heavy tailed distribution such as t_5 or t_8 for AR(1) and AR(2) process;
- distortion in size of test statistic from [18] for detecting change in mean from of a MA(1), MA(2), and ARMA(1,1) processes with errors from normal distribution;
- application of sieve bootstrap to detect change in mean of ARMA processes which provides consistency in size of the test statistic.
- increase in power to detect change with sieve bootstrap for errors following normal and $t-$distribution of an AR(1) and AR(2) process.
- increase in power to detect change with sieve bootstrap for errors following normal distribution of MA(1), MA(2), and ARMA(1,1) process.
- derivation of a new approach built further on sieve bootstrap by attaining the critical values in blocks. We refer to this approach as *Block Sieve Bootstrap*.
- illustration of consistency in size and increase in power by using critical values from *Block Sieve Bootstrap* approach which provides more stable critical values than sieve bootstrap.

1.3 Thesis Outline

The thesis is organized as follows.

In chapter 2, we introduce some of the most commonly used procedures to detect change in a given sequence of data. In addition, we discuss the Bayesian approach to a change-point problem and the inadequacy of the common procedures to detect change in time dependent data.

In chapter 3, we discuss bootstrap procedures and their application to time series data. In addition to usual I.I.D. and block bootstrap methods, we also discuss the novel sieve bootstrap approach for time series which is the method used in our proposal.

Chapter 4 outlines the change-point statistic of [18] for autoregressive processes, its derivation, and its asymptotic properties. We then propose our simulation procedure for determining the critical values for the change-point statistic from [18] using Sieve bootstrap.

In Chapter 5, we present the simulation results of our proposed method for a change in mean of an autoregressive process.

In Chapter 6, we use our procedure on real world data sets to illustrate its effectiveness to detect change.

In Chapter 7, we summarize our main contributions and provide a possible direction for future work on detecting change in dependent processes using sieve bootstrap.

Chapter 2

Overview of Change-Point detection

"*Nothing endures but change*" - Herculitus.

The main idea behind change-point detection is to find any change in the underlying distribution of the given data set. The change could be either in the mean, variance, covariance, parameter, or the actual distribution itself. In this chapter, we briefly introduce the topic and illustrate it in context of time series analysis.

2.1 Hypothesis for Change-Point framework

From the perspective of mathematical statistics, the problem of Change-Point detection can be classified into two categories: (**i**). On-line (Sequential) change-point detection, and (**ii**) Off-line change-point detection. The two categories are briefly discussed below.

2.1.1 On-line Change-Point Detection

The goal of on-line change-point detection is to determine the time of change in the sequence of observations as soon as it occurs. More specifically, let $\{X_1, X_2, \ldots, X_n\}$ be a sequence of incoming observations (dependent or independent) with density $f(X_1, X_2, \ldots, X_n; \theta, \eta)$ where $\theta \in \Omega_1 \subset \mathbb{R}^d$ is our parameter of interest with $d \geq 1$ and $\eta \in \Omega_2 \subset \mathbb{R}^q$ is a nuisance parameter with $q \geq 0$. Then our hypothesis is to detect the unknown time of change t_0 as soon as possible, where

$$\begin{aligned} H_0: \quad & \theta = \theta_0 && \text{for } , \forall i = 1, .., n \\ H_A: \quad & \theta = \theta_0 && \text{for } i = 1, .., t_0 - 1 \\ & \theta = \theta_1 && \text{for } i = t_0, .., n \end{aligned} \tag{2.1.1}$$

where $\theta_0 \neq \theta_1$. In the on-line change detection framework, we usually define a stopping criteria such as

$$t_0 = \inf\{n : \tau(X_1, X_2, \ldots, X_n) \geq \lambda\} \tag{2.1.2}$$

where $\tau(X_1, X_2, \ldots, X_n)$ is our statistic for detecting the change-point and λ is the threshold/critical value signifying the change in the sequence of observations. Once we detect the change time t_0, we can similarly start with the new subset $\{X_{t_0}, X_{t_0+1}, \ldots, X_n\}$ and the incoming observations $\{X_{n+1}, X_{n+2}, \ldots\}$ to detect the next change-point t_1.

2.1.2 Off-line Change-Point Detection

In off-line change-point detection, we start with a given set of data without any new incoming observations. The main idea is to only identify if a change has occurred. Suppose we are given a set of observations $\{X_1, X_2, \ldots, X_n\}$ (dependent or independent) with density $f(X_1, X_2, \ldots, X_n; \theta, \eta)$ where $\theta \in \Omega_1 \subset \mathbb{R}^d$ is our parameter of interest with $d \geq 1$ and $\eta \in \Omega_2 \subset \mathbb{R}^q$ is a nuisance parameter with $q \geq 0$. The hypothesis for off-line change-point detection can be stated as follows.

$$\begin{aligned} H_0: \quad & \theta = \theta_0 \quad \text{for } i = 1, .., n \\ H_A: \quad & \exists\, m \in \{1, ..., n\} \quad s.t. \\ & \theta = \theta_0 \quad \text{for } i = 1, .., m \\ & \theta = \theta_1 \quad \text{for } i = m + 1, .., n \end{aligned} \tag{2.1.3}$$

where $\theta_0 \neq \theta_1$. The challenge in off-line hypothesis testing comes when choosing the change detection statistic/algorithm. We have to decide between the ability to detect changes when they actually occur, and the ability to minimize false positive results. Problems of false positive results may arises in case of small changes to θ. If the change-point detection statistic is sensitive to minor change in the parameter θ, then it is likely to have more false alarms of detecting changes when they actually do not occur. The usual criterion in statistical hypothesis testing requires a trade-off between *Type I* error (reject the H_0 when it is true) and *Type II* error (accept the H_0 when it is false). Given the two contradictory requirements, the standard criterion in change-point detection is to maximize the probability of accepting H_A when its true (i.e. *the power*) subject to the constraint of holding a fixed probability of rejecting H_0 when its actually true.

Note that the above stated hypothesis is for detecting only one change in the given set of observation. If we can estimate the change-point index, then hypothesis test from (2.1.3) can be generalized to the case of more than one change-point. For instance,if we estimate a change in data $\{X_1, X_2, \ldots, X_n\}$ at t_0, where $t_0 \in \{1, \ldots, n\}$, then the hypothesis (2.1.3) can be recursively applied to subsets $\{X_1, X_2, \ldots, X_{t_0}\}$ and $\{X_{t_0+1}, X_{t_0+2}, \ldots, X_n\}$, to detect more changes in the overall data set $\{X_1, X_2, \ldots, X_n\}$.

In our thesis, we primarily focus on the off-line change-point framework.

2.2 Change-Point Statistics

In order to overcome the inefficiencies of control charts, Page's introduction of summation-type statistics [30] was a break through in the field of change-point detection. Various change-point statistics exist in literature such as M-type, R-type. We review some of the most commonly used statistics and illustrate their application on a constant mean model with at most one change (AMOC). Similar analogies can be drawn to more complicated models.

Assume a constant mean model for the sequence $\{Y_1, \ldots, Y_n\}$ with at most one change point

$$Y_i = \mu + \epsilon_i, \tag{2.2.4}$$

where $\mu \in \mathbb{R}$ is some constant and ϵ_i is the random error peculiar to the i^{th} observation. We want to test the hypothesis whether a change occurred in the series $\{Y_i\}_{i=1}^{n}$. Thus, (similar to off-line framework as in (2.1.3)) our hypotheses test would be

$$
\begin{aligned}
H_0: \quad & Y_i = \mu + \epsilon_i \quad i = 1,..,n \\
H_A: \quad & \exists \quad m \in \{1,...,n-1\} \quad s.t. \\
& Y_i = \mu + \epsilon_i, \quad i = 1,..,m \\
& Y_j = \mu + \delta + \epsilon_i \quad \delta \neq 0,\ j = m+1,..,n
\end{aligned} \tag{2.2.5}
$$

If a change occurred, it happened at some time $m \in (1, \ldots, n)$. We want to find such an m for which H_A is not rejected. There are various test statistics which can be used to test such a hypothesis. Three of the most common approaches are discussed below.

2.2.1 Ordinary Least-Squares Estimate

For the constant mean model (2.2.4) the first approach that comes to mind would be that of simple least squares. Note that the main parameters in our model are μ, δ and m which have the least squares estimates $\widehat{\mu_{LS}}$, $\widehat{\delta_{LS}}$, and $\widehat{m_{LS}}$ respectively.

Define S_k as

$$S_k = \sum_{i=1}^{k} (Y_i - \bar{Y_n}), \quad k = 1,..,n \tag{2.2.6}$$

The least squares approach is to minimize the sum

$$\min \left\{ \sum_{i=1}^{k} (Y_i - \mu)^2 + \sum_{i=k+1}^{n} (Y_i - \mu - \delta)^2 \right\} \tag{2.2.7}$$

where $k \in \{1,..,n-1\}$, and μ, $\delta \in \mathbb{R}$ and $m \in (1, \ldots, n)$. The above equation comes from the least squares approach where the parameters μ, σ, and δ are estimated such that the sum of squares of residuals is minimized. Solving (2.2.7) yields $\widehat{m_{LS}}$ to be

$$\widehat{m_{LS}} = \max \left\{ \sqrt{\frac{n}{k(n-k)}} |S_k|; \quad k \in \{1, \ldots, n\} \right\} \tag{2.2.8}$$

However, there are other test statistics which can be used to find such an estimate of m (denoted by $\widehat{m}$).

2.2.2 Cumulative Sum Test - CUSUM

Cumulative Sums (CUSUM) is one of the first statistic introduced by [30] which is based on partial sums to detect change in a given series. Define S_k to be the partial sums of the sequence $\{Y_i\}_{i=1}^n$ where σ_n is the standard deviation of the process and

$$S_k = \frac{1}{\sigma_n \sqrt{n}} \sum_{i=1}^{k} (Y_i - \bar{Y_n}), \qquad S_0 = 0, \qquad k = 1, \ldots, n$$
$$\text{where } \bar{Y_n} = \sum_{i=1}^{n} \frac{Y_i}{n} \tag{2.2.9}$$

Then, according to the CUSUM statistic, a change has occurred if $\forall\ k = 1, \ldots, n$

$$\tau_k > \lambda \quad \text{where}$$
$$\tau_k = S_k - \min_{1 \le i \le k} S_i \tag{2.2.10}$$

where λ is the threshold/critical value signifying the change. Thus, for example, with respect to hypothesis (2.2.5) the CUSUM estimator of the change-point $\widehat{m} = \min_k \tau_k > \lambda$. The choice of λ can be through either the boundaries of control charts (as it was originally intended) or it can be achieved through bootstrapping (discussed in the next chapter).

CUSUM statistic is the most commonly used statistic and preferred because of its simplicity. However, a drawback of using the CUSUM statistic is that in order to find a critical value, it is assumed that the mean is constant over the entire time series while trying to identify the change point. This increases the liklihood of reporting a false positive result. These drawbacks are overcome by other statistics introduced such as MOSUM estimators.

2.2.3 Moving Sums Test - MOSUM

The moving sums statistic (MOSUM) is an M-type estimator where for a certain predefined bandwidth G, the statistic is calculated recursively. More specifically, for $G \in (0, 1)$ we consider $\lceil G \times n \rceil$ consecutive units over the sequence $\{Y_i\}_{i=1}^n$ and calculate the MOSUM statistic over each segment. Given a process $\{Y_i\}_{i=1}^n$ with standard deviation σ_k of the process over k observations, the MOSUM statistic is defined as [8]

$$MSQ_k(Y_1, \dots, Y_n; G) = \breve{S}_k - \min_{1 \leq i \leq k} \breve{S}_i \tag{2.2.11}$$

where $\breve{S}_k$ denotes the partial sums

$$\breve{S}_k = \frac{1}{\sigma_n \sqrt{G}} \sum_{k=G+1}^{n-G} (Y_i - \bar{Y_G}), \quad k = 1, .., n$$
$$\text{where } \bar{Y_G} = \sum_{i=G+1}^{n-G} \frac{Y_i}{n} \tag{2.2.12}$$

First introduced by [8], it was illustrated that the MOSUM statistic is more robust than the CUSUM statistic (especially for large 'n') because the empirical fluctuations in the series $\{Y_i\}_{i=1}^{n}$ do not depend on the entire process but a fixed data window instead which is moved over the entire dataset. Because of the tuning parameter (i.e. bandwidth G) we only consider a proportion of data when calculating the statistic at each index k.

As shown by [14], it follows that the limiting process of MOSUM type statistic in (2.2.11) are increments of a Brownian Bridge.

$$MSQ_k(Y_1, \dots, Y_n; G) \stackrel{d}{=} W_n\left(\frac{\lfloor n \times k \rfloor - \lfloor G \times k \rfloor}{k}\right) - W_n\left(\frac{\lfloor n \times k \rfloor}{k}\right) \tag{2.2.13}$$

where $W(.)$ is the standard Brownian Motion.

2.3 Bayesian formulation of Change-Point Problem

Until now the literature we introduced on change-point analysis has been mainly viewed from a frequentist point of view. Typically, the Bayesian literature on the change-point detection problem commonly deals with detecting the point/index for a change in distribution. In this section we introduce the Bayesian formulation of the Change-Point problem.

Suppose that for a given experiment, a sample of independent sequential observations $\{x_n : n \geq 1\}$ is available. For known and different densities $f_0(x)$ and $f_1(x)$ we are interested in finding the change point "r" [27]

$$f_n(x|r) = \begin{cases} \prod_{i=1}^{r} f_0(x_i) \cdot \prod_{i=r+1}^{n} f_1(x_i) & 1 \leq r \leq n-1 \\ \prod_{i=1}^{r} f_0(x_i) & r = n \end{cases} \tag{2.3.14}$$

where the discrete unknown parameter r, indicates a change-point in the time-series. We want to test the null hypothesis of no change i.e.

$$\begin{aligned} H_0 &: f_n(x|n) = \prod_{i=1}^{n} f_0(x_i) \\ H_A &: f_n(x|r) \qquad \{1 \leq r \leq n-1\} \end{aligned} \tag{2.3.15}$$

From a Bayesian viewpoint, this is equivalent to solving the model selection problem. Thus, we test the hypothesis which is formulated as choosing between two models M_0 and M_1 where

$$\begin{aligned} M_0 &= f_n(x|n) \\ M_1 &= \{f_n(x|r), \pi(r|n)\} \end{aligned} \tag{2.3.16}$$

where $\pi(r|n)$ is a prior distribution on the set $\{1, 2, \cdots, n\}$.

As introduced by [27], if we let c_{ij} be the cost associated with choosing model M_i when the underlying model is M_j where $i, j \in \{0, 1\}$, then the optimal decision is to reject M_0 when the inequality

$$\frac{P(M_1|x)}{P(M_0|x)} \geq c \tag{2.3.17}$$

is satisfied where $P(M_0|x)$ and $P(M_1|x)$ are the posterior probabilities of each of two models involved and c is the ratio of costs associated with choosing each model i.e. $c = \frac{c_{10}}{c_{01}}$. Note that, from Bayes theorem

$$\mathbb{P}(M_1|x) = \frac{\sum_{r=1}^{n-1} f_n(x|r)\pi(r|n)P(M_1)}{f_n(x|r)P(M_0) + \sum_{r=1}^{n-1} f_n(x|r)\pi(r|n)P(M_1)} \tag{2.3.18}$$

where $P(M_0)$ and $P(M_1)$ are the prior probabilities of selecting each model respectively. Then, for prior $\pi(r|n)$, the Bayesian estimation of r is based on the posterior distribution.

$$\pi(r|x) = \frac{f_n(x_r)\pi(r|n)}{\sum_{r=1}^{n-1} f_n(x|r)\pi(r|n)} \tag{2.3.19}$$

Again, we illustrated the case for one change point in data set of size n, though similar analogies can drawn models with more than one change point.

2.4 Identifying Change-Point in Time Dependent Processes

The initial change-detection scheme introduced in [30] was primarily for IID random variables. However, most of the processes that we come across in application have dependency structure in them. For instance, parts being produced from a machine are likely to be autocorrelated where one flawed part will probably be followed by another. Such a behavior would indicate a break in the production process. As discussed in [43], earlier introduced methods such as Shewart control charts and the CUSUM test lose their effectiveness when the data are correlated. In this section we first introduce the concept of a autoregressive process and briefly discuss the various approaches in the literature to detect change in an autoregressive process. We also introduce Moving Average processes and then conclude the chapter with model for mixed Autoregressive Moving Average processes.

2.4.1 Autoregressive Processes

An *Autoregressive process* is a process where the current and future observations depend on the weighted sample from the past data. More specifically,

Definition 2.4.1. *A stochastic process is referred to as an Autoregressive Process of order 'p' (AR(p)) if it can be expressed in the form*

$$X_t = \mu + \phi_1 X_{t-1} + \phi_2 X_{t-2} + \cdots + X_{t-p} + \epsilon_t \tag{2.4.20}$$

where $\mu \in \mathbb{R}$ is a constant and ϵ_t is a white noise process, i.e. an IID sequence of random *variables with*

$$\mathbb{E}[\epsilon_t] = 0, \qquad Var[\epsilon_t] = \sigma^2 \tag{2.4.21}$$

Note that the parameter vector of an AR(p) process has a total of $(p+2)$ parameters and we will denote it as $\xi = (\mu, \sigma^2, \phi_1, \cdots, \phi_p)$. The condition from (2.4.21) defines most fundamental conditions for stationarity (or, strictly speaking, wide sense or second order stationary). Stationarity is a desirable trait of a process as it makes the analysis of the process more precise due to its stability. We can define the above conditions more generally by using the definition below.

Definition 2.4.2. *Let "B" be the backshift operator such that $B\ X_t = X_{t-1}$. Then an autoregressive process is said to be stationary if all the roots of its characteristic polynomial in terms of B*

$$\phi(B) = 1 - \phi_1 B - \phi_2 B^2 - \cdots - \phi_p B^p \tag{2.4.22}$$

has roots of $\phi(B) = 0$ greater than one in absolute value i.e. its roots must lie outside the unit circle.

For example, given definition (2.4.1), the stationarity condition for AR(1) process ($X_t = \mu + \phi_1 X_{t-1} + \epsilon_t$) is

$$|\phi_1| < 1 \tag{2.4.23}$$

For an AR(2) process, the stationarity conditions are

$$\begin{aligned} \phi_1 + \phi_2 &< 1 \\ \phi_2 - \phi_1 &< 1 \\ |\phi_2| &< 1 \end{aligned} \tag{2.4.24}$$

2.4.2 Change-Point Detection in Autoregressive Processes

Initial tests to detect change in an autoregressive process were tests that used CUSUM with likelihood ratios such as [25], [26] and , [33]. The main idea of Generalized Likelihood Ratios (GLR) was extended by [28] who suggested the following statistic

$$t_0 = \inf\left\{n : \max_{1\le k\le n}\sup_{\theta\in\Theta}\left[\sum_{i=k}^{n}\log\left\{\frac{f_\theta(X_i|X_1,\cdots,X_{i-1})}{f_{\theta_0}(X_i|X_1,\cdots,X_{i-1})}\right\}\right]\right\} \tag{2.4.25}$$

where t_0 denotes the index of the change point, θ is a $p\times 1$ parameter vector and Θ is a given subset of the parameter space Ω. However, it was noted afterwards by [7] that practical implementation of such GLR algorithms is not always possible as the computational complexity increases drastically.

Other classical tests considered in the literature were those such as Wald's Sequential Probability Ratio Test (SPRT) [42] with control charts where

$$t_0 = \inf\left\{n \ge 1 : \prod_{i=1}^{n}\frac{f_A(X_i)}{f_0(X_i)} < A \quad or \quad \frac{f_A(X_i)}{f_0(X_i)} > B\right\} \tag{2.4.26}$$

One of the most notable contributions was made by [24] which extended the classical change detection schemes to AR(p). It was also built on Page's idea of monitoring partial sums and calculating the expected value of stopping time. T. S. Lai [24] suggested a slightly modified version of window-limited GLR where windows of size γ over the n data points are considered. Given a data set $\{X_1,\cdots,X_n\}$, define the stopping time t_γ where

$$t_\gamma = \inf\left\{n : \max_{n-\gamma\le k\le n}\left\{\frac{(X_k+\cdots+X_n)^2}{2(n-k+1)}\right\} \ge c_\gamma\right\} \tag{2.4.27}$$

where $c_\gamma = \log\gamma + \frac{1}{2}\log(\log\gamma) + \log K + o(1)$ defines our threshold. Note that $K = \pi^{-1/2}\int_0^\infty x\ \psi^2(x)\ dx$ is a constant as defined in [24]. They show that the $E_0(t_\gamma) = \tilde{K}^{-1}c_\gamma^{-1/2}\exp\{c_\gamma\}$. This was a significant contribution as Lai's algorithm outperformed all the prior approaches as shown in [24].

In [18], the authors also illustrate the effectiveness of Lai's algorithm in detecting changes in parameter and compare its performance to their statistic. We primarily use the change detection statistic from [18] for monitoring change in the nuisance parameter ξ and illustrate how our approach can increase its effectiveness.

2.4.3 Moving Average Processes

A time series can also be represented as dependent on the white noise component i.e. a linear combination of "ϵ_t" on the right-hand side of the equation (2.4.20). Such processes are called *Moving Average processes* as defined below.

Definition 2.4.3. *A stochastic process is referred to as a Moving Average Process of order 'q' (MA(q)) if it can be expressed in the form*

$$X_t - \mu = \epsilon_t + \theta_1 \epsilon_{t-1} + \theta_2 \epsilon_{t-2} + \cdots + \theta_q \epsilon_{t-q} \tag{2.4.28}$$

where $\mu \in \mathbb{R}$ is a constant and ϵ_t is a white noise process satisfying conditions from equation (2.4.21).

The total number of parameters for a MA(q) process is (q+2) where $\xi = (\mu, \sigma^2, \theta_1, \cdots, \theta_q)$. A moving average process is always stationary since its linear filter representation has only finite number of weights. For any given moving average process, we only observe the actual time series $\{X_t\}_{t=1}^{n}$ and not the white noise component. Thus, as discussed in [1], there is no unique on-to-one correspondence between MA(q) process and its corresponding autocorrelation function. To mimic the stationary criterion of an AR(p) process, some moving average processes can be represented as $AR(\infty)$. Such processes are called *invertible* process and are defined below.

Definition 2.4.4. *Let "B" be the backshift operator such that $B\ X_t = X_{t-1}$. Then a moving average process of order 'q' can be expressed in terms of its characteristic polynomial $\theta(B)$*

$$X_t - \mu = \theta(B)\, \epsilon_t \tag{2.4.29}$$

where $\theta(B) = = 1 - \theta_1 B - \theta_2 B^2 - \cdots - \theta_q B^q$. Furthermore, the MA(q) process is said to be invertible if the roots of $\theta(B) = 0$ lie outside the unit circle.

Similar to the AR case, the invertibility of an MA(q) process imposes conditions on the parameters θ. For example, given definition (2.4.3), the invertibility conditions for MA(1) process ($X_t - \mu = \epsilon_t + \theta_1 \epsilon_{t-1}$) is

$$|\theta_1| < 1 \tag{2.4.30}$$

The invertibility conditions of a MA(2) process are

$$\begin{aligned}\theta_1 + \theta_2 &< 1 \\ \theta_2 - \theta_1 &< 1 \\ |\theta_2| &< 1\end{aligned} \tag{2.4.31}$$

Note that invertibility is a desirable trait because having an AR(∞) representation allows us to apply some of the methods discussed in section 2.4.2 to detect change in a moving average process. This advantage is further emphasized in Chapters 4 & 5.

2.4.4 Mixed Autoregressive-Moving Average Processes

In reality, not all the time series incurred are perfectly autoregressive or moving average processes. To have more flexibility in modeling the time series, sometimes the series is a modeled as a mixture of the two processes. Such models are referred to as *Autoregressive-Moving Average* process and are defined below.

Definition 2.4.5. *A stochastic process is referred to as an Autoregressive-Moving Average Process ARMA(p, q) if it can be written in the form*

$$\begin{aligned}X_t &= \mu + X_t + \phi_1 X_{t-1} + \phi_2 X_{t-2} + \cdots + \phi_p X_{t-p} + \epsilon_t + \theta_1 \epsilon_{t-1} + \theta_2 \epsilon_{t-2} + \cdots + \theta_q \epsilon_{t-q} \\ \implies \quad X_t &= \mu + \phi(B) X_t + \theta(B) \epsilon_t\end{aligned} \tag{2.4.32}$$

where $\mu \in \mathbb{R}$ is a constant and ϵ_t is a white noise process satisfying conditions from equation (2.4.21). For ARMA(p, q), 'p' and 'q' are non-negative integers representing the orders of the autoregressive and moving average component respectively.

ARMA representation is advantageous because it allows for parsimonious models when AR or MA models end up having a high order due to dynamic nature of the data. Since it is a mixed process, the stationarity of the ARMA(p, q) process depends on the stationarity of the AR(p) component of the process i.e. $\phi(B) = 1 - \phi_1 B - \phi_2 B^2 - \cdots - \phi_p B^p$ has roots of $\phi(B) = 0$ outside the unit circle. Similarly, the ARMA(p, q) is invertible if the MA(q) component is invertible i.e. $\theta(B)$ in equation (2.4.32) has roots outside the unit circle.

Since a MA(q) has an AR(∞) representation as discussed in the previous section, this allows to have an AR(∞) representation of an ARMA(p, q) process as well. Therefore, a stationary and invertible ARMA(p, q) process has an AR(∞) representation which gives the advantage of applying change-detection methods discussed in section 2.4.2. We discuss this in more detail in Chapters 4 & 5.

Chapter 3

Bootstrap Methods for Time Series

"*We must become the change we want to see.*" - Mahatma Gandhi

Sparked by Efron's research on Non-Parametric bootstrap method for independent identically distribute (IID) data [15], bootstrap methods have come to be a class of methods which can be used to resample the original data. Since then, the research on bootstrap methods has outgrown significantly with various cases showing the asymptotic consistency of bootstrap estimates. Bootstrapping techniques can be quite useful as it overcomes the limitations of insufficient data size or unknown theoretical distribution. However, in addition to being computationally intensive, it was realized that this method has its limitations when applied to dependent data such as time series data. These barriers can be overcome thorough methods such as Block bootstrap and Sieve bootstrap. We first introduce the concept of bootstrap and then further illustrate its subclasses such as Block bootstrap and Sieve bootstrap which provide an improvement over the original method introduced by Efron[15] for dependent data.

3.1 Bootstrap Resampling for Independent Data

In its most general form, given a data of size n with unknown theoretical distribution, we typically assume that each realized sample value is equally likely (i.e. sampled with probability $\frac{1}{n}$). The idea of a bootstrap technique is to substitute the unknown theoretical distribution of the sample with the known empirical distribution. Suppose we are given a sequence of IID random variables $\{X_1, X_2, \cdots, X_n\}$ from a common (possibly known) underlying distribution $\mathcal{F}$. Let $\mathcal{X}_n = \{X_1, X_2, \cdots, X_n\}$ and $\tau_n = T_n(\mathcal{X}_n; \mathcal{F})$ be a statistic of interest for $n \geq 1$ that can then be estimated based on the empirical distribution (e.g. estimators of mean $\bar{X}_n = \sum_{i=1}^{n} \frac{X_i}{n}$ or variance $s_n^2 = \frac{\sum_{i=1}^{n}(X_i - \bar{X}_n)^2}{n-1}$).

Typically, our goal is to find the approximate underlying distribution of τ_n or some underlying characteristic such as its standard deviation σ_τ. Then bootstrapping provides a non-parametric approach to achieve our goal without making any assumption about the underlying distribution $\mathcal{F}$. We resample from $\mathcal{X}_n$ by sampling m points with replacement, giving $\mathcal{X}_{m,n}^b = (X_1^b, X_2^b, \cdots, X_m^b)$ where $m \leq n$. Note that in our resampling protocol, we assume that each sample is equally likely

$$\mathbb{P}(X_j^b = X_i | \mathcal{X}_n) = \frac{1}{n}, \qquad \text{where } 1 \leq i \leq n, \quad 1 \leq j \leq m \tag{3.1.1}$$

where $\mathbb{P}$ is a probability measure defined on the space $\Omega \subset \mathbb{R}^d$ for $d \geq 1$. Typically, the resample size $m = n$, though there are examples where a different sample size could be desirable as explained in [10]. The bootstrap procedure can be defined as follows

1. Generate a bootstrap sample $\mathcal{X}_{m,n}^b = (X_1^b, X_2^b, \cdots, X_m^b)$ from $\mathcal{X}_n$ by sampling with replacement where $m \leq n$.
2. Compute $\hat{\tau}_{m,n}^b(X_1^b, X_2^b, \cdots, X_m^b)$ which is the estimate of τ_n from the bootstrap sample $\mathcal{X}_{m,n}^b$.
3. Repeat 1 and 2 "BS" times.

The statistic $\hat{\tau}_{m,n}^b$ is the bootstrap estimate of τ_n based on the b^{th} bootstrap sample $\mathcal{X}_{m,n}^b$ where $1 \leq b \leq BS$. Repeating the bootstrap procedure "BS" times gives a conditional distribution $\hat{H}_{m,n}^B$ of statistic $\hat{\tau}_{m,n}^b$ given $\mathcal{X}_n$ where $\hat{H}_{m,n}^B$ is the estimate of the unknown distribution H_n of τ_n.

We know that bootstrap estimation of the conditional distribution $\hat{H}_{m,n}^B$ of statistic $\hat{\tau}_{m,n}^b$ gives an approximation of H_n in the case of IID data sequence. A valid approximation is proved in [23] which shows that if $\mathcal{X}_n$ is IID with finite second moments ($\mathbb{E}[X_i^2] < \infty$), and the resample size $m = n$, then

Theorem 3.1.1. *If $X_1, X_2, \cdots$ are IID with $\mu = \mathbb{E}[X_i]$ and $\sigma^2 = Var(X_i) \in (0,\infty)$, then*

$$\sup_x |\ddot{\mathbb{P}}(\hat{\tau}_{m,n}^b \leq x) - \Phi(\frac{x-\mu}{\sigma})| = o(1) \ \textit{as } n \to \infty, \ \textit{a.s.} \tag{3.1.2}$$

$\ddot{\mathbb{P}}$ is a probability measure defined on the space $\ddot{\Omega} \subset \mathbb{R}^{\ddot{d}}$ for $\ddot{d} \geq 1$. Recall that $\Phi(.)$ denotes the standard normal distribution function and $a.s$ is the abbreviation for *almost surely*.

3.2 Inadequacy of Bootstrap for Dependent Data

While the above bootstrap method works well for an IID data sequence, it fails for data with dependent structure such as time series data as first pointed out in [39]. Before elaborating on inadequacy of Efron's IID bootstrap, we first review the properties of m-dependent data.

Definition 3.2.1. *A sequence of random variable $X_1, X_2, \cdots$ is $m-dependent$ if for some integer $m \geq 0$ the observation X_j is only dependent on $\{X_{j-1}, \cdots, X_{j-m}\}$ for all $j > m$. Thus, the subsets $\{X_1, X_2, \cdots, X_k\}$ and $\{X_{k+m+1}, X_{k+m+2}, \cdots\}$ are independent for all $k \geq 1$.*

Note that for a sequence of m-dependent variables, $\{X_1, X_2, \cdots, X_n\}$ the mean is still $\bar{X}_n = \sum_{i=1}^{n} \frac{X_i}{n}$, but the variance is $\sigma_m^2 = \sum_{i=1}^{m} Var(X_1) + 2\sum_{i=1}^{m-1} Cov(X_1, X_{1+i})$ for $\sigma_m^2 \in (0, \infty)$.

Define τ_n to be our statistic of interest and $\tau_{n,n}^B$ to be our bootstrapped estimate of the statistic based on B resamples of size n. Then the following corollary from [23] proves why Efron's bootstrap fails for dependent data.

Corollary 3.2.2. *Suppose that we have a sequence of stationary m-dependent random variables $\mathcal{X}_n^m = \{X_1, X_2, \cdots, X_n\}$ with $\mathbb{E}[X_i] = \mu$ and $\mathbb{E}[X_i^2] < \infty$ where $\mu \in \mathbb{R}$ and $1 \leq i \leq n$. If $\sum_{i=1}^{m} Cov(X_j, X_{j-m}) \neq 0$ for $m < j \leq n$ and $\sigma_m^2 \neq 0$, then for any $x \neq 0$*

$$\lim_{n\to\infty} [\mathbb{P}(\tau_{n,n}^B \leq x) - \mathbb{P}(\tau_n \leq x)] = [\Phi(x/\sigma) - \Phi(x/\sigma_m)] \neq 0 \qquad a.s. \tag{3.2.3}$$

Efron's bootstrap fails to account for lagged covariance terms in the asymptotic variance and, thus, completely ignores the dependence structure in the sequence of bootstrap generated data. As a result, the mean squared error (MSE) tends to a non-zero number as $n \to \infty$. Thus, the bootstrap estimator $\tau_{m,n}^B$ of the statistic of interest τ_n is inconsistent when the data generating process has an m-dependent structure for $m \geq 1$. This shortcoming is overcome by variation of Efron's IID bootstrap methods, Block bootstrap and Sieve Bootstrap, which are further discussed below.

3.3 Block Bootstrap

The concept of block bootstrap was first introduced by Carlstein (1986) and further advanced by Künsch (1989). In this method, where the idea is to preserve the dependency structure in the data by resampling from continuous blocks of data as opposed to selecting individual points. The advantage of such an approach is that the dependency structure in the observations within a single block is maintained.

There are several variations of block bootstrap that have been introduced over time. In this section we introduce the first block bootstrap method as proposed by Künsch (1989) , and discuss some of its variations.

3.3.1 Overlapping Block Bootstrap

The Overlapping Block Bootstrap (OBB) method was first introduced by Künsch (1989) which is also referred to as Moving Block Bootstrap. To motivate the illustration, suppose we have a subset $\mathcal{X}_n = \{X_1, X_2, \cdots, X_n\}$ from a sequence of stationary (possibly dependent) random variables. Let $\tau_n = T_n(\mathcal{X}_n; \mathcal{F})$ be a statistic of interest for $n \geq 1$ that can then be estimated based on the empirical distribution $\mathcal{F}_n$. Let $B_i = \{X_i, X_{i+1}, \cdots, X_{k+i-1}\}$ for $1 \leq i \leq N$ denote a subset from $\mathcal{X}_n$ where $1 \leq k \leq n$ is an integer representing the number of elements in this subset. We refer to B_i as a block of length k with elements obtained by sampling k consecutive elements from $\mathcal{X}_n$. Thus, such subsampling gives us a set of N blocks $\mathcal{B} = \{B_1, B_2, \cdots, B_N\}$ where $N = n - k + 1$. Then, to have a resampled set of size m where $m \leq n$, the moving block bootstrap procedure is

1. Generate a bootstrap sample $\mathcal{B}^b_{m,n} = (B^b_1, B^b_2, \cdots, B^b_N)$ from $\mathcal{B}_N$ by sampling with replacement where $m \leq n$.

2. Compute $\hat{\tau}^b_{m,n}(B^b_1, B^b_2, \cdots, B^b_m)$ which is the estimate of τ_n based on the block bootstrap sample $\mathcal{B}^b_{m,N} = (B^b_1, B^b_2, \cdots, B^b_N)$

3. Repeat 1 and 2 "BS" times.

where BS is the number of bootstrap iterations desired. Note that if the length k of each block is 1, then this procedure is similar to Efron's IID Bootstrap. Our bootstrap sample is $\mathcal{X}^b_n$ where

$$\mathcal{X}_n^b = B_1^b \cup B_2^b \cup \cdots \cup B_N^b \tag{3.3.4}$$

OBB provides the advantage over IID bootstrap by having dependency structure in the bootstrapped data. For each resampled block B_j^b

$$\mathbb{P}(B_j^b = B_i|\mathcal{B}_N) = \frac{1}{N}, \qquad \text{where } 1 \leq i, j \leq N \tag{3.3.5}$$

and the dependency structure is maintained within each block B_i. However, a major drawback of OBB is that not every point from the empirical distribution in $\mathcal{X}_n$ is sampled with equal probability. Note that from the construction of blocks $\mathcal{B}_N$ from $\mathcal{X}_N$, if each block is of length k then

$$\begin{aligned}
&B_1 = X_1, X_2, \cdots, X_k \\
&B_2 = X_2, X_3, \cdots, X_{k+1} \\
&\vdots \\
&B_N = X_{n-k+1}, \cdots, X_n
\end{aligned}$$

Each block B_i is sampled with equal probability, for block length $k > 1$. Points in $\mathcal{X}_n$ are not sampled with equal probability. For instance, the end points X_1 and X_n are least likely to be included in the bootstrap sample $\mathcal{X}_{m,n}^b$ with probability $\frac{1}{N}$. Compared to points X_2 and X_{n-1} are included in bootstrap samples $\mathcal{X}_{m,n}^b$ with probability $\frac{2}{N}$. Thus in OBB, points are resampled from the empirical distribution $\mathcal{X}_n$ with probability.

$$\mathbb{P}(X_i \in \mathcal{X}_n^b) = \frac{\sum_{i=1}^{N} \mathbb{I}_{Xi} B_i}{N} \qquad \text{where} \quad \mathbb{I}_{Xi} = \begin{cases} 1 & if X_i \in B_i \\ 0 & otherwise \end{cases} \tag{3.3.6}$$

This shortcoming gives biased OBB samples where the first and last few observations are least likely to be included in $\mathcal{X}_n^b$ compared to others. The use of non-overlapping blocks can be shown to overcome this drawback of OBB.

3.3.2 Non-Overlapping Block Bootstrap

The Non-Overlapping Block Bootstrap (NBB) is a variant of the OBB which was initially introduced by Carlstein (1986) [13]. Using the same motivating example as in section 3.3.1, let us suppose that $\mathcal{X}_n = \{X_1, X_2, \cdots, X_n\}$ is a sequence of stationary random variables, $\tau_n = T_n(\mathcal{X}_n; \mathcal{F})$ be a statistic of interest for $n \geq 1$, and T(-) is a real-valued function. Similarly to OBB, let the length of each block be k ($\in \mathbb{Z}^+$) and denote each NBB block by

$$\grave{B}_i = (X_{(i-1)k+1}, \cdots, X_{ik}), \qquad \text{i} = 1, \cdots, \grave{N} \tag{3.3.7}$$

Once we get $\grave{N}$ blocks, our sampling procedure is exactly the same as OBB.

1. Generate a bootstrap sample $\grave{\mathcal{B}}^b_{m,n} = (\grave{B}^b_1, \grave{B}^b_2, \cdots, \grave{B}^b_N)$ from $\mathcal{B}_N$ by sampling with replacement where $m \leq n$.
2. Compute $\hat{\grave{\tau}}^b_{m,n}(\grave{B}^b_1, \grave{B}^b_2, \cdots, \grave{B}^b_m)$ which is the estimate of $\grave{\tau}_n$ based on the block bootstrap sample $\grave{\mathcal{B}}^b_{m,N} = (\grave{B}^b_1, \grave{B}^b_2, \cdots, \grave{B}^b_N)$
3. Repeat 1 and 2 "BS" times.

where BS is the number of desired bootstrap iterations. It is clear that the NBB scheme is quite similar to OBB, but test statistics from each approach have very different properties. For instance, as shown by Lahiri (2003) [23], if the process $\{Y_n\}_{n\geq 1}$ satisfies standard moment and mixing conditions and our statistic of interest $\hat{\tau}_n$ is the simple case of sample mean, then $\mathbb{E}\{\mathbb{E}[\tau_{m,n}] - \mathbb{E}[\grave{\tau}_{m,n}]\}^2 = O(\frac{l}{n^2})$.

3.3.3 Circular Block Bootstrap

The Circular Block Bootstrap (CBB) was proposed by [32] to overcome the limitations of OBB from the boundary conditions as stated in section 3.3.1. Let that $\mathcal{X}_n = \{X_1, X_2, \cdots, X_n\}$ be a sequence of stationary random variables and $\tau_n = T_n(\mathcal{X}_n; \mathcal{F})$ be our statistic of interest for $n \geq 1$ and T(-) is a real-valued function. Then, the CBB approach is to construct a new dataset $\mathcal{Y}_n = \{Y\}^{\infty}_{i=1}$ by wrapping $\mathcal{X}_n$ around in a circle and define $X_i = Y_{i_N}$ where $1 \leq i \leq N$ and $i_N = i (\mod N)$.

While the block bootstrap methods have their advantages, there are some general drawbacks. For instance, the resampled blocks might not represent the actual behavior of the time-series under consideration. As a result of this, the bootstrapped series might have weaker dependency than that of original series. Such disadvantages can be overcome by another approach to the bootstrap method for time series called *Sieve Boostrap*.

3.4 Sieve Bootstrap

The sieve bootstrap procedure was first introduced by [12] which suggested a bootstrap method based on resampling from residuals of a fitted model as opposed to resampling the original data itself. [12] suggested to estimate the underlying process by a sequence of autoregressive processes of order $p = p(n)$ where $p(n) \to \infty$ as $n \to \infty$. Thus, when the data comes from a class of linear processes, sieve bootstrap provides consistent estimators with lower bias and mean squared error (MSE) compared to any of the block bootstrap approaches introduced in section 3.3 . In this section, we introduce sieve bootstrap for time series as introduced by [12] and elaborate on the Sieve bootstrap approach in [2] that we adopted in our proposed method.

Let $\mathcal{X}_n = \{X_i\}$ be a stationary time series for $i \in \mathbb{Z}^+$ and $\tau_n = T_n(\mathcal{X}_n; \mathcal{F})$ be our statistic of interest where $n \geq 1$ and T(-) is a real-valued function. Let $\mu_X = \mathbb{E}[X]$. Then by Wold's decomposition theorem [11], the process $\{X_t\}$ can be represented, as an infinite moving average process

$$X_t = \sum_{j=0}^{\infty} \psi_j \epsilon_{t-j} \tag{3.4.8}$$

where

- $\psi_0 = 1$ and $\sum_{j=0}^{\infty} \psi_j^2 < \infty$
- $\{\epsilon_t\} \sim \text{WN}(0, \sigma^2)$ is an uncorrelated sequence of innovations.

Further, if we assume that the MA(∞) process in (3.4.8) is invertible, then the process can be represented as one-sided infinite order autoregressive process

$$\sum_{j=0}^{\infty} \phi_j (X_{t-j} - \mu_X) = \epsilon_t \tag{3.4.9}$$

where

- $\phi_0 = 1$ and $\sum_{j=0}^{\infty} \phi_j^2 < \infty$
- $\{\epsilon_t\} \sim \text{WN}(0, \sigma^2)$.

The representation in (3.4.9) motivates the sieve approach which proceeds by fitting an autoregressive process of finite order to $\mathcal{X}_n$. We outline the sieve bootstrap approach below.

1. Given the sample $\mathcal{X}_n$, estimate the order 'p' of the process based on an order selection criterion for autoregressive processes such as Akaike's Information Criterion (AIC) [3], or Bayesian Information Criterion (BIC) [36], or the bias corrected AIC (AICC) [20] which was used in [2]. Note that given $\mathcal{X}_n$, the order $p(n)$ is a function of n where p(n) $\to \infty$ as $n \to \infty$.

2. Once we estimate the order $p(n)$, we estimate the autoregressive coefficients of the AR(p) process: $\hat{\phi} = (\hat{\phi}_1, \hat{\phi}_2, \cdots, \hat{\phi}_p)$. $\hat{\phi}$ can be estimated through least-squares or (more commonly) through Yule-Walker estimation.

3. The above yields the residuals from the fitted process

$$\hat{\epsilon}_t = \sum_{j=0}^{p} \hat{\phi}_j (X_{t-j} - \bar{X}_n), \qquad \hat{\phi}_0 = 1 \quad \text{and} \quad p(n) + 1 \leq t \leq n \tag{3.4.10}$$

where $\bar{X}_n = \frac{\sum_{i=1}^{n} X_i}{n}$

4. We obtain the centered residuals and their empirical distribution function

$$\hat{F}_{\tilde{\epsilon}}(X) = \sum_{t=p+1}^{n} \mathbb{I}_{\{\tilde{\epsilon}_t \leq X\}} \tag{3.4.11}$$

where $\tilde{\epsilon}_t = \hat{\epsilon}_t$ - $\bar{\epsilon}$ for $\bar{\epsilon} = \sum_{t=p+1}^{n} \frac{\epsilon_t}{n-p}$.

5. From the empirical distribution defined in (3.4.11) we can get a new set of resampled sieve bootstrap residuals ϵ_i^*, where $i \geq p + 1$.

6. Using $\{\epsilon^*\}$, we can generate sieve bootstrap observations X_t^* by defining the following recursive relation for an Autoregressive process of same order p as chosen in step 1.

$$(X_t^* - \bar{X}_n) = \sum_{j=1}^{p} \hat{\phi}_j (X_{t-j}^* - \bar{X}_n) + \epsilon_t^*, \qquad i \geq p+1 \tag{3.4.12}$$

where the starting p observations $X_1^*, X_2^*, \cdots, X_p^*$ can be set equal to $\bar{X}_n$ or 0 for large n.

To get more than one sample from the sieve bootstrap, we repeat steps 5 & 6 "BS" times (where BS is the desired number of bootstrapped samples). Thus, for our statistic of interest $\tau_n = T_n(\mathcal{X}_n; \mathcal{F})$, we can use the autoregressive sieve bootstrap estimator

$$\tau_{m,n}^* = T_m(X_1^*, \cdots, X_m^*), \qquad m > p(n) \tag{3.4.13}$$

which estimates the variance of the estimators more accurately than OBB, or NBB as shown by [12] and [2].

In our approach, we use the sieve bootstrap method illustrated since the primary purpose of our work is to detect a change in the mean of an AR(p), MA(q), and ARMA(p, q) processes.

Chapter 4

Sieve Bootstrap for Detecting Change in Time Series

"*God, grant me the serenity to accept the things I cannot change, the courage to change the things I can, and the wisdom to know the difference.*" - Reinhold Niebuhr

In chapter 2, detecting change in time-dependent series was discussed. As summarized in [24], typical statistics such as CUSUM or MOSUM fail to detect a change in the parameters. Bootstrapped critical values on the other hand provide asymptotic refinements and provide coverage and rejection probabilities that are more accurate than the first-order asymptotic critical values as discussed in [19] and [31]. Furthermore, [21] and [22] show that bootstrap critical values are more robust than the actual distribution of the change-point statistic and provide consistency in size and power.

In this chapter, we illustrate our main contribution by applying sieve bootstrap to detect a change-point using the statistics from [18]. We show how the change-point statistic from [18] under perform for heavy-tailed distributions and have inconsistency in size. We also introduce two different approaches for which we obtain sieve bootstrap critical values and demonstrate how they outperform the first order asymptotic distribution of the test statistic.

4.1 Detecting Change in Autoregressive processes

As previously discussed in chapter 2, when detecting change in time dependent series, the traditional sequential algorithms such as CUSUM and MOSUM fail. Most of the methods discussed in section 2.4.2 either end up having a distorted size or higher probability of false alarms as shown by [24]. The approach taken by [18] is also based on recursive partial sums, however, it makes use of efficient score vectors to detect a change in the parameters. First, we introduce the test statistic from [18] and discuss its derivation and asymptotic distribution below.

Let $\{X_i\}_{i=1}^n$ be a stationary autoregressive process as defined in section 2.4.1 and let θ be a the parameter of interest for which we want to test for its departure from an initially hypothesized value of θ_0. The typical approach for the test statistic (including [30]) is to consider the standardized partial sum of the process $\max_{1\leq j<k} \sum_{i=j}^{k} X_i$ for $k \in \{1, \cdots, n\}$. However, [18] considers the partial sums of the efficient score vectors instead of the series by defining

$$V_k(\xi) - V_j(\xi) = \nabla_\xi \sum_{i=1}^{k} \log f(X_i; \xi) - \nabla_\xi \sum_{i=1}^{j} \log f(X_i; \xi) \tag{4.1.1}$$

where $\xi = (\theta, \eta)$ is the parameter vector with θ being the parameter of interest and η denotes a $(p+1)$ dimensional vector which we estimate at each k. Thus, $\hat{\eta}_k$ denotes the maximum likelihood estimate of the parameters using observations $\{X_1, X_2, \cdots, X_k\}$ and leaving $\theta = \theta_0$. Standardizing V_k from equation (4.1.1) with the information matrix $I(\xi)$ gives the statistic $W_k(\theta_0, \hat{\eta}_k) = I^{-\frac{1}{2}}(\theta_0, \hat{\eta}_k) V_k(\theta_0, \hat{\eta}_k)$ which has uncorrelated components. Thus, the test proposed by [18] for monitoring change in θ can be given as follows.

TEST. *Given $\{X_1, \cdots, X_n\}$, conclude that H_0 is not supported by the data at first k, $1 < k \leq n$, when*

$$\max_{1<j<k} n^{-\frac{1}{2}} \left(W_k(\theta_0, \hat{\eta}_k) - W_j(\theta_0, \hat{\eta}_k) \right) \geq C(\alpha) \tag{4.1.2}$$

otherwise do not reject H_0.

The test above is one-sided test with critical value $C(\alpha)$ where α is the specified significance level. $W(.)$ is the standard brownian motion as shown in theorem (4.1.2) from [18]. The test statistics $W_k(.) - W_j(.)$ converges to the well known distribution

$$
\begin{aligned}
&\sup_{0\le s<t\le 1} W(t)-W(s)\\
&\overset{\mathrm{d}}{=} \sup_{0\le t\le 1} |W(t)|\\
&= \mathbb{P}\{\max_{0\le s\le t} |W(s)| < \beta | W(0)=0\} = \frac{1}{\sqrt{2\pi}} \sum_{h=-\infty}^{+\infty} (-1)^h \int_{\frac{-\beta+2h\beta}{\sqrt{t}}}^{\frac{\beta+2h\beta}{\sqrt{t}}} e^{-\frac{y^2}{2}}\, dy
\end{aligned}
\tag{4.1.3}
$$

where the sum and the integral converge as h $\to \infty$ shown by [29]. Note that the test is defined only for stationary AR(p) processes which satisfy the conditions specified in section 2.4.1. Thus, $C(0.10) = 1.96$, $C(0.05) = 2.24$, and $C(0.01) = 2.80$.

The following discussion is about the derivation of the test statistic for detecting change in mean of an AR(p) process. Given a sequence of observations $\mathcal{X} = \{X_1, X_2, \cdots, X_n\}$ from an autoregressive process (2.4.1) of order p we illustrate the hypothesis and test statistic under consideration.

4.1.1 Change in Mean

When testing for change in mean for an AR(p) process, we test the following hypothesis.

$$
\begin{aligned}
H_0:&\quad \mu_i = \mu_0, &&\sigma^2 \text{ and } \phi \text{ unknown, for all} && i \ge 1\\
H_A:&\quad \mu_i = \mu_0, &&\sigma^2 \text{ and } \phi \text{ unknown, for all} && i = 1,..,m\\
&\quad \mu_i = \mu_A, &&\sigma^2 \text{ and } \phi \text{ unknown, for all} && i = m+1,..,n
\end{aligned}
\tag{4.1.4}
$$

for $1 \le m \le n$. For the above hypothesis (4.1.4), only the nuisance parameters $\hat{\eta}_k = \left(\hat{\sigma^2}_k, \hat{\phi}_1, \cdots, \hat{\phi}_p\right)$ have to be estimated by their restricted maximum likelihood estimators and using $\mu = \mu_0$ for calculation along the entire sequence $\{X_1, X_2, \cdots, X_n\}$. The following theorem from [18] proves the required precision of the estimators of the nuisance parameter.

Theorem 4.1.1. *Let us assume there is no change in any of the parameters. Under the hypothesis $\phi = \phi_0$ or $\sigma^2 = \sigma_0^2$, and stationary condition in § 2.4.1*

$$
|\hat{\mu}_k - \mu| = O\left(\left(k^{-1}\log\log k\right)^{\frac{1}{2}}\right) \qquad a.s. \tag{4.1.5}
$$

For monitoring a change in mean, with σ^2 and ϕ are unknown, the standardized efficient score vector is

$$W_k\left(\mu_0, \hat{\sigma}_k^2, \hat{\phi}_k\right) = \frac{1}{\hat{\sigma}_k}\sum_{i=1}^{k}\left[(X_i - \mu_0) - \sum_{j=1}^{p}\hat{\phi}_{kj}(X_{i-j} - \mu_0)\right] \tag{4.1.6}$$

The test statistic $\max_{j<k} W_k\left(\mu_0, \hat{\sigma}_k^2, \hat{\phi}_k\right) - W_j\left(\mu_0, \hat{\sigma}_k^2, \hat{\phi}_k\right)$ for 4.1.2 is based on the following theorem from [18].

Theorem 4.1.2. *Under the hypothesis $H_0 : \mu = \mu_0$, with ϕ, and σ^2 unknown, stationarity conditions of (2.4.1), $\exists$ a Brownian Motion $W(.)$, such that*

$$\left|W_k\left(\mu_0, \hat{\sigma}_k^2, \hat{\phi}_k\right) - W(k)\right| \left(\overset{a.s.}{=}\right) \quad o\left(k^{\frac{1}{v}}\right) \tag{4.1.7}$$

for some $v > 2$.

The simulation results and performance of this test statistic under various AR(p) scenarios is illustrated in the next chapter.

4.2 Application of Sieve Bootstrap to obtain critical values

Typically the critical values used in change point hypothesis testing are based on the limiting behavior of the change-point test statistic. As stated earlier in Chapter 2, in the practice of off-line change-point detection hypothesis we fix the size of the test at level $\alpha = 0.05$ in our simulation simulations and try to maximize the power of our change-point test. However, the convergence to an asymptotic distribution is slow. In fact, the critical values from first order asymptotic distribution are based on large sample tests as $n \to \infty$. In practice most of the assumptions about the data do not hold, especially that of having a large data set.

In this case, the bootstrap approach turns out to be quite useful. Using sieve bootstrap, we obtained critical values as bootstrap provides asymptotic refinements. We discuss and apply two approaches below that we apply to the original data in question. We illustrate in Chapter 5 how both the sieve bootstrap approaches outperform the first-order asymptotic critical values and how sieve bootstrap critical values have stable size and improved power than that of the asymptotic critical values.

4.2.1 Block Sieve Bootstrap

In the Block Sieve bootstrap approach, we resample the data and obtain critical values by using blocks of test statistics calculated on each resampled data set. Before illustrating the algorithm we first define the following terms,

- MC :- refers to the number of Monte Carlo simulations done to obtain the power.
- B :- Number of iterations done for each block.
- BS_b :- The number of blocks we divide our bootstrapped test statistic into and take the $(1-\alpha)$ level quartile of each block.
- τ_i :- the test statistic calculated on i^{th} bootstrap sample.

The following algorithm is used to produce the results in Appendix A.

Algorithm 1

1: **for** $m = 1 \cdots MC$ **do**

2: Simulate Y_t from an AR(p) process

3: **for** $i = 1 \cdots BS_b$ **do**

4: **for** $b = 1 \cdots B$ **do**

5: Sieve Bootstrap $\{Y_t\}_{t=1}^n$ and get $\{Y_t^*\}_{t=1}^n$

6: Calculate the change-point test statistic $\tau_{i,b}$ on $\{Y_t^*\}_{t=1}^n$

7: **end for**

8: From the set of "ordered" change-point statistics $\{\tau_{i,1}, \tau_{i,2}, \cdots, \tau_{i,B}\}$ take its $(1-\alpha)$ level quartile.

9: **end for**

10: Given the set of change-point statistics $\{\tau_1, \tau_2, \cdots, \tau_N\}$, get the *bootstrap critical value* $\bar{\tau} = \sum_{i=1}^{n} \frac{\tau_i}{n}$

11: Simulate $\{X_t\}_{t=1}^n$ from an AR(p) process to test the H_0

12: Calculate the change-point statistic τ_m on X_t

13: Define P_m for each monte carlo iteration where

$$P_m = \begin{cases} 1 & if\ \tau_m > \bar{\tau} \\ 0 & otherwise \end{cases}$$

14: **end for**

15: Power $= \frac{\sum_{i=1}^{m} P_m}{MC}$

Note that the total number of sieve bootstrap iterations performed is $(BS_b \times B)$. For each $i \in \{1, \cdots, BS_b\}$, we bootstrap the original data set $\{Y_t\}_{t=1}^n$ B times and calculate the test statistic on each bootstrapped resample $Y_1^b, Y_2^b, \cdots, Y_n^b$. The main difference between Block Sieve Bootstrap and Naive Sieve Bootstrap (discussed below) is in Step 8, where the critical value is chosen by using the $(1-\alpha)^{th}$ level quartile of the test statistic calculated on each i^{th} block. Note that in Step 10, we take the mean of the critical value from each block that is chosen to be our bootstrap critical value. Another possible choice is by taking the median of the critical values in Step 10 since median is more robust than mean. The results of the proposed algorithm are illustrated in Chapter 5. We denote the Block Sieve Bootstrap approach with mean and median by BSB_{Mean} and BSB_{Median} respectively.

4.2.2 Naive Sieve Bootstrap Approach

Naive Sieve bootstrap approach is the simple application of sieve bootstrap to obtain the critical values. Using the same definition of terms as defined in 4.2.1, we illustrate the algorithm below.

Algorithm 2

---

for $m = 1 \cdots MC$ **do**
    Simulate Y_t from an AR(p) process
    for $i = 1 \cdots B$ **do**
        Sieve Bootstrap $\{Y_t\}_{t=1}^{n}$ and get $\{Y_t^*\}_{t=1}^{n}$
        Calculate the change-point test statistic $\tau_{i,b}$ on $\{Y_t^*\}_{t=1}^{n}$
    end for
    From the the set of "ordered" change-point statistics $\{\tau_1, \tau_2, \cdots, \tau_B\}$ take its $(1-\alpha)$ level quartile. This value will be our bootstrap critical value τ_B^*.
    Simulate $\{X_t\}_{t=1}^{n}$ from an AR(p) process to test the H_0
    Calculate the change-point statistic τ_m on X_t
    Define P_m for each monte carlo iteration where

$$P_m = \begin{cases} 1 & if\ \tau_m > \tau_B^* \\ 0 & otherwise \end{cases}$$

end for
Power $= \frac{\sum_{i=1}^{m} P_m}{MC}$

We denote the Naive Sieve Bootstrap approach by NSB and discuss its results later in Chapter 5. Note that the NSB approach is a special case of block sieve bootstrap where there is only 1 block and the number of bootstrap iterations within that 1 block is increased. Thus, the total number of bootstrap iterations in BSB and NSB are the same. Initially, we considered the NSB approach as it provided insight into the spread and stability of sieve bootstrap critical values. Furthermore, the NSB approach also assisted us with choosing the optimal number of monte carlo simulations and bootstrap repetitions.

Note that we use sieve bootstrap in both algorithms which allows us to test for change in not just an AR(p), but MA(q), and ARMA(p, q) processes as well. As discussed in section

3.4, sieve bootstrap estimates the underlying process as an autoregressive process of order $p = p(n)$ where $p(n) \to \infty$ as $n \to \infty$. Since an invertible moving average process has an $AR(\infty)$ representation, and the test proposed by [18] is for stationary autoregressive processes, this suggests that the proposed methods can also be applied to MA(q) and ARMA(p, q) processes.

In chapter 5, we discuss whether simple application of Sieve Bootstrap (NSB) to obtain critical values is effective or by dividing bootstrap critical values into blocks (BSB) gives any extra advantage in obtaining a higher power. Appendix A includes various tables for AR(1) and AR(2) process for errors from Normal and $t-$distribution. Furthermore, we applied our algorithms to detect change in MA(1), MA(2) and ARMA(1,1) processes. The results of all three approaches are discussed in chapter 5.

Chapter 5

Simulation Results

In this chapter, we discuss the results of applying sieve bootstrap to detect changes in the means of autoregressive moving average processes of first and second order. Our test compares the size and power of various sample sizes and errors from normal and $t-$distribution. We also extend our results to moving average processes of first and second order and mixed processes. Furthermore, we show that sieve bootstrap critical values outperform first-order asymptotic critical values even when there are delays in change indexes.

We use the following notation in the results for each approach.

- GS:- The test statistics from [18] as discussed in section 4.1.1.
- BSB_I:- Application of the block sieve bootstrap algorithm from (4.2.1) to test statistic from [18]. The "I" in the subscript is to denote that we take the *mean* of the critical values obtained in Step 10 of algorithm in section (4.2.1) i.e. the results from BSB_{Mean}
- BSB_{II}:-Similar to the above, the block sieve bootstrap algorithm from section 4.2.1 to test statistic from [18]. However the "II" in the subscript is to denote that we take the *median* of the critical values obtained in Step 10 of algorithm in section (4.2.1) i.e. it represents BSB_{Median}.
- NSB:- The results from this approach are based on application of naive sieve bootstrap algorithm from section (4.2.2) to the test statistic from [18].

For each bootstrap approach (BSB_I, BSB_{II}, and NSB) we performed 1500 bootstrap iterations. Initially, tests were performed with 1000, 1500, 2000, and 2500 bootstrap iterations. We found 1500 to be optimal as no significant deviations were observed in mean and variance of the critical values obtained. The number of Monte Carlo Iterations performed for each test was 1000. Again, tests were performed with 1000 and 1500 Monte Carlo iterations and no significant differences were observed in power and size of the tests. The chosen significance level for all the results was 5% ($\alpha = 0.05$).

The results are illustrated for sample sizes of 100, 200, and 300 data points in Appendix A. We show these results for an AR(1) and AR(2) process, with errors from $N(0,1)$ and t-distribution with 5 and 8 degrees of freedom. The results are also illustrated for MA(1), MA(2), and ARMA(1,1) processes with errors from $N(0,1)$.

For block sieve bootstrap approaches (BSB_I & BSB_{II}), the size of each block was 10. In general, blocks of bigger sizes may be used for larger samples and methods such as cross validation can be used for selecting an optimal block size.

5.1 Consistency in Size

As stated earlier in chapter 1, the challenge in change-point detection is to minimize the probability of Type I error while maximizing the power. Unfortunately, the two choices contradict each other. The standard criterion in off-line change-point hypothesis is to maximize the probability of accepting the H_A when it is actually true (i.e. the power = 1 - Type II error), with respect to the constraint of fixed probability of rejecting the H_0 when it is actually true (i.e. the size = Type I error).

5.1.1 Size for Autoregressive Processes

We checked size for an AR(1) process for the coefficients for which the stationarity conditions hold (stated in 2.4.23). Considering that our chosen significance level was 5%, Fig. 5.1 shows that results from BSB_I, BSB_{II}, and NSB are closer to the significance level.

Fig. 5.1 shows that even though the results from Bootstrap approach have a larger size, it is still closer to the chosen empirical significance level of $\alpha = 0.05$. This is true for all AR(1) coefficients. In Fig. 5.1, we see on the right tail end that bootstrap results deviate from the chosen significance level for ϕ_1 =0.8 & 0.9 since they are closer to unit root circle. However, they are still better than the results from first-order asymptotic critical values of [18]'s test

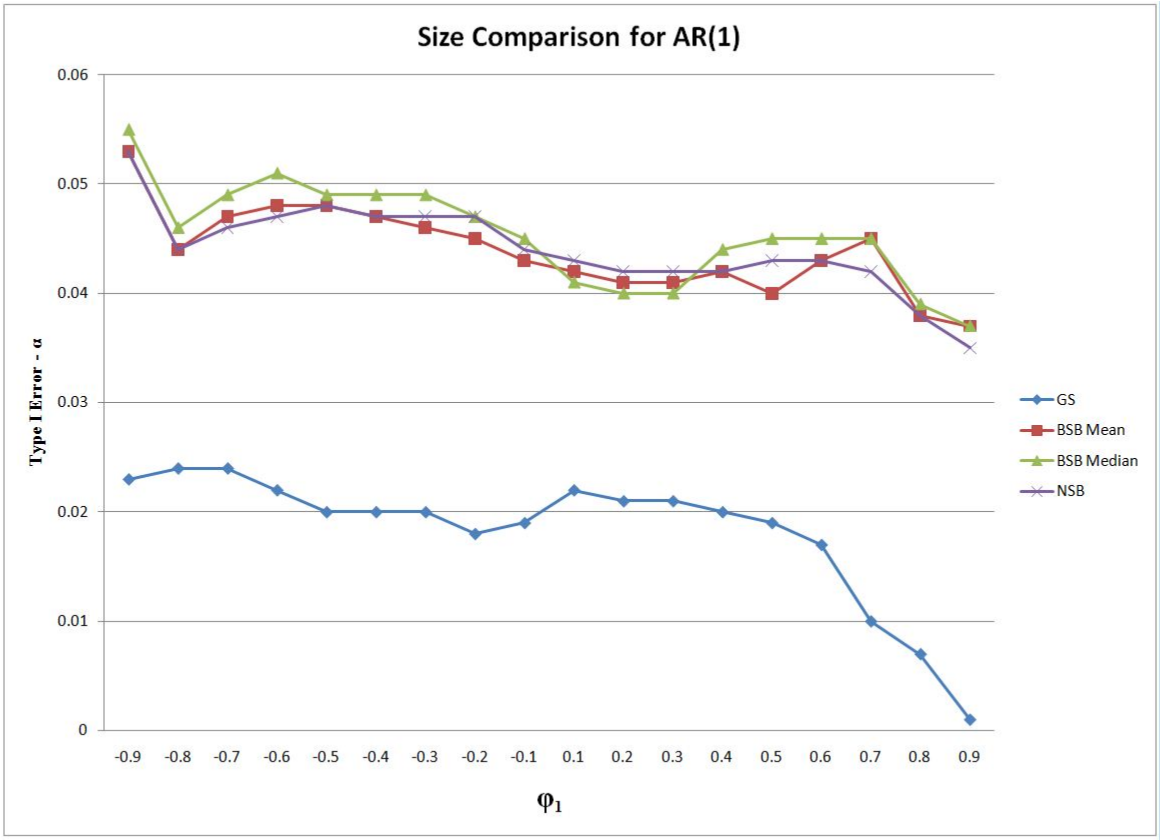

Figure 5.1: Distortion in size given by GS compared to the results from sieve bootstrap critical values when $\epsilon \sim N(0,1)$. The significance level at $\alpha = 0.05$. The number of Sieve Bootstrap Iterations = 1500, number of Monte Carlo Iterations = 1000, and number of points $n = 100$. GS: Test Statistic from Gombay and Serban (2009), BSB_{I}: Block Sieve Bootstrap with mean of critical values from each block, BSB_{II}: Block Sieve Bootstrap with median of critical values from each block, NSB: Naive Sieve Bootstrap approach

statistic. In fact, the results from GS appear to approach zero for $\phi_1 \geq 0.7$.

The results for $\epsilon \sim N(0,1)$ are also summarized in in Table (A.1). Consistency in size is also true for $n = 200$ points (Table A.5) and $n = 300$ points (Table A.9). The size from bootstrap critical values varies around 0.04, compared to the size from GS in both cases which tends to zero as $\phi_1 \to 1$.

When errors of an AR(1) process come from a heavy tailed distribution such as t-distribution, the size from bootstrap critical values is still closer to the chosen significance level of $\alpha = 0.05$ for all AR(1) coefficients compared to the results from GS. Figures 5.2 & 5.3 below show the results for an AR(1) process when $\epsilon \sim t_5$ and $\epsilon \sim t_8$.

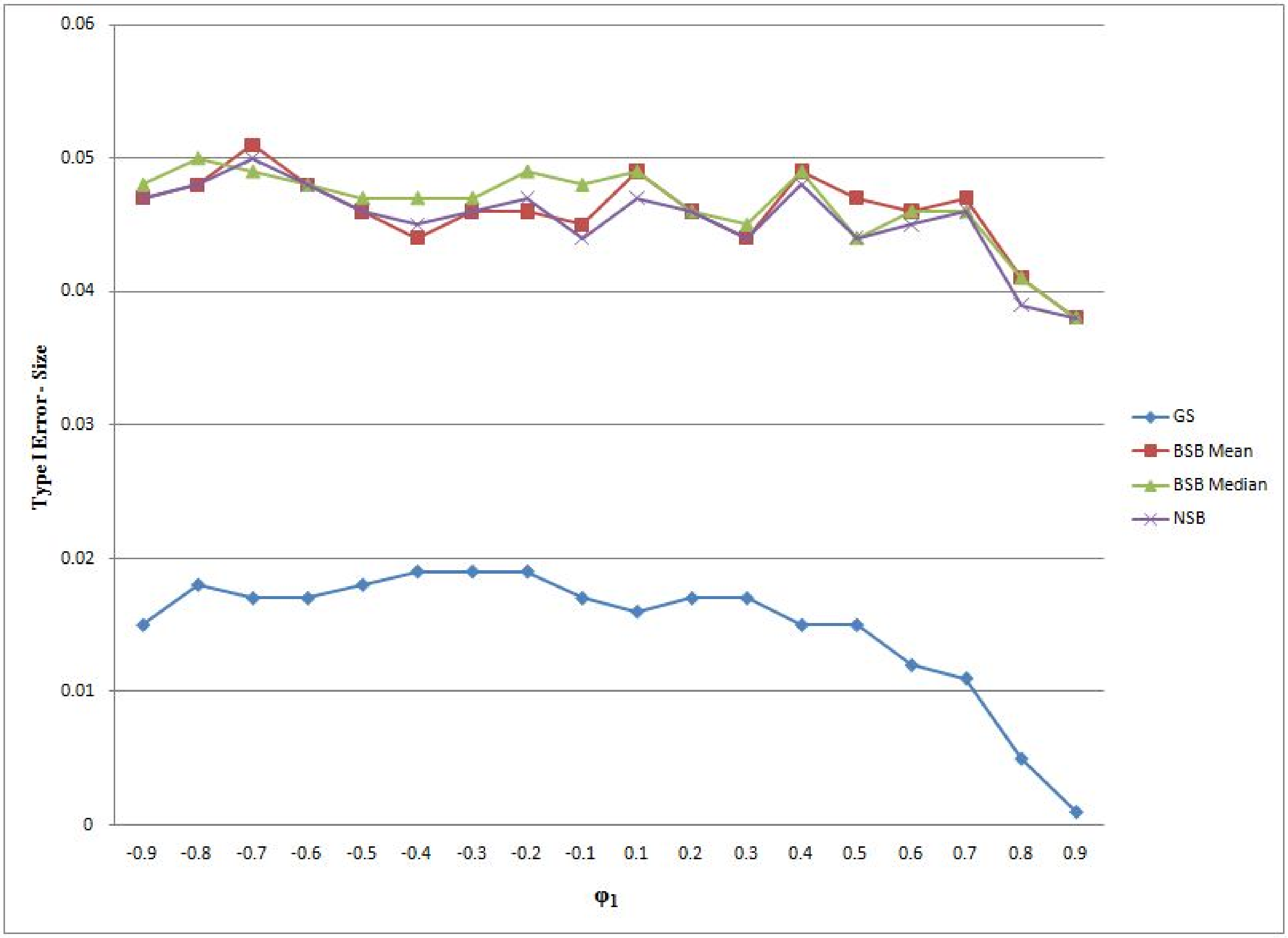

Figure 5.2: Distortion in size given by GS compared to the results from sieve bootstrap critical values when $\epsilon \sim t_5$. The significance level at $\alpha = 0.05$. The number of Sieve Bootstrap Iterations = 1500, number of Monte Carlo Iterations = 1000, and number of points $n = 100$. GS: Test Statistic from Gombay and Serban (2009), BSB_{I}: Block Sieve Bootstrap with mean of critical values from each block, BSB_{II}: Block Sieve Bootstrap with median of critical values from each block, NSB: Naive Sieve Bootstrap approach

The results are also summarized in Table A.21 for $\epsilon \sim t_5$ and Table A.25 for $\epsilon \sim t_8$ respectively. The size from GS approach is below 0.02 in both cases, even though the chosen significance level was of $\alpha = 0.05$. Compared to our proposed algorithm, Fig. 5.2 and Fig. 5.3 show consistent size for all tested AR(1) coefficients for t_5 and t_8 errors respectively.

We know that bootstrap critical values obtained are stable and consistent because they

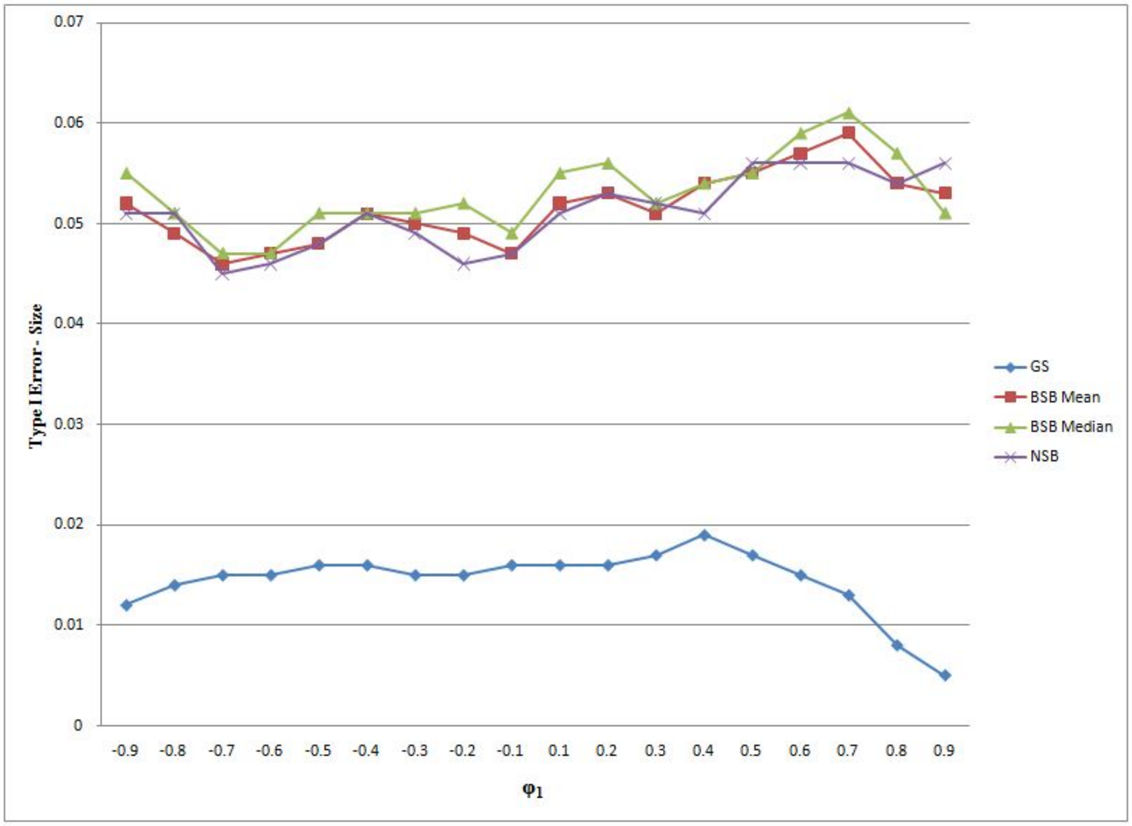

Figure 5.3: Distortion in size given by GS compared to the results from sieve bootstrap critical values when $\epsilon \sim t_8$. The significance level at $\alpha = 0.05$. The number of Sieve Bootstrap Iterations = 1500, number of Monte Carlo Iterations = 1000, and number of points $n = 100$. GS: Test Statistic from Gombay and Serban (2009), BSB_{I}: Block Sieve Bootstrap with mean of critical values from each block, BSB_{II}: Block Sieve Bootstrap with median of critical values from each block, NSB: Naive Sieve Bootstrap approach

have low standard deviation. For example, for the results in Table A.1 we show the mean and standard deviation for each bootstrap approach (BSB_I, BSB_{II}, NSB) in Table A.2, Table A.3, and Table A.4 respectively. We see that all the critical values obtained have a low standard deviation and vary with the AR(1) coefficient for each approach. Similarly for each set of results in the Appendix A we show the mean, variance and standard deviation of the critical values from BSB_{Mean}, BSB_{Median}, and NSB. We can conclude from the tables in Appendix A, that the results for AR(1) processes have stable bootstrap critical values from all three approaches and that size is consistent across various AR(1) coefficients when testing for a change in the mean of an AR(1) for both, Normal and t_{df} errors (for $df > 5$) respectively.

For an AR(2) process, the set of coefficients we choose also satisfy stationarity conditions as defined in equations (2.4.24). The results for an AR(2) process with $n = 100$ data points are shown in Table A.26 for $\epsilon \sim N(0, 1)$. We can see that the Type I error from GS is completely distorted as it is mostly around 0.01 for each set (ϕ_1, ϕ_2). Compared to the results from bootstrap critical values which stay mostly at 0.04 (closer to $\alpha = 0.05$). Furthermore, bootstrap critical values are consistent as shown by low standard deviation for BSB_I, BSB_{II}, and NSB in Tables A.27, A.28, and A.29. Similar conclusions hold, when $\epsilon \sim t_5$ & t_8 as shown in Tables A.30 and A.34 respectively.

5.1.2 Size for Moving Average Processes

We applied all three approaches to MA(1) and MA(2) processes as well. Their results are shown in Tables A.35 and A.40 respectively. In both cases, we approximated the underlying moving average process by an AR(p) process where the order p was selected based on the AIC. Then the test statistic from [18] and our 3 suggested approaches were applied to test for change in the estimated AR(p) process.

The results from Table A.35 show that for MA(1), *GS* doesn't work well since our chosen significance level was $\alpha = 0.05$ and the size from GS stays below 0.01 for all MA(1) coefficients. This is not surprising as GS is meant to test for changes in finite autoregressive stationary processes. Whereas, the results from, BSB and NSB are more stable and stay close to 0.05 for $-0.5 \leq \theta_1 \leq 0.8$. However, for values of θ_1 closer to the invertibility boundary condition ($-1 < \theta_1$) such as $\theta_1 = -0.8$, we see that size is below 0.01. This leads us to conclude that the size for a MA(1) process gets distorted as $\theta_1 \to -1$.

Table A.40 shows that MA(2) processes also have stable size relative to GS which is below 0.01. Compared to the other 3 approaches, BSB and NSB have relatively stable size for chosen set of (θ_1, θ_2) coefficients. For some of the values such as $(\theta_1,\ \theta_2) = (-0.5,\ -0.1)$, the size is low from all three approaches (close to 0.02). Otherwise, in general, the sieve bootstrap approach outperforms the GS for MA(2).

In case of both MA(1) and MA(2), the critical values from sieve bootstrap are stable as they have low standard deviation of < 0.1 in every case. This is shown in Tables A.36, A.37, & A.38 for MA(1). In both cases, we approximated the order of the underlying process using an AR approximation. From Tables A.39 and A.41, we see that the chosen order of the AR(p) approximation varies between 1 and 3.

5.1.3 Size of Autoregressive Moving Average Processes

Our results for ARMA(1, 1) are illustrated in Tables A.42 and A.43. The set of AR coefficients chosen satisfy the stationarity conditions and for the MA(1) we choose $\theta_1 = 0.8$ & 0.5. This ensured that the underlying ARMA(1,1) process is stationary & invertible and, thus, has an $AR(\infty)$ representation as discussed before in section 2.4.4. In Tables A.50 & A.51 we show the order of the underlying process as selected by AIC. There was no significant difference in results for $n = 200$ and 300 data points

As expected, GS approach has a distorted size close to 0.01 in every case. However, BSB_I, BSB_{II}, and NSB have a size varying between 0.04 and 0.05 which is preferred given that our chosen significance level was $\alpha = 0.05$. The order estimated by AIC for each underlying process usually varies between 1 & 3 for each set of values. This suggests that sieve bootstrap approach is stable in detecting change for ARMA(1,1) process for $\theta_1 \geq 0.5$. Tables A.42 and A.43 also show that there is no significant difference in the critical values obtained from each of the three approaches. Given any set of ARMA(1, 1) coefficients e.g. $\phi_1 = 0.5$ and $\theta_1 = 0.5$, we see that the maximum difference between size from BSB_I, BSB_{II}, and NSB is of 0.002.

We can conclude that with respect to criterion of fixing Type I error probability, sieve bootstrap critical values are more stable not only for various scenarios and errors distribution for an AR(1) and AR(2) process, but also for MA(1), MA(2), and ARMA(1, 1). Compared to the first-order asymptotic critical values of GS which provides lower and distorted size. While lower Type I error of GS might seem like an attractive choice, however, in addition to being inconsistent with practice of fixing Type I error probability, it offsets for a higher Type II error (β) as shown in the next section.

5.2 Increase in Power

The power of a statistical test is defined as the probability of rejecting H_0 when it is actually false i.e. 1 - Type II Error (β). In change- detection hypothesis testing, we want to maximize the power of our test and minimize the probability of false alarm as discussed in the beginning of section 5.1. In this section, we discuss our test results and illustrate how sieve bootstrap critical values increase the power of the test statistic.

5.2.1 Power for Autoregressive Processes

Fig. 5.4 shows that for a small change in the mean ($\mu_A = 0.1$), the GS approach hardly recognizes any change and has a very low power, mainly below 0.05. Since minor changes are very hard to detect, we see that three bootstrap approaches also have low power for $\mu_A = 0.1$. However, its capability to detect change is still much higher than that of GS. It is also evident that while the GS approach shows a decreasing pattern of power as $\phi_1 \to 0.9$, the results from bootstrap critical values are consistently higher between 0.12 and 0.14 till the ϕ_1 values get close to unit root stationary. Similar patterns can be observed in Fig. 5.4 for $\mu_A = 0.3$, 0.5, and 0.7. The power from bootstrap critical values is consistently higher than that of GS and converges faster to 1 as the magnitude of change increases.

For normal errors with sample size of 100, Table A.1 shows that the results from bootstrap values are close to each other for AR(1) coefficients and outperform the power of GS consistently. Similar results are expected when the sample size increases to n = 200 (Table A.5) and n = 300 (Table A.9).

The power also increases when errors come from t_5 and t_8 distributions. As shown in Tables A.21 and A.25, all three bootstrap approaches give consistently higher power than that of *GS*. Another observation we made was that when the change point lies the second half of the data (i.e. $\lceil \frac{n}{2} \rceil \leq \tau \leq n-1$), bootstrap critical values still give higher power compared to GS. This is shown in Tables A.13 and A.17. We can see from Tables A.18, A.19, and A.20 that even in this case, the critical values from bootstrap have low standard deviation. [18] suggested that their test statistics could also be used for on-line change detection. This finding suggests that the bootstrap critical value would also out perform if they are used in on-line change-point detection.

Similarly, when looking at results for AR(2) in the Appendix A, we can see from Tables A.26, A.30 and A.34 that bootstrap critical values again offer better results than GS and tables preceding them again show consistency and stability in bootstrap critical values.

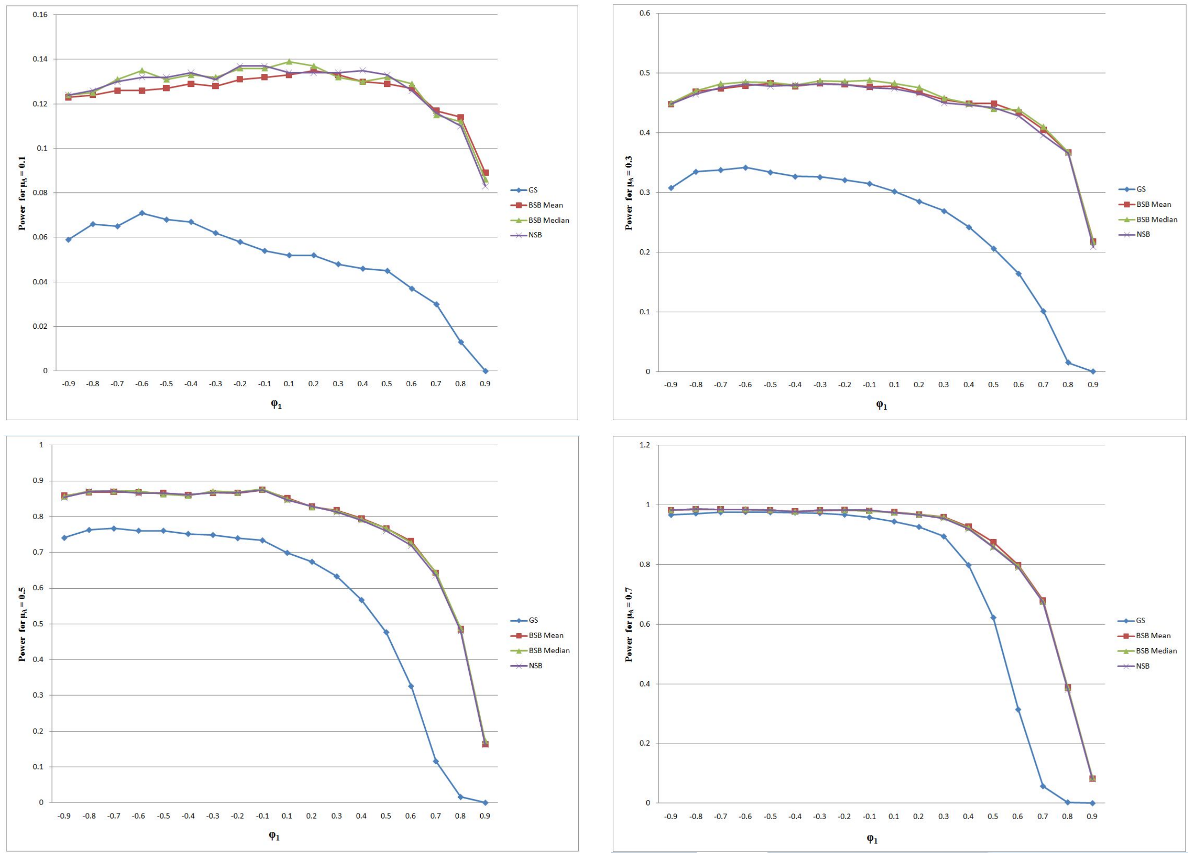

Figure 5.4: Power Comparison of the 4 approaches when for changes of different magnitude in μ_A. The significance level is $\alpha = 0.05$ and $\epsilon \sim N(0,1)$. The number of Sieve Bootstrap Iterations = 1500, number of Monte Carlo Iterations = 1000, and number of points $n = 100$. GS: Test Statistic from Gombay and Serban (2009), BSB_{I}: Block Sieve Bootstrap with mean of critical values from each block, BSB_{II}: Block Sieve Bootstrap with median of critical values from each block, NSB: Naive Sieve Bootstrap approach

5.2.2 Power for Moving Average Processes

Tables A.35 and A.40 show the power for MA(1) and MA(2) processes respectively. Again, all the three bootstrap approaches have stable critical values with low standard deviations.

For a MA(1) process, Tables A.36, A.37, & A.38 show that the maximum standard deviation we observe for the critical values is approximately 0.16. An increasing pattern in standard deviation can be noticed across all MA(1) coefficients as the alternative mean μ_A increases. The power from the sieve bootstrap approach is also higher than GS. However, the power stops increasing as the magnitude of change in mean gets higher. For example, for $\theta_1 = -0.5$, as μ_A increases from 0.7 to 0.9, we see that the power for BSB_I decreases from 0.827 to 0.789. Similar conclusion can be drawn for all the other MA(1) coefficients as well. This can be explained by the underlying process approximated by the sieve bootstrap method. Table A.41 shows that for $\theta_1 = -0.5$, the order of the approximated AR(p) process increases on average from 2.6 to 3.3 as the change in mean increases from 0.7 to 0.9.

A similar conclusion can be drawn for MA(2) processes as shown in Table A.40. Our bootstrap critical values obtained from each of the three approaches is stable and they outperform GS significantly. For example, for $(\theta_1, \theta_2) = (0.5, 0.1)$ and $\mu_A = 0.9$ the power from GS is 0.399 whereas all three bootstrap approaches give a power greater than 0.7. We can also see that (just like in case of MA(1)) as the magnitude of change increases, the underlying AR(p) process estimated is of higher order and, thus, has a lower power.

5.2.3 Power for Autoregressive Moving Average Processes

The results for ARMA(1, 1) are shown in Tables A.42 and A.43 for $\theta_1 = 0.8$ and 0.5 respectively. Our conclusion are similar to the case of moving average processes since we estimate the underlying process using AIC in both cases. Our bootstrap critical values are stable though their standard deviation does increase by a negligible amount as the magnitude of change increases.

The three bootstrap approaches outperform the GS approach from [18] as it is meant for finite autoregressive stationary processes with known order. We see an increasing pattern in power as μ_A increases. However, for an increase in the mean from 0.7 to 0.9 the power decreases for the chosen set of ARMA coefficients. Analogous the moving average processes, a plausible explanation for this is the underlying process approximated by the sieve bootstrap method using AIC.

Thus, we can conclude that application of sieve bootstrap to get critical values for detecting change in mean of a time series is much more effective and gives better results than using the first-order asymptotic critical values. Sieve bootstrap critical values do not only give a consistent size closer to our chosen empirical level but it also provides higher power for detecting change in AR(1) and AR(2) processes when errors follow normal or t-distribution. This conclusion can also be applied moving average processes as shown for MA(1) & MA(2), and mixed processes as shown for ARMA(1, 1).

5.3 Block Sieve Bootstrap V.S. Naive Sieve Bootstrap

The results show that Block Sieve Bootstrap (BSB) and Naive Sieve Bootstrap (NSB), both give stable critical values as shown by the variance and standard deviation of the critical values used in Appendix A. Both approaches give better results than the test statistic from [18] i.e. the GS approach. However, when compared with each other we see that BSB gives better results than NSB.

While the difference between the results might not be significant,but BSB is still closer to our criterion of fixing the size and maximizing the power. For instance looking at Table A.25 when $\mu_A = 0.3$, we can see that BSB_{Mean} (BSB_I) and BSB_{Median} (BSB_{II}) give higher power than NSB. The difference becomes more significant for $\phi_1 > 0$. Similar observation can be made in all the presented results.

The results of BSB_{Mean} and BSB_{Median} are quite similar. Mostly the difference in power and size does not exceed 0.05 in all the displayed results. We observe see that BSB_{Mean} gives slightly higher results than BSB_{Median}. However, this is not always the case e.g. in A.25 for $\mu_A = 0.3$ and $\phi_1 = -0.8$, BSB_I gives a power of 0.391 whereas BSB_{II} gives 0.393. Such minor differences can be ignored and overall we suggest that BSB_{Mean} should be the chosen approach when testing for changes in the mean of linear autoregressive processes. This is the approach we adapt in Chapter 6 as we apply our suggested approach to some real world data sets in the following chapter.

Chapter 6

Case Studies

"*There is nothing wrong with change, if it is in the right direction*" - Winston Churchill

In Chapter 5 we illustrated how application of sieve bootstrap can further enhance the power to detect a change in the mean of an autoregressive moving average time series. In this chapter we illustrate the application of our approach to some of the real world data sets. We present three case studies showing some of the areas where Block Sieve Bootstrap can be used effectively.

6.1 EUR/USD Exchange Rate

Since its inception as a legal tender, the European currency gained significant trade volume against the United States Dollar. As of year 2002, and even today, EUR/USD is the most widely traded pair in the Foreign Exchange market with its daily volume exceeding $300 million dollars [6]. We analyze the EUR/USD weekly data from 2002 to 2007 and see how the beginning recent financial crisis infirmed the myth of demolition of the United States dollar. Fig. 6.1 shows the plot of the data that we analyze (available in Appendix C).

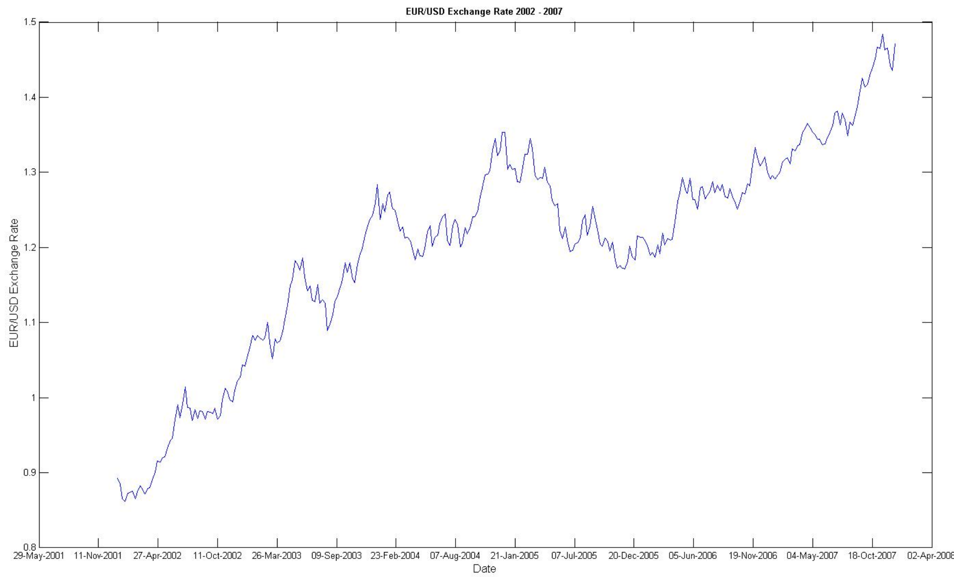

Figure 6.1: Plot of weekly closing Prices of EUR/USD from 2002 - 2007. The data consists of 311 data points.

Since we are dealing with financial time series, we preprocessed the data by taking the first difference of the log data to get a weakly stationary series. The resulting plot is shown in Fig. 6.2. The assumption of weak stationarity can be verified from Fig. 6.2 which shows zero mean and constant variance over the sampled period. This is also verified by the Augmented-Dickey Fuller Test which yields a p-value of less than 0.1.

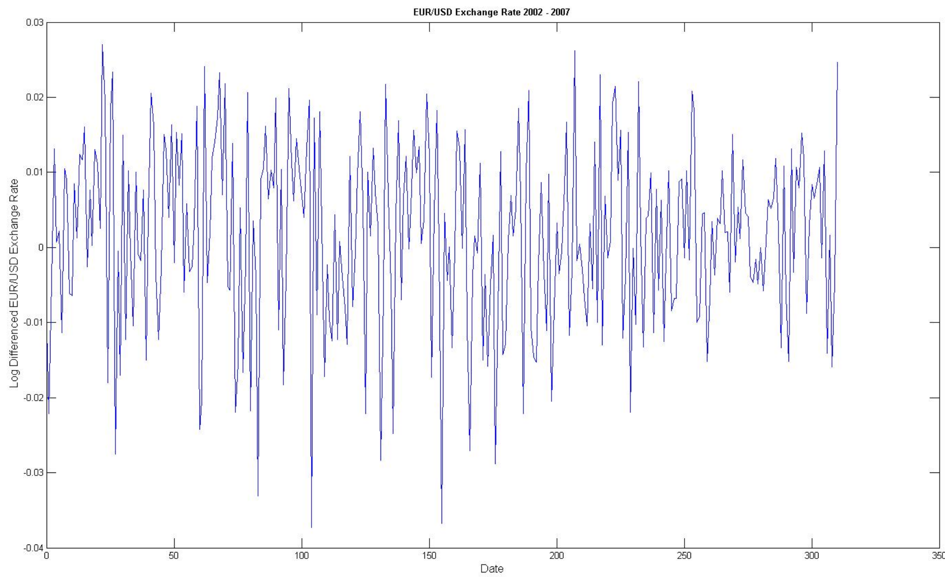

Figure 6.2: Plot of EUR/USD Exchange Rate data after variance stabilizing transformations.

First we apply the test statistic from [18] to test for a change in the mean. The test yields a critical value 2.2442 which is close to the test statistic's asymptotic critical value corresponding to $\alpha = 0.05$ significance level. Then we apply our Block Sieve Bootstrap approach where we choose the size of each block to be 10, similar to our simulation studies. After 1500 bootstrap samples we get the critical value of 0.7036 which is significantly lower than 2.442. We experimented with several numbers of bootstrap samples and found that 1500 bootstrap samples gave stable critical value with a standard deviation of 0.58.

We conclude from our finding that the rising trend of EUR/USD is likely to discontinue after 2007. The validity of our finding became evident as the exchange rate reach a low of 1.23 in 2008 and is trading around 1.19 since the beginning of June, 2010.

6.2 Quarterly Business Bankruptcy Filings in U.S.A.

In United States of America, businesses unable to meet their debt obligations file for chapter 11 bankruptcy, after which they are insolvent. In this example we analyze quarterly bankruptcy filings in USA from 1994 to first quarter of 2010. The raw plot of Bankruptcy filings in USA is given below. While it might seem obvious at first sight that there is a structural break in the data, not all tests indicate such a finding.

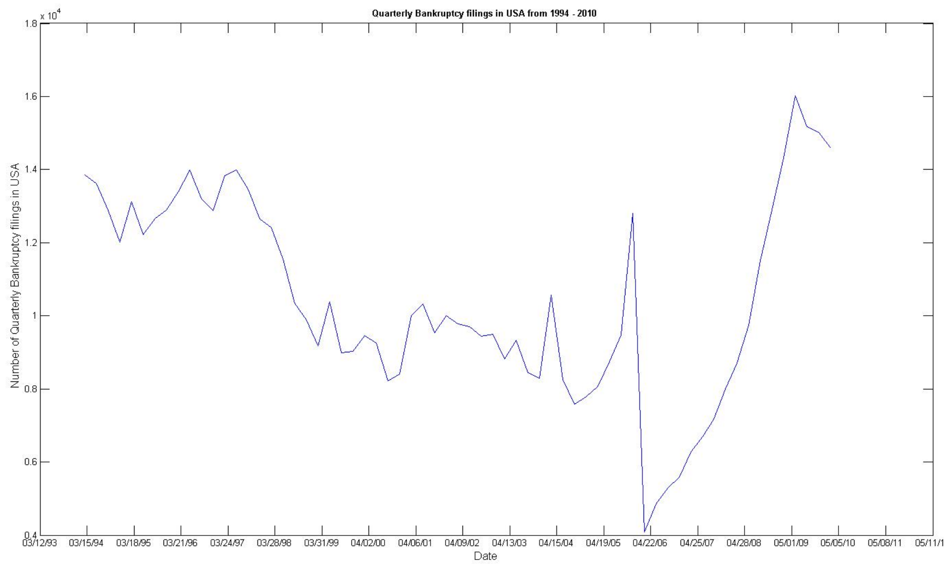

Figure 6.3: Plot of Quarterly business bankruptcy filings in USA from 1994 - 2010. The data consists of 64 data points.

Fig. 6.3 shows that the number of business bankruptcies did not significantly increase from 2000 - 2002 dot-com bubble burst which is the general perception. In fact, they decreased afterwards especially during 2005. First we performed a variance stabilizing transformation to the data to get a weakly stationary series shown in Fig. 6.4.

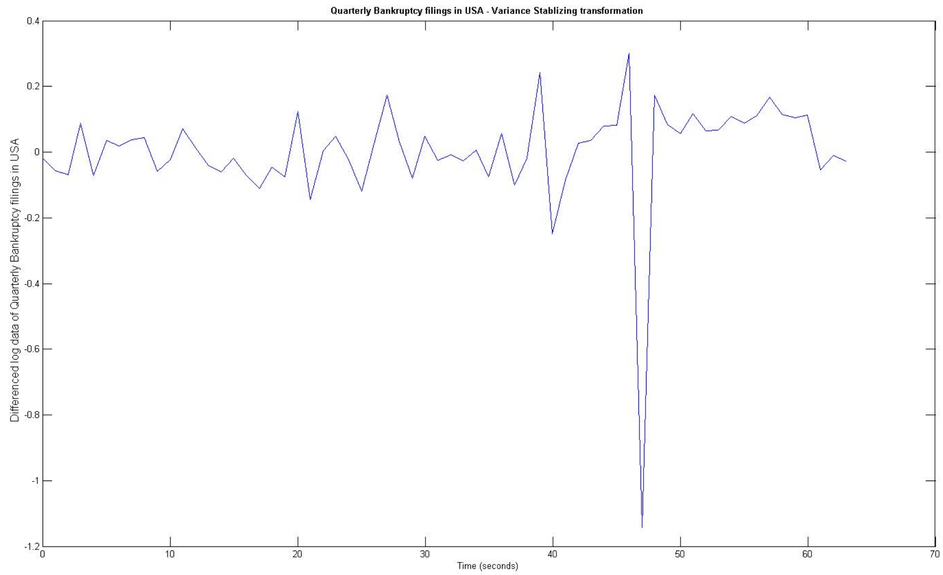

Figure 6.4: Plot of the first difference of logged Quarterly Business Bankruptcy filings in USA from 1994 - 2010.

Fig. 6.4 makes us question the stationarity because of one significant spike. However, applying Augmented-Dickey fuller test yielded a p-value of less than 0.1. Furthermore, the autocorrelation function (ACF) had no significant correlations, but the partial autocorrelation function (PACF) had first correlation significant. This lead us to fit an AR(1) model to the series based on AIC.

Initially, we applied the approach from [18] and we got a test statistic value of 1.0650. At 5% significance level, the first-order asymptotic critical value of the test statistic is 2.24 and using this approach would indicate that there has been no change in average bankruptcies in the United States. However, applying our Block Sieve Bootstrap approach, we get a bootstrap critical value of 0.8483 which is less than the test statistic value of 1.0650. We used 1500 bootstrap simulations where size of each block was 10 and we know that our bootstrap critical values were stable with the standard deviation of 0.76. Thus, according to our Block Sieve Bootstrap approach there was a change in the average number of bankruptcies in USA from 1994-2010. Our findings can be confirmed by recent economic events as the U.S. economy faced the worst credit crisis since the Great Depression.

Chapter 7

Conclusions and Future Work

"*The price of doing the same old thing is far higher than the price of change.*" - Bill Clinton

The purpose of this thesis was to illustrate how application of sieve bootstrap to detect change in time series can give us more stable size and higher power in our hypothesis testing. The test statistic we chose for illustration was from [18], which was built on the idea of using cumulative sums of efficient score vectors. We illustrated our results for detecting change in the mean of an autoregressive, moving average, and processes. We introduced two different approaches of applying sieve bootstrap to get the bootstrap critical values i.e. Block Sieve Bootstrap and Naive Sieve Bootstrap. From the simulation results illustrated in chapter 5 we concluded that Block Sieve Bootstrap gives better results than the Naive Sieve Bootstrap approach.

In chapter 6, we illustrated how the use of our approach is effective in detecting changes on two econometric data sets. In case of EUR/USD exchange rate, it showed that the GS approach [18] is equally effective in detecting the change. However, detecting change in quarterly US Bankruptcy filings, our approach does not give false positive results like their's.

Our research gave us more ideas which can be considered in the future to add more to the field of change-point analysis in time series. We are considering some of the following in the future.

- The sieve bootstrap approach from [2] approximates a linear time series using the AIC as model selection criterion. Another approach we would like to consider would be to apply Non-Linear Sieve Bootstrap Methods such as to Neural-Network Sieve Bootstrap

[16] to see if it adds to the performance of the change-point test statistics.

- In our thesis the algorithms we suggested enhance the power to detect change in the mean of an autoregressive moving average time series. We would further like to see if other methods or algorithms can be derived that can add similar improvements to detect a change in variance or parameter(s) of such processes.

- The processes under consideration in our thesis were stationary autoregressive, moving average, and mixed processes. We would like to extend the framework to see if application of sieve bootstrap will work for ARIMA(p,d,q) and FARIMA(p,d,q) processes.

- The case studies illustrated in chapter 6 considered the data at regularly spaced intervals. The frequency of the data analyzed was not very high (i.e. daily or weekly data). The problem of detecting change in time series that is not equally spaced and sampled at a higher frequency becomes much harder because of the statistical properties of such data. However, examples of high-frequency data are witnessed quite often in the field of financial risk management, high speed network monitoring, and real-time computing in information systems management to name a few. We would like to see if bootstrap methods can be found which are applicable to high-frequency data, and, thus enhance the power to detect change in on-line change-point framework.

The contributions of Bradley E. Efron and E.S. Page sparked extensive research into two new fields which are coming together now as more computing power becomes available to us. With the given interactions among various fields today, we can expect and hope to see further advancements in change-point detection for time series.

Appendix A

Tables for Change in Mean

μ_A	0				0.1				0.3			
ϕ_1	GS	BSB_{I}	BSB_{II}	NSB	GS	BSB_{I}	BSB_{II}	NSB	GS	BSB_{I}	BSB_{II}	NSB
-0.8	0.024	0.044	0.046	0.044	0.066	0.124	0.125	0.126	0.335	0.469	0.47	0.465
-0.5	0.02	0.048	0.049	0.048	0.068	0.127	0.131	0.132	0.334	0.483	0.484	0.478
-0.3	0.02	0.046	0.049	0.047	0.062	0.128	0.132	0.131	0.326	0.483	0.487	0.482
-0.1	0.019	0.043	0.045	0.044	0.054	0.132	0.136	0.137	0.315	0.477	0.488	0.476
0.1	0.022	0.042	0.041	0.043	0.052	0.133	0.139	0.134	0.302	0.478	0.483	0.474
0.3	0.021	0.041	0.04	0.042	0.048	0.133	0.132	0.134	0.269	0.455	0.458	0.45
0.5	0.019	0.04	0.045	0.043	0.045	0.129	0.132	0.133	0.206	0.449	0.44	0.442
0.8	0.007	0.038	0.039	0.038	0.013	0.114	0.112	0.11	0.015	0.367	0.368	0.366

μ_A	0.5				0.7				0.9			
ϕ_1	GS	BSB_{I}	BSB_{II}	NSB	GS	BSB_{I}	BSB_{II}	NSB	GS	BSB_{I}	BSB_{II}	NSB
-0.8	0.763	0.868	0.871	0.87	0.971	0.985	0.986	0.986	0.998	1	1	1
-0.5	0.761	0.866	0.863	0.865	0.975	0.982	0.982	0.982	0.997	0.999	0.999	0.999
-0.3	0.749	0.867	0.871	0.867	0.972	0.981	0.982	0.982	0.997	1	1	1
-0.1	0.734	0.875	0.877	0.875	0.958	0.98	0.98	0.982	0.996	0.998	0.998	0.998
0.1	0.699	0.852	0.849	0.846	0.944	0.976	0.974	0.973	0.993	0.997	0.997	0.997
0.3	0.633	0.818	0.816	0.813	0.894	0.959	0.958	0.955	0.964	0.984	0.984	0.984
0.5	0.477	0.767	0.768	0.761	0.622	0.875	0.86	0.858	0.585	0.842	0.836	0.834
0.8	0.016	0.485	0.488	0.48	0.003	0.387	0.388	0.382	0	0.216	0.219	0.212

Table A.1: Comparison for an AR(1) process for errors $\epsilon \sim N(0,1)$ with $n = 100$, change $\tau = 50$, and significance level at $\alpha = 0.05$. The number of Sieve Bootstrap Iterations = 1500 and number of Monte Carlo Iterations = 1000. GS: Test Statistic from Gombay and and Serban (2009), BSB_{I}: Block Sieve Bootstrap with Mean of Critical Values from each block, BSB_{II}: Block Sieve Bootstrap with Median of Critical Values from each block, NSB: Naive Sieve Bootstrap approach

μ_A	0			0.1			0.3			0.5		
ϕ_1	Mean	Var	Std. Dev.	Mean	Var	Std. Dev.	Mean	Var	Std. Dev.	Mean	Var	Std. Dev.
-0.8	1.9085	0.0303	0.1739	1.9136	0.0309	0.1756	1.9698	0.0309	0.1756	2.0847	0.039	0.1973
-0.5	1.899	0.0328	0.1811	1.9099	0.0324	0.1799	1.9658	0.0335	0.183	2.0784	0.0393	0.198
-0.3	1.9032	0.0316	0.1778	1.907	0.0317	0.178	1.9605	0.0311	0.1763	2.0561	0.0375	0.1937
-0.1	1.8959	0.0236	0.1535	1.9035	0.0224	0.1496	1.935	0.0223	0.1493	2.0123	0.0339	0.1842
0.1	1.8867	0.024	0.1549	1.8899	0.0248	0.1575	1.9321	0.0269	0.1638	2.0322	0.0385	0.196
0.3	1.8711	0.0274	0.1656	1.8755	0.0282	0.1678	1.9221	0.0288	0.1695	2.0086	0.039	0.1973
0.5	1.8464	0.0253	0.1588	1.8523	0.0256	0.16	1.8937	0.0273	0.1651	1.946	0.036	0.1896
0.8	1.7702	0.0159	0.1261	1.77	0.0166	0.1287	1.7599	0.0148	0.1216	1.7302	0.0143	0.1196

μ_A	0.7			0.9		
ϕ_1	Mean	Var	Std. Dev.	Mean	Var	Std. Dev.
-0.8	2.2239	0.0275	0.1656	2.3094	0.0061	0.0781
-0.5	2.2163	0.0281	0.1676	2.3031	0.0066	0.0808
-0.3	2.182	0.0344	0.1855	2.2671	0.0176	0.1325
-0.1	2.1355	0.04	0.1999	2.2539	0.0187	0.1366
0.1	2.1402	0.0382	0.1953	2.234	0.0215	0.1467
0.3	2.0914	0.0431	0.2074	2.1691	0.0341	0.1845
0.5	2.0047	0.0442	0.2101	2.0371	0.0467	0.2159
0.8	1.6872	0.0119	0.109	1.6446	0.0085	0.092

Table A.2: Mean, Variance, and Standard Deviation of the the critical values used in Table A.1 for BSB_{I}: Block Sieve Bootstrap with mean of critical values from each block. Number of Blocks = 10

μ_A	0			0.1			0.3			0.5		
ϕ_1	Mean	Var	Std. Dev.	Mean	Var	Std. Dev.	Mean	Var	Std. Dev.	Mean	Var	Std. Dev.
-0.8	1.9017	0.0338	0.1837	1.9067	0.0347	0.1861	1.9649	0.0357	0.1889	2.0824	0.0433	0.2079
-0.5	1.891	0.0356	0.1887	1.9017	0.0356	0.1887	1.9579	0.0377	0.1941	2.0764	0.0436	0.2088
-0.3	1.8945	0.0349	0.1867	1.8993	0.0349	0.1866	1.9529	0.0349	0.1867	2.0529	0.0414	0.2034
-0.1	1.8851	0.026	0.161	1.8951	0.025	0.158	1.9259	0.0253	0.1591	2.0061	0.0374	0.1934
0.1	1.8765	0.0259	0.1609	1.8811	0.0269	0.1639	1.9238	0.0299	0.1727	2.0292	0.0422	0.2053
0.3	1.8623	0.0294	0.1713	1.8673	0.03	0.1731	1.916	0.032	0.1788	2.0077	0.0431	0.2076
0.5	1.8402	0.0272	0.1649	1.8466	0.0277	0.1663	1.8901	0.0301	0.1735	1.9464	0.039	0.1974
0.8	1.77	0.0167	0.1292	1.7701	0.0174	0.1319	1.7596	0.0155	0.1242	1.73	0.0148	0.1216

μ_A	0.7			0.9		
ϕ_1	Mean	Var	Std. Dev.	Mean	Var	Std. Dev.
-0.8	2.2227	0.03	0.173	2.3093	0.0066	0.0808
-0.5	2.2155	0.0304	0.1742	2.3029	0.007	0.0835
-0.3	2.1809	0.0369	0.192	2.2666	0.0184	0.1354
-0.1	2.1329	0.0426	0.2064	2.2535	0.02	0.1412
0.1	2.1404	0.0407	0.2017	2.2357	0.0224	0.1496
0.3	2.0943	0.0459	0.2141	2.1733	0.0356	0.1886
0.5	2.0085	0.0468	0.2162	2.0412	0.0486	0.2203
0.8	1.6862	0.0121	0.11	1.6438	0.0087	0.093

Table A.3: Mean, Variance, and Standard Deviation of the the critical values used in Table A.1 for BSB_{II}: Block Sieve Bootstrap with median of critical values from each block. Number of Blocks = 10

μ_A	0			0.1			0.3			0.5		
ϕ_1	Mean	Var	Std. Dev.	Mean	Var	Std. Dev.	Mean	Var	Std. Dev.	Mean	Var	Std. Dev.
-0.8	1.9083	0.0328	0.1809	1.9141	0.0337	0.1836	1.9704	0.0341	0.1845	2.0867	0.0416	0.2039
-0.5	1.8974	0.0348	0.1864	1.9083	0.0346	0.1858	1.9649	0.0361	0.1899	2.0808	0.0416	0.204
-0.3	1.901	0.0337	0.1836	1.9053	0.0338	0.1838	1.9591	0.0332	0.1823	2.0571	0.0399	0.1998
-0.1	1.8925	0.0251	0.1582	1.9007	0.024	0.1548	1.9324	0.024	0.1547	2.0106	0.0359	0.1895
0.1	1.8826	0.0244	0.1561	1.886	0.0256	0.1598	1.9295	0.0283	0.1681	2.034	0.0407	0.2016
0.3	1.87	0.0283	0.1681	1.8744	0.0292	0.1707	1.9231	0.0308	0.1753	2.0126	0.0419	0.2046
0.5	1.8473	0.0264	0.1625	1.8526	0.0266	0.1629	1.896	0.0293	0.1712	1.952	0.0382	0.1953
0.8	1.7748	0.0164	0.1279	1.7751	0.017	0.1304	1.7638	0.0151	0.1226	1.7338	0.0146	0.1207

μ_A	0.7			0.9		
ϕ_1	Mean	Var	Std. Dev.	Mean	Var	Std. Dev.
-0.8	2.2254	0.0284	0.1684	2.31	0.0064	0.0798
-0.5	2.2186	0.0288	0.1696	2.3037	0.0068	0.0823
-0.3	2.1837	0.0354	0.188	2.2673	0.0181	0.1342
-0.1	2.1361	0.0412	0.2029	2.2551	0.0192	0.1384
0.1	2.1439	0.0393	0.1981	2.2372	0.0218	0.1474
0.3	2.0975	0.0445	0.211	2.1763	0.0344	0.1854
0.5	2.013	0.0458	0.214	2.0455	0.0478	0.2185
0.8	1.6899	0.0121	0.1099	1.6471	0.0086	0.0926

Table A.4: Mean, Variance, and Standard Deviation of the the critical values used in Table A.1 for NSB: Naive Sieve Bootstrap Approach

μ_A	0				0.1				0.3			
ϕ_1	GS	BSB_{I}	BSB_{II}	NSB	GS	BSB_{I}	BSB_{II}	NSB	GS	BSB_{I}	BSB_{II}	NSB
-0.8	0.024	0.044	0.046	0.044	0.066	0.124	0.125	0.126	0.335	0.469	0.47	0.465
-0.5	0.02	0.048	0.049	0.048	0.068	0.127	0.131	0.132	0.334	0.483	0.484	0.478
-0.3	0.02	0.046	0.049	0.047	0.062	0.128	0.132	0.131	0.326	0.483	0.487	0.482
-0.1	0.019	0.043	0.045	0.044	0.054	0.132	0.136	0.137	0.315	0.477	0.488	0.476
0.1	0.022	0.042	0.041	0.043	0.052	0.133	0.139	0.134	0.302	0.478	0.483	0.474
0.3	0.021	0.041	0.04	0.042	0.048	0.133	0.132	0.134	0.269	0.455	0.458	0.45
0.5	0.019	0.04	0.045	0.043	0.045	0.129	0.132	0.133	0.206	0.449	0.44	0.442
0.8	0.007	0.038	0.039	0.038	0.013	0.114	0.112	0.11	0.015	0.367	0.368	0.366

μ_A	0.5				0.7				0.9			
ϕ_1	GS	BSB_{I}	BSB_{II}	NSB	GS	BSB_{I}	BSB_{II}	NSB	GS	BSB_{I}	BSB_{II}	NSB
-0.8	0.763	0.868	0.871	0.87	0.971	0.985	0.986	0.986	0.998	1	1	1
-0.5	0.761	0.866	0.863	0.865	0.975	0.982	0.982	0.982	0.997	0.999	0.999	0.999
-0.3	0.749	0.867	0.871	0.867	0.972	0.981	0.982	0.982	0.997	1	1	1
-0.1	0.734	0.875	0.877	0.875	0.958	0.98	0.98	0.982	0.996	0.998	0.998	0.998
0.1	0.699	0.852	0.849	0.846	0.944	0.976	0.974	0.973	0.993	0.997	0.997	0.997
0.3	0.633	0.818	0.816	0.813	0.894	0.959	0.958	0.955	0.964	0.984	0.984	0.984
0.5	0.477	0.767	0.768	0.761	0.622	0.875	0.86	0.858	0.585	0.842	0.836	0.834
0.8	0.016	0.485	0.488	0.48	0.003	0.387	0.388	0.382	0	0.216	0.219	0.212

Table A.5: Comparison for an AR(1) process for errors $\epsilon \sim N(0,1)$ with $n = 200$, change $\tau = 100$, and significance level at $\alpha = 0.05$. The number of Sieve Bootstrap Iterations = 1500 and number of Monte Carlo Iterations = 1000. GS: Test Statistic from Gombay and and Serban (2009), BSB_{I}: Block Sieve Bootstrap with Mean of Critical Values from each block, BSB_{II}: Block Sieve Bootstrap with Median of Critical Values from each block, NSB: Naive Sieve Bootstrap approach

μ_A	0			0.1			0.3			0.5		
ϕ_1	Mean	Var	Std. Dev.	Mean	Var	Std. Dev.	Mean	Var	Std. Dev.	Mean	Var	Std. Dev.
-0.8	1.9085	0.0303	0.1739	1.9136	0.0309	0.1756	1.9698	0.0309	0.1756	2.0847	0.039	0.1973
-0.5	1.899	0.0328	0.1811	1.9099	0.0324	0.1799	1.9658	0.0335	0.183	2.0784	0.0393	0.198
-0.3	1.9032	0.0316	0.1778	1.907	0.0317	0.178	1.9605	0.0311	0.1763	2.0561	0.0375	0.1937
-0.1	1.8959	0.0236	0.1535	1.9035	0.0224	0.1496	1.935	0.0223	0.1493	2.0123	0.0339	0.1842
0.1	1.8867	0.024	0.1549	1.8899	0.0248	0.1575	1.9321	0.0269	0.1638	2.0322	0.0385	0.196
0.3	1.8711	0.0274	0.1656	1.8755	0.0282	0.1678	1.9221	0.0288	0.1695	2.0086	0.039	0.1973
0.5	1.8464	0.0253	0.1588	1.8523	0.0256	0.16	1.8937	0.0273	0.1651	1.946	0.036	0.1896
0.8	1.7702	0.0159	0.1261	1.77	0.0166	0.1287	1.7599	0.0148	0.1216	1.7302	0.0143	0.1196

μ_A	0.7			0.9		
ϕ_1	Mean	Var	Std. Dev.	Mean	Var	Std. Dev.
-0.8	2.2239	0.0275	0.1656	2.3094	0.0061	0.0781
-0.5	2.2163	0.0281	0.1676	2.3031	0.0066	0.0808
-0.3	2.182	0.0344	0.1855	2.2671	0.0176	0.1325
-0.1	2.1355	0.04	0.1999	2.2539	0.0187	0.1366
0.1	2.1402	0.0382	0.1953	2.234	0.0215	0.1467
0.3	2.0914	0.0431	0.2074	2.1691	0.0341	0.1845
0.5	2.0047	0.0442	0.2101	2.0371	0.0467	0.2159
0.8	1.6872	0.0119	0.109	1.6446	0.0085	0.092

Table A.6: Mean, Variance, and Standard Deviation of the the critical values used in Table A.5 for BSB_{I}: Block Sieve Bootstrap with mean of critical values from each block. Number of Blocks = 10

μ_A	0			0.1			0.3			0.5		
ϕ_1	Mean	Var	Std. Dev.	Mean	Var	Std. Dev.	Mean	Var	Std. Dev.	Mean	Var	Std. Dev.
-0.8	1.9017	0.0338	0.1837	1.9067	0.0347	0.1861	1.9649	0.0357	0.1889	2.0824	0.0433	0.2079
-0.5	1.891	0.0356	0.1887	1.9017	0.0356	0.1887	1.9579	0.0377	0.1941	2.0764	0.0436	0.2088
-0.3	1.8945	0.0349	0.1867	1.8993	0.0349	0.1866	1.9529	0.0349	0.1867	2.0529	0.0414	0.2034
-0.1	1.8851	0.026	0.161	1.8951	0.025	0.158	1.9259	0.0253	0.1591	2.0061	0.0374	0.1934
0.1	1.8765	0.0259	0.1609	1.8811	0.0269	0.1639	1.9238	0.0299	0.1727	2.0292	0.0422	0.2053
0.3	1.8623	0.0294	0.1713	1.8673	0.03	0.1731	1.916	0.032	0.1788	2.0077	0.0431	0.2076
0.5	1.8402	0.0272	0.1649	1.8466	0.0277	0.1663	1.8901	0.0301	0.1735	1.9464	0.039	0.1974
0.8	1.77	0.0167	0.1292	1.7701	0.0174	0.1319	1.7596	0.0155	0.1242	1.73	0.0148	0.1216

μ_A	0.7			0.9		
ϕ_1	Mean	Var	Std. Dev.	Mean	Var	Std. Dev.
-0.8	2.2227	0.03	0.173	2.3093	0.0066	0.0808
-0.5	2.2155	0.0304	0.1742	2.3029	0.007	0.0835
-0.3	2.1809	0.0369	0.192	2.2666	0.0184	0.1354
-0.1	2.1329	0.0426	0.2064	2.2535	0.02	0.1412
0.1	2.1404	0.0407	0.2017	2.2357	0.0224	0.1496
0.3	2.0943	0.0459	0.2141	2.1733	0.0356	0.1886
0.5	2.0085	0.0468	0.2162	2.0412	0.0486	0.2203
0.8	1.6862	0.0121	0.11	1.6438	0.0087	0.093

Table A.7: Mean, Variance, and Standard Deviation of the the critical values used in Table A.5 for BSB_{II}: Block Sieve Bootstrap with median of critical values from each block. Number of Blocks = 10

μ_A	0			0.1			0.3			0.5		
ϕ_1	Mean	Var	Std. Dev.	Mean	Var	Std. Dev.	Mean	Var	Std. Dev.	Mean	Var	Std. Dev.
-0.8	1.9083	0.0328	0.1809	1.9141	0.0337	0.1836	1.9704	0.0341	0.1845	2.0867	0.0416	0.2039
-0.5	1.8974	0.0348	0.1864	1.9083	0.0346	0.1858	1.9649	0.0361	0.1899	2.0808	0.0416	0.204
-0.3	1.901	0.0337	0.1836	1.9053	0.0338	0.1838	1.9591	0.0332	0.1823	2.0571	0.0399	0.1998
-0.1	1.8925	0.0251	0.1582	1.9007	0.024	0.1548	1.9324	0.024	0.1547	2.0106	0.0359	0.1895
0.1	1.8826	0.0244	0.1561	1.886	0.0256	0.1598	1.9295	0.0283	0.1681	2.034	0.0407	0.2016
0.3	1.87	0.0283	0.1681	1.8744	0.0292	0.1707	1.9231	0.0308	0.1753	2.0126	0.0419	0.2046
0.5	1.8473	0.0264	0.1625	1.8526	0.0266	0.1629	1.896	0.0293	0.1712	1.952	0.0382	0.1953
0.8	1.7748	0.0164	0.1279	1.7751	0.017	0.1304	1.7638	0.0151	0.1226	1.7338	0.0146	0.1207

μ_A	0.7			0.9		
ϕ_1	Mean	Var	Std. Dev.	Mean	Var	Std. Dev.
-0.8	2.2254	0.0284	0.1684	2.31	0.0064	0.0798
-0.5	2.2186	0.0288	0.1696	2.3037	0.0068	0.0823
-0.3	2.1837	0.0354	0.188	2.2673	0.0181	0.1342
-0.1	2.1361	0.0412	0.2029	2.2551	0.0192	0.1384
0.1	2.1439	0.0393	0.1981	2.2372	0.0218	0.1474
0.3	2.0975	0.0445	0.211	2.1763	0.0344	0.1854
0.5	2.013	0.0458	0.214	2.0455	0.0478	0.2185
0.8	1.6899	0.0121	0.1099	1.6471	0.0086	0.0926

Table A.8: Mean, Variance, and Standard Deviation of the the critical values used in Table A.5 for NSB: Naive Sieve Bootstrap Approach

μ_A	0				0.1				0.3			
ϕ_1	GS	BSB_{I}	BSB_{II}	NSB	GS	BSB_{I}	BSB_{II}	NSB	GS	BSB_{I}	BSB_{II}	NSB
-0.8	0.024	0.044	0.046	0.044	0.066	0.124	0.125	0.126	0.335	0.469	0.47	0.465
-0.5	0.02	0.048	0.049	0.048	0.068	0.127	0.131	0.132	0.334	0.483	0.484	0.478
-0.3	0.02	0.046	0.049	0.047	0.062	0.128	0.132	0.131	0.326	0.483	0.487	0.482
-0.1	0.019	0.043	0.045	0.044	0.054	0.132	0.136	0.137	0.315	0.477	0.488	0.476
0.1	0.022	0.042	0.041	0.043	0.052	0.133	0.139	0.134	0.302	0.478	0.483	0.474
0.3	0.021	0.041	0.04	0.042	0.048	0.133	0.132	0.134	0.269	0.455	0.458	0.45
0.5	0.019	0.04	0.045	0.043	0.045	0.129	0.132	0.133	0.206	0.449	0.44	0.442
0.8	0.007	0.038	0.039	0.038	0.013	0.114	0.112	0.11	0.015	0.367	0.368	0.366

μ_A	0.5				0.7				0.9			
ϕ_1	GS	BSB_{I}	BSB_{II}	NSB	GS	BSB_{I}	BSB_{II}	NSB	GS	BSB_{I}	BSB_{II}	NSB
-0.8	0.763	0.868	0.871	0.87	0.971	0.985	0.986	0.986	0.998	1	1	1
-0.5	0.761	0.866	0.863	0.865	0.975	0.982	0.982	0.982	0.997	0.999	0.999	0.999
-0.3	0.749	0.867	0.871	0.867	0.972	0.981	0.982	0.982	0.997	1	1	1
-0.1	0.734	0.875	0.877	0.875	0.958	0.98	0.98	0.982	0.996	0.998	0.998	0.998
0.1	0.699	0.852	0.849	0.846	0.944	0.976	0.974	0.973	0.993	0.997	0.997	0.997
0.3	0.633	0.818	0.816	0.813	0.894	0.959	0.958	0.955	0.964	0.984	0.984	0.984
0.5	0.477	0.767	0.768	0.761	0.622	0.875	0.86	0.858	0.585	0.842	0.836	0.834
0.8	0.016	0.485	0.488	0.48	0.003	0.387	0.388	0.382	0	0.216	0.219	0.212

Table A.9: Comparison for an AR(1) process for errors $\epsilon \sim N(0,1)$ with $n = 300$, change $\tau = 150$, and significance level at $\alpha = 0.05$. The number of Sieve Bootstrap Iterations = 1500 and number of Monte Carlo Iterations = 1000. GS: Test Statistic from Gombay and and Serban (2009), BSB_{I}: Block Sieve Bootstrap with Mean of Critical Values from each block, BSB_{II}: Block Sieve Bootstrap with Median of Critical Values from each block, NSB: Naive Sieve Bootstrap approach

μ_A	0			0.1			0.3			0.5		
ϕ_1	Mean	Var	Std. Dev.	Mean	Var	Std. Dev.	Mean	Var	Std. Dev.	Mean	Var	Std. Dev.
-0.8	1.9085	0.0303	0.1739	1.9136	0.0309	0.1756	1.9698	0.0309	0.1756	2.0847	0.039	0.1973
-0.5	1.899	0.0328	0.1811	1.9099	0.0324	0.1799	1.9658	0.0335	0.183	2.0784	0.0393	0.198
-0.3	1.9032	0.0316	0.1778	1.907	0.0317	0.178	1.9605	0.0311	0.1763	2.0561	0.0375	0.1937
-0.1	1.8959	0.0236	0.1535	1.9035	0.0224	0.1496	1.935	0.0223	0.1493	2.0123	0.0339	0.1842
0.1	1.8867	0.024	0.1549	1.8899	0.0248	0.1575	1.9321	0.0269	0.1638	2.0322	0.0385	0.196
0.3	1.8711	0.0274	0.1656	1.8755	0.0282	0.1678	1.9221	0.0288	0.1695	2.0086	0.039	0.1973
0.5	1.8464	0.0253	0.1588	1.8523	0.0256	0.16	1.8937	0.0273	0.1651	1.946	0.036	0.1896
0.8	1.7702	0.0159	0.1261	1.77	0.0166	0.1287	1.7599	0.0148	0.1216	1.7302	0.0143	0.1196

μ_A	0.7			0.9		
ϕ_1	Mean	Var	Std. Dev.	Mean	Var	Std. Dev.
-0.8	2.2239	0.0275	0.1656	2.3094	0.0061	0.0781
-0.5	2.2163	0.0281	0.1676	2.3031	0.0066	0.0808
-0.3	2.182	0.0344	0.1855	2.2671	0.0176	0.1325
-0.1	2.1355	0.04	0.1999	2.2539	0.0187	0.1366
0.1	2.1402	0.0382	0.1953	2.234	0.0215	0.1467
0.3	2.0914	0.0431	0.2074	2.1691	0.0341	0.1845
0.5	2.0047	0.0442	0.2101	2.0371	0.0467	0.2159
0.8	1.6872	0.0119	0.109	1.6446	0.0085	0.092

Table A.10: Mean, Variance, and Standard Deviation of the the critical values used in Table A.9 for BSB_{I}: Block Sieve Bootstrap with mean of critical values from each block. Number of Blocks = 10

μ_A	0			0.1			0.3			0.5		
ϕ_1	Mean	Var	Std. Dev.	Mean	Var	Std. Dev.	Mean	Var	Std. Dev.	Mean	Var	Std. Dev.
-0.8	1.9017	0.0338	0.1837	1.9067	0.0347	0.1861	1.9649	0.0357	0.1889	2.0824	0.0433	0.2079
-0.5	1.891	0.0356	0.1887	1.9017	0.0356	0.1887	1.9579	0.0377	0.1941	2.0764	0.0436	0.2088
-0.3	1.8945	0.0349	0.1867	1.8993	0.0349	0.1866	1.9529	0.0349	0.1867	2.0529	0.0414	0.2034
-0.1	1.8851	0.026	0.161	1.8951	0.025	0.158	1.9259	0.0253	0.1591	2.0061	0.0374	0.1934
0.1	1.8765	0.0259	0.1609	1.8811	0.0269	0.1639	1.9238	0.0299	0.1727	2.0292	0.0422	0.2053
0.3	1.8623	0.0294	0.1713	1.8673	0.03	0.1731	1.916	0.032	0.1788	2.0077	0.0431	0.2076
0.5	1.8402	0.0272	0.1649	1.8466	0.0277	0.1663	1.8901	0.0301	0.1735	1.9464	0.039	0.1974
0.8	1.77	0.0167	0.1292	1.7701	0.0174	0.1319	1.7596	0.0155	0.1242	1.73	0.0148	0.1216

μ_A	0.7			0.9		
ϕ_1	Mean	Var	Std. Dev.	Mean	Var	Std. Dev.
-0.8	2.2227	0.03	0.173	2.3093	0.0066	0.0808
-0.5	2.2155	0.0304	0.1742	2.3029	0.007	0.0835
-0.3	2.1809	0.0369	0.192	2.2666	0.0184	0.1354
-0.1	2.1329	0.0426	0.2064	2.2535	0.02	0.1412
0.1	2.1404	0.0407	0.2017	2.2357	0.0224	0.1496
0.3	2.0943	0.0459	0.2141	2.1733	0.0356	0.1886
0.5	2.0085	0.0468	0.2162	2.0412	0.0486	0.2203
0.8	1.6862	0.0121	0.11	1.6438	0.0087	0.093

Table A.11: Mean, Variance, and Standard Deviation of the the critical values used in Table A.9 for BSB_{II}: Block Sieve Bootstrap with median of critical values from each block. Number of Blocks = 10

μ_A	0			0.1			0.3			0.5		
ϕ_1	Mean	Var	Std. Dev.	Mean	Var	Std. Dev.	Mean	Var	Std. Dev.	Mean	Var	Std. Dev.
-0.8	1.9083	0.0328	0.1809	1.9141	0.0337	0.1836	1.9704	0.0341	0.1845	2.0867	0.0416	0.2039
-0.5	1.8974	0.0348	0.1864	1.9083	0.0346	0.1858	1.9649	0.0361	0.1899	2.0808	0.0416	0.204
-0.3	1.901	0.0337	0.1836	1.9053	0.0338	0.1838	1.9591	0.0332	0.1823	2.0571	0.0399	0.1998
-0.1	1.8925	0.0251	0.1582	1.9007	0.024	0.1548	1.9324	0.024	0.1547	2.0106	0.0359	0.1895
0.1	1.8826	0.0244	0.1561	1.886	0.0256	0.1598	1.9295	0.0283	0.1681	2.034	0.0407	0.2016
0.3	1.87	0.0283	0.1681	1.8744	0.0292	0.1707	1.9231	0.0308	0.1753	2.0126	0.0419	0.2046
0.5	1.8473	0.0264	0.1625	1.8526	0.0266	0.1629	1.896	0.0293	0.1712	1.952	0.0382	0.1953
0.8	1.7748	0.0164	0.1279	1.7751	0.017	0.1304	1.7638	0.0151	0.1226	1.7338	0.0146	0.1207

μ_A	0.7			0.9		
ϕ_1	Mean	Var	Std. Dev.	Mean	Var	Std. Dev.
-0.8	2.2254	0.0284	0.1684	2.31	0.0064	0.0798
-0.5	2.2186	0.0288	0.1696	2.3037	0.0068	0.0823
-0.3	2.1837	0.0354	0.188	2.2673	0.0181	0.1342
-0.1	2.1361	0.0412	0.2029	2.2551	0.0192	0.1384
0.1	2.1439	0.0393	0.1981	2.2372	0.0218	0.1474
0.3	2.0975	0.0445	0.211	2.1763	0.0344	0.1854
0.5	2.013	0.0458	0.214	2.0455	0.0478	0.2185
0.8	1.6899	0.0121	0.1099	1.6471	0.0086	0.0926

Table A.12: Mean, Variance, and Standard Deviation of the the critical values used in Table A.9 for NSB: Naive Sieve Bootstrap Approach

μ_A	0				0.1				0.3			
ϕ_1	GS	BSB_{I}	BSB_{II}	NSB	GS	BSB_{I}	BSB_{II}	NSB	GS	BSB_{I}	BSB_{II}	NSB
-0.8	0.017	0.042	0.042	0.043	0.044	0.094	0.096	0.092	0.21	0.329	0.335	0.329
-0.5	0.016	0.04	0.042	0.04	0.043	0.086	0.092	0.091	0.22	0.335	0.338	0.335
-0.3	0.014	0.042	0.041	0.038	0.042	0.091	0.095	0.092	0.209	0.329	0.331	0.329
-0.1	0.015	0.04	0.042	0.039	0.043	0.085	0.087	0.086	0.193	0.336	0.339	0.333
0.1	0.013	0.041	0.04	0.043	0.044	0.09	0.092	0.091	0.18	0.335	0.343	0.34
0.3	0.012	0.039	0.041	0.041	0.04	0.09	0.098	0.09	0.162	0.327	0.329	0.329
0.5	0.014	0.04	0.041	0.041	0.03	0.101	0.104	0.1	0.13	0.303	0.306	0.299
0.8	0.004	0.038	0.036	0.036	0.008	0.086	0.089	0.088	0.014	0.244	0.243	0.243
μ_A	0.5				0.7				0.9			
ϕ_1	GS	BSB_{I}	BSB_{II}	NSB	GS	BSB_{I}	BSB_{II}	NSB	GS	BSB_{I}	BSB_{II}	NSB
-0.8	0.546	0.679	0.68	0.671	0.871	0.923	0.926	0.921	0.979	0.989	0.989	0.989
-0.5	0.547	0.684	0.68	0.68	0.866	0.928	0.923	0.922	0.979	0.988	0.988	0.987
-0.3	0.536	0.68	0.681	0.678	0.848	0.918	0.916	0.918	0.975	0.985	0.985	0.984
-0.1	0.514	0.685	0.689	0.681	0.826	0.912	0.917	0.912	0.959	0.98	0.981	0.981
0.1	0.461	0.653	0.657	0.654	0.757	0.879	0.879	0.878	0.927	0.949	0.946	0.947
0.3	0.391	0.623	0.627	0.615	0.638	0.813	0.808	0.808	0.77	0.878	0.872	0.867
0.5	0.267	0.56	0.566	0.554	0.347	0.683	0.675	0.67	0.336	0.682	0.664	0.666
0.8	0.01	0.301	0.312	0.301	0	0.256	0.253	0.255	0	0.135	0.136	0.13

Table A.13: Comparison for an AR(1) process for errors $\epsilon \sim N(0,1)$ with $n = 100$, change $\tau = 60$, and significance level at $\alpha = 0.05$. The number of Sieve Bootstrap Iterations = 1500 and number of Monte Carlo Iterations = 1000. GS: Test Statistic from Gombay and Serban (2009), BSB_{I}: Block Sieve Bootstrap with Mean of Critical Values from each block, BSB_{II}: Block Sieve Bootstrap with Median of Critical Values from each block, NSB: Naive Sieve Bootstrap approach

μ_A	0			0.1			0.3			0.5		
ϕ_1	Mean	Var	Std. Dev.	Mean	Var	Std. Dev.	Mean	Var	Std. Dev.	Mean	Var	Std. Dev.
-0.8	1.9062	0.0331	0.182	1.9176	0.0294	0.1713	1.9621	0.031	0.1759	2.0498	0.0387	0.1966
-0.5	1.9038	0.0319	0.1784	1.9121	0.0305	0.1745	1.9604	0.0325	0.1802	2.0519	0.0381	0.195
-0.3	1.8974	0.0309	0.1758	1.9019	0.0307	0.1751	1.9546	0.031	0.1759	2.0359	0.0363	0.1904
-0.1	1.8924	0.0235	0.1532	1.8983	0.0228	0.1508	1.9326	0.0226	0.1503	1.9923	0.0312	0.1764
0.1	1.8829	0.0227	0.1506	1.888	0.0216	0.1468	1.919	0.0258	0.1606	1.9913	0.0351	0.1873
0.3	1.8671	0.0282	0.1678	1.8736	0.029	0.1702	1.9081	0.0307	0.175	1.9739	0.0363	0.1904
0.5	1.8411	0.0259	0.1607	1.8491	0.0263	0.1622	1.8814	0.027	0.1643	1.9306	0.0333	0.1825
0.8	1.7668	0.0159	0.126	1.7681	0.0158	0.1254	1.7634	0.0131	0.1145	1.7401	0.012	0.1095

μ_A	0.7			0.9		
ϕ_1	Mean	Var	Std. Dev.	Mean	Var	Std. Dev.
-0.8	2.1687	0.0355	0.1884	2.279	0.0144	0.12
-0.5	2.1679	0.0347	0.1861	2.2722	0.015	0.1224
-0.3	2.1337	0.0377	0.194	2.2335	0.0249	0.1577
-0.1	2.0904	0.0399	0.1996	2.2028	0.0303	0.174
0.1	2.0976	0.0395	0.1986	2.1973	0.0282	0.1678
0.3	2.0593	0.042	0.2049	2.1334	0.039	0.1975
0.5	1.9831	0.0413	0.2031	2.0221	0.0443	0.2105
0.8	1.6986	0.0103	0.1015	1.6586	0.008	0.089

Table A.14: Mean, Variance, and Standard Deviation of the the critical values used in Table A.13 for BSB_{I}: Block Sieve Bootstrap with mean of critical values from each block. Number of Blocks = 10

μ_A	0			0.1			0.3			0.5		
ϕ_1	Mean	Var	Std. Dev.	Mean	Var	Std. Dev.	Mean	Var	Std. Dev.	Mean	Var	Std. Dev.
-0.8	1.8968	0.0367	0.1914	1.908	0.0328	0.1811	1.9537	0.0349	0.1868	2.0455	0.0429	0.2072
-0.5	1.8959	0.0353	0.1877	1.9044	0.0342	0.1849	1.9545	0.0367	0.1914	2.0473	0.0428	0.2068
-0.3	1.8904	0.0342	0.1848	1.8944	0.0343	0.1851	1.9488	0.0354	0.188	2.0315	0.0404	0.201
-0.1	1.8838	0.026	0.1611	1.8897	0.0252	0.1588	1.9237	0.0257	0.1603	1.9847	0.0346	0.1859
0.1	1.8731	0.0247	0.157	1.8794	0.0236	0.1536	1.9112	0.0285	0.1688	1.9865	0.0384	0.1958
0.3	1.8607	0.0303	0.1741	1.868	0.031	0.1761	1.904	0.0336	0.1833	1.9723	0.0399	0.1998
0.5	1.8359	0.0277	0.1662	1.845	0.0284	0.1685	1.8782	0.0296	0.1718	1.9293	0.0365	0.1909
0.8	1.7642	0.0166	0.1286	1.7663	0.0165	0.1284	1.7639	0.0138	0.1173	1.7396	0.0126	0.112

μ_A	0.7			0.9		
ϕ_1	Mean	Var	Std. Dev.	Mean	Var	Std. Dev.
-0.8	2.1664	0.0387	0.1966	2.2785	0.0157	0.1253
-0.5	2.1659	0.0379	0.1947	2.2718	0.0161	0.1267
-0.3	2.1303	0.0412	0.2029	2.2317	0.0269	0.1639
-0.1	2.0852	0.0434	0.2083	2.2013	0.0325	0.1801
0.1	2.0951	0.0433	0.208	2.1993	0.0296	0.1721
0.3	2.0589	0.0461	0.2147	2.1367	0.0414	0.2034
0.5	1.9858	0.0446	0.2111	2.0261	0.0469	0.2166
0.8	1.6981	0.0105	0.1025	1.6585	0.0083	0.0908

Table A.15: Mean, Variance, and Standard Deviation of the the critical values used in Table A.13 for BSB_{II}: Block Sieve Bootstrap with median of critical values from each block. Number of Blocks = 10

μ_A	0			0.1			0.3			0.5		
ϕ_1	Mean	Var	Std. Dev.	Mean	Var	Std. Dev.	Mean	Var	Std. Dev.	Mean	Var	Std. Dev.
-0.8	1.9043	0.0351	0.1872	1.9152	0.0313	0.1768	1.9621	0.0334	0.1827	2.0516	0.0408	0.202
-0.5	1.9031	0.034	0.1843	1.9115	0.0328	0.181	1.9597	0.0349	0.1866	2.0527	0.0407	0.2017
-0.3	1.8969	0.033	0.1817	1.901	0.0327	0.1808	1.9547	0.0334	0.1825	2.0366	0.0386	0.1963
-0.1	1.8909	0.0246	0.1567	1.8967	0.024	0.1547	1.9317	0.024	0.1548	1.9913	0.0329	0.1813
0.1	1.8815	0.0235	0.1532	1.8863	0.0225	0.1497	1.9192	0.0274	0.1653	1.9927	0.0369	0.1921
0.3	1.868	0.0295	0.1718	1.874	0.0303	0.174	1.9105	0.0327	0.1807	1.9772	0.0385	0.1962
0.5	1.8422	0.0268	0.1638	1.8514	0.0278	0.1666	1.8844	0.0289	0.1698	1.9355	0.0354	0.1881
0.8	1.7704	0.0161	0.1267	1.7721	0.0161	0.1266	1.7677	0.0134	0.1157	1.7437	0.0123	0.1109

μ_A	0.7			0.9		
ϕ_1	Mean	Var	Std. Dev.	Mean	Var	Std. Dev.
-0.8	2.1699	0.037	0.1922	2.2804	0.0148	0.1214
-0.5	2.1693	0.0363	0.1905	2.2735	0.0154	0.1239
-0.3	2.1345	0.0395	0.1987	2.2337	0.0258	0.1606
-0.1	2.0895	0.0416	0.2038	2.2037	0.0314	0.1771
0.1	2.1001	0.0413	0.2033	2.2013	0.0288	0.1695
0.3	2.0642	0.0442	0.2101	2.1402	0.0401	0.2001
0.5	1.9912	0.0437	0.2091	2.0314	0.0464	0.2153
0.8	1.7017	0.0103	0.1015	1.6622	0.0081	0.0896

Table A.16: Mean, Variance, and Standard Deviation of the the critical values used in Table A.13 for NSB: Naive Sieve Bootstrap Approach

μ_A	0				0.1				0.3			
ϕ_1	GS	BSB_{I}	BSB_{II}	NSB	GS	BSB_{I}	BSB_{II}	NSB	GS	BSB_{I}	BSB_{II}	NSB
-0.8	0.026	0.053	0.06	0.053	0.048	0.093	0.098	0.096	0.138	0.244	0.25	0.248
-0.5	0.023	0.055	0.058	0.055	0.048	0.093	0.094	0.091	0.154	0.247	0.25	0.244
-0.3	0.026	0.055	0.054	0.055	0.048	0.099	0.099	0.097	0.149	0.249	0.249	0.247
-0.1	0.023	0.052	0.055	0.052	0.045	0.104	0.104	0.104	0.144	0.245	0.253	0.244
0.1	0.022	0.056	0.059	0.058	0.047	0.106	0.108	0.107	0.134	0.236	0.239	0.235
0.3	0.023	0.059	0.061	0.061	0.046	0.104	0.101	0.101	0.117	0.226	0.236	0.229
0.5	0.021	0.061	0.061	0.063	0.04	0.106	0.105	0.104	0.093	0.229	0.236	0.226
0.8	0.008	0.052	0.052	0.052	0.013	0.101	0.101	0.1	0.018	0.191	0.195	0.186

μ_A	0.5				0.7				0.9			
ϕ_1	GS	BSB_{I}	BSB_{II}	NSB	GS	BSB_{I}	BSB_{II}	NSB	GS	BSB_{I}	BSB_{II}	NSB
-0.8	0.333	0.46	0.458	0.453	0.612	0.704	0.708	0.706	0.835	0.875	0.875	0.872
-0.5	0.329	0.467	0.471	0.461	0.584	0.693	0.692	0.696	0.826	0.867	0.867	0.864
-0.3	0.314	0.464	0.467	0.46	0.561	0.697	0.698	0.696	0.8	0.854	0.853	0.851
-0.1	0.292	0.457	0.464	0.458	0.524	0.684	0.684	0.686	0.731	0.81	0.812	0.809
0.1	0.272	0.433	0.438	0.436	0.455	0.609	0.616	0.603	0.622	0.716	0.717	0.711
0.3	0.233	0.414	0.41	0.41	0.358	0.548	0.552	0.547	0.452	0.626	0.624	0.621
0.5	0.163	0.36	0.36	0.357	0.203	0.466	0.464	0.459	0.195	0.486	0.481	0.478
0.8	0.01	0.233	0.233	0.222	0.002	0.184	0.19	0.182	0.001	0.105	0.108	0.103

Table A.17: Comparison for an AR(1) process for errors $\epsilon \sim N(0,1)$ with $n = 100$, change $\tau = 70$, and significance level at $\alpha = 0.05$. The number of Sieve Bootstrap Iterations = 1500 and number of Monte Carlo Iterations = 1000. GS: Test Statistic from Gombay and Serban (2009), BSB_{I}: Block Sieve Bootstrap with Mean of Critical Values from each block, BSB_{II}: Block Sieve Bootstrap with Median of Critical Values from each block, NSB: Naive Sieve Bootstrap approach

μ_A	0			0.1			0.3			0.5		
ϕ_1	Mean	Var	Std. Dev.	Mean	Var	Std. Dev.	Mean	Var	Std. Dev.	Mean	Var	Std. Dev.
-0.8	1.9032	0.0306	0.1749	1.9051	0.031	0.1759	1.9479	0.0307	0.1751	2.0143	0.0352	0.1874
-0.5	1.8985	0.0308	0.1755	1.9029	0.0319	0.1784	1.9443	0.0318	0.1783	2.0056	0.0358	0.1893
-0.3	1.8904	0.0312	0.1765	1.8943	0.0323	0.1797	1.9347	0.0322	0.1794	1.9909	0.0336	0.1832
-0.1	1.8902	0.0254	0.1594	1.8925	0.0261	0.1614	1.9185	0.0261	0.1614	1.9673	0.0292	0.1708
0.1	1.8776	0.0225	0.15	1.8845	0.0232	0.1522	1.9157	0.0259	0.1607	1.9646	0.0309	0.1756
0.3	1.868	0.0268	0.1635	1.8709	0.0294	0.1713	1.9002	0.031	0.1759	1.9545	0.0358	0.1892
0.5	1.8433	0.0256	0.16	1.8469	0.0262	0.1617	1.8703	0.0282	0.1679	1.911	0.0312	0.1766
0.8	1.7734	0.0159	0.126	1.7683	0.0166	0.1286	1.7646	0.0163	0.1274	1.7458	0.0147	0.1212

μ_A	0.7			0.9		
ϕ_1	Mean	Var	Std. Dev.	Mean	Var	Std. Dev.
-0.8	2.1123	0.04	0.2	2.2129	0.0306	0.1748
-0.5	2.1054	0.0396	0.199	2.2076	0.0298	0.1725
-0.3	2.0819	0.0399	0.1996	2.1708	0.0357	0.1889
-0.1	2.0418	0.0373	0.1931	2.1377	0.0386	0.1965
0.1	2.0524	0.0401	0.2002	2.1416	0.0369	0.1919
0.3	2.0206	0.0407	0.2017	2.0904	0.0417	0.204
0.5	1.9582	0.0384	0.196	1.9947	0.0426	0.2063
0.8	1.7164	0.0129	0.1136	1.6789	0.0097	0.0982

Table A.18: Mean, Variance, and Standard Deviation of the the critical values used in Table A.13 for BSB_{I}: Block Sieve Bootstrap with mean of critical values from each block. Number of Blocks = 10

μ_A	0			0.1			0.3			0.5			0.7			0.9		
ϕ_1	Mean	Var	Std. Dev.	Mean	Var	Std. Dev.	Mean	Var	Std. Dev.	Mean	Var	Std. Dev.	Mean	Var	Std. Dev.	Mean	Var	Std. Dev.
-0.8	1.8968	0.034	0.1842	1.8979	0.0342	0.1849	1.9417	0.0349	0.1867	2.0114	0.0395	0.1988	2.1104	0.0435	0.2086	2.213	0.0323	0.1798
-0.5	1.8917	0.0334	0.1826	1.8955	0.0349	0.1868	1.9369	0.0359	0.1894	2.0015	0.04	0.2	2.1036	0.0429	0.2071	2.2071	0.0319	0.1785
-0.3	1.8827	0.0335	0.1829	1.8878	0.0352	0.1875	1.9262	0.0351	0.1872	1.9854	0.0373	0.1931	2.0792	0.0436	0.2087	2.1692	0.0383	0.1955
-0.1	1.8825	0.0276	0.166	1.8831	0.0282	0.1677	1.9098	0.0284	0.1686	1.96	0.0326	0.1806	2.0378	0.0412	0.2028	2.1356	0.0416	0.2038
0.1	1.8685	0.0244	0.1561	1.8748	0.0254	0.1591	1.9076	0.0286	0.1691	1.9581	0.0342	0.1849	2.0515	0.0438	0.2093	2.142	0.0396	0.199
0.3	1.8623	0.0295	0.1717	1.8654	0.0322	0.1794	1.8962	0.0345	0.1858	1.9518	0.0398	0.1995	2.0205	0.0445	0.2108	2.0927	0.0447	0.2113
0.5	1.8386	0.028	0.1671	1.8426	0.0286	0.169	1.8679	0.0309	0.1756	1.9097	0.0348	0.1865	1.9597	0.0419	0.2047	1.9985	0.0455	0.2132
0.8	1.7723	0.0168	0.1293	1.7668	0.0175	0.1323	1.7641	0.017	0.1304	1.7449	0.0153	0.1236	1.7156	0.0131	0.1143	1.6786	0.0098	0.0986

Table A.19: Mean, Variance, and Standard Deviation of the the critical values used in Table A.13 for BSB_{II}: Block Sieve Bootstrap with median of critical values from each block. Number of Blocks = 10

μ_A	0			0.1			0.3			0.5			0.7			0.9		
ϕ_1	Mean	Var	Std. Dev.	Mean	Var	Std. Dev.	Mean	Var	Std. Dev.	Mean	Var	Std. Dev.	Mean	Var	Std. Dev.	Mean	Var	Std. Dev.
-0.8	1.902	0.0323	0.1798	1.9047	0.033	0.1815	1.9482	0.0335	0.183	2.0166	0.0381	0.1952	2.1144	0.042	0.2049	2.2149	0.0316	0.1776
-0.5	1.8977	0.0324	0.1798	1.9019	0.0337	0.1834	1.9446	0.0344	0.1855	2.0076	0.0385	0.1962	2.1072	0.0415	0.2037	2.2089	0.0309	0.1758
-0.3	1.8895	0.0326	0.1805	1.8934	0.0342	0.1849	1.9338	0.0341	0.1845	1.9916	0.036	0.1896	2.0849	0.0416	0.2038	2.1716	0.0369	0.192
-0.1	1.8884	0.0265	0.1626	1.8909	0.0272	0.1649	1.9173	0.0272	0.1649	1.9663	0.0309	0.1756	2.0424	0.0392	0.1979	2.1391	0.04	0.2
0.1	1.8756	0.023	0.1517	1.8822	0.024	0.1547	1.9149	0.0272	0.1647	1.9651	0.0328	0.1809	2.0554	0.0425	0.206	2.1457	0.0381	0.1952
0.3	1.869	0.0284	0.1686	1.8722	0.0311	0.1763	1.9027	0.0333	0.1825	1.9577	0.0385	0.1963	2.0252	0.043	0.2073	2.0972	0.0433	0.2081
0.5	1.8455	0.0271	0.1645	1.8497	0.0276	0.1662	1.8733	0.03	0.173	1.9158	0.0338	0.1837	1.9645	0.0407	0.2016	2.0025	0.0445	0.2109
0.8	1.7772	0.0164	0.1278	1.7717	0.017	0.1302	1.7686	0.0167	0.1291	1.7497	0.0152	0.1231	1.7201	0.0133	0.1153	1.6823	0.0098	0.0988

Table A.20: Mean, Variance, and Standard Deviation of the the critical values used in Table A.13 for NSB: Naive Sieve Bootstrap Approach

μ_A	0				0.1				0.3			
ϕ_1	GS	BSB_{I}	BSB_{II}	NSB	GS	BSB_{I}	BSB_{II}	NSB	GS	BSB_{I}	BSB_{II}	NSB
-0.8	0.018	0.048	0.05	0.048	0.043	0.112	0.116	0.112	0.203	0.351	0.352	0.346
-0.5	0.018	0.046	0.047	0.046	0.046	0.108	0.106	0.106	0.217	0.351	0.354	0.349
-0.3	0.019	0.046	0.047	0.046	0.043	0.111	0.108	0.107	0.212	0.358	0.366	0.361
-0.1	0.017	0.045	0.048	0.044	0.043	0.113	0.114	0.108	0.198	0.357	0.368	0.363
0.1	0.016	0.049	0.049	0.047	0.038	0.111	0.115	0.111	0.184	0.356	0.364	0.36
0.3	0.017	0.044	0.045	0.044	0.035	0.103	0.104	0.1	0.156	0.353	0.357	0.352
0.5	0.015	0.047	0.044	0.044	0.027	0.096	0.098	0.092	0.132	0.352	0.35	0.346
0.8	0.005	0.041	0.041	0.039	0.011	0.083	0.082	0.081	0.019	0.289	0.287	0.286
μ_A	0.5				0.7				0.9			
ϕ_1	GS	BSB_{I}	BSB_{II}	NSB	GS	BSB_{I}	BSB_{II}	NSB	GS	BSB_{I}	BSB_{II}	NSB
-0.8	0.542	0.667	0.668	0.666	0.839	0.9	0.893	0.893	0.962	0.975	0.972	0.971
-0.5	0.546	0.69	0.697	0.689	0.844	0.904	0.904	0.902	0.959	0.975	0.973	0.972
-0.3	0.535	0.695	0.7	0.688	0.831	0.898	0.899	0.898	0.955	0.975	0.976	0.973
-0.1	0.521	0.693	0.703	0.692	0.811	0.898	0.896	0.896	0.949	0.969	0.97	0.971
0.1	0.493	0.671	0.672	0.668	0.784	0.881	0.882	0.876	0.936	0.962	0.96	0.956
0.3	0.45	0.635	0.632	0.635	0.702	0.855	0.855	0.85	0.883	0.945	0.942	0.94
0.5	0.322	0.608	0.609	0.601	0.508	0.793	0.789	0.79	0.592	0.84	0.837	0.829
0.8	0.02	0.456	0.45	0.443	0.009	0.481	0.485	0.471	0	0.414	0.411	0.396

Table A.21: Comparison for an AR(1) process for errors $\epsilon \sim t_5$ with $n = 100$, change $\tau = 50$, and significance level at $\alpha = 0.05$. The number of Sieve Bootstrap Iterations = 1500 and number of Monte Carlo Iterations = 1000. GS: Test Statistic from Gombay and and Serban (2009), BSB_{I}: Block Sieve Bootstrap with Mean of Critical Values from each block, BSB_{II}: Block Sieve Bootstrap with Median of Critical Values from each block, NSB: Naive Sieve Bootstrap approach

μ_A	0			0.1			0.3			0.5		
ϕ_1	Mean	Var	Std. Dev.	Mean	Var	Std. Dev.	Mean	Var	Std. Dev.	Mean	Var	Std. Dev.
-0.8	1.9018	0.0397	0.1991	1.9074	0.0378	0.1944	1.9513	0.0396	0.1989	2.0263	0.0413	0.2032
-0.5	1.8933	0.0393	0.1981	1.9012	0.0383	0.1956	1.9424	0.0371	0.1926	2.0141	0.0421	0.205
-0.3	1.8916	0.0395	0.1986	1.8977	0.0381	0.1952	1.9325	0.0369	0.192	2.0009	0.0409	0.2022
-0.1	1.8866	0.0338	0.1837	1.8902	0.0323	0.1795	1.9209	0.0323	0.1797	1.9692	0.0364	0.1907
0.1	1.8763	0.0344	0.1854	1.881	0.0344	0.1853	1.9126	0.034	0.1844	1.9724	0.039	0.1974
0.3	1.8567	0.037	0.1922	1.8617	0.0371	0.1926	1.8994	0.0383	0.1957	1.955	0.0399	0.1997
0.5	1.8341	0.0353	0.1878	1.8401	0.0362	0.1901	1.8687	0.0349	0.1868	1.916	0.0365	0.1911
0.8	1.763	0.0191	0.1382	1.761	0.019	0.1376	1.76	0.0178	0.1334	1.7451	0.0164	0.128

μ_A	0.7			0.9		
ϕ_1	Mean	Var	Std. Dev.	Mean	Var	Std. Dev.
-0.8	2.1232	0.0409	0.2021	2.2248	0.0281	0.1677
-0.5	2.1121	0.0433	0.2081	2.2111	0.0315	0.1773
-0.3	2.0834	0.0444	0.2106	2.1743	0.0393	0.1983
-0.1	2.0462	0.0443	0.2103	2.1433	0.0425	0.206
0.1	2.056	0.0452	0.2126	2.1424	0.0411	0.2026
0.3	2.026	0.0462	0.2148	2.0916	0.0461	0.2147
0.5	1.9698	0.0439	0.2094	2.003	0.0487	0.2206
0.8	1.7178	0.0151	0.1226	1.6812	0.0125	0.1118

Table A.22: Mean, Variance, and Standard Deviation of the the critical values used in Table A.21 for BSB_{I}: Block Sieve Bootstrap with mean of critical values from each block. Number of Blocks = 10

μ_A	0			0.1			0.3			0.5		
ϕ_1	Mean	Var	Std. Dev.	Mean	Var	Std. Dev.	Mean	Var	Std. Dev.	Mean	Var	Std. Dev.
-0.8	1.899	0.0446	0.211	1.9041	0.0428	0.2068	1.9508	0.0456	0.2135	2.0272	0.0471	0.2169
-0.5	1.8886	0.0435	0.2084	1.8961	0.0425	0.2062	1.9404	0.0419	0.2047	2.0131	0.047	0.2166
-0.3	1.8868	0.0432	0.2078	1.8932	0.0421	0.2052	1.9277	0.0411	0.2028	1.9988	0.046	0.2144
-0.1	1.8813	0.0373	0.1929	1.8848	0.0357	0.189	1.9148	0.0365	0.1911	1.9635	0.0404	0.201
0.1	1.8705	0.0376	0.1938	1.8752	0.0375	0.1935	1.9066	0.0372	0.1928	1.9693	0.0428	0.2067
0.3	1.8532	0.0399	0.1996	1.8581	0.0399	0.1997	1.8944	0.0414	0.2034	1.9526	0.0436	0.2088
0.5	1.8303	0.0372	0.1928	1.8365	0.0379	0.1947	1.8664	0.0373	0.1931	1.9152	0.0396	0.1989
0.8	1.7625	0.0201	0.1417	1.7607	0.0198	0.1405	1.7608	0.0186	0.1361	1.7452	0.017	0.1303

μ_A	0.7			0.9		
ϕ_1	Mean	Var	Std. Dev.	Mean	Var	Std. Dev.
-0.8	2.1239	0.0447	0.2114	2.2264	0.0295	0.1717
-0.5	2.1126	0.047	0.2167	2.2117	0.0333	0.1824
-0.3	2.0821	0.0481	0.2193	2.1725	0.0421	0.205
-0.1	2.0436	0.0478	0.2186	2.1425	0.0451	0.2123
0.1	2.054	0.049	0.2212	2.143	0.0438	0.2091
0.3	2.0258	0.0496	0.2226	2.0933	0.0488	0.2209
0.5	1.9715	0.0475	0.2178	2.0059	0.0518	0.2275
0.8	1.7171	0.0152	0.1232	1.6808	0.0129	0.1134

Table A.23: Mean, Variance, and Standard Deviation of the the critical values used in Table A.21 for BSB_{II}: Block Sieve Bootstrap with median of critical values from each block. Number of Blocks = 10

μ_A	0			0.1			0.3			0.5		
ϕ_1	Mean	Var	Std. Dev.	Mean	Var	Std. Dev.	Mean	Var	Std. Dev.	Mean	Var	Std. Dev.
-0.8	1.908	0.0446	0.2112	1.914	0.043	0.2073	1.9589	0.0445	0.2108	2.0342	0.0449	0.2118
-0.5	1.8968	0.0428	0.2067	1.9053	0.0418	0.2044	1.9476	0.0413	0.2032	2.019	0.0451	0.2124
-0.3	1.8938	0.0425	0.2061	1.9009	0.0413	0.203	1.9353	0.0402	0.2003	2.0037	0.0441	0.2098
-0.1	1.8875	0.0365	0.1909	1.8914	0.0347	0.1861	1.9231	0.0349	0.1868	1.9708	0.0385	0.1962
0.1	1.8778	0.0367	0.1915	1.8834	0.0368	0.1917	1.9137	0.0361	0.1898	1.9764	0.0415	0.2038
0.3	1.8585	0.039	0.1975	1.8644	0.039	0.1973	1.9025	0.0404	0.2009	1.9588	0.0423	0.2056
0.5	1.8365	0.0365	0.191	1.8431	0.0374	0.1934	1.8734	0.0366	0.1913	1.9205	0.0388	0.1968
0.8	1.7671	0.0196	0.1399	1.7652	0.0193	0.1387	1.7641	0.0181	0.1345	1.7493	0.0169	0.1298

μ_A	0.7			0.9		
ϕ_1	Mean	Var	Std. Dev.	Mean	Var	Std. Dev.
-0.8	2.1286	0.0432	0.2077	2.2285	0.0285	0.1688
-0.5	2.117	0.0452	0.2124	2.2143	0.0321	0.1791
-0.3	2.0868	0.0462	0.215	2.1758	0.0405	0.2011
-0.1	2.0484	0.0461	0.2145	2.1456	0.0438	0.2092
0.1	2.0598	0.0471	0.2169	2.1472	0.0421	0.2051
0.3	2.0313	0.0482	0.2195	2.0975	0.0474	0.2176
0.5	1.9768	0.0465	0.2155	2.0109	0.0508	0.2253
0.8	1.721	0.0151	0.1227	1.6842	0.0127	0.1126

Table A.24: Mean, Variance, and Standard Deviation of the the critical values used in Table A.21 for NSB: Naive Sieve Bootstrap Approach

μ_A	0				0.1				0.3			
ϕ_1	GS	BSB_{I}	BSB_{II}	NSB	GS	BSB_{I}	BSB_{II}	NSB	GS	BSB_{I}	BSB_{II}	NSB
-0.8	0.014	0.049	0.051	0.051	0.053	0.13	0.131	0.133	0.257	0.391	0.393	0.391
-0.5	0.016	0.048	0.051	0.048	0.059	0.129	0.131	0.13	0.264	0.404	0.412	0.408
-0.3	0.015	0.05	0.051	0.049	0.054	0.125	0.126	0.126	0.266	0.414	0.419	0.412
-0.1	0.016	0.047	0.049	0.047	0.054	0.131	0.13	0.13	0.26	0.417	0.422	0.419
0.1	0.016	0.052	0.055	0.051	0.051	0.126	0.129	0.128	0.239	0.418	0.42	0.42
0.3	0.017	0.051	0.052	0.052	0.051	0.123	0.122	0.123	0.214	0.407	0.405	0.402
0.5	0.017	0.055	0.055	0.056	0.051	0.128	0.127	0.125	0.176	0.381	0.381	0.378
0.8	0.008	0.054	0.057	0.054	0.021	0.141	0.141	0.138	0.032	0.318	0.317	0.314

μ_A	0.5				0.7				0.9			
ϕ_1	GS	BSB_{I}	BSB_{II}	NSB	GS	BSB_{I}	BSB_{II}	NSB	GS	BSB_{I}	BSB_{II}	NSB
-0.8	0.617	0.716	0.722	0.714	0.906	0.936	0.938	0.938	0.991	0.995	0.995	0.995
-0.5	0.629	0.741	0.737	0.735	0.897	0.93	0.93	0.929	0.988	0.996	0.996	0.996
-0.3	0.617	0.741	0.738	0.735	0.885	0.931	0.93	0.93	0.985	0.995	0.996	0.995
-0.1	0.592	0.747	0.753	0.748	0.868	0.935	0.939	0.933	0.975	0.995	0.995	0.995
0.1	0.557	0.728	0.73	0.727	0.838	0.909	0.909	0.909	0.961	0.987	0.987	0.986
0.3	0.502	0.707	0.705	0.707	0.78	0.883	0.878	0.879	0.912	0.959	0.957	0.955
0.5	0.382	0.673	0.672	0.659	0.554	0.815	0.807	0.798	0.602	0.854	0.844	0.841
0.8	0.02	0.457	0.453	0.456	0.004	0.427	0.432	0.418	0	0.297	0.29	0.289

Table A.25: Comparison for an AR(1) process for errors $\epsilon \sim t_8$ with $n = 100$, change $\tau = 50$, and significance level at $\alpha = 0.05$. The number of Sieve Bootstrap Iterations = 1500 and number of Monte Carlo Iterations = 1000. GS: Test Statistic from Gombay and and Serban (2009), BSB_{I}: Block Sieve Bootstrap with Mean of Critical Values from each block, BSB_{II}: Block Sieve Bootstrap with Median of Critical Values from each block, NSB: Naive Sieve Bootstrap approach

μ_A		0				0.1				0.3			
ϕ_1	ϕ_2	GS	BSB_{I}	BSB_{II}	NSB	GS	BSB_{I}	BSB_{II}	NSB	GS	BSB_{I}	BSB_{II}	NSB
0.5	0.1	0.012	0.042	0.041	0.041	0.029	0.123	0.123	0.124	0.107	0.393	0.398	0.388
0.5	-0.1	0.017	0.046	0.047	0.044	0.043	0.134	0.137	0.133	0.202	0.446	0.452	0.444
0.1	0.5	0.009	0.039	0.04	0.039	0.022	0.104	0.105	0.1	0.049	0.366	0.363	0.362
-0.5	-0.1	0.021	0.044	0.045	0.045	0.053	0.133	0.131	0.133	0.286	0.467	0.468	0.466
-0.5	-0.9	0.014	0.036	0.038	0.037	0.039	0.091	0.089	0.088	0.234	0.340	0.344	0.334
-0.9	-0.5	0.019	0.043	0.044	0.045	0.053	0.108	0.106	0.107	0.283	0.437	0.443	0.436
-0.9	-0.9	0.012	0.027	0.026	0.027	0.041	0.081	0.081	0.078	0.233	0.308	0.304	0.301
μ_A		0.5				0.7				0.9			
ϕ_1	ϕ_2	GS	BSB_{I}	BSB_{II}	NSB	GS	BSB_{I}	BSB_{II}	NSB	GS	BSB_{I}	BSB_{II}	NSB
0.5	0.1	0.143	0.638	0.642	0.629	0.071	0.672	0.673	0.673	0.018	0.599	0.592	0.584
0.5	-0.1	0.449	0.767	0.766	0.765	0.594	0.891	0.886	0.886	0.541	0.857	0.847	0.848
0.1	0.5	0.034	0.522	0.518	0.519	0.011	0.484	0.479	0.471	0	0.331	0.326	0.319
-0.5	-0.1	0.679	0.849	0.849	0.847	0.935	0.979	0.978	0.977	0.992	0.998	0.997	0.998
-0.5	-0.9	0.618	0.691	0.684	0.683	0.904	0.911	0.908	0.909	0.989	0.992	0.991	0.991
-0.9	-0.5	0.702	0.811	0.812	0.808	0.935	0.961	0.96	0.959	0.997	0.998	0.998	0.998
-0.9	-0.9	0.598	0.625	0.618	0.618	0.884	0.882	0.881	0.88	0.978	0.974	0.973	0.974

Table A.26: Comparison for an AR(2) process for errors $\epsilon \sim N(0,1)$ with $n = 100$, change $\tau = 50$, and significance level at $\alpha = 0.05$. The number of Sieve Bootstrap Iterations = 1500 and number of Monte Carlo Iterations = 1000. GS: Test Statistic from Gombay and Serban (2009), BSB_{I}: Block Sieve Bootstrap with Mean of Critical Values from each block, BSB_{II}: Block Sieve Bootstrap with Median of Critical Values from each block, NSB: Naive Sieve Bootstrap approach

μ_A		0			0.1			0.3			0.5		
ϕ_1	ϕ_2	Mean	Var	Std. Dev.	Mean	Var	Std. Dev.	Mean	Var	Std. Dev.	Mean	Var	Std. Dev.
0.5	0.1	1.8146	0.0136	0.1164	1.8143	0.0142	0.1191	1.8278	0.0165	0.1283	1.8337	0.0231	0.1518
0.5	-0.1	1.8426	0.0166	0.1288	1.8448	0.017	0.1301	1.8653	0.02	0.1415	1.9022	0.03	0.1731
0.1	0.5	1.7999	0.0193	0.1388	1.8008	0.0207	0.1437	1.81	0.0212	0.1454	1.7916	0.0232	0.1522
-0.5	-0.1	1.8781	0.0222	0.1487	1.8846	0.0218	0.1476	1.9195	0.0239	0.1545	1.9926	0.0352	0.1876
-0.5	-0.9	1.8852	0.0743	0.2725	1.9015	0.0741	0.2721	1.9996	0.0642	0.2533	2.1477	0.0373	0.193
-0.9	-0.5	1.9154	0.0296	0.1718	1.9255	0.0312	0.1766	1.9761	0.0326	0.1804	2.0727	0.0395	0.1987
-0.9	-0.9	1.959	0.0682	0.2612	1.9721	0.0683	0.2613	2.0705	0.0528	0.2297	2.1918	0.0262	0.1619

μ_A		0.7			0.9		
ϕ_1	ϕ_2	Mean	Var	Std. Dev.	Mean	Var	Std. Dev.
0.5	0.1	1.8215	0.0278	0.1667	1.7904	0.0274	0.1654
0.5	-0.1	1.9449	0.0428	0.2068	1.9863	0.0498	0.2231
0.1	0.5	1.746	0.0201	0.1418	1.6978	0.0147	0.1209
-0.5	-0.1	2.117	0.0426	0.2064	2.2284	0.0261	0.1614
-0.5	-0.9	2.2477	0.0132	0.1147	2.2949	0.0033	0.0567
-0.9	-0.5	2.1956	0.0291	0.1706	2.2808	0.0108	0.1039
-0.9	-0.9	2.2699	0.0084	0.0914	2.3065	0.0019	0.0431

Table A.27: Mean, Variance, and Standard Deviation of the the critical values used in Table A.26 for BSB_{I}: Block Sieve Bootstrap with mean of critical values from each block. Number of Blocks = 10

μ_A		0			0.1			0.3			0.5		
ϕ_1	ϕ_2	Mean	Var	Std. Dev.	Mean	Var	Std. Dev.	Mean	Var	Std. Dev.	Mean	Var	Std. Dev.
0.5	0.1	1.8114	0.0147	0.1212	1.8113	0.0152	0.1233	1.8267	0.018	0.1342	1.8336	0.0248	0.1575
0.5	-0.1	1.8392	0.0183	0.1353	1.8406	0.0189	0.1375	1.8628	0.0221	0.1486	1.9005	0.0324	0.1799
0.1	0.5	1.798	0.0204	0.1428	1.7987	0.0217	0.1473	1.8094	0.0222	0.149	1.7911	0.024	0.155
-0.5	-0.1	1.8696	0.0242	0.1553	1.8764	0.0243	0.1558	1.9126	0.0267	0.1632	1.9881	0.0387	0.1966
-0.5	-0.9	1.8907	0.083	0.2881	1.9064	0.0824	0.2869	2.0060	0.0716	0.2676	2.1572	0.0406	0.2014
-0.9	-0.5	1.9088	0.0337	0.1834	1.9211	0.036	0.1896	1.9738	0.0378	0.1943	2.0743	0.0443	0.2103
-0.9	-0.9	1.9653	0.0762	0.2759	1.9787	0.0762	0.2759	2.078	0.0587	0.2423	2.2014	0.0278	0.1666

μ_A		0.7			0.9		
ϕ_1	ϕ_2	Mean	Var	Std. Dev.	Mean	Var	Std. Dev.
0.5	0.1	1.822	0.0291	0.1705	1.7905	0.0284	0.1686
0.5	-0.1	1.9457	0.0454	0.2129	1.9891	0.0522	0.2285
0.1	0.5	1.7464	0.0208	0.144	1.6972	0.015	0.1225
-0.5	-0.1	2.1149	0.0457	0.2137	2.2281	0.0272	0.1649
-0.5	-0.9	2.2539	0.0136	0.1165	2.2969	0.0031	0.0552
-0.9	-0.5	2.1987	0.0314	0.1772	2.2821	0.0114	0.1064
-0.9	-0.9	2.2751	0.0084	0.0913	2.3079	0.0016	0.04

Table A.28: Mean, Variance, and Standard Deviation of the the critical values used in Table A.26 for BSB_{II}: Block Sieve Bootstrap with median of critical values from each block. Number of Blocks = 10

μ_A		0			0.1			0.3			0.5		
ϕ_1	ϕ_2	Mean	Var	Std. Dev.	Mean	Var	Std. Dev.	Mean	Var	Std. Dev.	Mean	Var	Std. Dev.
0.5	0.1	1.8167	0.0141	0.1185	1.8164	0.0145	0.1204	1.832	0.0174	0.1318	1.8377	0.0243	0.1558
0.5	-0.1	1.8448	0.0175	0.1322	1.8463	0.018	0.1342	1.8689	0.0211	0.1452	1.9063	0.0314	0.1772
0.1	0.5	1.8031	0.02	0.1413	1.804	0.0211	0.1451	1.8147	0.0216	0.1468	1.7955	0.0236	0.1536
-0.5	-0.1	1.8764	0.0231	0.152	1.8841	0.0231	0.1518	1.9178	0.0254	0.1594	1.9929	0.0373	0.1932
-0.5	-0.9	1.8988	0.0832	0.2884	1.9144	0.0818	0.2861	2.0147	0.0706	0.2656	2.1631	0.039	0.1973
-0.9	-0.5	1.9153	0.0329	0.1814	1.9269	0.0346	0.1859	1.9788	0.0365	0.191	2.0778	0.043	0.2073
-0.9	-0.9	1.9733	0.0753	0.2743	1.9882	0.0753	0.2744	2.0873	0.057	0.2387	2.2056	0.0269	0.1639

μ_A		0.7			0.9		
ϕ_1	ϕ_2	Mean	Var	Std. Dev.	Mean	Var	Std. Dev.
0.5	0.1	1.826	0.0288	0.1695	1.7947	0.0281	0.1675
0.5	-0.1	1.9506	0.0444	0.2106	1.9929	0.0511	0.2261
0.1	0.5	1.7496	0.0205	0.143	1.7	0.0148	0.1215
-0.5	-0.1	2.1178	0.0442	0.2102	2.2298	0.0265	0.1628
-0.5	-0.9	2.2569	0.0128	0.113	2.2985	0.0027	0.0519
-0.9	-0.5	2.2017	0.0301	0.1733	2.2834	0.0108	0.1039
-0.9	-0.9	2.2775	0.0075	0.0865	2.3089	0.0016	0.0393

Table A.29: Mean, Variance, and Standard Deviation of the the critical values used in Table A.26 for NSB: Naive Sieve Bootstrap Approach

μ_A		0				0.1				0.3			
ϕ_1	ϕ_2	GS	BSB_{I}	BSB_{II}	NSB	GS	BSB_{I}	BSB_{II}	NSB	GS	BSB_{I}	BSB_{II}	NSB
0.5	0.1	0.005	0.042	0.041	0.041	0.022	0.096	0.096	0.093	0.0700	0.3130	0.3160	0.3090
0.5	-0.1	0.01	0.045	0.047	0.044	0.032	0.1	0.099	0.101	0.032	0.1	0.099	0.101
0.1	0.5	0.003	0.04	0.042	0.04	0.013	0.087	0.087	0.084	0.035	0.274	0.278	0.276
0.1	-0.5	0.015	0.045	0.041	0.042	0.042	0.096	0.099	0.096	0.181	0.336	0.34	0.336
-0.5	-0.1	0.011	0.049	0.049	0.049	0.038	0.11	0.109	0.106	0.176	0.356	0.356	0.354
-0.5	-0.9	0.01	0.039	0.038	0.038	0.036	0.082	0.082	0.081	0.15	0.255	0.255	0.25
-0.9	-0.5	0.014	0.042	0.044	0.041	0.04	0.093	0.096	0.092	0.174	0.307	0.31	0.303
-0.9	-0.9	0.014	0.033	0.033	0.032	0.014	0.033	0.033	0.032	0.149	0.228	0.223	0.223
μ_A		0.5				0.7				0.9			
ϕ_1	ϕ_2	GS	BSB_{I}	BSB_{II}	NSB	GS	BSB_{I}	BSB_{II}	NSB	GS	BSB_{I}	BSB_{II}	NSB
0.5	0.1	0.122	0.54	0.536	0.536	0.113	0.671	0.668	0.665	0.082	0.679	0.669	0.663
0.5	-0.1	0.303	0.616	0.617	0.612	0.469	0.803	0.805	0.802	0.5040	0.8330	0.8280	0.8210
0.1	0.5	0.041	0.465	0.466	0.46	0.026	0.525	0.523	0.517	0.008	0.503	0.496	0.495
0.1	-0.5	0.497	0.654	0.656	0.65	0.782	0.869	0.866	0.861	0.941	0.962	0.96	0.959
-0.5	-0.1	0.473	0.666	0.671	0.666	0.758	0.884	0.879	0.877	0.929	0.961	0.957	0.958
-0.5	-0.9	0.41	0.503	0.503	0.499	0.701	0.744	0.739	0.734	0.901	0.908	0.904	0.903
-0.9	-0.5	0.483	0.624	0.621	0.619	0.782	0.861	0.858	0.857	0.937	0.959	0.959	0.959
-0.9	-0.9	0.384	0.436	0.426	0.426	0.682	0.71	0.707	0.706	0.873	0.88	0.877	0.878

Table A.30: Comparison for an AR(2) process for errors $\epsilon \sim t_5$ with $n = 100$, change $\tau = 50$, and significance level at $\alpha = 0.05$. The number of Sieve Bootstrap Iterations = 1500 and number of Monte Carlo Iterations = 1000. GS: Test Statistic from Gombay and Serban (2009), BSB_{I}: Block Sieve Bootstrap with Mean of Critical Values from each block, BSB_{II}: Block Sieve Bootstrap with Median of Critical Values from each block, NSB: Naive Sieve Bootstrap approach

μ_A		0			0.1			0.3			0.5		
ϕ_1	ϕ_2	Mean	Var	Std. Dev.	Mean	Var	Std. Dev.	Mean	Var	Std. Dev.	Mean	Var	Std. Dev.
0.5	0.1	1.8061	0.0192	0.1386	1.8066	0.0201	0.1415	1.8179	0.0204	0.1428	1.8372	0.0253	0.1588
0.5	-0.1	1.8364	0.0268	0.1637	1.8371	0.0284	0.1684	1.8371	0.0284	0.1684	1.8853	0.0312	0.1765
0.1	0.5	1.7842	0.0206	0.1434	1.7856	0.0222	0.1489	1.7951	0.0214	0.1461	1.7985	0.0251	0.1583
0.1	-0.5	1.8905	0.0365	0.191	1.895	0.036	0.1897	1.9323	0.0373	0.1931	2.0009	0.0425	0.2062
-0.5	-0.1	1.8664	0.0283	0.168	1.8717	0.0278	0.1667	1.9027	0.0279	0.167	1.9458	0.0346	0.1858
-0.5	-0.9	1.8808	0.0811	0.2847	1.8898	0.0795	0.282	1.958	0.0728	0.2697	2.0635	0.0567	0.2382
-0.9	-0.5	1.9147	0.035	0.1869	1.9178	0.0353	0.1878	1.9544	0.0367	0.1915	2.0226	0.04	0.2
-0.9	-0.9	1.9468	0.0722	0.2686	1.9468	0.0722	0.2686	2.028	0.0597	0.2443	2.1284	0.0412	0.2028

μ_A		0.7			0.9		
ϕ_1	ϕ_2	Mean	Var	Std. Dev.	Mean	Var	Std. Dev.
0.5	0.1	1.8333	0.029	0.1701	1.8235	0.0311	0.1763
0.5	-0.1	1.9177	0.0398	0.1995	1.9584	0.0460	0.2146
0.1	0.5	1.7815	0.026	0.161	1.7454	0.0228	0.151
0.1	-0.5	2.0979	0.0433	0.208	2.1893	0.0335	0.183
-0.5	-0.1	2.0217	0.0456	0.2134	2.1165	0.0467	0.216
-0.5	-0.9	2.1686	0.0331	0.1817	2.2447	0.0146	0.1206
-0.9	-0.5	2.1075	0.041	0.2023	2.1913	0.0329	0.1813
-0.9	-0.9	2.2154	0.0206	0.1433	2.2665	0.0102	0.1006

Table A.31: Mean, Variance, and Standard Deviation of the the critical values used in Table A.30 for BSB_{I}: Block Sieve Bootstrap with mean of critical values from each block. Number of Blocks = 10

μ_A		0			0.1			0.3			0.5		
ϕ_1	ϕ_2	Mean	Var	Std. Dev.	Mean	Var	Std. Dev.	Mean	Var	Std. Dev.	Mean	Var	Std. Dev.
0.5	0.1	1.8037	0.0204	0.1428	1.804	0.0211	0.1452	1.8168	0.0215	0.1465	1.8367	0.0265	0.1626
0.5	-0.1	1.8323	0.0292	0.1709	1.8337	0.0306	0.1747	1.8337	0.0306	0.1747	1.8836	0.0335	0.183
0.1	0.5	1.7837	0.0216	0.1467	1.7849	0.0228	0.1509	1.7952	0.0223	0.1492	1.799	0.0261	0.1615
0.1	-0.5	1.8875	0.0411	0.2027	1.8913	0.0404	0.201	1.9298	0.0419	0.2046	1.9996	0.0466	0.2158
-0.5	-0.1	1.8607	0.0309	0.1757	1.8669	0.0307	0.1751	1.8976	0.0311	0.1762	1.9422	0.0383	0.1956
-0.5	-0.9	1.8863	0.0898	0.2996	1.895	0.0881	0.2968	1.9643	0.0807	0.284	2.0717	0.0632	0.2512
-0.9	-0.5	1.9145	0.0401	0.2003	1.9173	0.0408	0.2018	1.956	0.0425	0.2062	2.0252	0.0452	0.2125
-0.9	-0.9	1.9542	0.0797	0.2823	1.9542	0.0797	0.2823	2.0366	0.0653	0.2556	2.1397	0.0447	0.2113

μ_A		0.7			0.9		
ϕ_1	ϕ_2	Mean	Var	Std. Dev.	Mean	Var	Std. Dev.
0.5	0.1	1.8331	0.0304	0.1742	1.8235	0.0324	0.1799
0.5	-0.1	1.9179	0.0425	0.206	1.9595	0.0485	0.2203
0.1	0.5	1.7817	0.0267	0.1635	1.7456	0.0232	0.1521
0.1	-0.5	2.1005	0.0461	0.2147	2.1915	0.0348	0.1866
-0.5	-0.1	2.0192	0.0491	0.2216	2.1164	0.0487	0.2206
-0.5	-0.9	2.179	0.0354	0.1882	2.2527	0.0145	0.1201
-0.9	-0.5	2.11	0.0447	0.2114	2.1944	0.0345	0.1855
-0.9	-0.9	2.2249	0.0214	0.1461	2.2715	0.0099	0.0994

Table A.32: Mean, Variance, and Standard Deviation of the the critical values used in Table A.30 for BSB_{II}: Block Sieve Bootstrap with median of critical values from each block. Number of Blocks = 10

μ_A		0			0.1			0.3			0.5		
ϕ_1	ϕ_2	Mean	Var	Std. Dev.	Mean	Var	Std. Dev.	Mean	Var	Std. Dev.	Mean	Var	Std. Dev.
0.5	0.1	1.8096	0.0199	0.141	1.8103	0.0205	0.1432	1.8217	0.0209	0.1445	1.8414	0.026	0.1612
0.5	-0.1	1.8391	0.0281	0.1675	1.8395	0.0297	0.1723	1.8395	0.0297	0.1723	1.8892	0.0329	0.1812
0.1	0.5	1.7885	0.0214	0.146	1.7907	0.0229	0.1511	1.7995	0.0219	0.1477	1.8028	0.0257	0.1602
0.1	-0.5	1.8926	0.04	0.1999	1.8973	0.0392	0.1978	1.9355	0.0404	0.201	2.0042	0.0449	0.2119
-0.5	-0.1	1.8688	0.0299	0.1728	1.8738	0.0296	0.172	1.9042	0.0297	0.1723	1.9483	0.0368	0.1917
-0.5	-0.9	1.8938	0.0892	0.2986	1.9019	0.0875	0.2957	1.9715	0.0797	0.2822	2.0789	0.0614	0.2477
-0.9	-0.5	1.9201	0.0392	0.1979	1.9239	0.04	0.1999	1.9609	0.0415	0.2037	2.0293	0.0442	0.2102
-0.9	-0.9	1.9596	0.079	0.281	1.9596	0.079	0.281	2.0417	0.0645	0.2538	2.1433	0.0434	0.2082

μ_A		0.7			0.9		
ϕ_1	ϕ_2	Mean	Var	Std. Dev.	Mean	Var	Std. Dev.
0.5	0.1	1.8384	0.0297	0.1723	1.8288	0.0322	0.1794
0.5	-0.1	1.9225	0.0414	0.2033	1.9647	0.0474	0.2178
0.1	0.5	1.7854	0.0264	0.1625	1.7489	0.0232	0.1523
0.1	-0.5	2.1043	0.0449	0.2119	2.1942	0.0338	0.1838
-0.5	-0.1	2.0236	0.0473	0.2175	2.1189	0.048	0.2191
-0.5	-0.9	2.185	0.0338	0.1837	2.2559	0.0136	0.1164
-0.9	-0.5	2.1148	0.0435	0.2084	2.1973	0.0334	0.1826
-0.9	-0.9	2.2284	0.0203	0.1425	2.2731	0.0097	0.0983

Table A.33: Mean, Variance, and Standard Deviation of the the critical values used in Table A.30 for NSB: Naive Sieve Bootstrap Approach

μ_A		0				0.1				0.3			
ϕ_1	ϕ_2	GS	BSB_{I}	BSB_{II}	NSB	GS	BSB_{I}	BSB_{II}	NSB	GS	BSB_{I}	BSB_{II}	NSB
0.5	0.1	0.013	0.059	0.059	0.059	0.034	0.121	0.123	0.12	0.092	0.349	0.349	0.343
0.5	-0.1	0.015	0.054	0.054	0.053	0.045	0.125	0.125	0.126	0.162	0.383	0.388	0.38
0.1	0.5	0.011	0.059	0.057	0.055	0.026	0.12	0.12	0.118	0.055	0.317	0.326	0.314
0.1	-0.5	0.014	0.043	0.044	0.046	0.048	0.114	0.113	0.117	0.229	0.367	0.372	0.369
-0.5	-0.1	0.013	0.047	0.052	0.048	0.047	0.127	0.127	0.125	0.223	0.403	0.411	0.409
-0.5	-0.9	0.015	0.048	0.049	0.046	0.046	0.097	0.094	0.096	0.188	0.279	0.282	0.274
-0.9	-0.5	0.015	0.042	0.045	0.043	0.046	0.104	0.105	0.104	0.228	0.35	0.357	0.349
-0.9	-0.9	0.011	0.039	0.036	0.036	0.038	0.083	0.086	0.083	0.179	0.239	0.239	0.237

μ_A		0.5				0.7				0.9			
ϕ_1	ϕ_2	GS	BSB_{I}	BSB_{II}	NSB	GS	BSB_{I}	BSB_{II}	NSB	GS	BSB_{I}	BSB_{II}	NSB
0.5	0.1	0.136	0.582	0.588	0.576	0.097	0.645	0.639	0.636	0.04	0.61	0.614	0.603
0.5	-0.1	0.35	0.678	0.674	0.675	0.507	0.822	0.819	0.813	0.564	0.862	0.855	0.859
0.1	0.5	0.041	0.461	0.462	0.46	0.014	0.484	0.487	0.475	0.002	0.413	0.422	0.415
0.1	-0.5	0.57	0.706	0.712	0.707	0.861	0.91	0.911	0.909	0.974	0.986	0.986	0.986
-0.5	-0.1	0.541	0.725	0.72	0.722	0.833	0.912	0.911	0.913	0.953	0.985	0.982	0.984
-0.5	-0.9	0.481	0.558	0.546	0.544	0.784	0.811	0.811	0.809	0.947	0.951	0.949	0.949
-0.9	-0.5	0.541	0.668	0.663	0.66	0.855	0.899	0.899	0.897	0.973	0.985	0.984	0.984
-0.9	-0.9	0.462	0.522	0.519	0.513	0.755	0.769	0.758	0.757	0.92	0.916	0.914	0.913

Table A.34: Comparison for an AR(2) process for errors $\epsilon \sim t_8$ with $n = 100$, change $\tau = 50$, and significance level at $\alpha = 0.05$. The number of Sieve Bootstrap Iterations = 1500 and number of Monte Carlo Iterations = 1000. GS: Test Statistic from Gombay and Serban (2009), BSB_{I}: Block Sieve Bootstrap with Mean of Critical Values from each block, BSB_{II}: Block Sieve Bootstrap with Median of Critical Values from each block, NSB: Naive Sieve Bootstrap approach

μ_A	0				0.1				0.3			
θ_1	GS	BSB_{I}	BSB_{II}	NSB	GS	BSB_{I}	BSB_{II}	NSB	GS	BSB_{I}	BSB_{II}	NSB
-0.8	0.001	0.007	0.009	0.007	0.002	0.022	0.021	0.021	0.008	0.077	0.079	0.075
-0.5	0.006	0.034	0.034	0.031	0.021	0.083	0.082	0.082	0.114	0.319	0.322	0.318
-0.3	0.012	0.042	0.042	0.042	0.038	0.114	0.116	0.117	0.212	0.406	0.408	0.405
-0.1	0.016	0.044	0.046	0.046	0.052	0.129	0.134	0.132	0.26	0.454	0.468	0.459
0.1	0.017	0.046	0.045	0.043	0.043	0.14	0.137	0.139	0.247	0.447	0.458	0.451
0.3	0.013	0.051	0.052	0.052	0.041	0.14	0.138	0.138	0.193	0.422	0.424	0.419
0.5	0.015	0.05	0.054	0.05	0.04	0.134	0.135	0.135	0.147	0.414	0.41	0.413
0.8	0.009	0.056	0.057	0.057	0.031	0.13	0.135	0.128	0.112	0.395	0.402	0.401

μ_A	0.5				0.7				0.9			
θ_1	GS	BSB_{I}	BSB_{II}	NSB	GS	BSB_{I}	BSB_{II}	NSB	GS	BSB_{I}	BSB_{II}	NSB
-0.8	0.045	0.22	0.221	0.221	0.109	0.409	0.42	0.412	0.22	0.613	0.62	0.615
-0.5	0.357	0.673	0.676	0.67	0.644	0.827	0.833	0.83	0.697	0.789	0.786	0.785
-0.3	0.521	0.755	0.76	0.758	0.673	0.792	0.796	0.795	0.579	0.679	0.675	0.68
-0.1	0.563	0.778	0.781	0.776	0.635	0.783	0.777	0.779	0.48	0.643	0.649	0.646
0.1	0.496	0.74	0.744	0.744	0.573	0.767	0.767	0.765	0.45	0.684	0.688	0.684
0.3	0.428	0.728	0.729	0.725	0.555	0.789	0.789	0.789	0.465	0.709	0.713	0.706
0.5	0.313	0.693	0.691	0.692	0.397	0.758	0.761	0.756	0.326	0.704	0.712	0.707
0.8	0.186	0.627	0.627	0.629	0.163	0.682	0.678	0.671	0.103	0.633	0.637	0.633

Table A.35: Comparison for MA(1) process for errors $\epsilon \sim N(0,1)$ with $n = 100$, change $\tau = 50$, and significance level at $\alpha = 0.05$. The number of Sieve Bootstrap Iterations = 1500 and number of Monte Carlo Iterations = 1000. GS: Test Statistic from Gombay and Serban (2009), BSB_{I}: Block Sieve Bootstrap with Mean of Critical Values from each block, BSB_{II}: Block Sieve Bootstrap with Median of Critical Values from each block, NSB: Naive Sieve Bootstrap approach

μ_A	0			0.1			0.3		
θ_1	$Mean$	Var	$Std.Dev.$	$Mean$	Var	$Std.Dev.$	$Mean$	Var	$Std.Dev.$
-0.8	1.7333	0.0102	0.1008	1.7355	0.0099	0.0993	1.7425	0.0095	0.0973
-0.5	1.807	0.009	0.0949	1.8076	0.0091	0.095	1.8317	0.0082	0.0902
-0.3	1.8491	0.0081	0.0898	1.8512	0.0079	0.0885	1.8739	0.0081	0.0899
-0.1	1.8723	0.0067	0.0814	1.8746	0.0064	0.0799	1.8874	0.0072	0.0847
0.1	1.8648	0.0059	0.0765	1.865	0.0061	0.078	1.8769	0.0075	0.0866
0.3	1.8369	0.0055	0.0739	1.8386	0.0055	0.074	1.8483	0.0057	0.0753
0.5	1.8104	0.0052	0.0718	1.8142	0.0056	0.0747	1.8241	0.0057	0.0753
0.8	1.8008	0.0059	0.0762	1.8003	0.0058	0.0762	1.8027	0.0058	0.0759

μ_A	0.5			0.7			0.9		
θ_1	$Mean$	Var	$Std.Dev.$	$Mean$	Var	$Std.Dev.$	$Mean$	Var	$Std.Dev.$
-0.8	1.7558	0.0087	0.093	1.773	0.0083	0.0906	1.7929	0.0086	0.0923
-0.5	1.8687	0.0104	0.1016	1.9081	0.0145	0.1202	1.958	0.02	0.1414
-0.3	1.9072	0.0105	0.1024	1.9548	0.0182	0.1348	1.9801	0.021	0.1448
-0.1	1.9161	0.011	0.1049	1.9513	0.0176	0.1326	1.9558	0.0208	0.144
0.1	1.9001	0.0103	0.1012	1.9139	0.0144	0.1197	1.9049	0.0168	0.1293
0.3	1.8666	0.0082	0.0903	1.8769	0.0119	0.1089	1.8709	0.0137	0.1168
0.5	1.8392	0.0079	0.0885	1.8387	0.0094	0.0966	1.8256	0.0108	0.1038
0.8	1.7945	0.0061	0.0778	1.772	0.0066	0.0813	1.7449	0.0064	0.08

Table A.36: Mean, Variance, and Standard Deviation of the the critical values used in Table A.35 for BSB_{I}: Block Sieve Bootstrap with mean of critical values from each block. Number of Blocks = 10

μ_A	0			0.1			0.3		
θ_1	*Mean*	*Var*	*Std.Dev.*	*Mean*	*Var*	*Std.Dev.*	*Mean*	*Var*	*Std.Dev.*
-0.8	1.727	0.0105	0.1024	1.7293	0.0102	0.1008	1.7341	0.0099	0.0993
-0.5	1.7971	0.0095	0.0973	1.7978	0.0097	0.0981	1.8222	0.0089	0.0939
-0.3	1.8389	0.0089	0.0939	1.8402	0.0082	0.0903	1.8645	0.0094	0.0967
-0.1	1.861	0.0076	0.0869	1.8644	0.0076	0.0872	1.8784	0.0088	0.0936
0.1	1.8539	0.0066	0.0809	1.8547	0.0069	0.0829	1.8685	0.0089	0.0943
0.3	1.8294	0.0061	0.0778	1.8308	0.0062	0.0783	1.8421	0.0066	0.0811
0.5	1.8032	0.0057	0.0755	1.8089	0.0061	0.0777	1.8184	0.0064	0.0799
0.8	1.795	0.0062	0.0782	1.7944	0.0061	0.0778	1.7979	0.0062	0.0785

μ_A	0.5			0.7			0.9		
θ_1	*Mean*	*Var*	*Std.Dev.*	*Mean*	*Var*	*Std.Dev.*	*Mean*	*Var*	*Std.Dev.*
-0.8	1.7473	0.009	0.0946	1.7642	0.0089	0.0939	1.7857	0.0094	0.0969
-0.5	1.8612	0.0122	0.1102	1.902	0.018	0.134	1.957	0.026	0.161
-0.3	1.8993	0.0134	0.1158	1.9534	0.0236	0.1534	1.9827	0.0275	0.1656
-0.1	1.9103	0.0142	0.1191	1.9501	0.022	0.1481	1.9557	0.0264	0.1625
0.1	1.8948	0.0126	0.1122	1.9127	0.0174	0.1316	1.903	0.0199	0.141
0.3	1.8619	0.0099	0.0991	1.8731	0.0138	0.1174	1.8688	0.016	0.1262
0.5	1.836	0.0091	0.0952	1.8354	0.0108	0.1037	1.8225	0.0119	0.109
0.8	1.7912	0.0065	0.0801	1.7704	0.0071	0.084	1.7442	0.0068	0.0825

Table A.37: Mean, Variance, and Standard Deviation of the the critical values used in Table A.35 for BSB_{II}: Block Sieve Bootstrap with median of critical values from each block. Number of Blocks = 10

μ_A	0			0.1			0.3		
θ_1	$Mean$	Var	$Std.Dev.$	$Mean$	Var	$Std.Dev.$	$Mean$	Var	$Std.Dev.$
-0.8	1.7331	0.0097	0.0984	1.7351	0.0093	0.0963	1.7414	0.0091	0.095
-0.5	1.805	0.0086	0.0926	1.8054	0.0088	0.0936	1.8284	0.0083	0.0907
-0.3	1.8443	0.0078	0.0881	1.8467	0.0074	0.0859	1.8716	0.0087	0.093
-0.1	1.8673	0.0066	0.0812	1.8706	0.0066	0.081	1.8843	0.008	0.089
0.1	1.8607	0.0059	0.0766	1.8612	0.006	0.0773	1.8749	0.0081	0.0899
0.3	1.8359	0.0055	0.0741	1.8369	0.0054	0.0735	1.8476	0.0061	0.0779
0.5	1.8103	0.005	0.0702	1.8137	0.0055	0.0738	1.8249	0.006	0.0769
0.8	1.8017	0.0058	0.0757	1.801	0.0058	0.0761	1.8043	0.0057	0.0754

μ_A	0.5			0.7			0.9		
θ_1	$Mean$	Var	$Std.Dev.$	$Mean$	Var	$Std.Dev.$	$Mean$	Var	$Std.Dev.$
-0.8	1.7539	0.0084	0.0912	1.7707	0.0081	0.0899	1.7914	0.0086	0.0925
-0.5	1.8663	0.0115	0.1069	1.9087	0.0172	0.1311	1.9644	0.0255	0.1596
-0.3	1.906	0.0124	0.111	1.9603	0.0228	0.1509	1.9899	0.0269	0.1638
-0.1	1.9154	0.0134	0.1155	1.9567	0.0215	0.1464	1.9622	0.0257	0.1603
0.1	1.8992	0.012	0.1095	1.9181	0.0169	0.1298	1.9084	0.0194	0.139
0.3	1.8677	0.0094	0.0969	1.8794	0.0134	0.1158	1.8744	0.0157	0.1252
0.5	1.841	0.0088	0.0936	1.8413	0.0102	0.101	1.8288	0.0117	0.1081
0.8	1.7962	0.0061	0.0781	1.7751	0.0066	0.0812	1.748	0.0064	0.0798

Table A.38: Mean, Variance, and Standard Deviation of the the critical values used in Table A.35 for NSB: Naive Sieve Bootstrap Approach

μ_A	0			0.1			0.3		
θ_1	*Mean*	*Median*	*Mode*	*Mean*	*Median*	*Mode*	*Mean*	*Median*	*Mode*
-0.8	3.5921	3	3	3.5729	3	3	3.4921	3	2
-0.5	2.5046	2	2	2.4967	2	2	2.3377	2	1
-0.3	1.9749	1	1	1.9647	1	1	1.9289	1	1
-0.1	1.6618	1	1	1.66	1	1	1.7004	1	1
0.1	1.6754	1	1	1.6825	1	1	1.799	1	1
0.3	1.9637	1	1	1.9679	1	1	2.0133	1	1
0.5	2.5067	2	2	2.5222	2	2	2.5701	2	1
0.8	3.6516	3	2	3.6625	3	3	3.7929	3	3

μ_A	0.5			0.7			0.9		
θ_1	*Mean*	*Median*	*Mode*	*Mean*	*Median*	*Mode*	*Mean*	*Median*	*Mode*
-0.8	3.3494	3	2	3.1908	3	2	3.0378	2	2
-0.5	2.3093	1	1	2.6064	1	1	3.3526	2	1
-0.3	2.1321	1	1	2.8558	1	1	3.8333	4	1
-0.1	2.0337	1	1	2.925	2	1	3.9399	4	1
0.1	2.2414	1	1	2.9453	2	1	3.681	3	1
0.3	2.3152	1	1	2.8099	2	1	3.3205	3	1
0.5	2.8289	2	1	3.1762	3	1	3.5333	3	3
0.8	4.0643	3	3	4.3072	3	3	4.4539	3	3

Table A.39: Average AR(p) order chosen by Sieve Bootstrap for the underlying process based on AIC Criterion. Values in reference to Table A.35 where the underlying process was MA(1).

μ_A		0				0.1				0.3			
θ_1	θ_2	GS	BSB_{I}	BSB_{II}	NSB	GS	BSB_{I}	BSB_{II}	NSB	GS	BSB_{I}	BSB_{II}	NSB
0.5	0.1	0.015	0.047	0.048	0.048	0.038	0.128	0.13	0.127	0.151	0.421	0.425	0.423
0.5	-0.1	0.01	0.056	0.055	0.056	0.039	0.13	0.129	0.129	0.14	0.417	0.427	0.417
0.1	0.5	0.008	0.041	0.045	0.044	0.035	0.116	0.116	0.115	0.071	0.367	0.363	0.368
0.1	-0.5	0.004	0.031	0.032	0.032	0.02	0.079	0.08	0.08	0.099	0.311	0.314	0.31
-0.5	-0.1	0.002	0.021	0.019	0.019	0.01	0.055	0.055	0.056	0.061	0.217	0.225	0.22
-0.8	0.1	0.002	0.02	0.021	0.02	0.011	0.051	0.052	0.05	0.06	0.21	0.208	0.203
0.7	0.2	0.013	0.052	0.054	0.053	0.039	0.141	0.139	0.142	0.129	0.406	0.413	0.406

μ_A		0.5				0.7				0.9			
θ_1	θ_2	GS	BSB_{I}	BSB_{II}	NSB	GS	BSB_{I}	BSB_{II}	NSB	GS	BSB_{I}	BSB_{II}	NSB
0.5	0.1	0.343	0.711	0.713	0.71	0.441	0.776	0.776	0.774	0.399	0.72	0.719	0.719
0.5	-0.1	0.281	0.684	0.681	0.677	0.321	0.735	0.742	0.74	0.2410	0.6590	0.6600	0.6630
0.1	0.5	0.119	0.616	0.612	0.603	0.091	0.679	0.681	0.671	0.036	0.666	0.674	0.666
0.1	-0.5	0.292	0.624	0.632	0.633	0.502	0.818	0.819	0.817	0.559	0.809	0.809	0.81
-0.5	-0.1	0.201	0.517	0.523	0.522	0.449	0.767	0.778	0.773	0.652	0.866	0.866	0.859
-0.8	0.1	0.171	0.482	0.495	0.486	0.401	0.735	0.747	0.732	0.605	0.828	0.83	0.829
0.7	0.2	0.24	0.676	0.679	0.674	0.261	0.732	0.736	0.731	0.213	0.699	0.704	0.695

Table A.40: Comparison for MA(2) process for errors $\epsilon \sim N(0,1)$ with $n = 100$, change $\tau = 50$, and significance level at $\alpha = 0.05$. The number of Sieve Bootstrap Iterations = 1500 and number of Monte Carlo Iterations = 1000. GS: Test Statistic from Gombay and Serban (2009), BSB_{I}: Block Sieve Bootstrap with Mean of Critical Values from each block, BSB_{II}: Block Sieve Bootstrap with Median of Critical Values from each block, NSB: Naive Sieve Bootstrap approach

μ_A		0			0.1			0.3		
θ_1	θ_2	*Mean*	*Median*	*Mode*	*Mean*	*Median*	*Mode*	*Mean*	*Median*	*Mode*
0.5	0.1	2.1825	1	1	2.1673	1	1	2.1671	1	1
0.5	-0.1	2.9699	2	2	2.9908	2	2	3.0892	3	3
0.1	0.5	3.2167	2	2	3.2298	2	2	3.2008	2	2
0.1	-0.5	3.1632	2	2	3.1712	2	2	3.136	2	2
-0.5	-0.1	2.9101	2	2	2.8879	2	2	2.746	2	2
-0.8	0.1	3.2635	3	2	3.2417	3	2	3.1253	2	2
0.7	0.2	2.5959	2	2	2.5865	2	2	2.6377	2	2

μ_A		0.5			0.7			0.9		
θ_1	θ_2	*Mean*	*Median*	*Mode*	*Mean*	*Median*	*Mode*	*Mean*	*Median*	*Mode*
0.5	0.1	2.3254	1	1	2.6728	1	1	3.009	3	1
0.5	-0.1	3.3623	3	3	3.7286	3	3	4.0636	3	3
0.1	0.5	3.4129	2	2	3.6684	2	2	3.8464	2	2
0.1	-0.5	3.1975	2	2	3.5142	3	2	4.0814	3	3
-0.5	-0.1	2.5471	2	2	2.5292	2	1	2.7427	1	1
-0.8	0.1	2.9582	2	2	2.92	2	2	3.0571	2	2
0.7	0.2	2.8051	2	1	3.0596	3	1	3.2711	3	1

Table A.41: Average AR(p) order chosen by Sieve Bootstrap for the underlying process based on AIC Criterion. Values in reference to Table A.40 where the underlying process was MA(2).

μ_A		0				0.1				0.3			
ϕ_1	θ_1	GS	BSB_{I}	BSB_{II}	NSB	GS	BSB_{I}	BSB_{II}	NSB	GS	BSB_{I}	BSB_{II}	NSB
-0.5	0.8	0.012	0.052	0.053	0.054	0.041	0.135	0.136	0.136	0.165	0.41	0.412	0.408
-0.3	0.8	0.018	0.044	0.046	0.043	0.053	0.138	0.14	0.139	0.263	0.457	0.465	0.453
-0.1	0.8	0.009	0.05	0.053	0.052	0.038	0.124	0.124	0.121	0.133	0.41	0.419	0.416
0.1	0.8	0.013	0.051	0.052	0.049	0.041	0.132	0.136	0.131	0.204	0.425	0.432	0.43
0.3	0.8	0.008	0.052	0.053	0.053	0.035	0.125	0.126	0.127	0.119	0.406	0.405	0.403
0.5	0.8	0.012	0.053	0.053	0.052	0.041	0.138	0.142	0.139	0.173	0.418	0.42	0.418

μ_A		0.5				0.7				0.9			
ϕ_1	θ_1	GS	BSB_{I}	BSB_{II}	NSB	GS	BSB_{I}	BSB_{II}	NSB	GS	BSB_{I}	BSB_{II}	NSB
-0.5	0.8	0.35	0.668	0.668	0.668	0.386	0.717	0.719	0.709	0.309	0.654	0.663	0.657
-0.3	0.8	0.542	0.754	0.763	0.758	0.586	0.767	0.771	0.766	0.463	0.663	0.665	0.665
-0.1	0.8	0.26	0.65	0.645	0.651	0.268	0.701	0.699	0.697	0.215	0.655	0.658	0.653
0.1	0.8	0.441	0.721	0.726	0.724	0.559	0.77	0.772	0.772	0.458	0.704	0.709	0.704
0.3	0.8	0.203	0.642	0.644	0.638	0.203	0.688	0.689	0.684	0.131	0.654	0.658	0.656
0.5	0.8	0.362	0.703	0.701	0.701	0.46	0.76	0.763	0.757	0.398	0.713	0.716	0.716

Table A.42: Comparison for ARMA(1,1) process for errors $\epsilon \sim N(0,1)$ with $\theta_1 = 0.8$, $n = 100$, change $\tau = 50$, and significance level at $\alpha = 0.05$. The number of Sieve Bootstrap Iterations = 1500 and number of Monte Carlo Iterations = 1000. GS: Test Statistic from Gombay and Serban (2009), BSB_{I}: Block Sieve Bootstrap with Mean of Critical Values from each block, BSB_{II}: Block Sieve Bootstrap with Median of Critical Values from each block, NSB: Naive Sieve Bootstrap approach

μ_A		0				0.1				0.3			
ϕ_1	θ_1	GS	BSB_{I}	BSB_{II}	NSB	GS	BSB_{I}	BSB_{II}	NSB	GS	BSB_{I}	BSB_{II}	NSB
-0.5	0.5	0.009	0.056	0.055	0.055	0.032	0.134	0.139	0.135	0.099	0.392	0.39	0.388
-0.3	0.5	0.01	0.051	0.053	0.051	0.042	0.135	0.135	0.135	0.129	0.416	0.42	0.417
-0.1	0.5	0.012	0.047	0.046	0.047	0.027	0.133	0.133	0.131	0.1	0.387	0.393	0.392
0.1	0.5	0.011	0.057	0.057	0.055	0.039	0.134	0.137	0.132	0.115	0.403	0.409	0.403
0.3	0.5	0.01	0.04	0.04	0.04	0.02	0.121	0.127	0.123	0.075	0.355	0.363	0.354
0.5	0.5	0.012	0.052	0.054	0.053	0.03	0.127	0.129	0.13	0.101	0.394	0.399	0.39

μ_A		0.5				0.7				0.9			
ϕ_1	θ_1	GS	BSB_{I}	BSB_{II}	NSB	GS	BSB_{I}	BSB_{II}	NSB	GS	BSB_{I}	BSB_{II}	NSB
-0.5	0.5	0.172	0.615	0.622	0.619	0.147	0.664	0.664	0.659	0.085	0.635	0.63	0.626
-0.3	0.5	0.28	0.69	0.693	0.688	0.332	0.743	0.745	0.74	0.458	0.704	0.709	0.704
-0.1	0.5	0.138	0.593	0.595	0.585	0.114	0.636	0.636	0.635	0.072	0.572	0.563	0.562
0.1	0.5	0.196	0.65	0.654	0.644	0.183	0.695	0.706	0.698	0.125	0.653	0.658	0.649
0.3	0.5	0.112	0.563	0.558	0.56	0.091	0.55	0.548	0.552	0.05	0.474	0.48	0.479
0.5	0.5	0.133	0.59	0.59	0.589	0.088	0.628	0.627	0.621	0.039	0.568	0.57	0.561

Table A.43: Comparison for ARMA(1,1) process for errors $\epsilon \sim N(0,1)$ with $\theta_1 = 0.5$, $n = 100$, change $\tau = 50$, and significance level at $\alpha = 0.05$. The number of Sieve Bootstrap Iterations = 1500 and number of Monte Carlo Iterations = 1000. GS: Test Statistic from Gombay and Serban (2009), BSB_{I}: Block Sieve Bootstrap with Mean of Critical Values from each block, BSB_{II}: Block Sieve Bootstrap with Median of Critical Values from each block, NSB: Naive Sieve Bootstrap approach

μ_A		0			0.1			0.3		
ϕ_1	θ_1	$Mean$	Var	$Std.Dev.$	$Mean$	Var	$Std.Dev.$	$Mean$	Var	$Std.Dev.$
-0.5	0.8	1.8194	0.0063	0.0791	1.8216	0.0063	0.0788	1.8385	0.0076	0.0871
-0.3	0.8	1.8727	0.0059	0.0766	1.8741	0.0059	0.0764	1.8852	0.0071	0.0838
-0.1	0.8	1.8085	0.006	0.0773	1.8107	0.0064	0.0798	1.8191	0.0065	0.0804
0.1	0.8	1.844	0.006	0.0775	1.8464	0.0061	0.0779	1.8584	0.0065	0.0806
0.3	0.8	1.8017	0.0059	0.0763	1.8025	0.0058	0.0758	1.8075	0.0059	0.0767
0.5	0.8	1.8166	0.0053	0.0727	1.819	0.0054	0.0734	1.8339	0.0061	0.0776

μ_A		0.5			0.7			0.9		
ϕ_1	θ_1	$Mean$	Var	$Std.Dev.$	$Mean$	Var	$Std.Dev.$	$Mean$	Var	$Std.Dev.$
-0.5	0.8	1.8538	0.0081	0.0898	1.8582	0.0106	0.1025	1.8383	0.0123	0.1108
-0.3	0.8	1.9134	0.0114	0.1065	1.935	0.0159	0.126	1.9273	0.0192	0.1383
-0.1	0.8	1.8239	0.0071	0.0838	1.8154	0.0086	0.0923	1.7908	0.0087	0.0928
0.1	0.8	1.8813	0.0093	0.096	1.8914	0.0128	0.1131	1.8838	0.0144	0.1197
0.3	0.8	1.8022	0.0062	0.0787	1.7861	0.0073	0.0855	1.7584	0.0068	0.0822
0.5	0.8	1.8521	0.0081	0.0897	1.8561	0.0102	0.101	1.8468	0.0124	0.1111

Table A.44: Mean, Variance, and Standard Deviation of the the critical values used in Table A.42 for BSB_{I}: Block Sieve Bootstrap with mean of critical values from each block. Number of Blocks = 10

μ_A		0			0.1			0.3		
ϕ_1	θ_1	$Mean$	Var	$Std.Dev.$	$Mean$	Var	$Std.Dev.$	$Mean$	Var	$Std.Dev.$
-0.5	0.5	1.7992	0.0059	0.0763	1.7975	0.0058	0.0758	1.7985	0.0056	0.0744
-0.3	0.5	1.8073	0.005	0.0706	1.8076	0.0055	0.0739	1.8157	0.0054	0.0735
-0.1	0.5	1.7932	0.0058	0.0761	1.7936	0.0059	0.0765	1.7857	0.0056	0.0742
0.1	0.5	1.8015	0.0048	0.0693	1.8007	0.0052	0.0718	1.8011	0.0051	0.0709
0.3	0.5	1.7865	0.0061	0.0777	1.7851	0.0063	0.0794	1.7704	0.0055	0.0736
0.5	0.5	1.7925	0.0047	0.0682	1.7896	0.0047	0.0683	1.7835	0.0045	0.0666

μ_A		0.5			0.7			0.9		
ϕ_1	θ_1	$Mean$	Var	$Std.Dev.$	$Mean$	Var	$Std.Dev.$	$Mean$	Var	$Std.Dev.$
-0.5	0.5	1.7853	0.0061	0.0777	1.7598	0.0062	0.0783	1.7319	0.0059	0.0763
-0.3	0.5	1.824	0.0067	0.0814	1.8199	0.0085	0.0918	1.8838	0.0144	0.1197
-0.1	0.5	1.7675	0.0055	0.074	1.7362	0.006	0.0772	1.7079	0.0058	0.0761
0.1	0.5	1.7989	0.0064	0.0799	1.7819	0.0071	0.0838	1.7564	0.0069	0.0831
0.3	0.5	1.7429	0.0057	0.0753	1.7108	0.0061	0.0776	1.6811	0.0051	0.0712
0.5	0.5	1.7654	0.0053	0.0727	1.7363	0.0057	0.0753	1.706	0.0049	0.0699

Table A.45: Mean, Variance, and Standard Deviation of the the critical values used in Table A.43 for BSB_{I}: Block Sieve Bootstrap with mean of critical values from each block. Number of Blocks = 10

μ_A		0			0.1			0.3		
ϕ_1	θ_1	$Mean$	Var	$Std.Dev.$	$Mean$	Var	$Std.Dev.$	$Mean$	Var	$Std.Dev.$
-0.5	0.8	1.8116	0.0069	0.0828	1.815	0.0068	0.0824	1.8332	0.0086	0.0928
-0.3	0.8	1.8616	0.007	0.0834	1.8633	0.0069	0.083	1.8758	0.0087	0.0929
-0.1	0.8	1.8032	0.0063	0.0794	1.8057	0.0069	0.0826	1.8137	0.0072	0.0848
0.1	0.8	1.8347	0.0068	0.0822	1.8373	0.0067	0.0817	1.8515	0.0075	0.0864
0.3	0.8	1.7962	0.0064	0.0794	1.797	0.0061	0.0778	1.8031	0.0065	0.0803
0.5	0.8	1.8093	0.0058	0.0758	1.812	0.0056	0.0742	1.8289	0.0068	0.0819
μ_A		0.5			0.7			0.9		
ϕ_1	θ_1	$Mean$	Var	$Std.Dev.$	$Mean$	Var	$Std.Dev.$	$Mean$	Var	$Std.Dev.$
-0.5	0.8	1.8498	0.0092	0.0959	1.8538	0.0122	0.1103	1.8352	0.0136	0.1165
-0.3	0.8	1.9097	0.0145	0.1203	1.9342	0.0193	0.1388	1.9266	0.0233	0.1526
-0.1	0.8	1.8197	0.0075	0.0866	1.8132	0.0093	0.0961	1.788	0.0091	0.0951
0.1	0.8	1.8763	0.0111	0.1053	1.8884	0.0153	0.1233	1.8814	0.0171	0.1305
0.3	0.8	1.7987	0.0067	0.0817	1.7839	0.0077	0.0876	1.7571	0.0072	0.0845
0.5	0.8	1.8486	0.0095	0.0973	1.8533	0.0119	0.1091	1.8452	0.0141	0.1186

Table A.46: Mean, Variance, and Standard Deviation of the the critical values used in Table A.42 for BSB_{II}: Block Sieve Bootstrap with median of critical values from each block. Number of Blocks = 10

μ_A		0			0.1			0.3		
ϕ_1	θ_1	*Mean*	*Var*	*Std.Dev.*	*Mean*	*Var*	*Std.Dev.*	*Mean*	*Var*	*Std.Dev.*
-0.5	0.5	1.7938	0.0062	0.0784	1.7926	0.0062	0.0783	1.7945	0.006	0.0771
-0.3	0.5	1.8007	0.0055	0.0737	1.8027	0.006	0.0773	1.8103	0.0062	0.0784
-0.1	0.5	1.788	0.0062	0.0783	1.7885	0.0063	0.0794	1.7818	0.0059	0.0765
0.1	0.5	1.7964	0.0053	0.0724	1.7963	0.0058	0.0756	1.7969	0.0054	0.0734
0.3	0.5	1.7823	0.0063	0.0794	1.7809	0.0066	0.0812	1.7681	0.0058	0.0758
0.5	0.5	1.7888	0.005	0.0706	1.7861	0.0049	0.0698	1.7815	0.0048	0.0688

μ_A		0.5			0.7			0.9		
ϕ_1	θ_1	*Mean*	*Var*	*Std.Dev.*	*Mean*	*Var*	*Std.Dev.*	*Mean*	*Var*	*Std.Dev.*
-0.5	0.5	1.7821	0.0063	0.0793	1.758	0.0066	0.081	1.7314	0.0063	0.079
-0.3	0.5	1.8212	0.0075	0.0864	1.8173	0.0096	0.0979	1.8814	0.0171	0.1305
-0.1	0.5	1.7654	0.0057	0.0749	1.7352	0.0061	0.0776	1.7064	0.0059	0.0763
0.1	0.5	1.7962	0.0071	0.0839	1.7794	0.0075	0.0866	1.755	0.0071	0.0842
0.3	0.5	1.7413	0.0059	0.0765	1.7102	0.0062	0.0785	1.6801	0.0051	0.0714
0.5	0.5	1.7631	0.0055	0.0742	1.7345	0.0057	0.0753	1.7044	0.005	0.0707

Table A.47: Mean, Variance, and Standard Deviation of the the critical values used in Table A.43 for BSB_{II}: Block Sieve Bootstrap with median of critical values from each block. Number of Blocks = 10

μ_A		0			0.1			0.3		
ϕ_1	θ_1	$Mean$	Var	$Std.Dev.$	$Mean$	Var	$Std.Dev.$	$Mean$	Var	$Std.Dev.$
-0.5	0.8	1.8173	0.0063	0.079	1.8204	0.0062	0.0785	1.8391	0.0082	0.0904
-0.3	0.8	1.8678	0.006	0.0772	1.8697	0.0059	0.0764	1.8819	0.0076	0.0869
-0.1	0.8	1.8082	0.006	0.0769	1.8102	0.0063	0.0793	1.8204	0.0068	0.0821
0.1	0.8	1.8425	0.0061	0.0776	1.8441	0.006	0.0774	1.8571	0.007	0.0833
0.3	0.8	1.8017	0.0057	0.0752	1.8028	0.0057	0.0755	1.8092	0.0059	0.0766
0.5	0.8	1.8165	0.0052	0.0718	1.8184	0.0052	0.072	1.8347	0.0062	0.0785

μ_A		0.5			0.7			0.9		
ϕ_1	θ_1	$Mean$	Var	$Std.Dev.$	$Mean$	Var	$Std.Dev.$	$Mean$	Var	$Std.Dev.$
-0.5	0.8	1.8558	0.009	0.0945	1.8606	0.0121	0.1097	1.8403	0.0136	0.1163
-0.3	0.8	1.9134	0.0137	0.1171	1.9401	0.019	0.1378	1.9326	0.0232	0.1521
-0.1	0.8	1.8257	0.0073	0.0855	1.8182	0.0091	0.0953	1.7932	0.0089	0.0941
0.1	0.8	1.8825	0.0108	0.1039	1.8946	0.0147	0.1212	1.8862	0.0166	0.1286
0.3	0.8	1.8047	0.0063	0.0794	1.7891	0.0073	0.0855	1.7615	0.0068	0.0825
0.5	0.8	1.8539	0.0089	0.0939	1.8586	0.0113	0.1061	1.8504	0.014	0.118

Table A.48: Mean, Variance, and Standard Deviation of the the critical values used in Table A.42 for NSB: Naive Sieve Bootstrap Approach

μ_A		0			0.1			0.3		
ϕ_1	θ_1	$Mean$	Var	$Std.Dev.$	$Mean$	Var	$Std.Dev.$	$Mean$	Var	$Std.Dev.$
-0.5	0.5	1.799	0.0058	0.0757	1.7986	0.0057	0.0752	1.8001	0.0056	0.0745
-0.3	0.5	1.8072	0.0048	0.0687	1.8076	0.0053	0.0728	1.8166	0.0056	0.0747
-0.1	0.5	1.794	0.0058	0.0757	1.7952	0.006	0.077	1.7877	0.0053	0.0725
0.1	0.5	1.8025	0.0047	0.0685	1.8019	0.0052	0.0719	1.803	0.0049	0.0698
0.3	0.5	1.7875	0.0059	0.0768	1.7863	0.006	0.0773	1.7725	0.0052	0.0718
0.5	0.5	1.7948	0.0045	0.0669	1.7919	0.0045	0.0668	1.7862	0.0045	0.0668

μ_A		0.5			0.7			0.9		
ϕ_1	θ_1	$Mean$	Var	$Std.Dev.$	$Mean$	Var	$Std.Dev.$	$Mean$	Var	$Std.Dev.$
-0.5	0.5	1.7875	0.0061	0.0776	1.7623	0.0062	0.0786	1.7352	0.006	0.0769
-0.3	0.5	1.825	0.0069	0.083	1.8232	0.0092	0.0959	1.8862	0.0166	0.1286
-0.1	0.5	1.7696	0.0053	0.0727	1.7389	0.006	0.0771	1.7103	0.0058	0.0756
0.1	0.5	1.8028	0.0066	0.081	1.7852	0.0075	0.0861	1.7592	0.0068	0.0819
0.3	0.5	1.7455	0.0056	0.0745	1.7133	0.0061	0.0777	1.6824	0.0051	0.071
0.5	0.5	1.7676	0.0053	0.0726	1.7397	0.0056	0.0746	1.7083	0.0049	0.0698

Table A.49: Mean, Variance, and Standard Deviation of the the critical values used in Table A.43 for NSB: Naive Sieve Bootstrap Approach

μ_A		0			0.1			0.3		
ϕ_1	θ_1	$Mean$	$Median$	$Mode$	$Mean$	$Median$	$Mode$	$Mean$	$Median$	$Mode$
-0.5	0.8	2.6248	2	1	2.6403	2	1	2.7875	2	1
-0.3	0.8	1.6169	1	1	1.6063	1	1	1.7035	1	1
-0.1	0.8	3.2784	3	2	3.3017	3	2	3.3878	3	3
0.1	0.8	1.9068	1	1	1.9026	1	1	2.0031	1	1
0.3	0.8	3.5811	3	2	3.6009	3	3	3.7027	3	3
0.5	0.8	2.3477	2	1	2.3528	2	1	2.401	2	1

μ_A		0.5			0.7			0.9		
ϕ_1	θ_1	$Mean$	$Median$	$Mode$	$Mean$	$Median$	$Mode$	$Mean$	$Median$	$Mode$
-0.5	0.8	3.1196	3	1	3.6657	3	3	4.1642	3	3
-0.3	0.8	2.1444	1	1	2.9653	2	1	3.8498	3	1
-0.1	0.8	3.6975	3	3	4.0968	3	3	4.3704	3	3
0.1	0.8	2.3888	1	1	2.9828	2	1	3.5862	3	1
0.3	0.8	3.9957	3	3	4.2919	3	3	4.4504	3	3
0.5	0.8	2.672	2	1	3.1022	3	1	3.5039	3	3

Table A.50: Average AR(p) order chosen by Sieve Bootstrap for the underlying process based on AIC Criterion. Values in reference to Table A.42 where the underlying process was ARMA(1, 1) with $\theta_1 = 0.8$.

μ_A		0			0.1			0.3		
ϕ_1	θ_1	*Mean*	*Median*	*Mode*	*Mean*	*Median*	*Mode*	*Mean*	*Median*	*Mode*
-0.5	0.5	3.6905	3	3	3.6973	3	3	3.8457	3	3
-0.3	0.5	2.667	2	2	2.6716	2	2	2.726	2	2
-0.1	0.5	3.6908	3	3	3.7174	3	3	3.8451	3	3
0.1	0.5	2.8595	2	2	2.8672	2	2	2.9578	2	2
0.3	0.5	3.6505	3	2	3.6905	3	2	3.7921	3	3
0.5	0.5	2.9623	2	2	2.9767	2	2	3.0582	2	2
μ_A		0.5			0.7			0.9		
ϕ_1	θ_1	*Mean*	*Median*	*Mode*	*Mean*	*Median*	*Mode*	*Mean*	*Median*	*Mode*
-0.5	0.5	4.0924	3	3	4.3157	3	3	4.416	3	3
-0.3	0.5	2.9292	2.5	1	3.2888	3	3	3.5862	3	1
-0.1	0.5	4.0828	3	3	4.2391	3	3	4.3214	3	3
0.1	0.5	3.1661	3	3	3.3904	3	3	3.5123	3	3
0.3	0.5	3.9727	3	3	4.092	3	3	4.1138	3	3
0.5	0.5	3.2313	3	3	3.3762	3	3	3.4381	3	3

Table A.51: Average AR(p) order chosen by Sieve Bootstrap for the underlying process based on AIC Criterion. Values in reference to Table A.43 where the underlying process was ARMA(1, 1) with $\theta_1 = 0.5$.

Appendix B

Matlab Code

```
% @ param: -) mu_0: Value of mean under Null
%          -) y: The data set for which we calculate the statistic
%          -) p: order of the AR process
%          -) n_max: index upto which we calculate the test statistic -> must be <=
%             numpoints
function TestStat = GombayTestMean(mu_0,y,p,n_max,CV)

y = reshape(y,length(y),1);

%% make X
% without subtracting mean from it.
Y = [0;y];
X = zeros(n_max,p);

for i = 1:p
    X(:,i) = Y(1:n_max);
    Y = [y(i);Y];
end

%% make M
Y = [0;y];
M = zeros(n_max,p);

for i = 1:n_max
    M(i,:) = Y(1:p)';
    Y = [y(i);Y];
end
%% calculate phi (Start of Loop)

i = p + 10; % we ignore the first ten points as recommended by Prof. Edit Gombay
TestStat = 0;

   while i <= n_max && TestStat < CV  % this loop is run for each subset of the data k


      Xh = X(1:i,:) - mu_0 * ones(i,p);
      Zh = y(1:i) - mu_0 * ones(i,1);
      phi_k =  (Xh'*Xh)\(Xh'*Zh);

      sigma_k = (  sum( (Zh - (M(1:i,:) - mu_0 * ones(i,p)) * phi_k).^2 )  )  /i;

      Sum_S = cumsum( y(1:i) - mu_0 * ones(i,1) - ((M(1:i,:) - mu_0 * ones(i,p)) * phi_k)↙
);

      D = (Sum_S(end)*ones(i-2,1) - Sum_S(2:end-1))./(sqrt(length(y)*sigma_k));

      TestStat = max(D);
      i= i+1;
   end
end
```

```
% @param: x = Original data set on which we have to apply bootstrap data.
%         bs = Number of bootstrap samples that have to be generated.
function [SBSamples] = SieveBootstrap(x,bs)
    [temp1,temp2,temp3,temp4,temp5, Order] = invest1(x,ceil(length(x)/2));    % invest1↙
function is used to estimate optimal AIC Order of the specified data
    %Order(2) contains AIC
    % Order(3) contains BIC
    NumPoints = length(x);
    O = max(Order(1,2),1); % Selected Order Finally chosen

    a=ar(x, O, 'yw'); % Step 2 i.e. Fit Yule Walker
    error=resid(a,x); % Step 3  -> Get the residuals
    centerResidual=error-mean(error); % Step 4 -> Demean the residuals
    % Step 5 that is resampling the residuals
    bsResiduals = randsample(centerResidual, bs, true)';
    for i = 1:NumPoints-1
         bsResiduals = [bsResiduals ; randsample(centerResidual, bs, true)'];
    end
    X = sim(a,bsResiduals(1:NumPoints,1));
    for j = 2:bs
       X = [X sim(a,bsResiduals(1:NumPoints,j))];
    end

    SBSamples = X;

end
```

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% The following function generates the ARMA(p,q) data for the model with
% the specified coefficients.
% @ param: NumPoints - Number of points for the series you want to generate. Try to keep
it greater than  p + q
%          NumSamples - Number of Samples  required
%          ARVector - The vector representing AR(p) coefficients.
%          MAVector - The moving average component of the process.
%          mu - mean of the process to be generated
%          sigma - standard deviation of the process to be generated
%          (optional) df - degrees of freedom for t-distribution, Currently
%                          set for df >= 4
%@ return:  An ARMA(p,q) process with N(0,1) errors. The returned matrix
%           will have dimensions [NumPoints x NumSamples]
% Note: The first 20-30 points will be discarded.
function Y = simARMA(NumPoints, NumSamples, ARVector, MAVector, mu, sigma, df)

A = [1 -ARVector];
B = [];
C = [1 MAVector];
D = [1];
F = [];

% if nargin >= 7
%    randn('seed',seed);
% end

ptsToDiscard = 20; % Initial Number of points that we discard when generating randomly

    if ((nargin >= 7) && (df >= 5))
        errors = mu*ones(NumPoints + ptsToDiscard,NumSamples) + sigma*trnd(df,NumPoints +
ptsToDiscard,NumSamples);
    else
        errors = mu*ones(NumPoints + ptsToDiscard,NumSamples) + sigma*randn(NumPoints +
ptsToDiscard,NumSamples);
    end

y=filter([1 MAVector],[1 -ARVector],errors);
Y=y( (ptsToDiscard+1):(NumPoints+ptsToDiscard),: );

end
```

Appendix C

Data Sets

C.1 EUR/USD Exchange Rate Data (2002 - 2007)

Data Source: Forex Capital Markets
http://www.FXCM.com

Date	EUR/USD Rate	Date	EUR/USD Rate	Date	EUR/USD Rate
04-Jan-02	0.8925	13-Sep-02	0.9814	23-May-03	1.1766
11-Jan-02	0.8844	20-Sep-02	0.9806	30-May-03	1.1699
18-Jan-02	0.865	27-Sep-02	0.9789	06-Jun-03	1.1862
25-Jan-02	0.8613	04-Oct-02	0.9864	13-Jun-03	1.1604
01-Feb-02	0.8727	11-Oct-02	0.9717	20-Jun-03	1.1426
08-Feb-02	0.8733	18-Oct-02	0.9762	27-Jun-03	1.1486
15-Feb-02	0.8752	25-Oct-02	0.9965	04-Jul-03	1.1296
22-Feb-02	0.8653	01-Nov-02	1.0128	11-Jul-03	1.1273
01-Mar-02	0.8744	08-Nov-02	1.0092	18-Jul-03	1.1508
08-Mar-02	0.8823	15-Nov-02	0.9969	25-Jul-03	1.126
15-Mar-02	0.8769	22-Nov-02	0.9943	01-Aug-03	1.13
22-Mar-02	0.8713	29-Nov-02	1.0094	08-Aug-03	1.1255
29-Mar-02	0.8787	06-Dec-02	1.0226	15-Aug-03	1.0889
05-Apr-02	0.8798	13-Dec-02	1.0267	22-Aug-03	1.0988
12-Apr-02	0.8908	20-Dec-02	1.0436	29-Aug-03	1.1105
19-Apr-02	0.9013	27-Dec-02	1.0415	05-Sep-03	1.1286
26-Apr-02	0.9159	03-Jan-03	1.0576	12-Sep-03	1.1359
03-May-02	0.9135	10-Jan-03	1.0664	19-Sep-03	1.1477
10-May-02	0.9205	17-Jan-03	1.0827	26-Sep-03	1.1568
17-May-02	0.9207	24-Jan-03	1.0763	03-Oct-03	1.1801
24-May-02	0.9328	31-Jan-03	1.0826	10-Oct-03	1.1672
31-May-02	0.9432	07-Feb-03	1.0791	17-Oct-03	1.1794
07-Jun-02	0.9456	14-Feb-03	1.0763	24-Oct-03	1.158
14-Jun-02	0.9715	21-Feb-03	1.0798	31-Oct-03	1.1533
21-Jun-02	0.9906	28-Feb-03	1.1003	07-Nov-03	1.178
28-Jun-02	0.9729	07-Mar-03	1.074	14-Nov-03	1.1915
05-Jul-02	0.991	14-Mar-03	1.0522	21-Nov-03	1.1989
12-Jul-02	1.0144	21-Mar-03	1.0779	28-Nov-03	1.2164
19-Jul-02	0.9869	28-Mar-03	1.0729	05-Dec-03	1.2292
26-Jul-02	0.9864	04-Apr-03	1.0751	12-Dec-03	1.2376
02-Aug-02	0.9698	11-Apr-03	1.088	19-Dec-03	1.2426
09-Aug-02	0.9844	18-Apr-03	1.1034	26-Dec-03	1.2591
16-Aug-02	0.9724	25-Apr-03	1.1227	02-Jan-04	1.2841
23-Aug-02	0.9824	02-May-03	1.1491	09-Jan-04	1.2371
30-Aug-02	0.9818	09-May-03	1.1572	16-Jan-04	1.2586
06-Sep-02	0.9716	16-May-03	1.1827	23-Jan-04	1.2473

Date	EUR/USD Rate		Date	EUR/USD Rate		Date	EUR/USD Rate
30-Jan-04	1.2701		08-Oct-04	1.2474		17-Jun-05	1.2101
06-Feb-04	1.2739		15-Oct-04	1.267		24-Jun-05	1.1947
13-Feb-04	1.2522		22-Oct-04	1.2796		01-Jul-05	1.1958
20-Feb-04	1.2493		29-Oct-04	1.2969		08-Jul-05	1.2041
27-Feb-04	1.2373		05-Nov-04	1.2976		15-Jul-05	1.206
05-Mar-04	1.222		12-Nov-04	1.3024		22-Jul-05	1.2123
12-Mar-04	1.2274		19-Nov-04	1.3293		29-Jul-05	1.235
19-Mar-04	1.2124		26-Nov-04	1.3455		05-Aug-05	1.2435
26-Mar-04	1.2134		03-Dec-04	1.3224		12-Aug-05	1.2163
02-Apr-04	1.2086		10-Dec-04	1.3285		19-Aug-05	1.2283
09-Apr-04	1.1989		17-Dec-04	1.353		26-Aug-05	1.2543
16-Apr-04	1.1835		24-Dec-04	1.3531		02-Sep-05	1.2411
23-Apr-04	1.198		31-Dec-04	1.3043		09-Sep-05	1.2232
30-Apr-04	1.1886		07-Jan-05	1.3102		16-Sep-05	1.2046
07-May-04	1.1879		14-Jan-05	1.3044		23-Sep-05	1.2018
14-May-04	1.1993		21-Jan-05	1.3045		30-Sep-05	1.2123
21-May-04	1.2212		28-Jan-05	1.2872		07-Oct-05	1.2083
28-May-04	1.2287		04-Feb-05	1.2866		14-Oct-05	1.195
04-Jun-04	1.2018		11-Feb-05	1.3067		21-Oct-05	1.2067
11-Jun-04	1.2139		18-Feb-05	1.3241		28-Oct-05	1.1822
18-Jun-04	1.2158		25-Feb-05	1.324		04-Nov-05	1.1727
25-Jun-04	1.232		04-Mar-05	1.3449		11-Nov-05	1.1765
02-Jul-04	1.2411		11-Mar-05	1.3312		18-Nov-05	1.1724
09-Jul-04	1.2447		18-Mar-05	1.2956		25-Nov-05	1.1718
16-Jul-04	1.2099		25-Mar-05	1.2907		02-Dec-05	1.1814
23-Jul-04	1.2023		01-Apr-05	1.2927		09-Dec-05	1.2013
30-Jul-04	1.2287		08-Apr-05	1.2917		16-Dec-05	1.1873
06-Aug-04	1.237		15-Apr-05	1.3063		23-Dec-05	1.1838
13-Aug-04	1.231		22-Apr-05	1.2868		30-Dec-05	1.2152
20-Aug-04	1.2009		29-Apr-05	1.2822		06-Jan-06	1.2131
27-Aug-04	1.206		06-May-05	1.2621		13-Jan-06	1.2136
03-Sep-04	1.2265		13-May-05	1.2559		20-Jan-06	1.21
10-Sep-04	1.218		20-May-05	1.2579		27-Jan-06	1.2019
17-Sep-04	1.226		27-May-05	1.2222		03-Feb-06	1.1894
24-Sep-04	1.241		03-Jun-05	1.2119		10-Feb-06	1.1932
01-Oct-04	1.2407		10-Jun-05	1.2275		17-Feb-06	1.1867

Date	EUR/USD Rate		Date	EUR/USD Rate		Date	EUR/USD Rate
24-Feb-06	1.2035		03-Nov-06	1.2845		13-Jul-07	1.382
03-Mar-06	1.1915		10-Nov-06	1.2824		20-Jul-07	1.3636
10-Mar-06	1.2192		17-Nov-06	1.3094		27-Jul-07	1.3785
17-Mar-06	1.2035		24-Nov-06	1.3335		03-Aug-07	1.3695
24-Mar-06	1.2117		01-Dec-06	1.3202		10-Aug-07	1.3488
31-Mar-06	1.21		08-Dec-06	1.3082		17-Aug-07	1.3666
07-Apr-06	1.2111		15-Dec-06	1.314		24-Aug-07	1.3621
14-Apr-06	1.2346		22-Dec-06	1.3201		31-Aug-07	1.3767
21-Apr-06	1.2614		29-Dec-06	1.3002		07-Sep-07	1.3876
28-Apr-06	1.2727		05-Jan-07	1.2915		14-Sep-07	1.4089
05-May-06	1.2927		12-Jan-07	1.296		21-Sep-07	1.4258
12-May-06	1.2772		19-Jan-07	1.2913		28-Sep-07	1.4133
19-May-06	1.2723		26-Jan-07	1.2964		05-Oct-07	1.4176
26-May-06	1.292		02-Feb-07	1.3005		12-Oct-07	1.4295
02-Jun-06	1.2639		09-Feb-07	1.3138		19-Oct-07	1.4389
09-Jun-06	1.2638		16-Feb-07	1.3164		26-Oct-07	1.4512
16-Jun-06	1.2509		23-Feb-07	1.3191		02-Nov-07	1.4668
23-Jun-06	1.2789		02-Mar-07	1.3113		09-Nov-07	1.4647
30-Jun-06	1.2814		09-Mar-07	1.3312		16-Nov-07	1.4836
07-Jul-06	1.2645		16-Mar-07	1.3286		23-Nov-07	1.4629
14-Jul-06	1.2694		23-Mar-07	1.3357		30-Nov-07	1.4653
21-Jul-06	1.2747		30-Mar-07	1.3373		07-Dec-07	1.4422
28-Jul-06	1.2874		06-Apr-07	1.353		14-Dec-07	1.4356
04-Aug-06	1.2729		13-Apr-07	1.3592		21-Dec-07	1.4714
11-Aug-06	1.2828		20-Apr-07	1.3648			
18-Aug-06	1.2755		27-Apr-07	1.3595			
25-Aug-06	1.2836		04-May-07	1.3531			
01-Sep-06	1.2676		11-May-07	1.351			
08-Sep-06	1.2655		18-May-07	1.3445			
15-Sep-06	1.2785		25-May-07	1.3445			
22-Sep-06	1.2678		01-Jun-07	1.3367			
29-Sep-06	1.2592		08-Jun-07	1.3379			
06-Oct-06	1.2507		15-Jun-07	1.3464			
13-Oct-06	1.2617		22-Jun-07	1.3534			
20-Oct-06	1.2733		29-Jun-07	1.3623			
27-Oct-06	1.2715		06-Jul-07	1.3786			

C.2 Quarterly U.S. Bankruptcy Filings

Source: American Bankruptcy Institute
http://www.abiworld.org/

Date	No. of Bankruptcies		Date	No. of Bankruptcies
Mar-94	13858		Jun-02	9695
Jun-94	13617		Sep-02	9433
Sep-94	12878		Dec-02	9500
Dec-94	12021		Mar-03	8814
Mar-95	13123		Jun-03	9331
Jun-95	12216		Sep-03	8446
Sep-95	12648		Dec-03	8294
Dec-95	12891		Mar-04	10566
Mar-96	13388		Jun-04	8249
Jun-96	13992		Sep-04	7574
Sep-96	13198		Dec-04	7778
Dec-96	12887		Mar-05	8063
Mar-97	13831		Jun-05	8736
Jun-97	13991		Sep-05	9476
Sep-97	13456		Dec-05	12798
Dec-97	12653		Mar-06	4086
Mar-98	12410		Jun-06	4858
Jun-98	11552		Sep-06	5284
Sep-98	10346		Dec-06	5586
Dec-98	9888		Mar-07	6280
Mar-99	9180		Jun-07	6705
Jun-99	10378		Sep-07	7167
Sep-99	8986		Dec-07	7985
Dec-99	9020		Mar-08	8713
Mar-00	9456		Jun-08	9743
Jun-00	9243		Sep-08	11504
Sep-00	8211		Dec-08	12901
Dec-00	8413		Mar-09	14319
Mar-01	10005		Jun-09	16014
Jun-01	10330		Sep-09	15177
Sep-01	9537		Dec-09	15020
Dec-01	10013		Mar-10	14607
Mar-02	9775			

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