Bounds and Approximations for Stochastic Fluid Networks

by

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I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

The success of modern networked systems has led to an increased reliance and greater demand of their services. To ensure that the next generation of networks meet these demands, it is critical that the behaviour and performance of these networks can be reliably predicted prior to deployment. Analytical modeling is an important step in the design phase to achieve both a qualitative and quantitative understanding of the system. This thesis contributes towards understanding the behaviour of such systems by providing new results for two fluid network models: The stochastic fluid network model and the flow level model.

The stochastic fluid network model is a simple but powerful modeling paradigm. Unfortunately, except for simple cases, the steady state distribution which is vital for many performance calculations, can not be computed analytically. A common technique to alleviate this problem is to use the so-called Heavy Traffic Approximation (HTA) to obtain a tractable approximation of the workload process, for which the steady state distribution can be computed. Though this begs the question: Does the steady-state distribution from the HTA correspond to the steady-state distribution of the original network model? It is shown that the answer to this question is yes. Additionally, new results for this model concerning the sample-path properties of the workload are obtained.

File transfers compose much of the traffic of the current Internet. They typically use the transmission control protocol (TCP) and adapt their transmission rate to the available bandwidth. When congestion occurs, users experience delays, packet losses and low transfer rates. Thus it is essential to use congestion control algorithms that minimize the probability of occurrence of such congestion periods. Flow level models hide the complex underlying packet-level mechanisms and simply represent congestion control algorithms as bandwidth sharing policies between flows. Balanced Fairness is a key bandwidth sharing policy that is efficient, tractable and insensitive. Unlike the stochastic fluid network model, an analytical formula for the steady-state distribution is known. Unfortunately, performance calculations for realistic systems are extremely time consuming. Efficient and tight approximations for performance calculations involving congestion are obtained.

Acknowledgements

The author acknowledges his debt to those who have helped along the way and influenced the formation of this thesis. In particular, I wish to express my gratitude to my supervisor, Professor Ravi Mazumdar. Without his guidance, knowledge and enthusiasm, I could not have completed this dissertation.

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Dedication

This thesis is dedicated to my family, who offered me unconditional love and support throughout the course of writing this thesis.

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Chapter 1

Introduction

The success of modern networked systems has led to an increased reliance and greater demand of their services. To ensure that the next generation of networks meet these demands reliably and within budget, it is critical that the behaviour and performance of these networks can be reliably predicted prior to deployment. Analytical modeling is an important step in the design phase to achieve both a qualitative and quantitative understanding of the system. This introduction provides some background and motivation for the researched questions investigated in this thesis.

A *probabilistic* approach, e.g. *Queueing Theory*, has been widely used to analyze and design communication, manufacturing and transportation systems. Probability theory provides a natural and powerful framework to capture the inherent uncertainty in many of the system parameters. Though, except for only the simplest network models, often a sacrifice between realistic assumptions and tractable analysis must be made.

A complementary *deterministic* approach, such as Network Calculus [45] for instance, has appeared in recent years. This approach has yielded bounds on system performance under minimal assumptions, but at the cost of precision. One of the recurring themes throughout this thesis is the use of deterministic bounds to simplify the analysis for many of the probabilistic results.

1.1 Fluid Inputs and Models

A critical part of network analysis is in choosing how to model the arrival process. This is often determined by the physical nature of the system and by the relevant time-scale. Most of the research focus has been on the traditional "'discrete"' queueing systems where arriving jobs are modeled as point processes.

Alternatively, one may model the arriving traffic as a continuous time process known as a *fluid* input. For example, consider a data network. The incoming traffic can be modeled at the bit level which is fundamentally discrete. But at coarser timescales, the input process can be approximated by a continuous time or fluid representation which is often suitable for performance calculations. In this thesis, only fluid inputs will be considered.

Irrespective of the type of the input model chosen, knowledge of the stationary distribution is critical to most performance calculations. For discrete queueing systems, little is known about the analytical form of the stationary distribution outside of the Markovian context. One must often resort to approximations or simulation.

For the two fluid models that will be studied in this thesis, the stochastic fluid network model and the flow level model, the situations are on different ends of the spectrum. For stochastic fluid networks, little is known about the structure of the stationary distribution, while for the specific flow level fluid network model studied, the exact form is explicitly known but unusable for realistic computations. It is the goal of this thesis to justify approximation techniques to help overcome these deficiencies.

1.1.1 Stochastic Fluid Network

Consider the problem of modeling the workload at each link in a packet network. Though packet arrivals are inherently discrete, the cumulative arrivals can be modeled as a fluid [49] (pg. 4). If the servers are reliable and can service the packets relatively quickly, then a simplifying assumption that server speed is constant can be made. This, in essence, is the stochastic fluid network model.

In regards to performance analysis, the model has distinct advantages and disadvantages which are best illustrated through the canonical model studied mainly by Kella and Whitt (e.g. [34, 38, 39]).

Let $\{J(t) : t \geq 0\}$ be a non-decreasing Lévy process representing the cumulative amount of fluid that has arrived at the links, P be the routing matrix of the network and r be a vector representing the service rates at each link. Then the workload $\{W(t) : t \geq 0\}$ can be simply represented by the following equation,

$$W(t) = W(0) + J(t) - (I - P')rt + (I - P')Z(t), \tag{1.1}$$

where $W(0)$ is the initial workload and $\{Z(t) : t \geq 0\}$ ensures that W is non-negative.

The advantage of this model is that one can analyze the model sample-path wise i.e. using deterministic methods. In Kella [34], a pathwise bound on the total workload was used to establish stability criteria for the network. In Kella and Whitt [39], a comparison theorem, used to establish monotonicity properties for the workload. Both of the theorems assume a fixed, and common in the case of the comparison theorem, routing matrix.

It is the goal of Chapter 2 to obtain similar pathwise bounds and comparison theorems when there is time- and state-dependence in the parameters r and P, and (in the case of comparison theorems) when the routing matrices are not the same.

Though one can establish stability criterion, it was shown in [34] that, except for trivial cases, the stationary distribution is never product form. Except for restricted class of network topologies, such as tandem networks [37], little is known about the stationary distribution.

In this situation, for both discrete and fluid inputs, a common technique to alleviate this problem is to use the so-called "'Heavy Traffic"' Approximation (HTA). The HTA essentially models a scaled version of the system as a *Reflected Brownian Motion*, for which the steady state distribution can be at least computed numerically. Though this begs the following question: Does the steady-state distribution from the HTA correspond to the steady-state distribution of the original network model?

This conjecture was first confirmed by Gamarnik and Zeevi [22] for Generalized Jackson Networks, followed by Budhiraja and Lee [17] for Generalized Jackson Networks under weaker assumptions and Zhang and Zwart [71] for limited processor sharing queues. To the authors knowledge, it has not been explored for fluid type inputs. This *interchange of limits* conjecture is confirmed to be true in Chapter 3 even when there is state-dependence in the routing matrix. The proofs utilize the pathwise results of Chapter 2.

1.1.2 Flow Level Model

File transfers compose much of the traffic of the current Internet. They typically use the transmission control protocol (TCP) and adapt their transmission rate to the available bandwidth. When congestion occurs, users experience delays, packet losses and low transfer rates. Thus it is essential to use congestion control algorithms that minimize the probability of occurrence of such congestion periods.

Flow level models hide the complex underlying packet-level mechanisms and simply represent congestion control algorithms as bandwidth sharing policies between flows. A natural approach is to treat bandwidth sharing as a utility maximization problem. A key bandwidth sharing policy of practical importance is *proportional fairness* [41], which seeks to maximize a logarithmic utility function. It has been shown by Low et. al. [46] that TCP Vegas is proportionally fair in equilibrium.

In general, analyzing the steady-state performance of a network operating under proportional fairness is quite difficult and can not be done analytically, except for simple network topologies [11]. It turns out that proportional fairness can be well approximated by the slightly different notion of *balanced fairness* [11]. Balanced Fairness is an insensitive bandwidth sharing policy for which an analytical formula for the stationary distribution is known.

In most of the literature on Balanced Fairness, it is assumed that flows utilize all the bandwidth alloted to them. In reality, they are often severely rate limited and heterogeneous. One can define congestion in such flow models as a flow not receiving its maximum bit rate. Balanced Fairness with rate limits has not been well studied in the literature, outside of the single link case [8, 15] or tree networks [14]. In fact, analysis of congestion in such networks has not been studied at all.

Chapter 4 develops analytic formulas for congestion metrics under the single link and so-called parking lot network topologies. Unfortunately, it was found that due to the structure of the stationary distribution, the calculations for even simple systems were extremely time consuming. Leveraging the large system asymptotic from loss networks [23], fast and tight closed-form approximations are introduced as well.

1.2 Outline of Thesis

The thesis is organized as follows: Chapter 2 investigates sample-path properties for stochastic fluid networks via comparison theorems, and discusses an application to Generalized Processor Sharing (GPS) networks. Chapter 3 affirms that the interchange of limits for stochastic fluid networks in heavy traffic holds. Chapter 4 analyzes congestion in networks operating under balanced fairness. Finally, Chapter 5 summarizes the main results in each of the previous chapters and suggests future lines of research.

Each chapter is organized in roughly the same manner: The introduction provides a literature review and a summary of the contributions. This is followed by a brief description of the external tools and techniques used in the chapter. Then a description of the model is given which includes a discussion of the assumptions. Lastly, the main results are stated and proven.

The results obtained in Chapter 2 can be found in [26] co-authored with Ravi Mazumdar and Francisco Piera. The results in Chapter 3 expands [24] and will appear in [25]. Both of the paper were co-authored with Ravi Mazumdar. Finally, the results in Section 4.5 will appear in a paper co-authored with Thomas Bonald and Ravi Mazumdar [9].

1.3 Contributions of this Thesis

The major original contributions of this thesis are as follows:

Chapter 2 New pathwise comparison theorems are proven for stochastic fluid networks with time- and state-dependent parameters. These comparison theorems provide new insight into the qualitative behaviour of stochastic fluid networks and generalize the results available in the literature.

Chapter 3 We prove the interchange of limits for stochastic fluid networks with state-dependent routing and Lévy type inputs. Of further interest, the proofs utilize the deterministic sample-path results of Chapter 2.

Chapter 4 Tight closed-form approximations for calculating congestion metrics in a single link are shown. The approximations are based on the large system asymptotic.

Chapter 4 A new criterion for identifying the states for which congestion occurs in a parking lot network operating under balanced fairness is shown. As well, upper bounds for the calculation of congestion metrics in parking lot networks are established. The calculation of these bounds are found to be extremely time consuming in even trivial systems with many flows classes, i.e. state-space explosion. To bypass this problem, tight closed-form approximations based on the large system asymptotic are introduced as well.

Chapter 2

Pathwise Results for Stochastic Fluid Networks

2.1 Introduction

The analytic analysis of networks in general has historically been a very difficult task. In fact, few concrete results are known outside of the Markovian setting. In recent years, pathwise analysis has provided invaluable insight into the general behavior of various classes of networks, especially for establishing bounds and proving stability.

Stochastic fluid networks (SFNs) are a simple but insightful class of network model for which arrivals are modeled as a fluid and service at the queues can be approximated as a deterministic fluid flow. The particular case of SFNs with fixed routing matrix has been extensively studied in a series of papers by Kella [33, 34], Kella & Whitt [37, 38, 39] and in the book by Whitt [65]. In particular, the papers of Kella [34] and Kella & Whitt [39] provide stability conditions for SFNs with Lévy and stationary increment inputs respectively, through the use of comparison theorems.

Most of the comparison results for stochastic fluid networks are through their association with reflected equations. Kella & Whitt [39] established several comparison results for reflected equations with constant reflection directions. A comparison result for reflected differential equations, with state- and time-dependent parameters, was proven by Ramasubramanian [55]. Piera and Mazumdar [50] established a similar comparison theorem for reflected diffusions with jumps.

The chapter is divided as follows: Section 2.2 gives an introduction to the Skorokhod Oblique Reflection Problem or just the Skorokhod Problem. Section 2.3 describes the fluid network model in more detail than the one given in the introduction. Section 2.4 proves and discusses the qualitative implications of various comparison theorems. The insights obtained from the previous section are then applied to show a comparison theorem for a multi-class stochastic fluid network under the Generalized Processor Sharing (GPS) service discipline in Section 2.5. Finally, future work is discussed in Section 5.1.

All results proven in Sections 2.4 and 2.5 are original contributions unless otherwise stated. The most important contribution in Section 2.4 is Theorem 2.4.1, which establishes conditions for comparing networks with different routing matrices. To the author's knowledge, this has not been considered before in the context of stochastic fluid networks.

2.1.1 Assumptions and Notation

This section specifies the assumptions and notation that will be used in the chapter. Unless otherwise stated the integer $N \geq 0$ will be fixed as the dimension. Let $D \equiv D([0,\infty),\mathbb{R}^N)$ be the space of càdlàg, $\mathbb{R}^N$-valued functions defined on the interval $[0,\infty)$. The space D will be endowed with the Skorokhod J_1 topology. Let the subsets $D_+ \equiv D_+([0,\infty),\mathbb{R}^N)$ be the non-negative càdlàg functions and $D_\uparrow \equiv D_\uparrow([0,\infty),\mathbb{R}^N)$ denote the non-negative, non-decreasing càdlàg functions. As well, we will denote $D_{\uparrow,0} \equiv D_{\uparrow,0}([0,\infty),\mathbb{R}^N)$ as the subset of functions in $D_\uparrow$ that are null at the origin. For any $x, y \in D_\uparrow([0,\infty),\mathbb{R}^N)$, the notation $x \prec y$ means that for all $s \geq t \geq 0$, $x(t) \leq y(t)$ and $x(s) - x(t) \leq y(s) - y(t)$. For any functions f, g, $f(N) \sim g(N)$ means $f(N)/g(N) \to 1$ when $N \to \infty$. Finally, the notation $x(t-)$ means the limit when x approaches t from the left.

Vectors and matrices are assumed to have real-valued entries. As well, vectors will be assumed to be column vectors. The transpose of a matrix A will be denoted by A' and I

will represent the identity matrix. For a constant C, we use $\vec{C}$ to mean an N-dimensional column vector with all its entries being equal to C. The notation x_i will mean the i^{th} entry of a vector x and likewise, $A_{i,j}$ will mean the $(i,j)^{th}$ entry of a matrix A. Note that the notation x_n will also refer to a possibly vectorial element in a sequence, though the meaning of the notation will be obvious. The space $\mathbb{R}^N$ will be equipped with the Euclidean metric. For $p \in [1, \infty]$, $|\cdot|_p$ will denote the standard vector and induced matrix p-norms. For simplicity we shall write $|\cdot| \equiv |\cdot|_1$. Comparisons are assumed to be component wise. As well, scalar operations on vectors are to be interpreted component wise. We denote by e_i the standard unit vector, i.e. the i^{th} component is 1 and the rest are 0. For any two scalars a and b, $a \wedge b$ is the minimum and $a \vee b$ is the maximum.

An $N \times N$ matrix R is said to be an M-matrix if it has positive diagonal entries, non-positive off-diagonal entries, and has a non-negative inverse [19](p. 164). In this thesis, routing matrix, denoted by P, will mean a substochastic matrix such that $(I - P')$ is an M-matrix.

2.2 Background

2.2.1 The Skorokhod Problem

Let $X \in D$ with $X(0) \in \mathbb{R}_+^N$ and let R be an M-matrix.

Definition 2.2.1. *The functions $(W, Z) \in D \times D_{\uparrow,0}$ are said to solve the Skorokhod Oblique Reflection Problem (SP) corresponding to (X, R) if the following conditions hold:*

1. $W(t) = X(t) + RZ(t) \in \mathbb{R}_+^N \quad \forall t \geq 0,$
2. $Z \in D_{\uparrow,0},$
3. $\displaystyle\int_0^t W_i(s) dZ_i(s) = 0, \quad i = 1 \ldots N.$

It is well known that the functions (W, Z), referred to as the reflected and regulator functions respectively, exist and are unique. Meaning for all $X \in D$ with $X(0) \in \mathbb{R}^N_+$, there exists a unique pair of mappings $\Phi_R, \Psi_R : D \to D$ such that $\Phi_R(X) = W$ and $\Psi_R(X) = Z$. The mapping Φ_R is known as the Skorokhod map. Both Φ_R and Ψ_R depend solely on the reflection matrix R. As well, they are Lipschitz continuous in the following sense:

Lemma 2.2.1. *Let $X^{(1)}, X^{(2)} \in D$, $X^{(1)}(0), X^{(2)}(0) \in \mathbb{R}^N_+$, and let R be an M-matrix. Also let $(W^{(i)}, Z^{(i)})$ solve the SP corresponding to $\left(X^{(i)}, R\right)$ (ie. $\Phi_R(X^{(i)}) = W^{(i)}$ and $\Psi_R(X^{(i)}) = Z^{(i)}$) for $i = 1, 2$. Then there exists a finite constant C whose value depends solely on the matrix R such that for any fixed $T > 0$,*

$$|W^{(1)}(t) - W^{(2)}(t)| + |Z^{(1)}(t) - Z^{(2)}(t)| \leq C \sup_{s\in[0,T]} |X^{(1)}(s) - X^{(2)}(s)| \tag{2.1}$$

for all $t \in [0, T]$.

Solutions of the SP possess useful pathwise properties that will often be exploited in the analysis. One such result is the following lemma which describes the regulator process as the minimal process that keeps the free process in the positive orthant.

Lemma 2.2.2. *Let $X \in D$, $X(0) \in \mathbb{R}^N_+$, R be an M-matrix and let (W, Z) solve the SP corresponding to (X, R). If there exists another $\tilde{Z} \in D_{\uparrow,0}$ such that $W(t) = X(t) + R\tilde{Z}(t) \in \mathbb{R}^N_+ \quad \forall t \geq 0$, then $\tilde{Z}(t) \geq Z(t) \quad \forall t \geq 0$.*

See Chapter 14 of Whitt [65] or Chapter 7 of Chen & Yao [19] for further details.

2.2.2 The Skorokhod Problem: Time- and State-Dependent Reflection

A generalization of the above classic version of the Skorokhod problem was studied by Ramasubramanian [55]. Given a càdlàg process $\{X(t); t \geq 0\}$ and functions $b : \mathbb{R}_+ \times \mathbb{R}^N_+ \times \mathbb{R}^N_+ \to \mathbb{R}^N$, $R : \mathbb{R}_+ \times \mathbb{R}^N_+ \times \mathbb{R}^N_+ \to \mathbb{R}^{N\times N}$ such that

1. Each component b_i, $1 \le i \le N$, is bounded continuous and $(z,w) \mapsto b_i(t,z,w)$ are Lipschitz continuous, uniformly in t.

2. Each component R_{ij}, $1 \le i,j \le N$, is bounded continuous and $(z,w) \mapsto R_{ij}(t,z,w)$ are Lipschitz continuous, uniformly in t. Moreover, $R_{ii} = 1$.

3. There exist a constant $V \in \mathbb{R}^{N \times N}$ such that $|R_{ij}(t,z,w)| \le V_{ij}$, for $i \neq j$, and $V_{ii} = 0$. As well, $\sigma(V) < 1$ where $\sigma(V)$ denotes the spectral radius of V.

The functions $(W,Z) \in D \times D_{\uparrow,0}$ are said to solve the Skorokhod Oblique Reflection Problem (SP) corresponding to $(X, b(t,z,w), R(t,z,w))$ if the following conditions hold:

1. $W(t) = X(t) + \int_0^t b(s, Z(s^-), W(s^-))ds + \int_0^t R(s, Z(s^-), W(s^-))dZ(s) \in \mathbb{R}_+^N \quad \forall t \ge 0$,

2. $Z \in D_{\uparrow,0}$,

3. $\int_0^t W_i(s)dZ_i(s) = 0, \quad i = 1 \dots N.$

Note that since V has spectral radius less than 1, there exists constants $a_i > 0$, $i = 1 \dots N$ and $0 < \alpha < 1$ such that for all $t \ge 0$, $w, z \in \mathbb{R}_+^N$, and $i = 1 \dots N$ the inequality

$$\sum_{j \neq i} a_j |R_{j,i}(t,z,w)| \le \sum_{j \neq i} a_j V_{j,i} < \alpha a_i, \tag{2.2}$$

is satisfied.

Let $c_1, c_2, T > 0$ be arbitrary constants, and let φ_T and ψ_T be the total variation norm and supremum norm over the interval $[0,T]$ respectively. Then for any

$$\left(W^{(1)}, Z^{(1)}\right), \left(W^{(2)}, Z^{(2)}\right) \in D([0,T], \mathbb{R}^N) \times D_{\uparrow,0}([0,T], \mathbb{R}^N)$$

the function

$$d_T\left(\left(W^{(1)}, Z^{(1)}\right), \left(W^{(2)}, Z^{(2)}\right)\right) \triangleq c_1 \sum_{i=1}^N a_i \varphi_T\left(Z_i^{(1)} - Z_i^{(2)}\right) + c_2 \sum_{i=1}^N a_i \psi_T\left(W_i^{(1)} - W_i^{(2)}\right) \tag{2.3}$$

is a metric. Moreover, the metric space $\big(D([0,T],\mathbb{R}^N)\times D_{\uparrow,0}([0,T],\mathbb{R}^N), d_T\big)$ is complete.

The following result is a comparison theorem due to Ramasubramanian [55]. This powerful comparison result will be used for some of the theorems and lemmas to follow.

Theorem 2.2.1 (Theorem 4.1,[55]). *Let $(W^{(i)}, Z^{(i)})$ solve the SP corresponding to $\big(X^{(i)}, b^{(i)}(t,z,w), R^{(i)}(t,z,w)\big)$ for $i = 1, 2$.*

Suppose that the following conditions are satisfied:

- $X^{(1)} \prec X^{(2)}$,
- $X^{(1)}(0) \leq X^{(2)}(0)$,
- $b_i^{(1)}(t, z_1, w_1) \leq b_i^{(2)}(t, z_2, w_2)$,
- $R_{ij}^{(1)}(t, z_1, w_1) \leq R_{ij}^{(2)}(t, z_2, w_2) \leq 0$, $i \neq j$,

whenever $w_1 \leq w_2$, $z_1 \geq z_2$ and $t \geq 0$.

Then

$$W^{(1)}(t) \leq W^{(2)}(t) \quad t \geq 0, \tag{2.4}$$

$$Z^{(2)} \prec Z^{(1)}. \tag{2.5}$$

2.3 Model

2.3.1 A Fluid Network Model

A Single Queue Model

Let $\{J(t) : t \geq 0\}$ be a non-negative, non-decreasing function such that for any time t, $J(t)$ is the amount of work offered to the queue in the interval $[0, t]$. When non-empty, fluid is drained from the queue at a constant rate r. Fluid that arrives into an empty queue gets processed and drained immediately. As well, unlimited storage capacity is assumed.

For a fluid model, the workload process $\{W(t) : t \geq 0\}$, defined by

$$W(t) \triangleq W(0) + J(t) - rt + rL(t) \in D_+, \tag{2.6}$$

where $W(0)$ is the initial workload (i.e. the initial amount of fluid in the queue) and $\{L(t) : t \geq 0\}$ is defined as

$$L(t) \triangleq \int_0^t \mathbf{1}_{W(s)=0} ds, \tag{2.7}$$

is the focus of many performance calculations. Define the regulator process $\{Z(t) : t \geq 0\}$ and the virtual workload process $\{X(t) : t \geq 0\}$ by

$$Z(t) = rL(t), \tag{2.8}$$

and

$$X(t) = W(0) + J(t) - rt, \tag{2.9}$$

respectively. Then (W, Z) is the solution to the SP corresponding to $(X, 1)$. It is straightforward to verify that

$$Z(t) = \sup_{s \leq t} -X(s) \vee 0. \tag{2.10}$$

Knowledge of the analytical form of the regulator process enables many performance calculations that have currently eluded the network models discussed in the next section.

Fluid Network

The section begins by looking at networks without any time- or state-dependencies in the parameters. It is assumed that the fluid network consists of N single server, work conserving queues naturally labeled $1, \ldots, N$. A stochastic fluid network can be uniquely characterized by the 4-tuple $(J, r, P, W(0))$: The input process $J \in D_{\uparrow,0}$, the processing or drain rate $r \in \mathbb{R}_+^N$, the routing matrix $P \in \mathbb{R}_+^{N \times N}$, and the initial workload $W(0) \in \mathbb{R}_+^N$.

The routing matrix P will be assumed to be substochastic such that $(I - P')$ is an M-matrix, which implies that the network is open. The cumulative amount of work that arrives externally to the system at queue $i = 1 \ldots N$ in the interval $[0, t]$ is modeled by

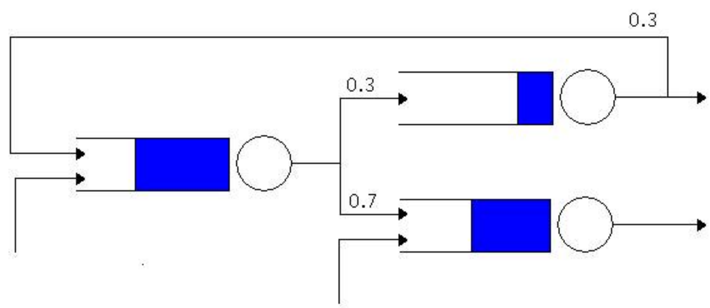

Figure 2.1: A Stochastic Fluid Network

$J_i(t)$. Work at queue i is drained as a fluid at rate $r_i > 0$, and routed to queue j at rate $P_{ij}r_i$. An example of a stochastic fluid network is given in Figure 2.1.

Let the virtual workload process $\{X(t) : t \geq 0\}$ be defined by

$$X(t) = W(0) + J(t) - (I - P')rt. \tag{2.11}$$

Then the workload and regulator processes (W, Z) are defined as the solution to the SP corresponding to $(X, I - P')$.

To include time- or state-dependency into the model, restrictions must be imposed on the routing matrix and processing rates. The routing matrix is assumed to be a function $P : \mathbb{R}_+ \times \mathbb{R}_+^N \times \mathbb{R}_+^N \to \mathbb{R}_+^{N\times N}$ and the processing rate $r : \mathbb{R}_+^N \to \mathbb{R}_+$, such that for fixed $t \geq 0$, $z, w \in \mathbb{R}_+^N$:

- Each component of the routing matrix P_{ij}, $1 \leq i, j \leq N$, is bounded continuous and $(z, w) \mapsto P_{ij}(t, z, w)$ are Lipschitz continuous, uniformly in t.

- $P_{ii} = 0$ and $P_{ij} \geq 0$.
- There exists a constant $V \in \mathbb{R}^{N\times N}$ such that $P_{ij}(t,z,w) \leq V_{ij}$, for $i \neq j$, and $V_{ii} = 0$. As well, $\sigma(V) < 1$ where $\sigma(V)$ denotes the spectral radius of V.
- $P(t,z,w)$ is a substochastic matrix such that $(I - P'(t,z,w))^{-1}$ exists and is non-negative.
- r is bounded continuous.

The workload and regulator processes (W,Z) are defined as the solution to the SP corresponding to $(W(0)+J, -(I-P'(t,z,w))\,r(t), I-P'(t,z,w))$. Explicitly, the workload is written as

$$W(t) = W(0) + J(t) - \int_0^t \big(I - P'(s, Z(s^-), W(s^-))\big)\, r(t)ds \qquad (2.12)$$
$$+ \int_0^t \big(I - P'(s, Z(s^-), W(s^-))\big) dZ(s).$$

The notation $(W,Z) = FN(W(0), J, r(t), P(t,z,w))$ will be used to indicate the workload and regulator processes of a particular fluid network. The dependence of r and P on time and state will always be made explicit in the notation, e.g. a constant routing matrix will be written as P while a time-dependent routing matrix will be written as $P(t)$.

2.4 Results

The first lemma uses the Ramasubramanian comparison theorem to highlight an important relationship between the service rates and the regulator process.

Lemma 2.4.1. *Let*

$$(W,Z) = FN\left(W(0), J, r(t), P\right).$$

Then

$$\{\int_0^t r(s)ds - Z(t) : t \geq 0\}$$

is a non-negative, non-decreasing process.

Proof. Let

$$\left(W^{(1)}, Z^{(1)}\right) = FN\left(W(0), J, r(t), P\right)$$

and

$$\left(W^{(2)}, Z^{(2)}\right) = FN\left(0, 0, r(t), P\right).$$

The solution to the second SP is

$$(W^{(2)}, Z^{(2)}) = (0, \int_0^{\cdot} r(s)ds).$$

The result follows by applying Theorem 2.2.1. □

The next result establishes monotonicity with respect to the initial workload, cumulative input, and the routing matrix.

Theorem 2.4.1. *Let*

$$\left(W^{(1)}, Z^{(1)}\right) = FN\left(W^{(1)}(0), J^{(1)}, r(t), P^{(1)}\right),$$

and

$$\left(W^{(2)}, Z^{(2)}\right) = FN\left(W^{(2)}(0), J^{(2)}, r(t), P^{(2)}\right).$$

As well suppose,

$$J^{(1)} \prec J^{(2)},$$

$$W^{(1)}(0) \leq W^{(2)}(0),$$

and assume that

$$P^{(1)}(t, z_1, w_1) \leq P^{(2)}(t, z_2, w_2) \quad \forall t \geq 0, z_1 \geq z_2 \text{ and } w_1 \leq w_2.$$

Then

$W^{(1)}(t) \leq W^{(2)}(t)$, $t \geq 0$,

$Z^{(1)}(t) \geq Z^{(2)}(t)$, $t \geq 0$,

$Z^{(2)} \prec Z^{(1)}$.

Proof. Let

$$R^i \equiv I - P^{(i)\prime} \text{ for } i = 1, 2.$$

For $(t, z, w) \in \mathbb{R}_+ \times D_\uparrow[0, \infty) \times D_+[0, \infty)$ define the mappings:

$$X^{(i)}(t, z, w) = W^{(i)}(0) + J^{(i)}(t) - \int_0^t R^{(i)}(s, z_{s^-}, w_{s^-})r(s)ds - \int_0^t P^{(i)\prime}(s, z_{s^-}, w_{s^-})dz_s,$$

$$T^{(i)}(t, z, w) = \sup_{0 \le s \le t} \max(0, -X^{(i)}(s, z, w)),$$

and

$$S^{(i)}(t, z, w) = X^{(i)}(t, z, w) + T^{(i)}(t, z, w)$$

where the sup and max operations are to be applied component wise and $i = 1, 2$. Note that the mappings $(T^{(i)}, S^{(i)})$ solves a SP with input $X^{(i)}$ and identity reflection matrix.

Choose processes

$$(z^{(i)}, w^{(i)}) \in D_\uparrow[0, \infty) \times D_+[0, \infty)$$

such that $w^{(2)} \ge w^{(1)}$, $z^{(i)}(0) = 0$, $z^{(1)}(t_2) - z^{(1)}(t_1) \ge z^{(2)}(t_2) - z^{(2)}(t_1)$ and $z^{(1)}(t_2) - z^{(1)}(t_1) \le \int_{t_1}^{t_2} r(s)ds \quad \forall 0 \le t_1 \le t_2$.

From the assumptions, it is straightforward to see that

$$X^{(2)}(0, z^{(2)}, w^{(2)}) \ge X^{(1)}(0, z^{(1)}, w^{(1)}),$$

and

$$X^{(2)}(t_2, z^{(2)}, w^{(2)}) - X^{(2)}(t_1, z^{(2)}, w^{(2)}) \ge X^{(1)}(t_2, z^{(1)}, w^{(1)}) - X^{(1)}(t_1, z^{(1)}, w^{(1)})$$

for all $t_2 \ge t_1 \ge 0$.

So by Theorem 2.2.1,

$S^{(1)}(t, z^{(1)}, w^{(1)}) \le S^{(2)}(t, z^{(2)}, w^{(2)})$, $t \ge 0$,

$T^{(1)}(t, z^{(1)}, w^{(1)}) \ge T^{(2)}(t, z^{(2)}, w^{(2)})$, $t \ge 0$,

$T^{(1)}(t_2, z^{(1)}, w^{(1)}) - T^{(1)}(t_1, z^{(1)}, w^{(1)}) \ge T^{(2)}(t_2, z^{(2)}, w^{(2)}) - T^{(1)}(t_1, z^{(2)}, w^{(2)}) \quad \forall t_2 \ge t_1 \ge 0$.

Note that

$$T^{(1)}(0, z^{(1)}, w^{(1)}) = T^{(2)}(0, z^{(2)}, w^{(2)}) = 0.$$

Also by Lemma 2.4.1,

$$T^{(1)}(t_2, z^{(1)}, w^{(1)}) - T^{(1)}(t_1, z^{(1)}, w^{(1)}) \leq \int_{t_1}^{t_2} r(s)ds \quad \forall t_2 \geq t_1 \geq 0.$$

The remainder of the proof follows mutatis mutandis from the proof of Theorem 3.7 in [55]: Essentially, there exists a time point $t_0 > 0$ such that, restricted to $[0, t_0]$, the maps $(T^{(i)}, S^{(i)})$ are contraction maps (using the metric d_{t_0} (2.3)) whose unique fixed point is the solution to the SP $(Z^{(i)}, W^{(i)})$. The procedure is then repeated starting at time t_0 and so forth. □

The previous result establishes a very intuitive notion. If the fluid leaving the network at each node decreases or a greater amount of fluid arrives at each point in time, then the workload at each queue should, and by Theorem 2.4.1 does indeed, increase. But, monotonicity of the workload with respect to all parameters was shown except for the service rate. So it is natural to wonder if a similar result with respect to the service rates can be found as well. In general the answer is no, and it is fairly straightforward to find examples of this. But Theorem 2.2.1 tells us that, under restrictive conditions, monotonicity can exist.

Lemma 2.4.2. *Let*

$$\left(W^{(1)}, Z^{(1)}\right) = FN\left(W(0), J, r^{(1)}(t), P(t, z, w)\right)$$

and

$$\left(W^{(2)}, Z^{(2)}\right) = FN\left(W(0), J, r^{(2)}(t), P(t, z, w)\right).$$

Assume that $R(t, z_1, w_1)r^{(1)}(t) \geq R(t, z_2, w_2)r^{(2)}(t)$ *and* $R(t, z_1, w_1) \leq R(t, z_2, w_2)$ *whenever* $z_1 \geq z_2$ *and* $w_1 \leq w_2$*, where* $R \equiv I - P'$*.*

Then:

$W^{(1)}(t) \leq W^{(2)}(t)$, $t \geq 0$,

$Z^{(1)}(t) \geq Z^{(2)}(t),\ t \geq 0,$

$Z^{(2)} \prec Z^{(1)}.$

Corollary 2.4.1. *Let*

$$\left(W^{(1)}, Z^{(1)}\right) = FN\left(W(0), J, r^{(1)}(t), P\right)$$

and

$$\left(W^{(2)}, Z^{(2)}\right) = FN\left(W(0), J, r^{(2)}(t), P\right).$$

If $(I - P')r^{(1)} \geq (I - P')r^{(2)}$ *then,*

$W^{(1)}(t) \leq W^{(2)}(t),\ t \geq 0,$

$Z^{(1)}(t) \geq Z^{(2)}(t),\ t \geq 0,$

$Z^{(2)} \prec Z^{(1)}.$

As mentioned above, increasing the service rates in the network does not necessarily correspond to a decrease in the workload at each queue. But as the next comparison theorems will show, they do decrease the total workload in the network. Theorem 2.4.2 and its proof are a generalization of Lemma 3.1 in [34].

Theorem 2.4.2. *Assume that* $W^{(1)}(0) \leq W^{(2)}(0)$*,* $J^{(1)} \prec J^{(2)}$*,* $P^{(1)} \geq P^{(2)}$ *and* $(I - P^{(1)\prime})^{-1}(I - P^{(2)\prime})r^{(2)}(t) \leq r^{(1)}(t)$*,* $\forall t \geq 0$*.*

Define

$$\left(W^{(1)}, Z^{(1)}\right) = FN\left(W^{(1)}(0), J^{(1)}, r^{(1)}(t), P^{(1)}\right)$$

and

$$\left(W^{(2)}, Z^{(2)}\right) = FN\left(W^{(2)}(0), J^{(2)}, r^{(2)}(t), P^{(2)}\right).$$

Then

$$\sum_{j=1\ldots N} W_j^{(1)}(t) \leq \sum_{j=1\ldots N} W_j^{(2)}(t), \quad \forall t \geq 0 \tag{2.13}$$

Proof. First assume that $W^{(1)}(0) = W^{(2)}(0) = 0$ and define $\{Z^*(t); t \geq 0\}$ such that

$$Z^*(t) = (I - P^{(1)'})^{-1}(J^{(2)}(t) - J^{(1)}(t)) + (I - P^{(1)'})^{-1}(I - P^{(2)'})Z^{(2)}(t) + \int_0^t r^{(1)}(s) - (I - P^{(1)'})^{-1}(I - P^{(2)'})r^{(2)}(s)dt.$$

Note that since $P^{(1)} \geq P^{(2)}$, $(I - P^{(1)'})^{-1}(I - P^{(2)'}) \geq I$. Z^* is also clearly a non-decreasing process with $Z^*(0) = 0$. So $J^{(1)}(t) - (I - P^{(1)'})r^{(1)}t + (I - P^{(1)'})Z^*(t) = W^{(2)} \geq 0$. Therefore $Z^* \geq Z^{(1)}$ by the minimality property of the regulator process.

This implies that $(I - P^{(1)'})^{-1}W^{(2)} \geq (I - P^{(1)'})^{-1}W^{(1)}$. Since P is substochastic, the result follows by multiplying both sides of the above inequality by a column vector of ones.

Now if $W^{(1)}(0) \leq W^{(2)}(0)$, one proves the results using the following method:

Set $J^{(1)}(t) = J^{(1)}(t) + W^{(1)}(0)$ and $J^{(2)}(t) = J^{(2)}(t) + W^{(2)}(0)$. Shift the starting time from 0 to $-t_0 < 0$ and defining $W^{(1)}(-t_0) = W^{(2)}(-t_0) = 0$ and $J^{(1)}(t) = J^{(2)}(t) = 0$ $\forall t \in [-t_0, 0)$, this implies that $W^{(1)}(t) = W^{(2)}(t) = 0$ $\forall t \in [-t_0, 0)$. The proof follows exactly the same as above but adjusting for the fact the time now starts at $-t_0$ instead of 0. □

For any stochastic fluid network, let $A \subset \{1, 2, \ldots, N\}$ and define $\mathcal{P}_A = \{j \in \{1 \ldots N\}$: $\exists$ a path from the output of queue j to queue i in A$\} \cup A$. Note that if $j \in \mathcal{P}_A$ and $m \notin \mathcal{P}_A$ then by definition $P_{m,j} = 0$.

Now we return to the problem of comparing two stochastic fluid networks. In Theorem 2.4.2, it was vital that $P^{(1)} \geq P^{(2)}$, as opposed to the much more natural comparison theorem condition that $P^{(1)} \leq P^{(2)}$. The following Lemma shows that in the latter case, a little more can be said.

Lemma 2.4.3. *Assume that* $W^{(1)}(0) \leq W^{(2)}(0)$*,* $J^{(1)} \prec J^{(2)}$*,* $P^{(1)} \leq P^{(2)}$ *and* $r^{(2)}(t) \leq r^{(1)}(t)$*,* $\forall t \geq 0$*.*

Define

$$\left(W^{(1)}, Z^{(1)}\right) = FN\left(W^{(1)}(0), J^{(1)}, r^{(1)}(t), P^{(1)}\right)$$

and

$$\left(W^{(2)}, Z^{(2)}\right) = FN\left(W^{(2)}(0), J^{(2)}, r^{(2)}(t), P^{(2)}\right).$$

Then for every set $A \subset \{1, 2, \ldots, N\}$*,*

$$\sum_{j \in \mathcal{P}_A^{(1)}} W_j^{(1)}(t) \leq \sum_{j \in \mathcal{P}_A^{(2)}} W_j^{(2)}(t), \quad \forall t \geq 0. \tag{2.14}$$

Proof. Combining Theorems 2.4.1 and 2.4.2 gives

$$\sum_{j=1\ldots N} W_j^{(1)}(t) \leq \sum_{j=1\ldots N} W_j^{(2)}(t), \quad \forall t \geq 0.$$

The condition $P^{(1)} \leq P^{(2)}$ implies that $\forall A \subset \{1, 2, \ldots, N\}$ $\mathcal{P}_A^{(1)} \subset \mathcal{P}_A^{(2)}$. Therefore

$$\sum_{j \in \mathcal{P}_A^{(2)}} W_j^{(2)}(t) \geq \sum_{j \in \mathcal{P}_A^{(2)}} W_j^{(1)}(t) \geq \sum_{j \in \mathcal{P}_A^{(1)}} W_j^{(1)}(t).$$

□

Before proceeding to the next section, a slight modification of the model, which will be labeled the "'state process"' model, needs to be introduced. To simplify the analysis, it will be assumed that the input is of the ON-OFF type and that the system knows the state of the input at time t, ie. whether $State(J_i(t)) = ON$ or $State(J_i(t)) = OFF$. Define the state process $S : \mathbb{R}_+ \to \{0, 1\}^N$ such that $S_i(t) = 1$ if fluid is flowing out of queue i at time t and $S_i(t) = 0$ otherwise.

For the remainder of the section we will assume that the service rates are dependent on the state process, i.e. $r : \{0, 1\}^N \to \mathbb{R}_+^N$. The workload process becomes

$$W(t) = J(t) - \int_0^t (I - P')r(S(s))dt + (I - P')Z(t).$$

From the physics of the fluid network, $S_i = 1$ if and only if there exists $j \in \mathcal{P}_i$ s.t. $W_j(t^-) > 0$ or $State(J_j(t)) = ON$. The implication is that if $S_j(t) = 1$, then $S_i(t) = 1$. So all permissible states may be strictly smaller then all combinations of $\{0, 1\}^{k \times n}$. Denote $\mathcal{S}$ as the set of permissible states of $S(t)$.

Theorem 2.4.3. *Assume that*

$$P^{(1)} \leq P^{(2)},$$

$$W^{(1)}(0) \leq W^{(2)}(0),$$

and

$$J^{(1)} \prec J^{(2)}.$$

As well, given $s^{(1)}, s^{(2)} \in \mathcal{S}$, *assume that if* $s^{(1)} \leq s^{(2)}$ *then* $r^{(1)}(s^{(1)}) \geq r^{(2)}(s^{(2)})$.

Define

$$W^{(1)}(t) = W^{(1)}(0) + J^{(1)}(t) - \int_0^t (I - P^{(1)\prime})r^{(1)}(S^{(1)}(s))dt + (I - P^{(1)\prime})Z^{(1)}(t),$$

and

$$W^{(2)}(t) = W^{(2)}(0) + J^{(2)}(t) - \int_0^t (I - P^{(2)\prime})r^{(2)}(S^{(2)}(s))dt + (I - P^{(2)\prime})Z^{(2)}(t).$$

Then $\forall t \geq 0$,

$$S^{(1)}(t) \leq S^{(2)}(t).$$

Furthermore, $\forall A \subset \{1, \ldots, N\}$,

$$\sum_{j \in \mathcal{P}_A} W_j^{(1)}(t) \leq \sum_{j \in \mathcal{P}_A} W_j^{(2)}(t).$$

Proof. Before proceeding, the following fact (Corollary 14.3.5 of [65] and Theorem 3.7 of [55]) is required: $\forall j \in \mathcal{P}_i$ states, $W^{(1)}$ and $W^{(2)}$ are continuous at all continuity points of the input processes $J^{(1)}$ and $J^{(2)}$ respectively.

The proof of the main result will follow by contradiction. From the assumptions, it is known that

$$S^{(1)}(0) \leq S^{(2)}(0).$$

So assume that there exists a time $T > 0$ such that

$$S^{(1)}(T) \nleq S^{(2)}(T).$$

Let i be a queue that has $S_i^{(1)}(T) = 1$ and $S_i^{(2)}(T) = 0$. Since $S_i^{(2)}(T) = 0$, this implies that $\forall j \in \mathcal{P}_{\{i\}}^{(2)}$,

$$W_j^{(2)}(T) = 0 \text{ and } state(J_j^{(2)}(T)) = OFF.$$

But since

$$r^{(1)}(S^{(1)}(t)) \geq r^{(2)}(S^{(2)}(t)) \quad \forall t \in [0, T),$$

Lemma 2.4.3 states that for all $t \in [0, T)$,

$$\sum_{j \in \mathcal{P}_{\{i\}}^{(2)}} W_j^{(2)}(t) \geq \sum_{j \in \mathcal{P}_{\{i\}}^{(1)}} W_j^{(1)}(t).$$

Non-negativity of the workload process and continuity imply that $\forall j \in \mathcal{P}_{\{i\}}^{(1)}$,

$$W_j^{(1)}(T) = 0.$$

This is a contradiction since

$$state(J_j^{(2)}(T)) = OFF \Rightarrow state(J_j^{(1)}(T)) = OFF$$

and $\forall j \in \mathcal{P}_{\{i\}}^{(1)}$, $W_j^{(1)}(T) = 0$ means that $S_i^{(1)}(T) = 0$. Therefore $\forall t \geq 0$,

$$S^{(1)}(t) \leq S^{(2)}(t).$$

The remainder follows by Lemma 2.4.3. □

2.5 Application

Multi-class networks are very difficult to analyze analytically. In this section we establish some comparison results for multi-class networks under the Generalized Processor Sharing service discipline. The results are established exploiting the single class state process model, which was defined in the previous section.

Generalized Processor Sharing or GPS is a service discipline that is used to imitate a (weighted) round robin process sharing at each queue. Assume the network has k classes and let $\phi_{c,i} > 0$ denote the "'weight"' of class $c \in \{1 \dots k\}$ at queue $i \in \{1 \dots N\}$. Without loss of generality, it will be assumed that for all $i \in \{1 \dots N\}$, $\displaystyle\sum_{c=1}^{k} \phi_{c,i} = 1$.

Each queue has service capacity C_i, and the service rate at time t for each class c at each queue i is

$$r_{c,i}(t) = \frac{S_{c,i}(t)\phi_{c,i}}{\phi_{c,i} + \sum_{\tilde{c} \neq c} S_{\tilde{c},i}(t)\phi_{\tilde{c},i}} C_i.$$

S is defined to be the $k \times N$ dimensional state process.

The workload process vector for all class $c \in \{1 \dots k\}$ is

$$W_c(t) = W_c(0) + J_c(t) - \int_0^t (I - P_c')r_c(S(s))ds + (I - P_c')Z_c(t).$$

Unfortunately, this model of the workload is not useful for our purposes. Define

$$\hat{r}_{c,i}(t) \triangleq \frac{\phi_{c,i}}{\phi_{c,i} + \sum_{\tilde{c} \neq c} S_{\tilde{c},i}(t)\phi_{\tilde{c},i}} C_i, \tag{2.15}$$

$$\check{r}_{c,i}(t) \triangleq \frac{(1 - S_{c,i}(t))\phi_{c,i}}{\phi_{c,i} + \sum_{\tilde{c} \neq c} S_{\tilde{c},i}(t)\phi_{\tilde{c},i}} C_i, \tag{2.16}$$

$$\hat{Z}_{c,i}(t) \triangleq \int_0^t \check{r}_{c,i}(S(s))ds + Z_{c,i}(t). \tag{2.17}$$

It is evident that

$$\begin{aligned} r_c(t) &= \hat{r}_c(t) - \check{r}_c(t), \\ \hat{Z}_{c,i}(0) &= 0, \\ \hat{Z}_{c,i} &\in D_{\uparrow}. \end{aligned}$$

Therefore we can now redefine the workload process as

$$W_c(t) = W_c(0) + J_c(t) - \int_0^t (I - P_c')r_c(S(s))ds + (I - P_c')Z_c(t), \tag{2.18}$$

$$W_c(t) = W_c(0) + J_c(t) - \int_0^t (I - P_c')\hat{r}_c(S(s))ds + (I - P_c')(\int_0^t \check{r}_c(S(s))ds + Z_c(t)), \tag{2.19}$$

$$W_c(t) = W_c(0) + J_c(t) - \int_0^t (I - P_c')\hat{r}_c(S(s))ds + (I - P_c')\hat{Z}_c(t). \tag{2.20}$$

Notice that $\hat{r}_c(S(t))$ has the very useful property that if, $s^{(1)}, s^{(2)} \in \{0,1\}^{k\times N}$ such that $s^{(1)} \leq s^{(2)}$, then $\hat{r}_c(s^{(1)}) \geq \hat{r}_c(s^{(2)})$. For simplicity, from now on we let $r_c \equiv \hat{r}_c$ and $Z_c \equiv \hat{Z}_c$. Finally, we now convert the multi-class network with k classes and N queues into a larger single class network $\{\tilde{J}, \tilde{r}, \tilde{P}, \tilde{W}(0)\}$ with $k * N$ queues using the following procedure:

- The multi-class processes $J_c, r_c, W_c(0)$, are mapped to the single class vectorial processes $\tilde{J}, \tilde{r}, \tilde{W}(0)$ using the mapping $(c, i) \to (c-1) * N + i$.
- The routing matrix $\tilde{P}$ is a $N * k \times N * k$ block diagonal matrix, with the routing matrices P_c, $c = 1 \dots k$ as the block diagonal elements.

Combining this setup and Theorem 2.4.3 establishes the following comparison theorem.

Theorem 2.5.1. *Assume that* $\forall c \in \{1 \dots k\}$*:*

$$P_c^{(1)} \leq P_c^{(2)}, \tag{2.21}$$

$$W_c^{(1)}(0) \leq W_c^{(2)}(0), \tag{2.22}$$

$$J^{(1)}(t) - J^{(1)}(s) \leq J^{(2)}(t) - J^{(2)}(s) \quad \forall t > s \geq 0, \tag{2.23}$$

$$C^{(1)} \geq C^{(2)}. \tag{2.24}$$

Then

$$S^{(1)}(t) \leq S^{(2)}(t), \tag{2.25}$$

$$\sum_{i=1}^{N} W_{c,i}^{(1)}(t) \leq \sum_{i=1}^{N} W_{c,i}^{(2)}(t) \quad \forall t \geq 0. \tag{2.26}$$

Descriptively the above theorem tells us that increasing congestion in one aspect of the network, increases the total workload in the network for each class. A simple corollary is that increasing the congestion in just one class adversely affects the other classes in terms of workload. Note that this statement is not true in general discrete queueing networks under GPS. In a discrete queueing network, very large arrivals of a certain class could potentially "‘clog"’ the routes of that class, which by the very nature of GPS would be beneficial to the other classes.

Chapter 3

Interchange of Limits

3.1 Introduction

It is often the case in the performance analysis of stochastic networks that the calculation of the stationary distribution of the workload process is of great importance. Unfortunately, besides exceptional cases such as Jackson networks, the stationary distribution is difficult to compute.

On the other hand heavy traffic analysis often leads to Reflected Brownian Motion (RBM) (more precisely Semi-martingale Reflected Brownian Motion (SRBM)) models for which a substantial theory exists and whose stationary distributions can be explicitly characterized in many interesting cases [19, 53] and in other situations can be numerically computed from the Fokker-Planck type equations [62]. The heavy traffic limit as an approximation for the stationary distribution rests on the notion that by studying the heavy traffic limits we can bound the actual performance of networks. This naturally leads to the question as to whether the stationary distribution of the diffusion limit, if it exists, is the limit of the stationary distributions of the pre-limits.

This "‘interchange of limits"’ conjecture was first answered by Gamarnik and Zeevi [22] for Generalized Jackson Networks, but under restrictive moment assumptions on the input sequences exist. The condition was relaxed in a recent paper by Budhiraja and Lee [17] who showed that the result was true assuming only the existence of the second moment. Under additional assumptions they proved that the moments of the stationary distribution can be interchanged as well.

In this chapter the interchange of limits approximation is justified for Stochastic Fluid Networks (SFN) with non-decreasing Lévy inputs. Akin to the paper of Budhiraja and Lee [17], only second moment assumptions are required to justify the interchange of limits and under further moment assumptions, the interchange of limits results for the moments of the stationary distribution is justified as well. Furthermore under stronger assumptions, it is shown that the interchange of limits also hold for state-dependent routing as well.

The stochastic fluid network model with non-decreasing Lévy inputs has been analyzed in a series of papers by Kella [33, 34], Kella and Whitt [36, 38, 39]. Results for the more general stochastic network with spectrally positive Lévy inputs can be found in Konstantopoulos et al. [43].

Diffusion, or heavy traffic approximations have been studied extensively. The SRBM model was introduced by Harrison and Reiman [27] and stability conditions were given in Harrison and Williams [28]. The convergence of networks in the heavy traffic limit to SRBM is by now well known, see the survey in [67]. The monograph of Whitt [65] is also a comprehensive reference.

The chapter is divided as follows: Section 3.2 gives a brief introduction to weak convergence of probability measures in a metric space. Section 3.3 describes the stochastic fluid network model and the heavy traffic approximation. Section 3.2.2 discusses and states some results from the single link stochastic fluid network model. Sections 3.4 and 3.5 justify the interchange of limits approximation for fixed and state-dependent routing respectively. Finally, a summary of chapters conclusions and future research directions are discussed in Section 5.2.

All results proven in Sections 3.4 and 3.5 are original contributions unless otherwise stated. Besides justifying the interchange of limits approximation for the SFN, an additional contribution of note was the use of primarily sample-path arguments to obtain the results.

3.1.1 Assumptions and Notation

This section specifies the assumptions and notation that will be used in the chapter. Unless otherwise stated the integer $N \geq 0$ will be fixed as the dimension. Let $D \equiv D([0,\infty), \mathbb{R}^N)$ be the space of càdlàg, $\mathbb{R}^N$-valued functions defined on the interval $[0,\infty)$. The space D will be endowed with the Skorokhod J_1 topology. Let the subsets $D_+ \equiv D_+([0,\infty), \mathbb{R}^N)$ be the non-negative càdlàg functions and $D_\uparrow \equiv D_\uparrow([0,\infty), \mathbb{R}^N)$ denote the non-negative, non-decreasing càdlàg functions. As well, we denote $D_{\uparrow,0} \equiv D_{\uparrow,0}([0,\infty), \mathbb{R}^N)$ as the subset of functions in $D_\uparrow$ that are null at the origin. For any $x, y \in D_\uparrow([0,\infty), \mathbb{R}^N)$, the notation $x \prec y$ means that for all $s \geq t \geq 0$, $x(t) \leq y(t)$ and $x(s) - x(t) \leq y(s) - y(t)$. Finally, the notation $x(t-)$ means the limit when x approaches t from the left, i.e. the left limit.

Vectors and matrices are assumed to have real-valued entries. As well, vectors will be assumed to be column vectors. The transpose of a matrix A will be denoted by A' and I will represent the identity matrix. For a constant C, we use $\vec{C}$ to mean an N-dimensional column vector with all its entries being equal to C. The notation x_i will mean the i^{th} entry of a vector x and likewise, $A_{i,j}$ will mean the $(i,j)^{th}$ entry of a matrix A. Note that the notation x_n will also refer to a possibly vectorial element in a sequence, though the meaning of the notation will be obvious. The space $\mathbb{R}^N$ will be equipped with the Euclidean metric. For $p \geq 1$, $|\cdot|_p$ will denote the standard vector and induced matrix p-norms. For simplicity we shall write $|\cdot| \equiv |\cdot|_1$. Comparisons are assumed to be component wise. As well, scalar operations on vectors are to be interpreted component wise. We denote by e_i the standard unit vector, i.e. the i^{th} component is 1 and the rest are 0. For any two scalars a and b, $a \wedge b$ is the minimum and $a \vee b$ is the maximum.

An $N \times N$ matrix R is said to be an M-matrix if it has positive diagonal entries, non-positive off-diagonal entries, and has a non-negative inverse [19](p. 164). In this thesis, routing matrix, denoted by P, will mean a substochastic matrix such that $(I - P')$ is an M-matrix.

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a stochastic basis. The filtration $(\mathcal{F}_t)_{t\geq 0}$ is right continuous and $\mathcal{F}_0$ contains all the $\mathbb{P}$-null sets of $\mathcal{F}$. All stochastic processes will be adapted to the filtration. The symbol $\Rightarrow$ will denote weak convergence for probability distributions and convergence in distribution for random elements. The abbreviation "'a.s."' denotes almost surely. We use the term subordinator to mean an N-dimensional (unless otherwise stated), a.s. non-decreasing Lévy process (i.e. subordinator). Furthermore, we shall impose the requirement that all components of the subordinator are independent and that their increments have finite second moments.

3.2 Background

3.2.1 Weak Convergence

This subsection about weak convergence is based on the books of Billingsley [7], Jacod and Shiryaev [30] and Whitt [65].

Let $\mathcal{X}$ be a Polish space (i.e. a complete, separable metric space) with metric d and let $\mathcal{B}$ be the Borel σ-field of subsets of $\mathcal{X}$ generated by the open sets. Suppose that $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space. A random element X in $\mathcal{X}$ is a measurable mapping from $(\Omega, \mathcal{F})$ into $(\mathcal{X}, \mathcal{B})$. A random element X has a corresponding probability measure π (on $\mathcal{X}$) known as the distribution defined by

$$\pi \triangleq \mathbb{P}\left(X \in \cdot\right).$$

A sequence of probability measures (π_n) converges weakly to π (i.e. $\pi_n \Rightarrow \pi$) if for all bounded, continuous real-valued functions f,

$$\lim_{n\to\infty} \int_{\mathcal{X}} f(x)\pi_n(dx) = \int_{\mathcal{X}} f(x)\pi(dx).$$

We say that a sequence of random elements (X_n) converges weakly to X (i.e. $X_n \Rightarrow X$) if the corresponding sequence of distributions converges weakly.

The following theorem provides equivalent characterizations of weak convergence.

Theorem 3.2.1 (Portmanteau)**.** *The following five conditions are equivalent:*

$$(i)\pi_n \Rightarrow \pi,$$
$$(ii) \lim_n \int f d\pi_n = \int f d\pi \text{ for all bounded, uniformly continuous } f,$$
$$(iii) \limsup_n \pi_n(F) \le \pi(F) \text{ for all closed sets } F,$$
$$(iv) \liminf_n \pi_n(G) \ge \pi(G) \text{ for all open sets } G, \text{and}$$
$$(v)\pi_n(A) \to \pi(A) \text{ for all sets } A \text{ such that the boundary } \partial A \text{ satisfies } \pi(\partial A) = 0.$$

The following important result states that weak convergence is preserved by continuous mappings. The continuity assumption can be relaxed, but that will not be required in this thesis.

Theorem 3.2.2 (Continuous Mapping Theorem)**.** *Let h be a continuous mapping from $\mathcal{X}$ into another metric space $\mathcal{X}'$, with metric d' and Borel σ-field $\mathcal{B}'$. If $\pi_n \Rightarrow \pi$, then $\pi_n h^{-1} \Rightarrow \pi h^{-1}$.*

Another important result is Prokhorov's theorem. A family of probability measures Π is *tight* if for each $\epsilon > 0$ there exists a compact set K such that $\pi(K) > 1-\epsilon$ for all $\pi \in \Pi$. A family Π of probability measures is *relatively compact* if every sequence contains a weakly convergent subsequence. The limit does not need to be in Π.

Theorem 3.2.3 (Prokhorov)**.** *A family of probability measures Π is tight if and only it is relatively compact.*

A sequence of $\mathcal{X}$-valued random variables is tight if and only if the corresponding sequence of distributions is tight. Now suppose that the metric space is (D, d_{J_1}) (which is Polish). An important notion in many of the proofs will be that of C-tightness. A sequence of D-valued random variables (X_n) is C-tight if it is tight and all limit points have a.s. continuous sample paths.

The following theorem from [30] (in Chapter VI, Proposition 3.26) gives a characterization of C-tightness.

Theorem 3.2.4. *A sequence of processes* (S_n) *is C-tight if and only if for all* $T, \alpha, \epsilon > 0$ *there are* $K, \theta > 0$ *and a positive integer* n_0 *s.t. for all* $n \geq n_0$*:*

1. $\mathbb{P}\left(\sup_{t\in[0,T]} |S_n(t)| > K\right) \leq \epsilon$,
2. $\mathbb{P}\left(w_T\left(S_n, \theta\right) > \alpha\right) \leq \epsilon$,

where $w_T(S_n, \theta) = \sup\{\sup_{r,s\in[t,t+\theta]} |S_n(r) - S_n(s)| : 0 < t < t + \theta < T\}$.

The first condition of Theorem 3.2.4 holds if the sequence (S_n) is simply tight (See Section VI, Theorem 3.2.1).

3.2.2 Properties of Reflected Lévy Processes

In this section, we state some useful properties about reflected Lévy processes that will be used in later sections. The proofs have been omitted since they are straightforward.

Let J be a 1-dimensional subordinator and r a non-negative constant such that $E[J(1)] < r$. Define the process X such that $X(t) = J(t) - rt$. Then X is a 1-dimensional finite variation Lévy process with no negative jumps and $E[X(1)] < 0$. We know from Kella [34] that the reflected process of X has a unique stationary and limiting distribution. Let W be a random variable whose law follows that stationary distribution.

Let $\alpha > 0$, $\phi(\alpha) = \ln E[e^{-\alpha X(1)}]$ and $\phi_k(\alpha) = \frac{d^k\phi(\alpha)}{d\alpha^k}$. It is known that (e.g. [38]) the Laplace-Stieltjes transform of W is $E[e^{-\alpha W}] = \frac{\alpha\phi_1(0)}{\phi(\alpha)}$. We begin by stating a simple recursive property of the Laplace-Stieltjes transform.

Lemma 3.2.1. *Define the functions*

$$f_0(\alpha) = \frac{\alpha}{\phi(\alpha)} = \frac{E[e^{-\alpha W}]}{\phi_1(0)}$$

and

$$f_k(\alpha) = \frac{df_{k-1}(\alpha)}{d\alpha}$$

for $k \geq 1$.

Then $f_k(\alpha) = \begin{cases} -\frac{f_0(\alpha)\phi_1(\alpha)-1}{\phi(\alpha)}, & k=1 \\ -\frac{\sum_{i=1}^{k}\binom{k}{i} f_{k-i}(\alpha)\phi_i(\alpha)}{\phi(\alpha)}, & k \geq 2. \end{cases}$

Let $\kappa_i = (-1)^i \phi_i(0)$ (i.e. the i^{th} cumulant of $X(1)$) which is finite if $E[J(1)^i] < \infty$. Fix a positive integer k. Noting that $\frac{d^k E[e^{-\alpha W}]}{d\alpha^k} = (-1)^k E[W^k e^{-\alpha W}]$ via the dominated convergence theorem and $\lim_{\alpha \to 0} E[W^k e^{-\alpha W}] = E[W^k]$ via the monotone convergence theorem, we obtain the following useful corollary:

Corollary 3.2.1. *Assume that* $E[J^{k+1}] < \infty$. *Then*

$$E[W^k] = \frac{-1}{(k+1)\kappa_1} \sum_{i=2}^{k+1} \binom{k+1}{i} E[W^{k+1-i}]\kappa_i < \infty.$$

In the case of $k = 1$, $E[W] = -\frac{Var(X(1))}{2E[X(1)]}$. *Alternatively there exists a multivariate polynomial* Q *such that* $E[W^k] = \frac{Q(\kappa_1,\ldots,\kappa_{k+1})}{\kappa_1^k}$.

An important consequence of Corollary 3.2.1 is that we can write the moments of the stationary distribution W in terms of the cumulants of $X(1)$. Cumulants of Lévy processes in general have the attractive property that they are linear with time (see Proposition 3.13 of [20]). Therefore, when κ_i exists, the i^{th} cumulant of $\frac{X(n)}{\sqrt{n}}$ is $n^{1-\frac{i}{2}}\kappa_i$.

3.3 Model

3.3.1 The Stochastic Fluid Network

In this chapter networks of N single server, work conserving queues are considered. The queues are naturally labeled as $1, \ldots, N$. A stochastic fluid network can be uniquely characterized by the 4-tuple $(J, r, P, W(0))$: An N-dimensional, $(\mathcal{F}_t)$-adapted stochastic process J, a vector $r \in \mathbb{R}_+^N$, a matrix $P \in \mathbb{R}_+^N \times \mathbb{R}_+^N$, and an N-dimensional non-negative random vector $W(0)$.

It is assumed that P is a routing matrix (ie. a substochastic matrix such that $(I - P')$ is an M-matrix), which implies that the network is open. The cumulative amount of work that arrives externally to the system at queue $i = 1 \ldots N$ is modeled by J_i. Work at queue i is drained as a fluid at rate $r_i > 0$, and routed to queue j at rate $P_{i,j} r_i$. The random vector $W(0)$ represents the initial amount of work in the system at time 0.

The cumulative input process J is assumed to be a subordinator, i.e. a non-decreasing Lévy process. We denote $\lambda = E[J(1)]$ where $J(1)$ denotes the amount of work that has arrived in the unit interval. Also, we assume that $W(0)$ is independent of J.

Let X be the virtual workload process, which is defined by the relation $X(t) = W(0) + J(t) - (I - P')rt$. Also let (W, Z) be the solution to the SP corresponding to $(X, I - P')$. The reflected process W, also known as the workload process, models the dynamics of the work in the network. Physically, $W_i(t)$ represents the amount of work at queue i at time t.

The workload W is known to be a strong Markov process (see Kella [34]). If $(I-P')^{-1}\lambda < r$, then from Theorem 3.1 of Kella [34] the process W is also ergodic (see pg. 94 of Bramson [16] for definition). We will represent the stationary distribution of the workload by π and let $W(\infty)$ denote a random vector with distribution π.

A recurring theme in this chapter is bounding the workload in the network by the workload of a simpler network. The following lemma due to Kella [34] (Lemma 3.1) that establishes conditions under which the total workload in a SFN can be bound by the workload of a simpler SFN consisting of N independent queues.

Lemma 3.3.1. *Assume that $(I - P')^{-1}\lambda < r$ and consider a vector $\tilde{\lambda} \in \mathbb{R}^N$ such that $\tilde{\lambda} > \lambda$ and $(I - P')^{-1}\tilde{\lambda} < r$. Let $(J, r, P, W(0))$ and $(J, \tilde{\lambda}, \boldsymbol{0}, W(0))$ be two stochastic fluid networks with respective workload processes W and $\tilde{W}$. Then $|W(t)| \leq |\tilde{W}(t)|$ a.s. for each $t \geq 0$.*

Since $\tilde{\lambda} > \lambda$, $\tilde{W}$ is ergodic. Let $\tilde{\pi}$ represent the stationary distribution of $\tilde{W}$, and $\tilde{W}(\infty)$ denote a random vector with distribution $\tilde{\pi}$. The following corollary of Lemma 3.3.1 will be useful in proving some of the later results.

Corollary 3.3.1. *Let $k = 1, 2, \ldots$ and assume that $E\left[J(1)^{k+1}\right] < \infty$. Then*

$$E\left[|W(\infty)|^k\right] \leq E\left[|\tilde{W}(\infty)|^k\right] < \infty. \tag{3.1}$$

Proof. Using Corollary 3.2.1, we have that $E\left[\tilde{W}(\infty)_i^k\right] < \infty$ for $i = 1, \ldots, N$. Therefore $E\left[|\tilde{W}(\infty)|^k\right] < \infty$ follows by expanding $E\left[|\tilde{W}(\infty)|^k\right]$ using the multinomial theorem and applying Hölder's inequality term wise.

Fix a positive integer M. From Lemma 3.3.1, for any $t \geq 0$ we have

$$E\left[\min(|W(t)|^k, M)\right] \leq E\left[\min(|\tilde{W}(t)|^k, M)\right].$$

Applying the Portmanteau theorem and the monotone convergence theorem to each side of the inequality, we obtain

$$E\left[|W(\infty)|^k\right] \leq E\left[|\tilde{W}(\infty)|^k\right].$$

□

3.3.2 Heavy Traffic Approximation

We now consider a sequence of stochastic fluid networks $(J_n, r_n, P, W_n(0))$. Each network in the sequence will have the same assumptions as in Section 3.3.1. We assume that J_n is a subordinator, $r_n > \vec{0}$, and $W_n(0)$ is a non-negative vector independent of J_n. As well, each network will have common routing matrix P. For each n we define λ_n and Γ_n as the mean and covariance matrix of $J_n(1)$ respectively. We will establish the heavy traffic limit for the sequence of workload processes by "stretching time by n" and "compressing space by $\sqrt{n}$".

For each $t \geq 0$ define

$$\overline{J}_n(t) = \frac{J_n(nt) - \lambda_n nt}{\sqrt{n}}, \tag{3.2}$$

$$\overline{W}_n(0) = \frac{W_n(0)}{\sqrt{n}} \tag{3.3}$$

and

$$\overline{X}_n(t) = \overline{W}_n(0) + \overline{J}_n(t) + \sqrt{n}(\lambda_n - (I - P')r_n)t. \tag{3.4}$$

Let (W_n, Z_n) be the sequence of workload and regulator processes corresponding to the sequence of SFNs $(J_n, r_n, P, W_n(0))$. Define

$$\overline{W}_n(t) = \frac{W_n(nt)}{\sqrt{n}} \tag{3.5}$$

and

$$\overline{Z}_n(t) = \frac{Z_n(nt)}{\sqrt{n}}. \tag{3.6}$$

Note that for each n, $(\overline{W}_n, \overline{Z}_n)$ is the solution to the SP corresponding to the sequence $(\overline{X}_n, I - P')$.

In order to establish the heavy traffic approximation, the following assumptions are required: There exists vectors $\lambda, r \in \mathbb{R}_+^N$, $\eta \in \mathbb{R}^N$, and a covariance matrix Γ such that

$$\begin{aligned} (I - P')^{-1}\lambda_n &< r_n, \\ \Gamma_n &\to \Gamma, \\ \lambda_n &\to \lambda, \\ r_n &\to r, \\ \sqrt{n}(\lambda_n - (I - P')r_n) &\to \eta. \end{aligned} \tag{3.7}$$

From the assumptions, we see that η satisfies $(I - P')^{-1}\eta < 0$. As well, for each fixed n, we will denote the unique stationary distribution of the ergodic Markov process $\overline{W}_n$ by π_n. Also, we define $\overline{W}_n(\infty)$ to be a random vector with distribution π_n.

Let B be a standard N-dimensional Brownian motion, $\overline{W}(0)$ a non-negative random vector and define $BM_{\overline{W}(0)}(\eta, \Gamma)$ where $BM_{\overline{W}(0)}(\eta, \Gamma)(t) = \overline{W}(0) + \eta t + \Gamma B(t)$. Assuming that $\overline{W}_n(0) \Rightarrow \overline{W}(0)$, the heavy traffic limit of the sequence $\overline{W}_n$ will be shown to be a Reflected Brownian Motion, which is the reflected process of the solution to the SP corresponding to $(BM_{\overline{W}(0)}(\eta, \Gamma), I - P')$. The reflected Brownian motion will be denoted by $RBM_{\overline{W}(0)}(\eta, \Gamma, I - P')$. We require the following result from Harrison and Williams

[28]: $RBM_{\overline{W}(0)}(\eta, \Gamma, I - P')$ possesses a unique stationary distribution if and only if $(I - P')^{-1}\eta < 0$. We will denote the stationary distribution by π_{RBM} and let $W_{RBM}(\infty)$ be a random vector with distribution π_{RBM}.

3.4 Results: Fixed Routing

The approach we use to prove the interchange of limits result follows the same line of reasoning as Gamarnik and Zeevi [22] and can roughly be divided into three steps. We first prove that the heavy traffic limit for our model is a reflected Brownian motion. Then we will prove that the sequence of stationary distributions $(\pi_n)_{n\in\mathbb{N}}$ is tight. Finally, we will prove that the sequence weakly converges to the stationary distribution of a reflected Brownian motion.

3.4.1 Convergence to Reflected Brownian Motion

Due to the stationary and independent increments property of the Lévy process, the weak convergence argument is straightforward.

Lemma 3.4.1. *Suppose that* $\overline{W}_n(0) \Rightarrow \overline{W}(0)$. *Then the sequence* $\left(\overline{X}_n\right)_{n\in\mathbb{N}}$ *converges in distribution to* $BM_{\overline{W}(0)}(\eta, \Gamma)$.

Proof. From the stationary and independent increments property, $J_n(n) = \sum_{i=1}^{n} \widetilde{J_n^i}(1)$ where $\widetilde{J_n^i}(1)$ are independent copies of $J_n(1)$. So from the central limit theorem, $\overline{J}_n(1) \Rightarrow N(0, \Gamma)$. Therefore by Corollary 3.6 of Jacod and Shiryaev [30] (Chapter VII), $\overline{J}_n \Rightarrow \Gamma B$. Since by assumption $\overline{W}_n(0)$ is independent of $\overline{J}_n$ and the sequence $(\sqrt{n}(\lambda_n - (I - P')r_n))$ is deterministic,

$$\left(\overline{W}_n(0), \overline{J}_n, \sqrt{n}(\lambda_n - (I - P')r_n)\right) \Rightarrow \left(\overline{W}(0), \Gamma B, \eta\right).$$

Therefore

$$\overline{X}_n \Rightarrow BM_{\overline{W}(0)}(\eta, \Gamma)$$

by applying the continuous mapping theorem. □

Due to the continuity of the Skorokhod map, one can prove the following theorem using the previous lemma and the continuous mapping theorem.

Theorem 3.4.1. *Suppose that $\overline{W}_n(0) \Rightarrow \overline{W}(0)$. Then the sequence $\left(\overline{W}_n\right)_{n\in\mathbb{N}}$ converges in distribution to $RBM_{\overline{W}(0)}(\eta, \Gamma, I - P')$.*

3.4.2 Tightness

The next step in proving the interchange of limits is to show that the sequence of stationary distributions is tight.

Lemma 3.4.2. *Suppose* $\sup_n E\left[J_n(1)^{k+1}\right] < \infty$. *Then for any $p \in [0, k]$, the sequence $\left(E[\overline{W}_n(\infty)^p]\right)_{n\in\mathbb{N}}$ is bounded.*

Proof. From Hölder's inequality, we only need to prove the lemma for $p = k$. Let $\overline{r}_n \equiv \sqrt{n}r_n$, $\overline{\lambda}_n \equiv \sqrt{n}\lambda_n$, $\epsilon_n = \dfrac{\min\limits_{i=1\ldots N}(\overline{r}_n - (I-P')^{-1}\overline{\lambda}_n)_i}{2\left|(I-P')^{-1}\right|_\infty}$ and $\tilde{\lambda}_n = \overline{\lambda}_n + \vec{\epsilon_n}$.

We will verify that $\tilde{\lambda}_n > \overline{\lambda}_n$ and $(I-P')^{-1}\tilde{\lambda}_n < \overline{r}_n$.

To verify the first inequality, it is enough to check that $\epsilon_n > 0$ which is true since $\overline{r}_n > (I-P')^{-1}\overline{\lambda}_n$.

To verify the second equality, note that

$$
\begin{aligned}
(I-P')^{-1}\vec{\epsilon_n} &= (I-P')^{-1}\vec{1}\frac{\min\limits_{i=1\ldots N}(\overline{r}_n - (I-P')^{-1}\overline{\lambda}_n)_i}{2\left|(I-P')^{-1}\right|_\infty}, \\
&\le \vec{1}\frac{\min\limits_{i=1\ldots N}(\overline{r}_n - (I-P')^{-1}\overline{\lambda}_n)_i}{2}, \\
&\le \frac{1}{2}(\overline{r}_n - (I-P')^{-1}\overline{\lambda}_n).
\end{aligned}
$$

Thus $(I-P')^{-1}\tilde{\lambda}_n \le \frac{1}{2}(I-P')^{-1}\overline{\lambda}_n + \frac{1}{2}\overline{r}_n < \overline{r}_n$.

For each n, define the process $\tilde{X}_n$ such that

$$\tilde{X}_n(t) = \overline{W}_n(0) + \overline{J}_n(t) + (\overline{\lambda}_n - \tilde{\lambda}_n)t.$$

Let $(\tilde{W}_n, \tilde{Z}_n)$ be the solution to the SP corresponding to $(\tilde{X}_n, I)$. Since $\epsilon_n > 0$ and $\lim_n \epsilon_n = \dfrac{\min_{i=1\ldots N}(-(I-P')^{-1}\eta)_i}{2\,|(I-P')^{-1}|_\infty} > 0$, we see that $\sup_n E[\tilde{W}_n(\infty)^k] < \infty$ from Corollary 3.2.1. The result follows by applying Corollary 3.3.1. □

Theorem 3.4.2. *The sequence of stationary distributions $(\pi_n)_{n\in\mathbb{N}}$ is tight.*

Proof. Since $\sup_n E\left[J_n(1)^2\right] < \infty$ by assumption, tightness follows by using the Markov inequality in conjunction with Lemma 3.4.2. □

We now complete the proof of the main result on the interchange of limits.

Theorem 3.4.3. *The sequence $(\pi_n)_{n\in\mathbb{N}}$ weakly converges to π_{RBM}.*

Proof. Suppose that $\overline{W}_n(0)$ has distribution π_n. Since the sequence (π_n) is tight, Prohorov's theorem says that for every subsequence there exists a further subsequence that converges. Let (π_{n_k}) be a convergent subsequence with weak limit π.

From Theorem 3.4.1, $\overline{W}_{n_k} \Rightarrow RBM_{\overline{W}(0)}(\eta, \Gamma, I-P')$, where $\overline{W}(0)$ has distribution π. So for any fixed time t, $\overline{W}_{n_k}(t) \Rightarrow RBM_{\overline{W}(0)}(\eta, \Gamma, I-P')(t)$. Since π_{n_k} is the stationary distribution of $\overline{W}_{n_k}$, for any $t \geq 0$, $\overline{W}_{n_k}(t)$ is equal in distribution to $\overline{W}_{n_k}(0)$. This implies that π is a stationary distribution of RBM. But the stationary distribution of RBM is unique, so π must be equal to π_{RBM}.

Since this was true for any arbitrary convergent subsequence, $\pi_n \Rightarrow \pi_{RBM}$.

□

Under an additional assumption, the next theorem will show that the interchange of limits holds for the moments as well.

Theorem 3.4.4. *Assume that* $\sup_n E\left[J_n(1)^{k+1}\right] < \infty$. *Then*

$$E[\overline{W}_n(\infty)^p] \to E[W_{RBM}(\infty)^p], \tag{3.8}$$

for all $p \in [0, k)$.

Proof. Lemma 3.4.2 implies uniform integrability of the sequence, which implies convergence of the p^{th} moments for any $p \in [0, k)$ (see [7] (pg. 32)).

□

3.5 Results: State Dependent Routing

Up until now, we have assumed fixed routing. With a few additional assumptions, we will show that the main results from the previous section hold when the routing is state dependent. First we will have to expand the meaning of the solution to the Skorokhod problem.

3.5.1 The Skorokhod Problem and the Stochastic Fluid Network Model

Let $X \in D$ with $X(0) \geq \vec{0}$, $b : \mathbb{R}_+^N \to \mathbb{R}^N$ with each entry being a Lipschitz continuous function, $\hat{R}$ be an M-matrix with $\hat{R}^{(i,i)} = 1$ for all $i = 1 \dots N$. Let $R : \mathbb{R}_+^N \to \mathbb{R}^{N \times N}$ be such that for each $w \in \mathbb{R}_+^N$, $R(w)$ is an M-matrix with $R(w)^{(i,i)} = 1$ for all $i = 1 \dots N$. As well suppose that $|R(w)^{(i,j)}| \leq |\hat{R}^{(i,j)}|$, and each entry $R^{(i,j)}$ is Lipschitz continuous for $i, j = 1 \dots N$ and $i \neq j$.

Definition 3.5.1. *The functions* $(W, Z) \in D^2$ *are said to solve the Skorokhod Problem corresponding to* $(X, b(w), R(w))$ *if the following conditions hold:*

1. $W(t) = X(t) + \int_0^t b(W(s))ds + \int_0^t R(W(s))dZ(s) \geq 0 \ \forall t \geq 0,$

2. $Z \in D_{\uparrow,0}$,

3. $\int_0^t W^{(i)}(s) dZ^{(i)}(s) = 0$ *for each* $i = 1 \ldots N$.

From Theorem 3.7 of Ramasubramanian [55], (W, Z) exist and are unique.

A stochastic fluid network with state dependent routing is defined almost the same way as its fixed routing counterpart. Let $\hat{P}$ be a routing matrix with diagonal entries equal to 0. Also let $P : \mathbb{R}_+^N \to \mathbb{R}_+^{N\times N}$ such that for each $w \in \mathbb{R}_+^N$, $P(w)$ is a routing matrix with diagonal entries equal to 0. Furthermore assume that $\sup_{w\in\mathbb{R}_+^N} P(w) \leq \hat{P}$ and $P^{(i,j)}$ is Lipschitz continuous for $i, j = 1 \ldots N$ and $i \neq j$. The matrix valued function P will model the state dependent routing.

As in Section 3.3.1, we define the vector of service rates by $r > \vec{0}$, the initial workload by the non-negative random vector $W(0)$ and the cumulative input process is modeled by a subordinator J. Furthermore, the initial workload $W(0)$ is independent of J. We denote the mean of $J(1)$ by λ. A stochastic fluid network with state dependent routing can be uniquely characterized by the 4-tuple $(J, r, P(w), W(0))$.

Define the process X such that $X(t) = W(0) + J(t)$. Let (W, Z) be the solution to the SP corresponding to $(X, -(I - P'(w))r, I - P'(w))$. The dynamics of the workload process will be modeled by the process W. Furthermore, from Theorem 6.1 in [55], W is strong Markov.

In the fixed routing case, bounding the workload of a SFN by the workload of another simpler network consisting of N independent queues played a critical role in proving the results of the previous section. Similarly, we will bound the workload of a SFN with state dependent routing with the workload of a SFN with fixed routing. Let $(\hat{W}, \hat{Z})$ be the solution to the SP corresponding to $(X, -(I - \hat{P}')r, I - \hat{P}')$. Under the assumptions of this section, we have the following useful lemma which is a special case of Theorem 2.4.1.

Lemma 3.5.1. *For all* $t \geq 0$, *the workload processes* W *and* $\hat{W}$ *satisfy the inequality* $W(t) \leq \hat{W}(t)$ *a.s.. Furthermore, the processes* Z *and* $\hat{Z}$ *satisfy the relation* $\hat{Z} \prec Z$ *a.s..*

A corollary of Lemma 3.5.1 is that Z is a continuous process since $\hat{Z}$ is a continuous process. Another application of Lemma 3.5.1 will be to provide a simple stability condition for SFN with state dependent routing.

Lemma 3.5.2. *Under the assumption $(I - \hat{P}')\lambda < r$, the workload process W is ergodic.*

Proof. From the proof of Theorem 3.1 in Kella [34], $\hat{W}$ is a positive recurrent Markov process with regeneration set $\{\vec{0}\}$. Furthermore, $\hat{W}$ admits coupling. Since, from Lemma 3.5.1, $\hat{W}(t) \geq W(t)$ a.s. for all $t \geq 0$, it follows that W is a positive recurrent Markov process with regeneration set $\{\vec{0}\}$. But the regeneration set is the single point $\vec{0}$, so W admits coupling as well. The result follows from Theorem 2.7 and Proposition 3.8 (iii) in [1] (Chapter VII). □

The definitions and terminology used in the previous result can be found in Chapter VII, Subsections 2 and 3 of Asmussen [1]. For a more detailed discussion about stability, see Chapter 4 of Bramson [16].

3.5.2 Heavy Traffic Approximation

As in Section 3.3.2, we will need to consider a sequence of SFNs $(J_n, r_n, \tilde{P}_n(w), W_n(0))$. For each network in the sequence, we assume that J_n is a subordinator, $r_n > \vec{0}$, and $W_n(0)$ is a non-negative vector independent of J_n. Each $\tilde{P}_n$ is a matrix valued function such that $\tilde{P}_n(w)$ is a routing matrix and each entry is a Lipschitz continuous function. Additionally, we assume that there exists a routing matrix $\hat{P}$ such that $\sup_{w \in \mathbb{R}_+^N} \tilde{P}_n(w) \leq \hat{P}$ for all n. As well, for each n, define λ_n and Γ_n to be the mean and covariance matrix of $J_n(1)$ respectively. Our goal will be to take the limit as $\tilde{P}_n$ approaches $\hat{P}$.

Let $(W_n, Z_n)_{n \in \mathbb{N}}$ be the sequence of workload and regulator processes corresponding to the sequence of SFNs $(J_n, r_n, \tilde{P}_n(w), W_n(0))_{n \in \mathbb{N}}$. We want to scale the workload and regulator processes by "stretching time by n" and "compressing space by $\sqrt{n}$".

For each $t \geq 0$ define

$$\overline{J}_n(t) = \frac{J_n(nt) - \lambda_n nt}{\sqrt{n}} \tag{3.9}$$

and

$$\overline{W}_n(0) = \frac{W_n(0)}{\sqrt{n}}. \tag{3.10}$$

As well let

$$\overline{W}_n(t) = \frac{W_n(nt)}{\sqrt{n}} \tag{3.11}$$

and

$$\overline{Z}_n(t) = \frac{Z_n(nt)}{\sqrt{n}}. \tag{3.12}$$

Writing out the scaled workload equation,

$$\begin{aligned}\frac{W(nt)}{\sqrt{n}} =& \overline{W}_n(0) + \overline{J}_n(t) + \int_0^t \sqrt{n}\left(\lambda_n - (I - \tilde{P}_n'(W(ns)))\right)ds + \\ & \int_0^t (I - \tilde{P}_n'(W(ns)))d\frac{Z(ns)}{\sqrt{n}}\end{aligned} \tag{3.13}$$

or

$$\begin{aligned}\overline{W}_n(t) = \overline{W}_n(0) + \overline{J}_n(t) + \int_0^t \sqrt{n}\left(\lambda_n - (I - \tilde{P}_n'(\sqrt{n}\overline{W}_n(s)))\right)ds + \\ \int_0^t (I - \tilde{P}_n'(\sqrt{n}\overline{W}_n(s)))d\overline{Z}_n(s).\end{aligned}$$

For simplicity, we write $P_n(w) \equiv \tilde{P}_n(\sqrt{n}w)$, $\eta_n \equiv \sqrt{n}(\lambda_n - (I - \hat{P}')r_n)$ and $p_n(w) \equiv \sqrt{n}(\hat{P} - P_n(w))r_n$. Note that $\sup_{w \in \mathbb{R}_+^N} P_n(w) \leq \hat{P}$ for all n. Also, define the process $\overline{X}_n$ such that

$$\overline{X}_n(t) = \overline{W}_n(0) + \overline{J}_n(t) + \eta_n t. \tag{3.14}$$

For each n, $(\overline{W}_n, \overline{Z}_n)$ is also the solution to the SP corresponding to $(\overline{X}_n, -p_n(w), I - P'(w))$.

We will require the following heavy traffic assumptions for all $w \in \mathbb{R}_+^N$:

There exists vectors $\lambda, r \in \mathbb{R}_+^N$, $\eta \in \mathbb{R}^N$, a covariance matrix Γ, and a Lipschitz continuous function $p : \mathbb{R}_+^N \to \mathbb{R}_+^N$ such that

$$\begin{aligned} (I - \hat{P}')^{-1}\lambda_n &< r_n, \\ \eta_n &\to \eta, \\ P_n(w) &\to \hat{P}, \\ p_n(w) &\to p(w), \\ \Gamma_n &\to \Gamma, \\ \lambda_n &\to \lambda, \\ r_n &\to r. \end{aligned} \tag{3.15}$$

Additionally, we need to assume that the sequence of functions p_n satisfies the following uniform linear growth property: There exists a constant $C_g > 0$ such that for each $w \in \mathbb{R}_+^N$, the inequality $|p_n(w)| \le C_g(1 + |w|)$ is satisfied for all n.

From the assumptions, we see that η satisfies $(I - P')^{-1}\eta < 0$. As well, for each fixed n, we will denote the unique stationary distribution of the ergodic Markov process $\overline{W}_n$ by π_n. Let $\overline{W}_n(\infty)$ be a random vector with distribution π_n.

We also define the sequence $(\hat{W}_n, \hat{Z}_n)_{n\in\mathbb{N}}$, which are the solutions to the SP corresponding to the sequence $(\overline{X}_n, 0, I - \hat{P}')_{n\in\mathbb{N}}$. Each $\hat{W}_n$ is the scaled workload process of a SFN with fixed routing characterized by $(J_n, r_n, \hat{P}, W_n(0))$. We will denote the unique stationary distribution of the ergodic Markov process $\hat{W}_n$ by $\hat{\pi}_n$. Let $\hat{W}_n(\infty)$ be a random vector with distribution $\hat{\pi}_n$.

As in Section 3.3.2, let B be a standard N-dimensional Brownian motion, $\overline{W}(0)$ a non-negative random vector and define $BM_{\overline{W}(0)}(\eta, \Gamma)$ where $BM_{\overline{W}(0)}(\eta, \Gamma)(t) = \overline{W}(0) + \eta t + \Gamma B(t)$. Assuming that $\overline{W}_n(0) \Rightarrow \overline{W}(0)$, the heavy traffic limit of the sequence $\overline{W}_n$ will be shown to be a reflected diffusion with state dependent drift, which is the reflected process of the solution to the SP corresponding to $(BM_{\overline{W}(0)}(\eta, \Gamma), -p(w), I - \hat{P}')$. For a discussion on the existence and uniqueness of the solution to the SP, see Theorem 2.1 in [2]. The reflected diffusion will be denoted by $RBM_{\overline{W}(0)}(\eta - p(w), \Gamma, I - \hat{P}')$. Since $(I - \hat{P}')\eta < 0$, there exists a unique limiting and stationary distribution. See Remark 4.3 in [50] and

Theorem 2.2 in [2]. We will denote the stationary distribution by π_{RBM} and let $W_{RBM}(\infty)$ be a random vector with distribution π_{RBM}.

3.5.3 Interchange of Limits for State-dependent Routing

Unlike the fixed routing case, the main difficulty will be showing weak convergence of the workload process under heavy traffic. Proving the interchange of limits result will be straightforward using Lemma 3.5.1. We will begin by proving that $\left(\overline{Z}_n\right)$ is C-tight, which means that the sequence is tight and for any convergent subsequence the limit has continuous sample paths.

Lemma 3.5.3. *Suppose that $\overline{W}_n(0) \Rightarrow \overline{W}(0)$. Then the sequence $\left(\overline{Z}_n\right)_{n\in\mathbb{N}}$ is C-tight.*

In this section we prove Lemma 3.5.3, which states that the sequence of regulator processes $\overline{Z}_n$ is C-tight. Before proving the lemma though, we will first state a simple lemma, which follows from the union bound, that will be used quite a bit in the proof of Lemma 3.5.3.

Lemma 3.5.4. *For some positive integer M, let X_i $i = 1 \ldots M$ be non-negative random variables. Then for all $K > 0$,*

$$\mathbb{P}\left(\sum_{i=1}^{M} X_i > K\right) \leq \sum_{i=1}^{M} \mathbb{P}\left(X_i > \frac{K}{M}\right). \tag{3.16}$$

Proof of Lemma 3.5.3. We will prove that the properties of C-tightness from Theorem 3.2.4 holds. All pathwise relations are to be interpreted almost surely. As well, throughout the proof we fix $\epsilon, \alpha, T > 0$.

To begin the proof, we first define the processes X_n, h_n such that $X_n(t) = \overline{X}_n(t) - \int_0^t p_n(\overline{W}_n(s))ds$ and $h_n(t) = \int_0^t \left(\hat{P}\prime - P\prime_n(\overline{W}_n(s))\right)d\overline{Z}_n(s)$. Note that from the uniqueness of the solution to the SP, we can say that $(\overline{W}_n, \overline{Z}_n)$ solves the SP corresponding to $(X_n + h_n, 0, I - \hat{P}\prime)$. Now let $(\tilde{W}_n, \tilde{Z}_n)$ be the solution to the SP corresponding to $(X_n, 0, I - \hat{P}\prime)$. Since $h_n \in D_{\uparrow,0}$, by applying Theorem 2.2.1 it is observed that $\overline{Z}_n \prec \tilde{Z}_n$. Furthermore from Proposition 3.35 of [30] (Chapter VI), if $\tilde{Z}_n$ is C-tight then so is $\overline{Z}_n$. The advantage of working with $\tilde{Z}_n$ is that we can apply the results of Section 2.2.1.

Now we will verify the first condition of Theorem 3.2.4. From Lemma 2.2.1 and the triangle inequality, there exists a constant $C_l > 0$ such that

$$\sup_{t\in[0,T]} \left|\tilde{Z}_n(t)\right| \le C_l \sup_{t\in[0,T]} \left|\overline{X}_n(t)\right| + C_l \sup_{t\in[0,T]} \left|\int_0^t p_n(\overline{W}_n(s))ds\right|.$$

From Lemma 3.4.1 and Prohorov's theorem, $\overline{X}_n$ is C-tight. By 3.2.4, there exists a constant K_1 and a positive integer n_1 such that for all $n \ge n_1$,

$$\mathbb{P}\left(C_l \sup_{t\in[0,T]} \left|\overline{X}_n(t)\right| > \frac{K_1}{2}\right) \le \frac{\epsilon}{2}.$$

We will now show that there exists a constant K_2 such that for all $n \ge n_1$,

$$\mathbb{P}\left(C_l \sup_{t\in[0,T]} \left|\int_0^t p_n(\overline{W}_n(s))ds\right| > \frac{K_2}{2}\right) \le \frac{\epsilon}{2}.$$

By applying the uniform linear growth property and Lemma 3.5.1 to $\left|\int_0^t p_n(\overline{W}_n(s))ds\right|$, we get that

$$\begin{aligned}
\left|\int_0^t p_n(\overline{W}_n(s))ds\right| &\le \int_0^t \left|p_n(\overline{W}_n(s))\right|ds, \\
&\le \int_0^t C_g \left(1 + \left|\overline{W}_n(s)\right|\right)ds, \\
&\le \int_0^t C_g \left(1 + \left|\hat{W}_n(s)\right|\right)ds, \\
&\le C_g t \left(1 + \sup_{s\in[0,t]} \left|\hat{W}_n(s)\right|\right).
\end{aligned}$$

Note that Lemma 2.2.1 says that there exists constant $\beta > 0$ such that

$$\sup_{t\in[0,T]} \left|\hat{W}_n(t)\right| \le \beta C_l \sup_{t\in[0,T]} \left|\overline{X}_n(t)\right|.$$

Therefore by setting a constant $K_2 = T(\beta K_1 + 2)$ and rearranging terms,

$$\mathbb{P}\left(C_l \sup_{t\in[0,T]} \left|\int_0^t p_n(\overline{W}_n(s))ds\right| > \frac{K_2}{2}\right) \leq \mathbb{P}\left(C_l \sup_{t\in[0,T]} \left|\overline{X}_n(t)\right| > \frac{K_1}{2}\right),$$
$$\leq \frac{\epsilon}{2}.$$

Setting

$$K = \frac{\max(K_1, K_2)}{2C_l}$$

and applying Lemma 3.5.4, we conclude that for all $n \geq n_1$,

$$\mathbb{P}\left(\sup_{t\in[0,T]} \left|\tilde{Z}_n(t)\right| > K\right) \leq \epsilon.$$

This verifies the first condition of Theorem 3.2.4. We will now proceed to show that the second condition of Theorem 3.2.4 is satisfied as well. First note that since $\tilde{Z}_n \in D_{\uparrow,0}$, we can write $w_T\left(\tilde{Z}_n, \theta\right) = \sup_{t\in[0,T-\theta]}\left|\tilde{Z}_n(t+\theta) - \tilde{Z}_n(t)\right|$ for any $\theta \in (0,T)$. Now fix $u \geq 0$ and let the mapping $\Psi_{I-\hat{P}'} \equiv \Psi$ be defined as in Section 2.2.1 (ie. $\tilde{Z}_n = \Psi(X_n)$). From the shift property of the regulator process (Property 4 on pg. 166 in [19]),

$$\tilde{Z}_n(u+\theta) - \tilde{Z}_n(u) = \Psi(\tilde{W}_n + X_n(u+\cdot) - X_n(u))(\theta).$$

Also from Theorem 2.2.1, $\Psi(X_n(u+\cdot) - X_n(u))(\theta) \geq \Psi(\tilde{W}_n + X_n(u+\cdot) - X_n(u))(\theta)$.

Now fix $\theta_1 \in (0,T)$. From Lemma 2.2.1, there exists a constant $C_l > 0$ such that

$$\left|\tilde{Z}_n(u+\theta_1) - \tilde{Z}_n(u)\right| \leq C_l \sup_{s\leq\theta_1} |X_n(u+s) - X_n(u)|.$$

Therefore

$$w_T\left(\tilde{Z}_n, \theta_1\right) \leq C_l \sup_{t\in[0,T-\theta_1]} \sup_{s\leq\theta_1} |X_n(t+s) - X_n(t)|.$$

From the triangle inequality,

$$w_T\left(\tilde{Z}_n, \theta_1\right) \leq w_T\left(C_l\overline{X}_n, \theta_1\right) + C_l \sup_{t\in[0,T-\theta_1]} \sup_{\theta\leq\theta_1} \left|\int_t^{t+\theta} p_n(\overline{W}_n(s))ds\right|.$$

Since $\overline{X}_n$ is C-tight, from Theorem 3.2.4 there exists a $\theta_2 \in (0,T)$ and a positive integer n_2 such that for all $n \geq n_2$, such that

$$\mathbb{P}\left(w_T\left(C_l\overline{X}_n, \theta_2\right) > \frac{\alpha}{2}\right) \leq \frac{\epsilon}{2}.$$

Moreover by applying the uniform linear growth property and Lemma 3.5.1 to

$$\left|\int_t^{t+\theta} p_n(\overline{W}_n(s))ds\right|,$$

we have that

$$C_l \sup_{t\in[0,T-\theta_1]} \sup_{\theta\leq\theta_1} \left|\int_t^{t+\theta} p_n(\overline{W}_n(s))ds\right| \leq C_l\theta_1 \sup_{t\in[0,T]} \left(C_g\left(1+\sup_{s\leq t}\left|\hat{W}_n(s)\right|\right)\right).$$

From Lemma 13.4.1 in [65], the supremum function is continuous. Therefore since $\hat{W}_n$ is tight (Theorem 3.4.1 and Prohorov's theorem), using the continuous mapping theorem we observe that $\sup_{s\in[0,\cdot]}\left|\hat{W}_n(s)\right|$ is tight as well. Again from Theorem 3.2.4, there exists a constant K_3 and a positive integer n_3 such that for all $n \geq n_3$

$$\mathbb{P}\left(\sup_{t\in[0,T]} C_lC_g(1+\sup_{s\in[0,t]}\left|\hat{W}_n(s)\right|) > K_3\right) \leq \frac{\epsilon}{2},$$

which gives us

$$\mathbb{P}\left(C_l \sup_{t\in[0,T-\theta_1]} \sup_{\theta\leq\theta_1} \left|\int_t^{t+\theta} p_n(\overline{W}_n(s))ds\right| > \theta_1 K_3\right) \leq \frac{\epsilon}{2}.$$

Since θ_1 was arbitrary, we select θ_1 from the interval $(0, \min(T, \frac{\alpha}{2K_3}))$ which gives us the desired inequality

$$\mathbb{P}\left(C_l \sup_{t\in[0,T-\theta_1]} \sup_{\theta\leq\theta_1} \left|\int_t^{t+\theta} p_n(\overline{W}_n(s))ds\right| > \frac{\alpha}{2}\right) \leq \frac{\epsilon}{2}.$$

Therefore by selecting $\theta_0 = \min(\theta_1, \theta_2)$, $n_0 = \max(n_1, n_2, n_3)$ and applying Lemma 3.5.4, we get that for all $n \geq n_0$,

$$\mathbb{P}\left(w_T\left(\tilde{Z}_n, \theta_0\right) > \alpha\right) \leq \epsilon.$$

We have satisfied the conditions from Theorem 3.2.4 to conclude that the sequence $\tilde{Z}_n$, and hence $\overline{Z}_n$, is C-tight.

$\square$

Theorem 3.5.1. *Suppose that* $\overline{W}_n(0) \Rightarrow \overline{W}(0)$. *Then the sequence* $\left(\overline{W}_n\right)_{n\in\mathbb{N}}$ *converges in distribution to* $RBM_{\overline{W}(0)}(\eta - p(w), \Gamma, I - \hat{P}')$.

Proof. From Lemma 3.4.1, $\overline{X}_n \Rightarrow BM_{\overline{W}(0)}(\eta, \Gamma)$. Therefore, the sequence $\left(\overline{X}_n\right)_{n\in\mathbb{N}}$ is C-tight from Prohorov's theorem. Since the sequence $\left(\overline{Z}_n\right)_{n\in\mathbb{N}}$ is also C-tight, using Corollary 3.33 of [30](Chapter VI) we observe that the tuple $\left(\overline{X}_n, \overline{Z}_n\right)$ is C-tight. Furthermore, since the paths of $\overline{Z}_n$ are in $D_{\uparrow,0}$, C-tightness of the sequence $\left(\overline{Z}_n\right)$ implies that the sequence is also predictably uniformly tight (See Section 6a of [30] (Chapter VI)).

Therefore from Theorem 6.9 of [30] (Chapter IX) and Prohorov's theorem, $\left(\overline{X}_n, \overline{Z}_n, \overline{W}_n\right)$ is tight. Thus, we can select a convergent subsequence $\left(\overline{X}_{n_k}, \overline{Z}_{n_k}, \overline{W}_{n_k}\right)$. Let the limit of that subsequence be $\left(BM_{\overline{W}(0)}(\eta, \Gamma), Z, W\right)$, where

$$W(t) = BM_{\overline{W}(0)}(\eta, \Gamma)(t) - \int_0^t p(W(s))ds + (I - \hat{P}')Z(t)$$

from Theorem 6.9 [30] (Chapter IX).

Since

$$\int_0^t \overline{W}_n(s)d\overline{Z}_n(s) = 0,$$

we also have

$$\int_0^t W(s)dZ(s) = 0$$

for all $t \geq 0$. Also, $W \in D_+$ and $Z \in D_{\uparrow,0}$. Therefore we have that (W, Z) solves the SP corresponding to $(BM_{\overline{W}(0)}(\eta, \Gamma), -p(w), I - \hat{P}')$. But the solution to the SP is unique, so all convergent subsequences converge to the same limit,

$$RBM_{\overline{W}(0)}(\eta - p(w), \Gamma, I - \hat{P}').$$

$\square$

As mentioned at the beginning of the section, after establishing the heavy traffic limit, the final result is quite straightforward to prove. The following theorem extends Theorems 3.4.3 and 3.4.4 to the state dependent routing model.

Theorem 3.5.2. *The sequence* $(\pi_n)_{n\in\mathbb{N}}$ *weakly converges to* π_{RBM}. *Furthermore, if*

$$\sup_n E\left[J_n(1)^{k+1}\right] < \infty$$

, then

$$E[\overline{W}_n(\infty)^p] \to E[W_{RBM}(\infty)^p] \tag{3.17}$$

for all $p \in [0, k)$.

Proof. From Lemma 3.4.2, the sequence $\left(E[\hat{W}_n(\infty)^k]\right)_{n\in\mathbb{N}}$ is bounded. It follows from Lemma 3.5.1 that the sequence $\left(E[\overline{W}_n(\infty)^k]\right)$ is bounded as well. The result is obtained by following the proofs of Theorems 3.4.2, 3.4.3 and 3.4.4. □

Chapter 4

Balanced Fairness

4.1 Introduction

File transfers compose much of the traffic of the current Internet. They typically use the transmission control protocol (TCP) and adapt their transmission rate to the available bandwidth. When congestion occurs, users experience delays, packet losses and low transfer rates. Thus it is essential to predict the probability of occurrence of such congestion periods.

A useful abstraction is to view each file transfer as a fluid elastic flow, whose rate adapts to the evolution of the number of other flows that share the same links. Under a separation of time scales assumption, the complex underlying packet-level mechanisms (e.g. congestion control algorithms, packet scheduling, buffer management) are then simply represented by some bandwidth sharing policy between ongoing flows.

In the study of flow-level models, one of the most critical concepts is that of "‘fair”’ bandwidth sharing (or allocation) between flows. For a single bottleneck link, flows are generally assumed to share the bandwidth equally, yielding the processor sharing model [29, 59, 4, 42]. This model relies on the assumption that the flows sharing the link are homogeneous. However, in practice, flows have different bandwidth requirements and constraints.

A natural approach is to treat bandwidth sharing as a utility maximization problem. A key bandwidth sharing policy is *proportional fairness*, introduced by Kelly et al. [41], which seeks to maximize a logarithmic utility function. The policy corresponds to a Nash bargaining solution [69] and can be implemented via a primal-dual mechanism, cf. [41]. Furthermore, it has been shown by Low et. al. [46] that TCP Vegas is (weighted) proportionally fair in equilibrium.

In general, analyzing the steady-state performance of a network operating under proportional fairness is quite difficult and can not be done analytically, except for simple network topologies [11, Theorem 3]. It turns out that proportional fairness can be well approximated by the slightly different notion of *balanced fairness* [10, 11, 48]. This bandwidth sharing policy has the attractive advantage of being both tractable and insensitive. Tractability means that the underlying dynamical system enters the class of Kelly-Whittle networks for which explicit analytical results are known [61, 65, 40]; insensitivity means that the stationary distribution does not depend on any flow-level traffic characteristics beyond the mean [11].

The goal of the chapter is to estimate congestion, roughly defined as a flow not being alloted its maximum bit rate, in *single link* and *parking lot* networks operating under balanced fairness. Since such calculations suffer from state space explosion, a more efficient method of computation, based on the large system scaling techniques used in loss systems [23] will be proposed.

The chapter is divided as follows: Section 4.4 introduces the flow model and provides basic results for general networks operating under balanced fairness. Section 4.3 gives an introduction to the large system scaling technique. Section 4.5 analyzes congestion for a single link operating under a balanced fair policy. The insights obtained from the single link are extended in Section 4.6 to the parking lot network topology. In Section 4.7, some numerical experiments are run to compare the approximate congestion calculations to the actual value and finally, future work is discussed in Section 5.3.

All results proven in Sections 4.5 and 4.6 are original contributions unless otherwise identified. Section 4.5 Balanced Fairness for networks with constraints on the maximum bit rates of flows has not been well studied except for the single link case and tree networks, cf. [8, 14] and references within. The main contribution in Section 4.5 is establishing that the large system scaling techniques used in loss systems can be used for this type of flow level model. The main contributions in Section 4.6 are identifying the states in which congestion occurs and establishing bounds for the congestion calculations using the insights and results of the single link case.

4.2 Assumptions and Notation

Vectors and matrices are assumed to have real-valued entries. As well, vectors will be assumed to be column vectors and will use the arrow notation, i.e. $\vec{x}$. The transpose of a vector $\vec{x}$ will be denoted by $\vec{x}'$. The notation x_i will mean the i^{th} entry of a vector $\vec{x}$. The space $\mathbb{R}^N$ will be equipped with the Euclidean metric. For $p \geq 1$, $|\cdot|_p$ will denote the standard vector and induced matrix p-norms. For simplicity we shall write $|\cdot| \equiv |\cdot|_1$. Comparisons are assumed to be component wise. As well, scalar operations on vectors are to be interpreted component wise. We denote by $\vec{e}_i$ the standard unit vector, i.e. the i^{th} component is 1 and the rest are 0. The notation $\vec{x} \cdot \vec{y}$ represents the dot or inner product of vectors $\vec{x}$ and $\vec{y}$.

For any set S, $|S|$ will mean the cardinality of the set. Also, for any functions f, g, $f(N) \sim g(N)$ means $f(N)/g(N) \to 1$ when $N \to \infty$. For any two scalars a and b, $a \wedge b$ is the minimum and $a \vee b$ is the maximum. Finally, the large system asymptotic notation used in Section 4.3.2, will be used throughout the chapter.

4.3 Background

4.3.1 Multi-rate Erlang Loss Systems

Consider a multi-rate circuit switching system consisting of C circuits which are accessed by M types of calls. Type-i calls arrive as an independent Poisson process with intensity

λ_i and request r_i circuits for an independent, exponentially distributed duration with parameter μ_i. We denote by $\beta_i = \lambda_i/\mu_i$ the corresponding traffic intensity in Erlangs.

This model is closely related to the one that will be studied in 4.4.3. The only difference is that calls are admitted in the system as long as the system state $\vec{x}$ satisfies $\vec{x} \cdot \vec{r} \leq C$ after each arrival; otherwise, the call is blocked and lost. Under elastic sharing, flows are always admitted in the system but adapt their rate to the level of congestion. We note that, in the absence of congestion, class-i flows have independent, exponentially distributed duration with parameter $\mu_i = r_i/v_i$. In particular, the normalized traffic intensity $\beta_i = \alpha_i/r_i$ that will be introduced in 4.4.3 coincides with the corresponding parameter $\beta_i = \lambda_i/\mu_i$ of the loss system.

The stationary distribution of the Markov process describing the evolution of the system state $\vec{x}$ is given by

$$\pi^B(\vec{x}) = \pi^B(\vec{0}) \prod_{i=1}^{M} \frac{\beta_i^{x_i}}{x_i!}$$

and the normalization constant will be denoted by

$$G^B = \frac{1}{\pi^B(\vec{0})} = \sum_{\vec{x}:\vec{x}\cdot\vec{r}\leq C} \prod_{i=1}^{M} \frac{\beta_i^{x_i}}{x_i!}.$$

The blocking probability of class-i calls then follows from PASTA [3]:

$$P_i^B = \sum_{\vec{x}:C-r_i<\vec{x}\cdot\vec{r}\leq C} \pi^B(\vec{x}), \tag{4.1}$$

Analysis of such a system is an extremely well studied problem. The blocking probabilities can be calculated exactly using the Kaufman-Roberts recursion [32, 60]. Unfortunately, the computation can be burdensome when dealing with large parameters, so one often resorts to asymptotic analysis.

4.3.2 Large Multi-rate Erlang Loss Systems

This section introduces the large system asymptotic (the term large system approximation will be used interchangeably). Consider a sequence of multi-rate Erlang loss models indexed by N, with arrival rates $\vec{\lambda}(N) = N\vec{\lambda}$ and $C(N) = NC$ circuits. By applying exponential centering around C and using a local limit theorem for sums of i.i.d. lattice random variables, Gazdzicki et al. [23] obtained closed-form expressions for calculating the asymptotic blocking probability in the three cases $\rho < 1$, $\rho = 1$, $\rho > 1$, where ρ denotes the system load, defined by (4.17) with $\alpha_i = \beta_i r_i$ for all $i = 1, \ldots, M$.

Theorem 4.3.1. *If $\rho < 1$, then for all $i = 1 \ldots M$,*

$$P_i^B(N) \sim \begin{cases} e^{-NI} e^{\tau d \epsilon(N)} \frac{d}{\sqrt{2\pi N}\sigma} \frac{1-e^{\tau r_i}}{1-e^{\tau d}} & \rho < 1, \\ \sqrt{\frac{2}{\pi N}} \frac{r_i}{\sigma} & \rho = 1, \\ 1 - e^{\tau r_i} & \rho > 1. \end{cases}$$

Where:

- *d is the greatest common divisor of $r_1, \ldots, r_M$,*
- $\epsilon(N) = \frac{NC}{d} - \left\lfloor \frac{NC}{d} \right\rfloor$,
- *τ is the unique solution to the equation* $\sum_{i=1}^{M} r_i \beta_i e^{\tau r_i} = C$,
- $I = C\tau - \sum_{i=1}^{M} \beta_i \left(e^{\tau r_i} - 1\right)$,
- $\sigma^2 = \sum_{i=1}^{M} r_i^2 \beta_i e^{\tau r_i}$.

4.4 Model

4.4.1 Flow-Level Model

Flow level models assume a separation of timescales such that the timescale of the packet level dynamics (e.g. the congestion control algorithms of TCP) is much smaller then the flow level dynamics (e.g. document arrivals and departures). This means that packet level details are ignored, there is no queueing or storage at the links and changes in network state are immediate (i.e. there is no delay in transmission). As well, flows are assumed to be fluid.

Consider a network as a set of links $\mathcal{L} = \{1, \ldots, K\}$, where each link $l \in \mathcal{L}$ has a finite capacity C_l bit/s. A random number of flows compete for access to these links. There are M flow classes indexed by $\mathcal{M} = \{1, \ldots, M\}$. Each class $m \in \mathcal{M}$ is uniquely identified by its route $p_m \subseteq \mathcal{L}$ and maximum bit rate r_m. Let $\mathcal{R}$ be the set of routes, let $\vec{r} = (r_1, \ldots, r_M)$ represent the maximum bit rate of the flow classes, and also let L_l be the set of flows that share link $l \in \mathcal{L}$. For convenience, $L_0 \triangleq \emptyset$. The maximum bit rate of a flow is always assumed to be less then the minimum capacity of its route, i.e. $r_m \leq \min_{l \in p_m} C_l$. The state of the network will be represented by the vector $\vec{x} = (x_1, \ldots, x_M)$, where x_i is the number of active flows of class $i \in \mathcal{M}$.

The aggregated capacity ϕ_m is the bandwidth allocated to all flows of class $m \in \mathcal{M}$. This bandwidth allocation depends only on the bandwidth sharing policy and the network state $\vec{x}$. Within a class m, the capacity ϕ_m is shared equally between flows, i.e. each flow of class i is given a bandwidth of ϕ_i/x_i. If $\vec{x} \notin \mathbb{Z}_+^M$, then $\Phi(\vec{x}) = 0$. For any state $\vec{x}$, the following link and rate constraint conditions must hold:

$$\sum_{i \in L_l} \phi_i(\vec{x}) \leq C_l \quad \forall l \in \mathcal{L}, \tag{4.2}$$

$$\phi_m(\vec{x}) \leq x_m r_m \quad \forall m \in \mathcal{M}. \tag{4.3}$$

Arrivals of class-m flows are modeled as an independent Poisson process with rate λ_m and have independent, exponentially distributed volumes with mean v_m. We refer to the product $\alpha_m = \lambda_m v_m$ as the traffic intensity of class m. The evolution of the system state $\vec{x}$ defines a Markov process $\{X(t) : t \geq 0\}$ with transition rates λ_m from state $\vec{x}$ to state $\vec{x} + \vec{e}_m$ and $\phi_m(\vec{x})/v_m$ from state $\vec{x}$ to state $\vec{x} - \vec{e}_m$, provided $x_m > 0$.

A necessary condition for stability in a flow-level model is that for all $l \in \mathcal{L}$,

$$\sum_{m \in L_l} \alpha_m \leq C_l. \tag{4.4}$$

This stability condition shall always be assumed to be satisfied throughout the chapter.

Finally, *congestion* in this thesis is defined as a class-i flow not being alloted it maximum bit rate, i.e. $\phi_m(\vec{x})/x_m < r_m$.

4.4.2 Insensitive Bandwidth Sharing Policies

A flow-level model can be also be modeled as a network of processor-sharing queues. Consider such a network where each queue corresponds to a flow class. Customers arrive at queue as a Poisson process with rate λ_i and i.i.d. exponential service requirements with mean v_i. They are served at the queue with state-dependent rate $\phi_i(\vec{x})$. Such networks are, in general, intractable unless the following balance property holds:

Definition 4.4.1 (Balance Property)**.**

$$\phi_i(\vec{x})\phi_j(\vec{x}-\vec{e}_i) = \phi_j(\vec{x})\phi_i(\vec{x}-\vec{e}_j) \quad \forall\, i, j \in \mathcal{M},\, \vec{x} : x_i, x_j > 0. \tag{4.5}$$

Processor sharing networks that satisfy the balance property are Kelly-Whittle networks [61] and the corresponding bandwidth allocation policies are labeled as being insensitive.

The balance property is equivalent to saying that the underlying Markov process X is reversible. Thus the stationary distribution π, when it exists, can be written in the product form

$$\pi(\vec{x}) = \pi(\vec{0})\Phi(\vec{x})\prod_{i=1}^{M}\alpha_i^{x_i} \quad \forall \vec{x}, \tag{4.6}$$

where the function $\Phi(\vec{x}) : \mathbb{Z}_+^M \to \mathbb{R}_+$ is known as the balance function. As such, the balance function plays a pivotal role in the analysis of balanced bandwidth sharing policies.

Let $< \vec{x}, \vec{x}-\vec{e}_{i_1}, \vec{x}-\vec{e}_{i_1}-\vec{e}_{i_2}, \ldots, 0 >$ be a direct path from state $\vec{x}$ to state 0. The balance function Φ is defined by $\Phi(0) = 1$, $\Phi(\vec{x}) = 0$ if $\vec{x} \notin \mathbb{Z}_+^M$ and

$$\Phi(\vec{x}) = \frac{1}{\phi_{i_1}(\vec{x})\phi_{i_2}(\vec{x}-e_{i_1})\cdots\phi_{i_n}(\vec{x}-e_{i_1}\cdots-e_{i_n})}$$

otherwise, where $n = |\vec{x}|$.

The balance function uniquely defines an insensitive allocation. One can recover the aggregate capacity ϕ_m of flow class $m \in \mathcal{M}$ by the relation

$$\phi_m(\vec{x}) = \frac{\Phi(\vec{x} - \vec{e}_m)}{\Phi(\vec{x})} \quad \forall\, \vec{x} \notin \mathbb{Z}_+^M. \tag{4.7}$$

The assumptions on the arrival and service requirements may seem restrictive, but are in fact being assumed for convenience. Due to the insensitivity of a Kelly-Whittle network, results in this chapter are applicable to much larger weaker assumptions. The service requirements, assumed to be exponentially distributed, can be replaced by a phase-type distribution (Chapter III, Section 4 of [1]) with the same mean. The set of phase-type distributions is dense in the set of positive-valued distributions, which means that any positive-valued distribution can be approximated by a phase-type. As well, the assumption that the arrival process of flows is Poisson, can be weakened to assuming that the arrival process of *sessions* are Poisson with the same rate. A session is composed of a succession of flows and separated by periods of inactivity referred to as *think-times*. Both the service requirements of the flows and the duration of the think-times are assumed to be phase-type. Note that no independence assumptions were made about the flows and think-times with a session. See [11] for further discussion.

The primary weakness of insensitive bandwidth sharing policies is that, unlike policies that maximize a utility function, they are not *Pareto efficient* in general. A policy is said to be Pareto efficient if one cannot increase the bandwidth allocated to one flow without reducing the bandwidth allocated to another flow. Though the lack of efficiency limits the applicability of most insensitive policies, there does exist a few that are "'efficient enough"'; the most prominent being Balanced Fairness.

4.4.3 Balanced Fairness

The balance function for Balanced Fairness is defined as

$$\Phi(\vec{x}) \triangleq \max\left(\max_{l \in \mathcal{L}} \frac{1}{C_l} \sum_{m \in L_l} \Phi(\vec{x} - \vec{e}_m), \max_{m:\, x_m > 0} \frac{\Phi(\vec{x} - \vec{e}_m)}{x_m r_m} \right). \tag{4.8}$$

A key property of Balanced fairness is that the balanced fairness is minimal.

Lemma 4.4.1 ([11, Lemma 1]). *Consider a positive function $\tilde{\Phi}$ such that $\tilde{\Phi}(0) = 1$ and the inequalities* (4.2),(4.3) *are satisfied. Then*

$$\tilde{\Phi}(\vec{x}) \geq \Phi(\vec{x}) \quad \forall \vec{x} \in \mathbb{Z}_{+}^{M}. \tag{4.9}$$

Balanced Fairness is considered the most "'efficient"' insensitive bandwidth sharing policy for several reasons. First, it is clear from the definition that a balanced fair allocation satisfies the link and rate constraints (4.2),(4.3) with at least one of the constraint inequalities being satisfied with equality. As well, if an allocation is insensitive and Pareto efficient, then that allocation coincides with one produced by Balanced Fairness.

When introducing the flow-level model, it was mentioned that the inequalities (4.4) were a necessary condition for the stationary distribution π of the underlying Markov process X to exist. In fact, under balanced fairness they are also sufficient.

Proposition 4.4.1 ([11, Theorem 2]). *The stationary distribution π exists if and only the inequalities* (4.4) *are satisfied.*

4.4.4 Congestion Metrics

As mentioned previously, flow-level congestion is defined as a flow not being alloted its maximum bit rate. For a general network topology, the states where congestion occurs can not be easily identified. Though, as will be seen in Sections 4.5 and 4.6, there exists simple and intuitive conditions to identify which states congestion occurs in the single link and parking lot networks. Once the congestion states are identified, then the steady-state congestion can be measured. In this chapter, two steady-state congestion metrics will be studied: The probability of congestion and the time-average congestion rate.

Let $\mathcal{C}_m$ be the set of states for which congestion occurs for flows of type $m \in \mathcal{M}$. The first congestion metric, the probability of congestion P_m, is defined in straightforward manner,

$$P_m \triangleq \mathbb{P}_{\pi}\left(X \in C_m\right). \tag{4.10}$$

There are two equivalent interpretations for P_m. It can be seen as the long term average the flows of class-m are congested. Alternatively, by the PASTA property, it can be seen as the steady-state probability that a flow of class-m enters a congested network.

The other congestion metric of interest, the time-average congestion rate, is a measure of the average fraction of time that an arrival does not receive its maximum bit rate during its time in the system. Let τ_m be the sojourn time of class-m arrivals in the system. Define

$$F_m \triangleq \frac{\mathrm{E}_m\left[\int_0^{\tau_m} \mathbf{1}_{\{X(t)\in\mathcal{C}_m\}}dt\right]}{\mathrm{E}_m[\tau_m]}, \tag{4.11}$$

where the expectation is taken with respect to the Palm measure for the point process of arrivals of class-m and $\vec{x}$ is the stationary state process. Then F_m denotes the ratio of the average time that a class-m flow spends in a congested state during its sojourn to the average sojourn time.

It follows respectively from the Swiss Army formula of Palm calculus, cf. [3], that

$$\mathrm{E}_\pi[X_m(0)] = \lambda_m \mathrm{E}_m[\tau_m]$$

and

$$\mathrm{E}_\pi[\mathbf{1}_{\{X(0)\in\mathcal{C}\}}X_m(0)] = \lambda_m \mathrm{E}_m\left[\int_0^{\tau_m} \mathbf{1}_{\{X(t)\in\mathcal{C}\}}dt\right].$$

Therefore

$$F_m = \frac{\displaystyle\sum_{\vec{x}\in\mathcal{C}} x_m \pi(\vec{x})}{\displaystyle\sum_{\vec{x}} x_m \pi(\vec{x})}. \tag{4.12}$$

Although the congestion metrics can be evaluated directly, the calculation is hardly feasible for high capacity links or a large number of classes. It is the goal of the chapter to give simple, tight approximations of these performance metrics for large systems. In particular, the complexity is independent of the number of classes. The approach relies on the corresponding results derived for loss systems.

4.5 Single Link

In this section, a single link operating under balanced fairness is investigated. Studying the single link case will provide invaluable insight into the parking lot network topology investigated in the next section. Since $|\mathcal{L}| = 1$, the capacity of the link will be written as $C \equiv C_1$.

The balance function Φ is recursively defined by $\Phi(\vec{0}) = 1$ and

$$\Phi(\vec{x}) = \max\left(\frac{1}{C}\sum_{m=1}^{M}\Phi(\vec{x}-\vec{e}_m),\ \max_{m:\,x_m>0}\frac{\Phi(\vec{x}-\vec{e}_m)}{x_m r_m}\right) \quad \forall\vec{x}\in\mathbb{Z}_+^M\setminus\{\vec{0}\}. \tag{4.13}$$

In fact, it can be shown [8] that for all $\vec{x}\in\mathbb{Z}_+^M$ the balance function can be simplified to

$$\Phi(\vec{x}) = \begin{cases} \displaystyle\prod_{m=1}^{M}\frac{1}{x_m!\,r_m^{x_m}} & \text{if } \vec{x}\cdot\vec{r}\le C,\\ \displaystyle\frac{1}{C}\sum_{m=1}^{M}\Phi(\vec{x}-\vec{e}_m) & \text{Otherwise.}\end{cases} \tag{4.14}$$

In particular, it follows that $\phi_m(\vec{x}) = x_m r_m$ if $\vec{x}\cdot\vec{r}\le C$, so that each flow gets its maximum bit rate in the absence of congestion; it will be shown in Lemma 4.5.1 that no flow gets its maximum bit rate in when $\vec{x}\cdot\vec{r} > C$.

As described in Section 4.4, the stationary distribution of the underlying Markov process is given by

$$\pi\left(\vec{x}\right) = \pi(\vec{0})\Phi(\vec{x})\prod_{m=1}^{M}\alpha_m^{x_m}. \tag{4.15}$$

By using (4.14), the stationary distribution can be rewritten as

$$\pi\left(\vec{x}\right) = \begin{cases} \displaystyle\pi(\vec{0})\prod_{m=1}^{M}\frac{{\beta_m}^{x_m}}{x_m!} & \text{if } \vec{x}\cdot\vec{r}\le C,\\ \displaystyle\sum_{m=1}^{M}\rho_m\pi(\vec{x}-\vec{e}_m) & \text{Otherwise.}\end{cases} \tag{4.16}$$

where $\beta_m = \alpha_m / r_m$ is the normalized traffic intensity and $\rho_m = \alpha_m / C$ is the load of class-m. The normalization constant, given by

$$G = \frac{1}{\pi(\vec{0})} = \sum_{\vec{x} \in \mathbb{Z}_+^M} \Phi(\vec{x}) \prod_{m=1}^{M} \alpha_m^{x_m},$$

is finite if and only if $\rho < 1$, where ρ denotes the aggregate link load, i.e.

$$\rho = \sum_{m=1}^{M} \rho_m = \sum_{m=1}^{M} \alpha_m / C. \tag{4.17}$$

As a reminder, the stability condition $\rho < 1$ will be assumed to be satisfied.

4.5.1 Congestion Events

It has been previously established that congestion will not occur for any class if the system state $\vec{x}$ satisfies the condition $\vec{x} \cdot \vec{r} \leq C$. The following lemma establishes that congestion will occur for all classes if $\vec{x} \cdot \vec{r} > C$, i.e. For all classes m, $\mathcal{C}_m = \{\vec{x} \in \mathbb{Z}_+^M : \vec{x} \cdot \vec{r} > C\}$.

Proposition 4.5.1. *If $\vec{x} \cdot \vec{r} > C$ then $\phi_m(\vec{x}) < x_m r_m$ for all classes $m = 1 \ldots M$ such that $x_m > 0$.*

Proof. Let m be such that $x_m > 0$. In view of (4.7), it is sufficient to show that

$$\Phi(\vec{x}) > \frac{\Phi(\vec{x} - \vec{e}_m)}{x_m r_m}.$$

The proof is split up into several cases.

First assume that $(\vec{x}-\vec{e}_m)\cdot\vec{r} > C$. If $x_m \geq 2$ then, in view of (4.13)

$$\begin{aligned}
\Phi(\vec{x}) &= \frac{1}{C}\sum_{n=1}^{M}\Phi(\vec{x}-\vec{e}_n),\\
&\geq \frac{1}{C}\left(\sum_{n\neq i}\frac{\Phi(\vec{x}-\vec{e}_m-\vec{e}_n)}{x_m r_m}+\frac{\Phi(\vec{x}-2\vec{e}_m)}{(x_m-1)r_m}\right),\\
&> \frac{1}{C}\sum_{n=1}^{M}\frac{\Phi(\vec{x}-\vec{e}_m-\vec{e}_n)}{x_m r_m},\\
&= \frac{\Phi(\vec{x}-\vec{e}_m)}{x_m r_m}.
\end{aligned}$$

Similarly, if $x_m = 1$ then

$$\begin{aligned}
\Phi(\vec{x}) &= \frac{1}{C}\sum_{n=1}^{M}\Phi(\vec{x}-\vec{e}_n),\\
&\geq \frac{1}{C}\left(\sum_{n\neq i}\frac{\Phi(\vec{x}-\vec{e}_m-\vec{e}_n)}{x_m r_m}+\Phi(\vec{x}-\vec{e}_m)\right),\\
&> \frac{1}{C}\sum_{n\neq i}\frac{\Phi(\vec{x}-\vec{e}_m-\vec{e}_n)}{x_m r_m},\\
&= \frac{\Phi(\vec{x}-\vec{e}_m)}{x_m r_m}.
\end{aligned}$$

Now assume that $(\vec{x}-\vec{e}_m)\cdot\vec{r} \leq C$. Then

$$\begin{aligned}
\Phi(x) &= \frac{1}{C}\sum_{n=1}^{M}\Phi(\vec{x}-\vec{e}_n),\\
&\geq \frac{1}{C}\left(\sum_{n\neq i}\frac{\Phi(\vec{x}-\vec{e}_m-\vec{e}_n)}{x_m r_m}+\Phi(\vec{x}-\vec{e}_m)\right),\\
&= \frac{1}{C}\left(\sum_{n\neq i}\frac{x_n r_n\Phi(\vec{x}-\vec{e}_m)}{x_m r_m}+\Phi(\vec{x}-\vec{e}_m)\right),\\
&= \frac{1}{C}\sum_{n=1}^{M}\frac{x_n r_n\Phi(\vec{x}-\vec{e}_m)}{x_m r_m},\\
&= \frac{\vec{r}\cdot\vec{x}}{C}\frac{\Phi(\vec{x}-\vec{e}_m)}{x_m r_m},\\
&> \frac{\Phi(\vec{x}-\vec{e}_m)}{x_m r_m}.
\end{aligned}$$

□

4.5.2 Probability of Congestion

Since, from Lemma 4.5.1, the states for which congestion occurs is the same for all flow classes, the probability of congestion will just be written as P. One can now write the probability of congestion as

$$P = \sum_{\vec{x}:\vec{x}\cdot\vec{r}>C}\pi\left(\vec{x}\right).$$

The following lemma due to Bonald and Virtamo [15] shows that the expressions can actually be written as a function of far fewer states. The proof is provided for the sake of completeness.

Lemma 4.5.1. *The probability of congestion can be written as*

$$P = \sum_{m=1}^{M}\frac{\rho_m B_m}{1-\rho},$$

with

$$B_m = \sum_{\vec{x}: C - r_m < \vec{x}\cdot\vec{r} \le C} \pi\left(\vec{x}\right).$$

Proof. In view of (4.16),

$$\begin{aligned}
P &= \sum_{\vec{x}:\vec{x}\cdot\vec{r}>C} \pi\left(\vec{x}\right), \\
&= \sum_{\vec{x}:\vec{x}\cdot\vec{r}>C} \sum_{m=1}^{M} \rho_m \pi(\vec{x} - \vec{e}_m), \\
&= \sum_{m=1}^{M} \rho_m \left(\sum_{\vec{x}:\vec{x}\cdot\vec{r}>C} \pi\left(\vec{x}\right) + \sum_{\vec{x}:C-r_m<\vec{x}\cdot\vec{r}\le C} \pi\left(\vec{x}\right) \right), \\
&= \sum_{m=1}^{M} \rho_m (P + B_m).
\end{aligned}$$

Hence

$$P = \sum_{m=1}^{M} \frac{\rho_m B_m}{1-\rho}.$$

□

Noting that the stationary distributions π and π^B are proportional on those states $\vec{x}$ such that $\vec{x}\cdot\vec{r} \le C$, it follows from (4.1) that

$$B_m = \frac{G^B}{G} P_m^B.$$

Thus the probability of congestion is:

$$P = \frac{G^B}{G} \sum_{m=1}^{M} \frac{\rho_m P_m^B}{1-\rho}.$$

In view of Theorem 4.3.1, a tight approximation under large system scaling using the blocking probabilities P_m^B can now be established. It remains to calculate the normalization constant, which can be unwieldy. As will be shown in the next two lemmas $G^B(N) \approx G(N)$ for large N, where $G^B(N)$ and $G(N)$ denote the normalization constants of the loss and the flow-level models respectively.

Lemma 4.5.2. *Let $\vec{x}_N$ be an M-dimensional random vector with mutually independent Poisson components with respective parameters $N\beta_1, \ldots, N\beta_M$. Then for any constant $K \in [0, C]$:*

$$\mathbb{P}\left(\vec{x}_N \cdot \vec{r} \geq NC - K\right) \to 0 \quad \textit{when } N \to \infty.$$

Proof. Let

$$F(a, h, N) = ha - N \sum_{m=1}^{M} \beta_m (e^{hr_m} - 1)$$

and

$$I\left(a, N\right) = \sup_{h \geq 0} F(a, h, N).$$

Since $F(NC - K, h, N)$ is concave with respect to h, there exists a unique maximum h_N. Looking at the first order conditions,

$$NC - K - N \sum_{m=1}^{M} \beta_m r_m e^{h_N r_m} = 0.$$

In particular,

$$\sum_{m=1}^{M} \beta_m r_m e^{h_N r_m} \to C \quad \text{when } N \to \infty.$$

Since

$$\rho = \sum_{n=1}^{M} \frac{\beta_m r_m}{C} < 1,$$

this implies $h_N > 0$ for sufficiently large N, say $N \geq N_0$. Therefore, $I\left(NC - K, N\right) > 0$ for all $N \geq N_0$.

Now, by the Chernoff bound,

$$\mathbb{P}\left(\vec{x}_N \cdot \vec{r} \geq NC - K\right) \leq e^{-I(NC - K, N)}.$$

The result then follows from the fact that

$$I\left(NC - K, N\right) \geq \frac{N}{N_0} I\left(N_0 C - K, N_0\right),$$

which grows to infinity when $N \to \infty$. □

Lemma 4.5.3.

$$\frac{G^B(N)}{G(N)} \to 1 \quad \textit{when } N \to \infty. \tag{4.18}$$

Proof. Let $\beta = \sum_{m=1}^{M} \beta_m$ and denote by $\vec{x}_N$ an M-dimensional random vector with mutually independent Poisson components with respective parameters $N\beta_1, \ldots, N\beta_M$.

$$\begin{aligned} G^B(N)e^{-N\beta} &= \sum_{\vec{x}:\vec{x}\cdot\vec{r}\le NC} \prod_{m=1}^{M} e^{-N\beta_m} \frac{(N\beta_m)^{x_m}}{x_m!}, \\ &= \mathbb{P}\left(\vec{x}_N \cdot \vec{r} \le NC\right), \\ &= 1 - \mathbb{P}\left(\vec{x}_N \cdot \vec{r} > NC\right). \end{aligned}$$

In view of Lemma 4.5.2,

$$G^B(N)e^{-N\beta} \to 1 \quad \text{when } N \to \infty.$$

Now let

$$P'(N) = \sum_{\vec{x}:\vec{x}\cdot\vec{r}>NC} \Phi_N(\vec{x}) \prod_{m=1}^{M} (N\alpha_m)^{x_m},$$

and for all $m = 1, \ldots, M$,

$$B'_m(N) = \sum_{\vec{x}:NC-r_m<\vec{x}\cdot\vec{r}\le NC} \Phi_N(\vec{x}) \prod_{m=1}^{M} (N\alpha_m)^{x_m}.$$

Note that $P'(N)$ and $B'_m(N)$ are the respective unnormalized versions of $P(N)$ and $B_m(N)$. In particular, it follows from Lemma 4.5.1 that

$$P'(N) = \sum_{m=1}^{M} \frac{\rho_m B'_m(N)}{1-\rho}.$$

Moreover, for all $m = 1, \ldots, M$,

$$
\begin{aligned}
B'_m(N)e^{-N\beta} &= \sum_{\vec{x}: NC - r_m < \vec{x}\cdot\vec{r} \le NC} \prod_{n=1}^{M} e^{-N\beta_n} \frac{(N\beta_n)^{x_n}}{x_n!}, \\
&= P\left(NC - r_m < \vec{x}_N \cdot \vec{r} \le NC\right), \\
&\le P\left(\vec{x}_N \cdot \vec{r} > NC - r_m\right).
\end{aligned}
$$

In view of Lemma 4.5.2,

$$
\forall m = 1, \ldots, M, \quad B'_m(N)e^{-N\beta} \to 0 \quad \text{when } N \to \infty,
$$

so that

$$
P'(N)e^{-N\beta} \to 0 \quad \text{when } N \to \infty.
$$

Noting that $G(N) = G^B(N) + P'(N)$, it is concluded that

$$
G(N)e^{-N\beta} \to 1 \quad \text{when } N \to \infty,
$$

and

$$
\frac{G^B(N)}{G(N)} = \frac{G^B(N)e^{-N\beta}}{G(N)e^{-N\beta}} \to 1.
$$

□

By combining the previous results, one arrives at the following conclusion.

Theorem 4.5.1. *Under the large system scaling,*

$$
P(N) \sim \sum_{m=1}^{M} \frac{\rho_m P_m^B(N)}{1-\rho}, \tag{4.19}
$$

where

$$
P_m^B(N) \sim e^{-NI} e^{\tau d \epsilon(N)} \frac{d}{\sqrt{2\pi N}\sigma} \frac{1 - e^{\tau r_m}}{1 - e^{\tau d}}
$$

- *d is the greatest common divisor of $r_1, \ldots, r_M$,*
- $\epsilon(N) = \frac{NC}{d} - \left\lfloor \frac{NC}{d} \right\rfloor$,

- *τ is the unique solution to the equation* $\sum_{m=1}^{M} r_m \beta_m e^{\tau r_m} = C$,
- $I = C\tau - \sum_{m=1}^{M} \beta_m \left(e^{\tau r_m} - 1\right)$,
- $\sigma^2 = \sum_{m=1}^{M} r_m^2 \beta_m e^{\tau r_m}$.

4.5.3 Time-Average Congestion Rates

Finally, the large system scaling will be applied to the time-average congestion rates (4.12). The following lemma due to Bonald and Virtamo [15] shows that the corresponding sums can be written as a function of far fewer states. Again, the proof is provided for completeness.

Lemma 4.5.4. *For all $m, n = 1, \ldots, M$, let*

$$Q_{m,n} = \sum_{\vec{x}: C - r_n < \vec{x}\cdot\vec{r} \le C} x_m \pi\left(\vec{x}\right),$$

and

$$Q_m = \sum_{\vec{x}: \vec{x}\cdot\vec{r} > C} x_m \pi\left(\vec{x}\right).$$

Then

$$Q_m = \frac{\rho_m P_m}{1-\rho} + \sum_{n=1}^{M} \frac{\rho_n Q_{m,n}}{1-\rho}.$$

Proof. Therefore,

$$
\begin{aligned}
Q_m &= \sum_{\vec{x}:\vec{x}\cdot\vec{r}>C} x_m \pi\,(\vec{x}),\\
&= \sum_{\vec{x}:\vec{x}\cdot\vec{r}>C} x_m \sum_{n=1}^{M} \rho_n \pi(\vec{x}-\vec{e}_n),\\
&= \sum_{n=1}^{M} \rho_n \sum_{\vec{x}:\vec{x}\cdot\vec{r}>C} x_m \pi(\vec{x}-\vec{e}_n),\\
&= \sum_{n=1}^{M} \rho_n \sum_{\vec{x}:\vec{x}\cdot\vec{r}>C-r_n} (x_m + 1_{\{n=i\}})\pi\,(\vec{x}),\\
&= \rho_m P_m + \sum_{n=1}^{M} \rho_n (Q_m + Q_{m,n}),
\end{aligned}
$$

from which the result follows. □

Now let $P^B_{m,n}$ be the class-n blocking probability in a multirate loss system with capacity $C - r_m$. Then:

Proposition 4.5.2. *Under large system scaling,*

$$P^B_{m,n}(N) \sim e^{-NI_m} e^{\tau_m d \epsilon_m(N)} \frac{d}{\sqrt{2\pi N}\sigma_m} \frac{1-e^{\tau_m r_n}}{1-e^{\tau_m d}},$$

where

- *d is the greatest common divisor of $r_1, \ldots, r_M$,*
- $\epsilon_m(N) = \frac{NC-r_m}{d} - \left\lfloor \frac{NC-r_m}{d} \right\rfloor$,
- *τ is the unique solution to the equation* $\sum_{n=1}^{M} r_n \beta_n e^{\tau r_n} = C$,
- $\sigma^2 = \sum_{n=1}^{M} r_n^2 \beta_n e^{\tau r_n}$,

- $\tau_m = \tau - \frac{r_m}{N\sigma^2}$,
- $I_m = \left(C - \frac{r_m}{N}\right)\tau_m - \sum_{n=1}^{M} \beta_n \left(e^{\tau_m r_n} - 1\right)$,
- $\sigma_m^2 = \sum_{n=1}^{M} r_m^2 \beta_n e^{\tau_m r_n}$.

Proof. In view of Theorem 4.3.1, it is sufficient to observe that the solution τ_m to the equation:

$$\sum_{n=1}^{M} r_n \beta_n e^{\tau_m r_n} = C - \frac{r_m}{N}$$

satisfies:

$$\tau_m = \tau - \frac{r_m}{N\sigma^2} + \left(\frac{1}{N}\right).$$

□

The following result, together with Theorem 4.5.1 and Proposition 4.5.2, provides the large system asymptotics of the time-average congestion rates.

Theorem 4.5.2. *Under large system scaling, for all $m = 1, \ldots, M$:*

$$F_m(N) \sim \frac{r_m}{NC(1-\rho)} P_m(N) + \sum_{n=1}^{M} \frac{\rho_n}{1-\rho} P_{m,n}^{B}(N).$$

Proof. First write F_m as

$$F_m = \frac{Q_m}{Q_m + S_m},$$

with

$$S_m = \sum_{\vec{x}:\vec{x}\cdot\vec{r}\le C} x_m \pi\left(\vec{x}\right).$$

In view of (4.16), for all states $\vec{x}$ such that $\vec{x}\cdot\vec{r} \le C$:

$$x_m \pi\left(\vec{x}\right) = \beta_m \pi(\vec{x} - \vec{e}_m).$$

In particular,

$$S_m = \beta_m \sum_{\vec{x}:\vec{x}\cdot\vec{r}\le C-r_m} \pi(\vec{x}),$$
$$= \beta_m \frac{G^B}{G}(1 - P_m^B).$$

In view of Theorem 4.3.1 and Lemma 4.5.3,

$$S_m(N) \sim N\beta_m.$$

Similarly, for all $n = 1, \ldots, M$:

$$Q_{m,n} = \beta_m \sum_{\vec{x}:C-r_m-r_n<\vec{x}\cdot\vec{r}\le C-r_m} \pi(\vec{x})$$
$$= \beta_m \frac{G^B}{G} P_{m,n}^B,$$

so that under large system scaling:

$$Q_{m,n}(N) \sim N\beta_m P_{m,n}^B(N).$$

By Lemma 4.5.4,

$$Q_m(N) \sim \frac{\rho_m P_m(N)}{1-\rho} + \sum_{n=1}^{M} N\beta_m \frac{\rho_n P_{m,n}^B(N)}{1-\rho}.$$

The proof then follows from the fact that:

$$Q_m(N) + S_m(N) \sim N\beta_m$$

and

$$\frac{\rho_m}{N\beta_m} = \frac{r_m}{NC}.$$

□

4.6 Parking Lot Network

In this section, the results of Section 4.5 will be shown to extend to the so-called *parking lot* network. An example of a parking lot network is given in Figure 4.1 courtesy of Bonald et al. [13]. The sets $(L_l)_{l\in\mathcal{L}}$ have a special structure, that is $L_l \subset L_{l+1}$. Recall the definition of the balance function, i.e.

$$\Phi(\vec{x}) = \max\left(\max_{l\in\mathcal{L}} \frac{1}{C_l}\sum_{m\in L_l}\Phi(\vec{x}-\vec{e}_m),\ \max_{m:\, x_m>0}\frac{\Phi(\vec{x}-\vec{e}_m)}{x_m r_m}\right). \tag{4.20}$$

Note that if there exists an $l \in \mathcal{L}$ such that $C_l > C_{l+1}$, then

$$\frac{1}{C_l}\sum_{m\in L_l}\Phi(\vec{x}-\vec{e}_m) < \frac{1}{C_{l+1}}\sum_{m\in L_{l+1}}\Phi(\vec{x}-\vec{e}_m) \quad \forall\vec{x},$$

which of course implies that link l is inconsequential in the maximization. Indeed if $C_l > C_{l+1}$, then the total allocation to all flows can never be greater then C_{l+1} and thus will always be less than C_l. So it shall always be assumed that $C_l \leq C_{l+1}$. As well, the balance function will often be indexed by the number of links, e.g. a network with n nodes will have balance function Φ_n.

Due the recursive nature of the network topology, notationally it will be more convenient to deal with sequences instead of vectors. So let $(r_m)_{m\in\mathbb{N}}$, $(C_n)_{n\in\mathbb{Z}_+}$, and $(L_n)_{n\in\mathbb{Z}_+}$ be the sequence analogues of the maximum bit rates, capacities, and index set of flows. For $n \geq 1$, Define $R_n \triangleq L_n \setminus L_{n-1}$ with $R_1 \triangleq L_1$. To avoid degeneracy, it will be assumed that $R_n \neq \emptyset$ for all n. Also for convenience, $C_0 = 0$ and $L_0 = \emptyset$.

4.6.1 Balanced Function

As was seen in Section 4.5, the particular form of the balance function (4.14) played a critical role in establishing the results. We will show that a similar form of the balance function exists for parking lot networks. Corollary 4.6.1, and therefore Proposition 4.6.1, can be established immediately from the Pareto efficiency of tree networks [14], for which

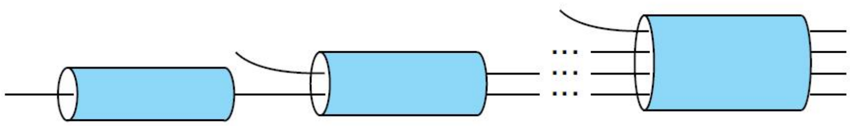

Figure 4.1: A Parking Lot Network

the parking lot network is a special case. The proofs in this section, which apply to the parking lot network only, have been included for completeness and additional insight into the behaviour of the balance function.

To begin, for any index set I_n, such as R_n, L_n, etc., define

$$\overline{\vec{x}}_{I_n} \triangleq \sum_{m \in I_n} x_m r_m.$$

For each n, and all states $\vec{x}$, the functions $\gamma_n : \mathbb{Z}_+^{|L_n|} \to \mathbb{Z}_+$, $y_n : \mathbb{Z}_+^{|L_n|} \to \mathbb{Z}_+$ and $\tilde{\Phi}_n : \mathbb{Z}_+^{|L_n|} \to \mathbb{Z}_+$ are defined recursively by:

$$\gamma_n(\vec{x}) \triangleq \prod_{m \in R_n} \frac{1}{r_m^{x_m} x_m!}, \tag{4.21}$$

$$y_n(\vec{x}) \triangleq \begin{cases} 1 & n = 0, \\ \overline{\vec{x}}_{R_n} + y_{n-1}(\vec{x}_{L_{n-1}}) \wedge C_{n-1} & \text{otherwise}, \end{cases} \tag{4.22}$$

$$\tilde{\Phi}_n(\vec{x}) \triangleq \begin{cases} 1 & n = 0 \text{ or } \vec{x} = \vec{0}, \\ \gamma_n(\vec{x}) \tilde{\Phi}_{n-1}(\vec{x}_{L_{n-1}}) & y_n(\vec{x}) \leq C_n, \\ \dfrac{1}{C_n} \displaystyle\sum_{m \in L_n} \tilde{\Phi}_n(\vec{x} - \vec{e}_m) & \text{otherwise}. \end{cases} \tag{4.23}$$

If $\vec{x}$ has a negative component or $n < 0$, then it will be understood that

$$\gamma_n(\vec{x}), y_n(\vec{x}), \tilde{\Phi}_n(\vec{x}) \triangleq 0.$$

To reduce the notational clutter, when the state $\vec{x}$ is unambiguous it will be dropped, e.g. $\gamma_n(\vec{x}) \equiv \gamma_n$. As well, explicit reference to the sub-vector will also be dropped, e.g. $y_{n-1}(\vec{x}_{L_{n-1}}) \equiv y_{n-1}(\vec{x}) \equiv y_{n-1}$ where the final equivalence will be used when the state vector $\vec{x}$ is unambiguous.

For any fixed n, $\tilde{\Phi}_n$ is a balance function. In fact, $\tilde{\Phi}_1$ coincides with the second form of the single link balance function (4.14). The goal of this section will be to show that for any n, $\tilde{\Phi}_n$ is the balance function for parking lot network with n links. The function y has a physical meaning as well. As the next lemma will show, $C_n \wedge y_n$ can be interpreted as the aggregate bandwidth allocation of all flows.

Lemma 4.6.1. *For all states $\vec{x}$ and $n \in \mathbb{N}$,*

$$\sum_{m \in L_n} \tilde{\Phi}_n(x - \vec{e}_m) = (C_n \wedge y_n)\, \tilde{\Phi}_n(\vec{x}). \tag{4.24}$$

Proof. **Base case:** If $n = 1$, note that $y_1 = \overline{\vec{x}}_{R_1}$. So if $y_1 > C_1$, then by (4.23)

$$\frac{1}{C_1} \sum_{m \in L_1} \tilde{\Phi}_1(\vec{x} - \vec{e}_m) = \tilde{\Phi}_1(\vec{x}).$$

So therefore

$$\sum_{m \in L_1} \tilde{\Phi}_1(\vec{x} - \vec{e}_m) = C_1 \tilde{\Phi}_1(\vec{x}) = (C_1 \wedge y_1)\, \tilde{\Phi}_1(\vec{x}).$$

Otherwise $y_1 \leq C_1$, for which (4.23) implies that

$$\sum_{m \in L_1} \tilde{\Phi}_1(\vec{x} - \vec{e}_m) = \overline{\vec{x}}_{R_1} \tilde{\Phi}_1(\vec{x}) = (C_1 \wedge y_1)\, \tilde{\Phi}_1(\vec{x}).$$

Induction step: Assume that

$$\sum_{m \in L_{n-1}} \tilde{\Phi}_{n-1}(\vec{x} - \vec{e}_m) = (C_{n-1} \wedge y_{n-1})\, \tilde{\Phi}_(\vec{x}) \quad \forall \vec{x}.$$

If $y_n > C_n$, then by definition (4.23),

$$\sum_{m \in L_n} \tilde{\Phi}_n(x - \vec{e}_m) = C_n \tilde{\Phi}_n(\vec{x}) = (C_n \wedge y_n)\, \tilde{\Phi}_n(x).$$

Otherwise if $y_n \leq C_n$,

$$\begin{aligned}
\sum_{m \in L_n} \tilde{\Phi}_n(\vec{x} - \vec{e}_m) &= \sum_{m \in R_n} \tilde{\Phi}_n(\vec{x} - \vec{e}_m) + \sum_{m \in L_{n-1}} \tilde{\Phi}_n(\vec{x} - \vec{e}_m) \\
&= \overline{x}_{R_1} \gamma_n \tilde{\Phi}_{n-1}(\vec{x}) + \gamma_n \sum_{m \in L_{n-1}} \tilde{\Phi}_{n-1}(\vec{x} - \vec{e}_m), \\
&= \overline{x}_{R_1} \gamma_n \tilde{\Phi}_{n-1}(\vec{x}) + (C_{n-1} \wedge y_{n-1})\, \gamma_n \tilde{\Phi}_{n-1}(\vec{x}), \\
&= y_n \gamma_n \tilde{\Phi}_{n-1}(\vec{x}), \\
&= y_n \tilde{\Phi}_n(\vec{x}), \\
&= (C_n \wedge y_n)\, \tilde{\Phi}_n(\vec{x}).
\end{aligned}$$

□

The next proposition establishes that for any fixed n, $\tilde{\Phi}_n(\vec{x})$ satisfies the link constraints (4.2) and rate constraints (4.3).

Proposition 4.6.1. *For any state $\vec{x}$ and non-negative integer n,*

$$\tilde{\Phi}_n(\vec{x}) \geq \frac{1}{C_l} \sum_{i \in L_l} \tilde{\Phi}_n(\vec{x} - \vec{e}_i), \tag{4.25}$$

$$\tilde{\Phi}_n(\vec{x}) \geq \frac{\tilde{\Phi}_n(\vec{x} - \vec{e}_m)}{x_m r_m}$$

for all $l = 1, \ldots, n$ and $m \in \mathbb{N}$ such that $x_m > 0$.

Proof. **Base case:** Since $\tilde{\Phi}_1(\vec{x})$ coincides with the balance function for a single link operating under balanced fairness, the inequalities (4.25) are automatically satisfied for all states $\vec{x}$ and $n = 1$. Also, note that since $\tilde{\Phi}_n(\vec{0}) = 1$, the inequalities are satisfied for all n and $|\vec{x}| = 0$ as well.

Induction step: Assume that for all $\vec{x}$ and $n_0 = 1, \ldots, n-1$, $\tilde{\Phi}_{n_0}$ satisfies the inequalities (4.25). Fix some positive integer k, and assume that $\tilde{\Phi}_n$ satisfies the inequalities (4.25) for all $\vec{x}$ such that $|\vec{x}| \leq k-1$.

Fix $\vec{x}$ such that $|\vec{x}| = k$ and $k_0 \in \{1, \ldots, n-1\}$. The remainder of the proof is split into two cases: $y_n \leq C_n$ and $y_n > C_n$.

First suppose that $y_n \leq C_n$. The inequality

$$\tilde{\Phi}_n(\vec{x}) \geq \frac{1}{C_n} \sum_{m \in L_n} \tilde{\Phi}_n(\vec{x} - \vec{e}_m),$$

is satisfied via Lemma 4.6.1. As well, by the induction assumption,

$$\tilde{\Phi}_{n-1}(\vec{x}) \geq \frac{1}{C_{n-k_0}} \sum_{m \in L_{n-k_0}} \tilde{\Phi}_{n-1}(\vec{x} - \vec{e}_m).$$

Therefore,

$$\begin{aligned}
\tilde{\Phi}_n(\vec{x}) &= \gamma_n \tilde{\Phi}_{n-1}(\vec{x}), \\
&\geq \gamma_n \frac{1}{C_{n-k_0}} \sum_{m \in L_{n-k_0}} \tilde{\Phi}_{n-1}(\vec{x} - \vec{e}_m), \\
&= \frac{1}{C_{n-k_0}} \sum_{m \in L_{n-k_0}} \tilde{\Phi}_n(\vec{x} - \vec{e}_m).
\end{aligned}$$

Now, for any m such that $x_m > 0$,

$$\tilde{\Phi}_{n-1}(x) \geq \frac{\tilde{\Phi}_{n-1}(x - \vec{e}_m)}{x_m r_m},$$

by the inductive assumption. So

$$\begin{aligned}
\tilde{\Phi}_n(\vec{x}) &= \gamma_n \tilde{\Phi}_{n-1}(\vec{x}), \\
&\geq \gamma_n \frac{\tilde{\Phi}_{n-1}(\vec{x} - \vec{e}_m)}{x_m r_m}, \\
&= \frac{\tilde{\Phi}_n(\vec{x} - \vec{e}_m)}{x_m r_m}.
\end{aligned}$$

Now suppose $y_n > C_n$. By (4.23),

$$\tilde{\Phi}_n(\vec{x}) = \frac{1}{C_n} \sum_{m \in L_n} \tilde{\Phi}_n(x - \vec{e}_m).$$

To show

$$\frac{1}{C_n} \sum_{m \in L_n} \tilde{\Phi}_n(x - \vec{e}_m) \geq \frac{1}{C_{n-k_0}} \sum_{m \in L_{n-k_0}} \tilde{\Phi}_n(\vec{x} - \vec{e}_m), \tag{1}$$

the strategy will be to expand the sums, eliminate common terms, and then show that the difference is non-negative.

The expansion of the terms will center around the sets

$$A_n \triangleq \{m \in L_n : y_n(\vec{x} - \vec{e}_m) \leq C_n\}.$$

Note that if $m \in A_n$ and $m \in R_k$ for some $k \in \{1, \dots n\}$, then $y_{k'}(\vec{x} - \vec{e}_m) \leq C_{k'}$ for all $k' \in \{k, \dots, n\}$.

To begin, the left hand side of (1) is split into two summations,

$$\frac{1}{C_n} \sum_{m \in L_n} \tilde{\Phi}_n(\vec{x} - \vec{e}_m) = \frac{1}{C_n} \sum_{m \in A_n} \tilde{\Phi}_n(\vec{x} - \vec{e}_m) + \frac{1}{C_n} \sum_{m \in A_n^c} \tilde{\Phi}_n(\vec{x} - \vec{e}_m). \tag{LHS}$$

The first summation can be further decomposed into

$$\begin{aligned}
&\frac{1}{C_n} \sum_{m \in A_n} \tilde{\Phi}_n(\vec{x} - \vec{e}_m) \qquad \text{(LHS } A_n\text{)} \\
&= \frac{1}{C_n} \left[\gamma_n \overline{\overline{x}}_{R_n \cap A_n} \tilde{\Phi}_{n-1}(\vec{x}) + \gamma_n \sum_{m \in A_n} \tilde{\Phi}_{n-1}(\vec{x} - \vec{e}_m) \right], \\
&\vdots, \\
&= \frac{1}{C_n} \left[\sum_{i=0}^{k_0 - 1} \left(\prod_{j=0}^{i} \gamma_{n-j} \right) \overline{\overline{x}}_{R_{n-m} \cap A_n} \tilde{\Phi}_{n-1-m}(\vec{x}) \right. \\
&\quad \left. + \left(\prod_{j=0}^{k_0 - 1} \gamma_{n-j} \right) \sum_{m \in L_{n-k_0} \cap A_n} \tilde{\Phi}_{n-k_0}(\vec{x} - \vec{e}_m) \right].
\end{aligned}$$

For the second summation, by applying the inductive assumption to each term,

$$\frac{1}{C_n}\sum_{m\in A_n^c}\tilde{\Phi}_n(\vec{x}-\vec{e}_m) \geq \frac{1}{C_n}\sum_{m\in A_n^c}\frac{1}{C_{n-k_0}}\sum_{j\in L_{n-k_0}}\tilde{\Phi}_n(\vec{x}-\vec{e}_m-\vec{e}_j). \qquad \text{(LHS } A_n^c\text{)}$$

The right hand side of (1) is split and expanded similarly:

$$\begin{aligned}&\frac{1}{C_{n-k_0}}\sum_{m\in L_{n-k_0}}\tilde{\Phi}_n(\vec{x}-\vec{e}_m) \qquad \text{(RHS)}\\ &\quad=\frac{1}{C_{n-k_0}}\sum_{m\in L_{n-k_0}\cap A_n}\tilde{\Phi}_n(\vec{x}-\vec{e}_m)+\frac{1}{C_{n-k_0}}\sum_{m\in L_{n-k_0}\cap A_n^c}\tilde{\Phi}_n(\vec{x}-\vec{e}_m),\end{aligned}$$

$$\frac{1}{C_{n-k_0}}\sum_{m\in L_{n-k_0}\cap A_n}\tilde{\Phi}_n(\vec{x}-\vec{e}_m)=\frac{1}{C_{n-k_0}}\left(\prod_{j=0}^{k_0-1}\gamma_{n-j}\right)\sum_{m\in L_{n-k_0}\cap A_n}\tilde{\Phi}_{n-k_0}(\vec{x}-\vec{e}_m), \quad \text{(RHS } A_n\text{)}$$

and

$$\frac{1}{C_{n-k_0}}\sum_{m\in L_{n-k_0}\cap A_n^c}\tilde{\Phi}_n(\vec{x}-\vec{e}_m)=\frac{1}{C_{n-k_0}}\sum_{m\in L_{n-k_0}\cap A_n^c}\frac{1}{C_n}\sum_{j\in L_n}\tilde{\Phi}_n(\vec{x}-\vec{e}_m-\vec{e}_j). \quad \text{(RHS } A_n^c\text{)}$$

Taking the difference between (LHS A_n^c) and (RHS A_n^c) yields

$$\begin{aligned}&\frac{1}{C_n}\sum_{m\in A_n^c}\tilde{\Phi}_n(\vec{x}-\vec{e}_m)-\frac{1}{C_{n-k_0}}\sum_{m\in L_{n-k_0}\cap A_n^c}\tilde{\Phi}_n(\vec{x}-\vec{e}_m)\\ &\geq \frac{1}{C_n}\sum_{m\in A_n^c}\frac{1}{C_{n-k_0}}\sum_{j\in L_{n-k_0}}\tilde{\Phi}_n(\vec{x}-\vec{e}_m-\vec{e}_j)\\ &\quad-\frac{1}{C_{n-k_0}}\sum_{m\in L_{n-k_0}\cap A_n^c}\frac{1}{C_n}\sum_{j\in L_n}\tilde{\Phi}_n(\vec{x}-\vec{e}_m-\vec{e}_j),\\ &=\frac{1}{C_n}\frac{1}{C_{n-k_0}}\left[\sum_{m\in A_n^c}\sum_{j\in L_{n-k_0}\cap A_n}\tilde{\Phi}_n(\vec{x}-\vec{e}_m-\vec{e}_j)\right.\\ &\quad\left.-\sum_{m\in L_{n-k_0}\cap A_n^c}\sum_{j\in A_n}\tilde{\Phi}_n(\vec{x}-\vec{e}_m-\vec{e}_j)\right].\end{aligned}$$

To find common terms, both double summations in the previous equation need to be analyzed further. Expanding the first double summation gives

$$\begin{aligned}
&\sum_{m\in A_n^c}\sum_{j\in L_{n-k_0}\cap A_n}\tilde{\Phi}_n(\vec{x}-\vec{e}_m-\vec{e}_j)\\
&=\sum_{l=0}^{k_0-1}\sum_{m\in L_{n-l}\cap A_n^c}\sum_{j\in L_{n-k_0}\cap A_n}\tilde{\Phi}_n(\vec{x}-\vec{e}_m-\vec{e}_j)\\
&\quad+\sum_{m\in L_{n-k_0}\cap A_n^c}\sum_{j\in L_{n-k_0}\cap A_n}\tilde{\Phi}_n(\vec{x}-\vec{e}_m-\vec{e}_j),\\
&=\left(\prod_{j=0}^{k_0-1}\gamma_{n-j}\right)\sum_{l=0}^{k_0-1}\overline{\vec{x}}_{R_{n-l}\cap A_n^c}\sum_{j\in L_{n-k_0}\cap A_n}\tilde{\Phi}_{n-k_0}(\vec{x}-\vec{e}_j)\\
&\quad+\sum_{m\in L_{n-k_0}\cap A_n^c}\sum_{j\in L_{n-k_0}\cap A_n}\tilde{\Phi}_n(\vec{x}-\vec{e}_m-\vec{e}_j).
\end{aligned}$$

Similarly expanding the second double summation,

$$\begin{aligned}
&\sum_{m\in L_{n-k_0}\cap A_n^c}\sum_{j\in A_n}\tilde{\Phi}_n(\vec{x}-\vec{e}_m-\vec{e}_j)\\
&=\sum_{l=0}^{k_0-1}\sum_{m\in L_{n-l}\cap A_n}\sum_{j\in L_{n-k_0}\cap A_n^c}\tilde{\Phi}_n(\vec{x}-\vec{e}_m-\vec{e}_j)\\
&\quad+\sum_{m\in L_{n-k_0}\cap A_n}\sum_{j\in L_{n-k_0}\cap A_n^c}\tilde{\Phi}_n(\vec{x}-\vec{e}_m-\vec{e}_j),\\
&=\sum_{l=0}^{k_0-1}\left(\prod_{j=0}^{l}\gamma_{n-j}\right)\overline{\vec{x}}_{R_{n-l}\cap A_n}\sum_{j\in L_{n-k_0}\cap A_n^c}\tilde{\Phi}_{n-1-l}(\vec{x}-\vec{e}_j)\\
&\quad+\sum_{m\in L_{n-k_0}\cap A_n}\sum_{j\in L_{n-k_0}\cap A_n^c}\tilde{\Phi}_n(\vec{x}-\vec{e}_m-\vec{e}_j).
\end{aligned}$$

Note the similarities between the final line of the previous equation and (LHS A_n). Taking the difference between (LHS) and (RHS A_n^c) yields the inequality

$$
\begin{aligned}
&\frac{1}{C_n}\sum_{m\in L_n}\tilde{\Phi}_n(\vec{x}-\vec{e}_m)-\frac{1}{C_{n-k_0}}\sum_{m\in L_{n-k_0}\cap A_n^c}\tilde{\Phi}_n(\vec{x}-\vec{e}_m)\\
&\geq \frac{1}{C_n}\left[\sum_{l=0}^{k_0-1}\left(\prod_{j=0}^{l}\gamma_{n-j}\right)\overline{\vec{x}}_{R_{n-l}\cap A_n}\tilde{\Phi}_{n-1-l}(\vec{x})\right.\\
&\quad\left.+\left(\prod_{j=0}^{k_0-1}\gamma_{n-j}\right)\sum_{m\in L_{n-k_0}\cap A_n}\tilde{\Phi}_{n-k_0}(\vec{x}-\vec{e}_m)\right]\\
&\quad+\frac{1}{C_n}\frac{1}{C_{n-k_0}}\left[\left(\prod_{j=0}^{k_0-1}\gamma_{n-j}\right)\sum_{l=0}^{k_0-1}\overline{\vec{x}}_{R_{n-l}\cap A_n^c}\sum_{j\in L_{n-k_0}\cap A_n}\tilde{\Phi}_{n-k_0}(\vec{x}-\vec{e}_j)\right.\\
&\quad\left.-\sum_{l=0}^{k_0-1}\left(\prod_{j=0}^{l}\gamma_{n-j}\right)\overline{\vec{x}}_{R_{n-l}\cap A_n}\sum_{j\in L_{n-k_0}\cap A_n^c}\tilde{\Phi}_{n-1-l}(\vec{x}-\vec{e}_j)\right],\\
&=\frac{1}{C_n}\left[\frac{1}{C_{n-k_0}}\left(\prod_{j=0}^{k_0-1}\gamma_{n-j}\right)\sum_{l=0}^{k_0-1}\overline{\vec{x}}_{R_{n-l}\cap A_n^c}\sum_{j\in L_{n-k_0}\cap A_n}\tilde{\Phi}_{n-k_0}(\vec{x}-\vec{e}_j)\right.\\
&\quad\left.+\left(\prod_{j=0}^{k_0-1}\gamma_{n-j}\right)\sum_{m\in L_{n-k_0}\cap A_n}\tilde{\Phi}_{n-k_0}(\vec{x}-\vec{e}_m)\right]\\
&\quad+\frac{1}{C_n}\sum_{l=0}^{k_0-1}\left(\prod_{j=0}^{l}\gamma_{n-j}\right)\overline{\vec{x}}_{R_{n-l}\cap A_n}\left(\tilde{\Phi}_{n-1-l}(\vec{x})-\frac{1}{C_{n-k_0}}\sum_{j\in L_{n-k_0}\cap A_n^c}\tilde{\Phi}_{n-1-l}(\vec{x}-\vec{e}_j)\right).
\end{aligned}
$$

Observe that by the inductive assumption, for all $l=0,\dots k_0-1$,

$$\tilde{\Phi}_{n-1-l}(\vec{x})\geq\frac{1}{C_{n-k_0}}\sum_{m\in L_{n-k_0}}\tilde{\Phi}_{n-1-l}(\vec{x}-\vec{e}_m).$$

This implies that

$$\begin{aligned}
&\tilde{\Phi}_{n-1-l}(\vec{x}) - \frac{1}{C_{n-k_0}} \sum_{m\in L_{n-k_0}\cap A_n^c} \tilde{\Phi}_{n-1-l}(\vec{x}-\vec{e}_m) \\
&\geq \frac{1}{C_{n-k_0}} \sum_{m\in L_{n-k_0}\cap A_n} \tilde{\Phi}_{n-1-l}(\vec{x}-\vec{e}_m), \\
&= \left(\prod_{j=l+1}^{k_0-1} \gamma_{n-j}\right) \frac{1}{C_{n-k_0}} \sum_{m\in L_{n-k_0}\cap A_n} \tilde{\Phi}_{n-k_0}(\vec{x}-\vec{e}_m).
\end{aligned}$$

Therefore

$$\begin{aligned}
&\frac{1}{C_n}\sum_{m\in L_n} \tilde{\Phi}_n(\vec{x}-\vec{e}_m) - \frac{1}{C_{n-k_0}} \sum_{m\in L_{n-k_0}\cap A_n^c} \tilde{\Phi}_n(\vec{x}-\vec{e}_m) \\
&\geq \frac{1}{C_n}\left[\frac{1}{C_{n-k_0}}\left(\prod_{j=0}^{k_0-1}\gamma_{n-j}\right)\sum_{l=0}^{k_0-1}\overline{\vec{x}}_{R_{n-l}\cap A_n^c} \sum_{j\in L_{n-k_0}\cap A_n} \tilde{\Phi}_{n-k_0}(\vec{x}-\vec{e}_j)\right. \\
&\quad \left. + \left(\prod_{j=0}^{k_0-1}\gamma_{n-j}\right) \sum_{m\in L_{n-k_0}\cap A_n} \tilde{\Phi}_{n-k_0}(\vec{x}-\vec{e}_m)\right] \\
&\quad + \frac{1}{C_n}\frac{1}{C_{n-k_0}}\left(\prod_{j=0}^{k_0-1}\gamma_{n-j}\right)\sum_{l=0}^{k_0-1}\overline{\vec{x}}_{R_{n-l}\cap A_n} \sum_{m\in L_{n-k_0}\cap A_n} \tilde{\Phi}_{n-k_0}(\vec{x}-\vec{e}_m), \\
&= \frac{1}{C_n}\frac{1}{C_{n-k_0}}\left(\prod_{j=0}^{k_0-1}\gamma_{n-j}\right)\left(\sum_{l=0}^{k_0-1}\overline{\vec{x}}_{R_{n-l}} \sum_{j\in L_{n-k_0}\cap A_n} \tilde{\Phi}_{n-k_0}(\vec{x}-\vec{e}_j) + C_{n-k_0}\right) \\
&\quad \left(\sum_{m\in L_{n-k_0}\cap A_n} \tilde{\Phi}_{n-k_0}(\vec{x}-\vec{e}_m)\right).
\end{aligned}$$

Finally, taking the difference of (LHS) and (RHS) gives

$$\begin{aligned}
&\frac{1}{C_n}\sum_{m\in L_n} \tilde{\Phi}_n(\vec{x}-\vec{e}_m) - \frac{1}{C_{n-k_0}} \sum_{m\in L_{n-k_0}} \tilde{\Phi}_n(\vec{x}-\vec{e}_m) \\
&\geq \frac{1}{C_n}\frac{1}{C_{n-k_0}}\left(\prod_{j=0}^{k_0-1}\gamma_{n-j}\right) \sum_{m\in L_{n-k_0}\cap A_n} \tilde{\Phi}_{n-k_0}(\vec{x}-\vec{e}_m)\left(\sum_{l=0}^{k_0-1}\overline{\vec{x}}_{R_{n-l}} + C_{n-k_0} - C_n\right).
\end{aligned}$$

But,

$$\begin{aligned}
\sum_{l=0}^{k_0-1} \overline{\vec{x}}_{R_{n-l}} + C_{n-k_0} &= \sum_{l=0}^{k_0-2} \overline{\vec{x}}_{R_{n-l}} + \left(\overline{\vec{x}}_{R_{n-k_0+1}} + C_{n-k_0}\right), \\
&\geq \sum_{l=0}^{k_0-2} \overline{\vec{x}}_{R_{n-l}} + \left(\overline{\vec{x}}_{R_{n-k_0+1}} + y_{n-k_0} \wedge C_{n-k_0}\right), \\
&= \sum_{l=0}^{k_0-2} \overline{\vec{x}}_{R_{n-l}} + y_{n-k_0+1}, \\
&= \sum_{l=0}^{k_0-3} \overline{\vec{x}}_{R_{n-l}} + \left(\overline{\vec{x}}_{R_{n-k_0+2}} + y_{n-k_0+1}\right), \\
&\geq \sum_{l=0}^{k_0-3} \overline{\vec{x}}_{R_{n-l}} + \left(\overline{\vec{x}}_{R_{n-k_0+2}} + y_{n-k_0+1} \wedge C_{n-k_0+1}\right), \\
&\vdots, \\
&\geq \overline{\vec{x}}_{R_n} + y_{n-1}, \\
&\geq \overline{\vec{x}}_{R_n} + y_{n-1} \wedge C_{n-1}, \\
&= y_n, \\
&> C_n.
\end{aligned}$$

Therefore

$$\frac{1}{C_n} \sum_{m \in L_n} \tilde{\Phi}_n(\vec{x} - \vec{e}_m) - \frac{1}{C_{n-k_0}} \sum_{m \in L_{n-k_0}} \tilde{\Phi}_n(\vec{x} - \vec{e}_m) \geq 0.$$

To show

$$\tilde{\Phi}_n(\vec{x}) \geq \frac{\tilde{\Phi}_n(\vec{x} - \vec{e}_m)}{x_m r_m}$$

for any m such that $x_m > 0$, the proof is again divided into different cases. Fix $\vec{x}$ and m such that $x_m > 0$.

The first case supposes $y(\vec{x}-\vec{e}_m) \leq C_n$ and $m \in R_n$. Then

$$\begin{aligned}
\tilde{\Phi}_n(\vec{x}) &= \frac{1}{C_n}\sum_{j\in L_n}\tilde{\Phi}_n(\vec{x}-\vec{e}_j), \\
&= \frac{1}{C_n}\left[\sum_{j\in L_n\setminus\{m\}}\tilde{\Phi}_n(\vec{x}-\vec{e}_j)+\tilde{\Phi}_n(\vec{x}-\vec{e}_m)\right], \\
&\geq \frac{1}{C_n}\left[\sum_{j\in L_n\setminus\{m\}}\frac{\tilde{\Phi}_n(\vec{x}-\vec{e}_m-\vec{e}_j)}{x_m r_m}+\tilde{\Phi}_n(\vec{x}-\vec{e}_m)\right], \\
&= \frac{\gamma_n(\vec{x}-\vec{e}_m)}{C_n}\left[\sum_{j\in R_n\setminus\{m\}}\frac{x_j r_j\tilde{\Phi}_{n-1}(\vec{x}-\vec{e}_m)}{x_m r_m}\right. \\
&\quad \left.+\sum_{j\in L_{n-1}}\frac{\tilde{\Phi}_{n-1}(\vec{x}-\vec{e}_m-\vec{e}_j)}{x_m r_m}+\frac{x_m r_m\tilde{\Phi}_{n-1}(\vec{x}-\vec{e}_m)}{x_m r_m}\right], \\
&= \frac{\gamma_n(\vec{x}-\vec{e}_m)}{x_m r_m C_n}\left[\overline{\vec{x}}_{R_n}\tilde{\Phi}_{n-1}(\vec{x}-\vec{e}_m)+C_{n-1}\wedge y_{n-1}(\vec{x}-\vec{e}_m)\tilde{\Phi}_{n-1}(\vec{x}-\vec{e}_m)\right] \\
&\quad \text{by Lemma 4.6.1,} \\
&= \frac{\gamma_n(\vec{x}-\vec{e}_m)}{x_m r_m C_n}\tilde{\Phi}_{n-1}(\vec{x}-\vec{e}_m)\left[\overline{\vec{x}}_{R_n}+C_{n-1}\wedge y_{n-1}(\vec{x})\right] \\
&\quad \text{since } y_{n-1}(\vec{x}) = y_{n-1}(\vec{x}-\vec{e}_m), \\
&= \frac{\gamma_n(\vec{x}-\vec{e}_m)}{x_m r_m}\tilde{\Phi}_{n-1}(\vec{x}-\vec{e}_m)\frac{y_n(\vec{x})}{C_n}, \\
&> \frac{\gamma_n(\vec{x}-\vec{e}_m)\tilde{\Phi}_{n-1}(\vec{x}-\vec{e}_m)}{x_m r_m}, \\
&= \frac{\tilde{\Phi}_n(\vec{x}-\vec{e}_m)}{x_m r_m}.
\end{aligned}$$

Now suppose $y(\vec{x}-\vec{e}_m) \le C_n$ and $m \notin R_n$, then almost exactly as the previous case,

$$\begin{aligned}
\tilde{\Phi}_n(\vec{x}) &= \frac{1}{C_n}\sum_{j\in L_n}\tilde{\Phi}_n(\vec{x}-\vec{e}_j), \\
&\ge \frac{1}{C_n}\left[\sum_{j\in L_n}\frac{\tilde{\Phi}_n(\vec{x}-\vec{e}_m-\vec{e}_j)}{x_m r_m}\right], \\
&= \frac{\gamma_n(\vec{x}-\vec{e}_m)}{C_n}\left[\sum_{j\in R_n}\frac{x_j r_j\tilde{\Phi}_{n-1}(\vec{x}-\vec{e}_m)}{x_m r_m} + \sum_{j\in L_{n-1}}\frac{\tilde{\Phi}_{n-1}(\vec{x}-\vec{e}_m-\vec{e}_j)}{x_m r_m}\right], \\
&= \frac{\gamma_n(\vec{x}-\vec{e}_m)}{x_m r_m C_n}\left[\overline{\overline{x}}_{R_n}\tilde{\Phi}_{n-1}(\vec{x}-\vec{e}_m) + C_{n-1}\wedge y_{n-1}(\vec{x}-\vec{e}_m)\tilde{\Phi}_{n-1}(\vec{x}-\vec{e}_m)\right], \\
&> \frac{\gamma_n(\vec{x}-\vec{e}_m)\tilde{\Phi}_{n-1}(\vec{x}-\vec{e}_m)}{x_m r_m}, \\
&= \frac{\tilde{\Phi}_n(\vec{x}-\vec{e}_m)}{x_m r_m}.
\end{aligned}$$

Finally, first suppose $y(\vec{x}-\vec{e}_m) > C_n$ and $x_m \ge 2$. Then,

$$\begin{aligned}
\tilde{\Phi}_n(\vec{x}) &= \frac{1}{C_n}\sum_{j\in L_n}\tilde{\Phi}_n(\vec{x}-\vec{e}_j), \\
&\ge \frac{1}{C_n}\left[\sum_{j\in L_n\setminus\{m\}}\frac{\tilde{\Phi}_n(\vec{x}-\vec{e}_m-\vec{e}_j)}{x_m r_m} + \tilde{\Phi}_n(\vec{x}-\vec{e}_m)\right], \\
&\ge \frac{1}{C_n}\left[\sum_{j\in L_n\setminus\{m\}}\frac{\tilde{\Phi}_n(\vec{x}-\vec{e}_m-\vec{e}_j)}{x_m r_m} + \frac{\tilde{\Phi}_n(\vec{x}-2\vec{e}_m)}{(x_m-1)r_m}\right], \\
&> \frac{1}{C_n}\left[\sum_{j\in L_n\setminus\{m\}}\frac{\tilde{\Phi}_n(\vec{x}-\vec{e}_m-\vec{e}_j)}{x_m r_m} + \frac{\tilde{\Phi}_n(\vec{x}-2\vec{e}_m)}{x_m r_m}\right], \\
&= \frac{1}{C_n}\sum_{j\in L_n}\frac{\tilde{\Phi}_n(\vec{x}-\vec{e}_m-\vec{e}_j)}{x_m r_m}, \\
&= \frac{\tilde{\Phi}_n(\vec{x}-\vec{e}_m)}{x_m r_m}.
\end{aligned}$$

Otherwise if $y(\vec{x} - \vec{e}_m) > C_n$ and $x_m = 1$, then

$$\begin{aligned}
\tilde{\Phi}_n(\vec{x}) &= \frac{1}{C_n} \sum_{j \in L_n} \tilde{\Phi}_n(\vec{x} - \vec{e}_j), \\
&\geq \frac{1}{C_n} \left[\sum_{j \in L_n \setminus \{m\}} \frac{\tilde{\Phi}_n(\vec{x} - \vec{e}_m - \vec{e}_j)}{x_m r_m} + \tilde{\Phi}_n(\vec{x} - \vec{e}_m) \right], \\
&> \frac{1}{C_n} \sum_{j \in L_n \setminus \{m\}} \frac{\tilde{\Phi}_n(\vec{x} - \vec{e}_m - \vec{e}_j)}{x_m r_m}, \\
&= \frac{\tilde{\Phi}_n(\vec{x} - \vec{e}_m)}{x_m r_m}.
\end{aligned}$$

□

Remark 4.6.1. *From the proof, it can be seen that for any fixed n, state $\vec{x}$ and m such that $x_m > 0$, the inequality $y_n(\vec{x}) > C_n$ implies that $\tilde{\Phi}_n(\vec{x}) > \frac{\tilde{\Phi}_n(\vec{x} - \vec{e}_m)}{x_m r_m}$. This fact will prove useful in the remaining sections.*

Using the previous proposition, we can now establish a more useful characterization of the balance function for parking lot networks.

Corollary 4.6.1. *For a parking lot network with n links, $\Phi_n(\vec{x}) = \tilde{\Phi}_n(\vec{x})$.*

Proof. By Proposition 4.6.1, all the link and rate constraints are satisfied. So

$$\Phi_n(\vec{x}) \leq \tilde{\Phi}_n(\vec{x}),$$

by Proposition 3 of [12]. It will be shown that $\Phi_n(\vec{x}) \geq \tilde{\Phi}_n(\vec{x})$ via induction.

Base case 1: For $n = 1$, it is immediate from (4.14) and (4.23) that $\Phi_1(\vec{x}) = \tilde{\Phi}_1(\vec{x})$ for all states $\vec{x}$.

Induction step 1: Assume that $\Phi_{n-1}(\vec{x}) \geq \tilde{\Phi}_{n-1}(\vec{x}) \quad \forall \vec{x}$.

Base case 2: For $|\vec{x}| = 0$, $\Phi_n(\vec{0}) = \tilde{\Phi}_n(\vec{0}) = 1$.

Induction step 2: Fix $k \in \mathbb{N}$ and assume $\Phi_n(\vec{x}) \geq \tilde{\Phi}_n(\vec{x})$ for all $|\vec{x}| \leq k - 1$.

Now fix $\vec{x}$ such that $|\vec{x}| = k$.

First suppose that $y_n \leq C_n$. If there exists an $m \in R_n$ such that $x_m > 0$, then

$$\begin{aligned}\Phi(\vec{x}) &\geq \frac{\Phi_n(\vec{x}-\vec{e}_m)}{x_m r_m}, \\ &\geq \frac{\tilde{\Phi}_n(\vec{x}-\vec{e}_m)}{x_m r_m}, \\ &= \tilde{\Phi}_n(\vec{x}).\end{aligned}$$

Otherwise

$$\begin{aligned}\tilde{\Phi}_n(\vec{x}) &= \tilde{\Phi}_{n-1}(\vec{x}), \\ &\leq \Phi_{n-1}(\vec{x}), \\ &= \Phi_n(\vec{x}),\end{aligned}$$

where the final equality follows from (4.20) and the fact that $x_m = 0$ for all $m \in R_n$.

If $y_n > C_n$ then,

$$\begin{aligned}\Phi_n(\vec{x}) \geq \frac{1}{C_n}\sum_{j\in L_n}\Phi(\vec{x}-\vec{e}_j) &\geq \frac{1}{C_n}\sum_{j\in L_n}\tilde{\Phi}_n(\vec{x}-\vec{e}_j) \\ &= \tilde{\Phi}_n(\vec{x}).\end{aligned}$$

□

The recursive form of the balance function allows for a simple characterization of the stationary distribution. Let π_n be the stationary distribution for a parking lot network with n links. Then it can be recursively defined by:

$$\pi_n(\vec{x}) = \begin{cases} \pi_n(\vec{0})\prod_{m\in R_n}\frac{\beta_m{}^{x_m}}{x_m!}\pi_{n-1}(\vec{x}) & \text{if } y_n(\vec{x}) \leq C_n, \\ \frac{1}{C_n}\sum_{m\in L_n}\alpha_m\pi_n(\vec{x}-\vec{e}_m) & \text{otherwise.}\end{cases} \tag{4.26}$$

4.6.2 Probability of congestion

For the remainder of this thesis, the number of links is fixed at some positive integer K. Recall that for a state $\vec{x}$, congestion occurs for a flow class m at that state when $\phi_m(\vec{x}) < r_m x_m$. For the single link case, all flow classes were congested over the same states $\vec{x}$ and that occurred when $\overline{\vec{x}} > C$. The former fact is not true in general for the parking lot network. But, as the following lemma will show, a simple criterion to identify the states where congestion occurs for each flow class does exist.

Lemma 4.6.2. *For all $n \in \{1, \dots, K\}$, flow class $m \in R_n$ and states $\vec{x}$, the total allocation to flows of class m are $\phi_m(\vec{x}) = x_m r_m$ if and only if $y_k \leq C_k$ for all $k = n \dots K$.*

Proof. Fix $n \in \{1, \dots, K\}$, $m \in R_n$, and $\vec{x}$. Since $\phi_m(\vec{x}) = 0$ if $x_m = 0$, then it will be assumed that $x_m > 0$. Recall from Corollary 4.6.1 that $\tilde{\Phi}_\cdot = \Phi_\cdot$.

To begin, assume that $y_k \leq C_k$ for all $k = n \dots K$. Then $\phi_m(\vec{x}) = x_m r_m$ follows directly from the definition of the balance function (4.23).

Now assume that $\phi_m(\vec{x}) = x_m r_m$. As discussed in Remark 4.6.1: If $y_K > C_K$ then $\Phi_K(\vec{x}) > \dfrac{\Phi_K(\vec{x} - \vec{e}_m)}{x_m r_m}$, which implies that $\phi_m(\vec{x}) < x_m r_m$. Therefore, $y_K \leq C_K$ by assumption. The remainder of the proof will show, via contradiction, that $y_k \leq C_k$ for all $k = n \dots K$.

Assume that there exist integers k such that $y_k > C_k$. Select the largest such integer and label it, for simplicity, as k. Also note that $K > k$. Then, via Remark 4.6.1, $\Phi_k(\vec{x}) > \dfrac{\Phi_k(\vec{x} - \vec{e}_m)}{x_m r_m}$. But from the definition of Φ, i.e. (4.23),

$$\Phi_K(\vec{x}) = \left(\prod_{j=k+1}^{K} \gamma_j \right) \Phi_k(\vec{x}).$$

So by (4.7),

$$\begin{aligned}\phi_m(\vec{x}) &= \frac{\Phi_K(\vec{x}-\vec{e}_m)}{\Phi_K(\vec{x})},\\ &= \frac{\left(\prod_{j=k+1}^{K}\gamma_j\right)\Phi_k(\vec{x}-\vec{e}_m)}{\left(\prod_{j=k+1}^{K}\gamma_j\right)\Phi_k(\vec{x})},\\ &= \frac{\Phi_k(\vec{x}-\vec{e}_m)}{\Phi_k(\vec{x})},\\ &< x_m r_m,\end{aligned}$$

which is a contradiction. □

The result agrees with intuition: Congestion occurs for a flow only if there exists a saturated link on its route. A link n is saturated if $y_n > C_n$. So for any flow class $m \in R_n$, the probability of congestion is now reduced to summing over all the states that have at least one of the links $n, \ldots, K$ saturated. For the majority of this section though, the attention will be focused on the quantity P_K which represents the probability that link K is saturated.

Akin to the single link case, to simplify the calculation of the congestion metrics the state space needs to be appropriately partitioned. Define the sets $(A_l^K)_{l=0,\ldots,K}$ such that

$$A_l^K \triangleq \{\vec{x} \in \mathbb{Z}_+^{|L_K|} : y_l(\vec{x}) \leq C_l\}, \tag{4.27}$$

and $A_0^0 = \emptyset$. The state space can be partitioned by the disjoint sets $(H_j^K)_{j=1,\ldots,K+1}$ defined by

$$H_j^K \triangleq \begin{cases} \left(A_K^K\right)^c & j = K+1,\\ \bigcap_{l=j}^{K} A_l^K \cap \left(A_{j-1}^K\right)^c & \text{otherwise.}\end{cases} \tag{4.28}$$

Partitioning the state space by the H_j^K's has numerous advantages. For any $1 < j < K+1$, let

$$\tilde{A}_{j,l}^K = \left\{ \vec{v} \in \mathbb{Z}_+^{|L_K \setminus L_{j-1}|} : \sum_{m \in L_K \setminus L_{j-1}} r_m \tilde{v}_{m,j} \leq C_l - C_{j-1} \right\},$$

where

$$\tilde{v}_{m,j} \equiv v_{m-|L_{j-1}|}.$$

Then

$$\sum_{\vec{x}\in H_j^K} \Phi_K(\vec{x}) \prod_{m\in L_K} \alpha_m^{x_m} = \left(\sum_{\vec{u}\in \left(A_{j-1}^{j-1}\right)^c} \Phi_{j-1}(\vec{u}) \prod_{m\in L_{j-1}} \alpha_m^{u_m} \right) \left(\sum_{\vec{v}\in \bigcap_{l=j}^K \tilde{A}_{j,l}^K} \prod_{m\in L_K\setminus L_{j-1}} \frac{\beta_m^{\tilde{v}_{m,j}}}{\tilde{v}_{m,j}!} \right). \tag{4.29}$$

Thus the normalization constant G_K can be written as

$$\begin{aligned} G_K &= \sum_{\vec{x}\in \mathbb{Z}_+^{|L_K|}} \Phi_K(\vec{x}) \prod_{m\in L_K} \alpha_m^{x_m}, \\ &= \sum_{j=1}^{K+1} \sum_{\vec{x}\in H_j^K} \Phi_K(\vec{x}) \prod_{m\in L_K} \alpha_m^{x_m}, \\ &= \sum_{j=2}^{K} \left(\sum_{\vec{u}\in \left(A_{j-1}^{j-1}\right)^c} \Phi_{j-1}(\vec{u}) \prod_{m\in L_{j-1}} \alpha_m^{u_m} \right) \left(\sum_{\vec{v}\in \bigcap_{l=j}^K \tilde{A}_{j,l}^K} \prod_{m\in L_K\setminus L_{j-1}} \frac{\beta_m^{\tilde{v}_{m,j}}}{\tilde{v}_{m,j}!} \right) \\ &\quad + \sum_{\vec{x}\in \bigcap_{l=1}^K A_l^K} \prod_{m\in L_K} \frac{\beta_m^{x_m}}{x_m!} + \sum_{\vec{x}\in \left(A_K^K\right)^c} \Phi_K(\vec{x}) \prod_{m\in L_K} \alpha_m^{x_m}. \end{aligned} \tag{4.30}$$

The term

$$P'_K \triangleq \sum_{\vec{x}\in \left(A_K^K\right)^c} \Phi_K(\vec{x}) \prod_{m\in L_K} \alpha_m^{x_m} \tag{4.31}$$

has a very familiar interpretation, from Lemma 4.6.2, it is the unnormalized probability that all flows are congested. Let

$$V_m^K \triangleq \{\vec{x} \in \mathbb{Z}_+^{|L_K|} : y_K(\vec{x} + \vec{e}_m) > C_K\}, \tag{4.32}$$

and

$$B'^K_m \triangleq \sum_{\vec{x}\in V_m^K \cap A_K^K} \Phi_K(\vec{x}) \prod_{m\in L_K} \alpha_m^{x_m}. \tag{4.33}$$

Following Lemma 4.5.1, P'_K can be rewritten as

$$\sum_{\vec{x}\in\left(A_K^K\right)^c} \Phi_K(\vec{x}) \prod_{m\in L_K} \alpha_m^{x_m} = \sum_{m\in L_K} \frac{\rho_m^{(K)} B'^K_m}{1-\rho^{(K)}}, \tag{4.34}$$

where $\rho_m^{(K)} = \frac{\alpha_m}{C_K}$ and $\rho^{(K)} = \sum_{m\in L_K} \rho_m^{(K)}$. Suppose $m \in R_n$. Then it is clear from the definitions of V_m^K and A_K^K that

$$\begin{aligned} V_m^K \cap A_K^K &= \bigcap_{j=1}^{K} H_j^K \cap V_m^K, \\ &= \bigcap_{j=1}^{n} H_j^K \cap V_m^K. \end{aligned} \tag{4.35}$$

For example, assume $H_{n+1}^K \cap V_m^K$ is not empty and select $\vec{x} \in H_j^K \cap V_m^K$. By definition of H_{n+1}^K and y_K, $C_K \geq y_K(\vec{x}) = \overline{\vec{x}}_{L_K\setminus L_{n+1}} + C_n$. Therefore $y_K(\vec{x}+\vec{e}_m) = y_K(\vec{x}) \leq C_K$ and so $\vec{x} \notin V_m^K$ which is a contradiction.

The advantage of this form is that the computation is over a smaller set of states. Unfortunately, when the capacities are "'large"', the computation can still be lengthy. To alleviate this problem the large system approximation is introduced again.

The first step is to analyze (4.31). Unfortunately, it is not readily amenable to the large system approximation. To work around this problem, an upper bound is established. Fix $m \in L_K$ and suppose $m \in R_n$. Let $\left(\overline{VH}_{m,j}^K\right)_{j=1\ldots n}$ be sets of vectors in $\mathbb{Z}_+^{|L_K|}$ such that for any $\vec{x} \in \overline{VH}_{m,j}^K$,

$$\begin{aligned} y_{j-1}(\vec{x}) &> C_{j-1}, \\ \overline{\vec{x}}_{L_K\setminus L_{j-1}} &\leq C_K - C_{j-1}, \\ \overline{\vec{x}}_{L_K\setminus L_{j-1}} + r_m &> C_K - C_{j-1}. \end{aligned} \tag{4.36}$$

Lemma 4.6.3. *For each flow class m,*

$$V_m^K \cap H_j^K \subseteq \overline{VH}_{m,j}^K, \tag{4.37}$$

where $j = 1, \ldots, n$ and $m \in R_n$.

Proof. Select $\vec{x} \in V_m^K \cap H_j^K$. Then by definition,

$$y_{j-1}(\vec{x}) > C_{j-1},$$

and

$$\begin{aligned} y_K(\vec{x}) &= \overline{\vec{x}}_{L_K \setminus L_{j-1}} + C_{j-1}, \\ &\leq C_K. \end{aligned}$$

As well, note that by definition of y_K, if $n = K$ then

$$\begin{aligned} y_K(\vec{x} + \vec{e}_m) &= \overline{\vec{x}}_{R_K} + r_m + C_{K-1} \wedge y_{K-1}(\vec{x}), \\ &\leq \overline{\vec{x}}_{R_K} + r_m + y_{K-1}(\vec{x}), \\ &\vdots \\ &\leq \overline{\vec{x}}_{L_K \setminus L_{j-1}} + r_m + C_{j-1} \wedge y_{j-1}(\vec{x}), \\ &\leq \overline{\vec{x}}_{L_K \setminus L_{j-1}} + r_m + C_{j-1}. \end{aligned}$$

Similarly if $n < K$,

$$\begin{aligned} y_K(\vec{x} + \vec{e}_m) &= \overline{\vec{x}}_{R_K} + C_{K-1} \wedge y_{K-1}(\vec{x} + \vec{e}_m), \\ &\leq \overline{\vec{x}}_{R_K} + r_m + y_{K-1}(\vec{x} + \vec{e}_m), \\ &\vdots \\ &\leq \overline{\vec{x}}_{L_K \setminus L_{j-1}} + r_m + C_{j-1} \wedge y_{j-1}(\vec{x}), \\ &\leq \overline{\vec{x}}_{L_K \setminus L_{j-1}} + r_m + C_{j-1}. \end{aligned}$$

Therefore,

$$\begin{aligned} C_K &< y_K(\vec{x} + \vec{e}_m), \\ &\leq \overline{\vec{x}}_{L_K \setminus L_{j-1}} + r_m + C_{j-1}. \end{aligned}$$

□

Let

$$\tilde{B}'^K_m \triangleq \sum_{j=1}^{K} \sum_{\vec{x}\in\overline{VH}^K_{m,j}} \Phi_K(\vec{x}) \prod_{m\in L_K} \alpha_m^{x_m}. \tag{4.38}$$

Then:

Lemma 4.6.4.

$$\tilde{B}'^K_m \geq B'^K_m. \tag{4.39}$$

Proof. Apply Lemma 4.6.3 with the fact that $\bigcup_{j=1}^{n} H_j^K \cap V_m^K = V_m^K \cap A_K^K$. □

Now, for any $1 < j \leq n$,

$$\sum_{\vec{x}\in\overline{VH}^K_{m,j}} \Phi_K(\vec{x}) \prod_{i\in L_K} \alpha_i^{x_i} = \left(\sum_{\vec{u}\in\left(A^{j-1}_{j-1}\right)^c} \Phi_{j-1}(\vec{u}) \prod_{i\in L_{j-1}} \alpha_i^{u_i} \right) \left(\sum_{\vec{v}\in T^K_{m,j}} \prod_{i\in L_K\setminus L_{j-1}} \frac{\beta_i^{\tilde{v}_{i,j}}}{\tilde{v}_{i,j}!} \right),$$

and for $j = 1$,

$$\sum_{\vec{x}\in\overline{VH}^K_{m,1}} \Phi_K(\vec{x}) \prod_{i\in L_K} \alpha_i^{x_i} = \left(\sum_{\vec{x}\in T^K_{m,1}} \prod_{i\in L_K} \frac{\beta_i^{x_i}}{x_i!} \right),$$

where

$$T^K_{m,j} \equiv \{\vec{v} \in \mathbb{Z}_+^{|L_K\setminus L_{j-1}|} : \overline{v}_j \leq C_K - C_{j-1} < \overline{v}_j + r_m\},$$

$$\overline{v}_j \equiv \sum_{m\in L_K\setminus L_{j-1}} v_{m-|L_{j-1}|} r_m,$$

and

$$\tilde{v}_{m,j} \equiv v_{m-|L_{j-1}|}.$$

Also, let

$$\tilde{P}'_K \triangleq \sum_{m\in L_K} \frac{\rho_m^{(K)} \tilde{B}'^K_m}{1-\rho^{(K)}}, \tag{4.40}$$

where, once again, $\rho_m^{(K)} = \frac{\alpha_m}{C_K}$ and $\rho^{(K)} = \sum_{m\in L_K} \rho_m^{(K)}$. Then it is clear from Lemma 4.6.4 that $\tilde{P}'_K \geq P'_K$. Let $\tilde{P}_K$ be the normalized version of $\tilde{P}'_K$.

Using the more amenable upper bound, the large systems approximation will now be analyzed. By induction over the number of links and Lemma 4.5.2 (via the proof of Lemma 4.5.3), $\tilde{P}'_K(N)e^{-N|\beta_K|} \to 0$. This leads to the following result:

Lemma 4.6.5.

$$G_K(N)e_K^{-N|\beta|} \to 1. \tag{4.41}$$

Finally, the main result of the section can now be stated.

Theorem 4.6.1. *The probability that all flows are congested is upper bounded by*

$$\tilde{P}_K(N) = \sum_{m \in L_K} \frac{\rho_m^{(K)} \tilde{B}_m(N)}{1 - \rho^{(K)}}, \tag{4.42}$$

where:

$$\tilde{P}_0(N) = 1,$$

and for each flow class $m \in R_n$*,*

$$\tilde{B}_m(N) \sim \sum_{j=1}^{n} \tilde{P}_{j-1}(N) P_{m,j}^{(K)}(N), \tag{4.43}$$

$$P_{m,j}^{(K)}(N) \sim \begin{cases} e^{-NI} e^{\tau d \epsilon(N)} \frac{d}{\sqrt{2\pi N}\sigma} \frac{1-e^{\tau r_m}}{1-e^{\tau d}} & \Delta C_j < \overline{\alpha}_j, \\ \sqrt{\frac{2}{\pi N}} \frac{r_m}{\sigma} & \Delta C_j = \overline{\alpha}_j, , \\ 1 - e^{\tau r_m} & \textit{otherwise.} \end{cases} \tag{4.44}$$

$\Delta C_j = C_K - C_{j-1}$,

$\overline{\alpha}_j = \sum_{m \in L_K \setminus L_{j-1}} \alpha_m$,

d *is the greatest common divisor of* $(r_m)_{m \in L_K \setminus L_{j-1}}$,

$\epsilon(N) = \frac{N\Delta C_j}{d} - \left\lfloor \frac{N\Delta C_j}{d} \right\rfloor$,

τ *is the unique solution to the equation* $\displaystyle\sum_{m \in L_K \setminus L_{j-1}} r_m \beta_m e^{\tau r_m} = \Delta C_j$,

$$I = \Delta C_j \tau - \sum_{m \in L_K \setminus L_{j-1}} \beta_m \left(e^{\tau r_m} - 1\right),$$

$$\sigma^2 = \sum_{m \in L_K \setminus L_{j-1}} r_m^2 \beta_m e^{\tau r_m}.$$

Now that an efficient formula to approximately calculate the probability that all flows are congested has been found, we now return to the problem of calculating the problem that a specified flow class is congested. Fix a flow class $m \in R_n$ and let $P_K^{(m)}$ be the probability of congestion for that flow, i.e.

$$P_K^{(m)} = \sum_{\vec{x} \in \bigcup_{l=n}^{K} \left(A_l^K\right)^c} \pi_K(\vec{x}).$$

But,

$$\bigcup_{l=n}^{K} \left(A_l^K\right)^c = \bigcup_{l=n+1}^{K+1} H_l^K.$$

So one can rewrite $P_K^{(m)}$ as

$$P_K^{(m)} = \sum_{l=n+1}^{K+1} \sum_{\vec{x} \in H_l^K} \pi_K(\vec{x}).$$

Then, from (4.29) and Lemma 4.6.5, the large system upper bound for the probability of congestion is

$$\begin{aligned} P_K^{(m)}(N) &\sim \sum_{j=n}^{K} P_j(N), \qquad (4.45) \\ &\leq \sum_{j=n}^{K} \tilde{P}_j(N). \end{aligned}$$

4.6.3 Time-Average Congestion Rates

The calculation of the time-average congestion rates follows very similarly to the previous section and so the same notation will carry over. Recall from (4.12) that the time-average congestion rate of a flow class $m \in R_k$ is

$$F_m^K = \frac{\sum_{\vec{x} \in \mathcal{C}_m} x_m \pi_K(\vec{x})}{\sum_{\vec{x}} x_m \pi_K(\vec{x})},$$

where $\mathcal{C}_m$ was identified in Lemma 4.6.2, i.e.

$$\mathcal{C}_m = \left(\bigcup_{j=k}^{K} A_j^K \right)^c.$$

Like the probability of congestion in the previous section, it will be fruitful to first investigate the states where all flows are congested. To that end, define Q_m^K by

$$Q_m^K \triangleq \sum_{\vec{x} \in (A_K^K)^c} x_m \pi_K(\vec{x}). \tag{4.46}$$

Following the proof of Lemma 4.5.4 mutatis mutandis,

$$Q_m^K = \frac{\rho_m^{(K)}}{1 - \rho^{(K)}} (P_K + B_m^K) + \sum_{j \in L_K} \frac{\rho_j^{(K)}}{1 - \rho^{(K)}} Q_{m,j}^K, \tag{4.47}$$

where B_m^K is $B_m'^K$ normalized and for $j \in R_n$,

$$\begin{aligned} Q_{m,j}^K &= \sum_{\vec{x} \in A_K^K \cap V_j^K} x_m \pi_K(\vec{x}), \\ &= \sum_{\vec{x} \in \bigcup_{l=1}^{K} \left(H_l^K \cap V_j^K \right)} x_m \pi_K(\vec{x}), \\ &= \sum_{\vec{x} \in \bigcup_{l=1}^{n} \left(H_l^K \cap V_j^K \right)} x_m \pi_K(\vec{x}), \\ &= \sum_{l=1}^{n} \sum_{\vec{x} \in H_l^K \cap V_j^K} x_m \pi_K(\vec{x}). \end{aligned} \tag{4.48}$$

The $Q_{m,j}^K$ can be further decomposed by analyzing the sets $H_l^K \cap V_j^K$. Fix $l = 1, \ldots, n$. If $k \geq n$ or $n > k \geq l$ then,

$$\begin{aligned}
\sum_{\vec{x} \in H_l^K \cap V_j^K} x_m \pi_K(\vec{x}) &= \beta_m \sum_{\vec{x} \in H_l^K \cap V_j^K} \pi_K(\vec{x} - \vec{e}_m), \\
&= \beta_m \sum_{\vec{x} + \vec{e}_m \in H_l^K \cap V_j^K} \pi_K(\vec{x}), \\
&\leq \beta_m \sum_{\vec{x} + \vec{e}_m \in \overline{VH}_{l,j}^K} \pi_K(\vec{x}), \\
&= \beta_m \left(\sum_{\vec{u} \in \left(A_{l-1}^{l-1}\right)^c} \pi_{|L_{l-1}|}(\vec{u}) \right) b_{lj}^K.
\end{aligned}$$

Otherwise, if $n \geq l > k$, then

$$\begin{aligned}
\sum_{\vec{x} \in H_l^K \cap V_j^K} x_m \pi_K(\vec{x}) &= \left(\sum_{\vec{u} \in \left(A_{l-1}^{l-1}\right)^c} x_m \pi_{|L_{l-1}|}(\vec{u}) \right) b_{lj}^K, \\
&= Q_m^{l-1} b_{lj}^K,
\end{aligned}$$

where

$$b_{lj}^K = \left(\sum_{\vec{v} \in \mathbb{Z}_+^{|L_K \setminus L_{l-1}|} : \overline{v} \leq C_K - C_{l-1} < \overline{v} + r_j} \pi_{|L_K \setminus L_{l-1}|}(\vec{v}) \right).$$

and for $l = 1$,

$$\left(\sum_{\vec{u} \in \left(A_{l-1}^{l-1}\right)^c} x_m \pi_{|L_{l-1}|}(\vec{u}) \right) = 1.$$

Of course, Q_m^1 is just the single link case and its computation was the focus of Section 4.5.3. Let

$$\tilde{Q}_m^K = \frac{\rho_m^{(K)}}{1 - \rho^{(K)}} (\tilde{P}_K + \tilde{B}_m^K) + \sum_{j \in L_K} \frac{\rho_j^{(K)}}{1 - \rho^{(K)}} \tilde{Q}_{m,j}^K, \tag{4.49}$$

where

$$\tilde{Q}_{m,j}^{K} = \begin{cases} \tilde{Q}_m^{l-1} b_{lj}^K & \text{if } n \geq l > k, \\ \beta_m \left(\sum_{\vec{u} \in \left(A_{l-1}^{l-1}\right)^c} \pi_{|L_{l-1}|}(\vec{u}) \right) b_{lj}^K & \text{otherwise.} \end{cases}$$

It is clear from the definition that $\tilde{Q}_m^K \geq Q_m^K$. Again, $\tilde{Q}_m^K$ can be efficiently calculated using the large system asymptotic (note that b_{lj}^K is equivalent to a single link calculation). Moreover, as derived in Theorem 4.5.2,

$$\begin{aligned} \sum_{\vec{x}} x_m \pi_K(\vec{x}) &\geq \sum_{\vec{x} \in \bigcap_{l=1}^K A_l^K} x_m \pi_K(\vec{x}), \\ &\sim N\beta_m \text{ in the large system asymptotic.} \end{aligned}$$

Let

$$\begin{aligned} f_m^K &= \frac{\displaystyle\sum_{\vec{x} \in \left(A_K^K\right)^c} x_m \pi_K(\vec{x})}{\displaystyle\sum_{\vec{x}} x_m \pi_K(\vec{x})}, \\ &= \frac{Q_m^K}{\displaystyle\sum_{\vec{x}} x_m \pi_K(\vec{x})}, \end{aligned}$$

and

$$\tilde{f}_m^K = \frac{\tilde{Q}_m^K}{\displaystyle\sum_{\vec{x}} x_m \pi_K(\vec{x})}.$$

Then using the same method as deriving (4.45), one gets

$$\begin{aligned} F_M^K(N) &\sim \sum_{j=n}^{K} f_m^j(N), \\ &\leq \sum_{j=n}^{K} \tilde{f}_m^j(N). \end{aligned}$$

4.7 Numerical Results

The chapter is concluded with a numerical comparison of the asymptotic formula of Theorems 4.5.1 and 4.6.1 with exact results for a single link and a two-link parking lot network with $M = 3$ classes of traffic. For the network case, the first flow class travels through both links while the other flows travels through the second link only. The rate limits for both networks are $r_1 = 1$, $r_2 = 1$ and $r_3 = 2$. As well the traffic intensities are $\alpha_1 = 1$, $\alpha_2 = 1$ and $\alpha_3 = 2$.

For the single link, the probability of congestion was computed. The numerical experiment was run twice. The first time the link capacity was set to $C = 4.8$, and the second it was set to $C = 4.3$ corresponding to a light and heavy load. The results are given in Table 4.1.

Table 4.1: Probability of Congestion - Single Link

(a) Light load

	Exact	Approximation
N		
10	2.21e-1	2.50e-1
20	9.51e-2	1.07e-1
30	4.72e-2	5.38e-2
40	2.49e-2	2.81e-2

(b) Heavy load

	Exact	Approximation
N		
10	9.34e-1	9.81e-1
20	6.14e-1	6.45e-1
30	4.66e-1	4.89e-1
40	3.75e-1	3.94e-1

Two immediate patterns become apparent when observing the tables. First, the heavier the load, the more conservative the approximation becomes. Secondly, the larger the scaling factor, i.e. N, the more accurate the approximation becomes. Not surprisingly, these points were noted by Gazdzicki et al. [23] in the loss networks context.

The two-link parking lot network had more interesting results. The quantity computed was the exact probability that all flows were congested and the approximate upper bound. The numerical experiment was run three times corresponding to the three cases in Theorem 4.6.1. The link capacities were set to $C = [1.2, 4.8], [1.4, 4.4]$ and $[1.5, 4.3]$. The results are given in Table 4.2.

The results suggest that the load at the final link is, unsurprisingly, the most important variable in determining the probability of congestion. Once again, the heavier the load, the more conservative the approximation. In fact, as it is an upper bound, the results seem to suggest that a simpler and tighter approximation could be achieved by simply ignoring the network structure and treating the system as a single link. Another interesting pattern that seems to be emerging is that the lighter the load at the first link, the tighter the approximation is to the exact value.

Table 4.2: Probability of Congestion - Parking Lot Network

(a) $C = [1.2, 4.8]$

	Exact $P_K(N)$	Approximation $\tilde{P}_K(N)$
N		
10	1.92e-1	4.02e-1
20	8.22e-2	1.52e-1
30	4.08e-2	7.11e-2
40	2.15e-2	3.60e-2

(b) $C = [1.4, 4.4]$

	Exact $P_K(N)$	Approximation $\tilde{P}_K(N)$
N		
10	6.41e-1	7.01e-1
20	4.04e-1	4.36e-1
30	2.92e-1	3.13e-1
40	2.23e-1	2.39e-1

(c) $C = [1.5, 4.3]$

	Exact $P_K(N)$	Approximation $\tilde{P}_K(N)$
N		
10	9.24e-1	1.19e0
20	6.13e-1	6.79e-1
30	4.66e-1	4.97e-1
40	3.75e-1	3.96e-1

Chapter 5

Conclusion and Future Work

5.1 Pathwise Results for Stochastic Fluid Networks

Sample-path comparison theorems were investigated to build insight into the behaviour of fluid networks. Some applications were discussed as well. The physics of the fluid model allows for strong pathwise conclusions that are not available in the discrete counterpart. From the comparison theorems discussed, several general conclusions can be drawn. Domination by the routing matrix, initial workload, or the input process ensures pathwise domination in the workload at each queue. Meanwhile decreasing the service rate can only ensure that total workload in the network increases.

An interesting future line of research direction would to be find conditions that would extend Lemma 2.4.2 to networks with a state-dependent routing matrix. It is the belief of the author that the following extension to Lemma 3.1 in [34] is most likely true.

Conjecture 5.1.1. *Suppose that* $\lim_{t\to\infty} \frac{J(t)}{t} = \lambda$ *and* $(I - P')^{-1}\lambda < r$*. Consider a vector* $\tilde{\lambda} \in \mathbb{R}^N$ *such that* $\tilde{\lambda} > \lambda$ *and* $(I - P')^{-1}\tilde{\lambda} < r$*. Let* $(J, r, P(w), W(0))$ *and* $(J, \tilde{\lambda}, \boldsymbol{0}, W(0))$ *be two stochastic fluid networks with respective workload processes* W *and* $\tilde{W}$*. Then* $|W(t)| \leq |\tilde{W}(t)|$ *for each* $t \geq 0$*.*

Proving such a conjecture would show, in great generality, that there exists a pathwise bound for the workload in a state-dependent network.

5.2 Interchange of Limits

The interchange of limits problem was shown to hold for stochastic fluid networks with both fixed and state-dependent routing matrices. The techniques used to prove the interchange results took advantage of the sample-path theorems developed in Chapter 2.

The method of proof used to prove the interchange of limits purposely avoided the powerful tools and techniques associated with Markov processes utilized by similar works [22], [17]. Since sample-path methods were used instead, the proofs can be adapted to more general inputs then the canonical Lévy setting. So an interesting future line of research is in proving the results for more general inputs. Unfortunately that is where the real difficulty lies since little is known beyond the Markovian setting.

Another line of potential future research is to remove the constraint in the state-dependent case that the routing matrix is upper bounded. Affirming Conjecture 5.1.1 would be a large step forward in proving such a result.

5.3 Balanced Fairness

In Chapter 4, congestion in networks operating under Balanced Fairness was investigated. For the single link case, the congestion metrics were shown to correspond to a multi-rate loss system. Through such a correspondence, the large system approximation for a single link loss system was applied to yield efficient congestion estimates. As well, an analysis of the qualitative properties for parking lot networks was given. It was shown that the results of the single link network could be applied to upper bound the congestion metrics in a parking lot network.

A future direction of research is to show that such results hold for more general networks, in particular a tree network. The tree topology is a generalization of the parking lot topology that possesses many of the latter's useful recursive properties. So many of the techniques used could potentially carry over to the tree network.

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