Modelling the dynamics of commodity prices for investment decisions under uncertainty

by

Shan Chen

A thesis
presented to the University of Waterloo
in fulfilment of the
thesis requirement for the degree of
Doctor of Philosophy
in
Economics

Waterloo, Ontario, Canada, 2010

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

This thesis consists of three essays on commodity-linked investment decisions under uncertainty. Specifically, the first essay investigates whether a regime switching model of stochastic lumber prices is a better model for the analysis of optimal harvesting problems in forestry than a more traditional single regime model. Prices of lumber derivatives are used to calibrate a regime switching model, with each of two regimes characterized by a different mean reverting process. A single regime, mean reverting process is also calibrated. The value of a representative stand of trees and optimal harvesting prices are determined by specifying a Hamilton-Jacobi-Bellman Variational Inequality, which is solved for both pricing models using a fully implicit finite difference approach. The regime switching model is found to more closely match the behaviour of futures prices than the single regime model. In addition, the optimal harvesting model indicates significant differences in terms of land value and optimal harvest thresholds between the regime switching and single regime models.

The second essay investigates whether convenience yield is an important factor in determining optimal decisions for a forestry investment. The Kalman filter method is used to estimate three different models of lumber prices: a mean reverting model, a simple geometric Brownian motion and the two-factor price model due to Schwartz (1997). In the latter model there are two correlated stochastic factors: spot price and convenience yield. The two-factor model is shown to provide a reasonable fit of the term structure of lumber futures prices. The impact of convenience yield on a forestry investment decision is examined using the Schwartz (1997) long-term model which transforms the two-factor price model into a single factor model with a composite price. Using the long-term model an optimal harvesting problem is analyzed, which requires the numerical solution of a Hamilton-Jacobi-Bellman

equation. I compare the results for the long-term model to those from single-factor mean reverting and geometric Brownian motion models. The inclusion of convenience yield through the long-term model is found to have a significant impact on land value and optimal harvesting decisions.

The third essay investigates the dynamics of recent crude oil prices by comparing and contrasting three different stochastic price models, which are a two-state regime switching model, a two-factor model analyzed in Schwartz (1997) and a two-factor model examined in Schwartz and Smith (2000). Prices of long-term crude oil futures contracts are used to calibrate and estimate the model parameters. The performances of the two-factor models are comparable in terms of fitting the market prices of the long-term oil futures contracts and more closely match the behavior of oil futures prices than the regime switching model.

Acknowledgements

First and foremost, I would like to express my sincere gratitude to my supervisors, Dr. Margaret Insley and Dr. Tony Wirjanto, for their constant support, motivation, guidance and encouragement throughout my doctoral study at the University of Waterloo and the writing of this dissertation. Their knowledge, experience and insights have been very influential in my research. I have been very lucky to be their student and having the opportunity to work with them has been wonderful.

My special thanks also go to the other committee members, Dr. Dinghai Xu, Dr. Alain-Desire Nimubona, Dr. Yuying Li and Dr. Pierre Lasserre for their precious advices and insightful ideas in developing my dissertation.

Last but not least, I want to thank my parents and friends, for bringing me endless encouragement and happiness during the past few years.

Dedication

To my parents, Xiaoyan Liu and Qimeng Chen, for their unconditional love.

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Chapter 1

Introduction

The past several decades have witnessed an increased interest by academics, investment professionals and others in commodity-related risk management and asset valuation. Markets for major commodities, such as lumber, crude oil and electricity, tend to be highly volatile with prices determined in global markets. It has long been recognized that a significant misallocation of resources can occur when management decisions are taken without reference to market volatility. An important research question raised is how best to characterize this uncertainty and what impact different assumptions about the nature of this uncertainty have on commodity-linked investment valuation.

The main purpose of this dissertation is to investigate the valuation of commodity-related investments and the derivation of the appropriate decision making rules. A key ingredient to valuing a commodity linked investment is the choice of a stochastic model for the commodity's price. Using the principle of equivalent risk-neutral valuation, the price process can be expressed in the appropriate risk neutral measure which is termed a Q-measure (see fu Huang and Litzenberger (1990)). One major component of this research is to investigate several model specifications under the Q-measure for both lumber and crude oil prices and examine the impacts

of different price models on investment decisions and valuations. In particular, all the models describing the behavior of commodity price processes examined in the dissertation are expressed in the risk-neutral world and the corresponding model parameters are calibrated or estimated using commodity derivatives, such as commodity futures or commodity options. Since commodity derivatives are actively traded in the market and play an important role for commodity-related risk management, another motivation of this dissertation is to use the informational content of lumber and oil derivatives prices in model calibration or estimation.

In devising better models for commodity prices we are faced with a tradeoff between increased realism through the addition of more stochastic factors, jumps, etc., and the added complexity and difficulty of solving for the value of related contingent claims. The focus of this dissertation is to find an approach to modeling commodity prices which, while adequately rich, still allows for a relatively simple solution of the related contingent claims using standard methods. In other words, the research seeks to find parsimonious models which can capture the main properties of commodity prices and are easily incorporated into the real option valuation problem.

One of my main interests in the dissertation is the analysis of forest industry investments. Forest products, including logs, lumber, and paper, are traded worldwide and Canada is a major player in this market, accounting for 14% of the value of world forest product exports in 2006.[1] Forest products are a significant component of Canada's balance of trade, with the corresponding exports amounting to $29 billion (Canadian) in 2006, which was 5.4% of total exports of goods and services. Note that this is down from a peak of $43 billion in 2000. The valuation of a forested land is an active research area in the academic literature. A more than thirty-year-long strand of this literature emphasizes the importance of valu-

[1]Source: FAOstat database, Food and Agricultural Organization of the United Nations, *http://faostat.fao.org/site/381/DesktopDefault.aspx?PageID=381*

ing managerial flexibility in the context of irreversible harvesting decisions when forest product prices are volatile relative to harvesting costs.[2] Furthermore, there has been a trend over the last two decades to view commercial timber lands as a suitable asset to diversify the portfolios of large investors. Institutional investors in the United States have significantly increased their holdings of timberlands, giving an added motivation for a better understanding of timber price dynamics and investment valuation.[3] An ongoing challenge is how best to model the dynamics of timber prices in determining optimal harvesting strategies and in estimating the value of forested lands. The model chosen to describe timber prices can have a significant effect on optimal harvesting decisions and land valuation. The issue is of importance to forest management, whether on publicly or privately owned land. Therefore, chapter II and III of my dissertation examine the performances of several promising stochastic models for describing lumber price dynamics and the effects of different model specifications on forestry investment decisions and land valuation.

In both of chapter II and chapter III, I use calibrated or estimated lumber price models to analyze a forestry investment problem. Harvesting a stand of trees generates revenue to the owner from log sales, but also incurs several costs such as harvesting costs and the loss of any additional timber volume that would accrue if the trees were allowed to grow for more periods. Given the stochastic prices, if the harvest is delayed until next period, the timber price may be higher or lower than in the current period. Holding the right to harvest, the land owner can delay cutting the trees until the prices are high enough to generate positive net revenue. In general, the opportunity to harvest a stand of trees may be treated as a real option, similar to an American call option, which can be exercised at any time before the expiration date. The rationale behind using real options to model real asset investment is that in the context of market uncertainty, in particular asset

[2]Hool (1966); Lembersky and Johnson (1975) are examples of some of the earlier literature.

[3]See Global Institute of Sustainable Forestry (2002) and Caulfield and Newman (1999) for a discussion of this shift in ownership.

price uncertainty, there is some value in having the possibility to delay the decision. The real options approach explicitly incorporates such managerial flexibility. In contrast a simple expected Net Present Value (NPV) approach, typically ignores the importance of options embedded in the investment decision which leads to a non-optimal solution.

Each of chapter II and chapter III begins with the calibration or estimation of specific stochastic lumber price models and the corresponding model fit is analyzed. In the second part of each chapter, the calibrated or estimated lumber price models are used to solve a representative optimal tree harvesting problem. Specifically, for Chapter II, two model specifications, a regime switching mean reverting model and a traditional one-factor mean reverting model, are calibrated using both lumber futures and lumber options. These models are compared in terms of fitting the market data for lumber derivatives. In the second part of this chapter, an optimal tree harvesting problem over infinite time is examined. For Chapter III, I investigate the impact on a forestry investment decision of modeling convenience yield. Three different stochastic models of lumber prices are estimated and compared: a mean reverting model, a geometric Brownian motion and the two-factor model analyzed in Schwartz (1997). The impact of convenience yield on an optimal tree harvesting problem is examined using the Schwartz (1998) long-term model which transforms the two-factor price model into a single factor model with a composite price.

In Chapter II, I apply a regime switching model to lumber prices and investigate whether it represents an improvement over a single regime model that has been previously used in the forestry literature. This task is motivated by two factors. First, a regime switching model first proposed by Hamilton (1989) appears to be a promising model for commodity prices. For example, Deng (2000), de Jong (2005), Chen and Forsyth (2008) all examine empirical models of regime switching in commodity prices (electricity or natural gas prices) and have shown promising results for their empirical applications. Second, the lumber industry is characterized

by periods of boom and bust which might point to the existence of two regimes.

The reason for using the Schwartz (1997) two-factor model in Chapter III for describing lumber prices is that for storable commodities and commodities that serve as inputs to production like lumber and oil, convenience yield plays an important role in the price formation. Convenience yield is said to arise from the benefit that producers obtain from physically holding inventories. This represents a benefit not available to individuals holding a futures or forward contract. Since convenience yield is very much like the dividend obtained from holding a company's stock, it also helps to explain the term structure of commodity futures prices. The term structure of futures prices is defined as the relationship between the spot price and the corresponding futures prices for any delivery date. It conveys useful information for hedging or investment decisions, because it synthesizes the information available in the market and reflects the operators' expectations concerning the future. The futures price can be greater or less than the commodity spot price, depending on the relationship between the (net) convenience yield[4] and the risk-free interest rate. This is explained by the cost of carry pricing model in which the forward/futures price is expressed as a function of the spot price and the cost of carry.[5] It is important to model convenience yield in order to make use of the valuable information conveyed by the commodity futures prices and reproduce the term structure of futures prices as accurately as possible.

In both Chapters II and III, the optimal choice of harvesting date for an even-aged stand of trees and the value of the option to harvest are modeled as a complementarity problem. The corresponding Hamilton-Jacobi-Bellman (HJB) equation characterizing the value of the option to harvest a stand is solved numerically using a combination of the fully implicit finite difference method, the semi-Lagarangian

[4]Net convenience yield is defined as the benefit of holding inventory minus physical storage costs. It is negative if the storage expense is higher. For simplicity, convenience yield mentioned in the rest of this document refers to net convenience yield.

[5]Cost of carry is defined as the physical storage cost plus the forgone interest. See Pindyck (2001)

method and the penalty method, since there is no analytical solution to this type of tree cutting problem. The value of the stand of trees and critical prices at which harvesting would be optimal are examined for various cases.

Energy prices and economic growth have been closely linked for decades. The development of energy derivatives markets has increased the ability of investors to hedge energy risk. After a period of relative stability in the 1990's, the post 2003 period has shown a marked change. Specifically, from 2003 until mid-2008, the world crude oil price rose from about \$35/barrel to over \$140/barrel. Then in September 2008 with the financial crisis initiated by the collapse of Lehman Brothers, the world oil price decreased sharply to around \$40/barrel followed by another sharp increase in 2009. In chapter IV, this thesis attempts to model recent patterns in world oil prices. A regime switching model based on the Schwartz (1997) single-factor model is proposed and its performance in terms of explaining the term structure of recent oil futures prices is compared with the widely used two-factor models proposed and analyzed in Schwartz (1997) and Schwartz and Smith (2000).

The choice of these three specifications for modeling recent crude oil price movements is motivated by the following considerations. First of all, Schwartz (1997) compares and contrasts one, two and three-factor models for explaining commodity prices including crude oil and shows that both two and three-factor models[6] are able to explain the term structure of commodity futures prices and generate lower estimation errors compared with a one-factor model. The regime switching model proposed in this chapter is based on the one-factor model applied in Schwartz (1997). By allowing the parameters in the one-factor model to be regime dependent, I wish to determine whether this revised one-factor model is rich enough to capture the main properties of the term structure of oil futures prices compared with the multi-factor models analyzed in Schwartz (1997). Secondly, the two-factor models

[6]Since both two-factor and three-factor models analyzed in Schwartz (1997) are empirically similar (see Schwartz (1998)), in this paper, I focus on the two-factor model which represents a considerable computational advantage in terms of oil-related investment valuation.

analyzed in Schwartz (1997) and Schwartz and Smith (2000) have been successfully used to model crude oil prices in the literature and some oil-related investments are valued based on these models. Given the very different oil price dynamics shown in recent years, it is worth exploring whether these two-factor models can still explain the main features of the recent oil price process. Furthermore, due to the lack of long-term crude oil futures data, Schwartz (1997) uses Enron long-term forward data. Since futures contracts are more regulated and standard than forward contracts, in this thesis, I use the available long-term[7] futures data to further explore the performance of both two-factor models.

In summary this dissertation contributes to our understanding of the modeling of commodity prices making use of the information contained in the prices of commodity derivatives, especially commodity futures prices. The thesis also contributes to our understanding of the valuation of commodity linked investments and the impact of key variables, such as volatility and convenience yield, on optimal decisions. This thesis is the first work to use a regime switching model for lumber prices and examine the impact on optimal decisions in forestry. It is also the first to apply a two-factor model of stochastic convenience yield and study its impact on a tree harvesting decision. Further, the thesis makes some methodological contributions in the implementation of real options models for natural resource investments. These include:

- Implementing numerical schemes for solving an optimal tree harvesting problem assuming a regime switching stochastic price process.
- Demonstrating the calibration, using derivatives price data, of regime switching price models for lumber prices and oil prices.

The main results of the thesis are briefly summarized below.

[7]The longest maturity of the chosen futures and forward contracts examined in Schwartz (1997) are less than 2 years and 9 years respectively. The longest maturity of the chosen oil futures contract in this chapter is up to 6 year.

- Regime switching models show promise as parsimonious models able to describe the dynamics of lumber prices. In an empirical example, price is shown to switch between two regimes, with each regime composed of a different mean reverting price process. The regime switching model generates reasonable values for the option to harvest a hypothetical stand of trees. The value of the stand and optimal harvesting prices are shown to differ depending on the regime that price currently resides in. The estimated stand values and optimal harvesting prices are significantly different than those calculated using a traditional single regime model.

- Convenience yield is found to play an important role in lumber price dynamics. A two factor price model with stochastic convenience yield and spot price is found to fit the lumber futures curve reasonably well. In the empirical tree harvesting example, the inclusion of convenience yield increases significantly the minimum stand age at which harvesting should occur. The estimated value of the stand of trees is also affected. It follows that forest owners and investors should take the dynamics of the convenience yield into account when making the forestry-related investment decisions.

- Multi-factor stochastic price models are found to be able to match oil futures prices reasonably well even given the increased volatility in world oil markets since 2003. I conclude that these models are good candidates for use in valuing oil linked investments.

Chapter 2

Regime switching in stochastic models of commodity prices: an application to an optimal tree harvesting problem

2.1 Introduction

The modelling of optimal tree harvesting and the valuation of land devoted to commercial timber harvesting is an active research area in the academic literature. An ongoing challenge is how best to model the dynamics of timber prices in determining optimal harvesting strategies and in estimating the value of forested lands. Over the past two decades some researchers have modeled lumber price as an exogenous factor described by a stochastic differential equation (see Thomson (1992); Plantinga (1998); Morck et al. (1989); Clarke and Reed (1989) for example). Others have used stand value (price of wood times quantity of wood), as a stochastic factor,

abstracting from physical tree growth, such as in Alvarez and Koskela (2007) and Alvarez and Koskela (2005). The model chosen to describe timber prices can have a significant effect on optimal harvesting decisions and land valuation. The issue is therefore of importance to forest management, whether on publicly or privately owned land. There has been a trend over the last two decades to view commercial timber lands as a suitable asset to diversify the portfolios of large investors. Institutional investors in the United States have significantly increased their holdings of timberlands, giving an added motivation for a better understanding of timber price dynamics and investment valuation.[1]

Several specifications have been proposed in the literature for modeling stochastic lumber prices, including geometric Brownian motion (GBM), mean reversion and jump processes. A number of researchers have solved optimal tree harvesting problems analytically, assuming prices follow GBM.[2] Some researchers have found that mean reversion rather than GBM provides a better characterization of the dynamics of the lumber prices (Brazee et al. (1999)). For commodities in general, it has been argued that mean reversion in price makes sense intuitively since any significant upturn in price will bring on additional supplies. Unfortunately it is difficult to conclude definitively whether the price of any particular commodity is stationary or not. As is noted in Insley and Rollins (2005) many different statistical tests exist, but none has been shown to be uniformly most powerful. In optimal tree harvesting problems, the assumption of a price process other than GBM generally requires a numerical solution. This can present significant challenges particularly if the researcher chooses to model the growing forest stand in a realistic fashion over multiple rotations or cutting cycles.

An added complication is that for many commodities, price appears to be characterized by discrete jumps. A recent insight in the literature suggests that instead

[1]See Global Institute of Sustainable Forestry (2002) and Caulfield and Newman (1999) for a discussion of this shift in ownership.

[2]Examples are Clarke and Reed (1989) and Yin and Newman (1997).

of modeling jumps, we may consider regime switching models, initially proposed by Hamilton (1989), to better capture the main characteristics of some commodity prices. Using a regime switching model, the observed stochastic behavior of a specific time series is assumed to be comprised of several separate regimes or states. For each regime or state, one can define a separate underlying stochastic process. The switching mechanism between each regime is typically assumed to be governed by an unknown random variable that follows a Markov chain. Various factors may contribute to the random shift between regimes, such as changes in government policies and weather conditions.

In this chapter I investigate whether a regime switching model is a good alternative for modeling stochastic timber prices. For simplicity I assume the existence of two states or regimes. In line with Chen and Forsyth (2008), I calibrate a regime switching model with timber price as the single stochastic factor which follows a different mean reverting process in each of two regimes. I compare this model (denoted the RSMR model) with a single regime mean reverting model (denoted the traditional mean reverting, or TMR, model) which has been used previously in the literature. For parameter calibration, these two models are expressed in the risk-neutral world and the corresponding parameters are calibrated using the prices of traded lumber derivatives, i.e. lumber futures and options on lumber futures. A benefit of calibrating model parameters in this way is that the parameters obtained are risk adjusted so that a forest investment can be valued using the risk-free interest rate.

In the second part of this chapter I use the calibrated RSMR and TMR models to solve an optimal harvesting problem. The optimal choice of harvesting date for an even-aged stand of trees and the value of the option to harvest are modeled as a Hamilton-Jacobi-Bellman variational inequality which is solved numerically using a fully implicit finite difference method. The approach is similar to that used in Insley and Lei (2007), except that the model must accommodate the different

regimes. I use the same cost and timber yield estimates as in Insley and Lei (2007) and hence I am able to compare my results with theirs.[3]

This chapter makes a methodological contribution to the literature. It demonstrates the numerical solution of a dynamic optimization problem in a natural resources context under the assumption of a regime switching stochastic state variable. In the future it is hoped that this methodology may be usefully applied to other types of natural resource investment problems, which are often sufficiently complex that closed-form solutions are unavailable. This chapter also makes an empirical contribution in the investigation of the dynamics of lumber prices. To my knowledge the parameterization of risk-adjusted lumber price models using lumber derivatives prices has not been done previously in the literature. Although I am limited by the short maturity dates of traded lumber futures, I find that the regime switching model shows promise as a parsimonious model of timber prices that can be incorporated into problems of forestry investment valuation using standard numerical solutions techniques. In the concluding section I discuss how this and other limitations of the current chapter point toward avenues for future research.

The remainder of the chapter will be organized as follows. Section 2.2 presents a brief literature review. Section 2.3 provides descriptive statistics and preliminary tests on a lumber price time series. Section 2.4 specifies the lumber price models that will be used in our analysis and details the methodology for calibrating the parameters of these models. Section 2.5 provides the results of the calibration. Section 2.6 specifies the forestry investment problem and its numerical solution. Section 2.7 uses the regime switching and single regime price models to solve for the optimal harvesting time and land value in a tree harvesting problem. Section 2.8 provides some concluding comments.

[3]In Insley and Lei (2007) parameter estimates of the price process were obtained by applying ordinary least squares on historical lumber price data only.

2.2 Modeling commodity prices: An overview of selected literature

Stochastic models of commodity prices play a central role for commodity-related risk management and asset valuation. As noted in Schwartz (1997), earlier research into valuing investments contingent on stochastic commodity prices generally adopted an assumption of geometric Brownian motion (GBM), $dP = aPdt + bPdz$, where P denotes commodity prices, a and b are constant, dz is the increment of a standard Winner process. This simple process allows the procedures developed for valuing financial options to be easily extended to valuing commodity based contingent claims.

Schwartz (1997) and Baker et al. (1998), among others, have emphasized the inadequacy of using GBM to model commodity prices. Under GBM the expected price level grows exponentially without bound. In contrast there is evidence that the real prices of many natural resource-based commodities have shown little upward trend. This is explained by the presence of substitutes as well as improvements in technology to harvest or extract a resource. In addition if a commodity's spot price is assumed to follow GBM, it can be demonstrated using Ito's lemma that the futures price will also follow GBM and both spot and futures prices will have the same constant volatility, (Geman, 2005). However, for most commodities, the volatility of futures prices decreases with maturity, so that the single factor log-normal model such as GBM is not consistent with reality (Pilipovic, 2007, page 233-234). In the literature on optimal tree harvesting, early papers adopting the GBM assumption include Reed and Clarke (1990), Clarke and Reed (1989), Yin and Newman (1995), and Morck et al. (1989).

It is not unreasonable to expect that the workings of supply and demand will result in commodity prices that exhibit some sort of mean reversion. There is also empirical research that supports this claim. For example Bessembinder et al. (1995)

find support for mean-reversion in commodity prices by comparing the sensitivity of long-maturity futures prices to changes in spot prices. One possible choice of mean reverting model is a common variation of Ornstein-Uhlenbeck process:

$$dP = \alpha(K - P)dt + \sigma P dz. \tag{2.1}$$

α is a constant and referred to as the speed of mean reversion. K represents the (constant) long run equilibrium price that P will tend towards. σ is a constant and dz is the increment of a Wiener process. The conditional variance of P depends on the level of P, thereby preventing P from becoming negative.

This process is adopted in Insley and Rollins (2005) and Insley and Lei (2007) to represent lumber prices in an optimal tree harvesting problem. Other optimal harvesting papers to adopt variations on this mean reverting process include Plantinga (1998) and Gong (1999). Mean reverting processes have also been used in modeling prices for oil, electricity, copper, and other minerals (see Cortazar and Schwartz (1994), Dixit and Pindyck (1994), Pilipovic (2007), Smith and McCardle (1998) and Lucia and Schwartz (2002) for example).

The mean reverting model of Equation (2.1), while an improvement over GBM, is not entirely satisfactory. It can be shown that under this model the implied volatility of futures prices decreases with maturity, which is a desirable property for modelling commodity prices. However volatility tends to zero for very long maturities, which is not consistent with what is observed in practice. In addition this model presumes a constant long run equilibrium price (K), when in reality K may be better characterized as a stochastic variable. Schwartz and Smith (2000) propose a two-factor model in which the equilibrium price level is assumed to evolve according to GBM and the short-term deviations are expected to revert toward zero following an Ornstein-Uhlenbeck process. In another variation, a commodity's convenience yield is modelled as additional stochastic factor which is assumed to

follow a MR process. Schwartz (1997) also develops a three-factor model with stochastic price, convenience yield and interest rate. Alternative versions of multi-factor models can be derived through variation of a number of dimensions. However the more factors incorporated into the model, the more complicated is the solution of the resulting partial differential equation that describes the value of contingent claims on the commodity.

A simple mean reverting model of price also ignores the presence of jumps. Saphores et al. (2002) find evidence of jumps in Pacific North West stumpage prices in the U.S. and demonstrate at the stand level that ignoring jumps can lead to significantly suboptimal harvesting decisions for old growth timber.

In devising better models for commodity prices we are faced with a tradeoff between increased realism through the addition of more stochastic factors, jumps, etc., and the added complexity and difficulty of solving for the value of related contingent claims. The optimal tree harvesting problem has the further complication that the asset (a stand of trees) is growing and being harvested over multiple rotations. The timing of harvest and hence the age of the stand depend on price, so that stand age is also stochastic. It is desirable to find an approach to modeling timber prices which, while adequately rich, still allows for the solution of the related contingent claim using standard approaches. It is towards this end that I investigate a regime switching model. The regime switching model with two regimes can readily be solved with a finite difference numerical approach.

Jumps in commodity prices are often driven by discrete events such as weather, disease, or economic booms and busts which may persist for months or years. Therefore the typical continuous time models with isolated and independent jumps may not provide a good description of stochastic commodity prices. The Markov regime switching (RS) model first proposed by Hamilton (1989) is a promising model for commodity prices. In a RS model, spot prices can jump discontinuously between different states governed by state probabilities and model parameters. The

RS model can be used to capture the shifts between "abnormal" and "normal" equilibrium states of supply and demand for a commodity.

Versions of the RS model have previously been applied to the investigation of business cycle asymmetry in Hamilton (1989) and Lam (1990), heteroscedasticity in time series of asset prices in Schwert (1996), the effects of oil prices on U.S. GDP growth in Raymond and Rich (1997). RS specifications for modeling stochastic commodity prices are studied in Deng (2000) and de Jong (2005) for electricity prices and in Chen and Forsyth (2008) for natural gas prices. Deng (2000) shows that by incorporating jumps and regime switching in modeling electricity prices, as opposed to the commonly used GBM model, the values of short-maturity out-of-the-money options approximate market prices very well. de Jong (2005) indicates that RS models are better able to capture the market dynamics than a GARCH(1,1) or Poisson jump model. Chen and Forsyth (2008) show that the RS model outperforms traditional one-factor MR model by solving the gas storage pricing problem using numerical techniques.

In this chapter, I examine the application of a RS model to lumber prices to investigate whether it represents an improvement over a single regime model that has been used previously in the forestry literature. I will use the prices of lumber derivatives to calibrate the parameters of the price process in each of two regimes, and compare with the results of assuming a single regime. Allowing for two regimes may be thought of as a generalization of the more restrictive one regime case. The two regimes may be seen as representing two distinct sets of parameter values, perhaps reflecting good and bad times, in which the volatility, long run equilibrium price level and speed of mean reversion are all able to change. It is hoped that the two regimes may be a rich enough description of timber prices so that the addition of other stochastic factors is unnecessary.

2.3 A first look at lumber markets and prices

Forest products, including logs, lumber, and paper, are traded worldwide and Canada is a major player in this market, accounting for 14% of the value of world forest product exports in 2006.[4] Canada's forest product exports are mainly destined for the United States (over 75% went to the U.S. in 2006) and Canada is the source of over 80% of U.S. lumber imports.[5]

Forest product prices in North America are affected by swings in housing starts and other demand sources, supply factors such as fire and pests that plague forests from time to time, regulatory changes and by the increased integration of forest product markets worldwide. In addition, forest operations in Canada have been severely affected by on-going trade disputes between Canada and the U.S. Forest product prices are almost all quoted in U.S. dollars, which is an added source of volatility for Canadian forest product producers who receive revenue in U.S. dollars but pay silviculture and harvesting costs in Canadian dollars. Participants in forest product markets can hedge some risks by buying or selling futures contracts. Lumber futures contracts with expiry dates for up to one year in the future have been traded on the Chicago Mercantile Exchange (CME) since 1969.

Real weekly spot prices for Canadian lumber are shown in Figure 2.1. Softwood lumber is the underlying commodity of the lumber futures traded at the CME. Periods of boom and bust are evident in the diagram, with the especially difficult time in the industry clearly apparent from mid-2004 onward. This reflects declining lumber prices in the United States as well as the appreciation of the Canadian dollar which rose from 0.772 $U.S./$Cdn in January 2004 to 0.998 $U.S./$Cdn in January 2008. Descriptive statistics for the lumber price time series and its corresponding

[4]Source: FAOstat database, Food and Agricultural Organization of the United Nations, *http://faostat.fao.org/site/381/DesktopDefault.aspx?PageID=381*

[5]Source: Random Lengths, "Yardstick" and Canada's Forests, Statistical Data, Natural Resources Canada, *http://canadaforests.nrcan.gc.ca/statsprofileCanada* (retrieved May 4, 2008).

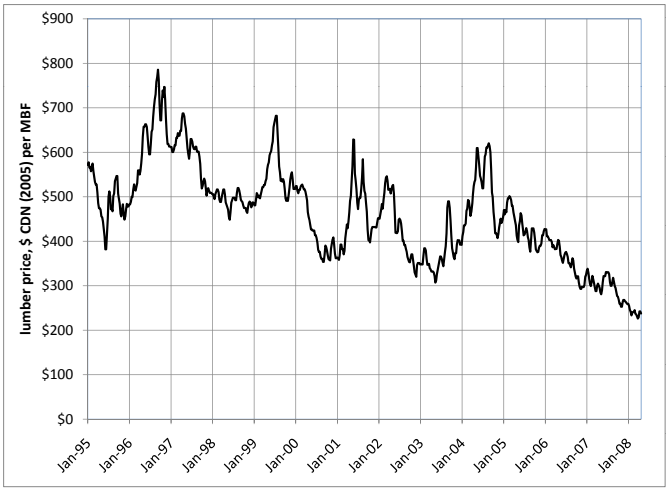


Figure 2.1: Real prices of softwood lumber, Toronto, Ontario. Weekly data from January 6th, 1995 to April 25th, 2008, $Cdn./MBF, (MBF $\equiv$ thousand board feet). Nominal prices deflated by the Canadian Consumer Price Index, base year = 2005. Source: Random Lengths.

Item	Max	Min	Mean	Std. Dev.	Skewness	Kurtosis
Cdn(2003)$/$m^3$	785.6	226.5	459.3	109.6	0.2151	2.711
Weekly Return	653.0 %	-644.5 %	-6.5 %	21.5 %	0.134	4.448

Table 2.1: Descriptive statistics for the lumber price time series (as shown in Figure 1) and its returns, from January 6th, 1995 to April 25th, 2008. The return is the continuously compounded return.

return are provided in Table 4.1. Return is calculated as $\ln(P_t/P_{t-1})$ where P_t referes to price at time t. Weekly data are used, however, the minimum, maximum, and mean returns as well as the standard deviation have been annualized. The returns of the price time series exhibit excess kurtosis, which implies that a pure GBM model is not able to fully describe the dynamics of lumber price process.[6] A formal tests of normality (the Jarque-Bera test) strongly rejects the null hypothesis that return follows a normal distribution.

2.4 Calibration of Lumber Spot Price Models

In this section I specify and parameterize the two timber price models that will be used in our optimal harvesting problem. The models I consider are a traditional mean reverting process (TMR) as used in Insley and Rollins (2005) and Insley and Lei (2007) and a regime switching model (the RSMR model) in which the spot price follows potentially two different mean reverting processes. I calibrate the two models using lumber derivatives prices and present evidence as to which can better describe timber prices.

[6]A GBM model implies that price follows a log normal distribution or the log returns are normal. For a normal distribution skewness is zero and kurtosis is three.

2.4.1 RSMR and TMR models

The RSMR model for lumber price, P, is given by the following stochastic differential equation (SDE):

$$dP = \alpha(s_t)(K(s_t) - P)dt + \sigma(s_t)PdZ \tag{2.2}$$

where s_t is a two-state continuous time Markov chain, taking two values 0 or 1. The value of s_t indicates the regime in which the lumber price resides at time t. Define a Poisson process $q^{s_t \to 1-s_t}$ with intensity $\lambda^{s_t \to 1-s_t}$. Then

$$\begin{aligned} dq^{s_t \to 1-s_t} &= 1 \qquad \text{with probability } \lambda^{s_t \to 1-s_t} dt \text{ over an infinitesimally small } dt \\ &= 0 \quad \text{with probability } 1 - \lambda^{s_t \to 1-s_t} dt \text{ over an infinitesimally small } dt \end{aligned}$$

In other words, the probability of regime shifts from s_t to $1 - s_t$ during the small time interval dt is $\lambda^{s_t \to 1-s_t} dt$. The probability of the lumber price staying in the current regime s_t is $1 - \lambda^{s_t \to 1-s_t} dt$.

In this RSMR model, each parameter in the equation is allowed to shift between two states implied by s_t. $K(s_t)$ is the long-run equilibrium level to which the price tends toward following any disturbance. I refer to $\alpha(s_t)$ as the mean reversion rate; the higher its value the more quickly price reverts to its long run mean value. $\sigma(s_t)$ denotes price volatility; dZ is the increment of the standard Wiener process. The stochastic factors for the two regimes are perfectly correlated. Therefore there is a common dZ for two different SDE.

The TMR model, which is calibrated for comparison with the RSMR model, is described by the following stochastic differential equation:

$$dP = \alpha(K - P)dt + \sigma PdZ \tag{2.3}$$

In contrast with RSMR model, the parameters in the above equation are constant, instead of being regime dependent,

Ideally I would rely on statistical tests to determine which of Equation (2.2) or Equation (2.3) is a better model of lumber prices. However, since the parameter $\lambda^{s_t \to 1-s_t} dt$ is defined only in relation to s_t in Equation (2.2) and is not present in (2.3), the traditional asymptotic tests such as the likelihood ratio, Lagrange multiplier and Wald tests do not have a standard asymptotic distribution and cannot be used (Davies (1977), Davies (1987)). As is detailed later in this section, I rely on the calibration procedure to determine which model best describes lumber prices.

For the regime switching model, Hamilton (1989) presents a nonlinear filter and smoother to get statistical estimates of the unobserved state, s_t, given observations on values of P_t. The marginal likelihood function of the observed variable is a byproduct of the recursive filter, allowing parameter estimation by maximizing this likelihood function. The parameters estimated in this way are under the P-measure implying that a corresponding market price of risk has to be estimated as well.

In contrast to Hamilton's method, in Chen and Forsyth (2008) the parameters of the risk-adjusted processes are calibrated by using natural gas derivative contracts, meaning that the parameters thus estimated are under the risk neutral probability measure, Q-measure, allowing the assumption of risk neutrality in the subsequent contingent investment valuation. In this chapter, I follow a similar procedure to Chen and Forsyth (2008) using lumber derivatives, and present the details here for the convenience of the reader. For all parameter values except the volatilities, lumber futures contracts are used in the calibration process. For reasons explained below, options on lumber futures are used to calibrate volatilities.

2.4.2 Calibration using futures prices

Ito's lemma is used to derive the partial differential equations characterizing lumber futures prices for the two price models. These partial differential equations are simplified to a system of ordinary differential equations which can be solved numerically to give futures prices consistent with different parameter values. The calibration procedure determines those parameter values (except for the volatilities) which produce calculated futures prices that most closely match a time series of market futures prices.

Beginning with the TMR model, let $F(P,t,T)$ denote the futures price at time t with maturity T. A futures contract is a contingent claim. From Ito's lemma, the PDE describing the futures price is given by Equation (2.4).

$$F_t + \alpha(K-P)F_P + \frac{1}{2}\sigma^2 P^2 F_{PP} = 0 \tag{2.4}$$

At the expiry date T the futures price will equal the spot price, which gives the boundary condition: $F(P,T,T) = P$

The solution of this PDE is known to have the form

$$F(P,t,T) = a(t,T) + b(t,T)P \tag{2.5}$$

Substituting Equation (2.5) into Equation (2.4), gives the following ODE system

$$\begin{aligned} a_t + \alpha K b &= 0 \\ b_t - \alpha b &= 0 \end{aligned} \tag{2.6}$$

where $a_t \equiv \partial a/\partial t$ and $b_t \equiv \partial b/\partial t$. The boundary conditions: $a(T,T) = 0$; $b(T,T) = 1$ are required in order for $F(P,T,T) = P$ to hold.

Next for the RSMR model, let $F(s_t,P,t,T)$ denote the lumber futures price at

time t with maturity T in regime s_t, where $s_t \in \{0, 1\}$. The no-arbitrage value $F(s_t, P, t, T)$ can be expressed as the risk neutral expectation of the spot price at T.

$$F(s_t, P, t, T) = E^Q[P(T)|P(t) = p, s_t] \tag{2.7}$$

The lumber futures price is a derivative contract whose value depends on the stochastic price and the corresponding regime. Using Ito's lemma for a jump process the conditional expectation satisfies two PDEs, one for each regime, given by:[7]

$$F(s_t)_t + \alpha(s_t)(K(s_t) - P)F(s_t)_P + \frac{1}{2}\sigma(s_t)^2 P^2 F(s_t)_{PP} + \lambda^{s_t \to (1-s_t)}(F(1-s_t) - F(s_t)) = 0 \tag{2.8}$$

with the boundary condition: $F(s_t, P, T, T) = P$.

The solution to these PDEs is known to have the form

$$F(s_t, P, t, T) = a(s_t, t, T) + b(s_t, t, T)P \tag{2.9}$$

This yields the following ordinary differential equation (ODE) system,[8]

$$\begin{aligned} a(s_t)_t + \lambda^{s_t \to (1-s_t)}(a(1 - s_t) - a(s_t)) + \alpha(s_t)K(s_t)b(s_t) = 0 \\ b(s_t)_t - (\alpha(s_t) + \lambda^{s_t \to (1-s_t)})b(s_t) + \lambda^{s_t \to (1-s_t)}b(1 - s_t) = 0 \end{aligned} \tag{2.10}$$

with boundary conditions $a(s_t, T, T) = 0$; $b(s_t, T, T) = 1$. $a(s_t)_t \equiv \partial a(s, t)/\partial t$ and $b(s_t)_t \equiv \partial b(s, t)/\partial t$. These ODEs will be solved numerically, which gives the model parameters. This is detailed in Section 2.5.

Note that the volatility σ does not appear in Equations (2.6) and (2.10). Hence I cannot use lumber futures prices to calibrate the spot price volatility. As in Chen and Forsyth (2008), lumber futures option prices are used to calibrate the volatility.

[7] $F(s_t) \equiv F(s_t, P, t, T)$

[8] $a(s_t) \equiv a(s_t, t, T)$ and $b(s_t) \equiv b(s_t, t, T)$.

A least squares approach is used for calibrating the risk-neutral parameter values. Let θ denote the set of parameters calibrated to the futures price data, where $\theta_{RSMR} = \{\alpha(s_t), K(s_t), \lambda^{s_t \to (1-s_t)} | s_t \in \{0,1\}\}$ and $\theta_{TMR} = \{\alpha, K\}$. In particular, at each observation day t, where $t \in \{1, ..., t^*\}$, there are T^* futures contracts with T^* different maturity dates. For the RSMR model the calibration is performed by solving the following optimization problems:

$$\min_{\theta_{RSMR}, s_t} \sum_t \sum_T (\hat{F}(s_t, P(t), t, T; \theta) - F(t,T))^2 \tag{2.11}$$

where $F(t,T)$ is the market futures price on the observation day t with maturity T. $\hat{F}(s_t, P(t), t, T; \theta)$ is the corresponding model implied futures price computed numerically and determined in equation (2.9) using the market spot price $P(t)$ and the parameter set θ, as well as the regime s_t.

This is a Mixed Integer problem, since the unknown parameters in θ are continuous variables and s_t is a binary variable which equals to 0 or 1 depending on the regime. It is known that some certain software packages provide a way of solving this Mixed Integer optimization problem. However in this thesis, I use an intuitive and reasonable way of calibrating these unknown model parameters. Specifically, this optimization program is implemented in Matlab which is a program specially devised for handling large vectors and performing matrix computations. I used the built-in function lsqnonlin to solve the problem. The lsqnonlin function take initial values of the parameters as inputs and then solves the problem iteratively by updating the parameters in the direction where the decline in the target function is the greatest. The calibrated parameter set θ and s_t will then minimize the distance between F and $\hat{F}$ for all t^*.

Similarly, for TMR model, the optimization problem becomes

$$\min_{\theta_{TMR}} \sum_t \sum_T (\hat{F}(P(t), t, T; \theta) - F(t,T))^2 \tag{2.12}$$

where $\hat{F}(P(t), t, T; \theta)$ is the model implied futures price.

2.4.3 Calibration of volatilities using options on futures

In this section, the spot price volatility is calibrated for the two different price models using market European call options on lumber futures. For the RSMR model, let $\bar{V}(s_t, F, t, T_v)$ denote the (theoretical) European call option value on the underlying lumber futures contract F at time t with maturity at T_v in regime s_t. $F(s_t, t, T)$ represents the value of the underlying futures contract at time t with maturity at T, where $T \geq T_v$. Let X be the strike price of option. In the risk-neutral world, $\bar{V}(s_t, F, t, T_v)$ can be expressed as

$$\bar{V}(s_t, F, t, T_v) = e^{-r(T_v - t)} E^Q[\max(F(s_T, T_v, T) - X, 0) | F(s_t, t, T) = F, s_t] \quad (2.13)$$

For the calibration I must assume that $T = T_v$, which implies that $\bar{V}(s_t, F, t, T_v) = \bar{V}(s_t, F, t, T)$ and $F(s_T, T_v, T) = F(s_T, T, T)$. Therefore the above equation can be transformed to

$$\begin{aligned} \bar{V}(s_t, F, t, T) &= e^{-r(T-t)} E^Q[\max(F(s_T, T, T) - X, 0) | F(s_t, t, T) = F, s_t] \\ &= e^{-r(T-t)} E^Q[\max(P(T) - X, 0) | a(s_t, t, T) + b(s_t, t, T) P(t) = F, s_t] \end{aligned} \quad (2.14)$$

where $P(T)$ is the lumber spot price and $F(s_T, T, T) = P(T)$ at the maturity date T.

For calibration purposes, a hypothetical European call option is needed. Let $V(s_t, P, t, T)$ denote such a call option on lumber at time t with maturity T in regime s_t. This option value can be expressed in the form of the risk-neutral expectation as

$$V(s_t, P, t, T) = e^{-r(T-t)} E^Q[\max(P(T) - X, 0) | P(t) = P, s_t] \quad (2.15)$$

Given that lumber price P follows RSMR, the option value $V(s_t, P, t, T)$ satisfies the coupled PDEs

$$V(s_t)_t + \alpha(s_t)(K(s_t) - P)V(s_t)_P + \frac{1}{2}\sigma(s_t)^2 P^2 V(s_t)_{PP} - rV(s_t) + \lambda^{s_t \to 1-s_t}[V(1-s_t) - V(s_t)] = 0 \qquad (2.16)$$

with the boundary condition: $V(s_t, P, T, T) = \max[P(T) - X, 0]$. The value of this hypothetical option $V(s_t, P, t, T)$ can be solved numerically by solving the above PDEs.

Comparing equations (2.14) and (2.15), the following relationship holds.

$$\bar{V}(s_t, F, t, T) = V(s_t, \frac{F - a(s_t, t, T)}{b(s_t, t, T)}, t, T) \qquad (2.17)$$

Therefore, after getting $V(s_t, P, t, T)$ by solving the equation (2.16), the theoretical lumber option value $\bar{V}(s_t, F, t, T)$ can be calculated using the interpolation method.

Similarly, for the TMR model, let $\bar{V}(F, t, T)$ and $V(P, t, T)$ represent the European call option on lumber futures and the hypothetical European call option on lumber respectively.[9] The corresponding PDE for characterizing $V(P, t, T)$ is expressed as

$$V_t + \alpha(K - P)V_P + \frac{1}{2}\sigma^2 P^2 V_{PP} - rV = 0 \qquad (2.18)$$

with boundary condition: $V(P, T, T) = \max[P(T) - X, 0]$. Given the relationship[10]

$$\bar{V}(F, t, T) = V(\frac{F - a(t, T)}{b(t, T)}, t, T) \qquad (2.19)$$

the model implied option value $\bar{V}(F, t, T)$ can be computed after getting $V(P, t, T)$ by solving the above PDE.

[9] $T_v \approx T$ in this model as well.

[10] This relationship is derived in the same way as equation (2.17).

A least squares approach is also used to calibrate the volatility. In particular for the RSMR model, I solve the following optimization problem:

$$\min_{\sigma^0,\sigma^1} \sum_K (\bar{V}(s_t, F(t,T_1), t, T_1; \theta, K, \sigma^0, \sigma^1) - V(t, T_1; K))^2 \tag{2.20}$$

where $\bar{V}(s_t, F, t, T_1; \theta, K, \sigma^0, \sigma^1)$ represents the corresponding model implied option value at time t with maturity T and strike price K and $V(t, T_1; K)$ is the market value of lumber call option on futures. T^* option contracts with T^* different strike prices are needed for volatility calibration. The calibrated parameter set $\{\sigma(0), \sigma(1)\}$ will minimize the square distance between $\bar{V}$ and V.

Similarly, for the TMR model, the optimization problem becomes:

$$\min_{\sigma} \sum_K (\bar{V}(F(t,T_1), t, T_1; \theta, K, \sigma) - V(t, T_1; K))^2 \tag{2.21}$$

2.5 Calibration results and model comparison

2.5.1 Data description: lumber futures and options on futures

Lumber market futures and options on futures are used to calculate the risk neutral spot price process. Four different futures contracts corresponding to each observation date for every Friday from January 6th, 1995 to April 25th, 2008 will be employed in the calibration. The average maturity days for these four futures contracts which trade on the Chicago Mercantile Exchange (CME), are about 30, 90, 150 and 210. Since I am interested in estimating the stochastic process for real lumber prices for a Canadian forestry problem, future prices were deflated by the

consumer price index and converted to Canadian dollars.[11]

The call options on futures used to calibrate volatilities are also from the CME. Two sets of six call options written on the same futures contract were chosen. The call options expire on October 31st, 2008 while the underlying futures contract expires on November 14, 2008. (At the CME, the lumber options expire the last business day in the month prior to the delivery month of the underlying futures contract.) The first set of six options was obtained on May 23rd, 2008 and the price of the corresponding futures contract was 260.8 $U.S./mbf. The second set was obtained on May 30th, 2008 and the futures price on that day was 260.9 $U.S./mbf. The strike prices of the six call options range from 260 to 310 $U.S./mbf.

In our case since the underlying futures contracts expires on November 14, 2008 and the options expire on October 31, 2008, $T_v < T$. For the calibration, I must assume that $T_v = T$ holds approximately. To justify this assumption, I appeal to the fact that options prices were retrieved in May 2008, some months before their expiry.

2.5.2 Calibration Results

A non-linear least squares approach is used to calibrate model parameters. Specifically given initial values of all the parameters, model implied futures prices of all the maturities at each date can be computed by solving the ODE for each of the two regimes. The differences of the model implied futures prices and market futures prices for all the futures contract at each date are computed for both regimes. The sum squared differences are used to determine the regime at each date. The sum squared difference of each date is then summed together over all the periods.

[11]For CME Random Length Lumber futures, the delivery contract months are as follows: January, March, May, July, September and November. There are six lumber futures on each day only the first four of which are actively traded. Therefore, only the first four futures contracts are used in parameter calibration. The last day of trading is the business day prior to the 16th calendar day of the contract month.

The optimal parameter values will be those that generate the lowest sum squared difference. Matlab is used for parameter calibration. There are 12 iterations in the calibration process and it took about 30 minutes to converge. When change in the residual is smaller than the specified tolerance, which is $5e^{(-3)}$ in this case, the program will stop.

It is important to check whether the obtained parameters are the only choices of parameter values which attain a reasonable in-sample fit. There could be the case that there exist several ranges of parameter values, all providing a reasonable fit to data. In other words, since the calibrated parameters are obtained by solving a nonlinear optimization optimization program, there is no guarantee that the obtained solution is a unique and global solution. This issue can be investigated by varying the initial values of the parameters used to initialize the calibration algorithm, and the upper bound and lower bound of the calibrated parameters used in the optimization process. If the optimal parameters are sensitive to changes in the initial values this should be taken to indicate that there are potentially several local optima. Following this argument, different sets of initial values as well as different combinations of upper and lower bounds are used to find the optimal solution and to check the stability of the calibrated parameter values. Given a certain set of upper and lower bounds the resulting estimates are insensitive to the initial values. Hence the calibrated parameters may be the local optimal. Since we used the bounds that seem economically reasonable, we believe that the resulting estimates are economically reasonable.

Table 2.2 and Table 2.3 present the calibration results for parameter values under the risk neutral measure in the RSMR model. In the table we observe two quite different regimes in the Q-measure. Regime 1 has a much higher equilibrium price level, $K(1)$, but a lower speed of mean reversion, $\alpha(1)$, compared to regime 0. The risk neutral intensity of switching out of regime 1 is very low at $\lambda^{1\to 0} = 0.39$ which implies that in the risk neutral world, prices are mostly in this regime with

RSMR Model

$\alpha(0)$	$\alpha(1)$	$K(0)$	$K(1)$	$\lambda^{0\to1}$	$\lambda^{1\to0}$
3.61	0.40	71.92	516.64	17.09	0.39

TMR Model

α	K
0.69	341.00

Table 2.2: Calibrated parameter values for the RSMR and TMR model, $K(0)$, $K(1)$ and K are in \$Cdn(2005)/cubic metre.

RSMR Model		TMR Model
$\sigma(0)$	$\sigma(1)$	σ
0.0038	0.2545	0.28

Table 2.3: Calibrated volatilities for the RSMR and TMR models

the higher equilibrium price.

These parameter estimates for the RSMR model describe a situation where price is mostly in regime 1 with the high long run equilibrium price and a moderate pace of mean reversion. Ignoring volatility and the risk of regime change, the mean reversion speed $\alpha(1) = 0.04$ implies the half-life for returning to the long run equilibrium is 1.7 years.[12] Occasionally price reverts to regime 0 which has a significantly lower equilibrium price and very little volatility. Regime 0 may be thought of as a depressed state, and in the risk neutral world this state is not expected to persist for long. The mean reversion rate is much higher in regime 0 than in regime 1.

Calibrated parameter values for the TMR model are also reported in Table 2.2 and Table 2.3. The long-run price level, K, and mean reversion rate α in the TMR model fall between regime 1 and regime 0 values in the RSMR model while volatility is close to that of regime 1.

[12]Solving the differential equation $dP = \alpha(K - P)dt$, the time to reduce $(P_t - K)$ by half is $-\ln(0.5)/\alpha$.

It is tempting to interpret these parameter estimates in terms of the behaviour of historical lumber prices, but this would be invalid since these are risk adjusted or Q-measure estimates. If I assume that in the real world, or under the P-measure, the spot price follows a process like Equation (2.2), then I can derive the relationship between P-measure and Q-measure parameters. I show in Appendix A1 that given assumptions about the signs of the speed of mean reversion, $\alpha(s_t)$, and the market price of risk for lumber price diffusion, denoted β_P, then the speed of mean reversion in the risk neutral world will exceed that of the real world. In addition, the long run equilibrium price $K(s_t)$ will be lower in the risk neutral world than the real world. It makes intuitive sense that the risk adjustment in moving to the Q-measure results in a price process which reverts at a faster rate to a lower long run equilibrium level. This would make the Q-measure process more pessimistic, as expected. It is also shown in Appendix A1 that volatility is the same in the real and risk-neutral worlds. Further, the risk neutral intensity of switching regimes, $\lambda^{s_t \to (1-s_t)}$, equals the market price of risk for regime switching, which I denote β_{sw}. Hence the calibrated risk adjusted probability $\lambda^{s_t \to (1-s_t)} dt$ may be quite different from the P-measure probability of switching regimes.

These calibration results allow me to determine the regime that is most likely reflected for any given date. From this I can derive an estimate of the physical probability of being in either regime, which may be contrasted with the risk neutral probabilities. Regimes in the period under consideration as implied by RSMR model are plotted in Figure 2.2. I assume that if the calibration error in a particular regime exceeds the error in the other, then price is most likely in the former regime. In the risk neutral world, regime 0 has a lower equilibrium price level, while regime 1 has the high equilibrium price level. It is shown in Appendix A1 that if I assume a positive and not a very large market price of stochastic price risk β_P, the high price regime in the risk neutral world is also the high price regime in the real world.

I observe in Figure 2.2 that price fluctuates between the two regimes and there

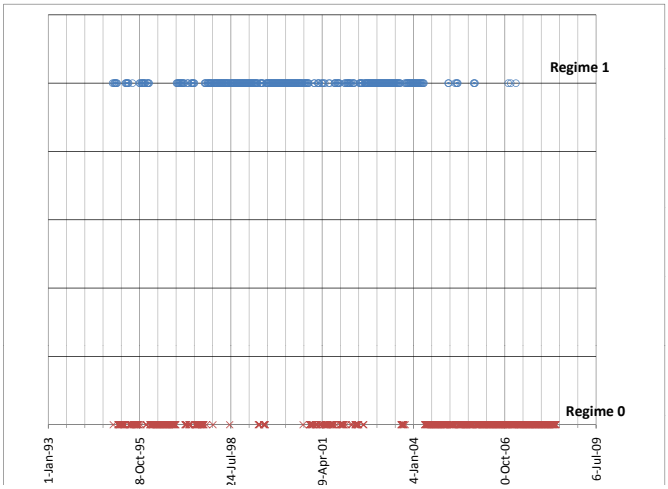


Figure 2.2: Implied regimes in the period under consideration by RSMR model. Blue O's on upper line indicate time steps in regime 1 and reddish X's on lower line indicate time steps in regime 0.

are distinct intervals when price appears to remain in one regime or the other. It is interesting to observe whether these regime shifts coincide with any particular events or shocks in lumber markets. For example, from the middle of the year 1998 to the beginning of year 2001, lumber prices mainly stay in high mean regime (regime 1). This period followed the signing of the five-year trade agreement between the United States and Canada in 1996. Under this Softwood Lumber Agreement, Canadian lumber exports to the United States were limited to a specified level that would be duty free. I hypothesize that this quantity restriction would support lumber prices remaining in the high price regime. The trade agreement expired in April 2001 and the two countries were unable to reach consensus on a replacement agreement. From Figure 2.2 I observe that during the period between middle 2001 to late 2002, lumber prices fluctuate between the two regimes. Even though, a new agreement between Canada and the United States was implemented

in 2006, this deal was criticized as "one-sided" and a "bad deal for Canada". After the middle of 2004 until 2008, lumber prices stay in the low mean regime most of time. The lumber industry has been severely affected by the global financial crisis that began in 2007 and precipitated a drastic fall in the number of new housing starts. The linking of the probability of being in either of the regimes to current events in lumber markets is just a rough intuitive analysis. However, the shifting that we observe between the two regimes lends support for a regime shifting model to account for the different circumstances faced by the industry in good times and bad times.

From the data used in Figure 2.2 we can estimate that over the 1995 to 2008 period, price is 51.4 percent of the time in regime 0 and 48.6 percent of the time in regime 1. In contrast the estimated λ's in the risk neutral world imply that price will be in the high price regime 98 percent of the time and in the low price regime only 2 percent of the time. It is surprising that that the risk adjusted probability of staying in the high price regime is larger than the actual probability, implying a more optimistic view in the Q-measure. However the impact of moving to the risk neutral world is reflected in adjustments to all of the parameters. I noted above that the speed of mean reversion will be higher and the equilibrium price level will be lower, which present a more pessimistic view of price.

2.5.3 Model comparison

Table 2.4 reports the mean absolute errors for the four futures contracts used to calibrate the RSMR and the TMR models. From the last column, it appears that the RSMR model outperforms the TMR model, since the overall average errors expressed in two different ways are lower in the RSMR model. The RSMR model also has lower errors for each of the four futures contracts individually. Figures 2.3 and 2.4 show plots of the the model implied futures prices and market futures prices

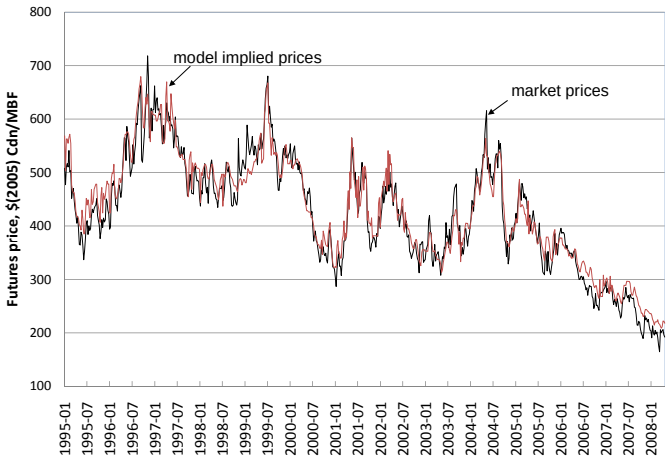
Mean absolute error					
T	30	90	150	210	Overall
RSMR model					
In dollars	22.23	18.50	18.97	20.56	20.07
In percentage	5.65	4.49	4.56	5.00	4.93
TMR model					
In dollars	39.33	30.90	30.49	34.48	33.80
In percentage	10.36	7.85	7.48	8.21	8.47

Table 2.4: Mean absolute errors for all the four different futures contracts in both RSMR and TMR models, expressed in dollars and in percentage. T refers to the number of days to maturity

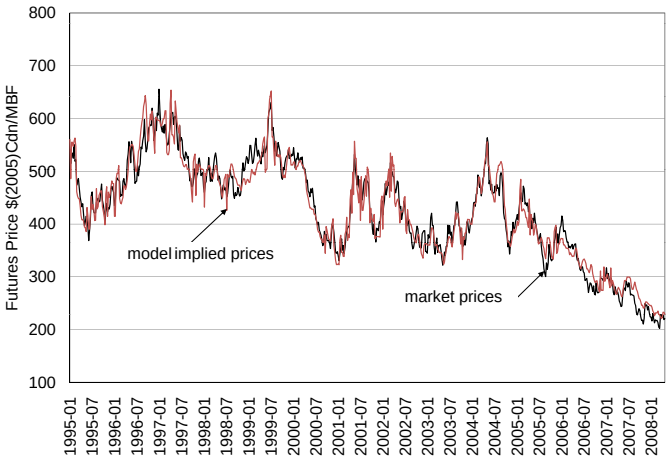
for the two futures contracts corresponding to the largest and smallest calibration errors from Table 2.4. The closer fit of the RSMR model to market data is noticeable through visual inspection of these graphs.

2.6 Specification of the optimal harvesting problem and its numerical solution

After analyzing the dynamics of the lumber price process and calibrating all the parameter values of the corresponding model, I am ready to solve for the value of a forestry investment. I will value a hypothetical stand of trees in Ontario's boreal forest using both price models examined in this chapter. I will investigate whether use of these models in a realistic optimal harvesting problem will result in different land values and optimal harvesting ages. I use the same investment problem as in Insley and Lei (2007). In Insley and Lei (2007) a TMR process was used and the estimation procedure was carried out through ordinary least squares on spot price data. I compare the regime switching model with the result of the single-factor mean reversion process and also the results from Insley and Lei (2007).

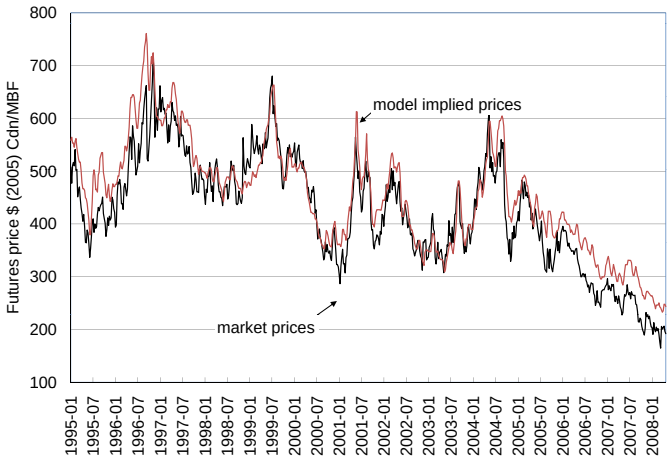


(a) f1: futures contracts with average 30 days to maturity.

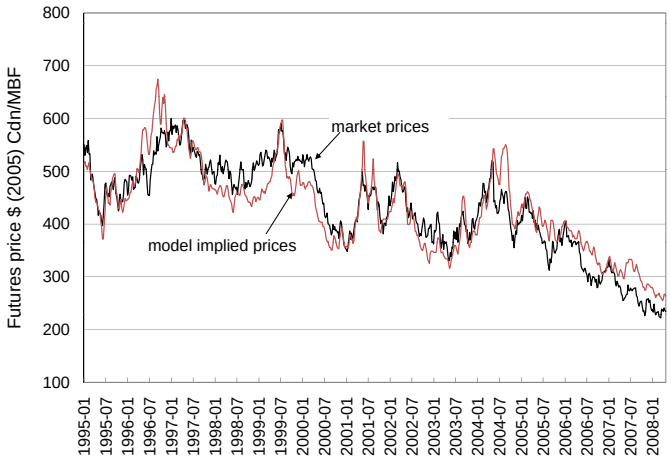


(b) f2: futures contract with average 60 days to maturity.

Figure 2.3: RSMR model implied futures prices and market futures prices for two futures contracts. f1 has the largest error while f2 has the smallest error in Table 2.4.



(a) f1: futures contracts with average 30 days to maturity.



(b) f3: futures contracts with average 90 days to maturity.

Figure 2.4: TMR model implied futures prices and market futures prices for two futures contracts, f1 has the largest error while f3 has the smallest error from Table 2.4.

In the following sections, a real options model of the forestry investment valuation will be developed assuming lumber prices follow the RSMR process. Coupled partial differential equations (PDEs) characterizing the values of the option to harvest the trees will be derived using contingent claim analysis. A finite difference method will be employed to solve the PDEs numerically given appropriate boundary conditions. The model and numerical solution scheme for the TMR price case is described in Insley and Rollins (2005).

2.6.1 Harvesting model for the RSMR case

I model the optimal decision of the owner of stand of trees who wants to maximize the value of the stand (or land value) by optimally choosing the harvest time. It is assumed that forestry is the best use for this land, so that once the stand is harvested it will be allowed to grow again for future harvesting. Since this is a multirotational optimal harvesting problem, it represents a path-dependent option. The value of the option to harvest the stand today depends on the quantity of lumber, which itself depends on the last time when the stand was harvested.

Lumber price is assumed to follow either the RSMR model or the TMR model detailed in the previous sections. In this section I derive the key partial differential equation that describes the value of the stand of trees for the RSMR case. Derivation of the key partial differential equation for the TMR case can be found in Insley and Lei (2007).

For now I write the RSMR model from Equation (2.2) in a more general form as:

$$dP(s_t) = a(s_t, P, t)dt + b(s_t, P, t)dZ \tag{2.22}$$

Denote $q^{s_t \to 1-s_t}$, the risk of regime shift, as a Poisson process, where $s_t \in \{0, 1\}$

indicates the regime.

$$dq^{s_t \to 1-s_t} = 1 \qquad \text{with probability } \lambda^{s_t \to 1-s_t} dt$$
$$dq^{s_t \to 1-s_t} = 0 \quad \text{with probability } 1 - \lambda^{s_t \to 1-s_t} dt$$

With probability λdt price changes regime during the small interval dt, and with probability $1 - \lambda dt$ price remains in the same regime.

There are two risks associated with this stochastic process. One is the standard continuous risk in the dZ term. The other, in discrete form, is due to the risk of regime switch. In order to hedge these two risks and value the stand of trees $V(s_t, P, \varphi)$, two other traded investment assets, which depend solely on lumber price, are needed. Let φ denote the age of the stand, defined as $\varphi = t - t_h$, where t_h represents the time of last harvest. φ in this case is another state variable, in addition to P. φ satisfies $d\varphi = dt$.

Assume that there exist investment assets which depend on the lumber price P and can be used to hedge the risk of our investment. Using standard arguments I set up a hedging portfolio that eliminates the two risks. I can derive the fundamental partial differential equation that characterizes the value of the stand of trees when it is optimal to refrain from harvesting.

$$V(s_t)_t + (a(s_t, P, t) - \beta_P b(s_t, P, t))V(s_t)_P + \frac{1}{2} b(s_t, P, t)^2 V(s_t)_{PP} + V(s_t)_\varphi - rV(s_t) + \beta_{sw}(V(1 - s_t) - V(s_t)) = 0 \qquad (2.23)$$

β_P and β_{sw} are parameters which represent market prices of risk for the diffusion risk and regime-switching risk respectively.

Our estimation method detailed in Sections 2.4 and 2.5 yields risk neutral pa-

rameter values. Therefore the following relationships hold

$$\begin{aligned} a(s_t, P, t) - \beta_P b(s_t, P, t) &= \alpha(s_t)(K(s_t) - P) \\ b(s_t, P, t) &= \sigma(s_t)P \\ \beta_{sw} &= \lambda^{s_t \to 1-s_t} \end{aligned}$$

Substituting these equations into the above PDE give

$$V(s_t)_t + \alpha(s_t)(K(s_t) - P)V(s_t)_P + \frac{1}{2}(\sigma(s_t)P)^2 V(s_t)_{PP} + V(s_t)_\varphi - rV(s_t) + \lambda^{s_t \to 1-s_t}(V(1 - s_t) - V(s_t)) = 0. \quad (2.24)$$

The complete harvesting problem which determines the optimal harvesting date can then be specified as a Hamilton-Jacobi-Bellman (HJB) variational inequality (VI). Define $\tau \equiv T - t$ as time remaining in the option's life. Rewrite the above PDE and define HV as

$$HV \equiv rV(s_t) - (V(s_t)_t + \alpha(s_t)(K(s_t) - P)V(s_t)_P + \frac{1}{2}(\sigma(s_t)P)^2 V(s_t)_{PP} + V(s_t)_\varphi + \lambda^{s_t \to 1-s_t}(V(1 - s_t) - V(s_t))) \quad (2.25)$$

Then the HJB VI is:

$$\begin{aligned} &(i) \quad HV \geq 0 \\ &(ii) \quad V(s_t, P, \varphi) - [(P - C_h)Q(\varphi) + V(s_t, P, 0)] \geq 0 \\ &(iii) \quad HV\Big[V(s_t, P, \varphi) - [(P - C_h)Q(\varphi) + V(s_t, P, 0)]\Big] = 0 \end{aligned} \quad (2.26)$$

where C_h is the cost per unit of lumber, $Q(\varphi)$ is the volume of the lumber which is a function of age, $Q = g(\varphi)$. $[(P - C_h)Q(\varphi) + V(s_t, P, 0)]$ is the payoff from harvesting immediately and consists of revenue from selling the harvested timber

plus the value of the bare land, $V(s_t, P, 0)$. The above HJB VI implies if the stand of trees is managed optimally either HV, $V(s_t, P, \varphi) - [(P - C_h)Q(\varphi) + V(s_t, P, 0)]$, or both will be equal to zero. If $HV = 0$, it is optimal for the investor to continue holding the option by delaying the decision to harvest. The growing stand of trees is earning the risk free return. If $V(s_t, P, \varphi) - [(P - C_h)Q(\varphi) + V(s_t, P, 0)] = 0$, then the value of the stand of trees just equals the value of immediate harvest and the investor should harvest the trees. If both terms are equal to zero, either strategy is optimal.

2.6.2 Numerical solution of the HJB VI equation

This section briefly describes the numerical methods used for solving the regime switching HJB VI, Equation (2.26). I also analyze the properties of the scheme, such as the stability and monotonicity. More details of the numerical solution are contained in Appendix A2.

General description of the numerical methods

The option to choose the optimal harvest time has no analytical solution. The HJB VI expressed in Equation (2.26) in this chapter is solved numerically using the combination of fully implicit finite difference method, semi-Lagrangian method and the penalty method. This approach is also used in Insley and Lei (2007) but for a single regime problem. The finite difference method is used to convert a differential equation into a set of discrete algebraic equations by replacing the differential operators in PDEs with finite difference operators.

For the optimal tree harvesting problem examined in this chapter, there are two state variables. One is the spot price P and the other is the stand age φ. Using the semi-Lagrangian method this two-factor problem can be reduced to a one factor

problem for each time step. After each time step, the true option value is obtained by using linear interpolation. For the details of this method, see Insley and Rollins (2005) and Morton and Mayers (1994).

There are several approaches to the numerical solution of the HJB VI. The penalty approach used here converts it into a nonlinear algebraic problem, which is then solved by Newton iteration. The penalty method has several benefits. It is more accurate than an explicit method and has good convergence properties. Another advantage is that at each iteration it generates a well-behaved sparse matrix, which can be solved using either direct or iterative methods.[13]

The penalty method used in this chapter is outlined here. Define $\tau = T - t$ and $V(s_t)_t = -V(s_t)_\tau$. The HJB VI[14] in Equation (2.26) can be expressed as a single equation:

$$V(s_t)_\tau - V(s_t)_\varphi = \alpha(s_t)(K(s_t) - P)V(s_t)_P + \frac{1}{2}(\sigma(s_t)P)^2 V(s_t)_{PP} - rV(s_t) + \lambda^{s_t \to 1-s_t}(V(1-s_t) - V(s_t)) + \Upsilon(s_t) \quad (2.27)$$

where $\Upsilon(s_t)$ on the right hand side of this equation is the penalty term, which satisfies

$$\Upsilon(s_t) > 0 \text{ if } V(s_t, P, \varphi) = [(P - C_h)Q(\varphi) + V(s_t, P, 0)] \quad (2.28)$$

$$= 0 \text{ if } V(s_t, P, \varphi) > [(P - C_h)Q(\varphi) + V(s_t, P, 0)] \quad (2.29)$$

Equation (2.28) implies that if value of the asset equals to the payoff, which is $[(P - C_h)Q(\varphi) + V(s_t, P, 0)]$[15], it is optimal to harvest the trees immediately, which is the first condition in HJB VI Equation (2.26). If the asset value is higher than the payoff, Equation (2.29) implies the harvest should be delayed which is the second

[13] See Zvan et al. (1998) and Fan et al. (1996) for more on the penalty method.
[14] This HJB VI characterizes the option value in regime s_t, $V(s_t)$.
[15] The payoff is defined as the net revenue of selling the trees plus the value of the bare land.

condition in the HJB VI equation. Since the investors can exercise the option to harvest the trees at any time, the penalty method in this way incorporates the characteristic of early exercise. This numerical method can also handle the case when the expiry date of the option $T \to \infty$. Basically T can be chosen to be a very large number so that the number is large enough that the option value is not sensitive to the increase of it.

A complicating factor in our problem is the presence of regime switching in the spot price process. I have two PDEs in the form of Equation (2.24), one for the value in each of the two regimes. Moreover, the value in one regime affects the value in the other regime[16]. I deal with this problem by stacking the discretized version of equation (2.27) for option values in two regimes and solving the two discretized PDEs together at each time step. In this manner the PDEs in the two regimes are fully coupled.

Discretization

This section illustrates the main results of finite difference discretization, the semi-Lagrangian method and penalty method of dealing with the HJB VI[17]. Prior to presenting the matrix form of the HJB VI discretization, some notations are introduced here.

For PDE discretization, unequally spaced grids in the directions of the two state variables P and φ are used. The grid points are represented by $[P_1, P_2, ..., P_{imax}]$ and $[\varphi_1, \varphi_2, ..., \varphi_{jmax}]$ respectively. I also discretize the time direction, represented as $\tau^N, ..., \tau^1$([18]). Define $V(s_t)_{ij}^{n+1}$ as an approximation of the exact solution $V(s_t, P_i, \varphi_j, \tau^{n+1})$, and $V^*(s_t)_{ij}^n$ as an approximation of $V(s_t, P_i, \varphi_j, \tau^n)$. Recall that

[16]i.e. The value in regime $(1 - s_t)$, $V(1 - s_t)$, appears in Equation (2.24) characterizing the option value in regime s_t, $V(s_t)$.

[17]Detailed discretization is provided in Appendix.

[18]The iteration starts from the final maturity date T and moves backward along the time direction until the current time 0.

$\tau = \tau^N$, $t = 0$ and at $\tau = \tau^1$, $t = T$. Based on the semi-Lagrangian method, the true solution of $V(s_t, P_i, \varphi_{j+\Delta\tau}, \tau^n)$ is obtained from $V^*(s_t)_{ij}^n$ using linear interpolation after each time step.

Denote ℓ a differential operator represented by

$$\ell V(s_t) = \alpha(s_t)(K(s_t) - P)V(s_t)_P + \frac{1}{2}(\sigma(s_t)P)^2 V(s_t)_{PP} - rV(s_t) + \lambda^{s_t \to 1-s_t} V(s_t)$$

Equation (2.27) can be rearranged as:

$$V(s_t)_\tau - V(s_t)_\varphi = \ell V(s_t) + \lambda^{s_t \to 1-s_t} V(1 - s_t) + \Upsilon(s_t) \tag{2.30}$$

Note that the right hand side of this equation has derivatives with respect to P only. Therefore this one-dimensional PDE for each φ_j is solved independently within each time step. After each time step is completed, using linear interpolation I will get $V(s_t, P_i, \varphi_{j+\Delta\tau}, \tau^n)$ from $V^*(s_t)_{ij}^n$. The discretized version of Equation (2.30) using the fully implicit method and the semi-Lagrangian method is written as

$$\frac{V(s_t)_{ij}^{n+1} - V^*(s_t)_{ij}^n}{\Delta\tau} = [\ell V(s_t)]_{ij}^{n+1} + \lambda^{s_t \to 1-s_t} V(1 - s_t)_{ij}^{n+1} + \pi(s_t)_{ij}^{n+1} \tag{2.31}$$

where the penalty term $\pi(s_t)_{ij}^{n+1}$ is defined as

$$\begin{aligned} \pi(s_t)_{ij}^{n+1} &= \frac{1}{\Delta\tau}(\text{payoff} - V(s_t)_{ij}^{n+1}) Large; \text{ if } V(s_t)_{ij}^{n+1} < \text{payoff} && (2.32) \\ &= 0; \text{ otherwise} && (2.33) \end{aligned}$$

The term 'Large' in equation (2.32) refers to a large number[19] and case dependent. The subscript ij refers to the point corresponding to (P_i, φ_j) and superscript n denotes the nth time step.

[19]For example, $Large = 10^6$ for some cases.

Rearranging Equation (2.31) and writing in a matrix form results in

$$W(s_t)V(s_t)^{n+1} - \Delta\tau\lambda^{s_t \to 1-s_t} V(1-s_t)^{n+1} = V^*(s_t)^n + \overline{\pi(s_t)}^{n+1}\text{payoff}(s_t)^{n+1} \quad (2.34)$$

where $W(s_t)$ is a sparse matrix containing all the parameters corresponding to the value in regime s_t. The other terms except $\Delta\tau\lambda^{s_t \to 1-s_t}$ are expressed in vector form. The ijth element in the penalty vector $\overline{\pi(s_t)}^{n+1}$ is defined as

$$\begin{aligned} \overline{\pi(s_t)}_{ij}^{n+1} &= Large;\ \text{if } V(s_t)_{ij}^{n+1} < \text{payoff} \\ &= 0;\ \text{otherwise} \end{aligned}$$

Equation (2.34) is the final discretized version of the HJB VI corresponding to $V(s_t)$. However, the value in the other regime $V(1-s_t)$ appears in this expression. In order to obtain both option values for all the grid points at each time step, the discretized HJB VI for $V(1-s_t)$ which is similar with the expression (2.34) is stacked with Equation (2.34) to form a system of equations, which can be written as

$$Z_{matrix}\begin{bmatrix} V(s_t) \\ V(1-s_t) \end{bmatrix}^{n+1} = \begin{bmatrix} V^*(s_t) \\ V^*(1-s_t) \end{bmatrix}^{n} + \begin{bmatrix} \overline{\pi(s_t)} \\ \overline{\pi(1-s_t)} \end{bmatrix}^{n+1} \begin{bmatrix} \text{payoff}(s_t) \\ \text{payoff}(1-s_t) \end{bmatrix}^{n+1} \quad (2.35)$$

Z_{matrix} is a large sparse matrix. This system of equations is solved iteratively at each time step. For simplicity, the more compact version of Equation (2.35) can be expressed as

$$Z_{matrix}[V]^{n+1} = [V^*]^n + [\overline{\pi}]^{n+1}[\text{payoff}]^{n+1} \quad (2.36)$$

This is the scheme I use to numerically solve the optimal tree harvesting problem.

Boundary conditions and pseudo code

In order to solve Equation (2.36), the appropriate boundary conditions as well as the terminal condition are specified below. These are the same as used in Insley and Rollins (2005).

1. **As $P \rightarrow 0$**, no specific boundary condition is needed. Substitute $P = 0$ into Equation (2.36) and discretize the resulted PDE.
2. **As $P \rightarrow \infty$**, I set $V(s_t)_{PP} = 0$. As price goes to infinity, I assume the option value is a linear function of P.
3. **As $\varphi \rightarrow 0$**, no specific boundary condition is needed since the PDE is first order hyperbolic in the φ direction, with outgoing characteristic in the negative φ direction.
4. **As $\varphi \rightarrow \infty$**, $V(s_t)_{\varphi} \rightarrow 0$, and hence no boundary condition is needed. Since as the stand age goes to infinity, I assume the wood volume in the stand has reached a steady state and the value of the option to harvest does not change with φ.
5. **Terminal condition.** $V(s_t, T) = 0$ This means when T gets very large, it has a negligible effect on the current option value.

Pseudo code for solving Equation (2.36) is provided as the follows[20].

[20] All programs are written in Matlab.

1. Set up tolerance level tol

2. $Large = \frac{1}{tol}$

3. for $\tau = 1 : N-1$; % time step iteration
   for $j = 1 : jmax$; % iterate along the age φ direction
   $([V]^{n+1})^0 = [V]^n$; % initial guess for $[V]^{n+1}$
   for $k = 0, ...$ until convergence; % penalty American constraint iteration

$$\begin{aligned}(\overline{\pi}^{n+1})^k &= Large; \text{if } V^{n+1} < \text{payoff} \\ &= 0; \text{ otherwise}\end{aligned}$$

$Z_{matrix}([V]^{n+1})^{k+1} = [V^*]^n + ([\overline{\pi}]^{n+1})^k[\text{payoff}]^{n+1}$

if $max_i \frac{|((V_i)^{n+1})^{k+1} - ((V_i)^{n+1})^k|}{max(1,|((V_i)^{n+1})^{k+1}|)} < tol$
quit;

endfor; % end penalty American constraint iteration
endfor; % end iteration along φ direction
$V(s_t, P_i, \varphi_{j+\Delta\tau}, \tau^n) = V^*(s_t)^n_{ij}$; % by linear interation
endfor; % end time step iteration

Properties of the numerical scheme

Since no closed-form solution exists for this optimal tree harvesting problem, the properties of my proposed numerical scheme have to been examined. In the case of nonlinear pricing problems, seemingly reasonable numerical schemes can converge to an incorrect solution[21]. A stable, consistent and monotone discretization will converge to the viscosity (i.e. reasonable) solution.[22]. Generally speaking, consis-

[21]See Pooley et al. (2003).

[22]See Barles (1997) for detailed proof. For the definition of viscosity solution, see d'Halluin et al. (2005). For the existence of a viscosity solution in the regime switching case, see Pemy and Zhang (2006).

tency is guaranteed if a reasonable discretization is used[23]. I use finite difference discretization which is a one of the standard discretization methods. In Appendix A3, I prove that this scheme is monotone and stable and thus converges to the viscosity solution which is financially reasonable.

2.7 Optimal harvesting problem: data and empirical results

2.7.1 Cost, wood volume and price data

I examine an optimal harvesting problem for a hypothetical stand of Jack Pine trees in Ontario's boreal forest. I consider the optimal harvesting decision and land value assuming that the stand will continue to be used for commercial forestry operations over multiple rotations. Values are calculated prior to any stumpage payments or taxes.

Timber volumes and harvesting costs are adopted from Insley and Lei (2007) and are repeated here for the convenience of the reader. Volume and silviculture cost data were kindly provided by Tembec Inc. The estimated volumes reflect 'basic' levels of forestry management which involves $1040 per hectare spent within the first five years on site preparation, planting and tending. These costs are detailed in Table 3.10. Note that in the Canadian context these basic silviculture expenses are mandated by government regulation for certain stands.

Volumes, estimated by product, are shown in Figure 2.5 for the basic regime.[24] SPF1 and SPF2 are defined as being greater than 12 centimeters at the small end,

[23]See d'Halluin et al. (2005).

[24]The yield curves were estimated by Margaret Penner of Forest Analysis Ltd., Huntsville, Ontario for Tembec Inc.

Item	Cost, $/ha	Age cost incurred
Site preparation	$200	1
Nursery stock	$360	1
Planting	$360	2
First tending	$120	5
Monitoring	$10	35

Table 2.5: Silviculture costs under a basic regime

Harvest and transportation cost	$47
Price of SPF1	$60
Price of SPF2	$55
Price of SPF3	$30
Price of poplar/birch	$20

Table 2.6: Assumed values for log prices and cost of delivering logs to the mill in $ per cubic meter

SPF3 is less than 12 centimeters, and 'other' refers to other less valuable species (poplar and birch). Data used to plot this graph is provided in Insley and Wirjanto (2008).

Assumptions for harvesting costs and current log prices at the millgate are given in Table 2.6. These prices are considered representative for 2003 prices at the millgate in Ontario's boreal forest. Average cost to deliver logs to the lumber mill in 2003 are reported as $55 per cubic meter in a recent Ontario government report Ontario Ministry of Natural Resources (2005). From this is subtracted $8 per cubic meter as an average stumpage charge in 2003 giving $47 per cubic meter.[25] It will be noted the lower valued items (SPF3 and poplar/birch) are harvested at a loss. These items must be harvested according to Ontario government regulation. The price for poplar/birch is at roadside, so there is no transportation cost to the mill.

[25]This consists of $35 per cubic meter for harvesting and $12 per cubic meter for transportation. Average stumpage charges are available from the Ontario Ministry of Natural Resources.

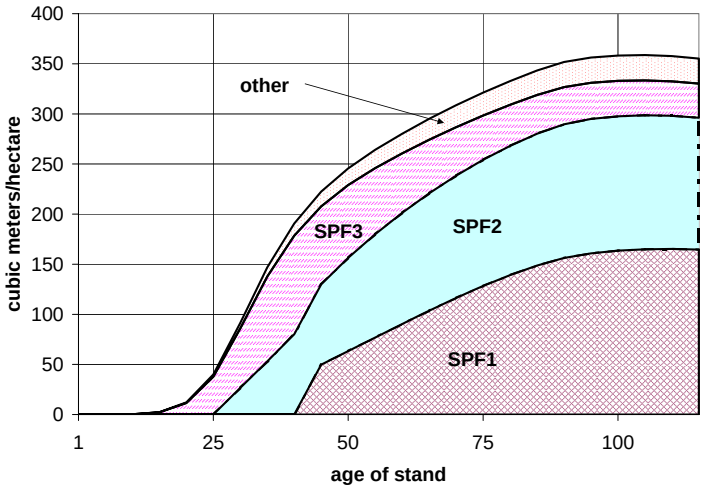


Figure 2.5: Volumes by product for hypothetical Jack Pine stands in Ontario's boreal forest under basic management

Land value in $ per hectare, Initial lumber price of $\$60/m^3$			
	RSMR model		TMR model
Initial Stand age	Regime 0	Regime 1	Single regime
Age 0	2858	2858	1404
Age 50	10593	10728	5617
Age 75	13406	13660	9078

Land value in $ per hectare, Initial lumber price of $\$100/m^3$			
	RSMR model		TMR model
Initial Stand age	Regime 0	Regime 1	Single regime
Age 0	2858	2858	1404
Age 50	11503	12242	7474
Age 75	15352	16619	13896

Table 2.7: Land values at the beginning of the first rotation for regime switching and traditional mean reversion models, $(2005)Canadian per hectare

2.7.2 Results for land value and critical harvesting prices

The parameter values of the RSMR model used to evaluate the investment are provided in previous sections. The equilibrium price levels in the two regimes, $K(s_t)$, as shown in Table 2.2, are stated in Canadian dollars at Toronto. In order to value our hypothetical stand of trees, the equilibrium prices need to be scaled to reflect prices at the millgate. Our estimate of price at the millgate in 2003 for SPF1 logs is Cdn.$60 per cubic meter. In 2003 the average spot price in Toronto was Cdn. $375 per MBF. I use the ratio of 375/60 as adjustment factor to scale the equilibrium price levels. The scaled long-run price levels become $K(0) = \$11.51$ and $K(1) = \$82.66$ per cubic metre. This rescaling accounts for transportation costs from Toronto to the mill and milling costs (as well as the conversion from MBF to m^3).

Land values calculated using the RSMR and TMR models are provided in Table 2.7 for three different initial stand ages and two initial lumber prices. For the

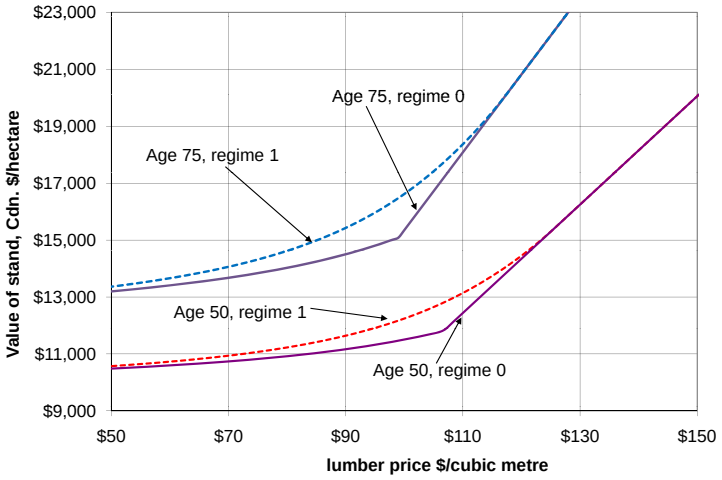


Figure 2.6: Land values for different aged stands in the RSMR case. Dashed lines: Regime 1, solid lines: Regime 0

RSMR model, the value of the opportunity to harvest a stand at the beginning of rotation (stand age of zero) is \$2858 per hectare in either regime 1 or 2 regime and for both initial price levels shown. This reflects the fact that at the beginning of the rotation the harvest date is many years away and regime switching will likely happen numerous times over the next few decades. Hence the current regime has little effect on land value at the beginning of the rotation. Similarly the current price has a negligible effect on the value of the bare land. For older stands for which the optimal harvesting time is nearer, the value of the stand does depend positively on the current price of lumber. Further, the stand value is slightly higher in regime 1 than in regime 0. In Table 2.7, we observe that at an initial price of $\$100/m^3$ the land value in regime 1 is approximately 8% higher than in regime 0. Another perspective on land values for older stands is given in Figure 2.6. Here we see that land values for 50 and 75 year old stands rise with lumber price and that for a range of prices values in regime 1 exceeds values in regime 0. As will be seen below, this price range is around the critical price level that would trigger optimal harvesting in regime 0. The apparent kink for regime 0 occurs at the critical harvesting price for that regime. The critical harvesting price for regime 1 occurs at the point of tangency between the two curves.

The value of land in the TMR regime, also shown in Table 2.7, is \$1404 per hectare at age 0, significantly lower than in the RSMR case. This is because, for the RSMR model, the calibrated mean price level in regime 1 is higher than that of the corresponding one-factor TMR model. Further, the price in the regime switching model stays in the high mean regime most of time giving a higher land value for the RSMR case.

For comparison purposes I note that the land value for the same stand at age 0 calculated in Insley and Lei (2007) was \$1630/ha. The analysis in Insley and Lei (2007) uses the same cost and yield data, with a TMR process. However the parameters of the TMR process were estimate through OLS on spot price data and

the market price of risk was estimated separately in a more simplistic manner.

Critical harvesting prices versus stand age are shown in Figure 2.7. For a stand of a given age, once the critical harvesting price is met or surpassed, harvesting of the stand and replanting for the next rotation are the optimal actions. Harvesting is not permitted in the model prior to age 35 until all silviculture expenditures have been made.

Critical prices are high during the earlier ages when the trees are still growing, but fall as the stand ages and eventually reach a steady state. Critical prices are highest for Regime 1 which is characterized by a high equilibrium level and a slower speed of mean reversion. Since volatility is at a moderate level of 0.25 and the probability of switching out of this regime is low, it is worthwhile delaying harvesting until a higher threshold is reached. In contrast in regime 0, the speed of mean reversion is faster and the equilibrium level is lower so that when in that regime it is expected that price will return fairly quickly to the low equilibrium level. In addition volatility in this regime is very low which reduces the value of delay. Offsetting this is a high likelihood of switching into the higher priced regime. Overall the critical prices of this regime are below those of Regime 1 at every age.

Critical prices for the TMR case are consistently below those of the two regimes in the RSMR model. This makes intuitive sense given that the long run equilibrium level is lower in the TMR case than in the high price regime (Regime 1) and that unlike in Regime 0, there is no potential to switch into a different regime with a higher long run equilibrium level.

In summary, the regime switching model results in different land values and leads to significantly different investment strategies than the corresponding single-factor models. The calibration results show the regime switching model outperforms the single regime model in terms of fitting lumber market prices. Moreover the regime switching model generates reasonable stand values as well as the critical prices. I would argue that the the regime switching model is preferred in the analysis of

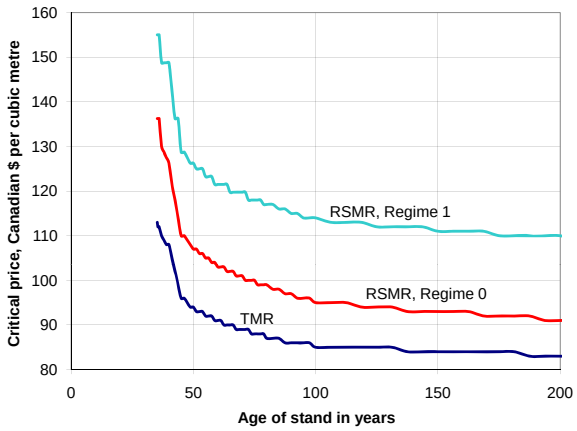


Figure 2.7: Critical harvesting prices for the RSMR and TMR cases

forestry investment decisions and land valuation.

2.8 Concluding remarks

Understanding forest valuation is important for policy makers, forestry firms and investors. In the Canadian situation, harvesting rights to specific areas of publicly owned forests are leased to private firms. Government regulators need to be aware of the value of these harvesting rights in order to ensure the public is compensated for the use of the resource and in order to gauge the impact of regulatory changes on the profitability of forestry operations. And of course private players in the industry also have an incentive to understand the impact of volatile prices on land values and optimal decisions, as well as changes that might result from regulatory decisions such as a requirement to increase spending on replanting or other conservation measures.

This chapter investigates a possible improvement in the modelling of stochastic timber prices in optimal tree harvesting problems. My goal is to find a modelling approach that is rich enough to capture the main characteristics of timber prices, while still being simple enough that the resulting price model can easily be incorporated into problems of forest investment valuation. I compare two different stochastic price process, a regime switching model with a different mean revering process in each regime (RSMR) and a traditional mean reverting model (TMR). The RSMR model allows for two states in lumber markets which we may characterize as being good times and bad times. The price models are calibrated using lumber futures prices and futures call option prices. The calibration process is able to find a reasonable fit for both models, but the mean absolute error is lower for the RSMR model.

In the second part of the chapter, I use the calibrated timber price models in a real options model of the optimal harvesting decision. PDEs characterizing the value of the stand of trees are derived using contingent claim analysis. A Hamilton-Jacobi-Bellman (HJB) variational inequality is then developed and solved using a fully implicit numerical method. I show that our numerical scheme converges to the viscosity solution (i.e. the financially reasonable solution.)

The empirical example is for a hypothetical stand of trees in Ontario's boreal forest. For the RSMR model, the estimated land value at the beginning of the rotation is insensitive to the particular regime and at $2858 per hectare is of a reasonable order of magnitude. The land value for the TMR model is $1404 per hectare. I also examined critical harvesting prices, which for the RSMR model differ depending on the current regime.

I conclude that the RSMR model shows some promise as a parsimonious model of timber prices, that can fairly easily be incorporated into optimal harvesting models. One limitation of this methodology is in the use of short term maturity contracts in the calibration exercise. The longest maturity of the chosen futures

contract is less than one year, but unfortunately this is all that is available. One may ask whether the calibrated parameter values are appropriate for long term forestry investment valuation problems. Schwartz and Smith (2000) has proposed a way of dealing with this issue. The applicability of his method for lumber prices is an area for future research.

Future research will also investigate the robustness of the RSMR model through comparison with other multi-factor models that have been used in the literature to value other commodity linked investments. I hope that other researchers will find the methodologies demonstrated here useful for the analysis of other types of investments, particularly those dependent on commodity prices where active futures markets exist.

Chapter 3

The impact of stochastic convenience yield on long-term forestry investment decisions

3.1 Introduction

The optimal management of natural resource investments typically depends on the ability of resource owners to interpret and react to volatile commodity prices. Owners of commercial forest land are no exception to this. Landowners are faced with decisions about when to harvest a stand of trees in an environment of highly uncertain timber prices which respond to news about the health of the economy, tariffs and trade barriers, as well as supply side factors such as fire and pests. A long strand of economics literature addresses the dual issues how best to model commodity prices and the determination of optimal resource management decisions under different representations of price. The literature has evolved significantly over the past few decades moving from deterministic models based on versions of

Hotelling's rule to stochastic models that draw on finance theory and contingent claims arguments. In addition to stochastic prices, the natural resources literature has investigated the impact of other key uncertain parameters, such as costs, interest rates, and convenience yield, on optimal natural resource management.

The focus of this chapter is on lumber prices and optimal decisions in forestry. A number of specifications have been proposed in the literature for modeling stochastic lumber prices, including geometric Brownian motion (GBM), mean reverting processes, jump processes and regime-switching models. For example, Clarke and Reed (1989) and Yin and Newman (1997) solve optimal tree harvesting problems analytically by assuming lumber prices follow GBM. Some researchers including Brazee et al. (1999) have found that mean reversion rather than GBM provides a better characterization of lumber prices. Saphores et al. (2002) find evidence of jumps in Pacific North West stumpage prices in the U.S. and demonstrate at the stand level that ignoring jumps can lead to significantly suboptimal harvesting decisions for old growth timber. A recent insight in the literature suggests that instead of modeling jumps in commodity prices, we may consider regime-switching models, initially proposed by Hamilton (1989), to better capture the main characteristics of lumber prices. Chen and Insley (2008) compare and contrast a two-state regime-switching mean reverting model and a traditional mean reverting model. They find that the regime-switching model outperforms the traditional one-factor mean reverting model in terms of fitting prices of market lumber derivatives.

For storable commodities and those that serve as inputs to production, such as lumber and oil, convenience yield[1] plays an important role in price formation. Convenience yield refers to the benefit that producers obtain from holding physical inventories, a benefit not available to individuals holding a futures or forward contract. Convenience yield is expected to be negatively correlated with inventories

[1]See Working (1948)

levels.[2] The seasonal harvesting of trees, as well as the importance of wood products as inputs to other industries, suggest that convenience yield may be important to understanding the dynamics of timber prices.

From a modelling perspective, convenience yield may be viewed as analogous to the dividend obtained from holding a company's stock. Convenience yield helps to explain the relationship between spot prices and futures prices - i.e. the term structure of commodity futures prices. The term structure conveys useful information for hedging or investment decisions, because it synthesizes the information available in the market and reflects the investors' expectations concerning the future. A futures price can be greater or less than the commodity spot price, depending on the relationship between the (net) convenience yield[3] and risk-free interest rate. This is explained by the cost of carry pricing model which expresses forward/futures price as a function of the spot price and the cost of carry.[4] Modelling of convenience yield is important for any analysis of futures prices.

Multi-factor models have been proposed in the literature to describe commodity price dynamics by including stochastic convenience yield to help explain the term structure of commodity futures prices. For example, Gibson and Schwartz (1990) first introduced a two-factor model, where spot prices are assumed to evolve according to GBM and the convenience yield follows a mean reverting stochastic process. Schwartz (1997) further explores this two-factor model in the context of a term structure model of commodity prices. This model provides a reasonable fit of the term structure of long-term forward prices which are essential for valuing long-term commodity linked investments. The Schwartz (1997) two-factor model has been successfully applied in the modelling of several key commercial commodities,

[2]See Brennan (1958), and Litzenberger and Rabinowitz (1995a).

[3]Net convenience yield is defined as the benefit of holding inventory minus physical storage costs. It is negative if the storage expense is higher. For simplicity, convenience yield mentioned in the rest of this chapter refers to net convenience yield.

[4]Cost of carry is defined as the physical storage cost plus the forgone interest. See Pindyck (2001).

including crude oil and copper. However, to the best of my knowledge no previous work has examined the impact of modeling stochastic convenience yield in an optimal harvesting problem applied to a renewable natural resource such as timber. In this chapter, I will investigate the implication of including stochastic convenience yield on the behavior of lumber prices and long-term forestry investment decisions.

The objective of this chapter is to further our understanding of the optimal valuation and harvesting of commercial forest land by using lumber future prices to estimate a two-factor model of lumber prices that includes stochastic convenience yield. The approach for estimating the price model uses the Kalman filter as is done in Schwartz (1997). I compare the ability of this two-factor model to match the term structure of lumber futures prices with that of two other simpler models which do not include stochastic convenience yield. These simpler models represent GBM and mean reverting processes. I then use the estimated price models in a multi-rotational optimal tree harvesting problem.

A real options model of the forestry investment valuation is developed assuming a joint stochastic process of lumber prices and convenience yield. The Hamilton-Jacobi-Bellman (HJB) equation characterizing the value of the option to harvest a stand contains three state variables: lumber prices, convenience yield and the stand age. To simplify the solution of the harvesting problem, I use the one-factor model introduced in Schwartz (1998), which retains most of the main characteristics of his two-factor model, especially its ability to fit long-term commodity futures prices. I call this one-factor model the "long-term model". The HJB equation derived using the long run model is solved numerically using the combination of the fully implicit finite difference method, the semi-Lagrangian method and the penalty method. The optimal harvesting decisions and land value computed for the long-term model are then compared with results using the simple GBM and mean reverting models.

The main conclusion is that modelling stochastic convenience yield improves our ability to match lumber futures prices and that the long run model provides

reasonable estimates of land value and optimal harvesting decisions. The long-run model gives significantly different results than the other simpler models that are used for comparison.

The remainder of this chapter is organized as follows. In Section 3.2 I give a brief description of lumber spot and futures prices. Section 3.3 describes the two-factor price model as well as the GBM and mean reverting models used for comparison. Section 3.4 describes the estimation of the price models using the Kalman filter. Section 3.5 presents the long-term model which is used as an approximation of the two-factor model. Section 3.6 describes the empirical results of the price model estimation and compares the ability to the different models to match the term structure of futures prices. Section 3.7 presents the real options model and analysis of an optimal tree harvesting problem. Section 3.8 concludes.

3.2 Lumber spot prices and futures prices

Forest products are traded worldwide and Canada is a major player in this market, accounting for 14% of the value of world forest product exports in 2006. There are two types of lumber, softwood and hardwood, with softwood lumber generally used in construction, building and housing purposes. It is also the underlying asset of lumber futures contracts traded in the futures market. There is no single spot market in which a uniform lumber product is traded, and therefore there is no unique lumber spot price. However, there is a single North American market for standard lumber futures contracts. Following the literature[5], the price of the futures contract which is closest to maturity is treated as the lumber spot price.

Lumber futures contracts were first traded on the Chicago Mercantile Exchange (CME) in 1969. The Random Length Lumber futures traded on the CME are

[5]See Gibson and Schwartz (1990).

Item	Max	Min	Mean	Std. Dev.	Skewness	Kurtosis
\$ Cdn/MBF	718.1	164.8	423.7	107.6	-0.09	2.50

Table 3.1: Descriptive statistics for the lumber price time series (as shown in Figure 3.1), from January 6th, 1995 to April 25th, 2008.

for on-track mill delivery of 110,000 board feet (plus or minus 5,000 board feet) of random length 8-foot to 20-foot nominal 2-inch $\times$ 4-inch pieces. The delivery contract months for CME Random Length Lumber futures are as follows: January, March, May, July, September and November. The last trading day of each contract is the business day prior to the 16th calendar day of the contract month.

Real spot prices, as approximated by the prices of the lumber futures contract closest to maturity, are shown in Figure 3.1[6]. These are weekly data, covering the period from January 1995 to April 2008. The original data in U.S. dollars were deflated by the CPI and converted to Canadian dollars. The transformation is made because our real options application is a hypothetical decision problem in Canada's boreal forest. In Figure 3.1 prior to 2006 there appears to be a tendency to revert to a mean between \$400 and \$500 (Cdn) per MBF. After 2006 we see a downward progression in price reflecting weak North American lumber markets as well as the impact of a strengthening Canadian dollar. We also observe a significant level of volatility. Summary statistics of the spot lumber prices are reported in Table 3.1.

There are six lumber futures contracts traded each day on the CME, the first four of which will be used in my analysis as these have the highest trading volumes and can be expected to provide more accurate information. Real weekly prices of the four selected lumber futures prices, ranging from January 1995 to April 2008, are shown in Figure 3.2. Summary statistics of these four time series are provided in Table 3.2. The mean price shown in Table 3.2 is lowest for the shortest maturity contract and rises with contract maturity. Conversely, the largest volatilities are for

[6]Data source: Chicago Mercantile Exchange.

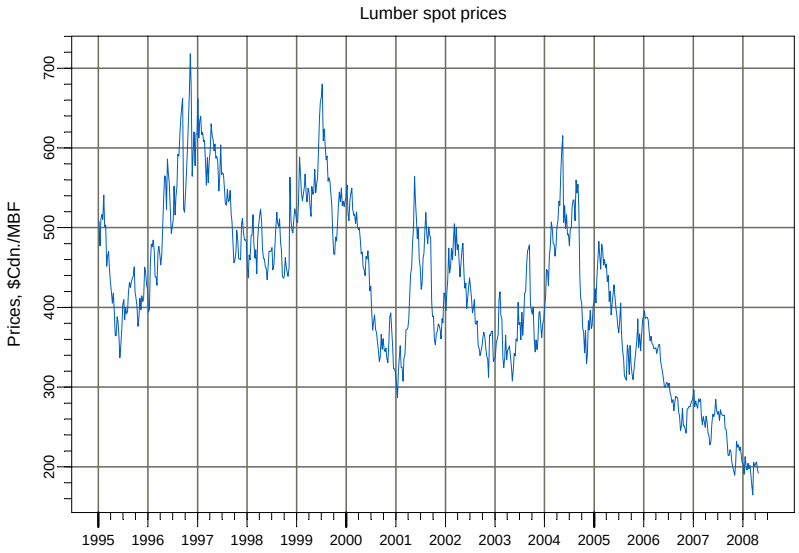


Figure 3.1: Real prices of lumber futures contract closed to maturity. Weekly data from January 6th, 1995 to April 25th, 2008, $Cdn./MBF, (MBF $\equiv$ thousand board feet). Nominal prices deflated by the Consumer Price Index, base year = 2005.

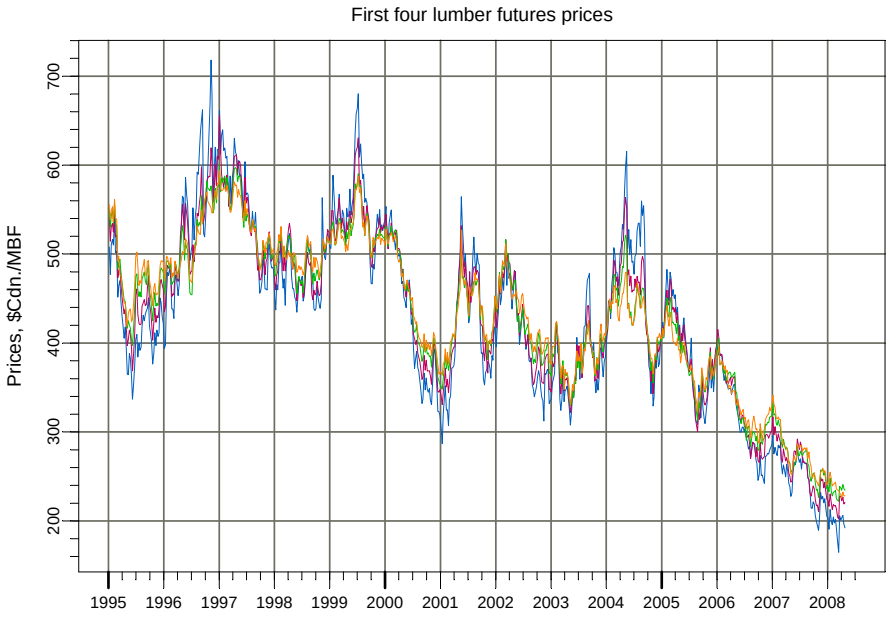


Figure 3.2: Real prices of four CME lumber futures,$Cdn./MBF (thousand board feet). Weekly data from January 6th, 1995 to April 25th, 2008. Nominal prices deflated by the Consumer Price Index, base year = 2005.

the prices of the short-term contracts, while the volatilities for the two longer term contracts are fairly close. The decreasing pattern of volatilities along the prices curve is often called "the Samuelson effect" in the literature. The term structure of lumber futures shown in Figure 3.3 provides an illustration of the Samuelson effect. In the diagram the spread of futures prices at 4 (F4) is smaller than at 1 (F1), indicating that the near term prices are more volatile. From this graph, we observe different shapes of the lumber term structure, from backwardation to contango for example.

Item	Number of observations	Mean	Std. Deviation	Maturity(on average)
F1	695	423.7	107.57	1 mon
F2	695	427.5	96.24	3 mon
F3	695	429.6	89.08	5 mon
F4	695	431.5	86.15	7 mon

Table 3.2: Summary statistics of four chosen CME lumber futures prices, $Cdn./MBF. Weekly data from January 6th, 1995 to April 25th, 2008.

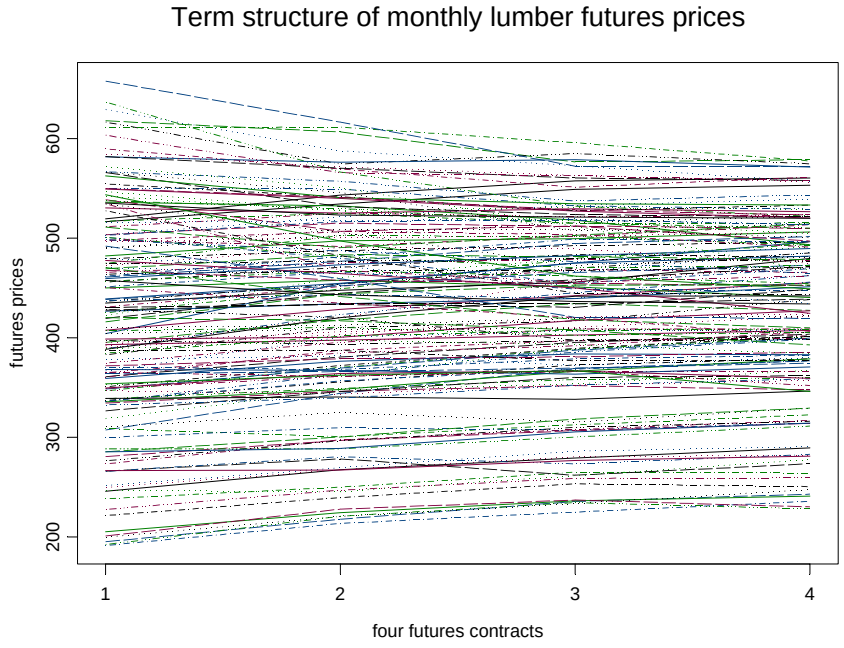


Figure 3.3: Term structure of lumber futures,$Cdn./MBF. Monthly data from January 6th, 1995 to April 25th, 2008.

3.3 Valuation models

The varying shapes of term structure of lumber futures prices shown in Figure 3.3 imply the need to model convenience yield in order to capture the main characteristics of lumber spot and futures prices. The two-factor model analyzed in Schwartz (1997) is one of the most popular models in this literature and it has been successfully employed to model several commodities, including crude oil and copper. For the convenience of the reader, in the next section I summarize the Schwartz (1997) two-factor model. I also present two one-factor models, GBM and mean reverting, to be used as comparison with the two-factor model. These one-factor models are also popular in the literature and are simpler to estimate and use in models of investment decisions than the two-factor model. I would like to determine whether the two-factor model does a substantially better job at modelling lumber prices and is therefore worth using despite its increased complexity.

3.3.1 The Schwartz (1997) two-factor model

The two-factor model analyzed in Schwartz (1997) is based on the model developed in Gibson and Schwartz (1990). Specifically, the spot price S follows a GBM process with a stochastic drift and the net convenience yield δ is formulated as a mean-reverting Ornstein-Uhlenbeck process. The joint stochastic process of the two state variables in Schwartz (1997) is given by:

$$\begin{aligned} dS &= (\mu - \delta)Sdt + \sigma_s S dz_s \\ d\delta &= \kappa(\alpha - \delta)dt + \sigma_\delta dz_\delta \\ dz_s dz_\delta &= \rho dt \end{aligned} \tag{3.1}$$

where μ is the expected return of spot prices, κ and α represent the mean reversion rate and the long-run equilibrium level of convenience yield respectively, σ_s and σ_δ

denote the volatilities of the two state variables, and ρ is the correlation coefficient between the two standard Brownian increments dz_s and dz_δ.

I note in the above specification that μ represents the total expected return from S and it remains constant. As the convenience yield changes the portion of total return that derives from capital gains, $(\mu - \delta)$, and the portion that derives from convenience yield adjusts stochastically while the total return is assumed fixed and determined by the market equilibrium return for that particular asset class.

I expect convenience yield and the commodity price to be positively correlated. Intuitively, when there is excess supply on the market, lumber inventories will rise and the spot price should fall. Convenience yield should also fall since the benefit of owning the commodity is smaller than when the commodity is scarce. A lower convenience yield implies it is more costly to carry commodity inventories. This will tend to drive up the futures price as it becomes more attractive to secure supply in the futures market rather than carrying inventory.

In Equation (3.1) convenience yield affects S through the correlation coefficient as well as through the drift term. With a positive ρ, a fall in S implies a fall in δ. This lower δ increases the drift rate for S, and hence S is pulled up again. Hence in the model specified in Equation (3.1), S is characterized by some reversion to the mean, but the mean is not constant.

The specification of convenience yield as a mean reverting process also makes intuitive sense. α represents a long run level that reflects the cost of storing the commodity and a benefit conveyed by having immediate access to inventories. δ will vary around α depending on commodity market conditions with $\delta > \alpha$ when markets are buoyant and the reverse when markets are depressed. The inverse relationship between the level of inventory and the convenience yield prevents the possibility that the net convenience yield goes to infinity. This relationship holds for storable commodities including lumber.

Under the equivalent martingale measure, the risk adjusted processes for the two state variables, the spot price S and convenience yield δ, are expressed as:

$$\begin{aligned} dS &= (r-\delta)Sdt + \sigma_s S dz_s^* \\ d\delta &= [\kappa(\alpha-\delta)-\lambda]dt + \sigma_\delta dz_\delta^* \\ dz_s^* dz_\delta^* &= \rho dt \end{aligned} \tag{3.2}$$

where λ is the market price of convenience yield risk. From the no-arbitrage condition, the risk-adjusted drift of the price process is $r-\delta$. The market price of convenience yield risk has to be incorporated in the risk neutral process of convenience yield, since convenience yield is not traded.

Applying Ito's Lemma, the log spot price $X = \ln S$ in this two-factor model can be derived as:

$$dX = (\mu - \frac{1}{2}\sigma_s - \delta)dt + \sigma_s dz_s \tag{3.3}$$

The partial differential equation (PDE)[7] characterizing the futures price $F(S,\delta,t,T)$ can be derived using Ito's Lemma and expressed as:

$$\frac{1}{2}\sigma_s^2 S^2 F_{ss} + (r-\delta)SF_s + \frac{1}{2}\sigma_\delta^2 F_{\delta\delta} + (\kappa(\alpha-\delta)-\lambda)F_\delta + \rho\sigma_s\sigma_\delta SF_{s\delta} - F_t = 0 \tag{3.4}$$

subject to boundary condition: $F(S,\delta,T,T) = S$, where T denotes the maturity date of the futures contract. The analytical solution of equation (4.15) is derived in Jamshidian and Fein (1990) and Bjerksund (1991) and can be expressed as:

$$F(S,\delta,0,T) = S\exp\left[A(T) - \delta\frac{1-e^{-\kappa T}}{\kappa}\right] \tag{3.5}$$

[7]For detailed derivation of this PDE, see Gibson and Schwartz (1990).

where

$$\begin{aligned} A(T) &= (r-\hat{\alpha}+\frac{1}{2}\frac{\sigma_\delta^2}{\kappa^2}-\frac{\sigma_s\sigma_\delta\rho}{\kappa})T+\frac{1}{4}\sigma_\delta^2\frac{1-e^{-2\kappa T}}{\kappa^3}+(\hat{\alpha}\kappa+\sigma_s\sigma_\delta\rho-\frac{\sigma_\delta^2}{\kappa})\frac{1-e^{-\kappa T}}{\kappa^2} \\ \hat{\alpha} &= \alpha-\frac{\lambda}{\kappa} \end{aligned} \tag{3.6}$$

The linear relationship between futures prices and spot prices can be found in the log form of futures prices:

$$\ln F(S,\delta,0,T)=\ln S+A(T)-\delta\frac{1-e^{-\kappa T}}{\kappa} \tag{3.7}$$

Equation (3.7) will be used for model estimation.

3.3.2 Single factor models

In order to analyze the impact of incorporating stochastic convenience yield on long-term forestry investment decisions, two single factor models are also estimated and compared in this chapter. These one-factor models are the log price mean reverting model analyzed in Schwartz (1997) and a GBM model with a constant convenience yield. Since the two-factor model analyzed in this chapter features mean reversion in the commodity's price, it seems reasonable to compare it with a single factor mean reverting model. I also use the GBM model for comparison since it is so widely used and the spot price in two-factor model follows an adjusted GBM process with stochastic convenience yield on the drift term. In this section, these one-factor models are briefly summarized.

The one-factor mean reverting model

This model is the same single factor model as analyzed in Schwartz (1997). The spot prices S are modeled as:

$$\frac{dS}{S} = \kappa_{MR}[\mu_{MR} - \ln S]dt + \sigma_{MR}dz \tag{3.8}$$

Applying Ito's Lemma, the log spot price $X = \ln S$ follows an Ornstein-Uhlenbeck process:

$$dX = \kappa_{MR}\big[(\mu_{MR} - \frac{\sigma_{MR}^2}{2\kappa_{MR}}) - X\big]dt + \sigma_{MR}dz \tag{3.9}$$

where the mean reverting rate is κ_{MR} and the long-run equilibrium log price level is $\mu_{MR} - \frac{\sigma_{MR}^2}{2\kappa_{MR}}$. The risk-adjusted version of this model can be expressed as:

$$dX = \kappa_{MR}[\alpha^* - X]dt + \sigma_{MR}dz^* \tag{3.10}$$

where $\alpha^* = \mu_{MR} - \frac{\sigma_{MR}^2}{2\kappa_{MR}} - \lambda_{MR}$ and λ_{MR} represents the market price of risk.

The corresponding futures price in log form, $\ln F(S, 0, T)$, can be expressed as[8]:

$$\ln F(S, 0, T) = e^{-\kappa_{MR}T} \ln S + (1 - e^{-\kappa_{MR}T})\alpha^* + \frac{\sigma_{MR}^2}{4\kappa_{MR}}(1 - e^{-2\kappa_{MR}T}) \tag{3.11}$$

This linear relationship between log futures prices and the state variable log spot prices will be used for model estimation.

The GBM model

The GBM model can be expressed as:

$$dS = [\mu_{GBM} - \delta_{GBM}]Sdt + \sigma_{GBM}Sdz \tag{3.12}$$

[8]See Schwartz (1997).

where δ_{GBM} refers to the constant convenience yield. Similarly, the log price $X = \ln S$ follows a normal distribution which can be expressed as:

$$dX = \big[\mu_{GBM} - \delta_{GBM} - \frac{\sigma^2_{GBM}}{2}\big]dt + \sigma_{GBM}dz \tag{3.13}$$

The risk-neutral version of this model is:

$$dX = \big[r - \delta_{GBM} - \frac{\sigma^2_{GBM}}{2}\big]dt + \sigma_{GBM}dz^* \tag{3.14}$$

The conditional mean of X under the equivalent martingale measure is $E_0[X(T)] = (r - \delta_{GBM} - \frac{\sigma^2_{GBM1}}{2})T + X_0$. Its conditional variance is $Var_0[X(T)] = \sigma^2_{GBM}T$.

Based on the properties of the log normal distribution, the futures price $F(S, 0, T)$ in this model can be expressed as:

$$F(S, 0, T) = e^{\ln S + (r - \delta_{GBM})T} \tag{3.15}$$

The log futures price can be derived as:

$$\ln F = \ln S + (r - \delta_{GBM})T \tag{3.16}$$

3.4 Model estimation: Kalman filter

When state variables are not observable, a practical method for estimating this type of model is by stating the problem in state space form and by using the Kalman filter based on an error prediction decomposition of the log-likelihood function. The Kalman filter is a recursive procedure for estimating unobserved state variables based on observations that depend on these variables (Kalman (1960)). Prediction errors, a by-product of the Kalman filter, can then be used to evaluate the likelihood

function and the model parameters are estimated by maximizing this likelihood function.

The state space form consists of a transition equation and a measurement equation. The transition equation describes the dynamics of an unobserved set of state variables. The measurement equation relates the unobserved variables to a vector of observables. In the two-factor model analyzed in this chapter, both the lumber spot price and convenience yield are assumed to be unobserved state variables.[9] The lumber spot prices in the single-factor models are also assumed to be unobserved. Futures prices with different maturities observed at different dates serve as observed variables and the measurement equation will specify the relationship between futures prices and the two state variables.

Specifically, the linear Gaussian state space model can be expressed as the following system of equations:

$$x_{t+1} = d_t + T_t x_t + \eta_t \tag{3.17}$$

$$y_t = C_t + Z_t x_t + \epsilon_t \tag{3.18}$$

where x denotes the vector of unobserved state variables and $y = \ln F$ denotes the observed log futures prices for all the models analyzed in this chapter.[10] Equation (3.17) represents the transition equation of the model, which describes the evolution of the non-observed state vector x_t over time. Equation (3.18) is the measurement equation describing the vector of observations y_t in terms of the state vector.

Two types of variables used recursively in the Kalman filter algorithm are called priori variables and posteriori variables. Define the observed data set at time t as

[9] In the commodity literature, since the exact meaning of commodity spot prices like electricity is difficult to pin down, when using Kalman filter to estimate parameters of the model containing spot price dynamics, researchers treat spot prices as unobserved state variable. See Schwartz (1997) for example.

[10] d_t, T_t, C_t, Z_t are terms containing corresponding model parameters which will be specified later in this chapter. η_t and ϵ_t denote the disturbances of the two equations.

$Y_t = (y_1, ..., y_t)$. Priori variables refer to the conditional mean, defined as $x_{t|t-1} = E[x_t|Y_{t-1}]$, and conditional variance, defined as $P_{t|t-1} = var[x_t|Y_{t-1}]$, of the state vector x_t based on information available at time $t-1$. Posteriori variables are the estimates for the mean and variance of the state vector conditional on the information available at time t, denoted as $x_{t|t} = E[x_t|Y_t]$ and $P_{t|t} = var[x_t|Y_t]$ respectively.

The first step of the Kalman filter is to compute one time step ahead priori variables $x_{t|t-1}$ and $P_{t|t-1}$ using the values of posteriori variables at time $t-1$ via the prediction equations:

$$x_{t|t-1} = d_{t-1} + T_{t-1}x_{t-1|t-1} \tag{3.19}$$

$$P_{t|t-1} = T_{t-1}P_{t-1|t-1}T'_{t-1} + Var(\eta) \tag{3.20}$$

Next, with the new observation y_t, the posteriori variables at time t can be updated using updating equations:

$$x_{t|t} = x_{t|t-1} + K_t v_t \tag{3.21}$$

$$P_{t|t} = P_{t|t-1} - P_{t|t-1}Z'_t K'_t \tag{3.22}$$

where

$$v_t = y_t - C_t - Z_t x'_{t|t-1} \tag{3.23}$$

$$F_t = Z_t P_{t|t-1} Z'_t + var(\epsilon) \tag{3.24}$$

$$K_t = P_{t|t-1}Z'_t F_t^{-1} \tag{3.25}$$

where v_t is the residual of the measurement equation (3.18) or prediction error. F_t is the variance of this prediction error, $F_t = var(v_t)$. K_t is the Kalman gain. This process is then repeated until the whole set of observations Y_N has been observed

and used in this recursive process. The resulting estimates of posteriori variables $x_{t|t}$ will be the filtered estimates of the state vector for each observation date t. The smoothed estimates of the state vector can be obtained by using all the information in the observation set Y_N.

Unknown parameters of the state space model can be estimated by maximizing the prediction error decomposition of the log-likelihood function, which is a by-product of the Kalman filter. The sample log-likelihood function is

$$\ln L = \sum_{t+1}^{N} \ln f(v_t) = c - \frac{1}{2} \sum_{t+1}^{N} (\ln |F_t| + v_t' F_t^{-1} v_t) \tag{3.26}$$

c is a constant and $f(v_t)$ denotes the probability density function of prediction error v_t. The standard errors of the maximum likelihood estimates of the parameters were calculated by taking the inverse of the sum of the outer product of the score functions evaluated at the maximum likelihood estimates for each observation.[11]

The two-factor model and single-factor models analyzed in this chapter are all written in the state space form and the corresponding model parameters are estimated using the Kalman Filter method. The state space form of each model is provided in this section.

3.4.1 Two-factor model

For the two-factor model, both the stochastic spot price and convenience yield serve as the unobserved state variables $x = [X, \delta]'$, where $X = \ln S$ denotes the log of the spot price. Based on equations (3.1) and (3.3), the terms of the transition equation

[11] S-plus was used to conduct the estimation process.

(3.17) in the state space form can be expressed as:

$$\begin{aligned} d_t &= [(\mu - \frac{1}{2}\sigma_s^2)\Delta t, \kappa\alpha\Delta t]' \\ T_t &= \begin{bmatrix} 1 & -\delta_t \\ 0 & 1-\kappa\delta_t \end{bmatrix} \end{aligned}$$

η_t in equation (4.19) denotes serially uncorrelated disturbances with mean zero for the equations characterizing the unobserved two state variables , and its covariance matrix is expressed as:

$$Var(\eta_t) = \begin{bmatrix} \sigma_s^2\Delta t & \rho\sigma_s\sigma_\delta\Delta t \\ \rho\sigma_s\sigma_\delta\Delta t & \sigma_\delta^2\Delta t \end{bmatrix}$$

Based on equation (3.7), the terms of the measurement equation (3.18) are given as:

$$\begin{aligned} C_t &= [A(T_i)] \\ Z_t &= \left[1, -\frac{1-e^{-\kappa T_i}}{\kappa}\right] \end{aligned}$$

$i = 1, ..., N$, where N is the number of futures contracts at each date t. ϵ_t in equation (3.17) represents a vector of serially uncorrelated disturbances with zero mean and identity variance-covariance matrix. The innovations in the transition equation η_t and those in the measurement equation ϵ_t are assumed to be independent in all the analyzed models in this chapter, which means $E[\eta_t\epsilon_t] = 0$.

3.4.2 One-factor mean reverting model

The spot price in this one-factor mean reverting model is the unobserved state variable, $x = [X]$. Based on equation (3.9), the terms of the transition equation

(3.17) in the state space form can be expressed as:

$$\begin{aligned} d_t &= [\kappa_{MR}(\mu_{MR} - \frac{\sigma_{MR}^2}{2\kappa_{MR}})\triangle t] \\ T_t &= [1 - \kappa_{MR}\triangle t] \end{aligned}$$

η_t in equation (3.17) denotes serially uncorrelated disturbances with mean zero, and its variance is $\sigma_{MR}^2\triangle t$.

Based on equation (3.11), the terms of the measurement equation (3.18) are given as:

$$\begin{aligned} C_t &= [(1 - e^{-\kappa_{MR}T_i})\alpha^* + \frac{\sigma_{MR}^2}{4\kappa_{MR}}(1 - e^{-2\kappa_{MR}T_i})] \\ Z_t &= [e^{-\kappa_{MR}T_i}] \end{aligned}$$

$i = 1, ..., N$. ϵ_t in equation (3.18) represents a vector of serially uncorrelated disturbances with zero mean and identity variance-covariance matrix.

3.4.3 GBM model

In this one-factor model, the spot price is the unobserved state variable, $x = [X]$. Based on equation (3.13), the terms of the transition equation (3.17) in the state space form can be expressed as:

$$\begin{aligned} d_t &= \left[(\mu_{GBM} - \delta_{GBM} - \frac{\sigma_{GBM}^2}{2})\triangle t\right] \\ T_t &= [1] \end{aligned}$$

η_t in equation (3.17) denotes serially uncorrelated disturbances with mean zero, and its variance is $\sigma_{GBM}^2\triangle t$.

Based on equation (3.16), the terms of the measurement equation (3.18) are

given as:

$$\begin{aligned} C_t &= [(r - \delta_{GBM})T_i] \\ Z_t &= [1] \end{aligned}$$

$i = 1, ..., N$. ϵ_t in equation (3.18) represents a vector of serially uncorrelated disturbances with zero mean and identity variance-covariance matrix.

3.5 Schwartz (1998) one-factor long-term model

Schwartz (1998) develops a one-factor model which is simpler than the two-factor model analyzed in Schwartz (1997) in terms of valuing long-term commodity-related investments, but it closely matches the performance of the two-factor model in terms of fitting the term structure of long term futures prices and the volatilities of all futures contracts. Schwartz (1998) calls it the long-term model. In this section, the one-factor long-term model is summarized.[12] All the parameters in this long-term model are derived from the parameter estimates in the Schwartz (1997) two-factor model.

The motivation for this one-factor long-term model is to match as closely as possible the risk-neutral distribution of the spot prices in the Schwartz (1997) two-factor model. In the risk-neutral world, the spot prices in the two-factor model are lognormally distributed with mean equal to the futures price and variance depending on the volatility of futures returns.[13] Schwartz (1998) applied his one-factor long-term model to oil and was able to fairly accurately generate long-term futures prices and the term structure of the futures volatilities.

[12]For the convenience of readers, the derivation of this long-term model is provided in Appendix B1.

[13]See Schwartz (1998).

The long-term model uses a composite price, denoted Z, as the single stochastic state variable.[14] Z depends on the two stochastic factors, spot price S and convenience yield δ, as follows:[15]

$$Z(S,\delta) = Se^{[\frac{c-\delta}{\kappa} - \frac{\sigma_\delta^2}{4\kappa^3}]} \tag{3.27}$$

c is defined as the constant convenience yield used to match the long-term rate of change in the futures prices and is expressed as

$$c = \alpha - \frac{\lambda}{\kappa} - \frac{\sigma_\delta^2}{2\kappa^2} + \frac{\rho\sigma_s\sigma_\delta}{\kappa} \tag{3.28}$$

Given S, δ, and the model parameters, Z can be calculated based on Equation (3.27). Z is defined in such a way that the futures prices of this one-factor model $F(Z,T)$ match the long-term futures prices of two-factor model $F(S,\delta,T)$, given the constant convenience yield expressed in Equation (3.28). It may be noted from Equation (3.27) that Z is increasing in S and decreasing in δ.

In order to match the volatility of futures returns between the one-factor long term model and the two-factor model, the stochastic differential equation followed by Z is given as:

$$\frac{dZ}{Z} = (r-c)dt + \sigma_F(t)dz \tag{3.29}$$

where $\sigma_F(t)$ represents the volatility of the futures returns based on the Schwartz (1997) two-factor model and is derived as

$$\sigma_F^2(t) = \sigma_s^2 + \sigma_\delta^2\frac{(1-e^{-\kappa t})^2}{\kappa^2} - 2\rho\sigma_s\sigma_\delta\frac{(1-e^{-\kappa t})}{\kappa} \tag{3.30}$$

Therefore, the futures price, F, with maturity T and the composite spot price Z,

[14] In Schwartz (1998), Z is referred to as the shadow price.

[15] This expression is slightly different than the corresponding Equation 17 in Schwartz (1998).

in this one-factor long-term model can be expressed as:

$$F(Z,T) = Ze^{(r-c)T} \tag{3.31}$$

This long-term model devised by Schwartz (1998) is much easier to use in valuing investment opportunities because there is only one stochastic variable, the composite price Z. Schwartz (1997) found that for oil prices the performance of this one-factor model in terms of fitting the long-term futures prices and the term structure of futures volatilities is comparable with that of the two-factor model. I investigate whether the long term model also works for lumber prices.

3.6 Estimation results

The prices of four lumber futures contracts are used for model estimation and their main characteristics are detailed in Section 3.2. In order to check the convergence of the estimated parameters, I used different sets of starting values for the maximization of the log-likelihood function of the model and obtained the same parameter estimates. In this section, the estimation results of one-factor and two-factor models analyzed in this chapter are presented and the corresponding model performance is examined. In addition the futures prices implied by the long-term model and the two-factor model are compared to determine whether the former provides a reasonable approximation of the latter.

3.6.1 The two-factor model

The parameter estimates of the two-factor model, Equation (3.1), using weekly futures prices are reported in Table 3.3. The estimate of the correlation coefficient, ρ, is above 0.9 and is statistically different from zero. This result implies that

	μ	κ	α	σ_s	σ_δ	ρ	λ
Estimates	-0.124	2.089	-0.142	0.397	0.824	0.934	-0.212
Std. Error	(0.107)	(0.136)	(0.107)	(0.014)	(0.049)	(0.008)	(0.227)
LL	6471.7						

Table 3.3: Parameter estimates of Schwartz (1997) two-factor model using Kalman filter. LL refers to the value of log-likelihood function. Weekly futures prices, from January 6th, 1995 to April 25th, 2008.

convenience yield is an important factor affecting lumber price dynamics. The positive estimate of ρ is consistent with the theory of storage and helps to explain the mean reverting feature of lumber prices observed in Figure 3.1. The estimate of the mean reversion rate, κ, for the convenience yield process is high and significant as well. $-ln(0.5)/\kappa$ can be interpreted as the half-life of the time it takes for δ to return to its long run value. With $\kappa = 2.089$ I expect the deviation δ from the long run value will halve in 0.33 years. The estimate of the equilibrium convenience yield level α is not significantly different from zero, which implies that on average, the net convenience yield of lumber is about zero. This result is consistent with the theoretical prediction that in equilibrium, the benefit of holding the physical commodity should be equal to the cost of storage, which leads to the zero net convenience yield. The estimate of market price of convenience yield risk, λ, is found to be not significant as well.

Model implied spot prices, S, and market lumber prices proxied by the futures contract closest to maturity, $F1$, are plotted in Figure 3.4. From this graph it appears that the model implied prices move very closely with the market spot prices.

Figure 3.5 plots the two model implied state variables, spot prices and convenience yield. The red line denotes the spot prices and blue line represents the convenience yield. This figure shows that spot prices and convenience yield tend to move together, confirming the estimation result of a high and positive correlation

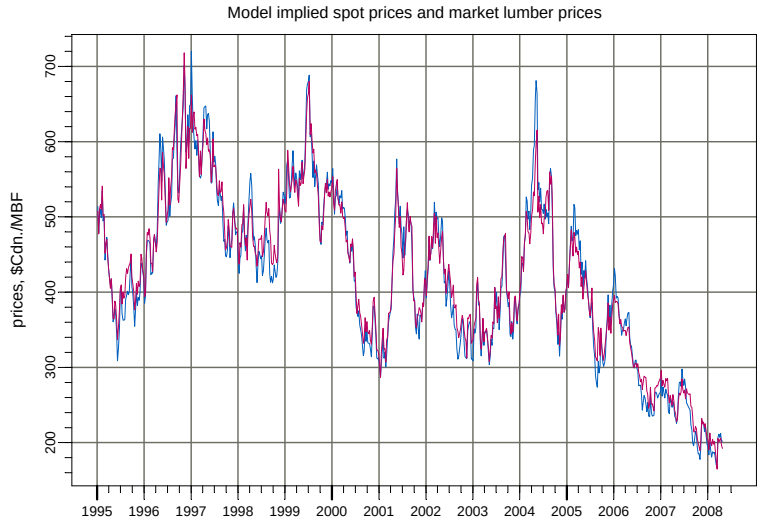


Figure 3.4: Plots of model implied (two-factor model) and market spot prices. Blue line: model implied prices. Red line: market prices.

Item	Max	Min	Mean	Std. Dev.
Net convenience yield	1.00	-0.94	-0.14	0.35

Table 3.4: Descriptive statistics for model implied net convenience yield.

coefficient ρ. From this figure, we also notice that the net convenience yield can be negative or positive and fluctuates around zero in the range of $[-1, 1]$. Whenever convenience yield exceeds higher than storage cost, net convenience yield is positive. Conversely, if storage cost exceeds convenience yield, net convenience yield will be negative. In the long-run, convenience yield is approximately equal to storage cost. Summary statistics of net convenience yield are provided in Table 3.4.

Model estimation errors of both futures prices and log futures prices including Root Mean Square Error (RMSE) and Mean Absolute Error (MAE) expressed in dollars per thousand board feet for four futures contracts are reported in Table 3.5. The overall average error of model implied futures prices from the last column is

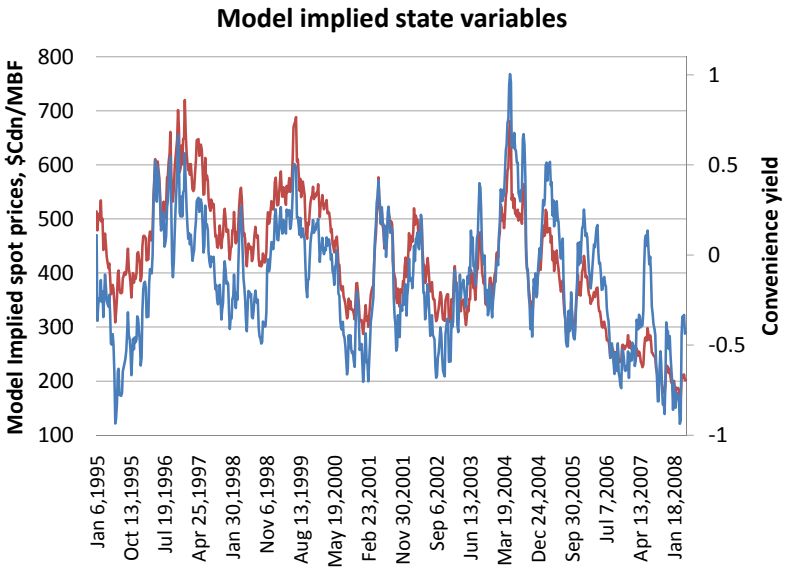


Figure 3.5: Red line: model implied spot prices. Blue line: model implied convenience yield.

Schwartz (1997) two-factor model					
	F1	F2	F3	F4	All
	Estimation errors of futures prices				
RMSE	13.664	1.764	4.719	3.600	7.501
MAE	9.701	1.327	3.547	2.711	4.322
	Estimation errors of log futures prices				
RMSE	0.031	0.004	0.011	0.009	0.017
MAE	0.023	0.003	0.008	0.006	0.010

Table 3.5: Average estimation errors of both futures prices and log futures prices of Schwartz (1997) two-factor model expressed as RMSE and MAE of 4 futures contracts, Cdn$/MBF.

less than $8/MBF which is about 1.8% of the mean lumber spot price. The overall average errors of log futures prices expressed in both ways are less than two cents per thousand board feet. It appears that the two-factor model provides a good tracking of the lumber futures time series. Plots of market futures prices and the model implied futures prices for the four futures contracts are shown in Figure 3.6. Again, the graphs display a reasonably close fit of the model prices versus actual prices.

3.6.2 One-factor mean reverting model

Parameter estimates for the single-factor model are reported in Table 3.6. From this table we find that all the model parameters are statistically significant except for the market price of risk λ_{MR}. The long-run equilibrium log price level $\mu_{MR} - \frac{\sigma_{MR}^2}{2\kappa_{MR}} = 6.206$ which implies a value for S of $496 per MBF. The mean reverting rate κ_{MR} is moderate at 0.229. Model implied and market lumber spot prices are plotted in Figure 3.7. The average error (RMSE) for all four futures maturities is $17.8 per MBF which is larger than for the two-factor model. More details are provided in Appendix B.

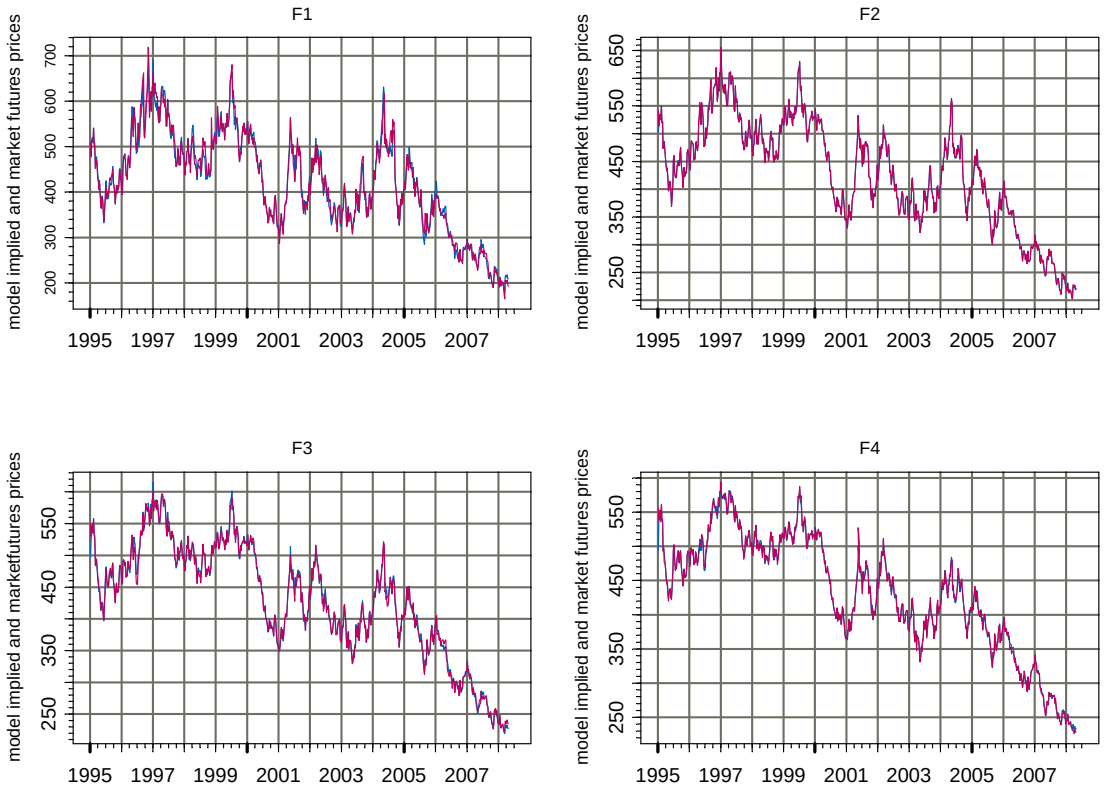


Figure 3.6: Plots of model implied and market futures prices for the two-factor model and the four chosen futures contracts. Weekly data from January 6th, 1995 to April 25th, 2008. Units are $Cdn per MBF. Blue line: model implied futures prices. Red line: market futures prices.

	μ_{MR}	κ_{MR}	σ_{MR}	λ_{MR}
Estimates	6.323	0.229	0.231	0.007
Std. Error	(0.297)	(0.031)	(0.009)	(0.284)
LL	5343.2			

Table 3.6: Parameter estimates of Schwartz (1997) one-factor model using Kalman filter. LL refers to the value of log-likelihood function. Weekly futures prices, from January 6th, 1995 to April 25th, 2008.

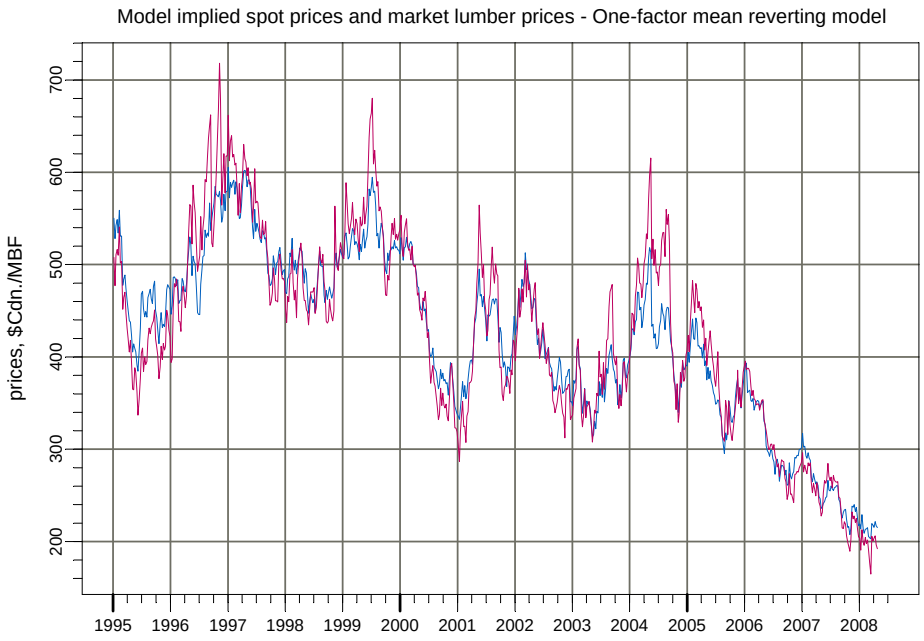


Figure 3.7: Plots of model implied and market spot prices: one-factor mean reverting model. Blue line: model implied prices. Red line: market prices.

	μ_{GBM}	σ_{GBM}	δ_{GBM}
Estimates	-0.065	0.215	-0.027
Std. Error	(0.058)	(0.006)	(0.004)
LL		5145.2	

Table 3.7: Parameter estimates of GBM model with convenience yield using Kalman filter. LL refers to the value of log-likelihood function. Weekly futures prices, from January 6th, 1995 to April 25th, 2008.

3.6.3 The GBM model

Parameter estimates for the GBM model with constant convenience yield are reported in Table 3.7. The drift term μ_{GBM} is negative, but not statistically significant. The constant convenience yield δ_{GBM} is small in magnitude. Model implied and market lumber spot prices are plotted in Figure 3.8. The average RMSE for all maturities is \$19.3 per MBF. Details are provided in Appendix B.

3.6.4 The one-factor long-term model

Since the single factor long-term model proposed in Schwartz (1998) is a mathematical transformation of the two-factor model, the model parameters are the same for the two models. Specifically, the constant convenience yield, c, based on Equation (3.28) is 0.028. Table 3.8 shows descriptive statistics for the composite spot price Z of this long-term model. Comparing this table with Table 3.1, we find that the range of the true spot price is wider than the composite spot price and the composite prices is less volatile than the market spot price. A plot of composite spot prices and model implied spot prices is provided in Figure 3.9. The composite price shown in this graph is less volatile than the model implied spot price.

For a given maturity T, model implied futures prices of both the two-factor model and the long-term model can be derived based on Equations (3.5) and (3.31) respectively. We are interested in the performance of the long-term model in terms

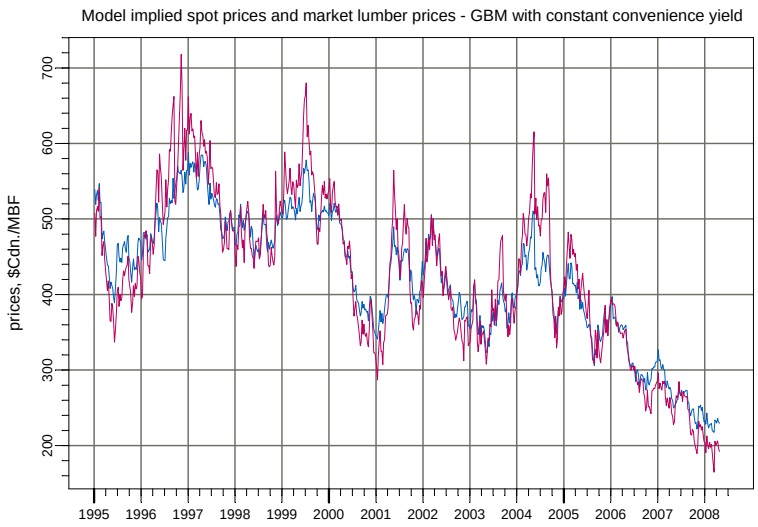


Figure 3.8: Plots of model implied and market spot prices: one-factor GBM model with constant convenience yield. Blue line: model implied prices. Red line: market prices.

	Max	Min	Mean	Std. Dev.	Skewness	Kurtosis
Cdn (2005) $/MBF	558.6	239.2	428.1	77.73	-0.48	2.37

Table 3.8: Descriptive statistics for the composite spot prices Z of one-factor long-term model.

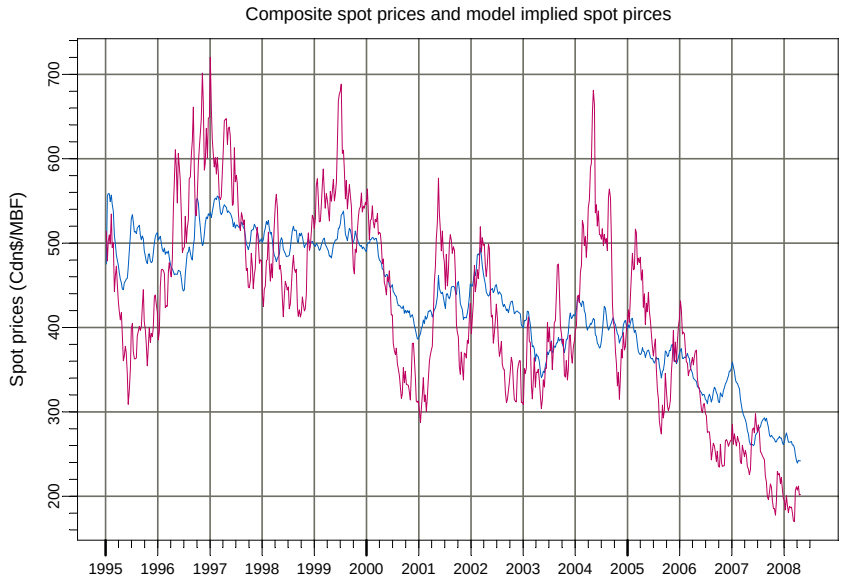


Figure 3.9: Plots of composite spot prices and model implied spot prices. Solid line: composite spot prices; dotted line: model implied spot prices.

Differences of futures prices in $/MBF: two-factor model and long-term model					
	F1	F2	F3	F4	F1-F4 average
RMSE	58.83	41.69	29.73	21.52	40.45
MAE	48.43	34.26	24.27	17.35	31.08
	F5	F6	F7	F8	F5-F8 average
RMSE	11.47	8.71	8.75	8.72	9.49
MAE	8.99	8.45	8.59	8.57	8.65
	F9	F10	F11	F12	F9-F12 average
RMSE	8.68	8.63	8.59	8.55	8.61
MAE	8.53	8.49	8.45	8.40	8.47

Table 3.9: Differences of model implied futures prices with different maturities for two models.

of fitting long-term commodity derivatives prices compared to that of the two-factor model. To this end, model implied futures prices with the maturities up to 8 years are calculated for both models. Note that beyond one year there are no actual futures prices that can be used for comparison. The differences expressed in RMSE and MAE of the model implied futures prices with different maturities between the two-factor model and the long-term model are reported in Table 3.9. The average difference for long term futures contracts (with maturities from 5 years to 8 years) is less than $9, which is about 2% of the mean futures prices. This result further confirms the close match of these two models in terms of generating long maturity futures prices. Plots of the model implied futures prices with different maturities for these two models are provided and compared in Appendix B3.

3.7 Analysis of a forestry investment

In this section I model the optimal decision of the owner of a stand of trees who seeks to maximize the value of the stand (or land value) by optimally choosing the harvest time. It is assumed that forestry is the best use for this land, so that once the stand is harvested it will be allowed to grow again for future harvesting. Since this

is a multi-rotational optimal harvesting problem, it represents a path-dependent option. This is because the value of the option to harvest the stand today depends on the quantity of lumber, which itself depends on the last time when the stand was harvested. The tree harvesting problem is impulse control problem (Phan, 2005). Determining the value of the stand and the optimal harvesting decision requires the numerical solution of a Hamilton-Jacobi-Bellman (HJB) variational inequality.

Some of the details of this timber harvesting problem such as costs and the growth curve for wood volume have been used in other papers including Chen and Insley (2008), Insley and Rollins (2005) and Insley and Wirjanto (2008). The latter two papers use a simple one factor mean reverting process for price, while Chen and Insley (2008) examine a regime-switching model.

3.7.1 Cost, wood volume and price data

I consider a harvesting problem for a hypothetical stand of Jack Pine trees in Ontario's boreal forest assuming that the stand is used for commercial forestry. Values are calculated prior to any stumpage payments or taxes.

Timber volumes and harvesting costs are adopted from Insley and Lei (2007) and are repeated here for the convenience of the reader. Volume and silviculture cost data were kindly provided by Tembec Inc. The estimated volumes reflect 'basic' levels of forestry management which involves $1040 per hectare spent within the first five years on site preparation, planting and tending. These costs are detailed in Table 3.10. Note that in the Canadian context these basic silviculture expenses are mandated by government regulation for certain stands. I assume that harvesting is not permitted before age 35 once all silvicultural expenditures have been made.

Volumes, estimated by product, are shown in Figure 3.10 for the basic silvicultural regime.[16] SPF1 and SPF2 are defined as being greater than 12 centimeters

[16]The yield curves were estimated by Margaret Penner of Forest Analysis Ltd., Huntsville,

Item	Cost, $/ha	Age cost incurred
Site preparation	$200	1
Nursery stock	$360	1
Planting	$360	2
First tending	$120	5
Monitoring	$10	35

Table 3.10: Silviculture costs under a basic regime

Harvest and transportation cost	$47
Price of SPF1	$60
Price of SPF2	$55
Price of SPF3	$30
Price of poplar/birch	$20

Table 3.11: Assumed values for log prices and cost of delivering logs to the mill in $ per cubic meter

at the small end, SPF3 is less than 12 centimeters, and 'other' refers to other less valuable species (poplar and birch). Data used to plot this graph is provided in Insley and Wirjanto (2008).

Assumptions for harvesting costs and current log prices at the millgate are given in Table 3.11. These prices are considered representative for 2003 prices at the millgate in Ontario's boreal forest. Average cost to deliver logs to the lumber mill in 2003 are reported as $55 per cubic meter in a recent Ontario government report (Ontario Ministry of Natural Resources, 2005). From this is subtracted $8 per cubic meter as an average stumpage charge in 2003 giving $47 per cubic meter.[17] It will be noted the lower valued items (SPF3 and poplar/birch) are harvested at a loss. These items must be harvested according to Ontario government regulation. The price for poplar/birch is at roadside, so there is no transportation cost to the mill.

Ontario for Tembec Inc.

[17]This consists of $35 per cubic meter for harvesting and $12 per cubic meter for transportation. Average stumpage charges are available from the Ontario Ministry of Natural Resources.

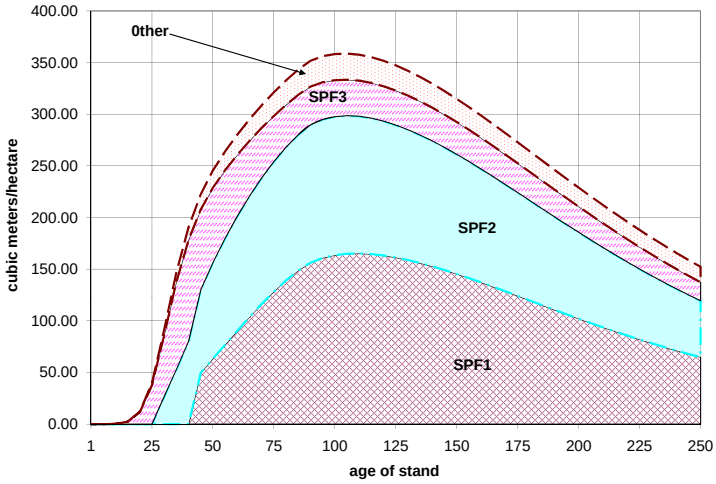


Figure 3.10: Volumes by product for hypothetical Jack Pine stands in Ontario's boreal forest under basic management

3.7.2 Optimal harvesting with different price models

Ideally I would solve the optimal harvesting problem using the two-factor model with stochastic price and convenience yield. However this requires the numerical solution of a complex HJB variational inequality with three state variables (price, volume and convenience yield) plus time. I have shown that the performance of the long-term model introduced in Schwartz (1998) is comparable to that of the two-factor model in terms of matching long run futures prices. I therefore analyze the forest investment problem using the long-term model with the composite price as the single stochastic variable. The results from the long-term model will be compared with those from the single factor mean reverting and GBM models. In the following sections, the HJB variational inequality is specified for the three different price models.

The long-term model

In the single-factor long-term model, based on the stochastic process describing the composite price Z in Equation (3.27), the value of the stand of trees is denoted as $V(Z, \varphi, t)$. At each period the stand owner makes the choice either to harvest the stand immediately or let the trees grow for another period and then reconsider whether or not harvesting should be undertaken. If the stand is harvested the stand owner receives revenue from selling the timber equal to $[(S - C_h)Q(\varphi) + V(Z, 0, t)]$. This it the price of timber, S, less per unit harvesting costs, C_h, times the quantity of timber, $Q(\varphi)$, which is a function of age, $Q = g(\varphi)$. In addition the stand owner receives an asset equal to $V(Z, 0, t)$ which refers to the value of the bare land when the stand is of age zero. If the stand owner chooses to delay harvesting for another period, he receives the value of the land, $V(Z, \varphi, t)$. Using standard no-arbitrage arguments, when it is optimal to delay harvesting (in the continuation region) the

value of the stand satisfies the following PDE:

$$V_t + (r-c)ZV_Z + \frac{1}{2}(\sigma_F(t)Z)^2 V_{ZZ} + V_\varphi - rV = 0 \tag{3.32}$$

where the variance of the futures returns $\sigma_F(t)$ is time dependent and is given in Equation (3.30). Rewrite the PDE Equation (3.32) as:

$$HV \equiv rV - (V_t + (r-c)ZV_Z + \frac{1}{2}(\sigma_F(t)Z)^2 V_{ZZ} + V_\varphi) \tag{3.33}$$

Then the HJB variational inequality characterizing the full optimal harvesting problem can be expressed as:

$$\begin{aligned} &(i) \quad HV \geq 0 \\ &(ii) \quad V(Z,\varphi,t) - [(S-C_h)Q(\varphi) + V(S,0,t)] \geq 0 \\ &(iii) \quad HV\Big[V(Z,\varphi,t) - [(S-C_h)Q(\varphi) + V(S,0,t)]\Big] = 0 \end{aligned} \tag{3.34}$$

Equation (3.34) implies if the stand of trees is managed optimally either HV, $V(Z,\varphi,t) - [(S-C_h)Q(\varphi) + V(S,0,t)]$, or both will be equal to zero. If $HV = 0$ and $V(Z,\varphi,t) - [(S-C_h)Q(\varphi) + V(S,0,t)] > 0$, it is optimal for the investor to continue holding the option by delaying the decision to harvest. In this case growing stand of trees is earning the risk free return and the value of the stand is greater than the payout the owner would receive from harvesting. On the other hand, if $HV < 0$ and $V(Z,\varphi,t) - [(S-C_h)Q(\varphi) + V(S,0,t)] = 0$, then the value of the stand of trees just equals the value of immediate harvest. The owner is not earning the risk free return from maintaining the standing timber and should harvest the trees. If both parts (i) and (ii) in Equation (3.34) are equal to zero, either strategy

is optimal. Equation (3.34) may be written more compactly as:

$$min\left\{ HV; \left[V(Z,\varphi,t) - [(S-C_h)Q(\varphi) + V(S,0,t)] \right] \right\} = 0 \tag{3.35}$$

No analytical solution exists for Equation (3.35). I solve it numerically, using the combination of the fully implicit finite difference method, the semi-Lagrangian method and the penalty method.[18]

Single-factor models

For the one-factor mean reverting model, the value of the stand of trees, $V(S,\varphi,t)$, satisfies the following PDE in the continuation region:

$$V_t + \kappa_{MR}(\mu_{MR} - \lambda_{MR} - \ln S)SV_S + \frac{1}{2}(\sigma_{MR}S)^2 V_{SS} + V_\varphi - rV = 0 \tag{3.36}$$

The HJB equation can be expressed as in Equation (3.35), except that the HV is defined as:

$$HV \equiv rV - (V_t + \kappa_{MR}(\mu_{MR} - \lambda_{MR} - \ln S)SV_S + \frac{1}{2}(\sigma_{MR}S)^2 V_{SS} + V_\varphi)$$

For the GBM model, the value of the stand of trees, $V(S,\varphi,t)$, satisfies the following PDE in the continuation region:

$$V_t + (r - \delta_{GBM})SV_S + \frac{1}{2}(\sigma_{GBM}S)^2 V_{SS} + V_\varphi - rV = 0 \tag{3.37}$$

[18]Details on this approach are provided in Insley and Rollins (2005). An introduction to numerical methods is provided in Wilmott (2006).

HV in this case is defined as:

$$HV \equiv rV - (V_t + (r - \delta_{GBM})SV_S + \frac{1}{2}(\sigma_{GBM}S)^2 V_{SS} + V_\varphi)$$

As with the long term model, the HJB equations for these single-factor models are solved numerically.

3.7.3 Results for land value and critical harvesting prices

In this section I present results for each of the lumber price models in terms of the value of the stand of trees (land value) and the critical prices at which it is optimal to harvest. As discussed earlier, the spot price data used to parameterize the models is approximated by the CME random lengths futures price for the nearest maturity date. To value a hypothetical stand of trees in Ontario, the long run equilibrium price (μ_{MR} in Equation (3.9)) needs to be scaled to reflect Ontario prices at the millgate. The estimate of price at the millgate in 2003 for SPF1 logs is Cdn.$60 per cubic meter. In 2003 the average spot price proxied by the price of futures contract closest to maturity was Cdn. $375 per MBF. I used the ratio of 375/60 as a rough adjustment factor to scale the equilibrium price levels. This rescaling accounts for transportation costs and milling costs (as well as the conversion from MBF to m^3).

For the one-factor long-term model, the middle curve in Figure 3.11 shows how the bare land value (a stand age of zero) changes with the composite lumber price, Z. We observe that land value increases with Z. For example, when the composite price is $\$50/m^3$, the land is worth $1147 per hectare. This rises to $1559/ha when the composite price is $60. This makes sense since Z is defined to be increasing in S and decreasing in δ (recall Equation (3.27)). In line with finance theory, the value of a call option increases with spot price and decreases with the dividend. In our forestry investment problem, the bare land value is like a call option and the convenience yield is like the dividend. Land values for different stand ages are

plotted in Figure 3.12. The land becomes more valuable as the trees mature and as Z increases.

I am more interested in the relationship of land value with spot price S rather than with our constructed composite price. One of the disadvantages of using the long term model is that this relationship is obscured. However we note from Table 3.4, that net convenience yield fluctuates in the range of $[-1, 1]$. Given the land value estimate for each Z, we can back out what the implied spot price would be when convenience yield is at either +1 or -1. This gives us a range for land values versus spot price which are shown as the dashed and dotted curves in Figure 3.11. For example, when $Z = \$50$, land value is \$1150. If $\delta = -1$, the spot price consistent with that Z and land value is \$31. If instead $\delta = 1$, the implied spot price must be higher at \$81. The logic here is that a higher convenience yield implies that it is more beneficial to hold the harvested lumber rather than trees "on the stump", so the option to harvest is actually worth less. Hence a higher spot price is required to be consistent with a land value of \$1150. Figure 3.11 also implies that given a certain level of convenience yield, land value increases with lumber prices. This can be explained as given a fixed net benefit of holding lumber in stock, the higher the lumber price is, the more the land value is. This result is consistent with the finance theory. Moreover, this graph indicates that the combination of high convenience yield and low spot price will lead to low land value and the combination of low convenience yield and high price will generate high land value. This result is also consistent with the finance theory, since with high convenience yield and low spot price, the option is less valuable and vice versa.

Figures 3.13 and 3.14 show land value versus the lumber spot price for the mean reverting and GBM models. We observe that for the MR process at a stand age of zero, land value is about \$5900 per hectare, which is insensitive to spot price. This follows from the fact that the estimated long run equilibrium price is constant and the speed of mean reversion is a moderate 0.229 (implying a half life to return to

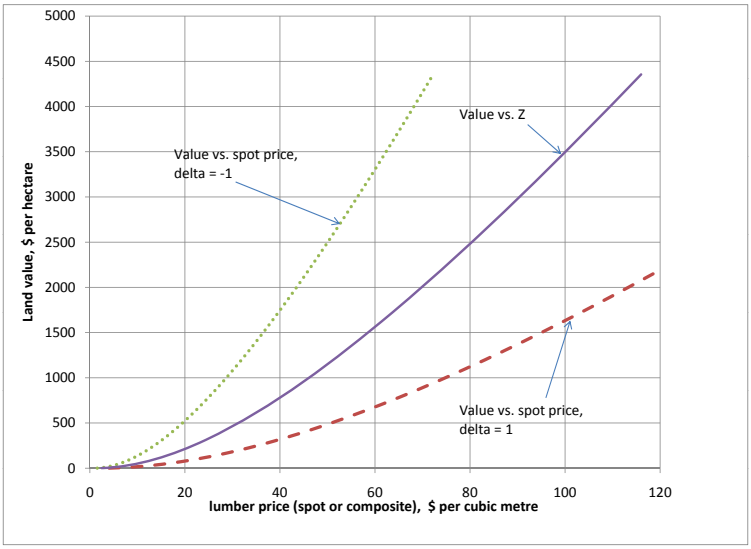


Figure 3.11: Land values v.s. composite price (Z) or spot price for stand age 0. For land values versus the spot price, δ is set at upper and lower limits of +1 and -1

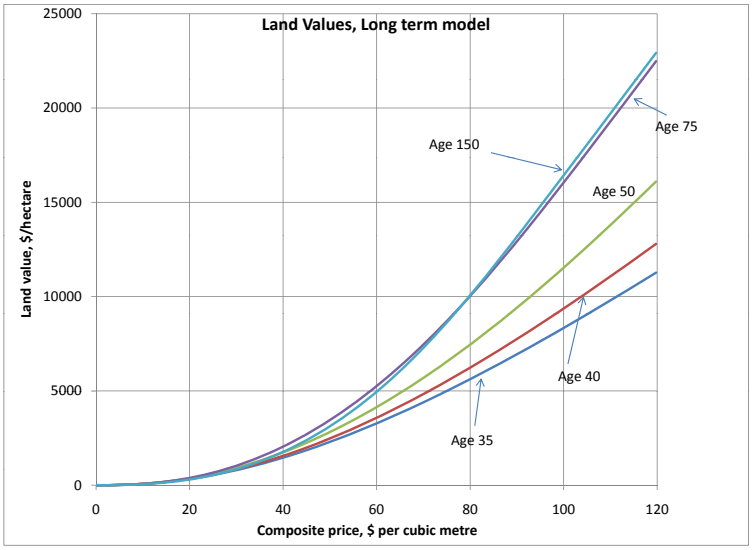


Figure 3.12: Land values v.s. composite prices for different stand ages.

this value of three years.) At a stand age of zero, the trees will not be harvested for at least 35 years, so that with this price model we expect to be back at the long run mean by the harvest date.[19] As the stand age increases, land value becomes positively related to spot price, since the stand may be harvested within a few years.

The GBM results are very different from the MR and long run models. At a lumber price of \$50, the GBM model gives a land value of \$199 million per hectare compared to around \$5900 for the MR model. The GBM land values are also much greater than the range given for the long-term model. At a \$50 spot price land value ranges from around \$500 for $\delta = -1$ to around \$2500 for $\delta = 1$. The large land value for GBM is consistent with the estimated parameter values. The risk adjusted drift rate for S in the GBM model is $r - \delta$ which works out to

[19]This result is consistent with the findings in Insley and Rollins (2005) in which a slightly different mean reverting process was used.

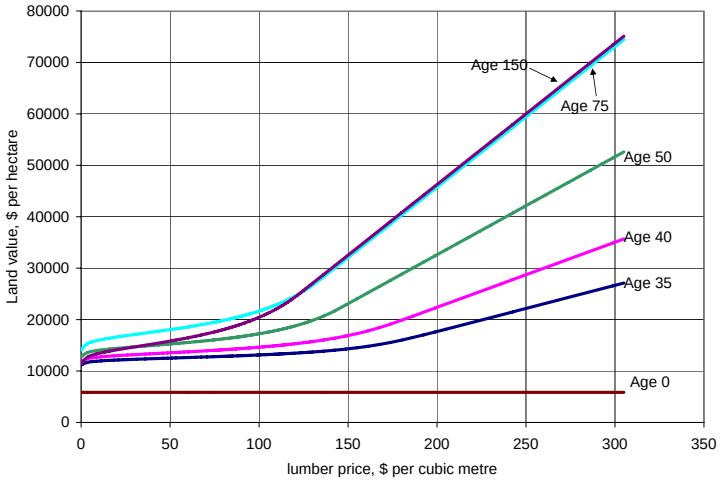


Figure 3.13: One-factor mean reverting model. Land values v.s. lumber spot prices for stands of various ages.

$[0.023-(-0.027)] = .05$ from the estimates reported in Table 3.7. This exceeds the assumed riskfree discount rate of 2.3%.

In addition to land value, I am also concerned with critical harvesting prices which indicate when it is optimal to harvest. The middle curve in Figure 3.15 shows the critical composite price versus stand age for the long-term model. We see the critical Z value is about \$120 per m^3 at age 70 and declines to reach a steady state of around \$90 per m^3. Based on the relationship amongst the spot price, convenience yield, δ, and composite price Z shown in Equation (3.27), I can calculate a range for the corresponding critical spot prices by substituting in the range of convenience yield. The upper and lower lines in Figure 3.15 show the corresponding upper and lower bounds of the critical spot prices for the long-term model. The upper line reflects critical S if $\delta = 1$ and the lower line reflects the critical S if $\delta = -1$.

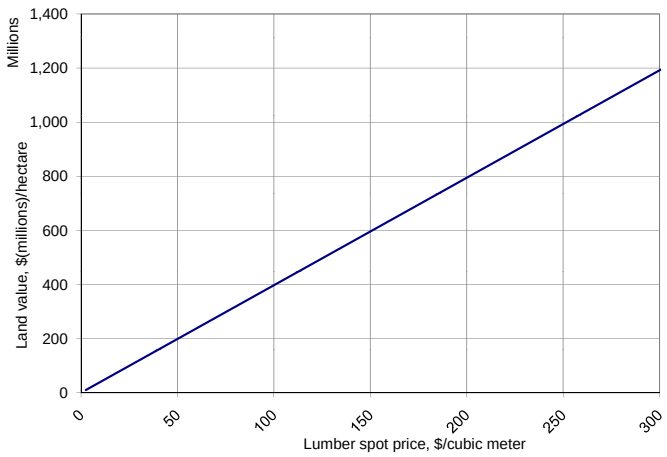


Figure 3.14: GBM model with constant convenience yield. Land values v.s. lumber spot prices for stand age 0.

The long term model is an approximation of the results that would be produced by the two-factor model. For intuition about the upper and lower bounds in Figure 3.15, I consider the relationship between the convenience yield and the spot price in the two-factor model. Referring to Equation (4.14), when δ is at its lower bound of -1, this implies the current drift rate of S is relatively high, but it is known that δ will be pulled up quickly in the future to its long run value. In this circumstance the critical prices are relatively low since the future reward for holding harvested lumber will increase while the reward for holding standing trees will decrease. With the high drift rate of S, given this lower bound of convenience yield δ, the land owner should also take advantage of the high future spot prices to harvest at a relatively low critical price. When $\delta = 1$ this implies the expected growth rate in S is relatively low, but it is expected that δ will revert back to its long run mean fairly quickly. This implies that the growth rate of S will increase in the future. Therefore, unless the current spot price is quite high, it is not optimal to harvest. By delaying the harvest the owner of the stand puts off paying harvesting costs and can take advantage of expected future growth in lumber prices.[20]

Figure 3.16 shows the critical harvesting prices for the MR single-factor model as well as the range of critical prices for the long-term model. Critical prices generated by the mean reverting and long-run models decrease with the stand age. When the trees are young and growing fairly rapidly it makes sense to delay harvesting, so that the critical prices that trigger harvesting are higher. Once tree growth declines we reach an approximate steady state for the critical harvest price.

The critical harvest prices for the MR case lie between the upper and lower bounds for the long run model case. For the MR model there are critical prices defined from age 35 and onward, whereas for the long run model critical prices are defined from age 70 onward. This implies that for the MR model if the spot price hits a very high value it is worthwhile harvesting even though the trees are

[20]We assume people are rational and forward looking in this economy.

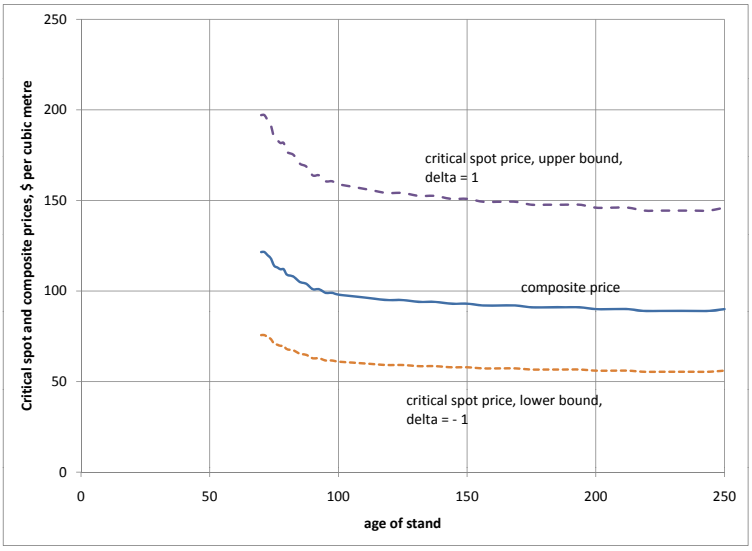


Figure 3.15: Critical composite prices and the calculated range of critical spot prices versus stand ages. Upper bound is associated with $\delta = 1$ and lower bound is associate with $\delta = -1$

still very young and growing rapidly. This follows from the assumption of a fixed equilibrium price in this model which make it beneficial to take advantage of any short term price surges. For the long run model, on the other hand, it would never be optimal to harvest before the trees are 70 years of age. In terms of the stochastic process followed by Z, Equation (3.29), the drift is a small negative number: $r - c = 0.023 - 0.028 = -0.005$. The expected return from holding the trees therefore comes from volume growth rather than any expected upward drift in Z. Hence with the long run model it is not optimal to harvest before age 70 while the volume growth rate is still strongly positive, no matter what the price. Referring to Figure 3.10 it may observed that volume growth is highest in the years before age 70.

There are no critical prices for the GBM case, implying it would never be optimal to harvest the stand. This is a result of the negative convenience yield which implies a drift rate in the risk neutral world that exceeds the riskless interest rate. (Note that after age 255 it is assumed that wood volume in the stand of trees remains constant.)

In reviewing the results I note that the long run model using the composite price Z gives significantly different land values and critical prices than the other two single factor models. It is interesting that for the GBM model, the parameter values that result from the Kalman filter estimation produce land values that are so different from the other two models. I obtained some land sale data for 2003 in the Ontario region which the timber volume curves apply. The land was marginal agricultural land which was being purchased for reforestation. The average land sale price was around $1100 per hectare. I therefore feel confident in concluding that the GBM results are not reasonable.

The land values given by the MR and long-term model are at least the right order of magnitude. It is significant that the MR model recommends harvesting at much younger stand ages than the long-term model. The composite price in

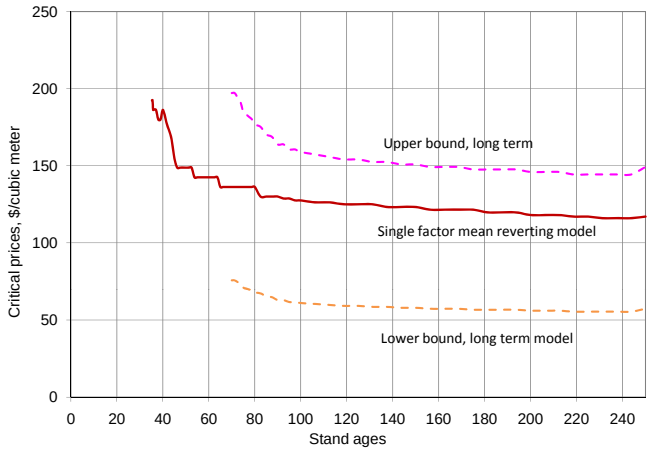


Figure 3.16: Critical spot prices for the MR and GBM models and a range of critical prices for the long-term model.

the long-term model is summarizing the long run relationship between convenience yield and spot price. We saw previously that the two-factor model provided a better match of market futures prices than the single factor mean reverting model. The performance of long-term model and two-factor model in terms of fitting long-term market data are comparable. In addition economic theory tells us convenience yield is an important consideration for pricing in commodity markets such as lumber. Hence it seems reasonable to have more confidence in the results of the long-run model, than in the simple MR model in which convenience yield is ignored.

3.7.4 Model comparison: regime switching model and the two-factor model

The forestry investment problem analyzed in Chapter 2 is the same as the one investigated in this chapter. Hence it is natural to compare the performances of the two price models examined in Chapter 2 and this chapter: regime switching model and the Schwartz (1997) two-factor model, in terms of their ability to price the land values as well as the critical prices generated by these two models.

At age 0, the land values generated by the regime switching model are indifferent with respect to the regime at which the lumber prices reside. However, for the two-factor model with stochastic convenience yield, the land value is an increasing function of the lumber prices given a certain level of convenience yield. The land values generated by both price models increase when the land ages.

The critical prices at which the lumber owner should harvest the trees exhibit the similar trend for these two models. They decrease with the age of stand. The critical prices generated by the regime switching model depend on the regime at which the lumber price resides, while those generated by the stochastic convenience yield two-factor model depend on the level of convenience yield.

Given the above analysis, we can not see which model is more suitable to be applied to solve this particular forestry investment problem. On one hand, the lumber price process exhibit two regimes. On the other hand, the economic theory suggests the importance of incorporating the convenience yield on the forestry investment analysis. Future research might further investigate this issue. For example, the collection of the land sales data can be used to find the actual land values.

3.8 Concluding remarks

This chapter investigates the importance of modeling the stochastic convenience yield of lumber in the context of an optimal tree harvesting problem. Schwartz (1997) proposed a stochastic model of commodity prices with both spot price and convenience yield as stochastic factors. In the first part of the chapter, I examine the performance of this two-factor model in terms of its ability to characterize the price of lumber derivatives . The estimation result shows that there is a positive and significant correlation between lumber prices and convenience yield. This two-factor model also provides a good model fit in terms of explaining the dynamics of lumber derivatives.

In the second part of the chapter, I examine the impact of stochastic convenience yield on a multi-rotational optimal harvesting problem. The HJB equation characterizing the value of the option to harvest a stand contains three stochastic variables: lumber prices, convenience yield and the stand age. To simplify the solution of the harvesting problem, we use the result of Schwartz (1998) who proposes a one-factor model (called the long-term model) which retains most of the characteristics of his two-factor model, especially its ability to fit long-term futures prices. The HJB equation derived using this one-factor model is solved numerically using the combination of the fully implicit finite difference method, the semi-Lagrangian method and the penalty method. I compare the results of the long term model with

two single factor models are common in the literature: a mean reverting model and geometric Brownian motion.

The result shows that including the effect of convenience yield through the long-term model has an important impact on long-term forestry investment decisions. Land values and critical harvesting prices were significantly different across the three models. The GBM model gave excessive land values. The single factor mean reverting model gave land values of a reasonable order of magnitude, but under MR model harvesting would potentially occur at much earlier stand ages than with the long-term model.

The results for the long-term model also showed that the critical harvesting prices varied significantly depending on the assumed value of the convenience yield. The higher the convenience yield, the higher the spot price that land owner requires to harvest the trees. This follows from the interaction of the convenience yield and the spot price. A high convenience yield today implies a lower convenience yield in the future and also a higher expected growth rate for the spot lumber price. Hence with a high convenience yield, the critical price that induces harvesting is relatively high, and we expect that the stand will be harvested at a later date than for a lower convenience yield.

A natural extension of this research is to solve the HJB equation for the full two-factor problem and compare with the long-run model results. This will be left for future research.

A criticism of both the two-factor model and the long-run model is that the forest owner is required to know what the convenience yield is to formulate his optimal harvesting strategy. Convenience yield is not easily observable, but it can be calculated from futures prices. More informally, one could imagine a forest owner taking into account convenience yield in a more intuitive fashion. If lumber inventories are very low and markets are buoyant, players in the market would be aware that there is a benefit to holding an inventory of logs - i.e. convenience yield

is high.

In conclusion, the results in this chapter demonstrate that convenience yield has an important effect on the optimal harvesting decision and that it is worthwhile using a richer model, such as the long-term model used in this chapter, when analyzing forest investment decisions, rather than relying on simple single factor models.

Chapter 4

The dynamics of crude oil prices: an analysis of recent evidence

4.1 Introduction

A casual observation of historical world oil prices shows a long period of stability from the post World War II era until 1973 when the turmoil of the OPEC oil crisis led to a greater than 10 fold increase in price by the peak in 1980. Prices collapsed in the 1980's and remained low until the end of the 1990's, but with much more volatility than the pre-1970 era. Since about 2003 we have witnessed another run up in oil prices, peaking at over $140 per barrel in July 2008. Figure 4.1 shows the price of West Texas Intermediate (WTI) since 1986.

Oil is a non-renewable resource and is a key input to the world's economy. This plus the highly volatile nature of oil prices since the 1970's has led a number of researchers to investigate different models of oil prices in an effort to better understand the dynamics of oil prices, for use in the valuation of oil-linked investments and portfolio risk management. This literature highlights the tradeoff between the

desire for added model complexity and realism versus the need to keep price models simple enough, so that they can be used to determine optimal decisions in complex oil related investments.

This chapter extends the literature on the modeling of oil prices by investigating the ability of three different models to describe the dynamics of oil futures prices. In particular, I propose a regime switching model in an effort to capture the historically observed effect of periods of lower but more stable prices followed by periods of high and volatile prices. I compare this regime-switching model with two well known commodity pricing models as described in Schwartz (1997) and Schwartz and Smith (2000). I estimate these latter two models using up-to-date data that includes the volatile post-2003 period and compare the estimation results with those presented in the earlier research. The objective of this chapter is to determine whether a regime switching model can provide a good fit of futures prices compared to the models proposed in Schwartz (1997) and Schwartz and Smith (2000). A second goal is to investigate the merits of the Schwartz (1997) and Schwartz and Smith (2000) models using more recent data.

In the literature, there are two main approaches which are used to explain the dynamics of commodity price processes, namely equilibrium models as in Deaton and Laroque (1992) and Chambers and Bailey (1996), and reduced form models as in Gibson and Schwartz (1990) and Schwartz (1998). Equilibrium models focus on the implications of total stock depletion, or stock-outs. In the presence of stock-outs, spot prices may be higher than expected future spot prices net of cost of carry.[1] This result in general suggests that the backwardation of the term structure of futures curve can occur only when there are stock-outs. However, this is not consistent with the empirical data. In Litzenberger and Rabinowitz (1995b) for instance, they find strong backwardation in the oil futures curve 77% of the time and stock-outs are the exception. One of the main drawbacks of the equilibrium

[1]See Newbery and Stiglitz (1981) and Bobenrieth et al. (2002).

model approach is that the models are highly stylized. Therefore we can not use an equilibrium model approach to analyze quantitative predictions about spot prices and the characteristics of the corresponding commodity derivatives, which is one of the main interests of practitioners.

In contrast, reduced form models explicitly specify the dynamics of a set of underlying state variables, such as the commodity spot price, the convenience yield or the instantaneous interest rate. The specified stochastic models can be used to capture the term structure of futures or forward curves and value sophisticated commodity derivatives.[2] The reduced form class of models has gained widespread acceptance and dominates the current literature and practice on energy derivatives, since it is flexible in capturing the relationship among several state variables, and it can be used for various commodity-based derivative valuation purposes. Therefore, in this chapter, I employ the reduced form method to analyze the crude oil price process.

Brennan and Schwartz (1985) proposed the first and simplest version of a reduced form commodity price model. The only stochastic factor in their model is the spot price which is assumed to follow a geometric Brownian motion (GBM). The convenience yield is treated as a constant dividend yield. A mean reverting process has also been proposed in the commodity literature in order to capture the notion that the workings of supply and demand will eventually result in commodity prices that exhibit some form of mean reversion. Bessembinder et al. (1995) find support for evidence of a mean reversion in commodity prices by comparing the sensitivity of long-maturity futures prices to changes in spot prices. One of the main drawbacks of these one-factor specifications is that they neglect the inventory-dependence property of the convenience yield by assuming that it is constant. Schwartz (1997) applied a one-factor model in which the logarithm of the

[2]See in particular Gibson and Schwartz (1990), Schwartz (1997), Schwartz and Smith (2000) and Casassus and Collin-Dufresne (2005).

commodity prices follows a mean reverting process and showed that this model is incapable of explaining the main properties of the term structure of commodity futures prices.

A two-factor model was first introduced in Gibson and Schwartz (1990) where the spot price is assumed to evolve according to a GBM and the convenience yield follows a mean reverting process. Mean reversion in commodity prices is therefore captured by letting the two stochastic factors to be positively correlated. For example, a positive shock to the spot price will typically be accompanied by a positive shock to the convenience yield which lowers the future expected return on the commodity. Schwartz (1997) further explores this two-factor model by explicitly incorporating the stochastic convenience yield into the drift part of the spot price dynamics and using the Kalman filter to estimate the model parameters. Schwartz and Smith (2000) introduce another type of two-factor model, the so-called short-term/long-term model, which does not explicitly specify the dynamics of convenience yield. The long-term equilibrium price level and short-term deviation are jointly modeled in their paper. The former follows a GBM process and the latter is assumed to fluctuate around zero and follows a mean-reverting process. The short-term/long-term model is mathematically equivalent to the stochastic convenience yield model proposed by Gibson and Schwartz (1990) because the underlying state variables in one model can be expressed as linear combinations of the state variables in the other. The idea of stochastically evolving short-term deviations and equilibrium long run prices seem more general and intuitive as well compared to the notion of convenience yields. These two types of two-factor specifications have been successfully used to model crude oil prices in the commodity literature and some oil-related investments are valued based on these models. Given the recent events in world oil markets, it is worth investigating whether these two models can still successfully match the term structure of oil futures prices. Schwartz (1997) and Schwartz and Smith (2000) use price data for Enron long-term forward contracts

in their estimation. In this chapter I use long-term futures contracts to calibrate the models and estimate the parameters. By estimating both of these models the robustness of the estimation method can be more readily assessed.

The Markov regime switching (RS) model first proposed by Hamilton (1989) is a promising model for commodity prices that has been used in the literature[3]. In a RS model, the observed stochastic behavior of commodity prices is assumed to consist of several regimes. For each regime, one can define a separate underlying stochastic process. The switching mechanism between each state is typically assumed to be governed by an unknown random variable that follows a Markov chain. The RS model can be used to capture the shifts between "abnormal" and "normal" equilibrium states of supply of and demand for a commodity. Resource industries tend to be characterized by times of boom and bust, which are often related to the world economy and demand for the resource as well as political events. Crude oil prices behave in a similar way as other commodities with a relatively large price swing in times of shortage and over-supply. Assuming for simplicity that there are only two regimes, I extend the one factor model of Schwartz (1997) allowing most of the model parameters to be regime dependent and calibrate this regime switching model using crude oil futures prices.

In summary, in this chapter I compare and contrast three different model specifications for oil prices: a regime switching model, the Schwartz (1997) two-factor model and the Schwartz and Smith (2000) two-factor model, in terms of their abilities to explain the main characteristics of trends in oil prices since 1997 (which is when the data on long-term futures prices become available). The remainder of this chapter is organized as follows. Section 4.2 describes the oil data examined in this chapter. Section 4.3 specifies the models used in the analysis and details the calibration and estimation methods. Section 4.4 presents the results of the model

[3]For example, Deng (2000), de Jong (2005), Chen and Forsyth (2008) all examine empirical models of regime switching in commodity prices (electricity or natural gas prices).

calibration and estimation. Section 4.5 provides some concluding comments.

4.2 The data

The models of interest in this chapter will be specified in the risk-neutral world and the model parameters will be estimated or calibrated using crude oil futures prices. The calibrated models can therefore be used in valuing oil-linked investments with the risk free interest as the discount rate. Before presenting the candidate models, in this section, both the crude oil spot prices and the corresponding futures prices examined in this chapter are discussed and analyzed.

4.2.1 Crude oil spot prices

Two major oil types often referenced in the literature are West Texas Intermediate (WTI) and Brent Crude. The crude oil spot price examined in this chapter is WTI, which is also considered as the major benchmark of crude oil in the Americas. It is of very high quality and is excellent for refining a larger portion of gasoline. WTI is a light and sweet crude, lighter and sweeter than Brent Crude. Its American Petroleum Institute (API) gravity[4] is 39.6 degrees which makes it a "light" crude oil, and it contains about 0.24% of sulfur which makes it a "sweet" crude oil. WTI is traded in the U.S. spot market at Cushing, Oklahoma and is also the underlying commodity of New York Mercantile Exchange (NYMEX)'s oil futures contracts. The price of WTI since 1986 is shown in Figure 4.1.[5]

In this chapter, I focus on explaining the prices ranging from January 24th, 1997 to May 29th, 2009, since the data for long-term oil futures contracts with maturities up to 6 years are available only from 1997. As the price models examined in

[4]API gravity is a measure of how heavy or light a petroleum liquid is compared to water.

[5]Data source: Energy Information Administration, official energy statistics from the U.S. government.

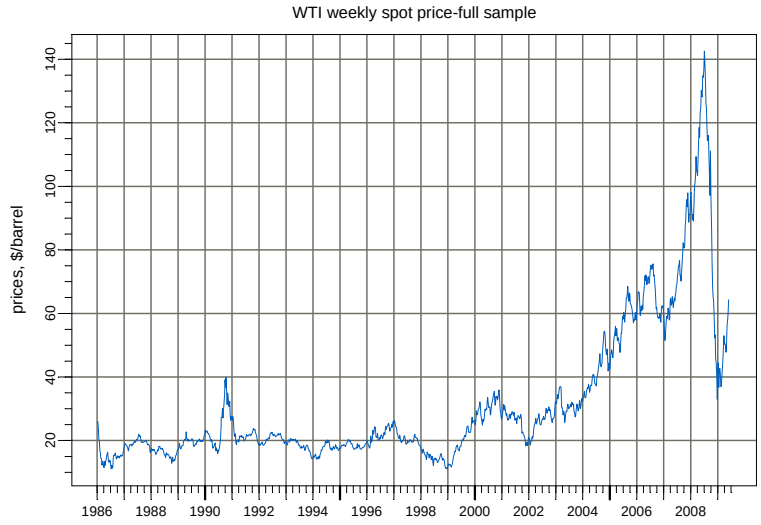


Figure 4.1: Weekly spot prices of WTI crude oil, from January 3rd, 1986 to May 29th, 2009.

this chapter are intended to be used to value an oil-related long-term project, the use of long-term futures price data is expected to be able to increase the accuracy and relevancy of the estimated model parameters for this sort of application. Due to the lack of long-term crude oil derivative prices, Schwartz (1997) and Schwartz and Smith (2000) use Enron long-term forward data to estimate the model parameters. Since futures contracts are more regulated and standard than the private forward contracts, the parameter estimates based on futures contracts should be more accurate.

As stated in Schwartz (1997), for some commodities the spot price is hard to obtain, so the futures contract closest to maturity is used as a proxy for the spot prices in the commodity literature.[6] I follow the literature and use the prices of crude oil futures with one month to maturity as a proxy for crude oil spot prices. Figure 4.2 plots the recent 13 years' spot prices examined in this chapter.

[6]See Jaillet et al. (2004) for example.

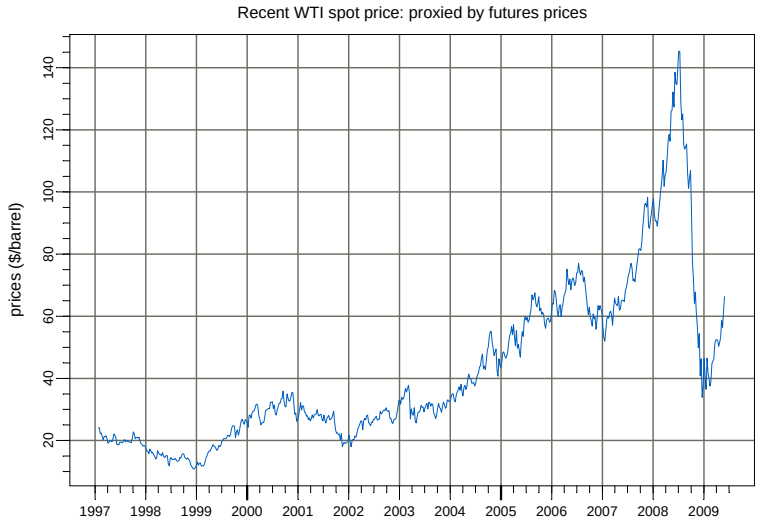


Figure 4.2: Weekly prices of WTI crude oil futures contract with one month to maturity, $/barrel, from January 24th, 1997 to May 29th, 2009.

Summary statistics of both spot prices and the corresponding log returns for three data sets are reported in Table 4.1. The full sample covers the period from January 24th 1997 to May 29th, 2009. Two sub samples cover the period before and after year 2003 when there appears to be a break in the series. Negative skewness and excess kurtosis of return series for all three data sets indicate the GBM model is not appropriate for modeling recent crude oil prices. The excess kurtosis of spot prices in both the full sample and the post-2003 sub-sample shows that the spot prices in these two periods are also not normally distributed.

Formal tests of normality, unit root and stationarity for spot prices and the corresponding returns for three data sets are performed and the results are reported in Table 4.2. The null hypothesis of normality is rejected for both the spot prices and returns for all three data sets. The unit root null hypothesis is strongly rejected for all three return processes. However, for the spot prices, the unit root hypothesis can not be rejected at a reasonable level of significance. This result is consistent

Item	Max	Min	Mean	Std. Dev.	Skewness	Kurtosis
Full sample						
Spot price	145.3	10.79	42.39	26.69	1.389	4.772
Weekly return	12.06	-15.61	7.88%	39.85%	-0.722	6.986
Sub-sample: before 2003						
Spot price	35.92	10.79	22.79	6.11	-0.075	1.98
Weekly return	8.84	-10.45	4.98%	36.93%	-0.297	4.318
Sub-sample: after 2003						
Spot price	145.3	25.67	60.52	25.54	1.106	4.026
Weekly return	12.06	-15.61	10.41%	42.48%	-0.977	8.206

Table 4.1: Summary statistics for WTI crude oil prices and the corresponding log returns for three data sets. Full sample: from January 24th, 1997 to May 29th, 2009; before 2003 sub-sample: from January 24th, 1997 to December 27th, 2002; after 2003 sub-sample: from January 3, 2003 to May 29th, 2009.

with the stationarity test for spot prices, which indicates that the stationarity null hypothesis is strongly rejected for spot price time series. These results are reassuring since the tests that have a unit root as the null hypothesis tend to over-reject the null hypothesis due to their poor size in finite samples. These results also confirm that the traditional one-factor GBM and mean reversion processes are unlikely to be able to capture the main properties of the recent crude oil spot prices.

4.2.2 Crude oil futures prices

A large data set of WTI crude oil futures traded in NYMEX are examined in this chapter. They are the world's most liquid and largest-volume futures contracts trading on a physical commodity. Each contract trades in units of 1,000 US barrels (42,000 gallons) of light, sweet crude oil and the delivery point is Cushing, Oklahoma, which is also accessible to the international spot markets via pipelines. The expiration of the contract is on the third business day prior to the 25th of each month. The contract is listed for up to 72 months. The availability of long

Tests	Spot prices	Returns
Normality test: Jarque-Bera test		
Full sample	291.68(0.00)	482.38(0.00)
Sub sample: before 2003	13.74(0.00)	26.92(0.00)
Sub sample: after 2003	83.02(0.00)	430.28(0.00)
Unit root test: Phillips-Perron test		
Full sample	-2.65(0.26)	-27.12(0.00)
Sub sample: before 2003	-2.38(0.39)	-17.34(0.00)
Sub sample: after 2003	-1.76(0.72)	-20.5(0.00)
Stationarity test: KPSS test		
Full sample	0.57**	0.07
Sub sample: before 2003	0.48**	0.16
Sub sample: after 2003	0.24**	0.10

Table 4.2: Model diagnostic tests for WTI crude oil prices and its log returns. Corresponding p-values are reported in parentheses.** denotes significant at 1% level.

maturity oil derivatives is necessary to improve the oil-related project investment valuation.

Data on 32 oil futures contracts with the maximum maturity up to 72 months are available. Among them, 10 futures contracts with different maturities for each day are selected and examined in this chapter. The choice of the futures contracts is consistent with the selected forward contracts used in Schwartz (1997). The maturities and the corresponding summary statistics for the 10 selected contracts are presented in Table 4.3. Term structure of futures prices on four selected days are presented in Figure 4.3. Both backwardation and contango are visible in this plot, which indicates the importance of convenience yield in modeling oil prices.

Item	Max ($)	Min ($)	Mean ($)	Std. Dev. ($)	Maturity (months)
F2	145.9	11.26	42.55	26.83	2
F5	146.5	12.10	42.48	27.19	5
F8	146.8	12.65	42.21	27.42	8
F12	146.1	13.12	41.83	27.58	12
F18	144.8	13.67	41.35	27.68	18
F24	143.9	14.16	41.02	27.67	24
F36	142.3	14.82	40.72	27.62	36
F48	141.7	15.35	40.49	27.61	48
F60	141.6	15.70	40.28	27.66	60
F72	141.7	16.02	40.23	27.79	72

Table 4.3: Summary statistics for 10 selected WTI crude oil futures prices, from January 31st, 1997 to April 25th, 2009. F2 represents the futures contract with 2 months maturity. The same notation applies to F5 to F72.

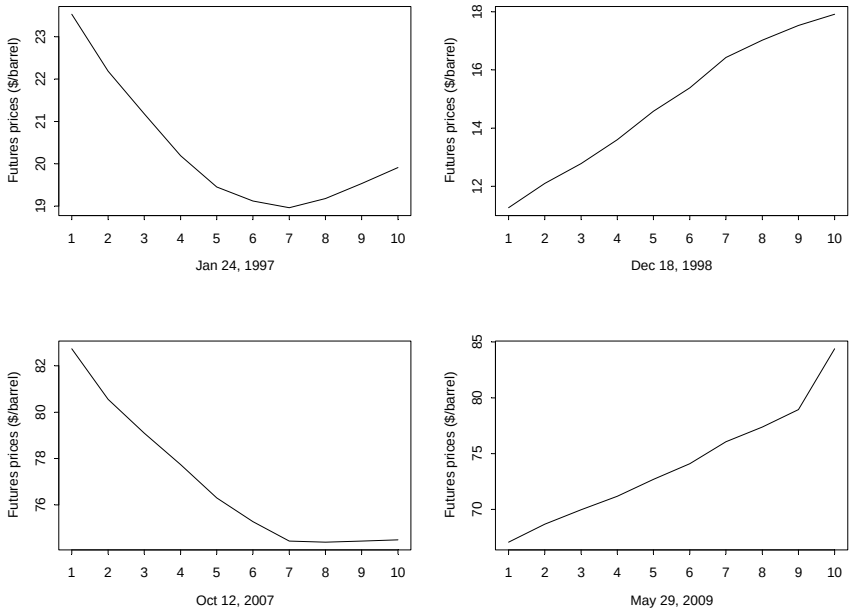


Figure 4.3: Term structure of futures prices for the 10 futures contracts on four selected days: Jan 24, 1997; Oct 18, 1998; Oct 12, 2007 and May 29, 2009.

4.3 The models and estimation methods

Three model specifications are examined and compared in this section. In particular I specify the newly proposed regime switching model and the two versions of two-factor models. For the regime switching model, the parameters are calibrated using the crude oil futures prices. The estimation of the two-factor models is carried out using the Kalman filter, which is a recursive estimation method.

4.3.1 Regime switching model

The model

The regime switching model proposed in this chapter is based on the one-factor mean reversion model analyzed in Schwartz (1997). The log commodity price in the one-factor model follows a mean reverting process. Specifically, the spot price S follows the following stochastic differential equation (SDE):

$$dS = \kappa(\mu - \ln S)Sdt + \sigma SdZ \tag{4.1}$$

Applying Ito's lemma, the log price $X = \ln S$ can be expressed as a mean-reverting Ornstein Uhlenbeck (OU)process:

$$dX = \kappa(\alpha - X)dt + \sigma dZ \tag{4.2}$$

where $\alpha = \mu - \frac{\sigma^2}{2\kappa}$. In equation (4.2), κ is the mean reversion rate, α denotes the long-run equilibrium level of log price X and σ refers to volatility. Euqation 4.1 is referred to as a log mean reverting process in this one-factor model.

Since the above one-factor model is not able to generate the various shapes of term structure of commodity futures prices, I extend this model to allow some

parameters in this model to be regime dependent assuming the existence of two states. Based on the data analysis in Section 4.2, in each of the two regimes, the crude oil spot price is assumed to follow a log mean reverting process as in equation (4.1).

Specifically, the regime switching model for crude oil spot price S is given by the following SDE:

$$dS = \kappa(\mu(s_t) - \ln S)Sdt + \sigma(s_t)SdZ \tag{4.3}$$

where s_t is a two-state continuous time Markov chain, taking two values 0 or 1. The value of s_t indicates the regime in which the oil price resides at time t. Define a Poisson process $q^{s_t \to 1-s_t}$ with intensity $\lambda^{s_t \to 1-s_t}$. Then

$$\begin{aligned} dq^{s_t \to 1-s_t} &= 1 \qquad \text{with probability } \lambda^{s_t \to 1-s_t} dt \text{ over an infinitesimally small } dt \\ &= 0 \quad \text{with probability } 1 - \lambda^{s_t \to 1-s_t} dt \text{ over an infinitesimally small } dt \end{aligned}$$

In other words, the probability of regime shifts from s_t to $1 - s_t$ over an infinitesimally small time interval dt is $\lambda^{s_t \to 1-s_t} dt$. The probability of the crude oil price staying in the current regime s_t over an infinitesimally small dt is $1 - \lambda^{s_t \to 1-s_t} dt$. The log price X in this model follows a regime switching stochastic process:

$$dX = \kappa(\alpha(s_t) - X)dt + \sigma(s_t)dZ \tag{4.4}$$

where $\alpha(s_t) = \mu(s_t) - \frac{\sigma(s_t)^2}{2\kappa}$.

In this regime switching model, both the equilibrium level of log price $\alpha(s_t)$ and the volatility $\sigma(s_t)$ are allowed to be regime dependent. However, for mathematical simplicity, the mean reversion rate κ is assumed to be the same for both states. As will be demonstrated in the following section, the calibration of the regime switching

model involves specifying partial differential equations satisfied by futures prices in each of the two regimes. When the constraint is imposed on κ, it is relatively easy to find a guess of the solution of the PDE's which transforms them into a system of ODE's. In spite of the imposition of this constraint, this regime switching model is still more flexible than the original one-factor model analyzed in Schwartz (1997).

Model calibration

Model parameters of the risk-adjusted process expressed in equation (4.3) are calibrated using oil futures prices. Specifically, the PDE characterizing oil futures prices can be simplified to a system of ODEs which can be solved numerically to give model implied futures prices consistent with different parameter values. A least squares approach is used to determine the parameter values which produce the calculated futures prices that most closely match a time series of market futures prices. A similar approach has been used in Chen and Forsyth (2008) and Chen and Insley (2008). All the parameters in the regime switching model proposed in this chapter can be calibrated simultaneously using futures data. This contrasts with the approaches in Chen and Forsyth (2008) and Chen and Insley (2008), where the volatilities have to be calibrated separately using data on options on futures which may incur more calibration errors.[7]

The crude oil future is a derivative contract whose value depends on the stochastic price S and the corresponding regime s_t. Let $F(s_t, S, t, T)$ denote oil futures prices at time t with maturity T in regime s_t, where $s_t \in \{0, 1\}$. The no-arbitrage value $F(s_t, S, t, T)$ can be expressed as a risk neutral expectation of the spot price at T:

$$F(s_t, S, t, T) = E^Q[S(T)|S(t), s_t] \tag{4.5}$$

[7]The regime switching models analyzed in Chen and Forsyth (2008) and Chen and Insley (2008) are more flexible than the one proposed in this chapter, since all the parameters are regime dependent in their papers.

Applying Ito's lemma, two PDEs characterizing the futures prices for both regimes can be derived and the one for regime s_t is expressed as:[8]

$$F(s_t)_t+\kappa(\mu(s_t)-\ln S)SF(s_t)_S+\frac{1}{2}\sigma(s_t)^2S^2F(s_t)_{SS}+\lambda^{s_t\to(1-s_t)}(F(1-s_t)-F(s_t))=0 \tag{4.6}$$

with the boundary condition: $F(s_t, S, T, T) = S$.

Given the assumption of constant mean reversion rate for both states, a guess for the solution of equation (4.6) has the following form:

$$F(s_t, S, t, T) = e^{[a(s_t,t,T)+b(t,T)\ln S]} \tag{4.7}$$

Note that in Equation (4.7) only the expression $a(s_t, t, T)$ is regime dependent as a direct consequence of our model specification. Substituting this solution guess into PDE (4.6) yields:[9]

$$\begin{aligned} F(s_t)\{\kappa b(\mu(s_t)-\ln S)+a(s_t)_t+b_t\ln S+\frac{1}{2}\sigma(s_t)^2[b^2-b]-\lambda^{s_t\to(1-s_t)}\} \\ +\lambda^{s_t\to(1-s_t)}F(1-s_t)=0 \end{aligned} \tag{4.8}$$

The following relationship between $F(s_t)$ and $F(1-s_t)$ holds:

$$F(1-s_t)=e^{[a(1-s_t)+b\ln S]}=F(s_t)e^{[a(1-s_t)-a(s_t)]} \tag{4.9}$$

Substituting equation (4.9) into the revised PDE (4.8) yields the following ODE system:

$$\begin{aligned} \kappa b\mu(s_t)+a(s_t)_t+\frac{1}{2}\sigma(s_t)^2[b^2-b]+\lambda^{s_t\to(1-s_t)}[e^{[a(1-s_t)-a(s_t)]}-1]=0 \\ b_t-\kappa b=0 \end{aligned} \tag{4.10}$$

[8] $F(s_t) \equiv F(s_t, S, t, T)$ and $F(1-s_t) \equiv F(1-s_t, S, t, T)$

[9] $a(s_t) \equiv a(s_t, t, T)$ and $b \equiv b(t, T)$ in the following equation, where $a(s_t)_t \equiv \partial a(s_t)/\partial t$ and $b_t \equiv \partial b/\partial t$.

with boundary conditions $a(s_t, T, T) = 0$; $b(T, T) = 1$. These ODEs will be solved numerically, which gives the model implied futures prices for each set of the parameter values.

A least squares approach is used for calibrating the risk-neutral parameter values. Let θ denote the set of parameters calibrated to the futures price data, where $\theta = \{\kappa, \mu(s_t), \sigma(s_t), \lambda^{s_t \to (1-s_t)} | s_t \in \{0, 1\}\}$. In particular, at each observation day t, where $t \in \{1, ..., t^*\}$, there are T^* futures contracts with T^* different maturity dates. The calibration is performed by solving the following optimization problems:

$$\min_{\theta, s_t} \sum_t \sum_T \{\hat{F}(s_t, S, t, T; \theta) - F(t, T)\}^2 \tag{4.11}$$

where $F(t, T)$ is the market futures price on the observation day t with maturity T. $\hat{F}(s_t, S, t, T; \theta)$ is the corresponding model implied futures price computed numerically and determined in equation (4.7) using the market spot price S and the parameter set θ, as well as the regime s_t.

This is a Mixed Integer problem, since the unknown parameters in θ are continuous variables and s_t is a binary variable which equals to 0 or 1 depending on the regime. It is known that some certain software packages provide a way of solving this Mixed Integer optimization problem. However in this thesis, I use an intuitive and reasonable way of calibrating these unknown model parameters. Specifically, this optimization program is implemented in Matlab which is a program specially devised for handling large vectors and performing matrix computations. I used the built-in function lsqnonlin to solve the problem. The lsqnonlin function take initial values of the parameters as inputs and then solves the problem iteratively by updating the parameters in the direction where the decline in the target function is the greatest. The calibrated parameter set θ and s_t will then minimize the distance between F and $\hat{F}$ for all t^*.

4.3.2 Schwartz (1997) two-factor model

The model

The two-factor model proposed in Gibson and Schwartz (1990) specifies that the spot price S follows a GBM process and the net convenience yield, δ, is formulated as a mean-reverting process. That is,

$$\begin{aligned} dS &= \mu S dt + \sigma_s S dz_s \\ d\delta &= \kappa(\alpha - \delta)dt + \sigma_\delta dz_\delta \\ dz_s dz_\delta &= \rho dt \end{aligned} \tag{4.12}$$

where μ is the expected return of spot prices, κ and α represent the mean reversion rate and the long-run equilibrium level of convenience yield respectively, σ_s and σ_δ denote the volatilities of the two state variables, and ρ is the correlation coefficient between the two standard Brownian increments dz_s and dz_δ.

Schwartz (1997) further explores this two-factor model by incorporating a stochastic convenience yield in the drift of price process. Specifically, the joint stochastic process of the two state variables in Schwartz (1997) is given by:

$$\begin{aligned} dS &= (\mu - \delta) S dt + \sigma_s S dz_s \\ d\delta &= \kappa(\alpha - \delta)dt + \sigma_\delta dz_\delta \\ dz_s dz_\delta &= \rho dt \end{aligned} \tag{4.13}$$

In this model, the convenience yield also follows a mean-reverting process. But the spot prices are assumed to follow an adjusted GBM process, the drift of which is stochastic instead of constant. Unlike a single factor GBM model, convenience yield will affect the price process through the correlation coefficient. Since oil is a storable commodity, based on the theory of storage, the changes in the two state

variables should be positively correlated. When inventory of crude oil rises, the spot price should decrease in response to the increasing supply and the convenience yield should also decrease since the benefit of owning the commodity is smaller compared with the time when the commodity is scarce.

Under the equivalent martingale measure, the risk adjusted processes for the two state variables, the spot price S and convenience yield δ, are expressed as:

$$\begin{aligned} dS &= (r-\delta)Sdt + \sigma_s S dz_s^* \\ d\delta &= [\kappa(\alpha-\delta)-\lambda]dt + \sigma_\delta dz_\delta^* \\ dz_s^* dz_\delta^* &= \rho dt \end{aligned} \tag{4.14}$$

where λ is the market price of convenience yield risk. From the no-arbitrage condition, the risk-adjusted drift of price process is $r-\delta$. The market price of convenience yield risk has to be incorporated in the risk neutral process of convenience yield, since the convenience yield is not traded.

Model estimation: Kalman filter

As mentioned before, there is no universal oil spot market. This renders the true oil prices not directly observable. Since both state variables are un-observable, an applicable method for estimating this type of model is by stating the problem in state space form and by using Kalman filter based on an error prediction decomposition of the log-likelihood function. The Kalman filter is a recursive procedure for estimating unobserved state variables based on observations that depend on these variables (Kalman (1960)). Prediction errors, a by-product of the Kalman filter, can then be used to evaluate the likelihood function and the model parameters are estimated by maximizing this likelihood function.

The state space form consists of a transition equation and a measurement equa-

tion. The transition equation describes the dynamics of an unobserved set of state variables. And the measurement equation relates the unobserved variables to a vector of observable. In this two-factor model, the oil spot price and convenience yield are two unobserved state variables. Futures prices with different maturities observed at different dates are served as observed variables and the measurement equation will specify the relationship between futures prices and the two state variables.

The partial differential equation (PDE)[10] characterizing the futures prices $F(S, \delta, t, T)$ can be derived using Ito's Lemma and expressed as:

$$\frac{1}{2}\sigma_s^2 S^2 F_{ss} + (r-\delta)SF_s + \frac{1}{2}\sigma_\delta^2 F_{\delta\delta} + (\kappa(\alpha-\delta)-\lambda)F_\delta + \rho\sigma_s\sigma_\delta SF_{s\delta} - F_t = 0 \quad (4.15)$$

subject to the following boundary condition: $F(S, \delta, T, T) = S$, where T denotes the maturity date of the futures contract. The analytical solution of equation (4.15) is derived in Jamshidian and Fein (1990) and Bjerksund (1991) and can be expressed as:

$$F(S, \delta, 0, T) = S \exp\left[A(T) - \delta\frac{1-e^{-\kappa T}}{\kappa}\right] \quad (4.16)$$

where

$$\begin{aligned} A(T) &= (r - \hat{\alpha} + \frac{1}{2}\frac{\sigma_\delta^2}{\kappa^2} - \frac{\sigma_s\sigma_\delta\rho}{\kappa})T + \frac{1}{4}\sigma_\delta^2\frac{1-e^{-2\kappa T}}{\kappa^3} + (\hat{\alpha}\kappa + \sigma_s\sigma_\delta\rho - \frac{\sigma_\delta^2}{\kappa})\frac{1-e^{-\kappa T}}{\kappa^2} \\ \hat{\alpha} &= \alpha - \frac{\lambda}{\kappa} \end{aligned} \quad (4.17)$$

The linear relationship between the futures prices and spot prices can be found in the log form of futures prices:

$$\ln F(S, \delta, 0, T) = \ln S + A(T) - \delta\frac{1-e^{-\kappa T}}{\kappa} \quad (4.18)$$

[10]For a detailed derivation of this PDE, see Gibson and Schwartz (1990).

Equation (4.18) relates the observed futures prices and the unobserved two state variables in this model and serves as measurement equation in the state space form of the model.

Specifically, the linear Gaussian state space model for Schwartz (1997) two-factor model can be represented as the following system of equations:

$$x_{t+1} = d_t + T_t x_t + \eta_t \tag{4.19}$$

$$y_t = C_t + Z_t x_t + \epsilon_t \tag{4.20}$$

where x denotes the vector of two unobserved state variables, $x = [X, \delta]'$, where $X = \ln S$ denotes the log of spot prices and $y = \ln F$ denotes the log futures prices.

Equation (4.19) represents the transition equation of the model, which describes the evolution of the non-observed state vector x_t over time. Based on equation (4.13), the expressions in this transition equation are given as:

$$d_t = [(\mu - \frac{1}{2}\sigma_s^2)\Delta t, \kappa\alpha\Delta t]'$$

$$T_t = \begin{bmatrix} 1 & -\delta_t \\ 0 & 1 - \kappa\delta_t \end{bmatrix}$$

η_t is serially uncorrelated disturbances with mean zero, and its variance is expressed as:

$$Var(\eta_t) = \begin{bmatrix} \sigma_s^2\Delta t & \rho\sigma_s\sigma_\delta\Delta t \\ \rho\sigma_s\sigma_\delta\Delta t & \sigma_\delta^2\Delta t \end{bmatrix}$$

Equation (4.20), describing the vector of observations y_t in terms of the state vector, is the measurement equation. Based on equation (4.18), the expressions in this

measurement equation are given as:

$$\begin{aligned} C_t &= [A(T_i)] \\ Z_t &= \left[1, -\frac{1-e^{-\kappa T_i}}{\kappa}\right] \end{aligned}$$

$i = 1, ..., N$. Hence N is the number of futures contracts at each date t and ϵ_t is a vector of serially uncorrelated disturbances with zero mean and identity variance-covariance matrix. The relationship between log-futures and the two unobserved state variables shown in this equation is linear. The innovations in the transition equation η_t and those in the measurement equation ϵ_t are assumed to be independent in our model, which means $E[\eta_t \epsilon_t] = 0$.

Two types of variables used recursively in the Kalman filter algorithm are called priori variables and posteriori variables. Define the observed data set at time t as $Y_t = (y_1, ..., y_t)$. Priori variables refer to the conditional mean, defined as $x_{t|t-1} = E[x_t|Y_{t-1}]$, and conditional variance, defined as $P_{t|t-1} = var[x_t|Y_{t-1}]$, of the state vector x_t based on information available at time $t-1$. Posteriori variables are the estimates for the mean and variance of the state vector conditional on the information available at time t, denoted as $x_{t|t} = E[x_t|Y_t]$ and $P_{t|t} = var[x_t|Y_t]$ respectively.

The first step of the Kalman filter is to compute one time step ahead priori variables $x_{t|t-1}$ and $P_{t|t-1}$ using the values of posteriori variables at time $t-1$ via the prediction equations:

$$x_{t|t-1} = d_{t-1} + T_{t-1} x_{t-1|t-1} \tag{4.21}$$

$$P_{t|t-1} = T_{t-1} P_{t-1|t-1} T'_{t-1} + Var(\eta) \tag{4.22}$$

Next, with the new observation y_t, the posteriori variables at time t can be updated

using the following updating equations:

$$x_{t|t} = x_{t|t-1} + K_t v_t \tag{4.23}$$

$$P_{t|t} = P_{t|t-1} - P_{t|t-1} Z_t' K_t' \tag{4.24}$$

where

$$v_t = y_t - C_t - Z_t x_{t|t-1}' \tag{4.25}$$

$$F_t = Z_t P_{t|t-1} Z_t' + var(\epsilon) \tag{4.26}$$

$$K_t = P_{t|t-1} Z_t' F_t^{-1} \tag{4.27}$$

where v_t is the residual of the measurement equation (4.20) or prediction error. F_t is the variance of this prediction error, $F_t = var(v_t)$. K_t is the Kalman gain. This process is then repeated until the whole set of observations Y_N has been observed and used in this recursive process. The resulting estimates of posteriori variables $x_{t|t}$ will be the filtered estimates of the state vector for each observation date t. The smoothed estimates of the state vector are obtained by using all the information in the observation set Y_N.

Lastly unknown parameters of the state space model can be estimated by maximizing the prediction error decomposition of the log-likelihood function, which is a by-product of the Kalman filter. The sample log-likelihood function is

$$\ln L = \sum_{t+1}^{N} \ln f(v_t) = c - \frac{1}{2} \sum_{t+1}^{N} (\ln |F_t| + v_t' F_t^{-1} v_t) \tag{4.28}$$

where c is constant, $f(v_t)$ denotes the probability density function of prediction error v_t. The resulting Quasi-Maximum likelihood estimates of the parameters are consistent and asymptotically normal.

4.3.3 Schwartz and Smith (2000) two-factor model

The model

The two state variables in the Schwartz and Smith (2000) model are short-term deviations which is defined as the difference between the spot and long-term prices, and the long-term equilibrium price level. The short-term deviation is assumed to follow a mean-reverting process and the equilibrium price level in their model evolves according to a GBM. The intuition is that there will be a long term drift in price in response to long run demand and supply conditions, but there will also be variations away from the long run trend due to temporary market conditions.

The spot prices are therefore determined jointly by these two state variables. Specifically, $S_t = e^{\chi_t + \xi_t}$, where χ_t denotes the short-term deviation and ξ_t represents the equilibrium price.

$$\begin{aligned} d\chi_t &= \kappa(0 - \chi_t)dt + \sigma_\chi dz_\chi \\ d\xi_t &= \mu dt + \sigma_\xi dz_\xi \\ dz_\chi dz_\xi &= \rho dt \end{aligned} \tag{4.29}$$

where κ is the mean-reversion rate of the short-term deviation, μ denotes the drift of equilibrium level, σ_χ and σ_ξ represent the volatilities of the two state variables.

The deviation χ is short-lived and not expected to persist. Therefore it is modeled as a process reverting to a mean of zero. The change in long-run price level is caused by permanent changes in supply or demand.

Estimation framework

The Kalman filter will be applied for model parameter estimation in this case as well. The two unobserved state variables in this model are short-term deviation

χ and long-term price level ξ. The observed futures prices will be used for the parameter estimation. Since futures prices depend on the risk-neutral spot price process, the risk-adjusted version of equation (4.29) can be written as:

$$\begin{aligned} d\chi_t &= (-\kappa\chi_t - \lambda_\chi)dt + \sigma_\chi dz^*_\chi \\ d\xi_t &= (\mu - \lambda_\xi)dt + \sigma_\xi dz^*_\xi \end{aligned} \tag{4.30}$$

where λ_χ and λ_ξ denote the market prices of short-term deviation risk and equilibrium price risk respectively. Based on this specification, the log-futures prices y_t, defined in the same way as in Schwartz (1997) model, can be expressed as:

$$y_t = e^{-\kappa T}\chi_0 + \xi_0 + A(T) \tag{4.31}$$

where χ_0 and ξ_0 represent the initial values of the two variables. In equation (4.31),

$$A(T) = \mu^* T - (1 - e^{-\kappa T})\frac{\lambda_\chi}{\kappa} + \frac{1}{2}((1 - e^{-2\kappa T})\frac{\sigma^2_\chi}{2\kappa} + \sigma^2_\xi T + 2(1 - e^{-\kappa T})\frac{\rho\sigma_\chi\sigma_\xi}{\kappa})$$

where $\mu^* = \mu - \lambda_\xi$.

Therefore the measurement equation can be expressed as:

$$y_t = C_t + Z_t[\chi_t, \xi_t]' + \epsilon_t \tag{4.32}$$

where

$$\begin{aligned} C_t &= [A(T_i)] \\ Z_t &= \left[e^{-\kappa T}, 1\right] \end{aligned} \tag{4.33}$$

And the transition equation is

$$[\chi_{t+1}, \xi_{t+1}]' = d_t + T_t[\chi_t, \xi_t]' + \eta_t \tag{4.34}$$

where

$$\begin{aligned} d_t &= [0, \mu\Delta t]' \\ T_t &= \begin{bmatrix} e^{-\kappa\Delta t} & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

Hence η_t is serially uncorrelated disturbances with mean zero, and the variance is expressed as:

$$Var(\eta_t) = \begin{bmatrix} (1 - e^{-2\kappa\Delta t})\frac{\sigma_\chi^2}{2\kappa} & (1 - e^{-\kappa\Delta t})\frac{\rho\sigma_\chi\sigma_\xi}{\kappa} \\ (1 - e^{-\kappa\Delta t})\frac{\rho\sigma_\chi\sigma_\xi}{\kappa} & \sigma_\xi^2\Delta t \end{bmatrix}$$

4.4 Empirical results

WTI crude oil futures prices are used for model calibration and estimation. The basic statistics and the term structure of 10 chosen futures contracts are shown in section 4.2. In this section, I present the calibration and estimation results, and compare the performance of all the three models examined in this chapter.

4.4.1 Calibration results of the regime switching model

For the regime switching model, a non-linear least squares approach is used to calibrate model parameters. Specifically given initial values of all the parameters, model implied futures prices of all the maturities at each date can be computed by solving the ODE for each of the two regimes. The differences of the model

implied futures prices and market futures prices for all the futures contract at each date are computed for both regimes. The sum squared differences are used to determine the regime at each date. The sum squared difference of each date is then summed together over all the period. The optimal parameter values will be those that generate the lowest sum squared difference. Matlab is used for parameter calibration. There are 21 iterations in this calibration process and it took about 100 minutes to converge. When the change in the residual is smaller than the specified tolerance, which is $5e^{(-3)}$ in this case, the program will stop.

It is important to check whether the obtained parameters are the only choices of parameter values which attain a reasonable in-sample fit. There could be the case that there exist several ranges of parameter values, all providing a reasonable fit to data. In other words, since the calibrated parameters are obtained by solving a nonlinear optimization program, there is no guarantee that the obtained solution is a unique and global solution. This issue can be investigated by varying the initial values of the parameters used to initialize the calibration algorithm. If the optimal parameters are sensitive to changes in the initial values this should be taken to indicate that there are potentially several local optima and, as a result, the optimal parameters will be unstable. Following this argument, different sets of initial values are used to find the optimal solution and to check the stability of the calibrated parameter values as a hint that the algorithm attains a global (instead of a local) minimum of the sum of squared function, and the estimates are generally robust to different initial values.

Table 4.4 presents the calibration results for parameter values in the proposed regime switching model for the spot price process given in equation (4.3).[11] In the table we observe two quite different regimes for the price process in the Q-measure.

[11]Since these parameters are calibrated in the Q-measure, it is not possible to interpret them in terms of the observed behavior of spot prices. However, if the market price of risk equals zero, the P-measure and Q-measure will coincide. If we believe that the market price of risk for a commodity is fairly low then we can draw some intuition about the P-measure process from our results.

Regime switching model						
κ	$\mu(0)$	$\mu(1)$	$\sigma(0)$	$\sigma(1)$	$\lambda^{0\to1}$	$\lambda^{1\to0}$
0.06	0.92	4.87	0.98	0.20	0.36	0.08

Table 4.4: Calibrated parameter values for the regime switching model.

Regime 0 has a much lower level of $\mu(0)$, but a higher volatility $\sigma(0)$ compared to regime 1 for the spot price process. The mean equilibrium level of log oil price in regime 0 $\alpha(0)$ is much lower than that of the log price in regime 1. The risk neutral intensity of switching out of regime 1 is lower at $\lambda^{1\to0} = 0.08$ which implies that in the risk neutral world prices are mostly in the regime with a higher long-run mean price level.

Plots of the model implied and market futures prices for four selected contracts are shown in Figure 4.4. The differences between the model implied futures prices and market data are smaller for contracts close to maturity, i.e. F2 and F8 (where F2 is the price on a futures contract with 2 months to maturity, etc.), and larger for long-term maturity contracts, i.e. F36 and F72. Model calibration errors including Root Mean Square Error (RMSE) and Mean Absolute Error (MAE) of both log futures prices and cash futures prices are reported in Table 4.5. From this table we observe that the longer the maturity the larger the calibration error, which implies that this regime switching model is better at capturing the dynamics of short-term contracts. The overall average errors for the futures prices expressed in two ways are less than 4.6 US$ by observing the last column of table.

4.4.2 Estimation results of Schwartz (1997) two-factor model

The parameter estimates for the Schwartz (1997) two-factor model using the Kalman filter are reported in Table 4.6.[12] For comparison purposes, the original Schwartz

[12]The subscript 97 for each parameter in this table indicates the estimated parameter for Schwartz (1997) two-factor model.

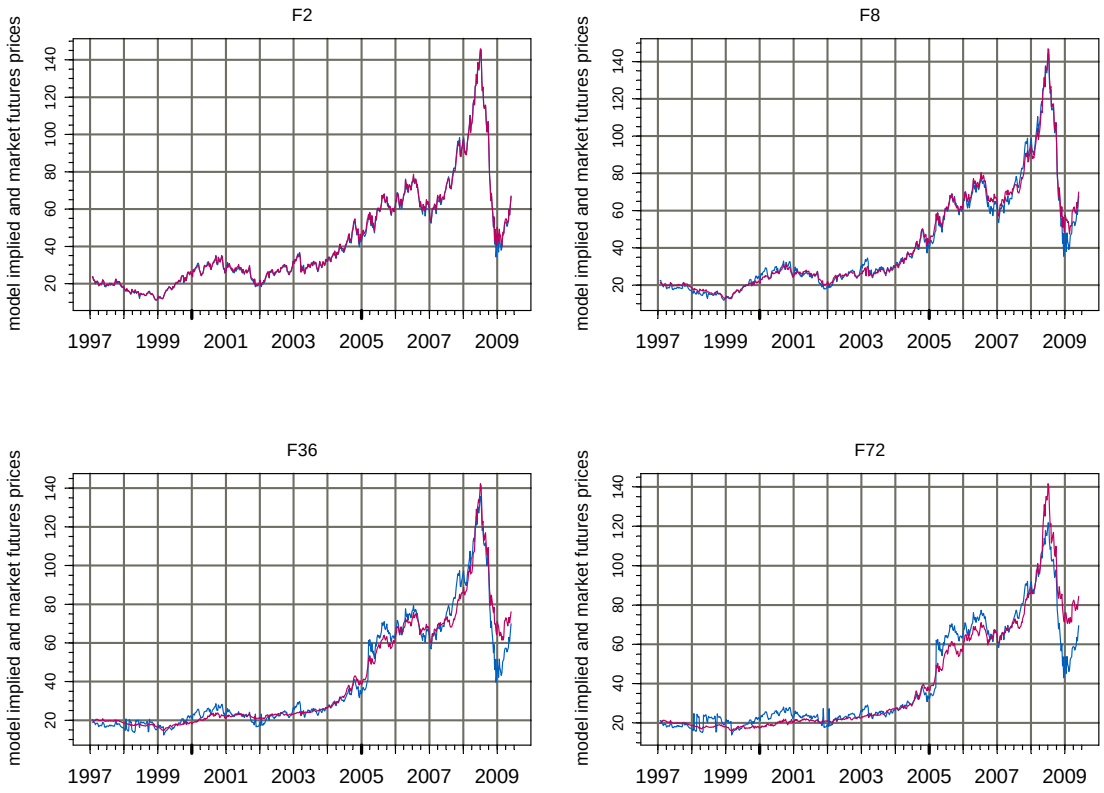


Figure 4.4: Regime switching model: plots of market futures and model implied futures prices for four different contracts, F2, F8, F36 and F72.

Calibration errors: Regime switching model											
Log futures prices											
	F2	F5	F8	F12	F18	F24	F36	F48	F60	F72	All
RMSE	0.02	0.05	0.07	0.08	0.10	0.11	0.12	0.12	0.13	0.15	0.10
MAE	0.02	0.04	0.05	0.06	0.07	0.08	0.09	0.10	0.10	0.11	0.07
Futures prices											
	F2	F5	F8	F12	F18	F24	F36	F48	F60	F72	All
RMSE	0.92	2.14	2.79	3.41	4.10	5.59	5.17	5.57	6.08	6.86	4.52
MAE	0.63	1.48	1.92	2.31	2.72	3.02	3.43	3.70	4.02	4.45	2.77

Table 4.5: Calibration errors of both log futures prices and cash futures prices for regime switching model expressed as RMSE and MAE of 10 futures contracts, US$/barrel.

(1997) results using private Enron forward data are also provided in this table. The correlation coefficient ρ_{97} is large and statistically significant from zero. The results from the current estimation and the Schwartz (1997) paper are generally consistent. The positive ρ_{97} is also consistent with the theory of storage and indicates the two state variables respond in the same direction in the presence of an unexpected change. The long-run equilibrium level of convenience yield α_{97} is higher than that reported in Schwartz (1997) and is statistically significant from zero. However α_{97} estimated using the long-term Enron forward data is not statistically significant. Both total expected return on the commodity prices μ_{97} and market price of convenience yield risk λ_{97} estimated using the more recent oil data are higher than those reported in Schwartz (1997) and (unlike in Schwartz) they are statistically significant. This result confirms the importance of accounting for changing convenience yield in oil price models.

Model implied futures prices and market futures prices for four selected oil futures contracts are plotted in Figure 4.5. The differences between the model implied and market futures prices are smaller for futures contracts with middle-length maturities, i.e. F8 and F36, and larger for a short-term future contract F2 and a long-term future contract F72. Two estimation errors for both log futures

Schwartz (1997) two-factor model							
This chapter's results using long-term futures data							
	μ_{97}	κ_{97}	α_{97}	σ_s	σ_δ	ρ_{97}	λ_{97}
Estimates	0.695	0.825	0.475	0.388	0.278	0.871	0.412
Std. Error	(0.115)	(0.008)	(0.099)	(0.011)	(0.008)	(0.010)	(0.084)
Original Schwartz (1997) results using Enron forward data							
Estimates	0.082	1.187	0.090	0.212	0.187	0.845	0.093
Std. Error	(0.120)	(0.026)	(0.086)	(0.011)	(0.012)	(0.024)	(0.101)

Table 4.6: Parameter estimates for Schwartz (1997) two-factor model using two different data sets. Weekly futures prices, from January 24th, 1997 to May 29th, 2009. Weekly Enron forward prices, from January 15, 1993 to May 16, 1996.

prices and cash futures prices, expressed as RMSE and MAE, are calculated for this model as well and reported in Table 4.7. The overall errors for the futures prices expressed in both ways are very small, lower than 1.2 US$/barrel. This model fits the futures with middle-length maturities the best, since the errors for F8 and F24 are close to zero. Comparing Tables 4.5 and 4.7, we can find lower errors generated by using Schwartz (1997) model than using a regime switching model for most futures contracts except for F2.

4.4.3 Estimation results of Schwartz and Smith (2000) model

The estimated parameter values for Schwartz and Smith (2000) model are reported in Table 4.8. For comparison purpose, the original Schwartz and Smith (2000) results using private Enron forward data are also provided in this table. The mean reversion rate in the short-term deviations κ_{00} is lower than that calculated in Schwartz and Smith (2000) using long-term forward Enron data. This implies that for recent oil data, it takes longer time which is about 10 months[13] for the short-term deviations to halve the distance from the long-run trend. The short-term

[13]Calculated based on the formula: $-\ln(0.5)/\kappa_{00}$.

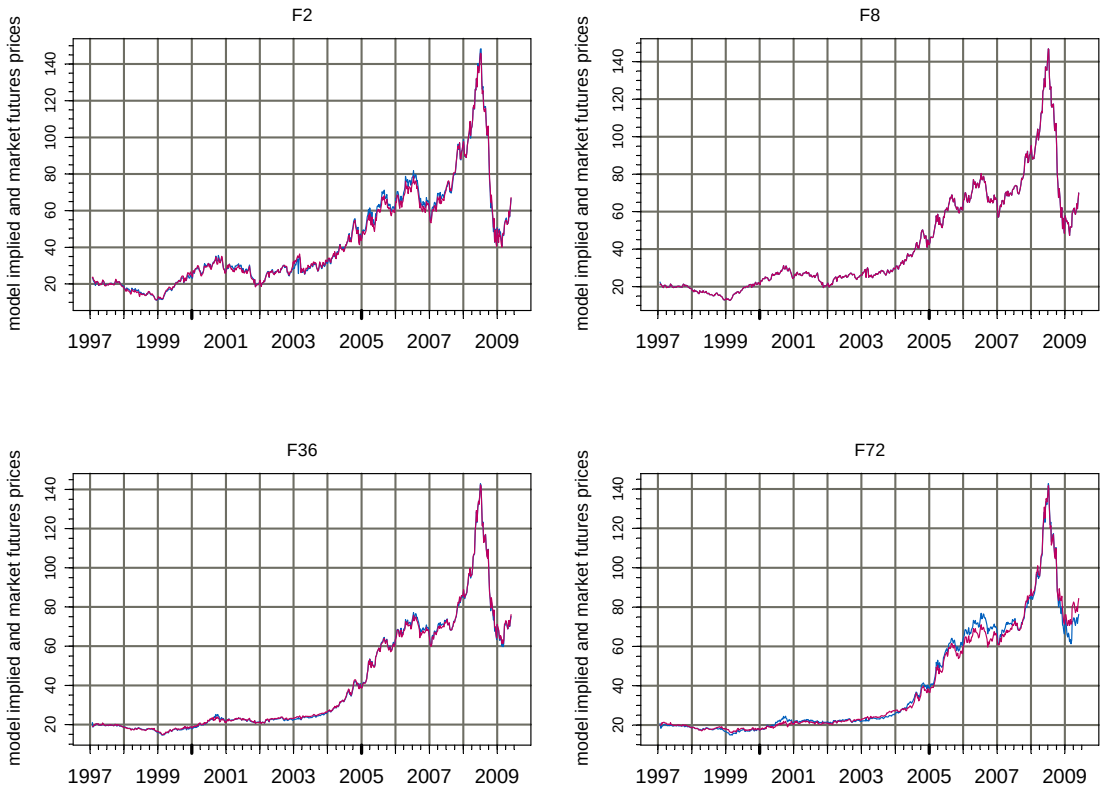


Figure 4.5: Schwartz (1997) two-factor model: plots of market futures and model implied futures prices for 4 different contracts, F2, F8, F36 and F72.

Estimation errors: Schwartz (1997) two-factor model											
Log futures prices											
	F2	F5	F8	F12	F18	F24	F36	F48	F60	F72	All
RMSE	0.03	0.01	0.00	0.01	0.01	0.00	0.02	0.03	0.04	0.05	0.03
MAE	0.03	0.01	0.00	0.01	0.01	0.00	0.01	0.02	0.03	0.04	0.02
Futures prices											
	F2	F5	F8	F12	F18	F24	F36	F48	F60	F72	All
RMSE	1.46	0.45	0.00	0.25	0.25	0.00	0.63	1.16	1.71	2.47	1.15
MAE	1.07	0.34	0.00	0.18	0.19	0.00	0.47	0.83	1.21	1.68	0.60

Table 4.7: Estimation errors of log and cash futures prices for Schwartz (1997) two-factor model expressed as RMSE and MAE of 10 futures contracts, US$/barrel.

deviation is more volatile than the long-run equilibrium. The risk adjusted drift rate μ^* is negative indicating a large positive market price of equilibrium price risk λ_ξ. The correlation coefficient ρ_{00} is statistically significant, which is consistent with the result in Schwartz and Smith (2000).

Model implied spot prices and the long-run equilibrium prices are plotted in Figure 4.6. The red line represents the spot prices and the blue line indicates model implied equilibrium price level. The spot prices fluctuate around the long-run price level until the year 2005, after which the spot prices are consistently below the long-run mean level. The recent higher equilibrium prices indicate that market participants expected higher world demand for crude oil and relatively shortage of oil supply.

Figure 4.7 shows model implied and market futures prices for four selected futures contracts. We can find a good fit for F8 and F36, since the differences between the two series are smaller compared with the very short futures contract F2 and the very long contract F72. Table 4.9 reports the estimations errors expressed as RMSE and MAE to the oil futures prices. This table also indicates that Schwartz and Smith (2000) model fits the mid-term futures contract the best since the errors of them are close to zero. In particular, the errors expressed in both ways for F8

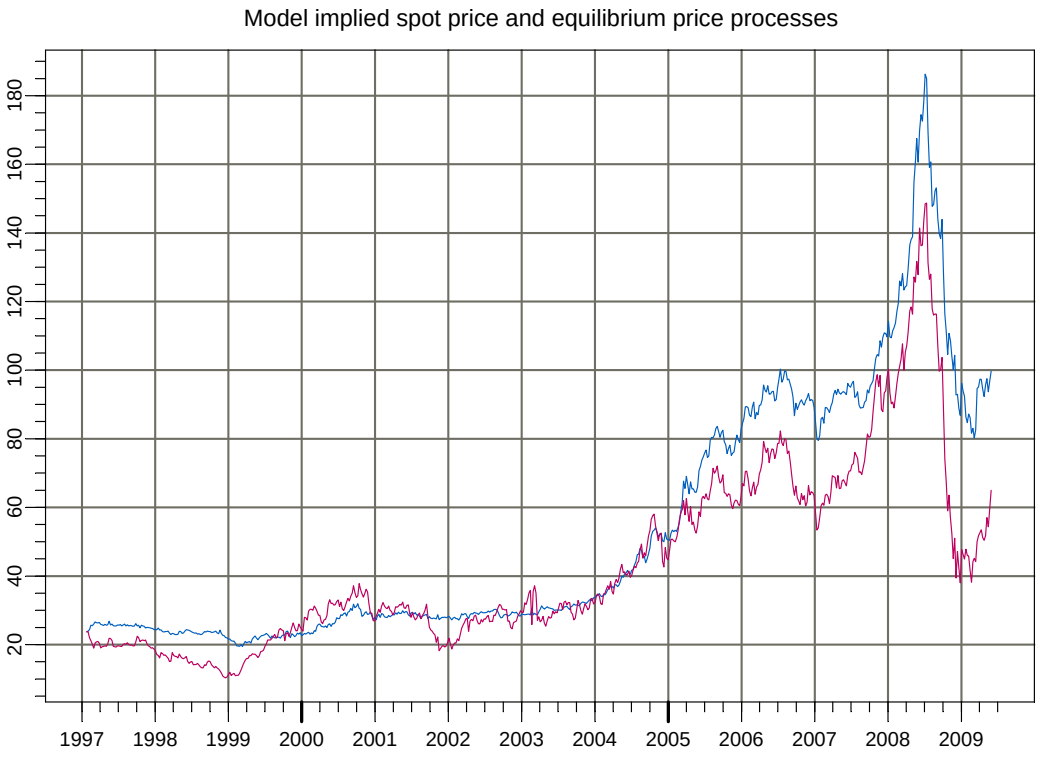


Figure 4.6: Schwartz and Smith (2000) model implied spot and equilibrium prices. Red line: spot prices. Blue line: equilibrium prices.

Schwartz and Smith (2000) model							
This chapter's results using long-term futures data							
	κ_{00}	σ_χ	μ_{00}	σ_ξ	ρ_{00}	λ_{00}	μ^*
Estimates	0.831	0.341	0.217	0.191	0.154	0.268	-0.017
Std. Error	(0.008)	(0.009)	(0.054)	(0.005)	(0.040)	(0.093)	(0.001)
Original Schwartz and Smith (2000) results using Enron data							
Estimates	1.19	0.158	-0.039	0.115	0.189	0.014	0.016
Std. Error	(0.03)	(0.009)	(0.073)	(0.006)	(0.096)	(0.082)	(0.001)

Table 4.8: Parameter estimates for Schwartz and Smith (2000) model using two different data sets. Weekly futures prices, from January 24th, 1997 to May 29th, 2009. Weekly Enron forward prices, from January 15, 1993 to May 16, 1996.

Estimation errors: Schwartz and Smith (2000) model											
Log futures prices											
	F2	F5	F8	F12	F18	F24	F36	F48	F60	F72	All
RMSE	0.03	0.01	0.00	0.01	0.01	0.00	0.02	0.03	0.04	0.05	0.03
MAE	0.03	0.01	0.00	0.01	0.01	0.00	0.01	0.02	0.03	0.04	0.02
Futures prices											
	F2	F5	F8	F12	F18	F24	F36	F48	F60	F72	All
RMSE	1.46	0.45	0.00	0.25	0.25	0.00	0.63	1.16	1.71	2.47	1.15
MAE	1.07	0.34	0.00	0.18	0.19	0.00	0.47	0.83	1.21	1.68	0.60

Table 4.9: Estimation errors of log futures prices for Schwartz and Smith (2000) model expressed as RMSE and MAE of 10 futures contracts, US$/barrel.

and F24 are almost zero. For the very-short and very-long futures contracts, the estimation errors are relatively large. The overall errors for the futures prices case expressed in two ways are lower than 1.2 US$/barrel indicating that this model produces a good fit for the recent oil futures. Furthermore, comparing Tables 4.7 and 4.9, we can see that the performances of both Schwartz and Smith (2000) model and Schwartz (1997) two-factor model are comparable in terms of fitting market futures prices.

Schwartz and Smith (2000) show that the two-factor model in Gibson and

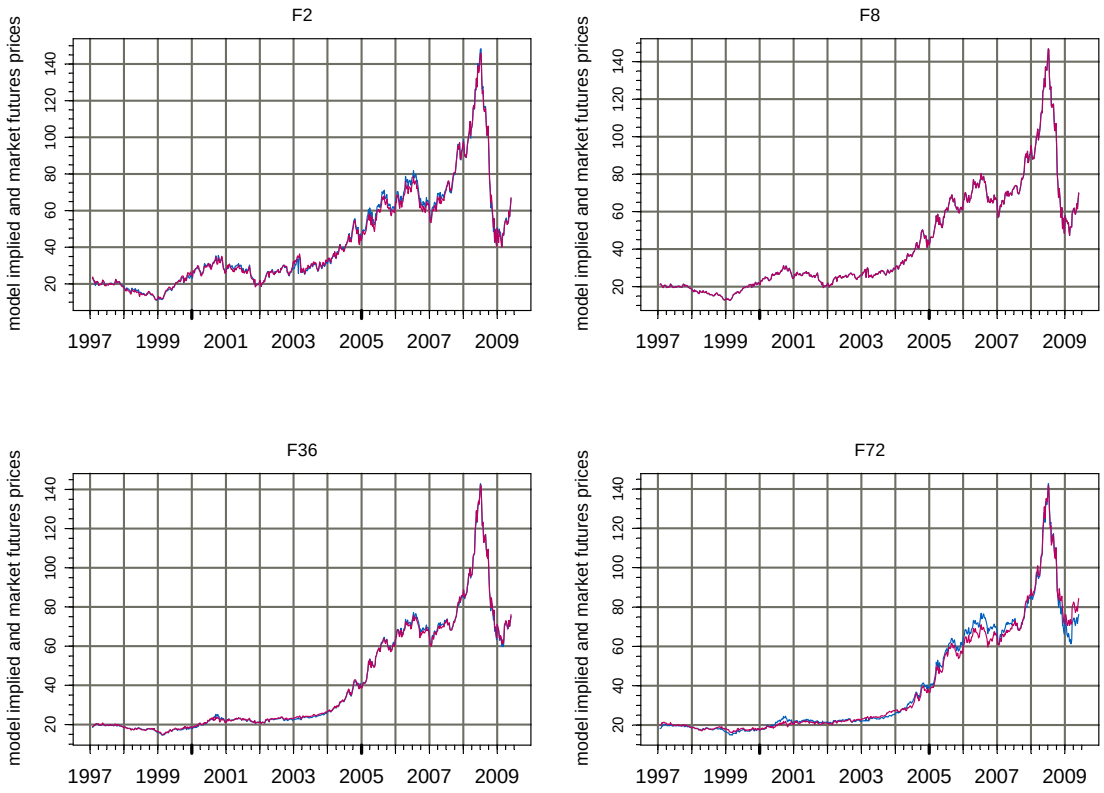


Figure 4.7: Schwartz and Smith (2000) model: plots of market futures and model implied futures prices for 4 different contracts, F2, F8, F36 and F72.

S97 two-factor model V.S. SS00 short-long model		
S97 parameters	S97	Using SS00 parameters
μ_{97}	0.695	0.537
κ_{97}	0.825	0.831
α_{97}	0.475	0.234
σ_s	0.388	0.417
σ_δ	0.278	0.283
ρ_{97}	0.871	0.891
λ_{97}	0.412	0.223

Table 4.10: Model equivalence check: Schwartz (1997) and Schwartz and Smith (2000).

Schwartz (1990) is mathematically the same as their short-term deviation and long-term dynamic model. Since the Schwartz (1997) two-factor model is a variation of Gibson-Schwartz 1990 model, they derive the relationships between the parameters of the two models. Using the parameter estimates reported in Table 4.8, the corresponding parameter values of Schwartz (1997) two-factor model are calculated based on the formulas derived in Schwartz and Smith (2000). These are presented in Table 4.10 where we see that most parameters in the Schwartz (1997) two-factor model including κ_{97} and ρ_{97} calculated using Schwartz and Smith (2000) parameter estimates (column 3 in Table 4.10) are close to Schwartz (1997) estimated parameter values (column 2 in Table 4.10). This result illustrates the closeness of these two models in terms of their similar performances in fitting the market futures prices.

4.4.4 Model comparison

To compare the performances of the above three models, I perform a cross-section out-of-sample test introduced in Schwartz (1997). The idea of this test is to compare the model calibration errors calculated by using the out-of-sample parameter estimates. Specifically, the parameters and the state variables are calibrated and estimated using the chosen 10 futures contracts over the period. And they will be

Out-of-Sample test						
	RMSE			MAE		
Model	RS	S97	SS00	RS	S97	SS00
F1	0.319	1.955	2.147	0.269	1.395	1.542
F4	1.786	0.611	0.727	1.198	0.429	0.520
F7	2.520	0.192	0.208	1.690	0.074	0.094
F10	3.014	0.170	0.206	2.004	0.086	0.124
F15	3.668	0.126	0.295	2.406	0.084	0.216
F21	4.268	0.162	0.156	2.783	0.121	0.119
F30	4.740	0.518	0.250	3.159	0.385	0.192
All	3.226	0.808	0.878	1.930	0.368	0.401

Table 4.11: Cross-section out-of-sample test between regime switching model (RS), Schwartz (1997) two-factor model (S97) and Schwartz and Smith (2000) model (SS00), expressed as RMSE and MAE of 7 futures contracts, US$/barrel.

applied to price 7 selected futures contracts which were not used for parameter estimation and calibration. The errors are then computed and compared. The average maturities (in months) of the 7 futures contracts are 1, 4, 7, 10, 15, 21 and 30.

Table 4.11 presents the results of the cross-section test for the three models examined in this chapter. The errors are expressed as RMSE and MAE. Comparing the overall errors which are reported in the last row of this table, we can find that the errors generated by using the regime switching model are the largest. The longer the maturity of a futures contract, the larger the errors for the regime switching model. Both Schwartz (1997) two-factor model and Schwartz and Smith (2000) fit the futures contracts the best and the errors for both near and long contracts are larger compared with those for the mid-term maturity contracts.

In this regime switching model specification, the mean reversion rates for the log price for the two regimes are kept the same. This restriction may be one of the reasons leading to the relatively larger calibration errors compared with the multi-factor models. However the calibration error expressed as RMSE on average

generated by the regime switching model is only about \$2.5/barrel higher than that obtained from the two multi-factor models. One obvious step for future research is to relax the constant mean reversion rate in this regime switching model and examine the performance of fitting the market data.

In general, the calibration and estimation errors of all three models are small in magnitude (less than \$4.6/barrel on average) indicating a good performance of all three models in terms of explaining the recent crude oil dynamics. Both two-factor models analyzed in Schwartz (1997) and Schwartz and Smith (2000) are shown to be able to explain the dynamics of recent crude oil prices using the long-term oil futures data. This result further confirms the importance of modeling the stochastic convenience yield on generating various term structures of oil futures prices. Hence for evaluating a long-term oil-linked investment, investors can apply these multi-factor models to make decisions. The proposed two-state regime switching model is worthy of further exploration.

4.5 Conclusions

The behavior of crude oil prices before and after year 2003 differs dramatically. This chapter investigates the dynamics of recent crude oil prices by comparing and contrasting three different stochastic models: a two-state regime switching model based on the one-factor model examined in Schwartz (1997), and two-factor models analyzed in Schwartz (1997) and Schwartz and Smith (2000) respectively. The regime switching model allows for two states in the crude oil markets represent good times and bad times. The importance of modeling stochastic convenience yield is further explored in this chapter by examining the Schwartz (1997) two-factor model. This chapter also illustrates the implication of the empirical equivalence of the Schwartz (1997) and Schwartz and Smith (2000) two-factor models. The goal of this chapter is to find a modeling approach that is rich enough to capture the

main features of the recent crude oil prices, while still being simple enough to be easily applied to evaluate a long-term oil-linked investment.

Long-term oil futures prices are used for model parameter calibration and estimation. A non-linear least squares approach is used to calibrate parameter values for the regime switching model and Kalman filter along with Quasi Maximum Likelihood method is applied to estimate the parameters for the two-factor models. The performances of these three different model specifications in terms of fitting market prices are then compared and analyzed. The estimation errors generated by the two multi-factor models are comparable and smaller in magnitude than the calibration errors generated by the proposed regime switching model. The implication of the mathematical equivalence of the two-factor models analyzed in Schwartz (1997) and Schwartz and Smith (2000) is empirically illustrated in this chapter as well.

Imposing the same mean reversion rate for the two regimes may be one of the reasons which leads to the relatively large calibration errors for the regime switching model. However the calibration errors expressed as RMSE generated by the regime switching model are small (less than \$4.6/barrel on average), indicating that this regime switching specification shows some promises as a parsimonious model which is able to explain the main properties of the recent crude oil prices. Future research will further investigate this regime switching model by relaxing the constant mean reversion rate in both regimes. Given the good fit of the multi-factor model in terms of matching the recent oil market data, in the future research, I will focus on applying the multi-factor model in a regime switching context to an analysis of oil-related investment decisions and asset valuation.

Chapter 5

Conclusion

The main purpose of this thesis is to propose and analyze stochastic models expressed in the risk neutral world for commodity prices in the context of valuing commodity linked investments. The goal is to find modeling approaches that are rich enough to capture the main characteristics of commodity prices, lumber and crude oil prices in particular, while still being simple enough that the resulting price model can easily be incorporated into problems of commodity investment valuation.

In Chapter 2, I investigate a possible improvement in the modeling of stochastic timber prices in optimal tree harvesting problems. In particular, I compare two different stochastic price processes, namely a regime switching model with a different mean reverting process in each regime (RSMR) and a traditional mean reverting model (TMR). The RSMR model allows for two states in lumber markets which we may characterize as being good times and bad times. These two price models are calibrated using lumber derivatives prices. I then use the calibrated timber price models to solve a forest investment problem for a hypothetical stand of trees in Ontario's boreal forest. A real options approach is used in this thesis to model the optimal harvesting decision. A Hamilton-Jacobi-Bellman (HJB) variational inequality is developed and solved numerically. I show that the RSMR model provides

a reasonably good and parsimonious model for lumber prices in terms of fitting the market prices of lumber derivatives. Moreover the regime switching model is shown to generate reasonable stand values and critical prices which serve as a useful input into the investment decision. Thus it is a preferred model to be used in the analysis of forestry investment decisions compared to the single-factor model which has been widely used in the forestry literature. Future research will investigate the robustness of the RSMR model for describing lumber price dynamics through a comparison with other multi-factor models that have been used in the literature to value other commodity linked investments.

For storable commodities and those that serve as inputs to production such as lumber, convenience yield plays an important role in price formation. The seasonal harvesting of trees, as well as the importance of wood products as inputs to other industries, suggest that convenience yield may be important to understanding the dynamics of timber prices. In chapter 3, I investigate whether convenience yield is an important factor in determining optimal decisions for a forestry investment. Three different stochastic models of lumber prices including a single-factor mean reverting model, a one-factor geometric Brownian motion model and a two-factor model analyzed in Schwartz (1997) are analyzed and compared in this chapter. The two-factor model is shown to provide a reasonable fit of the term structure of lumber futures prices. The impact of convenience yield on a forestry investment decision is examined using the Schwartz (1998) long-term model which transforms the two-factor price model into a single factor model with a composite price. Using the long-term model an optimal harvesting problem is analyzed, which requires a numerical solution of a Hamilton-Jacobi-Bellman (HJB) equation. I compare the results for the long-term model to those from the single-factor mean reverting and the geometric Brownian motion models. The inclusion of convenience yield through the long-term model is found to have a significant impact on land value and optimal harvesting decisions. When making the investment decisions, investors should take

the dynamics of the convenience yield into account. This chapter also shows that it is worthwhile applying the Schwartz (1998) long-term model when analyzing forestry investment decisions. A natural extension of this research is to solve the HJB equation for the full two-factor problem and compare it with the long-term model results.

Oil is a non-renewable resource and is a key input to the world's economy. Since about year 2003, we have witnessed a significant run up in oil prices, peaking at over $140 per barrel in July 2008. In chapter 4, I propose a new regime switching model to capture the recent crude oil dynamics and compare it with the widely used Schwartz (1997) and Schwartz and Smith (2000) two-factor models. Long-term oil futures data are used to calibrate and estimate the model parameters. The performances of the two-factor models are comparable in terms of fitting the market prices of the long-term oil futures contracts and are found to more closely match the behavior of oil futures prices than the regime switching model. Future research will further investigate the proposed regime switching model by relaxing the constant mean reversion rate. Given the good fit of the multi-factor model in terms of matching the recent oil market data, in the future research, I will also focus on applying the multi-factor model in a regime switching context to an analysis of oil-related investment decisions and asset valuation.

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Appendix

A Appendix to Chapter 2

A1 Relating P-measure and Q-measure parameters

Parameter estimates in Section 2.5 are all Q-measure or risk-adjusted estimates. It is natural to want to relate these estimates to real-world or P-measure parameter values. We can determine the relation between Q-measure and P-measure estimates if we make an assumption for the price process in the P-measure. Assume that the spot price model in the P-measure for the RSMR case is comparable to the Q-measure model and is given by:

$$dP = \alpha'(s_t)(K'(s_t) - P)dt + \sigma'(s_t)PdZ \tag{A1}$$

where s_t is a two-state continuous time Markov chain, taking two values 0 or 1. The value of s_t indicates the regime in which the lumber price resides at time t. Define a Poisson process $q^{s_t \to 1-s_t}$ with intensity $\lambda'^{[s_t \to 1-s_t]}$. Then

$$\begin{aligned} dq^{s_t \to 1-s_t} &= 1 \qquad \text{with probability } \lambda'^{[s_t \to 1-s_t]}dt \\ &= 0 \quad \text{with probability } 1 - \lambda'^{[s_t \to 1-s_t]}dt \end{aligned}$$

Observe that in the above equations, we have defined P-measure parameters, α', K', σ',and λ', to distinguish them from their counterparts in the Q-measure process.

Consider a futures contract on P, denoted $F(P, t, s_t)$ or just $F(s_t)$. Using Ito's lemma we can express dF as:

$$dF = \mu(s_t)dt + \sigma'(s_t)PF(s_t)_P dZ + \Delta F dq^{s_t \to 1-s_t} \tag{A2}$$

where

$$\mu(s_t) \equiv \alpha'(s_t)(K'(s_t) - P)F_P + \frac{\sigma'(s_t)^2 P^2}{2} F_{PP} + F_t \tag{A3}$$

$$\Delta F \equiv [F(1 - s_t) - F(s_t)] \tag{A4}$$

To find the value of F we create a hedging portfolio in the normal manner. Suppose we have three contracts, F_1, F_2 and F_3, which may be futures contracts with different maturities. We create a portfolio with these three securities choosing the quantity of each asset so that the portfolio is riskless. Following standard steps, this leads to the following condition that must hold under no-arbitrage assumptions for any contract $F(P, t)$:

$$\mu(s_t) = \beta_P \sigma'(s_t) P F_P - \beta_{sw} \Delta F \tag{A5}$$

β_P is the market price of risk for price diffusion risk and reflects the extra return over the risk free rate that the market requires for exposure to price risk. β_{sw} is the market price of risk for regime switching. Both of these terms may depend on P and t. Substituting in for $\mu(s_t)$ and ΔF gives

$$\alpha'(s_t)(K'(S_t) - P)F_P + \frac{\sigma'(s_t)^2 P^2}{2} F_{PP} + F_t = \beta_P \sigma'(s_t) P F_P - \beta_{sw}[F(1 - s_t) - F(s_t)] \tag{A6}$$

Further rearranging results in:

$$\alpha'(s_t)\left(1 + \frac{\beta_P \sigma'(s_t)}{\alpha'(s_t)}\right)\left(\frac{K'(S_t)}{1 + \frac{\beta_P \sigma'(s_t)}{\alpha'(s_t)}} - P\right) F_P + \frac{\sigma'(s_t)^2 P^2}{2} F_{PP} + F_t + \beta_{sw}[F(1 - s_t) - F(s_t)] = 0 \tag{A7}$$

Equation (A7) describes the behaviour of a futures contract that depends on the stochastic variable P, in terms of the parameters defined in the P-measure,

assuming the P-measure spot price is described by Equation (A1). Comparing Equation (A7) with Equation (4.6) we can see the relationship between P-measure and Q-measure parameters. In particular,

$$\alpha(s_t) = \alpha'(s_t)\left(1 + \frac{\beta_P \sigma'(s_t)}{\alpha'(s_t)}\right) \tag{A8}$$

$$K(s_t) = \frac{K'(s_t)}{1 + \frac{\beta_P \sigma'(s_t)}{\alpha'(s_t)}} \tag{A9}$$

$$\sigma(s_t) = \sigma'(s_t) \tag{A10}$$

$$\lambda^{s_t(1-s_t)} = \beta_{sw} \tag{A11}$$

For further comparison we make assumptions regarding the signs of the parameters in the above equations. We know that $\sigma'(s_t) > 0$. For the other two parameters the most likely case is that β_P and $\alpha'(s_t)$ are also positive. In this case it follows that $\alpha(s_t) > \alpha'(s_t)$ and $K(s_t) < K'(s_t)$. It makes intuitive sense that in moving from the real world to the risk neutral world, the risk adjustment implies a more rapid speed of mean reversion and a lower long run equilibrium level. Optimal actions are taken by assuming that lumber prices revert to a lower long run mean and at a faster rate than is actually the case.

Rearranging Equations (A8) and (A9), the mean reversion rate and the long run equilibrium price level under the P-measure, $\alpha'(s_t)$ and $K'(S_t)$, can be expressed as:

$$\alpha'(s_t) = \alpha(s_t) - \beta_P \sigma(s_t) \tag{A12}$$

$$K'(s_t) = \left(1 + \frac{\beta_P \sigma'(s_t)}{\alpha'(s_t)}\right) K(s_t) \tag{A13}$$

Based on the calibrated parameters presented in Tables 2.2 and 2.3, it is obvious from Equation (A12) that given a small positive β_P, $\alpha'(0) > \alpha'(1)$. Hence Equation (A13) implies that the high price regime in the real world is also the high price

regime in the risk neutral world, i.e. $K'(0) < K'(1)$.

Equation (A10) tells us that volatility is the same in the P and Q measures. Equation (A11) tells us that the intensity of regime switching, $\lambda^{s_t \to (1-s_t)}$, is equal to the market price of risk of regime switching. Hence the risk-adjusted probability of switching regimes $\lambda^{s_t \to (1-s_t)} dt$ may be quite different from the actual probability, $\lambda' dt$, as implied by historical price data.

A2 Numerical solution of HJB Variational Inequality

The basic linear complementarity problem of our optimal tree harvesting problem can be expressed as Equation (2.27)

$$V(s_t)_\tau - V(s_t)_\alpha = \alpha(s_t)(K(s_t) - P)V(s_t)_P + \frac{1}{2}(\sigma(s_t)P)^2 V(s_t)_{PP} - rV(s_t) + \lambda^{s_t \to 1-s_t}(V(1-s_t) - V(s_t)) + \Upsilon(s_t) \quad \text{(A14)}$$

This PDE is discretized using unequally spaced grids in the directions of P and α. Time direction is also discretized. Define nodes on the axes for P, α and τ by

$$\begin{aligned} P &= [P_1, P_2, ..., P_I] \\ \alpha &= [\alpha_1, \alpha_2, ..., \alpha_J] \\ \tau &= [\tau_1, \tau_2, ..., \tau_N] \end{aligned} \quad \text{(A15)}$$

Using fully implicit difference method, the difference scheme for Equation (A14) can be written as

$$\frac{V(s_t, P_i, \alpha_j, \tau^{n+1}) - V(s_t, P_i, \alpha_{j+\Delta\tau}, \tau^n)}{\Delta\tau} = \Big[\alpha(s_t)(K(s_t) - P)V(s_t)_P + \frac{1}{2}(\sigma(s_t)P)^2 V(s_t)_{PP} - rV(s_t) + \lambda^{s_t \to 1-s_t}(V(1-s_t) - V(s_t)) + \Upsilon(s_t)\Big]_{ij}^{n+1} \quad \text{(A16)}$$

For simplicity, define $V(s_t)_{ij}^{n+1} = V(s_t, P_i, \alpha_j, \tau^{n+1})$, $V^*(s_t)_{ij}^{n} = V(s_t, P_i, \alpha_{j+\Delta\tau}, \tau^n)$ and rewrite Equation (A16) as

$$\frac{V(s_t)_{ij}^{n+1} - V^*(s_t)_{ij}^{n}}{\Delta\tau} = \Big[\alpha(s_t)(K(s_t) - P)V(s_t)_P + \frac{1}{2}(\sigma(s_t)P)^2 V(s_t)_{PP} - rV(s_t) + \lambda^{s_t \to 1-s_t}(V(1-s_t) - V(s_t)) + \Upsilon(s_t)\Big]_{ij}^{n+1} \quad \text{(A17)}$$

Since the right hand side of Equation (A17) only contains the state variable P, this one-dimensional PDE is solved numerically for each stand age α_j within each time step. After one time step iteration completes, using linear interpolation to get $V(s_t, P_i, \alpha_{j+\Delta\tau}, \tau^n)$. Hence our only concern is the discretization of derivatives with respect to P.

A2.1 Discretization for interior points along P direction

For simplicity, the dependence of the regime s_t is dropped for discretization, except for $V(1 - s_t)$ in Equation (A17). Hence it can be further simplified as

$$\frac{V_{ij}^{n+1} - V_{ij}^{*n}}{\Delta\tau} = \Big[\alpha(K - P)V_P + \frac{1}{2}(\sigma P)^2 V_{PP} - rV + \lambda^{s_t \to 1-s_t}(V(1-s_t) - V) + \Upsilon\Big]_{ij}^{n+1} \quad \text{(A18)}$$

Central difference, forward difference and backward difference methods can be used to discretize the first derivative term V_P for interior points $i = [2, ..., I-1]$. We choose the difference method which will assure the positive coefficient scheme. If all these three methods can guarantee the positive coefficient scheme, central difference will be picked up for its faster convergence. For illustration purpose, the complete discretiztion equation will use central difference method for V_P.

$$\frac{V_{ij}^{n+1}-V_{ij}^{*n}}{\Delta\tau} = \left\{\frac{\sigma^2P^2}{2}\left[\frac{\frac{V_{i+1,j}-V_{ij}}{P_{i+1}-P_i}-\frac{V_{ij}-V_{i-1,j}}{P_i-P_{i-1}}}{\frac{P_{i+1}-P_{i-1}}{2}}\right]+\alpha(K-P)\left[\frac{V_{i+1,j}-V_{i-1,j}}{P_{i+1}-P_{i-1}}\right]\right.$$
$$-(r+\lambda^{s_t\to 1-s_t})V_{ij}+\lambda^{s_t\to 1-s_t}V(1-s_t)_{ij}$$
$$\left.+\frac{\overline{\pi_{ij}}}{\Delta\tau}[(P_i-C)Q_j+V_{i0}-V_{ij}]\right\}^{n+1} \quad \text{(A19)}$$

Equation (A19) can be simplified as

$$\frac{V_{ij}^{n+1}-V_{ij}^{*n}}{\Delta\tau} = a_iV_{i-1,j}^{n+1}+b_iV_{i+1,j}^{n+1}-[a_i+b_i+r+\lambda^{s_t\to 1-s_t}+\frac{\overline{\pi_{ij}}}{\Delta\tau}]V_{ij}^{n+1}$$
$$+\lambda^{s_t\to 1-s_t}V(1-s_t)_{ij}^{n+1}+\frac{\overline{\pi_{ij}}}{\Delta\tau}[(P_i-C)Q_j+V_{i0}-V_{ij}^{n+1}] \quad \text{(A20)}$$

where define $\alpha_i \equiv \frac{\sigma^2P_i^2}{P_{i+1}-P_{i-1}}$

1. **For central difference method**

$$a_i \equiv \frac{\alpha_i}{P_i-P_{i-1}}-\frac{\alpha(K-P_i)}{P_{i+1}-P_{i-1}}; \qquad b_i \equiv \frac{\alpha_i}{P_{i+1}-P_i}+\frac{\alpha(K-P_i)}{P_{i+1}-P_{i-1}}$$

2. **For forward difference method**

$$a_i \equiv \frac{\alpha_i}{P_i-P_{i-1}}; \qquad b_i \equiv \frac{\alpha_i}{P_{i+1}-P_i}+\frac{\alpha(K-P_i)}{P_{i+1}-P_i}$$

3. **For backward difference method**

$$a_i \equiv \frac{\alpha_i}{P_i-P_{i-1}}-\frac{\alpha(K-P_i)}{P_i-P_{i-1}}; \qquad b_i \equiv \frac{\alpha_i}{P_{i+1}-P_i}$$

A2.2 Discretization of boundary conditions for $i = 1$ and $i = I$

When $P = 0$, no specific boundary condition is needed. Substitute $P = 0$ into HJB Equation (A14) to get PDE for this boundary

$$V(s_t)_\tau - V(s_t)_\varphi = \alpha(s_t)K(s_t)V(s_t)_P - rV(s_t) + \lambda^{s_t \to 1-s_t}(V(1-s_t) - V(s_t)) + \Upsilon(s_t) \tag{A21}$$

Using forward discretization for $V(s_t)_P$, the discrete version of Equation (A21) can be written as

$$\begin{aligned}\frac{V_{1j}^{n+1} - V_{1j}^{*n}}{\Delta\tau} = b_1 V_{2,j}^{n+1} - [b_1 + r + \lambda^{s_t \to 1-s_t} + \frac{\overline{\pi_{1j}}}{\Delta\tau}]V_{1j}^{n+1} + \\ \lambda^{s_t \to 1-s_t} V(1-s_t)_{1j}^{n+1} + \frac{\overline{\pi_{1j}}}{\Delta\tau}[(P_1 - C)Q_j + V_{10} - V_{1j}^{n+1}]\end{aligned} \tag{A22}$$

where $b_1 = \frac{\alpha K}{P_1 - P_0}$.

When $P = P_I$, the option value is a linear function of the price. Hence the second derivative term $V(s_t)_{PP} = 0$. Guess the solution $V(s_t)_{Ij} = A(\tau) + B(\tau)P_I$. When $P \to \infty$, the term $B(\tau)P_I$ dominates and $V(s_t)_{Ij} \approx B(\tau)P_I$. For the first derivative term $\alpha(s_t)(K(s_t) - P)V(s_t)_P$, $P_I \gg K(s_t)$. Hence $\alpha(s_t)(K(s_t) - P)V(s_t)_P \approx -\alpha(s_t)PV(s_t)_P = -\alpha(s_t)V(s_t)$. The HJB equation (A14) in this boundary can then be expressed as

$$V(s_t)_\tau - V(s_t)_\varphi = -\alpha(s_t)V(s_t) - rV(s_t) + \lambda^{s_t \to 1-s_t}(V(1-s_t) - V(s_t)) + \Upsilon(s_t) \tag{A23}$$

The discrete version of Equation (A23) can be written as

$$\begin{aligned}\frac{V_{Ij}^{n+1} - V_{Ij}^{*n}}{\Delta\tau} = -[\alpha + r + \lambda^{s_t \to 1-s_t} + \frac{\overline{\pi_{Ij}}}{\Delta\tau}]V_{Ij}^{n+1} + \lambda^{s_t \to 1-s_t} V(1-s_t)_{Ij}^{n+1} + \\ \frac{\overline{\pi_{Ij}}}{\Delta\tau}[(P_I - C)Q_j + V_{I0} - V_{Ij}^{n+1}]\end{aligned} \tag{A24}$$

A2.3 Complete discretization

Combine Equations (A20), (A22) and (A24), and write them in matrix form as

$$\begin{aligned}&[(1+\Delta\tau(r+\lambda^{s_t\to 1-s_t}))I+W(s_t)+\overline{\pi}^{n+1}]V(s_t)^{n+1}\\ -\Delta\tau\lambda^{s_t\to 1-s_t}V(1-s_t)^{n+1}&=V(s_t)^{*n}+\overline{\pi(s_t)}^{n+1}[(P-C)Q+V(s_t)_0^{n+1}]\end{aligned}\quad\text{(A25)}$$

where $W(s_t)$ is a square sparse matrix which has the following elements:

$$W(s_t)=\begin{bmatrix}\Delta\tau b_1 & -\Delta\tau b_1 & 0 & ... & 0 & 0\\ -\Delta\tau a_2 & \Delta\tau(a_2+b_2) & -\Delta\tau b_2 & ... & 0 & 0\\ ... & ... & ... & ... & ... & ...\\ 0 & ... & 0 & -\Delta\tau a_{I-1} & \Delta\tau(b_{I-1}+b_{I-1}) & -\Delta\tau a_{I-1}\\ 0 & ... & 0 & 0 & 0 & \Delta\tau\alpha(s_t)\end{bmatrix}\quad\text{(A26)}$$

The above analysis for the option value in regime s_t can be used in the same way for the option value in the other regime $1-s_t$. The similar equation as Equation (A25) can be derived for $V(1-s_t)$ which can be written as

$$\begin{aligned}&[(1+\Delta\tau(r+\lambda^{1-s_t\to s_t}))I+W(1-s_t)+\overline{\pi}^{n+1}]V(1-s_t)^{n+1}-\\ \Delta\tau\lambda^{1-s_t\to s_t}V(s_t)^{n+1}&=V(1-s_t)^{*n}+\overline{\pi(1-s_t)}^{n+1}[(P-C)Q+V(1-s_t)_0^{n+1}]\end{aligned}\quad\text{(A27)}$$

Denote $AA(s_t)=[(1+\Delta\tau(r+\lambda^{s_t\to 1-s_t}))I+W(s_t)+\overline{\pi}^{n+1}]$. Then its counterpart for regime $1-s_t$ can be defined as $AA(1-s_t)=[(1+\Delta\tau(r+\lambda^{1-s_t\to s_t}))I+W(1-$

$s_t) + \overline{\pi}^{n+1}]$. Stack Equations (A25) and (A27) together and get

$$\begin{bmatrix} AA(s_t) & -\Delta\tau\lambda^{s_t\to 1-s_t} \\ -\Delta\tau\lambda^{1-s_t\to s_t} & AA(1-s_t) \end{bmatrix} \begin{bmatrix} V(s_t) \\ V(1-s_t) \end{bmatrix}^{n+1} = \begin{bmatrix} V^*(s_t) \\ V^*(1-s_t) \end{bmatrix}^{n} + \begin{bmatrix} \overline{\pi(s_t)} \\ \overline{\pi(1-s_t)} \end{bmatrix}^{n+1} \begin{bmatrix} payoff(s_t) \\ payoff(1-s_t) \end{bmatrix}^{n+1} \quad \text{(A28)}$$

where $Z_{matrix} = \begin{bmatrix} AA(s_t) & -\Delta\tau\lambda^{s_t\to 1-s_t} \\ -\Delta\tau\lambda^{1-s_t\to s_t} & AA(1-s_t) \end{bmatrix}$.

A3 Convergence to the viscosity solution

In this appendix, the monotonicity and stability properties of the discrete equations in our numerical scheme are analyzed. We claimed earlier that our scheme is consistent. A discretization that is consistent, monotone, and stable will converge to the viscosity solution.

Before proving the monotonicity and stability of our scheme, it is useful to gather together several results for the finite difference discretization.

Lemma A.1. *Z_{matrix} is an M matrix*[1]

Proof. Equation (2.30) is discretized using central difference, forward difference or backward difference methods to get a positive coefficient discretizations. The positive coefficient discretization means that Z_{matrix} has non-positive offdiagonal elements. Moreover, the sum of all elements in each row of Z_{matrix} is non-negative[2]. Hence Z_{matrix} is an M matrix. □

[1]For definition and properties of M matrix, see Varga (2000).
[2]This can be checked from detailed discretization in Appendix.

We follow d'Halluin et al. (2005)'s definition of monotone discretizations and rewrite the discrete Equation (2.27) at each pair of node (P_i, φ_j) as[3]

$$\begin{aligned} g_{ij}(V_{ij}^{n+1}, \{V_{i^-j}^{n+1}\}, \{V^n\}) &= -[ZV_j^{n+1}]_i + [\Phi^{n+1}V^n]_{ij} + [\overline{\pi}^{n+1}]_{ii}(\text{payoff}_{ij} - V_{ij}^{n+1}) \\ &= 0 \end{aligned} \tag{A29}$$

where $\{V_{i^-,j}^{n+1}\}$ denotes the set of values $V_{i^-,j}^{n+1}$ without the ith element V_{ij}^{n+1}. Φ^{n+1} in this expression is the Lagrange linear interpolant operator[4] used to deal with linear interpolation in the semi-lagrangian method.

$$[\Phi^{n+1}V^n]_{ij} = V(s_t, P_i, \varphi_{j+\Delta\tau}, \tau^n) + \text{ interpolation error}$$

Theorem A.2. *The discretization scheme* (A29) *is unconditionally monotone.*

Proof. In Lemma A.1 we have already showed that Z is an M-matrix. Therefore, $-[ZV_j^{n+1}]_i$ is a strictly decreasing function of V_{ij}^{n+1}, and a non-decreasing function of $\{V_{i^-,j}^{n+1}\}$. $[\Phi^{n+1}V^n]_{ij}$ is a non-decreasing function of $\{V^n\}$, since Φ^{n+1} is a linear interpolant operator. The last term in equation (A29) $[\overline{\pi}^{n+1}]_{ii}(payoff_{ij} - V_{ij}^{n+1})$ is a non-increasing function of V_{ij}^{n+1} since the elements in $[\overline{\pi}^{n+1}]_{ii}$ are non-negative. Therefore, this discretization scheme is monotone based on d'Halluin et al. (2005)'s definition. □

Theorem A.3. *The scheme satisfies*

$$||V^{n+1}||_\infty \leq \max\{||V^n||_\infty, ||payoff||_\infty\}$$

and is unconditionally stable.

[3]For simplicity, in this expression $V \equiv V(s_t)$ or $V \equiv V(1 - s_t)$.

[4]For details about Lagrange linear interpolation operator, seed'Halluin et al. (2005).

Proof. Write out the complete discretized version of Equation (2.27) as

$$-\Delta\tau V(s_t)^{n+1}_{i-1,j} + [1 + \Delta\tau(a(s_t)_i + b(s_t)_i + r + \lambda^{k\to 1-k}) + \overline{\pi(s_t)}^{n+1}_{ij}]V(s_t)^{n+1}_{ij}$$
$$-\Delta\tau b(s_t)_i V(s_t)^{n+1}_{i+1,j} - \Delta\tau\lambda^{k\to 1-k}V(1-s_t)^{n+1}_{ij} =$$
$$\sum_{ij} w_{ij}V(s_t)^n_{ij} + \overline{\pi(s_t)}^{n+1}_{ij} payoff_{ij} \quad \text{(A30)}$$

where w_{ij} is linear interpolant weight, satisfying $0 \leq w_{ij} \leq 1$ and $\sum w_{ij} = 1$. $a(s_t)_i$ and $b(s_t)_i$[5] are the components in Z matrix, which are non-negative. Denote $|V(s_t)^{n+1}_{m,j}| = ||V^{n+1}_j||_\infty$ where m is an index. Equation (A30) implies that

$$||V^{n+1}_j||_\infty(1 + r\Delta\tau + \overline{\pi}_{mm}) \leq ||V^n||_\infty + \overline{\pi}_{mm}||\text{payoff}||$$

which can be further simplified as

$$||V^{n+1}_j||_\infty(1 + r\Delta\tau + \overline{\pi}_{mm}) \leq \max\{||V^n||_\infty, ||payoff||_\infty\}(1 + \overline{\pi}_{mm}) \quad \text{(A31)}$$

Rearrange Equation (A31)

$$||V^{n+1}_j||_\infty \leq \max\{||V^n||_\infty, ||payoff||_\infty\}\frac{(1 + \overline{\pi}_{mm})}{(1 + r\Delta\tau + \overline{\pi}_{mm})}$$

Hence just as claimed, we get

$$||V^{n+1}||_\infty \leq \max\{||V^n||_\infty, ||payoff||_\infty\}$$

and the scheme is unconditionally stable

□

[5]The detail expression is in Appendix.

B Appendix to Chapter 3

B1 Derivation of Schwartz (1998) long-term model

Schwartz (1998) derives the one-factor long-term model based on the basic one-factor GBM model with constant convenience yield. Specifically, the spot price in the basic model follows GBM:

$$dS = (r - c)Sdt + \sigma SdZ \tag{B1}$$

where c is the constant convenience yield.[6] Hence the futures price of this basic one-factor model $F(S,T)$ can be derived as:

$$F(S,T) = Se^{(r-c)T} \tag{B2}$$

Based on Ito's Lemma, the futures return can be derived as $\frac{dF}{F} = \sigma dZ$. Its volatility is $\frac{Var(\frac{dF}{F})}{dt} = \frac{E[(\frac{dF}{F})^2]}{dt} = \sigma$, which is the same as the volatility of spot prices. The rate of change of the futures price[7] in this model is

$$\frac{\partial F/\partial T}{F} = r - c \tag{B3}$$

The futures price of the two-factor model $F(S,\delta,T)$ is given in Equation (4.16). The rate of change of the futures price in this two-factor model can be derived as:

$$\frac{\partial F/\partial T}{F} = r - \hat{\alpha} + \frac{\sigma_\delta^2}{2\kappa^2} - \frac{\rho\sigma_S\sigma_\delta}{\kappa} + \frac{e^{-2\kappa T}}{2\kappa^2} + [\hat{\alpha}\kappa + \rho\sigma_S\sigma_\delta - \frac{\sigma_\delta^2}{\kappa}]\frac{e^{-\kappa T}}{\kappa} - \delta e^{-\kappa T}$$

[6] All the stochastic processes in this part are expressed in the risk-neutral world.
[7] See Schwartz (1997).

As time goes to infinity $T \to \infty$, this rate will converge to:

$$\frac{\partial F/\partial T}{F}_{T\to\infty} = r - \hat{\alpha} + \frac{\sigma_\delta^2}{2\kappa^2} - \frac{\rho\sigma_S\sigma_\delta}{\kappa} \tag{B4}$$

Comparing Equations (B3) and (B4), if we define the constant convenience yield $c = \hat{\alpha} - \frac{\sigma_\delta^2}{2\kappa^2} + \frac{\rho\sigma_S\sigma_\delta}{\kappa}$ in the long-term model, the rate of change of futures prices in long-term model will converge to that in two-factor model.

With this rate of change $r - c$, the composite price $Z(S, \delta)$ is constructed to match the futures prices of two-factor model $F(S, \delta, T)$ based on the formula for futures prices[8] $F(Z, T) = Ze^{(r-c)T}$. Hence, Z can be derived as:

$$\begin{aligned} Z(S,\delta) &= \lim_{T\to\infty} e^{-(r-c)T} F(S,\delta,T) \\ &= Se^{\frac{c-\delta}{\kappa} - \frac{\sigma_\delta^2}{4\kappa^3}} \end{aligned} \tag{B5}$$

Given this composite price, Z, expressed in Equation (B5), combined with the defined constant convenience yield c, this long-term one-factor model can generate futures prices $F(Z, T)$ which closely match the long-term futures prices in the two-factor model $F(S, \delta, T)$.

Applying Ito's lemma to Equation (4.16), the futures return in the two-factor model can be derived as:

$$\frac{dF}{F} = \sigma_S dZ_S - \sigma_\delta \frac{1 - e^{-\kappa T}}{\kappa} dZ_\delta$$

Hence, the volatility of the futures return for this two-factor model is:

$$\sigma_F^2(T) = \frac{Var(\frac{dF}{F})}{dt} = \sigma_S^2 + \sigma_\delta^2 \frac{(1 - e^{-\kappa T})^2}{\kappa^2} - 2\rho\sigma_S\sigma_\delta \frac{1 - e^{-\kappa T}}{\kappa} \tag{B6}$$

[8] In this expression, $T \to \infty$ due to the convergence of rate of change to $r - c$.

Define the stochastic differential equation of composite price Z as:

$$dZ = (r - c)Zdt + \sigma_F(t)Zdz \tag{B7}$$

Therefore, the volatility of the futures return in this long-term model is the same as that in two-factor model.

B2 Model comparison

This section compares model performances of single-factor models with that of two-factor model in terms of fitting market prices. Model estimation errors including RMSE and MAE of the three one-factor models analyzed in this paper are provided here. Plots of model implied futures prices and market futures prices are also shown in this section.

B2.1 One-factor mean reverting model

Estimation errors of the one-factor mean reverting model including the Root Mean Square Error (RMSE) and Mean Absolute Error (MAE) are reported in Table B1. Comparing this table with Table 3.5 we find that except for the third futures contract F3, the errors of the futures contracts expressed in both ways for two-factor model are lower than those for the one-factor mean reverting model. This indicates the better performance of the two-factor model in terms of fitting market lumber derivative prices.

Plots of market futures prices and the model implied futures prices for the four futures contracts are shown in Figure B1. We observe the close match between these two time series. But comparing this figure with Figure 3.6 we find that except for the futures contract F3, the differences between the two futures prices for the other three futures contracts are higher for the one-factor mean reverting model.

Single-factor mean reverting model					
	F1	F2	F3	F4	All
	Calibration errors of futures prices				
RMSE	31.217	14.065	0.068	9.650	17.787
MAE	23.651	10.860	0.054	7.175	10.435
	Calibration errors of log futures prices				
RMSE	0.070	0.032	0.000	0.023	0.040
MAE	0.056	0.026	0.000	0.017	0.025

Table B1: Estimation errors of both futures prices and log futures prices of Schwartz (1997) single-factor model expressed as RMSE and MAE of 4 futures contracts, Cdn$/MBF.

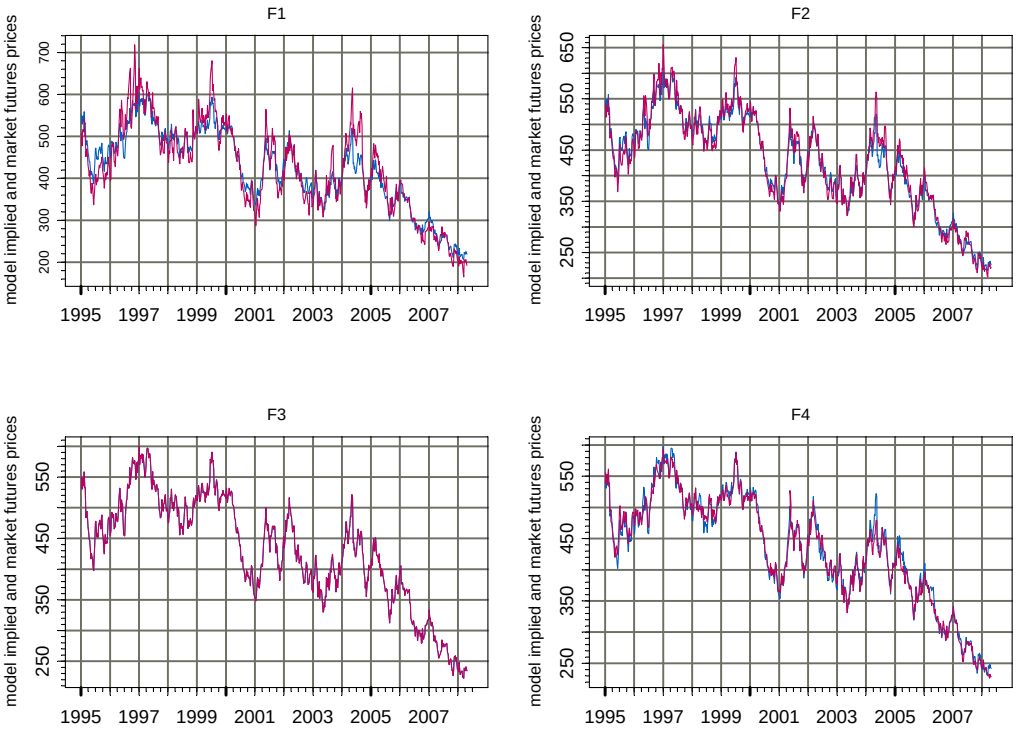

Figure B1: Plots of model implied and market futures prices for the single factor mean reverting model of the four chosen futures contracts. Weekly data from January 6th, 1995 to April 25th, 2008. Blue line: model implied futures prices. Red line: market futures prices.

Single-factor GBM model					
	F1	F2	F3	F4	All
	Calibration errors of futures prices				
RMSE	33.761	15.260	0.000	10.570	19.264
MAE	26.123	12.093	0.000	8.080	11.574
	Calibration errors of log futures prices				
RMSE	0.079	0.036	0.000	0.024	0.045
MAE	0.063	0.029	0.000	0.019	0.028

Table B2: Estimation errors of both futures prices and log futures prices of one-factor GBM model with constant convenience yield expressed as RMSE and MAE of 4 futures contracts, Cdn$/MBF.

B2.2 GBM model

Estimation errors of one-factor GBM model with constant convenience yield are reported in Table B2. We find that except for the third futures contract F3, the errors of the rest futures contracts expressed in both ways for the two-factor model are lower than those for the GBM model. This indicates the better performance of the two-factor model in terms of fitting market lumber derivative prices.

Plots of market futures prices and the model implied futures prices for the four futures contracts are shown in Figure B2. We can also find the close match between these two time series. But comparing this figure with figure 3.6 we find that except for the futures contract F3, the differences between the two futures prices for the rest three futures contracts are higher for the GBM model.

B3 Long-term model performance

Figures B3, B4 and B5 show the model implied futures prices with different maturities for the two-factor model and the long-term model. Comparing these three plots we observe that the differences between the two model implied futures prices are larger for contracts with short-term maturities and smaller for contracts with

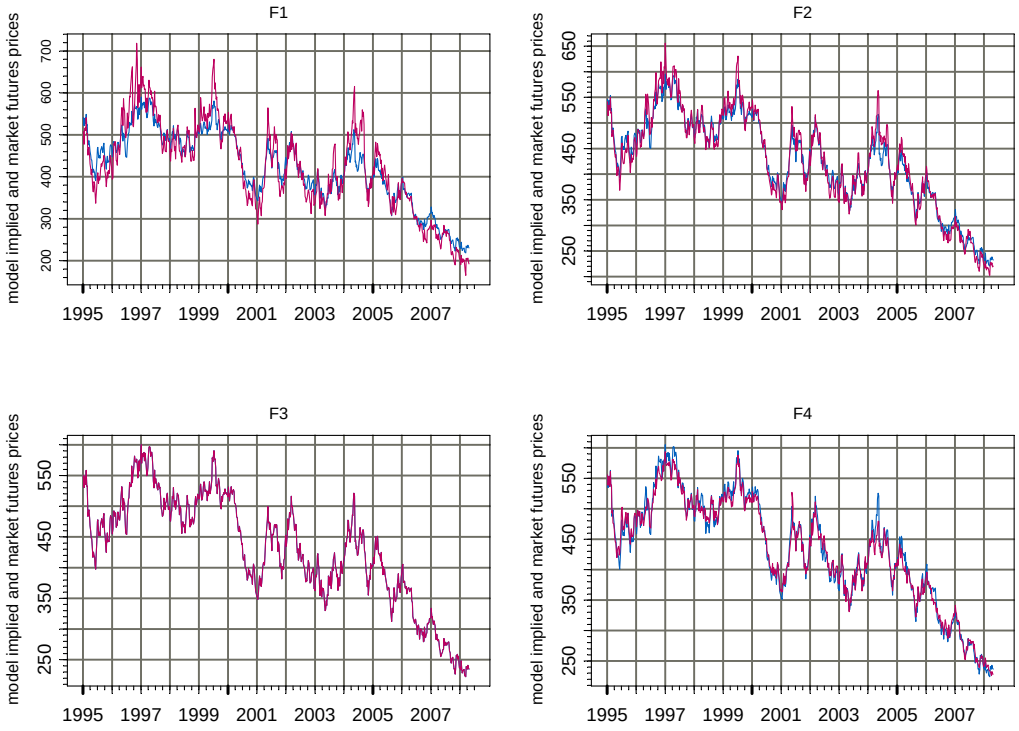

Figure B2: Plots of model implied and market futures prices for the one-factor GBM with constant convenience yield model of the four chosen futures contracts. Weekly data from January 6th, 1995 to April 25th, 2008. Blue line: model implied futures prices. Red line: market futures prices.

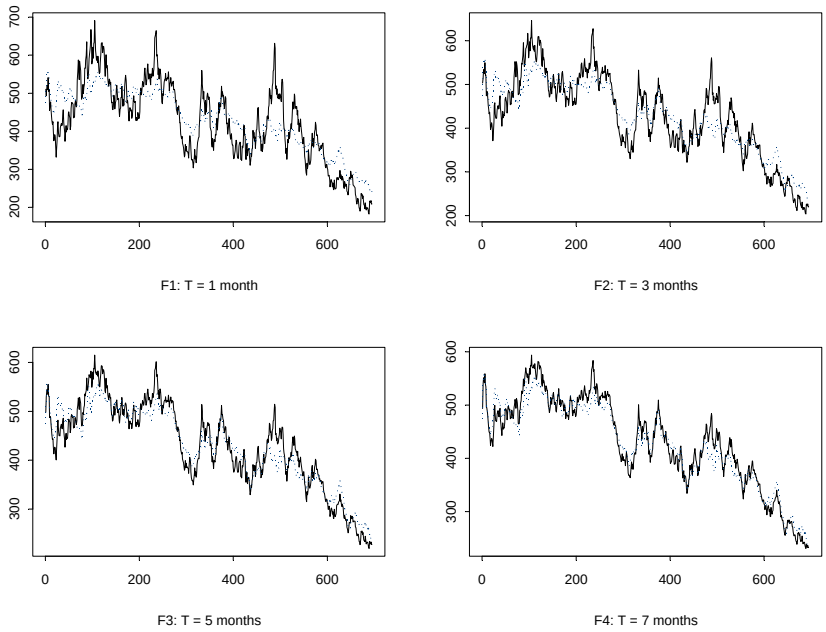

Figure B3: Prices in $/MBF of model implied futures contracts with four short-term maturities for Schwartz (1997) two-factor model and Schwartz (1998) long-term model.

long-term maturities. This result is consistent with the construction of the long-term model introduced in Schwartz (1998) since the purpose of the long-term model is to match the performance of the two-factor model analyzed in Schwartz (1997) in terms of fitting the long-term futures prices. The discrepancy between these two models in terms of generating the short-term futures prices is not as important in the analysis of a long-term forestry investment.

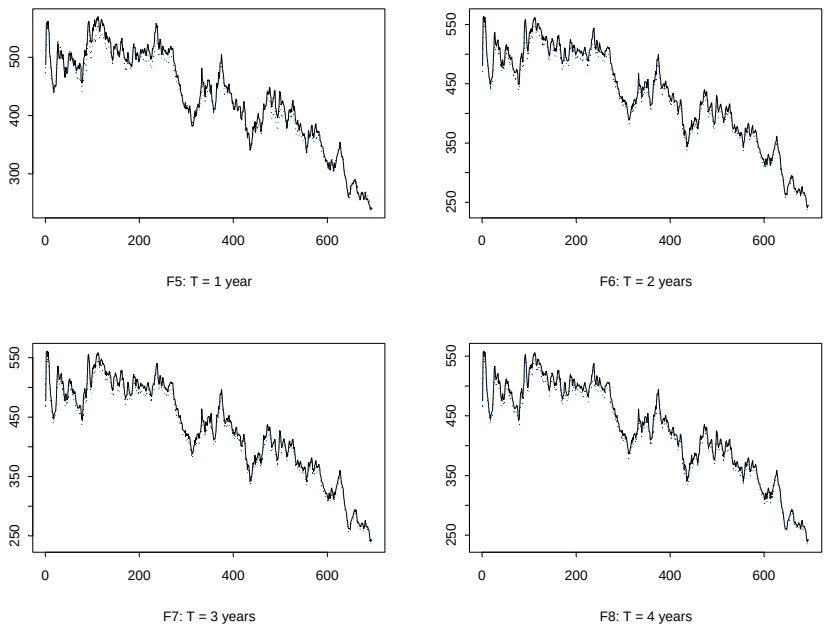

Figure B4: Prices in $/MBF of model implied futures contracts with four mid-term maturities for Schwartz (1997) two-factor model and Schwartz (1998) long-term model.

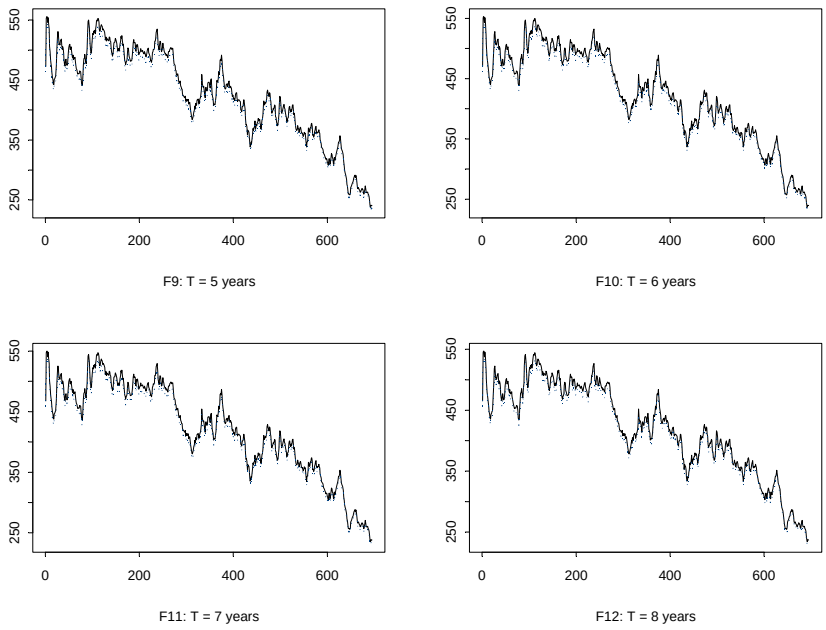

Figure B5: Prices in $/MBF of model implied futures contracts with four long-term maturities for Schwartz (1997) two-factor model and Schwartz (1998) long-term model.