Transmitting Quantum Information Reliably across Various Quantum Channels

by

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Abstract

Transmitting quantum information across quantum channels is an important task. However quantum information is delicate, and is easily corrupted. We address the task of protecting quantum information from an information theoretic perspective – we encode some message qudits into a quantum code, send the encoded quantum information across the noisy quantum channel, then recover the message qudits by decoding. In this dissertation, we discuss the coding problem from several perspectives.

The noisy quantum channel is one of the central aspects of the quantum coding problem, and hence quantifying the noisy quantum channel from the physical model is an important problem. We work with an explicit physical model – a pair of initially decoupled quantum harmonic oscillators interacting with a spring-like coupling, where the bath oscillator is initially in a thermal-like state. In particular, we treat the completely positive and trace preserving map on the system as a quantum channel, and study the truncation of the channel by truncating its Kraus set. We thereby derive the matrix elements of the Choi-Jamiolkowski operator of the corresponding truncated channel, which are truncated transition amplitudes. Finally, we give a computable approximation for these truncated transition amplitudes with explicit error bounds, and perform a case study of the oscillators in the off-resonant and weakly-coupled regime numerically.

In the context of truncated noisy channels, we revisit the notion of approximate error correction of finite dimension codes. We derive a computationally simple lower bound on the worst case entanglement fidelity of a quantum code, when the truncated recovery map of Leung et. al. is rescaled. As an application, we apply our bound to construct a family of multi-error correcting amplitude damping codes that are permutation-invariant. This demonstrates an explicit example where the specific structure of the noisy channel allows code design out of the stabilizer formalism via purely algebraic means.

We study lower bounds on the quantum capacity of adversarial channels, where we restrict the selection of quantum codes to the set of concatenated quantum codes. The adversarial channel is a quantum channel where an adversary corrupts a fixed fraction of qudits sent across a quantum channel in the most malicious way possible. The best

known rates of communicating over adversarial channels are given by the quantum Gilbert-Varshamov (GV) bound, that is known to be attainable with random quantum codes. We generalize the classical result of Thommesen to the quantum case, thereby demonstrating the existence of concatenated quantum codes that can asymptotically attain the quantum GV bound. The outer codes are quantum generalized Reed-Solomon codes, and the inner codes are random independently chosen stabilizer codes, where the rates of the inner and outer codes lie in a specified feasible region.

We next study upper bounds on the quantum capacity of some low dimension quantum channels. The quantum capacity of a quantum channel is the maximum rate at which quantum information can be transmitted reliably across it, given arbitrarily many uses of it. While it is known that random quantum codes can be used to attain the quantum capacity, the quantum capacity of many classes of channels is undetermined, even for channels of low input and output dimension. For example, depolarizing channels are important quantum channels, but do not have tight numerical bounds. We obtain upper bounds on the quantum capacity of some unital and non-unital channels – two-qubit Pauli channels, two-qubit depolarizing channels, two-qubit locally symmetric channels, shifted qubit depolarizing channels, and shifted two-qubit Pauli channels – using the coherent information of some degradable channels. We use the notion of twirling quantum channels, and Smith and Smolin's method of constructing degradable extensions of quantum channels extensively. The degradable channels we introduce, study and use are two-qubit amplitude damping channels. Exploiting the notion of covariant quantum channels, we give sufficient conditions for the quantum capacity of a degradable channel to be the optimal value of a concave program with linear constraints, and show that our two-qubit degradable amplitude damping channels have this property.

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Dedication

To my parents,
my late grandparents,
and
most unreservedly to
my favorite one.

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Chapter 1

Introduction

1.1 Motivation

The central object of this thesis is the quantum channel, which is a map from quantum states to quantum states. Quantum channels are building blocks of quantum computers and quantum networks. There are many explicit examples where the abstract notion of the quantum channel can be applied. For example, the perfect storage and transport of quantum information can also both be modeled using the trivial quantum channel, which is the identity map. Storage of quantum information necessarily occurs when there is idle time in the quantum information processing task, and quantum transport necessarily occurs when we have to move quantum information from one location to another.

However in most practical situations, quantum channels are noisy. Quantum information once exposed to the environment, often decoheres. The type of decoherence the quantum information undergoes determines the form of the noisy quantum channel. In a quantum cryptographic setting, an eavesdropper can eavesdrop on the quantum information that is transmitted between two parties, thereby introducing quantum noise into the communication channel. Even if the eavesdropper does not exist, two parties that are communicating quantum information may assume that their quantum channel belongs to a family of quantum channels, and then concoct strategies to protect their quantum information against all quantum channels from that family.

Hence we consider the canonical problem of quantum coding, where Alice wants to send quantum information to Bob over a noisy quantum channel. Alice's goal is to maximize her rate of transmitting quantum information reliably to Bob, given arbitrarily many identical uses of the noisy quantum channel. This rate is known as the quantum capacity, and is the direct quantum analog of the Shannon capacity. The advent of quantum information theory addressing the possibility of dealing with quantum channels and quantum information has greatly deepened our understanding on the limits of how one might harness the additional power that quantum mechanics offers us.

Unlike the classical setting however, the evaluation of the quantum capacity is an optimization problem of potentially unbounded dimensions, and the exact quantum capacity of even many low dimension quantum channels remains unknown. A notable example is the depolarizing channel which replaces the channel's input with the maximally mixed state with some probability, where tight bounds on its quantum capacity are still not available. Hence there is a need to obtain upper and lower bounds of the quantum capacity.

In a practical setting, determining the quantum capacity is even harder, because we often do not have precise knowledge of the exact form of the noisy quantum channel corrupting our quantum information. Evaluation of the transition amplitudes of a quantum channel in a specified basis is non-trivial, even for simple physical models. Thus, the optimization of quantum error correction procedures with respect to physical noise models that are not fully quantifiable is a problem.

These difficulties motivate the work of this thesis. In the first part of the thesis, we study the quantum dynamics of the simplest textbook model, the coupled harmonic oscillator, with the goal of approximating the dynamics on one of the quantum harmonic oscillators with error bounds. In particular, we discuss the utility of truncating a quantum channel to get a quantum operation. We also show how approximate knowledge of the truncated channel can give us worst case bounds on the performance of quantum error correction.

In the second part of the thesis, we study upper and lower bounds on the quantum capacity of various quantum channels. Adversarial quantum channels are those for which an adversary is permitted to corrupt a fixed fraction of blocks of quantum information in the worst way possible. We study the lower bounds of the quantum capacity of the adversarial quantum channels using concatenated codes. We also study upper bounds on quantum capacities of some low dimension quantum channels using the coherent information of other quantum channels.

1.2 Outline of the thesis

The original contributions of this thesis have two parts, each related to the different aspects of the problem of reliable transmission of quantum information across noisy quantum channels. The first part is about truncated quantum channels, and the second part is about upper and lower bounds on the quantum capacities of various channels.

In Chapter 2, we address the problem of quantifying the quantum channel in the coupled harmonic oscillator situation, where one harmonic oscillator defined to be the quantum system, couples through a spring-like interaction term to the environment – another harmonic oscillator in a thermal-like state. In doing so, we investigate the validity of this physical toy-model in accounting for the amplitude damping phenomenon.

In particular, we analyze the dynamics of our coupled harmonic oscillators by treating the completely positive and trace preserving map on the system as a quantum channel. We truncate the channel by truncating its Kraus set, and derive the matrix elements of the Choi-Jamiolkowski operator of the corresponding truncated channel, which are truncated transition amplitudes. These truncated transition amplitudes quantify the typical transitions of the physical model. We approximate the truncated transition amplitudes as weighted sums of computable integrals with convergent error bounds. We next numerically evaluate the approximate truncated transition amplitudes to study the behavior of off-resonant and weakly-coupled harmonic oscillators.

The numerical approximation of our truncated channel is also useful from the perspective of quantum error correction, because specialized recovery operations can be constructed even with knowledge of just approximations to the truncated channel. In Chapter 3, we study the notion of approximate error correction of finite dimension codes with respect to approximations of truncated quantum channels. We use the rescaled recovery map of Leung *et al.* to obtain a lower bound on the entanglement fidelity of a quantum code, given the truncated Kraus set of the noisy channel. This extends the Leung *et al.* result to the case where the set of Leung *et al.* code projectors are not orthogonal. As a consequence, we obtain worst case bounds on the entanglement fidelity of quantum codes with respect to channels with given approximate truncated representations.

In Chapter 4, we study the achievable performance of concatenated codes under blocks of quantum channels where a fixed fraction of the blocks are adversarially corrupted. The best known achievable rate at which information can be sent across these adversarially corrupted quantum qudit channels of dimension less than seven is known as the quantum Gilbert-Varshamov (GV) bound, and is a lower bound on the quantum capacity of adversarial quantum channels. While the quantum GV bound is known to be achievable using random codes, random codes have little structure. We generalize the classical result of Thommesen [Tho83] to the quantum case, demonstrating the existence of concatenated quantum codes that can asymptotically attain the quantum GV bound. The outer codes are quantum generalized Reed-Solomon codes, and the inner codes are random independently chosen stabilizer codes, where the rates of the inner and outer codes lie in a specified feasible region. The advantage of our construction is that the concatenated structure of our code construction leads to a speed-up in decoding time of our concatenated codes as compared to the decoding time of random codes.

In Chapter 5, we generalize the technical results of Smith and Smolin [SS08] pertaining to the use of degradable extensions to obtain upper bounds on the quantum capacity of channels in terms of the coherent information of other channels. Finite dimension degradable channels have quantum capacities that equal to the maximum value of their coherent informations optimized over their bounded domain, and are hence tractable to evaluate (see equation (18) of [DS05]). We extend Smith and Smolin's procedure to show that the quantum capacity of channel twirled over a particular unitary group is at most its coherent information of maximized over a strict subset of the entire state space, where this subset is the image of a channel that conjugates input states with unitaries uniformly at random from the chosen unitary group. Smith and Smolin's recipe is produced as a special case of our extension when the projective commutative unitary group is chosen to be the qubit Clifford group, where they provided an upper bound of the quantum capacity of the qubit depolarizing channel that was the coherent information of the qubit amplitude damping channel evaluated on the maximally mixed state. A consequence of our generalization is that degradable channels that are covariant with respect to diagonal Pauli matrices have quantum capacities that are their coherent information maximized over just the diagonal input states, which is just the maximization of a concave objective function subject to linear constraints.

As an application, we supply new upper bounds on the quantum capacity of some unital and non-unital channels – m-qubit depolarizing channels, two-qubit locally symmetric Pauli channels, and shifted qubit depolarizing channels. The main ingredients that we introduce to obtain these new upper bounds are our higher dimension amplitude damping channels that are degradable. These higher dimension amplitude damping channels are generalizations of the qubit amplitude damping channels.

Part I

Truncated quantum channels

Chapter 2

Truncated Quantum Channel Representations for Coupled Harmonic Oscillators

2.1 Introduction

One of the canonical physical models in quantum physics is that of quantum oscillators coupled with harmonic baths. The dynamics of such models and their variations has been extensively studied, using various techniques [FKM65, RLL67, EKN68, BBW73, Dav73, MH86, NRSS09, YUKG88, HPZ92, CYH08, MG12]. These techniques include Markovian master equations [Dav74], quantum stochastic processes and quantum Langevin equations [Dav69, Dav70, Dav71, FKM65, BK81, FLO88], Kossakowski-Lindblad equations [Kos72, Lin76], methods in density-functional theory [Par94], the standard techniques of perturbation theory, as well as many others [KK83, LCD+87].

Quantum channels [NC00] can be used to quantify the dynamics of a quantum system. There are at least two important representations of quantum channels – the Kraus representation [NC00] and the Choi-Jamiolkowski representation [Cho75]. Both of these representations fully describe the dynamics of quantum systems. The matrix elements of the Choi-Jamiolkowski operator quantify the probability amplitude that a specified matrix element of the input state's density operator contributes to another specified matrix element of the output state's density operator, and hence can be interpreted as transition amplitudes. In this chapter, we approximate the truncated transition amplitudes of a given channel, where the truncation is performed with respect to the quantum channel's set of Kraus operators. hilA truncated quantum channel quantifies the partial dynamics acting on the system, and its knowledge has utility – lower bounds on the performance of quantum error correction codes with its knowledge [LNCY97, BK02, FSW08, KSL08, BG09, Tys10, BO10, BO11, Ouy].

In this chapter, we work towards quantifying the approximate dynamics of a pair of initially decoupled quantum harmonic oscillators interacting with a spring-like coupling, where the bath oscillator is initially in a thermal-like state. We work with a truncated subset of the model's Kraus operators, and thereby approximate its truncated transition amplitudes, which are matrix elements of the Choi-Jamiolkowski operator of the truncated quantum channel. We note that the Kraus operators of oscillator-bath models have also been approximated by various authors [MH86, CLY97, LOMI04]. Recently, Holevo also gave a formal exact expression for the Choi-Jamiolkowski operator for Gaussian channels [Hol11], which describes the dynamics of coupled oscillators. Our contributions in this chapter, are the explicit upper bounds on the approximation error of the truncated transition amplitudes of two quantum harmonic oscillators coupled via a spring-like interaction, where the approximation is an explicit summation of a finite number of computable terms. Our results can be used to explicitly study this toy model with rigorous error bounds. In particular, we numerically demonstrate and provide lower bounds for the leakage error, and show how this leakage error is mitigated via quantum error correction.

The organization of the chapter is as follows. In Section 2.2, we introduce the preliminary material needed for this chapter. In particular, we review notions related to the $L^2(\mathbb{R})$ Hilbert space, quantum states, quantum channels, Hermite functions, and the linear canonical transformations for the quantum harmonic oscillator. In Section 2.3, we give a treatment of the truncated dynamics of two quantum harmonic oscillators interacting with a spring-like coupling, and give explicit bounds on the error term induced by approximating the truncated transition amplitudes with a finite sum in Theorem 2.3.1. In Section 2.4 we give bounds on Hermite functions that are needed for the proof of Theorem 2.3.1. Finally we apply our results explicitly in Section 2.5 in the case where the oscillators are off-resonant and weakly coupled.

2.2 Preliminaries

In this section, we review the theory of $L^2(\mathbb{R})$ Hilbert spaces, quantum states and various representations of quantum channels, Hermite polynomials and functions, and coupled harmonic oscillators.

2.2.1 The $L^2(\mathbb{R})$ Hilbert spaces

The theory of quantum mechanics builds upon the formalism of Hilbert spaces, where the dimensions of these Hilbert spaces are typically infinite in physically realistic scenarios. Hilbert spaces are complex inner product spaces for which the induced norm is complete. A separable space is one that admits a countable dense subset, and we restrict our attention to separable Hilbert spaces [RS72, NNB75], because Hilbert spaces are separable if and only if they admit countable bases, and countable bases are convenient to work with.

The infinite dimension Hilbert space that we work with in this chapter is the space $L^2(S)$ of square-integrable functions with respect to the Lebesgue measure over the set $S \subseteq \mathbb{R}$. While this chapter focuses on the case when the set S is the real line $\mathbb{R}$, other choices of S such as the unit interval may be more appropriate, depending on the physical model at hand. Examples of countable bases of $L^2(\mathbb{R})$ and $L^2([0,1])$ are the set of Hermite functions given in (2.2.11) and the set of sinusoidal functions $\{\sin n\pi x : x \in [0,1]\}_{n\in\mathbb{N}}$ respectively. We use the Dirac's 'ket' $|f_{\mathcal{H}}\rangle$ to denote a function f in the Hilbert space, and is typically called a 'wavefunction' in physics nonmenclature. We denote a generic countable basis of a separable Hilbert space $\mathcal{H}$ as $\{|j_{\mathcal{H}}\rangle\}_{j\in\mathbb{N}} \subset \mathcal{H}$.

In this chapter, we often work with tensor products of Hilbert spaces, and hence we use the notation $|f_{\mathcal{H}}, g_{\mathcal{K}}\rangle$ and $|f_{\mathcal{H}}\rangle|g_{\mathcal{K}}\rangle$ to denote $|f_{\mathcal{H}}\rangle \otimes |g_{\mathcal{K}}\rangle$. We will drop the explicit Hilbert space label on our 'kets' when the label is clear from the context. For Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, let $L(\mathcal{H}, \mathcal{K})$ denote the set of linear operators mapping $\mathcal{H}$ to $\mathcal{K}$, and let $L(\mathcal{H}) := L(\mathcal{H}, \mathcal{H})$. Let $\mathfrak{B}(\mathcal{H}, \mathcal{K})$ denote the set of bounded operators in $L(\mathcal{H}, \mathcal{K})$.

It is convenient to use the language of sesquilinear forms on an $L^2(\mathbb{R})$ Hilbert space as opposed to the language of linear operators. We believe that the formalism of sesquilinear forms is a natural one to elucidate some of the intricacies of the functional analysis of unbounded operators that we will often encounter in the context of the physics of quantum harmonic oscillators.

A sesquilinear form on a Hilbert space $\mathcal{H}$ is a map $\mathcal{H} \times \mathcal{H} \to \mathbb{C}$ that is linear in one argument and conjugate-linear in the other. Using the Dirac notation, the inner product on a Hilbert space $L^2(\mathbb{R})$ is a sesquilinear form that maps the function pair (f, g) to the Dirac bra-ket $\langle f|g\rangle_{\mathcal{H}} := \int_{\mathbb{R}} f(x)^* g(x) dx \in \mathbb{C}$. From this definition of the inner product, it is clear that $\langle f|g\rangle = \langle g|f\rangle^*$. We interpret the argument x of our functions in our $L^2(\mathbb{R})$ function space as a 'position coordinate'. Physically, the functions f and g are wavefunctions with wave amplitudes $f(x)$ and $g(x)$ at position $x \in \mathbb{R}$. We often use the Dirac notation

$$H = \sum_{j,k \in \mathbb{N}} h_{j,k} |j\rangle\langle k| \tag{2.2.1}$$

to denote a sesquilinear form that maps the function pair (f, g) to

$$\sum_{j,k \in \mathbb{N}} h_{j,k} \langle f|j\rangle\langle k|g\rangle, \tag{2.2.2}$$

where $\{|j\rangle\}_{j \in \mathbb{N}}$ denotes an orthonormal countable basis of $\mathcal{H}$ and $h_{j,k} \in \mathbb{C}$. However there might exist function pairs (f, g) for which the expression (2.2.2) is undefined, especially when $|h_{j,k}|$ is unbounded with respect to j and k. This motivates the definition of $\Gamma(H)$, the graph of a sesquilinear form H, which is the set of function pairs (f, g) for which $\langle f|H|g\rangle$ is defined. We say that a sesquilinear form H is densely defined on $\mathcal{H} \times \mathcal{H}$ if $\Gamma(H)$ is a dense subset of $\mathcal{H} \times \mathcal{H}$. When referring to sesquilinear forms, we adopt the convention where we implicitly refer to the graph on which these sesquilinear forms are defined.

Now let $\mathcal{H} = L^2(\mathbb{R})$. The physicist's position and momentum operators $\hat{x}_{\mathcal{H}}$ and $\hat{p}_{\mathcal{H}}$ are sesquilinear forms that map the function pair $(f, g) \in \mathcal{H} \times \mathcal{H}$ to $\langle f_{\mathcal{H}}|\hat{x}_{\mathcal{H}}|g_{\mathcal{H}}\rangle := \int_{\mathbb{R}} f(x)^* x g(x) dx$ and $\langle f_{\mathcal{H}}|\hat{p}_{\mathcal{H}}|g_{\mathcal{H}}\rangle := \int_{\mathbb{R}} f(x)^* \frac{\hbar \partial}{i \partial x} g(x) dx$ respectively. We define the Hermitian sesquilinear forms $\hat{x}^\dagger_{\mathcal{H}}$ and $\hat{p}^\dagger_{\mathcal{H}}$ to map the function pair (f, g) to $\langle f|\hat{x}^\dagger|g\rangle = \langle g|\hat{x}|f\rangle^*$ and $\langle f|\hat{p}^\dagger|g\rangle = \langle g|\hat{p}|f\rangle^*$ respectively. If h is a univariate power series defined on a real line (not necessarily in $L^2(\mathbb{R})$), we define the sesquilinear forms $h(\hat{x})$ and $h(\hat{p})$ to map the function pair (f, g) to $\int_{\mathbb{R}} f(x)^* h(x) g(x) dx$ and $\int_{\mathbb{R}} f(x)^* h(\frac{\hbar}{i}\frac{\partial}{\partial x}) g(x) dx$ respectively if they exist. The reduced Planck constant $\hbar$ makes an appearance in the definition of our momentum operator because we will work in SI units and thereby not make the assumption that $\hbar = 1$.

We now give an example of a sesquilinear function that is not everywhere defined. Consider the sesquilinear form $(\hat{x})^2$ which is proportional to the potential energy term in the Hamiltonian of a quantum harmonic oscillator and the function $f(x) = \frac{1}{\sqrt{x^2+1}}$. Note that while $\langle f|f\rangle = \pi$ which implies that $f \in L^2(\mathbb{R})$, the integral $\langle f|(\hat{x})^2|f\rangle$ is undefined. Intuitively, this is because the function f does not have tails that decay rapidly enough with respect to the sesquilinear form $(\hat{x})^2$. Now we denote $\{|j\rangle\}_{j\in\mathbb{N}}$ as the basis of Hermite functions, and let $|\psi\rangle = \sum_{j\in\mathbb{N}} a_j|j\rangle$ and $|\phi\rangle = \sum_{j\in\mathbb{N}} b_j|j\rangle$ where $\sum_{j\in\mathbb{N}} |a_j|^2 < \infty$ and $\sum_{j\in\mathbb{N}} |b_j|^2 < \infty$. Then $\langle\psi|f(\hat{x}) + g(\hat{p})|\phi\rangle$ is always defined for all polynomial functions f and g, because of the exponentially decaying tails of Hermite functions. In this sense, the sesquilinear form $f(\hat{x}) + g(\hat{p})$ is densely defined on $\mathcal{H} \times \mathcal{H}$ (but not everywhere defined in general). If $H = f(\hat{x}) + g(\hat{p})$ is the Hamiltonian of our physical system, then $\Gamma(H)$ describes the set of 'physical' input and output states.

Following the Dirac notation, we denote the map of a general sesquilinear form H on the function pair $(f, g) \in \mathcal{H} \times \mathcal{H}$ as $\langle f|H|g\rangle$. Given a sesquilinear form H on $\mathcal{H} \times \mathcal{H}$, if there exists some countable basis $\{|j_{\mathcal{H}}\rangle\}_{j\in\mathbb{N}} \subset \mathcal{H}$ and complex sequence $\{\lambda_j\}_{j\in\mathbb{N}}$ such that for every $j, k \in \mathbb{N}$, we have

$$\langle j|H|k\rangle = \delta_{j,k}\lambda_j, \tag{2.2.3}$$

then we say that H has a countable spectrum $\{\lambda_j\}_{j\in\mathbb{N}}$. Examples of sesquilinear forms that do not admit countable spectra include the position and the momentum operator. If the Hamiltonian H of a quantum system admits a countable spectrum and satisfies (2.2.3), we say that λ_j is its j-th energy eigenvalue with $|j\rangle$ being the corresponding energy eigenfunction.

2.2.2 Quantum states and channels

We refer the reader to [NC00] for an introduction to quantum states and channels. Define the set of quantum states on Hilbert space $\mathcal{H}$ to be $\mathfrak{D}(\mathcal{H})$, the set of all positive semi-definite and trace one operators in $\mathfrak{B}(\mathcal{H})$. When $\rho \in \mathfrak{D}(\mathcal{H} \otimes \mathcal{K})$, we denote the partial trace of ρ on Hilbert space $\mathcal{H}$ as $\mathrm{Tr}_{\mathcal{H}}(\rho) := \sum_j \langle j_{\mathcal{H}}|\rho|j_{\mathcal{H}}\rangle$.

A quantum channel $\Phi : L(\mathcal{H}) \to L(\mathcal{K})$ is a completely positive and trace-preserving (CPT) linear map, and its non-unique Kraus representation is [HK69, HK70, Kra83]

$$\Phi(\rho) = \sum_{\mathbf{K}\in\mathfrak{K}} \mathbf{K}\rho\mathbf{K}^\dagger,$$

where $\mathfrak{K} \subset \mathfrak{B}(\mathcal{K}, \mathcal{H})$ is called the Kraus set of Φ and the Kraus operators in the Kraus set satisfy the completeness relation

$$\sum_{\mathbf{K}\in\mathfrak{K}} \mathbf{K}^\dagger\mathbf{K} = \mathbb{1}_{\mathcal{H}}.$$

Note that in a universe with an underlying Hamiltonian that admits a countable spectrum, the Kraus set $\mathfrak{K}$ is countable, because the unitary operation describing the dynamics of the universe can be written in the form of (2.2.3). We denote the basis-dependent matrix elements of the Kraus operators by $\mathbf{K}_{j,j'}$ so that for all $\mathbf{K} \in \mathfrak{K}$,

$$\mathbf{K} = \sum_{j,j'} \mathbf{K}_{j,j'} |j'_{\mathcal{K}}\rangle\langle j_{\mathcal{H}}|.$$

We define the **transition amplitudes** of Φ with respect to the Kraus set $\mathfrak{K}$ to be

$$T_{\mathfrak{K}}^{(a,b)\to(a',b')} := \sum_{\mathbf{K}\in\mathfrak{K}} \mathbf{K}_{b,b'}\mathbf{K}^*_{a,a'}, \tag{2.2.4}$$

a sum of the product of two Kraus operators over the entire Kraus set. Now let $\rho \in \mathfrak{D}(\mathcal{H})$ and $\Phi(\rho) \in \mathfrak{D}(\mathcal{K})$ have the decompositions

$$\rho = \sum_{a,b} \rho_{a,b}|b_{\mathcal{H}}\rangle\langle a_{\mathcal{H}}|, \quad \Phi(\rho) = \sum_{a',b'} \rho'_{a',b'}|b'_{\mathcal{K}}\rangle\langle a'_{\mathcal{K}}|$$

so that in the Kraus representation,

$$\begin{aligned}
&\langle b'_{\mathcal{K}}|\Phi(\rho)|a'_{\mathcal{K}}\rangle = \langle b'_{\mathcal{K}}|\sum_{\mathbf{K}\in\mathfrak{K}} \mathbf{K}\rho\mathbf{K}^\dagger|a'_{\mathcal{K}}\rangle \\
=&\langle b'_{\mathcal{K}}|\sum_{\mathbf{K}\in\mathfrak{K}}\sum_{j,j'} \mathbf{K}_{j,j'}|j'_{\mathcal{K}}\rangle\langle j_{\mathcal{H}}|\sum_{a,b}\rho_{a,b}|b_{\mathcal{H}}\rangle\langle a_{\mathcal{H}}|\sum_{k,k'}\mathbf{K}^*_{k,k'}|k_{\mathcal{H}}\rangle\langle k'_{\mathcal{K}}|a'_{\mathcal{K}}\rangle \\
=&\sum_{\mathbf{K}\in\mathfrak{K}}\sum_{j,j'}\sum_{a,b}\sum_{k,k'} \mathbf{K}_{j,j'}\mathbf{K}^*_{k,k'}\rho_{a,b}\langle b'|j'\rangle_{\mathcal{K}}\langle j|b\rangle_{\mathcal{H}}\langle a|k\rangle_{\mathcal{H}}\langle k'|a'\rangle_{\mathcal{K}} \\
=&\sum_{a,b\in\mathbb{N}}\left(\sum_{\mathbf{K}\in\mathfrak{K}} \mathbf{K}_{b,b'}\mathbf{K}^*_{a,a'}\right)\rho_{a,b} = \sum_{a,b\in\mathbb{N}} T_{\Phi,\mathfrak{K}}^{(a,b)\to(a',b')}\rho_{a,b}.
\end{aligned} \tag{2.2.5}$$

Hence $T_{\mathfrak{K}}^{(a,b)\to(a',b')}$ quantifies the transition amplitudes of $\langle b_{\mathcal{H}}|\rho|a_{\mathcal{H}}\rangle$ to $\langle b'_{\mathcal{K}}|\Phi(\rho)|a'_{\mathcal{K}}\rangle$. In this chapter, we focus on the Choi-Jamiolkowski (CJ) representation of a channel. Define the stacking isomorphism $|\cdot\rangle\rangle : L(\mathcal{H},\mathcal{K}) \to \mathcal{K}\otimes\mathcal{H}$ to be a linear map such that

$$|\sum_{i,j} a_{i,j}|i_{\mathcal{K}}\rangle\langle j_{\mathcal{H}}|\rangle\rangle := \sum_{i,j} a_{i,j}|i_{\mathcal{K}}\rangle|j_{\mathcal{H}}\rangle.$$

Then the CJ operator of the channel Φ with Kraus set $\mathfrak{K}$ is the linear operator $X_\Phi \in L(\mathcal{K},\mathcal{H})$ where

$$X_\Phi := \sum_{\mathbf{K}\in\mathfrak{K}} |\mathbf{K}\rangle\rangle\langle\langle\mathbf{K}| = \sum_{j,j'}\sum_{k,k'}\left(\sum_{\mathbf{K}\in\mathfrak{K}} \mathbf{K}_{j,j'}\mathbf{K}^*_{k,k'}\right)|j_{\mathcal{K}}, j'_{\mathcal{H}}\rangle\langle k_{\mathcal{K}}, k'_{\mathcal{H}}|. \tag{2.2.6}$$

The CPT conditions for channel Φ in the CJ representation are that X_Φ is positive semidefinite and has a trace of one. Knowledge of the CJ operator X_Φ enables us to evaluate the image of Φ because

$$\begin{aligned}
&\langle b'_{\mathcal{K}}| \operatorname{Tr}_{\mathcal{H}}(X_\Phi(\mathbb{1}_{\mathcal{K}} \otimes \rho^T))|a'_{\mathcal{K}}\rangle = \langle b'_{\mathcal{K}}| \sum_{\mathbf{K}\in\mathfrak{K}} \operatorname{Tr}_{\mathcal{H}}(|\mathbf{K}\rangle\!\rangle\langle\!\langle\mathbf{K}|\rho^T)|a'_{\mathcal{K}}\rangle \\
&= \sum_{j,j'}\sum_{k,k'}\sum_{a,b}\left(\sum_{\mathbf{K}\in\mathfrak{K}} \mathbf{K}_{j,j'}\mathbf{K}^*_{k,k'}\right)\rho_{a,b}\langle b'_{\mathcal{K}}, b_{\mathcal{H}}|\left(|j'_{\mathcal{K}}, j_{\mathcal{H}}\rangle\langle k'_{\mathcal{K}}, k_{\mathcal{H}}|\right)|a'_{\mathcal{K}}, a_{\mathcal{H}}\rangle \\
&= \sum_{a,b}\left(\sum_{\mathbf{K}\in\mathfrak{K}} \mathbf{K}_{b,b'}\mathbf{K}^*_{a,a'}\right)\rho_{a,b} = \sum_{a,b\in\mathbb{N}} T^{(a,b)\to(a',b')}_{\Phi,\mathfrak{K}}\rho_{a,b}.
\end{aligned} \tag{2.2.7}$$

The equivalence of (2.2.5) and (2.2.7) implies that

$$\Phi(\rho) = \operatorname{Tr}_{\mathcal{H}}(X_\Phi(\mathbb{1}_{\mathcal{K}} \otimes \rho^T)) = \sum_{\mathbf{K}\in\mathfrak{K}} \mathbf{K}\rho\mathbf{K}^\dagger. \tag{2.2.8}$$

The matrix elements of X_Φ are also transition amplitudes, in the sense that

$$T^{(a,b)\to(a',b')}_{\mathfrak{K}} = \langle b'_{\mathcal{K}}, b_{\mathcal{H}}|X_\Phi|a'_{\mathcal{K}}, a_{\mathcal{H}}\rangle. \tag{2.2.9}$$

Now define the **transition operator** corresponding to the Kraus set $\mathfrak{K}$ to be $\mathbf{T}_{\mathfrak{K}} : \mathcal{H}\otimes\mathcal{H} \to \mathcal{K}\otimes\mathcal{K}$ where

$$\mathbf{T}_{\mathfrak{K}} := \sum_{a,b,a',b'} T^{(a,b)\to(a',b')}_{\mathfrak{K}}|a'_{\mathcal{H}}, b'_{\mathcal{H}}\rangle\langle a_{\mathcal{K}}, b_{\mathcal{K}}|.$$

Then the stacking isomorphsim $|\cdot\rangle\!\rangle$ and the transition operator are related by the equation

$$\mathbf{T}_{\mathfrak{K}}|\rho\rangle\!\rangle = |\Phi(\rho)\rangle\!\rangle. \tag{2.2.10}$$

For the purpose of quantum information processing, it may not be necessary to work with the full Kraus set $\mathfrak{K}$. In this chapter, we instead restrict our attention to the **truncated Kraus set** Ω, which is some appropriately chosen subset of the full Kraus set. This truncation procedure approximates the channel well if the truncated Kraus effects are ‘atypical’. For the purpose of quantum error correction, partial knowledge of the channel is already of great utility, and recovery channels can be constructed based on this partial information to give lower bounds on the entanglement fidelity of specifically chosen quantum codes. Hence in this chapter, the **truncated transition amplitudes** $T_{\Omega}^{(a,b)\to(a',b')}$ play a central role in quantifying the truncated dynamics of the channel Φ.

2.2.3 Hermite polynomials and functions

The energy eigenfunctions of the Hamiltonian of a quantum harmonic oscillator are Hermite functions, and hence arise naturally in the study of coupled ensembles of quantum harmonic oscillators. Hermite functions are products of Hermite polynomials with a gaussian function, and form a basis for the $L^2(\mathbb{R})$ function space. For $n \in \mathbb{N}$, define the Hermite polynomials $H_n(x)$ and the Hermite functions $\psi_n(x)$ to be

$$H_n(x) := (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}, \quad \psi_n(x) := \frac{e^{-\frac{1}{2}x^2} H_n(x)}{\sqrt{2^n n! \sqrt{\pi}}}. \tag{2.2.11}$$

For example, $H_0(x) = 1$ and $H_1(x) = 2x$. Here, we have used the physicist’s convention for the Hermite functions, as opposed to the probabilist’s convention. The properties of the Hermite polynomials and functions have been extensively studied, and a reference to more of their properties can be found in [AS64]. For $c > 0$, also define the rescaled Hermite function to be

$$\psi_{n,c}(x) = \langle \tilde{x}_{\mathcal{H}} | \psi_{n,c,\mathcal{H}} \rangle := \sqrt{c}\psi_n(cx). \tag{2.2.12}$$

When the rescaling constant c is chosen as an appropriate function of the mass and resonant frequency of our quantum harmonic oscillator, the rescaled Hermite function $\psi_{n,c}(x)$ is the n-th energy eigenfunction of the corresponding quantum harmonic oscillator’s Hamiltonian.

We often have to deal with infinite sums involving Hermite functions, and Mehler’s formula gives a closed form expression for one such sum. As stated by Watson [Wat33], Mehler's formula applies in the case when $|z| < 1$ and z is real, that is

$$\sum_{n=0}^{\infty} z^n \psi_n(x)\psi_n(y) = \frac{1}{\sqrt{\pi(1-z^2)}} \exp\left[\frac{4xyz - (x^2+y^2)(1+z^2)}{2(1-z^2)}\right]. \tag{2.2.13}$$

Mehler's formula also holds for all complex numbers $|z| < 1$, with the series converging uniformly and absolutely [Theorem 23.1 [Won98]]. However when $|z| = 1$ the sum is undefined, which makes evaluating the truncated transition amplitudes of our toy model as a closed form expression quite intractable.

2.2.4 A pair of harmonic oscillators and their linear canonical transformations

The classical model

The classical Hamiltonian $H(\mathbf{p}, \mathbf{q}, t)$ of a physical system quantifies the amount of its total energy, and is a function of its generalized coordinates $\mathbf{q}$, generalized momentum $\mathbf{p}$, and time t. The power of the Hamiltonian formalism in classical mechanics is demonstrated from the ease at which one can derive the corresponding equations of motion of the physical system from the classical Hamiltonian. We introduce notation related to the classical harmonic oscillator, with the goal of reviewing its corresponding quantum description. The interested reader may refer to [Sha94] for an introduction to the quantum harmonic oscillator where the Hamiltonian formalism is used.

Define the classical Hamiltonian of a classical harmonic oscillator with mass m, resonant frequency ω, position coordinate x and momentum coordinate p to be

$$H_{m,\omega;x,p} := \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2. \tag{2.2.14}$$

The model we study has the classical Hamiltonian

$$H = H_{m_{\mathrm{x}},\omega_{\mathrm{x},\mathbf{o}};x,p_{\mathrm{x}}} + H_{m_{\mathrm{y}},\omega_{\mathrm{y},\mathbf{o}};y,p_{\mathrm{y}}} + H_{\mathrm{int},\mathbf{o}}.$$

where $H_{\text{int},\mathbf{o}} := \frac{1}{2}k(x-y)^2$ is the classical Hamiltonian representing the spring-like interaction between the oscillators where $k \geq 0$. The spring-like interaction $H_{\text{int},\mathbf{o}}$ introduces quadratic terms $\frac{kx^2}{2}$ and $\frac{ky^2}{2}$ into H, effectively renormalizing the oscillator frequencies from $\omega_{\text{x},\mathbf{o}}$ and $\omega_{\text{y},\mathbf{o}}$ to $\omega_{\text{x}} := \sqrt{\omega_{\text{x},\mathbf{o}}^2 + \frac{k}{m_{\text{x}}}}$ and $\omega_{\text{y}} := \sqrt{\omega_{\text{y},\mathbf{o}}^2 + \frac{k}{m_{\text{y}}}}$ respectively. Hence when $H_{\text{int}} := -kxy$,

$$H = H_{m_{\text{x}},\omega_{\text{x}};x,p_{\text{x}}} + H_{m_{\text{y}},\omega_{\text{y}};y,p_{\text{y}}} + H_{\text{int}}. \tag{2.2.15}$$

In an experimental setup, it may be impossible to turn off the interaction between the two oscillators. Then the physically measured oscillator frequencies correspond to the renormalized frequencies. Therefore, we work with the renormalized representation of the model Hamiltonian given by (2.2.15).

The quantized model

Define the Hamiltonian of a quantum harmonic oscillator with associated Hilbert space $\mathcal{H}$, mass $M > 0$, resonant frequency $\omega > 0$, position operator $\hat{x}_{\mathcal{H}}$ and momentum operator $\hat{p}_{\mathcal{H}}$ to be

$$\mathbf{H}_{M,\omega;\hat{x}_{\mathcal{H}},\hat{p}_{\mathcal{H}}} := \frac{\hat{p}_{\mathcal{H}}^2}{2M} + \frac{1}{2}M\omega^2\hat{x}_{\mathcal{H}}^2. \tag{2.2.16}$$

The set of rescaled Hermite functions $\{|\psi_{n,\sqrt{\frac{M\omega}{\hbar}},\mathcal{H}}\rangle\}_{n\in\mathbb{N}}$ is the set of energy eigenfunctions of the Hamiltonian $\mathbf{H}_{(M,\omega;\hat{x}_{\mathcal{H}},\hat{p}_{\mathcal{H}})}$. Let the Hilbert space of the first and second oscillators be $\mathcal{X}$ and $\mathcal{Y}$ respectively, both isomorphic to $L^2(\mathbb{R})$. Define $\hat{x} := \hat{x}_{\mathcal{X}} \otimes \mathbb{1}_{\mathcal{Y}}$, $\hat{y} := \mathbb{1}_{\mathcal{X}} \otimes \hat{x}_{\mathcal{Y}}$, $\hat{p}_x := \hat{p}_{\mathcal{X}} \otimes \mathbb{1}_{\mathcal{Y}}$ and $\hat{p}_y := \mathbb{1}_{\mathcal{X}} \otimes \hat{p}_{\mathcal{Y}}$. Then the quantized model Hamiltonian (2.2.15) is

$$\begin{aligned}\mathbf{H} &= \mathbf{H}_{m_{\text{x}},\omega_{\text{x}};\hat{x}_{\mathcal{X}},\hat{p}_{\mathcal{X}}} \otimes \mathbb{1}_{\mathcal{Y}} + \mathbb{1}_{\mathcal{X}} \otimes \mathbf{H}_{m_{\text{y}},\omega_{\text{y}};\hat{x}_{\mathcal{Y}},\hat{p}_{\mathcal{Y}}} + \mathbf{H}_{\text{int}} \\ &= \mathbf{H}_{m_{\text{x}},\omega_{\text{x}};\hat{x},\hat{p}_x} + \mathbf{H}_{m_{\text{y}},\omega_{\text{y}};\hat{y},\hat{p}_y} + \mathbf{H}_{\text{int}}\end{aligned} \tag{2.2.17}$$

where $\mathbf{H}_{\text{int}} := -k\hat{x}\hat{y}$ is the quantized interaction.

The coupled quantum harmonic oscillators can be decoupled by a linear canonical transformation of the oscillator positions and momenta [JMM11]. Define the rotation matrix, the rotation angle, and the rescaled mass by

$$\mathbf{R} := \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \qquad \theta := \frac{1}{2}\tan^{-1}\left(\frac{2k/m}{\omega_{\rm y}^2 - \omega_{\rm x}^2}\right), \quad m := \sqrt{m_{\rm x} m_{\rm y}} \tag{2.2.18}$$

respectively. The use of straightforward trigonometry then gives

$$\cos\theta = \frac{1}{\sqrt{2}}\left(1 + \frac{1}{\sqrt{1 + \frac{4k^2/m^2}{(\omega_{\rm y}^2 - \omega_{\rm x}^2)^2}}}\right)^{1/2}, \quad \sin\theta = \frac{1}{\sqrt{2}}\left(1 - \frac{1}{\sqrt{1 + \frac{4k^2/m^2}{(\omega_{\rm y}^2 - \omega_{\rm x}^2)^2}}}\right)^{1/2}. \tag{2.2.19}$$

where

$$\frac{4k^2/m^2}{(\omega_{\rm y}^2 - \omega_{\rm x}^2)} = 4\left(\frac{\sqrt{m_{\rm x} m_{\rm y}}}{k}(\omega_{\rm y,\mathbf{o}}^2 - \omega_{\rm x,\mathbf{o}}^2) + \frac{m_{\rm x} - m_{\rm y}}{\sqrt{m_{\rm x} m_{\rm y}}}\right)^{-2}.$$

Note that the rotation angle quantifies the strength of the coupling, in the sense that $\cos\theta \approx 1$ and $\sin\theta \approx 0$ when the coupling constant k is small and the oscillators are off-resonant. Define the normalization parameter $\mu := \sqrt[4]{\frac{m_{\rm x}}{m_{\rm y}}}$. Then we choose the transformed position and momenta operators to be given by

$$\begin{aligned} (\hat{u}, \hat{v}) &:= \mathbf{R}(\mu^{-1}\hat{x}, \mu\hat{y}), \\ (\hat{p}_u, \hat{p}_v) &:= \mathbf{R}(\mu\hat{p}_x, \mu^{-1}\hat{p}_y) \end{aligned} \tag{2.2.20}$$

where $(x_1, x_2, ...)$ denotes a column vector. The quantized Hamiltonian is $\mathbf{H} = \mathbf{H}_{m,\omega_u;\hat{u},\hat{p}_u} + \mathbf{H}_{m,\omega_v;\hat{v},\hat{p}_v}$ where

$$\omega_{\rm u} = \sqrt{\begin{pmatrix} \omega_{\rm x}^2 \\ \omega_{\rm y}^2 \end{pmatrix} \cdot \begin{pmatrix} \cos^2\theta \\ \sin^2\theta \end{pmatrix} - \frac{k}{m}\sin(2\theta)}, \quad \omega_{\rm v} = \sqrt{\begin{pmatrix} \omega_{\rm x}^2 \\ \omega_{\rm y}^2 \end{pmatrix} \cdot \begin{pmatrix} \sin^2\theta \\ \cos^2\theta \end{pmatrix} + \frac{k}{m}\sin(2\theta)}. \tag{2.2.21}$$

Note that the frequencies $\omega_{\rm u}$ and $\omega_{\rm v}$ are real as long as the original oscillator frequencies $\omega_{\rm x,\mathbf{o}}$ and $\omega_{\rm y,\mathbf{o}}$ before renormalization are also real, because the renormalized frequencies $\omega_{\rm x}$ and $\omega_{\rm y}$ increase as the coupling strength k increases.

The linear canonical transformation that decouples the pair of harmonic oscillators is not unique. We chose the transformation that gives the same mass m for the decoupled oscillators, so that the only parameter different between them are the frequencies ω_u and ω_v. Since the transformation we have performed is canonical, $[\hat{u},\hat{v}] = [\hat{p}_u,\hat{p}_v] = [\hat{u},\hat{p}_v] = [\hat{v},\hat{p}_u] = 0$, $[\hat{u},\hat{p}_u] = i\hbar$ and $[\hat{v},\hat{p}_v] = i\hbar$. Hence there exist Hilbert spaces $\mathcal{U},\mathcal{V}$ isomorphic to $L^2(\mathbb{R})$ such that $\mathcal{X}\otimes\mathcal{Y} = \mathcal{U}\otimes\mathcal{V}$, $\hat{u} = \hat{x}_{\mathcal{U}}\otimes\mathbb{1}_{\mathcal{V}}$, $\hat{v} = \mathbb{1}_{\mathcal{U}}\otimes\hat{x}_{\mathcal{V}}$, $\hat{p}_u = \hat{p}_{\mathcal{U}}\otimes\mathbb{1}_{\mathcal{V}}$ and $\hat{p}_v = \mathbb{1}_{\mathcal{U}}\otimes\hat{p}_{\mathcal{V}}$.

Let $c_{\mathrm{u}} := \sqrt{\frac{m\omega_{\mathrm{u}}}{\hbar}}$ and $c_v := \sqrt{\frac{m\omega_v}{\hbar}}$, $c_{\mathrm{x}} := \sqrt{\frac{m_{\mathrm{x}}\omega_{\mathrm{x}}}{\hbar}}$ and $c_{\mathrm{y}} := \sqrt{\frac{m_{\mathrm{y}}\omega_{\mathrm{y}}}{\hbar}}$. Then the set of eigenstates of the uncoupled Hamiltonian $\mathbf{H}_{m_{\mathrm{x}},\omega_{\mathrm{x}};\hat{x},\hat{p}_{\mathrm{x}}} + \mathbf{H}_{m_{\mathrm{y}},\omega_{\mathrm{y}};\hat{y},\hat{p}_{\mathrm{y}}}$ and the full Hamiltonian $\mathbf{H}$ are $\{|\psi_{\kappa,c_{\mathrm{u}},\mathcal{U}},\psi_{\chi,c_{\mathrm{v}},\mathcal{V}}\rangle\}_{\kappa,\chi\in\mathbb{N}}$ and $\{|\psi_{j,c_{\mathrm{x}},\mathcal{X}},\psi_{\ell,c_{\mathrm{y}},\mathcal{Y}}\rangle\}_{j,\ell\in\mathbb{N}}$ respectively.

2.3 Truncated dynamics of the interacting system

This section highlights the main results of our chapter. We provide a computable approximation to our physical model's truncated channel with corresponding error bounds that are simple to describe.

2.3.1 The general model

The Hilbert space of our model has the general form $\mathcal{H} = \mathcal{X}\otimes\mathcal{Y}$ where $\mathcal{X}$ and $\mathcal{Y}$ are separable Hilbert spaces of the system and the environment respectively. Our model's Hamiltonian is

$$\mathbf{H} := \mathbf{H}_{\mathrm{x}}\otimes\mathbb{1}_{\mathcal{Y}} + \mathbb{1}_{\mathcal{X}}\otimes\mathbf{H}_{\mathrm{y}} + \mathbf{H}_{\mathrm{int}}$$

where $\mathbf{H},\mathbf{H}_{\mathrm{x}},\mathbf{H}_{\mathrm{y}}$ and $\mathbf{H}_{\mathrm{int},j}$ are (typically unbounded) Hermitian operators in the sets $L(\mathcal{H}), L(\mathcal{X}), L(\mathcal{Y})$ and $L(\mathcal{H})$ respectively. The Hamiltonians $\mathbf{H}_{\mathrm{x}}$ and $\mathbf{H}_{\mathrm{y}}$ describes the bare dynamics on $\mathcal{X}$ and $\mathcal{Y}$ respectively, and $\mathbf{H}_{\mathrm{int}}$ describes the system-bath interaction.

Let the initial state of the entire model be $\rho_{\mathrm{all}} := \rho_0\otimes\sigma_{\mathrm{y}}$, where $\rho_0\in\mathfrak{D}(\mathcal{X})$ and $\sigma_{\mathrm{y}} = \sum_{\ell\in\mathbb{N}} p_\ell|\ell_{\mathcal{Y}}\rangle\langle\ell_{\mathcal{Y}}|\in\mathfrak{D}(\mathcal{Y})$. Let the time evolution operator of the entire model at time t be the unitary operator $\mathbf{U}_t\in\mathfrak{B}(\mathcal{H})$. Then the time evolved state of system $\mathcal{X}$ at time t

is

$$
\begin{aligned}
\rho_t := \Phi_t(\rho_0) &= \operatorname{Tr}_{\mathcal{Y}} \mathbf{U}_t \rho_{\text{all}} \mathbf{U}_t^\dagger \\
&= \sum_{j,j',\ell,\ell' \in \mathbb{N}} \langle \ell'_{\mathcal{Y}} | \mathbf{U}_t | j_{\mathcal{Y}} \rangle \langle j_{\mathcal{Y}} | \rho_0 \otimes p_\ell | \ell_{\mathcal{Y}} \rangle \langle \ell_{\mathcal{Y}} | j'_{\mathcal{Y}} \rangle \langle j'_{\mathcal{Y}} | \mathbf{U}_t^\dagger | \ell'_{\mathcal{Y}} \rangle \\
&= \sum_{\ell,\ell' \in \mathbb{N}} \langle \ell'_{\mathcal{Y}} | \mathbf{U}_t | \ell_{\mathcal{Y}} \rangle p_\ell \rho_0 \langle \ell_{\mathcal{Y}} | \mathbf{U}_t^\dagger | \ell'_{\mathcal{Y}} \rangle. \qquad (2.3.1)
\end{aligned}
$$

Using (2.2.4), a feasible Kraus set and transition amplitudes for Φ_t are

$$
\mathfrak{K}_t := \left\{ \sqrt{p_\ell} \langle \ell'_{\mathcal{Y}} | \mathbf{U}_t | \ell_{\mathcal{Y}} \rangle : \ell, \ell' \in \mathbb{N} \right\} \qquad (2.3.2)
$$

$$
T_{\mathfrak{K}_t}^{(a,b)\to(a',b')} = \sum_{\ell,\ell' \in \mathbb{N}} p_\ell \langle b'_{\mathcal{X}}, \ell'_{\mathcal{Y}} | \mathbf{U}_t | b_{\mathcal{X}}, \ell_{\mathcal{Y}} \rangle \langle a_{\mathcal{X}}, \ell_{\mathcal{Y}} | \mathbf{U}_t^\dagger | a'_{\mathcal{X}}, \ell'_{\mathcal{Y}} \rangle. \qquad (2.3.3)
$$

The transition amplitudes (2.3.3) of the full quantum channel might not have closed form expressions and hence be impossible to evaluate. In view of this, we can instead evaluate the truncated transition amplitudes by truncating the infinite summation. These truncated transition amplitudes are the transition amplitudes of the truncated quantum channel.

2.3.2 Coupled harmonic oscillators

The approximate dynamics of coupled harmonic oscillators is still actively studied [CC07, CC09]. In our chapter, we use the model as described in Section 2.2.4. Let $z_u := e^{-i\omega_u t}$ and $z_v := e^{-i\omega_v t}$. For $j \in \mathbb{N}$, define $|j_{\text{x}}\rangle := |\psi_{j,c_{\text{x}},\mathcal{X}}\rangle, |j_{\text{y}}\rangle := |\psi_{j,c_{\text{y}},\mathcal{Y}}\rangle, |j_{\text{u}}\rangle := |\psi_{j,c_{\text{u}},\mathcal{U}}\rangle$, and $|j_{\text{v}}\rangle := |\psi_{j,c_{\text{v}},\mathcal{V}}\rangle$. Then the unitary operator $\mathbf{U}_t$ has the spectral decomposition

$$
\mathbf{U}_t = \sum_{\kappa,\chi \in \mathbb{N}} \sqrt{z_u z_v} z_u^\kappa z_v^\chi |\kappa_{\text{u}}, \chi_{\text{v}}\rangle \langle \kappa_{\text{u}}, \chi_{\text{v}}|. \qquad (2.3.4)
$$

Let $r = \exp(-\frac{\hbar\omega_y}{k_B T}) \in [0,1)$, where k_B is the Boltzmann constant and $0 \le T < \infty$ is the effective temperature of the bath. The state of the bath with a Boltzmannian distribution is

$$\sigma_{\mathrm{y}} = \sum_{\ell \in \mathbb{N}} r^{\ell}(1-r)|\ell_{\mathrm{y}}\rangle\langle\ell_{\mathrm{y}}|. \tag{2.3.5}$$

Thus $p_\ell = r^\ell(1-r)$ in equations (2.3.2) and (2.3.3).

Kraus operators and transition amplitudes

Using (2.3.2), the matrix elements of our Kraus operator $\mathbf{K} \in \mathfrak{K}_t$ indexed by $\ell, \ell' \in \mathbb{N}$ are

$$\begin{aligned}\langle j'_{\mathrm{x}}|\mathbf{K}|j_{\mathrm{x}}\rangle &= \sqrt{p_\ell}\langle j'_{\mathrm{x}}, \ell'_{\mathrm{y}}|\mathbf{U}_t|j'_{\mathrm{x}}, \ell_{\mathrm{y}}\rangle \\ &= \sqrt{p_\ell}\langle j'_{\mathrm{x}}, \ell'_{\mathrm{y}}| \sum_{\kappa,\chi \in \mathbb{N}} z_{\mathrm{u}}^{\kappa+\frac{1}{2}} z_{\mathrm{v}}^{\chi+\frac{1}{2}} |\kappa_{\mathrm{u}}, \chi_{\mathrm{v}}\rangle\langle\kappa_{\mathrm{u}}, \chi_{\mathrm{v}}|j_{\mathrm{x}}, \ell_{\mathrm{y}}\rangle\end{aligned} \tag{2.3.6}$$

The goal is now to find an expression for the truncated transition amplitudes for small values of a, b, a' and b'. We first give an expression for the transition amplitude with respect to the full Kraus set $\mathfrak{K}_t$, which is

$$\begin{aligned}T_{\mathfrak{K}_t}^{(a,b)\to(a',b')} &= \sum_{\ell,\ell' \in \mathbb{N}} r^\ell(1-r)\langle b'_{\mathrm{x}}, \ell'_{\mathrm{y}}|\mathbf{U}_t|b_{\mathrm{x}}, \ell_{\mathrm{y}}\rangle\langle a_{\mathrm{x}}, \ell_{\mathrm{y}}|\mathbf{U}_t^\dagger|a'_{\mathrm{x}}, \ell'_{\mathrm{y}}\rangle \\ &= \sum_{\ell,\ell' \in \mathbb{N}} r^\ell(1-r)\langle b'_{\mathrm{x}}, \ell'_{\mathrm{y}}| \sum_{\kappa,\chi \in \mathbb{N}} z_u^{\kappa+\frac{1}{2}} z_v^{\chi+\frac{1}{2}} |\kappa_{\mathrm{u}}, \chi_{\mathrm{v}}\rangle\langle\kappa_{\mathrm{u}}, \chi_{\mathrm{v}}|b_{\mathrm{x}}, \ell_{\mathrm{y}}\rangle \\ &\qquad \times \langle a_{\mathrm{x}}, \ell_{\mathrm{y}}| \sum_{\kappa',\chi' \in \mathbb{N}} z_u^{-\kappa'-\frac{1}{2}} z_v^{-\chi'-\frac{1}{2}} |\kappa'_{\mathrm{u}}, \chi'_{\mathrm{v}}\rangle\langle\kappa'_{\mathrm{u}}, \chi'_{\mathrm{v}}|a'_{\mathrm{x}}, \ell'_{\mathrm{y}}\rangle.\end{aligned} \tag{2.3.7}$$

The matrix elements in the expression above can be simplified by expressing them in the x and y coordinates of the original oscillators. In particular, the expression above becomes an integral of the product of rescaled Hermite functions. To simplify notion, let $u_{x,y} := c_{\mathrm{u}}(\frac{x}{\mu c_{\mathrm{x}}}\cos\theta + \frac{\mu y}{c_{\mathrm{y}}}\sin\theta)$ and $v_{x,y} := c_{\mathrm{v}}(-\frac{x}{\mu c_{\mathrm{x}}}\sin\theta + \frac{\mu y}{c_{\mathrm{y}}}\cos\theta)$ denote the coordinates of the decoupled oscillators in the basis of the original oscillators. By making appropriate substitutions, we have that

$$
\begin{aligned}
u_{x,y} &= \sqrt{\frac{m_{\mathrm{y}}\omega_{\mathrm{u}}}{m_{\mathrm{x}}\omega_{\mathrm{x}}}}x\cos\theta + \sqrt{\frac{m_{\mathrm{x}}\omega_{\mathrm{u}}}{m_{\mathrm{y}}\omega_{\mathrm{y}}}}y\sin\theta \\
v_{x,y} &= -\sqrt{\frac{m_{\mathrm{y}}\omega_{\mathrm{v}}}{m_{\mathrm{x}}\omega_{\mathrm{x}}}}x\sin\theta + \sqrt{\frac{m_{\mathrm{x}}\omega_{\mathrm{v}}}{m_{\mathrm{y}}\omega_{\mathrm{y}}}}y\cos\theta.
\end{aligned} \tag{2.3.8}
$$

The summation indices in the transition amplitude corresponding to the full Kraus set $\mathfrak{K}_t$ in (2.3.7) are ℓ and ℓ' respectively. In this chapter, we choose our truncated Kraus set to be $\Omega_{L,t}$, where only the summation over ℓ' is truncated.

By applying Mehler's formula (2.2.13) on the variable ℓ, the expression for the truncated transition amplitude is

$$
T_{\Omega_{L,t}}^{(a,b)\to(a',b')} = \sum_{\ell'\le L} \sum_{\substack{\kappa,\chi\in\mathbb{N} \\ \kappa',\chi'\in\mathbb{N}}} z_u^{\kappa-\kappa'} z_v^{\chi-\chi'} f[a,b,a',b';\ell',\kappa,\kappa',\chi,\chi'] \frac{\omega_{\mathrm{u}}\omega_{\mathrm{v}}}{\omega_{\mathrm{x}}\omega_{\mathrm{y}}} \sqrt{\frac{1-r}{\pi(1+r)}} \tag{2.3.9}
$$

where the path-dependent and time-independent transition amplitudes $f[a,b,a',b';\ell',\kappa,\kappa',\chi,\chi']$ are

$$
\begin{aligned}
f[a,b,a',b';\ell',\kappa,\kappa',\chi,\chi'] &:= \int_{\mathbf{x},\mathbf{y}\in\mathbb{R}^4} d\mathbf{x}\, d\mathbf{y} \exp\left[-\frac{1+r^2}{2(1-r^2)}\left(y_3^2 - \frac{4ry_3y_4}{1+r^2} + y_4^2\right)\right] \\
&\times \Big(\psi_{a'}(x_1)\psi_{b'}(x_2)\psi_b(x_3)\psi_a(x_4)\Big)\Big(\psi_{\ell'}(y_1)\psi_{\ell'}(y_2)\Big) \\
&\quad\times \psi_{\kappa'}(u_{x_1,y_1})\psi_{\chi'}(v_{x_1,y_1})\psi_\kappa(u_{x_2,y_2})\psi_\chi(v_{x_2,y_2}) \\
&\quad\quad\times \psi_\kappa(u_{x_3,y_3})\psi_\chi(v_{x_3,y_3})\psi_{\kappa'}(u_{x_4,y_4})\psi_{\chi'}(v_{x_4,y_4}).
\end{aligned} \tag{2.3.10}
$$

where $\mathbf{x} = (x_1, x_2, x_3, x_4)$ and $\mathbf{y} = (y_1, y_2, y_3, y_4)$ are the rescaled position coordinates of the system and environment oscillator respectively. The integral f can be more easily evaluated if we express it as a product of three integrals, in the sense that

$$
f[a,b,a',b';\ell',\kappa,\kappa',\chi,\chi'] = I_{a',\ell',\kappa',\chi'} I_{b',\ell',\kappa,\chi} J_{b,a,\kappa,\chi,\kappa',\chi',r} \tag{2.3.11}
$$

where the integrals are

$$I_{a',\ell',\kappa',\chi'} := \int_{\mathbb{R}^2} \Theta_{x,y}(a';\ell')\psi_{\kappa'}(u_{x,y})\psi_{\chi'}(v_{x,y})\ dx\ dy \tag{2.3.12}$$

$$J_{b,a,\kappa,\chi,\kappa',\chi',r} := \int_{\mathbb{R}^4} \Theta_{w,x,y,z}(b,a;r)\psi_{\kappa}(u_{w,y})\psi_{\chi}(v_{w,y})\psi_{\kappa'}(u_{x,z})\psi_{\chi'}(v_{x,z})\ dw\ dx\ dy\ dz, \tag{2.3.13}$$

and the kernels are

$$\Theta_{x,y}(i;j) := \psi_i(x)\psi_j(y) \tag{2.3.14}$$

$$\Theta_{w,x,y,z}(i,j;r) := \psi_i(w)\psi_j(x)\exp\left[-\frac{1+r^2}{2(1-r^2)}\left(y^2-\frac{4r}{1+r^2}yz+z^2\right)\right]. \tag{2.3.15}$$

Now our truncated transition amplitude is still a sum over an infinite number of integrals, and hence we intend to approximate it by the finite sum

$$A^{(a,b)\to(a',b')}_{L,N,t} = \sum_{\ell'\leq L}\sum_{\substack{\kappa,\chi\leq N\\ \kappa',\chi'\leq N}} z_u^{\kappa-\kappa'}z_v^{\chi-\chi'}f[a,b,a',b';\ell',\kappa,\kappa',\chi,\chi']\frac{\omega_{\mathrm{u}}\omega_{\mathrm{v}}}{\omega_{\mathrm{x}}\omega_{\mathrm{y}}}\sqrt{\frac{1-r}{\pi(1+r)}} \tag{2.3.16}$$

for some positive integer N. In Theorem 2.3.1, we prove that the absolute value of the error term by approximating the truncated transition amplitude (2.3.9) with (2.3.16) vanishes as N becomes large while a, b, a', b' and L remain small. Restricting a, b and a', b' to be small corresponds to truncating the dimension of the input space and the dimension of the output space respectively. The integer L corresponds to the cutoff in the truncation of the quantum channel's Kraus set. The proof of our theorem uses mainly the Cauchy-Schwarz inequality, Lemma 2.4.3 and Lemma 2.4.4. The intuition behind our technical lemmas arises from the behavior of order n Hermite functions in the oscillatory interval $[-\sqrt{n}, \sqrt{n}]$ and outside of it. Within the oscillatory interval, Hermite functions have amplitudes that vanish as n gets large. Outside the oscillatory region, Hermite functions have exponentially small amplitudes as their arguments becomes large. Thus we construct our upper bounds for the integral of the product of Hermite functions by performing the integration separately in two overlapping regions, as depicted in Figure 2.1.

Theorem 2.3.1. *Let $m_{\mathrm{x}}, m_{\mathrm{y}}, \omega_{\mathrm{x}}, \omega_{\mathrm{y}}, t > 0, k \geq 0$ and $0 \leq r < 1$ be real numbers, and D, L and N be positive integers. Let C be a constant that depends on $m_{\mathrm{x}}, m_{\mathrm{y}}, \omega_{\mathrm{x}}, \omega_{\mathrm{y}}$ and k (see (2.3.21)), and $A, \tilde{A}, B, \tilde{B}$ be constants that depend on D and L (see (2.3.22)). Let $T^{(a,b)\to(a',b')}_{\Omega_{L,t}}$ be the truncated transition amplitude defined in (2.3.9) have approximation $A^{(a,b)\to(a',b')}_{L,N,t}$ given by (2.3.16). Then for all integers $0 \leq a, b, a', b' \leq D$,*

$$
\begin{aligned}
&\left|T^{(a,b)\to(a',b')}_{\Omega_{L,t}} - A^{(a,b)\to(a',b')}_{L,N,t}\right| \\
&\leq \Bigg(\frac{4\sqrt{A^2B}}{9(N-\frac{1}{2})^3} + \frac{4\sqrt{A\tilde{A}B}}{3N^{5/4}(N-\frac{1}{2})^{3/2}}\frac{e^{-NC/2}}{1-e^{-C/2}} + \frac{\sqrt{\tilde{A}^2B}}{N^{5/2}}\frac{e^{-NC}}{(1-e^{-C/2})^2} + \frac{\sqrt{A^2\tilde{B}}}{N^{5/2}}\frac{e^{-NC}}{(1-e^{-C/2})^2} \\
&\quad + \frac{2\sqrt{A\tilde{A}\tilde{B}}}{N^{5/4}}\frac{e^{-3NC/2}}{(1-e^{-C/2})(1-e^{-C})} + \sqrt{\tilde{A}^2\tilde{B}}\frac{e^{-2NC}}{(1-e^{-C})^2}\Bigg)^2 \frac{\omega_{\mathrm{u}}\omega_{\mathrm{v}}}{\omega_{\mathrm{x}}\omega_{\mathrm{y}}}\sqrt{\frac{1-r}{\pi(1+r)}}(L+1).
\end{aligned}
\tag{2.3.17}
$$

Remark 2.3.2. When the ratio between the coupling constant k and the difference between the square of resonance frequencies of the oscillators is small, the the rotation angle θ (given in 2.2.18) is small, which causes C to be very large. We call this the off-resonant and weakly-coupled regime. In this situation, the upper bound of the above theorem is dominated by the expression

$$
\frac{4A^2B}{81(N-\frac{1}{2})^6}\left(\frac{\omega_{\mathrm{u}}\omega_{\mathrm{v}}}{\omega_{\mathrm{x}}\omega_{\mathrm{y}}}\sqrt{\frac{1-r}{\pi(1+r)}}(L+1)\right) \leq \frac{14.7103(L+1)\tilde{n}_D^4\tilde{n}_L^2}{(N-\frac{1}{2})^6}\left(\frac{\omega_{\mathrm{u}}\omega_{\mathrm{v}}}{\omega_{\mathrm{x}}\omega_{\mathrm{y}}}\sqrt{\frac{1-r}{1+r}}\right)
$$

as N becomes large (and $\tilde{n}_i$ is defined as $(\max_{0\leq j\leq i}\|\psi_j\|_1)$.

Remark 2.3.3. If the parity of $a+b$ differs from that of $a'+b'$, the approximate truncated amplitude is necessarily identically zero for all positive integers L and N. This is a result of a simple parity counting argument after noting that the I-type integrals (2.3.12) and J-type integrals (2.3.13) are zero whenever the parity of the sum of their indices are odd. Hence parity is conserved with regards to our physical model.

Remark 2.3.4. The bounds of Theorem 2.3.1 can be substantially tightened using information pertaining to the I-type integrals (2.3.12), which are substantially simplier to evaluate than the J-type integrals (2.3.13). Using bounds for the J-type integrals (2.3.24), we have that for integer N' greater than N,

$$\left|T_{\Omega_{L,t}}^{(a,b)\to(a',b')} - A_{L,N,t}^{(a,b)\to(a',b')}\right| \le \sum_{0\le\ell\le L} \sum_{\substack{N<\kappa,\kappa'\le N' \\ N<\chi,\chi'\le N'}} |I_{a',\ell',\kappa',\chi'} I_{b',\ell',\kappa,\chi}| B(\kappa\kappa'\chi\chi')^{-5/2} + \left|T_{\Omega_{L,t}}^{(a,b)\to(a',b')} - A_{L,N',t}^{(a,b)\to(a',b')}\right| \tag{2.3.18}$$

Proof of Theorem 2.3.1. Our goal is to obtain upper bounds on each of the integrals I and J defined in (2.3.12) and (2.3.13). Applying the Cauchy-Schwarz inequality on $|I_{a',\ell',\kappa',\chi'}|$ gives

$$|I_{a',\ell',\kappa',\chi'}| \le \sqrt{\int_{\mathbb{R}^2} \left|\Theta_{x,y}(a';\ell')\right| \psi_{\kappa'}(u_{x,y})^2 \, dx \, dy} \sqrt{\int_{\mathbb{R}^2} \left|\Theta_{x,y}(a';\ell')\right| \psi_{\chi'}(v_{x,y})^2 \, dx \, dy}. \tag{2.3.19}$$

We similarly use the Cauchy-Schwarz inequality to obtain an upper bound of the absolute value of (2.3.13), which is

$$|J_{b,a,\kappa,\chi,\kappa',\chi',r}| \le \sqrt{\int_{\mathbb{R}^2} \left|\Theta_{w,x,y,z}(b,a;r)\right| \psi_{\kappa}(u_{w,y})^2 \psi_{\kappa'}(u_{x,z})^2 \, dx \, dy} \times \sqrt{\int_{\mathbb{R}^2} \left|\Theta_{w,x,y,z}(b,a;r)\right| \psi_{\chi}(v_{w,y})^2 \psi_{\chi'}(v_{x,z})^2 \, dx \, dy}. \tag{2.3.20}$$

For the purpose of using Lemma 2.4.3 and Lemma 2.4.4, define the constants

$$\begin{aligned} c_1 &= \min\left\{ \sqrt{\frac{m_\mathrm{y}\omega_\mathrm{u}}{m_\mathrm{x}\omega_\mathrm{x}}} \cos\theta, \sqrt{\frac{m_\mathrm{x}\omega_\mathrm{u}}{m_\mathrm{y}\omega_\mathrm{y}}} \sin\theta \right\} \\ c_2 &= \min\left\{ \sqrt{\frac{m_\mathrm{y}\omega_\mathrm{v}}{m_\mathrm{x}\omega_\mathrm{x}}} \sin\theta, \sqrt{\frac{m_\mathrm{x}\omega_\mathrm{v}}{m_\mathrm{y}\omega_\mathrm{y}}} \cos\theta \right\} \\ C &= \min\{1/(4c_1^2), 1/(4c_2^2)\} \end{aligned} \tag{2.3.21}$$

and

$$A = 1.74^2 \left(\max_{0 \le j \le D} \|\psi_j\|_1 \right) \left(\max_{0 \le j \le L} \|\psi_j\|_1 \right), \quad \tilde{A} = (4.74)(2e^{-\frac{1}{2}})^{D+L} \sqrt{D! D^D L! L^L}$$

$$B = 57.6 \left(\max_{0 \le j \le D} \|\psi_j\|_1 \right)^2, \quad \tilde{B} = (39.6)(2e^{-1})^D D! D^D. \tag{2.3.22}$$

Noting that $\sin\theta$ and $\cos\theta$ are positive by definition (see (2.2.19)), and using the definitions of c_1, c_2 and C with Lemma 2.4.3, we have that

$$|I_{a',\ell',\kappa',\chi'}| \le \sqrt{A(\kappa')^{-5/2} + \tilde{A}e^{-\kappa' C}} \sqrt{A(\chi')^{-5/2} + \tilde{A}e^{-\chi' C}} \tag{2.3.23}$$

and we have a similar upper bound of $|I_{b',\ell',\kappa,\chi}|$. Using Lemma 2.4.4, we have that

$$|J_{b,a,\kappa,\chi,\kappa',\chi',r}| \le \sqrt{B(\kappa\kappa')^{-5/2} + \tilde{B}e^{-(\kappa+\kappa')C}} \sqrt{B(\chi\chi')^{-5/2} + \tilde{B}e^{-(\chi+\chi')C}}. \tag{2.3.24}$$

By expanding out the terms of the products of the upper bounds given by (2.3.23) and (2.3.24), our upper bound on the absolute value of (2.3.11)

$$\left| f[a, b, a', b'; \ell', \kappa, \kappa', \chi, \chi'] \right| \le \sqrt{W_{\kappa,\kappa'} W_{\chi,\chi'}}$$

where

$$\begin{aligned} W_{\kappa,\kappa'} :=& \frac{A^2 B}{\kappa^5 (\kappa')^5} + \frac{A\tilde{A}B}{\kappa^5 (\kappa')^{5/2}} e^{-\kappa' C} + \frac{A\tilde{A}B}{\kappa^{5/2} (\kappa')^5} e^{-\kappa C} + \frac{\tilde{A}^2 B}{\kappa^5 (\kappa')^{5/2}} e^{-(\kappa+\kappa')C} \\ &+ \frac{A^2 \tilde{B}}{\kappa^{5/2} (\kappa')^{5/2}} e^{-(\kappa+\kappa')C} + \frac{A\tilde{A}\tilde{B}}{\kappa^{5/2}} e^{-(\kappa+2\kappa')C} + \frac{A\tilde{A}\tilde{B}}{(\kappa')^{5/2}} e^{-(2\kappa+\kappa')C} + \tilde{A}^2 \tilde{B} e^{-(2\kappa+2\kappa')C}. \end{aligned}$$

By the subadditivity of the square root function, we have that

$$\begin{aligned} \sqrt{W_{\kappa,\kappa'}} \le& \frac{\sqrt{A^2 B}}{\kappa^{5/2} (\kappa')^{5/2}} + \frac{\sqrt{A\tilde{A}B}}{\kappa^{5/2} (\kappa')^{5/4}} e^{-\kappa' C/2} + \frac{\sqrt{A\tilde{A}B}}{\kappa^{5/4} (\kappa')^{5/2}} e^{-\kappa C/2} + \frac{\sqrt{\tilde{A}^2 B}}{\kappa^{5/2} (\kappa')^{5/4}} e^{-(\kappa+\kappa')C/2} \\ &+ \frac{\sqrt{A^2 \tilde{B}}}{\kappa^{5/4} (\kappa')^{5/4}} e^{-(\kappa+\kappa')C/2} + \frac{\sqrt{A\tilde{A}\tilde{B}}}{\kappa^{5/4}} e^{-(\kappa+2\kappa')C/2} + \frac{\sqrt{A\tilde{A}\tilde{B}}}{(\kappa')^{5/4}} e^{-(2\kappa+\kappa')C/2} + \sqrt{\tilde{A}^2 \tilde{B}} e^{-(\kappa+\kappa')C}. \end{aligned}$$

The summation of the above expression over κ and κ' can be seen as an inner product of vectors with exponentially decaying terms and polynomially decaying terms respectively. We hence apply Hölder's inequality for sequence spaces on the summation of the above expression over κ and κ'. We thereby obtain an upper bound of the sum in terms of the one-norm of the vector with exponentially decaying terms, and the infinity-norm of the vector with polynomially decaying terms:

$$\sum_{\kappa,\kappa'\geq N}\sqrt{W_{\kappa,\kappa'}} \leq \sqrt{A^2B}\Big(\sum_{X\geq N}X^{-5/2}\Big)^2 + \frac{2\sqrt{A\tilde{A}B}}{N^{5/4}}\Big(\sum_{X\geq N}X^{-5/2}\Big)\frac{e^{-NC/2}}{1-e^{-C/2}} + \frac{\sqrt{\tilde{A}^2B}}{N^{5/2}}\frac{e^{-NC}}{(1-e^{-C/2})^2}$$
$$+ \frac{\sqrt{A^2\tilde{B}}}{N^{5/2}}\frac{e^{-NC}}{(1-e^{-C/2})^2} + \frac{2\sqrt{A\tilde{A}\tilde{B}}}{N^{5/4}}\frac{e^{-3NC/2}}{(1-e^{-C/2})(1-e^{-C})} + \sqrt{\tilde{A}^2\tilde{B}}\frac{e^{-2NC}}{(1-e^{-C})^2}.$$

Now the integral $\int_{N-\frac{1}{2}}^{\infty} x^{-5/2}dx$ is an upper bound of the sum $\sum_{X\geq N}^{\infty} X^{-5/2}$ by the convexity of the integrand. Hence $\sum_{X\geq N}^{\infty} X^{-5/2} \leq \frac{2}{3}(N-\frac{1}{2})^{-3/2}$, and we can subsitute this bound into our upper bound of the square of $\sum_{\kappa,\kappa'\geq N} W_{\kappa,\kappa'}$ summed over the index $0 \leq \ell' \leq L$ to get the result. □

2.4 Bounds on Hermite functions

This section provides the main technical lemmas that are used to obtain error bounds on our approximation to our truncated transition amplitudes. The main technical tools that we use in this section are Alzer's sharp bounds on the gamma function [Alz09] and bounds on the Dominici's asymptotic approximation of Hermite functions with error estimates by Kerman, Huang and Brannan [RKB09].

Lemma 2.4.1. *For all positive integers n and reals $x \in [-\sqrt{n}, \sqrt{n}]$, we have $|\psi_n(x)| < 1.74n^{-5/4}$.*

Proof. This proof combines Alzer's sharp bounds on the gamma function [Alz09] with uniform bounds on the envelope of the Hermite functions in the oscillatory region by

Kerman, Huang and Brannan [RKB09]. Using Kerman, Huang and Brannan's result (see equation (2.1) and (1.4) in [RKB09]), for $x \in [-\sqrt{n}, \sqrt{n}]$ we have

$$|\psi_n(x)| \leq 2^{-3/4}\sqrt{35}\frac{\sqrt{n!}\pi^{-1/4}2^{-n/2}}{\sqrt{2}\Gamma((n/2)+1)}n^{-1}.$$

Alzer's sharp bounds for the gamma function are that for all $n > 0$,

$$1 < \frac{\Gamma(n+1)}{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n\left(n\sinh\frac{1}{n}\right)^{n/2}} < 1 + \frac{1}{1620}n^5.$$

Note that for real $n \geq 1$, we have $1 \leq \left(n\sinh\frac{1}{n}\right)^{n/2} < 1.085$. Hence we have that for positive integers n,

$$\frac{\sqrt{\Gamma(n+1)}}{\Gamma(\frac{n}{2}+1)} < \frac{\sqrt{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n(1.085)(1+\frac{1}{1620n^5})}}{\sqrt{\pi n}\left(\frac{n}{2e}\right)^{n/2}} < 0.9308(2^{n/2})n^{-1/4}.$$

Hence Kerman, Huang and Brannan's upper bound on the envelope of the Hermite function for $x \in [-\sqrt{n}, \sqrt{n}]$ and $n \geq 1$ becomes

$$|\psi_n(x)| \leq 2^{-3/4}\sqrt{35}\frac{\pi^{-1/4}2^{-n/2}}{\sqrt{2}}\frac{\sqrt{\Gamma(n+1)}}{\Gamma(\frac{n}{2}+1)}n^{-1} < 1.74n^{-5/4}.$$

□

The next lemma provides a rather coarse upper bound on the absolute value of the Hermite function, with maximum utility in the monotonic region of the Hermite function.

Lemma 2.4.2. *For all reals $|x| > 1$ and integers $n \geq 0$, we have $|\psi_n(x)| \leq 2^n\sqrt{\frac{n!n^n}{e^n\sqrt{\pi}}}e^{-x^2/4}$.*

Proof. Using the Maclaurin decomposition of the Hermite polynomial $H_n(x)$ [Erd85], we get

$$|\psi_n(x)| \le n!n(2|x|)^n \frac{e^{-x^2/2}}{\sqrt{2^n n!\sqrt{\pi}}} = \sqrt{\frac{2^n n!}{\sqrt{\pi}}} n|x|^n e^{-x^2/2}.$$

It is easy to verify that $\sup_{x\in\mathbb{R}}\{|x|^n e^{-x^2/4}\} = \left(\frac{2n}{e}\right)^{n/2}$. Hence when $|x| > 1$,

$$|\psi_n(x)| \le 2^n \sqrt{\frac{n!n^n}{e^n\sqrt{\pi}}} e^{-x^2/4}.$$

□

Lemma 2.4.3 provides upper bounds on the one-norm of the product of Hermite functions in terms of the order of the Hermite functions, and is crucial in obtaining upper bounds on our error estimates.

Lemma 2.4.3. *For real numbers $a, b \ne 0$, let $c = \min(|a|, |b|)$. Let $j, k, n \in \mathbb{N}$ and $n \ge 1$. Then*

$$\int_{\mathbb{R}^2} \psi_n(ax+by)^2 |\psi_j(x)\psi_k(y)| \; dxdy \le \frac{(1.74^2)\|\psi_j\|_1\|\psi_k\|_1}{n^{5/2}} + (4.74)2^{j+k}\sqrt{\frac{j!k!j^j k^k}{e^{j+k}}} e^{-n/(8c^2)}.$$

Proof. We split the region over which the integral is performed into two overlapping regions A_1 and A_2 (see Figure 2.1), where

$$A_1 = \{(x,y) : |ax| + |by| \le \sqrt{n}\}$$

and

$$A_2 = \{(x,y) : x^2 + y^2 > \frac{\sqrt{n}}{\sqrt{2}c}\}.$$

Then using the Charlier-Cramér bound [Erd85] which states that $\sup_{n,x} |\psi_n(x)| \le \frac{1.086435}{\pi^{1/4}}$, and the uniform upper bound of the envelope of the Hermite polynomial in the oscillatory

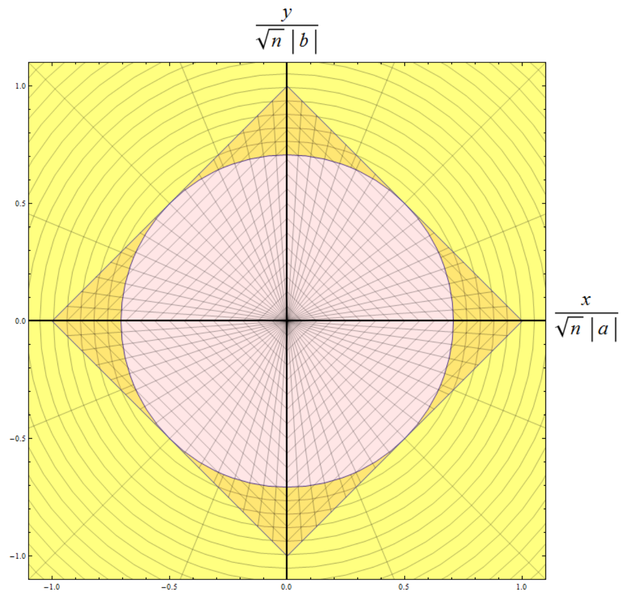

Figure 2.1: The area inside the square corresponds to the region A_1 (where $|\psi_n(ax+by)| \leq 1.74n^{-5/4}$), and the area outside the circle corresponds to the region A_2 (where all Hermite functions decay exponentially).

region as stated in Lemma 2.4.1, we get

$$\int_{\mathbb{R}^2} \psi_n(ax+by)^2 |\psi_j(x)\psi_k(y)|\ dxdy$$
$$\leq \int_{A_1} \frac{1.74^2}{n^{5/2}} |\psi_j(x)\psi_k(y)|\ dxdy + \int_{A_2} \frac{1.086435^2}{\sqrt{\pi}} |\psi_j(x)\psi_k(y)|\ dxdy.$$

The integral of $|\psi_j(x)\psi_k(y)|$ over the region A_1 is at most

$$\frac{(1.74^2)\|\psi_j\|_1 \|\psi_k\|_1}{n^{5/2}}.$$

Using the exponential upper bound of Lemma 2.4.2, the integral over the annulus region

A_2 is at most

$$\begin{aligned}
&\frac{1.086435^2}{\sqrt{\pi}}2^{j+k}\sqrt{\frac{j!k!j^jk^k}{e^{j+k}\pi}}\int_{A_2} e^{-(x^2+y^2)/4}\ dxdy \\
\leq&\frac{1.086435^2}{\sqrt{\pi}}2^{j+k}\sqrt{\frac{j!k!j^jk^k}{e^{j+k}\pi}}2\pi\int_{r>\frac{\sqrt{n}}{\sqrt{2}c}} re^{-r^2/4}\ dr \\
=&1.086435^2 2^{j+k+1}\sqrt{\frac{j!k!j^jk^k}{e^{j+k}}}(2e^{-n/(8c^2)}).
\end{aligned}$$

Combining the upper bounds for region A_1 and A_2 then gives the result. $\square$

The following lemma is needed to obtain upper bounds on the absolute value of $|J_{b,a,\kappa,\chi,\kappa',\chi',r}|$, and is similar to the preceding lemma.

Lemma 2.4.4. *Let $0 \neq a, b \in \mathbb{R}$, $c = \min(|a|, |b|)$ and $j, k, n, n' \in \mathbb{N}$ where $n, n' \geq 1$. Then*

$$\begin{aligned}
&\int_{\mathbb{R}^2} \psi_n(aw+by)^2\psi_{n'}(ax+bz)^2|\Theta_{w,x,y,z}(j,k;r)|\ dxdy \\
\leq&\frac{57.6\|\psi_j\|_1\|\psi_k\|_1}{(nn')^{5/2}} + (39.6)2^{j+k}\sqrt{\frac{j!k!j^jk^k}{e^{j+k}}}e^{-(n+n')/(8c^2)}.
\end{aligned}$$

Proof. We split the region over which the integral is performed into two overlapping regions A_1 and A_2 (just as in the proof of Lemma 2.4.3), where

$$A_1 = \{(w,x,y,z) : |aw| + |by| \leq \sqrt{n}, |ax| + |bz| \leq \sqrt{n'}\}$$

and

$$A_2 = \{(w,x,y,z) : w^2 + y^2 > \frac{\sqrt{n}}{\sqrt{2}c}, x^2 + z^2 > \frac{\sqrt{n'}}{\sqrt{2}c}\}.$$

Then using the Charlier-Cramér bound, Lemma 2.4.1, and Lemma 2.4.2, we get

$$\begin{aligned}
&\int_{\mathbb{R}^2} \psi_n(aw+by)^2\psi_{n'}(ax+bz)^2|\Theta_{w,x,y,z}(j,k;r)|\ dw\ dx\ dy\ dz \\
\leq&\int_{A_1}\frac{1.74^4}{(nn')^{5/2}}|\Theta_{w,x,y,z}(j,k;r)|\ dw\ dx\ dy\ dz + \int_{A_2}\frac{1.086435^4}{\pi}|\Theta_{w,x,y,z}(j,k;r)|\ dw\ dx\ dy\ dz.
\end{aligned}$$

Now observe that we can express the kernel $\Theta_{w,x,y,z}(j,k;r)$ as

$$\Theta_{w,x,y,z}(j,k;r) = e^{-\frac{1}{2}(y^2+z^2)} \exp\left[-\frac{r}{1-r^2}(y-z)^2\right] \psi_j(w)\psi_k(x)$$

and hence the absolute value of our kernel has an upper bound that factorizes, in the sense that

$$|\Theta_{w,x,y,z}(j,k;r)| \leq e^{-\frac{1}{2}(y^2+z^2)}|\psi_j(w)\psi_k(x)|.$$

Hence the integral over the region A_1 is at most

$$\frac{1.74^4(2\pi)\|\psi_j\|_1\|\psi_k\|_1}{(nn')^{5/2}}.$$

The upper bound on the kernel also allows us to find that the integral over the region A_2 is at most

$$\begin{aligned}
&\frac{1.086435^4}{\pi}2^{j+k}\sqrt{\frac{j!k!j^jk^k}{e^{j+k}\pi}}\int_{A_2} e^{-(w^2+x^2)/4}e^{-\frac{1}{2}(y^2+z^2)}\ dw\ dx\ dy\ dz \\
\leq\ &\frac{1.086435^4}{\pi^{3/2}}2^{j+k}\sqrt{\frac{j!k!j^jk^k}{e^{j+k}}}(2\pi)^2\int_{r_1\geq\frac{\sqrt{n}}{\sqrt{2}c}} e^{-r_1^2/4}dr_1\int_{r_2\geq\frac{\sqrt{n'}}{\sqrt{2}c}} e^{-r_2^2/4}dr_2 \\
\leq &\frac{1.086435^4(2\pi)^2}{\pi^{3/2}}2^{j+k}\sqrt{\frac{j!k!j^jk^k}{e^{j+k}}}4e^{-(n+n')/(8c^2)} \\
\leq &(39.6)2^{j+k}\sqrt{\frac{j!k!j^jk^k}{e^{j+k}}}e^{-(n+n')/(8c^2)}.
\end{aligned}$$

Combining our upper bounds on the integrals in the regions A_1 and A_2 thereby gives the result. $\square$

2.5 A case study: Off-resonant weakly coupled oscillators

In this section, we perform a case study of the dynamics of our physical model when the oscillators are off-resonant and weakly coupled. In agreement with the standard results of perturbation theory, we find negligible amplitude damping in the truncated dynamics of our system. Moreover, we show that leakage error is the dominant error process.

2.5.1 Parameters of the physical model and the truncated channel

The amount of truncation in our truncated channel is quantified by the parameter $L = 2$, and the order of our approximation to the truncated channel is quantified by the parameter $N = 6$ (see (2.3.16) for the definition of the truncated transition amplitudes). We restrict the analysis of our truncated channels to a 4-level system by setting the parameter $D = 3$.

Table 2.1 shows the parameters used for our physical model. We use the SI units. The output parameters are numerically computed using floating point numbers with 1024 bits of precision, and are shown up to ten decimal places. For our choice of parameters,

	Input parameters		Output parameters		More bounds
m_{x}	10^{-6} (kg)	ω_{x}	1000499.8750624610 (Hz)	$\|\psi_0\|_1$	≤ 1.882792528
m_{y}	2×10^{-6} (kg)	ω_{y}	10000024.9999687501 (Hz)	$\|\psi_1\|_1$	≤ 2.1245038641
$\omega_{\mathrm{x},\mathbf{o}}$	10^{6} (Hz)	ω_{u}	1001004.5438332595(Hz)	$\|\psi_2\|_1$	≤ 2.2853242243
$\omega_{\mathrm{y},\mathbf{o}}$	10^{7} (Hz)	ω_{v}	10000025.0001779494 (Hz)	$\|\psi_3\|_1$	≤ 2.4102377590
k	100	u_1	10000025.0001779494		
r	0	u_2	0.0000031958		
		v_1	-0.0000451621		
		v_2	1.0000000000		

Table 2.1: We tabulate the important parameters of our physical model. Here, u_1, u_2, v_1 and v_2 are defined implicitly in the equations $u_{x,y} = u_1 x + u_2 y$ and $v_{x,y} = v_1 x + v_2 y$ (see (2.3.8)). Also note that $\|\psi_n\|_1 := \int_{\mathbb{R}} |\psi_n(x)| dx$.

the positive constants $A\tilde{A}B, \tilde{A}^2B, A^2\tilde{B}, A\tilde{A}\tilde{B}, \tilde{A}^2\tilde{B} \leq 10^{-288}$, and are negligible, and hence Remark 2.3.2 holds.

2.5.2 Approximate dynamics of the truncated channel

We numerically evaluate approximate truncated transition amplitude (2.3.16) corresponding to the transitions within the lowest energy levels of the system. The evaluation of each such approximate truncated transition amplitude $A_{2,6,t}^{(a,b)\to(a',b')}$ is a sum of 7203 terms for our choice of $L = 2$ and $N = 6$.

We obtain the corresponding error bounds of our approximation from Remark 2.3.4 with $N' = 15$. The error of each of our approximate truncated transition amplitude is at most 0.00084 – a negligible amount. We plot magnitudes of the non-negligible truncated transition amplitudes in Figure 2.2.

Numerically, we find that the non-negligible terms of $A_{2,6,t}^{(a,b)\to(a',b')}$ have values of $0 \leq a, b \leq 1$ and $0 \leq a', b' \leq 3$ that satisfy the relation

$$\frac{a'-a}{2}, \frac{b'-b}{2} \in \mathbb{Z}. \tag{2.5.1}$$

The above relation holds for two reasons. First, the conservation of the parity of all truncated transition amplitudes as stated in Remark 2.3.3 implies that the parity of $a + b$ equals the parity of $a' + b'$. Second, the other negligible transitions are in agreement with the results of using pertubation theory on off-resonant and weakly coupled harmonic oscillators. Also note that the negligible damping from the first excited state to the ground state of our truncated channel suggests that the coupled oscillator model is inconsistent with the phenomenon of amplitude damping [NC00] even at zero temperature.

The Choi-Jamiolkowski (CJ) operator of the finite input and output dimension channel Φ with Kraus set $\mathfrak{K}$ is the linear operator

$$\chi_\Phi := \sum_{\mathbf{K}\in\mathfrak{K}} |\mathbf{K}\rangle\!\rangle\langle\!\langle\mathbf{K}| = \sum_{j,j'}\sum_{k,k'}\left(\sum_{\mathbf{K}\in\mathfrak{K}} \mathbf{K}_{j,j'}\mathbf{K}^*_{k,k'}\right)|j_\mathcal{K}, j'_\mathcal{H}\rangle\langle k_\mathcal{K}, k'_\mathcal{H}|. \tag{2.5.2}$$

where the linear map $|\cdot\rangle\rangle : L(\mathcal{H},\mathcal{K}) \to \mathcal{K}\otimes\mathcal{H}$ is a stacking isomorphism such that

$$|\sum_{i,j} a_{i,j}|i_{\mathcal{K}}\rangle\langle j_{\mathcal{H}}|\rangle\rangle := \sum_{i,j} a_{i,j}|i_{\mathcal{K}}\rangle|j_{\mathcal{H}}\rangle.$$

We now construct a quantum operation $\mathcal{A}$ to approximate the truncated channel $\mathcal{N}$. Abbreviating our approximate truncated transition amplitudes as $A_{aba'b'} := A_{2,6,t}^{(a,b)\to(a',b')}$, our approximation to the CJ operator of our truncated channel is

$$\chi := \begin{pmatrix} A_{0000} & A_{0002} & A_{0101} & A_{0103} \\ A_{0020} & A_{0022} & A_{1012} & A_{0123} \\ A_{1010} & A_{1012} & A_{1111} & A_{1113} \\ A_{1030} & A_{1032} & A_{1131} & A_{1133} \end{pmatrix} \tag{2.5.3}$$

where the bases labeled by the rows and columns are $|0,0\rangle$, $|2,0\rangle$, $|1,1\rangle$, $|3,1\rangle$ and $\langle 0,0|$, $\langle 2,0|$, $\langle 1,1|$, $\langle 3,1|$ respectively. The first and second entries of our bras and kets correspond to the output and input Hilbert spaces of the truncated map respectively. We plot the absolute value of some of these non-negligible matrix elements in Figure 2.2. Let χ have spectral decomposition $\chi = \sum_{i=1}^4 \lambda_i|\lambda_i\rangle\langle\lambda_i|$ where $|\lambda_i\rangle\langle\lambda_i|$ are orthogonal projectors and the eigenvalues λ_i are in non-increasing order [1]. For $\lambda_i \geq 0$, let A_i be the image of the inverse map of the linear operator $|\cdot\rangle\rangle$ acting on $\sqrt{\lambda_i}|\lambda_i\rangle$. Numerically diagonalizing the matrix χ shows that it has only one dominant eigenvalue. Hence we use the quantum operation $\mathcal{A}(v) := A_1 v A_1^\dagger$ to approximate the truncated channel $\mathcal{N}$ where $A_1 = \sqrt{\lambda_1}(k_0|0\rangle\langle 0| + k_2|2\rangle\langle 0| + k_1|1\rangle\langle 1| + k_3|3\rangle\langle 1|)$ and $|\lambda_1\rangle = k_0|0,0\rangle + k_2|2,0\rangle + k_1|1,1\rangle + k_3|3,1\rangle$. Observe that for $i, j \in \{0,1,2,3\}$ we have

$$\mathcal{A}(|i\rangle\langle j|) = \lambda_1\big(k_i|i\rangle + k_{i+2}|i+2\rangle\big)\big(k_j^*\langle j| + k_{j+2}^*\langle j+2|\big). \tag{2.5.4}$$

[1] The matrix χ is symmetric and hence has real eigenvalues.

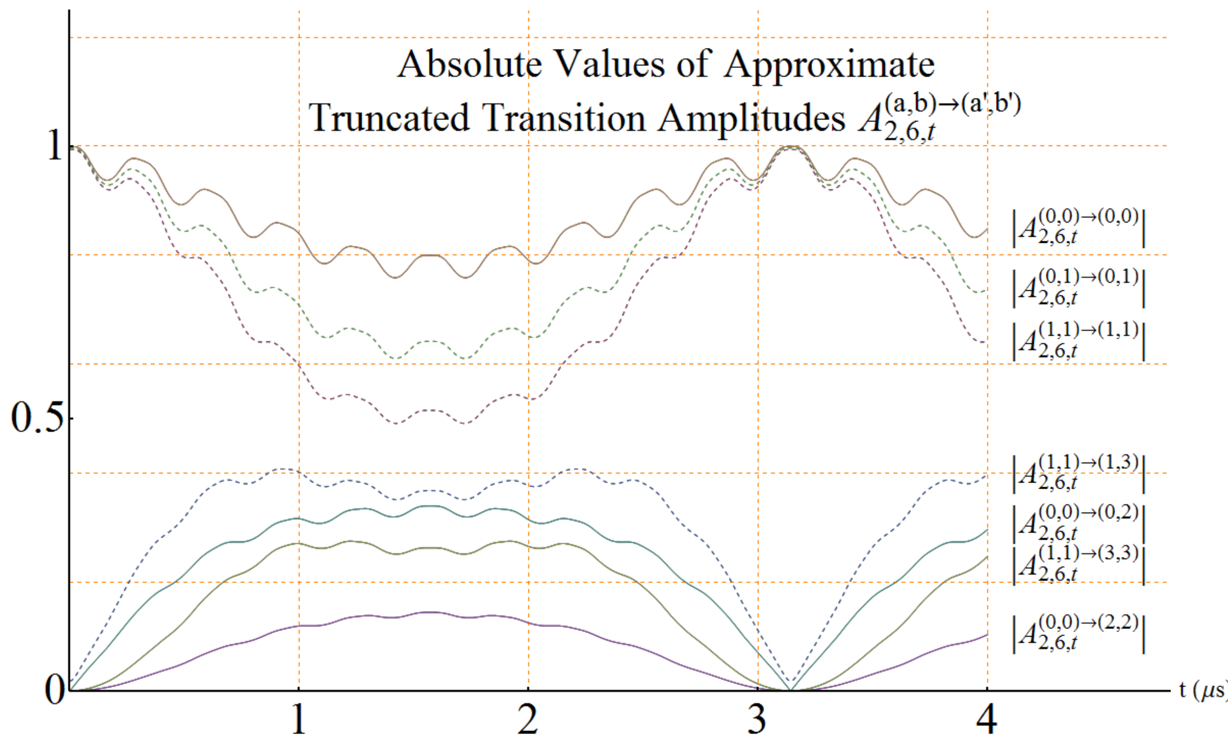

Figure 2.2: Absolute values of approximations to our truncated amplitudes are depicted for our model with parameters given by Table (2.1).

Leakage error

Leakage error, a dominant process of our model, occurs when low energy states transition into higher energy states within the quantum system. We derive lower bounds on the minimum amount of qubit leakage in our system, when the quantum channel is Φ_t with Kraus set $\mathfrak{K}_t$ given by (2.3.2). For qubit leakage to occur, it suffices to have the strict inequality

$$\text{Leakage}(\Phi_t, \rho) := \sum_{i=2}^{\infty} \langle i_{\mathcal{X}} | \Phi_t(\rho) | i_{\mathcal{X}} \rangle > 0, \tag{2.5.5}$$

for some density operator ρ supported on the span of $|0_{\mathcal{X}}\rangle$ and $|1_{\mathcal{X}}\rangle$. The above expression quantifies the amount of leakage from the qubit state space of our system.

We proceed to obtain a strictly positive lower bound on Leakage(Φ_t, ρ). If $\Phi_t = \mathcal{N} + \mathcal{M}$ for some completely positive maps $\mathcal{N}$ and $\mathcal{M}$, then the complete positivity of $\mathcal{N}$ and $\mathcal{M}$ implies that Leakage$(\Phi_t, \rho) \geq$ Leakage$(\mathcal{N}, \rho)$. Hence it suffices to obtain a lower bound for Leakage$(\mathcal{N}, \rho)$. For our application, $\mathcal{N}$ is our truncated channel.

Assume that the initial state of the system is $\rho = \frac{1}{2}\big(|0_{\mathcal{X}}\rangle\langle 0_{\mathcal{X}}| + |1_{\mathcal{X}}\rangle\langle 1_{\mathcal{X}}|\big)$, the maximally mixed state in the qubit space. Then the leakage of the truncated channel $\mathcal{N}$ is at least $\frac{1}{2}\Big(T^{(0,0)\to(2,2)}_{\Omega_{L,t}} + T^{(1,1)\to(3,3)}_{\Omega_{L,t}}\Big)$. Thus we have that Leakage$(\mathcal{N}, \rho) \geq \frac{1}{2}\Big(\big|A^{(0,0)\to(2,2)}_{L,N,t}\big| + \big|A^{(1,1)\to(3,3)}_{L,N,t}\big|\Big) - 2\epsilon$ where ϵ is given the upper bound in Theorem 2.3.1. In view of the data given in Figure 2.2, the amount of qubit leakage can actually be quite substantial. In particular, when $t = 5 \times 10^{-6}$, the amount of leakage of Φ_t is at least 0.4 which is substantially larger than zero.

The large amount of leakage from our qubit state highlights the importance of accounting for transitions of low energy states into excited states in oscillator systems, and also the problem of using the lowest two energy eigenstates as a basis to encode our qubit.

Quantum error correction versus no quantum error correction

We consider a universe with four harmonic oscillators $\mathcal{X}_1, \mathcal{X}_2, \mathcal{Y}_1$ and $\mathcal{Y}_2$. We identify the oscillators $\mathcal{X}_1$ and $\mathcal{X}_2$ as x-type oscillators and the oscillators $\mathcal{Y}_1$ and $\mathcal{Y}_2$ as y-type oscillators, with their parameters given by Table 2.1. Assume that there are only $\mathcal{X}_1$-$\mathcal{Y}_1$ couplings and $\mathcal{X}_2$-$\mathcal{Y}_2$ couplings in our universe. Suppose that a maximally entangled two-qubit state is initialized in the $\mathcal{X}_1$ and $\mathcal{X}_2$ oscillators supported on their two lowest energy levels. We assume that the oscillators $\mathcal{Y}_1$ and $\mathcal{Y}_2$ are initialized in the ground state. We obtain a lower bound on the fidelity of the time-evolved states when instantaneous, identical and independent recovery operations are performed on oscillators $\mathcal{X}_1$ and $\mathcal{X}_2$. Denoting our truncated channel and recovery channel on a single system oscillator by $\mathcal{N}$

and $\mathcal{R}$ respectively, our lower bound on the output fidelity is

$$f_{\mathcal{R}} = \langle\Phi|(\mathcal{R}\otimes\mathcal{R})\circ(\mathcal{N}\otimes\mathcal{N})(|\Phi\rangle\langle\Phi|)|\Phi\rangle, \tag{2.5.6}$$

where $|\Phi\rangle = (|0_{\mathcal{X}_1}, 0_{\mathcal{X}_2}\rangle + |1_{\mathcal{X}_1}, 1_{\mathcal{X}_2}\rangle)/\sqrt{2}$. We now proceed to obtain a lower bound on the entanglement fidelity of our time-evolved maximally entangled state, with and without recovery. The maximally entangled state on two qubits written as a density matrix is

$$|\Phi\rangle\langle\Phi| = \frac{1}{2}\sum_{i,j\in\{0,1\}} |i,i\rangle\langle j,j|. \tag{2.5.7}$$

By linearity, the action of our truncated channels on the maximally entangled state gives

$$(\mathcal{N}\otimes\mathcal{N})|\Phi\rangle\langle\Phi| = \frac{1}{2}\sum_{i,j\in\{0,1\}} \mathcal{N}(|i\rangle\langle j|)\otimes\mathcal{N}(|i\rangle\langle j|). \tag{2.5.8}$$

Let us denote the error of the truncated transition amplitude of $A_{iji'j'}$ to be $\epsilon_{iji'j'}$ so that

$$\begin{aligned}(\mathcal{N}\otimes\mathcal{N})|\Phi\rangle\langle\Phi| &= \frac{1}{2}\sum_{i,j\in\{0,1\}} \mathcal{N}(|i\rangle\langle j|)\otimes\mathcal{N}(|i\rangle\langle j|)\\ &= \frac{1}{2}\sum_{\substack{i,j\in\{0,1\}\\ i_1,j_1\in\{0,1,2,3\}\\ i_2,j_2\in\{0,1,2,3\}}} (A_{iji_1j_1}+\epsilon_{iji_1j_1})(A_{iji_2j_2}+\epsilon_{iji_2j_2})|i_1,i_2\rangle\langle j_1,j_2| \end{aligned}\tag{2.5.9}$$

If we perform no recovery operation, $\mathcal{R}$ is just the identity map $\mathcal{I}$, and we have

$$\begin{aligned}f_{\mathcal{I}} &= \langle\Phi|(\mathcal{N}\otimes\mathcal{N})(|\Phi\rangle\langle\Phi|)|\Phi\rangle\\ &= \langle\Phi|\left(\sum_{(i,j)\in\{0,1\}} \mathcal{N}(|i_{\mathcal{X}_1}\rangle\langle j_{\mathcal{X}_1}|)\otimes\mathcal{N}(|i_{\mathcal{X}_2}\rangle\langle j_{\mathcal{X}_2}|)\right)|\Phi\rangle/2. \end{aligned}\tag{2.5.10}$$

Dropping the labels on the Hilbert spaces of our bras and kets, we can use (2.5.9) to find that

$$f_{\mathcal{I}} = \frac{1}{2}(\langle 0,0| + \langle 1,1|)(\mathcal{N}\otimes\mathcal{N})|\Phi\rangle\langle\Phi|(|0,0\rangle + |1,1\rangle)$$

$$= \frac{1}{4}\sum_{\substack{i,j\in\{0,1\}\\ i_1,j_1\in\{0,1,2,3\}\\ i_2,j_2\in\{0,1,2,3\}}} (A_{iji_1j_1} + \epsilon_{iji_1j_1})(A_{iji_2j_2} + \epsilon_{iji_2j_2})|i_1,i_2\rangle\langle j_1,j_2| \tag{2.5.11}$$

$$\geq \frac{1}{4}(A_{0000}^2 + A_{1111}^2 + A_{0101}^2 + A_{1010}^2) - 8(2\epsilon + \epsilon^2) \tag{2.5.12}$$

where $\epsilon \leq 0.00084$. Note that $A_{0101}^2 + A_{1010}^2$ can be negative. Equation (2.5.12) gives a lower bound for f_I, which is the fidelity of our initially entangled state after time t if we are to use no quantum recovery operation. We plot the lower bound for f_I in Figure 2.3. An upper bound for f_I can similarly be obtained from (2.5.11).

The Barnum-Knill recovery operator $\mathcal{R}^{\mathrm{BK}}$ [BK02] and the Tyson-Beny-Oreshkov quadratic recovery operator [Tys10, BO10] are near optimal recovery operators defined with respect to a quantum operation $\mathcal{A}$ and a state ρ, and are equivalent when $\mathcal{A}$ has only one Kraus operator. For our application, we study the Barnum-Knill recovery, with $\rho = (|0\rangle\langle 0| + |1\rangle\langle 1|)/2$ and quantum operation $\mathcal{A}$ approximating the truncated channel $\mathcal{N}$. Our choice of ρ shows that our quantum information is encoded in the trivial quantum code (no encoding). For our application, the Barnum-Knill recovery operator which is also a quantum operation is defined as

$$\mathcal{R}^{\mathrm{BK}}(v) = A_1^\dagger(A_1A_1^\dagger)^{-1/2^+}v(A_1A_1^\dagger)^{-1/2^+}A_1 \tag{2.5.13}$$

where $(A_1A_1^\dagger)^{-1/2^+}$ is the square root of the psuedo-inverse of the operator $(A_1A_1^\dagger)$. When we use the Barnum-Knill recovery operation $\mathcal{R}^{\mathrm{BK}}$, the fidelity of recovery is

$$f_{\mathrm{BK}} = \langle\Phi|(\mathcal{R}^{\mathrm{BK}}\otimes\mathcal{R}^{\mathrm{BK}})\circ(\mathcal{N}\otimes\mathcal{N})(|\Phi\rangle\langle\Phi|)|\Phi\rangle. \geq \lambda_1^2 - \frac{|\lambda_2| + |\lambda_3| + |\lambda_4|}{4} - \frac{1}{4}(2\epsilon + \epsilon^2). \tag{2.5.14}$$

We plot lower bounds of the fidelity with and without Barnum-Knill recovery in Figure 2.3, and demonstrate that a fidelity of more than 60% is still possible in spite of the leakage error and the use of truncated quantum channels in our analysis.

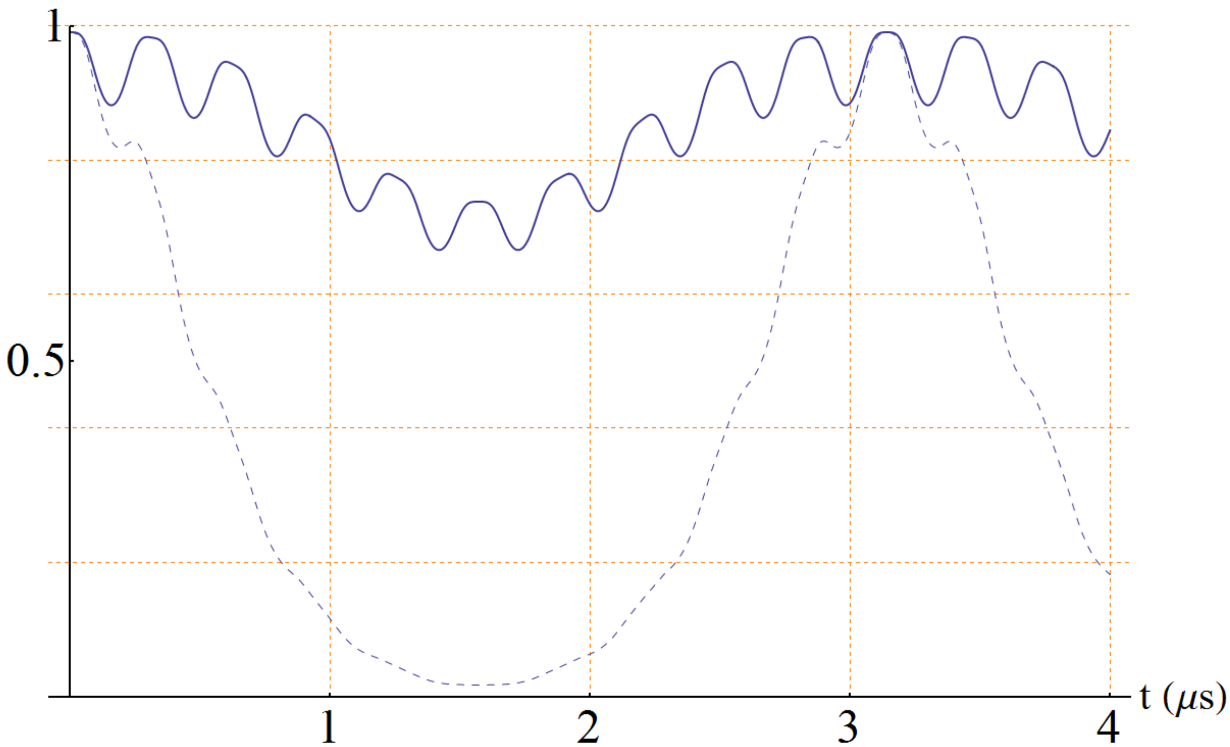

Figure 2.3: Lower bounds on the fidelity of an entangled state without recovery $f_{\mathcal{I}}$ (dashed line) and with Barnum-Knill recovery f_{BK} (solid line) are plotted with respect to time.

2.6 Discussions

The system we consider is described by a quantum harmonic oscillator coupled through a spring-like interaction to another initially decoupled harmonic oscillator. We provide approximations to the truncated transition amplitudes of such a system. The converging error bound of such approximations is our main result. Properties of the integrals of products of Hermite functions lies at the heart of the proof. It is also worth noting that any ensemble of harmonic oscillators with spring-like coupling can be analyzed similarly.

We also show numerically that in agreement with intuition from perturbation theory, when the oscillators are off-resonant and weakly coupled, amplitude damping is a negligible physical process. We also use our truncated channel representation to show that qubit leakage can be a dominant physical process, and how Barnum-Knill recovery can help protect a maximally entangled state stored in two oscillators each coupled independently to distinct zero-temperature harmonic baths, in the paradigm of off-resonant and weak coupling between the system and a zero temperature bath.

Chapter 3

The Perturbed Error-Correction Criterion and Rescaled Truncated Recovery

3.1 Introduction

Quantum information, when left unprotected, often decoheres because of its inevitable interaction with the environment. The field of quantum error correction arose from the need to combat decoherence in quantum systems, and treats the decoherence as a noisy quantum channel. An important problem in quantum error correction is that of determining the utility of a given code with respect to the noisy quantum channel. The quantum error correction conditions of Knill and Laflamme [KL97] are equations from which one can determine whether a quantum code is entirely robust against a given set of Kraus effects of the noisy channel. The Knill-Laflamme conditions lie at the foundations of Gottesman's stabilizer formalism [Got97] from which quantum error correction codes are designed and studied.

In this chapter, we revisit the approximate error correction of finite dimension codes via a perturbation of the Knill-Laflamme conditions. We derive a computationally simple lower bound on the worst case entanglement fidelity of a quantum code, when the truncated recovery map of Leung *et al.* [LNCY97] is rescaled to guarantee its validity as a quantum operation. Our lower bound arises from repeated application of the Gershgorin circle theorem on the relevant matrices.

The simplicity of our bound comes at a price – we do not have the near-optimal guarantees that the methods of Barnum-Knill [BK02] and Tyson-Beny-Oreshkov [Tys10, BO10, BO11] yield. However in this trade-off, we are able to construct a family of multi-error correcting amplitude damping qubit codes that are permutation-invariant. We thereby demonstrate an example where the specific structure of the noisy channel allows code design out of the stabilizer formalism via purely algebraic means, as opposed to optimization techniques [KSL08, FSW08, Yam09, TKL10] and other approaches [LS07, SSSZ11, DJZ10]. Our qubit permutation-invariant codes also extend the existing theory of qubit permutation-invariant codes [CLY97, Rus00, PR04, WB07]; while no qubit permutation-invariant code corrects arbitrary single qubit errors, there exist qubit permutation-invariant codes that correct multiple amplitude damping errors.

3.1.1 Organization

In Section 5.2, we introduce notation and concepts needed for this chapter, including quantum channels, quantum codes and the entanglement fidelity of a code. In Section 3.3, we address the perturbed Knill-Laflamme conditions, revisit the Leung *et al.* recovery map and determine using Lemma 3.3.3 when the Leung *et al.* recovery can be rescaled to a quantum operation. In Section 3.3.3, we prove our algebraic lower bound on the worst case entanglement fidelity (Theorem 3.3.4). Finally in Section 3.4, we apply our lower bound to construct a family of qubit permutation-invariant codes that correct multiple amplitude damping errors (Theorem 3.4.3).

3.2 Preliminaries

For any integer k and non-negative integer ℓ, define the falling factorial $k_{(\ell)}$ to be

$$k_{(\ell)} := \prod_{i=0}^{\ell}(k-i).$$

For all integers n, k where $k \geq 0$, define the binomial coefficient $\binom{n}{k} := n_{(k)}/k!$.

3.2.1 Quantum channels

For a complex separable Hilbert space $\mathcal{H}$, let $\mathfrak{B}(\mathcal{H})$ be the set of bounded linear operators mapping $\mathcal{H}$ to $\mathcal{H}$. Define the set of quantum states on Hilbert space $\mathcal{H}$ to be $\mathfrak{D}(\mathcal{H})$ where $\mathfrak{D}(\mathcal{H})$ is the set of all positive semi-definite and trace one operators in $\mathfrak{B}(\mathcal{H})$. For any subspace $\mathcal{C} \subset \mathcal{H}$ with orthonormal basis $\mathcal{B}_{\mathcal{C}}$, define $\mathfrak{D}(\mathcal{C})$ to be the set of all elements in $\mathfrak{D}(\mathcal{H})$ that are invariant under conjugation by the projector $\Pi = \sum_{|\beta\rangle \in \mathcal{B}_{\mathcal{C}}} |\beta\rangle\langle\beta|$.

A quantum operation $\Phi : \mathfrak{B}(\mathcal{H}) \to \mathfrak{B}(\mathcal{H}')$ is a linear map that is completely positive and trace non-increasing. A quantum channel Φ is a trace preserving quantum operation. In this chapter, the Hilbert spaces $\mathcal{H}$ and $\mathcal{H}'$ are always isomorphic. A quantum operation $\Phi : \mathfrak{B}(\mathcal{H}) \to \mathfrak{B}(\mathcal{H})$ can always be expressed in the Kraus representation [Kra83]:

$$\Phi(\rho) = \sum_{\mathbf{A} \in \mathfrak{K}_\Phi} \mathbf{A}\rho\mathbf{A}^\dagger, \quad \mathbb{1}_{\mathcal{H}} \geq \sum_{\mathbf{A} \in \mathfrak{K}_\Phi} \mathbf{A}^\dagger\mathbf{A}$$

where $\rho \in \mathfrak{B}(\mathcal{H})$, $\mathfrak{K}_\Phi \subset \mathfrak{B}(\mathcal{H})$ is a set of Kraus operators of quantum operation Φ, and $\mathbb{1}_{\mathcal{H}}$ is the identity operator on complex Hilbert space $\mathcal{H}$.

3.2.2 Quantum codes and Entanglement Fidelity

Define the minimum eigenvalue of a finite dimension Hermitian matrix $\mathbf{H}$ restricted to subspace $\mathcal{C}$ to be

$$\lambda_{\min,\mathcal{C}}(\mathbf{H}) := \min_{\substack{|\beta\rangle\in\mathcal{C} \\ \langle\beta|\beta\rangle=1}} \langle\beta|\mathbf{H}|\beta\rangle. \tag{3.2.1}$$

The entanglement fidelity of a state ρ with respect to the quantum channel $\mathcal{N}$ is

$$F_e(\rho, \mathcal{N}) = \sum_{\mathbf{B}\in\mathfrak{K}_{\mathcal{N}}} |\mathrm{Tr}(\mathbf{B}\rho)|^2 \tag{3.2.2}$$

where the set of Kraus operators of $\mathcal{N}$ is $\mathfrak{K}_{\mathcal{N}}$ [NC00]. The entanglement fidelity of a state ρ with respect to the quantum channel $\mathcal{N} = \mathcal{R} \circ \mathcal{A}$ quantifies how well the entanglement consistent with state ρ is preserved when the noisy channel is $\mathcal{A}$ and the recovery map is $\mathcal{R}$.

3.3 Rescaled truncated recovery

In this section, we analyze the performance of the rescaled truncated recovery map. Firstly in Section 3.3.1, we study the diagonalization of the Knill-Laflamme conditions. We next analyze the rescaling of the truncated recovery of Leung *et al.* [LNCY97] into a quantum operation. In Section 3.3.3, we prove Theorem 3.3.4, the main result of this chapter.

3.3.1 The perturbed Knill-Laflamme conditions

In this subsection, we apply the canonical procedure [NC00] to diagonalize the error correction perturbed Knill-Laflamme conditions. We also introduce notation that is used both in the statement and the proof of Theorem 3.3.4. We start by introducing the following notation.

N1. *Let the M-dimension code $\mathcal{C}$ be a subspace of the Hilbert space $\mathcal{H}$ with projector Π and orthonormal basis $\mathcal{B}_{\mathcal{C}}$. Let $\mathcal{A} : \mathfrak{B}(\mathcal{H}) \to \mathfrak{B}(\mathcal{H})$ be a quantum channel with truncated Kraus set Ω.*

Using the notation of **N1**, define an orthonormal basis $\{|\mathbf{E}\rangle : \mathbf{E} \in \Omega\} \subseteq \mathbb{C}^{|\Omega|}$ labeling the effects in Ω. For all $\mathbf{A}, \mathbf{B} \in \Omega$ and $|\alpha\rangle, |\beta\rangle \in \mathcal{B}_{\mathcal{C}}$, define

$$\epsilon(\mathbf{A}, \mathbf{B}, |\alpha\rangle, |\beta\rangle) := \langle\alpha|\mathbf{A}^{\dagger}\mathbf{B}|\beta\rangle - g_{\mathbf{A},\mathbf{B}}\delta_{|\alpha\rangle,|\beta\rangle}. \tag{3.3.1}$$

to quantify the perturbation to the Knill-Laflamme condition, where

$$g_{\mathbf{A},\mathbf{B}} := \frac{1}{M} \sum_{|\beta\rangle \in \mathcal{B}_{\mathcal{C}}} \langle\beta|\mathbf{A}^{\dagger}\mathbf{B}|\beta\rangle. \tag{3.3.2}$$

Define the Hermitian matrix

$$\mathbf{G} := \sum_{\mathbf{A},\mathbf{B} \in \Omega} g_{\mathbf{A},\mathbf{B}} |\mathbf{A}\rangle\langle\mathbf{B}|. \tag{3.3.3}$$

The hermiticity of $\mathbf{G}$ implies the existence of a unitary matrix $\mathbf{V}$ and diagonal matrix $\mathbf{D}$ such that

$$\mathbf{V} = \sum_{\mathbf{E},\mathbf{F} \in \Omega} v_{\mathbf{E},\mathbf{F}} |\mathbf{E}\rangle\langle\mathbf{F}| \tag{3.3.4}$$

$$\mathbf{D} := \mathbf{V}\mathbf{G}\mathbf{V}^{\dagger} = \sum_{\mathbf{E} \in \Omega} d_{\mathbf{E}} |\mathbf{E}\rangle\langle\mathbf{E}| \tag{3.3.5}$$

which implies that

$$\sum_{\mathbf{F},\mathbf{F}' \in \Omega} v_{\mathbf{E},\mathbf{F}} v^{*}_{\mathbf{E}',\mathbf{F}'} g_{\mathbf{F},\mathbf{F}'} = d_{\mathbf{E}} \delta_{\mathbf{E},\mathbf{E}'}. \tag{3.3.6}$$

For all $\mathbf{A} \in \Omega$, define the transformed Kraus operators

$$\widetilde{\mathbf{A}} := \sum_{\mathbf{F} \in \Omega} v_{\mathbf{A},\mathbf{F}} \mathbf{F}. \tag{3.3.7}$$

Substituting (3.3.6) into (3.3.7) gives

$$\langle \alpha | \widetilde{\mathbf{A}}^{\dagger} \widetilde{\mathbf{B}} | \beta \rangle = d_{\mathbf{A}} \delta_{\mathbf{A},\mathbf{B}} \delta_{|\alpha\rangle,|\beta\rangle} + \widetilde{\epsilon}\big(\mathbf{A}, \mathbf{B}, |\alpha\rangle, |\beta\rangle\big) \tag{3.3.8}$$

where

$$\widetilde{\epsilon}\big(\mathbf{A}, \mathbf{B}, |\alpha\rangle, |\beta\rangle\big) := \sum_{\mathbf{F},\mathbf{F}' \in \Omega} (v^{*}_{\mathbf{A},\mathbf{F}'} v_{\mathbf{B},\mathbf{F}}) \epsilon\big(\mathbf{F}', \mathbf{F}, |\alpha\rangle, |\beta\rangle\big). \tag{3.3.9}$$

Equation (3.3.8) gives the 'diagonalized' form of the perturbed Knill-Laflamme conditions. The transformed error is quantified by (3.3.9). Let $\epsilon_{|\alpha\rangle,|\beta\rangle} := \max_{\mathbf{A},\mathbf{B} \in \Omega} \left| \epsilon\Big(\mathbf{A}, \mathbf{B}, |\alpha\rangle, |\beta\rangle\Big) \right|$. Then the Cauchy-Schwarz inequality and normalization of the rows of $\mathbf{V}$ implies that

$$|\widetilde{\epsilon}\big(\mathbf{A}, \mathbf{B}, |\alpha\rangle, |\beta\rangle\big)| \leq \sum_{\mathbf{F},\mathbf{F}' \in \Omega} |v^{*}_{\mathbf{A},\mathbf{F}'} v_{\mathbf{B},\mathbf{F}}| \epsilon_{|\alpha\rangle,|\beta\rangle} \leq |\Omega| \epsilon_{|\alpha\rangle,|\beta\rangle}. \tag{3.3.10}$$

3.3.2 The Leung *et al.* truncated recovery

The truncated recovery of Leung *et al.* [LNCY97] gives an algebraically simple lower bound on the entanglement fidelity of a code with respect to a noisy channel, under certain assumptions on the noisy channel and the quantum code. To understand this truncated recovery, we need the following notation.

N2. *We use the notation of* **N1**. *For all* $\mathbf{A} \in \Omega$, *define* $\Pi_{\mathbf{A}} := \mathbf{U}_{\mathbf{A}} \Pi \mathbf{U}_{\mathbf{A}}^{\dagger}$ *where* $\mathbf{U}_{\mathbf{A}}$ *is the unitary in the polar decomposition* $\mathbf{A}\Pi = \mathbf{U}_{\mathbf{A}} \sqrt{\Pi \mathbf{A}^{\dagger} \mathbf{A} \Pi}$. *Define the completely positive but not necessarily trace preserving linear operator* $\mathcal{R}_{\Omega,\mathcal{C}} : \mathfrak{B}(\mathcal{H}) \rightarrow \mathfrak{B}(\mathcal{H})$ *where for all*

$\mu \in \mathfrak{B}(\mathcal{H})$,

$$\mathcal{R}_{\Omega,\mathcal{C}}(\mu) := \sum_{\mathbf{A}\in\Omega} \mathbf{R}_{\mathbf{A}} \mu \mathbf{R}_{\mathbf{A}}^{\dagger}, \quad \mathbf{R}_{\mathbf{A}} := \mathbf{U}_{\mathbf{A}}^{\dagger} \Pi_{\mathbf{A}}. \tag{3.3.11}$$

Without loss of generality, we pick Ω *such that* $\sqrt{\Pi \mathbf{A}^{\dagger} \mathbf{A} \Pi}$ *is never the zero operator.*

We call the completely positive map $\mathcal{R}_{\Omega,\mathcal{C}}$ a *truncated recovery map*, because Ω is a truncated Kraus set of the quantum channel $\mathcal{A}$. The truncated recovery map $\mathcal{R}_{\Omega,\mathcal{C}}$ is also a quantum operation when the projectors $\Pi_{\mathbf{A}}$ in **N2** are orthogonal, that is

$$\Pi_{\mathbf{A}} \Pi_{\mathbf{B}} = \Pi_{\mathbf{A}} \delta_{\mathbf{A},\mathbf{B}} \quad \forall \mathbf{A}, \mathbf{B} \in \Omega. \tag{3.3.12}$$

Lemma 3.3.1 (Leung *et al.* [LNCY97])**.** *In the notation of* **N1** *and* **N2***, for any* $\rho \in \mathfrak{D}(\mathcal{C})$*, the bound* $\sum_{\mathbf{A}\in\Omega} |\mathrm{Tr}(\mathbf{R}_{\mathbf{A}} \mathbf{A} \rho)|^2 \geq \sum_{\mathbf{A}\in\Omega} \lambda_{\min,\mathcal{C}}(\mathbf{A}^{\dagger}\mathbf{A})$ *holds.*

Proof. For $\mathbf{A} \in \Omega$, define $\mu_{\mathbf{A}} := \lambda_{\min,\mathcal{C}}(\mathbf{A}^{\dagger}\mathbf{A})$ and the positive semidefinite residue operator

$$\pi_{\mathbf{A}} := \sqrt{\Pi \mathbf{A}^{\dagger} \mathbf{A} \Pi} - \sqrt{\mu_{\mathbf{A}}} \Pi.$$

Substituting $\pi_{\mathbf{A}}$ into the polar decomposition of $\mathbf{A}\Pi$ gives

$$\begin{aligned} \mathbf{A}\Pi &= \mathbf{U}_{\mathbf{A}} \sqrt{\Pi \mathbf{A}^{\dagger} \mathbf{A} \Pi} \\ &= \mathbf{U}_{\mathbf{A}} (\sqrt{\Pi \mathbf{A}^{\dagger} \mathbf{A} \Pi} - \sqrt{\mu_{\mathbf{A}}} \Pi + \sqrt{\mu_{\mathbf{A}}} \Pi) \\ &= \mathbf{U}_{\mathbf{A}} (\pi_{\mathbf{A}} + \sqrt{\mu_{\mathbf{A}}} \Pi) \\ &= \mathbf{U}_{\mathbf{A}} (\pi_{\mathbf{A}} + \sqrt{\mu_{\mathbf{A}}} \mathbb{1}_{\mathcal{H}}) \Pi. \end{aligned} \tag{3.3.13}$$

The spectral decomposition of ρ in the basis $\mathcal{B}_\mathcal{C}$ and equation (3.3.13) imply that

$$\sum_{\mathbf{A}\in\Omega} |\mathrm{Tr}(\mathbf{R_A A}\rho)|^2 \geq \sum_{\mathbf{A}\in\Omega} \left| \sum_{|\beta\rangle\in\mathcal{B}_\mathcal{C}} p_{|\beta\rangle} \langle\beta|\mathbf{R_A} A|\beta\rangle \right|^2$$

$$= \sum_{\mathbf{A}\in\Omega} \left| \sum_{|\beta\rangle\in\mathcal{B}_\mathcal{C}} p_{|\beta\rangle} \langle\beta|(\Pi\mathbf{U}_\mathbf{A}^\dagger)(\mathbf{U_A}(\sqrt{\mu_\mathbf{A}}\mathbb{1}_\mathcal{H} + \pi_\mathbf{A})\Pi)|\beta\rangle \right|^2 . \tag{3.3.14}$$

Using $\Pi|\beta\rangle = |\beta\rangle$ and $\mathbf{U}_\mathbf{A}^\dagger\mathbf{U_A} = \mathbb{1}_\mathcal{H}$, (3.3.14) becomes

$$\sum_{\mathbf{A}\in\Omega} \Big(\sum_{|\beta\rangle\in\mathcal{B}_\mathcal{C}} p_{|\beta\rangle} \langle\beta|(\sqrt{\mu_\mathbf{A}}\mathbb{1}_\mathcal{H} + \pi_\mathbf{A})|\beta\rangle \Big)^2 .$$

Moreover $\pi_\mathbf{A}$ is positive semi-definite, hence the above expression is at least

$$\sum_{\mathbf{A}\in\Omega} \Big(\sum_{|\beta\rangle\in\mathcal{B}_\mathcal{C}} p_{|\beta\rangle} \langle\beta|(\sqrt{\mu_\mathbf{A}}\mathbb{1}_\mathcal{H})|\beta\rangle \Big)^2 = \sum_{\mathbf{A}\in\Omega} \mu_\mathbf{A}.$$

□

Leung *et al.* proved that when the orthogonality condition (3.3.12) holds, the truncated recovery map $\mathcal{R}_{\Omega,\mathcal{C}}$ is also quantum operation, and thus Lemma 3.3.1 gives a lower bound on the worst case entanglement fidelity. We detail this in Lemma 3.3.2.

Lemma 3.3.2 (Leung et.al. [LNCY97]). *We use the notation of* **N1** *and* **N2**. *Suppose that (3.3.12) holds. Then the truncated recovery map $\mathcal{R}_{\Omega,\mathcal{C}}$ is a quantum operation and*

$$\min_{\rho\in\mathfrak{D}(\mathcal{C})} F_e(\rho, \mathcal{R}_{\Omega,\mathcal{C}} \circ \mathcal{A}) \geq \sum_{\mathbf{A}\in\Omega} \lambda_{\min,\mathcal{C}}(\mathbf{A}^\dagger\mathbf{A}). \tag{3.3.15}$$

Proof. Observe that

$$\sum_{\mathbf{A}\in\Omega} \mathbf{R}_\mathbf{A}^\dagger\mathbf{R_A} = \sum_{\mathbf{A}\in\Omega} \Pi_\mathbf{A}\mathbf{U_A}\mathbf{U}_\mathbf{A}^\dagger\Pi_\mathbf{A} = \sum_{\mathbf{A}\in\Omega} \Pi_\mathbf{A}. \tag{3.3.16}$$

Orthogonality of the projectors $\Pi_{\mathbf{A}}$ implies that

$$\left(\sum_{\mathbf{A}\in\Omega}\Pi_{\mathbf{A}}\right)\left(\sum_{\mathbf{A}'\in\Omega}\Pi_{\mathbf{A}'}\right)=\sum_{\mathbf{A},\mathbf{A}'\in\Omega}\Pi_{\mathbf{A}}\delta_{\mathbf{A},\mathbf{A}'}=\sum_{\mathbf{A}\in\Omega}\Pi_{\mathbf{A}}. \tag{3.3.17}$$

Hence $\sum_{\mathbf{A}\in\Omega}\mathbf{R}_{\mathbf{A}}^{\dagger}\mathbf{R}_{\mathbf{A}}$ is also a projector. The map $\mathcal{R}_{\Omega,\mathcal{C}}$ is also completely positive, that is $\sum_{\mathbf{A}\in\Omega}\mathbf{R}_{\mathbf{A}}\rho\mathbf{R}_{\mathbf{A}}^{\dagger}\geq 0$, because $\mathbf{U}_{\mathbf{A}}^{\dagger}\rho\mathbf{U}_{\mathbf{A}}\geq 0$ which implies that $\mathbf{R}_{\mathbf{A}}\rho\mathbf{R}_{\mathbf{A}}^{\dagger}=\Pi\mathbf{U}_{\mathbf{A}}^{\dagger}\rho\mathbf{U}_{\mathbf{A}}\Pi\geq 0$. Obmitting non-negative terms in the sum pertaining to the entanglement fidelity, we get

$$\mathcal{F}_e(\rho,\mathcal{R}_{\Omega,\mathcal{C}}\circ\mathcal{A})\geq\sum_{\mathbf{A}\in\Omega}|\operatorname{Tr}(\mathbf{R}_{\mathbf{A}}\mathbf{A}\rho)|^2. \tag{3.3.18}$$

Applying Lemma 3.3.1 gives the result. □

Rescaled maps have been used in the study of near-optimal quantum recovery operations, including the Barnum-Knill recovery map [BK02] and the Tyson-Beny-Oreshkov quadratic recovery map [Tys10, BO10]. In the notation of **N1** and **N2**, the completely positive map $\mathcal{R}_{\Omega,\mathcal{C}}$ might increase trace and hence not be a quantum operation. Fortunately a bounded $\mathcal{R}_{\Omega,\mathcal{C}}$ can be rescaled to the quantum operation $\mathcal{R}_{\Omega,\mathcal{C},\eta}:=(1+\eta)^{-1}\mathcal{R}_{\Omega,\mathcal{C}}$.

Lemma 3.3.3. *Using the notation of* **N1** *and* **N2**, *let* $\eta\geq|\Omega|^2\max_{\mathbf{A}\neq\mathbf{B}\in\Omega}\left\|\Pi\mathbf{U}_{\mathbf{A}}^{\dagger}\mathbf{U}_{\mathbf{B}}\Pi\right\|_2$. *Then* $\mathcal{R}_{\Omega,\mathcal{C},\eta}$ *is a quantum operation.*

Proof. It suffices to show that $\left\|\sum_{\mathbf{A}\in\Omega}\mathbf{R}_{\mathbf{A}}^{\dagger}\mathbf{R}_{\mathbf{A}}\right\|_2\leq 1+\eta$. First observe that

$$\sum_{\mathbf{A}\in\Omega}\mathbf{R}_{\mathbf{A}}^{\dagger}\mathbf{R}_{\mathbf{A}}=\sum_{\mathbf{A}\in\Omega}\Pi_{\mathbf{A}}\mathbf{U}_{\mathbf{A}}\mathbf{U}_{\mathbf{A}}^{\dagger}\Pi_{\mathbf{A}}=\sum_{\mathbf{A}\in\Omega}\Pi_{\mathbf{A}}.$$

The projectors $\Pi_{\mathbf{A}}$ may not be orthogonal, so

$$\begin{aligned}\left(\sum_{\mathbf{A}\in\Omega}\mathbf{R}_{\mathbf{A}}^{\dagger}\mathbf{R}_{\mathbf{A}}\right)^2&=\Big(\sum_{\mathbf{A}\in\Omega}\Pi_{\mathbf{A}}\Big)\Big(\sum_{\mathbf{B}\in\Omega}\Pi_{\mathbf{B}}\Big)=\sum_{\mathbf{A},\mathbf{B}\in\Omega}\mathbf{U}_{\mathbf{A}}\Pi\mathbf{U}_{\mathbf{A}}^{\dagger}\mathbf{U}_{\mathbf{B}}\Pi\mathbf{U}_{\mathbf{B}}^{\dagger}\\&=\sum_{\mathbf{A}\in\Omega}\Pi_{\mathbf{A}}+\sum_{\mathbf{A}\neq\mathbf{B}\in\Omega}\mathbf{U}_{\mathbf{A}}\Pi\mathbf{U}_{\mathbf{A}}^{\dagger}\mathbf{U}_{\mathbf{B}}\Pi\mathbf{U}_{\mathbf{B}}^{\dagger}\end{aligned}$$

and hence

$$\left\|\left(\sum_{\mathbf{A}\in\Omega}\mathbf{R}_{\mathbf{A}}^{\dagger}\mathbf{R}_{\mathbf{A}}\right)^2\right\|_2 = \left\|\sum_{\mathbf{A}\in\Omega}\mathbf{R}_{\mathbf{A}}^{\dagger}\mathbf{R}_{\mathbf{A}}\right\|_2^2 \leq \left\|\sum_{\mathbf{A}\in\Omega}\mathbf{R}_{\mathbf{A}}^{\dagger}\mathbf{R}_{\mathbf{A}}\right\|_2 + |\Omega|^2 \max_{\mathbf{A}\neq\mathbf{B}\in\Omega}\left\|\Pi\mathbf{U}_{\mathbf{A}}^{\dagger}\mathbf{U}_{\mathbf{B}}\Pi\right\|_2.$$

Let $\epsilon = \left\|\sum_{\mathbf{A}\in\Omega}\mathbf{R}_{\mathbf{A}}^{\dagger}\mathbf{R}_{\mathbf{A}}\right\|_2 - 1 \geq 0$. Then substituting ϵ and η into the above inequality gives

$$1 + 2\epsilon + \epsilon^2 \leq 1 + \epsilon + \eta,$$

which implies that $\epsilon \leq \eta$. Hence $\left\|\sum_{\mathbf{A}\in\Omega}\mathbf{R}_{\mathbf{A}}^{\dagger}\mathbf{R}_{\mathbf{A}}\right\|_2 \leq 1 + \eta$, and the completely positive map $\mathcal{R}_{\Omega,\mathcal{C},\eta}$ is a quantum operation. □

3.3.3 Algebraic lower bounds on the worst case entanglement fidelity

Here, we prove algebraic lower bounds on the worst case entanglement fidelity of a code using a rescaled truncated recovery map, given partial knowledge of the noisy quantum channel. Our main technical result is the following:

Theorem 3.3.4. *Let the M-dimension code $\mathcal{C}$ with an orthonormal basis $\mathcal{B}_{\mathcal{C}}$ be a subspace of the Hilbert space $\mathcal{H}$. Let $\mathcal{A} : \mathfrak{B}(\mathcal{H}) \to \mathfrak{B}(\mathcal{H})$ be a quantum channel with truncated Kraus set Ω. Define $\mathbf{G} = \sum_{\mathbf{A},\mathbf{B}\in\Omega} g_{\mathbf{A},\mathbf{B}}$ where $g_{\mathbf{A},\mathbf{B}} := \frac{1}{M}\sum_{|\beta\rangle\in\mathcal{B}_{\mathcal{C}}}\langle\beta|\mathbf{A}^{\dagger}\mathbf{B}|\beta\rangle$, and suppose that $\lambda_{\min}(\mathbf{G}) > 0$. Suppose that for all Kraus effects $\mathbf{A},\mathbf{B}$ in the truncated Kraus set Ω and for all distinct orthonormal basis vectors $|\alpha\rangle, |\beta\rangle \in \mathcal{B}_{\mathcal{C}}$,*

$$\left|g_{\mathbf{A},\mathbf{B}} - \langle\alpha|\mathbf{A}^{\dagger}\mathbf{B}|\alpha\rangle\right| \leq \epsilon, \quad \left|\langle\alpha|\mathbf{A}^{\dagger}\mathbf{B}|\beta\rangle\right| \leq \epsilon. \tag{3.3.19}$$

Then the minimum entanglement fidelity of our code $\mathcal{C}$ with respect to the noisy channel $\mathcal{A}$ is at least $(\operatorname{Tr}\mathbf{G} - M|\Omega|^2\epsilon)\left(1 + \frac{M|\Omega|^3\epsilon}{\lambda_{\min}(\mathbf{G})}\right)^{-1}$.

The proof of Theorem 3.3.4 follows from the direct application of Lemma 3.3.3 which gives us the important properties of our rescaled truncated recovery, and the repeated application of the Gershgorin circle theorem.

Proof of Theorem 3.3.4. From (3.3.1), we have

$$\frac{1}{M}\sum_{|\alpha\rangle\in\mathcal{B}_{\mathcal{C}}}\epsilon\Big(\mathbf{A},\mathbf{B},|\alpha\rangle,|\alpha\rangle\Big)=0$$

which implies that

$$\frac{1}{M}\sum_{|\alpha\rangle\in\mathcal{B}_{\mathcal{C}}}\widetilde{\epsilon}\Big(\mathbf{A},\mathbf{B},|\alpha\rangle,|\alpha\rangle\Big)=0.$$

Hence $\lambda_{\max}(\Pi\widetilde{\mathbf{A}}^\dagger\widetilde{\mathbf{A}}\Pi)\geq\frac{1}{M}\sum_{|\alpha\rangle\in\mathcal{B}_{\mathcal{C}}}\langle\alpha|\widetilde{\mathbf{A}}^\dagger\widetilde{\mathbf{A}}|\alpha\rangle\geq d_{\mathbf{A}}$ which is at least $\lambda_{\min}(\mathbf{G})$.

Applying the Gershgorin circle theorem on the matrix $\Pi\widetilde{\mathbf{A}}^\dagger\widetilde{\mathbf{B}}\Pi$ with the error estimate (3.3.10), we get $\lambda_{\max}(\Pi\widetilde{\mathbf{A}}^\dagger\widetilde{\mathbf{B}}\Pi)\leq M|\Omega|\epsilon$ for distinct $\mathbf{A},\mathbf{B}\in\Omega$.

For distinct $\mathbf{A},\mathbf{B}\in\Omega$, let $\widetilde{\mathbf{A}}\Pi$ and $\widetilde{\mathbf{B}}\Pi$ have polar decompositions $\widetilde{\mathbf{A}}\Pi=\mathbf{U}_{\widetilde{\mathbf{A}}}\sqrt{\Pi\widetilde{\mathbf{A}}^\dagger\widetilde{\mathbf{A}}\Pi}$ and $\widetilde{\mathbf{B}}\Pi=\mathbf{U}_{\widetilde{\mathbf{B}}}\sqrt{\Pi\widetilde{\mathbf{B}}^\dagger\widetilde{\mathbf{B}}\Pi}$ respectively. Then

$$\Pi\widetilde{\mathbf{A}}^\dagger\widetilde{\mathbf{B}}\Pi=\sqrt{\Pi\widetilde{\mathbf{A}}^\dagger\widetilde{\mathbf{A}}\Pi}\mathbf{U}_{\widetilde{\mathbf{A}}}^\dagger\mathbf{U}_{\widetilde{\mathbf{B}}}\sqrt{\Pi\widetilde{\mathbf{B}}^\dagger\widetilde{\mathbf{B}}\Pi}=\sqrt{\Pi\widetilde{\mathbf{A}}^\dagger\widetilde{\mathbf{A}}\Pi}(\Pi\mathbf{U}_{\widetilde{\mathbf{A}}}^\dagger\mathbf{U}_{\widetilde{\mathbf{B}}}\Pi)\sqrt{\Pi\widetilde{\mathbf{B}}^\dagger\widetilde{\mathbf{B}}\Pi}.$$

Hence by the sub-multiplicative property for norms, $\|\Pi\mathbf{U}_{\widetilde{\mathbf{A}}}^\dagger\mathbf{U}_{\widetilde{\mathbf{B}}}\Pi\|_2$ is at most

$$\left\|\sqrt{\Pi\widetilde{\mathbf{A}}^\dagger\widetilde{\mathbf{A}}\Pi}\right\|_2^{-1}\left\|\Pi\widetilde{\mathbf{A}}^\dagger\widetilde{\mathbf{B}}\Pi\right\|_2\left\|\sqrt{\Pi\widetilde{\mathbf{B}}^\dagger\widetilde{\mathbf{B}}\Pi}\right\|_2^{-1}\leq\frac{\|\Pi\widetilde{\mathbf{A}}^\dagger\widetilde{\mathbf{B}}\Pi\|_2}{\min\limits_{\mathbf{F}\in\{\mathbf{A},\mathbf{B}\}}\lambda_{\max}(\Pi\widetilde{\mathbf{F}}^\dagger\widetilde{\mathbf{F}}\Pi)}\leq\frac{M|\Omega|\epsilon}{\lambda_{\min}(\mathbf{G})}.$$

Hence by Lemma 3.3.3, the map $\mathcal{R}_{\Omega,\mathcal{C},\eta}$ is a quantum operation whenever $\eta\geq\frac{M|\Omega|^3\epsilon}{\lambda_{\min}(\mathbf{G})}$. By the Gershgorin circle theorem, $\lambda_{\min,\mathcal{C}}(\widetilde{\mathbf{A}}^\dagger\widetilde{\mathbf{A}})$ is at least $d_{\mathbf{A}}-M|\Omega|\epsilon$, and hence

$$\sum_{\mathbf{A}\in\Omega}\lambda_{\min,\mathcal{C}}(\widetilde{\mathbf{A}}^\dagger\widetilde{\mathbf{A}})\geq\sum_{\mathbf{A}\in\Omega}(d_A-M|\Omega|\epsilon)=\operatorname{Tr}\mathbf{D}-M|\Omega|^2\epsilon=\operatorname{Tr}\mathbf{G}-M|\Omega|^2\epsilon.$$

Use of Lemma 3.3.1 then gives the result. □

Under stronger assumptions on the set of conditions the truncated set of our noisy channel must satisfy, we can obtain the following corollary.

Corollary 3.3.5. *Let t be a positive integer, and $\gamma > 0$ be an error parameter, and assume that the requirements of Theorem 3.3.4 hold with $\epsilon = \gamma^{2t+1}$. Suppose that every $\mathbf{A} \in \Omega$ is of order $\mathcal{O}(\gamma^{s_{\mathbf{A}}})$ for some non-negative integer $s_{\mathbf{A}} \leq t$, and all Kraus effects not in Ω are of order $\mathcal{O}(\gamma^{t+1})$. Then the minimum entanglement fidelity of the code $\mathcal{C}$ with respect to the noisy channel $\mathcal{A}$ is at least $1 - \mathcal{O}(\gamma^{t+1})$.*

Proof. Note that $\operatorname{Tr} \mathbf{G} = 1 - \mathcal{O}(\gamma^{t+1})$, and $\lambda_{\min}(\mathbf{G}) = \mathcal{O}(\gamma^{t}) \neq 0$. Substitution of these parameters into Theorem 3.3.4 gives the result. □

3.4 Application: Permutation-invariant amplitude damping codes

In this section, we apply Theorem 3.3.4 to prove the existence of family of multiple amplitude error correcting permutation-invariant codes.

First we define permutation-invariant codes encoding a single qubit. Define $P_{m,a}$ to be the set of all length m binary vectors of weight a, and the corresponding permutation-invariant qubit states to be $|P_{m,a}\rangle := \sum_{\mathbf{x} \in P_{m,a}} |\mathbf{x}\rangle \binom{m}{a}^{-1/2}$. The permutation-invariant codes we consider have the logical codewords $|j_L\rangle := \sum_{a=0}^{m} \sqrt{\lambda_{j,a}} |P_{m,a}\rangle$ where $j \in \{0,1\}$ and $|j_L\rangle$ are linearly independent. Moreover the non-negative weights $\lambda_{j,a}$ are normalized so that $\sum_{a=0}^{m} \lambda_{j,a} = 1$.

Now we introduce notation relevant to the amplitude damping channel. The amplitude damping channel $\mathcal{A}_\gamma$ has Kraus operators $\mathbf{A}_0 = |0\rangle\langle 0| + \sqrt{1-\gamma}|1\rangle\langle 1|$ and $\mathbf{A}_1 = \sqrt{\gamma}|0\rangle\langle 1|$, where the non-negative parameter $\gamma \leq 1$ quantifies the amount of amplitude damping of $\mathcal{A}_\gamma$. We denote the Kraus operators of $\mathcal{A}_\gamma^{\otimes m}$ by $\mathbf{A}_{\mathbf{k}} := \mathbf{A}_{k_1} \otimes ... \otimes \mathbf{A}_{k_m}$, where $\mathbf{k} = (k_1, ..., k_m)$ is a binary vector. For positive integers t, we say that a code $\mathcal{C}$ is a t-amplitude damping code (or a t-AD code) if there exists a quantum operation $\mathcal{R}$ such that

$$\max_{\rho \in \mathfrak{D}(\mathcal{C})} (1 - F_e(\rho, \mathcal{R} \circ \mathcal{A}_\gamma^{\otimes m})) = o(\gamma^t). \tag{3.4.1}$$

The recovery operation that we use in this section is the rescaled recovery map of Leung *et al.*

For all $i \in [m]$, let $\mathcal{H}_i$ denote two-dimension complex Euclidean spaces, which is the i-th qubit space in a block of length m. Given a subset $S \subseteq [m]$, we use $|\psi\rangle_S$ to denote a state $|\psi\rangle \in \bigotimes_{i \in S} \mathcal{H}_i$. Let $\bar{S} := \{i \in [m] : i \notin S\}$ denote the complement of S with respect to $[m]$. Let an *indicator vector* $\mathfrak{I}_S$ be a length-m binary vector with components equal to 1 on the indices indexed by S and zero everywhere else. Let $\mathbf{k} = \mathfrak{I}_S$ where S has a size of k. Then we have

$$A_{\mathbf{k}}|j_L\rangle = \sqrt{\gamma^k} \sum_{a=0}^{m} \sqrt{\lambda_{j,a}} \sqrt{(1-\gamma)^{a-k}} \binom{m}{a}^{-\frac{1}{2}} |0\rangle_S \sum_{x \in P_{m-k,a-k}} |x\rangle_{\bar{S}}. \tag{3.4.2}$$

Let S' be another subset of $[m]$ with size k', and let $\mathbf{k'} = \mathfrak{I}_{S'}$. Now observe that

$$\sum_{\substack{x' \in P_{m-k',a'-k'} \\ x \in P_{m-k,a-k}}} \langle 0|_{S'} \langle x'|_{\bar{S}'} |0\rangle_S |x\rangle_{\bar{S}}. = \delta_{a'-k',a-k} \binom{m - |S \cup S'|}{a-k} \tag{3.4.3}$$

where $0 \leq a - k \leq m - |S \cup S'|$. Hence if $\lambda_j, a = 0$ for all $j \in \{0,1\}$ and for all integers $a \notin [k, m-k-k']$ we can apply (3.4.3) to get

$$\begin{aligned}
\langle j'_L|A_{\mathbf{k'}}^\dagger A_{\mathbf{k}}|j_L\rangle &= \sum_{a=0}^{m} \sum_{a'=0}^{m} \sqrt{\frac{\lambda_{j',a'}\lambda_{j,a}}{\binom{m}{a'}\binom{m}{a}}} \sqrt{\gamma^{k+k'}} \sqrt{(1-\gamma)^{a'-k'}(1-\gamma)^{a-k}} \delta_{a'-k',a-k} \binom{m-|S \cup S'|}{a-k} \\
&= \sum_{a=0}^{m} \sqrt{\frac{\lambda_{j',a+k'-k}\lambda_{j,a}}{\binom{m}{a+k'-k}\binom{m}{a}}} \sqrt{\gamma^{k+k'}} (1-\gamma)^{a-k} \binom{m-|S \cup S'|}{a-k}.
\end{aligned} \tag{3.4.4}$$

The expression (3.4.4) allows us to deduce algebraically sufficient conditions on $\lambda_{j,a}$ where the matrix elements of $\langle j_L|A^\dagger_{\mathbf{k}} A_{\mathbf{k}'}|j_L\rangle$ are approximately equivalent to the matrix element $g_{A_{\mathbf{k}},A_{\mathbf{k}'}}$, which is the essence of Lemma 3.4.1. Lemma 3.4.1 is in turn one of the key technical lemmas to show the existence of t-AD permutation-invariant codes.

Lemma 3.4.1. *Let m and t be positive integers, with $m > t$. Let $\lambda_{j,b} = 0$ for all integers $b \in (m-t, m]$ and $j \in \{0,1\}$. Further suppose that for all non-negative integers $c \le t$ and $\ell \le 2t$, we have*

$$\sum_{b=0}^{m} \lambda_{0,b} \binom{m-b}{c} \binom{b}{\ell} = \sum_{b=0}^{m} \lambda_{1,b} \binom{m-b}{c} \binom{b}{\ell}.$$

Let $\mathbf{k}$ and $\mathbf{k}'$ be binary vectors of equal weight k where $k \le t$. Then

$$\langle 0_L|A^\dagger_{\mathbf{k}} A_{\mathbf{k}'}|0_L\rangle = \langle 1_L|A^\dagger_{\mathbf{k}} A_{\mathbf{k}'}|1_L\rangle + \mathcal{O}(\gamma^{2t+1}).$$

Proof. For non-negative integers m, k, b, c such that $b \le m - c$ and $c \le k$, we have

$$\frac{\binom{m-k-c}{b-k}}{\binom{m}{b}} = \frac{\binom{b}{k}}{\binom{m}{k-c}} \binom{m-b}{c} \binom{k}{c} (c!)^2. \tag{3.4.5}$$

Using the identity (3.4.5) with (3.4.4), we get

$$
\begin{aligned}
\langle j_L | A_{\mathbf{k}}^\dagger A_{\mathbf{k}'} | j_L \rangle &= \sum_{b=0}^{m} \lambda_{j,b} \gamma^k (1-\gamma)^{b-k} \frac{\binom{b}{k}}{\binom{m}{k-c}} \binom{m-b}{c} \binom{k}{c} (c!)^2 \\
&= \sum_{b=0}^{m} \lambda_{j,b} \gamma^k \sum_{\beta=0}^{b-k} \binom{b-k}{\beta} \gamma^\beta (-1)^\beta \frac{\binom{b}{k}}{\binom{m}{k-c}} \binom{m-b}{c} \binom{k}{c} (c!)^2 \\
&= \binom{k}{c} (c!)^2 \binom{m}{k-c}^{-1} \gamma^k \sum_{b=0}^{m} \lambda_{j,b} \binom{m-b}{c} \sum_{\beta=0}^{b-k} \frac{b_{(k)} (b-k)_{(\beta)}}{k! \beta!} \gamma^\beta (-1)^\beta \\
&= \binom{k}{c} (c!)^2 \binom{m}{k-c}^{-1} \gamma^k \sum_{b=0}^{m} \lambda_{j,b} \binom{m-b}{c} \sum_{\beta=0}^{b-k} \binom{b}{k+\beta} \binom{k+\beta}{k} \gamma^\beta (-1)^\beta .
\end{aligned}
\tag{3.4.6}
$$

Hence the coefficient of $\gamma^{\beta+k}$ in the expression above is

$$
\binom{k}{c} (c!)^2 \binom{m}{k-c}^{-1} \left(\sum_{b=0}^{m} \lambda_{j,b} \binom{m-b}{c} \binom{b}{k+\beta} \right) \binom{k+\beta}{k} (-1)^\beta .
$$

By the assumption of the lemma, the desired result follows. □

We also need the following combinatorial lemma:

Lemma 3.4.2. *Let t be a positive integer. Then for any non-negative $\alpha \le t+1$ we have*

$$
\sum_{i=0}^{t+1} \binom{t+1}{i} i^\alpha (-1)^i = 0.
$$

Proof. Let $g(x) := (1-x)^{t+1}$ so that $g^{(\alpha)}(x) = (t+1)_{(\alpha)} (1-x)^{t+1-\alpha}$ and $g^{(\alpha)}(1) = 0$.

Substituting $x = 1$ into the expansion

$$g^{(\alpha)}(x) = \sum_{i=0}^{t+1} \binom{t+1}{i} (-1)^i i_{(\alpha)} x^{i-\alpha},$$

we get the binomial identity

$$\sum_{i=0}^{t+1} \binom{t+1}{i} (-1)^i i_{(\alpha)} = 0. \tag{3.4.7}$$

We prove our lemma by induction on α. The binomial identity (3.4.7) implies that our lemma is true for the base cases $\alpha = 0, 1$. The identity (3.4.7) also implies that

$$\sum_{i=0}^{t+1} \binom{t+1}{i} (-1)^i i_{(\alpha'+1)} = 0.$$

Expanding the falling factorial $i_{(\alpha'+1)}$ into a sum of monomials in i when $\alpha' \leq t$, we get

$$\sum_{i=0}^{t+1} \binom{t+1}{i} i^{\alpha'+1} (-1)^i = \sum_{\beta=0}^{\alpha'} b_\beta \left(\sum_{i=0}^{t+1} \binom{t+1}{i} i^\beta (-1)^i \right)$$

for some choice of constants $b_\beta \in \mathbb{R}$. Note that the bracketed term in the equation above is zero by the hypothesis that our lemma is true for $\alpha = \alpha'$, where $1 \leq \alpha' \leq t$. Hence the lemma also holds for $\alpha = \alpha' + 1$. □

The existence of permutation-invariant amplitude damping codes is given by the following theorem:

Theorem 3.4.3. *Let t be any positive integer and $m = 9t^2 + 4t$. For all $j \in \{0, 1\}$ and integers $b \leq m$ divisible by $3t$ let*

$$\lambda_{j,b} = \binom{3t+1}{b/(3t)} 2^{-3t} \frac{1 + (-1)^{b+j}}{2} \tag{3.4.8}$$

and all other values of $\lambda_{j,b}$ be zero. Then the span of $|0_L\rangle = \sum_{b=0}^{m} \sqrt{\lambda_{0,b}}|P_{m,b}\rangle$ and $|1_L\rangle = \sum_{b=0}^{m} \sqrt{\lambda_{1,b}}|P_{m,b}\rangle$ is a t-AD permutation-invariant code.

Proof. By definition, $\lambda_{j,b}$ satisfy the normalization condition $\sum_{b=0}^{m} \lambda_{j,b} = 1$, so $\lambda_{j,b}$ define valid logical codewords $|j_L\rangle$. Moreover, $|0_L\rangle$ and $|1_L\rangle$ have distinct support, and are hence linearly independent spanning a two-dimension codespace.

Now define the truncated Kraus set of amplitude damping effects $\Omega := \{\mathbf{A_k} : \mathbf{k} \in \mathbb{Z}_2^m, \|\mathbf{k}\|_1 \leq t\}$, so that Ω satisfies the order constraints of Corollary 3.3.5. Define the matrix $\mathbf{G}_j$ to be $\sum_{\mathbf{E},\mathbf{F}\in\Omega}\langle j_L|\mathbf{E}^\dagger\mathbf{F}|j_L\rangle|\mathbf{E}\rangle\langle\mathbf{F}|$, and $\mathbf{G} := \frac{\mathbf{G}_0+\mathbf{G}_1}{2}$. To use Corollary 3.3.5, we first have to prove that the matrix $\mathbf{G}$ is positive definite.

Let $\widetilde{\mathbf{A}} = \sum_{\mathbf{F}\in\Omega} v_{\mathbf{A},\mathbf{F}}\mathbf{F}$ and $\mathbf{V} := \sum_{\mathbf{E},\mathbf{F}\in\Omega} v_{\mathbf{E},\mathbf{F}}|\mathbf{E}\rangle\langle\mathbf{F}|$ be as defined in (3.3.7) and (3.3.4) respectively. and correspondingly define the vector $|\Psi_\mathbf{A}\rangle := \sum_{\mathbf{F}\in\Omega} v_{\mathbf{A},\mathbf{F}}|\mathbf{F}\rangle$. Observe that $\lambda_{\min}(\mathbf{G}) = \lambda_{\min}(\mathbf{VGV}^\dagger) = \min_{\mathbf{A}\in\Omega}\langle\mathbf{A}|\mathbf{VGV}^\dagger|\mathbf{A}\rangle = \min_{\mathbf{A}\in\Omega}\langle\Psi_\mathbf{A}|\mathbf{G}|\Psi_\mathbf{A}\rangle = \min_{\mathbf{A}\in\Omega}\frac{1}{2}\langle\Psi_\mathbf{A}|(\mathbf{G}_0+\mathbf{G}_1)|\Psi_\mathbf{A}\rangle = \min_{\mathbf{A}\in\Omega}\frac{1}{2}\sum_{j=0}^{1}\langle j_L|\widetilde{\mathbf{A}}^\dagger\widetilde{\mathbf{A}}|j_L\rangle = \min_{\mathbf{A}\in\Omega}\frac{1}{2}\sum_{j=0}^{1}\|\widetilde{\mathbf{A}}|j_L\rangle\|_2$. Now the Kraus elements in Ω annihilate at most t excitations, but the logical states $|j_L\rangle$ are permutation-invariant with support containing at least $3t$ excitations. Hence $\|\widetilde{\mathbf{A}}|j_L\rangle\|_2 > 0$ which implies that $\lambda_{\min}(\mathbf{G}) > 0$.

Our choice of $\lambda_{j,b}$ implies that $\langle j_L|\mathbf{A}_\mathbf{k}^\dagger\mathbf{A}_{\mathbf{k}'}|j_L\rangle = 0$ and $\langle 0_L|\mathbf{A}_\mathbf{k}^\dagger\mathbf{A}_{\mathbf{k}'}|1_L\rangle = 0$ when $\mathbf{k},\mathbf{k}' \in \mathbb{Z}_2^m$ such that $\|\mathbf{k}\|_1 = \|\mathbf{k}'\|_1 \leq t$. Now we set t to $3t$ in Lemma 3.4.2, and note that the coefficient of $\lambda_{j,b}$ in the assumption of Lemma 3.4.1 is a polynomial of order no more than $3t$ in the variable b. Then applying Lemma 3.4.2 to Lemma 3.4.1 shows that the conditions of Theorem 3.3.4 are satisfied with $\epsilon = \mathcal{O}(\gamma^{2t})$. □

Part II

Upper and lower bounds on the quantum capacity of various quantum channels

Chapter 4

Concatenated Quantum Codes attaining the Quantum Gilbert-Varshamov Bound

4.1 Introduction

A quantum code is a subspace of a Hilbert space with a possibly infinite dimension. In the context of this chapter, we restrict ourselves to finite dimension Hilbert spaces which are isomorphic to complex Euclidean spaces. Qubit quantum block codes are subspaces of an n-fold tensor product of two-dimension Hilbert spaces. Here n denotes the block length of the quantum block code. We study q-ary quantum codes which are quantum block codes, and are subspaces of an n-fold tensor product of q-dimension Hilbert spaces. The dimension of a q-ary quantum code quantifies the amount of quantum information that the code can encode, and its logarithm to base q is the code's rate. The distance of a q-ary code is the minimum number of blocks that can be corrupted such that one can be fooled to believe that there was in fact no corruption of quantum information. If a q-ary code has a distance d, there would exist procedures to reverse the corruption of up to $\frac{d-1}{2}$ blocks of quantum information perfectly.

A family of q-ary quantum codes [Rai99b] of increasing block length is defined to be *good* if the ratio of its distance to its block length approaches a non-zero constant and has a strictly positive rate. Designing good quantum codes is highly nontrivial, just as it is in the classical case. The quantum Gilbert-Varshamov (GV) bound [Got97, ABKL00, AK01, FM04, Ma08, JX11] is a lower bound on an achievable relative distance of a quantum code of a fixed rate, and is attainable for various families of random quantum codes [Got97, AK01, Ma08]. Explicit families of quantum codes, both unconcatenated [ALT01, Mat02] and concatenated [CLX01, Fuj06, Ham08, LXW09], have been studied, but do not attain the quantum GV bound for $q < 7$ [Nie07]. We show that concatenated quantum codes can attain the quantum GV bound.

We are motivated by the historical development of the idea of concatenating a sequence of increasingly long classical Reed-Solomon (RS) outer codes with various types of classical inner codes. In both cases where the inner codes are all identical [MS77] or all distinct [Jus72], the resultant sequence of concatenated codes while asymptotically good nonetheless fail to attain the GV bound. A special case of Thommesen's result [Tho83] shows that even if the inner codes all have a rate of one, if they are chosen uniformly at random, the resultant sequence of concatenated codes almost surely attains the GV bound. Our work extends this classical observation to the quantum case.

We show the quantum analog of Thommesen's result – the sequence of concatenated quantum codes with the outer code being a quantum generalized RS code [GGB99, GBR04, LXW08, LXW09] and random inner stabilizer codes almost surely attains the quantum GV bound when the rates of the inner and outer codes lie in feasible region defined by Figure 4.1. The property of the outer code that we need is that the normalizer of its stabilizer is a classical generalized RS code [LXW08]. Our work is closest in spirit to that of Fujita [Fuj06], where quantum equivalents of the Zyablov and the Blokh-Zyablov bounds are obtained (not attaining the quantum GV bound) by choosing a quantum RS code with essentially random inner codes.

In the proof of the classical result, Thommesen uses a random coding argument to compute the probability that any codeword of weight less than the target minimum distance belongs to the random code. Subsequently, he uses the union bound, properties of the

Reed-Solomon outer code, and properties of the q-ary entropy function (defined in 4.2.1), to prove that the proposed random code almost surely does not contain any codeword of weight less than the prescribed minimum distance.

The proof of our quantum result follows a similar strategy, with codewords replaced by elements of the normalizer not in the stabilizer. However the feasible region for the rates of the inner and outer codes for the classical and the quantum result are not analogous, because we use a slightly different property of the q-ary entropy function.

The organization of this chapter is as follows: Section 5.2 introduces the notation and preliminary material used in this chapter. The formalism of concatenating stabilizer codes, which is crucial to the proof of the main result, is carefully laid out in this section. We state our main result in Theorem 4.3.1 of Section 5.5, and the remainder of the chapter is dedicated to its proof.

4.2 Preliminaries

Let $L(\mathbb{C}^q)$ denote the set of complex q by q matrices. Define $\mathbb{1}_q$ to be a size q identity matrix and $\omega_q := e^{2\pi i/q}$ to be a primitive q-th root of unity, where $q \geq 2$ is an prime power. Define $0 \log_q 0 := 0$. Define the q-ary entropy function and its inverse to be $H_q : [0,1] \to [0,1]$ and $H_q^{-1} : [0,1] \to [0, \frac{q-1}{q}]$ respectively where

$$H_q(x) := x \log_q(q-1) - x \log_q x - (1-x) \log_q(1-x). \tag{4.2.1}$$

The q-ary entropy function is important because it helps us to count the size of sets with q symbols. The base-q logarithm of the number of vectors over $\mathbb{F}_q$ of length n that differ in at most xn components from the zero-vector is dominated by $nH_q(x)$ as n becomes large.

For a ground set Ω, define $|\Omega|$ to be its cardinality. For all n-tuples $\mathbf{x} \in \Omega^n$, define x_j to be j-th element of the n-tuple $\mathbf{x}$. Given tuples $\mathbf{x} \in \Omega^n$ and $\mathbf{y} \in \Omega^m$, define the pasting of the tuples $\mathbf{x}$ and $\mathbf{y}$ to be $(\mathbf{x}|\mathbf{y}) := (x_1, ..., x_n, y_1, ..., y_m)$. When M_1 and M_2 are matrices with the same number of columns, define $(M_1; M_2) := \begin{pmatrix} M_1 \\ M_2 \end{pmatrix}$. For positive integer ℓ, define $[\ell] := \{1, ..., \ell\}$. Define the Hamming distance $d_H(\mathbf{x}, \mathbf{y})$ between $\mathbf{x} \in \Omega^n$ and $\mathbf{y} \in \Omega^n$ as the number of indices on which $\mathbf{x}$ and $\mathbf{y}$ differ. Define the minimum distance of any subset $C \subset \Omega^n$ $\min_{\mathbf{x},\mathbf{y} \in C} \{d_H(\mathbf{x}, \mathbf{y}) : \mathbf{x} \neq \mathbf{y}\}$.

A code over a vector field $\mathbb{F}_q^n$ is q-ary linear code of length n if it is a subspace of $\mathbb{F}_q^n$. A classical q-ary linear code [MS77] of block length n and k generators with minimum distance of d is said to be an $[n, k]_q$ code or an $[n, k, d]_q$ code. A classical $[n, k, d]_q$ code is maximally distance separated (MDS) if $d = n - k + 1$. A quantum q-ary stabilizer code [Rai99b] of block length n encoding k qudits is said to be an $[\![n, k]\!]_q$ code. The rates of an $[\![n, k]\!]_q$ code and an $[n, k]_q$ code are both defined to be $\frac{k}{n}$.

4.2.1 Finite Fields and q-ary Error Bases

In this section, we review the connection between finite fields and q-ary error bases [AK01]. Let $q = p^k$ where p is a prime number and k is a positive integer. Let $\mathbb{b} := (\beta_1, ..., \beta_k)$ have components that form a basis for $\mathbb{F}_q$. For any $\alpha^{(j)}, \beta^{(j)} \in \mathbb{F}_q$ where j is a dummy index, we have $\alpha^{(j)} = (\mathbf{a}^{(j)})^T \mathbb{b}$ and $\beta^{(j)} = (\mathbf{b}^{(j)})^T \mathbb{b}$ where the coefficients vectors are $\mathbf{a}^{(j)}, \mathbf{b}^{(j)} \in \mathbb{F}_p^k$. Let

$$X := \sum_{j=0}^{p-1} |(j+1) \mod p\rangle\langle j|$$

$$Z := \sum_{j=0}^{p-1} (\omega_p)^j |j\rangle\langle j| \tag{4.2.2}$$

be generalizations of the qubit Pauli matrices satisfying the commutation property $X^a Z^b = (\omega_q)^{ab} Z^b X^a$ for integers a and b. The matrix defined as

$$X_{\mathbf{a}^{(j)}} Z_{\mathbf{b}^{(j)}} := X^{a_1^{(j)}} Z^{b_1^{(j)}} \otimes ... \otimes X^{a_k^{(j)}} Z^{b_k^{(j)}} \tag{4.2.3}$$

is an element of a q-ary error basis, and is naturally identified with $(\alpha^{(j)}, \beta^{(j)}) \in \mathbb{F}_q^2$ and also $\alpha^{(j)} + \beta^{(j)}\gamma \in \mathbb{F}_{q^2}$ where $\{1, \gamma\}$ is a $\mathbb{F}_q$-linear basis of $\mathbb{F}_{q^2}$.

Elements of a q-ary error basis on n qudits have the form

$$X_{(\mathbf{a}^{(1)}|...|\mathbf{a}^{(n)})} Z_{(\mathbf{b}^{(1)}|...|\mathbf{b}^{(n)})}$$

and can be identified with $(\alpha^{(1)}|...|\alpha^{(n)}|\beta^{(1)}|...|\beta^{(n)}) \in \mathbb{F}_q^{2n}$ and $(\alpha^{(1)}+\beta^{(1)}\gamma, ..., \alpha^{(n)}+\beta^{(n)}\gamma) \in \mathbb{F}_{q^2}^n$, where the vertical bars denote the pasting operation that we have defined earlier in the preliminaries. The matrices $X_\mathbf{a} Z_\mathbf{b}$ and $X_{\mathbf{a}'} Z_{\mathbf{b}'}$ commute if and only if the vectors $(\mathbf{a}|\mathbf{b})$ and $(\mathbf{a}'|\mathbf{b}')$ are orthogonal with respect to a well-chosen scalar product[1] over $\mathbb{F}_q^{2n}$ (see (22) and (26) of [AK01]) that we denote as $*$. When we use the field $\mathbb{F}_{q^2}^n$, we consider a different notion of orthogonality using the Hermitian scalar product (see (28) of [AK01]), that maps the vectors $\mathbf{x}, \mathbf{y} \in \mathbb{F}_{q^2}^n$ to $\sum_{i=1}^n (x_i)^q y_i$. The scalar product is called Hermitian because taking an element of $\mathbb{F}_{q^2}$ to the q-th power is analogous to conjugation over the complex field. For all non-zero $x \in \mathbb{F}_{q^2}$, $x^q \neq x$ and $(x^q)^q = x^{q^2} = x$.

4.2.2 Stabilizer Codes

In this section, we define terminology related to stabilizer codes with a special type of structure in the context of finite fields and error bases [Got97, AK01].

Consider the generator matrix

$$G = (G_S; G_X; G_Z)$$

over $\mathbb{F}_q$ with $n+k$ rows and $2n$ columns where the stabilizer generator $G_S = (s^{(1)}; ...; s^{(n-k)})$, the logical-X generator $G_X = (x^{(1)}; ...; x^{(k)})$, and the logical-Z generator $G_Z = (z^{(1)}; ...; z^{(k)})$ are submatrices of G, each of full rank. We also require $G = (G_S; G_X; G_Z)$ to satisfy the following properties:

[1] In [AK01], the authors use the phrase *'inner product' to refer to the scalar product in an abuse of terminology; the scalar product does not satisfy the properties required for an inner product when the vector space is a finite field.*

1. Each row of G_S is orthogonal to every row of G.

2. For all $i \in [k]$, the i-th row of G_X is orthogonal to every row of G except the i-th row of G_Z. In particular, $x^{(i)} * z^{(i)} = c$ for some fixed non-zero $c \in \mathbb{F}_q^{2n}$. (The choice of c may be set to 1 without loss of generality.)

We also define $\tilde{G} = (\tilde{G}_S; \tilde{G}_X; \tilde{G}_Z)$ to be a similarly defined matrix over $\mathbb{F}_{q^2}$, with $n + k$ rows and n columns. In particular, the matrices $\tilde{G}_S, \tilde{G}_X$, and $\tilde{G}_Z$ are the $\mathbb{F}_{q^2}$ analogs of the matrices G_S, G_X and G_Z.

We denote the classical codes generated by G_S and G by C_S and C_N respectively, which we call the stabilizer and normalizer over $\mathbb{F}_q$ respectively. We also denote the classical codes generated by $\tilde{G}_S$ and $\tilde{G}$ by $\tilde{C}_S$ and $\tilde{C}_N$ respectively, and call them the stabilizer and normalizer over $\mathbb{F}_{q^2}$ respectively.

An $[\![n, k]\!]_q$ stabilizer code is a subspace of $(\mathbb{C}^q)^{\otimes n}$ of dimension q^k that is left invariant by the action of error base elements corresponding to the elements in C_S. The error basis elements corresponding to the rows of G_X and G_Z are generators for logical operations that can be applied on the stabilizer code. Here, we refer to the language of finite fields to work with stabilizer codes developed by Ashikhmin and Knill [AK01]. In this chapter, we use primarily the representation of stabilizer codes with the generator matrix $\tilde{G}$.

The distance of an $[\![n, k]\!]_q$ stabilizer code generated by G is the minimum distance of the punctured classical code $\tilde{C}_N \backslash \tilde{C}_S := \{x \in \tilde{C}_N : x \notin \tilde{C}_S\}$ [AK01], and has a lower bound given by the minimum distance of the code $\tilde{C}_N$. When the lower bound is met with equality, the stabilizer code said to be degenerate. We denote an $[\![n, k]\!]_q$ stabilizer code with distance d as $[\![n, k, d]\!]_q$.

In this chapter, a *random stabilizer code* with parameters $[\![n, k]\!]_q$ is a stabilizer code with its generator matrix $G = (G_S; G_X; G_Z)$ chosen uniformly at random from all possible generator matrices with $n + k$ rows and $2n$ columns.

Let us have two convergent sequences $\{r_n\}_{n\in\mathbb{N}}, \{\delta_n\}_{n\in\mathbb{N}} \subset [0, 1]$, where $\lim_{n\to\infty} r_n = r$ and $\lim_{n\to\infty} \delta_n = \delta$. Here r and δ are to be interpreted as the parameters of the codes of interest, which are the codes' rate and relative distance respectively. The asymptotic q-ary classical GV bound is the inequality

$$\delta \geq H_q^{-1}(1-r), \tag{4.2.4}$$

and the asymptotic quantum q-ary GV bound is the inequality

$$\delta \geq H_{q^2}^{-1}\left(\frac{1-r}{2}\right). \tag{4.2.5}$$

We say that a sequence of $[n, r_n, \delta_n n]_q$ classical linear q-ary codes attains the classical GV bound if the inequality (4.2.4) is satisfied. Similarly we say that a sequence of $[\![n, nr_n, n\delta_n]\!]_q$ quantum stabilizer codes attains the quantum GV bound if (4.2.5) is satisfied.

The classical GV bound and the quantum GV bound can be attained almost surely by sequences of random linear codes and sequences of random stabilizer codes respectively. Classical and quantum random codes almost surely do not have (4.2.4) and (4.2.5) holding with a strict inequality. Hence the classical and quantum GV bounds characterize the typical performance of random classical linear codes and quantum stabilizer codes tightly. However given a random code, its distance is hard to evaluate, and the corresponding encoding and decoding procedure is inefficient because of its lack of structure. For $q < 7$, there are no known efficiently encodable and decodable q-ary quantum stabilizer codes that satisfy the quantum GV bound strictly. The condition $q < 7$ is necessary because the q-ary quantum Goppa codes [Nie07] satisfy the quantum GV bound strictly for $q \geq 7$. Finding q-ary quantum codes that attain the quantum GV bound that are more efficiently encodable and decodable than purely random stabilizer codes for $q < 7$ remains an important problem.

To decode a $[\![n, k]\!]_q$ quantum stabilizer code, one performs error correction by measuring $n-k$ times, where each measurement which has q possible outcomes corresponds to a distinct row of G_S. Hence there are q^{n-k} total syndromes. In the absence of any structure in the stabilizer code, one would have to construct a lookup table with q^{n-k} rows to perform maximum likelihood decoding. Hence the decoding complexity of a random stabilizer code using this naïve strategy is $O(q^{n(1-r)})$, where the rate of the code $r = k/n$ is viewed as a fixed parameter, and n is the varying large parameter.

4.2.3 Concatenation of Stabilizer Codes

In this section we only consider the notion of concatenation with respect to stabilizer codes. Concatenation is a procedure that makes a longer code out of an appropriately chosen set of shorter codes, and allows both the encoding and decoding procedure each to be broken down into two steps. Let $q = p^k$ where p is prime.

1. **Encoding:** The quantum message which is a q^K-dimension (or equivalently p^{kK}-dimension) quantum state is encoded into $[\![N, K]\!]_q$ stabilizer code. We call this code an *outer code*. The outer code comprises of N blocks of dimension q complex Euclidean spaces, and each of these N blocks is further encoded by possibly distinct $[\![n, k]\!]_p$ codes. These codes with n blocks of dimension p complex Euclidean spaces are called *inner codes*. The resultant code is a concatenated code with parameters $[\![nN, kK]\!]_p$.

2. **Decoding:** The quantum state that is to be decoded comprises of nN blocks of p-dimension complex Euclidean spaces. Each consecutive block of n dimension p complex Euclidean spaces is decoded using the corresponding decoding procedure for the corresponding $[\![n, k]\!]_p$ codes. After this initial layer of decoding, the output is N blocks of q-dimension complex Euclidean spaces. These N blocks are decoded using the decoding procedure of the $[\![N, K]\!]_q$ outer code.

Since the encoding and decoding process can be broken down into two layers, concatenated codes potentially have better encoding and decoding time complexities than codes without such a structure. For our main result, we choose our outer code to be a quantum generalized RS code.

Let our $[\![N, K]\!]_q$ outer code be generated by $G^{(\mathrm{out})} = (G_S^{(\mathrm{out})}; G_X^{(\mathrm{out})}; G_Z^{(\mathrm{out})})$ and the N possibly distinct $[\![n, k]\!]_p$ inner codes be generated by $G^{(\mathrm{in},j)} = (G_S^{(\mathrm{in},j)}; G_X^{(\mathrm{in},j)}; G_Z^{(\mathrm{in},j)})$ for

$j \in [N]$. The concatenation of the $[\![N,K]\!]_q$ outer code with N possibly distinct inner codes is an $[\![nN,kK]\!]_p$ code generated by $G^{(\text{concat})} = (G_S^{(\text{concat})}; G_X^{(\text{concat})}; G_Z^{(\text{concat})})$. In the remaining part of this section, we describe how $G^{(\text{concat})}$ is constructed using $G^{(\text{out})}$ and $G^{(\text{in,j})}$ for $j \in [N]$.

Define the direct product of the stabilizers of the inner codes to be

$$\begin{aligned}\tilde{C}_S^{(\text{in},1:N)} &:= \tilde{C}_S^{(\text{in},1)} \times .... \times \tilde{C}_S^{(\text{in},N)}\\ &= \left\{(\sigma^{(1)}|...|\sigma^{(N)}) : \sigma^{(j)} \in \tilde{C}_S^{(\text{in},j)}, j \in [N]\right\}\end{aligned}$$

and denote the logical generators of the j-th inner code as

$$\begin{aligned}\tilde{G}_X^{(\text{in,j})} &:= (x^{(\text{in},j),1}; ...; x^{(\text{in},j),k})\\ \tilde{G}_Z^{(\text{in},j)} &:= (z^{(\text{in},j),1}; ...; z^{(\text{in},j),k}).\end{aligned}$$

Now let $\sigma = (\alpha^{(1)}+\beta^{(1)}\gamma_q, ..., \alpha^{(N)}+\beta^{(N)}\gamma_q)$ be an element of $\mathbb{F}_{q^2}^N$, where $\{1, \gamma_q\}$ is a $\mathbb{F}_q$-linear basis of $\mathbb{F}_{q^2}$, and $\alpha^{(i)}, \beta^{(i)} \in \mathbb{F}_q$ for all $i \in [k]$. For all $i \in [k]$, also let $\alpha^{(i)} = (a_1^{(i)}, ..., a_k^{(i)})^T \mathbb{b}$ and $\beta^{(i)} = (b_1^{(i)}, ..., b_k^{(i)})^T \mathbb{b}$ where $\mathbb{b} \in \mathbb{F}_q^k$ has components that form a basis for $\mathbb{F}_q$, and $a_j^{(i)}, b_j^{(i)} \in \mathbb{F}_p$. Then we define the map $\pi : \mathbb{F}_{q^2}^N \to \mathbb{F}_{p^2}^{nN}$ where

$$\pi(\sigma) := \sum_{\ell=1}^{k} \left(a_\ell^{(1)} x^{(\text{in},1),\ell} + b_\ell^{(1)} z^{(\text{in},1),\ell}|...|a_\ell^{(N)} x^{(\text{in},N),\ell} + b_\ell^{(N)} z^{(\text{in},N),\ell}\right). \tag{4.2.6}$$

Then the stabilizer generator, X-logical generator and the Z-logical generator of our concatenated code are given by

$$\begin{aligned}\tilde{G}_S^{(\text{concat})} &= \left(\pi(\tilde{G}_S^{(\text{out})}); \begin{pmatrix}\tilde{G}_S^{(\text{in},1)} & \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \tilde{G}_S^{(\text{in},2)} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \ddots & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \tilde{G}_S^{(\text{in},N)}\end{pmatrix}\right)\\ \tilde{G}_X^{(\text{concat})} &= \pi(\tilde{G}_X^{(\text{out})}), \quad \tilde{G}_Z^{(\text{concat})} = \pi(\tilde{G}_Z^{(\text{out})})\end{aligned} \tag{4.2.7}$$

respectively, where the map π acts component-wise on a set of finite-field elements. The classical codes associated with the stabilizers and the normalizers of the concatenated code over $\mathbb{F}_{p^2}$ are

$$\begin{aligned}\tilde{C}_S^{(\text{concat})} &:= \pi(\tilde{C}_S^{(\text{out})}) + \tilde{C}_S^{(\text{in},1:N)}\\ \tilde{C}_N^{(\text{concat})} &:= \pi(\tilde{C}_N^{(\text{out})}) + \tilde{C}_S^{(\text{concat})},\end{aligned}$$

In this chapter, we use some of the quantum codes of Li, Xing and Wang [LXW08] as the outer codes of our concatenated codes. The key feature of their code that we use is that their constructed stabilizer $\tilde{C}_S$ and normalizer $\tilde{C}_N$ are classical MDS codes. While the codes in [GBR04] have the same parameters as the ones that we use, they need not have the property that their stabilizer and normalizers are classical MDS codes.

Theorem 4.2.1 (Li, Xing, Wang [LXW08])**.** *Let N be a prime power. Then for all positive integers $D \leq \frac{N}{2}$, there exists an $[\![N, N-2(D-1), D]\!]_N$ code with stabilizer $\tilde{C}_S$ and normalizer $\tilde{C}_N$ that are classical MDS codes, where $\tilde{C}_N$ is the Hermitian dual of $\tilde{C}_S$.*

4.3 The Main Result

Theorem 4.3.1. *Let $q = N = p^k$ where p is prime and k is a positive integer. Let $0 \leq R < 1$ and $0 \leq r \leq 1$ be rationals such that $R = \frac{q-2(D-1)}{q}$, $r = \frac{k}{n}$, k, n, D are positive integers with $D \geq 2$, and the inequality*

$$0 \leq 1 - r \leq -\log_{p^2}\left(1 - H_{p^2}^{-1}\left(\frac{1-R}{2}\right)\right). \tag{4.3.1}$$

is satisfied. Then an $[\![N, RN]\!]_N$ outer code of Theorem 4.2.1 concatenated with N random $[\![n, rn]\!]_p$ inner quantum codes is an $[\![nN, rRnN, d]\!]_p$ quantum code, where with probability *at least* $1-(p-p^{-1})^{-1}p^{-N(1-R)/2}$,

$$\frac{d}{nN} > H_{p^2}^{-1}\left(\frac{1-rR}{2}\right) - \frac{c(p,r)}{n}$$

and

$$c(p,r) := \frac{\log_p 2 + 1}{\log_{p^2}\left((p^2-1)\left(\frac{1}{H_{p^2}^{-1}(\frac{1+r}{2})} - 1\right)\right)}. \tag{4.3.2}$$

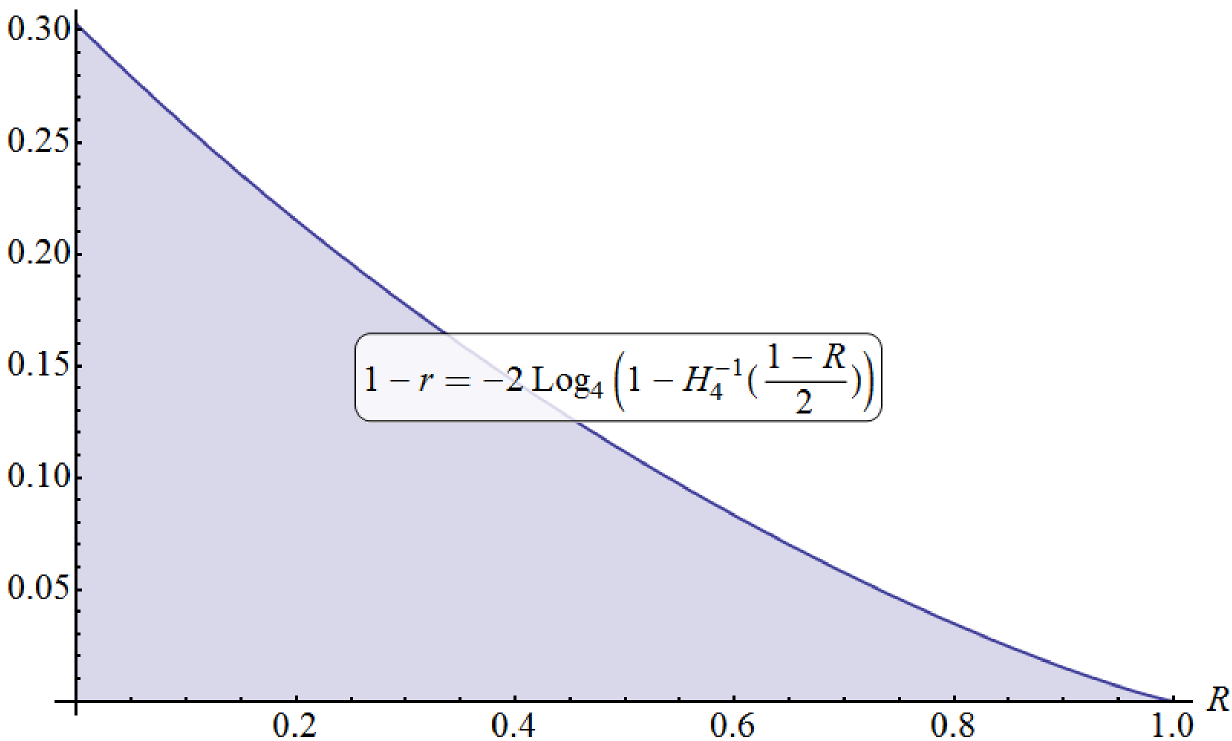

Figure 4.1: The shaded region indicates the feasible region of $(R, 1-r)$ for which Theorem 4.3.1 applies. Here R is the asymptotic rate of the outer code, r is the rate of each of the inner codes, and $p = 2$.

Let R_0 and r_0 be feasible asymptotic rates (see Figure 4.1) of the outer and inner codes respectively with respect to the inequality (4.3.1). When n becomes arbitrarily large, there will always exist an epsilon ball about R_0 and r_0 that contains feasible a (r, R) tuple for Theorem 4.3.1 to hold, where epsilon becomes arbitrarily small. Thus Theorem 4.3.1 implies that concatenated codes can attain the quantum GV bound.

The significance of the result is a potential saving in decoding complexity of quantum GV bound attaining codes with a concatenated structure as compared to those that do not. The decoding complexity of a random $[\![nN, nNrR]\!]_q$ stabilizer code is $O(p^{nN(1-rR)}) = O(p^{np^{rn}(1-rR)})$. On the other hand, the worst decoding complexity of our concatenated code is $O((p^{n(1-r)}(p^k)^{N(1-R)})) = O(p^{n(1-r)+nrp^{rn}(1-R)})$, which outperforms the decoding of random codes without a concatenated structure when $r > 0$.

In this section, the outer and inner codes are of the type stipulated by Theorem 4.3.1, and their notation follow that of Section 4.2.3.

4.3.1 Technical Lemmas

Proposition 4.3.2. *Consider a random $[\![n, k]\!]_p$ stabilizer code generated by $G = (G_S; G_X; G_Z)$. Let $\mathbf{a}, \mathbf{b} \in \mathbb{F}_p^k$ where $(\mathbf{a}, \mathbf{b}) \neq \mathbf{0}$. The probability that nonzero $\sigma \in \mathbb{F}_p^{2n}$, belongs to the set $\mathbf{a}^T G_X + \mathbf{b}^T G_Z + C_S$ is at most $p^{-(n+k)}$.*

Proof. Every dimension ℓ subspace of $\mathbb{F}_q^{2n}$ has $q^{2n-\ell}$ cosets, each of size q^ℓ. Given a set $\mathfrak{X}$ of x linearly independent vectors, the number of ways to pick the the generating rows of a feasible G_S that corresponds to a $[\![n, k]\!]_p$ code such that $\mathfrak{X} \subseteq C_N$ is $\prod_{i=0}^{n-k-1}(p^{2n-x-i} - p^i)$. Given that a feasible G_S is picked, the number of ways to pick G_X and G_Z is $\prod_{j=n-k}^{n+k} p^{2n-x-j}$. Hence the required probability is

$$\frac{\prod_{i=0}^{n-k-1}(p^{2n-1-i} - p^i)\prod_{j=n-k}^{n+k} p^{2n-1-j}}{\prod_{i=0}^{n-k-1}(p^{2n-i} - p^i)\prod_{j=n-k}^{n+k} p^{2n-j}}$$

which is at most $p^{-(n+k)}$. □

Lemma 4.3.3. *Let $W \in \tilde{C}_N^{(\mathrm{out})}$ have weight $w \geq D'$, and let h be a positive positive integer* *such that* $h \leq \frac{p^2-1}{p^2}nw$*. Then*

$$\Pr\left[\min_{\sigma\in\pi(W)+\tilde{C}^{(\mathrm{in},1:\mathrm{N})}} \mathrm{wt}(\sigma) \leq h\right] \leq (p^2)^{nwH_{p^2}(\frac{h}{nw})}p^{-(n+k)}.$$

Proof. Applying Proposition 4.3.2, we find that for all $\sigma \in \pi(W) + \tilde{C}^{(\mathrm{in},1:\mathrm{N})}$, the inequality $\Pr[\mathrm{wt}(\sigma) \leq h] \leq p^{-(n+k)\ \mathrm{supp}(W)}$ holds, where $\mathrm{supp}(W)$ is the number of components of the vector W that are non-zero. Then application of the union bound gives the result, where we have used an upper bound on the size of a p^2-ary Hamming ball [MS77] which holds when $h \leq \frac{p^2-1}{p^2}nw$. □

Now we proceed to prove our main result, Theorem 4.3.1.

Proof of Theorem 4.3.1. Let A_w be the number of codewords in the code $\tilde{C}_N^{(\mathrm{out})}$ with weight w. Since $\tilde{C}_N^{(\mathrm{out})}$ is a classical MDS code [LXW08] of alphabet size p^{2k}, all of the A_w's are known exactly [MS77] and we can use Thommesen's result [Tho83] to conclude that

$$A_w \leq \binom{N}{w}(p^{2k})^{w-D'+1}, \quad D' \leq w \leq N. \tag{4.3.3}$$

where D' is the distance $\tilde{C}_N^{(\mathrm{out})}$. Observe that

$$\begin{aligned}
\Pr[d \leq h] &= \Pr[\min\{\mathrm{wt}(\sigma) : \sigma \in \tilde{C}_N^{(\mathrm{concat})} \backslash \tilde{C}_S^{(\mathrm{concat})}\} \leq h] \\
&\leq \sum_{W\in\tilde{C}_N^{(\mathrm{out})}\backslash\tilde{C}_S^{(\mathrm{out})}} \Pr\left[\min\{\mathrm{wt}(\sigma) : \sigma \in \pi(W) + \tilde{C}^{(\mathrm{in},1:\mathrm{N})}\} \leq h\right] \\
&\leq \sum_{w=D'}^{N} 2^N (p^{2k})^{w-D'+1}(p^2)^{nwH_{p^2}(\frac{h}{nw})-\frac{n+k}{2}w}
\end{aligned}$$

The first inequality is from the union bound, and the second inequality comes from simultaneous application of Lemma 4.3.3, (4.3.3) and the fact that $\binom{N}{w} \leq 2^N$. Hence

$$\begin{aligned}\Pr[d \leq h] &\leq \sum_{w=D'}^{N} (p^2)^{-nw\gamma(h,w)} \\ &\leq (p^2)^{-nD'\bar{\eta}} \frac{1}{1-p^{-2n\bar{\eta}}}\end{aligned}$$

where $\bar{\eta} \in (0, \min_{w\in[D',N]} \eta(h,w)]$ and

$$\eta(h,w) := -\epsilon - r\theta - H_{p^2}\left(\frac{h}{nw}\right) + \frac{1+r}{2}, \tag{4.3.4}$$

$\epsilon = \frac{N}{w}\frac{\log_p 2}{n}$ and $\theta = 1 - \frac{D}{w} + \frac{1}{w}$. Here $R_N = \frac{1+R}{2}$ is the rate of $\tilde{C}_N^{(\text{out})}$. Then $1 \geq \frac{w}{N} = \frac{1-R_N}{1-\theta} \geq \frac{1-R}{2}$. Our strategy is to first fix a positive $\bar{\eta}$, then select a large feasible h for which the inequality $\eta(h,w) \geq \bar{\eta}$ holds for all $w \in [D', N]$. We pick $\bar{\eta} = \frac{1}{n}$.

The inequality $\eta(h,w) \geq \bar{\eta}$, the equation (4.3.4), and the monotone increasing property of the inverse entropy function on the open unit interval imply that

$$\frac{h}{nN} \leq \frac{1-R_N}{1-\theta} H_{p^2}^{-1}\left(\frac{1+r}{2} - r\theta - \epsilon - \bar{\eta}\right) \tag{4.3.5}$$

for all $w \in [D', N]$. We need to obtain a lower bound on the right hand side of the inequality (4.3.5) for all $w \in [D', N]$. The right hand side of the inequality satisfies the following lower bounds.

$$\begin{aligned}&\frac{1-R_N}{1-\theta} H_{p^2}^{-1}\left(\frac{1+r}{2} - r\theta - \epsilon - \bar{\eta}\right) \\ \geq &\frac{1-R_N}{1-\theta} H_{p^2}^{-1}\left(\frac{1+r}{2} - r\theta\right) - \frac{w}{N}(\epsilon + \bar{\eta}) \frac{c(p,r)}{\log_p 2 + 1} \\ \geq &\frac{1-R_N}{1-\theta} H_{p^2}^{-1}\left(\frac{1+r}{2} - r\theta\right) - \frac{c(p,r)}{n}.\end{aligned} \tag{4.3.6}$$

Using Lemma 4.4.2 on the right hand side of (4.3.5) gives us the first inequality. The second inequality comes from the monotonicity of the inverse p^2-ary entropy function and the inequality

$$\frac{w}{N}(\epsilon(w)+\bar{\eta}) = \frac{\log_p 2}{n} + \frac{w}{N}\frac{1}{n} \leq \frac{\log_p 2 + 1}{n}. \tag{4.3.7}$$

Since $0 \leq \theta \leq R_N$ and $1 + 2\log_{p^2}(1 - H_{p^2}^{-1}(\frac{1-R}{2})) \leq r \leq 1$, we can use Lemma 4.4.1 to get

$$\begin{aligned} \min_{0\leq\theta\leq R_N} \frac{1-R_N}{1-\theta} H_{p^2}^{-1}\left(\frac{1+r}{2} - r\theta\right) &= H_{p^2}^{-1}\left(\frac{1+r}{2} - r\frac{1+R}{2}\right) \\ &= H_{p^2}^{-1}\left(\frac{1-rR}{2}\right). \end{aligned} \tag{4.3.8}$$

Thus it suffices to have

$$\frac{h}{nN} \leq H_{p^2}^{-1}\left(\frac{1-rR}{2}\right) - \frac{c(p,r)}{n} \tag{4.3.9}$$

for our concatenated code to have a distance of at most h with probability at most $p^{-D'}\frac{1}{1-p^{-2}}$. This is equivalent to saying that the distance of our concatenated code is strictly greater than h with probability strictly larger than $1 - p^{-D'}\frac{1}{1-p^{-2}}$. Now $D' = N - \frac{N+NR}{2} + 1 = N\frac{1-R}{2} + 1$, and thus the result follows. □

4.4 Appendix : The q-ary Entropy and its Inverse

In this section, we derive properties of the q-ary entropy function and its inverse. Since H_q is a concave function strictly increasing on $(0, \frac{q-1}{q})$, H_q^{-1} is a convex function strictly increasing on the open interval $(0,1)$. Observe that for $x \in (0,1)$,

$$H_q'(x) := \frac{d}{dx}H_q(x) = \log_q(q-1) - \log_q x + \log_q(1-x) \tag{4.4.1}$$

$$(1-x)H_q'(1-x) = H_q(1-x) + \log_q x. \tag{4.4.2}$$

Since $H_q(y)$ is a continuously differentiable function for $y \in (0, 1-\frac{1}{q})$, by the inverse function theorem, we have that

$$(H_q^{-1})'(y) := \frac{d}{dy} H_q^{-1}(y) = \frac{1}{H_q'(H_q^{-1}(y))} \tag{4.4.3}$$

for $y \in (0,1)$.

Lemma 4.4.1. *The function*

$$\frac{1}{1-\theta} H_q^{-1}\left(\frac{1+r}{2} - r\theta\right)$$

is non-increasing with respect to θ *for* $0 \le \theta \le \frac{1+R}{2}$ *when* $0 \le R \le 1$ *and*

$$1 \ge r \ge 1 + 2\log_q\left(1 - H_q^{-1}\left(\frac{1-R}{2}\right)\right). \tag{4.4.4}$$

Proof. First observe that by making the substitution $1 - f = H_q^{-1}(\frac{1+r}{2} - r\theta)$, we have

$$\begin{aligned}
&\frac{d}{d\theta}\left(\frac{1}{1-\theta} H_q^{-1}\left(\frac{1+r}{2} - r\theta\right)\right) \\
=&\frac{1}{(1-\theta)^2}(1-f) - \frac{r}{1-\theta}(H_q^{-1})'\left(\frac{1+r}{2} - r\theta\right) \\
=&\frac{1}{1-\theta}\left(\frac{1-f}{1-\theta} - \frac{r}{H_q'(1-f)}\right)
\end{aligned} \tag{4.4.5}$$

where the second equality comes from applying (4.4.3). The expression (4.4.5) is non-positive if and only if

$$\begin{aligned}
\frac{1-f}{1-\theta} &\le \frac{r}{H_q'(1-f)} \\
(1-f)H_q'(1-f) &\le r(1-\theta).
\end{aligned} \tag{4.4.6}$$

Using (4.4.2), we find that

$$\begin{aligned}(1-f)H_q'(1-f) &= H_q(1-f) + \log_q f \\ &= \left(\frac{1+r}{2} - r\theta\right) + \log_q f.\end{aligned} \tag{4.4.7}$$

Thus the inequality (4.4.6) holds if and only if

$$\begin{aligned}\frac{1+r}{2} - r\theta + \log_q f &\le r(1-\theta) \\ r\left(\frac{1}{2} - \theta\right) + \frac{1}{2} + \log_q f &\le r(1-\theta) \\ 1 + 2\log_q f &\le r.\end{aligned} \tag{4.4.8}$$

Thus it suffices to obtain an upper bound on $1+2\log_q f$ that holds for all $\theta \in [0, \frac{1+R}{2}]$. The monotonicity of the inverse q-ary function and the restrictions on the domains of r and R imply that

$$\begin{aligned}1 + 2\log_q f &\ge 1 + 2\log_q\left(1 - H_q^{-1}\left(\frac{1+r}{2} - r\frac{1+R}{2}\right)\right) \\ &= 1 + 2\log_q\left(1 - H_q^{-1}\left(\frac{1-rR}{2}\right)\right) && (4.4.9) \\ &\ge 1 + 2\log_q\left(1 - H_q^{-1}\left(\frac{1-R}{2}\right)\right), && (4.4.10)\end{aligned}$$

thereby proving the result. □

Lemma 4.4.2. *For all integer $q \ge 2$ and for all $y, y-\epsilon \in (0, \frac{q-1}{q})$ we have*

$$H_q^{-1}(y-\epsilon) \ge H_q^{-1}(y) - \epsilon\left(\log_q(q-1) + \log_q\left(\frac{1}{H_q^{-1}(y)} - 1\right)\right)^{-1}.$$

Proof. Let $g = H_q^{-1}$. Since g is convex and continuously differentiable on the open interval $(0,1)$, for all $y, y-\epsilon$ in the open interval $(0,1)$ we have $g(y-\epsilon) \ge g(y) - \epsilon g'(y)$. The equations

(4.4.3) and (4.4.1) imply that

$$g'(y) = \frac{1}{(H_q)'(g(y))} = \left(\log_q(q-1) + \log_q\left(\frac{1}{g(y)} - 1\right)\right)^{-1} \tag{4.4.11}$$

for $y \in (0,1)$. Hence the result follows. $\square$

Chapter 5

The Quantum Capacity of Degradable Covariant Channels and their Convex Combinations

5.1 Introduction

The quantum capacity of a quantum channel is the maximum rate at which quantum information can be transmitted reliably across it, given arbitrarily many uses of it [Wil11]. However, evaluating the quantum capacity of a quantum channel is in general an infinite dimension optimization problem, and hence difficult, even for quantum channels with low dimension input and output states. The quantum capacity of even the simply described family of depolarizing channels is undetermined, and upper bounds of the quantum capacity of depolarizing channels has only been studied in the qubit case, of which the best-known upper bounds were obtained by Smith and Smolin [SS08].

The contributions of this chapter are new upper bounds on the quantum capacity of several families of simply described channels. The key ingredient Smith and Smolin use in their recipe to obtain upper bounds on the quantum capacity of the qubit depolarizing channel [SS08] is the family of degradable qubit amplitude damping channels. To obtain new upper bounds on the quantum capacity of other families of low dimension channels, we introduce a new ingredient – degradable amplitude damping channels of dimension four – in our extension of Smith and Smolin recipe .

In our extension of Smith and Smolin's recipe, we prove that the quantum capacity of a degradable channel twirled over a projective commutative unitary group at most its coherent information of the degradable channels maximized over a contracted input state space (Theorem 5.4.2). Smith and Smolin's recipe is produced as a special case of our extension when the projective commutative unitary group is chosen to be the full qubit Clifford group.

We find that many degradable quantum channels have quantum capacities that are just their coherent informations maximized over the diagonal input states. This finding is a direct consequence of our Theorem 5.4.2, because these degradable channels are covariant with respect to diagonal Pauli matrices.

We use our main result to produce explicit upper bounds for the quantum capacity for several families of quantum channels. These include the m-qubit depolarizing channels (Corollary 5.5.4, Figure 5.1), the two-qubit locally symmetric channels (Corollary 5.5.6, Figure 5.2), and the shifted qubit depolarizing channels (Corollary 5.5.7, Figure 5.3). The main ingredients that we introduce to obtain these new upper bounds are our higher dimension amplitude damping channels that are degradable.

We organize this chapter in the following order. In Section 5.2 we review background material needed for this chapter, introducing concepts such as the quantum capacity, complementary channels, and the degradable extension channels of Smith and Smolin. In Section 5.4, we present the main structural result of the chapter, which is Theorem 5.4.2, placed in the context of channel twirlings and channel covariance. In Section 5.5, we present our explicit upper bounds on the quantum capacity of some mostly low dimension unital and non-unital channels.

5.2 Preliminaries

Define $\eta(z) := -z \log_2 z$ where $z \in [0,1]$ and $\eta(0) := 0$. Define the Pauli matrices to be

$$\mathbb{1} := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \mathbf{X} := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \mathbf{Z} := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \mathbf{Y} := i\mathbf{X}\mathbf{Z}.$$

Define the Pauli group on m qubits modulo phases, to be $\mathcal{P}_m := \{\mathbb{1}, \mathbf{X}, \mathbf{Y}, \mathbf{Z}\}^{\otimes m}$. For all $\mathbf{P} \in \mathcal{P}_m$, define the weight of $\mathbf{P}$ to be the number of qubits that the operator $\mathbf{P}$ acts non-trivially on.

For a complex separable Hilbert space $\mathcal{H}$, let $\mathfrak{B}(\mathcal{H})$ be the set of bounded linear operators mapping $\mathcal{H}$ to $\mathcal{H}$. In this chapter, we only deal with finite dimension Hilbert spaces. A quantum channel $\Phi : \mathfrak{B}(\mathcal{H}_A) \to \mathfrak{B}(\mathcal{H}_B)$ is a completely positive and trace-preserving (CPT) linear map, and can be written in its Kraus form [Kra83]

$$\Phi(\rho) = \sum_k \mathbf{A}_k \rho \mathbf{A}_k^\dagger,$$

where the completeness relation $\sum_k \mathbf{A}_k^\dagger \mathbf{A}_k = \mathbb{1}_{d_A}$ is satisfied. Here, $d_A = \dim(\mathcal{H}_A)$ and $\mathbb{1}_{d_A}$ is a dimension d_A identity matrix. We can also write down the action of a quantum channel Φ in terms of an isometry on the input state. Now define an isometry $\mathbf{W} : \mathfrak{B}(\mathcal{H}_A) \to \mathfrak{B}(\mathcal{H}_E \otimes \mathcal{H}_B)$

$$\mathbf{W} = \sum_k |k\rangle \otimes \mathbf{A}_k.$$

Here $\{|k\rangle\}$ is an orthornormal set, and spans a Hilbert space $\mathcal{H}_E$ that we interpret to be the environment. Then

$$\mathbf{W}\rho\mathbf{W}^\dagger = \sum_{j,k} |j\rangle\langle k| \otimes \mathbf{A}_j \rho \mathbf{A}_k^\dagger$$

and

$$\mathrm{Tr}_{\mathcal{H}_E}(\mathbf{W}\rho\mathbf{W}^\dagger) = \Phi(\rho).$$

Then we can define the **complementary channel** $\Phi^C : \mathfrak{B}(\mathcal{H}_A) \to \mathfrak{B}(\mathcal{H}_E)$ [DS05] as

$$\Phi^C(\rho) = \mathrm{Tr}_{\mathcal{H}_B}(\mathbf{W}\rho\mathbf{W}^\dagger).$$

Since we are free to choose the orthornormal basis of the environment $\mathcal{H}_E$, Φ^C is only defined up to a unitary. We use the above definition as our canonical one. Let $\Phi^C(\rho) = \sum_\mu \mathbf{R}_\mu \rho \mathbf{R}_\mu{}^\dagger$. The j-th row of $\mathbf{R}_\mu$ is the μ-th row of $\mathbf{A}_j$, where $\mathbf{R}_\mu = \sum_j |j\rangle\langle\mu|\mathbf{A}_j$ [KMNR07]. To see this, observe that

$$\begin{aligned}
\Phi^C(\rho) &= \mathrm{Tr}_{\mathcal{H}_B}(\mathbf{W}\rho\mathbf{W}^\dagger) \\
&= \mathrm{Tr}_{\mathcal{H}_B}\Big(\sum_{j,k} |j\rangle\langle k| \otimes \mathbf{A}_j\rho\mathbf{A}_k^\dagger\Big) \\
&= \sum_{j,k} |j\rangle\langle k| \,\mathrm{Tr}\left(\mathbf{A}_j\rho\mathbf{A}_k^\dagger\right) \\
&= \sum_{j,k} |j\rangle \sum_\mu \langle\mu| \left(\mathbf{A}_j\rho\mathbf{A}_k^\dagger\right) |\mu\rangle\langle k| \\
&= \sum_\mu \Big(\sum_j |j\rangle\langle\mu|\mathbf{A}_j\Big)\rho\Big(\sum_k \mathbf{A}_k^\dagger|\mu\rangle\langle k|\Big) \\
&= \sum_\mu \mathbf{R}_\mu\rho\mathbf{R}_\mu^\dagger.
\end{aligned}$$

5.2.1 Quantum Capacity

For a quantum channel $\Phi : \mathfrak{B}(\mathcal{H}_A) \to \mathfrak{B}(\mathcal{H}_B)$, define

$$I_{coh}(\Phi, \rho) := S(\Phi(\rho)) - S(\Phi^C(\rho))$$

and

$$I_{coh}(\Phi) := \max_\rho I_{coh}(\Phi, \rho)$$

where the maximization of ρ is over all quantum states – trace one positive semidefinite operators in $\mathfrak{B}(\mathcal{H}_A)$ – and

$$S(\rho) := -\operatorname{Tr}(\rho \log_2 \rho)$$

is the von Neumann entropy of a given density matrix ρ. I_{coh} is a function called the coherent information first introduced by Schumacher and Nielsen[SN96]. Lloyd [Llo97], Shor[Sho02] and Devetak [Dev05] showed that the quantum capacity of Φ is

$$Q(\Phi) = \lim_{n\to\infty} \frac{1}{n} I_{coh}(\Phi^{\otimes n}), \tag{5.2.1}$$

and the expression on the right hand side of (5.2.1) exists [BNS98].

5.2.2 Degradable Channels

A channel $\mathcal{N}$ is **degradable** [DS05] if it can be composed with another quantum channel Ψ to become equivalent to its complementary channel $\mathcal{N}^C$, that is $\mathcal{N}^C = \Psi \circ \mathcal{N}$. Physically, this means that the environment associated with channel $\mathcal{N}$ can be simulated using its output quantum state. Conversely, $\mathcal{N}$ is **antidegradable** if its complementary channel $\mathcal{N}^C$ is degradable. A channel $\mathcal{N}_{\text{ext}}$ is a **degradable extension** [SS08] of channel $\mathcal{N}$ if $\mathcal{N}_{\text{ext}}$ is degradable and there exists a quantum operation Ψ such that $\Psi \circ \mathcal{N}_{\text{ext}} = \mathcal{N}$.

If $\mathcal{N}_{\text{ext}}$ is a degradable extension of some channel, then $Q(\mathcal{N}_{\text{ext}}) = I_{coh}(\mathcal{N}_{\text{ext}})$. Degradable extensions allow us to construct upper bounds of the quantum capacity of quantum channels. In this chapter, all degradable extensions of $\mathcal{N}$ have the partial trace as the degrading map, which is possible because of the following lemma by Smith and Smolin.

Theorem 5.2.1 (Smith, Smolin (Lemma 4 in [SS08]))**.** *Let $\sum_{i=1}^k \lambda_i \mathcal{N}_i$ be a convex combination of degradable channels $\mathcal{N}_i$. Then the channel*

$$\mathcal{N}_{\text{ext}}(\rho) = \sum_{i=1}^{k} \lambda_i \mathcal{N}_i(\rho) \otimes |i\rangle\langle i|$$

is a degradable extension of $\sum_{i=1}^k \lambda_i \mathcal{N}_i$ with the degrading map being the partial trace on *the extending space. Moreover, the quantum capacity is a convex function with respect to degradable channels in the sense that*

$$Q\left(\sum_{i=1}^{k} \lambda_i \mathcal{N}_i\right) \leq Q(\mathcal{N}_{\text{ext}}) \leq \sum_{i=1}^{k} \lambda_i Q(\mathcal{N}_i). \tag{5.2.2}$$

The key utility of Smith and Smolin's result above is the convexity of the upper bounds obtained from degradable extensions (see Section IIC, [SS08]). Moreover the no cloning bounds of Cerf [Cer00] admit degradable extensions too (see Section IIB, [SS08]).

5.3 New Degradable Channels: Higher Dimension Amplitude Damping Channels

In this section, we generalize the qubit amplitude damping channel, a channel with Kraus operators $|0\rangle\langle 0| + \sqrt{1-\gamma}|1\rangle\langle 1|$ and $\sqrt{\gamma}|0\rangle\langle 1|$, to its higher dimension counterparts. We say that a channel with dimension d input and output states is an **amplitude damping channel** if it admits a Kraus representation such that (i) one Kraus operator is a diagonal matrix, and (ii) all the other Kraus operators are strictly upper triangular matrices. Physically, this corresponds to the case where only transitions from excited states to less excited states are allowed, thereby modeling the phenomenon of spontaneous decay.

Determining whether an amplitude damping channel is degradable is in general a nontrivial task. In this section, we construct two families of amplitude damping channels that are degradable.

5.3.1 Uniformly Amplitude Damping Channels

We first define a **uniformly amplitude damping channel** of dimension d and damping strength $0 \leq \gamma \leq 1$ to be the quantum channel $\mathcal{A}_{\gamma,d}$ with the Kraus operators

$$|0\rangle\langle 0| + \sum_{i=1}^{d-1} \sqrt{1-\gamma}|i\rangle\langle i|,$$
$$\sqrt{\gamma}|0\rangle\langle j|, \quad \text{for all } 1 \leq j \leq d-1.$$

The complementary channel of $\mathcal{A}_{\gamma,d}$ is also a d-dimension uniformly amplitude damping channel $\mathcal{A}_{1-\gamma,d}$. Moreover, when $0 \leq \gamma \leq \frac{1}{2}$, the uniformly amplitude dapming channel is degradable with degrading map $\mathcal{A}_{\frac{1-2\gamma}{1-\gamma}}$. Hence the uniformly amplitude damping channel $\mathcal{A}_{\gamma,d}$ is a degradable channel when $0 \leq \gamma \leq \frac{1}{2}$, and is anti-degradable when $\frac{1}{2} \leq \gamma \leq 1$.

5.3.2 Four-Dimension Amplitude Damping Channels

Qubit amplitude damping quantum channels are examples of qubit degradable channels that are non-unital. These channels model spontaneous decay in two-level quantum systems, and hence knowledge of their quantum capacity is a physically relevant problem [GF05]. An interesting fact is that these channels are covariant [Hol93] with respect to the diagonal Pauli matrices [WPG07], where the covariance of channels is defined in (5.4.2). This notion of covariance can help us simplify the evaluation of the quantum capacity of these channels.

Prior to this work, explicit non-trivial upper bounds on the quantum capacity has only been performed on single qubit channels. In this chapter, we extend the study of explicit upper bounds on the quantum capacity to four-dimension channels. Since the qubit amplitude damping channel has been crucial to obtain explicit upper bounds on the quantum capacity of the qubit depolarizing channel, one might expect that a four-dimension amplitude damping channel would also be crucial to obtain explicit upper bounds on the four-dimension depolarizing channel.

Hence in this section, we introduce a two-qubit amplitude damping channel which contains the tensor product of single-qubit amplitude damping channels as special cases.

Define the linear map $\Phi_{\mathbf{a}}(\rho) := \sum_{i=0}^{3} \mathbf{K}_i \rho \mathbf{K}_i^\dagger$ with two-qubit input and output states, to have Kraus operators

$$\begin{aligned}
\mathbf{K}_0 &= \sum_{i=0}^{3} a_{0,i} |i\rangle\langle i| \\
\mathbf{K}_1 &= a_{1,1}|0\rangle\langle 1| + a_{1,2}|2\rangle\langle 3| \\
\mathbf{K}_2 &= a_{2,1}|0\rangle\langle 2| + a_{2,2}|1\rangle\langle 3| \\
\mathbf{K}_3 &= a_{3,1}|0\rangle\langle 3|
\end{aligned} \tag{5.3.1}$$

where $\mathbf{a} = (a_{0,0}, a_{0,1}, a_{0,2}, a_{0,3}, a_{1,1}, a_{1,2}, a_{2,1}, a_{2,2}, a_{3,1}) \geq \mathbf{0}$. When

$$a_{0,0} = 1, \quad a_{0,1}^2 + a_{1,1}^2 = 1, \quad a_{0,2}^2 + a_{2,1}^2 = 1, \quad a_{0,3}^2 + a_{1,2}^2 + a_{2,2}^2 + a_{3,1}^2 = 1, \tag{5.3.2}$$

then $\Phi_{\mathbf{a}}$ is also a quantum channel.

For $x, y, z \geq 0, 1 - 2y - z \geq 0$, define $\Phi_{x,y,z} : \mathfrak{B}(\mathbb{C}^4) \to \mathfrak{B}(\mathbb{C}^4)$ where

$$\Phi_{x,y,z} := \Phi_{(1,s_1,s_1,s_2,\sqrt{x},\sqrt{y},\sqrt{x},\sqrt{y},\sqrt{z})} \tag{5.3.3}$$

to be a quantum channel with Kraus operators

$$\begin{aligned}
\mathbf{A}_0 &= |0\rangle\langle 0| + s_1(|1\rangle\langle 1| + |2\rangle\langle 2|) + s_2|3\rangle\langle 3| \\
\mathbf{A}_1 &= \sqrt{x}|0\rangle\langle 1| + \sqrt{y}|2\rangle\langle 3| \\
\mathbf{A}_2 &= \sqrt{x}|0\rangle\langle 2| + \sqrt{y}|1\rangle\langle 3| \\
\mathbf{A}_3 &= \sqrt{z}|0\rangle\langle 3|
\end{aligned} \tag{5.3.4}$$

where $s_1 = \sqrt{1-x}, s_2 = \sqrt{1-2y-z}$. For $x, y, z \geq 0, 1 - 2y - z \geq 0$, we call $\Phi_{x,y,z}$ a two-qubit amplitude damping channel. Observe $\Phi_{\gamma,\gamma(1-\gamma),\gamma^2} = \Phi_\gamma \otimes \Phi_\gamma$ where Φ_γ is the qubit amplitude damping channel with Kraus operators $\sqrt{\gamma}|0\rangle\langle 1|$ and $|0\rangle\langle 0| + \sqrt{1-\gamma}|1\rangle\langle 1|$

for $\gamma \in [0, \frac{1}{2}]$.

Let $\mathfrak{K}$ be the family of four-dimension amplitude damping channels with Kraus operators of the form (5.3.1). Then we prove that the set of such channels is closed under complementation. This is the content of the following proposition.

Proposition 5.3.1. *$\mathcal{K} \in \mathfrak{K}$ if and only if $\mathcal{K}^C \in \mathfrak{K}$.*

Proof. Using the recipe of King, Matsumoto, Nathanson and Ruskai [KMNR07], if the Kraus operators of $\mathcal{K}$ have the form of (5.3.1), then the channel $\mathcal{K}^C$ has the Kraus operators

$$\begin{aligned}
\mathbf{K}'_0 &= a_{0,0}|0\rangle\langle 0| + \sum_{i=1}^{3} a_{i,1}|i\rangle\langle i| \\
\mathbf{K}'_1 &= a_{0,1}|0\rangle\langle 1| + a_{2,2}|2\rangle\langle 3| \\
\mathbf{K}'_2 &= a_{0,2}|0\rangle\langle 2| + a_{1,2}|1\rangle\langle 3| \\
\mathbf{K}'_3 &= a_{0,3}|0\rangle\langle 3| \qquad (5.3.5)
\end{aligned}$$

which is also of the form (5.3.1). This proves the forward direction. Since $(\mathcal{K}^C)^C = \mathcal{K}$, the reverse implication also holds. □

The linear map $\Phi_{x,y,z}$ can also be a degradable quantum channel when x, y and z satisfy the following inequalities.

$$\begin{aligned}
x, y, z \geq 0, \quad 2y + z < 1, \quad x < \frac{1}{2}, \\
2z \leq 1 - 2y\Big(2 - \frac{x}{1-x}\Big). \qquad (5.3.6)
\end{aligned}$$

We prove this result in the lemma below, and the essence of the lemma's proof is that we construct the degrading map of $\Phi_{x,y,z}$ to its complementary channel explicitly. In this case, the degrading map and the complementary channels are also hilfour-dimension amplitude damping channels.

Lemma 5.3.2. $\Phi_{x,y,z}$ *is a degradable channel with degrading map* $\Phi_{g,h,k}$ *when* x, y *and* z *satisfy the inequalities (5.3.6) and where*

$$g = \frac{1-2x}{1-x}, \quad h = \frac{gy}{(1-2y-z)}$$
$$k = 1 - 2h - \frac{z}{1-2y-z}. \tag{5.3.7}$$

Proof. Note that $\Phi_{x,y,z}$ is a quantum channel for $x, y, z \geq 0$ and $2y + z < 1$ which Kraus operators given by (5.3.4). Also note that its complementary channel $\Phi^C_{x,y,z} = \Phi_{1-x,y,1-2y-z}$ has the Kraus operators

$$\mathbf{R}_0 = |0\rangle\langle 0| + \sqrt{x}|1\rangle\langle 1| + \sqrt{x}|2\rangle\langle 2| + \sqrt{z}|3\rangle\langle 3|$$
$$\mathbf{R}_1 = \sqrt{1-x}|0\rangle\langle 1| + \sqrt{y}|2\rangle\langle 3|$$
$$\mathbf{R}_2 = \sqrt{1-x}|0\rangle\langle 2| + \sqrt{y}|1\rangle\langle 3|$$
$$\mathbf{R}_3 = \sqrt{1-2y-z}|0\rangle\langle 3|.$$

Let us define $\mathcal{G} := \Phi_{g,h,k}$ with Kraus operators

$$\mathbf{G}_0 = |0\rangle\langle 0| + \sqrt{1-g}(|1\rangle\langle 1| + |2\rangle\langle 2|) + \sqrt{1-2h-k}|3\rangle\langle 3|$$
$$\mathbf{G}_1 = \sqrt{g}|0\rangle\langle 1| + \sqrt{h}|2\rangle\langle 3|$$
$$\mathbf{G}_2 = \sqrt{g}|0\rangle\langle 2| + \sqrt{h}|1\rangle\langle 3|$$
$$\mathbf{G}_3 = \sqrt{k}|0\rangle\langle 3|.$$

When the inequalities in (5.3.6) are satisfied, $0 \leq g, h, k \leq 1$ and hence $\mathcal{G}$ is a valid quantum operation.

We want to find the conditions where $\mathcal{G} \circ \Phi_{x,y,z} = \Phi^C_{x,y,z}$ which means that $\mathcal{G}$ is a degrading map that takes the output state of $\Phi_{x,y,z}$ to the output state of $\Phi^C_{x,y,z}$. By the Kraus representation,

$$\mathcal{G}(\Phi_{x,y,z}(\rho)) = \sum_{k,\ell \in \{0,1,2,3\}} \mathbf{G}_k \mathbf{A}_\ell \rho \mathbf{A}_\ell^\dagger \mathbf{G}_k^\dagger.$$

Hence in this representation $\Phi^C_{x,y,z} = \mathcal{G} \circ \Phi_{x,y,z}$ is a quantum channel with the sixteen Kraus operators $\mathbf{G}_k\mathbf{A}_\ell$ for $k, \ell \in \mathbb{Z}_4$. Now we evaluate $\mathbf{G}_k\mathbf{A}_\ell$ explicitly.

$$\mathbf{G}_1\mathbf{A}_3 = \mathbf{G}_1\mathbf{A}_1 = 0, \mathbf{G}_1\mathbf{A}_2 = \sqrt{\frac{1-2x}{1-x}y}|0\rangle\langle 3|$$
$$\mathbf{G}_2\mathbf{A}_3 = \mathbf{G}_2\mathbf{A}_2 = 0, \mathbf{G}_2\mathbf{A}_1 = \sqrt{\frac{1-2x}{1-x}y}|0\rangle\langle 3|$$
$$\mathbf{G}_3\mathbf{A}_3 = \mathbf{G}_3\mathbf{A}_2 = \mathbf{G}_3\mathbf{A}_1 = 0.$$

Also we have

$$\mathbf{G}_1\mathbf{A}_0 = \sqrt{1-2x}|0\rangle\langle 1| + \sqrt{\frac{1-2x}{1-x}y}|2\rangle\langle 3|$$
$$\mathbf{G}_2\mathbf{A}_0 = \sqrt{1-2x}|0\rangle\langle 2| + \sqrt{\frac{1-2x}{1-x}y}|1\rangle\langle 3|$$
$$\mathbf{G}_3\mathbf{A}_0 = \sqrt{\frac{1-x-2y(2-3x)}{1-x}}|0\rangle\langle 3|.$$

Moreover

$$\mathbf{G}_0\mathbf{A}_1 = \sqrt{x}|0\rangle\langle 1| + \sqrt{\frac{xy}{1-x}}|2\rangle\langle 3|$$
$$\mathbf{G}_0\mathbf{A}_2 = \sqrt{x}|0\rangle\langle 2| + \sqrt{\frac{xy}{1-x}}|1\rangle\langle 3|$$
$$\mathbf{G}_0\mathbf{A}_3 = \sqrt{z}|0\rangle\langle 3|.$$

Observe then that $\mathbf{G}_0\mathbf{A}_1 = \sqrt{\frac{x}{1-2x}}\mathbf{G}_1\mathbf{A}_0$ and $\mathbf{G}_0\mathbf{A}_2 = \sqrt{\frac{x}{1-2x}}\mathbf{G}_2\mathbf{A}_0$. Thus applying the Kraus operators $\mathbf{G}_i\mathbf{A}_0$ and $\mathbf{G}_0\mathbf{A}_i$ is equivalent to applying the Kraus operator $\mathbf{R}_i$ for $i \in \{1, 2\}$. Similarly, applying the Kraus operators $\mathbf{G}_1\mathbf{A}_2, \mathbf{G}_2\mathbf{A}_1$ and $\mathbf{G}_3\mathbf{A}_0$ is equivalent to applying the Kraus operator $\mathbf{R}_3$. Moreover, since $1-g = \frac{x}{1-x}$ and $(1-2h-k)(1-2y-z) = z$, we have that $\mathbf{G}_0\mathbf{A}_0 = \mathbf{R_0}$. Therefore we have shown that $\Phi_{x,y,z}$ is degradable with degrading map $\mathcal{G}$ when x, y and z satisfy the inequalities in (5.3.6). $\square$

5.4 Twirling, Contraction and Covariance

Let $\mathcal{N}$ be a channel with d-dimension input and output states, and $\mathcal{V}$ be a set of d-dimension unitary operators. We define the **$\mathcal{V}$-contraction channel** to be the channel $\mathcal{V}_{\triangleright}$ with Kraus set $\frac{\mathcal{V}}{\sqrt{|\mathcal{V}|}}$, so that

$$\mathcal{V}_{\triangleright}(\rho) := \frac{1}{|\mathcal{V}|}\sum_{\mathbf{V}\in\mathcal{V}} \mathbf{V}\rho\mathbf{V}^{\dagger}.$$

If $\mathcal{V}$ is the set of m-qubit Pauli matrices, then the contraction channel $\mathcal{V}_{\triangleright}$ maps all input states to the maximally mixed state $\mathbb{1}_{2^m}/2^m$. If $\mathcal{V}$ is the set of diagonal m-qubit Pauli matrices, then the contraction channel $\mathcal{V}_{\triangleright}$ is the m-fold tensor product of the maximally dephasing qubit channel, and maps the set of m-qubit input states to the set of m-qubit diagonal states.

The $\mathcal{V}$-twirl of $\mathcal{N}$ is the channel

$$\mathcal{N}_{\ltimes\mathcal{V}\rtimes}(\rho) := \frac{1}{|\mathcal{V}|}\sum_{\mathbf{V}\in\mathcal{V}} \mathbf{V}^{\dagger}\Phi(\mathbf{V}\rho\mathbf{V}^{\dagger})\mathbf{V}.$$

When the set $\mathcal{V}$ is the m-qubit Pauli set $\mathcal{P}_m$, the $\mathcal{V}$-twirl of a channel is also called the Pauli-twirl of a channel. Note that channels that are Pauli-twirled are Pauli channels [DCEL09], because their Kraus operators can all be expressed as m-qubit Pauli matrices $\mathbf{P}$ multiplied by the coefficient

$$\frac{1}{2^m}\sqrt{\sum_{\mathbf{K}\in K_{\mathcal{N}}} \left|\mathrm{Tr}(\mathbf{PK})\right|^2}, \tag{5.4.1}$$

where $K_{\mathcal{N}}$ is the set of Kraus operators of the channel $\mathcal{N}$.

We say that the channel $\mathcal{N}$ is $\mathcal{V}$-covariant if the equation

$$\Phi(\mathbf{V}\rho\mathbf{V}^{\dagger}) = \mathbf{V}\Phi(\rho)\mathbf{V}^{\dagger} \tag{5.4.2}$$

holds for every unitary matrix $\mathbf{V}$ in the set $\mathcal{V}$ and for every input state ρ. Properties of quantum channels covariant with respect to a locally compact group were studied by Holevo [Hol93].

Because the $\mathcal{V}$ is a set of unitary matrices, a channel that is $\mathcal{V}$-covariant is also invariant under $\mathcal{V}$-twirling (however it is not clear whether the converse holds). A $\mathcal{V}$-covariant channel is hence equal to its $\mathcal{V}$-twirl, and this is how the notion of covariance connects with the notion of twirling.

This section contains the main structural result of this chapter, which is Theorem 5.4.2. Our theorem generalizes Smith and Smolin's technique of obtaining upper bounds of the Clifford-twirl of a qubit degradable channel (see Lemma 8 in [SS08]). Theorem 5.4.2 gives an upper bound on the $\mathcal{V}$-twirl of a degradable channel in terms of its coherent information maximized over the output states of the $\mathcal{V}$-contraction channel.

The covariance of qubit amplitude damping channels with respect to diagonal Pauli matrices is a well-known fact, and has been used by Wolf and Pérez-García to prove that the qubit amplitude damping channel's quantum capacity is just the maximum coherent information over all **diagonal** qubit states [WPG07].

We prove that this property is not restricted just to qubit amplitude damping channels, which is the essence of Corollary 5.4.3. This in turn is a simple consequence of Theorem 5.4.2, which applies for all degradable channels invariant under the twirling of diagonal Pauli matrices. We give examples of such degradable channels in Section 5.4.2.

5.4.1 The Quantum Capacity of Covariant and Twirled Channels

In this section, let $\mathcal{N}$ be a degradable channel with d-dimension input and output states, and $\mathcal{V}$ be a finite projective group of d-dimension unitary matrices. We say that a set of d-dimension unitary matrices is a finite projective group if the set satisfies the following additional properties.

1. **Projective Property:** No two distinct elements of $\mathcal{V}$ are equivalent up to a constant.

2. **Group Property:** For all $\mathbf{V}$ and $\mathbf{W}$ in the set $\mathcal{V}$, there exists complex number of unit amplitude $z_{\mathbf{V},\mathbf{W}^\dagger}$ such that $z_{\mathbf{V},\mathbf{W}^\dagger}\mathbf{V}\mathbf{W}^\dagger$ is also an element of $\mathcal{V}$.

Note that the complex number $z_{\mathbf{V},\mathbf{W}^\dagger}$ in the second property above defines the binary operation of the projective unitary group $\mathcal{V}$, in the sense that

$$\mathbf{V} \star \mathbf{W} := z_{\mathbf{V},\mathbf{W}}\mathbf{V}\mathbf{W}$$

for all $\mathbf{V}$ and $\mathbf{W}$ that are elements of $\mathcal{V}$. Important examples of finite projective unitary groups include the m-qubit Clifford group, the set of m-qubit Pauli matrices, and the set of diagonal m-qubit Pauli matrices.

In this section, we also define $\widetilde{\mathcal{N}}$ to be a particular extension of the $\mathcal{V}$-twirl of $\mathcal{N}$, where

$$\widetilde{\mathcal{N}}(\rho) := \sum_{\mathbf{V}\in\mathcal{V}} \frac{1}{|\mathcal{V}|}\mathbf{V}^\dagger\mathcal{N}(\mathbf{V}\rho\mathbf{V}^\dagger)\mathbf{V} \otimes |\mathbf{V}\rangle\langle\mathbf{V}|. \tag{5.4.3}$$

By using the obvious isometric extensions of $\mathcal{N}$ and $\widetilde{\mathcal{N}}$, one can show that

$$\widetilde{\mathcal{N}}^C(\rho) = \sum_{\mathbf{V}\in\mathcal{V}} \frac{1}{|\mathcal{V}|}\mathcal{N}^C(\mathbf{V}\rho\mathbf{V}^\dagger) \otimes |\mathbf{V}\rangle\langle\mathbf{V}|. \tag{5.4.4}$$

We state this fact formally in Proposition 5.4.1.

Proposition 5.4.1. *Let $\mathcal{N}$ be a quantum channel with d-dimension input and output states, $\mathcal{V}$ be a set of d-dimension unitary matrices, and $\widetilde{\mathcal{N}}$ be as defined in (5.4.3). Then equation (5.4.4) holds.*

Proof. Let $K_{\mathcal{N}}$ denote the Kraus set of the channel $\mathcal{N}$. Using the canonical definition of the complementary channel of $\mathcal{N}$ from its canonical isometric extension, we have for all $\mathbf{V} \in \mathcal{V}$,

$$\mathcal{N}^C(\mathbf{V}\rho\mathbf{V}^\dagger) = \mathrm{Tr}_{\mathcal{H}_B}\left(\Big(\sum_{\mathbf{A},\mathbf{A}'\in K_{\mathcal{N}}} \mathbf{A}\mathbf{V}\rho\mathbf{V}^\dagger\mathbf{A}'^\dagger\Big)_{\mathcal{H}_B} \otimes |\mathbf{A}\rangle\langle\mathbf{A}'|\right). \tag{5.4.5}$$

Similarly, the canonical complementary channel of $\widetilde{\mathcal{N}}$ is

$$\begin{aligned}
\widetilde{\mathcal{N}}^C(\rho) &= \mathrm{Tr}_{\mathcal{H}_B \otimes \mathcal{H}_C}\left(\frac{1}{|\mathcal{V}|}\sum_{\substack{V,V'\in\mathcal{V}\\ A,A'\in K_{\mathcal{N}}}}\Big(\mathbf{V}^\dagger\mathbf{A}\mathbf{V}\rho\mathbf{V}^\dagger\mathbf{A}'^\dagger\mathbf{V}'\Big)_{\mathcal{H}_B}\otimes\Big(|\mathbf{V}\rangle\langle\mathbf{V}'|\Big)_{\mathcal{H}_C}\otimes|\mathbf{A}\rangle\langle\mathbf{A}'|\otimes|\mathbf{V}\rangle\langle\mathbf{V}'|\right)\\
&= \frac{1}{|\mathcal{V}|}\sum_{V\in\mathcal{V}}\mathrm{Tr}_{\mathcal{H}_B}\left(\sum_{A,A'\in K_{\mathcal{N}}}\Big(\mathbf{V}^\dagger\mathbf{A}\mathbf{V}\rho\mathbf{V}^\dagger\mathbf{A}'^\dagger\mathbf{V}\Big)_{\mathcal{H}_B}\otimes|\mathbf{A}\rangle\langle\mathbf{A}'|\right)\otimes|\mathbf{V}\rangle\langle\mathbf{V}|\\
&= \frac{1}{|\mathcal{V}|}\sum_{V\in\mathcal{V}}\mathcal{N}^C(\mathbf{V}\rho\mathbf{V}^\dagger)\otimes|\mathbf{V}\rangle\langle\mathbf{V}|
\end{aligned}$$

where we have used the unitary invariance of the partial trace. □

With equation (5.4.4), one can verify that $\widetilde{\mathcal{N}}$ is a degradable channel with a degrading channel

$$\sum_{\mathbf{W}\in\mathcal{V}}\Psi(\mathbf{W}^\dagger\rho\mathbf{W})\otimes|\mathbf{W}\rangle\langle\mathbf{W}|$$

where Ψ is the degrading channel for the degradable channel $\mathcal{N}$. Hence $\widetilde{\mathcal{N}}$ is a degradable extension of the $\mathcal{V}$-twirl of $\mathcal{N}$.

Let the set of d-dimension quantum states be M_d. Define the image of the $\mathcal{V}$-contraction map to be $\mathrm{Im}(\mathcal{V}_\triangleright) := \{\sigma = \mathcal{V}_\triangleright(\rho) : \rho \in M_d\}$. Now define the $\mathcal{V}$-contracted coherent information of a channel $\mathcal{N}$ to be

$$I_{coh}(\mathcal{N},\mathcal{V}_\triangleright) := \max_{\rho\in\mathrm{Im}(\mathcal{V}_\triangleright)} I_{coh}(\mathcal{N},\rho).$$

We now state the main structural result of this chapter.

Theorem 5.4.2 (Twirling and Contraction)**.** *Let $\mathcal{V}$ be a projective group of d-dimension unitary matrices, $\mathcal{N}$ be a degradable channel with d-dimension input and output states, and $\widetilde{\mathcal{N}}$ be a degradable extension of $\mathcal{N}$ as defined in (5.4.3). Then*

$$Q(\mathcal{N}_{\ltimes\mathcal{V}\rtimes}) \le Q(\widetilde{\mathcal{N}}) \le I_{coh}(\mathcal{N},\mathcal{V}_\triangleright).$$

Proof. Observe that for all $\mathbf{V} \in \mathcal{V}$,

$$\widetilde{\mathcal{N}}(\mathbf{V}\rho\mathbf{V}^\dagger) = \sum_{\mathbf{V}'\in\mathcal{V}} \frac{1}{|\mathcal{V}|}\mathbf{V}\mathbf{V}^\dagger\mathbf{V}'^\dagger\mathcal{N}(\mathbf{V}'\mathbf{V}\rho\mathbf{V}^\dagger\mathbf{V}'^\dagger)\mathbf{V}'\mathbf{V}\mathbf{V}^\dagger \otimes |\mathbf{V}'\rangle\langle\mathbf{V}'|$$
$$= \sum_{\mathbf{V}'\in\mathcal{V}} \frac{1}{|\mathcal{V}|}\mathbf{V}(z_{\mathbf{V}',\mathbf{V}}\mathbf{V}'\mathbf{V})^\dagger\mathcal{N}((z_{\mathbf{V}',\mathbf{V}}\mathbf{V}'\mathbf{V})\rho(z_{\mathbf{V},\mathbf{V}'}\mathbf{V}'\mathbf{V})^\dagger)(z_{\mathbf{V}',\mathbf{V}}\mathbf{V}'\mathbf{V})\mathbf{V}^\dagger \otimes |\mathbf{V}'\rangle\langle\mathbf{V}'|.$$

By the projective group property of $\mathcal{V}$, we can make the substitution $\mathbf{R} = z_{\mathbf{V}'\mathbf{V}}\mathbf{V}'\mathbf{V} = \mathbf{V}' \star \mathbf{V}$ and replace the index of the summation so that

$$\begin{aligned}\widetilde{\mathcal{N}}(\mathbf{V}\rho\mathbf{V}^\dagger) =& \sum_{\mathbf{R}\in\mathcal{V}} \frac{1}{|\mathcal{V}|}\mathbf{V}\mathbf{R}^\dagger\mathcal{N}(\mathbf{R}\rho\mathbf{R}^\dagger)\mathbf{R}\mathbf{V}^\dagger \otimes |\mathbf{R}\star\mathbf{V}^\dagger\rangle\langle\mathbf{R}\star\mathbf{V}^\dagger| \\ =&(\mathbf{V}\otimes\mathbf{U}_\mathbf{V})\widetilde{\mathcal{N}}(\rho)(\mathbf{V}^\dagger\otimes\mathbf{U}_\mathbf{V}^\dagger) \end{aligned} \tag{5.4.6}$$

where $\mathbf{U}_\mathbf{V} := \sum_{\mathbf{R}\in\mathcal{V}} |\mathbf{R}\rangle\langle\mathbf{R}\star\mathbf{V}^\dagger|$ is a unitary matrix. Now we can use the isometric extensions of the channels $\mathcal{N}$ and $\widetilde{\mathcal{N}}$ to show that (see Proposition 5.4.1)

$$\widetilde{\mathcal{N}}^C(\rho) = \sum_{\mathbf{V}\in\mathcal{V}} \frac{1}{|\mathcal{V}|}\mathcal{N}^C(\mathbf{V}\rho\mathbf{V}^\dagger)\otimes|\mathbf{V}\rangle\langle\mathbf{V}|.$$

By a similar argument as in (5.4.6),

$$\widetilde{\mathcal{N}}^C(\mathbf{V}\rho\mathbf{V}^\dagger) = (\mathbb{1}_{d_E}\otimes\mathbf{U}_\mathbf{V})\widetilde{\mathcal{N}}^C(\rho)(\mathbb{1}_{d_E}\otimes\mathbf{U}_\mathbf{V}^\dagger), \tag{5.4.7}$$

where d_E is the dimension of the output states of the complementary channel $\mathcal{N}^C$. Note that the von-Neumann entropy is additive with respect to each block in a block diagonal matrix, and is also invariant under unitary conjugation of its argument. Hence the coherent information of the degradable extension $\widetilde{\mathcal{N}}$ evaluated on the input state ρ is

$$\begin{aligned} S(\widetilde{\mathcal{N}}(\rho)) - S(\widetilde{\mathcal{N}}^C(\rho)) &= \left(\sum_{\mathbf{V}\in\mathcal{V}} \frac{1}{|\mathcal{V}|} S\big(\mathcal{N}(\mathbf{V}\rho\mathbf{V}^\dagger)\big)\right) - \left(\sum_{\mathbf{V}\in\mathcal{V}} \frac{1}{|\mathcal{V}|} S\big(\mathcal{N}^C(\mathbf{V}\rho\mathbf{V}^\dagger)\big)\right) \\ &= \sum_{\mathbf{V}\in\mathcal{V}} \frac{1}{|\mathcal{V}|} I_{coh}\big(\mathcal{N}, \mathbf{V}\rho\mathbf{V}^\dagger\big) \\ &\leq I_{coh}\Big(\mathcal{N}, \sum_{\mathbf{V}\in\mathcal{V}} \frac{1}{|\mathcal{V}|} \mathbf{V}\rho\mathbf{V}^\dagger\Big) \end{aligned}$$

where the inequality above results from the concavity of the coherent information of degradable channels with respect to the input state [YHD08]. Hence the coherent information of the degradable channel $\mathcal{N}$ maximized over all output states of the $\mathcal{V}$-contraction channel upper bounds the coherent information and quantum capacity of the degradable extension $\widetilde{\mathcal{N}}$. □

Theorem 5.4.2 has several important consequences. Firstly, when the finite projective unitary group $\mathcal{V}$ is chosen to be the set of m-qubit Pauli matrices, then Theorem 5.4.2 implies that quantum capacity of the Pauli-twirl of a degradable channel $\mathcal{N}$ is at most the coherent information of the channel $\mathcal{N}$ evaluated on the maximally mixed state.

The second important consequence of Theorem 5.4.2 applies when the degradable channel $\mathcal{N}$ is $\mathcal{V}$-covariant, which means that $\mathcal{N}$ is invariant under the $\mathcal{V}$-twirl. Application of Theorem 5.4.2 gives the following corollary.

Corollary 5.4.3 (Degradable and Covariant Channels)**.** *Let $\mathcal{V}$ be a finite projective unitary group and $\mathcal{N}$ be a degradable channel. If $\mathcal{N}$ is also $\mathcal{V}$-covariant, then*

$$Q(\mathcal{N}) = I_{coh}(\mathcal{N}, \mathcal{V}_{\triangleright}).$$

Proof. Since $\mathcal{N}$ is a degradable channel, Theorem 5.4.2 implies that the $\mathcal{V}$-twirl of $\mathcal{N}$ has a quantum capacity that is at most the coherent information of $\mathcal{N}$ maximized over all output states of the $\mathcal{V}$-contraction map. Now the $\mathcal{V}$-twirl of $\mathcal{N}$ is precisely the untwirled channel

$\mathcal{N}$. Moreover the quantum capacity of $\mathcal{N}$ is its coherent information maximized over all its input states. Combining these facts gives the result. □

An immediate consequence of Corollary 5.4.3 is that degradable channels that are covariant with respect to m-qubit diagonal Pauli matrices have quantum capacities that are their coherent information maximized over just their diagonal input states, instead of the entire set of feasible input states. This simplifies the evaluation of the quantum capacity of degradable channels with this property. The next section gives examples of many channels that are covariant with respect to the diagonal Pauli matrices.

5.4.2 Examples of Degradable Channels that are Covariant

In this section, we give sufficient conditions for a m-qubit channel to be covariant with respect to the m-qubit diagonal Pauli matrices. Many degradable channels have this covariance property, such as Hadamard channels, all qubit degradable channels, and higher dimension amplitude damping channels.

Hadamard channels are complementary channels of entanglement breaking channels, and map a quantum state to their Hadamard product with some matrix [KMNR07, BHTW10]. Let $\rho_{i,j}$ and $m_{i,j}$ be matrix elements a quantum state ρ and a matrix M respectively. Then a Hadamard channel parameterized by M maps $\rho_{i,j}$ to $\rho_{i,j}m_{i,j}$. Of course, the matrix M has to be carefully chosen for the Hadamard channel to be a valid quantum channel.

Proposition 5.4.4 (Hadamard channels)**.** *An m-qubit Hadamard channel is covariant with respect to m-qubit diagonal Pauli matrices.*

Proof. We prove a stronger fact – that a Hadamard channel is covariant with respect to diagonal matrices. Notice that the effect of conjugating a density matrix with diagonal matrices is equivalent to that of applying some Hadamard product to the density matrix. Since Hadamard multiplication is commutative, the result immediately follows. □

We define a quantum channel Φ to be **almost Pauli diagonal**, if it admits a Kraus decomposition with all of its Kraus having the form $\mathbf{K}_j = \mathbf{D}_j\mathbf{P}_j$ where $\mathbf{D}_j$ is a size 2^m diagonal matrix and $\mathbf{P}_j \in \mathcal{P}_m$. **Almost Pauli diagonal** channels are covariant with respect to the m-qubit diagonal Pauli matrices because

$$(\mathbf{D}_j\mathbf{P}_j)(\Lambda\mathbf{W}\Lambda)(\mathbf{P}_j\mathbf{D}_j^\dagger) = \Lambda(\mathbf{D}_j\mathbf{P}_j)\mathbf{W}(\mathbf{P}_j\mathbf{D}_j^\dagger)\Lambda$$

for all Paulis $\mathbf{W}$ and diagonal Paulis $\Lambda \in \{\mathbb{1}, \mathbf{Z}\}^{\otimes m}$. The above equality can be proved by commuting the Λ's 'outwards', by using firstly the fact that Pauli matrices either commute or anti-commute, and secondly the fact that diagonal matrices commute.

All qubit degradable channels are also covariant with respect to the diagonal Pauli matrices because they are almost Pauli diagonal.

Proposition 5.4.5 (Qubit degradable channels)**.** *All qubit degradable channels are covariant with respect to diagonal Pauli matrices.*

Proof. All qubit degradable channels necessarily have Kraus operators of the following form [WPG07, CRS08]

$$\begin{pmatrix} \cos\alpha & 0 \\ 0 & \cos\beta \end{pmatrix}, \quad \begin{pmatrix} 0 & \sin\beta \\ \sin\alpha & 0 \end{pmatrix} = \begin{pmatrix} \sin\beta & 0 \\ 0 & \sin\alpha \end{pmatrix}\mathbf{X}.$$

Hence these channels are of the almost Pauli diagonal form, and since we have shown earlier in this section that almost Pauli diagonal channels are covariant with respect to diagonal Pauli matrices, the result follows. □

The four-dimension amplitude damping channels that we study in this chapter are also almost Pauli diagonal, and hence covariant with respect to diagonal two-qubit Pauli matrices.

Proposition 5.4.6 (Four-dimension amplitude damping channels)**.** *If the linear map $\Phi_{\mathbf{a}}$ defined by (5.3.1) is a quantum channel, then it is also covariant with respect to two-qubit diagonal Pauli matrices.*

Proof. It suffices to show that every Kraus operator of Φ can be written in the form $\mathbf{K}_i = \mathbf{D}_i \mathbf{P}_i$ with $\mathbf{D}_i$ being diagonal and $\mathbf{P}_i$ being a two-qubit Pauli. We define the vectors $|0\rangle, |1\rangle, |2\rangle, |3\rangle$ to be the two qubit states $|0,0\rangle, |0,1\rangle, |1,0\rangle, |1,1\rangle$ respectively. One can verify using equations (5.7.14), (5.7.15), (5.7.12), (5.7.13), (5.7.11) that a suitable choice of the matrices $\mathbf{D}_i$ and $\mathbf{P}_i$ is given by

$$\begin{aligned}
\mathbf{D}_0 &= \sum_{i=0}^{3} a_{0,i}|i\rangle\langle i|, & \mathbf{P}_0 &= \mathbb{1} \otimes \mathbb{1} \\
\mathbf{D}_1 &= a_{1,1}|0\rangle\langle 0| - a_{1,2}|2\rangle\langle 2|, & \mathbf{P}_1 &= \mathbf{Z} \otimes \mathbf{X} \\
\mathbf{D}_2 &= a_{2,1}|0\rangle\langle 0| - a_{2,2}|1\rangle\langle 1|, & \mathbf{P}_2 &= \mathbf{X} \otimes \mathbf{Z} \\
\mathbf{D}_3 &= |0\rangle\langle 0|, & \mathbf{P_3} &= \mathbf{X} \otimes \mathbf{X}.
\end{aligned}$$

□

5.5 New Upper Bounds of the Quantum Capacity

This section presents explicit bounds for the quantum capacity of some low dimension channels. We first introduce the ingredient – a four-dimension degradable amplitude damping channel – on which we apply our main structural results to obtain our upper bounds. the main results of this chapter. The qubit amplitude damping channel was used to give the best known upper bounds for the quantum capacity of the depolarizing channel [SS08], and hence it is natural to expect that four-dimension amplitude damping channels can give good upper bounds for the quantum capacity of some four-dimension quantum channels, such as the four-dimension depolarizing channel (see Figure 5.1).

5.5.1 Four-Dimension Amplitude Damping Channels

Theorem 5.5.1. *When x, y and z satisfy the inequalities in (5.3.6), the quantum capacity of the four-dimension amplitude damping channel $\Phi_{x,y,z}$ defined by (5.3.3) is the optimal* *value of the following concave program with linear constraints.*

$$
\begin{aligned}
\max \quad & \eta(p_1 + p_2 x + p_3 x + p_4 z) + \eta(p_2(1-x) + p_4 y) \\
& + \eta(p_3(1-x) + p_4 y) + \eta(p_4(1-2y-z)) \\
& - \eta(p_1 + p_2(1-x) + p_3(1-x) + p_4(1-2y-z)) - \eta(p_2 x + p_4 y) \\
& - \eta(p_3 x + p_4 y) - \eta(p_4 z) \\
\text{subject to} \quad & p_1 + p_2 + p_3 + p_4 = 1 \\
& p_1, p_2, p_3, p_4 \geq 0
\end{aligned}
\tag{5.5.1}
$$

Proof. For the choice of x, y and z, the linear map $\Phi_{x,y,z}$ is a degradable four-dimension amplitude damping channel (Lemma 5.3.2). By Proposition 5.4.6, the degradable channel $\Phi_{x,y,z}$ is also covariant under the diagonal Pauli matrices. Hence we apply Corollary 5.4.3 to find that the quantum capacity of $\Phi_{x,y,z}$ is its maximum coherent information over all output states of the contraction map associated with the diagonal Pauli matrices. Since such a contraction map is just an m-fold tensor product of the maximally dephasing channel, its output states are all the feasible diagonal states. It remains to show that the optimization problem stated in the theorem is equivalent to the coherent information of the degradable ampliude damping channel $\Phi_{x,y,z}$ maximized over all diagonal input states.

Firstly note that the objective function of (5.5.1) is concave [YHD08] because $\Phi_{x,y,z}$ is a degradable channel. The output state of the four-dimension amplitude damping channel $\Phi_{x,y,z}$ evaluated on the diagonal input state $\text{diag}(p_1, p_2, p_3, p_4)$ is

$$
\begin{aligned}
&(p_1 + p_2 x + p_3 x + p_4 z)|0\rangle\langle 0| + (p_2(1-x) + p_4 y)|1\rangle\langle 1| \\
&+ (p_3(1-x) + p_4 y)|2\rangle\langle 2| + p_4(1-2y-z)|3\rangle\langle 3|.
\end{aligned}
$$

The complementary channel of $\Phi_{x,y,z}$ is also a four-dimension amplitude damping channel (Proposition 5.3.1), and in particular, $\Phi^C_{x,y,z} = \Phi_{1-x,y,1-2y-z}$. Hence the output state of

$\Phi^C_{x,y,z}$ evaluated on the input diagonal state $\mathrm{diag}(p_1, p_2, p_3, p_4)$ is

$$(p_1 + p_2(1-x) + p_3(1-x) + p_4(1-2y-z))|0\rangle\langle 0| + (p_2 x + p_4 y)|1\rangle\langle 1| \\ + (p_3 x + p_4 y)|2\rangle\langle 2| + p_4 z|3\rangle\langle 3|.$$

Therefore using the definition of the coherent information and the von Neumann entropy, the coherent information of the degradable amplitude damping channel $\Phi_{x,y,z}$ evaluated on the diagonal input state $\mathrm{diag}(p_1, p_2, p_3, p_4)$ is the objective function in (5.5.1). □

5.5.2 Two-Qubit Pauli Channels

In this section, we provide upper bounds on the quantum cpacity of some two-qubit Pauli channels. Theorem 5.5.3 gives upper bounds on the quantum capacity of the Pauli twirl of our two-qubit amplitude damping channels. Using Theorem 5.5.3, we obtain upper bounds on the quantum capacities of other channels, including the two-qubit depolarizing channel in Corollary 5.5.4, locally symmetric two-qubit channels in Corollary 5.5.6, and other non-unital channels in the subsequent subsection. Proposition 5.5.2 states the effect of twirling $\Phi_{x,y,z}$ to a Pauli channel.

Proposition 5.5.2. *Let x, y and z be nonnegative real numbers such that $\Phi_{x,y,z}$ defined by (5.3.3) is a four-dimension amplitude damping channel. Then the Pauli-twirl of $\Phi_{x,y,z}$ has* *the Kraus operators*

$$\left(\frac{1+2\sqrt{1-x}+\sqrt{1-2y}}{4}\right)\mathbb{1}\otimes\mathbb{1},$$
$$\left(\frac{1-\sqrt{1-2y}}{4}\right)\mathbf{P}, \qquad \mathbf{P}\in\{\mathbb{1}\otimes\mathbf{Z},\quad \mathbf{Z}\otimes\mathbb{1}\}$$
$$\left|\frac{1-2\sqrt{1-x}+\sqrt{1-2y}}{4}\right|\mathbf{Z}\otimes\mathbf{Z}$$
$$\left|\frac{\sqrt{x}+\sqrt{y}}{4}\right|\mathbf{P}, \qquad \mathbf{P}\in\{\mathbb{1}\otimes\mathbf{X},\quad \mathbb{1}\otimes\mathbf{Y},\quad \mathbf{X}\otimes\mathbb{1},\quad \mathbf{Y}\otimes\mathbb{1}\}$$
$$\left|\frac{\sqrt{x}-\sqrt{y}}{4}\right|\mathbf{P}, \qquad \mathbf{P}\in\{\mathbf{Z}\otimes\mathbf{X},\quad \mathbf{Z}\otimes\mathbf{Y},\quad \mathbf{X}\otimes\mathbf{Z},\quad \mathbf{Y}\otimes\mathbf{Z}\}$$
$$\frac{\sqrt{z}}{2}\mathbf{P}, \qquad \mathbf{P}\in\{\mathbf{X}\otimes\mathbf{X},\quad \mathbf{X}\otimes\mathbf{Y},\quad \mathbf{Y}\otimes\mathbf{X},\quad \mathbf{Y}\otimes\mathbf{Y}\}$$

with probability of weight i Pauli's being P_i where

$$P_0=\left(\frac{1+2s_1+s_2}{4}\right)^2$$
$$P_1=2\left(\frac{1-s_2}{4}\right)^2+\left(\frac{\sqrt{x}+\sqrt{y}}{2}\right)^2$$
$$P_2=\left(\frac{1-2s_1+s_2}{4}\right)^2+\left(\frac{\sqrt{x}-\sqrt{y}}{2}\right)^2+\frac{z}{4}$$

with $s_1=\sqrt{1-x}$ and $s_2=\sqrt{1-2y-z}$.

Proof. Substituting the equations (5.7.1) to (5.7.10) into the equations (5.3.4), we can express the Kraus operators of $\Phi_{x,y,z}$ in the Pauli basis to get

$$\begin{aligned}
\mathbf{A}_1 &= \frac{\sqrt{x}+\sqrt{y}}{4}(\mathbb{1}\otimes\mathbf{X}+i\mathbb{1}\otimes\mathbf{Y})+\frac{\sqrt{x}-\sqrt{y}}{4}(\mathbf{Z}\otimes\mathbf{X}+i\mathbf{Z}\otimes\mathbf{Y})\\
\mathbf{A}_2 &= \frac{\sqrt{x}+\sqrt{y}}{4}(\mathbf{X}\otimes\mathbb{1}+i\mathbf{Y}\otimes\mathbb{1})+\frac{\sqrt{x}-\sqrt{y}}{4}(\mathbf{X}\otimes\mathbf{Z}+i\mathbf{Y}\otimes\mathbf{Z})\\
\mathbf{A}_3 &= \frac{\sqrt{z}}{4}(\mathbf{X}\otimes\mathbf{X}-\mathbf{Y}\otimes\mathbf{Y}+i(\mathbf{X}\otimes\mathbf{Y}+\mathbf{Y}\otimes\mathbf{X}))\\
\mathbf{A}_0 &= \frac{1+2\sqrt{1-x}+\sqrt{1-2y}}{4}\mathbb{1}\otimes\mathbb{1}+\frac{1-\sqrt{1-2y}}{4}(\mathbf{Z}\otimes\mathbb{1}+\mathbb{1}\otimes\mathbf{Z})\\
&\quad+\frac{1-2\sqrt{1-x}+\sqrt{1-2y}}{4}\mathbf{Z}\otimes\mathbf{Z}.
\end{aligned}$$

Applying (5.4.1) on the above decomposition of the Kraus operators of $\Phi_{x,y,z}$ in the Pauli basis, we can derive the probabilities that the Pauli twirl of $\Phi_{x,y,z}$ applies each two-qubit Pauli matrix. Subsequently, we obtain the probabilities of the Pauli twirl of $\Phi_{x,y,z}$ having Pauli errors of the weights zero, one and two. $\square$

Given the above proposition, we can determine the explicit form of four-dimension quantum channels for which we have non-trivial upper bounds on the quantum capacity. These non-trivial upper bounds come from applying Theorem 5.4.2 on our degradable four-dimension amplitude damping channels.

Theorem 5.5.3. *Let x, y and z satisfy the inequalities in (5.3.6). Then there is a channel $\mathcal{N}_{x,y,z}$, a degradable extension of the Pauli twirl of $\Phi_{x,y,z}$, such that the quantum capacity of the Pauli twirl of $\Phi_{x,y,z}$ is at most*

$$\begin{aligned}
Q(\mathcal{N}_{x,y,z}) \le\ &\eta\left(\frac{1+2x+z}{4}\right)+2\eta\left(\frac{1-x+y}{4}\right)+\eta\left(\frac{1-2y-z}{4}\right)\\
&-\eta\left(1-\frac{2x+2y+z}{4}\right)-2\eta\left(\frac{x+y}{4}\right)-\eta\left(\frac{z}{4}\right). \qquad (5.5.2)
\end{aligned}$$

Proof. The quantum channel $\Phi_{x,y,z}$ defined in (5.3.3) is degradable for the stipulated values of x, y and z by Lemma 5.3.2. The two-qubit Pauli-contraction channel maps all input states to the maximally mixed state. Hence Theorem 5.4.2 implies that there exists a degradable extension $\mathcal{N}_{x,y,z}$ of the Pauli-twirl of $\Phi_{x,y,z}$, such that the quantum capacity of the Pauli-twirl of $\Phi_{x,y,z}$ is at most the quantum capacity of $\mathcal{N}_{x,y,z}$, which is at most the coherent information of $\Phi_{x,y,z}$ evaluated on the maximally mixed state. □

Depolarizing Channels

Depolarizing channels are often used as toy-models for the noisy quantum channel. In the Kraus representation, an m-qubit depolarizing channel applies the identity m-qubit Pauli with probability $1-p$, and each of the other non-trivial m-qubit Paulis with probability $\frac{p}{2^{2m}-1}$. Here, p quantifies the depolarizing probability, and varies between 0 and 1. The m-qubit depolarizing channel is a d-dimension depolarizing channel with $d = 2^m$. The d-dimension depolarizing channel of depolarizing probability p can be described as a quantum channel that maps an m-qubit input state to a convex combination of the maximally mixed m-qubit state and the input state, and is defined as

$$\mathcal{D}_{p,d}(\rho) = \rho \left(1 - p\frac{d^2-1}{d^2}\right) + \frac{\mathbb{1}_d}{d}\left(p\frac{d^2-1}{d^2}\right)\mathrm{Tr}(\rho).$$

Upper bounds [Cer00, Rai99a, Rai01, SSW08, SS08] and lower bounds [AC97, DSS98, SS07, FW08] on the quantum capacity of qubit depolarizing channels, the simplest type of depolarizing channels, have been studied. However these bounds are not tight when the depolarizing probability is in the interval $(0, \frac{1}{4})$. Even less is known about the quantum capacity of higher dimension depolarizing channels. The goal of this section is to tighten the upper bounds for the quantum capacity of d-dimension depolarizing channels.

The obvious upper bounds for the quantum capacity of the depolarizing channel comes using Cerf's no-cloning bounds [Cer00] for depolarizing channels with Smith and Smolin's result (Theorem 5.2.1). By Cerf's result, a d-dimension depolarizing channel of depolarizing probability p is both degradable and anti-degradable when

$$p = \frac{d}{2d+2}\frac{d^2-1}{d^2} = \frac{d^2-1}{2d(d+1)} = \frac{d-1}{2d}. \tag{5.5.3}$$

Hence applying Smith and Smolin's technique of degradable extensions [SS08] immediately gives the upper bound of

$$Q(\mathcal{D}_{p,d}) \leq (\log_2 d)\left(1 - p\frac{2d}{d-1}\right) \tag{5.5.4}$$

for depolarizing probability $0 \leq p \leq \frac{2d}{d-1}$. We call this upper bound the no-cloning upper bound for the quantum capacity of the depolarizing channel.

An obvious lower bound for the quantum capacity of the d-dimension depolarizing channel of noise stength p is $\max(0, \log_2 d + (1-p)\log_2(1-p) + p\log_2(\frac{p}{d^2-1})$, which is its coherent information evaluated on the maximally mixed state.

Picking a d-dimension channel to twirl to get improvements over the no-cloning upper bounds for the quantum capacity of the depolarizing channel is non-trivial for two reasons. Firstly, we have to verify that the channel that we pick is degradable, and checking for the degradability of a quantum channel is not an entirely straightforward problem. Secondly, the coherent information of the channel evaluated on the maximally mixed state has to be sufficiently low, in order to produce an improvement on the no-cloning upper bound of the quantum capacity of the depolarizing channel.

The four-dimension amplitude damping channel $\Phi_{x,0,x}$ can be used to improve on the no-cloning upper bound of the four-dimension depolarizing channel (see Figure 5.1).

With the d-dimension uniformly amplitude damping channel, we can obtain non-trivial upper bounds for the quantum capacity of the d-dimension depolarizing channels, which is the statement of the following corollary.

Corollary 5.5.4 (m-qubit Depolarizing Channels)**.** *Let $d = 2^m$ be the dimension of our m-qubit depolarizing channel of depolarizing probability $0 \leq p \leq \frac{d-1}{2d}$. the quantum capacity of our d-dimension depolarizing channel with depolarizing probability p is at most the convex hull (see Figure 5.1)*

$$\operatorname{conv}\left(I_{coh}\left(\mathcal{A}_{\gamma,d}, \frac{\mathbb{1}}{d}\right), (\log_2 d)\left(1 - p\frac{2d}{d-1}\right)\right)$$

where

$$I_{coh}\Big(\mathcal{A}_{\gamma,d},\frac{\mathbb{1}}{d}\Big)=\eta\Big(\frac{1+(d-1)\gamma}{d}\Big)+(d-1)\eta\Big(\frac{1-\gamma}{d}\Big)-\eta\Big(1-\frac{(d-1)\gamma}{d}\Big)-(d-1)\eta\Big(\frac{\gamma}{d}\Big)$$

and $\gamma=\frac{2d}{(d-1)^2}(\sqrt{1-p}-(1-\frac{pd}{2})..$

Proof. The only Kraus operator of the d-dimension uniformly amplitude damping channel with damping parameter γ that is not traceless is $\mathbf{A}_0$, which has a trace of $1+(d-1)\sqrt{1-\gamma}$. Hence using equation (5.4.1), the complete Clifford-twirl of our uniformly amplitude damping is the m-qubit depolarizing channel of depolarizing probability p, where

$$1-p=\left(\frac{1+(d-1)\sqrt{1-\gamma}}{d}\right)^2.$$

Taking the non-negative solution for γ of the above equation for the feasible values of p and d, we get

$$\gamma=\frac{2d}{(d-1)^2}\Big(\sqrt{1-p}-(1-\frac{pd}{2})\Big)$$

as required in our corollary. Hence using our Theorem 5.4.2 pertaining to twirling of degradable channels and the contraction channel, there is a degradable extension of the d-dimension depolarizing channel $\mathcal{D}_{p,d}$ with an upper bound that is the coherent information of the d-dimension uniformly amplitude damping channel with damping paramater γ evaluated on the maximally mixed state. Taking convex combinations of the upper bound of the quantum capacity of the twirl of our amplitude damping channels and the no-cloning upper bounds then gives the result. □

Locally Clifford Covariant Channels

To obtain locally symmetric Pauli channels, we introduce the notion of localized Clifford twirling. Instead of twirling our channel over the entire Clifford group over all the qubits [DLT02], we can twirl the channel with respect to the Clifford group for individual qubits

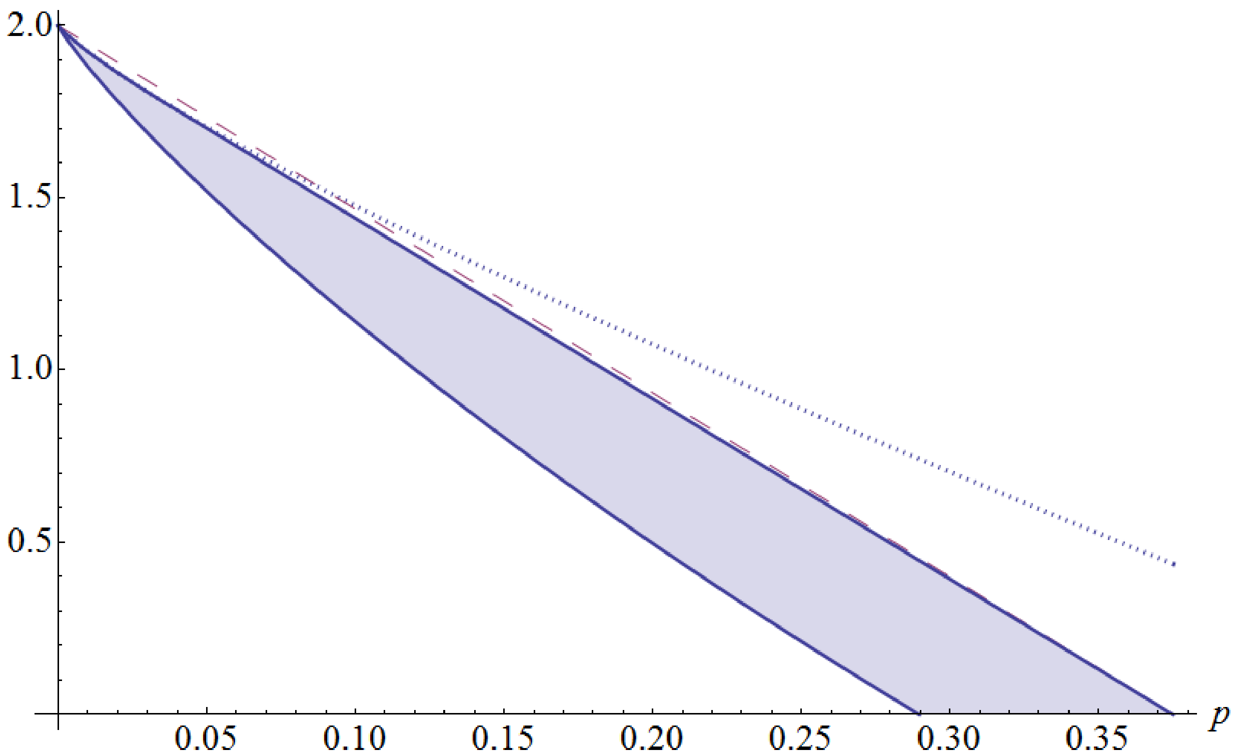

Figure 5.1: This figure depicts bounds on the quantum capacity for the four-dimension depolarizing channel of depolarizing probability p, where the upper bounds are given in Corollary 5.5.4. The upper and lower boundaries of the shaded region depict the upper and lower bounds for the quantum capacity of the four-dimension depolarizing channel respectively. The dotted line is an upper bound that comes from Cerf's no-cloning bound, and the dashed line is an upper bound that comes from twirling our four-dimension amplitude damping channel.

independently. The material below is an explicit discussion on the notion of localized Clifford twirling.

Now define the set of non-trivial Pauli matrices to be $\mathcal{P}_1^* := \{\mathbf{X}, \mathbf{Y}, \mathbf{Z}\}$. We study a set of automorphisms on the non-trivial Pauli matrices. To define this set of automorphisms, we first define a Hermitian and traceless qubit operator

$$\mathbf{H}_{\tau_1,\tau_2} := \frac{\tau_1 + \tau_2}{\sqrt{2}}$$

for all non-trivial Pauli matrices τ_1 and τ_2, which is just the Hadamard matrix in an arbitary Pauli basis. For all non-trivial Pauli matrices $\mathbf{W}$, conjugation of $\mathbf{W}$ with $\mathbf{H}_{\tau_1,\tau_2}$ gives the following.

$$\mathbf{H}_{\tau_1,\tau_2}\mathbf{W}\mathbf{H}_{\tau_1,\tau_2} = \begin{cases} \tau_1 & , \quad \mathbf{W} = \tau_2 \\ \tau_2 & , \quad \mathbf{W} = \tau_1 \\ -\mathbf{W} & , \quad \mathbf{W} \notin \{\tau_1, \tau_2\} \end{cases}$$

Hence the automorphism associated with the generalized Hadamards $\mathbf{H}_{\tau_1,\tau_2}$ on the set of non-trivial Pauli matrices swaps τ_1 and τ_2. The size of the set of all automorphisms on the set of non-trivial Pauli matrices is the size of the symmetric group of order 3, which is 6. Hence we consider the set

$$\mathcal{B} := \{\mathbb{1}, \mathbf{H}_{\mathbf{X},\mathbf{Y}}, \mathbf{H}_{\mathbf{X},\mathbf{Z}}, \mathbf{H}_{\mathbf{Y},\mathbf{Z}}, \mathbf{H}_{\mathbf{X},\mathbf{Z}}\mathbf{H}_{\mathbf{X},\mathbf{Y}}, \mathbf{H}_{\mathbf{X},\mathbf{Y}}\mathbf{H}_{\mathbf{X},\mathbf{Z}}\} \tag{5.5.5}$$

with six qubit operators, each operator corresponding to a distinct automorphism of the set of non-trivial Pauli matrices. For all $\mathbf{P}, \mathbf{V} \in \mathcal{P}_1$, observe that

$$\frac{1}{6}\sum_{\mathbf{B}\in\mathcal{B}}(\mathbf{B}^\dagger\mathbf{P}\mathbf{B})\mathbf{V}(\mathbf{B}^\dagger\mathbf{P}\mathbf{B}) = \begin{cases} \frac{1}{3}\sum_{\mathbf{P}'\in\mathcal{P}_1^*}\mathbf{P}'\mathbf{V}\mathbf{P}' & , \quad \mathbf{P} \in \mathcal{P}_1^* \\ \mathbf{V} & , \quad \mathbf{P} = \mathbb{1} \end{cases} . \tag{5.5.6}$$

Lemma 5.5.5 (Localized Clifford Twirling)**.** *Let $\mathcal{N}$ be a two-qubit Pauli diagonal channel that applies the two-qubit Paulis $\mathbf{P}\otimes\mathbf{P}'$ with probabilities $a_{\mathbf{P}\otimes\mathbf{P}'}$. Then $(\mathcal{N}_{\ltimes \mathbb{1}\otimes\mathcal{B}\rtimes})_{\ltimes\mathcal{B}\otimes\mathbb{1}\rtimes}$ is a two-qubit Pauli channel that applies the identity Pauli operator with probability $a_{\mathbb{1}\otimes\mathbb{1}}$, each weight one Pauli operator supported on the first and second qubits with probabilities $\sum_{\mathbf{R}\in\mathcal{P}_1^*}\frac{1}{3}a_{\mathbf{R}\otimes\mathbb{1}}$ and $\sum_{\mathbf{R}\in\mathcal{P}_1^*}\frac{1}{3}a_{\mathbb{1}\otimes\mathbf{R}}$ respectively, and each weight two Pauli operator with proba-*

bility $\sum_{\mathbf{R},\mathbf{R}'\in\mathcal{P}_1^*} \frac{1}{9} a_{\mathbf{R}\otimes\mathbf{R}'}$.

Proof. Let $\mathbf{V}$ and $\mathbf{W}$ be single qubit Pauli matrices. Then using (5.5.6) we get

$$\begin{aligned}
\mathcal{N}_{\wr\mathcal{B}\otimes\mathbb{1}\wr}(\mathbf{V}\otimes\mathbf{W}) =& \frac{1}{6}\sum_{\mathbf{B}\in\mathcal{B}}\sum_{\mathbf{P},\mathbf{P}'\in\mathcal{P}_1} \mathbf{B}^\dagger\mathbf{P}\mathbf{B}\mathbf{V}\mathbf{B}^\dagger\mathbf{P}\mathbf{B}\otimes\mathbf{P}'\mathbf{W}\mathbf{P}' a_{\mathbf{P}\otimes\mathbf{P}'} \\
=& \frac{1}{6}\sum_{\mathbf{P},\mathbf{P}'\in\mathcal{P}_1}\Big(\sum_{\mathbf{B}\in\mathcal{B}}(\mathbf{B}^\dagger\mathbf{P}\mathbf{B})\mathbf{V}(\mathbf{B}^\dagger\mathbf{P}\mathbf{B})\Big)\otimes\mathbf{P}'\mathbf{W}\mathbf{P}' a_{\mathbf{P}\otimes\mathbf{P}'} \\
=& \sum_{\mathbf{P}'\in\mathcal{P}_1}\mathbf{V}\otimes\mathbf{P}'\mathbf{W}\mathbf{P}' a_{\mathbb{1}\otimes\mathbf{P}'} + \frac{1}{3}\sum_{\mathbf{P}\in\mathcal{P}_1^*}\Big(\sum_{\mathbf{R}\in\mathcal{P}_1^*}\mathbf{R}\mathbf{V}\mathbf{R}\Big)\otimes\sum_{\mathbf{P}'\in\mathcal{P}_1}\mathbf{P}'\mathbf{W}\mathbf{P}' a_{\mathbf{P}\otimes\mathbf{P}'}.
\end{aligned}$$

By rearranging the terms above, we get

$$\mathcal{N}_{\wr\mathcal{B}\otimes\mathbb{1}\wr}(\mathbf{V}\otimes\mathbf{W}) = \mathbf{V}\otimes\sum_{\mathbf{P}'\in\mathcal{P}_1}\mathbf{P}'\mathbf{W}\mathbf{P}' a_{\mathbb{1}\otimes\mathbf{P}'} + \Big(\sum_{\mathbf{R}\in\mathcal{P}_1^*}\mathbf{R}\mathbf{V}\mathbf{R}\Big)\otimes\sum_{\mathbf{P}'\in\mathcal{P}_1}\mathbf{P}'\mathbf{W}\mathbf{P}'\sum_{\mathbf{P}\in\mathcal{P}_1^*}\frac{a_{\mathbf{P}\otimes\mathbf{P}'}}{3}.$$

Similarly,

$$\begin{aligned}
(\mathcal{N}_{\wr\mathcal{B}\otimes\mathbb{1}\wr})_{\wr\mathbb{1}\otimes\mathcal{B}\wr}(\mathbf{V}\otimes\mathbf{W}) =& \mathbf{V}\otimes\frac{1}{6}\sum_{\mathbf{B}\in\mathcal{B}}\sum_{\mathbf{P}'\in\mathcal{P}_1}(\mathbf{B}^\dagger\mathbf{P}'\mathbf{B})\mathbf{W}(\mathbf{B}^\dagger\mathbf{P}'\mathbf{B}) a_{\mathbb{1}\otimes\mathbf{P}'} \\
&+\Big(\sum_{\mathbf{R}\in\mathcal{P}_1^*}\mathbf{R}\mathbf{V}\mathbf{R}\Big)\otimes\frac{1}{6}\sum_{\mathbf{B}\in\mathcal{B}}\sum_{\mathbf{P}'\in\mathcal{P}_1}(\mathbf{B}^\dagger\mathbf{P}'\mathbf{B})\mathbf{W}(\mathbf{B}^\dagger\mathbf{P}'\mathbf{B})\Big(\sum_{\mathbf{P}\in\mathcal{P}_1^*}\frac{a_{\mathbf{P}\otimes\mathbf{P}'}}{3}\Big) \\
=& a_{\mathbb{1}\otimes\mathbb{1}}\mathbf{V}\otimes\mathbf{W} + \mathbf{V}\otimes\Big(\sum_{\mathbf{R}'\in\mathcal{P}_1^*}\mathbf{R}'\mathbf{W}\mathbf{R}'\Big)\Big(\sum_{\mathbf{P}'\in\mathcal{P}_1^*}\frac{a_{\mathbb{1}\otimes\mathbf{P}'}}{3}\Big) \\
&+\Big(\sum_{\mathbf{R}\in\mathcal{P}_1^*}\mathbf{R}\mathbf{V}\mathbf{R}\Big)\otimes\mathbf{W}\Big(\sum_{\mathbf{P}\in\mathcal{P}_1^*}\frac{a_{\mathbf{P}\otimes\mathbb{1}}}{3}\Big) \\
&+\Big(\sum_{\mathbf{R}\in\mathcal{P}_1^*}\mathbf{R}\mathbf{V}\mathbf{R}\Big)\otimes\Big(\sum_{\mathbf{R}'\in\mathcal{P}_1^*}\mathbf{R}'\mathbf{W}\mathbf{R}'\Big)\sum_{\mathbf{P},\mathbf{P}'\in\mathcal{P}_1^*}\frac{a_{\mathbf{P}\otimes\mathbf{P}'}}{9}.
\end{aligned}$$

□

Corollary 5.5.6 gives upper bounds on the quantum capacity of locally symmetric Pauli channels. Such channels are simple examples of two-qubit quantum channels that need not have a tensor product structure.

Corollary 5.5.6 (Locally Symmetric Two-qubit Pauli Channels)**.** *When x, y and z satisfy the inequalities in (5.3.6), the quantum capacity of a two-qubit Pauli channel that applies the weight zero, weight one, and weight two Pauli operators with probabilities q_0, q_1 and q_2 respectively has a quantum capacity at most the right hand side of the inequality (5.5.2), and is depicted in Figure 5.2, where*

$$q_0 = \left(\frac{1 + 2\sqrt{1-x} + \sqrt{1-2y-z}}{4}\right)^2$$
$$q_1 = \frac{(1 - \sqrt{1-2y-z})^2}{8} + \frac{(\sqrt{x} + \sqrt{y})^2}{4}$$
$$q_2 = \frac{(1 - 2\sqrt{1-x} + \sqrt{1-2y-z})^2}{16} + \frac{(\sqrt{x} - \sqrt{y})^2}{4} + \frac{z}{4}.$$

Proof. Using Proposition 5.5.2 for Pauli-twirling and Lemma 5.5.5 for localized Clifford twirling, the $(\mathbb{1} \otimes \mathcal{B})$-$(\mathcal{B} \otimes \mathbb{1})$-twirl of the Pauli twirl of the degradable four-dimension amplitude damping channel $\Phi_{x,y,z}$ is the Pauli channel as stipulated in our corollary. By Theorem 5.5.3, the Pauli-twirl of the four-dimension amplitude damping channel has a degradable extension $\mathcal{N}_{x,y,z}$ with quantum capacity at most the right hand side of the inequality (5.5.2). Since channel twirling is a special way of taking convex combinations of channels, we can apply Theorem 5.2.1 on the degradable extension $\mathcal{N}_{x,y,z}$ to find that there exists a degradable extension $\mathcal{N}_{\text{ext}}$ of the $(\mathbb{1} \otimes \mathcal{B})$-$(\mathcal{B} \otimes \mathbb{1})$-twirl of the Pauli twirl of the degradable four-dimension amplitude damping channel $\Phi_{x,y,z}$ constructed from flagged extensions of $\mathcal{N}_{x,y,z}$, where the quantum capacity of $\mathcal{N}_{\text{ext}}$ is no more than the quantum capacity of $\mathcal{N}_{x,y,z}$. □

5.5.3 Non-unital Channels

We give upper bounds for the quantum capacity of certain non-unital and non-degradable channels [RSW02, Fuk05, GLR05, CGMR08]. Non-unital channels cannot be Pauli-twirled channels, because Pauli-twirled channels are necessarily unital. However we can still

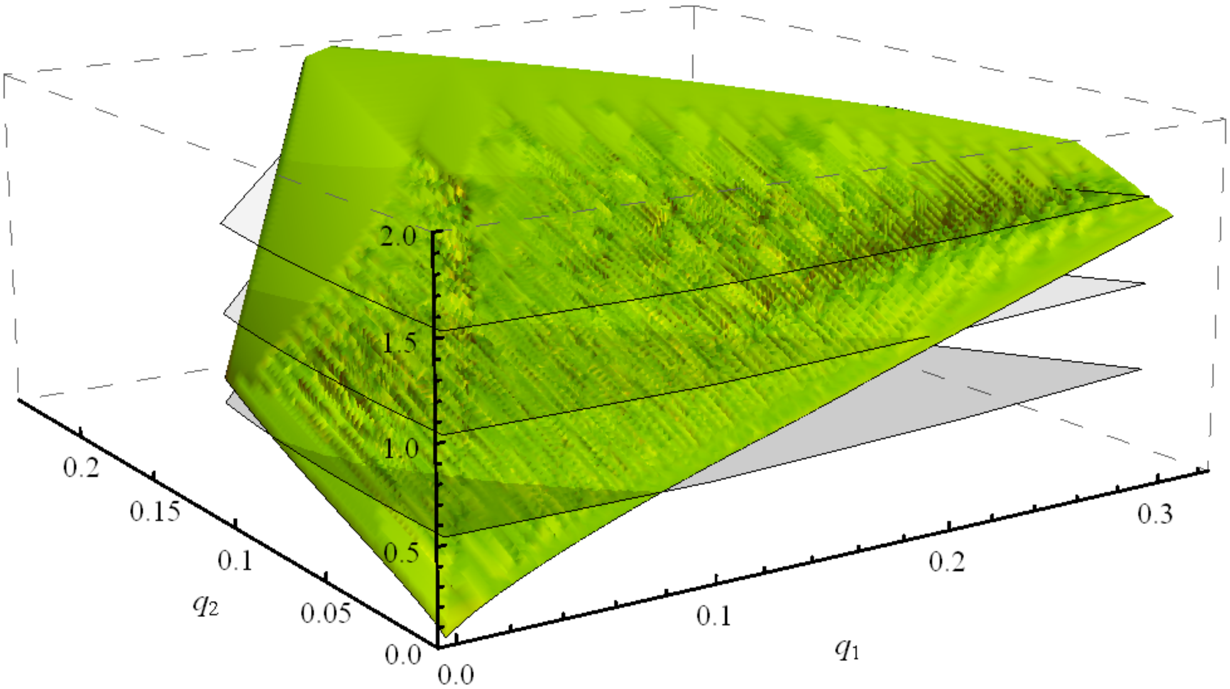

Figure 5.2: The concave roof of the depicted dimpled surface is our lower bound for two minus the quantum capacity of the locally-symmetric Pauli channel (see Corollary 5.5.6). The locally symmetric channel applies some weight one and weight two Pauli error with probabilities q_1 and q_2 respectively.

construct upper bounds on non-unital channels that are convex combinations of twirled degradable channels. In this case, it is necessary to use twirls weaker than the Pauli-twirl.

In this section, we illustrate how one can obtain upper bounds for the quantum capacity of the shifted qubit depolarizing channel, which is the content of Corollary 5.5.7. The shifted depolarizing channel [Fuk05, GLR05] of dimension d is defined by

$$\mathcal{D}_{p,d,\mathbf{A}}(\rho) := \mathcal{D}_{p,d}(\rho) + \mathbf{A}\mathrm{Tr}(\rho) \tag{5.5.7}$$

where $\mathbf{A}$ is a d-dimension Hermitian traceless matrix such that $\mathcal{D}_{p,d,\mathbf{A}}$ is a completely positive map and hence still a quantum channel. Here, the operator $\mathbf{A}$ quantifies the amount by which the depolarizing channel $\mathcal{D}_{p,d}$ is shifted. The shifted depolarizing channel $\mathcal{D}_{p,d,\mathbf{A}}$ can also be interpreted as a channel that transmits a state ρ perfectly with probability $1 - p\frac{d^2-1}{d^2}$, and with probability $p\frac{d^2-1}{d^2}$ transmits the state $(\frac{\mathbb{1}_d}{d} + \frac{d^2\mathbf{A}}{p(d^2-1)})\,\mathrm{Tr}(\rho)$. In the following corollary, we provide explicit upper bounds for the quantum capacity of the shifted depolarizing channel (see also Figure 5.3).

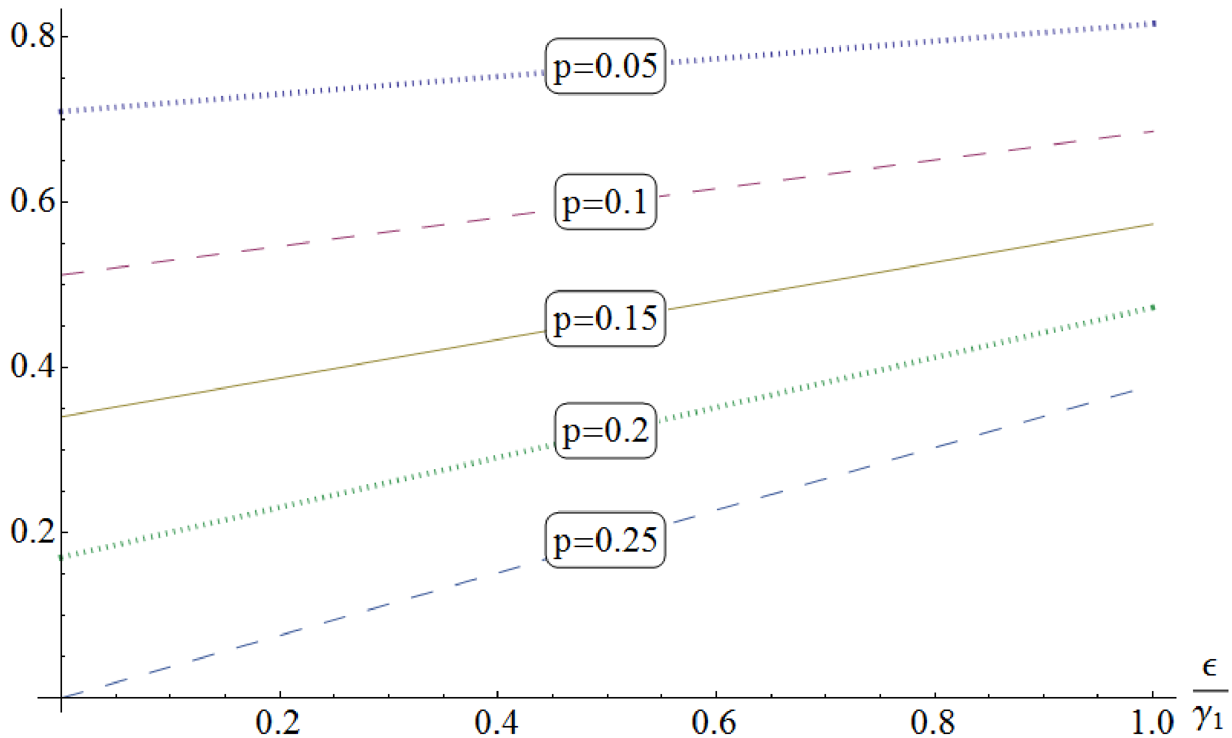

Figure 5.3: Upper bounds on the shifted qubit depolarizing channel $\mathcal{D}_{p,2,\epsilon\mathbf{Z}}$ are depicted in this plot for different values of depolarizing probability p, and γ_1 is as defined in Corollary 5.5.7.

Corollary 5.5.7 (Shifted Qubit Depolarizing Channels)**.** *Let $0 < p \leq \frac{1}{4}$ be the depolarizing probability and non-negative number ϵ quantify the amount of shifting for a shifted qubit depolarizing channel $\mathcal{D}_{p,2,\epsilon\mathbf{Z}}$. Let $\gamma_1 = \sqrt{16-9p} + \frac{9p-16}{4}$, and $\gamma_2 = 4\sqrt{1-p}(1-\sqrt{1-p})$ be amplitude damping parameters dependent on the depolarizing probability p. Let Φ_{γ_1} be the qubit amplitude damping channel with Kraus operators $\gamma_1|0\rangle\langle 1|$ and $|0\rangle\langle 0| + \sqrt{1-\gamma_1}|1\rangle\langle 1|$*

and

$$F_{ss}(p) := \operatorname{conv}\left(1 - H_2(p), H_2\Big(\frac{1-\gamma_2}{2}\Big) - H_2\Big(\frac{\gamma_2}{2}\Big), 1 - 4p\right)$$

be Smith and Smolin's upper bound for the quantum capacity of the qubit depolarizing channel [SS08], where $H_2(q) := \eta(q) + \eta(1-q)$ *is the binary entropy function. Then for* $\epsilon \le \gamma_1$*, the quantum capacity of the shifted qubit depolarizing channel* $\mathcal{D}_{p,2,\epsilon \mathbf{Z}}$ *is at most*

$$\epsilon\gamma_1^{-1} I_{coh}(\Phi_{\gamma_1}, \{\mathbb{1}, \mathbf{Z}\}_{\triangleright}) + (1 - \epsilon\gamma_1^{-1}) F_{ss}(p).$$

Proof. Let $\mathcal{U}$ be the set of unitaries $\{\mathbb{1}, \mathbf{H}_{\mathbf{X},\mathbf{Z}}, \mathbf{H}_{\mathbf{Y},\mathbf{Z}}\}$. Then the $\mathcal{U}$-twirl of Φ_{γ_1} is a shifted depolarizing channel, in the sense that

$$\begin{aligned}(\Phi_{\gamma_1})_{\ltimes \mathcal{U} \rtimes}(\mathbb{1}) &= \mathbb{1} + \gamma_1 \mathbf{Z} \\ (\Phi_{\gamma_1})_{\ltimes \mathcal{U} \rtimes}(\mathbf{P}) &= \frac{2\sqrt{1-\gamma_1} + (1-\gamma_1)}{3}\mathbf{P}\end{aligned}$$

for all non-trivial Paulis $\mathbf{P} \in \{\mathbf{X}, \mathbf{Y}, \mathbf{Z}\}$. Thus the $\mathcal{U}$-twirl of Φ_{γ_1} is the qubit depolarizing channel of depolarizing probability $p = \frac{4}{3}\left(1 - \frac{2\sqrt{1-\gamma_1}+(1-\gamma_1)}{3}\right)$ shifted by $\gamma_1 \mathbf{Z}$. Solving γ_1 in terms of p, and imposing the condition that γ_1 is non-negative for $p \in (0, \frac{1}{4}]$, we get $\gamma_1 = \sqrt{16 - 9p} + \frac{9p-16}{4}$, which is less than γ_2 on the interval $p \in (0, \frac{1}{4}]$. Then the shifted qubit depolarizing channel is the following convex combination of channels the $\mathcal{U}$-twirled qubit amplitude damping channel and the qubit depolarizing channel. Hence

$$\mathcal{D}_{p,2,\epsilon \mathbf{Z}} = \epsilon\gamma_1^{-1}(\Phi_{\gamma_1})_{\ltimes \mathcal{U} \rtimes} + (1 - \epsilon\gamma_1^{-1})\mathcal{D}_{p,2}.$$

Hence applying Theorem 5.2.1 on the degradable extensions of the $\mathcal{U}$-twirl of Φ_{γ_1} and Smith and Smolin's degradable extension of the qubit depolarizing channel gives the result. □

Similarly, it is also possible to obtain upper bounds on the quantum capacity of some four-dimension non-unital channels. For example, let $\mathcal{U}$ be any set of four-dimension unitary matrices, and x, y, z and x', y', z' satisfy the inequalities in (5.3.6). Then the quantum capacity of the convex combination of the $\mathcal{U}$-twirl of the four-dimension amplitude damping channel $\Phi_{x,y,z}$ and the Pauli-twirl of another two qubit amplitude damping channel $\Phi_{x',y',z'}$ is at most the convex combination of the coherent information of $\Phi_{x,y,z}$ and the coherent information of $\Phi_{x',y',z'}$ evaluated on the maximally mixed state.

5.6 Discussions

In this chapter, we have generalized Smith and Smolin's result (Lemma 8 of [SS08]) to our Theorem 5.4.2, which is the main tool that we use to provide new upper bounds for the quantum capacity of several families of quantum channels. We provide upper bounds on the quantum capacity of some non-unital channels – the shifted depolarizing channels, and some shifted two-qubit Pauli channels – thereby demonstrating the potential of our sharpening of Smith and Smolin's technique.

Upper bounds for the quantum-capacities of the two-qubit locally symmetric channels were originally investigated in this chapter in hope of improving the upper bound on the quantum capacity of the qubit-depolarizing channel. However numerical evidence indicates that this is not impossible, and improving on the upper bound of the qubit depolarizing channel remains an open problem.

5.7 Miscellaneous

Observe that

$$4|0\rangle\langle 3| = \mathbf{X}\otimes\mathbf{X} - \mathbf{Y}\otimes\mathbf{Y} + i(\mathbf{X}\otimes\mathbf{Y} + \mathbf{Y}\otimes\mathbf{X}) \tag{5.7.1}$$

$$4|1\rangle\langle 2| = \mathbf{X}\otimes\mathbf{X} + \mathbf{Y}\otimes\mathbf{Y} + i(-\mathbf{X}\otimes\mathbf{Y} + \mathbf{Y}\otimes\mathbf{X}) \tag{5.7.2}$$

$$4|0\rangle\langle 2| = \mathbf{X}\otimes\mathbb{1} + \mathbf{X}\otimes\mathbf{Z} + i(\mathbf{Y}\otimes\mathbb{1} + \mathbf{Y}\otimes\mathbf{Z}) \tag{5.7.3}$$

$$4|1\rangle\langle 3| = \mathbf{X}\otimes\mathbb{1} - \mathbf{X}\otimes\mathbf{Z} + i(\mathbf{Y}\otimes\mathbb{1} - \mathbf{Y}\otimes\mathbf{Z}) \tag{5.7.4}$$

$$4|0\rangle\langle 1| = \mathbb{1}\otimes\mathbf{X} + \mathbf{Z}\otimes\mathbf{X} + i(\mathbb{1}\otimes\mathbf{Y} + \mathbf{Z}\otimes\mathbf{Y}) \tag{5.7.5}$$

$$4|2\rangle\langle 3| = \mathbb{1}\otimes\mathbf{X} - \mathbf{Z}\otimes\mathbf{X} + i(\mathbb{1}\otimes\mathbf{Y} - \mathbf{Z}\otimes\mathbf{Y}). \tag{5.7.6}$$

Also

$$4|0\rangle\langle 0| = \mathbb{1}\otimes\mathbb{1} + \mathbb{1}\otimes\mathbf{Z} + \mathbf{Z}\otimes\mathbb{1} + \mathbf{Z}\otimes\mathbf{Z} \tag{5.7.7}$$

$$4|1\rangle\langle 1| = \mathbb{1}\otimes\mathbb{1} - \mathbb{1}\otimes\mathbf{Z} + \mathbf{Z}\otimes\mathbb{1} - \mathbf{Z}\otimes\mathbf{Z} \tag{5.7.8}$$

$$4|2\rangle\langle 2| = \mathbb{1}\otimes\mathbb{1} + \mathbb{1}\otimes\mathbf{Z} - \mathbf{Z}\otimes\mathbb{1} - \mathbf{Z}\otimes\mathbf{Z} \tag{5.7.9}$$

$$4|3\rangle\langle 3| = \mathbb{1}\otimes\mathbb{1} - \mathbb{1}\otimes\mathbf{Z} - \mathbf{Z}\otimes\mathbb{1} + \mathbf{Z}\otimes\mathbf{Z}. \tag{5.7.10}$$

We can also rewrite the above matrices in the following form.

$$|0\rangle\langle 3| = (|0\rangle\langle 0|)(\mathbf{X}\otimes\mathbf{X}) \tag{5.7.11}$$

$$|0\rangle\langle 2| = (|0\rangle\langle 0|)(\mathbf{X}\otimes\mathbf{Z}) \tag{5.7.12}$$

$$|1\rangle\langle 3| = (-|1\rangle\langle 1|)(\mathbf{X}\otimes\mathbf{Z}) \tag{5.7.13}$$

$$|0\rangle\langle 1| = (|0\rangle\langle 0|)(\mathbf{Z}\otimes\mathbf{X}) \tag{5.7.14}$$

$$|2\rangle\langle 3| = (-|2\rangle\langle 2|)(\mathbf{Z}\otimes\mathbf{X}) \tag{5.7.15}$$

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