

# Angles, Majorization, Wielandt Inequality and Applications

by

Minghua Lin

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## Abstract

In this thesis we revisit two classical definitions of angle in an inner product space: real-part angle and Hermitian angle. Special attention is paid to Kreĭn's inequality and its analogue. Some applications are given, leading to a simple proof of a basic lemma for a trace inequality of unitary matrices and also its extension. A brief survey on recent results of angles between subspaces is presented. This naturally brings us to the world of majorization. After introducing the notion of majorization, we present some classical as well as recent results on eigenvalue majorization. Several new norm inequalities are derived by making use of a powerful decomposition lemma for positive semidefinite matrices. We also consider coneigenvalue majorization. Some discussion on the possible generalization of the majorization bounds for Ritz values is presented. We then turn to a basic notion in convex analysis, the Legendre-Fenchel conjugate. The convexity of a function is important in finding the explicit expression of the transform for certain functions. A sufficient convexity condition is given for the product of positive definite quadratic forms. When the number of quadratic forms is two, the condition is also necessary. The condition is in terms of the condition number of the underlying matrices. The key lemma in our derivation is found to have some connection with the generalized Wielandt inequality. A new inequality between angles in inner product spaces is formulated and proved. This leads directly to a concise statement and proof of the generalized Wielandt inequality, including a simple description of all cases of equality. As a consequence, several recent results in matrix analysis and inner product spaces are improved.

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# Table of Contents

<b>Notation</b>	<b>vii</b>
<b>1 Introduction</b>	<b>1</b>
1.1 Outline . . . . .	2
<b>2 Preliminaries</b>	<b>4</b>
2.1 Real-part angle and Hermitian-part angle . . . . .	4
2.2 Kreĭn’s inequality . . . . .	6
2.3 A Cauchy-Schwarz inequality . . . . .	9
2.4 Applications . . . . .	11
2.5 Angles between subspaces . . . . .	14
<b>3 Some block-matrix majorization inequalities</b>	<b>18</b>
3.1 Classical results . . . . .	20
3.2 Recent results . . . . .	25
3.3 A decomposition lemma for positive definite matrices . . . . .	33
3.4 Several norm inequalities . . . . .	34
3.5 Positive definite matrices with Hermitian blocks . . . . .	42
3.5.1 2-by-2 blocks . . . . .	43
3.5.2 Quaternions and 4-by-4 blocks . . . . .	45

3.6	Discussion . . . . .	50
3.7	Majorization inequalities for normal matrices . . . . .	55
3.8	Majorization inequalities for coneigenvalues . . . . .	62
<b>4</b>	<b>When is a product of positive definite quadratic forms convex</b>	<b>69</b>
4.1	Motivation and the convexity condition . . . . .	69
4.2	Auxiliary results and the proof . . . . .	72
<b>5</b>	<b>Generalized Wielandt inequalities</b>	<b>77</b>
5.1	Kantorovich inequality and Wielandt inequality . . . . .	77
5.2	Some more background and applications . . . . .	80
5.3	Generalized Wielandt inequality in inner product spaces . . . . .	83
5.4	Formulation in terms of matrices . . . . .	92
<b>6</b>	<b>Summary</b>	<b>98</b>
	<b>Bibliography</b>	<b>100</b>
	<b>Index</b>	<b>110</b>

## Notation

We will use the following notation in this work:

$\mathbb{R}$ : the real field;

$\mathbb{C}$ : the complex field;

$\mathbb{F}$ :  $\mathbb{R}$  or  $\mathbb{C}$ ;

$\mathbb{F}^n$ :  $n$ -dimensional real or complex vector space;

$\operatorname{Re} z$ : real part of a complex number  $z$ ;

$\operatorname{Im} z$ : imaginary part of a complex number  $z$ ;

$\bar{z}$ : conjugate of a complex number  $z$ ;

$\mathbb{R}_+^n$ : the set of  $n$ -dimensional real vectors with positive entries;

$M_{m \times n}(\mathbb{F})$ : the set of real or complex matrices of size  $m \times n$ ;

$M_n(\mathbb{F})$ : the set of real or complex matrices of size  $n \times n$ ;

$H_n$ : the set of Hermitian matrices of size  $n \times n$ ;

$H_n^+$ : the set of positive semidefinite Hermitian matrices of size  $n \times n$ ;

$H_n^{++}$ : the set of positive definite Hermitian matrices of size  $n \times n$ ;

$I_n$ : identity matrix of size  $n \times n$ ;

$A^T$ : transpose of a matrix  $A$ ;

$A^*$ : transpose conjugate of a matrix  $A$ ;

$\bar{A}$ : entrywise conjugate of a matrix  $A$ ;

$\operatorname{Re} A$ : Hermitian part of a complex square matrix  $A$ , i.e.,  $\operatorname{Re} A = (A + A^*)/2$ ;

$\operatorname{Im} A$ : skew-Hermitian part of a complex square matrix  $A$ , i.e.,  $\operatorname{Im} A = (A - A^*)/2i$ ;

$\operatorname{Diag}(A)$ : diagonal part of a square matrix  $A$ ;

$A^p$ :  $p$ th root of a positive semidefinite matrix  $A$ , which is also positive semidefinite;

$|A|$ : absolute value of a matrix, i.e.,  $|A| = (A^*A)^{1/2}$ .

$\langle x, y \rangle$ : inner product of  $x$  and  $y$ ;

$\|x\|$ : vector norm induced by inner product, i.e.,  $\|x\| = \sqrt{\langle x, x \rangle}$ ;

$\|A\|$ : (any) symmetric norm of a square matrix  $A$ ;  
 $\|A\|_\infty$ : operator norm of a square matrix  $A$ ;  
 $\prec$ : majorization;  
 $\prec_w$ : weak majorization;  
 $\prec_{\log}$ : log majorization;  
 $\prec_{w\log}$ : weak log majorization;  
 $\oplus$ : direct sum;  
 $\det A$ : determinant of a square matrix  $A$ ;  
 $\text{Tr}$ : trace of a square matrix  $A$ ;  
 $\otimes$ : Kronecker product;  
 $W(A)$ : numerical range of a square matrix  $A$ ;  
 $w(A)$ : numerical radius of a square matrix  $A$ ;  
 $A\sharp B$ : geometric mean of two positive definite matrices  $A$  and  $B$ ;  
 $\mathbb{H}$ : the ring of quaternions;



# Chapter 1

## Introduction

The theme of this thesis consists of two main topics. One is majorization inequalities, the other is the generalized Wielandt inequality.

Majorization inequalities is an interesting area of study, both from the theoretical and applied point of view. A comprehensive survey on this topic can be found in [80]. The notion of majorization has its roots in matrix theory and mathematical inequalities. Loosely speaking, for two vectors  $x, y \in \mathbb{R}^n$  with equal summation of components, we say that  $x$  is majorized by  $y$  if the components in  $x$  are “less spread out” than the components in  $y$ . This may be expressed in terms of linear inequalities for the partial sums in these vectors. This notion arises in a wide range of contexts in mathematical areas, e.g., in combinatorics, probability, matrix theory, numerical analysis.

Majorization turns out to be an underlying structure for several classes of inequalities. One such simple example is the classical arithmetic-geometric mean inequality. Another example is a majorization order between the diagonal entries and the eigenvalues of a real symmetric matrix. Actually, several interesting inequalities arise by applying some ordering-preserving function to a suitable majorization ordering.

Majorization is also studied in connection with familiar network structures (trees and transportation matrices); see, e.g., [27, 28].

In this thesis we investigate topics on eigenvalue majorization, which is also a very basic concept in matrix theory. Due to the important applications of eigenvalue majorization, any new inequality of this type will have a flow of consequences and applications.

The Wielandt and generalized Wielandt inequalities bound how much angles can change under a given invertible matrix transformation of  $\mathbb{C}^n$ . The bound is given in terms of the condition number of the matrix. Wielandt, in [109], gave a bound on the resulting angles when orthogonal complex lines are transformed. Subsequently, Bauer and Householder, in [11], extended the inequality to include arbitrary starting angles. These basic inequalities of matrix analysis were introduced to give bounds on convergence rates of iterative projection methods [77], and have further found a variety of applications in numerical methods, especially eigenvalue estimation. They are also applied in multivariate analysis, where angles between vectors correspond to statistical correlation. See, for example, [11], [33], [35], [51] and [53]. There are also matrix-valued versions of the inequality that are receiving attention, especially in the context of statistical analysis. See [16], [76], [105], and [114]. We noticed that in [63], the equality condition for the generalized Wielandt inequality was established, but the proof was rather involved and complicated. Our main contribution to this topic is a new inequality between angles in inner product spaces. It leads directly to a concise statement and proof of the generalized Wielandt inequality. Legendre-Fenchel conjugate is a basic notion in convex optimization. It is shown in [117, 118] that the convexity of a function is important in finding the explicit expression of the transform for certain functions. As an interesting application of the generalized Wielandt inequality, we shall show that it can be used, in a very elegant way, to derive a sufficient condition for the convexity of the product of positive definite quadratic forms.

Chapter 5 of this thesis includes specific applications of the results we obtain in the field of matrix inequalities. Our results also have potential applications in more applied areas. We provide a brief description of these potential application areas in the introductions of Chapters 3 and 4, but this thesis does not discuss specific applications in applied areas.

## 1.1 Outline

The thesis is organized as follows. In Chapter 2, we revisit two classical definitions of angle in inner product space: real-part angle and Hermitian angle. Special attention is paid to Kreĭn's inequality and its analogue. Some applications are given, leading to a simple proof of a basic lemma for a trace inequality of unitary matrices and also its extension. A brief survey on recent results of angles between subspaces is presented. This naturally bring us to the world of majorization. In Chapter 3, after introducing the notion of majorization, I

present some classical as well as recent results on eigenvalue majorization. Several new norm inequalities are derived by making use of a powerful decomposition lemma for positive semidefinite matrices. We also consider coneigenvalue majorization. Some discussion on the possible generalization of the majorization bounds for Ritz values is presented. In Chapter 4, we turn to a basic notion in convex analysis, the Legendre-Fenchel conjugate. The convexity of a function is important in finding the explicit expression of the transform for certain functions. A sufficient condition is given for the product of positive definite quadratic forms. When the number of quadratic forms is two, the condition is also necessary. The condition is in terms of the condition number of the underlying matrices. The key lemma in our derivation is found to have some connection with the generalized Wielandt inequality that is discussed in Chapter 5. In Chapter 5, a new inequality between angles in inner product spaces is formulated and proved. It leads directly to a concise statement and proof of the generalized Wielandt inequality, including a simple description of all cases of equality. As a consequence, several recent results in matrix analysis and inner product spaces are improved. In Chapter 6, we summarize the main contributions of the thesis.

# Chapter 2

## Preliminaries

In this chapter we survey some results on angles between complex vectors and canonical angles between subspaces in  $\mathbb{C}^n$ .

### 2.1 Real-part angle and Hermitian-part angle

We let  $\mathbb{F}$  denote the field of real numbers  $\mathbb{R}$  or the field of complex numbers  $\mathbb{C}$ .

For  $x = (x_1, x_2, \dots, x_n)^T \in \mathbb{F}^n$ ,  $x^T$  (resp.  $x^*$ ) denotes the transpose (resp. conjugate transpose) of  $x$ . If  $\mathbb{F}^n = \mathbb{C}^n$ , the real part (resp. imaginary part) of  $x$  is denoted by  $\operatorname{Re} x = (\operatorname{Re} x_1, \operatorname{Re} x_2, \dots, \operatorname{Re} x_n)^T$  (resp.  $\operatorname{Im} x = (\operatorname{Im} x_1, \operatorname{Im} x_2, \dots, \operatorname{Im} x_n)^T$ ).

In a real inner product space  $(V, \langle \cdot, \cdot \rangle)$ , the angle  $\theta_{xy}$  between two nonzero vectors  $x, y$  is defined by  $0 \leq \theta_{xy} \leq \pi$  and

$$\cos \theta_{xy} = \frac{\langle x, y \rangle}{\|x\| \|y\|}. \quad (2.1)$$

Here  $\|x\| = \sqrt{\langle x, x \rangle}$  is the norm induced by the standard inner product. When considered in a complex inner product space, the situation becomes less intuitive. There is some ambiguity in the definition of angle between complex vectors. Scharnhorst [98] lists several angle concepts between complex vectors: Euclidean (embedded) angle, complex-valued angle, Hermitian angle, Kasner's pseudo angle, Kähler angle (or Wirtinger angle, slant

angle, etc). Among them, two are familiar to the linear algebra community<sup>1</sup>. One is the *Euclidean angle*, the other one is the *Hermitian angle*.

The Euclidean angle  $\varphi_{xy}$  between two nonzero vectors  $x, y \in \mathbb{C}^n$  is defined by  $0 \leq \varphi_{xy} \leq \pi$  and

$$\cos \varphi_{xy} = \frac{\langle \tilde{x}, \tilde{y} \rangle}{\|\tilde{x}\| \|\tilde{y}\|}, \quad (2.2)$$

where we choose to determine the components of the vectors  $\tilde{x}, \tilde{y} \in \mathbb{R}^{2n}$  by means of the relation  $\tilde{x}_{2k-1} = \operatorname{Re} x_k$  and  $\tilde{x}_{2k} = \operatorname{Im} x_k$ ,  $k = 1, \dots, n$ .

The Hermitian angle  $\psi_{xy}$  between two nonzero vectors  $x, y \in \mathbb{C}^n$  is defined by  $0 \leq \psi_{xy} \leq \pi/2$  and

$$\cos \psi_{xy} = \frac{|\langle x, y \rangle|}{\|x\| \|y\|}. \quad (2.3)$$

As in any real vector space, the cosine of the Hermitian angle between nonzero vectors  $x, y \in \mathbb{C}^n$  can be defined to be the ratio between the length of the orthogonal projection of, say, the vector  $x$  onto the vector  $y$  to the length of the vector  $x$  itself (this projection vector is equal to  $|\langle x, y \rangle| / \|y\|$ ).

It is easy to observe that (2.2) is equivalent to

$$\cos \varphi_{xy} = \frac{\operatorname{Re} \langle x, y \rangle}{\|x\| \|y\|}. \quad (2.4)$$

In this sense, we use the terminology “*real-part angle*” instead of “Euclidean angle” from now on.

Neither the real-part angle nor the Hermitian angle seems perfectly satisfactory. For the previous one, the law of cosines<sup>2</sup> holds, but  $\varphi_{xy} = \frac{\pi}{2}$  does not imply  $\langle x, y \rangle = 0$ . For the

---

<sup>1</sup>A historical remark [50]: In the 1950s and 1960s, linear algebra was generally seen as a dead subject which all mathematicians must know, but hardly a topic for research. However, in the 1970s there was another way to look at the field, as an essential ingredient of many mathematical areas (at least during some stage of their development) and that this would lead to new results in linear algebra. An example of such an applied area active in the 1970s is control theory. Now new results arise because of connections to such applied topics as compressed sensing and quantum information.

<sup>2</sup>Recall that in trigonometry, the law of cosines (also known as the cosine formula or cosine rule) relates the lengths of the sides of a plane triangle to the cosine of one of its angles. More precisely, for a triangle with sides  $a, b, c$ , the law of cosines says  $c^2 = a^2 + b^2 - 2ab \cos \gamma$ , where  $\gamma$  denotes the angle contained between the sides of lengths  $a$  and  $b$  and opposite the side of length  $c$ .

latter,  $\psi_{xy} = \frac{\pi}{2}$  if and only if  $\langle x, y \rangle = 0$ , but the law of cosines does not hold. Nevertheless, these two notions are used for different purpose.

## 2.2 Kreĭn's inequality

For the real-part angle, Kreĭn [64] in 1969 discovered the following interesting relation between the angles of three nonzero vectors, say  $x, y, z \in \mathbb{C}^n$ :

$$\varphi_{xz} \leq \varphi_{xy} + \varphi_{yz}. \quad (2.5)$$

Kreĭn himself did not include a proof in [64]. A proof can be found in [44, p. 56]. (The proof there was suggested by T. Ando.) Since a part of Ando's proof will be useful for our purposes, we include a proof here.

*Proof.* (of (2.5)) Without loss of generality, we assume that  $x, y, z$  are unit vectors. Let

$$\langle x, y \rangle = a_1 + ib_1, \quad \langle y, z \rangle = a_2 + ib_2, \quad \langle x, z \rangle = a_3 + ib_3,$$

where  $a_j, b_j \in \mathbb{R}$  and  $|a_j|^2 + |b_j|^2 \leq 1$  for  $j = 1, 2, 3$ . We have  $\cos \varphi_{xy} = a_1$ ,  $\cos \varphi_{yz} = a_2$  and  $\cos \varphi_{xz} = a_3$ . Since  $\cos \alpha$  is a decreasing function of  $\alpha \in [0, \pi]$ , we need only to prove

$$\begin{aligned} \cos \varphi_{xz} &\geq \cos(\varphi_{xy} + \varphi_{yz}) \\ &= \cos \varphi_{xy} \cos \varphi_{yz} - \sin \varphi_{xy} \sin \varphi_{yz}, \end{aligned}$$

or equivalently,

$$a_3 \geq a_1 a_2 - \sqrt{1 - a_1^2} \sqrt{1 - a_2^2}.$$

Thus, we need

$$\sqrt{1 - a_1^2} \sqrt{1 - a_2^2} \geq a_1 a_2 - a_3. \quad (2.6)$$

We are done if the right hand side of (2.6) is negative. Otherwise, we need prove

$$(1 - a_1^2)(1 - a_2^2) \geq (a_1 a_2 - a_3)^2$$

or

$$1 - a_1^2 - a_2^2 - a_3^2 + 2a_1 a_2 a_3 \geq 0.$$

Since the matrix

$$G = \begin{bmatrix} \langle x, x \rangle & \langle x, y \rangle & \langle x, z \rangle \\ \langle y, x \rangle & \langle y, y \rangle & \langle y, z \rangle \\ \langle z, x \rangle & \langle z, y \rangle & \langle z, z \rangle \end{bmatrix} \quad (2.7)$$

is positive semidefinite<sup>3</sup>, as is its real part<sup>4</sup>, i.e., the matrix  $\begin{bmatrix} 1 & a_1 & a_3 \\ a_1 & 1 & a_2 \\ a_3 & a_2 & 1 \end{bmatrix}$  is positive semidefinite, we conclude that its determinant is nonnegative, and the desired result follows.  $\square$

It is of interest to know whether an analogue relation for Hermitian angles  $\psi_{xz}, \psi_{xy}, \psi_{yz}$  holds as well. The answer is yes and the next result gives an analogue of Kreĭn's inequality (2.5).

$$\psi_{xz} \leq \psi_{xy} + \psi_{yz}. \quad (2.8)$$

We present two proofs for this result.

The first proof uses part of Ando's proof of *Kreĭn's inequality*, but we first need to invoke an interesting property on positive semidefinite matrices.

The set of  $n \times n$  positive semidefinite matrices is denoted by  $H_n^+$ .

**Lemma 2.1.** [79] *Let  $A = [a_{ij}] \in H_3^+$ . Then*

$$|A| := [|a_{ij}|] \in H_3^+. \quad (2.9)$$

*Proof.* 1-by-1 and 2-by-2 principal minors of  $|A|$  are easily seen to be nonnegative. It suffices to show that  $\det|A| \geq 0$ . Note that  $\det A \geq 0$  implies  $a_{11}a_{22}a_{33} + a_{12}a_{23}\bar{a}_{13} + a_{13}\bar{a}_{12}\bar{a}_{23} - a_{11}|a_{23}|^2 - a_{22}|a_{13}|^2 - a_{33}|a_{12}|^2 \geq 0$ . Moreover,

$$\begin{aligned} a_{12}a_{23}\bar{a}_{13} + a_{13}\bar{a}_{12}\bar{a}_{23} &= 2 \operatorname{Re} a_{12}a_{23}\bar{a}_{13} \\ &\leq 2|a_{12}a_{23}a_{13}| = 2|a_{12}||a_{23}||a_{13}|. \end{aligned}$$

---

<sup>3</sup>As is well known, a Hermitian matrix is positive semidefinite if and only if it is a *Gram matrix*, see e.g., [51, p.407].

<sup>4</sup>Obviously, here the real part of a matrix is understood as the entrywise real part.

<sup>5</sup>Unless otherwise specified, in this chapter  $|A|$  is used to stand for entrywise absolute value.

Hence,

$$\begin{aligned} 0 &\leq a_{11}a_{22}a_{33} + 2|a_{12}||a_{23}||a_{13}| \\ &\quad - a_{11}|a_{23}|^2 - a_{22}|a_{13}|^2 - a_{33}|a_{12}|^2 \\ &= \det|A|. \end{aligned}$$

Showing  $|A| := [|a_{ij}|] \in H_3^+$ . □

**Remark 2.2.** Lemma 2.1 can fail for matrices of size larger than 3 as the following example shows

$$B = \begin{bmatrix} 1 & \frac{1}{\sqrt{3}} & 0 & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} & 1 & \frac{1}{\sqrt{3}} & 0 \\ 0 & \frac{1}{\sqrt{3}} & 1 & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} & 0 & \frac{1}{\sqrt{3}} & 1 \end{bmatrix} \in H_4^+,$$

but  $\det|B| < 0$ . The example was also given in [79] with acknowledgement to R. C. Thompson.

*First proof of (2.8).* Assume that  $x, y, z$  are unit vectors. Let

$$|\langle x, y \rangle| = a, \quad |\langle y, z \rangle| = b, \quad |\langle x, z \rangle| = c.$$

Then we need to show

$$c \geq ab - \sqrt{1-a^2}\sqrt{1-b^2}.$$

It suffices to show

$$(1-a^2)(1-b^2) \geq (ab-c)^2,$$

or

$$1 - a^2 - b^2 - c^2 + 2abc \geq 0. \tag{2.10}$$

By Lemma 2.1, we know

$$|G| = \begin{bmatrix} |\langle x, x \rangle| & |\langle x, y \rangle| & |\langle x, z \rangle| \\ |\langle y, x \rangle| & |\langle y, y \rangle| & |\langle y, z \rangle| \\ |\langle z, x \rangle| & |\langle z, y \rangle| & |\langle z, z \rangle| \end{bmatrix}$$

is positive semidefinite, so its determinant is nonnegative, which is just (2.10). □



The second proof (suggested by G. Sinnamon) makes a clever use of Kreĭn's inequality (2.5).

*Second proof of (2.8).* Note that

$$\Psi_{xy} = \inf_{\alpha, \beta \in \mathbb{C} \setminus \{0\}} \varphi_{\alpha x \beta y} = \inf_{\alpha \in \mathbb{C} \setminus \{0\}} \varphi_{\alpha xy} = \inf_{\beta \in \mathbb{C} \setminus \{0\}} \varphi_{x \beta y}.$$

Using (2.5) we have, for any nonzero vectors  $x, y, z \in \mathbb{C}^n$ ,

$$\begin{aligned} \inf_{\alpha, \beta \in \mathbb{C} \setminus \{0\}} \varphi_{\alpha x \beta z} &\leq \inf_{\alpha, \beta \in \mathbb{C} \setminus \{0\}} (\varphi_{\alpha xy} + \varphi_y \beta z) \\ &= \inf_{\alpha \in \mathbb{C} \setminus \{0\}} \varphi_{\alpha xy} + \inf_{\beta \in \mathbb{C} \setminus \{0\}} \varphi_y \beta z, \end{aligned}$$

so

$$\Psi_{xz} \leq \Psi_{xy} + \Psi_{yz}.$$

□

Some remarks are in order.

**Remark 2.3.** *It is easy to see from (2.8) that the Hermitian angle defines a metric on  $\mathbb{C}^n$ .*

**Remark 2.4.** *The inequality (2.8) has been rediscovered many times. The earliest one on record seems to be from Wedin [107]. Alternative proofs can be found in Qiu & Davison [93] and Vinnicombe [103] in the engineering literature, and it is demonstrated that the use of the Hermitian angle as a metric is essential in engineering applications. However, the proof in terms of Kreĭn's inequality seems to be new; see [72].*

**Remark 2.5.** *The derivation shows that the inequality for the Hermitian angle (2.8) is in a sense weaker than that for the real-part angle (2.5).*

## 2.3 A Cauchy-Schwarz inequality

The three angle definitions we have seen, i.e., (2.1), (2.2) and (2.3), more or less depend on an underlying *Cauchy-Schwarz inequality*<sup>6</sup>. In [3], the following Cauchy-Schwarz type inequality is stated for triples of real vectors.

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<sup>6</sup>Preferably, we would like to see that the value of the cosine is no larger than 1. However, this is not the case in complex analysis as the Liouville theorem states: every bounded entire function must be constant.

**Proposition 2.6.** *If  $x, y, z$  are nonzero vectors in  $\mathbb{R}^n$ ,  $n \geq 3$ , then*

$$\frac{\langle x, y \rangle^2}{\|x\|^2 \|y\|^2} + \frac{\langle y, z \rangle^2}{\|y\|^2 \|z\|^2} + \frac{\langle z, x \rangle^2}{\|z\|^2 \|x\|^2} \leq 1 + 2 \frac{\langle x, y \rangle \langle y, z \rangle \langle z, x \rangle}{\|x\|^2 \|y\|^2 \|z\|^2}, \quad (2.11)$$

*with equality if, and only if, the vectors  $x, y, z$  are linearly dependent.*

If we consider complex vector spaces, then (2.11) will have three versions.

**Proposition 2.7.** *If  $x, y, z$  are nonzero vectors in  $\mathbb{C}^n$ ,  $n \geq 3$ , then*

$$\frac{|\langle x, y \rangle|^2}{\|x\|^2 \|y\|^2} + \frac{|\langle y, z \rangle|^2}{\|y\|^2 \|z\|^2} + \frac{|\langle z, x \rangle|^2}{\|z\|^2 \|x\|^2} \leq 1 + 2 \operatorname{Re} \frac{\langle x, y \rangle \langle y, z \rangle \langle z, x \rangle}{\|x\|^2 \|y\|^2 \|z\|^2}, \quad (2.12)$$

$$\frac{|\langle x, y \rangle|^2}{\|x\|^2 \|y\|^2} + \frac{|\langle y, z \rangle|^2}{\|y\|^2 \|z\|^2} + \frac{|\langle z, x \rangle|^2}{\|z\|^2 \|x\|^2} \leq 1 + 2 \frac{|\langle x, y \rangle| |\langle y, z \rangle| |\langle z, x \rangle|}{\|x\|^2 \|y\|^2 \|z\|^2}, \quad (2.13)$$

$$\frac{(\operatorname{Re} \langle x, y \rangle)^2}{\|x\|^2 \|y\|^2} + \frac{(\operatorname{Re} \langle y, z \rangle)^2}{\|y\|^2 \|z\|^2} + \frac{(\operatorname{Re} \langle z, x \rangle)^2}{\|z\|^2 \|x\|^2} \leq 1 + 2 \frac{\operatorname{Re} \langle x, y \rangle \operatorname{Re} \langle y, z \rangle \operatorname{Re} \langle z, x \rangle}{\|x\|^2 \|y\|^2 \|z\|^2}. \quad (2.14)$$

*Equality holds in either (2.12) or (2.13), if, and only if, the vectors  $x, y, z$  are linearly dependent. While equality holds in (2.14) if and only if  $\det(G + \bar{G}) = 0$ , where  $G$  is given in (2.7).*

*Proof.* The proof, like the proof of (2.11), is to write out the determinant of certain  $3 \times 3$  positive semidefinite matrices. Consider the Gram matrix  $G$  given in (2.7). Then (2.12), (2.13) and (2.14) follow from  $\det G \geq 0$ ,  $\det |G| \geq 0$  and  $\det(G + \bar{G}) \geq 0$ , respectively.

Equality holds in (2.12) if and only if  $\det G = 0$ , or equivalently,  $x, y, z$  are linearly dependent.

If  $x, y, z$  are linearly dependent, then it is easy to check that equality holds in (2.13). Conversely, if equality holds in (2.13), then equality holds in (2.12) as well, showing  $x, y, z$  are linearly dependent.  $\square$

The following proposition is well known. We include a proof for completeness.

**Proposition 2.8.** *If  $A, B \in H_n^+$ , then*

$$\det(A + B) \geq \det A + \det B.$$

*Proof.* This is weaker than the Minkowski determinant inequality which asserts

$$\det(A+B)^{1/n} \geq \det A^{1/n} + \det B^{1/n},$$

for any  $A, B \in H_n^+$ ; see e.g., [51]. □

Note that both  $G$  and  $\overline{G}$  are positive semidefinite. Therefore, by Proposition 2.8, we have  $\det(G + \overline{G}) = 0$  occurs only if both  $\det G = 0$  and  $\det \overline{G} = 0$ , or  $x, y, z$  are linearly dependent. However,  $x, y, z$  being linearly dependent is insufficient to imply  $\det(G + \overline{G}) = 0$ , as the following example shows.

**Example 2.9.** Let  $G = \begin{bmatrix} 1 & i & 0 \\ -i & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ . Then  $G \in H_3^+$  and  $\det G = 0$ . Suppose the Gram matrix  $G$  is formed by  $x, y, z \in \mathbb{C}^n$ , then  $\det G = 0$  implies  $x, y, z$  are linearly dependent. However, we have  $\det(G + \overline{G}) = 8$ .

It is readily seen that (2.12) is stronger than (2.13). Using the notion of Hermitian angle and real-part angle, (2.13) and (2.14) can be restated as

$$\cos^2 \psi_{xy} + \cos^2 \psi_{yz} + \cos^2 \psi_{zx} \leq 1 + 2 \cos \psi_{xy} \cos \psi_{yz} \cos \psi_{zx} \quad (2.15)$$

and

$$\cos^2 \varphi_{xy} + \cos^2 \varphi_{yz} + \cos^2 \varphi_{zx} \leq 1 + 2 \cos \varphi_{xy} \cos \varphi_{yz} \cos \varphi_{zx}, \quad (2.16)$$

respectively.

We have used these two inequalities in the proof of Kreĭn's inequality (2.5) and its analogue (2.8).

## 2.4 Applications

Originally, Kreĭn's inequality (2.5) was used to establish a property of *deviation*. Recall that the deviation of an operator  $T$  on a Hilbert space  $\mathcal{H}$ , denoted by  $\text{dev}(T)$ , is given by  $\text{dev}(T) = \sup_{x \in \mathcal{H}} \phi(Tx, x)$ , where  $\phi(Tx, x)$ ,  $0 \leq \phi \leq \pi$ , is defined by the equation

$$\cos(\phi(Tx, x)) = \frac{\text{Re}\langle Tx, x \rangle}{\|Tx\| \|x\|}.$$

**Proposition 2.10.** [44] *Let  $A$  and  $B$  be bounded invertible operators on a Hilbert space. Then*

$$\text{dev}(AB) \leq \text{dev}(A) + \text{dev}(B). \quad (2.17)$$

If we bring in a new object  $\underline{\text{dev}}(T)$ , which is given by  $\underline{\text{dev}}(T) = \sup_{x \in \mathcal{H}} \underline{\phi}(Tx, x)$ , where  $\underline{\phi}(Tx, x)$ ,  $0 \leq \underline{\phi} \leq \pi/2$ , is defined by the equation

$$\cos(\underline{\phi}(Tx, x)) = \frac{|\langle Tx, x \rangle|}{\|Tx\| \|x\|},$$

then analogously, we have

**Proposition 2.11.** *Let  $A$  and  $B$  be bounded invertible operators on a Hilbert space. Then*

$$\underline{\text{dev}}(AB) \leq \underline{\text{dev}}(A) + \underline{\text{dev}}(B). \quad (2.18)$$

*Proof.* Obviously,  $\underline{\text{dev}}(A) = \underline{\text{dev}}(A^{-1})$ . By (2.8), we have

$$\begin{aligned} \underline{\phi}(ABx, x) &\leq \underline{\phi}(ABx, A^{-1}x) + \underline{\phi}(A^{-1}x, x) \\ &= \underline{\phi}(Bx, x) + \underline{\phi}(A^{-1}x, x). \end{aligned}$$

Their suprema bear the same relation, so one has

$$\underline{\text{dev}}(AB) \leq \underline{\text{dev}}(A) + \underline{\text{dev}}(B).$$

□

In [106] (see also [115, p.195]), the following elegant inequality was derived to prove a trace inequality for unitary matrices.

**Proposition 2.12.** *For any unit vectors  $x, y$  and  $z \in \mathbb{C}^n$ , we have*

$$\sqrt{1 - |\langle x, z \rangle|^2} \leq \sqrt{1 - |\langle x, y \rangle|^2} + \sqrt{1 - |\langle y, z \rangle|^2}. \quad (2.19)$$

Let  $U, V$  be  $n \times n$  unitary matrices. By (2.19), it is clear that

$$\sqrt{1 - \left| \frac{1}{n} \text{Tr} UV \right|^2} \leq \sqrt{1 - \left| \frac{1}{n} \text{Tr} U \right|^2} + \sqrt{1 - \left| \frac{1}{n} \text{Tr} V \right|^2}.$$

The next result is to give an interesting application of the inequalities (2.5) and (2.8), from which (2.19) follows immediately.

We start with a simple lemma, with obvious geometric meaning: it can be regarded as the triangle inequality for the chordal metric on the circle.

**Lemma 2.13.** Let  $\alpha \in [0, \pi]$ ,  $\beta, \gamma \in [0, \frac{\pi}{2}]$  with  $\alpha \leq \beta + \gamma$ . Then

$$\sin \alpha \leq \sin \beta + \sin \gamma. \quad (2.20)$$

*Proof.* If  $0 \leq \beta + \gamma \leq \frac{\pi}{2}$ , obviously,

$$\begin{aligned} \sin \alpha &\leq \sin(\beta + \gamma) \\ &= \sin \beta \cos \gamma + \sin \gamma \cos \beta \\ &\leq \sin \beta + \sin \gamma. \end{aligned}$$

If  $\frac{\pi}{2} \leq \beta + \gamma \leq \pi$ , then  $\beta \geq \frac{\pi}{2} - \gamma$ ,

$$\begin{aligned} \sin \beta + \sin \gamma &\geq \sin\left(\frac{\pi}{2} - \gamma\right) + \sin \gamma \\ &= \cos \gamma + \sin \gamma \\ &= \sqrt{2} \sin\left(\gamma + \frac{\pi}{4}\right) \geq 1 \geq \sin \alpha. \end{aligned}$$

□

With (2.8) and Lemma 2.13, we have

$$\sin \psi_{xz} \leq \sin \psi_{xy} + \sin \psi_{yz}, \quad (2.21)$$

which is just a restatement of (2.19). Moreover, we have

**Proposition 2.14.** For any unit vectors  $x, y$  and  $z \in \mathbb{C}^n$ ,

$$\sqrt{1 - (\operatorname{Re}\langle x, z \rangle)^2} \leq \sqrt{1 - (\operatorname{Re}\langle x, y \rangle)^2} + \sqrt{1 - (\operatorname{Re}\langle y, z \rangle)^2} \quad (2.22)$$

*Proof.* If  $\operatorname{Re}\langle x, y \rangle \leq 0$ , we replace  $x$  by  $-x$ ; if  $\operatorname{Re}\langle y, z \rangle \leq 0$ , we replace  $z$  by  $-z$ . That is to say, we may always let  $\varphi_{xy}, \varphi_{yz} \in [0, \frac{\pi}{2}]$  and  $\varphi_{xz} \in [0, \pi]$ , thus the condition in Lemma 2.13 is satisfied. Therefore  $\sin \varphi_{xz} \leq \sin \varphi_{xy} + \sin \varphi_{yz}$ , i.e., (2.22) holds. □

To end this section, we present a unified extension of (2.19) and (2.22).

**Proposition 2.15.** [72] Let  $p > 2$ . Then for any unit vectors  $x, y$  and  $z \in \mathbb{C}^n$  we have

$$\sqrt[p]{1 - |\langle x, z \rangle|^p} \leq \sqrt[p]{1 - |\langle x, y \rangle|^p} + \sqrt[p]{1 - |\langle y, z \rangle|^p} \quad (2.23)$$

and

$$\sqrt[p]{1 - |\operatorname{Re}\langle x, z \rangle|^p} \leq \sqrt[p]{1 - |\operatorname{Re}\langle x, y \rangle|^p} + \sqrt[p]{1 - |\operatorname{Re}\langle y, z \rangle|^p} \quad (2.24)$$

*Proof.* <sup>7</sup> Fix  $p > 2$  and set  $f(t) = (1 - (1 - t^2)^{p/2})^{1/p}$  for  $t \in [0, 1]$ . Then simple calculation shows

$$\frac{d}{dt}f(t) = (1 - (1 - t^2)^{p/2})^{1/p-1}(1 - t^2)^{p/2-1}t \geq 0.$$

and

$$\frac{d}{dt} \frac{f(t)}{t} = t^{-2}(1 - (1 - t^2)^{p/2})^{1/p-1}((1 - t^2)^{p/2-1} - 1) \leq 0.$$

Since  $f(t)$  is increasing and  $f(t)/t$  is decreasing, if  $a, b, c \in [0, 1]$  and  $0 \leq a \leq b + c \leq 1$ , then

$$\begin{aligned} f(a) &\leq f(b+c) \\ &= b \frac{f(b+c)}{b+c} + c \frac{f(b+c)}{b+c} \\ &\leq b \frac{f(b)}{b} + c \frac{f(c)}{c} \\ &= f(b) + f(c). \end{aligned}$$

If  $a, b, c \in [0, 1]$  and  $0 \leq a \leq 1 \leq b + c$ , then we can choose  $b', c'$  such that  $0 \leq b' \leq b$  and  $0 \leq c' \leq c$  and  $b' + c' = 1$ . Again, we have  $f(a) \leq f(1) \leq f(b') + f(c') \leq f(b) + f(c)$ , i.e.,  $f(a) \leq f(b) + f(c)$ . Taking  $a = \sqrt{1 - |\langle x, z \rangle|^2}$ ,  $b = \sqrt{1 - |\langle x, y \rangle|^2}$ , and  $c = \sqrt{1 - |\langle y, z \rangle|^2}$ , we get

$$\begin{aligned} (1 - |\langle x, z \rangle|^p)^{1/p} &= f(a) \\ &\leq f(b) + f(c) \\ &= (1 - |\langle x, y \rangle|^p)^{1/p} + (1 - |\langle y, z \rangle|^p)^{1/p}. \end{aligned}$$

This proves (2.23). Inequality (2.24) can be proved by taking  $a = \sqrt{1 - |\operatorname{Re}\langle x, z \rangle|^2}$ ,  $b = \sqrt{1 - |\operatorname{Re}\langle x, y \rangle|^2}$ , and  $c = \sqrt{1 - |\operatorname{Re}\langle y, z \rangle|^2}$ .  $\square$

## 2.5 Angles between subspaces

In this section, we shall survey some remarkable results related to the angle between subspaces. To begin, let us recall the notion of *canonical angles* (in the literature, the terminology “principal angles” is occasionally used).

<sup>7</sup>The elegant proof was suggested by Gord Sinnamon, to whom I am indebted.

Let  $\mathcal{X}, \mathcal{Y}$  be  $m$ -dimensional subspaces of  $\mathbb{C}^n$  (of course,  $m \leq n$ ). One may define a vector

$$\Psi(\mathcal{X}, \mathcal{Y}) = (\Psi_1(\mathcal{X}, \mathcal{Y}), \dots, \Psi_m(\mathcal{X}, \mathcal{Y}))$$

of  $m$  angles describing the relative position between these two subspaces (see e.g., [43]) as follows:

Let

$$\cos \Psi_m(\mathcal{X}, \mathcal{Y}) := \max_{x \in \mathcal{X}, y \in \mathcal{Y}} |\langle x, y \rangle|, \quad \|x\| = \|y\| = 1.$$

This defines the smallest angle  $\Psi_k(\mathcal{X}, \mathcal{Y})$  between  $\mathcal{X}$  and  $\mathcal{Y}$ . The maximum is achieved for some  $x_m \in \mathcal{X}$  and  $y_m \in \mathcal{Y}$ . Now “remove”  $x_m$  from  $\mathcal{X}$  by considering the orthogonal complement of  $x$  in  $\mathcal{X}$  and do the same for  $y_m$  in  $\mathcal{Y}$ . Repeat the definition for the  $m - 1$  dimensional subspaces

$$\{x \in \mathcal{X} : x \perp x_m\} \quad \text{and} \quad \{y \in \mathcal{Y} : y \perp y_m\},$$

and then keep going in the same fashion until reaching empty spaces. After completion the above procedure defines recursively the  $m$  canonical angles

$$\pi/2 \geq \Psi_1(\mathcal{X}, \mathcal{Y}) \geq \dots \geq \Psi_m(\mathcal{X}, \mathcal{Y}) \geq 0.$$

The angle  $\Psi_m(\mathcal{X}, \mathcal{Y})$  is called the *minimal angle* between  $\mathcal{X}$  and  $\mathcal{Y}$ , which is of particular interest. In practice, one is also interested in the maximal angle  $\Psi_1(\mathcal{X}, \mathcal{Y})$ , since it gives a better idea of “how far away” the spaces are from each other. In this sense,  $\Psi_1(\mathcal{X}, \mathcal{Y})$  is usually called the *gap* between  $\mathcal{X}$  and  $\mathcal{Y}$  and is sometimes used as a measure of the “distance” between  $\mathcal{X}$  and  $\mathcal{Y}$ .

The next example serves as a description of minimal angle and maximal angle in  $\mathbb{R}^3$ .

**Example 2.16.** Given two planes  $\mathcal{P}_1, \mathcal{P}_2$  in  $\mathbb{R}^3$ . Without loss of generality, they pass through the origin. If they are the same, then we say the angle between them is 0. Assume  $\mathcal{P}_1 = \text{span}\{u, v\}$ , with  $u \perp v$  and  $\mathcal{P}_2 = \text{span}\{u, w\}$  with  $u \perp w$ .

In this case, we know the minimal canonical angle between the subspaces  $\text{span}\{u, v\}$  and  $\text{span}\{u, w\}$  is 0. Then we go on to find the second smallest canonical angle (by considering the **orthogonal complement** of  $u$  in  $\text{span}\{u, v\}$  and  $\text{span}\{u, w\}$ , respectively): it is the angle between  $v$  and  $w$ . Thus the angle between two planes  $\mathcal{P}_1, \mathcal{P}_2$  in the usual sense is the maximal canonical angle.

**Remark 2.17.** *Note that the canonical angles are defined in terms of Hermitian angle between complex vectors. What if we use real-part angle instead? It turns out with little surprise that they are the same, since we consider angles between **subspaces**; see [41, Lemma 6].*

Let the columns of  $X, Y \in M_{n \times m}(\mathbb{F})$  be any two orthonormal bases for the  $m$  dimensional subspaces  $\mathcal{X}, \mathcal{Y}$ , respectively. Singular value decomposition tells us that we can take unitary matrices  $U$  and  $V$  such that

$$U^* X^* Y V = \text{Diag}(\sigma_1, \dots, \sigma_m),$$

where the singular values are written from largest to smallest. It is easy to observe that the cosines of the canonical angles between  $\mathcal{X}$  and  $\mathcal{Y}$  are precisely the singular values of the matrix  $X^* Y$ . Note here that these singular values are always the same regardless of the initial choice of bases  $\mathcal{X}$  and  $\mathcal{Y}$ , that is, the angles depend on the subspaces but not on the choice of bases. Generally we have

$$\cos \Psi(\mathcal{X}, \mathcal{Y}) = \sigma(X^* Y),$$

with  $X, Y$  being any orthonormal bases for  $\mathcal{X}, \mathcal{Y}$ , respectively.

It is natural to ask whether Kreĭn's inequality (2.5) or its analogue (2.8) has some extension in the setting of angles between subspaces. Taking into account Remark 2.17, we consider the original version of canonical angle (i.e., defined in terms of Hermitian angle). Indeed, a conjecture on this was announced by Qiu at the 10th ILAS conference in 2002. Later, Qiu et al [94] proved his conjecture. Their result can be stated as

**Theorem 2.18.** *Let  $\mathcal{X}, \mathcal{Y}$  and  $\mathcal{Z} \subset \mathbb{C}^n$  be subspaces of the same dimension, say  $m$ . Then*

$$\sum_{j=1}^k \Psi_j(\mathcal{X}, \mathcal{Z}) \leq \sum_{j=1}^k \Psi_j(\mathcal{X}, \mathcal{Y}) + \sum_{j=1}^k \Psi_j(\mathcal{Y}, \mathcal{Z}) \quad (2.25)$$

and

$$\sum_{j=1}^k \Psi_{i_j}(\mathcal{X}, \mathcal{Z}) \leq \sum_{j=1}^k \Psi_{i_j}(\mathcal{X}, \mathcal{Y}) + \sum_{j=1}^k \Psi_j(\mathcal{Y}, \mathcal{Z}) \quad (2.26)$$

for any  $1 \leq i_1 < \dots < i_k \leq m$  and  $1 \leq k \leq m$ .

**Remark 2.19.** *In [94], the authors showed something more. It is clear that (2.26) is stronger than (2.25), both of them can be regarded as an extension of (2.8).*



Results of the form (2.25) or (2.26) are called (weak) majorization relation (they are also proved using techniques of majorization) whose definition is given in the beginning of the next chapter. That is, (2.25) can be rewritten equivalently as

$$\Psi(\mathcal{X}, \mathcal{Z}) \prec_w \Psi(\mathcal{Y}, \mathcal{Z}) + \Psi(\mathcal{X}, \mathcal{Y}).$$

Similarly, (2.26) corresponds with (but is slightly stronger than)

$$|\Psi(\mathcal{X}, \mathcal{Z}) - \Psi(\mathcal{Y}, \mathcal{Z})| \prec_w \Psi(\mathcal{X}, \mathcal{Y}).$$

Regarding the extension of (2.21) to the multidimensional setting, Knyazev and Argenti [61] obtained the following result.

**Theorem 2.20.** *Let  $\mathcal{X}, \mathcal{Y}$  and  $\mathcal{Z} \subset \mathbb{C}^n$  be subspaces of the same dimension. Then*

$$|\sin \Psi(\mathcal{X}, \mathcal{Z}) - \sin \Psi(\mathcal{Y}, \mathcal{Z})| \prec_w \sin \Psi(\mathcal{X}, \mathcal{Y}) \quad (2.27)$$

and

$$|\cos \Psi(\mathcal{X}, \mathcal{Z}) - \cos \Psi(\mathcal{Y}, \mathcal{Z})| \prec_w \sin \Psi(\mathcal{X}, \mathcal{Y}).$$

Obviously, (2.27) implies

$$\sin \Psi(\mathcal{X}, \mathcal{Z}) \prec_w \sin \Psi(\mathcal{X}, \mathcal{Y}) + \sin \Psi(\mathcal{Y}, \mathcal{Z}).$$

There is also one additional important breakthrough related to the topic of majorization bounds for angles in subspaces. It was conjectured in [8] and proved in [62] by Knyazev and Argenti that the following extension of Ruhe's result [97] holds true.

**Theorem 2.21.** *Let  $\mathcal{X}, \mathcal{Y}$  be subspaces of  $\mathbb{C}^n$  having the same dimension, with orthonormal bases given by the columns of the matrices  $X$  and  $Y$ , respectively. Also, let  $A \in H_n$  and let  $\mathcal{X}$  be  $A$ -invariant. Then*

$$|\lambda(X^*AX) - \lambda(Y^*AY)| \prec_w \operatorname{spr}(A) \sin^2 \Psi(\mathcal{X}, \mathcal{Y}). \quad (2.28)$$

Here, the spread of a matrix  $X \in M_n(\mathbb{F})$  with spectrum  $\{\lambda_1(X), \dots, \lambda_n(X)\}$  is defined by  $\operatorname{spr}(X) = \max_{j,k} |\lambda_j(X) - \lambda_k(X)|$ .

It is not the purpose of this thesis to exposit the proof of the above majorization results. For Theorem 2.21, a nice exposition has appeared in a recent PhD thesis [89]. As I mentioned, majorization results are usually (if not mainly) proved using majorization techniques. This is especially the case for the proof of Theorem 2.21 (originally a conjecture). What interests me in this thesis is the majorization technique or tools that underly the beautiful results I will expand on in the next chapter.

# Chapter 3

## Some block-matrix majorization inequalities

In this chapter, we survey some classical and recent results on majorization inequalities. Special attention is given to majorization results of block-matrices. Matrix inequalities by means of the techniques on block matrices; usually they are  $2 \times 2$  in most applications. The  $2 \times 2$ , ordinary or partitioned, matrices play an important role in various matrix problems, particularly in deriving matrix inequalities. Besides the many applications of majorization inequalities listed in the introduction, here we mention that it also plays a significant role in solving communication and information theoretic problems in wireless communications; see [57].

For a real vector  $x = (x_1, x_2, \dots, x_n)$ , let  $x^\downarrow = (x_1^\downarrow, x_2^\downarrow, \dots, x_n^\downarrow)$  be the vector obtained by rearranging the coordinates of  $x$  in nonincreasing order. Thus  $x_1^\downarrow \geq x_2^\downarrow \geq \dots \geq x_n^\downarrow$ .

The set of  $m \times n$  matrices with entries from  $\mathbb{F}$  is denoted by  $M_{m \times n}(\mathbb{F})$ . Also, we identify  $M_n(\mathbb{F}) = M_{n \times n}(\mathbb{F})$ . The set of  $n \times n$  Hermitian matrices is denoted by  $H_n$ .  $H_n^{++}$  ( $H_n^+$ ) means the set of  $n \times n$  positive definite (semidefinite) matrices.

We start with the notion of *majorization* relations between two real vectors.

**Definition 3.1.** Let  $x, y \in \mathbb{R}^n$ . Then we say that  $x$  is weakly majorized by  $y$ , denoted by  $x \prec_w y$  (the same as  $y \succ_w x$ ), if  $\sum_{j=1}^k x_j^\downarrow \leq \sum_{j=1}^k y_j^\downarrow$  for all  $k = 1, 2, \dots, n$ . We say that  $x$  is majorized by  $y$ , denoted by  $x \prec y$  (or  $y \succ x$ ), if further  $\sum_{j=1}^n x_j = \sum_{j=1}^n y_j$ .

Majorization is a powerful, easy-to-use and flexible mathematical tool which can be applied to a wide variety of problems in pure and applied mathematics. For example, applications in quantum mechanics can be found in e.g., [86, 87, 88]. The results and techniques to be described in this chapter may also have applications in this area.

A well known and useful characterization of majorization is in terms of *doubly stochastic matrices*. Recall that a doubly stochastic matrix is a square (entrywise) nonnegative matrix whose row sums and column sums are all equal to 1. In symbols,  $A \in M_n(\mathbb{R})$  is doubly stochastic if  $A$  is nonnegative and for  $e = (1, \dots, 1)^T \in \mathbb{R}^n$ ,

$$Ae = e \text{ and } e^T A = e^T.$$

A *doubly substochastic matrix* is a square nonnegative matrix whose row and column sums are each at most 1, i.e.,

$$Ae \leq e \text{ and } e^T A \leq e^T.$$

**Proposition 3.2.** [115] *Let  $x, y \in \mathbb{R}^n$ . Then  $x \prec y$  if and only if there is a doubly stochastic matrix  $A \in M_n(\mathbb{R})$  such that  $x = Ay$ .  $x \prec_w y$  if and only if there is a doubly substochastic matrix  $A \in M_n(\mathbb{R})$  such that  $x = Ay$ .*

Another useful characterization of majorization is related to convex functions.

**Proposition 3.3.** [115] *Let  $x, y \in \mathbb{R}^n$ . Then*

1.  $x \prec y \Leftrightarrow \sum_{j=1}^n f(x_j) \leq \sum_{j=1}^n f(y_j)$  for all convex functions  $f : \mathbb{R} \rightarrow \mathbb{R}$ .
2.  $x \prec_w y \Leftrightarrow \sum_{j=1}^n f(x_j) \leq \sum_{j=1}^n f(y_j)$  for all increasing convex functions  $f : \mathbb{R} \rightarrow \mathbb{R}$ .

For many specific applications, a weaker form is generally enough. Here, we let  $f(x) := (f(x_1), \dots, f(x_n))$ .

**Proposition 3.4.** [115] *Let  $x, y \in \mathbb{R}^n$ . If  $f : \mathbb{R} \rightarrow \mathbb{R}$  is convex, then*

$$x \prec y \Rightarrow f(x) \prec_w f(y);$$

*if  $f : \mathbb{R} \rightarrow \mathbb{R}$  is increasing and convex, then*

$$x \prec_w y \Rightarrow f(x) \prec_w f(y).$$

An interesting corollary of Proposition 3.4 is the following.

**Corollary 3.5.** *If  $x, y \in \mathbb{R}_+^n$  and  $x \prec y$ , then*

$$\prod_{j=k}^n x_j \geq \prod_{j=k}^n y_j,$$

for  $k = 1, \dots, n$ .

*Proof.* Note that  $f(t) = -\log t$  is convex for  $t \in (0, \infty)$ . □

The vector of eigenvalues of a matrix  $A \in M_n(\mathbb{F})$  is denoted by

$$\lambda(A) = (\lambda_1(A), \lambda_2(A), \dots, \lambda_n(A)).$$

When the eigenvalues are real, they are ordered

$$\lambda_1(A) \geq \lambda_2(A) \geq \dots \geq \lambda_n(A)$$

otherwise, the real parts satisfy

$$\operatorname{Re} \lambda_1(A) \geq \operatorname{Re} \lambda_2(A) \geq \dots \geq \operatorname{Re} \lambda_n(A).$$

We mainly focus on majorization between the eigenvalues of some matrices.

### 3.1 Classical results

This section is devoted to some classical results on eigenvalue majorization. In most cases, we include the proof for completeness.

The diagonal part of a square matrix  $A$  is denoted by  $\operatorname{Diag}(A)$ , i.e.,  $\operatorname{Diag}(A)$  is obtained by replacing off-diagonal entries of  $A$  by zeros. The *direct sum* of two matrices  $A \in M_m(\mathbb{F})$  and  $B \in M_n(\mathbb{F})$  is a larger block matrix  $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ , denoted by  $A \oplus B$ . Zeros here are understood as zero matrices of appropriate size. A remark on the notation: in the sequel, if  $A \in M_n(\mathbb{F})$  and  $B \in M_m(\mathbb{F})$  with  $m < n$ , then  $\lambda(A) \prec \lambda(B)$  really means that  $\lambda(A) \prec \lambda(B \oplus 0)$  by adding a zero matrix such that the size of  $A$  and  $B \oplus 0$  agree. Since the zero eigenvalues are not so important in our consideration, we may simply write  $\lambda(A) = \lambda(A \oplus 0)$  for any zero square matrix  $0$ .

**Proposition 3.6. Schur (1923):** If  $A \in H_n$ , then

$$\text{Diag}(A) \prec \lambda(A).$$

*Proof.* The proof adapted here can be found in [80]. Since  $A$  is Hermitian, there exists a unitary matrix  $U = [u_{ij}]$  such that  $A = UDU^*$ , where  $D$  is diagonal with diagonal entries  $\lambda_1(A), \dots, \lambda_n(A)$ . The diagonal elements  $a_{11}, a_{22}, \dots, a_{nn}$  of  $A$  are

$$a_{ii} = \sum_{j=1}^n u_{ij} \bar{u}_{ij} \lambda_j(A) = \sum_{j=1}^n p_{ij} \lambda_j(A), \quad i = 1, \dots, n,$$

where  $p_{ij} = u_{ij} \bar{u}_{ij}$ . Because  $U$  is unitary, the matrix  $P = [p_{ij}]$  is doubly stochastic. Consequently,

$$(a_{11}, a_{22}, \dots, a_{nn}) = (\lambda_1(A), \lambda_2(A), \dots, \lambda_n(A))P,$$

so that by Proposition 3.2, the assertion follows.  $\square$

Schur's result can be extended to the block case.

**Proposition 3.7. Fan (1954):** If  $\begin{bmatrix} A & X \\ X^* & B \end{bmatrix} \in H_{m+n}$ , then

$$\lambda(A \oplus B) \prec \lambda \left( \begin{bmatrix} A & X \\ X^* & B \end{bmatrix} \right). \quad (3.1)$$

*Proof.* Let  $A = U^* D_1 U$ ,  $B = V^* D_2 V$ , where  $D_1, D_2$  are diagonal matrices, be the spectral decomposition of  $A, B$ , respectively. Then

$$\begin{aligned} \lambda \left( \begin{bmatrix} A & X \\ X^* & B \end{bmatrix} \right) &= \lambda \left( \begin{bmatrix} D_1 & UXV^* \\ VX^*U^* & D_2 \end{bmatrix} \right) \\ &\succ \lambda(D_1 \oplus D_2) \\ &= \lambda(A \oplus B), \end{aligned}$$

where the majorization is by Proposition 3.6.  $\square$

Another well known result by Fan on majorization is the following

**Proposition 3.8. Fan (1949):** If  $A, B \in H_n$ , then

$$\lambda(A + B) \prec \lambda(A) + \lambda(B). \quad (3.2)$$

A stronger result than (3.2) is obtained by Thompson.

**Theorem 3.9. Thompson (1971):** *Let  $A, B \in H_n$ . Then for any sequence  $1 \leq i_1 < \dots < i_k \leq n$ ,*

$$\sum_{t=1}^k \lambda_{i_t}(A) + \sum_{t=1}^k \lambda_{n-k+t}(B) \leq \sum_{t=1}^k \lambda_{i_t}(A+B) \leq \sum_{t=1}^k \lambda_{i_t}(A) + \sum_{t=1}^k \lambda_t(B) \quad (3.3)$$

This theorem is proved by using the min-max expression for the sum of eigenvalues. The proof is delicate and I decide not to spend room on the proof. A detailed proof can be found in [115, p.281].

Theorem 3.9 leads to another well known result due to Lidskii, see [70] or [14, p. 69].

**Proposition 3.10. Lidskii (1950):** *If  $A, B \in H_n$ , then*

$$\lambda(A) - \lambda(B) \prec \lambda(A - B). \quad (3.4)$$

*Proof.* The proof adapted here is from the standard reference [115]. Write  $A = B + (A - B)$ . By Theorem 3.9,

$$\sum_{t=1}^n \lambda_{i_t}(A) \leq \sum_{t=1}^n \lambda_{i_t}(B) + \sum_{t=1}^k \lambda_t(A - B)$$

which yields, for  $k = 1, 2, \dots, n$ ,

$$\max_{1 \leq i_1 < \dots < i_k \leq n} \sum_{t=1}^k (\lambda_{i_t}(A) - \lambda_{i_t}(B)) \leq \sum_{t=1}^k \lambda_t(A - B),$$

that is,

$$\lambda(A) - \lambda(B) \prec_w \lambda(A - B).$$

As equality holds when  $k = n$ , the desired majorization follows.  $\square$

A very basic and useful result was obtained by Rotfel'd and independently by Thompson.

**Proposition 3.11. Rotfel'd (1969); Thompson (1977):** *If  $A, B \in H_n^+$ , then*

$$\lambda(A \oplus B) \prec \lambda(A + B). \quad (3.5)$$

*Proof.* The proof adapted here is from [80]. Since  $A$  and  $B$  are positive semidefinite, they can be written in the form  $A = MM^*$ ,  $B = NN^*$ , for some  $M, N \in M_n(\mathbb{F})$ . If  $X = [M, N]$ , then  $A + B = XX^*$ . Furthermore, the nonzero eigenvalues of  $XX^*$  coincide with the nonzero eigenvalues of

$$X^*X = \begin{bmatrix} M^*M & M^*N \\ N^*M & N^*N \end{bmatrix},$$

It follows from (3.1) that

$$\begin{aligned} \lambda(A \oplus B) &= (\lambda(A), \lambda(B)) \\ &= (\lambda(MM^*), \lambda(NN^*)) \\ &= (\lambda(M^*M), \lambda(N^*N)) \\ &\prec \lambda(X^*X) = \lambda(XX^*) = \lambda(A + B). \end{aligned}$$

This completes the proof. □

Another result complementary to Fan's (3.1) is the following, which can be found in [52, p. 217, Problem 22].

**Proposition 3.12.** *If  $\begin{bmatrix} A & X \\ X^* & B \end{bmatrix} \in H_{m+n}^+$ , then*

$$\lambda \left( \begin{bmatrix} A & X \\ X^* & B \end{bmatrix} \right) \prec \lambda(A) + \lambda(B). \quad (3.6)$$

*Proof.* Since  $\begin{bmatrix} A & X \\ X^* & B \end{bmatrix}$  positive semidefinite, then we have

$$\begin{bmatrix} A & X \\ X^* & B \end{bmatrix} = [M, N]^*[M, N],$$

for some  $M \in M_{m+n,m}(\mathbb{F})$  and  $N \in M_{m+n,n}(\mathbb{F})$ . Therefore  $A = M^*M$ ,  $B = N^*N$  and so

$$\begin{aligned}
\lambda \left( \begin{bmatrix} A & X \\ X^* & B \end{bmatrix} \right) &= \lambda([M, N]^*[M, N]) \\
&= \lambda([M, N][M, N]^*) \\
&= \lambda(MM^* + NN^*) \\
&\prec \lambda(MM^*) + \lambda(NN^*) \\
&= \lambda(M^*M) + \lambda(N^*N) \\
&= \lambda(A) + \lambda(B),
\end{aligned}$$

where the majorization is by (3.2). This completes the proof.  $\square$

In view of (3.4), the above proposition can be slightly improved.

**Proposition 3.13.** *If  $\begin{bmatrix} A & X \\ X^* & B \end{bmatrix} \in H_{m+n}^+$ , then*

$$\lambda \left( \begin{bmatrix} A & X \\ X^* & B \end{bmatrix} \right) - \lambda(A) \prec \lambda(B). \tag{3.7}$$

*Proof.* As above, write

$$\begin{bmatrix} A & X \\ X^* & B \end{bmatrix} = [M, N]^*[M, N],$$

for some  $M \in M_{m+n,m}(\mathbb{F})$  and  $N \in M_{m+n,n}(\mathbb{F})$ . Then

$$\begin{aligned}
\lambda \left( \begin{bmatrix} A & X \\ X^* & B \end{bmatrix} \right) - \lambda(A) &= \lambda([M, N]^*[M, N]) - \lambda(M^*M) \\
&= \lambda(MM^* + NN^*) - \lambda(MM^*) \\
&\prec \lambda(NN^*) = \lambda(N^*N) = \lambda(B),
\end{aligned}$$

where the majorization is by (3.4). This completes the proof.  $\square$

**Remark 3.14.** *Proposition 3.12 can be generalized to  $m \times m$  block matrices by simple induction, so one may wonder whether Proposition 3.13 also has such an extension. What will be the correct form? Generally, we don't have  $\lambda(X+Y+Z) - \lambda(X) - \lambda(Y) \prec \lambda(Z)$  for  $X, Y, Z \in H_n$ . For example, one may take  $Z = 0$ , reducing to  $\lambda(X+Y) - \lambda(X) - \lambda(Y) \prec 0$ , which clearly does not hold.*



Comparing (3.1), (3.5) and (3.6), it is natural to ask the question: if  $\begin{bmatrix} A & X \\ X^* & B \end{bmatrix} \in H_{2n}^+$ , do we have

$$\lambda \left( \begin{bmatrix} A & X \\ X^* & B \end{bmatrix} \right) \prec \lambda(A+B)?$$

Generally, the answer is no as the following examples shows.

**Example 3.15.** Let  $A = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$ ,  $B = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$  and  $X = \begin{bmatrix} 1 & 0 \\ 2 & 2 \end{bmatrix}$ . Then

$$\begin{aligned} \lambda(A+B) &= (4+\sqrt{2}, 4-\sqrt{2}), \\ \lambda \left( \begin{bmatrix} A & X \\ X^* & B \end{bmatrix} \right) &= (4+\sqrt{5}, 4-\sqrt{5}), \end{aligned}$$

where the spectrum of  $\begin{bmatrix} A & X \\ X^* & B \end{bmatrix}$  ensures the block matrix is positive semidefinite.

However, if we add an additional requirement that the off block diagonal  $X$  is Hermitian, then the answer is affirmative. This is the main result of the next section.

## 3.2 Recent results

The following result has been published in [75]. It is joint work with Wolkowicz. We shall provide two proofs. Only after the publication of [75] and [20] did we get informed by K. Audenaert that Hiroshima [49] proved a more general result obtained from the consideration of quantum information science. However, the line of proof is quite different.

**Theorem 3.16.** If  $\begin{bmatrix} A & X \\ X^* & B \end{bmatrix} \in H_{2n}^+$  with  $X$  being Hermitian, then

$$\lambda \left( \begin{bmatrix} A & X \\ X^* & B \end{bmatrix} \right) \prec \lambda(A+B). \quad (3.8)$$

We need a simple lemma in our first proof.

**Lemma 3.17.** *If  $A, B \in H_n$ , then*

$$2\lambda(A) \prec \lambda(A+B) + \lambda(A-B). \quad (3.9)$$

*Proof.* The lemma is easily seen to be equivalent to Fan's majorization inequality (3.2), i.e.,  $\lambda(A+B) \prec \lambda(A) + \lambda(B)$ . A proof can be found in [51, Theorem 4.3.27].  $\square$

*First proof of (3.8).* Since  $\begin{bmatrix} A & X \\ X & B \end{bmatrix}$  is positive semidefinite, as before we may write

$$\begin{bmatrix} A & X \\ X & B \end{bmatrix} = [M, N]^* [M, N],$$

for some  $M \in M_{2n,n}(\mathbb{F})$  and  $N \in M_{2n,n}(\mathbb{F})$ . Therefore, we have  $A = M^*M$ ,  $B = N^*N$  and  $X = M^*N = N^*M$ . Note that  $\lambda\left(\begin{bmatrix} A & X \\ X & B \end{bmatrix}\right) = \lambda([M, N][M, N]^*) = \lambda(MM^* + NN^*)$ . The conclusion is then equivalent to showing

$$M^*N = N^*M \implies \lambda(MM^* + NN^*) \prec \lambda(M^*M + N^*N). \quad (3.10)$$

First, note that

$$\begin{aligned} (M + iN)^*(M + iN) &= M^*M + N^*N + i(M^*N - N^*M) \\ &= M^*M + N^*N \end{aligned}$$

$$\begin{aligned} (M - iN)^*(M - iN) &= M^*M + N^*N - i(M^*N - N^*M) \\ &= M^*M + N^*N \end{aligned}$$

$$(M + iN)(M + iN)^* = MM^* + NN^* - i(MN^* - NM^*)$$

$$(M - iN)(M - iN)^* = MM^* + NN^* + i(MN^* - NM^*).$$

Therefore we see that

$$\begin{aligned} \lambda(M^*M + N^*N) &= \frac{1}{2} \{ \lambda((M + iN)^*(M + iN)) + \lambda((M - iN)^*(M - iN)) \} \\ &= \frac{1}{2} \{ \lambda((M + iN)(M + iN)^*) + \lambda((M - iN)(M - iN)^*) \} \\ &\succ \lambda(MM^* + NN^*), \end{aligned}$$

where the majorization is by applying Lemma 3.17 with  $A = (MM^* + NN^*)$ ,  $B = i(MN^* - NM^*)$ .  $\square$

For  $A, B \in H_n$ , we write  $A \succeq B$  (the same as  $B \preceq A$ ) to mean  $A - B$  is positive semidefinite. Thus  $A \succeq 0$  is the same as saying  $A \in H_n^+$ . This notion is the so called *Löwner partial order*; see e.g., [14].  $A^{1/2}$  means the unique square root of  $A \in H_n^+$ , which is also positive semidefinite. Now we can introduce the *absolute value* of a general matrix  $A \in M_{m \times n}(\mathbb{F})$ , defined by  $|A| = (A^*A)^{1/2}$ .

After defining the object  $|A|$ , the authors of the book [95] warn “The reader should be wary of the emotional connotations of the symbol  $|\cdot|$ ”. This is due to negative answers to some plausible inequalities. For example, among other things, the prospective triangle inequality

$$|A + B| \preceq |A| + |B|$$

is not true in general. Also,  $|A - B| \preceq |A| + |B|$  is not true for  $A, B \in H_n^+$ , see, e.g., [17]. Let  $A, B \in H_n^{++}$ . Their *geometric mean*  $A\sharp B$  is defined by two quite natural requirements:

- $AB = BA$  implies  $A\sharp B = \sqrt{AB}$ ,
- $(X^*AX)\sharp(X^*BX) = X^*(A\sharp B)X$  for any invertible  $X$ .

Then, we must have

$$A\sharp B = A^{1/2}(I\sharp A^{-1/2}BA^{-1/2})A^{1/2} = A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2},$$

i.e.,  $A\sharp B := A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}$ . This is the commonly accepted definition of the geometric mean of two positive definite matrices.

In recent years there has been added interest in this object because of its connections with Riemannian geometry, e.g., [15].

A differentiable function  $\gamma: [0, 1] \rightarrow H_n^{++}$  is called a curve, its tangent vector at  $t$  is  $\gamma'(t)$  and the length of the curve is

$$\int_0^1 \sqrt{g_{\gamma(t)}(\gamma'(t), \gamma'(t))} dt.$$

Here the inner product on the tangent space at  $A \in H_n^{++}$  is  $g_A(H_1, H_2) = \text{Tr} A^{-1}H_1A^{-1}H_2$ . Note that this geometry has many symmetries: each similarity transformation of the matrices becomes a symmetry. Namely,

$$g_{S^{-1}AS^{-1}}(S^{-1}H_1S^{-1}, S^{-1}H_2S^{-1}) = g_A(H_1, H_2).$$

Given  $A, B \in H_n^{++}$  the curve

$$\gamma(t) = A^{1/2}(A^{-1/2}BA^{-1/2})^t A^{1/2} \quad (0 \leq t \leq 1)$$

connects the following two points:  $\gamma(0) = A$ ,  $\gamma(1) = B$ . This is the shortest curve connecting the two points, and is called a geodesic, e.g., [91]. Thus, the geometric mean  $A\sharp B$  is just the mid point of the geodesic curve.

A remarkable property of the geometric mean is a *maximal characterization* by Pusz-Woronowicz [92]:

**Proposition 3.18.** *Let  $A, B \in H_n^{++}$ . Then*

$$A\sharp B = \max \left\{ X : \begin{bmatrix} A & X \\ X & B \end{bmatrix} \succeq 0, X = X^* \right\}.$$

*The maximization here is in the sense of Löwner partial order.*

*Proof.* Since  $\begin{bmatrix} A & X \\ X & B \end{bmatrix}$  is positive semidefinite, then  $B \succeq XA^{-1}X$ , and hence

$$A^{-1/2}BA^{-1/2} \succeq A^{-1/2}XA^{-1}XA^{-1/2} = (A^{-1/2}XA^{-1/2})^2.$$

By the operator monotonicity<sup>1</sup> of the square root functions (see, e.g., [15]), this leads to

$$A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2} \succeq X.$$

On the other hand, if  $X = A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}$ , then  $B = XA^{-1}X$ . This shows the maximality property of  $A\sharp B$ .  $\square$

In the above proof, with  $A, B \in H_n^+$ , we only require that  $A$  is positive definite. Therefore, Proposition 3.18 tells us that  $A\sharp B$  is the largest positive matrix  $X$  such that  $\begin{bmatrix} A & X \\ X & B \end{bmatrix}$  is positive semidefinite. This can be used as the definition of  $A\sharp B$  for non-invertible  $A$ . An equivalent possibility is

$$A\sharp B := \lim_{\varepsilon \rightarrow 0} (A + \varepsilon I)\sharp B.$$

An immediate consequence of Theorem 3.16 is the following

---

<sup>1</sup>A real-valued continuous function  $f(t)$  defined on a real interval  $\Lambda$  is said to be operator monotone if  $A \preceq B$  implies  $f(A) \preceq f(B)$  for all such Hermitian matrices  $A, B$  of all orders whose eigenvalues are contained in  $\Lambda$ , see e.g., [113].

**Corollary 3.19.** *Let  $A, B \in H_n^+$ , then*

$$\lambda \left( \begin{bmatrix} A & A\sharp B \\ A\sharp B & B \end{bmatrix} \right) \prec \lambda(A + B). \quad (3.11)$$

To my best knowledge, we know fairly little about  $\lambda \left( \begin{bmatrix} A & A\sharp B \\ A\sharp B & B \end{bmatrix} \right)$  besides (3.11).

The second proof of (3.8) is made possible by a powerful decomposition lemma for positive definite matrices, which is of independent interest. I will present the decomposition lemma in a separate section, followed by the second proof. The remaining part of this section is devoted to several applications of Theorem 3.16.

As we can see from the first proof of (3.8), a special case of Theorem 3.16 can be stated as follows.

**Corollary 3.20.** *Let  $M, N \in M_n(\mathbb{F})$  with  $M^*N$  Hermitian. Then we have*

$$\lambda(MM^* + NN^*) \prec \lambda(M^*M + N^*N).$$

**Corollary 3.21.** *Let  $k \geq 1$  be an integer. If  $A, B \in H_n$ , then we have*

$$\lambda(A^2 + (BA)^k(AB)^k) \prec \lambda(A^2 + (AB)^k(BA)^k).$$

*Proof.* Let  $M = A$  and  $N = (BA)^k$ . Then  $M^*N = A(BA)^k$  is Hermitian. The result now follows from Corollary 3.20.  $\square$

**Corollary 3.22.** *Let  $k \geq 1$  be an integer,  $p \in [0, \infty)$  and let  $A, B \in H_n$ . Then we have*

1.  $\text{Tr}[(A^2 + (AB)^k(BA)^k)^p] \geq \text{Tr}[(A^2 + (BA)^k(AB)^k)^p], \quad p \geq 1;$
2.  $\text{Tr}[(A^2 + (AB)^k(BA)^k)^p] \leq \text{Tr}[(A^2 + (BA)^k(AB)^k)^p], \quad 0 \leq p \leq 1.$

*Proof.* Since  $f(x) = x^p$  is a convex function for  $p \geq 1$  and concave for  $0 \leq p \leq 1$ , Corollary 3.22 follows from Corollary 3.21 and Proposition 3.3.  $\square$

**Corollary 3.23.** *Let  $A, B \in H_n^{++}$ , then*

$$\text{Tr}[(A^2 + AB^2A)^{-1}] \geq \text{Tr}[(A^2 + BA^2B)^{-1}].$$

*Proof.* Note that  $g(x) = x^{-1}$  is a convex function on  $(0, \infty)$ . Corollary 3.23 follows from Corollary 3.21 and Proposition 3.3.  $\square$

**Corollary 3.24.** *If  $A, B \in H_n^+$ , then*

$$\det(A^2 + AB^2A) \leq \det(A^2 + BA^2B).$$

*Proof.* By Corollary 3.21, we have  $\lambda(A^2 + AB^2A) \succ \lambda(A^2 + BA^2B)$ . Applying Corollary 3.5 with  $k = 1$ , we get  $\prod_{j=1}^n \lambda_j(A^2 + AB^2A) \leq \prod_{j=1}^n \lambda_j(A^2 + BA^2B)$ , i.e.,  $\det(A^2 + AB^2A) \leq \det(A^2 + BA^2B)$ . This completes the proof.  $\square$

**Remark 3.25.** *A slightly different argument can be found in [82].*

In [39], the following conjecture was posed.

**Conjecture 3.26.** *If  $X, Y \in H_n^+$  and  $p \in \mathbb{R}$ , then*

$$(i) \quad \text{Tr}[(I + X + Y + Y^{1/2}XY^{1/2})^p] \leq \text{Tr}[(I + X + Y + XY)^p], \quad p \geq 1.$$

$$(ii) \quad \text{Tr}[(I + X + Y + Y^{1/2}XY^{1/2})^p] \geq \text{Tr}[(I + X + Y + XY)^p], \quad 0 \leq p \leq 1.$$

We firstly note that the matrix  $I + X + Y + XY = (I + X)(I + Y)$  is generally not positive semidefinite. However, the eigenvalues of the matrix  $(I + X)(I + Y)$  are the same as those of the positive semidefinite matrix  $(I + X)^{1/2}(I + Y)(I + X)^{1/2}$ . Therefore the expression  $\text{Tr}[(I + X + Y + XY)^p]$  makes sense.

We easily find that the equality for (i) and (ii) in Conjecture 3.26 holds in the case of  $p = 1$ . In addition, the case of  $p = 2$  was proven by elementary calculations in [39].

Putting  $A = (I + X)^{1/2}$  and  $B = Y^{1/2}$ , Conjecture 3.26 can be equivalently reformulated as the following one (now a theorem), because we have

$$\begin{aligned} \text{Tr}[(I + X + Y + XY)^p] &= \text{Tr}[(A^2 + A^2B^2)^p] = \text{Tr}[(A^2(I + B^2))^p] \\ &= \text{Tr}[(A(I + B^2)A)^p] = \text{Tr}[(A^2 + AB^2A)^p]. \end{aligned}$$

**Theorem 3.27.** *If  $A, B \in H_n^+$  and  $p \in \mathbb{R}$ , then*

$$\text{Tr}[(A^2 + BA^2B)^p] \leq \text{Tr}[(A^2 + AB^2A)^p], \quad p \geq 1;$$

$$\text{Tr}[(A^2 + BA^2B)^p] \geq \text{Tr}[(A^2 + AB^2A)^p], \quad 0 \leq p \leq 1.$$

It is then clear that this is just the case where  $k = 1$  in Corollary 3.22. Thus Conjecture 3.26 will be referred to as Theorem 3.26.

In statistical mechanics, Golden [42] has proved that if  $A, B \in H_n^+$  then the inequality

$$\operatorname{Tr} e^A e^B \geq \operatorname{Tr} e^{A+B} \quad (3.12)$$

holds. Independently, Thompson [101] proved (3.12) for Hermitian  $A, B$  without the requirement of definiteness. As an application of Theorem 3.26, we shall give a one-parameter extension of the *Golden-Thompson inequality*.

We define  $\exp_\nu(x) \equiv (1 + \nu x)^{\frac{1}{\nu}}$  if  $1 + \nu x > 0$ , and otherwise it is undefined. It is clear that  $\lim_{\nu \rightarrow 0} \exp_\nu(x) = e^x$ .

We also need the following proposition proved in [38]. For completeness, we include the simple proof.

**Proposition 3.28.** [38] For  $A, B \in H_n^+$ , and  $\nu \in (0, 1]$ , we have

$$\operatorname{Tr}[\exp_\nu(A + B)] \leq \operatorname{Tr}[\exp_\nu(A + B + \nu B^{1/2} A B^{1/2})]; \quad (3.13)$$

$$\operatorname{Tr}[\exp_\nu(A + B + \nu A B)] \leq \operatorname{Tr}[\exp_\nu(A) \exp_\nu(B)]. \quad (3.14)$$

*Proof.* Since  $B^{1/2} A B^{1/2} \in H_n^+$ , we have

$$I + \nu(A + B) \leq I + \nu(A + B + \nu(B^{1/2} A B^{1/2})).$$

Proposition 3.4 tells us

$$\operatorname{Tr}[(I + \nu(A + B)^\nu)^{1/\nu}] \leq \operatorname{Tr}[(I + \nu(A + B + \nu(B^{1/2} A B^{1/2})))^{1/\nu}]$$

for  $0 < \nu \leq 1$ . This proves the first claim. For the second one, the *Lieb-Thirring inequality* [71] says  $\operatorname{Tr}[(XY)^{1/\nu}] \leq \operatorname{Tr}[X^{1/\nu} Y^{1/\nu}]$  for any  $X, Y \in H_n^+$ . Now putting  $X = I + \nu A$ ,  $Y = I + \nu B$ , we have

$$\operatorname{Tr}[(I + \nu A)(I + \nu B)]^{1/\nu} \leq \operatorname{Tr}[(I + \nu A)^{1/\nu} (I + \nu A)^{1/\nu}],$$

as desired. □

By Theorem 3.26 and Proposition 3.28, we have the following proposition.

**Proposition 3.29.** For  $A, B \in H_n^+$  and  $\nu \in (0, 1]$ , we have

$$\mathrm{Tr}[\exp_\nu(A + B)] \leq \mathrm{Tr}[\exp_\nu(A) \exp_\nu(B)]. \quad (3.15)$$

*Proof.* It suffices to show the RHS (i.e., right hand side) of (3.13) is bounded from above by the LHS (i.e., left hand side) of (3.14). Putting  $A_1 = \nu A$ ,  $B_1 = \nu B$  and  $p = \frac{1}{\nu}$ , one obtains

$$\begin{aligned} \mathrm{Tr} \left[ \exp_\nu(A + B + \nu B^{1/2} A B^{1/2}) \right] &= \mathrm{Tr} \left[ \left\{ I + \nu(A + B + \nu B^{1/2} A B^{1/2}) \right\}^{\frac{1}{\nu}} \right] \\ &= \mathrm{Tr} \left[ (I + A_1 + B_1 + B_1^{1/2} A_1 B_1^{1/2})^p \right] \\ &\leq \mathrm{Tr} \left[ (I + A_1 + B_1 + A_1 B_1)^p \right] \\ &= \mathrm{Tr} \left[ \left\{ I + \nu(A + B + \nu AB) \right\}^{\frac{1}{\nu}} \right] \\ &= \mathrm{Tr}[\exp_\nu(A + B + \nu AB)], \end{aligned}$$

This completes the proof.  $\square$

**Remark 3.30.** Though we have a positivity requirement on  $A, B$ , in the proof, we only need  $I + \nu A > 0$  and  $I + \nu B > 0$  for  $0 < \nu \leq 1$ . As  $\nu \rightarrow 0^+$ , we necessarily have  $I + \nu A > 0$  and  $I + \nu B > 0$  for Hermitian matrices  $A$  and  $B$ . In this sense, inequality (3.15) can be regarded as a kind of one-parameter extension of the Golden-Thompson inequality.

The simplest proof of Golden-Thompson inequality (3.12) appeals to the Lieb-Thirring inequality and the following exponential product formula for matrices

**Proposition 3.31.** For any  $A, B \in M_n(\mathbb{F})$ ,

$$\lim_{p \rightarrow \infty} (e^{A/p} e^{B/p})^p = \lim_{p \rightarrow \infty} (e^{B/2p} e^{A/p} e^{B/2p})^p = e^{A+B}.$$

It is worthwhile to note that [25] contains interesting historical remarks concerning the previous proposition.

**Remark 3.32.** A remarkable extension of the Golden-Thompson inequality is due to Cohen et al. [25], which says for any  $A, B \in M_n(\mathbb{F})$ ,

$$\mathrm{Tr} e^{(A+A^*)/2} e^{(B+B^*)/2} \geq |\mathrm{Tr} e^{A+B}|.$$

We end this section with a question.

**Question 3.33.** Let  $A, B \in M_n$  such that  $A^* B$  is Hermitian. Is it true that

$$\lambda(|A^*| + |B^*|) \prec \lambda(|A| + |B|)?$$



### 3.3 A decomposition lemma for positive definite matrices

For positive block-matrices,

$$\begin{bmatrix} A & X \\ X^* & B \end{bmatrix} \in H_{n+m}^+, \quad \text{with } A \in H_n^+, B \in H_m^+,$$

we have a remarkable *decomposition lemma* for elements in  $H_{n+m}^+$  observed in [19]:

**Lemma 3.34.** *For every matrix in  $H_{n+m}^+$  written in blocks, we have the decomposition*

$$\begin{bmatrix} A & X \\ X^* & B \end{bmatrix} = U \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} U^* + V \begin{bmatrix} 0 & 0 \\ 0 & B \end{bmatrix} V^*$$

for some unitaries  $U, V \in M_{n+m}(\mathbb{F})$ .

The motivation for such a decomposition is various inequalities for convex or concave functions of positive operators partitioned in blocks. These results are extensions of some classical majorization, Rofel'd and Minkowski type inequalities. Lemma 3.34 actually implies a host of such inequalities as shown in the recent papers [18] and [19] where a proof of Lemma 3.34 can be found too. Here we also include the simple proof. Positivity of  $\begin{bmatrix} A & X \\ X^* & B \end{bmatrix}$  tells us there is a Hermitian matrix  $\begin{bmatrix} C & Y \\ Y^* & D \end{bmatrix}$ , conformally partitioned such that

$$\begin{bmatrix} A & X \\ X^* & B \end{bmatrix} = \begin{bmatrix} C & Y \\ Y^* & D \end{bmatrix} \begin{bmatrix} C & Y \\ Y^* & D \end{bmatrix}$$

and observe that it can be written as

$$\begin{bmatrix} C & 0 \\ Y^* & 0 \end{bmatrix} \begin{bmatrix} C & Y \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & Y \\ 0 & D \end{bmatrix} \begin{bmatrix} 0 & 0 \\ Y^* & D \end{bmatrix} = T^*T + S^*S.$$

Then, use the fact that  $T^*T$  and  $S^*S$  are unitarily congruent to

$$TT^* = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad SS^* = \begin{bmatrix} 0 & 0 \\ 0 & B \end{bmatrix}.$$

### 3.4 Several norm inequalities

Most of the result in this section has appeared in [20]. It is joint work with Bourin and Lee.

If  $A$  is a linear operator on  $\mathbb{F}^n$ , the *operator norm* of  $A$ , denoted by  $\|\cdot\|_\infty$ , is defined as

$$\|A\|_\infty = \sup_{\|x\|=1} \|Ax\|.$$

A norm  $\|\cdot\|$  on  $M_n(\mathbb{F})$  is called symmetric if for  $A, B, C \in M_n(\mathbb{F})$

$$\|BAC\| \leq \|B\|_\infty \|A\| \|C\|_\infty.$$

Classical *symmetric norms* include *Ky Fan  $k$ -norms*, denoted by  $\|\cdot\|_k$ ,  $k = 1, 2, \dots, n$ , where  $n$  is the size of the matrix<sup>2</sup>; and the usual *Schatten  $p$ -norms*, denoted by  $\|\cdot\|_p$ ,  $1 \leq p < \infty$ ; see, e.g., [52].

**Proposition 3.35.** [14, p. 94] *A norm on  $M_n(\mathbb{F})$  is symmetric if and only if it is unitarily invariant, i.e.,  $\|UAV\| = \|A\|$  for any unitaries  $U, V \in M_n(\mathbb{F})$ .*

By the *Fan dominance theorem* [14], given  $A, B \in H_n^+$ , the following two conditions are equivalent:

- (i)  $\|A\| \leq \|B\|$  for all symmetric norms.
- (ii)  $\sum_{j=1}^k \lambda_j(A) \leq \sum_{j=1}^k \lambda_j(B)$  for all  $k = 1, 2, \dots, n$ .

In particular,  $\lambda(A) \prec \lambda(B)$  implies  $\|A\| \leq \|B\|$  for every symmetric norm for  $A, B \in H_n^+$ . Most of the corollaries below are rather straightforward consequences of Lemma 3.34, except Corollary 3.42 which also requires some more elaborate estimates. If we first use a *unitary congruence* with

$$J = \frac{1}{\sqrt{2}} \begin{bmatrix} I & -I \\ I & I \end{bmatrix}$$

where  $I$  is the identity of  $M_n$ , we observe that

$$J^* \begin{bmatrix} A & X \\ X^* & B \end{bmatrix} J = \begin{bmatrix} \frac{A+B}{2} + \operatorname{Re}X & \star \\ \star & \frac{A+B}{2} - \operatorname{Re}X \end{bmatrix}$$

where  $\star$  stands for unspecified entries and  $\operatorname{Re}X = (X + X^*)/2$ , the so called *Hermitian part* of a square matrix  $X$ .

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<sup>2</sup>When  $k = 1, n$ , it is just operator norm, trace norm, respectively.

**Remark 3.36.** *If we take*

$$K = \frac{1}{\sqrt{2}} \begin{bmatrix} I & I \\ I & -I \end{bmatrix},$$

*again, we have*

$$K \begin{bmatrix} A & X \\ X^* & B \end{bmatrix} K^* = \begin{bmatrix} \frac{A+B}{2} + \operatorname{Re}X & \star \\ \star & \frac{A+B}{2} - \operatorname{Re}X \end{bmatrix}.$$

*A special case of  $K$ , i.e.,  $\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$  is called the Hadamard gate. It sends the basis vectors into uniform superposition and vice versa. For more information of this kind of matrix; see [91, p. 122].*

Thus Lemma 3.34 yields:

**Proposition 3.37.** *For every matrix in  $H_{2n}^+$  written in blocks of the same size, we have a decomposition*

$$\begin{bmatrix} A & X \\ X^* & B \end{bmatrix} = U \begin{bmatrix} \frac{A+B}{2} + \operatorname{Re}X & 0 \\ 0 & 0 \end{bmatrix} U^* + V \begin{bmatrix} 0 & 0 \\ 0 & \frac{A+B}{2} - \operatorname{Re}X \end{bmatrix} V^*$$

*for some unitaries  $U, V \in M_{2n}(\mathbb{F})$ .*

This is equivalent to Proposition 3.38 below by the obvious unitary congruence

$$\begin{bmatrix} iI & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A & X \\ X^* & B \end{bmatrix} \begin{bmatrix} iI & 0 \\ 0 & I \end{bmatrix}^* = \begin{bmatrix} A & iX \\ -iX^* & B \end{bmatrix}.$$

**Proposition 3.38.** *For every matrix in  $H_{2n}^+$  written in blocks of the same size, we have a decomposition*

$$\begin{bmatrix} A & X \\ X^* & B \end{bmatrix} = U \begin{bmatrix} \frac{A+B}{2} + \operatorname{Im}X & 0 \\ 0 & 0 \end{bmatrix} U^* + V \begin{bmatrix} 0 & 0 \\ 0 & \frac{A+B}{2} - \operatorname{Im}X \end{bmatrix} V^*$$

*for some unitaries  $U, V \in M_{2n}(\mathbb{F})$ .*

Here  $\operatorname{Im}X = (X - X^*)/2i$ , the so called *skew-Hermitian part* of  $X$ . The decomposition allows us to obtain some norm estimates depending on how far the full matrix is from a block-diagonal matrix.

*Second proof of (3.8).* From Proposition 3.37 and Proposition 3.38, we know that if  $X$  is skew-Hermitian or Hermitian, i.e.,  $\operatorname{Re} X = 0$  or  $\operatorname{Im} X = 0$ , by using Fan's inequality (3.2), (3.8) follows immediately.  $\square$

Now, by noticing that  $\operatorname{Im} X \preceq |\operatorname{Im} X| = \frac{1}{2}|X - X^*|$ , we have:

**Proposition 3.39.** *For every matrix in  $H_{2n}^+$  written in blocks of the same size, we have*

$$\begin{bmatrix} A & X \\ X^* & B \end{bmatrix} \preceq \frac{1}{2} \left\{ U \begin{bmatrix} A+B+|X-X^*| & 0 \\ 0 & 0 \end{bmatrix} U^* + V \begin{bmatrix} 0 & 0 \\ 0 & A+B+|X-X^*| \end{bmatrix} V^* \right\}$$

for some unitaries  $U, V \in M_{2n}(\mathbb{F})$ .

The remaining part of this section is devoted to inequalities. Since a symmetric norm on  $M_{n+m}(\mathbb{F})$  induces a symmetric norm on  $M_n(\mathbb{F})$ , we may assume that our norms are defined on all spaces  $M_n(\mathbb{F})$ ,  $n \geq 1$ .

We start with a lemma.

**Lemma 3.40.** *If  $S, T \in H_n^+$  and if  $f : [0, \infty) \rightarrow [0, \infty)$  is concave, then, for some unitary  $U, V \in M_n(\mathbb{F})$ ,*

$$f(S+T) \preceq Uf(S)U^* + Vf(T)V^*. \quad (3.16)$$

The lemma can be found in [7] (for a proof see also, [19, Section 3]).

(3.16) is a matrix version of the scalar inequality  $f(a+b) \leq f(a) + f(b)$  for positive concave functions  $f$  on  $[0, \infty)$ . This inequality via unitary orbits considerably improves the famous Rotfel'd trace inequality for non-negative concave functions and positive operators,

$$\operatorname{Tr} f(A+B) \leq \operatorname{Tr} f(A) + \operatorname{Tr} f(B),$$

and its symmetric norm version

$$\|f(A+B)\| \leq \|f(A)\| + \|f(B)\|,$$

Combined with Lemma 3.34, (3.16) entails a recent result of Lee [66], which states: Let  $f(t)$  be a non-negative concave function on  $[0, \infty)$ . Then, given an arbitrary partitioned positive semi-definite matrix,

$$\left\| f \left( \begin{bmatrix} A & X \\ X^* & B \end{bmatrix} \right) \right\| \leq \|f(A)\| + \|f(B)\|,$$

for all symmetric norms.

**Remark 3.41.** Specified to  $f(x) = |x|$ , obviously, the condition of Lemma 3.40 is satisfied. In this context, a remarkable property due to Thompson says that for any  $A, B \in M_n(\mathbb{F})$ , there are unitary matrices  $U$  and  $V$  such that

$$|A + B| \preceq U|A|U^* + V|B|V^*.$$

We refer to [115, p. 289] for a proof.

**Proposition 3.42.** For every matrix in  $H_{2n}^+$  written in blocks of the same size and for all symmetric norms, we have

$$\left\| \begin{bmatrix} A & X \\ X^* & B \end{bmatrix}^p \right\| \leq 2^{p-1} \{ \|(A+B)^p\| + \| |X - X^*|^p \| \}$$

for all  $p > 0$ .

*Proof.* We first show the case  $0 < p < 1$ . Applying (3.16) to  $f(t) = t^p$  and the RHS of Proposition 3.39 with

$$S = \frac{1}{2}U \begin{bmatrix} A+B+|X-X^*| & 0 \\ 0 & 0 \end{bmatrix} U^*, \quad T = \frac{1}{2}V \begin{bmatrix} 0 & 0 \\ 0 & A+B+|X-X^*| \end{bmatrix} V^*$$

we obtain

$$\left\| \begin{bmatrix} A & X \\ X^* & B \end{bmatrix}^p \right\| \leq 2^{1-p} \{ \|(A+B+|X-X^*|)^p\| \}.$$

Applying again (3.16) with  $f(t) = t^p$ ,  $S = A+B$  and  $T = |X - X^*|$  yields the result for  $0 < p < 1$ .

To get the inequality for  $p \geq 1$ , it suffices to use in the RHS of Proposition 3.39 the elementary inequality, for  $S, T \in H_n^+$ ,

$$\left\| \left( \frac{S+T}{2} \right)^p \right\| \leq \frac{\|S^p\| + \|T^p\|}{2} \quad (3.17)$$

With

$$S = U \begin{bmatrix} A+B+|X-X^*| & 0 \\ 0 & 0 \end{bmatrix} U^*,$$

$$T = V \begin{bmatrix} 0 & 0 \\ 0 & A+B+|X-X^*| \end{bmatrix} V^*$$

we get from Proposition 3.39 and (3.17)

$$\left\| \begin{bmatrix} A & X \\ X^* & B \end{bmatrix}^p \right\| \leq \|(A + B + |X - X^*|)^p\|$$

and another application of (3.17) with  $S = 2(A + B)$  and  $T = 2|X - X^*|$  completes the proof.  $\square$

**Proposition 3.43.** *For any matrix in  $H_{2n}^+$  written in blocks of the same size such that the right upper block  $X$  is accretive (i.e.,  $\operatorname{Re}X$  is positive semidefinite), we have*

$$\left\| \begin{bmatrix} A & X \\ X^* & B \end{bmatrix} \right\| \leq \|A + B\| + \|\operatorname{Re}X\|$$

for all symmetric norms.

*Proof.* By Proposition 3.38, for all Ky Fan  $k$ -norms  $\|\cdot\|_k$ ,  $k = 1, \dots, 2n$ , we have

$$\left\| \begin{bmatrix} A & X \\ X^* & B \end{bmatrix} \right\|_k \leq \left\| \begin{bmatrix} \frac{A+B}{2} + \operatorname{Re}X & 0 \\ 0 & 0 \end{bmatrix} \right\|_k + \left\| \begin{bmatrix} 0 & 0 \\ 0 & \frac{A+B}{2} \end{bmatrix} \right\|_k.$$

Equivalently,

$$\left\| \begin{bmatrix} A & X \\ X^* & B \end{bmatrix} \right\|_k \leq \left\| \left( \frac{A+B}{2} + \operatorname{Re}X \right)^\downarrow \right\|_k + \left\| \left( \frac{A+B}{2} \right)^\downarrow \right\|_k$$

where  $Z^\downarrow$  stands for the diagonal matrix listing the eigenvalues of  $Z \in H_n^+$  in decreasing order. By using the triangle inequality for  $\|\cdot\|_k$  and the fact that

$$\|Z_1^\downarrow\|_k + \|Z_2^\downarrow\|_k = \|(Z_1^\downarrow + Z_2^\downarrow)\|_k$$

for all  $Z_1, Z_2 \in H_n^+$  we infer

$$\left\| \begin{bmatrix} A & X \\ X^* & B \end{bmatrix} \right\|_k \leq \left\| (A+B)^\downarrow + (\operatorname{Re}X)^\downarrow \right\|_k.$$

Hence

$$\left\| \begin{bmatrix} A & X \\ X^* & B \end{bmatrix} \right\| \leq \left\| (A+B)^\downarrow + (\operatorname{Re}X)^\downarrow \right\|$$

for all symmetric norms. The triangle inequality completes the proof.  $\square$

Recall that the *field of values*<sup>3</sup>  $W(\cdot)$  is a set of complex numbers naturally associated with a given  $n$ -by- $n$  matrix  $A$ :

$$W(A) = \{x^*Ax : x \in \mathbb{C}^n, x^*x = 1\}.$$

The spectrum (i.e., the set of eigenvalues) of a matrix is a discrete point set, while the field of values can be a continuum; it is always a compact convex set. The *numerical radius* of  $A \in M_n(\mathbb{F})$  is

$$w(A) = \max\{|z| : z \in W(A)\}.$$

It is easy to observe that for all  $A \in M_n(\mathbb{F})$ ,  $W(\operatorname{Re}A) = \operatorname{Re}W(A)$ ; see e.g., [52, p. 9]. This immediately leads to

**Corollary 3.44.** *Let  $A \in M_n(\mathbb{F})$ . Then  $W(A)$  is contained in RHP (i.e., right half plane) if and only if  $\operatorname{Re}A$  is positive definite.*

We need the following interesting fact that characterizes whether the origin is in the field of values.

**Lemma 3.45.** *Let  $A \in M_n(\mathbb{F})$  be given. Then  $0 \notin W(A)$  if and only if there exists a complex number  $z$  such that  $\operatorname{Re}zA$  is positive definite.*

*Proof.* The proof here is adapted from [52, p. 21]. If  $\operatorname{Re}zA$  is positive definite for some  $z \in \mathbb{C}$ , then  $0 \notin W(A)$  by Corollary 3.44. Conversely, suppose  $0 \notin W(A)$ . By the separating hyperplane theorem (see e.g., [24]), there is a line  $L$  in the plane such that each of the two nonintersecting compact convex sets  $\{0\}$  and  $W(A)$  lies entirely within exactly one of the two open half-planes determined by  $L$ . The coordinate axes may now be rotated so that the line  $L$  is carried into a vertical line in the right half-plane with  $W(A)$  strictly to the right of it, that is, for some  $z \in \mathbb{C}$ ,  $W(zA) = zW(A) \subset \text{RHP}$ , so  $W(zA)$  is positive definite by Corollary 3.44.  $\square$

A classical bound for numerical radius in terms of operator norms can be found, e.g., in [52, p. 44]:

$$\frac{1}{2}\|A\|_\infty \leq w(A) \leq \|A\|_\infty,$$

and both bounds are sharp.

---

<sup>3</sup>This is the same as the term “*numerical range*”.

**Proposition 3.46.** *For any matrix in  $H_{2n}^+$  written in blocks of the same size such that  $0 \notin W(X)$ , the numerical range of the right upper block  $X$ , we have*

$$\left\| \begin{bmatrix} A & X \\ X^* & B \end{bmatrix} \right\| \leq \|A + B\| + \|X\|$$

for all symmetric norms.

*Proof.* By Lemma 3.45 the condition  $0 \notin W(X)$  means that  $zX$  is accretive for some complex number  $z$  in the unit circle. Making use of the unitary congruence

$$\begin{bmatrix} A & X \\ X^* & B \end{bmatrix} \simeq \begin{bmatrix} A & zX \\ \bar{z}X^* & B \end{bmatrix}$$

we obtain the result from Corollary 3.43. □

The condition  $0 \notin W(X)$  in the previous corollary can obviously be relaxed to 0 does not belong to the relative interior<sup>4</sup> of  $X$ , denoted by  $W_{int}(X)$ . In case of the usual operator norm  $\|\cdot\|_\infty$ , this can be restated with the numerical radius  $w(X)$ :

**Corollary 3.47.** *For any matrix in  $H_{2n}^+$  written in blocks of same size such that  $0 \notin W_{int}(X)$ , the relative interior of the numerical range of the right upper block  $X$ , we have*

$$\left\| \begin{bmatrix} A & X \\ X^* & B \end{bmatrix} \right\|_\infty \leq \|A + B\|_\infty + w(X).$$

In case of the operator norm, we also infer from Proposition 3.37 the following result:

**Corollary 3.48.** *For any matrix in  $H_{2n}^+$  written in blocks of the same size, we have*

$$\left\| \begin{bmatrix} A & X \\ X^* & B \end{bmatrix} \right\|_\infty \leq \|A + B\|_\infty + 2w(X).$$

Once again, the proof follows by replacing  $X$  by  $zX$  where  $z$  is a complex number of modulus 1 such that  $w(X) = w(zX) = \|\operatorname{Re}(zX)\|_\infty$  and then by applying Proposition 3.37.

---

<sup>4</sup>Intuitively, the relative interior of a set contains all points which are not on the “edge” of the set, relative to the smallest subspace in which this set lies.



**Example 3.49.** By letting

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad X = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

we have an equality case in the previous corollary. This example also gives an equality case in Proposition 3.42 for the operator norm and any  $p \geq 1$ . (For any  $0 < p < 1$  and for the trace norm, equality occurs in Proposition 3.42 with  $A = B$  and  $X = 0$ .)

Letting  $X = 0$  in the above corollary we get the basic inequality (3.5). We also have the last two corollaries.

**Corollary 3.50.** Given any matrix in  $H_{2n}^+$  written in blocks of the same size, we have

$$\left\| \begin{bmatrix} A & X \\ X^* & B \end{bmatrix} \oplus \begin{bmatrix} A & X \\ X^* & B \end{bmatrix} \right\| \leq 2\|A \oplus B\|$$

for all symmetric norms.

*Proof.* This follows from (3.5) and the obvious unitary congruence

$$\begin{bmatrix} A & X \\ X^* & B \end{bmatrix} \oplus \begin{bmatrix} A & X \\ X^* & B \end{bmatrix} \simeq \begin{bmatrix} A & X \\ X^* & B \end{bmatrix} \oplus \begin{bmatrix} A & -X \\ -X^* & B \end{bmatrix}$$

□

The previous corollary entails the last one:

**Corollary 3.51.** Given any matrix in  $H_{2n}^+$  written in blocks of same size, we have

$$\left\| \begin{bmatrix} A & X \\ X^* & B \end{bmatrix} \right\|_p \leq 2^{1-1/p} (\|A\|_p^p + \|B\|_p^p)^{1/p}$$

for all  $p \in [1, \infty)$ .

*Proof.*

$$\begin{aligned} 2 \left\| \begin{bmatrix} A & X \\ X^* & B \end{bmatrix} \right\|_p^p &= \left\| \begin{bmatrix} A & X \\ X^* & B \end{bmatrix} \oplus \begin{bmatrix} A & X \\ X^* & B \end{bmatrix} \right\|_p^p \\ &\leq 2^p \|A \oplus B\|_p^p \\ &= 2^p (\|A\|_p^p + \|B\|_p^p). \end{aligned}$$

Taking the  $p$ th root on both sides, this completes the proof. □

Note that if  $A = X = B$  we have an equality case in Corollary 3.51.

**Remark 3.52.** *Lemma 3.34 is still valid for compact operators on a Hilbert space, by taking  $U$  and  $V$  as partial isometries. A similar remark holds for the subadditivity inequality (3.16). Hence the symmetric norm inequalities in this section may be extended to the setting of normed ideals of compact operators.*

We have made the following conjecture in [20]:

**Conjecture 3.53.** *If  $\begin{bmatrix} A & N \\ N^* & B \end{bmatrix} \in H_{2n}^+$  with  $N \in M_n(\mathbb{F})$  being normal, then*

$$\lambda \left( \begin{bmatrix} A & N \\ N^* & B \end{bmatrix} \right) \prec \lambda(A+B).$$

C.-K. Li has sent us a counterexample. Each block in his example is  $3 \times 3$ . Thus we would ask

**Question 3.54.** *If  $\begin{bmatrix} A & N \\ N^* & B \end{bmatrix} \in H_4^+$  with  $N \in M_2(\mathbb{F})$  being normal, is it true that*

$$\lambda \left( \begin{bmatrix} A & N \\ N^* & B \end{bmatrix} \right) \prec \lambda(A+B)?$$

Also, we would like to ask

**Question 3.55.** *If  $\begin{bmatrix} A & N \\ N^* & B \end{bmatrix} \in H_{2n}^+$  with  $N \in M_n(\mathbb{F})$  being normal and  $AB = BA$ , is it true that*

$$\lambda \left( \begin{bmatrix} A & N \\ N^* & B \end{bmatrix} \right) \prec \lambda(A+B)?$$

### 3.5 Positive definite matrices with Hermitian blocks

The result of this section has been published in [21]. It is joint work with Bourin and Lee.

### 3.5.1 2-by-2 blocks

For partitions of positive matrices, the diagonal blocks play a special role.

**Theorem 3.56.** *Given any matrix in  $H_{2n}^+(\mathbb{C})$  partitioned into blocks in  $H_n(\mathbb{C})$  with Hermitian off-diagonal blocks, we have*

$$\begin{bmatrix} A & X \\ X & B \end{bmatrix} = \frac{1}{2} \{U(A+B)U^* + V(A+B)V^*\}$$

for some isometries  $U, V \in M_{2n \times n}(\mathbb{C})$ .

We detail here how it follows from Lemma 3.34.

*Proof.* Taking the unitary matrix

$$W = \frac{1}{\sqrt{2}} \begin{bmatrix} -iI & iI \\ I & I \end{bmatrix},$$

where  $I$  is the identity of  $M_n$ , then

$$W^* \begin{bmatrix} A & X \\ X & B \end{bmatrix} W = \frac{1}{2} \begin{bmatrix} A+B & * \\ * & A+B \end{bmatrix}$$

where  $*$  stands for unspecified entries. By Lemma 3.34, there are two unitaries  $U, V \in M_{2n}$  partitioned into equally sized matrices,

$$U = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix}, \quad V = \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix}$$

such that

$$\frac{1}{2} \begin{bmatrix} A+B & * \\ * & A+B \end{bmatrix} = \frac{1}{2} \left\{ U \begin{bmatrix} A+B & 0 \\ 0 & 0 \end{bmatrix} U^* + V \begin{bmatrix} 0 & 0 \\ 0 & A+B \end{bmatrix} V^* \right\}.$$

Therefore

$$\frac{1}{2} \begin{bmatrix} A+B & * \\ * & A+B \end{bmatrix} = \frac{1}{2} \left\{ \tilde{U}(A+B)\tilde{U}^* + \tilde{V}(A+B)\tilde{V}^* \right\}$$

where

$$\tilde{U} = \begin{bmatrix} U_{11} \\ U_{21} \end{bmatrix} \quad \text{and} \quad \tilde{V} = \begin{bmatrix} V_{12} \\ V_{22} \end{bmatrix}$$

are isometries. The proof is complete by assigning  $W\tilde{U}$ ,  $W\tilde{V}$  to new isometries  $U, V$ , respectively.  $\square$

As a consequence of this inequality we have a refinement of a well-known determinantal inequality,

$$\det(I+A)\det(I+B) \geq \det(I+A+B)$$

for all  $A, B \in H_n^+$ .

**Corollary 3.57.** *Let  $A, B \in H_n^+$ . For any Hermitian  $X \in M_n$  such that  $H = \begin{bmatrix} A & X \\ X & B \end{bmatrix}$  is positive semi-definite, we have*

$$\det(I+A)\det(I+B) \geq \det(I+H) \geq \det(I+A+B).$$

*Note that equality obviously occurs in the first inequality when  $X = 0$ , and equality occurs in the second inequality when  $AB = BA$  and  $X = A^{1/2}B^{1/2}$ .*

*Proof.* The left inequality is a special case of Fischer's inequality,

$$\det X \det Y \geq \det \begin{bmatrix} X & Z \\ Z^* & Y \end{bmatrix}$$

for any partitioned positive semi-definite matrix. Now we prove the second inequality. Indeed, the majorization  $\lambda(S) \prec \lambda(T)$  in  $H_n^+$  entails the trace inequality

$$\operatorname{Tr} f(S) \geq \operatorname{Tr} f(T) \tag{3.18}$$

for all concave functions  $f(t)$  defined on  $[0, \infty)$ . Using (3.18) with  $f(t) = \log(1+t)$  and the relation  $\lambda(H) \prec \lambda(A+B)$  we have

$$\begin{aligned} \det(I+H) &= \exp \operatorname{Tr} \log(I+H) \\ &\geq \exp \operatorname{Tr} \log(I + ((A+B) \oplus 0_n)) \\ &= \det(I+A+B). \end{aligned}$$

□

Theorem 3.56 says more than the eigenvalue majorization in Theorem 3.16. We have a few other eigenvalue inequalities as follows.

**Corollary 3.58.** Let  $H = \begin{bmatrix} A & X \\ X & B \end{bmatrix} \in H_{2n}^+$  be partitioned into Hermitian blocks in  $M_n$ . Then, we have

$$\lambda_{1+2k}(H) \leq \lambda_{1+k}(A+B)$$

for all  $k = 0, \dots, n-1$ .

*Proof.* Together with Theorem 3.56, the alleged inequalities follow immediately from a simple fact, Weyl's theorem: if  $Y, Z \in H_m$ , then

$$\lambda_{r+s+1}(Y+Z) \leq \lambda_{r+1}(Y) + \lambda_{s+1}(Z)$$

for all nonnegative integers  $r, s$  such that  $r+s \leq m-1$ . □

**Corollary 3.59.** Let  $S, T \in H_n$ . Then

$$\lambda_{1+2k}(T^2 + ST^2S) \leq \lambda_{1+k}(T^2 + TS^2T)$$

for all  $k = 0, \dots, n-1$ .

*Proof.* The nonzero eigenvalues of  $T^2 + ST^2S = \begin{bmatrix} T & ST \end{bmatrix} \begin{bmatrix} T & ST \end{bmatrix}^*$  are the same as those of

$$\begin{bmatrix} T & ST \end{bmatrix}^* \begin{bmatrix} T & ST \end{bmatrix} = \begin{bmatrix} T^2 & TST \\ TST & TS^2T \end{bmatrix}.$$

This block-matrix is of course positive and has its off-diagonal blocks Hermitian. Therefore, the eigenvalue inequalities follow from Corollary 3.58. □

### 3.5.2 Quaternions and 4-by-4 blocks

Theorem 3.56 refines Hiroshima's theorem in case of 2-by-2 blocks. In this section, we introduce quaternions to deal with 4-by-4 partitions. This approach leads to the following theorem.

**Theorem 3.60.** Let  $H = [A_{s,t}] \in H_{\beta n}^+(\mathbb{C})$  be partitioned into Hermitian blocks in  $M_n(\mathbb{C})$  with  $\beta \in \{3, 4\}$  and let  $\Delta = \sum_{s=1}^{\beta} A_{s,s}$  be the sum of its diagonal blocks. Then,

$$H \oplus H = \frac{1}{4} \sum_{k=1}^4 V_k (\Delta \oplus \Delta) V_k^*$$

for some isometries  $V_k \in M_{2\beta n \times 2n}(\mathbb{C})$ ,  $k = 1, 2, 3, 4$ .

Note that, for  $\alpha = \beta \in \{3, 4\}$ , Theorem 3.60 considerably improves Theorem 3.64. Indeed, Theorem 3.60 implies the majorization  $\|H \oplus H\| \leq \|\Delta \oplus \Delta\|$  which is equivalent to the majorisation of Theorem 3.64,  $\|H\| \leq \|\Delta\|$ .

As for Theorem 3.56, we must consider isometries with complex entries, even for a full matrix  $H$  with real entries. The isometries are then with real coefficients, but the proof is more intricate and the result is not so simple since it requires direct sums of sixteen copies: we obtain a decomposition of  $\oplus^{16}H$  in term of  $\oplus^{16}\Delta$ .

Before turning to the proof, we recall some facts about quaternions.

The algebra  $\mathbb{H}$  of quaternions is an associative real division algebra of dimension four containing  $\mathbb{C}$  as a sub-algebra. Quaternions  $q$  are usually written as

$$q = a + bi + cj + dk$$

with  $a, b, c, d \in \mathbb{R}$  and  $a + bi \in \mathbb{C}$ . The quaternion units  $1, i, j, k$  satisfy

$$i^2 = j^2 = k^2 = ijk = -1.$$

The algebra  $\mathbb{H}$  can be represented as the real sub-algebra of  $M_2$  consisting of matrices of the form

$$\begin{pmatrix} z & -\bar{w} \\ w & \bar{z} \end{pmatrix}$$

by the identification map

$$a + bi + cj + dk \mapsto \begin{pmatrix} a + bi & ic - d \\ ic + d & a - ib \end{pmatrix}.$$

The quaternion units  $1, i, j, k$  are then represented by the matrices (related to the Pauli matrices),

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad (3.19)$$

that we will use in the following proof of Theorem 3.60.

We will work with matrices in  $M_{8n}$  partitioned in 4-by-4 blocks in  $M_{2n}$ .

*Proof.* It suffices to consider the case  $\beta = 4$ ; the case  $\beta = 3$  follows by completing  $H$  with some zero columns and rows.

First, replace the positive block matrix  $H = [A_{s,t}]$  where  $1 \leq s, t, \leq 4$  and all blocks are Hermitian by a bigger one in which each block is counted two times :

$$G = [G_{s,t}] := [A_{s,t} \oplus A_{s,t}].$$

Thus  $G \in_{8n}(\mathbb{C})$  is written in 4-by-4 blocks in  $M_{2n}(\mathbb{C})$ . Then perform a unitary congruence with the matrix

$$W = E_1 \oplus E_2 \oplus E_3 \oplus E_4$$

where the  $E_i$  are the analogues of quaternion units, that is, with  $I$  the identity of  $M_n(\mathbb{C})$ ,

$$E_1 = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}, \quad E_2 = \begin{bmatrix} iI & 0 \\ 0 & -iI \end{bmatrix}, \quad E_3 = \begin{bmatrix} 0 & iI \\ iI & 0 \end{bmatrix}, \quad E_4 = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}.$$

Note that  $E_s E_t^*$  is skew-Hermitian whenever  $s \neq t$ . A direct matrix computation then shows that the block matrix

$$\Omega := W G W^* = [\Omega_{s,t}]$$

has the following property for its off-diagonal blocks : For  $1 \leq s < t \leq 4$

$$\Omega_{s,t} = -\Omega_{t,s}.$$

Using this property we compute the unitary congruence implemented by

$$R_2 = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \otimes \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

and we observe that  $R_2 \Omega R_2^*$  has its four diagonal blocks  $(R_2 \Omega R_2^*)_{k,k}$ ,  $1 \leq k \leq 4$ , all equal to the matrix  $D \in M_{2n}(\mathbb{C})$ ,

$$D = \frac{1}{4} \sum_{s=1}^4 A_{s,s} \oplus A_{s,s}.$$

Let  $\Gamma = D \oplus 0_{6n} \in M_{8n}$ . Thanks to the decomposition of Lemma 3.34, there exist some unitaries  $U_i \in M_{8n}(\mathbb{C})$ ,  $1 \leq i \leq 4$ , such that

$$\Omega = \sum_{i=1}^4 U_i \Gamma U_i^*.$$

That is, since  $\Omega$  is unitarily equivalent to  $H \oplus H$ , and  $\Gamma = WDW^*$  for some isometry  $W \in M_{8n \times 2n}(\mathbb{C})$ ,

$$H \oplus H = \sum_{s=1}^4 V_k D V_k^*$$

for some isometries  $V_k \in M_{8n \times 2n}(\mathbb{C})$ . Since  $D = \frac{1}{4}\Delta \oplus \Delta$ , the proof is complete.  $\square$

In the same vein, we have the following consequences.

**Corollary 3.61.** *Let  $H = [A_{s,t}] \in H_{\beta n}^+$  be written in Hermitian blocks in  $H_n$  with  $\beta \in \{3, 4\}$  and let  $\Delta = \sum_{s=1}^{\beta} A_{s,s}$  be the sum of its diagonal blocks. Then,*

$$\prod_{s=1}^{\beta} \det(I + A_{ss}) \geq \det(I + H) \geq \det\left(I + \sum_{s=1}^{\beta} A_{ss}\right).$$

**Corollary 3.62.** *Let  $H = [A_{s,t}] \in H_{\beta n}^+$  be written in Hermitian blocks in  $M_n$  with  $\beta \in \{3, 4\}$  and let  $\Delta = \sum_{s=1}^{\beta} A_{s,s}$  be the sum of its diagonal blocks. Then,*

$$\lambda_{1+4k}(H) \leq \lambda_{1+k}(A + B)$$

for all  $k = 0, \dots, n-1$ .

**Corollary 3.63.** *Let  $T \in H_n$  and let  $\{S_i\}_{i=1}^{\beta} \in \mathcal{E}_n$  be commuting Hermitian matrices with  $\beta \in \{3, 4\}$ . Then,*

$$\left\| \sum_{i=1}^{\beta} S_i T^2 S_i \right\| \leq \left\| \sum_{i=1}^{\beta} T S_i^2 T \right\|$$

for all symmetric norms, and

$$\lambda_{1+4k}\left(\sum_{i=1}^{\beta} S_i T^2 S_i\right) \leq \lambda_{1+k}\left(\sum_{i=1}^{\beta} T S_i^2 T\right)$$

for all  $k = 0, \dots, n-1$ .

The proofs of these corollaries are quite similar of those of Section 2. We give details only for the norm inequality of Corollary 3.63.



*Proof.* We may assume that  $\beta = 4$  by completing, if necessary with  $S_4 = 0$ . So, let  $T \in H_n^+$  and let  $\{S_i\}_{i=1}^4$  be four commuting Hermitian matrices in  $M_n$ . Then

$$H = XX^* = \begin{bmatrix} TS_1 \\ TS_2 \\ TS_3 \\ TS_4 \end{bmatrix} \begin{bmatrix} S_1T & S_2T & S_3T & S_4T \end{bmatrix}$$

is positive and partitioned into Hermitian blocks with diagonal blocks  $TS_i^2T$ ,  $1 \leq i \leq 4$ . Thus, from Theorem 3.60, for all symmetric norms,

$$\|H \oplus H\| \leq \left\| \left\{ \sum_{i=1}^4 TS_i^2T \right\} \oplus \left\{ \sum_{i=1}^4 TS_i^2T \right\} \right\|$$

or equivalently

$$\|H\| \leq \left\| \sum_{i=1}^4 TS_i^2T \right\|$$

Since  $H = XX^*$  and  $X^*X = \sum_{i=1}^4 S_iT^2S_i$ , the norm inequality of Corollary 3.63 follows.  $\square$

Bourin and Lee have continued some works in this direction; for more details, we refer to [22].

We end this section by recording Hiroshima's beautiful result, which contains Theorem 3.16 as a special case.

**Theorem 3.64.** [49] *Let  $H = [A_{s,t}] \in H_{\beta n}^+(\mathbb{C})$  be partitioned into Hermitian blocks in  $M_n(\mathbb{C})$  with any positive integer  $\beta$  and let  $\Delta = \sum_{s=1}^{\beta} A_{s,s}$  be the sum of its diagonal blocks. Then,*

$$\lambda(H) \prec \lambda(\Delta).$$

By recognizing that every  $H \in H_m^+$  can be written as  $H = M^*M$  for some  $M \in M_m(\mathbb{C})$ . Theorem 3.64 has the following appealing variant.

**Theorem 3.65.** *Let  $X_1, \dots, X_k \in M_{m \times n}(\mathbb{C})$  such that  $X_s^*X_t$  is Hermitian for all  $1 \leq s, t \leq k$ . Then*

$$\lambda \left( \sum_{s=1}^k X_s X_s^* \right) \prec \lambda \left( \sum_{s=1}^k X_s^* X_s \right). \quad (3.20)$$

## 3.6 Discussion

In this section, we present some discussion and questions for further investigation.

As before, the absolute value is defined as  $|X| = (X^*X)^{1/2}$  and the geometric mean between two positive definite matrices  $A, B$  is given by  $A\sharp B = A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}$ .

**Question 3.66.** *Let  $A, B \in H_n$ . Is it true that*

$$\lambda \left( \begin{bmatrix} A & X \\ X^* & B \end{bmatrix} \right) \succ \lambda \left( \begin{bmatrix} A & Y \\ Y^* & B \end{bmatrix} \right)$$

*if  $\lambda(|X|) \succ_w \lambda(|Y|)$ ?*

When  $A = B = 0$ , obviously, the answer is yes.

The question is motivated by the following fact, see [78, Theorem 2.10].

**Proposition 3.67.** *If  $A, B \in H_n^+$ , then*

$$\lambda(|A^{1/2}B^{1/2}|) \succ_w \lambda(A\sharp B).$$

It is easy to see that for positive definite matrices  $A, B$

$$\lambda \left( \begin{bmatrix} A & A^{1/2}B^{1/2} \\ B^{1/2}A^{1/2} & B \end{bmatrix} \right) = \lambda(A + B).$$

**Remark 3.68.** *In [78], the authors showed something stronger than Proposition 3.67, that is*

$$\lambda(|A^{1/2}B^{1/2}|) \succ_{\log} \lambda(A\sharp B).$$

*The definition of log majorization is given in Section 3.8.*

Thus if Question 3.66 is true, then the assertion of Theorem 3.16 follows immediately.

However, the answer to Question 3.66 is no. Below is a concrete example adapted from [110] showing

$$\begin{bmatrix} A & X \\ X^* & B \end{bmatrix} \geq 0 \not\Rightarrow \begin{bmatrix} A & X^* \\ X & B \end{bmatrix} \geq 0,$$

let alone the spectrums of  $\begin{bmatrix} A & X \\ X^* & B \end{bmatrix}$  and  $\begin{bmatrix} A & X^* \\ X & B \end{bmatrix}$  are the same.

**Example 3.69.** Taking  $A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ ,  $X = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$  and  $B = \begin{bmatrix} 1 & 1 \\ 1 & x \end{bmatrix}$  for  $x \in \mathbb{R}$ , then

$$\begin{bmatrix} A & X \\ X^* & B \end{bmatrix} \geq 0 \text{ for any } x \geq 2.$$

But  $\begin{bmatrix} A & X^* \\ X & B \end{bmatrix} \not\geq 0$  for any  $x \in \mathbb{R}$ .

**Question 3.70.** Let  $A, B \in H_n$ . Is it true that

$$\lambda \left( \begin{bmatrix} A & X \\ X^* & B \end{bmatrix} \right) \succ \lambda \left( \begin{bmatrix} A & Y \\ Y^* & B \end{bmatrix} \right)$$

if  $|X| \geq |Y|$ ?

One interesting piece of evidence to support Question 3.70 is the following fact, which can be found in [80, p. 309].

**Proposition 3.71.** Let  $A, B \in H_n$  and  $1 \geq t_1 \geq t_2 \geq 0$ . Then

$$\lambda \left( \begin{bmatrix} A & t_1 X \\ t_1 X^* & B \end{bmatrix} \right) \succ \lambda \left( \begin{bmatrix} A & t_2 X \\ t_2 X^* & B \end{bmatrix} \right).$$

*Proof.* It is sufficient to prove the result with  $t_1 = 1$ . Write

$$\begin{bmatrix} A & t_2 X \\ t_2 X^* & B \end{bmatrix} = t_2 \begin{bmatrix} A & X \\ X^* & B \end{bmatrix} + (1 - t_2) \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}.$$

By Fan's inequality (3.1) and (3.2), we get

$$\begin{aligned} \lambda \left( \begin{bmatrix} A & t_2 X \\ t_2 X^* & B \end{bmatrix} \right) &\prec t_2 \lambda \left( \begin{bmatrix} A & X \\ X^* & B \end{bmatrix} \right) + (1 - t_2) \lambda(A \oplus B) \\ &\prec t_2 \lambda \left( \begin{bmatrix} A & X \\ X^* & B \end{bmatrix} \right) + (1 - t_2) \lambda \left( \begin{bmatrix} A & X \\ X^* & B \end{bmatrix} \right) \\ &= \lambda \left( \begin{bmatrix} A & X \\ X^* & B \end{bmatrix} \right). \end{aligned}$$

□

Unfortunately, the answer to Question 3.70 is still no. There are examples

$$\begin{bmatrix} A & U \\ U^* & B \end{bmatrix} \geq 0 \not\Rightarrow \begin{bmatrix} A & U^* \\ U & B \end{bmatrix} \geq 0$$

with  $U$  being unitary.

**Question 3.72.** Let  $A, B, X, Y \in H_n$  and  $|X| \geq |Y|$ . Is it true

$$\lambda \left( \begin{bmatrix} A & X \\ X & B \end{bmatrix} \right) \prec \lambda \left( \begin{bmatrix} A & Y \\ Y & B \end{bmatrix} \right)?$$

I have run extensive numerical experiments for the following special case of Question 3.72:

**Conjecture 3.73.** If  $X \geq Y \geq 0$ , then

$$\lambda \left( \begin{bmatrix} A & X \\ X & B \end{bmatrix} \right) \prec \lambda \left( \begin{bmatrix} A & Y \\ Y & B \end{bmatrix} \right).$$

Without loss of generality, we may assume both  $\begin{bmatrix} A & X \\ X & B \end{bmatrix}, \begin{bmatrix} A & Y \\ Y & B \end{bmatrix}$  are positive definite, then Conjecture 3.73 has a familiar equivalent reformulation.

The next result is due to Bapat and Sunder.

**Theorem 3.74.** [10] If  $A \in H_n$  and  $D_1, \dots, D_m \in M_n(\mathbb{F})$  such that  $\sum_{k=1}^m D_k D_k^* = \sum_{k=1}^m D_k^* D_k = I$ , then

$$\lambda \left( \sum_{k=1}^m D_k A D_k^* \right) \prec \lambda(A). \quad (3.21)$$

**Remark 3.75.** The idea of the proof in [10] is to find a doubly stochastic matrix.

Specified to  $m = 2$ , we have

**Corollary 3.76.** If  $A \in H_n$  and  $D_1, D_2 \in M_n(\mathbb{F})$  such that  $A = A^*$  and  $D_1^* D_1 + D_2^* D_2 = D_1 D_1^* + D_2 D_2^* = I$ , then

$$\lambda \left( \sum_{k=1}^2 D_k A D_k^* \right) \prec \lambda(A).$$

We may assume without loss of generality that  $A$  is positive definite by a shift. Then

$$\begin{aligned} \lambda \left( \sum_{k=1}^2 D_k A D_k^* \right) &= \lambda \left( [D_1 \ D_2] \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix} \begin{bmatrix} D_1^* \\ D_2^* \end{bmatrix} \right) \\ &= \lambda \left( \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix} \begin{bmatrix} D_1^* \\ D_2^* \end{bmatrix} [D_1 \ D_2] \right) \\ &= \lambda \left( \begin{bmatrix} A^{1/2} D_1^* D_1 A^{1/2} & A^{1/2} D_1^* D_2 A^{1/2} \\ A^{1/2} D_2^* D_1 A^{1/2} & A^{1/2} D_2^* D_2 A^{1/2} \end{bmatrix} \right). \end{aligned}$$

It would be very good if  $D_1^* D_2$  is Hermitian, then using (3.8), one would have

$$\lambda \left( \begin{bmatrix} A^{1/2} D_1^* D_1 A^{1/2} & A^{1/2} D_1^* D_2 A^{1/2} \\ A^{1/2} D_2^* D_1 A^{1/2} & A^{1/2} D_2^* D_2 A^{1/2} \end{bmatrix} \right) \prec \lambda \left( A^{1/2} (D_1^* D_1 + D_2^* D_2) A^{1/2} \right) = \lambda(A).$$

Unfortunately,  $D_1^* D_2$  is not Hermitian generally, e.g., taking  $D_1 = \frac{1}{\sqrt{2}}U$ ,  $D_2 = \frac{1}{\sqrt{2}}V$  for any two unitaries  $U, V \in M_n(\mathbb{F})$ . However, we have the following proposition.

**Proposition 3.77.** *If  $D_1, D_2 \in M_n(\mathbb{F})$  are such that  $D_1^* D_1 + D_2^* D_2 = D_1 D_1^* + D_2 D_2^* = I$ , then  $D_1^* D_2$  is normal.*

*Proof.* Pre and post multiplying

$$D_1 D_1^* + D_2 D_2^* = I \tag{3.22}$$

by  $D_1^*$  and  $D_1$ , respectively, we get

$$(D_1^* D_1)^2 + D_1^* D_2 D_2^* D_1 = D_1^* D_1,$$

i.e.,

$$(D_1^* D_1)^2 - D_1^* D_1 = -D_1^* D_2 D_2^* D_1.$$

Pre and post multiplying (3.22) by  $D_2^*$  and  $D_2$ , respectively, we get

$$D_2^* D_1 D_1^* D_2 + (D_2^* D_2)^2 = D_2^* D_2,$$

i.e.,

$$(D_2^* D_2)^2 - D_2^* D_2 = -D_2^* D_1 D_1^* D_2.$$

Pre multiplying

$$D_1^*D_1 + D_2^*D_2 = I \quad (3.23)$$

by  $D_1^*D_1$ , we get

$$(D_1^*D_1)^2 - D_1^*D_1 = -D_1^*D_1D_2^*D_2.$$

Post multiplying (3.23) by  $D_2^*D_2$ , we get

$$(D_2^*D_2)^2 - D_2^*D_2 = -D_1^*D_1D_2^*D_2.$$

Thus  $D_1^*D_2D_2^*D_1 = D_2^*D_1D_1^*D_2$ , i.e.,  $D_1^*D_2$  is normal.  $\square$

The next result is complementary to Theorem 3.74.

**Theorem 3.78.** *If  $A \in H_n$  and  $D_1, \dots, D_m \in M_n(\mathbb{F})$ , are such that  $\sum_{k=1}^m D_k D_k^* = I$ , then*

$$\lambda(A) \prec \sum_{k=1}^m \lambda(D_k^* A D_k).$$

*Proof.* We may assume  $A$  is positive semidefinite by a shift. Let  $D = [D_1 \ D_2 \ \dots \ D_m]$ , then the diagonal blocks of  $D^*AD$  are  $D_k^*AD_k$  for  $k = 1, \dots, m$ , so by Proposition 3.12, we have  $\lambda(D^*AD) \prec \sum_{k=1}^m \lambda(D_k^*AD_k)$ . Moreover,  $\lambda(D^*AD) = \lambda(ADD^*) = \lambda(A)$ . This completes the proof.  $\square$

We remark that Theorem 3.74 has some connection with the following property, see [34, Example 2.4].

**Proposition 3.79.** *Let  $X$  and  $Y$  be two arbitrary  $n \times n$  symmetric matrices. Then the inequality*

$$\lambda(X) \succ \lambda(Y)$$

*holds if and only if  $X$  is expressed as the following convex combination*

$$Y = \sum_{i=1}^m c_i U_i X U_i^*$$

*for some integer  $m$ , some positive reals  $c_1, \dots, c_m$  satisfying  $\sum_{i=1}^m c_i = 1$ , and some unitary matrices  $U_1, \dots, U_m \in M_n(\mathbb{F})$ .*

### 3.7 Majorization inequalities for normal matrices

Firstly, I would like to extend some classical majorization results to the *normal matrix* case. Recall that a square matrix  $N \in M_n(\mathbb{F})$  is normal if  $NN^* = N^*N$ . There are several key differences between normal matrices and Hermitian matrices. For example, the sum (or difference) of two normal matrices is not necessarily normal. The principal submatrices of a normal matrix are not necessarily normal, either. For the latter, the following proposition illustrates this point.

**Proposition 3.80.** *Let  $N = \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix} \in M_n(\mathbb{F})$  be partitioned such that the diagonal blocks are square. Then  $N_{11}$  (resp.  $N_{22}$ ) is normal if and only if  $N_{12}N_{12}^* = N_{21}^*N_{21}$  (resp.  $N_{21}N_{21}^* = N_{12}^*N_{12}$ ).*

*Proof.* This follows immediately from comparing

$$NN^* = \begin{bmatrix} N_{11}N_{11}^* + N_{12}N_{12}^* & N_{11}N_{21}^* + N_{12}N_{22}^* \\ N_{21}N_{11}^* + N_{22}N_{12}^* & N_{21}N_{21}^* + N_{22}N_{22}^* \end{bmatrix}$$

and

$$N^*N = \begin{bmatrix} N_{11}^*N_{11} + N_{21}^*N_{21} & N_{11}^*N_{12} + N_{21}^*N_{22} \\ N_{12}^*N_{11} + N_{22}^*N_{21} & N_{12}^*N_{12} + N_{22}^*N_{22} \end{bmatrix}.$$

□

However,  $N_{12}N_{12}^* = N_{21}^*N_{21}$  does not hold for normal matrices  $N$  generally. Here is a concrete example:

**Example 3.81.** *Let  $N_{11} = N_{22} = N_{12}^* = N_{21}^* = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ . Then  $N_{12}N_{12}^* \neq N_{21}^*N_{21}$ . Indeed, one can check that  $\begin{bmatrix} Z & Z^* \\ Z^* & Z \end{bmatrix}$  is normal for any square matrix  $Z$ .*

The following lemma plays an important role in our investigation.

**Lemma 3.82.** *Let  $A \in M_n(\mathbb{F})$ . Then*

$$\operatorname{Re} \lambda(A) \prec \lambda(\operatorname{Re} A).$$

*Proof.* It can be found in, e.g., [52]. We include a simple proof here. The *Schur decomposition lemma* [51] tells us that there is a unitary matrix  $U$  such that  $U^*AU = T$ , where  $T$  is an upper triangular matrix. Note that the real parts of the eigenvalues of  $T$  coincide with the diagonal entries of  $\frac{T+T^*}{2}$ . Thus

$$\begin{aligned} \operatorname{Re} \lambda(A) &= \operatorname{Re} \lambda(T) \\ &\prec \lambda \left( \frac{T+T^*}{2} \right) \\ &= \lambda \left( \frac{A+A^*}{2} \right) = \lambda(\operatorname{Re}A), \end{aligned}$$

where the majorization is by Proposition 3.6. □

Concerning the eigenvalues of a normal matrix, one important property is that the real parts of the eigenvalues coincide with the eigenvalues of its Hermitian parts. That is,  $\operatorname{Re} \lambda(N) = \lambda(\operatorname{Re}N)$ , whenever  $N$  is normal.

The next proposition shows that (3.2) can be extended to the case of normal matrices, i.e., we have

**Proposition 3.83.** *Let  $A, B \in M_n(\mathbb{F})$  be normal matrices. Then*

$$\operatorname{Re} \lambda(A+B) \prec \operatorname{Re} \lambda(A) + \operatorname{Re} \lambda(B). \quad (3.24)$$

*Proof.*

$$\begin{aligned} \operatorname{Re} \lambda(A+B) &\prec \lambda \operatorname{Re}(A+B) \\ &= \lambda(\operatorname{Re}A + \operatorname{Re}B) \\ &\prec \lambda(\operatorname{Re}A) + \lambda(\operatorname{Re}B) \\ &= \operatorname{Re} \lambda(A) + \operatorname{Re} \lambda(B), \end{aligned}$$

where the first majorization is by Lemma 3.82 and the second majorization is by (3.2). □

It is natural to ask whether (3.4) also has such an analogue, i.e., if  $A, B \in M_n(\mathbb{F})$  are normal matrices, do we have

$$\operatorname{Re} \lambda(A) \succ \operatorname{Re} \lambda(A+B) - \operatorname{Re} \lambda(B)? \quad (3.25)$$

Unfortunately, the answer is no as the following example shows.



**Example 3.84.** Taking

$$A = \begin{bmatrix} 0 & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

obviously,  $A, B$  are normal. Simple calculation gives

$$\lambda(A) = \{\sqrt{3}i/2, -\sqrt{3}i/2\}, \quad \lambda(B) = \{1, -1\}, \quad \lambda(A+B) = \{1/2, -1/2\}.$$

Thus

$$\operatorname{Re} \lambda(A) = (0, 0), \quad \operatorname{Re} \lambda(A+B) - \operatorname{Re} \lambda(B) = (1/2, -1/2).$$

I would like to thank F. Zhang for this simple counterexample.

Proposition 3.7 also possesses an extension to normal matrices:

**Proposition 3.85.** Let  $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \in M_n(\mathbb{F})$  be normal and partitioned such that the diagonal blocks are square. Then

$$\operatorname{Re} \lambda(A_{11} \oplus A_{22}) \prec \operatorname{Re} \lambda(A).$$

*Proof.*

$$\begin{aligned} \operatorname{Re} \lambda(A_{11} \oplus A_{22}) &\prec \lambda(\operatorname{Re}(A_{11} \oplus A_{22})) \\ &= \lambda(\operatorname{Re}A_{11} \oplus \operatorname{Re}A_{22}) \\ &\prec \lambda \begin{bmatrix} \operatorname{Re}A_{11} & (A_{12} + A_{21}^*)/2 \\ (A_{21} + A_{12}^*)/2 & \operatorname{Re}A_{22} \end{bmatrix} \\ &= \lambda(\operatorname{Re}A) = \operatorname{Re} \lambda(A), \end{aligned}$$

where the first majorization is by Lemma 3.82 and the second majorization is by Proposition 3.7. □

A normal version of Theorem 3.74 can be stated as follows:

**Proposition 3.86.** If  $A \in M_n(\mathbb{F})$  is normal and  $D_1, \dots, D_m \in M_n(\mathbb{F})$  such that  $\sum_{k=1}^m D_k D_k^* = \sum_{k=1}^m D_k^* D_k = I$ . Then

$$\operatorname{Re} \lambda \left( \sum_{k=1}^m D_k A D_k^* \right) \prec \operatorname{Re} \lambda(A).$$

*Proof.*

$$\begin{aligned}
\operatorname{Re} \lambda \left( \sum_{k=1}^m D_k A D_k^* \right) &\prec \lambda \left( \operatorname{Re} \sum_{k=1}^m D_k A D_k^* \right) \\
&= \lambda \left( \sum_{k=1}^m D_k (\operatorname{Re} A) D_k^* \right) \\
&\prec \lambda (\operatorname{Re} A) = \operatorname{Re} \lambda (A),
\end{aligned}$$

where the first majorization is by Lemma 3.82 and the second majorization is by (3.21).  $\square$

**Remark 3.87.** In [10], the authors remarked that Theorem 3.74 has a normal version with the understanding that for  $x, y \in \mathbb{C}^n$ ,  $x \prec y$  is to be interpreted as  $x = My$  for some doubly stochastic matrix  $M$ . The reader should be able to observe that this is indeed the same as the above proposition, but our proof seems much easier.

In the remaining part of this section, we revisit Theorem 2.21 and present some related results.

In [60], Knyazev and Argentati proved the following result:

**Proposition 3.88.** *Let  $x, y \in \mathbb{C}^n$  be two unit vectors and let  $A \in H_n$ . Then*

$$|x^* A x - y^* A y| \leq \operatorname{spr}(A) \sin \psi_{xy}.$$

Proposition 3.88 has some applications, for example, it can be used to analyze the convergence rate of preconditioned iterative methods for large scale symmetric eigenvalue problems [58]. Proposition 3.88 was soon generalized by them [61] to the majorization bound:

**Theorem 3.89.** *Let  $\mathcal{X}, \mathcal{Y}$  be subspaces of  $\mathbb{C}^n$  having the same dimension  $k$ , with orthonormal bases given by the columns of the matrices  $X$  and  $Y$ , respectively. Also, let  $A \in H_n$ . Then*

$$|\lambda(X^* A X) - \lambda(Y^* A Y)| \prec_w \operatorname{spr}(A) \sin \Psi(\mathcal{X}, \mathcal{Y}). \quad (3.26)$$

An early result due to Ruhe [97] asserts that if  $Ax = ax$ , i.e.,  $x$  is an eigenvector of  $A$  corresponding to the eigenvalue  $a$ , assume further that  $x, y$  are unit vectors, then

$$|a - y^* A y| = |x^* A x - y^* A y| \leq \operatorname{spr}(A) \sin^2 \psi_{xy}. \quad (3.27)$$

Hence, Theorem 2.21 is an extension of Ruhe's result to a multidimensional setting.

Knyazev and Argenti's proof of Theorem 2.21 (see [62]) makes an ingenious manipulation of the basic majorization relation between vectors, Ky Fan's result (3.1) and Lidskii's result (3.4). Also, the following proposition plays a vital role.

**Proposition 3.90.** *(A special case of [62, Theorem 4.5]) Let  $B, M \in H_n$  and suppose that all the eigenvalues of  $M$  lie in the interval  $[0, 1]$ . Then*

$$\lambda \left( M^{1/2} B M^{1/2} \oplus (I - M)^{1/2} B (I - M)^{1/2} \right) \prec \lambda(B).$$

Now we extend this proposition.

**Proposition 3.91.** *Let  $A \in M_n(\mathbb{F})$  be normal and suppose  $D_1, \dots, D_m \in M_n(\mathbb{F})$  are such that  $\sum_{k=1}^m D_k D_k^* = I$ . Then*

$$\operatorname{Re} \lambda \left( \bigoplus_{k=1}^m D_k^* A D_k \right) \prec \operatorname{Re} \lambda(A).$$

*Proof.* Let  $D = [D_1 \ D_2 \ \dots \ D_m]$ . Then

$$\begin{aligned} & \operatorname{Re} \lambda \left( \bigoplus_{k=1}^m D_k^* A D_k \right) \\ & \prec \lambda \left( \operatorname{Re} \left( \bigoplus_{k=1}^m D_k^* A D_k \right) \right) \\ & \prec \lambda \left( \operatorname{Re} D^* A D \right) \\ & = \lambda \left( D^* (\operatorname{Re} A) D \right) = \lambda \left( (\operatorname{Re} A) D D^* \right) \\ & = \lambda (\operatorname{Re} A) = \operatorname{Re} \lambda(A). \end{aligned}$$

□

**Question 3.92.** *Comparing Proposition 3.91 with Proposition 3.86, it is natural to ask (under the possible condition  $A$  is normal and accretive) whether*

$$\operatorname{Re} \lambda \left( \bigoplus_{k=1}^m D_k^* A D_k \right) \prec \operatorname{Re} \lambda \left( \sum_{k=1}^m D_k A D_k^* \right)?$$

Our next result is to show that Proposition 3.88 can be extended to normal matrices. I am indebted to Gord Sinnamon for suggesting the concise argument.

**Theorem 3.93.** *Let  $x, y \in \mathbb{C}^n$  be two unit vectors and let  $A \in M_n(\mathbb{F})$  be normal. Then*

$$|x^* A x - y^* A y| \leq \operatorname{spr}(A) \sin \psi_{xy}. \quad (3.28)$$

*Proof.* Without loss of generality, we assume  $A$  to be a diagonal matrix (since every normal matrix is unitarily equivalent to a diagonal matrix) with diagonal entries  $z_1, \dots, z_n \in \mathbb{C}$ . Write  $x = (x_1, \dots, x_n)^T$ ,  $y = (y_1, \dots, y_n)^T$ , then (3.28) becomes

$$\left| \sum_{j=1}^n z_j (|x_j|^2 - |y_j|^2) \right|^2 \leq \max_{j,k} |z_j - z_k|^2 \left( 1 - \left| \sum_{j=1}^n \bar{x}_j y_j \right|^2 \right) \quad (3.29)$$

with  $\sum_{j=1}^n |x_j|^2 = \sum_{j=1}^n |y_j|^2 = 1$ .

Fix  $x$  and  $y$  and let  $J = \{j : |x_j| > |y_j|\}$ . Suppose that

$$\left( \sum_{j \in J} (|x_j|^2 - |y_j|^2) \right)^2 \leq 1 - \left| \sum_{j=1}^n \bar{x}_j y_j \right|^2 \quad (3.30)$$

Set  $\sigma = \sum_{j \in J} (|x_j|^2 - |y_j|^2)$  and note that  $\sigma = \sum_{j \notin J} (|x_j|^2 - |y_j|^2)$  as well. Now fix complex numbers  $z_1, \dots, z_n$  and observe that the diameter of their convex hull is  $\max_{j,k} |z_j - z_k|$ . It follows that

$$\left| \sum_{j \in J} z_j \frac{|x_j|^2 - |y_j|^2}{\sigma} - \sum_{j \notin J} z_j \frac{|x_j|^2 - |y_j|^2}{\sigma} \right| \leq \max_{j,k} |z_j - z_k|.$$

Multiplying through by  $\sigma^2$  and using the inequality (3.30), we have

$$\left| \sum_{j=1}^n z_j (|x_j|^2 - |y_j|^2) \right|^2 \leq \left( \max_{j,k} |z_j - z_k|^2 \right) \left( 1 - \left| \sum_{j=1}^n \bar{x}_j y_j \right|^2 \right). \quad (3.31)$$

On the other hand, setting  $z_j = 1$  for  $j \in J$  and  $z_j = 0$  otherwise reduces (3.31) to (3.30). In particular, we know (3.31) holds for real  $z_1, \dots, z_n$  (by Proposition 3.88), i.e., (3.30) is true. Then (3.31) holds in general. This completes the proof.  $\square$

**Remark 3.94.** *Ruhe's result (3.27) can be extended to normal matrices as well.*

With the evidence of the one-dimensional case, we would like to propose two conjectures as possible generalizations of Theorem 3.89 and Theorem 2.21 in Chapter one.

**Conjecture 3.95.** *Let  $\mathcal{X}, \mathcal{Y}$  be subspaces of  $\mathbb{C}^n$  having the same dimension  $k$ , with orthonormal bases given by the columns of the matrices  $X$  and  $Y$ , respectively. Also, let  $N \in M_n(\mathbb{F})$  be normal. Then*

$$|\lambda(X^*NX) - \lambda(Y^*NY)| \prec_w \text{spr}(N) \sin^2 \Psi(\mathcal{X}, \mathcal{Y}). \quad (3.32)$$

**Conjecture 3.96.** Let  $\mathcal{X}, \mathcal{Y}$  be subspaces of  $\mathbb{C}^n$  having the same dimension  $k$ , with orthonormal bases given by the columns of the matrices  $X$  and  $Y$ , respectively. Also, let  $N \in M_n(\mathbb{F})$  be normal, and  $\mathcal{X}$  be  $N$ -invariant. Then

$$|\operatorname{Re} \lambda(X^*NX) - \operatorname{Re} \lambda(Y^*NY)| \prec_w \operatorname{spr}(\operatorname{Re} N) \sin^2 \Psi(\mathcal{X}, \mathcal{Y}). \quad (3.33)$$

**Remark 3.97.** Under the same conditions of Conjecture 3.96, by Theorem 3.89, we have

$$|\lambda(X^*(\operatorname{Re} N)X) - \lambda(Y^*(\operatorname{Re} N)Y)| \prec_w \operatorname{spr}(\operatorname{Re} N) \sin^2 \Psi(\mathcal{X}, \mathcal{Y}).$$

Conjecture 3.96 would be a direct consequence of Theorem 3.89 if one has

$$|\operatorname{Re} \lambda(X^*NX) - \operatorname{Re} \lambda(Y^*NY)| \prec_w |\lambda(X^*(\operatorname{Re} N)X) - \lambda(Y^*(\operatorname{Re} N)Y)|. \quad (3.34)$$

However, (3.34) is not true in general, here is an example:

**Example 3.98.** Let  $P = \begin{bmatrix} Z & Z^* \\ Z^* & Z \end{bmatrix}$  with  $Z = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$  and  $Q = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ . Taking  $N = P \oplus Q$  (obviously,  $N$  is normal),  $X = \begin{bmatrix} I_2 \\ 0 \end{bmatrix}$ ,  $Y = \begin{bmatrix} 0 \\ I_2 \end{bmatrix}$ , where  $0$  means a zero matrix of size  $6 \times 2$ . Since  $\det N = 16 \neq 0$ ,  $X$  is  $N$ -invariant. A short calculation shows

$$\begin{aligned} |\operatorname{Re} \lambda(X^*NX) - \operatorname{Re} \lambda(Y^*NY)| &= |\operatorname{Re} \lambda(Z) - \operatorname{Re} \lambda(Q)| \\ &= |(0, 0) - (1, -1)| = (1, 1); \end{aligned}$$

$$\begin{aligned} |\lambda(X^*(\operatorname{Re} N)X) - \lambda(Y^*(\operatorname{Re} N)Y)| &= |\lambda(\operatorname{Re} Z) - \lambda(\operatorname{Re} Q)| \\ &= |(1, -1) - (1, -1)| = (0, 0). \end{aligned}$$

Thus, (3.34) does not hold.

**Remark 3.99.** We have seen that Ky Fan's result (3.1) can be extended to normal matrices while Lidskii's result (3.4) cannot. Proposition 3.90 has such an extension. However, principal submatrices of normal matrices are not normal generally. Thus, the line of proof in [62] does not work here.

### 3.8 Majorization inequalities for coneigenvalues

The result of this section has appeared in [29]. It is joint work with De Sterck.

We need the notion of (weak) *log-majorization* in this section.

**Definition 3.100.** Let  $x = (x_1, x_2, \dots, x_n)$  and  $y = (y_1, y_2, \dots, y_n)$  be two vectors with non-negative entries. Then we say that  $x$  is weakly log-majorized by  $y$ , denoted by  $x \prec_{w \log} y$  (the same as  $y \succ_{w \log} x$ ), if  $\prod_{j=1}^k x_j^\downarrow \leq \prod_{j=1}^k y_j^\downarrow$  for all  $k = 1, 2, \dots, n$ . We say that  $x$  is majorized by  $y$ , denoted by  $x \prec_{\log} y$  (or  $y \succ_{\log} x$ ), if further  $\prod_{j=1}^n x_j = \prod_{j=1}^n y_j$ .

A classical result connecting (weak) log-majorization and (weak) majorization is the following

**Proposition 3.101.** Let  $x, y \in \mathbb{R}_n^+$ . Then

$$x \prec_{w \log} y \Rightarrow x \prec_w y.$$

For a proof of this proposition, we refer to [115, p. 345].

For a complex vector  $x = (x_1, x_2, \dots, x_n)$ , its entrywise absolute value is defined by

$$|x| = (|x_1|, |x_2|, \dots, |x_n|).$$

**Definition 3.102.** A matrix  $A \in M_n(\mathbb{F})$  is said to be *conjugate-normal* if

$$AA^* = \overline{A^*A}.$$

In particular, complex symmetric, skew-symmetric, and unitary matrices are special subclasses of *conjugate-normal matrices*. It seems that the term ‘conjugate-normal matrices’ was first introduced in [104]. For more properties and characterizations of this kind of matrix, we refer to [36].

For  $A \in M_n(\mathbb{C})$ , define  $B = \overline{AA}$ . An early result of Djoković [31] says  $B$  is similar to  $R^2$ , where  $R$  is a real matrix. Thus  $\lambda(B) = \{\lambda_1, \lambda_2, \dots, \lambda_n\}$  is symmetric with respect to the real axis and the negative eigenvalues of  $B$  (if any) are of even algebraic multiplicity, see also [52].

**Definition 3.103.** [56] The coneigenvalues of  $A \in M_n(\mathbb{F})$  are  $n$  scalars  $\mu_1, \mu_2, \dots, \mu_n$  obtained as follows:

1. If  $\lambda_k \in \lambda(B)$  does not lie on the negative real semiaxis, then the corresponding coneigenvalue  $\mu_k$  is defined as the square root of  $\lambda_k$  with a nonnegative real part. The multiplicity of  $\mu_k$  is set equal to that of  $\lambda_k$ .
2. With a real negative  $\lambda_k \in \lambda(B)$ , we associate two conjugate purely imaginary coneigenvalues (i.e., the two square roots of  $\lambda_k$ ). The multiplicity of each coneigenvalue is set equal to half the multiplicity of  $\lambda_k$ .

For  $A \in M_n(\mathbb{F})$ , the vector of its coneigenvalues will be denoted by

$$\mu(A) = (\mu_1(A), \mu_2(A), \dots, \mu_n(A)).$$

In the sequel, we will briefly review some known properties related to coneigenvalues.

Define the matrix

$$\widehat{A} = \begin{bmatrix} 0 & A \\ \overline{A} & 0 \end{bmatrix}.$$

**Proposition 3.104.** [56] *If  $\mu_1, \mu_2, \dots, \mu_n$  are the coneigenvalues of an  $n \times n$  matrix  $A$ , then*

$$\lambda(\widehat{A}) = (\mu(A), -\mu(A)).$$

**Proposition 3.105.** [56] *Let  $A$  be a conjugate-normal matrix. Then the coneigenvalues of the matrices  $\frac{A+A^T}{2}$  and  $\frac{A-A^T}{2}$  are the real and imaginary parts, respectively, of the coneigenvalues of  $A$ .*

The purpose of this section is to extend some classical eigenvalue majorization results to the coneigenvalue case. We restate the classical results here for convenience.

**Theorem 3.106.** (see, e.g., [52]) *Let  $A \in M_n(\mathbb{F})$ . Then*

$$\lambda(\operatorname{Re}(A)) \succ \operatorname{Re}(\lambda(A)), \quad (3.35)$$

$$\sigma(A) \succ_{\log} |\lambda(A)|. \quad (3.36)$$

**Theorem 3.107.** (see, e.g., [52]) *Let  $A, B \in H_n$ . Then*

$$\lambda(A) + \lambda(B) \succ \lambda(A + B), \quad (3.37)$$

$$\lambda(A) \succ \lambda(A + B) - \lambda(B). \quad (3.38)$$

We start with some observations.

**Observation 1.** *The coneigenvalues of a complex symmetric matrix are nonnegative, the coneigenvalues of a complex skew symmetric matrix are purely imaginary.*

*Proof.* If  $A$  is complex symmetric, then  $\bar{A}A = \overline{A^T}A = A^*A$ , thus the coneigenvalues of  $A$  coincide with the singular values of  $A$  and are thus all nonnegative. The case  $A$  being complex skew symmetric can be proved similarly.  $\square$

**Observation 2.** *Let  $A \in M_n(\mathbb{C})$ . Then  $|\det(A)| = \prod_{k=1}^n \mu_k(A)$ . However, we generally do not have  $\text{Tr}A = \sum_{k=1}^n \mu_k(A)$  or  $|\text{Tr}A| = \sum_{k=1}^n \mu_k(A)$ .*

*Proof.* By definition of coneigenvalues,  $\prod_{k=1}^n \mu_k^2(A) = \det(A\bar{A}) = |\det(A)|^2$ . Moreover,  $\text{Re}(\mu_k(A)) \geq 0$  for all  $k$  and the multiplicity of  $\bar{\mu}_k(A)$  coincides with that of  $\mu_k(A)$ . Thus  $\prod_{k=1}^n \mu_k(A) \geq 0$ . Taking the square root leads to the first claim. For the second claim, we take  $A = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$ . Then  $\text{Tr}A = 1 + i$ ,  $|\text{Tr}A| = \sqrt{2}$  and  $\sum_{k=1}^2 \mu_k(A) = 2$ .  $\square$

**Lemma 3.108.** *Let  $x, y$  be two nonnegative vectors of the same size. Denote  $\hat{x} = (x, -x)$ ,  $\hat{y} = (y, -y)$ . If  $\hat{x} \prec \hat{y}$ , then*

$$x \prec_w y.$$

*Proof.* This is trivial by definition of majorization.  $\square$

**Lemma 3.109.** *Let  $x, y$  be two nonnegative vectors of the same size. Denote  $\hat{x} = (x, x)$ ,  $\hat{y} = (y, y)$ . If  $\hat{x} \prec_{\log} \hat{y}$ , then*

$$x \prec_{\log} y.$$

*Proof.* Trivial.  $\square$

**Theorem 3.110.** *Let  $A \in M_n(\mathbb{F})$ . Then*

$$\mu \left( \frac{A + A^T}{2} \right) \succ_w \text{Re}(\mu(A)). \quad (3.39)$$



*Proof.* It is clear that the left hand side of (3.39) is a nonnegative vector, since  $\frac{A+A^T}{2}$  is complex symmetric.

$$\begin{aligned} \operatorname{Re} \lambda \left( \begin{bmatrix} 0 & A \\ \bar{A} & 0 \end{bmatrix} \right) &\prec \lambda \operatorname{Re} \left( \begin{bmatrix} 0 & A \\ \bar{A} & 0 \end{bmatrix} \right) \\ &= \lambda \left( \begin{bmatrix} 0 & \frac{A+(\bar{A})^*}{2} \\ \frac{\bar{A}+A^*}{2} & 0 \end{bmatrix} \right) \\ &= \lambda \left( \begin{bmatrix} 0 & \frac{A+A^T}{2} \\ \frac{A+A^T}{2} & 0 \end{bmatrix} \right). \end{aligned}$$

That is,

$$\lambda \left( \begin{bmatrix} 0 & \frac{A+A^T}{2} \\ \frac{A+A^T}{2} & 0 \end{bmatrix} \right) \succ \operatorname{Re} \lambda \left( \begin{bmatrix} 0 & A \\ \bar{A} & 0 \end{bmatrix} \right).$$

By Lemma 3.108, the desired result holds.  $\square$

We cannot replace “ $\succ_w$ ” by “ $\succ$ ” in (3.39) as the following example shows

**Example 3.111.** Let  $A = \begin{bmatrix} 1 & 2i \\ 0 & 1 \end{bmatrix}$ , then  $\mu(A) = (1, 1)$ ,  $\mu \left( \frac{A+A^T}{2} \right) = \sigma \left( \frac{A+A^T}{2} \right) = (\sqrt{2}, \sqrt{2})$ .

Thus  $\sum_{k=1}^2 \mu_k \left( \frac{A+A^T}{2} \right) > \sum_{k=1}^2 \operatorname{Re}(\mu(A))$  in this case.

**Theorem 3.112.** Let  $A \in M_n(\mathbb{F})$ . Then

$$\sigma(A) \succ_{\log} |\mu(A)|. \quad (3.40)$$

*Proof.* By Proposition 3.104, we have

$$\begin{aligned} (|\mu(A)|, |\mu(A)|) &= |\lambda(\widehat{A})| \prec_{\log} \sigma(\widehat{A}) \\ &= \lambda^{1/2} \left( \begin{bmatrix} 0 & A \\ \bar{A} & 0 \end{bmatrix}^* \begin{bmatrix} 0 & A \\ \bar{A} & 0 \end{bmatrix} \right) \\ &= \lambda^{1/2} \left( \begin{bmatrix} \overline{A^*A} & 0 \\ 0 & A^*A \end{bmatrix} \right) \\ &= (\sigma(A), \sigma(A)), \end{aligned}$$

where the majorization is by (3.36). Here  $x^r$  ( $r \geq 0$ ) means the entrywise  $r$ th power of a nonnegative vector  $x$ . Then Lemma 3.109 gives the desired result.  $\square$

By Proposition 3.101, we have the following corollary, which was the first majorization result discovered on coneigenvalues.

**Corollary 3.113.** [56] *Let  $A \in M_n(\mathbb{F})$ . Then for any  $p \geq 0$ ,*

$$\sigma^p(A) \succ_w |\mu^p(A)|. \quad (3.41)$$

The next corollary is an analogue of the generalized *Schur inequality* [90] with coneigenvalues involved.

**Corollary 3.114.** *Let  $A = [a_{jk}] \in M_n(\mathbb{C})$ . Then for any  $0 \leq p \leq 2$ ,*

$$\sum_{j,k=1}^n |a_{jk}|^p \geq \sum_{k=1}^n \mu_k^p(A). \quad (3.42)$$

*Proof.* Note that the right hand side of (3.42) is real. Mond and Pečarić [85] have showed that

$$\sum_{j,k=1}^n |a_{jk}|^p \geq \sum_{k=1}^n \sigma_k^p(A) \quad (3.43)$$

for  $0 \leq p \leq 2$ . Thus (3.42) follows immediately by (3.41).  $\square$

**Remark 3.115.** *Though Petri and Ikramov [90] only presented (3.43) for  $p \geq 1$  and later a much simpler proof was given in [55], the proofs given there held also for  $0 \leq p < 1$ .*

**Theorem 3.116.** *Let  $A, B \in M_n(\mathbb{F})$  be conjugate normal matrices. Then*

$$\operatorname{Re} \mu(A) + \operatorname{Re} \mu(B) \succ_w \operatorname{Re} \mu(A + B). \quad (3.44)$$

*Proof.* By Theorem 3.110, we have

$$\begin{aligned} \operatorname{Re}(\mu(A + B)) &\prec_w \mu\left(\frac{A + B + (A + B)^T}{2}\right) \\ &= \sigma\left(\frac{A + B + (A + B)^T}{2}\right) \\ &\prec_w \sigma\left(\frac{A + A^T}{2}\right) + \sigma\left(\frac{B + B^T}{2}\right) \\ &= \mu\left(\frac{A + A^T}{2}\right) + \mu\left(\frac{B + B^T}{2}\right) \\ &= \operatorname{Re} \mu(A) + \operatorname{Re} \mu(B) \end{aligned}$$

where the last equality is by Proposition 3.105.  $\square$

**Corollary 3.117.** *Let  $A, B \in M_n(\mathbb{F})$  be symmetric matrices, then*

$$\mu(A) + \mu(B) \succ_w \mu(A + B). \quad (3.45)$$

**Remark 3.118.** *Readers should be able to observe that (3.45) is the same as  $\sigma(A) + \sigma(B) \succ_w \sigma(A + B)$  (for symmetric matrices).*

**Theorem 3.119.** *Let  $A, B \in M_n(\mathbb{F})$  be symmetric matrices, then*

$$\mu(A) \succ_w |(\mu(A + B) - \mu(B))|. \quad (3.46)$$

*Proof.* Since  $A, B$  are symmetric, (3.46) is the same as

$$\sigma(A) \succ_w |\sigma(A + B) - \sigma(B)|. \quad (3.47)$$

(3.47) is the singular value counterpart of (3.38) and can be found in, e.g., [2]. □

To end this section, we give a definition of *consingular value*. For  $A \in M_n(\mathbb{F})$ , we know that one alternative definition for singular values of  $A$  is the nonnegative eigenvalues of the augmented matrix  $\begin{bmatrix} 0 & A \\ A^* & 0 \end{bmatrix}$ . Given the present notion of *coneigenvalue*, the notion of its counterpart, say *consingular value*, seems lacking. What would be a possible definition for *consingular value*? We provide one here, analogous to the definition of singular values in terms of eigenvalues of an augmented matrix.

**Definition 3.120.** *Let  $A \in M_n(\mathbb{C})$ . The consingular values of  $A$  are the  $n$  scalars  $\gamma_1(A), \gamma_2(A), \dots, \gamma_n(A)$  defined by the coneigenvalues of  $\begin{bmatrix} 0 & A \\ A^T & 0 \end{bmatrix}$ , with each consingular value taking half the multiplicity of the corresponding coneigenvalue.*

We can see that, since  $\begin{bmatrix} 0 & A \\ A^T & 0 \end{bmatrix}$  is symmetric,

$$\begin{aligned} \mu \left( \begin{bmatrix} 0 & A \\ A^T & 0 \end{bmatrix} \right) &= \sigma \left( \begin{bmatrix} 0 & A \\ A^T & 0 \end{bmatrix} \right) \\ &= \lambda^{1/2} \left( \begin{bmatrix} 0 & A \\ A^T & 0 \end{bmatrix}^* \begin{bmatrix} 0 & A \\ A^T & 0 \end{bmatrix} \right) \\ &= \lambda^{1/2} \left( \begin{bmatrix} (AA^*)^T & 0 \\ 0 & A^*A \end{bmatrix} \right) \\ &= \sigma \left( \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix} \right). \end{aligned}$$

Thus, with our definition, we have

*The consingular values of a matrix are exactly its singular values.*

Theorem 3.112 can thus be rephrased as

*The consingular values of a matrix log majorize its coneigenvalues in absolute value.*

Majorization relations for eigenvalues or singular values are currently still an active area of study. It is expected that more results on coneigenvalue majorization will be discovered in the near future.

# Chapter 4

## When is a product of positive definite quadratic forms convex

In this chapter, we consider finite products of positive definite quadratic forms. The field we work on here is restricted to real numbers, considering the practical background. The main result is a sufficient condition for the convexity of a finite product of positive definite quadratic forms given in terms of the condition numbers of the underlying matrices. When only two factors are involved, the condition is also necessary.

The result of this chapter has been published in [73]. It is joint work with Sinnamon.

### 4.1 Motivation and the convexity condition

Given a function  $h : \mathbb{R}^n \rightarrow \mathbb{R}$ , its *Legendre-Fenchel conjugate* (LF-conjugate for short), which is also widely referred to as the Legendre-Fenchel transform of  $h$  [4, 5, 13, 26, 48], is defined as

$$h^*(x) = \sup_{y \in \mathbb{R}^n} x^T y - h(y).$$

The LF-conjugate has a significant impact in many areas. It plays an essential role in developing convex optimization theory and algorithms (e.g., [6, 24, 96]); it is also widely used in matrix analysis and eigenvalue optimization [67, 68, 69].

If  $A$  is a real symmetric positive definite matrix we let  $q_A$  denote the quadratic form

$$q_A(y) = \frac{1}{2}y^T A y.$$

It is easy to verify that  $q_A$  is a convex function on  $\mathbb{R}^n$ , and the following fact is not hard to verify (see, e.g., [96]):

**Proposition 4.1.** *The LF-conjugate of  $q_A$  is also a positive definite quadratic form; specifically,*

$$q_A^*(x) = \frac{1}{2}x^T A^{-1}x.$$

*Proof.* Clearly,  $f(y) = x^T y - \frac{1}{2}y^T A y$  is concave and differentiable. The maximum is achieved at its stationary points. From  $\nabla f(y) = x - A y = 0$ , we get  $y = A^{-1}x$ . Thus

$$q_A^*(x) = x^T y - \frac{1}{2}y^T A y = \frac{1}{2}x^T A^{-1}x.$$

□

From a fast computation and practical application point of view, it is interesting and important to know the LF-conjugate of the product of two positive definite quadratic forms. This problem was posed by Hiriart-Urruty as an open question in the field of nonlinear analysis and optimization [47] and recently studied by Y. B. Zhao in [117]. Zhao also considered the LF-conjugate for the products of finitely many positive definite quadratic forms in [118]. Before introducing his result, we need to introduce some notation.

Let  $\kappa(A)$  denote the *condition number* of  $A$ . If  $A \succ 0$ , then  $\kappa(A) = \frac{\lambda_{\max}(A)}{\lambda_{\min}(A)}$ , the ratio of its largest and smallest eigenvalues. Fix  $m \geq 2$ ,  $n \times n$  real matrices  $A_1, \dots, A_m \succ 0$ , and let  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  be the product  $q_{A_1} \dots q_{A_m}$ , i.e.,

$$f(y) = \prod_{i=1}^m \frac{1}{2}y^T A_i y.$$

For  $f$  to be a convex function on  $\mathbb{R}^n$  it is necessary and sufficient that the *Hessian matrix*  $\nabla^2 f(y)$  of  $f$  be positive semi-definite at each point  $y$ . This fact can be found in, e.g., [12, 244]. For  $y \neq 0$ , the gradient and the Hessian matrix of  $f$  are given by,

$$\begin{aligned} \nabla f(y) &= 2f(y) \sum_{i=1}^m \frac{A_i y}{y^T A_i y}, \\ \nabla^2 f(y) &= 2f(y) \left( \sum_{i=1}^m \frac{A_i}{y^T A_i y} + 2 \sum_{i=1}^m \sum_{j \neq i} \frac{A_i y y^T A_j}{y^T A_i y y^T A_j y} \right). \end{aligned}$$

Since  $f(y) > 0$  whenever  $y \neq 0$ , the convexity of  $f$  reduces to showing that

$$\sum_{i=1}^m \frac{x^T A_i x}{y^T A_i y} + 2 \sum_{i=1}^m \sum_{j \neq i} \frac{x^T A_i y}{y^T A_i y} \frac{x^T A_j y}{y^T A_j y} \geq 0 \quad (4.1)$$

for all  $x, y \in \mathbb{R}^n$  with  $y \neq 0$ . (When  $y = 0$ ,  $\nabla^2 f(0) = 0$  is positive semi-definite for any choice of  $A_1, \dots, A_m$ .)

If all  $A_i$  are equal, obviously (4.1) is true. However, (4.1) does not hold for general  $A_i$ . In Theorem 3.6 of [117], Zhao gave an explicit formula for the LF-conjugate of  $f$ , provided  $f$  is known to be convex. So it is important to have simple, easily verified conditions that ensure the convexity of  $f$ . Zhao obtained the following sufficient condition for the convexity of  $f$ .

**Proposition 4.2.** [118] *Let  $A_i \succ 0$ ,  $i = 1, \dots, m$  be real  $n \times n$  matrices. If*

$$\kappa(A_j^{-\frac{1}{2}} A_i A_j^{-\frac{1}{2}}) \leq \frac{\sqrt{4m-2} + 2}{\sqrt{4m-2} - 2}$$

for all  $i, j = 1, \dots, m$ ,  $i \neq j$ , then the product of  $m$  quadratic forms  $f = \prod_{i=1}^m q_{A_i}$  is convex.

As a consequence of our main result, Theorem 4.9 below, we give the following improvement of Proposition 4.2. The proof will be given in the next section.

**Theorem 4.3.** *Let  $A_i \succ 0$ ,  $i = 1, \dots, m$  be real  $n \times n$  matrices. If*

$$\kappa(A_j^{-\frac{1}{2}} A_i A_j^{-\frac{1}{2}}) \leq \left( \frac{\sqrt{2m-2} + 1}{\sqrt{2m-2} - 1} \right)^2 \quad (4.2)$$

for all  $i, j = 1, \dots, m$ ,  $i \neq j$ , then the product of the  $m$  quadratic forms  $f = \prod_{i=1}^m q_{A_i}$  is convex. If  $m = 2$  the condition (4.2) is also necessary for the convexity of  $f$ .

**Remark 4.4.** For  $m \geq 2$ ,  $2\sqrt{2m-2} > \sqrt{4m-2}$  so

$$\begin{aligned} \left( \frac{\sqrt{2m-2} + 1}{\sqrt{2m-2} - 1} \right)^2 &> \frac{\sqrt{2m-2} + 1}{\sqrt{2m-2} - 1} \\ &> \frac{\sqrt{4m-2} + 2}{\sqrt{4m-2} - 2}. \end{aligned}$$

This shows that (4.2) is strictly weaker than the hypothesis of Proposition 4.2. When  $m = 2$ , the upper bound in Theorem 4.3, i.e.,  $17 + 12\sqrt{2}$ , was already known to be the greatest possible right-hand-side value such that (4.2) could ensure the convexity of the product of two positive definite quadratic forms. See Remark 2.7 in [118].

**Corollary 4.5.** *If  $A \succ 0$  is a real  $n \times n$  matrix, then the Kantorovich function  $(x^T Ax)(x^T A^{-1}x)$ , where  $x \in \mathbb{R}^n$ , is convex if and only if  $\kappa(A) \leq 3 + 2\sqrt{2}$ .*

*Proof.* Let  $m = 2$ ,  $A_1 = A$  and  $A_2 = A^{-1}$  in Theorem 4.3. Then  $\kappa(A_2^{-\frac{1}{2}}A_1A_2^{-\frac{1}{2}}) \leq (3 + 2\sqrt{2})^2$  is equivalent to  $\kappa(A^2) \leq (3 + 2\sqrt{2})^2$ , i.e.,  $\kappa(A) \leq 3 + 2\sqrt{2}$ .  $\square$

The result of the corollary in the case  $n = 2$  as well as the necessity of the condition on  $\kappa$  for general  $n$  was given in [119].

## 4.2 Auxiliary results and the proof

We start with a simple but useful lemma. It may be viewed as a sharp version of Theorem 4.3 in the case of two  $2 \times 2$  matrices.

**Lemma 4.6.** *If  $\kappa \geq 1$  and  $\eta = ((\sqrt{\kappa} - 1)/(\sqrt{\kappa} + 1))^2$  then*

$$\eta(\kappa + s^2)(1 + t^2) + \eta(\kappa + t^2)(1 + s^2) + 2(\kappa + st)(1 + st) \geq 0$$

*for all  $s, t \in \mathbb{R}$ . Equality holds if and only if  $s = -t = \pm\kappa^{1/4}$  or  $\kappa = 1$  and  $st = -1$ .*

*Proof.* For any  $s, t$ , and  $z$  we may factor out  $z^2 + 1$  and complete the square on  $z$  to get,

$$\begin{aligned} & (z - 1)^2(z^2 + s^2)(1 + t^2) + (z - 1)^2(z^2 + t^2)(1 + s^2) + 2(z + 1)^2(z^2 + st)(1 + st) \\ &= (z^2 + 1)(4 + (s + t)^2) \left( \left( z - \frac{(s - t)^2}{4 + (s + t)^2} \right)^2 + \frac{4(s + t)^2(1 + st)^2}{(4 + (s + t)^2)^2} \right). \end{aligned}$$

The second expression is non-negative and vanishes if and only if either  $s + t = 0$  and  $z = s^2$ , or  $st = -1$  and  $z = 1$ . In the first expression, divide through by  $(z + 1)^2$  and take  $z = \sqrt{\kappa}$  to obtain the conclusion of the lemma.  $\square$

The next lemma essentially gives a reduction of the case of two  $n \times n$  matrices to the case of two  $2 \times 2$  matrices, and then applies the previous result.

**Lemma 4.7.** *Suppose  $A, B \succ 0$  are real  $n \times n$  matrices and let  $\kappa = \kappa(A^{-1/2}BA^{-1/2})$ . Then for  $x, y \in \mathbb{R}^n$ , with  $y \neq 0$ , we have*

$$2 \frac{x^T Ay x^T By}{y^T Ay y^T By} \geq - \left( \frac{\sqrt{\kappa} - 1}{\sqrt{\kappa} + 1} \right)^2 \left( \frac{x^T Ax}{y^T Ay} + \frac{x^T Bx}{y^T By} \right).$$

*The inequality is sharp.*



*Proof.* Since  $A^{-1/2}BA^{-1/2} \succ 0$ , there exists an orthogonal matrix  $U$  such that  $U^T A^{-1/2}BA^{-1/2}U$  is a diagonal matrix with diagonal entries  $\lambda_1 \geq \dots \geq \lambda_n > 0$ . Note that  $\kappa = \lambda_1/\lambda_n$ . Let  $\eta = ((\sqrt{\kappa} - 1)/(\sqrt{\kappa} + 1))^2$ . If we replace  $x$  by  $A^{-1/2}Ux$  and  $y$  by  $A^{-1/2}Uy$ , an invertible change of variable, the statement of the lemma reduces to showing,

$$2 \frac{\sum_{i=1}^n x_i y_i}{\sum_{i=1}^n y_i^2} \frac{\sum_{j=1}^n \lambda_j x_j y_j}{\sum_{j=1}^n \lambda_j y_j^2} \geq -\eta \left( \frac{\sum_{i=1}^n x_i^2}{\sum_{i=1}^n y_i^2} + \frac{\sum_{j=1}^n \lambda_j x_j^2}{\sum_{i=j}^n \lambda_j y_j^2} \right), \quad (4.3)$$

for all  $x$  and  $y$  in  $\mathbb{R}^n$  with  $y \neq 0$ . Multiplying through to eliminate the denominators, we see that this is equivalent to showing  $\sum_{j=1}^n \lambda_j r_j \geq 0$ , where

$$r_j = \eta x_j^2 \sum_{i=1}^n y_i^2 + \eta y_j^2 \sum_{i=1}^n x_i^2 + 2x_j y_j \sum_{i=1}^n x_i y_i.$$

Because  $\sum_{j=1}^n \lambda_j r_j$  is continuous in both  $x$  and  $y$ , it is enough to show that it is non-negative for all  $x$  and  $y$  such that  $x_1, x_n, y_1, y_n$  are all non-zero. Fix  $x$  and  $y$  satisfying that condition and partition  $\{1, \dots, n\}$  into subsets  $J_1$  and  $J_2$  as follows:  $1 \in J_1, n \in J_2$  and for  $2 \leq j \leq n-1$ ,  $j \in J_1$  if  $r_j \leq 0$  and  $j \in J_2$  otherwise. This ensures that  $\lambda_j r_j \geq \lambda_1 r_j$  for  $j \in J_1$  and  $\lambda_j r_j \geq \lambda_n r_j$  for  $j \in J_2$ . Thus,

$$\sum_{j=1}^n \lambda_j r_j \geq \lambda_1 \sum_{j \in J_1} r_j + \lambda_n \sum_{j \in J_2} r_j.$$

Now for  $p = 1, 2$ , define  $u_p$  and  $v_p$  by,

$$u_p^2 = \left( \frac{\sum_{i \in J_p} x_i^2}{\sum_{i \in J_p} y_i^2} \right)^{1/2} \sum_{i \in J_p} x_i y_i$$

and

$$v_p^2 = \left( \frac{\sum_{i \in J_p} y_i^2}{\sum_{i \in J_p} x_i^2} \right)^{1/2} \sum_{i \in J_p} x_i y_i,$$

ensuring that  $u_p \geq 0$  and choosing the sign of  $v_p$  so that  $u_p v_p = \sum_{i \in J_p} x_i y_i$ . The Cauchy-Schwarz inequality shows  $u_p^2 \leq \sum_{i \in J_p} x_i^2$  and  $v_p^2 \leq \sum_{i \in J_p} y_i^2$ , and it follows from the definition of  $r_j$  that,

$$\sum_{j \in I_p} r_j \geq \eta u_p^2 (v_1^2 + v_2^2) + \eta v_p^2 (u_1^2 + u_2^2) + 2u_p v_p (u_1 v_1 + u_2 v_2).$$

These estimates complete the proof, as

$$\begin{aligned}
\sum_{j=1}^n \lambda_j r_j &\geq \lambda_1 \sum_{j \in J_1} r_j + \lambda_n \sum_{j \in J_2} r_j \\
&= \lambda_n \left( \kappa \sum_{j \in J_1} r_j + \sum_{j \in J_2} r_j \right) \\
&\geq \lambda_n (\eta (\kappa u_1^2 + u_2^2) (v_1^2 + v_2^2) + \eta (\kappa v_1^2 + v_2^2) (u_1^2 + u_2^2) \\
&\quad + 2(\kappa u_1 v_1 + u_2 v_2) (u_1 v_1 + u_2 v_2)) \\
&= \lambda_n u_1^2 v_1^2 [\eta (\kappa + s^2) (1 + t^2) + \eta (\kappa + t^2) (1 + s^2) + 2(\kappa + st) (1 + st)],
\end{aligned}$$

where  $s = u_2/u_1$  and  $t = v_2/v_1$ . The last expression is non-negative by Lemma 4.6.

To see that the inequality of the lemma is sharp it is enough to find  $(x_1, \dots, x_n)$  and  $(y_1, \dots, y_n)$  such that equality is achieved in (4.3). Since  $\kappa = \lambda_1/\lambda_n$  it is routine to verify that the choice,  $x_1 = 1$ ,  $x_n = \kappa^{1/4}$ ,  $y_1 = 1$ ,  $y_n = -\kappa^{1/4}$  and  $x_2 = \dots = x_{n-1} = y_2 = \dots = y_{n-1} = 0$  will suffice.

□

**Remark 4.8.** *It turns out Lemma 4.7 can be proved using a simple consequence of the generalized Wielandt inequality that we present in the next chapter.*

The next theorem gives the main result of the chapter, a readily computed condition for a product of positive definite quadratic forms to be a convex function. The condition is expressed in terms of the condition numbers of the matrices involved.

**Theorem 4.9.** *Let  $A_1, A_2, \dots, A_m \succ 0$  be real  $n \times n$  matrices and let  $\kappa_{i,j} = \kappa(A_i^{-1/2} A_j A_i^{-1/2})$  for  $i, j = 1, \dots, m$ . If*

$$\sum_{j=1}^m \left( \frac{\sqrt{\kappa_{i,j}} - 1}{\sqrt{\kappa_{i,j}} + 1} \right)^2 \leq \frac{1}{2} \tag{4.4}$$

*for  $i = 1, 2, \dots, m$ , then  $f = \prod_{i=1}^m q_{A_i}$  is convex. If  $m = 2$  the condition is also necessary for the convexity of  $f$ .*

*Proof.* Note that  $\kappa_{i,j} = \kappa_{j,i}$  and  $\kappa_{i,i} = 1$ . Let  $\eta_{i,j} = \left(\frac{\sqrt{\kappa_{i,j}}-1}{\sqrt{\kappa_{i,j}}+1}\right)^2$  and apply Lemma 4.7 to get

$$\begin{aligned} \sum_{i=1}^m \frac{x^T A_i x}{y^T A_i y} + 2 \sum_{i=1}^m \sum_{j \neq i} \frac{x^T A_i y}{y^T A_i y} \frac{x^T A_j y}{y^T A_j y} &\geq \sum_{i=1}^m \frac{x^T A_i x}{y^T A_i y} - \sum_{i=1}^m \sum_{j \neq i} \eta_{i,j} \left( \frac{x^T A_i x}{y^T A_i y} + \frac{x^T A_j x}{y^T A_j y} \right) \\ &= \sum_{i=1}^m \frac{x^T A_i x}{y^T A_i y} - 2 \sum_{i=1}^m \left( \sum_{j \neq i} \eta_{i,j} \right) \frac{x^T A_i x}{y^T A_i y} \\ &= \sum_{i=1}^m \frac{x^T A_i x}{y^T A_i y} \left( 1 - 2 \sum_{j=1}^m \eta_{i,j} \right) \geq 0. \end{aligned}$$

As pointed out in (4.1) this shows that  $f$  is convex.

If  $m = 2$ , the convexity of  $f$  implies, via (4.1), that

$$2 \frac{x^T A_1 y}{y^T A_1 y} \frac{x^T A_2 y}{y^T A_2 y} \geq -\frac{1}{2} \left( \frac{x^T A_1 x}{y^T A_1 y} + \frac{x^T A_2 x}{y^T A_2 y} \right)$$

for all  $x$  and non-zero  $y$ . Combining this with the sharpness of the inequality of Lemma 4.7 gives,

$$\left( \frac{\sqrt{\kappa_{1,2}}-1}{\sqrt{\kappa_{1,2}}+1} \right)^2 \leq \frac{1}{2},$$

showing that (4.4) is necessary for convexity.  $\square$

*Proof of Theorem 4.3.* We verify the condition of the Theorem 4.9. Recall that  $\eta_{i,i} = 0$  and calculate as follows,

$$\sum_{j=1}^m \left( \frac{\sqrt{\kappa_{i,j}}-1}{\sqrt{\kappa_{i,j}}+1} \right)^2 \leq (m-1) \left( \frac{\frac{\sqrt{2m-2}+1}{\sqrt{2m-2}-1}-1}{\frac{\sqrt{2m-2}+1}{\sqrt{2m-2}-1}+1} \right)^2 = \frac{1}{2}.$$

So (4.4) is satisfied and therefore  $f$  is convex. If  $m = 2$ , an easy calculation shows that the conditions (4.2) and (4.4) coincide so (4.2) is also necessary for convexity.

**Remark 4.10.** *The proof of Theorem 4.9 suggests the following weakening of condition (4.4). Since*

$$\frac{1}{\kappa(A_i)} \frac{x^T x}{y^T y} \leq \frac{x^T A_i x}{y^T A_i y} \leq \kappa(A_i) \frac{x^T x}{y^T y},$$

*if we define,*

$$L = \left\{ i : \sum_{j=1}^m \eta_{i,j} \leq \frac{1}{2} \right\}$$

and

$$G = \left\{ i : \sum_{j=1}^m \eta_{i,j} > \frac{1}{2} \right\}$$

then the proof goes through with condition (4.4) replaced by,

$$\sum_{i \in L} \frac{1}{\kappa(A_i)} \left( 1 - 2 \sum_{j=1}^m \eta_{i,j} \right) + \sum_{i \in G} \kappa(A_i) \left( 1 - 2 \sum_{j=1}^m \eta_{i,j} \right) \geq 0. \quad (4.5)$$

This condition is weaker than (4.4) and still implies that  $f$  is convex, but is complicated and rather unwieldy. It can be applied, however, as we see in the next example where it is used to show that the condition (4.4) is not necessary when  $m > 2$ .

**Example 4.11.** With  $m = 3$ , take  $A_1$  and  $A_2$  to be  $2 \times 2$  identity matrices, and  $A_3$  to be a  $2 \times 2$  diagonal matrix with diagonal entries  $(3 + \delta)^2$  and 1. Calculations show that for sufficiently small positive  $\delta$ , (4.4) fails but (4.5) holds. (Any positive  $\delta < 0.18$  will do.) Thus, the sufficient condition of Theorem 4.9 is not necessary for general  $m$ .

# Chapter 5

## Generalized Wielandt inequalities

### 5.1 Kantorovich inequality and Wielandt inequality

The Kantorovich inequality, first published in 1948, aroused a considerable amount of interest. It was originally advanced to provide an estimate of the rate of convergence of the steepest descent method for minimizing a quadratic problem with a positive definite Hessian. For more information, see [54, 1]. There are many generalizations and new proofs. We state here the original form of the *Kantorovich* and *Wielandt inequalities*, including simple proofs.

Let  $A \in H_n^{++}$  with largest and smallest eigenvalues  $\lambda_1$  and  $\lambda_n$ , respectively. Then

#### Kantorovich inequality

$$(x^*Ax)(x^*A^{-1}x) \leq \frac{(\lambda_1 + \lambda_n)^2}{4\lambda_1\lambda_n}(x^*x)^2 \quad (5.1)$$

for any  $x \in \mathbb{C}^n$ ;

*Proof.* We may assume  $A = \text{Diag}(\lambda_1, \dots, \lambda_n)$  and  $x$  is a unit vector. Then (5.1) reduces to

$$\sum_{j=1}^n \lambda_j x_j^2 \sum_{j=1}^n \frac{1}{\lambda_j} x_j^2 \leq \frac{(\lambda_1 + \lambda_n)^2}{4\lambda_1\lambda_n}. \quad (5.2)$$

Obviously,

$$\sum_{i=1}^n x_i^2 (\lambda_1 - \lambda_i) \geq \sum_{i=1}^n \frac{\lambda_n}{\lambda_i} x_i^2 (\lambda_1 - \lambda_i).$$

Expanding it, we have

$$\lambda_1 - \sum_{i=1}^n \lambda_i x_i^2 \geq \lambda_1 \lambda_n \left( \sum_{i=1}^n \frac{1}{\lambda_i} x_i^2 \right) - \lambda_n.$$

That is, we have

$$\begin{aligned} \lambda_1 + \lambda_n &\geq \lambda_1 \lambda_n \left( \sum_{i=1}^n \frac{1}{\lambda_i} x_i^2 \right) + \sum_{i=1}^n \lambda_i x_i^2 \\ &\geq 2\sqrt{\lambda_1 \lambda_n} \sqrt{\left( \sum_{i=1}^n \frac{1}{\lambda_i} x_i^2 \right) \left( \sum_{i=1}^n \lambda_i x_i^2 \right)}, \end{aligned}$$

where the second inequality is by the arithmetic mean-geometric mean inequality. Thus

$$\frac{\lambda_1 + \lambda_n}{2\sqrt{\lambda_1 \lambda_n}} \geq \sqrt{\left( \sum_{i=1}^n \frac{1}{\lambda_i} x_i^2 \right) \left( \sum_{i=1}^n \lambda_i x_i^2 \right)}.$$

Taking the square on both sides gives (5.2). This completes the proof.  $\square$

**Remark 5.1.** We actually proved a stronger inequality than (5.1), i.e.,

$$x^*Ax + \lambda_1 \lambda_n x^*A^{-1}x \leq (\lambda_1 + \lambda_n)x^*x \quad (5.3)$$

for any  $A \in H_n^{++}$  and  $x \in \mathbb{C}^n$ . (5.3) was first observed in Mond's note [84], which was a special case of Marshall and Olkin's result, see [81].

### Wielandt inequality

$$|x^*Ay|^2 \leq \left( \frac{\lambda_1 - \lambda_n}{\lambda_1 + \lambda_n} \right)^2 (x^*Ax)(y^*Ay), \quad (5.4)$$

where  $x, y \in \mathbb{C}^n$  such that  $x^*y = 0$ .

*Proof.* When  $n = 2$ , write  $A = \begin{bmatrix} a & b \\ \bar{b} & c \end{bmatrix}$  and let  $\alpha$  and  $\beta$  be the eigenvalues of  $A$  with  $\alpha \geq \beta$ .

Observe that

$$\alpha, \beta = \frac{(a+c) \pm \sqrt{(a-c)^2 + 4|b|^2}}{2}.$$

It is easy to verify that

$$|b|^2 \leq \left( \frac{\alpha - \beta}{\alpha + \beta} \right)^2 ac. \quad (5.5)$$

Consider the 2-by-2 matrix

$$M = \begin{bmatrix} x^*Ax & x^*Ay \\ y^*Ax & y^*Ay \end{bmatrix}.$$

Then  $M = (x, y)^*A(x, y)$  is bounded from below by  $\lambda_n(x, y)^*(x, y)$  and from above by  $\lambda_1(x, y)^*(x, y)$ .

We may assume that  $x$  and  $y$  are orthonormal by scaling both sides of (5.4). Then  $\lambda_n I \preceq M \preceq \lambda_1 I$  and thus the eigenvalues  $\gamma$  and  $\delta$  of  $M$  with  $\gamma \geq \delta$  are contained in  $[\lambda_n, \lambda_1]$ . Therefore  $\frac{\gamma - \delta}{\gamma + \delta} \leq \frac{\lambda_1 - \lambda_n}{\lambda_1 + \lambda_n}$  since  $\frac{t-1}{t+1}$  is monotone in  $t$ . An application of (5.5) to  $M$  results in

$$|x^*Ay|^2 \leq \left( \frac{\gamma - \delta}{\gamma + \delta} \right)^2 (x^*Axy^*Ay) \leq \left( \frac{\lambda_1 - \lambda_n}{\lambda_1 + \lambda_n} \right)^2 (x^*Axy^*Ay).$$

□

As is noticed in [11], see also [53], taking  $y = (I - xx^*)A^{-1}x$  reduces (5.4) to (5.1). But only 40 years later, the equivalence between these two inequalities was established, see [16] and [116].

In this section, we give an alternative proof that the Kantorovich inequality implies the Wielandt inequality.

*The proof.* The homogeneous appearance of (5.1) and (5.4) enable us to assume  $x^*x = y^*y = 1$ , so in the following, we shall require that  $x, y$  be orthonormal vectors. (5.1) can be written as

$$\frac{4\lambda_1\lambda_n}{(\lambda_1 + \lambda_n)^2} \leq \frac{1}{(x^*Ax)(x^*A^{-1}x)}. \quad (5.6)$$

Note that  $\frac{4\lambda_1\lambda_n}{(\lambda_1 + \lambda_n)^2} + \left( \frac{\lambda_1 - \lambda_n}{\lambda_1 + \lambda_n} \right)^2 = 1$ , so to show that (5.1) implies (5.4), it suffices to show

$$1 - \frac{1}{(x^*Ax)(x^*A^{-1}x)} \geq \frac{|x^*Ay|^2}{(x^*Ax)(y^*Ay)},$$

i.e.,

$$((x^*A^{-1}x)^2 x^*Ax - x^*A^{-1}x)(y^*Ay) \geq (x^*A^{-1}x)^2 |x^*Ay|^2. \quad (5.7)$$

Let  $B$  be a positive definite square root of  $A$ . Note that  $(B^{-1}x)^*(By) = 0$ , so

$$\begin{aligned} (x^*A^{-1}x)^2|x^*Ay|^2 &= |(x^*A^{-1}x(Bx) - (B^{-1}x)^*By)|^2 \\ &\leq \|x^*A^{-1}x(Bx) - (B^{-1}x)^*\|^2\|By\|^2 \\ &= ((x^*A^{-1}x)^2x^*Ax - x^*A^{-1}x)(y^*Ay), \end{aligned}$$

where  $\|\cdot\|$  means the Euclidean norm. This completes the proof.  $\square$

**Remark 5.2.** *If we consider the real case only, there is an alternative argument for (5.7). Again, we let  $B$  be the positive definite square root of  $A$ , also let  $S = B(yx^T - xy^T)B$ . Obviously,  $S$  is skew symmetric, so the eigenvalues of  $S$  are of the form  $\pm it$ ,  $t \in \mathbb{R}$ , which implies*

$$2\|S\|^2 \leq \|S\|_F^2,$$

where  $\|S\|$ ,  $\|S\|_F$  means the spectral norm (the largest singular value), Frobenius norm of  $S$ , respectively. The Frobenius norm of  $S$  is

$$\|S\|_F = \sqrt{\text{Tr}(S^T S)} = \sqrt{2[(x^T Ax)(y^T Ay) - (x^T Ay)^2]}.$$

Observe that  $By = SB^{-1}x$ , and we get

$$\|By\|_2 = \|SB^{-1}x\|_2 \leq \|S\|_2\|B^{-1}x\|_2 \leq \frac{\|S\|_F}{\sqrt{2}}\|B^{-1}x\|_2,$$

i.e.,  $\|By\|_2 \leq \frac{\|S\|_F}{\sqrt{2}}\|B^{-1}x\|_2$ . This proves (5.7).

We are not sure whether (5.4) and (5.3) are equivalent. Thus we leave the following question for interested readers.

**Question 5.3.** *Is it possible to show that the Wielandt inequality (5.4) implies (5.3)?*

## 5.2 Some more background and applications

The Wielandt and generalized Wielandt inequalities control how much angles can change under a given invertible matrix transformation of  $\mathbb{C}^n$ . The control is given in terms of the condition number of the matrix. Wielandt, in [109], gave a bound on the resulting angles when orthogonal complex lines are transformed. Subsequently, Bauer and Householder,



in [11], extended the inequality to include arbitrary starting angles. These basic inequalities of matrix analysis were introduced to give bounds on convergence rates of iterative projection methods but have found a variety of applications in numerical methods, especially eigenvalue estimation. They are also applied in multivariate analysis, where angles between vectors correspond to statistical correlation. See, for example, [11], [33], [35], [51] and [53]. There are also matrix-valued versions of the inequality that are receiving attention, especially in the context of statistical analysis. See [16], [76], [105], and [114].

The condition number of an invertible matrix  $A$  is  $\kappa(A) = \|A\| \|A^{-1}\|$ , where  $\|\cdot\|$  denotes the operator norm. If  $A$  is positive definite and Hermitian,  $\kappa(A)$  is easily seen to be the ratio of the largest and smallest eigenvalues of  $A$ . The following statement of the generalized Wielandt inequality is taken from [53].

**Theorem 5.4.** *Let  $A$  be an invertible  $n \times n$  matrix. If  $x, y \in \mathbb{C}^n$  and  $\Phi, \Psi \in [0, \pi/2]$  satisfy*

$$|y^*x| \leq \|x\| \|y\| \cos \Phi \quad \text{and} \quad \cot(\Psi/2) = \kappa(A) \cot(\Phi/2),$$

*then*

$$|(Ay)^*(Ax)| \leq \|Ax\| \|Ay\| \cos \Psi.$$

The generalized Wielandt inequality can be difficult to apply for several reasons. First, despite having various equivalent formulations, the inequality seems always to be expressed in ways that hide the natural symmetry coming from the invertible transformation involved. Next, the conditions for equality are known, see [63], but are unwieldy and hard to apply. Finally, the angles involved are angles between complex lines<sup>1</sup> rather than between individual vectors.

Although the last point seems minor, we found it to be the key to a symmetric formulation and a simple description of the cases of equality. In Theorem 5.8 and its matrix analytic counterpart, Theorem 5.14, we present a new inequality that gives sharp upper and lower bounds for the angle between a pair of transformed vectors. The conditions for equality are simple and easy to apply. This new inequality relates angles between vectors rather than between complex lines but it immediately implies a result for angles between

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<sup>1</sup>A complex line is a one-dimensional affine subspace of a vector space over the complex numbers. A common point of confusion is that while a complex line has dimension one over  $\mathbb{C}$  (hence the term “line”), it has dimension two over the real numbers  $\mathbb{R}$ , and is topologically equivalent to a real plane, not a real line, see [108].

complex lines that is equivalent to the generalized Wielandt inequality. Moreover, this version of the generalized Wielandt inequality retains the simple form of the new inequality and (most of) the simplicity of its conditions for equality.

In Section 5.3 we work in the context of an arbitrary real or complex vector space having two inner products. This approach preserves symmetry by avoiding the distinction between angles before and after a fixed transformation. Also, the main result is not restricted to  $\mathbb{C}^n$  but holds for vectors in infinite-dimensional spaces. As an application of the unrestricted result, we improve a metric space inequality from [32]. The main results are then formulated in the language of matrix analysis in Section 5.4, and we apply them to improve inequalities from [112] and [73], and to settle a conjecture from [111].

To begin, a short discussion of angles in inner product spaces is in order. Recall that in a real inner product space  $(V, \langle \cdot, \cdot \rangle)$  the angle  $\theta = \theta(u, v)$  between two non-zero vectors is defined by,  $0 \leq \theta \leq \pi$  and

$$\cos \theta = \frac{\langle u, v \rangle}{\|u\| \|v\|}.$$

Here  $\|u\| = \sqrt{\langle u, u \rangle}$  is the norm induced by the inner product. The angle between subsets  $S$  and  $T$  of  $V$  is the infimum of the angles between non-zero elements of  $S$  and  $T$ , so

$$\Theta(S, T) = \inf\{\theta(u, v) : 0 \neq u \in S, 0 \neq v \in T\}.$$

With this definition it is easy to check that the angle  $\Theta = \Theta(\mathbb{R}u, \mathbb{R}v)$  between the lines  $\mathbb{R}u$  and  $\mathbb{R}v$  satisfies  $0 \leq \Theta \leq \pi/2$  and

$$\cos \Theta = \frac{|\langle u, v \rangle|}{\|u\| \|v\|}.$$

A complex inner product space  $(V, \langle \cdot, \cdot \rangle)$  may be viewed as the real inner product space  $(V_{\mathbb{R}}, \operatorname{Re}\langle \cdot, \cdot \rangle)$  where  $V_{\mathbb{R}} = V$  with the scalars restricted to  $\mathbb{R}$ . Since  $\operatorname{Re}\langle v, v \rangle = \langle v, v \rangle$  for all  $v \in V$ , lengths in  $V$  are preserved and therefore so are angles. Thus, this real inner product is used to define the angle  $\theta$  between the vectors  $u$  and  $v$ , and a computation gives the formula for the angle  $\Theta$  between the complex lines  $\mathbb{C}u$  and  $\mathbb{C}v$ . We have,

$$\cos \theta = \frac{\operatorname{Re}\langle u, v \rangle}{\|u\| \|v\|} \quad \text{and} \quad \cos \Theta = \frac{|\langle u, v \rangle|}{\|u\| \|v\|}.$$

The second formula is often used as a definition of the angle between vectors  $u$  and  $v$  in a complex inner product space. (Angles defined this way do not determine angles in triangles

correctly but they have the advantage that complex orthogonality, namely  $\langle u, v \rangle = 0$ , is equivalent to the angle between  $u$  and  $v$  being  $\pi/2$ .)

We will make use of the simple observation that if  $|\alpha| = 1$ , then

$$\Theta(\mathbb{C}u, \mathbb{C}v) = \theta(\alpha u, v) \quad \text{if and only if} \quad |\langle u, v \rangle| = \alpha \langle u, v \rangle. \quad (5.8)$$

(Note that our inner products are taken to be linear in the first variable.) The above observation remains valid for  $\Theta(\mathbb{R}u, \mathbb{R}v)$  in a real inner product space, where  $\alpha = \pm 1$ .

### 5.3 Generalized Wielandt inequality in inner product spaces

The result of this section has been published in [74]. It is joint work with Sinnamon.

Suppose  $V$  is a non-trivial real or complex vector space. Let  $\langle \cdot, \cdot \rangle_1$  and  $\langle \cdot, \cdot \rangle_2$  be inner products on  $V$  and define  $m, V_m, M, V_M, E_1$  and  $E_2$  by,

$$\left\{ \begin{array}{l} m = \inf_{0 \neq v \in V} \|v\|_2 / \|v\|_1, \quad V_m = \{v \in V : \|v\|_2 = m\|v\|_1\}, \\ M = \sup_{0 \neq v \in V} \|v\|_2 / \|v\|_1, \quad V_M = \{v \in V : \|v\|_2 = M\|v\|_1\}, \\ E = E_j = \left\{ (u, v) : \frac{u}{\|u\|_j} + \frac{v}{\|v\|_j} \in V_m, \frac{u}{\|u\|_j} - \frac{v}{\|v\|_j} \in V_M \right\}, \end{array} \right. \quad (5.9)$$

for  $j = 1, 2$ . Here, as usual,  $\|v\|_1 = \sqrt{\langle v, v \rangle_1}$  and  $\|v\|_2 = \sqrt{\langle v, v \rangle_2}$ . We anticipate the result of Corollary 5.7 in the definition of  $E$  above.

Evidently  $0 \leq m \leq M \leq \infty$ ,  $0 \in V_m$  and  $0 \in V_M$ . (The convention  $0 \cdot \infty = 0$  ensures that  $0 \in V_M$  when  $M = \infty$ .) A standard compactness argument shows that if  $V$  is finite dimensional then  $0 < m \leq M < \infty$  and  $V_m \neq \{0\} \neq V_M$ . If  $m = M$  then  $V_m = V_M = V$  and, by polarization,  $\langle u, v \rangle_2 = m^2 \langle u, v \rangle_1$  for all  $u, v \in V$ .

**Lemma 5.5.** *Let  $V$  be a real vector space equipped with inner products  $\langle \cdot, \cdot \rangle_1$  and  $\langle \cdot, \cdot \rangle_2$ . Let (5.9) hold. If  $m < M$ , then  $V_m$  and  $V_M$  are subspaces and the two are mutually orthogonal with respect to both inner products.*

*Proof.* Suppose  $u$  is a non-zero vector in  $V_m$  and  $v \in V$  is not a multiple of  $u$ . Then

$$f(t) = \frac{\|u + tv\|_2^2}{\|u + tv\|_1^2} = \frac{\langle u, u \rangle_2 + 2t \langle u, v \rangle_2 + t^2 \langle v, v \rangle_2}{\langle u, u \rangle_1 + 2t \langle u, v \rangle_1 + t^2 \langle v, v \rangle_1}$$

is defined and differentiable for  $t \in \mathbb{R}$ . Since  $f$  achieves its minimum value at  $t = 0$ ,  $f'(0) = 0$ . That is,  $\langle u, v \rangle_2 \langle u, u \rangle_1 = \langle u, u \rangle_2 \langle u, v \rangle_1$ . Thus, for all  $u \in V_m$  and all  $v \in V$ ,

$$\langle u, v \rangle_2 = m^2 \langle u, v \rangle_1.$$

(The excluded case,  $u = 0$  or  $v$  a multiple of  $u$ , is easily verified.) It follows that if  $v \in V_m$  then  $f$  is the constant function with value  $m^2$ . In particular,  $f(1) = m^2$ , so  $u + v \in V_m$ . Since it is clearly closed under scalar multiplication,  $V_m$  is a subspace.

Repeating the argument for  $V_M$  shows that it, too, is a subspace and that for all  $v \in V_M$  and  $u \in V$ ,

$$\langle u, v \rangle_2 = M^2 \langle u, v \rangle_1.$$

If  $u \in V_m$  and  $v \in V_M$  then  $m^2 \langle u, v \rangle_1 = \langle u, v \rangle_2 = M^2 \langle u, v \rangle_1$  and hence  $\langle u, v \rangle_1 = \langle u, v \rangle_2 = 0$ . Thus  $u$  and  $v$  are orthogonal with respect to both inner products. This completes the proof.  $\square$

**Corollary 5.6.** *Let  $V$  be a real vector space equipped with inner products  $\langle \cdot, \cdot \rangle_1$  and  $\langle \cdot, \cdot \rangle_2$ . Let (5.9) hold. If  $V$  is two-dimensional, then there is a basis of  $V$  that is orthogonal with respect to both inner products.*

*Proof.* If  $m = M$  then the two inner products are multiples of each other and any orthogonal basis will do. Otherwise, let  $0 \neq b \in V_m$  and  $0 \neq B \in V_M$ . Then  $\{b, B\}$  is the desired basis.  $\square$

The next result justifies the use of  $E$  to denote either  $E_1$  or  $E_2$ .

**Corollary 5.7.** *Let  $V$  be a real vector space equipped with inner products  $\langle \cdot, \cdot \rangle_1$  and  $\langle \cdot, \cdot \rangle_2$ . Let (5.9) hold. Then  $E_1 = E_2$ .*

*Proof.* By symmetry it is enough to show that  $E_1 \subseteq E_2$ . For  $(u, v) \in E_1$ , let

$$w = \frac{u}{\|u\|_1} + \frac{v}{\|v\|_1} \in V_m$$

and

$$W = \frac{u}{\|u\|_1} - \frac{v}{\|v\|_1} \in V_M.$$

By Lemma 5.5,  $w$  and  $W$  are orthogonal with respect to  $\langle \cdot, \cdot \rangle_2$ , so

$$\begin{aligned}\|u\|_2^2 / \|u\|_1^2 &= \frac{1}{4} \|w + W\|_2^2 \\ &= \frac{1}{4} (\|w\|_2^2 + \|W\|_2^2) \\ &= \frac{1}{4} \|w - W\|_2^2 = \|v\|_2^2 / \|v\|_1^2.\end{aligned}$$

Thus

$$\frac{u}{\|u\|_2} + \frac{v}{\|v\|_2} = \frac{\|u\|_1}{\|u\|_2} w \in V_m$$

and

$$\frac{u}{\|u\|_2} - \frac{v}{\|v\|_2} = \frac{\|u\|_1}{\|u\|_2} W \in V_m$$

and so  $(u, v) \in E_2$ . □

Having two inner products, the space  $V$  has two differing notions of the angle between vectors. Our main result provides a comparison between these angles in terms of the quantities  $m$  and  $M$  defined in (5.9).

**Theorem 5.8.** *Let  $V$  be a real or complex vector space equipped with inner products  $\langle \cdot, \cdot \rangle_1$  and  $\langle \cdot, \cdot \rangle_2$ . Let (5.9) hold. For independent vectors  $u$  and  $v$  in  $V$  let  $\varphi$  and  $\psi$  be defined by,  $0 \leq \varphi \leq \pi$ ,  $0 \leq \psi \leq \pi$ ,*

$$\cos \varphi = \frac{\operatorname{Re} \langle u, v \rangle_1}{\|u\|_1 \|v\|_1} \quad \text{and} \quad \cos \psi = \frac{\operatorname{Re} \langle u, v \rangle_2}{\|u\|_2 \|v\|_2}.$$

Then

$$(m/M) \tan(\varphi/2) \leq \tan(\psi/2) \leq (M/m) \tan(\varphi/2). \quad (5.10)$$

Equality holds in the right-hand inequality if and only if  $(u, v) \in E$ . Equality holds in the left-hand inequality if and only if  $(u, -v) \in E$ .

*Proof.* First consider the case that  $V$  is a real vector space. Note that the assumption of independence ensures  $0 < \varphi < \pi$  and  $0 < \psi < \pi$ .

By Corollary 5.6, the span of  $u$  and  $v$  has a basis  $\{b, B\}$  that is orthogonal with respect to both inner products. Without loss of generality we may assume that  $\|b\|_1 = \|B\|_1 = 1$ . For notational convenience, set  $n = \|b\|_2$  and  $N = \|B\|_2$  and suppose, by interchanging  $b$  and  $B$  if necessary, that  $n \leq N$ . Note that the definitions of  $m$  and  $M$  ensure that  $m \leq n$  and

$N \leq M$ . Write  $u = u_b b + u_B B$  and  $v = v_b b + v_B B$  for some real numbers  $u_b, u_B, v_b,$  and  $v_B$ . In terms of these coordinates we have,

$$\begin{aligned}\|u\|_1^2 \|v\|_1^2 \sin^2 \varphi &= \|u\|_1^2 \|v\|_1^2 - \langle u, v \rangle_1^2 \\ &= (u_b^2 + u_B^2)(v_b^2 + v_B^2) - (u_b v_b + u_B v_B)^2 \\ &= (u_b v_B - u_B v_b)^2\end{aligned}$$

and

$$\begin{aligned}\|u\|_2^2 \|v\|_2^2 \sin^2 \psi &= \|u\|_2^2 \|v\|_2^2 - \langle u, v \rangle_2^2 \\ &= (n^2 u_b^2 + N^2 u_B^2)(n^2 v_b^2 + N^2 v_B^2) - (n^2 u_b v_b + N^2 u_B v_B)^2 \\ &= n^2 N^2 (u_b v_B - u_B v_b)^2.\end{aligned}$$

Thus,

$$\|u\|_2 \|v\|_2 \sin \psi = nN \|u\|_1 \|v\|_1 \sin \varphi. \quad (5.11)$$

The derivative of

$$g(x) = (u_b^2 + x u_B^2)^{1/2} (v_b^2 + x v_B^2)^{1/2} + (u_b v_b + x u_B v_B)$$

is

$$g'(x) = \frac{1}{2} \left( u_B \left( \frac{v_b^2 + x v_B^2}{u_b^2 + x u_B^2} \right)^{1/4} + v_B \left( \frac{u_b^2 + x u_B^2}{v_b^2 + x v_B^2} \right)^{1/4} \right)^2 \geq 0,$$

so  $g(1) \leq g(N^2/n^2)$ . Multiplying both sides of this by  $n^2$  gives,

$$n^2 \|u\|_1 \|v\|_1 (1 + \cos \varphi) \leq \|u\|_2 \|v\|_2 (1 + \cos \psi). \quad (5.12)$$

Combining (5.11) and (5.12) gives,

$$\tan(\psi/2) = \frac{\sin \psi}{(1 + \cos \psi)} \leq \frac{nN \sin \varphi}{n^2 (1 + \cos \varphi)} = (N/n) \tan(\varphi/2), \quad (5.13)$$

with equality if and only if  $g'(x) = 0$  for  $x \in (1, N^2/n^2)$ . Since  $m \leq n \leq N \leq M$ , (5.13) proves the right-hand inequality of (5.10).

If equality holds in the right-hand inequality of (5.10), then equality holds in (5.13) and  $n = m, N = M, b \in V_m,$  and  $B \in V_M$ . If  $m = M$  then  $V_m = V_M = V$  and  $\varphi = \psi$  so the last two

statements of the theorem are trivial. Otherwise, equality in (5.13) implies that  $g'$  is zero on the non-trivial interval  $(1, M^2/m^2)$ . That is,

$$u_B \left( \frac{v_b^2 + xv_B^2}{u_b^2 + xu_B^2} \right)^{1/4} + v_B \left( \frac{u_b^2 + xu_B^2}{v_b^2 + xv_B^2} \right)^{1/4} = 0$$

and hence  $u_B^2 v_b^2 = v_B^2 u_b^2$ . Since  $u$  and  $v$  are independent, both  $u_B$  and  $v_B$  are non-zero, they have opposite signs, and  $u_B v_b = -v_B u_b$ . Therefore,

$$\begin{aligned} \frac{u}{\|u\|_1} + \frac{v}{\|v\|_1} &= \frac{u_b b + u_B B}{\sqrt{u_b^2 + u_B^2}} + \frac{v_b b + v_B B}{\sqrt{v_b^2 + v_B^2}} \\ &= \pm \left( \frac{(u_b/u_B)b + B}{\sqrt{(u_b/u_B)^2 + 1}} - \frac{(v_b/v_B)b + B}{\sqrt{(v_b/v_B)^2 + 1}} \right) \\ &= \pm \frac{2(u_b/u_B)b}{\sqrt{(u_b/u_B)^2 + 1}} \in V_m \end{aligned}$$

and

$$\begin{aligned} \frac{u}{\|u\|_1} - \frac{v}{\|v\|_1} &= \frac{u_b b + u_B B}{\sqrt{u_b^2 + u_B^2}} - \frac{v_b b + v_B B}{\sqrt{v_b^2 + v_B^2}} \\ &= \pm \left( \frac{(u_b/u_B)b + B}{\sqrt{(u_b/u_B)^2 + 1}} + \frac{(v_b/v_B)b + B}{\sqrt{(v_b/v_B)^2 + 1}} \right) \\ &= \pm \frac{2B}{\sqrt{(u_b/u_B)^2 + 1}} \in V_M. \end{aligned}$$

That is,  $(u, v) \in E_1 = E$ .

Conversely, suppose that  $(u, v) \in E$ , set

$$w = \frac{u}{\|u\|_1} + \frac{v}{\|v\|_1} \in V_m$$

and

$$W = \frac{u}{\|u\|_1} - \frac{v}{\|v\|_1} \in V_M,$$

and observe that  $w + W$  is in the direction of  $u$  and  $w - W$  is in the direction of  $v$ . By Lemma 5.5,  $w$  and  $W$  are orthogonal with respect to both inner products. Thus,

$$\begin{aligned} \cos \varphi &= \frac{\langle w + W, w - W \rangle_1}{\|w + W\|_1 \|w - W\|_1} \\ &= \frac{\|w\|_1^2 - \|W\|_1^2}{\|w\|_1^2 + \|W\|_1^2} \end{aligned}$$

and

$$\tan^2(\varphi/2) = \frac{1 - \cos \varphi}{1 + \cos \varphi} = \frac{\|W\|_1^2}{\|w\|_1^2}.$$

A similar calculation yields the corresponding formula for  $\psi$  and leads to the conclusion,

$$\tan^2(\psi/2) = \frac{\|W\|_2^2}{\|w\|_2^2} = \frac{M^2 \|W\|_1^2}{m^2 \|w\|_1^2} = (M/m)^2 \tan^2(\varphi/2).$$

Taking square roots establishes equality in the right-hand inequality of (5.10).

Applying the right-hand inequality of (5.10) to the vectors  $u$  and  $-v$  replaces  $\varphi$  by  $\pi - \varphi$  and  $\psi$  by  $\pi - \psi$  to give the conclusion,

$$\begin{aligned} \cot(\psi/2) &= \tan(\pi/2 - \psi/2) \\ &\leq (M/m) \tan(\pi/2 - \varphi/2) = (M/m) \cot(\varphi/2). \end{aligned}$$

This proves the left-hand inequality of (5.10), with equality if and only if  $(u, -v) \in E$ . This completes the proof in the case that  $V$  is a real vector space.

If  $V$  is a complex space and  $\langle \cdot, \cdot \rangle_1$  and  $\langle \cdot, \cdot \rangle_2$  are complex inner products, the conclusion of the theorem follows by applying the result just proved to the real vector space  $V_{\mathbb{R}}$  equipped with the real inner products  $\operatorname{Re}\langle \cdot, \cdot \rangle_1$  and  $\operatorname{Re}\langle \cdot, \cdot \rangle_2$ . This completes the proof.  $\square$

The angle between two subsets of  $V$  is defined as an infimum of angles between pairs of vectors. The inequality (5.10) remains valid when we take an infimum of all three terms so we have the following result. Note that since the cosine function is decreasing, the cosine of an infimum of angles is achieved by taking the supremum of their cosines.

**Corollary 5.9.** *Let  $V$  be a real or complex vector space equipped with inner products  $\langle \cdot, \cdot \rangle_1$  and  $\langle \cdot, \cdot \rangle_2$ . Let (5.9) hold. For  $S, T \subseteq V$ , each containing at least one non-zero vector, let  $\Phi$  and  $\Psi$  be the angles between the subsets  $S$  and  $T$  with respect to  $\langle \cdot, \cdot \rangle_1$  and  $\langle \cdot, \cdot \rangle_2$ , respectively. That is,  $0 \leq \Phi \leq \pi$ ,  $0 \leq \Psi \leq \pi$ ,*

$$\cos \Phi = \sup_{\substack{0 \neq u \in S \\ 0 \neq v \in T}} \frac{\operatorname{Re}\langle u, v \rangle_1}{\|u\|_1 \|v\|_1}, \quad \text{and} \quad \cos \Psi = \sup_{\substack{0 \neq u \in S \\ 0 \neq v \in T}} \frac{\operatorname{Re}\langle u, v \rangle_2}{\|u\|_2 \|v\|_2}. \quad (5.14)$$

Then

$$(m/M) \tan(\Phi/2) \leq \tan(\Psi/2) \leq (M/m) \tan(\Phi/2).$$



The following theorem is our version of the generalized Wielandt inequality in inner product spaces. As pointed out earlier, the angles between the (real or complex) lines determined by  $u$  and  $v$  are often taken as alternative definitions of the angle between vectors themselves. We show that with this definition the results of Theorem 5.8 still hold, but the conditions for equality become slightly more complicated.

**Theorem 5.10.** *Let  $V$  be a real or complex vector space equipped with inner products  $\langle \cdot, \cdot \rangle_1$  and  $\langle \cdot, \cdot \rangle_2$ . Let (5.9) hold. For independent vectors  $u$  and  $v$  in  $V$  let  $\Phi$  and  $\Psi$  be defined by,  $0 \leq \Phi \leq \pi/2$ ,  $0 \leq \Psi \leq \pi/2$ ,*

$$\cos \Phi = \frac{|\langle u, v \rangle_1|}{\|u\|_1 \|v\|_1} \quad \text{and} \quad \cos \Psi = \frac{|\langle u, v \rangle_2|}{\|u\|_2 \|v\|_2}.$$

*Then*

$$(m/M) \tan(\Phi/2) \leq \tan(\Psi/2) \leq (M/m) \tan(\Phi/2). \quad (5.15)$$

*Let  $\alpha_1$  and  $\alpha_2$  be solutions to  $|\langle u, v \rangle_1| = \alpha_1 \langle u, v \rangle_1$  and  $|\langle u, v \rangle_2| = \alpha_2 \langle u, v \rangle_2$ . Equality holds in the right-hand inequality of (5.15) if and only if  $(\alpha_1 u, v) \in E$  and either  $\alpha_1 = \alpha_2$  or  $\langle u, v \rangle_2 = 0$ . Equality holds in the left-hand inequality of (5.15) if and only if  $(\alpha_2 u, -v) \in E$  and either  $\alpha_1 = \alpha_2$  or  $\langle u, v \rangle_1 = 0$ .*

*Proof.* Apply Corollary 5.9 to the lines  $S = \mathbb{C}u$  and  $T = \mathbb{C}v$  ( $S = \mathbb{R}u$  and  $T = \mathbb{R}v$  in the real case) to obtain (5.15). By (5.8),  $\Phi$  is the angle between  $\alpha_1 u$  and  $v$  with respect to  $\langle \cdot, \cdot \rangle_1$  and  $\Psi$  is the angle between  $\alpha_2 u$  and  $v$  with respect to  $\langle \cdot, \cdot \rangle_2$ . To analyse the right-hand inequality of (5.15), let  $\theta$  be the angle between  $\alpha_1 u$  and  $v$  with respect to  $\langle \cdot, \cdot \rangle_2$ . The infimum definition of  $\Psi$  and Theorem 5.8 show that

$$\tan(\Psi/2) \leq \tan(\theta/2) \leq (M/m) \tan(\Phi/2). \quad (5.16)$$

By (5.8), the first of these is equality if and only if either  $\alpha_1 = \alpha_2$  or  $\langle u, v \rangle_2 = 0$ . By Theorem 5.8, the second is equality if and only if  $(\alpha_1 u, v) \in E$ . Thus equality holds in the right-hand inequality of (5.15) if and only if  $(\alpha_1 u, v) \in E$  and either  $\alpha_1 = \alpha_2$  or  $\langle u, v \rangle_2 = 0$ .

To analyse the left-hand inequality of (5.15), let  $\theta$  be the angle between  $\alpha_2 u$  and  $v$  with respect to  $\langle \cdot, \cdot \rangle_1$ . The infimum definition of  $\Phi$  and Theorem 5.8 show that

$$(m/M) \tan(\Phi/2) \leq (m/M) \tan(\theta/2) \leq \tan(\Psi/2). \quad (5.17)$$

By (5.8), the first of these is equality if and only if either  $\alpha_1 = \alpha_2$  or  $\langle u, v \rangle_1 = 0$ . By Theorem 5.8, the second is equality if and only if  $(\alpha_2 u, -v) \in E$ . Thus equality holds

in the left-hand inequality of (5.15) if and only if  $(\alpha_2 u, -v) \in E$  and either  $\alpha_1 = \alpha_2$  or  $\langle u, v \rangle_1 = 0$ .  $\square$

The inequalities (5.10) and (5.15) can be expressed in various equivalent forms. In terms of cosines (5.10) becomes, with  $\chi = (M^2 - m^2)/(M^2 + m^2)$ ,

$$\frac{-\chi + \cos \varphi}{1 - \chi \cos \varphi} \leq \cos \psi \leq \frac{\chi + \cos \varphi}{1 + \chi \cos \varphi}. \quad (5.18)$$

Replace  $\varphi$  and  $\psi$  by  $\Phi$  and  $\Psi$  to get the expression for (5.15). In terms of inner products instead of angles, the inequalities (5.10) of Theorem 5.8 and (5.15) of Theorem 5.10 become, in the case  $\|u\|_1 = \|v\|_1 = 1$ ,

$$\frac{-\chi + \operatorname{Re}\langle u, v \rangle_1}{1 - \chi \operatorname{Re}\langle u, v \rangle_1} \leq \frac{\operatorname{Re}\langle u, v \rangle_2}{\|u\|_2 \|v\|_2} \leq \frac{\chi + \operatorname{Re}\langle u, v \rangle_1}{1 + \chi \operatorname{Re}\langle u, v \rangle_1}. \quad (5.19)$$

and

$$\frac{-\chi + |\langle u, v \rangle_1|}{1 - \chi |\langle u, v \rangle_1|} \leq \frac{|\langle u, v \rangle_2|}{\|u\|_2 \|v\|_2} \leq \frac{\chi + |\langle u, v \rangle_1|}{1 + \chi |\langle u, v \rangle_1|}, \quad (5.20)$$

respectively.

The special case  $\Phi = \pi/2$  in Theorem 5.10 gives an inner product formulation of Wielandt's inequality that includes all cases of equality. Note that the right-hand inequality of (5.20) is equivalent to the left-hand inequality of (5.15).

**Corollary 5.11.** *Let  $V$  be a real or complex vector space equipped with inner products  $\langle \cdot, \cdot \rangle_1$  and  $\langle \cdot, \cdot \rangle_2$ . Let (5.9) hold. Suppose the non-zero vectors  $u, v \in V$  are orthogonal with respect to  $\langle \cdot, \cdot \rangle_1$  and  $\alpha$  satisfies  $|\langle u, v \rangle_2| = \alpha \langle u, v \rangle_2$ . Then,*

$$\frac{|\langle u, v \rangle_2|}{\|u\|_2 \|v\|_2} \leq \frac{M^2 - m^2}{M^2 + m^2} \quad (5.21)$$

with equality if and only if  $(\alpha u, -v) \in E$ .

The following theorem gives upper and lower bounds on the difference between the cosines of  $\varphi$  and  $\psi$ . It improves the estimates given in Theorems 1 and 2 of [32].

**Theorem 5.12.** *Let  $V$  be a real or complex vector space equipped with inner products  $\langle \cdot, \cdot \rangle_1$  and  $\langle \cdot, \cdot \rangle_2$ . Let (5.9) hold. For independent vectors  $u$  and  $v$  in  $V$ ,*

$$-2 \frac{M - m}{M + m} \leq \frac{\operatorname{Re}\langle u, v \rangle_2}{\|u\|_2 \|v\|_2} - \frac{\operatorname{Re}\langle u, v \rangle_1}{\|u\|_1 \|v\|_1} \leq 2 \frac{M - m}{M + m} \quad (5.22)$$

and, if  $\operatorname{Re}\langle u, v \rangle_1 \geq 0$ , then

$$\frac{\operatorname{Re}\langle u, v \rangle_2}{\|u\|_2\|v\|_2} - \frac{\operatorname{Re}\langle u, v \rangle_1}{\|u\|_1\|v\|_1} \leq \frac{M^2 - m^2}{M^2 + m^2}. \quad (5.23)$$

Also,

$$-\frac{M^2 - m^2}{M^2 + m^2} \leq \frac{|\langle u, v \rangle_2|}{\|u\|_2\|v\|_2} - \frac{|\langle u, v \rangle_1|}{\|u\|_1\|v\|_1} \leq \frac{M^2 - m^2}{M^2 + m^2}. \quad (5.24)$$

*Proof.* Suppose  $\varphi$  and  $\psi$  are the angles between  $u$  and  $v$  with respect to  $\langle \cdot, \cdot \rangle_1$  and  $\langle \cdot, \cdot \rangle_2$ . Since,

$$\cos \psi - \cos \varphi = 2/(1 + \tan^2(\psi/2)) - 2/(1 + \tan^2(\varphi/2)),$$

Theorem 5.8 gives

$$\frac{2}{1 + (M/m)^2 x} - \frac{2}{1 + x} \leq \cos \psi - \cos \varphi \leq \frac{2}{1 + (m/M)^2 x} - \frac{2}{1 + x},$$

where  $x = \tan^2(\varphi/2)$ . A little calculus shows that the minimum value, over all  $x \in [0, \infty]$ , of the expression on the left occurs at  $x = m/M$  and the maximum value, over all  $x \in [0, \infty]$ , of the expression on the right occurs at  $x = M/m$ . This gives (5.22). If  $\operatorname{Re}\langle u, v \rangle_1 \geq 0$  then  $\varphi \leq \pi/2$  and so  $x = \tan^2(\varphi/2) \leq 1$ . The maximum value on the right now occurs at  $x = 1$ , giving (5.23).

The same analysis, applied to the angles  $\Phi$  and  $\Psi$  between the lines  $\mathbb{C}u$  and  $\mathbb{C}v$  (or  $\mathbb{R}u$  and  $\mathbb{R}v$  in the real case) includes the restriction  $\tan^2(\Phi/2) \leq 1$  and gives the right-hand inequality in (5.24). The left-hand inequality follows from the right-hand one by interchanging the inner products  $\langle \cdot, \cdot \rangle_1$  and  $\langle \cdot, \cdot \rangle_2$ . Besides interchanging the angles  $\varphi$  and  $\psi$ , this has the effect of replacing  $m$  by  $1/M$  and  $M$  by  $1/m$  to give

$$\frac{|\langle u, v \rangle_1|}{\|u\|_1\|v\|_1} - \frac{|\langle u, v \rangle_2|}{\|u\|_2\|v\|_2} \leq \frac{(1/m)^2 - (1/M)^2}{(1/m)^2 + (1/M)^2} = \frac{M^2 - m^2}{M^2 + m^2}.$$

Multiplying through by  $-1$  completes the proof.  $\square$

In our notation, Dragomir's results from [32] are

$$1 - \frac{M^2}{m^2} \leq \frac{|\langle u, v \rangle_2|}{\|u\|_2\|v\|_2} - \frac{|\langle u, v \rangle_1|}{\|u\|_1\|v\|_1} \leq 1 - \frac{m^2}{M^2},$$

and, if  $\operatorname{Re}\langle u, v \rangle_1 \geq 0$ , then

$$1 - \frac{M^2}{m^2} \leq \frac{\operatorname{Re}\langle u, v \rangle_2}{\|u\|_2 \|v\|_2} - \frac{\operatorname{Re}\langle u, v \rangle_1}{\|u\|_1 \|v\|_1} \leq 1 - \frac{m^2}{M^2}.$$

Since

$$1 - \frac{M^2}{m^2} \leq -2 \frac{M-m}{M+m} \leq -\frac{M^2-m^2}{M^2+m^2} \quad \text{and} \quad \frac{M^2-m^2}{M^2+m^2} \leq 1 - \frac{m^2}{M^2},$$

Theorem 5.12 improves on both of these statements.

The estimate (5.22), on the difference between the cosines of  $\varphi$  and  $\psi$  readily gives a lower bound on the product of those cosines.

**Corollary 5.13.** *Let  $V$  be a real or complex vector space equipped with inner products  $\langle \cdot, \cdot \rangle_1$  and  $\langle \cdot, \cdot \rangle_2$ . Let (5.9) hold. For independent vectors  $u$  and  $v$  in  $V$ ,*

$$\frac{\operatorname{Re}\langle u, v \rangle_1}{\|u\|_1 \|v\|_1} \frac{\operatorname{Re}\langle u, v \rangle_2}{\|u\|_2 \|v\|_2} \geq -\left(\frac{M-m}{M+m}\right)^2. \quad (5.25)$$

*Proof.* Let  $\mu = (M-m)/(M+m)$ ,

$$x = \frac{\operatorname{Re}\langle u, v \rangle_1}{\|u\|_1 \|v\|_1}, \quad \text{and} \quad y = \frac{\operatorname{Re}\langle u, v \rangle_2}{\|u\|_2 \|v\|_2}.$$

Note that  $0 \leq \mu < 1$ . By the Cauchy-Schwarz inequality and (5.22), the point  $(x, y)$  lies in the region defined by  $-1 \leq x \leq 1$ ,  $-1 \leq y \leq 1$ , and  $-2\mu \leq x - y \leq 2\mu$ . Minimizing  $xy$  over this hexagonal region easily yields  $(x, y) = (-\mu, \mu)$  or  $(x, y) = (\mu, -\mu)$ . Thus,  $xy \geq -\mu^2$  as required.  $\square$

## 5.4 Formulation in terms of matrices

Recall that the angle  $\theta$  between vectors  $x, y \in \mathbb{C}^n$  is defined by  $0 \leq \theta \leq \pi$  and

$$\cos \theta = \frac{\operatorname{Re} y^* x}{\|x\| \|y\|}$$

and the angle  $\Theta$  between the complex lines  $\mathbb{C}x$  and  $\mathbb{C}y$  satisfies  $0 \leq \Theta \leq \pi/2$  and

$$\cos \Theta = \frac{|y^* x|}{\|x\| \|y\|},$$

in contrast to the terminology “real-part angle” and “Hermitian angle” used in Chapter 1.

Let  $A$  be an invertible  $n \times n$  matrix and consider the two inner products

$$\langle x, y \rangle_1 = y^* x \quad \text{and} \quad \langle x, y \rangle_2 = (Ay)^*(Ax) \quad (5.26)$$

on  $\mathbb{C}^n$ . Then the definitions in (5.9) show that  $M = \|A\|$  and  $1/m = \|A^{-1}\|$  so the condition number of  $A$  is  $\kappa(A) = M/m$ . Theorem 5.8 becomes the following.

**Theorem 5.14.** *Let  $A$  be an invertible  $n \times n$  matrix. For independent  $x, y \in \mathbb{C}^n$  let  $\varphi$  be the angle between  $x$  and  $y$  and let  $\psi$  be the angle between  $Ax$  and  $Ay$ . Then,*

$$\kappa(A)^{-1} \tan(\varphi/2) \leq \tan(\psi/2) \leq \kappa(A) \tan(\varphi/2).$$

*Let  $\lambda_n$  and  $\lambda_1$  denote the smallest and largest eigenvalues of  $A^*A$ . Then equality holds in the right-hand inequality above if and only if  $x/\|x\| + y/\|y\|$  is in the  $\lambda_n$ -eigenspace of  $A^*A$  and  $x/\|x\| - y/\|y\|$  is in the  $\lambda_1$ -eigenspace of  $A^*A$ . Also, equality holds in the left-hand inequality above if and only if  $x/\|x\| - y/\|y\|$  is in the  $\lambda_n$ -eigenspace of  $A^*A$  and  $x/\|x\| + y/\|y\|$  is in the  $\lambda_1$ -eigenspace of  $A^*A$ .*

Theorem 5.10 gives a concise reformulation of the generalized Wielandt inequality. Since  $\kappa(A) = \kappa(A^{-1})$ , the symmetry between the angles  $\Phi$  and  $\Psi$  is clear.

**Theorem 5.15.** *Let  $A$  be an invertible  $n \times n$  matrix. For independent  $x, y \in \mathbb{C}^n$  let  $\Phi$  be the angle between the complex lines  $\mathbb{C}x$  and  $\mathbb{C}y$  and let  $\Psi$  be the angle between the complex lines  $\mathbb{C}(Ax)$  and  $\mathbb{C}(Ay)$ . Then*

$$\kappa(A)^{-1} \tan(\Phi/2) \leq \tan(\Psi/2) \leq \kappa(A) \tan(\Phi/2).$$

It takes a bit of care to show the equivalence of this theorem with Theorem 5.4 because the angles  $\Phi$  and  $\Psi$  represent subtly different concepts in the two statements. In Theorem 5.15,  $\Phi$  and  $\Psi$  represent angles between given complex lines, while in Theorem 5.4 they represent bounds on those angles rather than the angles themselves. Also, one must apply Theorem 5.4 to  $A$  and to  $A^{-1}$  (or else to  $x, y$  and to  $x, -y$ ) to obtain both sides of the inequality above.

The conclusion of Theorems 5.14 and 5.15 may be rewritten as

$$\frac{-\chi + \cos \varphi}{1 - \chi \cos \varphi} \leq \cos \psi \leq \frac{\chi + \cos \varphi}{1 + \chi \cos \varphi}, \quad (5.27)$$

where  $\chi = (\kappa(A)^2 - 1)/(\kappa(A)^2 + 1)$ . (Of course,  $\varphi$  and  $\psi$  should be replaced by  $\Phi$  and  $\Psi$  when rewriting Theorem 5.15.)

We have omitted the characterization of the cases of equality in Theorem 5.15 but they can be readily obtained from Theorem 5.10. Conditions for equality in Theorem 5.8 are simpler than those in Theorem 5.10 because the former deals with angles between a single pair of vectors and the latter with an infimum of angles between vectors in two one-dimensional subspaces. To recognize when equality occurs in Theorem 5.8 one only has to consider the placement of the vectors  $u$  and  $v$  relative to the eigenspaces  $V_m$  and  $V_M$ . But equality in Theorem 5.10 requires that this infimum of angles be achieved for  $u$  and  $v$  in addition to requiring their correct placement with respect to these eigenspaces. In [63], Kolotilina gave the following characterization of the cases of equality in the generalized Wielandt inequality, without explicit recognition of this two-stage requirement. We give an alternative proof using Theorem 5.10. (Notice that the complex numbers  $\xi$  and  $\eta$  appearing in the Theorem of [63] are unnecessary as they may be absorbed into the eigenvectors  $x_1$  and  $x_n$ .)

**Proposition 5.16.** *Let  $B$  be an  $n \times n$  invertible Hermitian matrix, suppose  $\lambda_1 > \lambda_n > 0$  are its largest and smallest eigenvalues, respectively, and set  $\chi = (\lambda_1 - \lambda_n)/(\lambda_1 + \lambda_n)$ . Fix independent  $x, y \in \mathbb{C}^n$  and let  $\cos \varphi = |y^*x|/(\|x\|\|y\|)$ . Then*

$$|y^*Bx| = \frac{\chi + \cos \varphi}{1 + \chi \cos \varphi} \sqrt{x^*Bx} \sqrt{y^*By} \quad (5.28)$$

if and only if

$$\frac{x}{\|x\|} = \frac{1}{\sqrt{2}} (\sqrt{1 + \cos \varphi} x_1 + \sqrt{1 - \cos \varphi} x_n) \quad (5.29)$$

and

$$\frac{y}{\|y\|} = \frac{\varepsilon}{\sqrt{2}} (\sqrt{1 + \cos \varphi} x_1 - \sqrt{1 - \cos \varphi} x_n)$$

for some complex number  $\varepsilon$  of unit modulus and some unit eigenvectors  $x_1$  and  $x_n$  satisfying  $Bx_1 = \lambda_1 x_1$  and  $Bx_n = \lambda_n x_n$ .

*Proof.* With  $A = B^{1/2}$  we have  $B = A^*A$ . Apply Theorem 5.10 to the inner products (5.26) and note that  $M = \lambda_1$  and  $m = \lambda_n$  so  $V_M$  and  $V_m$  are the  $\lambda_1$ - and  $\lambda_n$ -eigenspaces of  $B$ , respectively. Using (5.18), we see that (5.28) is equivalent to equality in the left hand inequality of (5.15). Thus, Theorem 5.10 shows that (5.28) holds if and only if  $(\alpha_2 x, -y) \in$

$E$  and either  $\alpha_1 = \alpha_2$  or  $y^*x = 0$ . As in Theorem 5.10,  $|y^*x| = \alpha_1 y^*x$  and  $|(Ay)^*(Ax)| = \alpha_2 (Ay)^*(Ax)$ .

First suppose that  $x$  and  $y$  satisfy (5.29). A calculation, using the fact that  $x_1$  and  $x_n$  are orthogonal, shows that  $\varepsilon y^*x \geq 0$  and  $\varepsilon (Ay)^*(Ax) \geq 0$ . It follows that either  $\alpha_1 = \alpha_2 = \varepsilon$  or  $y^*x = 0$ . Also,

$$\frac{\varepsilon x}{\|\varepsilon x\|} + \frac{-y}{\|-y\|} = \sqrt{2\varepsilon} \sqrt{1 - \cos \varphi} x_n \in V_m$$

and

$$\frac{\varepsilon x}{\|\varepsilon x\|} - \frac{-y}{\|-y\|} = \sqrt{2\varepsilon} \sqrt{1 + \cos \varphi} x_1 \in V_M$$

so  $(\alpha_2 x, -y) \in E$ .

Conversely, suppose that  $(\alpha_2 x, -y) \in E$  and either  $\alpha_1 = \alpha_2$  or  $y^*x = 0$ . Set  $\varepsilon = \alpha_2$ . Then there exist  $w \in V_m$  and  $W \in V_M$  such that

$$\frac{\varepsilon x}{\|x\|} - \frac{y}{\|y\|} = w \quad \text{and} \quad \frac{\varepsilon x}{\|x\|} + \frac{y}{\|y\|} = W.$$

Since  $w$  and  $W$  are orthogonal, the parallelogram law gives  $\|W\|^2 + \|w\|^2 = 4$  and the definition of  $\varphi$  gives  $\|W\|^2 - \|w\|^2 = 4 \cos \varphi$ . Solving these two equations yields,  $\|W\| = \sqrt{2} \sqrt{1 + \cos \varphi}$  and  $\|w\| = \sqrt{2} \sqrt{1 - \cos \varphi}$ . With  $x_1 = \bar{\varepsilon} W / \|W\|$  and  $x_n = \bar{\varepsilon} w / \|w\|$  we have (5.29). This completes the proof.  $\square$

In Theorem 3 of [112], Yeh gave a different generalization of the Wielandt inequality for angles between complex lines. Here we show that Theorem 5.15 gives a stronger inequality.

**Theorem 5.17.** [112] *Let  $A$  be an invertible  $n \times n$  matrix. For independent  $x, y \in \mathbb{C}^n$  let  $\Phi$  be the angle between the complex lines  $\mathbb{C}x$  and  $\mathbb{C}y$  and let  $\Psi$  be the angle between the complex lines  $\mathbb{C}(Ax)$  and  $\mathbb{C}(Ay)$ . Define  $\theta$  by  $0 \leq \theta \leq \pi/2$  and  $\cot(\theta/2) = \kappa(A)$ . If  $\cos \Phi \leq 1/\kappa(A)^2$ , then*

$$\cos \Psi \leq \cos \theta + 2 \cos^2(\theta/2) \cos \Phi. \quad (5.30)$$

*Proof.* By Theorem 5.15 and (5.27), it is enough to show that

$$\frac{\chi + \cos \Phi}{1 + \chi \cos \Phi} \leq \cos \theta + (1 + \cos \theta) \cos \Phi,$$

where

$$\chi = \frac{\kappa(A)^2 - 1}{\kappa(A)^2 + 1} = \frac{\cot^2(\theta/2) - 1}{\cot^2(\theta/2) + 1} = \cos \theta.$$

But both  $\chi$  and  $\cos \Phi$  are positive, so

$$\frac{\chi + \cos \Phi}{1 + \chi \cos \Phi} \leq \chi + \cos \Phi \leq \chi + (1 + \chi) \cos \Phi$$

as required.  $\square$

In Theorem 3.1 of [111], Yan generalized the Wielandt inequality for real symmetric matrices as follows.

**Theorem 5.18.** [111] *Let  $B$  be a real  $n \times n$  symmetric positive definite matrix with eigenvalues  $\lambda_1 \geq \lambda_2 \cdots \geq \lambda_n > 0$ . For independent  $x, y \in \mathbb{R}^n$  define  $\Phi$  by  $0 \leq \Phi \leq \pi/2$  and  $\|x\|\|y\| \cos \Phi = |y^T x|$ . Then,*

$$|x^T B y| \leq \left( \max_{i,j} \frac{\lambda_i \cos^2(\Phi/2) - \lambda_j \sin^2(\Phi/2)}{\lambda_i \cos^2(\Phi/2) + \lambda_j \sin^2(\Phi/2)} \right) \sqrt{x^T B x} \sqrt{y^T B y}. \quad (5.31)$$

It was left as a conjecture in [111] that the theorem remains true for complex vectors  $x$  and  $y$  and a positive definite Hermitian matrix  $B$ .

It is routine to verify that the expression

$$\frac{s \cos^2(\Phi/2) - t \sin^2(\Phi/2)}{s \cos^2(\Phi/2) + t \sin^2(\Phi/2)}$$

is increasing in  $s$  and decreasing in  $t$ . Thus, the maximum in (5.31) is achieved when  $i = 1$  and  $j = n$ , where it takes the value,

$$\frac{\lambda_1 \cos^2(\Phi/2) - \lambda_n \sin^2(\Phi/2)}{\lambda_1 \cos^2(\Phi/2) + \lambda_n \sin^2(\Phi/2)} = \frac{\chi + \cos \Phi}{1 + \chi \cos \Phi}.$$

Here  $\chi = (\lambda_1/\lambda_n - 1)/(\lambda_1/\lambda_n + 1)$ . If  $A = B^{1/2}$ , then  $\kappa(A)^2 = \kappa(B) = \lambda_1/\lambda_n$  so Theorem 5.15 and (5.27) implies that Theorem 5.18 holds in both the real and complex cases, confirming Yan's conjecture.

We end with an improvement of Lemma 4.7, see also [73, Lemma 2.2]. It follows directly from Corollary 5.13 with  $\langle x, y \rangle_1 = y^T A x$  and  $\langle x, y \rangle_2 = y^T B x$ .

**Lemma 5.19.** *Suppose  $A$  and  $B$  are real symmetric positive definite  $n \times n$  matrices and let  $\kappa = \kappa(A^{-1/2} B A^{-1/2})$ . Then for  $x, y \in \mathbb{R}^n$  with  $y \neq 0$ ,*

$$\frac{y^T A x}{\sqrt{x^T A x} \sqrt{y^T A y}} \frac{y^T B x}{\sqrt{x^T B x} \sqrt{y^T B y}} \geq - \left( \frac{\sqrt{\kappa} - 1}{\sqrt{\kappa} + 1} \right)^2.$$



The above inequality followed by the AM-GM inequality give the conclusion of Lemma 4.7:

$$\begin{aligned} 2 \frac{y^T A x}{y^T A y} \frac{y^T B x}{y^T B y} &\geq -2 \left( \frac{\sqrt{\kappa} - 1}{\sqrt{\kappa} + 1} \right)^2 \left( \frac{x^T A x}{y^T A y} \frac{x^T B x}{y^T B y} \right)^{1/2} \\ &\geq - \left( \frac{\sqrt{\kappa} - 1}{\sqrt{\kappa} + 1} \right)^2 \left( \frac{x^T A x}{y^T A y} + \frac{x^T B x}{y^T B y} \right). \end{aligned}$$

# Chapter 6

## Summary

In this chapter, we summarize the main contributions of the thesis. An open problem is included at the end.

In Chapter 2, we presented two new proofs to an analogue of Kreĭn's inequality. Also, we extended an inequality of Wang & Zhang. This is published in [72].

In Chapter 3, independent of Hiroshima's work in [49], Wolkowicz and I proved an eigenvalue majorization inequality for 2-by-2 block positive semidefinite block matrices. It is worth mentioning we bring in a new line of proof. This is published in [75]. As an application, we proved a trace inequality conjectured by Furuichi and then further extended this inequality. This is joint work with Furuichi, [40]. Various norm inequalities and eigenvalue inequalities are derived using a decomposition lemma due to Bourin and Lee. This is published in [20, 21] and is joint work with Bourin and Lee.

Further majorization inequalities for coneigenvalues are presented in Chapter 3, including a new notion: consingular value. This is published in [29]. It is joint work with De Sterck.

In Chapter 4, we gave a condition for the convexity of the product of positive definite quadratic forms. When the number of positive definite quadratic forms is two, the condition is also necessary. It is shown in [117, 118] that the convexity of a function is important in finding the explicit expression of the transform for certain functions. This is joint work with Sinnamon, [73].

In Chapter 5, a new version of the generalized Wielandt inequality was formulated and

proved, leading to improvements of several results on matrix theory, including a resolution of a conjecture of Yan. As an interesting application of the generalized Wielandt inequality, we showed that it could be used in a very elegant way, to derive a sufficient condition for the convexity of the product of positive definite quadratic forms. This is published in [74] and is joint work with Sinnamon.

In conclusion, we formulate the following open problem for future investigation.

**Open Problem 1.** *We know from Example 4.11 that (4.2) in Theorem 4.3 is only a sufficient condition for the convexity of the product of  $m$  quadratic forms. What is a necessary and sufficient condition? Is there a necessary and sufficient condition in terms of the omega condition number introduced in [30]?*

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# Index

- absolute value, 27
- canonical angles, 14
- Cauchy-Schwarz inequality, 9
- complex symmetric, 62
- condition number, 70, 81
- coneigenvalues, 62, 64, 67
- conjugate-normal matrices, 62
- consingular value, 67
- decomposition lemma, 33
- deviation, 11
- direct sum, 20
- doubly stochastic matrix, 19
- doubly substochastic matrix, 19
- eigenvalues, 20
- Euclidean angle, 5
- Fan, 21, 59
- Fan dominance theorem, 34
- field of values, 39
- gap, 15
- generalized Wielandt inequality, 80
- geodesic, 28
- geometric mean, 27
- Golden-Thompson inequality, 31
- Gram matrix, 7
- Hadamard gate, 35
- Hermitian angle, 5
- Hermitian part, 34
- Hessian matrix, 70
- Kantorovich function, 72
- Kantorovich inequality, 77
- Kreĭn's inequality, 7
- Ky Fan  $k$ -norms, 34
- Löwner partial order, 27
- LF-conjugate, 69–71
- Lidskii, 22, 59
- Lieb-Thirring inequality, 31
- log-majorization, 62
- majorization, 18
- maximal characterization, 28
- minimal angle, 15
- normal matrix, 55
- numerical radius, 39
- numerical range, 39
- operator norm, 34
- real-part angle, 5
- Riemannian geometry, 27
- Rotfel'd, 22
- Schatten  $p$ -norms, 34
- Schur, 21

Schur decomposition lemma, 56  
Schur inequality, 66  
singular values, 68  
skew-Hermitian part, 35  
skew-symmetric, 62  
spectrum, 39  
symmetric norms, 34

Thompson, 22

unitarily invariant, 34  
unitary congruence, 34  
unitary matrices, 62

Wielandt inequality, 77